CHAPTER 1

NUMBER SYSTEMS

1.1 Introduction

In your earlier classes, you have learnt about the number line and how to represent various types of numbers on it (see Fig. 1.1).

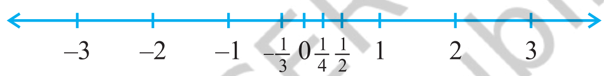


Fig. 1.1 : The number line

Just imagine you start from zero and go on walking along this number line in the positive direction. As far as your eyes can see, there are numbers, numbers and numbers!

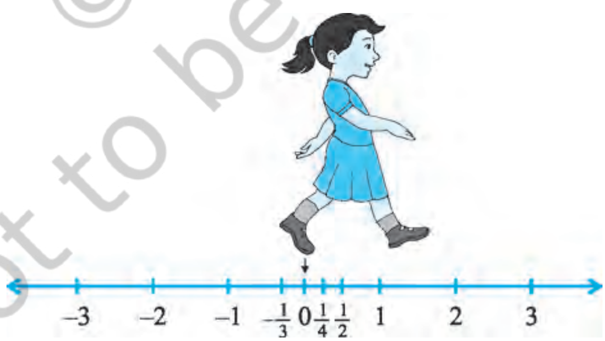
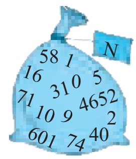


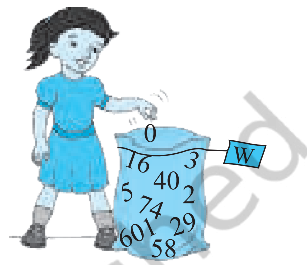
Fig. 1.2

Now suppose you start walking along the number line, and collecting some of the numbers. Get a bag ready to store them!

You might begin with picking up only natural numbers like 1, 2, 3, and so on. You know that this list goes on for ever. (Why is this true?) So, now your bag contains infinitely many natural numbers! Recall that we denote this collection by the symbol **N**.

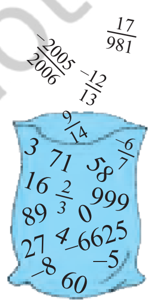


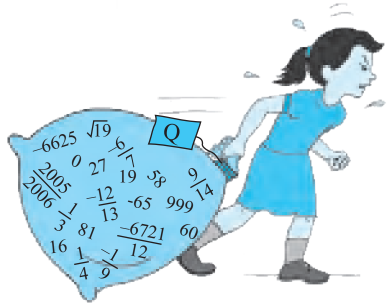
Now turn and walk all the way back, pick up zero and put it into the bag. You now have the collection of *whole numbers* which is denoted by the symbol **W.**



Now, stretching in front of you are many, many negative integers. Put all the negative integers into your bag. What is your new collection? Recall that it is the collection of all *integers*, and it is denoted by the symbol **Z**.

Are there some numbers still left on the line? Of course! There are numbers like $\frac{1}{2}, \frac{3}{4}$, or even $\frac{-2005}{2006}$. If you put all such numbers also into the bag, it will now be the





collection of *rational numbers*. The collection of rational numbers is denoted by **Q**. 'Rational' comes from the word 'ratio', and Q comes from the word 'quotient'.

You may recall the definition of rational numbers:

A number 'r' is called a *rational number,* if it can be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. (Why do we insist that $q \neq 0$?)

Notice that all the numbers now in the bag can be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. For example, –25 can be written as $\frac{-25}{1}$; here $p = -25$ and $q = 1$. Therefore, the rational numbers also include the natural numbers, whole numbers and integers.

You also know that the rational numbers do not have a unique representation in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. For example, $\frac{1}{2} = \frac{2}{4} = \frac{10}{20} = \frac{25}{50} = \frac{47}{94}$, and so on. These are *equivalent rational numbers* (*or fractions*). However, when we say that $\frac{p}{q}$ is a rational number, or when we represent $\frac{p}{q}$ on the number line, we assume that $q \neq 0$ and that p and q have no common factors other than 1 (that is, p and q are *co-prime*). So, on the number line, among the infinitely many fractions equivalent to $\frac{1}{2}$, we will choose $\frac{1}{2}$ to represent all of them.

Now, let us solve some examples about the different types of numbers, which you have studied in earlier classes.

Example 1 : Are the following statements true or false? Give reasons for your answers.

(i) Every whole number is a natural number.

(ii) Every integer is a rational number.

(iii) Every rational number is an integer.

Solution : (i) False, because zero is a whole number but not a natural number.

(ii) True, because every integer m can be expressed in the form $\frac{m}{1}$, and so it is a rational number.

(iii) False, because $\frac{3}{5}$ is not an integer.

Example 2 : Find five rational numbers between 1 and 2.

We can approach this problem in at least two ways.

Solution 1 : Recall that to find a rational number between r and s, you can add r and s and divide the sum by 2, that is $\frac{r+s}{2}$ lies between r and s. So, $\frac{3}{2}$ is a number between 1 and 2. You can proceed in this manner to find four more rational numbers between 1 and 2. These four numbers are $\frac{5}{4}, \frac{11}{8}, \frac{13}{8}$ and $\frac{7}{4}$.

Solution 2 : The other option is to find all the five rational numbers in one step. Since we want five numbers, we write 1 and 2 as rational numbers with denominator 5 + 1, i.e., $1 = \frac{6}{6}$ and $2 = \frac{12}{6}$. Then you can check that $\frac{7}{6}, \frac{8}{6}, \frac{9}{6}, \frac{10}{6}$ and $\frac{11}{6}$ are all rational numbers between 1 and 2. So, the five numbers are $\frac{7}{6}, \frac{4}{3}, \frac{3}{2}, \frac{5}{3}$ and $\frac{11}{6}$.

Remark : Notice that in Example 2, you were asked to find five rational numbers between 1 and 2. But, you must have realised that in fact there are infinitely many rational numbers between 1 and 2. In general, **there are infinitely many rational numbers between any two given rational numbers**.

Let us take a look at the number line again. Have you picked up all the numbers? Not, yet. The fact is that there are infinitely many more numbers left on the number line! There are gaps in between the places of the numbers you picked up, and not just one or two but infinitely many. The amazing thing is that there are infinitely many numbers lying between any two of these gaps too!

So we are left with the following questions:

1. What are the numbers, that are left on the number line, called?
2. How do we recognise them? That is, how do we distinguish them from the rationals (rational numbers)?

These questions will be answered in the next section.

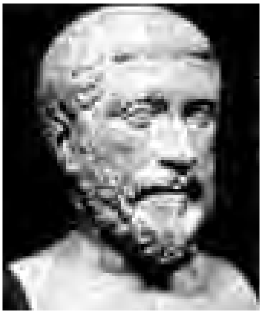
EXERCISE 1.1

1. Is zero a rational number? Can you write it in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$?
2. Find six rational numbers between 3 and 4.
3. Find five rational numbers between $\frac{3}{5}$ and $\frac{4}{5}$.
4. State whether the following statements are true or false. Give reasons for your answers.
   (i) Every natural number is a whole number.
   (ii) Every integer is a whole number.
   (iii) Every rational number is a whole number.

1.2 Irrational Numbers

We saw, in the previous section, that there may be numbers on the number line that are not rationals. In this section, we are going to investigate these numbers. So far, all the numbers you have come across, are of the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. So, you may ask: are there numbers which are not of this form? There are indeed such numbers.

The Pythagoreans in Greece, followers of the famous mathematician and philosopher Pythagoras, were the first to discover the numbers which were not rationals, around 400 BC. These numbers are called *irrational numbers* (*irrationals*), because they cannot be written in the form of a ratio of integers. There are many myths surrounding the discovery of irrational numbers by the Pythagorean, Hippacus of Croton. In all the myths, Hippacus has an unfortunate end, either for discovering that $\sqrt{2}$ is irrational or for disclosing the secret about $\sqrt{2}$ to people outside the secret Pythagorean sect!



Pythagoras
(569 BCE – 479 BCE)
Fig. 1.3

Let us formally define these numbers.

A number 's' is called *irrational,* if it cannot be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

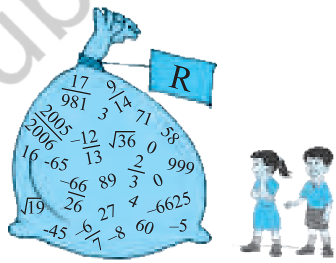
You already know that there are infinitely many rationals. It turns out that there are infinitely many irrational numbers too. Some examples are:

$$\sqrt{2}, \sqrt{3}, \sqrt{15}, \pi, 0.10110111011110...$$

Remark : Recall that when we use the symbol $\sqrt{\ }$, we assume that it is the positive square root of the number. So $\sqrt{4} = 2$, though both 2 and –2 are square roots of 4.

Some of the irrational numbers listed above are familiar to you. For example, you have already come across many of the square roots listed above and the number π.

The Pythagoreans proved that $\sqrt{2}$ is irrational. Later in approximately 425 BC, Theodorus of Cyrene showed that $\sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{10}, \sqrt{11}, \sqrt{12}, \sqrt{13}, \sqrt{14}, \sqrt{15}$ and $\sqrt{17}$ are also irrationals. Proofs of irrationality of $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, etc., shall be discussed in Class X. As to π, it was known to various cultures for thousands of years, it was proved to be irrational by Lambert and Legendre only in the late 1700s. In the next section, we will discuss why 0.10110111011110... and π are irrational.



Let us return to the questions raised at the end of the previous section. Remember the bag of rational numbers. If we now put all irrational numbers into the bag, will there be any number left on the number line? The answer is no! It turns out that the collection of all rational numbers and irrational numbers together make up what we call the collection of *real numbers*, which is denoted by **R**. Therefore, a real number is either rational or irrational. So, we can say that **every real number is represented by a unique point on the number line. Also, every point on the number line represents a unique real number**. This is why we call the number line, the *real number line*.

In the 1870s two German mathematicians, Cantor and Dedekind, showed that : Corresponding to every real number, there is a point on the real number line, and corresponding to every point on the number line, there exists a unique real number.

R. Dedekind (1831-1916)
Fig. 1.4

G. Cantor (1845-1918)
Fig. 1.5

Let us see how we can locate some of the irrational numbers on the number line.

Example 3 : Locate $\sqrt{2}$ on the number line.

Solution : It is easy to see how the Greeks might have discovered $\sqrt{2}$. Consider a square OABC, with each side 1 unit in length (see Fig. 1.6). Then you can see by the Pythagoras theorem that OB = $\sqrt{1^2 + 1^2} = \sqrt{2}$. How do we represent $\sqrt{2}$ on the number line?

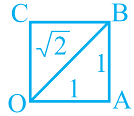


Fig. 1.6

This is easy. Transfer Fig. 1.6 onto the number line making sure that the vertex O coincides with zero (see Fig. 1.7).

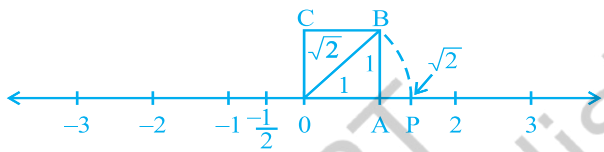


Fig. 1.7

We have just seen that OB = $\sqrt{2}$. Using a compass with centre O and radius OB, draw an arc intersecting the number line at the point P. Then P corresponds to $\sqrt{2}$ on the number line.

Example 4 : Locate $\sqrt{3}$ on the number line.

Solution : Let us return to Fig. 1.7.

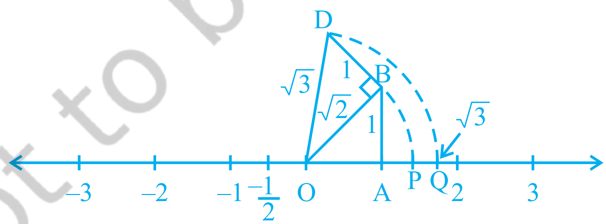


Fig. 1.8

Construct BD of unit length perpendicular to OB (as in Fig. 1.8). Then using the Pythagoras theorem, we see that OD = $\sqrt{\left(\sqrt{2}\right)^2 + 1^2} = \sqrt{3}$. Using a compass, with centre O and radius OD, draw an arc which intersects the number line at the point Q. Then Q corresponds to $\sqrt{3}$.

In the same way, you can locate $\sqrt{n}$ for any positive integer n, after $\sqrt{n-1}$ has been located.

EXERCISE 1.2

1. State whether the following statements are true or false. Justify your answers.
   (i) Every irrational number is a real number.
   (ii) Every point on the number line is of the form $\sqrt{m}$, where m is a natural number.
   (iii) Every real number is an irrational number.
2. Are the square roots of all positive integers irrational? If not, give an example of the square root of a number that is a rational number.
3. Show how $\sqrt{5}$ can be represented on the number line.
4. **Classroom activity (Constructing the 'square root spiral') :** Take a large sheet of paper and construct the 'square root spiral' in the following fashion. Start with a point O and draw a line segment OP_1 of unit length. Draw a line segment P_1P_2 perpendicular to OP_1 of unit length (see Fig. 1.9). Now draw a line segment P_2P_3 perpendicular to OP_2. Then draw a line segment P_3P_4 perpendicular to OP_3. Continuing in this manner, you can get the line segment $P_{n-1}P_n$ by drawing a line segment of unit length perpendicular to OP_{n-1}. In this manner, you will have created the points $P_2, P_3, \ldots, P_n, \ldots$, and joined them to create a beautiful spiral depicting $\sqrt{2}, \sqrt{3}, \sqrt{4}, \ldots$

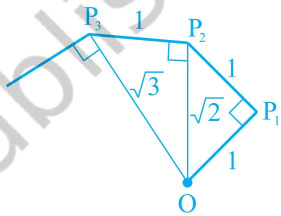


Fig. 1.9 : Constructing square root spiral

1.3 Real Numbers and their Decimal Expansions

In this section, we are going to study rational and irrational numbers from a different point of view. We will look at the decimal expansions of real numbers and see if we can use the expansions to distinguish between rationals and irrationals. We will also explain how to visualise the representation of real numbers on the number line using their decimal expansions. Since rationals are more familiar to us, let us start with them. Let us take three examples : $\frac{10}{3}, \frac{7}{8}, \frac{1}{7}$.

Pay special attention to the remainders and see if you can find any pattern.

Example 5 : Find the decimal expansions of $\frac{10}{3}$, $\frac{7}{8}$ and $\frac{1}{7}$.

Solution :

$$
\begin{array}{r|l}
 & 3.333... \\ \hline
3 & 10 \\
 & \underline{\ 9\ } \\
 & \ 10 \\
 & \ \underline{\ 9\ } \\
 & \ \ 10 \\
 & \ \ \underline{\ 9\ } \\
 & \ \ \ 10 \\
 & \ \ \ \underline{\ 9\ } \\
 & \ \ \ \ 1
\end{array}
\qquad
\begin{array}{r|l}
 & 0.875 \\ \hline
8 & 7.0 \\
 & \underline{64} \\
 & \ 60 \\
 & \ \underline{56} \\
 & \ \ 40 \\
 & \ \ \underline{40} \\
 & \ \ \ 0
\end{array}
\qquad
\begin{array}{r|l}
 & 0.142857... \\ \hline
7 & 1.0 \\
 & \ \underline{7} \\
 & \ 30 \\
 & \ \underline{28} \\
 & \ \ 20 \\
 & \ \ \underline{14} \\
 & \ \ \ 60 \\
 & \ \ \ \underline{56} \\
 & \ \ \ \ 40 \\
 & \ \ \ \ \underline{35} \\
 & \ \ \ \ \ 50 \\
 & \ \ \ \ \ \underline{49} \\
 & \ \ \ \ \ \ 1
\end{array}
$$

Remainders : 1, 1, 1, 1, 1...
Divisor : 3

Remainders : 6, 4, 0
Divisor : 8

Remainders : 3, 2, 6, 4, 5, 1, 3, 2, 6, 4, 5, 1,...
Divisor : 7

What have you noticed? You should have noticed at least three things:

(i) The remainders either become 0 after a certain stage, or start repeating themselves.

(ii) The number of entries in the repeating string of remainders is less than the divisor (in $\frac{10}{3}$ one number repeats itself and the divisor is 3, in $\frac{1}{7}$ there are six entries 326451 in the repeating string of remainders and 7 is the divisor).

(iii) If the remainders repeat, then we get a repeating block of digits in the quotient (for $\frac{10}{3}$, 3 repeats in the quotient and for $\frac{1}{7}$, we get the repeating block 142857 in the quotient).

Although we have noticed this pattern using only the examples above, it is true for all rationals of the form $\frac{p}{q}$ $(q \neq 0)$. On division of p by q, two main things happen – either the remainder becomes zero or never becomes zero and we get a repeating string of remainders. Let us look at each case separately.

Case (i) : The remainder becomes zero

In the example of $\frac{7}{8}$, we found that the remainder becomes zero after some steps and the decimal expansion of $\frac{7}{8} = 0.875$. Other examples are $\frac{1}{2} = 0.5$, $\frac{639}{250} = 2.556$. In all these cases, the decimal expansion terminates or ends after a finite number of steps. We call the decimal expansion of such numbers *terminating*.

Case (ii) : The remainder never becomes zero

In the examples of $\frac{10}{3}$ and $\frac{1}{7}$, we notice that the remainders repeat after a certain stage forcing the decimal expansion to go on for ever. In other words, we have a repeating block of digits in the quotient. We say that this expansion is non-terminating recurring. For example, $\frac{10}{3} = 3.3333...$ and $\frac{1}{7} = 0.142857142857142857...$

The usual way of showing that 3 repeats in the quotient of $\frac{10}{3}$ is to write it as $3.\overline{3}$.

Similarly, since the block of digits 142857 repeats in the quotient of $\frac{1}{7}$, we write $\frac{1}{7}$ as $0.\overline{142857}$, where the bar above the digits indicates the block of digits that repeats. Also 3.57272... can be written as $3.5\overline{72}$. So, all these examples give us *non-terminating recurring* (*repeating*) decimal expansions.

Thus, we see that the decimal expansion of rational numbers have only two choices: either they are terminating or non-terminating recurring.

Now suppose, on the other hand, on your walk on the number line, you come across a number like 3.142678 whose decimal expansion is terminating or a number like 1.272727... that is, $1.\overline{27}$, whose decimal expansion is non-terminating recurring, can you conclude that it is a rational number? The answer is yes!

We will not prove it but illustrate this fact with a few examples. The terminating cases are easy.

Example 6 : Show that 3.142678 is a rational number. In other words, express 3.142678 in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

Solution : We have $3.142678 = \frac{3142678}{1000000}$, and hence is a rational number.

Now, let us consider the case when the decimal expansion is non-terminating recurring.

Example 7 : Show that $0.3333... = 0.\overline{3}$ can be expressed in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

Solution : Since we do not know what $0.\overline{3}$ is , let us call it 'x' and so

$$x = 0.3333...$$

Now here is where the trick comes in. Look at

$$10\,x = 10 \times (0.333...) = 3.333...$$

Now, $$3.3333... = 3 + x, \text{ since } x = 0.3333...$$

Therefore, $$10\,x = 3 + x$$

Solving for x, we get

$$9x = 3, \text{ i.e., } x = \frac{1}{3}$$

Example 8 : Show that $1.272727... = 1.\overline{27}$ can be expressed in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

Solution : Let $x = 1.272727...$ Since two digits are repeating, we multiply x by 100 to get

$$100\,x = 127.2727...$$

So, $$100\,x = 126 + 1.272727... = 126 + x$$

Therefore, $$100\,x - x = 126, \text{ i.e., } 99\,x = 126$$

i.e., $$x = \frac{126}{99} = \frac{14}{11}$$

You can check the reverse that $\frac{14}{11} = 1.\overline{27}$.

Example 9 : Show that $0.2353535... = 0.2\overline{35}$ can be expressed in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

Solution : Let $x = 0.2\overline{35}$. Over here, note that 2 does not repeat, but the block 35 repeats. Since two digits are repeating, we multiply x by 100 to get

$$100\, x = 23.53535...$$

So, $$100\, x = 23.3 + 0.23535... = 23.3 + x$$

Therefore, $$99\, x = 23.3$$

i.e., $$99\, x = \frac{233}{10}, \text{ which gives } x = \frac{233}{990}$$

You can also check the reverse that $\frac{233}{990} = 0.2\overline{35}$.

So, every number with a non-terminating recurring decimal expansion can be expressed in the form $\frac{p}{q}$ $(q \neq 0)$, where p and q are integers. Let us summarise our results in the following form :

The decimal expansion of a rational number is either terminating or non-terminating recurring. Moreover, a number whose decimal expansion is terminating or non-terminating recurring is rational.

So, now we know what the decimal expansion of a rational number can be. What about the decimal expansion of irrational numbers? Because of the property above, we can conclude that their decimal expansions are *non-terminating non-recurring*.

So, the property for irrational numbers, similar to the property stated above for rational numbers, is

The decimal expansion of an irrational number is non-terminating non-recurring. Moreover, a number whose decimal expansion is non-terminating non-recurring is irrational.

Recall s = 0.10110111011110... from the previous section. Notice that it is non-terminating and non-recurring. Therefore, from the property above, it is irrational. Moreover, notice that you can generate infinitely many irrationals similar to s.

What about the famous irrationals $\sqrt{2}$ and π? Here are their decimal expansions up to a certain stage.

$$\sqrt{2} = 1.4142135623730950488016887242096...$$

$$\pi = 3.14159265358979323846264338327950...$$

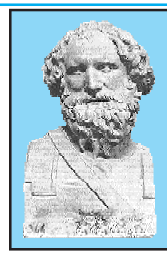
(Note that, we often take $\frac{22}{7}$ as an approximate value for π, but $\pi \neq \frac{22}{7}$.)

Over the years, mathematicians have developed various techniques to produce more and more digits in the decimal expansions of irrational numbers. For example, you might have learnt to find digits in the decimal expansion of $\sqrt{2}$ by the division method. Interestingly, in the Sulbasutras (rules of chord), a mathematical treatise of the Vedic period (800 BC - 500 BC), you find an approximation of $\sqrt{2}$ as follows:

$$\sqrt{2} = 1 + \frac{1}{3} + \left(\frac{1}{4} \times \frac{1}{3}\right) - \left(\frac{1}{34} \times \frac{1}{4} \times \frac{1}{3}\right) = 1.4142156$$

Notice that it is the same as the one given above for the first five decimal places. The history of the hunt for digits in the decimal expansion of π is very interesting.

The Greek genius Archimedes was the first to compute digits in the decimal expansion of π. He showed $3.140845 < \pi < 3.142857$. Aryabhatta (476 – 550 C.E.), the great Indian mathematician and astronomer, found the value of π correct to four decimal places (3.1416). Using high speed computers and advanced algorithms, π has been computed to over 1.24 trillion decimal places!



Archimedes (287 BCE – 212 BCE)
Fig. 1.10

Now, let us see how to obtain irrational numbers.

Example 10 : Find an irrational number between $\frac{1}{7}$ and $\frac{2}{7}$.

Solution : We saw that $\frac{1}{7} = 0.\overline{142857}$. So, you can easily calculate $\frac{2}{7} = 0.\overline{285714}$.

To find an irrational number between $\frac{1}{7}$ and $\frac{2}{7}$, we find a number which is non-terminating non-recurring lying between them. Of course, you can find infinitely many such numbers.

An example of such a number is 0.150150015000150000...

EXERCISE 1.3

1. Write the following in decimal form and say what kind of decimal expansion each has :

   (i) $\frac{36}{100}$ (ii) $\frac{1}{11}$ (iii) $4\frac{1}{8}$

   (iv) $\frac{3}{13}$ (v) $\frac{2}{11}$ (vi) $\frac{329}{400}$

2. You know that $\frac{1}{7} = 0.\overline{142857}$. Can you predict what the decimal expansions of $\frac{2}{7}$, $\frac{3}{7}$, $\frac{4}{7}$, $\frac{5}{7}$, $\frac{6}{7}$ are, without actually doing the long division? If so, how?

   [**Hint :** Study the remainders while finding the value of $\frac{1}{7}$ carefully.]

3. Express the following in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

   (i) $0.\overline{6}$ (ii) $0.4\overline{7}$ (iii) $0.\overline{001}$

4. Express 0.99999 .... in the form $\frac{p}{q}$. Are you surprised by your answer? With your teacher and classmates discuss why the answer makes sense.

5. What can the maximum number of digits be in the repeating block of digits in the decimal expansion of $\frac{1}{17}$? Perform the division to check your answer.

6. Look at several examples of rational numbers in the form $\frac{p}{q}$ $(q \neq 0)$, where p and q are integers with no common factors other than 1 and having terminating decimal representations (expansions). Can you guess what property q must satisfy?

7. Write three numbers whose decimal expansions are non-terminating non-recurring.

8. Find three different irrational numbers between the rational numbers $\frac{5}{7}$ and $\frac{9}{11}$.

9. Classify the following numbers as rational or irrational :

   (i) $\sqrt{23}$ (ii) $\sqrt{225}$ (iii) 0.3796

   (iv) 7.478478... (v) 1.101001000100001...

1.4 Operations on Real Numbers

You have learnt, in earlier classes, that rational numbers satisfy the commutative, associative and distributive laws for addition and multiplication. Moreover, if we add, subtract, multiply or divide (except by zero) two rational numbers, we still get a rational number (that is, rational numbers are 'closed' with respect to addition, subtraction, multiplication and division). It turns out that irrational numbers also satisfy the commutative, associative and distributive laws for addition and multiplication. However, the sum, difference, quotients and products of irrational numbers are not *always* irrational. For example, $\left(\sqrt{6}\right)+\left(-\sqrt{6}\right), \left(\sqrt{2}\right)-\left(\sqrt{2}\right), \left(\sqrt{3}\right)\cdot\left(\sqrt{3}\right)$ and $\frac{\sqrt{17}}{\sqrt{17}}$ are rationals.

