Lecture 13: Matching and Market Design

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What is Matching and Market Design?

- Recently economists have been using economics to design institutions successfully, such as
  1. student placement in schools.
  2. labor markets where workers and firms are matched matched.
  3. organizing organ donation network.

- Many more are being proposed and implemented, such as
  1. course allocation in schools.
  2. cadets to branches in the military, and so on.
- The economics of "matching and market design" analyzes and designs real-life institutions.
- A lot of emphasis is placed on concrete markets and details so that we can offer practical solutions.

Labor Markets: The case of American hospital-intern markets.

- Medical students in many countries work as residents (interns) at hospitals.
- In the U.S. more than 20,000 medical students and 4,000 hospitals are matched through a clearinghouse, called NRMP (National Resident Matching program).
- Doctors and hospitals submit preference rankings to the clearinghouse, and the clearinghouse uses a specified rule (computer program) to decide who works where.
- Some markets succeeded while others failed. What is a "'good way" to match doctors and hospitals?

Kidney Exchange

- Kidney transplant is a preferred method to save kidney-disease patients.
- There are lots of kidney shortages, and willing donor may be incompatible with the donor.
- Kidney Exchange tries to solve this by matching donor-patient pairs.
- What is a “good way” to match donor-patient pairs?

School Choice

- In many countries, especially in the past, children were automatically sent to a school in their neighborhoods.
- Recently, more and more cities in the United States and in other countries employ school choice programs: school authorities take into account preferences of children and their parents.

School Choice

- Because school seats are limited (for popular schools), school districts should decide who is admitted.
- How should school districts decide placements of students in schools?
- Typical goals of school authorities are:
  1. Efficient placement.
  2. Fairness of outcomes.
  3. Easy for participants to understand and use, etc.
- To study these questions (and others), we will study the theory of matching beginning today.

One-to-one matching: the marriage model

- In these lectures we will examine in detail the two-sided matching market without money that arises when each agent may be matched with (at most) one agent of the opposite set.
- This model is often called a "marriage market," with the two sets of agents being referred to as "men" and "women" instead of students and colleges, firms and workers, or physicians and hospitals.
- We will follow this practice here.

The formal cooperative model

- The elements of the formal model are as follows. There are two finite and disjoint sets M and W : $M = \{m_1, m_2, \ldots, m_n\}$ is the set of men, and $W = \{w_1, w_2, \ldots, w_p\}$ is the set of women.
- Each man has preferences over the women, and each woman has preferences over the men.
- These preferences may be such that, say, a man m would prefer to remain single rather than be married to some woman w he doesn't care for.

- An individual's preferences are meant to represent how he or she would choose among different alternatives, if he or she were faced with a choice.
- So when we say some individual prefers alternative a to alternative b, we mean that if that individual were faced with a choice between the two, he or she would choose a and not b, and if faced with a choice from a set of alternatives that included b, then he or she would not choose b if a were also available.

- When we say an individual is indifferent between the two, we mean that he or she might choose either one.
- We will say an individual likes a at least as well as b if he or she either prefers a to b or is indifferent between them.
- To express these preferences concisely, the preferences of each man m will be represented by an ordered list of preferences, $P(m)$, on the set $W \cup \{m\}$.
- That is, a man m's preferences might be of the form

$$P(m) = w_1, w_2, m, w_3, \ldots, w_p$$

indicating that his first choice is to be married to woman w_1, his second choice is to be married to woman w_2, and his third choice is to remain single.

- A man m' may be indifferent between several possible mates.
- This will be denoted by brackets in the preference list, so for example the list

$$P\left(m'\right) = w_2, \left[w_1, w_7\right], m', w_3, \ldots, w_k$$

indicates that man m' prefers woman w_2 to w_1, but that he is indifferent between w_1 and w_7, and he prefers remaining single to marrying anyone else.

