Lecture 11: Market-clearing prices

Indiana University,

Emerson Melo

- Our goal: understanding why market-clearing prices must always exist.
- We are going to do this by taking an arbitrary set of buyer valuations, and describing a procedure that arrives at market-clearing prices.
- The method will in fact be a kind of auction, not a single-item auction, but a more general kind taking into account the fact that there are multiple things being auctioned, and multiple buyers with different valuations.

- This particular auction procedure was described by the economists Demange, Gale, and Sotomayor (1986), but its actually equivalent to a con- struction of market-clearing prices discovered by the Hungarian mathematician Egerváry seventy years earlier, in 1916

Definition

We define the neighbor set of S, denoted $N(S)$, to be the collection of all neighbors of S.

Definition

We say that a set S on the right-hand side is constricted if S is strictly larger than $N(S)$-that is, S contains strictly more nodes than $N(S)$ does.

Definition

We will call the seller or sellers that maximize the payoff for buyer j the preferred sellers of buyer j, provided the payoff from these sellers is not negative. We say that buyer j has no preferred seller if the payoffs $v_{ij} - p_i$ are negative for all choices of i.

- We note that the set of preferred sellers for bujer j, may be identified as a demand function:

$$D_j(p) = \{i \in S : v_{ij} - p_i = \max_{i' \in N(S)} \{v_{i'j} - p_{i'}\}\}$$

- Here's how the auction works.
- Initially all sellers set their prices to 0.
- Buyers react by choosing their preferred seller(s), and we look at the resulting preferred-seller graph.
- Otherwise and this is the key point there is a constricted set of buyers S.
- Consider the set of neighbors $N(S)$, which is a set of sellers.

- The buyers in S only want what the sellers in $N(S)$ have to sell, but there are fewer sellers in $N(S)$ than there are buyers in S.
- So the sellers in $N(S)$ are in high demand too many buyers are interested in them.
- They respond by each raising their prices by one unit, and the auction then continues.
- There is one more ingredient, which is a reduction operation on the prices. It will be useful to have our prices scaled so that the smallest one is 0.

- Thus, if we ever reach a point where all prices are strictly greater than 0-suppose the smallest price has value $p > 0$- then we reduce the prices by subtracting p from each one.
- This drops the lowest price to 0, and shifts all other prices by the same relative amount.
- A general round of the auction looks like what we've just described.

(i) At the start of each round, there is a current set of prices, with the smallest one equal to 0.

(ii) We construct the preferred-seller graph and check whether there is a perfect matching.

(iii) If there is, we are done: the current prices are market-clearing.

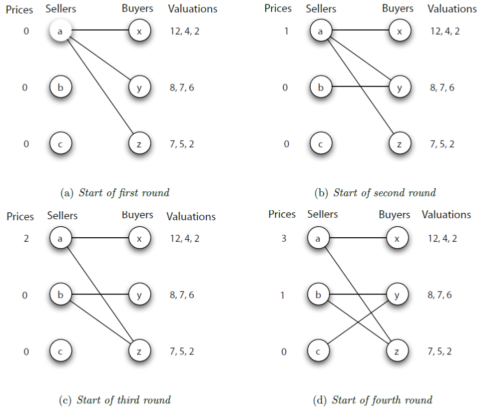
(iv) If not, we find a constricted set of buyers S and their neighbors $N(S)$.

(v) Each seller in $N(S)$ (simultaneously) raises his price by one unit.

(vi) If necessary, we reduce the prices-the same amount is subtracted from each price so that the smallest price becomes zero.

(vii) We now begin the next round of the auction, using these new prices.

- The example in Figure 10.6 illustrates two aspects of this auction that should be emphasized.
- First, in any round where the set of "*over-demanded*" sellers $N(S)$ consists of more than one individual, all the sellers in this set raise their prices simultaneously.
- For example, in the third round in Figure 10.6, the set $N(S)$ consists of both a and b, and so they both raise their prices so as to produce the prices used for the start of the fourth round.
- Second, while the auction procedure shown in Figure 10.6 produces the market-clearing prices shown in Figure 10.5(d), we know from Figure 10.5(b) that there can be other market-clearing prices for the same set of buyer valuations.



