S520 Instructor's Solutions

Spring 2023 STAT-S 520

January 17th, 2023

1.

We'll assume that $S = \{s : s \in \mathbb{N}\}$ instead (as was really intended) then:

a. $(F \cap P) \cup (F \cap Q) = \{2, 3, 5, 13\} \cup \{0, 1\} = \{0, 1, 2, 3, 5, 13\}$

b. Using the property shown in part c (below):

$$\begin{aligned}
(F \cup P^c) \cap (F \cup Q^c) &= F \cup (P^c \cap Q^c) \\
&= F \cup (P \cup Q)^c \\
&= \{0, 1, 2, 3, 5, 8, 13\} \cup \{0, 1, 2, 3, 4, 5, 7, 9, 11, 13, 16, 17, 19\}^c \\
&= \{0, 1, 2, 3, 5, 8, 13\} \cup \{6, 8, 10, 12, 14, 15, 18, 20\} \\
&= \{0, 1, 2, 3, 5, 6, 8, 10, 12, 13, 14, 15, 18, 20\}
\end{aligned}$$

c.

$$\begin{aligned}
(P \cup Q)^c &= \{0, 1, 2, 3, 4, 5, 7, 9, 11, 13, 16, 17, 19\}^c \\
&= \{6, 8, 10, 12, 14, 15, 18, 20\} \\
P^c \cap Q^c &= \{0, 1, 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20\} \cap \{2, 3, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 17, 18, 19, 20\} \\
&= \{6, 8, 10, 12, 14, 15, 18, 20\}
\end{aligned}$$

If, instead, we assume $S = \{s : s \in \mathbb{R}\}$ as given in the problem then:

a. It's the same as before
b. $F \cup (P \cup Q)^c = \{s : s \notin W\}$ where $W = (F \cup (P \cup Q)^c)^c = \{4, 7, 9, 11, 16, 17, 19\}$
c. The sets needed are too messy to describe properly, so let's just show the result in general (this was not required for you to do). First we show that if $s \in (P \cup Q)^c$ then $s \in (P^c \cup Q^c)$. For any outcome $s \in (P \cup Q)^c \Rightarrow s \notin (P \cup Q)$ (complement) $\Rightarrow s \notin P$ and $s \notin Q$ (union) $\Rightarrow s \in P^c$ and $s \in Q^c$, respectively (complement) $\Rightarrow s \in P^c \cap Q^c$ (intersection). Second, we show that if $s \notin (P \cup Q)^c$ then $s \notin (P^c \cup Q^c)$. For any outcome $s \notin (P \cup Q)^c$ we get $s \in (P \cup Q)$ (complement) $\Rightarrow s \in P$ or $s \in Q$ or both. If $s \in P \Rightarrow s \notin P^c \Rightarrow s \notin P^c \cap Q^c$. A similar argument can be made if $s \in Q$. First and second arguments together make $(P \cup Q)^c \equiv P^c \cup Q^c$

2.

Let's provide two solutions.

First solution: A longer solution, but perhaps more intuitive for some of you, would be to get to find the number of series based on the total number of games. If we first work with series that A wins:

- If A wins in 4 games: 1 outcome only, AAAA.
- If A wins in 5 games, for example AANAA, the 5th game should be won by A, and from the previous 4, we get

$$\binom{4}{3} = \frac{4!}{3! \cdot 1!} = 4$$

- If A wins in 6 games, the 6th game should be won by A, and from the previous 5, we get

$$\binom{5}{3} = \frac{5!}{3! \cdot 2!} = 10$$

.
- For 7 games we get

$$\binom{6}{3} = \frac{6!}{3! \cdot 3!} = 20$$

So there are $1 + 4 + 10 + 20 = 35$ ways for A to win and $2 \cdot 35 = 70$ total possible series outcomes. Here is the problem solved in R:

```
choose(3,3); choose(4,3); choose(5,3); choose(6,3)
```

```
## [1] 1

## [1] 4

## [1] 10

## [1] 20
```

```
# or simply
choose(3:6,3)
```

```
## [1]  1  4 10 20
```

