Lecture 12: The assignment game

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- There are two finite disjoint sets of players P and Q, containing m and n players, respectively.
- Members of P will sometimes be called P-agents and members of Q called Q-agents, and the letters i and j will be reserved for P and Q agents, respectively.
- Associated with each possible partnership $(i, j) \in P \times Q$ is a non-negative real number α_{ij}.

- A game in coalitional function form with sidepayments is determined by (P, Q, α), with the numbers α_{ij} being equal to the worth of the coalitions $\{i, j\}$ consisting of one P agent and one Q agent.
- The worth of larger coalitions is determined entirely by the worth of the pairwise combinations that the coalition members can form.

- That is, the coalitional function v is given;
$v(S) = \alpha_{ij}$ if $S = \{i, j\}$ for $i \in P$ and $j \in Q$;
$v(S) = 0$ if S contains only P agents or only Q agents; and
$v(S) = \max\{v(i_1, j_1) + v(i_2, j_2) + \cdots + v(i_k, j_k))\}$ for arbitrary coalitions S, with the maximum to be taken over all arrangements of $2k$ distinct players $i_1, i_2, \ldots, i_k$ belonging to S_P and $j_1, j_2, \ldots, j_k$ belonging to S_Q, where S_P and S_Q denote the sets of P and Q agents in S (i.e. the intersection of the coalition S with P and with Q), respectively.

- So the rules of the game are that any pair of agents $(i, j) \in P \times Q$ can together obtain α_{ij}, and any larger coalition is valuable only insofar as it can organize itself into such pairs.
- The members of any coalition may divide among themselves their collective worth in any way they like. An imputation of this game is thus a non-negative vector $(u, v) \in \mathbb{R}^m_+ \times \mathbb{R}^n_+$ such that $\sum_{i \in P} u_i + \sum_{j \in Q} v_j = v(P \bigcup Q)$.

- The easiest way to interpret this is to take the quantities α_{ij} to be amounts of money, and to assume that agents' preferences are concerned only with their monetary payoffs.
- We might think of P as a set of potential buyers of some objects offered for sale by the set Q of potential sellers, and each seller owns and each buyer wants exactly one indivisible object.

- If each seller j has a reservation price c_j, and each buyer i has a reservation price r_{ij} for object j, we may take α_{ij} to be the potential gains from trade between i and j.
- That is, $\alpha_{ij} = \max\{0, r_{ij} - c_j\}$.
- If buyer i buys object j from seller j at a price p, and if no other monetary transfers are made, the utilities are $u_i = r_{ij} - p$ and $v_j = p - c_j$.

- So, when no other monetary transfers are made, $u_i + v_j = \alpha_{ij}$ when i buys from j. But note that transfers between agents are not restricted to those between buyers and sellers; e.g. buyers may make transfers among themselves.
- We can also think of the P and Q agents as being firms and workers, etc.

- We look here at the simple case of one-to-one matching, with firms constrained to hire at most one worker.
- In such a case, the α_{ij}'s represent some measure of the joint productivity of the firm and worker, while transfers between a matched firm and worker represent salary.
- Transfers can also take place between workers (as when workers form a labor union in which the dues of employed members help pay unemployment benefits to unemployed members), or between firms.

- The maximization problem to determine $v(S)$ for a given matrix α is called an *assignment problem*, so games of this form are called *assignment games*.
- We will be particularly interested in the coalition $P \bigcup Q$, since $v(P \bigcup Q)$ is the maximum total payoff available to the players, and hence determines the Pareto set and the set of imputations.

- Consider the following linear programming (LP) problem P_1

$$\text{Maximize} \quad \sum_{i=1}^{m}\sum_{j=1}^{n} \alpha_{ij}x_{ij} \qquad (P_1)$$

$$\text{Subject to} \quad \sum_{i=1}^{n} x_{ij} = 1, \quad j = 1,\ldots,m \qquad (1)$$

$$\sum_{j=1}^{m} x_{ij} = 1, \quad i = 1,\ldots,n \qquad (2)$$

$$x_{ij} \geq 0, \quad i = \ldots,m, j = 1,\ldots,n. \qquad (3)$$

- We may interpret x_{ij} as, for example, the probability that a partnership (i, j) will form.
- Then the linear inequalities of type (1), one for each j in Q, say that the probability that j will be matched to some i cannot exceed 1.
- The inequalities of form (2), one for each $i \in P$, say the same about the probability that i will be matched.