- Similarly, each woman w in W has an ordered list of preterences, $P(w)$ on the set $M \cup \{w\}$.
- We will usually describe an agent's preferences by writing only the ordered set of people that the agent prefers to being single.
- Thus the preferences $P(m)$ just described will be abbreviated by

$$P(m) = w_1, w_2$$

- We will denote by P the set of preference lists $P = \left\{P(m_1), \ldots, P(m_n), P(w_1), \ldots, P(w_p)\right\}$ one for each man and woman.
- A specific marriage market will be denoted by the triple (M, W, P).
- We write $w >_m w'$ to mean m prefers w to w' and $w \geq_m w'$ to mean m likes w at least as well as w'.

- Similarly we write $m >_w m'$ and $m \geq_w m'$.
- Woman w is acceptable to man m if he likes her at least as well as remaining single, that is, if $w \geq_m m$.
- Analogously, m is acceptable to w if $m \geq_w w$.
- If an individual is not indifferent between any two acceptable alternatives, he or she has *strict preferences*.
- We assume that preferences are rational, i.e., they are complete and transitive.

Definition

A matching μ is a one-to-one correspondence from the set $M \cup W$ onto itself of order two (that is, $\mu^2(x) = x$) such that if $\mu(m) \neq m$ then $\mu(m) \in W$ and if $\mu(w) \neq w$ then $\mu(w) \in M$. We refer to $\mu(x)$ as the mate of x

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Definition

An outcome of the game is a matching:$\mu : M \bigcup W \longrightarrow M \bigcup W$ such that $w = \mu(m)$ iff $m = \mu(w)$, and for all m and w either $\mu(w)$ is in M or $\mu(w) = w$, and either $\mu(m)$ is in W or $\mu(m) = m$.

- Note that $\mu^2(x) = x$ means that if man m is matched to woman w (i.e., if $\mu(m) = w$), then woman w is matched to man m (i.e., $\mu(w) = m$).
- The definition also requires that individuals who are not single be matched with agents of the opposite set - that is, men are matched with women.
- These two requirements explain why matchings can be thought of as sets marriages.
- A matching will sometimes be represented as a set of matched pairs

Stable Matching

- We can now start to consider the elements of a theory about which matchings are likely to occur, and which are not.
- This is where the rules of the game play a critical role.
- The first element of our theory is that since the rules specify that no agent may be compelled to marry, we will not observe any matchings that could only result from compulsion of one of the agents.

- Specifically, consider a matching μ that matches a pair (m, w) who are not mutually acceptable.
- Then at least one of the individuals m and w would prefer to be single rather than be matched to the other.
- Such a matching μ will be said to be **blocked** by the unhappy individual.
- If man m, say, prefers remaining single to marrying woman w, this means that if he were faced with a choice, he would never choose to marry woman w so long as remaining single was one of his alternatives. But the rules of the game insure that the option of remaining single is always available.

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Definition

The matching μ is individually rational if each agent is acceptable to his or her mate. That is, a matching is individually rational if it is not blocked by any (individual) agent.

- Consider a matching μ such that there exist a man m and a woman w who are not matched to one another at μ, but who prefer each other to their assignments at μ.
- That is, suppose that $w >_m \mu(m)$ and $m >_w \mu(w)$.
- The man and woman (m,w) will be said to block the matching μ.

Definition

A matching μ is stable if it is not blocked by any individual or any pair of agents.

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Example

- There are three men and three women, with the following preferences.

$$
\begin{array}{ll}
P(m_1) = w_2, w_1, w_3 & P(w_1) = m_1, m_3, m_2 \\
P(m_2) = w_1, w_3, w_2 & P(w_2) = m_3, m_1, m_2 \\
P(m_3) = w_1, w_2, w_3 & P(w_3) = m_1, m_3, m_2
\end{array}
$$

- All possible matchings are individually rational (since all pairs (m,w) are mutually acceptable).
- The matching μ given by:

$$\mu = \begin{matrix} w_1 & w_2 & w_3 \\ m_1 & m_2 & m_3 \end{matrix}$$

is unstable, since (m_1, w_2) is a blocking pair.

- However the matching

$$\mu' = \begin{matrix} w_1 & w_2 & w_3 \\ m_1 & m_3 & m_2 \end{matrix}$$

is stable.

Some properties of stable matchings

- Do stable matching always exist?