(a) *Start of first round*

(b) *Start of second round*

(c) *Start of third round*

(d) *Start of fourth round*

Figure 10.6: The auction procedure applied to the example from Figure 10.5. Each separate picture shows steps (i) and (ii) of successive rounds, in which the preferred-seller graph for that round is constructed.

- Here is a key property of the auction procedure we've defined: the only way it can come to end is if it reaches a set of market-clearing prices; otherwise, the rounds continue.
- So if we can show that the auction must come to an end for any set of buyer valuations-i.e. that the rounds cannot go on forever-then we've shown that market-clearing prices always exist.

Emerson Melo (emelo@iu.edu)

- It's not immediately clear, however, why the auction must always come to an end.
- Consider, for example, the sequence of steps the auction follows in Figure 10.6: prices change, different constricted sets form at diferent points in time, and eventually the auction stops with a set of market-clearing prices.
- But why should this happen in general? Why couldn't there be a set of valuations that cause the prices to constantly shift around so that some set of buyers is always constricted, and the auction never stops?

- In fact, the prices can't shift forever without stopping; the auction must always come to an end. '
- The way we're going to show this is by identifying a precise sense in which a certain kind of "potential energy" is draining out of the auction as it runs; since the auction starts with only a bounded supply of this potential energy at the beginning, it must eventually run out.

- Here is how we define this notion of potential energy precisely.
- For any current set of prices, define the *potential of a buyer* to be the maximum payoff she can currently get from any seller.
- This is the buyer's potential payoff; the buyer will actually get this payoff if the current prices are market-clearing prices.
- We also define the *potential of a seller* to be the current price he is charging.
- This is the seller's potential payoff; the seller will actually get this payoff if the current prices are market-clearing prices.
- Finally, we define the potential energy of the auction to be the sum of the potential of all participants, both buyers and sellers.

- How does the potential energy of the auction behave as we run it?
- It begins with all sellers having potential 0, and each buyer having a potential equal to her maximum valuation for any house-so the potential energy of the auction at the start is some whole number $P_0 \geq 0$.
- Also, notice that at the start of each round of the auction, everyone has potential at least 0.
- The sellers always have potential at least 0 since the prices are always at least 0.

- Because of the price-reduction step in every round, the lowest price is always 0, and therefore each buyer is always doing at least as well as the option of buying a 0-cost item, which gives a payoff of at least 0 (This also means that each buyer has at least one preferred seller at the start of each round.).
- Finally, since the potentials of the sellers and buyers are all at least 0 at the start of each round, so is the potential energy of the auction.

- Now, the potential only changes when the prices change, and this only happens in steps (v) and (vi).
- Notice that the reduction of prices, as defined above, does not change the potential energy of the auction: if we subtract p from each price, then the potential of each seller drops by p, but the potential of each buyer goes up by p-it all cancels out.
- Finally, what happens to the potential energy of the auction in step (v), when the sellers in $N(S)$ all raise their prices by one unit?

- Each of these sellers' potentials goes up by one unit.
- But the potential of each buyer in S goes down by one unit, since all their preferred houses just got more expensive.
- Since S has strictly more nodes than $N(S)$ does, this means that the potential energy of the auction goes down by at least one unit more than it goes up, so it strictly decreases by at least one unit.

- So what we've shown is that in each step that the auction runs, the potential energy of the auction decreases by at least one unit. It starts at some fixed value P_0, and it can't drop below 0, so the auction must come to an end within P_0 steps-and when it comes to an end, we have our market-clearing prices.

- There is a very natural way to view the single-item auction-both the outcome and the procedure itself-as a special case of the bipartite graph auction we've just defined. We can do this as follows.
- Suppose we have a set of n buyers and a single seller auctioning an item; let buyer j have valuation v_j for the item.

