Random Variables 1

STAT-S520

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These slides complement ISI Section 3.5

Random variable

label

- A random variable (RV) is a function that assigns a real number to each outcome of an experiment.

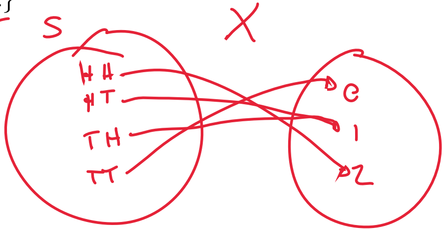
  object
  - We use uppercase letters, sometimes with sub-indices, to denote random variables; e.g., X, Y, X_1, Z_3 etc.
  - If X is a random variable we can write $X : S \to \mathbb{R}$

Example 1

- Experiment: Toss a fair coin twice and observe the top faces
- The sample space can be given by $S = \{HH, HT, TH, TT\}$
- Let X be a random variable that assigns to each outcome its total number of heads. We then have:
  - $X(HH) = 2$,
  - $X(HT) = 1$,
  - $X(TH) = 1$, and
  - $X(TT) = 0$

The range of a random variable

- The range of a random variable is the set of all the numbers assigned to each possible outcome
  - If X is a random variable with corresponding sample space S, we write $X(S)$ to denote the range of X
  - In our previous example $S = \{HH, HT, TH, TT\}$ and $X(S) = \{0, 1, 2\}$



Events and random variables

- We can define events based on random variables
- Let S be the sample space and X a random variable
- For any real number $y \in \mathbb{R}$ we can define

$$\underbrace{\{s \in S : X(s) \leq y\}}$$

- The expression above is an event (set of outcomes) which depends on y
  - A random variable requires that for any number y the event defined above has a well defined probability
  - This is always true if the sample space if finite

Exercise 1

Using the example above, recall that if we toss a fair coin twice, $S = \{HH, HT, TH, TT\}$, and X assigns the number of heads to each outcome. Let's find the probability for event

$$\{s \in S : X(s) \le y\}$$

for different values of y

- If $y = 10$ then $P\left(\overbrace{\{s \in S : X(s) \le 10\}}^{\{HH, HT, TH, TT\}}\right) = 1$
- If $y = -3$ then $P\left(\{s \in S : X(s) \le -3\}\right) = 0$ $\quad \to \emptyset$
- If $y = \underbrace{\pi \cdot (1/2)^2}_{\frac{\pi}{4}}$ then $P\left(\underbrace{\{s \in S : X(s) \le \pi \cdot (1/2)^2\}}\right) =$

$P(\{TT\}) = 0.25$

Cumulative distribution function (CDF)

The exercise above illustrates a very useful function. Let X be a random variable. The cumulative distribution function (CDF) of X,

$$\underline{F : \mathbb{R} \to [0, 1]}$$

is defined as

$$F(y) = \underbrace{\underline{P(\{s \in S : X(s) \leq y\})}}_{P(X \leq y)}$$

for any $y \in \mathbb{R}$

The probability that X assigns values that are less than or equal to y

Example 1 continued

We toss a fair coin twice, $S = \{HH, HT, TH, TT\}$, and X assigns the number of heads to each outcome. Then

- $F(10) =$ $P(X \leq y) = P(\{s \in S : X(s) \leq 10\}) = P(\{HH, HT, TH, TT\})$
  $= P(S) = 1$
- $F(-3) =$ 0
- $F(\pi \cdot (1/2)^2) =$ 0.25

$F(e) = 1$

≈ 2.7

$F(\sqrt{2}) = 0.75$

≈ 1.4

$F(\sqrt{3}) = 0.75$

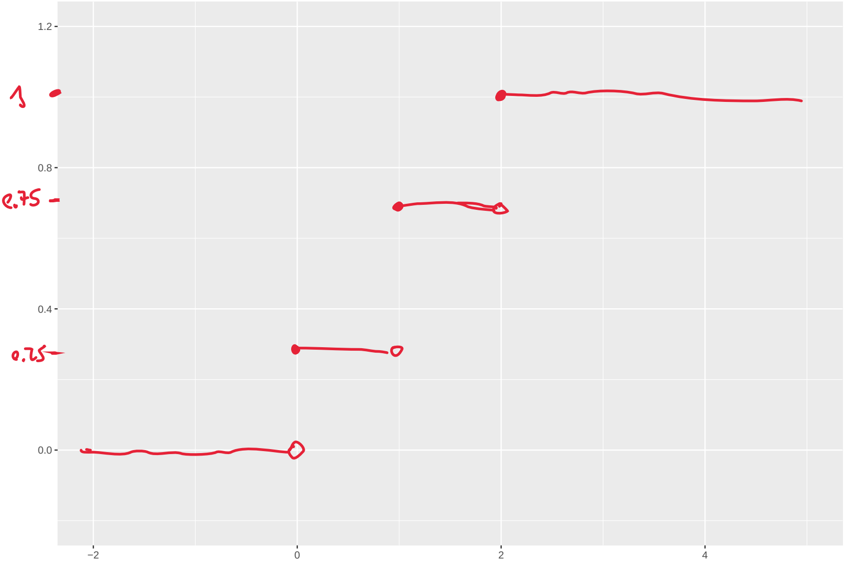
≈ 1.7

Example 1 continued: The function F for the coin example

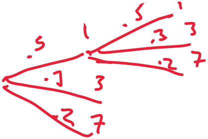
$$\{s \in S : X(s) \leq y\}$$

$$F(y) = \begin{cases} 0 & -\infty < y < 0 \\ 0.25 & 0 \leq y < 1 \\ 0.75 & 1 \leq y < 2 \\ 1 & 2 \leq y < \infty \end{cases}$$

Example 1 continued: The graph of F for the coin example



Exercise 1



$P(\{(1,1)\}) = 0{,}25$

- Experiment: Draw 2 tickets with replacement from the urn

$$[1, 1, 1, 1, 1, 3, 3, 3, 7, 7]$$

and let Y be the random variable that assigns the sum of both tickets

a. What is the sample space, S?
   - $S = \{(1,1), (1,3), (1,7), (3,1), \ldots\}$

b. What is the range of Y, $Y(S)$?
   - $Y(S) = \{2, 4, 6, 8, 10, 14\}$

c. If F_Y is the CDF of Y, what is $F(3\pi)$?
   - $F(3\pi) = P(Y \leq 3\pi) = P(\{(1,1), (1,3), (1,7), (3,1), (3,3), (7,1)\})$

     ≈ 9.4

     $= 0.25 + 0.3 + 0.09 + 0.2 = 0.84$