Lecture 7: Traffic games

Indiana University,

Emerson Melo

- In a Nash equilibrium, each player strategy is a best response to the other player's strategies.
- In other words, the players are optimizing individually.
- But this does not mean that, as a group, the players will necessarily reach an outcome that is in any sense good.

Definition

A choice of strategies-one by each player-is Pareto-optimal if there is no other choice of strategies in which all players receive payoffs at least as high, and at least one player receives a strictly higher payoff.

- To see the intuitive appeal of Pareto-optimality, let's consider a choice of strategies that is not Pareto-optimal.
- In this case, there is an alternate choice of strategies that makes at least one player better off without harming any player.
- In basically any reasonable sense, this alternate choice is superior to what's currently being played.
- If the players could jointly agree on what to do, and make this agreement binding, then surely they would prefer to move to this superior choice of strategies.

- The motivation here relies crucially on the idea that the players can construct a binding agreement to actually play the superior choice of strategies: if this alternate choice is not a Nash equilibrium, then absent a binding agreement, at least one player would want to switch to a different strategy.

- A stronger condition that is even simpler to state is social optimality.

Definition

A choice of strategies one by each player is a social welfare maximizer (or socially optimal) if it maximizes the sum of the players' payoffs.

- Let's begin by developing a model of a transportation network and how it responds to traffic congestion; with this in place, we can then introduce the game-theoretic aspects of the problem.
- We represent a transportation network by a directed graph: we consider the edges to be highways, and the nodes to be exits where you can get on or off a particular highway.
- There are two particular nodes, which we'll call A and B, and we'll assume everyone wants to drive from A to B.

- For example, we can imagine that A is an exit in the suburbs, B is an exit downtown, and we are looking at a large collection of morning commuters.
- Finally, each edge has a designated travel time that depends on the amount of traffic it contains.
- To make this concrete, consider the graph in Figure 8.1.

- The label on each edge gives the travel time (in minutes) when there are x cars using the edge.
- In this simplified example, the A-D and C-B edges are insensitive to congestion: each takes 45 minutes to traverse regardless of the number of cars traveling on them.
- On the other hand, the A-C and D-B edges are highly sensitive to congestion: for each one, it takes $x/100$ minutes to traverse when there are x cars using the edge.

- Now, suppose that 4000 cars want to get from A to B as part of the morning commute.
- There are two possible routes that each car can choose: the upper route through C, or the lower route through D.
- For example, if each car takes the upper route (through C), then the total travel time for everyone is 85 minutes, since $4000/100+45=85$.

- The same is true if everyone takes the lower route.
- On the other hand, if the cars divide up evenly between the two routes, so that each carries 2000 cars, then the total travel time for people on both routes is $2000/100 + 45 = 65$.

- So what do we expect will happen?
- The traffic model we've described is really a game in which the players correspond to the drivers, and each player's possible strategies consist of the possible routes from A to B.
- In our example, this means that each player only has two strategies; but in larger networks, there could be many strategies for each player.
- The payoff for a player is the negative of his or her travel time (we use the negative since large travel times are bad).

- The game does have Nash equilibria, however: as we will see next, any list of strategies in which the drivers balance themselves evenly between the two routes (2000 on each) is a Nash equilibrium, and these are the only Nash equilibria.
- Why does equal balance yield a Nash equilibrium, and why do all Nash equilibria have equal balance?

- To answer the first question, we just observe that with an even balance between the two routes, no driver has an incentive to switch over to the other route.
- For the second question, consider a list of strategies in which x drivers use the upper route and the remaining $4000x$ drivers use the lower route.
- Then if x is not equal to 2000, the two routes will have unequal travel times, and any driver on the slower route would have an incentive to switch to the faster one.

- Hence any list of strategies in which x is not equal to 2000 cannot be a Nash equilibrium; and any list of strategies in which $x = 2000$ is a Nash equilibrium.

- In Figure 8.1, everything works out very cleanly: self-interested behavior by all drivers causes them-at equilibrium-to balance perfectly between the available routes.
- But with only a small change to the network, we can quickly find ourselves in truly counterintuitive territory.
- The change is as follows: suppose that the city government decides to build a new, very fast highway from C to D, as indicated in Figure 8.2.

- To keep things simple, we'll model its travel time as 0, regardless of the number of cars on it, although the resulting effect would happen even with more realistic (but small) travel times.
- It would stand to reason that people's travel time from A to B ought to get better after this edge from C to D is added. Does it?
- There is a unique Nash equilibrium in this new highway network.

- This new equilibrium it leads to a worse travel time for everyone
- At equilibrium, every driver uses the route through both C and D; and as a result, the travel time for every driver is 80 (since $4000/100 + 0 + 4000/100 = 80$).
- To see why this is an equilibrium, note that no driver can benefit by changing their route: with traffic snaking through C and D the way it is, any other route would now take 85 minutes.

- And to see why it's the only equilibrium, you can check that the creation of the edge from C to D has in fact made the route through C and D a dominant strategy for all drivers: regardless of the current traffic pattern, you gain by switching your route to go through C and D.
- In other words, once the fast highway from C to D is built, the route through C and D acts like a "vortex" that draws all drivers into it-to the detriment of all.
- In the new network there is no way, given individually self-interested behavior by the drivers, to get back to the even-balance solution that was better for everyone.

- This phenomenon that adding resources to a transportation network can sometimes hurt performance at equilibrium was first articulated by Dietrich Braess in 1968, and it has become known as Braess's Paradox

Definition

We define the price of anarchy (POA) to be the ratio between the travel time in an equilibrium and the minimum-possible average travel time.