- To map this to our model based on perfect matchings, we need an equal number of buyers and sellers, but this is easily dealt with: we create $n-1$ "fake" additional sellers (who conceptually represent $n-1$ different ways to fail to acquire the item), and we give buyer j a valuation of 0 for the item offered by each of these fake sellers.
- With the real seller labeled 1, this means we have $v_{1j} = v_j$, the valuation of buyer j for the real item; and $v_{ij} = 0$ for larger values of i.

- Now we have a genuine instance of our bipartite graph model: from a perfect matching of buyers to sellers, we can see which buyer ends up paired with the real seller (this is the buyer who gets the item), and from a set of market-clearing prices, we will see what the real item sells for.
- Moreover, the price-raising procedure to produce market-clearing prices based on finding constricted sets-has a natural meaning here as well. The execution of the procedure on a simple example is shown in Figure 10.7.

- Initially, all buyers will identify the real seller as their preferred seller (assuming that they all have positive valuations for the item).
- The first constricted set S we find is the set of all buyers, and $N(S)$ is just the single real seller. Thus, the seller raises his price by one unit.
- This continues as long as at least two buyers have the real seller as their unique preferred seller: they form a constricted set S with $N(S)$ equal to the real seller, and this seller raises his price by a unit.
- The prices of the fake items remain fixed at 0 throughout the auction.

- Finally, when all but one buyer has identified other sellers as preferred sellers, the graph has a perfect matching. This happens at precisely the moment that the buyer with the second-highest valuation drops out, in other words, the buyer with the highest valuation gets the item, and pays the second-highest valuation.
- So the bipartite graph procedure precisely implements an ascending bid (English) auction.

(a) In the first round, all prices start at 0. The set of all buyers forms a constricted set S, with $N(S)$ equal to the seller a. So a raises his price by one unit and the auction continues to the second round.

(b) In the second round, the set of buyers consisting of x and z forms a constricted set S, with $N(S)$ again equal to the seller a. Seller a again raises his price by one unit and the auction continues to the third round. (Notice that in this round, we could alternately have identified the set of all buyers as a different constricted set S, in which case $N(S)$ would have been the set of sellers a and b. There is no problem with this-it just means that there can be multiple options for how to run the auction procedure in certain rounds, with any of these options leading to market-clearing prices when the auction comes to an end.)

(c) In the third round, the set of all buyers forms a constricted set S, with $N(S)$ equal to the set of two sellers a and b. So a and b simultaneously raise their prices by one unit each, and the auction continues to the fourth round.

(d) In the fourth round, when we build the preferred-seller graph, we find it contains a perfect matching. Hence, the current prices are market-clearing and the auction comes to an end.

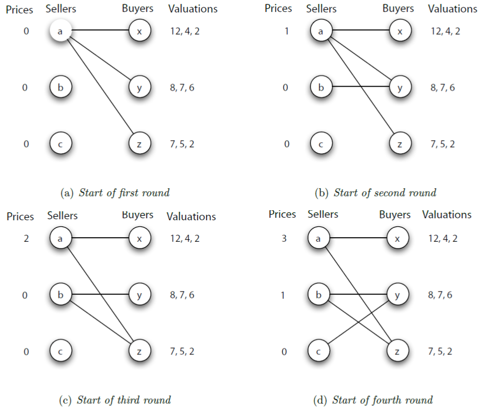


Figure 10.6: The auction procedure applied to the example from Figure 10.5. Each separate picture shows steps (i) and (ii) of successive rounds, in which the preferred-seller graph for that round is constructed.

(a) *Start of the Auction* (b) *End of the Auction*

Figure 10.7: A single-item auction can be represented by the bipartite graph model: the item is represented by one seller node, and then there are additional seller nodes for which all buyers have 0 valuation. (a) The start of the bipartite graph auction. (b) The end of the bipartite graph auction, when buyer x gets the item at the valuation of buyer y.

Theorem (Demange, Gale, and Sotomayor (1986))

Let p the price vector obtained from the multi-unit auction procedure and let q be any other competitive price. Then $p \leq q$.

Theorem (Demange, Gale, and Sotomayor (1986))

if p is the minimum competitive price then there is a matching μ^* such that (p, μ^*) is an equilibrium.