CS B551, Fall 2016, Probability practice problems (1)

1. You have a red die, a white die, and a blue die. The red and white dice are "fair" in that the probability of any given number being rolled is one-sixth. The blue die is biased so that the probability of rolling a 1 is one-half while the probability of rolling each of the other numbers is one-tenth.

   (a) You roll the blue die. What is the probability of rolling an even number?
   $P(2) + P(4) + P(6) = 0.1 + 0.1 + 0.1 = 0.3$

   (b) You roll the red and white dice. What is the probability of rolling "snake eyes" (two ones)?
   $\left(\frac{1}{6}\right)\left(\frac{1}{6}\right) = \frac{1}{36}$

   (c) You roll the red and white dice. What is the probability of rolling an odd number on the red die and an even number on the white die?
   $\left(\frac{3}{6}\right)\left(\frac{3}{6}\right) = \frac{9}{36} = \frac{1}{4}$

   (d) You roll the red and white dice. What is the probability that at least one die shows an even number?

   Let R be the event that the red die is even, and W be the event that the white die is even.

$$\begin{aligned} P(R \cup W) &= P(R) + P(W) - P(R \cap W) \\ &= \frac{3}{6} + \frac{3}{6} - \left(\frac{3}{6}\right)\left(\frac{3}{6}\right) \\ &= \frac{27}{36} = \frac{3}{4} \end{aligned}$$

   (e) You roll the red and white dice. What is the probability that at least one die is 5, given that the sum of the dice is an odd number?

   Let F be the event that at least one die is five. There are eleven possible outcomes in F. Let O be the event that the sum is odd. There are eighteen outcomes in O. There are six outcomes in the intersection of F and O, $F \cap O = \{(2,5), (4,5), (5,6), (5,2), (5,4), (6,5)\}$

$$\begin{aligned} P(F|O) &= \frac{P(F,O)}{P(O)} \\ &= \frac{\frac{6}{36}}{\frac{18}{36}} = \frac{1}{3} \end{aligned}$$

(f) You roll all three dice. What is the probability that the sum of the red and blue dice is 7, given that the white die is odd?

Of the 36 possible outcomes, six have a sum of 7: (1,6), (2,5), (3,4), (4,3), (5,2), (6,1). The outcome (6,1) has probability $\frac{1}{6} \times \frac{1}{2}$, while the other five outcomes have probability $\frac{1}{6} \times \frac{1}{10}$. Thus

$$P(\text{sum of red and blue is 7}|\text{white odd}) = \frac{1}{12} \times 5\frac{1}{60} = \frac{1}{6}.$$

(The outcome of the white die is independent from the sum of the other dice.)

(g) You roll all three dice. What is the probability that all three show the same number?

Of the $6^3 = 216$ possible outcomes, six have the same number on all three die. The outcome $(1,1,1)$ has probability $\frac{1}{6} \times \frac{1}{6} \times \frac{1}{2} = \frac{1}{72}$, while the other five outcomes have probability $\frac{1}{6} \times \frac{1}{6} \times \frac{1}{10} = \frac{1}{360}$. Thus $P(\text{all three die the same}) = \frac{1}{72} + \frac{5}{360} = \frac{1}{36}$.

2. Suppose that you have a friend, Mary, who lives in Seattle. She tells the truth 80% of the time and lies the other 20% of the time. The weather in Seattle on any given day is either sunny or cloudy; 30% of days are sunny and 70% are cloudy.

(a) Mary calls you one day and says that the weather in Seattle is cloudy. What is the probability that the weather is actually cloudy?

Let C and S denote the events that the weather is cloudy and sunny, respectively, and let MC and MS denote the events that Mary reports that the weather is cloudy and sunny, respectively. We are interested in finding the probability that it is actually cloudy given that Marys says it is cloudy, $P(C|MC)$. By Bayes' Law,

$$P(C|MC) = \frac{P(MC|C)P(C)}{P(MC)}.$$

We know that $P(C) = 0.7$ from the problem statement. The probability that Mary says it is cloudy given that it actually is cloudy is just the probability that she tells the truth, $P(MC|C) = 0.8$. The probability that Mary says cloudy is given by,

$$P(MC) = P(MC|C)P(C) + P(MC|S)P(S) = (0.8)(0.7) + (0.2)(0.3) = 0.62.$$

Substituting these values into the expression for $P(C|MC)$,

$$\begin{aligned} P(C|MC) &= \frac{P(MC|C)P(C)}{P(MC)} \\ &= \frac{(0.8)(0.7)}{0.62} \approx 90.3 \end{aligned}$$

(b) Suppose that you have another friend in Seattle, Sue, who tells the truth 90% of the time. Sue tells you that the weather is cloudy, while on the same day Mary says that it is sunny. What is the probability that the weather is actually sunny? You may assume that whether or not Sue lies is independent of whether or not Mary lies.

