Network games and shock propagation

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- Most decisions that people make, from which products to buy to whom to vote for, are influenced by the choices of their friends and acquaintances.
- The decision of an individual to whether buy or not a new product, attend a meeting, commit a crime, find a job is often influenced by the choices of his or her friends and acquaintances (be they social or professional).

- We can see these settings as special cases of game theory.
- Then some results from the game theory literature directly apply: for example, existence of various forms of equilibria can be deduced from standard results.

- The interest is in whether there is anything we can deduce that holds systematically regarding how play in game depends on the network structure of interactions.
- For example: Can we say anything about who is the most influential individual in a network where people look to their peers in choosing an effort level in education?

- The main challenge that faced in studying strategic interaction in social settings is the inherent complexity of networks.
- Without focusing on specific structures in terms of the games, it is hard to draw any conclusions.

- The literature has primarily taken three approaches to this challenge.
  1. One approach involves looking at games of strategic complements and strategic substitutes, where the interaction in payoffs between players satisfies some natural and useful monotonicity properties.
  2. A second approach relies on looking at quite tractable "*linear-quadratic*" setting where agents choose a continuous level of activity.
  3. A third approach considers settings with an uncertain pattern of interactions, where players make choices without being certain about with whom they will interact.

- Let N be the set of players, where $N = \{1, \dots, n\}$.
- For all $i \in N$, let X_i be player i's strategy set.
- Let $\mathbf{L}$ denote the set of links.
- Let $\mathbf{G} = (N, \mathbf{L})$ be the network representing players relationships.
- $g_{ij} = 1$ whenever i and j are connected.

- Player i's payoff function is denoted

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- A Nash equilibrium is a profile $\mathbf{x}^*$ such that:

$$u_i(x_i^*;\mathbf{x}_{-i}^*;\mathbf{G}) \geq u_i(x_i,\mathbf{x}_{-i}^*;\mathbf{G}) \quad \forall x_i \in X_i.$$

- Players actions space are $X_i = \{0, 1\}$.
- In this particular game, if more than one half of i's neighbors choose action 1, then it is best for player i to choose 1.
- if fewer than one half of i's neighbors choose action 1 then it is best for player i to choose action 0.
- Specifically, the payoff to a player from taking action 1 compared to action 0 depends on the fraction of neighbors who choose action 1, such that:

$$u_i(1,\mathbf{x}_{-i};\mathbf{G}) > u_i(0,\mathbf{x}_{-i};\mathbf{G}) \quad if \quad \frac{\sum_{j=1}^{n} g_{ij}x_j}{\sum_{j=1}^{n} x_j} > \frac{1}{2},$$

and

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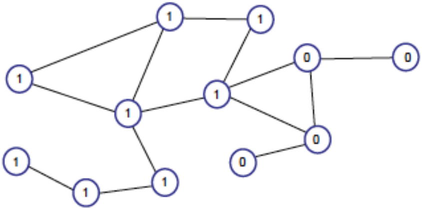
This game has multiple equilibria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- For instance, the action might be learning how to do something.
- The information is easily communicated,
- For instance, buying a book or other product that is easily lent from one player to another.
- Taking the action 1 is costly and if any of a player's neighbors takes the action then the player is better off not taking the action.
- Taking the action and paying the cost is better than having nobody in a player's neighborhood take the action.

$$u_i(x_i, \mathbf{x}_{-i}; \mathbf{G}) = \begin{cases} 1 - c & \text{if } x_i = 1 \text{ and } 0 < c < 1 \\ 1 & \text{if } x_i = 0 \text{ and } x_j = 1 \text{ for some } (i, j) \in \mathbf{L} \\ 0 & \text{if } x_i = 0 \text{ and } x_j = 0 \text{ for all } (i, j) \in \mathbf{L} \end{cases}$$

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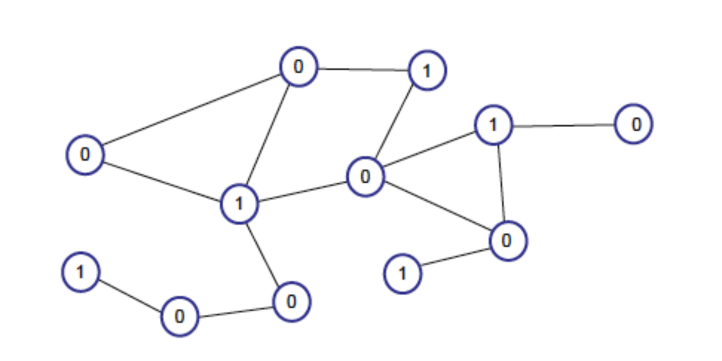
There are many possible equilibria in this game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- The equilibria in this game correspond exactly to having the set of players who choose action 1 form a maximal independent set of nodes in the network (Bramoulle and Kranton, JET (2007))
- The equilibria form a maximal set of nodes that have no links to each other in the network

- In games of strategic complements, an increase in the actions of other players leads a given player's higher actions to have relatively higher payoffs compared to that player's lower actions.
- Examples of such games include situations like the adoption of a technology, human capital decisions, and criminal efforts.