Let us look at what happens when we add and multiply a rational number with an irrational number. For example, $\sqrt{3}$ is irrational. What about $2+\sqrt{3}$ and $2\sqrt{3}$? Since $\sqrt{3}$ has a non-terminating non-recurring decimal expansion, the same is true for $2+\sqrt{3}$ and $2\sqrt{3}$. Therefore, both $2+\sqrt{3}$ and $2\sqrt{3}$ are also irrational numbers.

Example 11 : Check whether $7\sqrt{5}$, $\frac{7}{\sqrt{5}}$, $\sqrt{2}+21$, $\pi-2$ are irrational numbers or not.

Solution : $\sqrt{5} = 2.236...$, $\sqrt{2} = 1.4142...$, $\pi = 3.1415...$

Then $7\sqrt{5} = 15.652...$, $\frac{7}{\sqrt{5}} = \frac{7\sqrt{5}}{\sqrt{5}\sqrt{5}} = \frac{7\sqrt{5}}{5} = 3.1304...$

$\sqrt{2} + 21 = 22.4142...$, $\pi - 2 = 1.1415...$

All these are non-terminating non-recurring decimals. So, all these are irrational numbers.

Now, let us see what generally happens if we add, subtract, multiply, divide, take square roots and even *n*th roots of these irrational numbers, where *n* is any natural number. Let us look at some examples.

Example 12 : Add $2\sqrt{2}+5\sqrt{3}$ and $\sqrt{2}-3\sqrt{3}$.

Solution :
$$\left(2\sqrt{2}+5\sqrt{3}\right)+\left(\sqrt{2}-3\sqrt{3}\right) = \left(2\sqrt{2}+\sqrt{2}\right)+\left(5\sqrt{3}-3\sqrt{3}\right)$$
$$= (2+1)\sqrt{2}+(5-3)\sqrt{3} = 3\sqrt{2}+2\sqrt{3}$$

Example 13 : Multiply $6\sqrt{5}$ by $2\sqrt{5}$.

Solution : $6\sqrt{5} \times 2\sqrt{5} = 6 \times 2 \times \sqrt{5} \times \sqrt{5} = 12 \times 5 = 60$

Example 14 : Divide $8\sqrt{15}$ by $2\sqrt{3}$.

Solution : $8\sqrt{15} \div 2\sqrt{3} = \dfrac{8\sqrt{3} \times \sqrt{5}}{2\sqrt{3}} = 4\sqrt{5}$

These examples may lead you to expect the following facts, **which are true**:

(i) The sum or difference of a rational number and an irrational number is irrational.

(ii) The product or quotient of a non-zero rational number with an irrational number is irrational.

(iii) If we add, subtract, multiply or divide two irrationals, the result may be rational or irrational.

We now turn our attention to the operation of taking square roots of real numbers. Recall that, if a is a natural number, then $\sqrt{a} = b$ means $b^2 = a$ and $b > 0$. The same definition can be extended for positive real numbers.

Let $a > 0$ be a real number. Then $\sqrt{a} = b$ means $b^2 = a$ and $b > 0$.

In Section 1.2, we saw how to represent $\sqrt{n}$ for any positive integer n on the number line. We now show how to find $\sqrt{x}$ for any given positive real number x geometrically. For example, let us find it for $x = 3.5$, i.e., we find $\sqrt{3.5}$ geometrically.

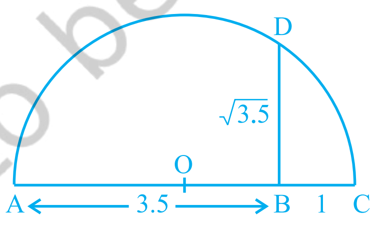


Fig. 1.11

Mark the distance 3.5 units from a fixed point A on a given line to obtain a point B such that AB = 3.5 units (see Fig. 1.11). From B, mark a distance of 1 unit and mark the new point as C. Find the mid-point of AC and mark that point as O. Draw a semicircle with centre O and radius OC. Draw a line perpendicular to AC passing through B and intersecting the semicircle at D. Then, BD = $\sqrt{3.5}$.

More generally, to find $\sqrt{x}$, for any positive real number x, we mark B so that AB = x units, and, as in Fig. 1.12, mark C so that BC = 1 unit. Then, as we have done for the case $x = 3.5$, we find BD = $\sqrt{x}$ (see Fig. 1.12). We can prove this result using the Pythagoras Theorem.

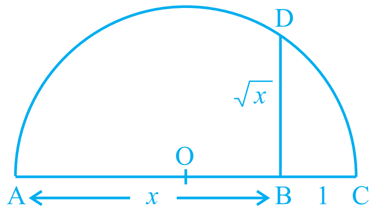


Fig. 1.12

Notice that, in Fig. 1.12, Δ OBD is a right-angled triangle. Also, the radius of the circle is $\frac{x+1}{2}$ units.

Therefore, OC = OD = OA = $\frac{x+1}{2}$ units.

Now, OB = $x - \left(\frac{x+1}{2}\right) = \frac{x-1}{2}$.

So, by the Pythagoras Theorem, we have

$$BD^2 = OD^2 - OB^2 = \left(\frac{x+1}{2}\right)^2 - \left(\frac{x-1}{2}\right)^2 = \frac{4x}{4} = x.$$

This shows that BD = $\sqrt{x}$.

This construction gives us a visual, and geometric way of showing that $\sqrt{x}$ exists for all real numbers $x > 0$. If you want to know the position of $\sqrt{x}$ on the number line, then let us treat the line BC as the number line, with B as zero, C as 1, and so on. Draw an arc with centre B and radius BD, which intersects the number line in E (see Fig. 1.13). Then, E represents $\sqrt{x}$.

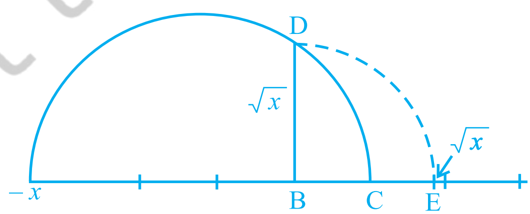


Fig. 1.13

We would like to now extend the idea of square roots to cube roots, fourth roots, and in general nth roots, where n is a positive integer. Recall your understanding of square roots and cube roots from earlier classes.

What is $\sqrt[3]{8}$? Well, we know it has to be some positive number whose cube is 8, and you must have guessed $\sqrt[3]{8} = 2$. Let us try $\sqrt[5]{243}$. Do you know some number b such that $b^5 = 243$? The answer is 3. Therefore, $\sqrt[5]{243} = 3$.

From these examples, can you define $\sqrt[n]{a}$ for a real number $a > 0$ and a positive integer n?

Let $a > 0$ be a real number and n be a positive integer. Then $\sqrt[n]{a} = b$, if $b^n = a$ and $b > 0$. Note that the symbol '$\sqrt{\ }$' used in $\sqrt{2}, \sqrt[3]{8}, \sqrt[n]{a}$, etc. is called the *radical sign*.

We now list some identities relating to square roots, which are useful in various ways. You are already familiar with some of these from your earlier classes. The remaining ones follow from the distributive law of multiplication over addition of real numbers, and from the identity $(x + y)(x - y) = x^2 - y^2$, for any real numbers x and y.

Let a and b be positive real numbers. Then

(i) $\sqrt{ab} = \sqrt{a}\sqrt{b}$ (ii) $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

(iii) $\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right) = a - b$ (iv) $\left(a+\sqrt{b}\right)\left(a-\sqrt{b}\right) = a^2 - b$

(v) $\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{c}+\sqrt{d}\right) = \sqrt{ac}+\sqrt{ad}+\sqrt{bc}+\sqrt{bd}$

(vi) $\left(\sqrt{a}+\sqrt{b}\right)^2 = a + 2\sqrt{ab} + b$

Let us look at some particular cases of these identities.

Example 15 : Simplify the following expressions:

(i) $\left(5+\sqrt{7}\right)\left(2+\sqrt{5}\right)$ (ii) $\left(5+\sqrt{5}\right)\left(5-\sqrt{5}\right)$

(iii) $\left(\sqrt{3}+\sqrt{7}\right)^2$ (iv) $\left(\sqrt{11}-\sqrt{7}\right)\left(\sqrt{11}+\sqrt{7}\right)$

Solution : (i) $\left(5+\sqrt{7}\right)\left(2+\sqrt{5}\right)=10+5\sqrt{5}+2\sqrt{7}+\sqrt{35}$

(ii) $\left(5+\sqrt{5}\right)\left(5-\sqrt{5}\right)=5^2-\left(\sqrt{5}\right)^2=25-5=20$

(iii) $\left(\sqrt{3}+\sqrt{7}\right)^2=\left(\sqrt{3}\right)^2+2\sqrt{3}\sqrt{7}+\left(\sqrt{7}\right)^2=3+2\sqrt{21}+7=10+2\sqrt{21}$

(iv) $\left(\sqrt{11}-\sqrt{7}\right)\left(\sqrt{11}+\sqrt{7}\right)=\left(\sqrt{11}\right)^2-\left(\sqrt{7}\right)^2=11-7=4$

Remark : Note that 'simplify' in the example above has been used to mean that the expression should be written as the sum of a rational and an irrational number.

We end this section by considering the following problem. Look at $\frac{1}{\sqrt{2}}\cdot$ Can you tell where it shows up on the number line? You know that it is irrational. May be it is easier to handle if the denominator is a rational number. Let us see, if we can 'rationalise' the denominator, that is, to make the denominator into a rational number. To do so, we need the identities involving square roots. Let us see how.

Example 16 : Rationalise the denominator of $\frac{1}{\sqrt{2}}\cdot$

Solution : We want to write $\frac{1}{\sqrt{2}}$ as an equivalent expression in which the denominator is a rational number. We know that $\sqrt{2}\cdot\sqrt{2}$ is rational. We also know that multiplying $\frac{1}{\sqrt{2}}$ by $\frac{\sqrt{2}}{\sqrt{2}}$ will give us an equivalent expression, since $\frac{\sqrt{2}}{\sqrt{2}}=1$. So, we put these two facts together to get

$$\frac{1}{\sqrt{2}}=\frac{1}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}=\frac{\sqrt{2}}{2}\cdot$$

In this form, it is easy to locate $\frac{1}{\sqrt{2}}$ on the number line. It is half way between 0 and $\sqrt{2}$.

Example 17 : Rationalise the denominator of $\frac{1}{2+\sqrt{3}}$.

Solution : We use the Identity (iv) given earlier. Multiply and divide $\frac{1}{2+\sqrt{3}}$ by $2-\sqrt{3}$ to get $\frac{1}{2+\sqrt{3}} \times \frac{2-\sqrt{3}}{2-\sqrt{3}} = \frac{2-\sqrt{3}}{4-3} = 2-\sqrt{3}$.

Example 18 : Rationalise the denominator of $\frac{5}{\sqrt{3}-\sqrt{5}}$.

Solution : Here we use the Identity (iii) given earlier.

So, $$\frac{5}{\sqrt{3}-\sqrt{5}} = \frac{5}{\sqrt{3}-\sqrt{5}} \times \frac{\sqrt{3}+\sqrt{5}}{\sqrt{3}+\sqrt{5}} = \frac{5\left(\sqrt{3}+\sqrt{5}\right)}{3-5} = \left(\frac{-5}{2}\right)\left(\sqrt{3}+\sqrt{5}\right)$$

Example 19 : Rationalise the denominator of $\frac{1}{7+3\sqrt{2}}$.

Solution : $$\frac{1}{7+3\sqrt{2}} = \frac{1}{7+3\sqrt{2}} \times \left(\frac{7-3\sqrt{2}}{7-3\sqrt{2}}\right) = \frac{7-3\sqrt{2}}{49-18} = \frac{7-3\sqrt{2}}{31}$$

So, when the denominator of an expression contains a term with a square root (or a number under a radical sign), the process of converting it to an equivalent expression whose denominator is a rational number is called *rationalising the denominator*.

EXERCISE 1.4

1. Classify the following numbers as rational or irrational:

(i) $2-\sqrt{5}$ (ii) $\left(3+\sqrt{23}\right)-\sqrt{23}$ (iii) $\frac{2\sqrt{7}}{7\sqrt{7}}$

(iv) $\frac{1}{\sqrt{2}}$ (v) 2π

2. Simplify each of the following expressions:

(i) $\left(3+\sqrt{3}\right)\left(2+\sqrt{2}\right)$ (ii) $\left(3+\sqrt{3}\right)\left(3-\sqrt{3}\right)$

(iii) $\left(\sqrt{5}+\sqrt{2}\right)^2$ (iv) $\left(\sqrt{5}-\sqrt{2}\right)\left(\sqrt{5}+\sqrt{2}\right)$

3. Recall, π is defined as the ratio of the circumference (say c) of a circle to its diameter (say d). That is, $\pi = \frac{c}{d}\cdot$ This seems to contradict the fact that π is irrational. How will you resolve this contradiction?

4. Represent $\sqrt{9.3}$ on the number line.

5. Rationalise the denominators of the following:

(i) $\frac{1}{\sqrt{7}}$ (ii) $\frac{1}{\sqrt{7}-\sqrt{6}}$

(iii) $\frac{1}{\sqrt{5}+\sqrt{2}}$ (iv) $\frac{1}{\sqrt{7}-2}$

1.5 Laws of Exponents for Real Numbers

Do you remember how to simplify the following?

(i) $17^2 \,.\, 17^5 =$ (ii) $(5^2)^7 =$

(iii) $\frac{23^{10}}{23^7} =$ (iv) $7^3 \,.\, 9^3 =$

Did you get these answers? They are as follows:

(i) $17^2 \,.\, 17^5 = 17^7$ (ii) $(5^2)^7 = 5^{14}$

(iii) $\frac{23^{10}}{23^7} = 23^3$ (iv) $7^3 \,.\, 9^3 = 63^3$

To get these answers, you would have used the following laws of exponents, which you have learnt in your earlier classes. (Here a, n and m are natural numbers. Remember, a is called the base and m and n are the exponents.)

(i) $a^m \,.\, a^n = a^{m+n}$ (ii) $(a^m)^n = a^{mn}$

(iii) $\frac{a^m}{a^n} = a^{m-n},\ m > n$ (iv) $a^m b^m = (ab)^m$

What is $(a)^0$? Yes, it is 1! So you have learnt that $(a)^0 = 1$. So, using (iii), we can get $\frac{1}{a^n} = a^{-n}$. We can now extend the laws to negative exponents too.

So, for example :

(i) $17^2 \cdot 17^{-5} = 17^{-3} = \frac{1}{17^3}$ (ii) $(5^2)^{-7} = 5^{-14}$

(iii) $\frac{23^{-10}}{23^7} = 23^{-17}$ (iv) $(7)^{-3} \cdot (9)^{-3} = (63)^{-3}$

Suppose we want to do the following computations:

(i) $2^{\frac{2}{3}} \cdot 2^{\frac{1}{3}}$ (ii) $\left(3^{\frac{1}{5}}\right)^4$

(iii) $\frac{7^{\frac{1}{5}}}{7^{\frac{1}{3}}}$ (iv) $13^{\frac{1}{5}} \cdot 17^{\frac{1}{5}}$

How would we go about it? It turns out that we can extend the laws of exponents that we have studied earlier, even when the base is a positive real number and the exponents are rational numbers. (Later you will study that it can further to be extended when the exponents are real numbers.) But before we state these laws, and to even make sense of these laws, we need to first understand what, for example $4^{\frac{3}{2}}$ is. So, we have some work to do!

We define $\sqrt[n]{a}$ for a real number $a > 0$ as follows:

Let $a > 0$ be a real number and n a positive integer. Then $\sqrt[n]{a} = b$, if $b^n = a$ and $b > 0$.

In the language of exponents, we define $\sqrt[n]{a} = a^{\frac{1}{n}}$. So, in particular, $\sqrt[3]{2} = 2^{\frac{1}{3}}$.

There are now two ways to look at $4^{\frac{3}{2}}$.

$$4^{\frac{3}{2}} = \left(4^{\frac{1}{2}}\right)^3 = 2^3 = 8$$

$$4^{\frac{3}{2}} = \left(4^3\right)^{\frac{1}{2}} = (64)^{\frac{1}{2}} = 8$$

Therefore, we have the following definition:

Let $a > 0$ be a real number. Let m and n be integers such that m and n have no common factors other than 1, and $n > 0$. Then,

$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$

We now have the following extended laws of exponents:

Let $a > 0$ be a real number and p and q be rational numbers. Then, we have

(i) $a^p \cdot a^q = a^{p+q}$ (ii) $(a^p)^q = a^{pq}$

(iii) $\dfrac{a^p}{a^q} = a^{p-q}$ (iv) $a^p b^p = (ab)^p$

You can now use these laws to answer the questions asked earlier.

Example 20 : Simplify (i) $2^{\frac{2}{3}} \cdot 2^{\frac{1}{3}}$ (ii) $\left(3^{\frac{1}{5}}\right)^4$

(iii) $\dfrac{7^{\frac{1}{5}}}{7^{\frac{1}{3}}}$ (iv) $13^{\frac{1}{5}} \cdot 17^{\frac{1}{5}}$

Solution :

(i) $2^{\frac{2}{3}} \cdot 2^{\frac{1}{3}} = 2^{\left(\frac{2}{3}+\frac{1}{3}\right)} = 2^{\frac{3}{3}} = 2^1 = 2$ (ii) $\left(3^{\frac{1}{5}}\right)^4 = 3^{\frac{4}{5}}$

(iii) $\dfrac{7^{\frac{1}{5}}}{7^{\frac{1}{3}}} = 7^{\left(\frac{1}{5}-\frac{1}{3}\right)} = 7^{\frac{3-5}{15}} = 7^{\frac{-2}{15}}$ (iv) $13^{\frac{1}{5}} \cdot 17^{\frac{1}{5}} = (13 \times 17)^{\frac{1}{5}} = 221^{\frac{1}{5}}$

EXERCISE 1.5

1. Find : (i) $64^{\frac{1}{2}}$ (ii) $32^{\frac{1}{5}}$ (iii) $125^{\frac{1}{3}}$
2. Find : (i) $9^{\frac{3}{2}}$ (ii) $32^{\frac{2}{5}}$ (iii) $16^{\frac{3}{4}}$ (iv) $125^{\frac{-1}{3}}$
3. Simplify : (i) $2^{\frac{2}{3}} \cdot 2^{\frac{1}{5}}$ (ii) $\left(\frac{1}{3^3}\right)^7$ (iii) $\dfrac{11^{\frac{1}{2}}}{11^{\frac{1}{4}}}$ (iv) $7^{\frac{1}{2}} \cdot 8^{\frac{1}{2}}$

1.6 Summary

In this chapter, you have studied the following points:

1. A number r is called a rational number, if it can be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

2. A number s is called a irrational number, if it cannot be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.
3. The decimal expansion of a rational number is either terminating or non-terminating recurring. Moreover, a number whose decimal expansion is terminating or non-terminating recurring is rational.
4. The decimal expansion of an irrational number is non-terminating non-recurring. Moreover, a number whose decimal expansion is non-terminating non-recurring is irrational.
5. All the rational and irrational numbers make up the collection of real numbers.

6. If r is rational and s is irrational, then $r + s$ and $r - s$ are irrational numbers, and rs and $\frac{r}{s}$ are irrational numbers, $r \neq 0$.
7. For positive real numbers a and b, the following identities hold:

   (i) $\sqrt{ab} = \sqrt{a}\sqrt{b}$ (ii) $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

   (iii) $\left(\sqrt{a} + \sqrt{b}\right)\left(\sqrt{a} - \sqrt{b}\right) = a - b$ (iv) $\left(a + \sqrt{b}\right)\left(a - \sqrt{b}\right) = a^2 - b$

   (v) $\left(\sqrt{a} + \sqrt{b}\right)^2 = a + 2\sqrt{ab} + b$

8. To rationalise the denominator of $\frac{1}{\sqrt{a} + b}$, we multiply this by $\frac{\sqrt{a} - b}{\sqrt{a} - b}$, where a and b are integers.
9. Let $a > 0$ be a real number and p and q be rational numbers. Then

   (i) $a^p \cdot a^q = a^{p+q}$ (ii) $(a^p)^q = a^{pq}$

   (iii) $\frac{a^p}{a^q} = a^{p-q}$ (iv) $a^p b^p = (ab)^p$

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CHAPTER 2

POLYNOMIALS

2.1 Introduction

You have studied algebraic expressions, their addition, subtraction, multiplication and division in earlier classes. You also have studied how to factorise some algebraic expressions. You may recall the algebraic identities :

$$(x + y)^2 = x^2 + 2xy + y^2$$

$$(x - y)^2 = x^2 - 2xy + y^2$$

and $$x^2 - y^2 = (x + y)(x - y)$$

and their use in factorisation. In this chapter, we shall start our study with a particular type of algebraic expression, called *polynomial,* and the terminology related to it. We shall also study the *Remainder Theorem* and *Factor Theorem* and their use in the factorisation of polynomials. In addition to the above, we shall study some more algebraic identities and their use in factorisation and in evaluating some given expressions.

2.2 Polynomials in One Variable

Let us begin by recalling that a variable is denoted by a symbol that can take any real value. We use the letters x, y, z, etc. to denote variables. Notice that $2x$, $3x$, $-x$, $-\frac{1}{2}x$ are algebraic expressions. All these expressions are of the form (a constant) × x. Now suppose we want to write an expression which is (a constant) × (a variable) and we do not know what the constant is. In such cases, we write the constant as a, b, c, etc. So the expression will be ax, say.

However, there is a difference between a letter denoting a constant and a letter denoting a variable. The values of the constants remain the same throughout a particular situation, that is, the values of the constants do not change in a given problem, but the value of a variable can keep changing.

Now, consider a square of side 3 units (see Fig. 2.1). What is its perimeter? You know that the perimeter of a square is the sum of the lengths of its four sides. Here, each side is 3 units. So, its perimeter is 4×3, i.e., 12 units. What will be the perimeter if each side of the square is 10 units? The perimeter is 4×10, i.e., 40 units. In case the length of each side is x units (see Fig. 2.2), the perimeter is given by $4x$ units. So, as the length of the side varies, the perimeter varies.

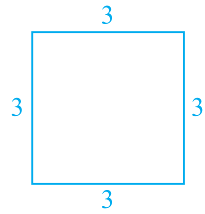

Fig. 2.1

Can you find the area of the square PQRS? It is $x \times x = x^2$ square units. x^2 is an algebraic expression. You are also familiar with other algebraic expressions like $2x$, $x^2 + 2x$, $x^3 - x^2 + 4x + 7$. Note that, all the algebraic expressions we have considered so far have only whole numbers as the exponents of the variable. Expressions of this form are called *polynomials in one variable*. In the examples above, the variable is x. For instance, $x^3 - x^2 + 4x + 7$ is a polynomial in x. Similarly, $3y^2 + 5y$ is a polynomial in the variable y and $t^2 + 4$ is a polynomial in the variable t.

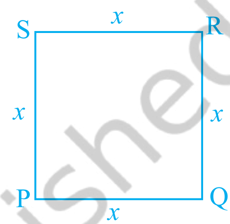

Fig. 2.2

In the polynomial $x^2 + 2x$, the expressions x^2 and $2x$ are called the **terms** of the polynomial. Similarly, the polynomial $3y^2 + 5y + 7$ has three terms, namely, $3y^2$, $5y$ and 7. Can you write the terms of the polynomial $-x^3 + 4x^2 + 7x - 2$? This polynomial has 4 terms, namely, $-x^3$, $4x^2$, $7x$ and -2.

Each term of a polynomial has a **coefficient**. So, in $-x^3 + 4x^2 + 7x - 2$, the coefficient of x^3 is -1, the coefficient of x^2 is 4, the coefficient of x is 7 and -2 is the coefficient of x^0 (Remember, $x^0 = 1$). Do you know the coefficient of x in $x^2 - x + 7$? It is -1.

2 is also a polynomial. In fact, 2, -5, 7, etc. are examples of *constant polynomials*. The constant polynomial 0 is called the **zero polynomial**. This plays a very important role in the collection of all polynomials, as you will see in the higher classes.

Now, consider algebraic expressions such as $x + \frac{1}{x}$, $\sqrt{x} + 3$ and $\sqrt[3]{y} + y^2$. Do you know that you can write $x + \frac{1}{x} = x + x^{-1}$? Here, the exponent of the second term, i.e., x^{-1} is -1, which is not a whole number. So, this algebraic expression is not a polynomial.

Again, $\sqrt{x} + 3$ can be written as $x^{\frac{1}{2}} + 3$. Here the exponent of x is $\frac{1}{2}$, which is not a whole number. So, is $\sqrt{x} + 3$ a polynomial? No, it is not. What about $\sqrt[3]{y} + y^2$? It is also not a polynomial (Why?).

If the variable in a polynomial is x, we may denote the polynomial by $p(x)$, or $q(x)$, or $r(x)$, etc. So, for example, we may write :

$$p(x) = 2x^2 + 5x - 3$$
$$q(x) = x^3 - 1$$
$$r(y) = y^3 + y + 1$$
$$s(u) = 2 - u - u^2 + 6u^5$$

A polynomial can have any (finite) number of terms. For instance, $x^{150} + x^{149} + ... + x^2 + x + 1$ is a polynomial with 151 terms.

Consider the polynomials $2x$, 2, $5x^3$, $-5x^2$, y and u^4. Do you see that each of these polynomials has only one term? Polynomials having only one term are called *monomials* ('mono' means 'one').

Now observe each of the following polynomials:

$p(x) = x + 1$, $\quad q(x) = x^2 - x$, $\quad r(y) = y^9 + 1$, $\quad t(u) = u^{15} - u^2$

How many terms are there in each of these? Each of these polynomials has only two terms. Polynomials having only two terms are called *binomials* ('bi' means 'two').

