Math Preliminaries

STAT-S520

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This slides complement material presented in ISI Ch2

Sets

- The universe is the collection of all objects of interest
  - It is denoted by S
- Set: A collection of objects. We use uppercase letters, e.g., $A, B, C, \ldots$
  - If A is a set, all the objects in A are in S, so $A \subset S$

Example 1

curly brackets

- $S = \{$ the natural numbers$\} = \mathbb{N} = \{x : x \in \mathbb{N}\}$

Some sets:

$A = \{1, 2, 5\}, B = \{5\}, C = \{$ odd numbers$\}$

Example 2

It's often easier to visualize the sets if there is an underlying experiment. For example, if the experiment is to roll a six sided fair die and we observe the top face, then

- $S = \{\text{ faces of a die }\} = \{⚀, ⚁, ⚂, ⚃, ⚄, ⚅\}$ $= \{1, 2, 3, 4, 5, 6\}$

While we could have used numbers here too, the objects can be represented differently. Some sets:

$A = \{⚀, ⚁, ⚄\}$, $B = \{⚄\}$,

$C = \{\text{ odd number of pips}\} = \{⚀, ⚂, ⚄\}$

Example 3

- $S = \{$ Marvel Movies $\}$

Some sets:

$A = \{$ Directed by Jon Favreau$\}$,

$B = \{$ Includes Spiderman$\}$,

$C = \{$ Box office, worldwide, was greater than 1 billion dollars $\}$

Some definitions

- If x is an object that is part of set A, we say that x is in A and write $x \in A$. Otherwise, x is not in A and we write $x \notin A$
  - The set of all the elements that are not in A is call the complement of A and we write A^c. So, $A^c = \{x : x \notin A\}$
- A set with no object is called the empty set and we write $\emptyset$

  $\emptyset$
  - Observe that $S^c = \emptyset$
- If all the object in A are also object in B, we say the A is a subset of B and write $A \subset B$
  - Observe that $A \subset S$ for any set A

More definitions

- A and B are disjoint or mutually exclusive if $A \cap B = \emptyset$
- By convention, for any sets A and B:

$$\emptyset \subset (A \cap B) \subset A \subset (A \cup B) \subset S$$

Common operations with sets

For any two sets A, B, these are common operations

- The union: $\underline{A \cup B} = \{x : x \in A \text{ or } x \in B\}$
- The intersection: $\underline{A \cap B} = \{x : x \in A \text{ and } x \in B\}$

Exercise 1

Let $S = \{x : x \in \mathbb{N}\}$, $A = \{ \text{ odd numbers}\}$,
$D = \{ \text{ square numbers}\}$, and $G = \{x : x \leq 12\}$. What is

- $D \cap G$ $= \{1, 4, 9\}$
- $A \cap (D \cap G)$ $= \{1, 9\}$
- $(A \cap G) \cup (D \cap G)$ $= \{1,3,5,7,9,11\} \cup \{1,4,9\} = \{1,3,4,5,7,9,11\}$
- $\underline{(D \cap G) \cup A^c}$ $= \{1,9\} \cup \{\text{ even numbers }\}$

Counting: Multiplication Principle

From ISI page 29:

Suppose that two decisions are to be made and that there are n_1 possible outcomes of the first decision. If, for each outcome of the first decision, there are n_2 possible outcomes of the second decision, then there are $n_1 \cdot n_2$ possible outcomes of the pair of decisions.