- Games of strategic substitutes are such that the opposite is true: an increase in other players' actions leads to relatively lower payoffs to higher actions of a given player.
- Applications of strategic substitutes include, for example, local public good provision and information gathering.

- **This section: Games with Strategic Substitutes, Continuous Action Spaces and Linear Best-Replies**

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- **Games with strategic substitutes are difficult to characterize and multiple equilibria rather than unique equilibrium are the rule**

- Players' payoffs functions are given by:

$$u_i(x_i, \mathbf{x}_{-i}; \mathbf{G}) = x_i - \frac{1}{2}x_i^2 - \delta \sum_{j=1}^{n} g_{ij} x_j x_i \quad \text{for all } i \in N,$$

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- The first order condition is

$$\frac{\partial u_i(x_i; \mathbf{x}_{-i}; \mathbf{G})}{\partial x_i} = 1 - x_i - \delta \sum_{j=1}^{n} g_{ij} x_j \leq 0$$

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- In particular

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- A Nash equilibrium is given by Bonacich centrality measure.

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- The potential function approach is given by:

$$P(\mathbf{x};\delta,\mathbf{G}) = \mathbf{x}^T\mathbf{1} - \frac{1}{2}\mathbf{x}^T[\mathbf{I}+\delta\mathbf{G}]\mathbf{x}.$$

Proposition

If $\delta < \frac{-1}{\lambda_{min}(\mathbf{G})}$, then the potential function P is strictly concave

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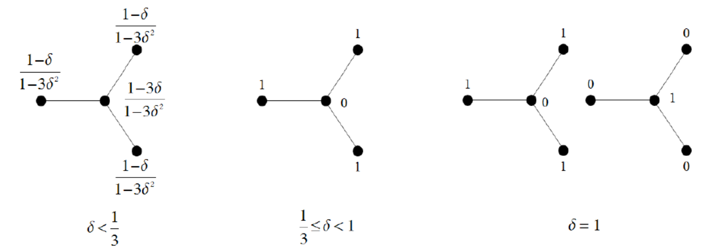
The results depend on the sparsity of the network $\mathbf{G}$, which is captured by $\lambda_{min}(\mathbf{G}) < 0$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$\delta = 1$

Corollary

For regular graphs, there is a unique equilibrium if and only if $\delta < \frac{-1}{\lambda_{min}(\mathbf{G})}$ and this equilibrium is interior. If $\delta \geq \frac{-1}{\lambda_{min}(\mathbf{G})}$ there are both interior and corner equilibria.

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Main insight from this approach: Connection with spectral analysis and the lowest eigenvalue of a network

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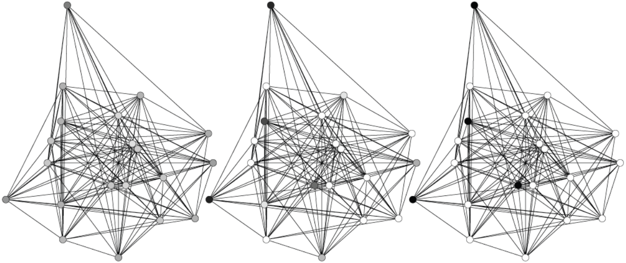
Complex networks can be analyzed



(a) $\delta = 0.15$, unique equilibrium (b) $\delta = 0.35$, 3 equilibria (c) $\delta = 0.55$, 31 equilibria

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- Agent i's returns from x_i are higher when total crime levels are lower, capturing the possibility that criminals compete for victims or territory.
- Agent i's costs are lower when i's friends engage in more crime, capturing peer effects.
- Together, player i's payoffs are given by:

$$u_i(x_i, \mathbf{x}_{-i}; \alpha, \varphi, \mathbf{G}) = x_i(1 - \sum_{j=1}^{n} x_j) - cx_i(1 - \varphi \sum_{j=1}^{n} g_{ij} x_j)$$

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- $0 \leq c < 1$ is a cost parameter.
- $\beta \geq 0$ such that $\beta \leq \alpha$ is the impact of friends' crime.
- Maximizing u_i subject to $x_i \geq 0$, we get

$$x_i = \max\{x_0 - \sum_{j=1}^{n} h_{ij} g_{ij}, 0\}$$

where $x_0 = (1-c)/2\alpha$ is individually optimal when no other agent engages in crime and $h_{ij} = \frac{1}{2}(1 - \frac{c\beta}{\alpha} g_{ij})$.