The roommate problem (Gale and Shapley)

- There is a single set of n people who can be matched in pairs (to be roommates in a college dormitory, or partners in paddling a canoe).
- Each person in the set ranks the $n-1$ others in order of preference. An outcome is a matching, which is a partition of the people into pairs. (To keep things simple, suppose the number n of people is even.)
- A stable matching is a matching such that no two persons who are not roommates both prefer each other to their actual partners.

- Consider four people: a, b, c, and d, with the following preferences:

$$
\begin{aligned}
P(a) &= b, c, d \\
P(b) &= c, a, d \\
P(c) &= a, b, d \\
P(d) &= \text{arbitrary}
\end{aligned}
$$

- Person d is the last choice of everyone else. (Perhaps you know someone like that.) Each of the other people is someone else's first choice.
- So no matching will be stable, since any matching must pair someone with agent d, and that someone will be able to find another person to make a blocking pair.

$$\mu_1 = \begin{matrix} c & a \\ b & d \end{matrix}, \quad \mu_2 = \begin{matrix} a & d \\ b & c \end{matrix}, \quad \mu_3 = \begin{matrix} b & a \\ d & c \end{matrix}.$$

- Person d is the last choice of everyone else. (Perhaps you know someone like that.) Each of the other people is someone else's first choice.
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$$\mu_1 = \begin{array}{cc} c & a \\ b & d \end{array}, \quad \mu_2 = \begin{array}{cc} a & d \\ b & c \end{array}, \quad \mu_3 = \begin{array}{cc} b & a \\ d & c \end{array}.$$

- And $(c,a),(b,c)$, and (a,b) block μ_1,μ_2 and μ_3, respectively.

- Returning now to the marriage problem, a natural way to prove that stable matchings always exist would be to demonstrate some procedure or algorithm that, when applied to any marriage problem, would produce a stable matching.
- One procedure that might suggest itself is to start with an arbitrary matching, and if it is blocked by, say, man m and woman w, make a new matching in which m and w are matched to one another.
- Since there are only a finite number of possible matchings, we might hope this procedure would eventually lead to a stable matching.
- However, the procedure can form a cycle, and thus repeat itself indefinitely without reaching a stable matching.

Theorem (Gale and Shapley (1962))

A stable matching exists for every marriage market.

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Gale and Shapley approached this problem from a purely theoretical perspective, but proved this theorem via a constructive algorithm of the kind that has subsequently turned up at the heart of a variety of clearinghouses.

Proof. Deferred Acceptance Algorithm, with men proposing (roughly the Gale-Shapley 1962 version).

- To start, each man proposes to his favorite woman, that is, to the first woman on his preference list of acceptable women.
- Each woman rejects the proposal of any man who is unacceptable to her, and each woman who receives more than one proposal rejects all but her most preferred of these.
- Any man whose proposal is not rejected at this point is kept "engaged."
- At any step any man who was rejected at the previous step proposes to his next choice (i.e., to his most preferred woman among those who have not yet rejected him), so long as there remains an acceptable woman to whom he has not yet proposed.

- (If at any step of the procedure a man has already proposed to, and been rejected by, all of the women he finds acceptable, then he issues no further proposals.)
- Each woman receiving proposals rejects any from unacceptable men, and also rejects all but her most preferred among the group consisting of the new proposers together with any man she may have kept engaged from the previous step.
- The algorithm stops after any step in which no man is rejected.

- At this point, every man is either engaged to some woman or has been rejected by every woman on his list of acceptable women.
- The marriages are now consummated, with each man being matched to the woman to whom he is engaged.
- Women who did not receive any acceptable proposal, and men who were rejected by all women acceptable to them, will stay single. □

- This completes the description of the algorithm, except that we have described it as if all agents have strict preferences.
- The modification required in case some man or woman is *indifferent* between two or more possible mates is simple.
- At any step of the algorithm at which some agent must indicate a choice between two mates who are equally well liked, introduce some fixed "tie-breaking" rule (e.g., when an agent is indifferent, proceed as if the preferences are according to alphabetical order of family names, or as if agents prefer mates who are closer to them in age, etc.).

- Such a tie-breaking rule therefore specifies, arbitrarily, to which woman a man will propose when he is indifferent about his next proposal, and which man a woman will keep engaged when she is indifferent among more than one most fayored suitors.