In addition to events C, S, MC, and MS, defined as in the last problem, let SC and SS denote the events that Sue says cloudy and sunny, respectively. From Bayes' Law we have,

$$P(S|MS, SC) = \frac{P(MS, SC|S)P(S)}{P(MS, SC)}.$$

The assumption that Mary and Sue decide whether or not to lie independently lets us factor $P(MS, SC|S)$ as,

$$P(MS, SC|S) = P(MS|S)P(SC|S) = (0.8)(0.1) = 0.08.$$

We can write the denominator as $P(MS, SC) = P(MS, SC|S) + P(MS, SC|C)$, and then further factor each term using the independence assumption,
$P(MS, SC) = P(MS|S)P(SC|S)P(S) + P(MS|C)P(SC|C)P(C) = (0.8)(0.1)(0.3) + (0.2)(0.9)(0.7) = 0.15$.
Substituting these values into the equation for Bayes' Law gives the answer,

$$P(S|MS, SC) = \frac{P(MS, SC|S)P(S)}{P(MS, SC)} = \frac{(0.08)(0.3)}{0.15} = 0.16.$$

3. The *Powerball* is a weekly multi-state lottery featuring a large jackpot of typically several tens of millions of dollars. Each Wednesday at 10:59PM, the lottery selects 5 white balls from a drum with 55 white balls, numbered 1 through 55. They also select 1 red ball from a drum of 42 red balls, numbered 1 through 42. To win the jackpot, a player must correctly guess the numbers on the five white balls, in any order, and the number on the red ball. There is a second prize of $200,000 for correctly guessing

the numbers on the five white balls but incorrectly guess the number on the red ball. Assume that the lottery chooses the balls uniformly at random without replacement. That is, it is not possible for two white balls of the same number to be selected during the same week.

(a) If you played the Powerball once, what is your probability of winning at least second prize?

Let J be the event that you win the jackpot. Let S be the event you win second place but not the jackpot. Let W be the event that you win at least second place.

$$\begin{aligned} P(W) &= P(J \cup S) \\ &= P(J) + P(S) - P(J \cap S) \\ &= \frac{1}{\binom{55}{5}} \times \frac{1}{42} + \frac{1}{\binom{55}{5}} \times \frac{41}{42} \\ &= \frac{1}{\binom{55}{5}} \approx 2.87 \times 10^{-7} \end{aligned}$$

(b) If you play the Powerball two weeks in a row, what is the probability that the first week you win no prize, but the second week you win the jackpot?

$$\begin{aligned} P(\text{no prize first week, jackpot second week}) &= (1 - P(W)) \times P(J) \\ &= \left(1 - \frac{1}{\binom{55}{5}}\right) \times \frac{1}{\binom{55}{5}} \times \frac{1}{42} \\ &\approx 6.84 \times 10^{-9} \end{aligned}$$

(c) If you play the Powerball three weeks in a row, what is the probability that you will win second prize at least once, but not the jackpot?

Let N_i denote the event that you win nothing on the i-th week, and let S_i denote the event that you win second prize but not the jackpot on the i-th week. Then there are seven ways of winning second prize at least once while never winning the jackpot: $S_1N_2N_3$, $N_1S_2N_3$, $N_1N_2S_3$, $S_1S_2N_3$, $S_1N_2S_3$, $N_1S_2S_3$, $S_1S_2S_3$. In three of these outcomes you win second prize exactly once, in three of the outcomes you win second prize exactly twice, and in one of the outcomes you win second prize exactly three times, so the probability of winning second prize at least once but never winning the jackpot is,

$$= 3P(S)P(N)^2 + 3P(N)P(S)^2 + P(S)^3$$

$$
\begin{aligned}
&= (3)\frac{41}{42\binom{55}{5}}\left(1-\frac{1}{\binom{55}{5}}\right)^2 + (3)\left(\frac{41}{42\binom{55}{5}}\right)^2\left(1-\frac{1}{\binom{55}{5}}\right) + \left(\frac{41}{42\binom{55}{5}}\right)^3 \\
&\approx 8.42 \times 10^{-7}
\end{aligned}
$$

(d) If you play the Powerball every week for a year, what is the probability that you will win the jackpot at least once? (Assume there are 52 weeks in a year.)