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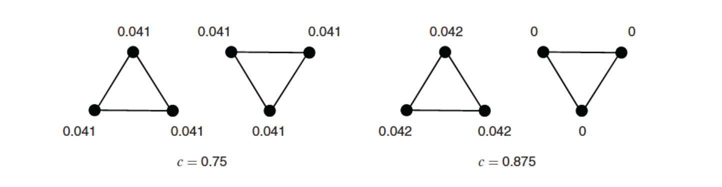
- The index $\lambda_{min}(\mathbf{H})$ shapes the equilibrium set.
- Studying the friendships, $\mathbf{G}$, and costs, c, as they feed into $\lambda_{min}(\mathbf{H})$ yields predictions as to levels and patterns of crime.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$c = 0.875$

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- It is possible to extend and generalize the analysis borrowing ideas from the theory of variational inequalities
- The lowest eigenvalue still useful as a way of determining uniqueness
- The extension does not rely on potential games

- Our main goal is to develop a unified framework for the study of how network interactions can function as a mechanism for propagation and amplification of microeconomic shocks
- We can use a reduced form approach
- This general framework, allows us to provide simple formulas to understand how local shocks may propagate to the entire economy

- Consider an economy consisting of n agents indexed by $N = \{1, \ldots, n\}$
- Let $x_i \in \mathbb{R}$ be agent i's state, which captures the agent's choice of action (e.g., output or investment) or some other economic variable of interest (such as the solvency of a financial institution)
- The states of different agents are *interlinked* through a network

- To model how the state of any given agent i depends on the states of other agents we model follow a reduced form approach
- Formally, we assume that

$$x_i = \varphi\left(\sum_{j=1}^{n} w_{ij}x_j + \varepsilon_i\right), \tag{1}$$

  where φ is a continuous and increasing function, which we refer to as the economy's interaction function
- This function represents the nature of interactions between the agents in the economy

- The variable ε_i is an “agent-level” shock, which captures stochastic disturbances to i’s state
- We assume that these shocks are *i.i.d.* (so that they corresponds to “idiosyncratic shocks”) and have mean zero and variance σ^2
- The scalar $w_{ij} \geq 0$ in (1) captures the extent of interaction between agents i and j
- In particular, a higher w_{ij} means that the state of agent i is more sensitive to the state of agent j, whereas $w_{ij} = 0$ implies that agent j does not have a direct impact on i’s state

- Without loss of generality, assume that $\sum_{j=1}^{n} w_{ij} = 1$, which guarantees that the extent to which the state of each agent depends on the rest of the agents is constant
- We say that the economy is *symmetric* if $w_{ij} = w_{ji}$ for all pair of agents i and j
- Thus for a given φ, the interactions between agents can be also represented by a weighted, directed graph on n vertices, which we refer to as the economy's interaction network

- Each node in this network corresponds to an agent and a directed edge from node j to node i is present if $w_{ij} > 0$, that is, if the state of agent i is directly affected by the state of agent j
- Finally, define *the macro state* of the economy as

$$y = f(h(x_1) + \cdots + h(x_n)), \tag{2}$$

where $f, h : \mathbb{R} \longrightarrow \mathbb{R}$

- y represents some macroeconomic outcome of interest that is obtained by aggregating the individual states of all agents
- We refer to f as the economy's aggregation function

Definition

Given the realization of the shocks $(\varepsilon_1, \ldots, \varepsilon_n)$, an equilibrium of the economy is a collection of states $(x_1, \ldots, x_n)$ such that equation (1) holds for all agents i simultaneously

- This solution concept is an *ex post* equilibrium notion, in the sense that agents' states are determined after the shocks are realized
- This notion enables us to study how the equilibrium varies as a function of the shock realization
- This framework assumes that the equilibrium is *interior*

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- This framework assumes that the equilibrium is *interior*
- Too strong!!

Assumption

$\varphi(0) = f(0) = h(0)$

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This normalization guarantees that, in the absence of shocks, the equilibrium state of all agents and the economy's macro state are equal to zero

- Previous framework nests a general class of network games as a special case
- Consider, for example, an n-player, complete information game, in which the utility function of agent i is given by

$$u_i(x_i, \ldots, x_n) = x_i \varphi\left(\sum_{j=1}^{n} w_{ij} x_j + \varepsilon_i\right) - \frac{1}{2} x_i^2, \tag{3}$$

where x_i denotes the action of player i and ε_i is the realization of some shock to her payoffs

- That is, the payoff of player i depends not only on her own action, but also on those of her neighbors via the interaction function φ
- The underlying network, encoded in terms of coefficients w_{ij} , captures the pattern and strength of strategic interactions between various players in the game
- It is immediate to verify that as long as the interaction function φ satisfies certain regularity conditions-essentially to ensure that one can use the first-order conditions-and that w_{ii} for all i, the best-response of player i as a function of the actions of other players is given by equation (1)