- It can be shown that there exists a solution of this LP problem which involves only values of zero and one.
- Thus the fractions artificially introduced in the LP formulation disappear in the solution and the (continuous) LP problem is equivalent to the (discrete) assignment problem for the coalition of all players, that is, the determination of $v(P\bigcup Q)$.
- Then $v(P\bigcup Q) = \sum_{i,j} \alpha_{ij} x_{ij}$, where x is an optimal solution of the LP problem

Definition

A feasible assignment for (P, Q, α) is a matrix $x = (x_{ij})$ (of zeros and ones) that satisfies (1), (2) and (3) above. An optimal assignment is a feasible assignment x such that $\sum_{i,j} \alpha_{ij} x_{ij} \geq \sum_{i,j} \alpha_{ij} x'_{ij}$, for all feasible assignments x'.

- So if x is a feasible assignment, $x_{ij} = 1$ if i and j form a partnership and $x_{ij} = 0$ otherwise.
- If $\sum_{j=1}^{n} x_{ij} = 0$, then i is unassigned, and if $\sum_{i=1}^{m} x_{ij} = 0$, then j is likewise unassigned.
- A feasible assignment x corresponds exactly to a matching μ as in the definition of feasibility, with $\mu(i) = j$ if and only if $x_{ij} = 1$.

Definition

The pair of vectors (u,v), $u \in \mathbb{R}^m_+$ and $v \in \mathbb{R}^n_+$ is called a feasible payoff for (P,Q,α) if there is a feasible assignment x such that

$$\sum_{i \in P} u_i + \sum_{j \in Q} v_j = \sum_{i,j} \alpha_{ij} x_{ij}.$$

- In this case we say (u, v) and x are compatible with each other, and we call $((u, v); x)$ a feasible outcome. Note again that a feasible payoff vector may involve monetary transfers between agents who are not assigned to one another.

Definition

A feasible outcome $((u,v);x)$ is stable (or the payoff (u,v) with an assignment x is stable if

(i) $u_i \geq 0, v_j \geq 0$,

(ii) $u_i + v_j \geq \alpha_{ij}$ for all $(i,j) \in P \times Q$.

- Condition (i) (individual rationality) reflects that a player always has the option of remaining unmatched.
- Condition (ii) requires that the outcome cannot be improved by any pair: if (ii) is not satisfied for some agents i and j, then it would pay them to break up their present partnership(s) (either with one another or with other agents) and form a new partnership together, because this could give them each a higher payoff.

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- Similarly, we can say that a matching μ is stable if it is not blocked by any individual or any pairs of agents.

Lemma

Let $((u,v),x)$ be a stable outcome for (P,Q,α). Then

(i) $u_i + v_j = \alpha_{ij}$ for all pairs (i,j) such that x_{ij};

(ii) $u_i = 0$ for all unassigned i, and $v_j = 0$ for all unassigned j at x.

- The lemma implies that at a stable outcome, the only monetary transfers that occur are between P and Q agents who are matched to each other. (Note that this is an implication of stability, not an assumption of the model.)

- Now consider the LP problem P_1^* that is the dual of P_1, i.e. the LP problem of finding a pair of vectors (u, v), $u \in \mathbb{R}_+^m$, $v \in \mathbb{R}_+^n$, that minimizes the sum

$$\sum_{i \in P} u_i + \sum_{j \in Q} v_i$$

subject, for all $i \in P$ and $j \in Q$, to

(a^*) $u_i \geq 0, v_j \geq 0,$

(b^*) $u_i + v_j \geq \alpha_{ij}.$

- Because we know that P_1 has a solution, we know also that p_1^* must have an optimal solution.
- A fundamental duality theorem asserts that the objective functions of these dual LPs must attain the same value.
- That is, if x is an optimal assignment and (u, v) is a solution of P_1^*, we have that

$$\sum_{i \in P} u_i + \sum_{j \in Q} v_i = \sum_{P \times Q} \alpha_{ij} x_{ij} = v(P \bigcup Q). \tag{4}$$

- This means that $((u,v),x)$ is a feasible outcome.
- Moreover, $((u,v),x)$ is a stable outcome for (P,Q,α), since (a^*) ensures individual rationality and $u_i + v_j \geq \alpha_{ij}$ for all $(i,j) \in P \times Q$ by (b^*).
- It follows, by the definition of $v(S)$, that for any coalition $S = S_P \bigcup S_Q$, where S_P e is contained in P and S_Q in Q,

$$\sum_{i \in S_P} u_i + \sum_{j \in S_Q} v_j \geq v(S_P \bigcup S_Q). \tag{5}$$

- But (4) and (5) are exactly how the core of the game is determined: (4) ensures the feasibility of (u, v) and (5) ensures its nonimprovability by any coalition.
- Conversely, any payoff vector in the core, i.e. satisfying (4) and (5), satisfies the conditions for a solution to P_1^*.
- Hence we have shown

Theorem (Shapley and Shubik (1972))

Let (P, Q, α) be an assignment game. Then

(a) the set of stable outcomes and the core of (P, Q, α) are the same;

(b) the core of (P, Q, α) is the (nonempty) set of solutions of the dual LP of the corresponding assignment problem.

Corollary

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References

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