Lecture 6: Games

Indiana University,

Emerson Melo

Definition

A strategic game consists of

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- For each player, a set of actions
- For each player, payoffs over the set of action profiles.

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Players choose their actions **simultaneously**
We are working with the notion of **normal form games**

Figure: Payoff Matrix for the prisoner's dilemma.

		Player 2	
		C	D
Player 1	C	2 2	3 0
	D	0 3	1 1

Example

Two people wish to go out together. Two concerts are available: one of music by Bach (B), and one of music by Stravinsky (S). One person prefers Bach and the other prefers Stravinsky. If they go to different concerts, each of them is equally unhappy listening to the music of either composer.

Figure: Payoff Matrix for Bach or Stravinsky.

Player 2

		B	S
Player 1	B	1 2	0 0
	S	0 0	2 1

Example

Two people choose, simultaneously, whether to show the Head (H) or the Tail (T) of a coin. If they show the same side, person 2 pays person 1 a dollar; if they show different sides, person 1 pays person 2 a dollar. Each person cares only about the amount of money she receives, and prefers to receive more than less.

Figure: Payoff Matrix for the matching pennies game.

Player 2

Player 1	H	T
H	1, -1	-1, 1
T	-1, 1	1, -1

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In this course we will use the notion of **Nash equilibrium** (and its extensions).

Intuitively a Nash equilibrium consists of two components:

1. Each player chooses her action according to the model of rational choice, given her belief about the other players' actions.
2. Every player's belief about the other players' actions is correct.

Informally, a Nash equilibrium can be defined as:

A Nash equilibrium is an action profile a^* with the property that no player i do better by choosing an action different from a_i^*, given that every other player j adheres to a_j^* .

Definition (Nash equilibrium)

The action profile a^* in a strategic game is a Nash equilibrium if, for every player i and every action a_i of player i, a^* is at least as good according to player i's preferences as the action profile (a_i, a^*_{-i}) in which player i chooses a_i while every other player j chooses a^*_j. Equivalently, for every player i,

$$u_i(a^*_i, a^*_{-i}) \geq u_i(a_i, a^*_{-i}) \quad \text{for every action } a_i \text{ if player i,}$$

where u_i is a payoff function that represents player i's preferences.

- In order to solve a game we need to find a procedure.
- The idea is to define a mathematical object which captures players' incentives.
- We will introduce the notion of **best response functions**.

Definition

The best response function for player i is defined by

$$B_i(a_{-i}) = \{a_i \in A_i : u(a_i, a_{-i}) \geq u_i(a'_i, a_{-i}) \quad \forall a'_i \in A_i\}$$

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$$B_i(a_{-i}) = \{a_i \in A_i : u(a_i, a_{-i}) \geq u_i(a_i', a_{-i}) \quad \forall a_i' \in A_i\}$$

Any action in $B_i(a_{-i})$ is at least as good for player i as every other action of player i when the other players' actions are given by a_{-i}.

- The best response map B_i is a set valued map.
- Every member of the set $B_i(a_{-i})$ is a best response of player i to action a_{-i}.
- We use the functions B_i's to find the Nash equilibria of the game.

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Player 2

Player 1	H	T
H	1, -1	-1, 1
T	-1, 1	1, -1

The functions B_i's help us to implement the following procedure:

- Find the best response functions of each player
- Find the action that satisfies

$$a_i^* \in B_i(a_{-i}^*).$$

- Consider a two-player game.