- The collection of $(x_1, \ldots, x_n)$ that solves the system (1) is a NE for the game
- Note that since f is increasing, the players face a game of *strategic complements* over the network: the benefit of taking a higher action to player i increases the higher the actions of her neighbors are
- Examples of such network games include research collaboration among firms (Goyal and Moraga-Gonzalez, (2001)), crime networks (Ballester, Calvo-Armengol, and Zenou, (2006)), peer effect and education decisions in social networks (Calvo-Armengol, Patacchini, and Zenou, (2009)), and local consumption externalities (Candogan, Bimpikis, and Ozdaglar, (2012))

- Two natural candidates for the economy's macro state in this context are
  1. $y_{agg} = x_1 + \cdots + x_n$
  2. $y_{sw} = \frac{1}{2}\sum_{i=1}^{n} x_i^2$, which corresponds to $f(z) = z$ and $h(z) = \frac{z^2}{2}$

- Previous setup also nests a class of models that focus on the propagation of shocks in the real economy
- Next example provides a model along the lines of Long and Plosser (1983) and Acemoglu, Carvalho, Ozdaglar, and Tahbaz-Salehi (2012)
- Consider an economy consisting of n competitive firms (or sectors) denoted by $\{1, \ldots, n\}$ each of which producing a distinct product
- Each product can be either consumed by a mass of consumers or used as an input for production of other goods

- Firms employ Cobb-Douglas production technologies with constant returns to scale that transform labor and intermediate goods to final products
- Production is subject to some idiosyncratic technology shock
- Formally, the output of firm i, which is denoted by X_i, is equal to

$$X_i = b_i A_i^{\alpha} l_i^{1-\alpha} \left(\prod_{i=1}^{n} X_{ij}^{w_{ij}} \right)^{\alpha}, \tag{4}$$

where A_i is the corresponding productivity shock; l_i is the amount of labor hired by firm i; X_{ij} is the amount of good j used for production of good i; b_i is a constant; and $\alpha \in (0,1)$ is the share of intermediate goods in production

- The exponent $w_{ij} \geq 0$ captures the share of good j in the production technology of good i: a higher w_{ij} means that good j is more important in producing i, whereas $w_{ij} = 0$ implies that good j is not a required input for i's production technology
- The assumption that firms employ constant returns to scale technologies implies that $\sum_{j=1}^{n} w_{ij} = 1$ for all i
- The economy contains a unit mass of identical consumers
- Each consumer is endowed with one unit of labor which can be hired by the firms for the purpose of production

- The representative consumer has symmetric Cobb-Douglas preferences over the n goods produced in the economy
- Formally,

$$u(c_1, \ldots, c_n) = \tilde{b} \prod_{i=1}^{n} c_i^{\frac{1}{n}}, \tag{5}$$

where c_i is the amount of good i consumed and $\tilde{b}$ is some positive constant

- Consider the first-order conditions corresponding to firm i' problem

$$X_{ij} = \alpha w_{ij} p_i \frac{X_i}{p_j}, \quad (6)$$

$$l_i = (1-\alpha) p_i \frac{X_i}{w}, \quad (7)$$

where w denotes the market wage and p_i is the price of good i

- The market clearing condition for good i, given by $c_i + \sum_{j=1}^{n} X_{ji} = X_i$, implies that

$$s_i = \frac{w}{n} + \alpha \sum_{j=1}^{n} w_{ji} s_j, \quad (8)$$

where $s_i = p_i X_i$ is the equilibrium sales of firm i

- Previous equality defines a linear system of equations in terms of the equilibrium sales of different firms
- It is straightforward to show that $s_i = p_i X_i = \xi_i w$ for some ξ_i
- Replacing for equilibrium price p_i in equations (6) and (7) in terms of the output of firm i yields

$$X_{ij} = \alpha w_{ij} \xi_i \frac{X_j}{\xi_j},$$

and

$$l_i = (1-\alpha)\xi_i$$

- Plugging these quantities back into the production function of firm i leads to

$$X_i = b_i \xi_i (1-\alpha)^{1-\alpha} A_i^{\alpha} \prod_{j=1}^{n} (\alpha w_{ij} X_i / \xi_j)^{\alpha w_{ij}}$$

- With the proper choice of constants b_i, the log output of firm i, denoted by $x_i = log(X_i)$, satisfies

$$x_i = \alpha \sum_{j=1}^{n} w_{ij} x_j + \alpha \varepsilon_i, \qquad (9)$$

  where $\varepsilon_i = log(A_i)$ is the log productivity shock to firm i

- In other words, the interactions between different firms can be cast as a special case of our general framework in equation (1) with linear interaction function $\varphi(z) = \alpha z$