Similarly, polynomials having only three terms are called *trinomials* ('tri' means 'three'). Some examples of trinomials are

$p(x) = x + x^2 + \pi$, $\quad q(x) = \sqrt{2} + x - x^2$,

$r(u) = u + u^2 - 2$, $\quad t(y) = y^4 + y + 5$.

Now, look at the polynomial $p(x) = 3x^7 - 4x^6 + x + 9$. What is the term with the highest power of x ? It is $3x^7$. The exponent of x in this term is 7. Similarly, in the polynomial $q(y) = 5y^6 - 4y^2 - 6$, the term with the highest power of y is $5y^6$ and the exponent of y in this term is 6. We call the highest power of the variable in a polynomial as the *degree of the polynomial*. So, the degree of the polynomial $3x^7 - 4x^6 + x + 9$ is 7 and the degree of the polynomial $5y^6 - 4y^2 - 6$ is 6. **The degree of a non-zero constant polynomial is zero**.

Example 1 : Find the degree of each of the polynomials given below:

(i) $x^5 - x^4 + 3$ $\quad$ (ii) $2 - y^2 - y^3 + 2y^8$ $\quad$ (iii) 2

Solution : (i) The highest power of the variable is 5. So, the degree of the polynomial is 5.

(ii) The highest power of the variable is 8. So, the degree of the polynomial is 8.

(iii) The only term here is 2 which can be written as $2x^0$. So the exponent of x is 0. Therefore, the degree of the polynomial is 0.

Now observe the polynomials $p(x) = 4x + 5$, $q(y) = 2y$, $r(t) = t + \sqrt{2}$ and $s(u) = 3 - u$. Do you see anything common among all of them? The degree of each of these polynomials is one. A polynomial of degree one is called a *linear polynomial*. Some more linear polynomials in one variable are $2x - 1$, $\sqrt{2}\,y + 1$, $2 - u$. Now, try and find a linear polynomial in x with 3 terms? You would not be able to find it because a linear polynomial in x can have at most two terms. So, any linear polynomial in x will be of the form $ax + b$, where a and b are constants and $a \neq 0$ (why?). Similarly, $ay + b$ is a linear polynomial in y.

Now consider the polynomials :

$$2x^2 + 5,\ 5x^2 + 3x + \pi,\ x^2 \text{ and } x^2 + \frac{2}{5}x$$

Do you agree that they are all of degree two? A polynomial of degree two is called a *quadratic polynomial*. Some examples of a quadratic polynomial are $5 - y^2$, $4y + 5y^2$ and $6 - y - y^2$. Can you write a quadratic polynomial in one variable with four different terms? You will find that a quadratic polynomial in one variable will have at most 3 terms. If you list a few more quadratic polynomials, you will find that any quadratic polynomial in x is of the form $ax^2 + bx + c$, where $a \neq 0$ and a, b, c are constants. Similarly, quadratic polynomial in y will be of the form $ay^2 + by + c$, provided $a \neq 0$ and a, b, c are constants.

We call a polynomial of degree three a *cubic polynomial*. Some examples of a cubic polynomial in x are $4x^3$, $2x^3 + 1$, $5x^3 + x^2$, $6x^3 - x$, $6 - x^3$, $2x^3 + 4x^2 + 6x + 7$. How many terms do you think a cubic polynomial in one variable can have? It can have at most 4 terms. These may be written in the form $ax^3 + bx^2 + cx + d$, where $a \neq 0$ and a, b, c and d are constants.

Now, that you have seen what a polynomial of degree 1, degree 2, or degree 3 looks like, can you write down a polynomial in one variable of degree n for any natural number n? A polynomial in one variable x of degree n is an expression of the form

$$a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

where $a_0, a_1, a_2, \ldots, a_n$ are constants and $a_n \neq 0$.

In particular, if $a_0 = a_1 = a_2 = a_3 = \ldots = a_n = 0$ (all the constants are zero), we get the **zero polynomial**, which is denoted by 0. What is the degree of the zero polynomial? The degree of the zero polynomial is *not defined*.

So far we have dealt with polynomials in one variable only. We can also have polynomials in more than one variable. For example, $x^2 + y^2 + xyz$ (where variables are x, y and z) is a polynomial in three variables. Similarly $p^2 + q^{10} + r$ (where the variables are p, q and r), $u^3 + v^2$ (where the variables are u and v) are polynomials in three and two variables, respectively. You will be studying such polynomials in detail later.

EXERCISE 2.1

1. Which of the following expressions are polynomials in one variable and which are not? State reasons for your answer.

   (i) $4x^2 - 3x + 7$ (ii) $y^2 + \sqrt{2}$ (iii) $3\sqrt{t} + t\sqrt{2}$ (iv) $y + \frac{2}{y}$

   (v) $x^{10} + y^3 + t^{50}$

2. Write the coefficients of x^2 in each of the following:

   (i) $2 + x^2 + x$ (ii) $2 - x^2 + x^3$ (iii) $\frac{\pi}{2}x^2 + x$ (iv) $\sqrt{2}x - 1$

3. Give one example each of a binomial of degree 35, and of a monomial of degree 100.

4. Write the degree of each of the following polynomials:

   (i) $5x^3 + 4x^2 + 7x$ (ii) $4 - y^2$

   (iii) $5t - \sqrt{7}$ (iv) 3

5. Classify the following as linear, quadratic and cubic polynomials:

   (i) $x^2 + x$ (ii) $x - x^3$ (iii) $y + y^2 + 4$ (iv) $1 + x$

   (v) $3t$ (vi) r^2 (vii) $7x^3$

2.3 Zeroes of a Polynomial

Consider the polynomial $p(x) = 5x^3 - 2x^2 + 3x - 2$.

If we replace x by 1 everywhere in $p(x)$, we get

$$p(1) = 5 \times (1)^3 - 2 \times (1)^2 + 3 \times (1) - 2$$
$$= 5 - 2 + 3 - 2$$
$$= 4$$

So, we say that the value of $p(x)$ at $x = 1$ is 4.

Similarly, $p(0) = 5(0)^3 - 2(0)^2 + 3(0) - 2$

$$= -2$$

Can you find $p(-1)$?

Example 2 : Find the value of each of the following polynomials at the indicated value of variables:

(i) $p(x) = 5x^2 - 3x + 7$ at $x = 1$.

(ii) $q(y) = 3y^3 - 4y + \sqrt{11}$ at $y = 2$.

(iii) $p(t) = 4t^4 + 5t^3 - t^2 + 6$ at $t = a$.

Solution : (i) $p(x) = 5x^2 - 3x + 7$

The value of the polynomial $p(x)$ at $x = 1$ is given by

$$p(1) = 5(1)^2 - 3(1) + 7$$
$$= 5 - 3 + 7 = 9$$

(ii) $q(y) = 3y^3 - 4y + \sqrt{11}$

The value of the polynomial $q(y)$ at $y = 2$ is given by

$$q(2) = 3(2)^3 - 4(2) + \sqrt{11} = 24 - 8 + \sqrt{11} = 16 + \sqrt{11}$$

(iii) $p(t) = 4t^4 + 5t^3 - t^2 + 6$

The value of the polynomial $p(t)$ at $t = a$ is given by

$$p(a) = 4a^4 + 5a^3 - a^2 + 6$$

Now, consider the polynomial $p(x) = x - 1$.

What is $p(1)$? Note that : $p(1) = 1 - 1 = 0$.

As $p(1) = 0$, we say that 1 is a *zero* of the polynomial $p(x)$.

Similarly, you can check that 2 is a *zero* of $q(x)$, where $q(x) = x - 2$.

In general, we say that a *zero* of a polynomial $p(x)$ is a number c such that $p(c) = 0$.

You must have observed that the zero of the polynomial $x - 1$ is obtained by equating it to 0, i.e., $x - 1 = 0$, which gives $x = 1$. We say $p(x) = 0$ is a polynomial equation and 1 is the *root of the polynomial* equation $p(x) = 0$. So we say 1 is the zero of the polynomial $x - 1$, or a *root* of the polynomial equation $x - 1 = 0$.

Now, consider the constant polynomial 5. Can you tell what its zero is? It has no zero because replacing x by any number in $5x^0$ still gives us 5. In fact, *a non-zero constant polynomial has no zero*. What about the zeroes of the zero polynomial? By convention, *every real number is a zero of the zero polynomial*.

Example 3 : Check whether –2 and 2 are zeroes of the polynomial $x + 2$.

Solution : Let $p(x) = x + 2$.

Then $p(2) = 2 + 2 = 4$, $p(-2) = -2 + 2 = 0$

Therefore, –2 is a zero of the polynomial $x + 2$, but 2 is not.

Example 4 : Find a zero of the polynomial $p(x) = 2x + 1$.

Solution : Finding a zero of $p(x)$, is the same as solving the equation

$$p(x) = 0$$

Now, $$2x + 1 = 0 \text{ gives us } x = -\frac{1}{2}$$

So, $-\frac{1}{2}$ is a zero of the polynomial $2x + 1$.

Now, if $p(x) = ax + b$, $a \neq 0$, is a linear polynomial, how can we find a zero of $p(x)$? Example 4 may have given you some idea. Finding a zero of the polynomial $p(x)$, amounts to solving the polynomial equation $p(x) = 0$.

Now, $p(x) = 0$ means $$ax + b = 0,\ a \neq 0$$

So, $$ax = -b$$

i.e., $$x = -\frac{b}{a}.$$

So, $x = -\frac{b}{a}$ is the only zero of $p(x)$, i.e., a *linear polynomial has one and only one zero*.

Now we can say that 1 is *the* zero of $x - 1$, and -2 is *the* zero of $x + 2$.

Example 5 : Verify whether 2 and 0 are zeroes of the polynomial $x^2 - 2x$.

Solution : Let $$p(x) = x^2 - 2x$$

Then $$p(2) = 2^2 - 4 = 4 - 4 = 0$$

and $$p(0) = 0 - 0 = 0$$

Hence, 2 and 0 are both zeroes of the polynomial $x^2 - 2x$.

Let us now list our observations:

(i) A zero of a polynomial need not be 0.

(ii) 0 may be a zero of a polynomial.

(iii) Every linear polynomial has one and only one zero.

(iv) A polynomial can have more than one zero.

EXERCISE 2.2

1. Find the value of the polynomial $5x - 4x^2 + 3$ at

(i) $x = 0$ (ii) $x = -1$ (iii) $x = 2$

2. Find $p(0)$, $p(1)$ and $p(2)$ for each of the following polynomials:

(i) $p(y) = y^2 - y + 1$ (ii) $p(t) = 2 + t + 2t^2 - t^3$

(iii) $p(x) = x^3$ (iv) $p(x) = (x - 1)(x + 1)$

3. Verify whether the following are zeroes of the polynomial, indicated against them.

(i) $p(x) = 3x + 1,\ x = -\frac{1}{3}$ (ii) $p(x) = 5x - \pi,\ x = \frac{4}{5}$

(iii) $p(x) = x^2 - 1,\ x = 1, -1$ (iv) $p(x) = (x + 1)(x - 2),\ x = -1, 2$

(v) $p(x) = x^2,\ x = 0$ (vi) $p(x) = lx + m,\ x = -\frac{m}{l}$

(vii) $p(x) = 3x^2 - 1,\ x = -\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}$ (viii) $p(x) = 2x + 1,\ x = \frac{1}{2}$

4. Find the zero of the polynomial in each of the following cases:

(i) $p(x) = x + 5$ (ii) $p(x) = x - 5$ (iii) $p(x) = 2x + 5$

(iv) $p(x) = 3x - 2$ (v) $p(x) = 3x$ (vi) $p(x) = ax,\ a \neq 0$

(vii) $p(x) = cx + d,\ c \neq 0,\ c, d$ are real numbers.

2.4 Factorisation of Polynomials

Let us now look at the situation of Example 10 above more closely. It tells us that since the remainder, $q\left(-\frac{1}{2}\right) = 0$, $(2t + 1)$ is a factor of $q(t)$, i.e., $q(t) = (2t + 1)\, g(t)$ for some polynomial $g(t)$. This is a particular case of the following theorem.

Factor Theorem : If $p(x)$ is a polynomial of degree $n \geq 1$ and a is any real number, then (i) $x - a$ is a factor of $p(x)$, if $p(a) = 0$, and (ii) $p(a) = 0$, if $x - a$ is a factor of $p(x)$.

Proof: By the Remainder Theorem, $p(x)=(x - a)\, q(x) + p(a)$.

(i) If $p(a) = 0$, then $p(x) = (x - a)\, q(x)$, which shows that $x - a$ is a factor of $p(x)$.

(ii) Since $x - a$ is a factor of $p(x)$, $p(x) = (x - a)\, g(x)$ for same polynomial $g(x)$. In this case, $p(a) = (a - a)\, g(a) = 0$.

Example 6 : Examine whether $x + 2$ is a factor of $x^3 + 3x^2 + 5x + 6$ and of $2x + 4$.

Solution : The zero of $x + 2$ is -2. Let $p(x) = x^3 + 3x^2 + 5x + 6$ and $s(x) = 2x + 4$

Then,
$$\begin{aligned} p(-2) &= (-2)^3 + 3(-2)^2 + 5(-2) + 6 \\ &= -8 + 12 - 10 + 6 \\ &= 0 \end{aligned}$$

So, by the Factor Theorem, $x + 2$ is a factor of $x^3 + 3x^2 + 5x + 6$.

Again, $s(-2) = 2(-2) + 4 = 0$

So, $x + 2$ is a factor of $2x + 4$. In fact, you can check this without applying the Factor Theorem, since $2x + 4 = 2(x + 2)$.

Example 7 : Find the value of k, if $x - 1$ is a factor of $4x^3 + 3x^2 - 4x + k$.

Solution : As $x - 1$ is a factor of $p(x) = 4x^3 + 3x^2 - 4x + k$, $p(1) = 0$

Now, $p(1) = 4(1)^3 + 3(1)^2 - 4(1) + k$

So, $4 + 3 - 4 + k = 0$

i.e., $k = -3$

We will now use the Factor Theorem to factorise some polynomials of degree 2 and 3. You are already familiar with the factorisation of a quadratic polynomial like $x^2 + lx + m$. You had factorised it by splitting the middle term lx as $ax + bx$ so that $ab = m$. Then $x^2 + lx + m = (x + a)(x + b)$. We shall now try to factorise quadratic polynomials of the type $ax^2 + bx + c$, where $a \neq 0$ and a, b, c are constants.

Factorisation of the polynomial $ax^2 + bx + c$ **by splitting the middle term** is as follows:

$\frac{3x^2}{x} = 3x$ = first term of quotient

Let its factors be $(px + q)$ and $(rx + s)$. Then

$$ax^2 + bx + c = (px + q)(rx + s) = pr\,x^2 + (ps + qr)\,x + qs$$

Comparing the coefficients of x^2, we get $a = pr$.

Similarly, comparing the coefficients of x, we get $b = ps + qr$.

And, on comparing the constant terms, we get $c = qs$.

This shows us that b is the sum of two numbers ps and qr, whose product is $(ps)(qr) = (pr)(qs) = ac$.

Therefore, to factorise $ax^2 + bx + c$, we have to write b as the sum of two numbers whose product is ac. This will be clear from Example 13.

Example 8 : Factorise $6x^2 + 17x + 5$ by splitting the middle term, and by using the Factor Theorem.

Solution 1 : (By splitting method) **:** If we can find two numbers p and q such that $p + q = 17$ and $pq = 6 \times 5 = 30$, then we can get the factors.

So, let us look for the pairs of factors of 30. Some are 1 and 30, 2 and 15, 3 and 10, 5 and 6. Of these pairs, 2 and 15 will give us $p + q = 17$.

$$\text{So, } 6x^2 + 17x + 5 = 6x^2 + (2 + 15)x + 5$$
$$= 6x^2 + 2x + 15x + 5$$
$$= 2x(3x + 1) + 5(3x + 1)$$
$$= (3x + 1)(2x + 5)$$

Solution 2 : (Using the Factor Theorem)

$6x^2 + 17x + 5 = 6\left(x^2 + \frac{17}{6}x + \frac{5}{6}\right) = 6\,p(x)$, say. If a and b are the zeroes of $p(x)$, then $6x^2 + 17x + 5 = 6(x - a)(x - b)$. So, $ab = \frac{5}{6}\cdot$ Let us look at some possibilities for a and b. They could be $\pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{5}{3}, \pm\frac{5}{2}, \pm 1$. Now, $p\left(\frac{1}{2}\right) = \frac{1}{4} + \frac{17}{6}\left(\frac{1}{2}\right) + \frac{5}{6} \neq 0$. But $p\left(\frac{-1}{3}\right) = 0$. So, $\left(x + \frac{1}{3}\right)$ is a factor of $p(x)$. Similarly, by trial, you can find that $\left(x + \frac{5}{2}\right)$ is a factor of $p(x)$.

Therefore,
$$6x^2 + 17x + 5 = 6\left(x + \frac{1}{3}\right)\left(x + \frac{5}{2}\right)$$
$$= 6\left(\frac{3x + 1}{3}\right)\left(\frac{2x + 5}{2}\right)$$
$$= (3x + 1)(2x + 5)$$

For the example above, the use of the splitting method appears more efficient. However, let us consider another example.

Example 9 : Factorise $y^2 - 5y + 6$ by using the Factor Theorem.

Solution : Let $p(y) = y^2 - 5y + 6$. Now, if $p(y) = (y - a)(y - b)$, you know that the constant term will be ab. So, $ab = 6$. So, to look for the factors of $p(y)$, we look at the factors of 6.

The factors of 6 are 1, 2 and 3.

Now, $p(2) = 2^2 - (5 \times 2) + 6 = 0$

So, $y - 2$ is a factor of $p(y)$.

Also, $p(3) = 3^2 - (5 \times 3) + 6 = 0$

So, $y - 3$ is also a factor of $y^2 - 5y + 6$.

Therefore, $y^2 - 5y + 6 = (y - 2)(y - 3)$

Note that $y^2 - 5y + 6$ can also be factorised by splitting the middle term $-5y$.

Now, let us consider factorising cubic polynomials. Here, the splitting method will not be appropriate to start with. We need to find at least one factor first, as you will see in the following example.

Example 10 : Factorise $x^3 - 23x^2 + 142x - 120$.

Solution : Let $p(x) = x^3 - 23x^2 + 142x - 120$

We shall now look for all the factors of -120. Some of these are $\pm 1, \pm 2, \pm 3,$

$\pm 4, \pm 5, \pm 6, \pm 8, \pm 10, \pm 12, \pm 15, \pm 20, \pm 24, \pm 30, \pm 60$.

By trial, we find that $p(1) = 0$. So $x - 1$ is a factor of $p(x)$.

Now we see that $x^3 - 23x^2 + 142x - 120 = x^3 - x^2 - 22x^2 + 22x + 120x - 120$

$$= x^2(x - 1) - 22x(x - 1) + 120(x - 1) \quad \text{(Why?)}$$

$$= (x - 1)(x^2 - 22x + 120) \quad [\text{Taking } (x - 1) \text{ common}]$$

We could have also got this by dividing $p(x)$ by $x - 1$.

Now $x^2 - 22x + 120$ can be factorised either by splitting the middle term or by using the Factor theorem. By splitting the middle term, we have:

$$x^2 - 22x + 120 = x^2 - 12x - 10x + 120$$

$$= x(x - 12) - 10(x - 12)$$

$$= (x - 12)(x - 10)$$

So, $$x^3 - 23x^2 - 142x - 120 = (x - 1)(x - 10)(x - 12)$$

EXERCISE 2.3

1. Determine which of the following polynomials has $(x + 1)$ a factor :

   (i) $x^3 + x^2 + x + 1$ (ii) $x^4 + x^3 + x^2 + x + 1$

   (iii) $x^4 + 3x^3 + 3x^2 + x + 1$ (iv) $x^3 - x^2 - \left(2 + \sqrt{2}\right)x + \sqrt{2}$

2. Use the Factor Theorem to determine whether $g(x)$ is a factor of $p(x)$ in each of the following cases:

   (i) $p(x) = 2x^3 + x^2 - 2x - 1,\ g(x) = x + 1$

(ii) $p(x) = x^3 + 3x^2 + 3x + 1, g(x) = x + 2$

(iii) $p(x) = x^3 - 4x^2 + x + 6, g(x) = x - 3$

3. Find the value of k, if $x - 1$ is a factor of $p(x)$ in each of the following cases:

(i) $p(x) = x^2 + x + k$ (ii) $p(x) = 2x^2 + kx + \sqrt{2}$

(iii) $p(x) = kx^2 - \sqrt{2}x + 1$ (iv) $p(x) = kx^2 - 3x + k$

4. Factorise :

(i) $12x^2 - 7x + 1$ (ii) $2x^2 + 7x + 3$

(iii) $6x^2 + 5x - 6$ (iv) $3x^2 - x - 4$

5. Factorise :

(i) $x^3 - 2x^2 - x + 2$ (ii) $x^3 - 3x^2 - 9x - 5$

(iii) $x^3 + 13x^2 + 32x + 20$ (iv) $2y^3 + y^2 - 2y - 1$

2.5 Algebraic Identities

From your earlier classes, you may recall that an algebraic identity is an algebraic equation that is true for all values of the variables occurring in it. You have studied the following algebraic identities in earlier classes:

Identity I : $(x + y)^2 = x^2 + 2xy + y^2$

Identity II : $(x - y)^2 = x^2 - 2xy + y^2$

Identity III : $x^2 - y^2 = (x + y)(x - y)$

Identity IV : $(x + a)(x + b) = x^2 + (a + b)x + ab$

You must have also used some of these algebraic identities to factorise the algebraic expressions. You can also see their utility in computations.

Example 11 : Find the following products using appropriate identities:

(i) $(x + 3)(x + 3)$ (ii) $(x - 3)(x + 5)$

Solution : (i) Here we can use Identity I : $(x + y)^2 = x^2 + 2xy + y^2$. Putting $y = 3$ in it, we get

$$(x + 3)(x + 3) = (x + 3)^2 = x^2 + 2(x)(3) + (3)^2$$
$$= x^2 + 6x + 9$$

(ii) Using Identity IV above, i.e., $(x + a)(x + b) = x^2 + (a + b)x + ab$, we have

$$(x - 3)(x + 5) = x^2 + (-3 + 5)x + (-3)(5)$$
$$= x^2 + 2x - 15$$

Example 12 : Evaluate 105 × 106 without multiplying directly.

Solution :

$$\begin{aligned} 105 \times 106 &= (100 + 5) \times (100 + 6) \\ &= (100)^2 + (5 + 6)(100) + (5 \times 6), \text{ using Identity IV} \\ &= 10000 + 1100 + 30 \\ &= 11130 \end{aligned}$$

You have seen some uses of the identities listed above in finding the product of some given expressions. These identities are useful in factorisation of algebraic expressions also, as you can see in the following examples.

Example 13 : Factorise:

(i) $49a^2 + 70ab + 25b^2$ (ii) $\frac{25}{4}x^2 - \frac{y^2}{9}$

Solution : (i) Here you can see that

$$49a^2 = (7a)^2,\ 25b^2 = (5b)^2,\ 70ab = 2(7a)(5b)$$

Comparing the given expression with $x^2 + 2xy + y^2$, we observe that $x = 7a$ and $y = 5b$.

Using Identity I, we get

$$49a^2 + 70ab + 25b^2 = (7a + 5b)^2 = (7a + 5b)(7a + 5b)$$

(ii) We have $\frac{25}{4}x^2 - \frac{y^2}{9} = \left(\frac{5}{2}x\right)^2 - \left(\frac{y}{3}\right)^2$

Now comparing it with Identity III, we get

$$\begin{aligned} \frac{25}{4}x^2 - \frac{y^2}{9} &= \left(\frac{5}{2}x\right)^2 - \left(\frac{y}{3}\right)^2 \\ &= \left(\frac{5}{2}x + \frac{y}{3}\right)\left(\frac{5}{2}x - \frac{y}{3}\right) \end{aligned}$$

So far, all our identities involved products of binomials. Let us now extend the Identity I to a trinomial $x + y + z$. We shall compute $(x + y + z)^2$ by using Identity I.

Let $x + y = t$. Then,

$$\begin{aligned} (x + y + z)^2 &= (t + z)^2 \\ &= t^2 + 2tz + t^2 && \text{(Using Identity I)} \\ &= (x + y)^2 + 2(x + y)z + z^2 && \text{(Substituting the value of } t\text{)} \end{aligned}$$

$$= x^2 + 2xy + y^2 + 2xz + 2yz + z^2 \qquad \text{(Using Identity I)}$$

$$= x^2 + y^2 + z^2 + 2xy + 2yz + 2zx \qquad \text{(Rearranging the terms)}$$

So, we get the following identity:

Identity V : $(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$

Remark : We call the right hand side expression **the expanded form** of the left hand side expression. Note that the expansion of $(x + y + z)^2$ consists of three square terms and three product terms.

Example 14 : Write $(3a + 4b + 5c)^2$ in expanded form.

Solution : Comparing the given expression with $(x + y + z)^2$, we find that

$$x = 3a,\ y = 4b \text{ and } z = 5c.$$

Therefore, using Identity V, we have

$$\begin{aligned}(3a + 4b + 5c)^2 &= (3a)^2 + (4b)^2 + (5c)^2 + 2(3a)(4b) + 2(4b)(5c) + 2(5c)(3a)\\ &= 9a^2 + 16b^2 + 25c^2 + 24ab + 40bc + 30ac\end{aligned}$$

Example 15 : Expand $(4a - 2b - 3c)^2$.

Solution : Using Identity V, we have

$$\begin{aligned}(4a - 2b - 3c)^2 &= [4a + (-2b) + (-3c)]^2\\ &= (4a)^2 + (-2b)^2 + (-3c)^2 + 2(4a)(-2b) + 2(-2b)(-3c) + 2(-3c)(4a)\\ &= 16a^2 + 4b^2 + 9c^2 - 16ab + 12bc - 24ac\end{aligned}$$

Example 16 : Factorise $4x^2 + y^2 + z^2 - 4xy - 2yz + 4xz$.

