Quantiles

STAT-S520

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- These slides complement material from ISI Chapter 6

Quantile

→ continuous or discrete.

Let X be a random variable and $\alpha \in (0,1)$. Let $q = q(X;\alpha)$ a function such that:

$$P(X < q) \leq \alpha \qquad \text{and} \qquad P(X > q) \leq 1 - \alpha$$

then q is called the α-quantile of X.

For continuous random variables

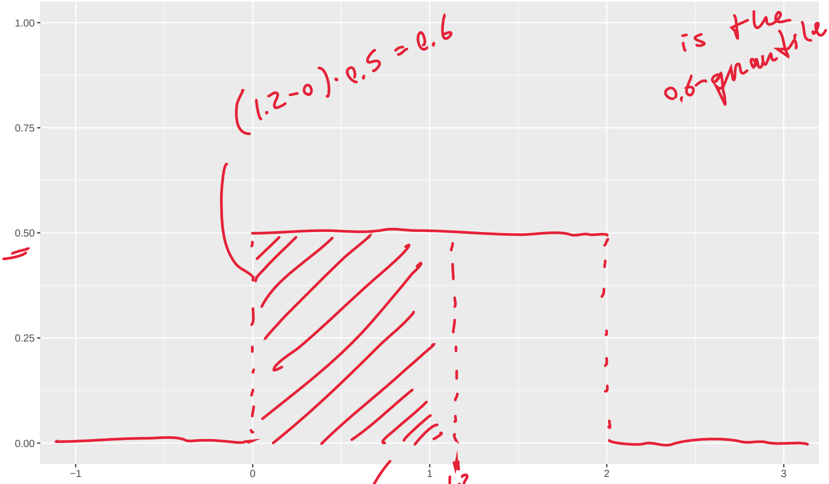
$$P(X < q) = \alpha \qquad P(X > q) = 1 - \alpha$$

Example 1:

α

Let $X \sim \mathit{Uniform}(0, 2)$, let's find the 0.6-quantile of X.

$q = 1.2$ is the 0.6-quantile of X

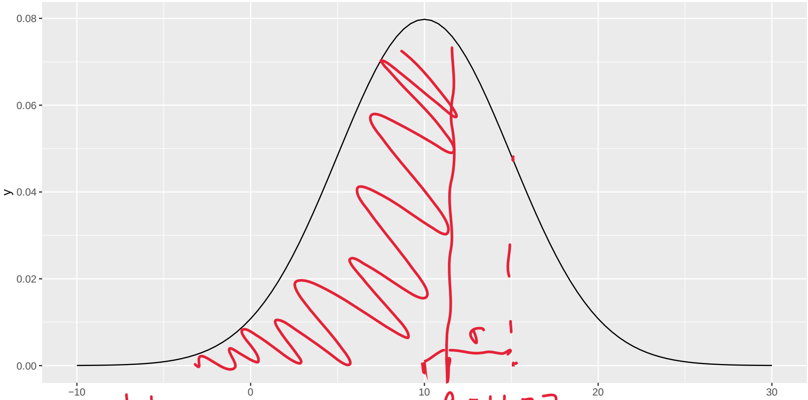


$$P(X < q) \le 0.6 \checkmark \qquad \underline{P(X > q)} \le 1 - 0.6 = 0.4 \checkmark$$

Example 2:

μ σ^2

Let $Y \sim Normal(10, 25)$, let's find the 0.6-quantile of Y. In R we use `qnorm()`.



q = 11,27

quantile.

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qnorm(p = 0.6, mean = 10, sd = sqrt(25))
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α   μ   σ

Example 3:

Let X be discrete with PMF

$$f(x) = \begin{cases} 0.4 & x = 1 \\ 0.4 & x = 2 \\ 0.2 & x = 3 \\ 0 & \text{otherwise} \end{cases}$$

Range of X

$X(S) = \{1, 2, 3\}$

What are the 0.6, 0.7, and 0.8-quantiles of X?

0.6-quantile -

$q = 1.5$ $\quad P(X < 1.5) = P(X = 1) = 0.4 \leq 0.6$ ✓

$P(X > 1.5) = P(X=2) + P(X=3) = 0.4 + 0.2 = 0.6 \leq \cancel{1-0.6}$ 0.4 ✗

$q = 2$ $\quad P(X < 2) = 0.4 \leq 0.6$ ✓ $\quad P(X > 2) = 0.2 \leq 0.4$ ✓

2 is a 0.6-quantile of X

0.7-quantile?

$q = 2$ $\quad P(X < 2) = 0.4 \leq 0.7$ ✓ $\quad P(X > 2) = 0.2 \leq 0.3$ ✓

2 is also a 0.7 quantile of X

0.8-quantile? $\quad q = 2$ $\quad q = 3$ $\quad P(X < q) = 0.8$ $\quad P(X > 3) = 0 \leq 0.2$

Both 2 and 3 are 0.8-quantiles

Note

- If the random variable is continuous, there is a single q for each α (one-to-one correspondence).
- If the random variable is discrete, the one-to-one breaks down for certain regions, for both q and α.

Commonly used terminology

- Quartiles: $\overset{\text{0.25-quantile}}{q_1(X)}$, $\overset{\text{0.5-quantile}}{q_2(X)}$, and $\overset{\text{0.75-quantile.}}{q_3(X)}$
  - They divide the range of X in four equal parts.
  - E.g., the first quartile is the 0.25-quantile
  - The median is the second quartile, $q_2(X)$ = 0.5-quantile.
- Percentiles: They divide the range of X in 100 equal parts
  - There are 99 percentiles: the 1st, 2nd, . . . , 99th.
  - E.g., the 57th percentile is the 0.57-quantile
- The interquartilerange (IQR) of X is

$$iqr(X) = \underbrace{q_3(X)}_{\text{0.75-quantile}} - \underbrace{q_1(X)}_{\text{0.25-quantile.}}$$

This is a measure of dispersion.

Example 4:

= 0.83-quantile.

Let $Y \sim Normal(10, 25)$, let's find the 83th percentile and the IQR of Y