- The natural candidate for the economy's macro state y, can be expressed in terms of expression (2)
- Because of the constant returns to scale assumption, firms make zero profits in equilibrium, all the surplus in the economy goes to the consumers, and as a consequence, value added is simply equal to the market wage w
- Choosing the ideal price index as the numeraire, $\frac{n}{\tilde{b}}(p_1 \cdot \cdot p_n)^{\frac{1}{n}} = 1$, and using the fact that $p_i = \xi_i w / X_i$ we obtain that the log real value added in the economy is equal to

$$log(w) = \frac{1}{n}\sum_{i=1}^{n} log(X_i) - \frac{1}{n}\sum_{i=1}^{n} log(\xi_i) + log(\tilde{b}/n)$$

- Therefore, with the appropriate choice of $\tilde{b}$, we can rewrite $log(GDP)$ as

$$y = log(GDP) = \frac{1}{n}\sum_{i=1}^{n} x_i \tag{10}$$

- We may establish the existence and (generic) uniqueness of equilibrium
- In general, the set of equilibria not only depends on the economy's interaction network, but also on the properties of the interaction function
- We impose the following regularity assumption on φ

Assumption

There exists $\beta \leq 1$ such that $|\varphi(z) - \varphi(\tilde{z})| \leq \beta |z - \tilde{z}|$ for all $z, \tilde{z} \in \mathbb{R}$. Furthermore if $\beta = 1$, then there exists $\delta > 0$ such that $|f(z)| < \delta$ for all $z \in \mathbb{R}$

- Previous assumption guarantees that the economy's interaction function φ is either
  i) a contraction with Lipschitz constant $\beta < 1$; or
  ii) a bounded non-expansive mapping

- Previous assumption guarantees that the economy's interaction function φ is either
  i) a contraction with Lipschitz constant $\beta < 1$; or
  ii) a bounded non-expansive mapping
- Either way, it is easy to establish that an equilibrium always exists
- When $\beta < 1$, the contraction mapping theorem implies that (1) always has a fixed point, whereas if φ is bounded, the existence of equilibrium is guaranteed by the Brouwer fixed point theorem

Theorem

Suppose that Assumption 1 is satisfied. Then, an equilibrium always exists and is generically unique

- The economy may have multiple equilibria, however, when $\beta = 1$
- Nevertheless, previous theorem guarantees that the equilibrium is *generically unique*, in the sense that the economy has multiple equilibria only for a measure zero set of realizations of agents-level shocks

- We may carry out a first-order approximation to the agents' equilibrium states around the point where $\varepsilon_i = 0$ for all i
- If the size of the agent-level shocks are small, such an approximation captures the dominant effects of how shocks shape the economy's macro state
- Let us first use the implicit function theorem to differentiate both sides of the interaction equation (1) with respect to the shock to agent r

- Mathematically we get:

$$\frac{\partial x_i}{\partial \varepsilon_r} = \varphi'\left(\sum_{m=1}^{n} w_{im} x_m + \varepsilon_i\right) \cdot \left(\sum_{m=1}^{n} w_{im} \frac{\partial x_m}{\partial \varepsilon_r} + \mathbf{1}_{\{r=i\}}\right) \quad (11)$$

- Evaluating at 0, expression (11) is equal to

$$\frac{\partial x_i}{\partial \varepsilon_r} = \varphi'(0) \cdot \left(\sum_{m=1}^{n} w_{im} \frac{\partial x_m}{\partial \varepsilon_r} + \mathbf{1}_{\{r=i\}} \right)$$

- This equation can be rewritten in matrix form as

$$\frac{\partial x}{\partial \varepsilon_r} = \varphi'(0) \cdot \mathbf{W} \cdot \frac{\partial x}{\partial \varepsilon_r} + \varphi'(0)\mathbf{e}_r,$$

where $x = (x_1, \ldots, x_n)^T$ is the vector of agents' states, $\mathbf{e}_r$ represents the r- unit vector, and $\mathbf{W}$ is the connection matrix

- It follows that

$$\frac{\partial x}{\partial \varepsilon_r} = \varphi'(0) \cdot [\mathbf{I} - \varphi'(0) \cdot \mathbf{W}]^{-1} \cdot \mathbf{e}_r \qquad (12)$$

- Note that, as long as $\varphi'(0) < 1$, the matrix $[\mathbf{I} - \varphi'(0) \cdot \mathbf{W}]^{-1}$, implying that the right-hand side of (12) is well-defined.