Solution : We have $4x^2 + y^2 + z^2 - 4xy - 2yz + 4xz = (2x)^2 + (-y)^2 + (z)^2 + 2(2x)(-y)$

$$\begin{aligned}&+ 2(-y)(z) + 2(2x)(z)\\ &= [2x + (-y) + z]^2 \qquad \text{(Using Identity V)}\\ &= (2x - y + z)^2 = (2x - y + z)(2x - y + z)\end{aligned}$$

So far, we have dealt with identities involving second degree terms. Now let us extend Identity I to compute $(x + y)^3$. We have:

$$\begin{aligned}(x + y)^3 &= (x + y)\,(x + y)^2\\ &= (x + y)(x^2 + 2xy + y^2)\\ &= x(x^2 + 2xy + y^2) + y(x^2 + 2xy + y^2)\\ &= x^3 + 2x^2y + xy^2 + x^2y + 2xy^2 + y^3\\ &= x^3 + 3x^2y + 3xy^2 + y^3\\ &= x^3 + y^3 + 3xy(x + y)\end{aligned}$$

So, we get the following identity:

Identity VI : $(x + y)^3 = x^3 + y^3 + 3xy\ (x + y)$

Also, by replacing y by $-y$ in the Identity VI, we get

Identity VII : $(x - y)^3 = x^3 - y^3 - 3xy(x - y)$

$$= x^3 - 3x^2y + 3xy^2 - y^3$$

Example 17 : Write the following cubes in the expanded form:

(i) $(3a + 4b)^3$ (ii) $(5p - 3q)^3$

Solution : (i) Comparing the given expression with $(x + y)^3$, we find that

$$x = 3a \text{ and } y = 4b.$$

So, using Identity VI, we have:

$$(3a + 4b)^3 = (3a)^3 + (4b)^3 + 3(3a)(4b)(3a + 4b)$$
$$= 27a^3 + 64b^3 + 108a^2b + 144ab^2$$

(ii) Comparing the given expression with $(x - y)^3$, we find that

$$x = 5p,\ y = 3q.$$

So, using Identity VII, we have:

$$(5p - 3q)^3 = (5p)^3 - (3q)^3 - 3(5p)(3q)(5p - 3q)$$
$$= 125p^3 - 27q^3 - 225p^2q + 135pq^2$$

Example 18 : Evaluate each of the following using suitable identities:

(i) $(104)^3$ (ii) $(999)^3$

Solution : (i) We have

$$(104)^3 = (100 + 4)^3$$
$$= (100)^3 + (4)^3 + 3(100)(4)(100 + 4)$$

(Using Identity VI)

$$= 1000000 + 64 + 124800$$
$$= 1124864$$

(ii) We have

$$(999)^3 = (1000 - 1)^3$$
$$= (1000)^3 - (1)^3 - 3(1000)(1)(1000 - 1)$$

(Using Identity VII)

$$= 1000000000 - 1 - 2997000$$
$$= 997002999$$

Example 19 : Factorise $8x^3 + 27y^3 + 36x^2y + 54xy^2$

Solution : The given expression can be written as

$$(2x)^3 + (3y)^3 + 3(4x^2)(3y) + 3(2x)(9y^2)$$
$$= (2x)^3 + (3y)^3 + 3(2x)^2(3y) + 3(2x)(3y)^2$$
$$= (2x + 3y)^3 \qquad \text{(Using Identity VI)}$$
$$= (2x + 3y)(2x + 3y)(2x + 3y)$$

Now consider $(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

On expanding, we get the product as

$x(x^2 + y^2 + z^2 - xy - yz - zx) + y(x^2 + y^2 + z^2 - xy - yz - zx)$
$+ z(x^2 + y^2 + z^2 - xy - yz - zx) = x^3 + xy^2 + xz^2 - x^2y - xyz - zx^2 + x^2y$
$+ y^3 + yz^2 - xy^2 - y^2z - xyz + x^2z + y^2z + z^3 - xyz - yz^2 - xz^2$

$$= x^3 + y^3 + z^3 - 3xyz \qquad \text{(On simplification)}$$

So, we obtain the following identity:

Identity VIII : $\mathbf{x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)}$

Example 20 : Factorise : $8x^3 + y^3 + 27z^3 - 18xyz$

Solution : Here, we have

$$8x^3 + y^3 + 27z^3 - 18xyz$$
$$= (2x)^3 + y^3 + (3z)^3 - 3(2x)(y)(3z)$$
$$= (2x + y + 3z)[(2x)^2 + y^2 + (3z)^2 - (2x)(y) - (y)(3z) - (2x)(3z)]$$
$$= (2x + y + 3z)(4x^2 + y^2 + 9z^2 - 2xy - 3yz - 6xz)$$

EXERCISE 2.4

1. Use suitable identities to find the following products:
   (i) $(x + 4)(x + 10)$ (ii) $(x + 8)(x - 10)$ (iii) $(3x + 4)(3x - 5)$
   (iv) $(y^2 + \frac{3}{2})(y^2 - \frac{3}{2})$ (v) $(3 - 2x)(3 + 2x)$
2. Evaluate the following products without multiplying directly:
   (i) 103×107 (ii) 95×96 (iii) 104×96
3. Factorise the following using appropriate identities:
   (i) $9x^2 + 6xy + y^2$ (ii) $4y^2 - 4y + 1$ (iii) $x^2 - \frac{y^2}{100}$

4. Expand each of the following, using suitable identities:

(i) $(x + 2y + 4z)^2$ (ii) $(2x - y + z)^2$ (iii) $(-2x + 3y + 2z)^2$

(iv) $(3a - 7b - c)^2$ (v) $(-2x + 5y - 3z)^2$ (vi) $\left[\frac{1}{4}a - \frac{1}{2}b + 1\right]^2$

5. Factorise:

(i) $4x^2 + 9y^2 + 16z^2 + 12xy - 24yz - 16xz$

(ii) $2x^2 + y^2 + 8z^2 - 2\sqrt{2}\,xy + 4\sqrt{2}\,yz - 8xz$

6. Write the following cubes in expanded form:

(i) $(2x + 1)^3$ (ii) $(2a - 3b)^3$ (iii) $\left[\frac{3}{2}x + 1\right]^3$ (iv) $\left[x - \frac{2}{3}y\right]^3$

7. Evaluate the following using suitable identities:

(i) $(99)^3$ (ii) $(102)^3$ (iii) $(998)^3$

8. Factorise each of the following:

(i) $8a^3 + b^3 + 12a^2b + 6ab^2$ (ii) $8a^3 - b^3 - 12a^2b + 6ab^2$

(iii) $27 - 125a^3 - 135a + 225a^2$ (iv) $64a^3 - 27b^3 - 144a^2b + 108ab^2$

(v) $27p^3 - \frac{1}{216} - \frac{9}{2}p^2 + \frac{1}{4}p$

9. Verify : (i) $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$ (ii) $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

10. Factorise each of the following:

(i) $27y^3 + 125z^3$ (ii) $64m^3 - 343n^3$

[**Hint :** See Question 9.]

11. Factorise : $27x^3 + y^3 + z^3 - 9xyz$

12. Verify that $x^3 + y^3 + z^3 - 3xyz = \frac{1}{2}(x + y + z)\left[(x - y)^2 + (y - z)^2 + (z - x)^2\right]$

13. If $x + y + z = 0$, show that $x^3 + y^3 + z^3 = 3xyz$.

14. Without actually calculating the cubes, find the value of each of the following:

(i) $(-12)^3 + (7)^3 + (5)^3$

(ii) $(28)^3 + (-15)^3 + (-13)^3$

15. Give possible expressions for the length and breadth of each of the following rectangles, in which their areas are given:

Area : $25a^2 - 35a + 12$	Area : $35y^2 + 13y - 12$
(i)	(ii)

16. What are the possible expressions for the dimensions of the cuboids whose volumes are given below?

(i)	(ii)
Volume : $3x^2 - 12x$	Volume : $12ky^2 + 8ky - 20k$

2.6 Summary

In this chapter, you have studied the following points:

1. A *polynomial* $p(x)$ in one variable x is an algebraic expression in x of the form

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0,$$

where $a_0, a_1, a_2, \ldots, a_n$ are constants and $a_n \neq 0$.

$a_0, a_1, a_2, \ldots, a_n$ are respectively the *coefficients* of $x^0, x, x^2, \ldots, x^n$, and n is called the *degree of the polynomial*. Each of $a_n x^n, a_{n-1} x^{n-1}, \ldots, a_0$, with $a_n \neq 0$, is called a *term* of the polynomial $p(x)$.

2. A polynomial of one term is called a monomial.

3. A polynomial of two terms is called a binomial.

4. A polynomial of three terms is called a trinomial.

5. A polynomial of degree one is called a linear polynomial.

6. A polynomial of degree two is called a quadratic polynomial.

7. A polynomial of degree three is called a cubic polynomial.

8. A real number 'a' is a *zero* of a polynomial $p(x)$ if $p(a) = 0$. In this case, a is also called a *root* of the equation $p(x) = 0$.

9. Every linear polynomial in one variable has a unique zero, a non-zero constant polynomial has no zero, and every real number is a zero of the zero polynomial.

10. Factor Theorem : $x - a$ is a factor of the polynomial $p(x)$, if $p(a) = 0$. Also, if $x - a$ is a factor of $p(x)$, then $p(a) = 0$.

11. $(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$

12. $(x + y)^3 = x^3 + y^3 + 3xy(x + y)$

13. $(x - y)^3 = x^3 - y^3 - 3xy(x - y)$

14. $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

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Chapter 3

COORDINATE GEOMETRY

What's the good of Mercator's North Poles and Equators, Tropics, Zones and Meridian Lines?' So the Bellman would cry; and crew would reply ' They are merely conventional signs!'

LEWIS CARROLL, *The Hunting of the Snark*

3.1 Introduction

You have already studied how to locate a point on a number line. You also know how to describe the position of a point on the line. There are many other situations, in which to find a point we are required to describe its position with reference to more than one line. For example, consider the following situations:

I. In Fig. 3.1, there is a main road running in the East-West direction and streets with numbering from West to East. Also, on each street, house numbers are marked. To look for a friend's house here, is it enough to know only one reference point? For instance, if we only know that she lives on Street 2, will we be able to find her house easily? Not as easily as when we know two pieces of information about it, namely, the number of the street on which it is situated, and the house number. If we want to reach the house which is situated in the 2nd street and has the number 5, first of all we would identify the 2nd street and then the house numbered 5 on it. In Fig. 3.1, H shows the location of the house. Similarly, P shows the location of the house corresponding to Street number 7 and House number 4.

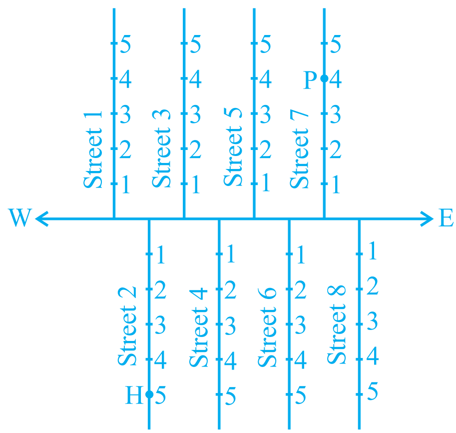


Fig. 3.1

II. Suppose you put a dot on a sheet of paper [Fig.3.2 (a)]. If we ask you to tell us the position of the dot on the paper, how will you do this? Perhaps you will try in some such manner: "The dot is in the upper half of the paper", or "It is near the left edge of the paper", or "It is very near the left hand upper corner of the sheet". Do any of these statements fix the position of the dot precisely? No! But, if you say " The dot is nearly 5 cm away from the left edge of the paper", it helps to give some idea but still does not fix the position of the dot. A little thought might enable you to say that the dot is also at a distance of 9 cm above the bottom line. We now know exactly where the dot is!

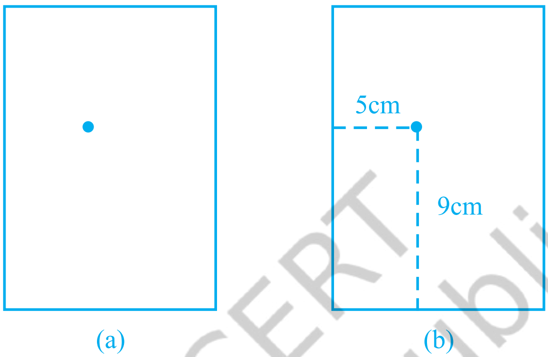


Fig. 3.2

For this purpose, we fixed the position of the dot by specifying its distances from two fixed lines, the left edge of the paper and the bottom line of the paper [Fig.3.2 (b)]. In other words, we need **two** independent informations for finding the position of the dot.

Now, perform the following classroom activity known as 'Seating Plan'.

Activity 1 (Seating Plan) : Draw a plan of the seating in your classroom, pushing all the desks together. Represent each desk by a square. In each square, write the name of the student occupying the desk, which the square represents. Position of each student in the classroom is described precisely by using two independent informations:

(i) the column in which she or he sits,

(ii) the row in which she or he sits.

If you are sitting on the desk lying in the 5^{th} column and 3^{rd} row (represented by the shaded square in Fig. 3.3), your position could be written as (5, 3), first writing the column number, and then the row number. Is this the same as (3, 5)? Write down the names and positions of other students in your class. For example, if Sonia is sitting in the 4^{th} column and 1^{st} row, write S(4,1). The teacher's desk is not part of your seating plan. We are treating the teacher just as an observer.

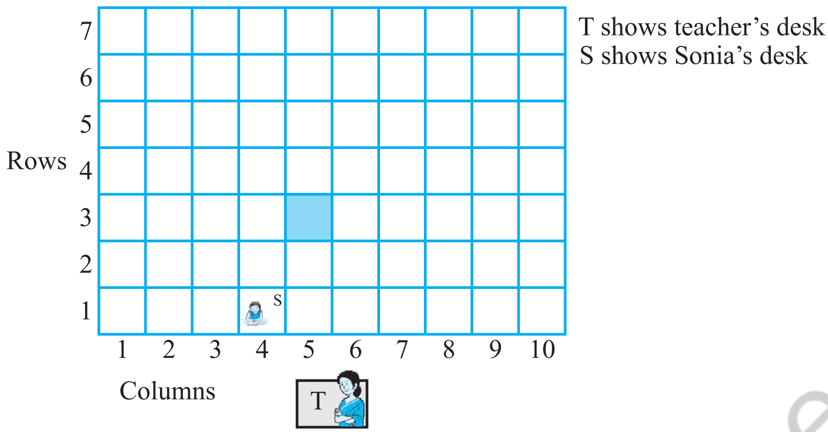


Fig. 3.3

In the discussion above, you observe that position of any object lying in a plane can be represented with the help of two perpendicular lines. In case of 'dot', we require distance of the dot from bottom line as well as from left edge of the paper. In case of seating plan, we require the number of the column and that of the row. This simple idea has far reaching consequences, and has given rise to a very important branch of Mathematics known as *Coordinate Geometry*. In this chapter, we aim to introduce some basic concepts of coordinate geometry. You will study more about these in your higher classes. This study was initially developed by the French philosopher and mathematician *René Déscartes*.

René Déscartes, the great French mathematician of the seventeenth century, liked to lie in bed and think! One day, when resting in bed, he solved the problem of describing the position of a point in a plane. His method was a development of the older idea of latitude and longitude. In honour of Déscartes, the system used for describing the position of a point in a plane is also known as the *Cartesian system*.

René Déscartes (1596 -1650)
Fig. 3.4

EXERCISE 3.1

1. How will you describe the position of a table lamp on your study table to another person?
2. **(Street Plan) :** A city has two main roads which cross each other at the centre of the city. These two roads are along the North-South direction and East-West direction.

All the other streets of the city run parallel to these roads and are 200 m apart. There are 5 streets in each direction. Using 1cm = 200 m, draw a model of the city on your notebook. Represent the roads/streets by single lines.

There are many cross- streets in your model. A particular cross-street is made by two streets, one running in the North - South direction and another in the East - West direction. Each cross street is referred to in the following manner : If the 2^{nd} street running in the North - South direction and 5^{th} in the East - West direction meet at some crossing, then we will call this cross-street (2, 5). Using this convention, find:

(i) how many cross - streets can be referred to as (4, 3).

(ii) how many cross - streets can be referred to as (3, 4).

3.2 Cartesian System

You have studied the *number line* in the chapter on 'Number System'. On the number line, distances from a fixed point are marked in equal units positively in one direction and negatively in the other. The point from which the distances are marked is called the *origin*. We use the number line to represent the numbers by marking points on a line at equal distances. If one unit distance represents the number '1', then 3 units distance represents the number '3', '0' being at the origin. The point in the positive direction at a distance r from the origin represents the number r. The point in the negative direction at a distance r from the origin represents the number $-r$. Locations of different numbers on the number line are shown in Fig. 3.5.

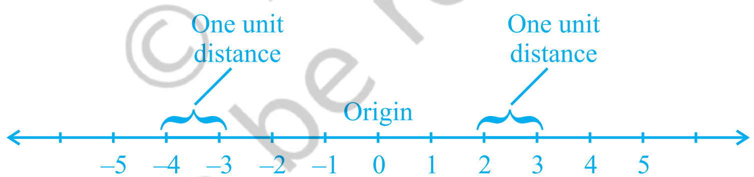


Fig. 3.5

Descartes invented the idea of placing two such lines perpendicular to each other on a plane, and locating points on the plane by referring them to these lines. The perpendicular lines may be in any direction such as in Fig.3.6. But, when we choose

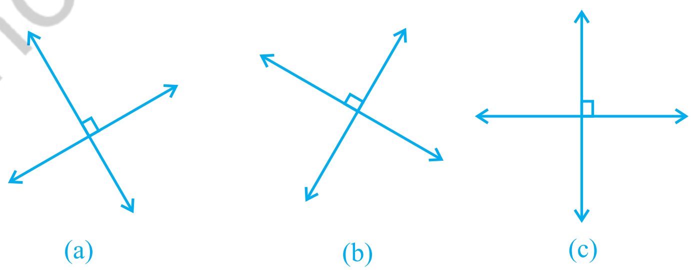
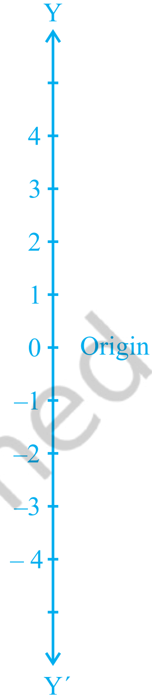
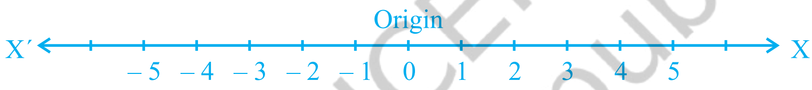


Fig. 3.6

these two lines to locate a point in a plane *in this chapter*, one line will be horizontal and the other will be vertical, as in Fig. 3.6(c).

These lines are actually obtained as follows : Take two number lines, calling them X′X and Y′Y. Place X′X horizontal [as in Fig. 3.7(a)] and write the numbers on it just as written on the number line. We do the same thing with Y′Y except that Y′Y is vertical, not horizontal [Fig. 3.7(b)].





(a)

(b)

Fig. 3.7

Combine both the lines in such a way that the two lines cross each other at their zeroes, or origins (Fig. 3.8). The horizontal line X′X is called the x - axis and the vertical line YY′ is called the y - axis. The point where X′X and Y′Y cross is called the **origin**, and is denoted by O. Since the positive numbers lie on the directions OX and OY, OX and OY are called the **positive directions** of the x - axis and the y - axis, respectively. Similarly, OX′ and OY′ are called the **negative directions** of the x - axis and the y - axis, respectively.

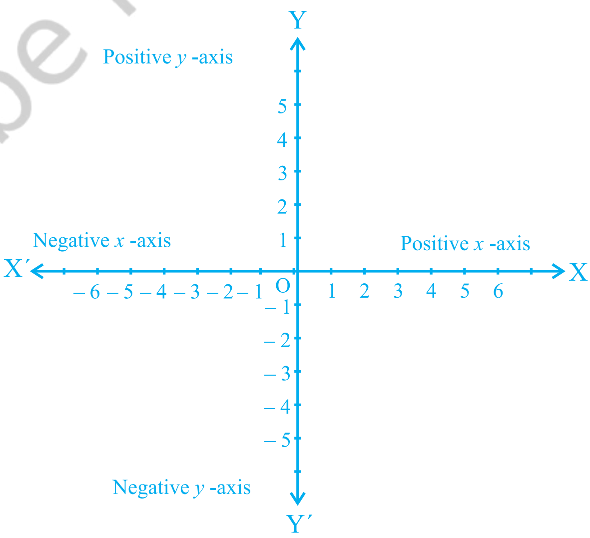


Fig. 3.8

You observe that the axes (plural of the word 'axis') divide the plane into four parts. These four parts are called the *quadrants* (one fourth part), numbered I, II, III and IV anticlockwise from OX (see Fig.3.9). So, the plane consists of the axes and these quadrants. We call the plane, the *Cartesian plane,* or the *coordinate plane,* or the *xy-plane.* The axes are called the *coordinate axes*.

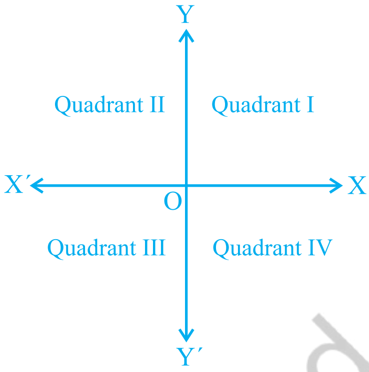


Fig. 3.9

Now, let us see why this system is so basic to mathematics, and how it is useful. Consider the following diagram where the axes are drawn on graph paper. Let us see the distances of the points P and Q from the axes. For this, we draw perpendiculars PM on the x - axis and PN on the y - axis. Similarly, we draw perpendiculars QR and QS as shown in Fig. 3.10.

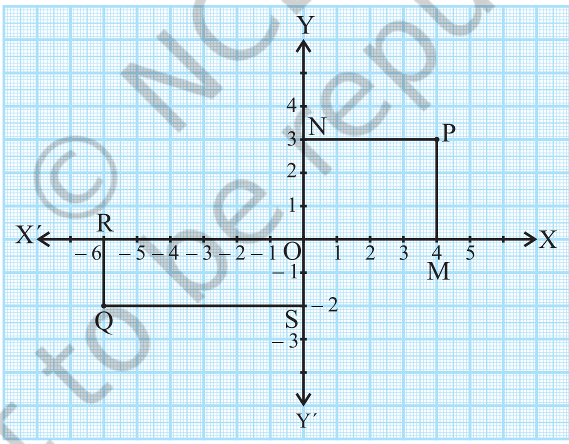


Fig.3.10

You find that

(i) The perpendicular distance of the point P from the y - axis measured along the positive direction of the x - axis is PN = OM = 4 units.

(ii) The perpendicular distance of the point P from the x - axis measured along the positive direction of the y - axis is PM = ON = 3 units.

(iii) The perpendicular distance of the point Q from the y - axis measured along the negative direction of the x - axis is OR = SQ = 6 units.

(iv) The perpendicular distance of the point Q from the x - axis measured along the negative direction of the y - axis is OS = RQ = 2 units.

Now, using these distances, how can we describe the points so that there is no confusion?

We write the coordinates of a point, using the following conventions:

(i) The *x - coordinate* of a point is its perpendicular distance from the y - axis measured along the x -axis (positive along the positive direction of the x - axis and negative along the negative direction of the x - axis). For the point P, it is + 4 and for Q, it is – 6. The x - coordinate is also called the *abscissa*.

(ii) The *y - coordinate* of a point is its perpendicular distance from the x - axis measured along the y - axis (positive along the positive direction of the y - axis and negative along the negative direction of the y - axis). For the point P, it is + 3 and for Q, it is –2. The y - coordinate is also called the *ordinate*.

(iii) In stating the coordinates of a point in the coordinate plane, the x - coordinate comes first, and then the y - coordinate. We place the coordinates in brackets.

Hence, the coordinates of P are (4, 3) and the coordinates of Q are (– 6, – 2).

Note that the coordinates describe a point in the plane *uniquely.* (3, 4) is not the same as (4, 3).

Example 1 : See Fig. 3.11 and complete the following statements:

(i) The abscissa and the ordinate of the point B are _ _ _ and _ _ _, respectively. Hence, the coordinates of B are (_ _, _ _).

(ii) The x-coordinate and the y-coordinate of the point M are _ _ _ and _ _ _, respectively. Hence, the coordinates of M are (_ _, _ _).

(iii) The x-coordinate and the y-coordinate of the point L are _ _ _ and _ _ _, respectively. Hence, the coordinates of L are (_ _, _ _).

(iv) The x-coordinate and the y-coordinate of the point S are _ _ _ and _ _ _, respectively. Hence, the coordinates of S are (_ _, _ _).

