Lecture 10: Optimal Assignments

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- The problem of bipartite matching from the previous lecture illustrates some aspects of a market in a very simple form: *individuals express preferences in the form of acceptable options; a perfect matching then solves the problem of allocating objects to individuals according to these preferences; and if there is no perfect matching, it is because of a "constriction" in the system that blocks it.*

- We now want to extend this model to introduce some additional features. First, rather than expressing preferences simply as binary "acceptable-or-not" choices, we allow each individual to express how much they'd like each object, in numerical form.
- We can define valuations whenever we have a collection of individuals evaluating a collection of objects. And using these valuations, we can evaluate the quality of an assignment of objects to individuals, as follows: *it is the sum of each individual's valuation for what they get.*

Definition

An optimal assignment maximizes the total happiness of everyone for what they get.

- It is worth noticing that, while the optimal assignment maximizes total happiness, it does not necessarily give everyone their favorite item.
- In a very concrete sense, the problem of finding an optimal assignment also forms a natural generalization of the bipartite matching problem.

- Specifically, it contains the bipartite matching problem as a special case.
- Here is why. Suppose, that there are an equal number of students and rooms, and each student simply submits a list of acceptable rooms without providing a numerical valuation; this gives us a bipartite graph.
- We would like to know if this bipartite graph contains a perfect matching, and we can express precisely this question in the language of valuations and optimal assignments as follows.
- We give each student a valuation of 1 for each room they included on their acceptable list, and a valuation of 0 for each room they omitted from their list.

- Now, there is a perfect matching precisely when we can find an assignment that gives each student a room that he or she values at 1 rather than 0-that is, precisely when the optimal assignment has a total valuation equal to the number of students.
- This simple translation shows how the problem of bipartite matching is implicit in the broader problem of finding an optimal assignment.

- Thus far, we have been using the metaphor of a central "administrator" who determines a perfect matching, or an optimal assignment, by collecting data from everyone and then performing a centralized computation.
- And while there are clearly instances of market-like activity that function this way (such as our example of students and dorm rooms), a more standard picture of a market involves much less central coordination, with individuals making decisions based on prices and their own valuations.

- Capturing this latter idea brings us to the crucial step in our formulation of matching markets: understanding the way in which prices can serve to decentralize the market.
- We will see that if we replace the role of the central administrator by a particular scheme for pricing items, then allowing individuals to follow their own self-interest based on valuations and prices can still produce optimal assignments.

- To describe this, let's change the housing metaphor slightly, from students and dorm rooms to one where the role of prices is more natural.
- Suppose that we have a collection of sellers, each with a house for sale, and an equal-sized collection of buyers, each of whom wants a house.
- By analogy with the previous section, each buyer has a valuation for each house, and as before, two different buyers may have very different valuations for the same houses.

- The valuation that a buyer j has for the house held by seller i will be denoted v_{ij} , with the subscripts i and j indicating that the valuation depends on both the identity of the seller i and the buyer j.
- We also assume that each valuation is a non-negative whole number $(0, 1, 2, \ldots)$.
- We assume that sellers have a valuation of 0 for each house; they care only about receiving payment from buyers, which we define next.

- Suppose that each seller i puts his house up for sale, offering to sell it for a price $p_i \geq 0$.
- If a buyer j buys the house from seller i at this price, we will say that the buyer's payoff is her valuation for this house, minus the amount of money she had to pay: $v_{ij} - p_i$.
- So given a set of prices, if buyer j wants to maximize her payoff, she will buy from the seller i for which this quantity $v_{ij} - p_i$ is maximized-with the following caveats.

- First, if this quantity is maximized in a tie between several sellers, then the buyer can maximize her payoff by choosing any one of them.
- Second, if her payoff $v_{ij} - p_i$ is negative for every choice of seller i, then the buyer would prefer not to buy any house: we assume she can obtain a payoff of 0 by simply not transacting.
- We will call the seller or sellers that maximize the payoff for buyer j the preferred sellers of buyer j, provided the payoff from these sellers is not negative. We say that buyer j has no preferred seller if the payoffs $v_{ij} - p_i$ are negative for all choices of i.