Definition

The Leontief matrix of the economy with parameter $\alpha \in [0,1)$ is $\mathbf{L} = [\mathbf{I} - \alpha \cdot \mathbf{W}]^{-1}$, where $\mathbf{W} = [w_{ij}]$ is the economy's interaction matrix (network)

- It follows that (12) may be rewritten as:

$$\frac{\partial x_i}{\partial \varepsilon_r} = \alpha \cdot l_{ir}, \tag{13}$$

where $\alpha = \varphi'(0)$, and l_{ir} is the (i, r) element of the economy's Leontief matrix with parameter α

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- Thus when the agent-level shocks are small, the economy's Leontief matrix serves as a *sufficient statistic* for the network's role in determining the state of agent i
- More specifically, the impact of a shock to agent r on the equilibrium state of agent i is simply captured by l_{ir}

- Then for small perturbations, the equilibrium state of agent i can be linearly approximated as

$$x_i = \alpha \sum_{r=1}^{n} l_{ir} \varepsilon_r \tag{14}$$

- In simple, when the agent-level shocks are small (so that we can rely on a linear approximation), the economy's Leontief matrix serves as a sufficient statistic for the network's role in determining the state of agent i
- Thus the impact of a shock to agent r on the equilibrium state of agent i is simply captured by l_{ir}

- The (i, r) element of the matrix, not only captures the *direct interaction* between agents i and r, but also accounts for all possible *indirect interactions* between the two
- To see this, note that l_{ir} can be rewritten as

$$l_{ir} = 1 + \alpha w_{ir} + \alpha^2 \sum_{k=1}^{n} w_{ik} w_{kr} + \cdots, \tag{15}$$

where the higher-order terms account for the possibility of indirect interactions between i and r

- Thus, essentially, equation (15) shows that a shock to agent r impacts agent i not only through their direct interaction term w_{ir}, but also via indirect interactions with the rest of the agents: such a shock may impact the state of some agent k and then indirectly propagate to agent i
- Note that the impact of a shock to agent r on i's state is deflated by a factor $\alpha < 1$ whenever the length of the indirect interaction chain between the two agents is increased by one

- Our next goal is to derive an approximation to the economy's macro state y in the presence of small schocks
- Differentiating (2) with respect to ε_r yields

$$\frac{\partial y}{\partial \varepsilon_r} = f'\left(h(x_1) + \cdots + h(x_n)\right) \sum_{m=1}^{n} h'(x_m) \frac{\partial x_m}{\partial \varepsilon_r} \tag{16}$$

- Evaluating this expression at $\varepsilon = 0$ and using (13) we get

$$\frac{\partial y}{\partial \varepsilon_r} = \alpha \varphi'(0) h'(0) \sum_{m=1}^{n} l_{mr}, \tag{17}$$

where we have again used that fact that, in the absence of shocks, $x_m = 0$ for all m and that $h(0) = 0$

Definition

For a given parameter $\alpha \in [0,1)$, the Bonacich centrality of agent i is $v_i = \sum_{m=1}^{n} l_{mi}$ where $\mathbf{L} = [l_{ij}]$ is the corresponding Leontief matrix of the economy

- More generally, in any given interaction network, agent i' Bonacich centrality can be written recursively in terms of the centralities of the rest of the agents in the economy:

$$v_i = 1 + \alpha \sum_{j=1}^{n} v_j w_{ji} \tag{18}$$

- This expression shows that i has a higher centrality (and hence a more pronounced impact on the rest of the agents) if it interacts strongly with agents that are themselves central

Theorem

Suppose that $\varphi'(0) < 1$. Then, the first-order approximation to the macro state of the economy is

$$\frac{\partial y}{\partial \varepsilon_r} = \varphi'(0) f'(0) h'(0) \sum_{i=1}^{n} v_i \varepsilon_i \tag{19}$$

where v_i is the Bonacich centrality of agent i with parameter $\alpha = \varphi'(0)$

Let

$$\kappa \equiv \frac{1}{2} f'(0) \sum_{i=1}^{n} \sum_{j=1}^{n} [h'(0)\varphi''(0) \sum_{m=1}^{n} v_m l_{mi} l_{mj} + [\varphi'(0)]^2 h''(0) \sum_{m=1} l_{mi} l_{mj}] \varepsilon_i \varepsilon_j$$

Theorem

Suppose that $\varphi'(0) < 1$. Then, the second-order approximation to the macro state of the economy is given by

$$\begin{aligned} y^{2nd} &= \varphi'(0) f'(0) h'(0) \sum_{i=1}^{n} v_i \varepsilon_i \\ &+ \frac{1}{2} f''(0) [\varphi'(0) h'(0)] \cdot \sum_{i=1}^{n} \sum_{j=1}^{n} v_i v_j \varepsilon_i \varepsilon_j \\ &+ \kappa \end{aligned} \tag{20}$$

where $\mathbf{L} = [l_{ij}]$ is the economy's Leontief matrix with parameter $\alpha = \varphi'(0)$ and v_i is the corresponding Bonacich centrality of agent i