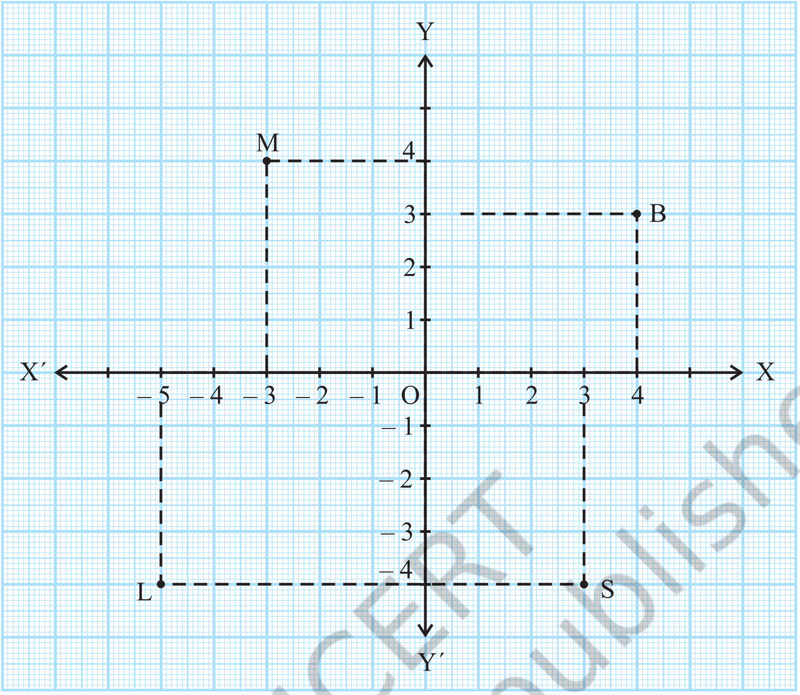


Fig. 3.11

Solution: (i) Since the distance of the point B from the y - axis is 4 units, the x - coordinate or abscissa of the point B is 4. The distance of the point B from the x - axis is 3 units; therefore, the y - coordinate, i.e., the ordinate, of the point B is 3. Hence, the coordinates of the point B are (4, 3).

As in (i) above :

(ii) The x - coordinate and the y - coordinate of the point M are –3 and 4, respectively. Hence, the coordinates of the point M are (–3, 4).

(iii) The x - coordinate and the y - coordinate of the point L are –5 and – 4, respectively. Hence, the coordinates of the point L are (–5, – 4).

(iv) The x - coordinate and the y- coordinate of the point S are 3 and – 4, respectively. Hence, the coordinates of the point S are (3, – 4).

Example 2 : Write the coordinates of the points marked on the axes in Fig. 3.12.

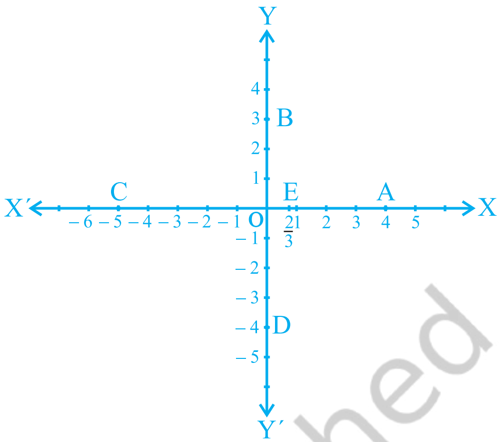


Fig. 3.12

Solution : You can see that :

(i) The point A is at a distance of + 4 units from the y - axis and at a distance zero from the x - axis. Therefore, the x - coordinate of A is 4 and the y - coordinate is 0. Hence, the coordinates of A are (4, 0).

(ii) The coordinates of B are (0, 3). Why?

(iii) The coordinates of C are (– 5, 0). Why?

(iv) The coordinates of D are (0, – 4). Why?

(v) The coordinates of E are $\left(\frac{2}{3}, 0\right)$. Why?

Since every point on the x - axis has no distance (zero distance) from the x - axis, therefore, the y - coordinate of every point lying on the x - axis is always zero. Thus, the coordinates of any point on the x - axis are of the form (x, 0), where x is the distance of the point from the y - axis. Similarly, the coordinates of any point on the y - axis are of the form (0, y), where y is the distance of the point from the x - axis. Why?

What are the coordinates of the ***origin* O**? *It has zero distance from both the axes so that its abscissa and ordinate are both zero. Therefore, the coordinates of the origin are* **(0, 0)**.

In the examples above, you may have observed the following relationship between the signs of the coordinates of a point and the quadrant of a point in which it lies.

(i) If a point is in the 1st quadrant, then the point will be in the form (+, +), since the 1st quadrant is enclosed by the positive x - axis and the positive y - axis.

(ii) If a point is in the 2nd quadrant, then the point will be in the form (–, +), since the 2nd quadrant is enclosed by the negative x - axis and the positive y - axis.

(iii) If a point is in the 3rd quadrant, then the point will be in the form (–, –), since the 3rd quadrant is enclosed by the negative x - axis and the negative y - axis.

(iv) If a point is in the 4th quadrant, then the point will be in the form (+, –), since the 4th quadrant is enclosed by the positive x - axis and the negative y - axis (see Fig. 3.13).

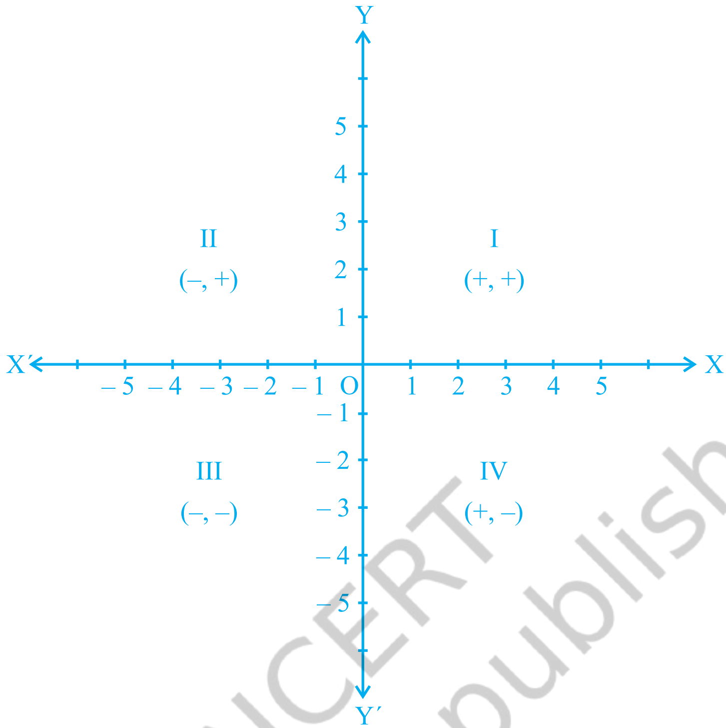


Fig. 3.13

Remark : The system we have discussed above for describing a point in a plane is only a convention, which is accepted all over the world. The system could also have been, for example, the ordinate first, and the abscissa second. However, the whole world sticks to the system we have described to avoid any confusion.

EXERCISE 3.2

1. Write the answer of each of the following questions:

(i) What is the name of horizontal and the vertical lines drawn to determine the position of any point in the Cartesian plane?

(ii) What is the name of each part of the plane formed by these two lines?

(iii) Write the name of the point where these two lines intersect.

2. See Fig.3.14, and write the following:

(i) The coordinates of B.

(ii) The coordinates of C.

(iii) The point identified by the coordinates (–3, –5).

(iv) The point identified by the coordinates (2, – 4).

(v) The abscissa of the point D.

(vi) The ordinate of the point H.

(vii) The coordinates of the point L.

(viii) The coordinates of the point M.

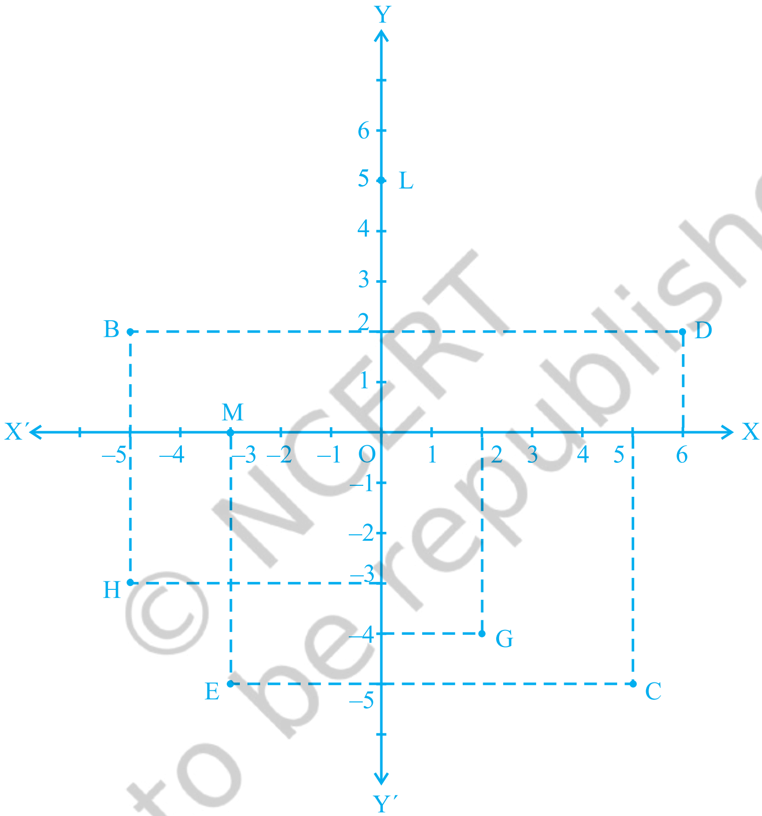


Fig. 3.14

3.3 Summary

In this chapter, you have studied the following points :

1. To locate the position of an object or a point in a plane, we require two perpendicular lines. One of them is horizontal, and the other is vertical.
2. The plane is called the Cartesian, or coordinate plane and the lines are called the coordinate axes.
3. The horizontal line is called the x -axis, and the vertical line is called the y - axis.

4. The coordinate axes divide the plane into four parts called quadrants.

5. The point of intersection of the axes is called the origin.

6. The distance of a point from the y - axis is called its x-coordinate, or abscissa, and the distance of the point from the x-axis is called its y-coordinate, or ordinate.

7. If the abscissa of a point is x and the ordinate is y, then (x, y) are called the coordinates of the point.

8. The coordinates of a point on the x-axis are of the form $(x, 0)$ and that of the point on the y-axis are $(0, y)$.

9. The coordinates of the origin are (0, 0).

10. The coordinates of a point are of the form (+ , +) in the first quadrant, (–, +) in the second quadrant, (–, –) in the third quadrant and (+, –) in the fourth quadrant, where + denotes a positive real number and – denotes a negative real number.

11. If $x \neq y$, then $(x, y) \neq (y, x)$, and $(x, y) = (y, x)$, if $x = y$.

CHAPTER 4

LINEAR EQUATIONS IN TWO VARIABLES

The principal use of the Analytic Art is to bring Mathematical Problems to Equations and to exhibit those Equations in the most simple terms that can be.

—***Edmund Halley***

4.1 Introduction

In earlier classes, you have studied linear equations in one variable. Can you write down a linear equation in one variable? You may say that $x + 1 = 0$, $x + \sqrt{2} = 0$ and $\sqrt{2}\,y + \sqrt{3} = 0$ are examples of linear equations in one variable. You also know that such equations have a unique (i.e., one and only one) solution. You may also remember how to represent the solution on a number line. In this chapter, the knowledge of linear equations in one variable shall be recalled and extended to that of two variables. You will be considering questions like: Does a linear equation in two variables have a solution? If yes, is it unique? What does the solution look like on the Cartesian plane? You shall also use the concepts you studied in Chapter 3 to answer these questions.

4.2 Linear Equations

Let us first recall what you have studied so far. Consider the following equation:

$$2x + 5 = 0$$

Its solution, i.e., the root of the equation, is $-\frac{5}{2}$. This can be represented on the number line as shown below:

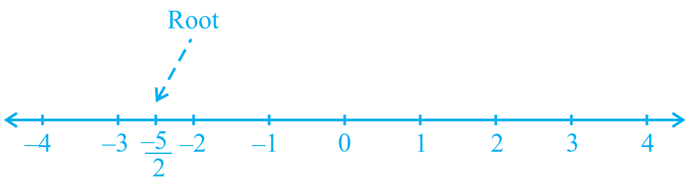


Fig. 4.1

While solving an equation, you must always keep the following points in mind:

The solution of a linear equation is not affected when:

(i) the same number is added to (or subtracted from) both the sides of the equation.

(ii) you multiply or divide both the sides of the equation by the same non-zero number.

Let us now consider the following situation:

In a One-day International Cricket match between India and Sri Lanka played in Nagpur, two Indian batsmen together scored 176 runs. Express this information in the form of an equation.

Here, you can see that the score of neither of them is known, i.e., there are two unknown quantities. Let us use x and y to denote them. So, the number of runs scored by one of the batsmen is x, and the number of runs scored by the other is y. We know that

$$x + y = 176,$$

which is the required equation.

This is an example of a linear equation in two variables. It is customary to denote the variables in such equations by x and y, but other letters may also be used. Some examples of linear equations in two variables are:

$$1.2s + 3t = 5,\ p + 4q = 7,\ \pi u + 5v = 9 \text{ and } 3 = \sqrt{2}\,x - 7y.$$

Note that you can put these equations in the form $1.2s + 3t - 5 = 0$, $p + 4q - 7 = 0$, $\pi u + 5v - 9 = 0$ and $\sqrt{2}\,x - 7y - 3 = 0$, respectively.

So, any equation which can be put in the form $ax + by + c = 0$, where a, b and c are real numbers, and a and b are not both zero, is called a *linear equation in two variables*. This means that you can think of many many such equations.

Example 1 : Write each of the following equations in the form $ax + by + c = 0$ and indicate the values of a, b and c in each case:

(i) $2x + 3y = 4.37$ (ii) $x - 4 = \sqrt{3}\,y$ (iii) $4 = 5x - 3y$ (iv) $2x = y$

Solution : (i) $2x + 3y = 4.37$ can be written as $2x + 3y - 4.37 = 0$. Here $a = 2$, $b = 3$ and $c = -4.37$.

(ii) The equation $x - 4 = \sqrt{3}\,y$ can be written as $x - \sqrt{3}\,y - 4 = 0$. Here $a = 1$, $b = -\sqrt{3}$ and $c = -4$.

(iii) The equation $4 = 5x - 3y$ can be written as $5x - 3y - 4 = 0$. Here $a = 5$, $b = -3$ and $c = -4$. Do you agree that it can also be written as $-5x + 3y + 4 = 0$? In this case $a = -5$, $b = 3$ and $c = 4$.

(iv) The equation $2x = y$ can be written as $2x - y + 0 = 0$. Here $a = 2$, $b = -1$ and $c = 0$.

Equations of the type $ax + b = 0$ are also examples of linear equations in two variables because they can be expressed as

$$ax + 0.y + b = 0$$

For example, $4 - 3x = 0$ can be written as $-3x + 0.y + 4 = 0$.

Example 2 : Write each of the following as an equation in two variables:

(i) $x = -5$ (ii) $y = 2$ (iii) $2x = 3$ (iv) $5y = 2$

Solution : (i) $x = -5$ can be written as $1.x + 0.y = -5$, or $1.x + 0.y + 5 = 0$.

(ii) $y = 2$ can be written as $0.x + 1.y = 2$, or $0.x + 1.y - 2 = 0$.

(iii) $2x = 3$ can be written as $2x + 0.y - 3 = 0$.

(iv) $5y = 2$ can be written as $0.x + 5y - 2 = 0$.

EXERCISE 4.1

1. The cost of a notebook is twice the cost of a pen. Write a linear equation in two variables to represent this statement.

   (Take the cost of a notebook to be ₹ x and that of a pen to be ₹ y).

2. Express the following linear equations in the form $ax + by + c = 0$ and indicate the values of a, b and c in each case:

   (i) $2x + 3y = 9.3\overline{5}$ (ii) $x - \frac{y}{5} - 10 = 0$ (iii) $-2x + 3y = 6$ (iv) $x = 3y$

   (v) $2x = -5y$ (vi) $3x + 2 = 0$ (vii) $y - 2 = 0$ (viii) $5 = 2x$

4.3 Solution of a Linear Equation

You have seen that every linear equation in one variable has a unique solution. What can you say about the solution of a linear equation involving two variables? As there are two variables in the equation, a solution means a pair of values, one for x and one for y which satisfy the given equation. Let us consider the equation $2x + 3y = 12$. Here, $x = 3$ and $y = 2$ is a solution because when you substitute $x = 3$ and $y = 2$ in the equation above, you find that

$$2x + 3y = (2 \times 3) + (3 \times 2) = 12$$

This solution is written as an ordered pair (3, 2), first writing the value for x and then the value for y. Similarly, (0, 4) is also a solution for the equation above.

On the other hand, (1, 4) is not a solution of $2x + 3y = 12$, because on putting $x = 1$ and $y = 4$ we get $2x + 3y = 14$, which is not 12. Note that (0, 4) is a solution but not (4, 0).

You have seen at least two solutions for $2x + 3y = 12$, i.e., (3, 2) and (0, 4). Can you find any other solution? Do you agree that (6, 0) is another solution? Verify the same. In fact, we can get many many solutions in the following way. Pick a value of your choice for x (say $x = 2$) in $2x + 3y = 12$. Then the equation reduces to $4 + 3y = 12$, which is a linear equation in one variable. On solving this, you get $y = \frac{8}{3}$. So $\left(2, \frac{8}{3}\right)$ is another solution of $2x + 3y = 12$. Similarly, choosing $x = -5$, you find that the equation becomes $-10 + 3y = 12$. This gives $y = \frac{22}{3}$. So, $\left(-5, \frac{22}{3}\right)$ is another solution of $2x + 3y = 12$. So there is no end to different solutions of a linear equation in two variables. That is, *a linear equation in two variables has infinitely many solutions.*

Example 3 : Find four different solutions of the equation $x + 2y = 6$.

Solution : By inspection, $x = 2, y = 2$ is a solution because for $x = 2, y = 2$

$$x + 2y = 2 + 4 = 6$$

Now, let us choose $x = 0$. With this value of x, the given equation reduces to $2y = 6$ which has the unique solution $y = 3$. So $x = 0, y = 3$ is also a solution of $x + 2y = 6$. Similarly, taking $y = 0$, the given equation reduces to $x = 6$. So, $x = 6, y = 0$ is a solution of $x + 2y = 6$ as well. Finally, let us take $y = 1$. The given equation now reduces to $x + 2 = 6$, whose solution is given by $x = 4$. Therefore, (4, 1) is also a solution of the given equation. So four of the infinitely many solutions of the given equation are:

$$(2, 2), (0, 3), (6, 0) \text{ and } (4, 1).$$

Remark : Note that an easy way of getting a solution is to take $x = 0$ and get the corresponding value of y. Similarly, we can put $y = 0$ and obtain the corresponding value of x.

Example 4 : Find two solutions for each of the following equations:

(i) $4x + 3y = 12$

(ii) $2x + 5y = 0$

(iii) $3y + 4 = 0$

Solution : (i) Taking $x = 0$, we get $3y = 12$, i.e., $y = 4$. So, (0, 4) is a solution of the given equation. Similarly, by taking $y = 0$, we get $x = 3$. Thus, (3, 0) is also a solution.

(ii) Taking $x = 0$, we get $5y = 0$, i.e., $y = 0$. So (0, 0) is a solution of the given equation. Now, if you take $y = 0$, you again get (0, 0) as a solution, which is the same as the earlier one. To get another solution, take $x = 1$, say. Then you can check that the corresponding value of y is $-\frac{2}{5}\cdot$ So $\left(1, -\frac{2}{5}\right)$ is another solution of $2x + 5y = 0$.

(iii) Writing the equation $3y + 4 = 0$ as $0.x + 3y + 4 = 0$, you will find that $y = -\frac{4}{3}$ for any value of x. Thus, two solutions can be given as $\left(0, -\frac{4}{3}\right)$ and $\left(1, -\frac{4}{3}\right)$.

EXERCISE 4.2

1. Which one of the following options is true, and why?

$y = 3x + 5$ has

(i) a unique solution, (ii) only two solutions, (iii) infinitely many solutions

2. Write four solutions for each of the following equations:

(i) $2x + y = 7$ (ii) $\pi x + y = 9$ (iii) $x = 4y$

3. Check which of the following are solutions of the equation $x - 2y = 4$ and which are not:

(i) $(0, 2)$ (ii) $(2, 0)$ (iii) $(4, 0)$

(iv) $\left(\sqrt{2}, 4\sqrt{2}\right)$ (v) $(1, 1)$

4. Find the value of k, if $x = 2$, $y = 1$ is a solution of the equation $2x + 3y = k$.

4.4 Summary

In this chapter, you have studied the following points:

1. An equation of the form $ax + by + c = 0$, where a, b and c are real numbers, such that a and b are not both zero, is called a linear equation in two variables.

2. A linear equation in two variables has infinitely many solutions.

3. Every point on the graph of a linear equation in two variables is a solution of the linear equation. Moreover, every solution of the linear equation is a point on the graph of the linear equation.

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CHAPTER 5

INTRODUCTION TO EUCLID'S GEOMETRY

5.1 Introduction

The word 'geometry' comes form the Greek words 'geo', meaning the 'earth', and 'metrein', meaning 'to measure'. Geometry appears to have originated from the need for measuring land. This branch of mathematics was studied in various forms in every ancient civilisation, be it in Egypt, Babylonia, China, India, Greece, the Incas, etc. The people of these civilisations faced several practical problems which required the development of geometry in various ways.

For example, whenever the river Nile overflowed, it wiped out the boundaries between the adjoining fields of different land owners. After such flooding, these boundaries had to be redrawn. For this purpose, the Egyptians developed a number of geometric techniques and rules for calculating simple areas and also for doing simple constructions. The knowledge of geometry was also used by them for computing volumes of granaries, and for constructing canals and pyramids. They also knew the correct formula to find the volume of a truncated pyramid (see Fig. 5.1). You know that a pyramid is a solid figure, the base of which is a triangle, or square, or some other polygon, and its side faces are triangles converging to a point at the top.

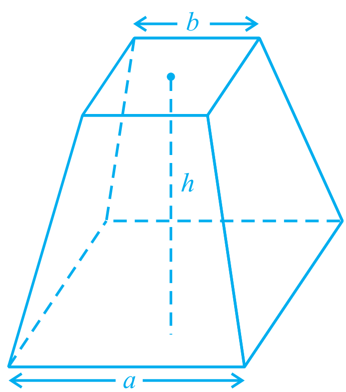


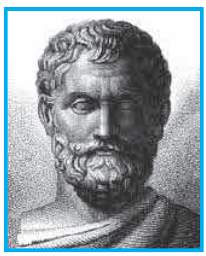
Fig. 5.1 : A Truncated Pyramid

In the Indian subcontinent, the excavations at Harappa and Mohenjo-Daro, etc. show that the Indus Valley Civilisation (about 3000 BCE) made extensive use of geometry. It was a highly organised society. The cities were highly developed and very well planned. For example, the roads were parallel to each other and there was an underground drainage system. The houses had many rooms of different types. This shows that the town dwellers were skilled in mensuration and practical arithmetic. The bricks used for constructions were kiln fired and the ratio length : breadth : thickness, of the bricks was found to be 4 : 2 : 1.

In ancient India, the *Sulbasutras* (800 BCE to 500 BCE) were the manuals of geometrical constructions. The geometry of the Vedic period originated with the construction of altars (or *vedis*) and fireplaces for performing Vedic rites. The location of the sacred fires had to be in accordance to the clearly laid down instructions about their shapes and areas, if they were to be effective instruments. Square and circular altars were used for household rituals, while altars whose shapes were combinations of rectangles, triangles and trapeziums were required for public worship. The *sriyantra* (given in the *Atharvaveda*) consists of nine interwoven isosceles triangles. These triangles are arranged in such a way that they produce 43 subsidiary triangles. Though accurate geometric methods were used for the constructions of altars, the principles behind them were not discussed.

These examples show that geometry was being developed and applied everywhere in the world. But this was happening in an unsystematic manner. What is interesting about these developments of geometry in the ancient world is that they were passed on from one generation to the next, either orally or through palm leaf messages, or by other ways. Also, we find that in some civilisations like Babylonia, geometry remained a very practical oriented discipline, as was the case in India and Rome. The geometry developed by Egyptians mainly consisted of the statements of results. There were no general rules of the procedure. In fact, Babylonians and Egyptians used geometry mostly for practical purposes and did very little to develop it as a systematic science. But in civilisations like Greece, the emphasis was on the *reasoning* behind why certain constructions work. The Greeks were interested in establishing the truth of the statements they discovered using deductive reasoning (see Appendix 1).

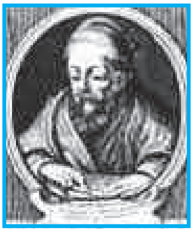
A Greek mathematician, Thales is credited with giving the first known proof. This proof was of the statement that a circle is bisected (i.e., cut into two equal parts) by its diameter. One of Thales' most famous pupils was Pythagoras (572 BCE), whom you have heard about. Pythagoras and his group discovered many geometric properties and developed the theory of geometry to a great extent. This process continued till 300 BCE. At that time Euclid, a teacher of mathematics at Alexandria in Egypt, collected all the known work and arranged it in his famous treatise,



Thales
(640 BCE – 546 BCE)
Fig. 5.2

called 'Elements'. He divided the 'Elements' into thirteen chapters, each called a book. These books influenced the whole world's understanding of geometry for generations to come.

In this chapter, we shall discuss Euclid's approach to geometry and shall try to link it with the present day geometry.



Euclid (325 BCE – 265 BCE)
Fig. 5.3

5.2 Euclid's Definitions, Axioms and Postulates

The Greek mathematicians of Euclid's time thought of geometry as an abstract model of the world in which they lived. The notions of point, line, plane (or surface) and so on were derived from what was seen around them. From studies of the space and solids in the space around them, an abstract geometrical notion of a solid object was developed. A solid has shape, size, position, and can be moved from one place to another. Its boundaries are called **surfaces.** They separate one part of the space from another, and are said to have no thickness. The boundaries of the surfaces are **curves** or straight **lines.** These lines end in **points**.

Consider the three steps from solids to points (solids-surfaces-lines-points). In each step we lose one extension, also called a **dimension**. So, a solid has three dimensions, a surface has two, a line has one and a point has none. Euclid summarised these statements as definitions. He began his exposition by listing 23 definitions in Book 1 of the 'Elements'. A few of them are given below :

1. A **point** is that which has no part.
2. A **line** is breadthless length.
3. The ends of a line are points.
4. A **straight line** is a line which lies evenly with the points on itself.
5. A **surface** is that which has length and breadth only.
6. The edges of a surface are lines.
7. A **plane surface** is a surface which lies evenly with the straight lines on itself.