Exercise 2

Assume that 30 students want to be part of a committee of 3 people. Use the multiplication principle to determine the number of ways to form the committee in the following cases:

a. The committee needs 1 president, 1 vicepresident, and 1 secretary.
b. The committee needs 3 members without any given roles.
c. The committee needs a commitee chair and 2 additional members.

a) $\underline{30 \cdot 29 \cdot 28} \cdot \dfrac{27 \cdot 26 \cdots 3 \cdot 2 \cdot 1}{27 \cdot 26 \cdots 3 \cdot 2 \cdot 1} = \dfrac{30!}{(30-3)!}$

b) $\underline{30 \cdot 29 \cdot 28} \cdot \dfrac{1}{6} \cdot \dfrac{30!}{(30-3)} \cdot \dfrac{1}{3!} = \dfrac{30!}{(30-3)!\,3!}$

c) $30 \cdot \binom{29}{2}$

a, b, c
b, a, c
a, c, b
b, c, a
c, a, b
c, b, a

$\underline{3 \cdot 2 \cdot 1 = 6}$

Permutations

The number of permutations (ordered choices) of r objects from n objects is

$$P(n,r) = n \times (n-1) \times \cdots \times (n-r+1) = \frac{n!}{(n-r)!}$$

Combinations

The number of ~~permutations~~ combinations (unordered choices) of r objects from n objects is

Check also ISI examples 2.4 and 2.5

$$C(n,r) = \binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

n choose r

ISI 2.5 Exercises 6 and 7

6a) $\underline{1} \cdot \underline{1} \cdot \underline{1} \cdot \underline{1} = 1$

6b) $\underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} = 4! = 24$

7a) __ __ __ __ __

$10 \cdot 1 = 10$

$\underline{4}\ \underline{\ \ }\ \underline{4}\ \underline{\ \ }\ \underline{4}$

$\underline{4}\ \underline{4}\ \underline{4}\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }$

$$\binom{5}{3} \cdot \binom{2}{2}$$

$$\frac{5!}{3!2!}$$

7b) $\binom{5}{1}\binom{4}{2}\binom{2}{2} = \binom{5}{2}\binom{3}{2}\binom{1}{1}$

Functions

A function is a rule that assigns labels to objects

3 questions:

- What are the objects? Domain
- What are the labels? Image (Range)
- What is the assignment rule?

Notation

Function are denote by letters, often times Greek letters. Sometimes we

$$F : \mathbb{R} \to [0,1]$$

→ For each object there can only be one label

$(0,1)$

$\{0,1\}$

$0,1$

Example

Let the function $\phi : \mathbb{R} \to \mathbb{R}$, i.e. the *objects* are real numbers and the *labels* are also real numbers. The rule of assignment is given by

$$\phi(x) = 5x^2$$

So $\phi(3) = 45$ and $\phi(\sqrt{2}) = 10$

$5 \cdot 3^2$

$5(\sqrt{2})^2$

$\phi(-\sqrt{2}) = 5 \cdot (-\sqrt{2})^2 = 10$

Exercises (and definitions)

Inverse function:

- $\phi^{-1}(80) =$ $\{-4, 4\}$

Image of subset:

- $\phi([-1, 5]) =$ $[0, 5] \cup [0, 125]$
  $[0, 125]$

Inverse image:

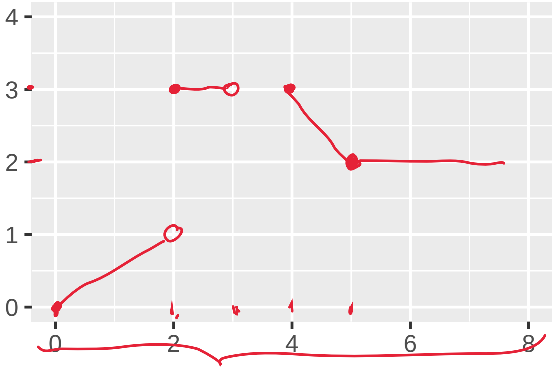
- $\phi^{-1}([5, 20]) =$ $[-2, -1] \cup [1, 2]$

Graphs of functions

Let's graph the function $f : \mathbb{R} \to \mathbb{R}$, given by:

$$f(x) = \begin{cases} \frac{1}{2}x & 0 \le x < 2 \\ 3 & 2 \le x < 3 \\ 7 - x & 4 \le x < 5 \\ 2\cancel{x} & x \ge 5 \end{cases}$$

$[0,1) \cup [2,3]$



$[0,3) \cup [4,\infty)$