- Note that the first line of (20) is simply the first-order approximation, y^{1st}, characterized in (19)
- The rest of the terms, which depend on the curvatures of the interaction and aggregation functions, capture the second-order aggregate effects

- These terms depend simply on Bonacich centralities, the v_i terms
- This is due to the fact that as long as the interaction function φ is linear, the total influence of agent i on the rest of the agents in the economy is given by the Bonacich centrality of agent $i, v_i = \sum_{m=1}^{n} l_{mi}$
- Other network statistics-in particular, $\sum_{m=1}^{n} v_m l_{mi} l_{mj}$ also play a key role in how shocks propagate throughout the economy

Definition

An economy outperforms another if $\mathbb{E}(y)$ is larger in the former than the latter

- In order to obtain a comparison between the performance of different economies in the presence of small shocks is to compare their first-order approximations
- Recall that the first-order approximation of an economy's macro state is equal to a linear combination of agent-level shocks with the corresponding weights given by the agents' Bonacich centralities, i.e., $y^{1st} = \varphi'(0) f'(0) h'(0) \sum_{i=1}^{n} v_i \varepsilon_i$
- It follows then

Corollary

$\mathbb{E}(y^{1st}) = 0$

- This simple corollary shows that the economy exhibits a certainty equivalence property from an ex ante perspective up to a first-order approximation: *the expected value of the economy's macro state is equal to its unperturbed value when no shocks are present, regardless of the nature of pairwise interactions or the shape of the interaction and aggregation functions*
- In other words, the linear approximation *is not informative* about the comparative performance of different economies, even in the presence of small shocks
- Rather, a meaningful comparison between the ex ante performance of two economies requires that we also take the higher-order terms into account

- We first consider an economy with a general, potentially non-linear aggregation function φ while assuming that f and h are increasing, linear functions
- In this case, the ex-ante performance of the economy is given by

$$\mathbb{E}(y) = \mathbb{E}(g(x_1 + \cdots + x_n))$$

- This observation highlights that the curvature f essentially captures the extent to which society cares about volatility, for instance because of risk-aversion at the aggregate level

- The expected value of the economy's macro state, up to a second-order approximation, is given by

$$\mathbb{E}(y^{2nd}) = \frac{1}{2}\sigma^2 f''(0)[\varphi'(0)h'(0)]^2 \sum_{i=1}^{n} v_i^2 \tag{21}$$

- We are using the assumption that all shocks are independent with mean zero and variance σ^2 and the assumption that functions φ and h are linear

- From (21) it follows that not all economies have identical performances once second-order terms are taken into account
- In this case, the economy's ex ante performance depends on $\sum_{i=1}^{n} v_i^2$, which in turn, can be rewritten as

$$\sum_{i=1}^{n} v_i^2 = nvar(v_1, \ldots, v_n) + \frac{n}{1-\alpha},$$

where $\alpha = \varphi'(0)$

Proposition

Suppose that the aggregation function f is concave (convex). An economy's ex ante performance decreases (increases) in $var(v_1, \ldots, v_n)$

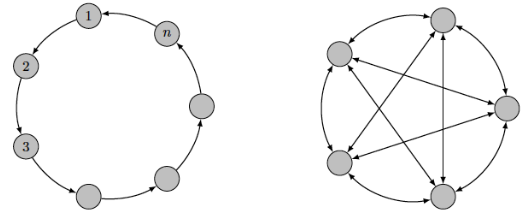
- This proposition implies that, if f is concave, networks in which agents exhibit a less heterogenous distribution of Bonacich centralities *outperform* those with a more unequal distribution

- This is due to the fact that a more equal distribution of Bonacich centralities means that shocks to different agents have a more homogenous impact on the economy's macro state, and thus wash each other out more effectively at the aggregate level

- On the other hand, a more unequal distribution of centralities implies that shocks to some agents play a disproportionally larger role in shaping y and as a result, are not canceled out by the rest of the agent-level shocks, increasing the overall volatility and reducing the value of $\mathbb{E}(y)$ whenever f is concave

Definition

An economy is regular if $\sum_{j=1}^{n} w_{ji} = 1$ for all agents i



(a) The ring financial network (b) The complete financial network

Lemma

In any regular economy, all agents have identical Bonacich centralities.

- Thus $var(v_1, \ldots, v_n)$ is minimized for all regular economies, implying that with a concave f, they out perform all other economies from an ex ante perspective: all agent-level shocks in such an economy take symmetric roles in determining the macro state, and minimize the overall volatility of $x_1 + \cdots + x_n$ and thus increase $\mathbb{E}(y)$

Proposition

Suppose that aggregation function f is concave (convex). Any regular economy outperforms (underperforms) all other economies, whereas the economy with the star interaction network underperforms (outperforms) all others.