If you carefully study these definitions, you find that some of the terms like part, breadth, length, evenly, etc. need to be further explained clearly. For example, consider his definition of a point. In this definition, 'a part' needs to be defined. Suppose if you define 'a part' to be that which occupies 'area', again 'an area' needs to be defined. So, to define one thing, you need to define many other things, and you may get a long chain of definitions without an end. For such reasons, mathematicians agree to leave some geometric terms *undefined*. However, we do have a intuitive feeling for the geometric concept of a point than what the 'definition' above gives us. So, we represent a point as a dot, even though a dot has some dimension.

A similar problem arises in Definition 2 above, since it refers to breadth and length, neither of which has been defined. Because of this, a few terms are kept undefined while developing any course of study. So, in geometry, we *take a point, a line and a plane* (*in Euclid's words a plane surface*) *as undefined terms*. The only thing is that we can represent them intuitively, or explain them with the help of 'physical models'.

Starting with his definitions, Euclid assumed certain properties, which were not to be proved. These assumptions are actually 'obvious universal truths'. He divided them into two types: axioms and postulates. He used the term '**postulate**' for the assumptions that were specific to geometry. Common notions (often called **axioms**), on the other hand, were assumptions used throughout mathematics and not specifically linked to geometry. For details about axioms and postulates, refer to Appendix 1. Some of **Euclid's axioms**, not in his order, are given below :

(1) Things which are equal to the same thing are equal to one another.

(2) If equals are added to equals, the wholes are equal.

(3) If equals are subtracted from equals, the remainders are equal.

(4) Things which coincide with one another are equal to one another.

(5) The whole is greater than the part.

(6) Things which are double of the same things are equal to one another.

(7) Things which are halves of the same things are equal to one another.

These 'common notions' refer to magnitudes of some kind. The first common notion could be applied to plane figures. For example, if an area of a triangle equals the area of a rectangle and the area of the rectangle equals that of a square, then the area of the triangle also equals the area of the square.

Magnitudes of the same kind can be compared and added, but magnitudes of different kinds cannot be compared. For example, a line cannot be compared to a rectangle, nor can an angle be compared to a pentagon.

The 4th axiom given above seems to say that if two things are identical (that is, they are the same), then they are equal. In other words, everything equals itself. It is the justification of the principle of superposition. Axiom (5) gives us the definition of 'greater than'. For example, if a quantity B is a part of another quantity A, then A can be written as the sum of B and some third quantity C. Symbolically, A > B means that there is some C such that A = B + C.

Now let us discuss **Euclid's five postulates.** They are :

Postulate 1 : *A straight line may be drawn from any one point to any other point.*

Note that this postulate tells us that at least one straight line passes through two distinct points, but it does not say that there cannot be more than one such line. However, in his work, Euclid has frequently assumed, without mentioning, that there is a *unique* line joining two distinct points. We state this result in the form of an axiom as follows:

Axiom 5.1 : *Given two distinct points, there is a unique line that passes through them.*

How many lines passing through P also pass through Q (see Fig. 5.4)? Only one, that is, the line PQ. How many lines passing through Q also pass through P? Only one, that is, the line PQ. Thus, the statement above is self-evident, and so is taken as an axiom.

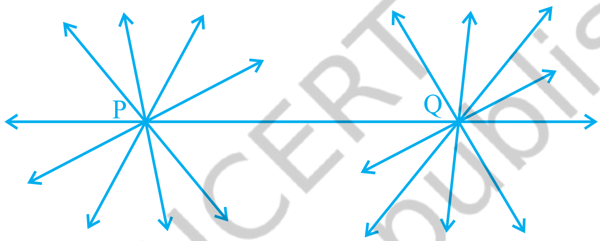


Fig. 5.4

Postulate 2 : *A terminated line can be produced indefinitely.*

Note that what we call a line segment now-a-days is what Euclid called a terminated line. So, according to the present day terms, the second postulate says that a line segment can be extended on either side to form a line (see Fig. 5.5).

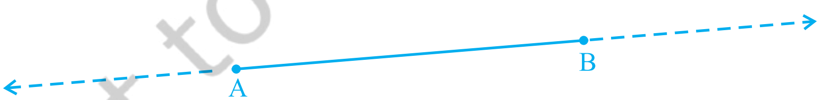


Fig. 5.5

Postulate 3 : *A circle can be drawn with any centre and any radius.*

Postulate 4 : *All right angles are equal to one another.*

Postulate 5 : *If a straight line falling on two straight lines makes the interior angles on the same side of it taken together less than two right angles, then the two straight lines, if produced indefinitely, meet on that side on which the sum of angles is less than two right angles.*

For example, the line PQ in Fig. 5.6 falls on lines AB and CD such that the sum of the interior angles 1 and 2 is less than 180° on the left side of PQ. Therefore, the lines AB and CD will eventually intersect on the left side of PQ.

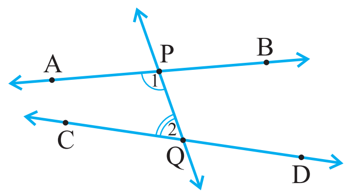


Fig. 5.6

A brief look at the five postulates brings to your notice that Postulate 5 is far more complex than any other postulate. On the other hand, Postulates 1 through 4 are so simple and obvious that these are taken as 'self-evident truths'. However, it is not possible to prove them. So, these statements are accepted without any proof (see Appendix 1). Because of its complexity, the fifth postulate will be given more attention in the next section.

Now-a-days, 'postulates' and 'axioms' are terms that are used interchangeably and in the same sense. 'Postulate' is actually a verb. When we say "let us postulate", we mean, "let us make some statement based on the observed phenomenon in the Universe". Its truth/validity is checked afterwards. If it is true, then it is accepted as a 'Postulate'.

A system of axioms is called **consistent** (see Appendix 1), if it is impossible to deduce from these axioms a statement that contradicts any axiom or previously proved statement. So, when any system of axioms is given, it needs to be ensured that the system is consistent.

After Euclid stated his postulates and axioms, he used them to prove other results. Then using these results, he proved some more results by applying deductive reasoning. The statements that were proved are called **propositions or theorems**. Euclid deduced 465 propositions in a logical chain using his axioms, postulates, definitions and theorems proved earlier in the chain. In the next few chapters on geometry, you will be using these axioms to prove some theorems.

Now, let us see in the following examples how Euclid used his axioms and postulates for proving some of the results:

Example 1 : If A, B and C are three points on a line, and B lies between A and C (see Fig. 5.7), then prove that AB + BC = AC.

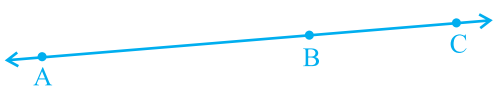


Fig. 5.7

Solution : In the figure given above, AC coincides with AB + BC.

Also, Euclid's Axiom (4) says that things which coincide with one another are equal to one another. So, it can be deduced that

$$AB + BC = AC$$

Note that in this solution, it has been assumed that there is a unique line passing through two points.

Example 2 : Prove that an equilateral triangle can be constructed on any given line segment.

Solution : In the statement above, a line segment of any length is given, say AB [see Fig. 5.8(i)].

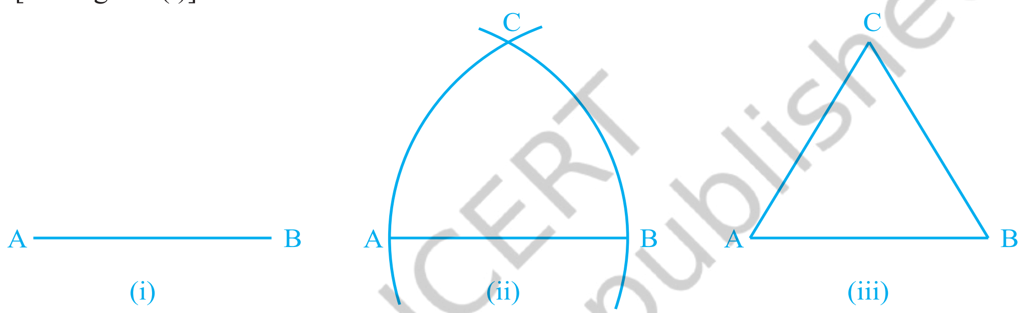


Fig. 5.8

Here, you need to do some construction. Using Euclid's Postulate 3, you can draw a circle with point A as the centre and AB as the radius [see Fig. 5.8(ii)]. Similarly, draw another circle with point B as the centre and BA as the radius. The two circles meet at a point, say C. Now, draw the line segments AC and BC to form Δ ABC [see Fig. 5.8 (iii)].

So, you have to prove that this triangle is equilateral, i.e., AB = AC = BC.

Now, AB = AC, since they are the radii of the same circle (1)

Similarly, AB = BC (Radii of the same circle) (2)

From these two facts, and Euclid's axiom that things which are equal to the same thing are equal to one another, you can conclude that AB = BC = AC.

So, Δ ABC is an equilateral triangle.

Note that here Euclid has assumed, without mentioning anywhere, that the two circles drawn with centres A and B will meet each other at a point.

Now we prove a theorem, which is frequently used in different results:

Theorem 5.1 : *Two distinct lines cannot have more than one point in common.*

Proof : Here we are given two lines l and m. We need to prove that they have only one point in common.

For the time being, let us suppose that the two lines intersect in two distinct points, say P and Q. So, you have two lines passing through two distinct points P and Q. But this assumption clashes with the axiom that only one line can pass through two distinct points. So, the assumption that we started with, that two lines can pass through two distinct points is wrong.

From this, what can we conclude? We are forced to conclude that two distinct lines cannot have more than one point in common.

EXERCISE 5.1

1. Which of the following statements are true and which are false? Give reasons for your answers.
   (i) Only one line can pass through a single point.
   (ii) There are an infinite number of lines which pass through two distinct points.
   (iii) A terminated line can be produced indefinitely on both the sides.
   (iv) If two circles are equal, then their radii are equal.
   (v) In Fig. 5.9, if AB = PQ and PQ = XY, then AB = XY.

Fig. 5.9

2. Give a definition for each of the following terms. Are there other terms that need to be defined first? What are they, and how might you define them?
   (i) parallel lines (ii) perpendicular lines (iii) line segment
   (iv) radius of a circle (v) square
3. Consider two 'postulates' given below:
   (i) Given any two distinct points A and B, there exists a third point C which is in between A and B.
   (ii) There exist at least three points that are not on the same line.

   Do these postulates contain any undefined terms? Are these postulates consistent? Do they follow from Euclid's postulates? Explain.

4. If a point C lies between two points A and B such that AC = BC, then prove that $AC = \frac{1}{2}AB$. Explain by drawing the figure.

5. In Question 4, point C is called a mid-point of line segment AB. Prove that every line segment has one and only one mid-point.

6. In Fig. 5.10, if AC = BD, then prove that AB = CD.

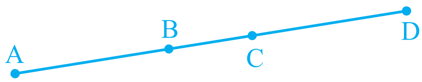


Fig. 5.10

7. Why is Axiom 5, in the list of Euclid's axioms, considered a 'universal truth'? (Note that the question is not about the fifth postulate.)

5.3 Summary

In this chapter, you have studied the following points:

1. Though Euclid defined a point, a line, and a plane, the definitions are not accepted by mathematicians. Therefore, these terms are now taken as undefined.
2. Axioms or postulates are the assumptions which are obvious universal truths. They are not proved.
3. Theorems are statements which are proved, using definitions, axioms, previously proved statements and deductive reasoning.
4. Some of Euclid's axioms were :
   (1) Things which are equal to the same thing are equal to one another.
   (2) If equals are added to equals, the wholes are equal.
   (3) If equals are subtracted from equals, the remainders are equal.
   (4) Things which coincide with one another are equal to one another.
   (5) The whole is greater than the part.
   (6) Things which are double of the same things are equal to one another.
   (7) Things which are halves of the same things are equal to one another.
5. Euclid's postulates were :

   Postulate 1 : A straight line may be drawn from any one point to any other point.

   Postulate 2 : A terminated line can be produced indefinitely.

   Postulate 3 : A circle can be drawn with any centre and any radius.

   Postulate 4 : All right angles are equal to one another.

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CHAPTER 6

LINES AND ANGLES

6.1 Introduction

In Chapter 5, you have studied that a minimum of two points are required to draw a line. You have also studied some axioms and, with the help of these axioms, you proved some other statements. In this chapter, you will study the properties of the angles formed when two lines intersect each other, and also the properties of the angles formed when a line intersects two or more parallel lines at distinct points. Further you will use these properties to prove some statements using deductive reasoning (see Appendix 1). You have already verified these statements through some activities in the earlier classes.

In your daily life, you see different types of angles formed between the edges of plane surfaces. For making a similar kind of model using the plane surfaces, you need to have a thorough knowledge of angles. For instance, suppose you want to make a model of a hut to keep in the school exhibition using bamboo sticks. Imagine how you would make it? You would keep some of the sticks parallel to each other, and some sticks would be kept slanted. Whenever an architect has to draw a plan for a multistoried building, she has to draw intersecting lines and parallel lines at different angles. Without the knowledge of the properties of these lines and angles, do you think she can draw the layout of the building?

In science, you study the properties of light by drawing the ray diagrams. For example, to study the refraction property of light when it enters from one medium to the other medium, you use the properties of intersecting lines and parallel lines. When two or more forces act on a body, you draw the diagram in which forces are represented by directed line segments to study the net effect of the forces on the body. At that time, you need to know the relation between the angles when the rays (or line segments) are parallel to or intersect each other. To find the height of a tower or to find the distance of a ship from the light house, one needs to know the angle formed between the horizontal and the line of sight. Plenty of other examples can be given where lines and angles are used. In the subsequent chapters of geometry, you will be using these properties of lines and angles to deduce more and more useful properties.

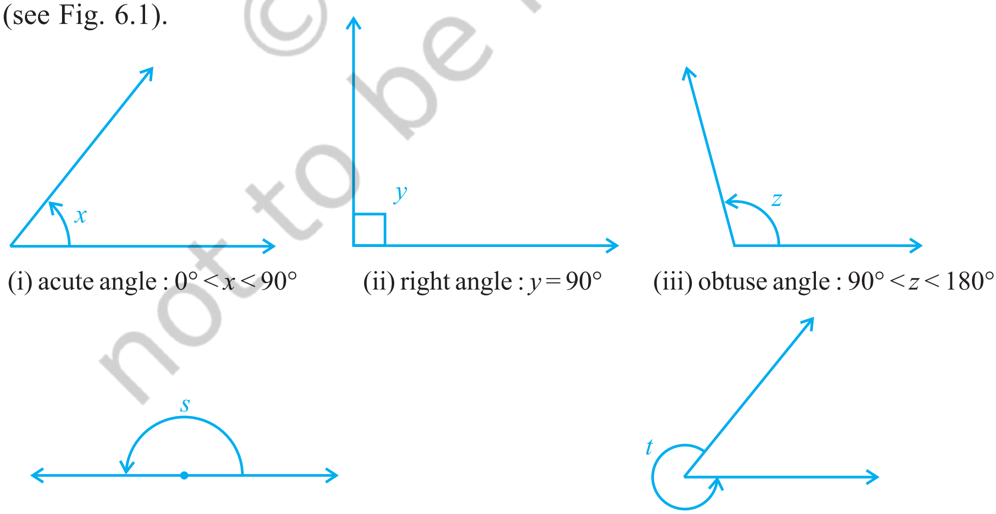
Let us first revise the terms and definitions related to lines and angles learnt in earlier classes.

6.2 Basic Terms and Definitions

Recall that a part (or portion) of a line with two end points is called a **line-segment** and a part of a line with one end point is called a **ray**. Note that the line segment AB is denoted by $\overline{\text{AB}}$, and its length is denoted by AB. The ray AB is denoted by $\overrightarrow{\text{AB}}$, and a line is denoted by $\overleftrightarrow{\text{AB}}$. However, **we will not use these symbols**, and will denote the line segment AB, ray AB, length AB and line AB by the same symbol, AB. The meaning will be clear from the context. Sometimes small letters *l*, *m*, *n*, etc. will be used to denote lines.

If three or more points lie on the same line, they are called **collinear points**; otherwise they are called **non-collinear points**.

Recall that an **angle** is formed when two rays originate from the same end point. The rays making an angle are called the **arms** of the angle and the end point is called the **vertex** of the angle. You have studied different types of angles, such as acute angle, right angle, obtuse angle, straight angle and reflex angle in earlier classes (see Fig. 6.1).



(i) acute angle : $0° < x < 90°$ (ii) right angle : $y = 90°$ (iii) obtuse angle : $90° < z < 180°$

(iv) straight angle : $s = 180°$ (v) reflex angle : $180° < t < 360°$

Fig. 6.1 : Types of Angles

An **acute** angle measures between 0° and 90°, whereas a **right angle** is exactly equal to 90°. An angle greater than 90° but less than 180° is called an **obtuse angle**. Also, recall that a **straight angle** is equal to 180°. An angle which is greater than 180° but less than 360° is called a **reflex angle**. Further, two angles whose sum is 90° are called **complementary angles**, and two angles whose sum is 180° are called **supplementary angles**.

You have also studied about adjacent angles in the earlier classes (see Fig. 6.2). Two angles are **adjacent**, if they have a common vertex, a common arm and their non-common arms are on different sides of the common arm. In Fig. 6.2, ∠ ABD and ∠ DBC are adjacent angles. Ray BD is their common arm and point B is their common vertex. Ray BA and ray BC are non common arms. Moreover, when two angles are adjacent, then their sum is always equal to the angle formed by the two non-common arms. So, we can write

$$\angle ABC = \angle ABD + \angle DBC.$$

Note that ∠ ABC and ∠ ABD are not adjacent angles. Why? Because their non-common arms BD and BC lie on the same side of the common arm BA.

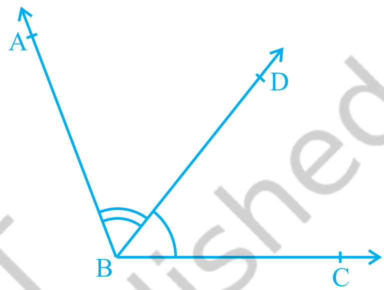


Fig. 6.2 : Adjacent angles

If the non-common arms BA and BC in Fig. 6.2, form a line then it will look like Fig. 6.3. In this case, ∠ ABD and ∠ DBC are called **linear pair of angles**.

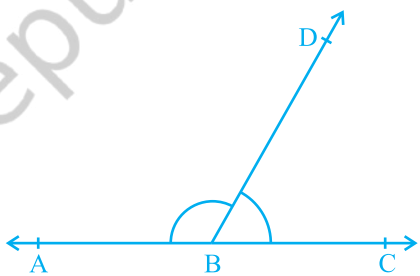


Fig. 6.3 : Linear pair of angles

You may also recall the **vertically opposite angles** formed when two lines, say AB and CD, intersect each other, say at the point O (see Fig. 6.4). There are two pairs of vertically opposite angles.

One pair is ∠AOD and ∠BOC. Can you find the other pair?

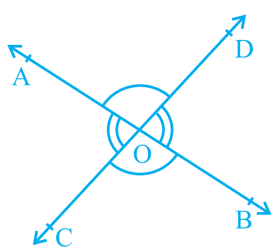
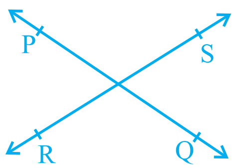


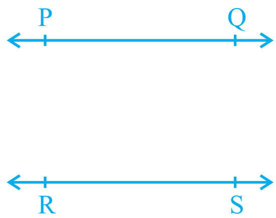
Fig. 6.4 : Vertically opposite angles

6.3 Intersecting Lines and Non-intersecting Lines

Draw two different lines PQ and RS on a paper. You will see that you can draw them in two different ways as shown in Fig. 6.5 (i) and Fig. 6.5 (ii).



(i) Intersecting lines



(ii) Non-intersecting (parallel) lines

Fig. 6.5 : Different ways of drawing two lines

Recall the notion of a line, that it extends indefinitely in both directions. Lines PQ and RS in Fig. 6.5 (i) are intersecting lines and in Fig. 6.5 (ii) are parallel lines. Note that the lengths of the common perpendiculars at different points on these parallel lines is the same. This equal length is called the *distance between two parallel lines*.

6.4 Pairs of Angles

In Section 6.2, you have learnt the definitions of some of the pairs of angles such as complementary angles, supplementary angles, adjacent angles, linear pair of angles, etc. Can you think of some relations between these angles? Now, let us find out the relation between the angles formed when a ray stands on a line. Draw a figure in which a ray stands on a line as shown in Fig. 6.6. Name the line as AB and the ray as OC. What are the angles formed at the point O? They are $\angle$ AOC, $\angle$ BOC and $\angle$ AOB.

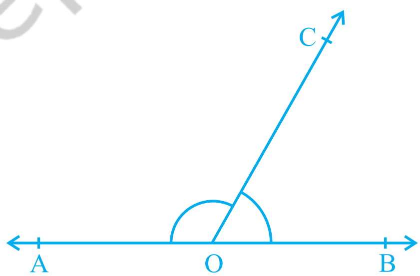


Fig. 6.6 : Linear pair of angles

Can we write $\angle$ AOC + $\angle$ BOC = $\angle$ AOB? (1)

Yes! (Why? Refer to adjacent angles in Section 6.2)

What is the measure of $\angle$ AOB? It is 180°. (Why?) (2)

From (1) and (2), can you say that $\angle$ AOC + $\angle$ BOC = 180°? Yes! (Why?)

From the above discussion, we can state the following Axiom:

Axiom 6.1 : *If a ray stands on a line, then the sum of two adjacent angles so formed is* 180°.

Recall that when the sum of two adjacent angles is 180°, then they are called a **linear pair of angles**.

In Axiom 6.1, it is given that 'a ray stands on a line'. From this 'given', we have concluded that 'the sum of two adjacent angles so formed is 180°'. Can we write Axiom 6.1 the other way? That is, take the 'conclusion' of Axiom 6.1 as 'given' and the 'given' as the 'conclusion'. So it becomes:

(A) If the sum of two adjacent angles is 180°, then a ray stands on a line (that is, the non-common arms form a line).

Now you see that the Axiom 6.1 and statement (A) are in a sense the reverse of each others. We call each as converse of the other. We do not know whether the statement (A) is true or not. Let us check. Draw adjacent angles of different measures as shown in Fig. 6.7. Keep the ruler along one of the non-common arms in each case. Does the other non-common arm also lie along the ruler?

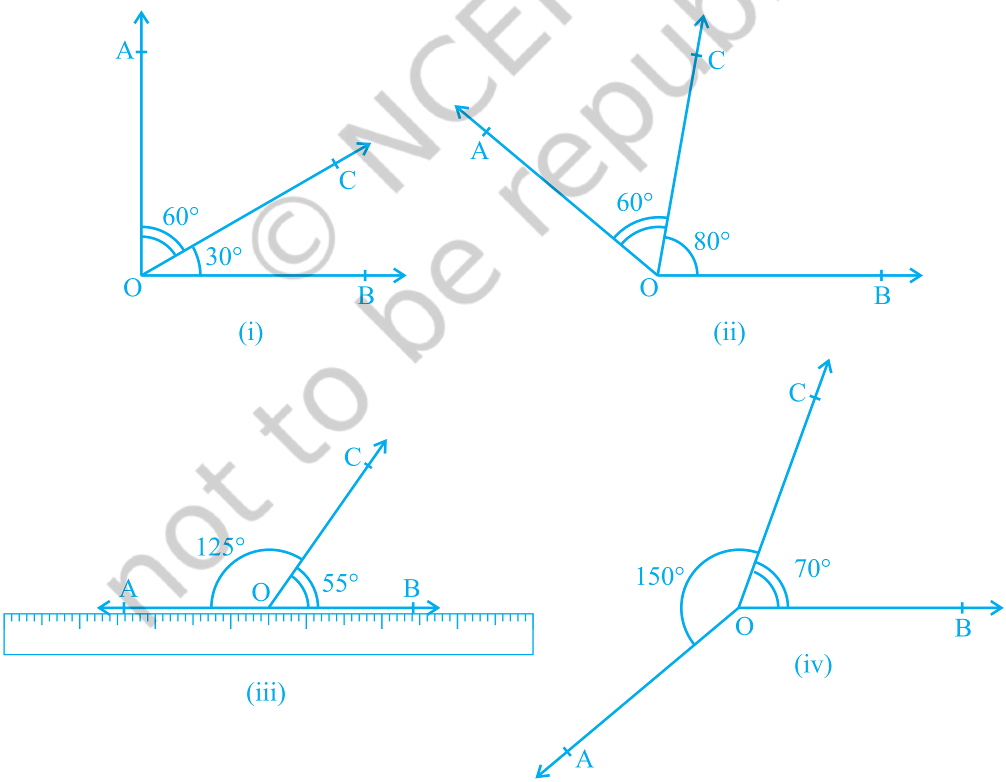


Fig. 6.7 : Adjacent angles with different measures

You will find that only in Fig. 6.7 (iii), both the non-common arms lie along the ruler, that is, points A, O and B lie on the same line and ray OC stands on it. Also see that $\angle$ AOC + $\angle$ COB = 125° + 55° = 180°. From this, you may conclude that statement (A) is true. So, you can state in the form of an axiom as follows:

Axiom 6.2 : *If the sum of two adjacent angles is* 180°, *then the non-common arms of the angles form a line.*

For obvious reasons, the two axioms above together is called the **Linear Pair Axiom**.

Let us now examine the case when two lines intersect each other.

Recall, from earlier classes, that when two lines intersect, the vertically opposite angles are equal. Let us prove this result now. See Appendix 1 for the ingredients of a proof, and keep those in mind while studying the proof given below.

