Lecture 9: Matching

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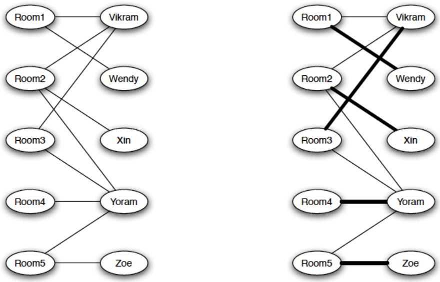
- When we consider markets creating opportunities for interaction among buyers and sellers, there is an implicit network encoding the access between these buyers and sellers
- Matching markets have a long history of study in economics, operations research, and other areas because they embody, in a very clean and stylized way, a number of basic principles:
  i) the way in which people may have different preferences for different kinds of goods
  ii) the way in which prices can decentralize the allocation of goods to people,and
  iii) the way in which such prices can in fact lead to allocations that are socially optimal

- The model we start with is called the bipartite matching problem, and we can motivate it via the following scenario
- Suppose that the administrators of a college dormitory are assigning rooms to returning students for a new academic year
- each room is designed for a single student, and each student is asked to list several acceptable options for the room they'd like to get

- Students can have different preferences over rooms; some people might want larger rooms, quieter rooms, sunnier rooms, and so forth-and so the lists provided by the students may overlap in complex ways
- We can model the lists provided by the students using a graph, as follows
- There is a node for each student, a node for each room, and an edge connecting a student to a room if *the student has listed the room as an acceptable option*



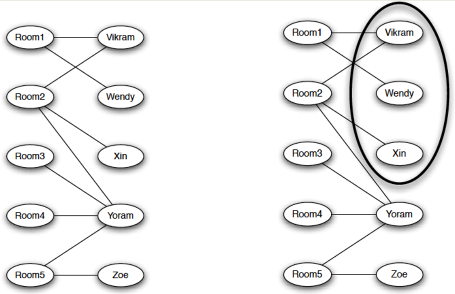
(a) *Bipartite Graph* (b) *A Perfect Matching*

Figure 10.1: (a) An example of a bipartite graph. (b) A perfect matching in this graph, indicated via the dark edges.

- In a bipartite graph the nodes are divided into two categories, and each edge connects a node in one category to a node in the other category
- In this case, the two categories are students and rooms

- Let's return to the task that the college dorm administrators were trying to solve: assigning each student a room that they'd be happy to accept
- This task has a natural interpretation in terms of the graph we've just drawn: since the edges represent acceptable options for students, we want to assign a distinct room to each student, so that each student is assigned a room to which he or she is connected by an edge

- We will refer to such an assignment as a perfect matching
- When there are an equal number of nodes on each side of a bipartite graph, a perfect matching is an assignment of nodes on the left to nodes on the right, in such a way that
  i) each node is connected by an edge to the node it is assigned to, and
  ii) no two nodes on the left are assigned to the same node on the right



(a) *Bipartite graph with no perfect matching*

(b) *A constricted set demonstrating there is no perfect matching*

Figure 10.2: (a) A bipartite graph with no perfect matching. (b) A constricted set demonstrating there is no perfect matching.

- We call the set of three students in this example a constricted set, since their edges to the other side of the bipartite graph "*constrict*" the formation of a perfect matching
- This example points to a general phenomenon, which we can make precise by defining in general what it means for a set to be constricted, as follows

- First, for any set of nodes S on the right-hand side of a bipartite graph, we say that a node on the left-hand side is a neighbor of S if it has an edge to some node in S
- We define the neighbor set of S, denoted $N(S)$, to be the collection of all neighbors of S
- Finally, we say that a set S on the right-hand side is constricted if S is strictly larger than $N(S)$-that is, S contains strictly more nodes than $N(S)$ does

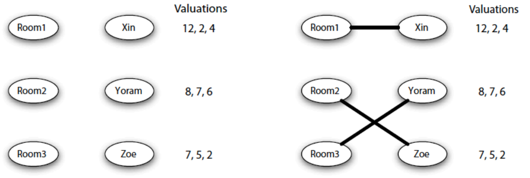
- Any time there is a constricted set S in a bipartite graph, it immediately shows that there can be no perfect matching: each node in S would have to be matched to a different node in $N(S)$, but there are more nodes in S than there are in $N(S)$, so this is not possible

Theorem (Matching Theorem)

If a bipartite graph (with equal numbers of nodes on the left and right) has no perfect matching, then it must contain a constricted set.

- The problem of bipartite matching from the previous section illustrates some aspects of a market in a very simple form
- Individuals express preferences in the form of acceptable options
- A perfect matching then solves the problem of allocating objects to individuals according to these preferences
- And if there is no perfect matching, it is because of a "*constriction*" in the system that blocks it

- Rather than expressing preferences simply as binary "acceptable-or-not" choices, we allow each individual to express how much they'd like each object, in numerical form
- In our example of students and dorms, suppose that rather than specifying a list of acceptable rooms, each student provides a numerical score for each room, indicating how happy they'd be with it
- We will refer to these numbers as the students' valuations for the respective rooms



(a) *A set of valuations* (b) *An optimal assignment*

Figure 10.3: (a) A set of valuations. Each person's valuations for the objects appears as a list next to them. (b) An optimal assignment with respect to these valuations.

- We can define valuations whenever we have a collection of individuals evaluating a collection of objects
- And using these valuations, we can evaluate the *quality* of an assignment of objects to individuals, as follows:
- it is the sum of each individual's valuation for what they get
- Thus, for example, the quality of the assignment illustrated in Figure 10.3 is $12+6+5=23$

- If the dorm administrators had accurate data on each student's valuations for each room, then a reasonable way to assign rooms to students would be to choose the assignment of maximum possible quality.
- We will refer to this as the optimal assignment, since it maximizes the total happiness of everyone for what they get
- You can check that the assignment in Figure 10.3(b) is in fact the optimal assignment for this set of valuations
- Of course, while the optimal assignment maximizes total happiness, it does not necessarily give everyone their favorite item; for example, in Figure 10.3(b), all the students think Room 1 is the best, but it can only go to one of them