Inference Part 2

STAT-S520

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- These slides complement material from ISI Chapter 9

Hypothesis Testing

- A claim about μ is made (null hypothesis) about the mean of the population
- A random sample from this population is obtained and used to obtain the sample mean $\bar{x}_n$
- Assuming the claim about μ is correct, we determine how likely is to observe a value as extreme as or more extreme than $\bar{x}_n$
  - If, getting $\bar{x}_n$ is very unlikely, we reject the claim (reject the null hypothesis)

Hypotheses

- The null hypothesis, H_0, is what we initially assume to be true. It's a statement about μ.
  - For procedural reasons, we **always** include the equal sign in the statement under H_0
- The alternative hypothesis, H_1, is what we would conclude (about μ) if we were to reject H_0.

ISI Example 9.3

$X_1, X_2, \ldots, X_{150} \sim P$

(Under H_0)

$E X_i = \mu = 15$

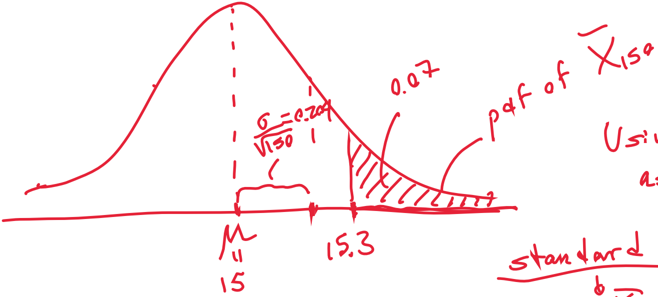
$\text{Var}\, X_i = \sigma^2$

$\bar{X}_{150} \,\dot\sim\, \text{Normal}\left(\mu, \frac{\sigma^2}{150}\right)$

$E \bar{X}_{150} = \mu$

$\text{Var}\, \bar{X}_{150} = \frac{\sigma^2}{150}$

$SD\, \bar{X}_{150} = \sqrt{\frac{\sigma^2}{150}} = \frac{\sigma}{\sqrt{150}}$



Using $s = 2.5$ as an estimate of σ

standard error of $\bar{X}_{150}$

$SD\, \bar{X}_{150} = \frac{s}{\sqrt{150}} = \frac{2.5}{\sqrt{150}} = 0.204$

Hypotheses:

Null hypothesis μ_0

$H_0: \mu \leq 15$

$H_1: \mu > 15$

Alternative hypothesis

City traffic office perspective.

$H_0: \mu \geq 15$

$H_1: \mu < 15$

parent's perspective

Aside:

$H_0: \mu = 15$

$H_1: \mu \neq 15$

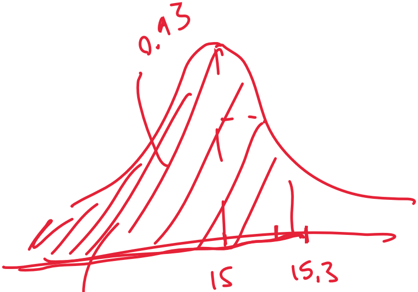
Parents' perspective

$H_0: \mu \geq 15$

$H_1: \mu < 15$

$\bar{x} = 15.3$

$s = 2.5$



pnorm(15.3, 15, 2.5/sqrt(150))

=

Test Statistic

The statistic is the method we use to assess the claim made under the null hypothesis. Many statistics exist, the one we encounter often looks like this:

$$\underline{statistic} = \frac{\overset{\bar{x}}{estimator} - \overbrace{parameter\ under\ H_0}^{\mu_0}}{\underbrace{standard\ error}_{SD\ \bar{X}_n}}$$

Hypothesis about μ when σ is known

When the hypothesis is made about μ, and σ is known, the test statistic is given by

$$Z = \frac{\bar{X}_n - \mu_0}{\sigma/\sqrt{n}}$$

and due to the CLT, $Z \sim N(0, 1)$.

Hypothesis about μ when σ is unknown

When σ is unknown (as in most real-life problems), the test statistic is given by

$$T = \frac{\bar{X}_n - \mu_0}{S_n / \sqrt{n}}$$

where $S_n = \sqrt{\frac{\sum (X_i - \bar{X}_n)^2}{n-1}}$ is the sample standard deviation, another estimator. Given the added uncertainty of S_n, T is no longer normal, but T follows a T-distribution with $n-1$ degrees of freedom and we write $T \sim T_{n-1}$

Observed test statistic

Once you collect a (random) sample of n observations, the sample observed is $\vec{x} = (x_1, \ldots, x_n)$. If σ is known, the observed test statistic is

$$\boxed{z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}}$$

and if σ is unknown, the observed test statistic is

$$\boxed{t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}}$$

where $s = \sqrt{\frac{\sum(x_i - \bar{x})^2}{n-1}}$ is the sample standard deviation.

Back to the example
office's perspective

$$t = \frac{15.3 - 15}{2.5/\sqrt{150}}$$

Significance Probability (p-value)

- The p-value is the probability of observing a test statistic that is as extreme or more extreme than the one observed in our data, when we assume that H_0 is true.
- The p-value depends on the hypotheses statements made

Example

$1 - pt(\underline{t}, 150-1)$

Conclusion and Interpretation of results

- We want to reject H_0 if the observed estimate is highly unlikely to have been obtained by chance
  - The smaller the p-value, the more evidence to reject H_0.
- While how small p-value needs to be is somewhat arbitrary, it should be guided by how important is not to make a mistake by rejecting H_0 when it is actually true (Type I Error) or by failing to reject H_0 when it is actually false (Type II Error).

Reasonable ranges for the *p*-value

To guide you in the decision process, some reasonable values (although still arbitrary) can be:

- If p-value > 0.1, do not reject H_0.
- If p-value < 0.001, reject H_0.
- If $0.001 \leq p$-value ≤ 0.1, decide based on your own perception of this uncertainty (different people may make different decisions).
- Alternatively, come up with a significance level, α, such that if p-value $\leq \alpha$, we reject H_0, and if p-value $> \alpha$, we fail to reject H_0.
  - The choice of α should be set before collecting and/or observing the random sample.

ISI Example 9.3 (continued)