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- The best response functions may be defined as:

$$B_1 : A_2 \longrightarrow A_1$$

and

$$B_2 : A_1 \longrightarrow A_2$$

Then if $a^* = (a_1^*, a_2^*)$ we have:

$$a_1^* \in B_1(a_2^*)$$

and

$$a_2^* \in B_2(a_1^*)$$

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- $a^* = (a_1^*, \ldots, a_n^*)$ is Nash Equilibrium iff $a^* = B(a^*)$.
- In other words, $a^* = (a_1^*, \ldots, a_n^*)$ is a Nash equilibrium iff

$$a_i^* = B_i(a_{-i}^*) \quad \forall i.$$

Definition

The pure strategy profile $a^* = (a_1^*, a_2^*)$ is a Nash Equilibrium iff a_i^* is a best response to a_{-i}^* for $i = 1, 2$, that is

$$u_1(a_1^*, a_2^*) \geq u_1(a_1', a_2^*) \quad \forall a_1' \in A_1,$$

and

$$u_2(a_1^*, a_2^*) \geq u_1(a_1^*, a_2') \quad \forall a_2' \in A_2,$$

Definition

The pure strategy profile $a^* = (a_1^*, \ldots, a_n^*)$ is a Nash Equilibrium iff a_i^* is a best response to a_{-i}^* for all i, that is

$$u_i(a_i^*, a_{-i}^*) \geq u_i(a_i', a_{-i}^*) \quad \forall a_i' \in A_i.$$

Proposition

The action profile a^* is a Nash equilibrium of a strategic game if and only if every player's action is a best response to the other players' actions:

$$a_i^* = B_i(a_{-i}^*) \quad \text{for every player } i.$$

- The standard approach, developed by von Neumann and Morgenstern (1944), allows us to conclude that the decision-maker's preferences can be represented by an expected payoff function
- This approach allows us to conclude that there is a ***payoff function*** u ***over deterministic outcomes*** such that the decision-maker's preference relation over lotteries is represented by the function

$$U(p_1, \ldots, p_k) = \sum_{k=1}^{K} p_k u_k(a_k),$$

where a_k is the k-th outcome of the lottery

- The decision-maker prefers the lottery $(p_1, \ldots, p_K)$ to the lottery $(p'_1, \ldots, p'_K)$ if and only if

$$\sum_{k=1}^{K} p_k u(a_k) > \sum_{k=1}^{K} p'_k u(a_k)$$

- That is, the decision-maker evaluates a lottery by its expected payoff according to the function u, which is known as the decision-maker's Bernoulli payoff function.

- We denote Δ_i as the set of probability measures over the set A_i
- Formally,

$$\Delta_i = \{\alpha_i : \alpha_i(a_1) + \cdots + \alpha_i(a_K) = 1, \alpha_i(a_k) \geq 0, k = 1, \ldots, K\}$$

Definition

A **belief** for player i is given by a probability distribution over the strategies of his opponents.

Definition

A strategic game (with vNM preferences) consists of

- A set of players
- For each player, a set of actions Δ_i
- For each player, preferences regarding lotteries over action profiles that may be represented by the expected value of a ("Bernoulli") payoff function over action profiles

Definition

The mixed strategy profile α^* is a (mixed strategy)Nash equilibrium if, for each player i and every mixed strategy α_i of player i, the expected payoff to player i of α^* is at least as large as the expected payoff to player i of $(\alpha_i, \alpha^*_{-i})$ according to a payoff function whose expected value represents player is preferences over lotteries. Equivalently, for each player i,

$$U_i(\alpha^*) \geq U_i(\alpha_{-i}, \alpha^*_{-i}),$$

for every mixed strategy α_i of player i, where $U_i(\alpha)$ is player i's expected payoff to the mixed strategy profile α.

1. Each player chooses her action according to the model of rational choice, given her belief about the other players' actions.
2. Every player's belief about the other players' actions is correct.

Definition

For a given α_{-i}, player i's best response function is given by:

$$B_i(\alpha_{-i}) = \{\alpha_i \in \Delta_i : U(\alpha_i, \alpha_{-i}) \geq U_i(\alpha_i', \alpha_{-i}), \text{for } \alpha_i' \in \Delta_i\} \quad (1)$$

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In other words: the mixed strategy profile α^* is a mixed strategy Nash equilibrium if and only if $\alpha_i^* \in B_i(\alpha_{-i}^*)$ for every player i.