The total number of possible series outcomes is given by:

```
sum(choose(3:6,3))*2
```

```
## [1] 70
```

Second solution A simpler solution, but perhaps less intuitive, is to simply notice that all we need is to place four As in 7 slots, and the outcome would have been

$$\binom{7}{4} = \frac{7!}{4! \cdot 3!} = 35$$

We double this to account for those times that N wins, so $35 \cdot 2 = 70$. Here is the code in R:

```
# 2nd method
choose(7,4)
```

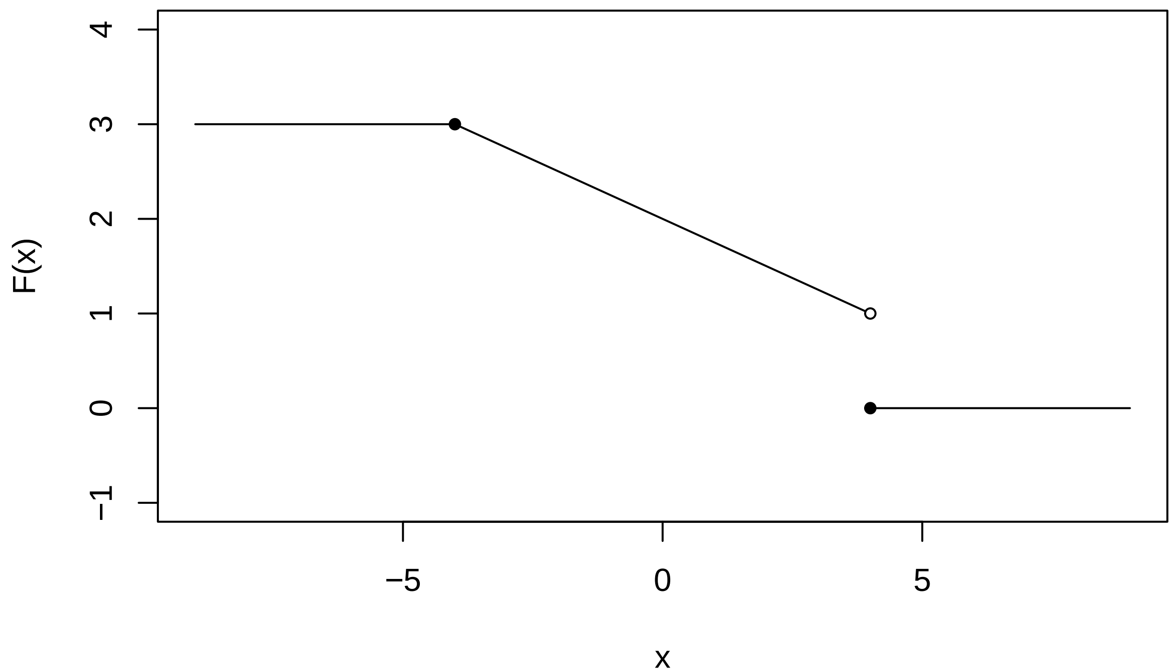
```
## [1] 35
```

```r
choose(7,4)*2
```

```
## [1] 70
```

3

a.



Domain: $\mathbb{R}$. Range: $\{0\} \cup (1, 3]$ so

$$F : \mathbb{R} \to \{0\} \cup (1, 3]$$

b.

$$F(y) = \begin{cases} 0 & -\infty < y < 2 \\ 0.5 & 2 \leq y < 3 \\ 1 & 3 \leq y < \infty \end{cases}$$

4

a.
$$\phi(6) = 4096$$

b.
$$\phi(-3) = \frac{1}{64}$$

c.
$$\phi(\mathbb{R}) = \{4^x : x \in \mathbb{R}\} = (0, \infty)$$

d.
$$\phi^{-1}(16) = 2$$

e.
$$\phi^{-1}(1/4) = -1$$

f.
$$\phi^{-1}([2, 32]) = \left[\frac{1}{2}, \frac{5}{2}\right]$$

5

a.
$$2^8 = 256$$

b.
$$\binom{8}{5} = 56$$

c.
$$\text{At least one head } = 2^8 - 1 = 255$$