Theorem 6.1 : *If two lines intersect each other, then the vertically opposite angles are equal.*

Proof : In the statement above, it is given that 'two lines intersect each other'. So, let AB and CD be two lines intersecting at O as shown in Fig. 6.8. They lead to two pairs of vertically opposite angles, namely,

(i) $\angle$ AOC and $\angle$ BOD (ii) $\angle$ AOD and $\angle$ BOC.

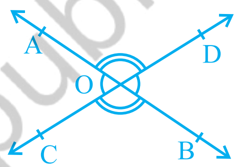


Fig. 6.8 : Vertically opposite angles

We need to prove that $\angle$ AOC = $\angle$ BOD and $\angle$ AOD = $\angle$ BOC.

Now, ray OA stands on line CD.

Therefore, $\angle$ AOC + $\angle$ AOD = 180° (Linear pair axiom) (1)

Can we write $\angle$ AOD + $\angle$ BOD = 180°? Yes! (Why?) (2)

From (1) and (2), we can write

$$\angle AOC + \angle AOD = \angle AOD + \angle BOD$$

This implies that $\angle$ AOC = $\angle$ BOD (Refer Section 5.2, Axiom 3)

Similarly, it can be proved that $\angle$AOD = $\angle$BOC

Now, let us do some examples based on Linear Pair Axiom and Theorem 6.1.

Example 1 : In Fig. 6.9, lines PQ and RS intersect each other at point O. If $\angle$ POR : $\angle$ ROQ = 5 : 7, find all the angles.

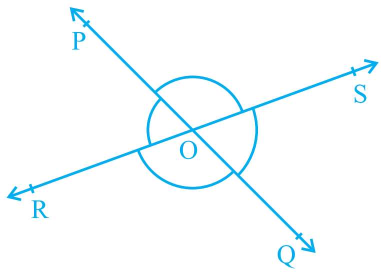


Fig. 6.9

Solution : $\angle$ POR + $\angle$ ROQ = 180° (Linear pair of angles)

But $\angle$ POR : $\angle$ ROQ = 5 : 7 (Given)

Therefore, $\angle \text{POR} = \frac{5}{12} \times 180° = 75°$

Similarly, $\angle \text{ROQ} = \frac{7}{12} \times 180° = 105°$

Now, $\angle$ POS = $\angle$ROQ = 105° (Vertically opposite angles)

and $\angle$ SOQ = $\angle$POR = 75° (Vertically opposite angles)

Example 2 : In Fig. 6.10, ray OS stands on a line POQ. Ray OR and ray OT are angle bisectors of $\angle$ POS and $\angle$ SOQ, respectively. If $\angle$ POS = x, find $\angle$ ROT.

Solution : Ray OS stands on the line POQ.

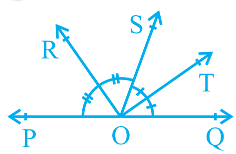


Fig. 6.10

Therefore, $\angle$ POS + $\angle$ SOQ = 180°

But, $\angle$ POS = x

Therefore, x + $\angle$ SOQ = 180°

So, $\angle$ SOQ = 180° – x

Now, ray OR bisects $\angle$ POS, therefore,

$$\angle \text{ROS} = \frac{1}{2} \times \angle \text{POS}$$

$$= \frac{1}{2} \times x = \frac{x}{2}$$

Similarly, $$\angle \text{SOT} = \frac{1}{2} \times \angle \text{SOQ}$$

$$= \frac{1}{2} \times (180° - x)$$

$$= 90° - \frac{x}{2}$$

Now, $\angle \text{ROT} = \angle \text{ROS} + \angle \text{SOT}$

$$= \frac{x}{2} + 90° - \frac{x}{2}$$

$$= 90°$$

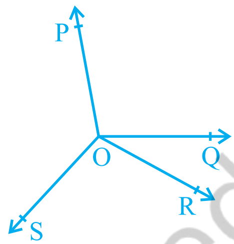


Fig. 6.11

Example 3 : In Fig. 6.11, OP, OQ, OR and OS are four rays. Prove that $\angle$ POQ + $\angle$ QOR + $\angle$ SOR + $\angle$ POS = 360°.

Solution : In Fig. 6.11, you need to produce any of the rays OP, OQ, OR or OS backwards to a point. Let us produce ray OQ backwards to a point T so that TOQ is a line (see Fig. 6.12).

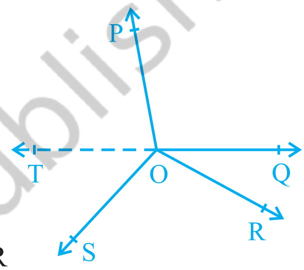


Fig. 6.12

Now, ray OP stands on line TOQ.

Therefore, $\angle \text{TOP} + \angle \text{POQ} = 180°$ (1)

(Linear pair axiom)

Similarly, ray OS stands on line TOQ.

Therefore, $\angle \text{TOS} + \angle \text{SOQ} = 180°$ (2)

But $\angle \text{SOQ} = \angle \text{SOR} + \angle \text{QOR}$

So, (2) becomes

$$\angle \text{TOS} + \angle \text{SOR} + \angle \text{QOR} = 180° \quad (3)$$

Now, adding (1) and (3), you get

$$\angle \text{TOP} + \angle \text{POQ} + \angle \text{TOS} + \angle \text{SOR} + \angle \text{QOR} = 360° \quad (4)$$

But $\angle \text{TOP} + \angle \text{TOS} = \angle \text{POS}$

Therefore, (4) becomes

$$\angle \text{POQ} + \angle \text{QOR} + \angle \text{SOR} + \angle \text{POS} = 360°$$

EXERCISE 6.1

1. In Fig. 6.13, lines AB and CD intersect at O. If $\angle$ AOC + $\angle$ BOE = 70° and $\angle$ BOD = 40°, find $\angle$ BOE and reflex $\angle$ COE.

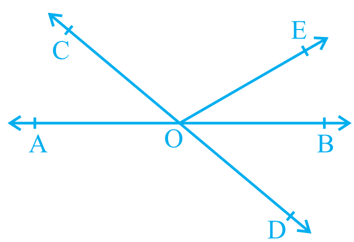


Fig. 6.13

2. In Fig. 6.14, lines XY and MN intersect at O. If $\angle$ POY = 90° and $a : b = 2 : 3$, find c.

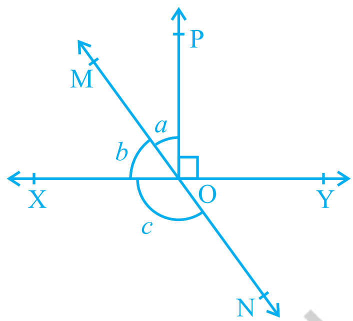


Fig. 6.14

3. In Fig. 6.15, $\angle$ PQR = $\angle$ PRQ, then prove that $\angle$ PQS = $\angle$ PRT.

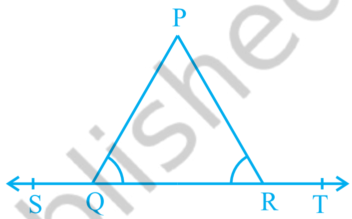


Fig. 6.15

4. In Fig. 6.16, if $x + y = w + z$, then prove that AOB is a line.

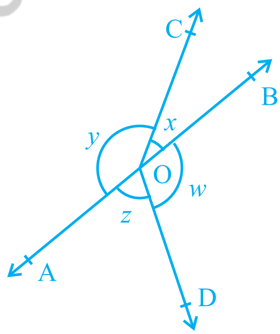


Fig. 6.16

5. In Fig. 6.17, POQ is a line. Ray OR is perpendicular to line PQ. OS is another ray lying between rays OP and OR. Prove that

$$\angle \text{ROS} = \frac{1}{2}(\angle \text{QOS} - \angle \text{POS}).$$

6. It is given that $\angle$ XYZ = 64° and XY is produced to point P. Draw a figure from the given information. If ray YQ bisects $\angle$ ZYP, find $\angle$ XYQ and reflex $\angle$ QYP.

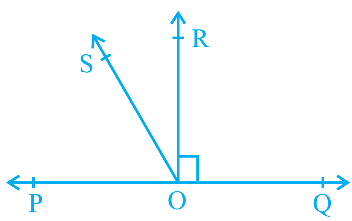


Fig. 6.17

6.5 Lines Parallel to the Same Line

If two lines are parallel to the same line, will they be parallel to each other? Let us check it. See Fig. 6.18 in which line $m \parallel$ line l and line $n \parallel$ line l.

Let us draw a line t transversal for the lines, l, m and n. It is given that line $m \parallel$ line l and line $n \parallel$ line l.

Therefore, $\angle 1 = \angle 2$ and $\angle 1 = \angle 3$

(Corresponding angles axiom)

So, $\angle 2 = \angle 3$ (Why?)

But $\angle 2$ and $\angle 3$ are corresponding angles and they are equal.

Therefore, you can say that

Line $m \parallel$ Line n

(Converse of corresponding angles axiom)

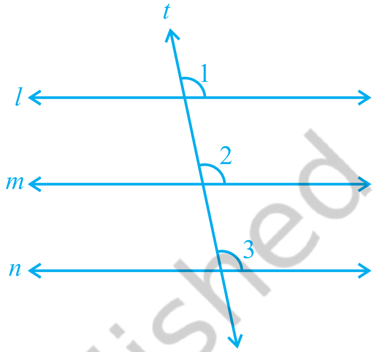


Fig. 6.18

This result can be stated in the form of the following theorem:

Theorem 6.6 : *Lines which are parallel to the same line are parallel to each other.*

Note : The property above can be extended to more than two lines also.

Now, let us solve some examples related to parallel lines.

Example 4 : In Fig. 6.19, if PQ || RS, $\angle$ MXQ = 135° and $\angle$ MYR = 40°, find $\angle$ XMY.

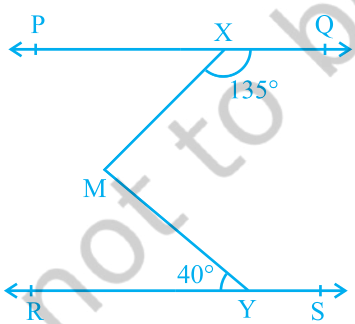


Fig. 6.19

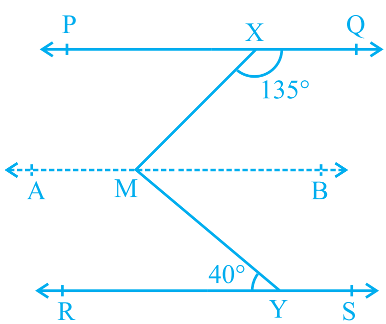


Fig. 6.20

Solution : Here, we need to draw a line AB parallel to line PQ, through point M as shown in Fig. 6.20. Now, AB || PQ and PQ || RS.

Therefore, $AB \parallel RS$ (Why?)

Now, $\angle QXM + \angle XMB = 180°$

(AB || PQ, Interior angles on the same side of the transversal XM)

But $\angle QXM = 135°$

So, $135° + \angle XMB = 180°$

Therefore, $\angle XMB = 45°$ (1)

Now, $\angle BMY = \angle MYR$ (AB || RS, Alternate angles)

Therefore, $\angle BMY = 40°$ (2)

Adding (1) and (2), you get

$$\angle XMB + \angle BMY = 45° + 40°$$

That is, $\angle XMY = 85°$

Example 5 : If a transversal intersects two lines such that the bisectors of a pair of corresponding angles are parallel, then prove that the two lines are parallel.

Solution : In Fig. 6.21, a transversal AD intersects two lines PQ and RS at points B and C respectively. Ray BE is the bisector of $\angle ABQ$ and ray CG is the bisector of $\angle BCS$; and $BE \parallel CG$.

We are to prove that $PQ \parallel RS$.

It is given that ray BE is the bisector of $\angle ABQ$.

Therefore, $\angle ABE = \frac{1}{2} \angle ABQ$ (1)

Similarly, ray CG is the bisector of $\angle BCS$.

Therefore, $\angle BCG = \frac{1}{2} \angle BCS$ (2)

But $BE \parallel CG$ and AD is the transversal.

Therefore, $\angle ABE = \angle BCG$

(Corresponding angles axiom) (3)

Substituting (1) and (2) in (3), you get

$$\frac{1}{2} \angle ABQ = \frac{1}{2} \angle BCS$$

That is, $\angle ABQ = \angle BCS$

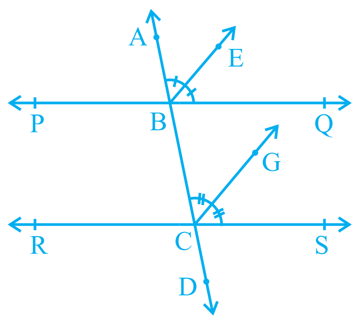


Fig. 6.21

But, they are the corresponding angles formed by transversal AD with PQ and RS; and are equal.

Therefore, PQ || RS

(Converse of corresponding angles axiom)

Example 6 : In Fig. 6.22, AB || CD and CD || EF. Also EA ⊥ AB. If ∠ BEF = 55°, find the values of x, y and z.

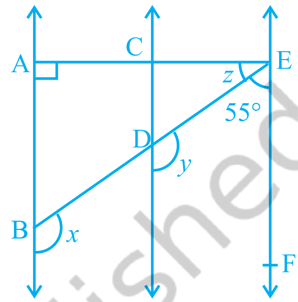


Fig. 6.22

Solution : $y + 55° = 180°$

(Interior angles on the same side of the transversal ED)

Therefore, $y = 180° – 55° = 125°$

Again $x = y$

(AB || CD, Corresponding angles axiom)

Therefore $x = 125°$

Now, since AB || CD and CD || EF, therefore, AB || EF.

So, ∠ EAB + ∠ FEA = 180° (Interior angles on the same side of the transversal EA)

Therefore, $90° + z + 55° = 180°$

Which gives $z = 35°$

EXERCISE 6.2

1. In Fig. 6.23, if AB || CD, CD || EF and $y : z = 3 : 7$, find x.

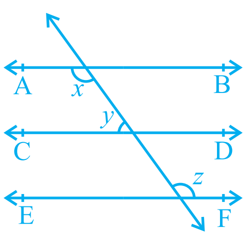


Fig. 6.23

2. In Fig. 6.24, if AB || CD, EF ⊥ CD and ∠ GED = 126°, find ∠ AGE, ∠ GEF and ∠ FGE.

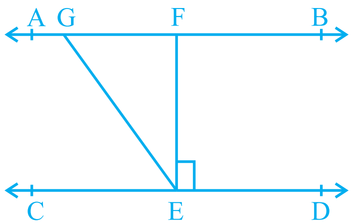


Fig. 6.24

3. In Fig. 6.25, if PQ || ST, ∠ PQR = 110° and ∠ RST = 130°, find ∠ QRS.
[**Hint :** Draw a line parallel to ST through point R.]

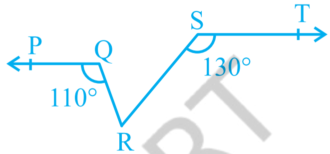


Fig. 6.25

4. In Fig. 6.26, if AB || CD, ∠ APQ = 50° and ∠ PRD = 127°, find x and y.

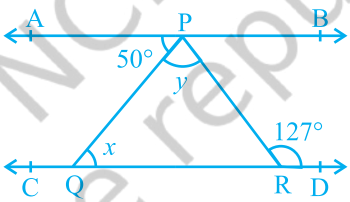


Fig. 6.26

5. In Fig. 6.27, PQ and RS are two mirrors placed parallel to each other. An incident ray AB strikes the mirror PQ at B, the reflected ray moves along the path BC and strikes the mirror RS at C and again reflects back along CD. Prove that AB || CD.

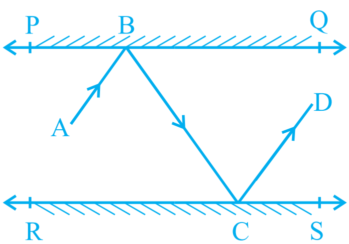


Fig. 6.27

6.6 Summary

In this chapter, you have studied the following points:

1. If a ray stands on a line, then the sum of the two adjacent angles so formed is 180° and vice-versa. This property is called as the Linear pair axiom.
2. If two lines intersect each other, then the vertically opposite angles are equal.
3. Lines which are parallel to a given line are parallel to each other.

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CHAPTER 7

TRIANGLES

7.1 Introduction

You have studied about triangles and their various properties in your earlier classes. You know that a closed figure formed by three intersecting lines is called a triangle. ('Tri' means 'three'). A triangle has three sides, three angles and three vertices. For example, in triangle ABC, denoted as Δ ABC (see Fig. 7.1); AB, BC, CA are the three sides, $\angle$ A, $\angle$ B, $\angle$ C are the three angles and A, B, C are three vertices.

In Chapter 6, you have also studied some properties of triangles. In this chapter, you will study in details about the congruence of triangles, rules of congruence, some more properties of triangles and inequalities in a triangle. You have already verified most of these properties in earlier classes. We will now prove some of them.

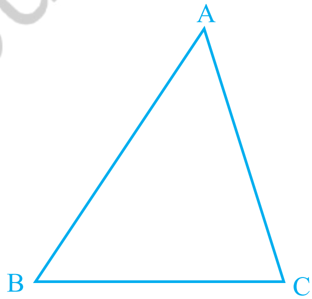

Fig. 7.1

7.2 Congruence of Triangles

You must have observed that two copies of your photographs of the same size are identical. Similarly, two bangles of the same size, two ATM cards issued by the same bank are identical. You may recall that on placing a one rupee coin on another minted in the same year, they cover each other completely.

Do you remember what such figures are called? Indeed they are called **congruent figures** ('congruent' means equal in all respects or figures whose shapes and sizes are both the same).

Now, draw two circles of the same radius and place one on the other. What do you observe? They cover each other completely and we call them as congruent circles.

Repeat this activity by placing one square on the other with sides of the same measure (see Fig. 7.2) or by placing two equilateral triangles of equal sides on each other. You will observe that the squares are congruent to each other and so are the equilateral triangles.

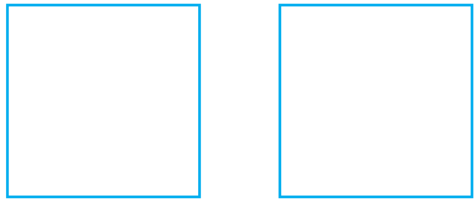

Fig. 7.2

You may wonder why we are studying congruence. You all must have seen the ice tray in your refrigerator. Observe that the moulds for making ice are all congruent. The cast used for moulding in the tray also has congruent depressions (may be all are rectangular or all circular or all triangular). So, whenever identical objects have to be produced, the concept of congruence is used in making the cast.

Sometimes, you may find it difficult to replace the refill in your pen by a new one and this is so when the new refill is not of the same size as the one you want to remove. Obviously, if the two refills are identical or congruent, the new refill fits.

So, you can find numerous examples where congruence of objects is applied in daily life situations.

Can you think of some more examples of congruent figures?

Now, which of the following figures are not congruent to the square in Fig 7.3 (i) :

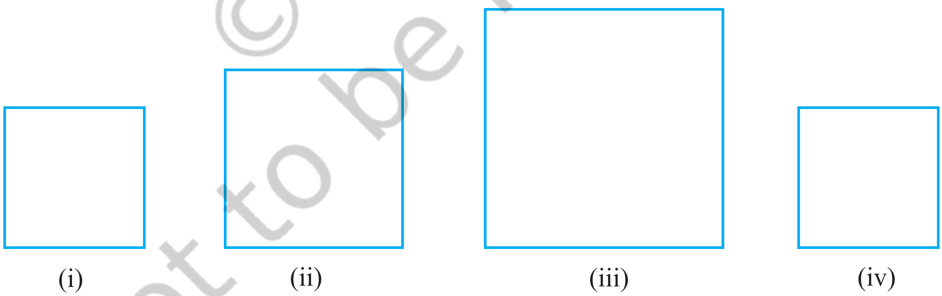

Fig. 7.3

The large squares in Fig. 7.3 (ii) and (iii) are obviously not congruent to the one in Fig 7.3 (i), but the square in Fig 7.3 (iv) is congruent to the one given in Fig 7.3 (i).

Let us now discuss the congruence of two triangles.

You already know that two triangles are congruent if the sides and angles of one triangle are equal to the corresponding sides and angles of the other triangle.

Now, which of the triangles given below are congruent to triangle ABC in Fig. 7.4 (i)?

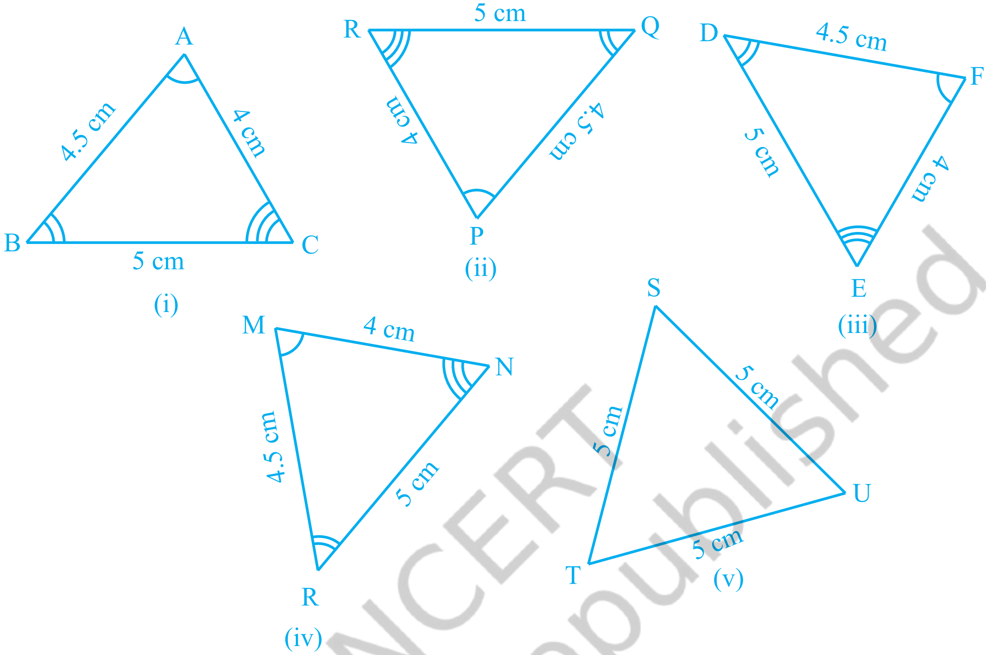

Fig. 7.4

Cut out each of these triangles from Fig. 7.4 (ii) to (v) and turn them around and try to cover Δ ABC. Observe that triangles in Fig. 7.4 (ii), (iii) and (iv) are congruent to Δ ABC while Δ TSU of Fig 7.4 (v) is not congruent to Δ ABC.

If Δ PQR is congruent to Δ ABC, we write Δ PQR ≅ Δ ABC.

Notice that when Δ PQR ≅ Δ ABC, then sides of Δ PQR fall on corresponding equal sides of Δ ABC and so is the case for the angles.

That is, PQ covers AB, QR covers BC and RP covers CA; ∠ P covers ∠ A, ∠ Q covers ∠ B and ∠ R covers ∠ C. Also, there is a one-one correspondence between the vertices. That is, P corresponds to A, Q to B, R to C and so on which is written as

$$P \leftrightarrow A,\ Q \leftrightarrow B,\ R \leftrightarrow C$$

Note that under this correspondence, Δ PQR ≅ Δ ABC; but it will not be correct to write ΔQRP ≅ Δ ABC.

Similarly, for Fig. 7.4 (iii),

FD ↔ AB, DE ↔ BC and EF ↔ CA

and F ↔ A, D ↔ B and E ↔ C

So, Δ FDE ≅ Δ ABC but writing Δ DEF ≅ Δ ABC is not correct.

Give the correspondence between the triangle in Fig. 7.4 (iv) and Δ ABC.

So, it is necessary to write the correspondence of vertices correctly for writing of congruence of triangles in symbolic form.

Note that in **congruent triangles corresponding parts** are **equal** and we write in short 'CPCT' for *corresponding parts of congruent triangles*.

7.3 Criteria for Congruence of Triangles

In earlier classes, you have learnt four criteria for congruence of triangles. Let us recall them.

Draw two triangles with one side 3 cm. Are these triangles congruent? Observe that they are not congruent (see Fig. 7.5).

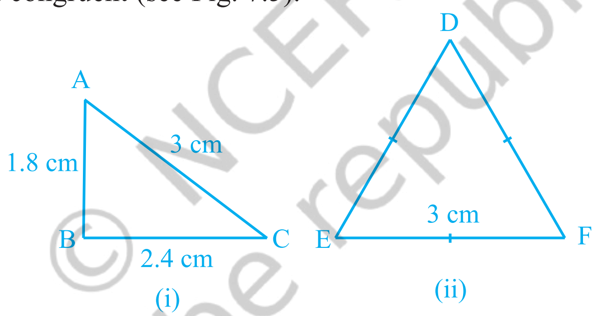

Fig. 7.5

Now, draw two triangles with one side 4 cm and one angle 50° (see Fig. 7.6). Are they congruent?

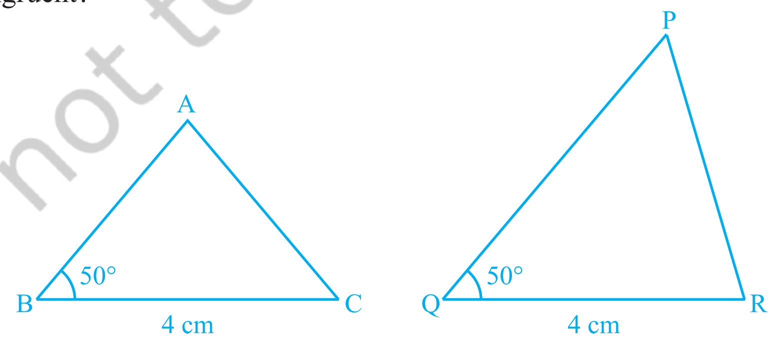

Fig. 7.6

See that these two triangles are not congruent.