- Formally we can define

$$POA = \frac{\text{Equilibrium Outcome}}{\text{Social Optimum}}$$

- In Braess's paradox, suppose we allow the graph to be arbitrary.
- Assume that the travel time on each edge depends in a linear way on the number of cars traversing it-that is, all travel times across edges have the form

$$c(x) = ax + b,$$

where each of a and b is either 0 or a positive number.

- In this case, elegant results of Tim Roughgarden and Eva Tardos can be used to show that if we *add* edges to a network with an equilibrium pattern of traffic, there is always an equilibrium in the new network whose travel time is no more than $4/3$ times as large.
- Moreover, $4/3$ is the factor increase that we'd get in the example from Figures 8.1 and 8.2, if we replace the two travel times of 45 with 40. (In that case, the travel time at equilibrium would jump from 60 to 80 when we add the edge from C to D.)

- There is an even simpler traffic network in which the POA is $4/3$, first discussed in 1920 by Pigou.
- In Pigou's example (Figure 2(a)), every driver has a dominant strategy to take the lower link-even when congested with all of the traffic, it is no worse than the alternative.
- Thus, in equilibrium all drivers use the lower edge and experience travel time 1.

- Can we do better?
- Sure-any other solution is better!
- A benevolent social planner would minimize the average travel time by splitting the traffic equally between the two links.
- This results in an average travel time of 34 , showing that the POA in Pigou's example is $4/3$.

- In the nonlinear Pigou's example (Figure 2(b)), we replace the previous cost function $c(x) = x$ of the lower edge with the function $c(x) = x^p$, with p large.
- The lower edge remains a dominant strategy, and the equilibrium travel time remains 1.
- What's changed is that the optimal solution is now much better.
- If we again split the traffic equally between the two links, then the average travel time tends to $1/2$ as $p \longrightarrow \infty$-traffic on the bottom edge gets to t nearly instantaneously.

- We can do even better by routing $1-\varepsilon$ traffic on the bottom link, where $\varepsilon \longrightarrow 0$ as $p \longrightarrow \infty$.
- Then almost all of the traffic gets to t with travel time $(1-\varepsilon)^p$, which is close to 0 when p is sufficiently large, and the upper the upper edge contribute little to the average travel time.
- We conclude that the POA in the nonlinear Pigou's example is unbounded as $p \longrightarrow \infty$.

Theorem (Roughgarden (2003))

Among all networks with cost functions in a set $\mathscr{C}$, the largest POA is achieved in a Pigou-like network.

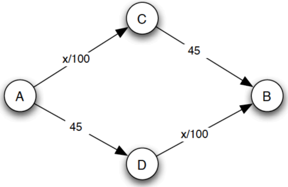


Figure 8.1: A highway network, with each edge labeled by its travel time (in minutes) when there are x cars using it. When 4000 cars need to get from A to B, they divide evenly over the two routes at equilibrium, and the travel time is 65 minutes.

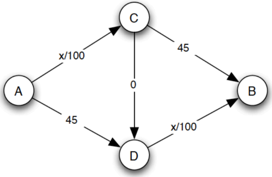
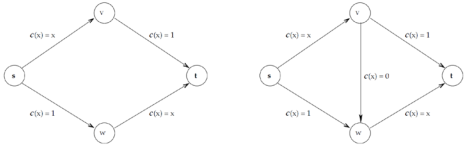


Figure 8.2: The highway network from the previous figure, after a very fast edge has been added from C to D. Although the highway system has been "upgraded," the travel time at equilibrium is now 80 minutes, since all cars use the route through C and D.



(a) Initial network (b) Augmented network

Figure 1: Braess's Paradox. After the addition of the (v, w) edge, the price of anarchy is 4/3.

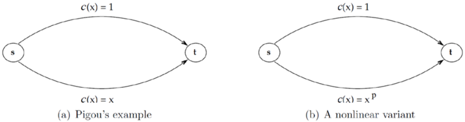


Figure 2: Pigou's example and a nonlinear variant. The cost function $c(x)$ describes the cost incurred by users of an edge, as a function of the amount of traffic routed on the edge.

- There are 1000 cars which must travel from town A to town B. There are two possible routes that each car can take: the upper route through town C or the lower route through town D. Let x be the number of cars traveling on the edge AC and let y be the number of cars traveling on the edge DB. The directed graph in Figure 8.8 indicates that travel time per car on edge AC is $x/100$ if x cars use edge AC, and similarly the travel time per car on edge DB is $y/100$ if y cars use edge DB. The travel time per car on each of edges CB and AD is 12 regardless of the number of cars on these edges. Each driver wants to select a route to minimize his travel time. The drivers make simultaneous choices.

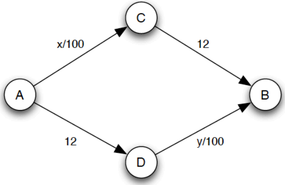


Figure 8.8: Traffic Network.

Questions

1. Find Nash equilibrium values of x and y.
2. Now the government builds a new (one-way) road from town C to town D. The new road adds the path $ACDB$ to the network. This new road from C to D has a travel time of 0 per car regardless of the number of cars that use it. Find a Nash equilibrium for the game played on the new network. What are the equilibrium values of x and y? What happens to total cost-of-travel (the sum of total travel times for the 1000 cars) as a result of the availability of the new road?

3. Suppose now that conditions on edges CB and AD are improved so that the travel times on each edge are reduced to 5. The road from C to D that was constructed in part (b) is still available. Find a Nash equilibrium for the game played on the network with the smaller travel times for CB and AD. What are the equilibrium values of x and y? What is the total cost-of-travel? What would happen to the total cost-of-travel if the government closed the road from C to D?