- We now focus on the role of non-linear interactions in shaping the economy's ex ante performance
- We consider an economy with a general, non-linear interaction function φ, while assuming that f and h are increasing, linear functions
- The ex ante performance of such an economy is given by

$$\mathbb{E}(y) = \sum_{i=1}^{n} \mathbb{E}(x_i) = \sum_{i=1}^{n} \mathbb{E}\left(\varphi\left(\sum_{j=1}^{n} w_{ij}x_j + \varepsilon_i\right)\right)$$

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- Previous equation highlights that the curvature of the interaction function φ captures the extent of "*risk-aversion*" at the micro-level

- We focus on the set of symmetric, regular economies: $w_{ij} = w_{ji}$ for all i, j

Proposition

Suppose that there are no self-interaction terms, that is, $w_{ii} = 0$. If the interaction function φ is concave (convex), then the complete network outperforms (underperforms) all other symmetric economies

- The aim of this section is to provide an introduction to the literature on financial contagion on networks
- We consider contagion via transmission of shocks, i.e., an abrupt drop in the flow of revenue to one firm, which affects other firms connected to it through financial linkages

- Let there be N financial firms (say, banks)
- Each firm has liabilities, equal to l towards external investors, and assets, given by claims to the returns on projects
- There are N projects and the return on each project i is subject to shocks: it is equal to R if there is no shock, and to $R - s_i \in [0, R)$ if a shock hits
- We want to understand the effects of the presence of financial linkages among firms on their financial situation, and in particular on their solvency

- We can model these linkages in a general way by saying that the value v_i the assets of a firm i may be related to the value v_j of the assets of any other firm $j \neq i$ as they both depend on the vector $\mathbf{r}$ which describes the realizations of the returns of the N projects
- For each r_i we have $r_i = R_i - s_i$ or $r_i = R_i$
- This relationship is modeled as follows:

$$v = f(A; \mathbf{r}),$$

where $f : \mathbb{R}_+^{N \times N} \times \mathbb{R}_+^N \longrightarrow \mathbb{R}_+^N$ and A is $N \times N$ non negative matrix with generic entry a_{ij}

- The matrix A describes the pattern of the linkages among the N firms and the function $f(\cdot,\cdot)$ the effect of these linkages on the value of the firms' assets
- If the value v_i of the assets of firm i is lower than the value l of its liabilities, the firm default
- Because of the presence of linkages among Ěfirms, default events are correlated

- Since default of a firm is costly, one of the main objectives of this type of analysis is indeed to analyze the extent of default in the system and whether generalized default, or contagion, may occur in the presence of such linkages.

- The linkages among firms may have a different nature
- They may arise from the mutual ownership of the claims to the returns of the underlying projects: that is, the returns on the assets of a generic firm i are given by a certain linear combination of the returns of the N projects with weights given by the i-row of the matrix A: $\sum_{j=1}^{n} a_{ij} r_j$

- The terms a_{ij} describe the ownership by firm i of claims entitling the owner to a fraction of the returns of project j
- These claims are obtained via a sequence of rounds of exchanges of assets by each firm i; initially endowed with full ownership of project i; with a subset of other firms (constituting its immediate neighbors)
- The pattern of exchanges at each round is described by the matrix B; where the nonzero elements of row i describe i's trades with its immediate neighbors

- Hence we have (when the number of rounds of these exchanges is given by k)

$$A = B^k \quad \text{and} \quad f(A;\mathbf{r}) = A \cdot \mathbf{r} \tag{22}$$

- We may consider consider linkages among both assets and liabilities of firms, arising from mutual lending and borrowing relationships among them, via standard debt contracts
- In this case a_{ij} denotes the payments due from i to firm j

- The value of firm i's liabilities is then augmented now to include the value of the payments due to other firms $l_i + \sum_{j \neq i} a_{ij}$
- Firm i is again endowed with a 100% share of the returns generated by project i
- The book-value value of its assets is similarly augmented by the book-value of the loans it granted, $r_i + \sum_{j \neq i} a_{ji}$
- To establish the solvency of firm i however what matters is not the book-value but the actual value of the firms' assets, which reflects the actual payments made by its debtors and may be less than the due payments when they are in default

- Hence the actual payments to firms i depend on the value of the assets of this firms debtors, that is on the payments they in turn receive from their own debtors
- This creates an interdependence among the actual value of the assets of all firms in the system
- For any vector $\mathbf{r}$ of realized returns, a final (equilibrium) repayment vector is then obtained as a solution of the following system

$$p_i(\mathbf{r})(\sum_{j\neq i} a_{ij} + l_i) = \max\left\{\sum_{j\neq i} a_{ij} + l_i, \sum_{j\neq i} p_j(\mathbf{r}) a_{ji} + r_i\right\} \forall i \quad (23)$$