The DA algorithm and stable matching

- To see that the matching μ produced by the algorithm is stable, suppose some man m and woman w are not married to each other at μ, but m prefers w to his own mate at μ.
- Then woman w must be acceptable to man m, and so he must have proposed to her before proposing to his current mate (or before being rejected by all of the women he finds acceptable).
- Since he was not engaged to w when the algorithm stopped, he must have been rejected by her in favor of someone she liked at least as well.

- Therefore w is matched at μ to a man whom she likes at least as well as man m, since preferences are transitive (and hence acyclic), and so m end w do not block the matching μ.
- Since the matching is not blocked by any individual or by any pair, it is stable.

An example of the DA algorithm

- Consider a Marriage Market where $M=\{m_1,m_2,m_3,m_4,m_5\}$, $W=\{w_1,w_2,w_3,w_4\}$, and

$$\begin{aligned}
P(m_1) &= w_1,w_2,w_3,w_4\\
P(m_2) &= w_4,w_2,w_3,w_1\\
P(m_3) &= w_4,w_3,w_1,w_2\\
P(m_4) &= w_1,w_4,w_3,w_2\\
P(m_5) &= w_1,w_2,w_4
\end{aligned} \tag{1}$$

$$
\begin{aligned}
P(w_1) &= m_2, m_3, m_1, m_4, m_5 \\
P(w_2) &= m_3, m_1, m_2, m_4, m_5 \\
P(w_3) &= m_5, m_4, m_1, m_2, m_3 \\
P(w_4) &= m_1, m_4, m_5, m_2, m_3
\end{aligned} \tag{2}
$$

First Step

- m_1, m_4, and m_5 propose to w_1, and m_2 and m_3 propose to w_4
- w_1 rejects m_4 and m_5 and keeps m_1 engaged.
- w_4 rejects m_3 and keeps m_2 engaged.
- We indicate this in the following manner:

$$\begin{array}{cccc} w_1 & w_2 & w_3 & w_4 \\ m_1 & & & m_2 \end{array}$$

Second Step

- m_3, m_4, and m_5 propose to their second choice, that is, to w_3, w_4, and w_2, respectively.
- w_4 rejects m_2 and keeps m_4 engaged:

$$\begin{array}{cccc} w_1 & w_2 & w_3 & w_4 \\ m_1 & m_5 & m_3 & m_4 \end{array}$$

Third Step

- m_2 proposes to his second choice, w_2, who rejects m_5 and keeps m_2 engaged:

$$\begin{array}{cccc} w_1 & w_2 & w_3 & w_4 \\ m_1 & m_2 & m_3 & m_4 \end{array}$$

Fourth Step

- m_5 proposes to his third choice, w_4, who rejects m_5 and continues with m_4 engaged.
- Since m_5 has been rejected by every woman on his list of acceptable women, he stays single, that is, matched with himself, and the stable matching obtained is:

$$\mu_M = \begin{matrix} w_1 & w_2 & w_3 & w_4 & (m_5) \\ m_1 & m_2 & m_3 & m_4 & m_5 \end{matrix}$$

Definition

For a given marriage market (M,W,P), a stable matching μ is M-optimal if every man likes it at least as well as any other stable matching; that is, if for every other stable matching $\mu', \mu \geq_M \mu'$. Similarly, a stable matching ν is W - optimal if every woman likes it at least as well as any other stable matching, that is, if for every other stable matching $\nu', \nu \geq_W \nu'$

Theorem (Gale and Shapley (1962))

When all men and women have strict preferences, there always exists an M-optimal stable matching (that every man likes at least as well as any other stable matching), and a W-optimal stable matching. Furthermore, the matching μ_M produced by the deferred acceptance algorithm with men proposing is the M-optimal stable matching. The W-optimal stable matching is the matching μ_W produced by the algorithm when the women propose.

Theorem (Roth and Sotomayor (1990))

No stable matching mechanism exists for which stating the true preferences is a dominant strategy for every agent.

Theorem (Dubins and Freedman, Roth and Sotomayor (1990))

The mechanism that yields the M-optimal stable matching (in terms of the stated preferences) makes it a dominant strategy for each man to state his true preferences.

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