Either you win the jackpot at least once during the year, or you win the jackpot zero times. Thus we can compute the probability of winning at least once by calculating the probability of never winning the jackpot. Let L denote the event that you lose the jackpot on a given week.

$$P(L) = 1 - P(J) = 1 - \frac{1}{\binom{55}{5}} \times \frac{1}{42}$$

$$
\begin{aligned}
P(\text{win jackpot at least once}) &= 1 - P(L)^{52} = 1 - \frac{1}{\binom{55}{5}} \times \frac{1}{42} \\
&\approx 3.559 \times 10^{-7}
\end{aligned}
$$

4. Suppose someone receives roughly equal amounts of spam (S) and non-spam ($\bar{S}$) e-mail, $p(S) = p(\bar{S}) = 0.5$. Suppose that in a corpus of 1000 spam messages, 250 messages contain the word `pharmacy`, 350 contain the word `offer`, 15 contain the word `program`, and 25 contain the word `research`. In a corpus of 1000 non-spam messages, 10 messages contain the word `pharmacy`, 25 contain the word `offer`, 100 contain the word `program`, and 200 contain the word `research`.

(a) For each word $w \in \{\text{pharmacy}, \text{offer}, \text{program}, \text{research}\}$, what are the probabilities $p(w|S)$ and $p(w|\bar{S})$?

$$
\begin{aligned}
P(\text{pharmacy}|S) = \frac{250}{1000} = 0.25, P(\text{pharmacy}|\bar{S}) = \frac{10}{1000} = 0.01 \\
P(\text{offer}|S) = \frac{350}{1000} = 0.35, P(\text{offer}|\bar{S}) = \frac{25}{1000} = 0.025 \\
P(\text{program}|S) = \frac{15}{1000} = 0.015, P(\text{program}|\bar{S}) = \frac{100}{1000} = 0.1 \\
P(\text{research}|S) = \frac{25}{1000} = 0.025, P(\text{research}|\bar{S}) = \frac{200}{1000} = 0.2
\end{aligned}
$$

(b) For each word $w \in \{\text{pharmacy}, \text{offer}, \text{program}, \text{research}\}$, what is the probability that an e-mail containing the word is spam?

Recall that $P(w) = P(w|S)P(S) + P(w|\bar{S})P(\bar{S})$. We know that $P(S) = 0.5$, so

$$\begin{aligned}
P(\text{pharmacy}) &= 0.25(0.5) + 0.01(0.5) = 0.13 \\
P(\text{offer}) &= 0.25(0.5) + 0.01(0.5) = 0.1875 \\
P(\text{program}) &= 0.25(0.5) + 0.01(0.5) = 0.0575 \\
P(\text{research}) &= 0.025(0.5) + 0.2(0.5) = 0.1125
\end{aligned}$$

$$\begin{aligned}
P(S|\text{pharmacy}) &= \frac{P(\text{pharmacy}|S)P(S)}{P(\text{pharmacy})} = \frac{(0.25)(0.5)}{0.13} \approx 0.961 \\
P(S|\text{offer}) &= \frac{P(\text{offer}|S)P(S)}{P(\text{offer})} = \frac{(0.35)(0.5)}{0.1875} \approx 0.933 \\
P(S|\text{program}) &= \frac{P(\text{program}|S)P(S)}{P(\text{program})} = \frac{(0.015)(0.5)}{0.0575} \approx 0.1304 \\
P(S|\text{research}) &= \frac{P(\text{research}|S)P(S)}{P(\text{research})} = \frac{(0.025)(0.5)}{0.1125} \approx 0.111
\end{aligned}$$

5. You come home one day to discover that your roommate has two children from a previous marriage, and that the extended family has decided to pay him (and you) a friendly visit. Having not yet met the children, you suppose that either child is equally likely to be male or female, and that the gender of one child is independent of the gender of the other. For each of the following cases, what is the probability that your roommate's second child is a boy?

Given the information so far, we know that there are four possible outcomes: the first (older) child is a boy and the second child is a boy (BB), the first child is a boy and the second is a girl (BG), the first is a girl and the second a boy (GB), or both are girls (GG). Each of these four outcomes is equally probable, because the probability of having a boy and the probability of having a girl are both 0.5.

(a) A boy walks into the room. Your roommate says, "That's my older child."

In this case we know the first child is a boy, so only two outcomes are possible now: BG and BB. The probability that the second child is a boy is thus 0.5.

(b) A boy walks into the room. Your roommate says, "That's one of my children."

In this case we know that at least one of the children is a boy, so there are three possible outcomes: BB, BG, and GB. In two of the three cases the second child is a boy, so the probability is $\frac{2}{3}$.

(c) Your roommate claims to come from a culture where male children are always introduced first, in descending order of age, and then female children are introduced. Your roommate beckons, "John, come here!" [Yes, John is a boy.]

In this case we again know that there is at least one boy, so the possible outcomes are BB, BG, and GB. In two of the three cases the second child is a boy, so the probability is $\frac{2}{3}$.

(d) You accompany your roommate to his first PTA meeting, where the principal asks everyone who has at least one son to raise their hand. Your roommate does so.

Once again there are three possible outcomes, BB, BG, and GB, and in two cases the second child is a boy, so the probability is $\frac{2}{3}$.