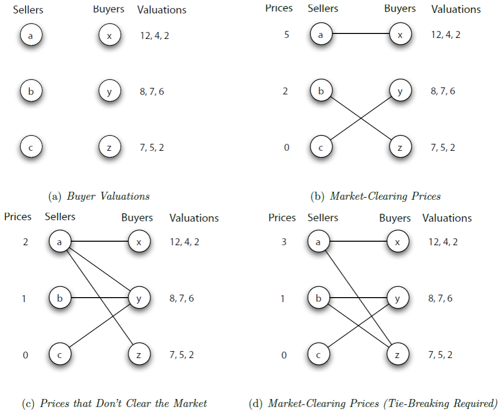

Figure 10.5: (a) Three sellers (a, b, and c) and three buyers (x, y, and z). For each buyer node, the valuations for the houses of the respective sellers appear in a list next to the node. (b) Each buyer creates a link to her preferred seller. The resulting set of edges is the preferred-seller graph for this set of prices. (c) The preferred-seller graph for prices 2, 1, 0. (d) The preferred-seller graph for prices 3, 1, 0.

- In Figures 10.5(b)-10.5(d), you can see the results of three different sets of prices for the same set of buyer valuations.
- Note how the sets of preferred sellers for each buyer change depending on what the prices are. So for example, in Figure 10.5(b), buyer x would receive a payoff of $125 = 7$ if she buys from a, a payoff of $42 = 2$ if she buys from b, and $20 = 2$ if she buys from c.
- This is why a is her unique preferred seller. We can similarly determine the payoffs for buyers y (3,5, and 6) and z (2,3, and 2) for transacting with sellers a,b,and c respectively.

- Figure 10.5(b) has the particularly nice property that if each buyer simply claims the house that she likes best, each buyer ends up with a different house: somehow the prices have perfectly resolved the contention for houses.
- And this happens despite the fact that each of the three buyers value the house of seller a the highest; it is the high price of 5 that dissuades buyers y and z from pursuing this house.

- We will call such a set of prices market-clearing, since they cause each house to get bought by a different buyer.
- In contrast, Figure 10.5(c) shows an example of prices that are not market-clearing, since buyers x and z both want the house offered by seller a -so in this case, when each buyer pursues the house that maximizes their payoff, the contention for houses is not resolved. (Notice that although each of a, b, and c is a preferred seller for y, since they all give y equal payoffs, this does not help with the contention between x and z.)

- Figure 10.5(d) illustrates one further subtlety in the notion of market-clearing prices.
- Here, if the buyers coordinate so that each chooses the appropriate preferred seller, then each buyer gets a different house. (This requires that y take c's house and z take b's house.)
- Since it is possible to eliminate contention using preferred sellers, we will say that this set of prices is market-clearing as well, even though a bit of coordination is required due to ties in the maximum payoffs. In some cases, ties like this may be inevitable: for example, if all buyers have the same valuations for everything, then no choice of prices will break this symmetry.

- Given the possibility of ties, we will think about market-clearing prices more generally as follows.
- For a set of prices, we define the preferred-seller graph on buyers and sellers by simply constructing an edge between each buyer and her preferred seller or sellers. (There will be no edge out of a buyer if she has no preferred seller.)
- So in fact, Figures 10.5(b)-10.5(d) are just drawings of preferred-seller graphs for each of the three indicated sets of prices. Now we simply say: a set of prices is market-clearing if the resulting preferred-seller graph has a perfect matching.

- In a way, market-clearing prices feel a bit too good to be true: if sellers set prices the right way, then self-interest runs its course and (potentially with a bit of coordination over tie-breaking) all the buyers get out of each other's way and claim different houses.
- We've seen that such prices can be achieved in one very small example; but in fact, something much more general is true:

  Property 1-Existence of Market-Clearing Prices: *For any set of buyer valuations, there exists a set of market-clearing prices.*

- So market-clearing prices are not just a fortuitous outcome in certain cases; they are always present. This is far from obvious, and we will turn shortly to a method for constructing market-clearing prices that, in the process, proves they always exist.
- Before doing this, we consider another natural question: the relationship between market-clearing prices and social welfare. Just because market-clearing prices resolve the contention among buyers, causing them to get different houses, does this mean that the total valuation of the resulting assignment will be good?
- In fact, there is something very strong that can be said here as well: market-clearing prices (for this buyer-seller matching problem) always provide socially optimal outcomes:

- ***Property 2-Optimality of Market-Clearing Prices:*** *For any set of market-clearing prices, a perfect matching in the resulting preferred-seller graph has the maximum total valuation of any assignment of sellers to buyers.*

- Compared with the previous claim on the existence of market-clearing prices, this fact about optimality can be justified by a much shorter, if somewhat subtle, argument.
- The argument is as follows. Consider a set of market-clearing prices, and let M be a perfect matching in the preferred-seller graph. Now, consider the total payoff of this matching, defined simply as the sum of each buyer's payoff for what she gets. Since each buyer is grabbing a house that maximizes her payoff individually, M has the maximum total payoff of any assignment of houses to buyers.

- Now how does total payoff relate to total valuation, which is what we're hoping that M maximizes? If buyer j chooses house i, then her valuation is v_{ij} and her payoff is $v_{ij} - p_i$. Thus, the total payoff to all buyers is simply the total valuation, minus the sum of all prices:

$$\text{Total Payoff of } M = \text{Total Valuation of } M \text{-Sum of all prices.} \tag{1}$$

- But the sum of all prices is something that doesn't depend on which matching we choose (it's just the sum of everything the sellers are asking for, regardless of how they get paired up with buyers). So a matching M that maximizes the total payoff is also one that maximizes the total valuation. This completes the argument.

- There is another important way of thinking about the optimality of market-clearing prices, which turns out to be essentially equivalent to the formulation we've just described.
- Suppose that instead of thinking about the total valuation of the matching, we think about the total of the payoffs received by all participants in the market-both the sellers and the buyers.
- For a buyer, her payoff is defined as above: it is her valuation for the house she gets minus the price she pays.
- A seller's payoff is simply the amount of money he receives in payment for his house.

- Therefore, in any matching, the total of the payoffs to all the sellers is simply equal to the sum of the prices (since they all get paid, and it doesn't matter which buyer pays which seller).
- Above, we just argued that the total of the payoffs to all the buyers is equal to the total valuation of the matching M, minus the sum of all prices.
- Therefore, the total of the payoffs to all participants-both the sellers and the buyers-is exactly equal to the total valuation of the matching M; the point is that the prices detract from the total buyer payoff by exactly the amount that they contribute to the total seller payoff, and hence the sum of the prices cancels out completely from this calculation.

- Therefore, to maximize the total payoffs to all participants, we want prices and a matching that lead to the maximum total valuation, and this is achieved by using market-clearing prices and a perfect matching in the resulting preferred-seller graph. We can summarize this as follows:

  Optimality of Market-Clearing Prices (equivalent version): *A set of market-clearing prices, and a perfect matching in the resulting preferred-seller graph, produces the maximum possible sum of payoffs to all sellers and buyers.*

- **The Assignment Problem:** Suppose we have n resources to which we want to assign to n tasks on a one-to-one basis. Suppose also that we know the cost (valuation) of assigning a given resource to a given task. We wish to ïňĄnd an optimal assignment-one which minimizes total cost (or maximizes total valuation).

  The Mathematical Model: Let c_{ij} be the cost of assigning the ith resource to the jth task. We define the cost matrix to be the n-by-n matrix

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \dots & c_{1n} \\ c_{21} & c_{22} & c_{23} & \dots & c_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & c_{n3} & \dots & c_{nn} \end{bmatrix}$$

- An assignment is a set of n entry positions in the cost matrix, no two of which lie in the same row or column. The sum of the n entries of an assignment is its cost. An assignment with the smallest possible cost is called an optimal assignment.

  A mathematical model for the assignment problem is given as follows:

$$
\begin{aligned}
\text{Minimize} \quad & \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} \\
\text{Subject to} \quad & \sum_{j=1}^{n} x_{ij} = 1, \quad i = 1,\ldots,n \\
& \sum_{i=1}^{n} x_{ij} = 1, \quad i = 1,\ldots,n \\
& x_{ij} = 0 \quad \text{or} \quad 1, \quad i,j = 1\ldots,n. \qquad (2)
\end{aligned}
$$

- The underlaying graph is bipartite, and so the assignment problem is referred to as a "*minimum weighted matching problem on a bipartite graph*," with graphs.

- **The Hungarian Method:** The Hungarian method is an algorithm which finds an optimal assignment for a given cost matrix.