Repeat this activity with some more pairs of triangles.

So, equality of one pair of sides or one pair of sides and one pair of angles is not sufficient to give us congruent triangles.

What would happen if the other pair of arms (sides) of the equal angles are also equal?

In Fig 7.7, BC = QR, $\angle$ B = $\angle$ Q and also, AB = PQ. Now, what can you say about congruence of Δ ABC and Δ PQR?

Recall from your earlier classes that, in this case, the two triangles are congruent. Verify this for Δ ABC and Δ PQR in Fig. 7.7.

Repeat this activity with other pairs of triangles. Do you observe that the equality of two sides and the included angle is enough for the congruence of triangles? Yes, it is enough.

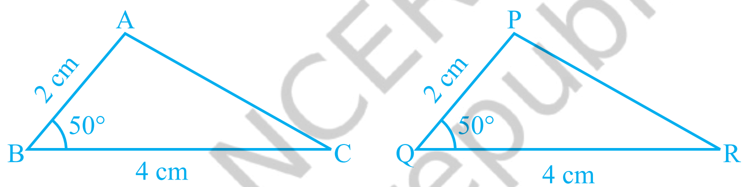

Fig. 7.7

This is the first criterion for congruence of triangles.

Axiom 7.1 (SAS congruence rule) : *Two triangles are congruent if two sides and the included angle of one triangle are equal to the two sides and the included angle of the other triangle.*

This result cannot be proved with the help of previously known results and so it is accepted true as an axiom (see Appendix 1).

Let us now take some examples.

Example 1 : In Fig. 7.8, OA = OB and OD = OC. Show that

(i) Δ AOD $\cong$ Δ BOC and (ii) AD || BC.

Solution : (i) You may observe that in Δ AOD and Δ BOC,

$$\left.\begin{aligned} OA &= OB \\ OD &= OC \end{aligned}\right\} \quad \text{(Given)}$$

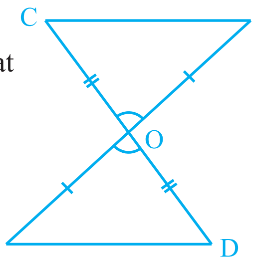

Fig. 7.8

Also, since $\angle$ AOD and $\angle$ BOC form a pair of vertically opposite angles, we have

$$\angle \text{AOD} = \angle \text{BOC}.$$

So, Δ AOD $\cong$ Δ BOC (by the SAS congruence rule)

(ii) In congruent triangles AOD and BOC, the other corresponding parts are also equal.

So, $\angle$ OAD = $\angle$ OBC and these form a pair of alternate angles for line segments AD and BC.

Therefore, AD || BC.

Example 2 : AB is a line segment and line l is its perpendicular bisector. If a point P lies on l, show that P is equidistant from A and B.

Solution : Line $l \perp$ AB and passes through C which is the mid-point of AB (see Fig. 7.9). You have to show that PA = PB. Consider Δ PCA and Δ PCB.

We have AC = BC (C is the mid-point of AB)

$\angle$ PCA = $\angle$ PCB = 90° (Given)

PC = PC (Common)

So, Δ PCA $\cong$ Δ PCB (SAS rule)

and so, PA = PB, as they are corresponding sides of congruent triangles.

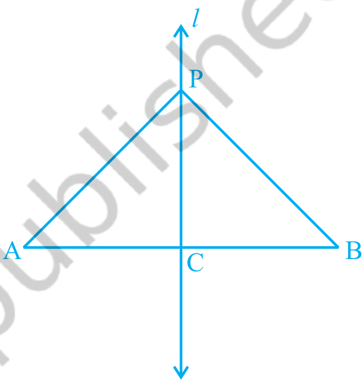

Fig. 7.9

Now, let us construct two triangles, whose sides are 4 cm and 5 cm and one of the angles is 50° and this angle is not included in between the equal sides (see Fig. 7.10). Are the two triangles congruent?

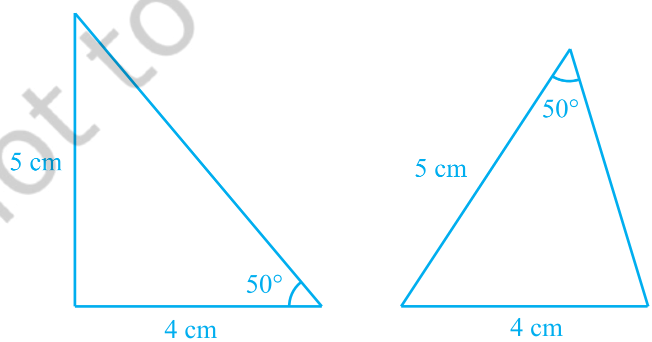

Fig. 7.10

Notice that the two triangles are not congruent.

Repeat this activity with more pairs of triangles. You will observe that for triangles to be congruent, it is very important that the equal angles are included between the pairs of equal sides.

So, SAS *congruence rule holds* but not ASS or SSA rule.

Next, try to construct the two triangles in which two angles are 60° and 45° and the side included between these angles is 4 cm (see Fig. 7.11).

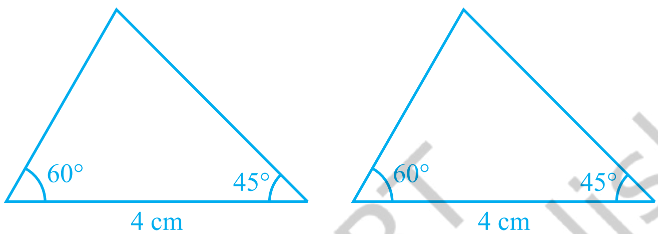

Fig. 7.11

Cut out these triangles and place one triangle on the other. What do you observe? See that one triangle covers the other completely; that is, the two triangles are congruent. Repeat this activity with more pairs of triangles. You will observe that equality of two angles and the included side is sufficient for congruence of triangles.

This result is the **Angle-Side-Angle** criterion for congruence and is written as **ASA** criterion. You have verified this criterion in earlier classes, but let us state and prove this result.

Since this result can be proved, it is called a theorem and to prove it, we use the SAS axiom for congruence.

Theorem 7.1 (ASA congruence rule) : *Two triangles are congruent if two angles and the included side of one triangle are equal to two angles and the included side of other triangle.*

Proof : We are given two triangles ABC and DEF in which:

$$\angle B = \angle E, \angle C = \angle F$$

and $\quad BC = EF$

We need to prove that $\quad \Delta ABC \cong \Delta DEF$

For proving the congruence of the two triangles see that three cases arise.

Case (i) : Let AB = DE (see Fig. 7.12).

Now what do you observe? You may observe that

$$AB = DE \quad \text{(Assumed)}$$

$$\angle B = \angle E \quad \text{(Given)}$$

$$BC = EF \quad \text{(Given)}$$

So, $$\Delta\, ABC \cong \Delta\, DEF \quad \text{(By SAS rule)}$$

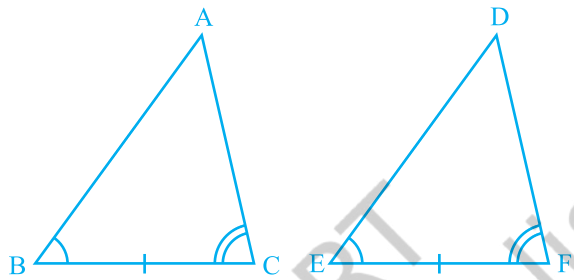

Fig. 7.12

Case (ii) : Let if possible AB > DE. So, we can take a point P on AB such that PB = DE. Now consider Δ PBC and Δ DEF (see Fig. 7.13).

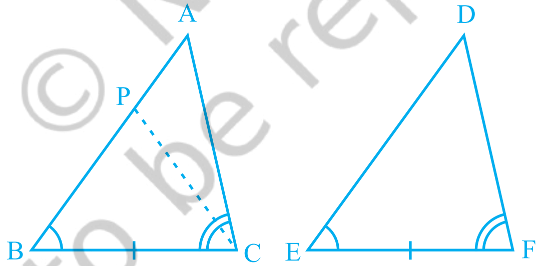

Fig. 7.13

Observe that in Δ PBC and Δ DEF,

$$PB = DE \quad \text{(By construction)}$$

$$\angle B = \angle E \quad \text{(Given)}$$

$$BC = EF \quad \text{(Given)}$$

So, we can conclude that:

Δ PBC ≅ Δ DEF, by the SAS axiom for congruence.

Since the triangles are congruent, their corresponding parts will be equal.

So, $\angle$ PCB = $\angle$ DFE

But, we are given that

$\angle$ ACB = $\angle$ DFE

So, $\angle$ ACB = $\angle$ PCB

Is this possible?

This is possible only if P coincides with A.

or, BA = ED

So, Δ ABC $\cong$ Δ DEF (by SAS axiom)

Case (iii) : If AB < DE, we can choose a point M on DE such that ME = AB and repeating the arguments as given in Case (ii), we can conclude that AB = DE and so, Δ ABC $\cong$ Δ DEF.

Suppose, now in two triangles two pairs of angles and one pair of corresponding sides are equal but the side is not included between the corresponding equal pairs of angles. Are the triangles still congruent? You will observe that they are congruent. Can you reason out why?

You know that the sum of the three angles of a triangle is 180°. So if two pairs of angles are equal, the third pair is also equal (180° – sum of equal angles).

So, two *triangles are congruent if any two pairs of angles and one pair of corresponding sides are equal*. We may call it as the **AAS Congruence Rule**.

Now let us perform the following activity :

Draw triangles with angles 40°, 50° and 90°. How many such triangles can you draw?

In fact, you can draw as many triangles as you want with different lengths of sides (see Fig. 7.14).

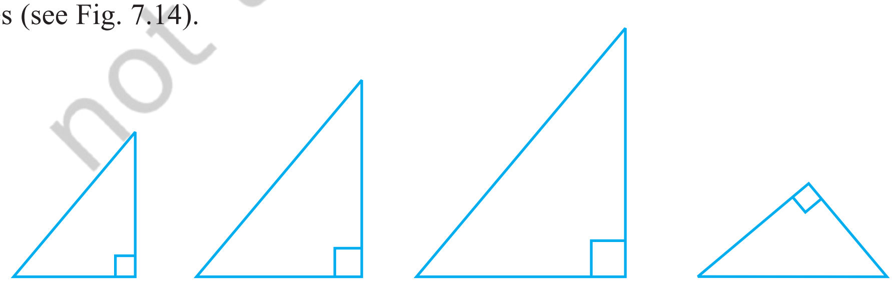

Fig. 7.14

Observe that the triangles may or may not be congruent to each other.

So, equality of three angles is not sufficient for congruence of triangles. Therefore, for congruence of triangles out of three equal parts, one has to be a side.

Let us now take some more examples.

Example 3 : Line-segment AB is parallel to another line-segment CD. O is the mid-point of AD (see Fig. 7.15). Show that (i) $\Delta AOB \cong \Delta DOC$ (ii) O is also the mid-point of BC.

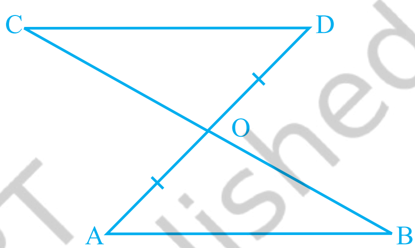

Fig. 7.15

Solution : (i) Consider Δ AOB and Δ DOC.

$\angle ABO = \angle DCO$

(Alternate angles as AB || CD and BC is the transversal)

$\angle AOB = \angle DOC$

(Vertically opposite angles)

OA = OD (Given)

Therefore, $\Delta AOB \cong \Delta DOC$ (AAS rule)

(ii) OB = OC (CPCT)

So, O is the mid-point of BC.

EXERCISE 7.1

1. In quadrilateral ACBD, AC = AD and AB bisects $\angle$ A (see Fig. 7.16). Show that $\Delta ABC \cong \Delta ABD$. What can you say about BC and BD?

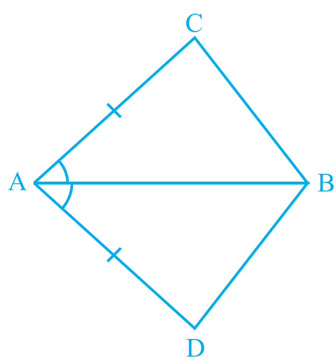

Fig. 7.16

2. ABCD is a quadrilateral in which AD = BC and $\angle$ DAB = $\angle$ CBA (see Fig. 7.17). Prove that
   (i) Δ ABD $\cong$ Δ BAC
   (ii) BD = AC
   (iii) $\angle$ ABD = $\angle$ BAC.

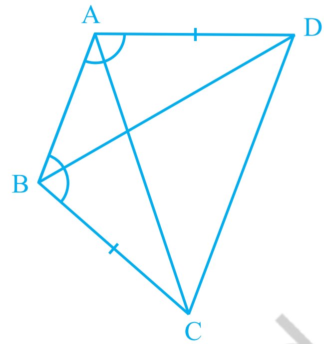

Fig. 7.17

3. AD and BC are equal perpendiculars to a line segment AB (see Fig. 7.18). Show that CD bisects AB.

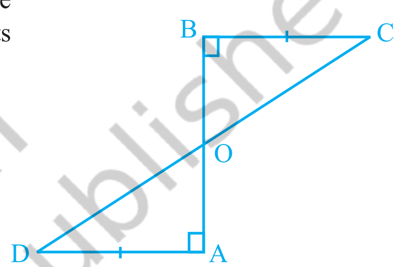

Fig. 7.18

4. *l* and *m* are two parallel lines intersected by another pair of parallel lines *p* and *q* (see Fig. 7.19). Show that Δ ABC $\cong$ Δ CDA.

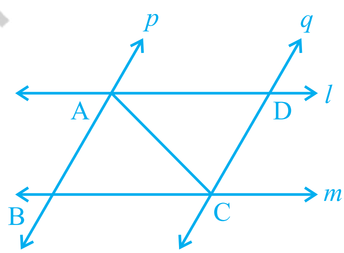

Fig. 7.19

5. Line *l* is the bisector of an angle $\angle$ A and B is any point on *l*. BP and BQ are perpendiculars from B to the arms of $\angle$ A (see Fig. 7.20). Show that:
   (i) Δ APB $\cong$ Δ AQB
   (ii) BP = BQ or B is equidistant from the arms of $\angle$ A.

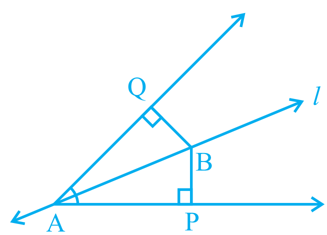

Fig. 7.20

6. In Fig. 7.21, AC = AE, AB = AD and $\angle$ BAD = $\angle$ EAC. Show that BC = DE.

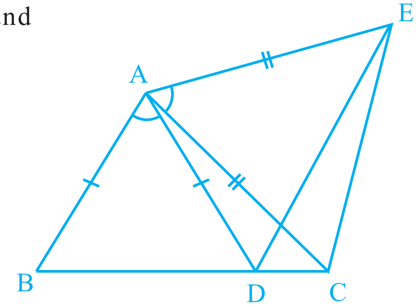

Fig. 7.21

7. AB is a line segment and P is its mid-point. D and E are points on the same side of AB such that $\angle$ BAD = $\angle$ ABE and $\angle$ EPA = $\angle$ DPB (see Fig. 7.22). Show that

(i) Δ DAP $\cong$ Δ EBP

(ii) AD = BE

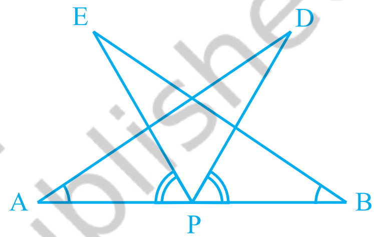

Fig. 7.22

8. In right triangle ABC, right angled at C, M is the mid-point of hypotenuse AB. C is joined to M and produced to a point D such that DM = CM. Point D is joined to point B (see Fig. 7.23). Show that:

(i) Δ AMC $\cong$ Δ BMD

(ii) $\angle$ DBC is a right angle.

(iii) Δ DBC $\cong$ Δ ACB

(iv) CM = $\frac{1}{2}$ AB

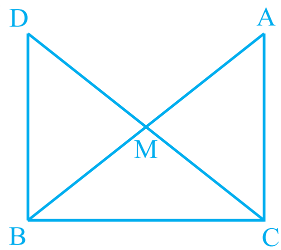

Fig. 7.23

7.4 Some Properties of a Triangle

In the above section you have studied two criteria for congruence of triangles. Let us now apply these results to study some properties related to a triangle whose two sides are equal.

Perform the activity given below:

Construct a triangle in which two sides are equal, say each equal to 3.5 cm and the third side equal to 5 cm (see Fig. 7.24). You have done such constructions in earlier classes.

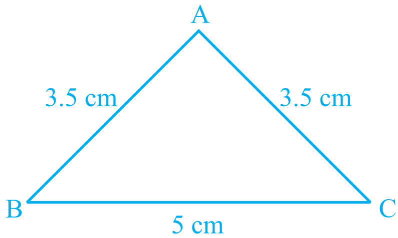

Fig. 7.24

Do you remember what is such a triangle called?

A triangle in which two sides are equal is called an **isosceles triangle**. So, Δ ABC of Fig. 7.24 is an isosceles triangle with AB = AC.

Now, measure $\angle$ B and $\angle$ C. What do you observe?

Repeat this activity with other isosceles triangles with different sides.

You may observe that in each such triangle, the angles opposite to the equal sides are equal.

This is a very important result and is indeed true for any isosceles triangle. It can be proved as shown below.

Theorem 7.2 : *Angles opposite to equal sides of an isosceles triangle are equal.*

This result can be proved in many ways. One of the proofs is given here.

Proof : We are given an isosceles triangle ABC in which AB = AC. We need to prove that $\angle$ B = $\angle$ C.

Let us draw the bisector of $\angle$ A and let D be the point of intersection of this bisector of $\angle$ A and BC (see Fig. 7.25).

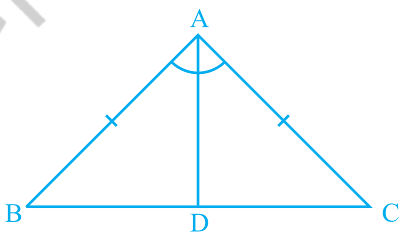

Fig. 7.25

In Δ BAD and Δ CAD,

	AB = AC	(Given)
	$\angle$ BAD = $\angle$ CAD	(By construction)
	AD = AD	(Common)
So,	Δ BAD $\cong$ Δ CAD	(By SAS rule)

So, $\angle$ ABD = $\angle$ ACD, since they are corresponding angles of congruent triangles.

So, $\angle B = \angle C$

Is the converse also true? That is:

If two angles of any triangle are equal, can we conclude that the sides opposite to them are also equal?

Perform the following activity.

Construct a triangle ABC with BC of any length and $\angle$ B = $\angle$ C = 50°. Draw the bisector of $\angle$ A and let it intersect BC at D (see Fig. 7.26).

Cut out the triangle from the sheet of paper and fold it along AD so that vertex C falls on vertex B.

What can you say about sides AC and AB?

Observe that AC covers AB completely

So, AC = AB

Repeat this activity with some more triangles. Each time you will observe that the sides opposite to equal angles are equal. So we have the following:

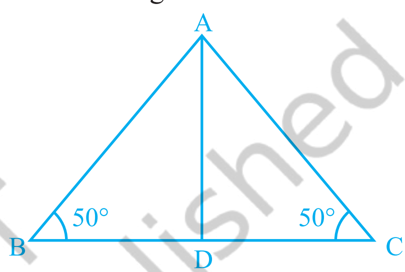

Fig. 7.26

Theorem 7.3 : *The sides opposite to equal angles of a triangle are equal.*

This is the converse of Theorem 7.2.

You can prove this theorem by ASA congruence rule.

Let us take some examples to apply these results.

Example 4 : In Δ ABC, the bisector AD of $\angle$ A is perpendicular to side BC (see Fig. 7.27). Show that AB = AC and Δ ABC is isosceles.

Solution : In ΔABD and ΔACD,

$\angle$ BAD = $\angle$ CAD (Given)

AD = AD (Common)

$\angle$ ADB = $\angle$ ADC = 90° (Given)

So, Δ ABD ≅ Δ ACD (ASA rule)

So, AB = AC (CPCT)

or, Δ ABC is an isosceles triangle.

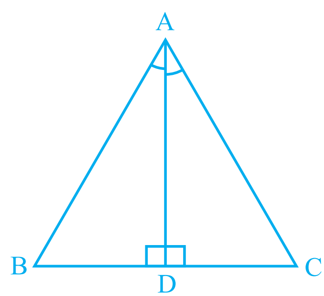

Fig. 7.27

Example 5 : E and F are respectively the mid-points of equal sides AB and AC of Δ ABC (see Fig. 7.28). Show that BF = CE.

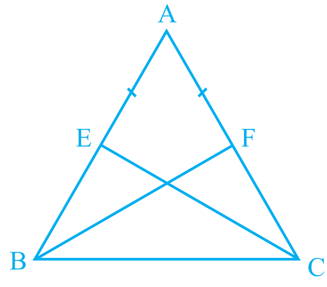

Fig. 7.28

Solution : In Δ ABF and Δ ACE,

AB = AC (Given)

∠ A = ∠ A (Common)

AF = AE (Halves of equal sides)

So, Δ ABF ≅ Δ ACE (SAS rule)

Therefore, BF = CE (CPCT)

Example 6 : In an isosceles triangle ABC with AB = AC, D and E are points on BC such that BE = CD (see Fig. 7.29). Show that AD = AE.

Solution : In Δ ABD and Δ ACE,

AB = AC (Given) (1)

∠ B = ∠ C

(Angles opposite to equal sides) (2)

Also, BE = CD

So, BE – DE = CD – DE

That is, BD = CE (3)

So, Δ ABD ≅ Δ ACE

(Using (1), (2), (3) and SAS rule).

This gives AD = AE (CPCT)

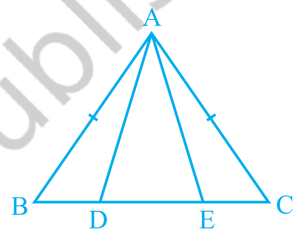

Fig. 7.29

EXERCISE 7.2

1. In an isosceles triangle ABC, with AB = AC, the bisectors of ∠ B and ∠ C intersect each other at O. Join A to O. Show that :

   (i) OB = OC (ii) AO bisects ∠ A

2. In Δ ABC, AD is the perpendicular bisector of BC (see Fig. 7.30). Show that Δ ABC is an isosceles triangle in which AB = AC.

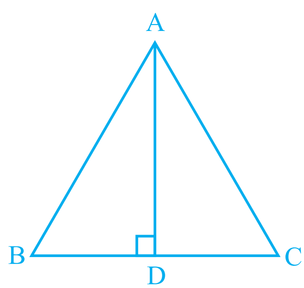

Fig. 7.30

3. ABC is an isosceles triangle in which altitudes BE and CF are drawn to equal sides AC and AB respectively (see Fig. 7.31). Show that these altitudes are equal.

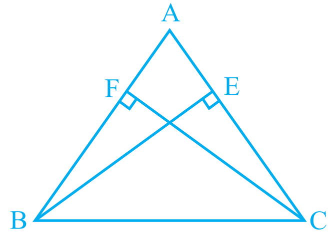

Fig. 7.31

4. ABC is a triangle in which altitudes BE and CF to sides AC and AB are equal (see Fig. 7.32). Show that

(i) Δ ABE $\cong \Delta$ ACF

(ii) AB = AC, i.e., ABC is an isosceles triangle.

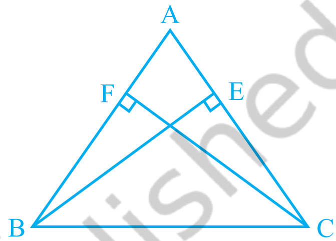

Fig. 7.32

5. ABC and DBC are two isosceles triangles on the same base BC (see Fig. 7.33). Show that $\angle$ ABD = $\angle$ ACD.

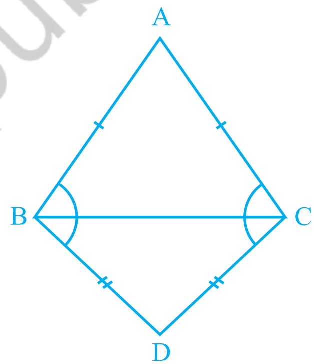

Fig. 7.33

6. ΔABC is an isosceles triangle in which AB = AC. Side BA is produced to D such that AD = AB (see Fig. 7.34). Show that $\angle$ BCD is a right angle.

7. ABC is a right angled triangle in which $\angle$ A = 90° and AB = AC. Find $\angle$ B and $\angle$ C.

8. Show that the angles of an equilateral triangle are 60° each.

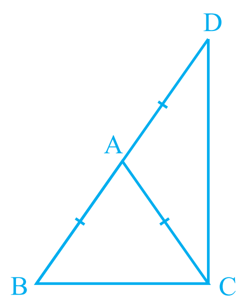

Fig. 7.34

7.5 Some More Criteria for Congruence of Triangles

You have seen earlier in this chapter that equality of three angles of one triangle to three angles of the other is not sufficient for the congruence of the two triangles. You may wonder whether equality of three sides of one triangle to three sides of another triangle is enough for congruence of the two triangles. You have already verified in earlier classes that this is indeed true.

To be sure, construct two triangles with sides 4 cm, 3.5 cm and 4.5 cm (see Fig. 7.35). Cut them out and place them on each other. What do you observe? They cover each other completely, if the equal sides are placed on each other. So, the triangles are congruent.

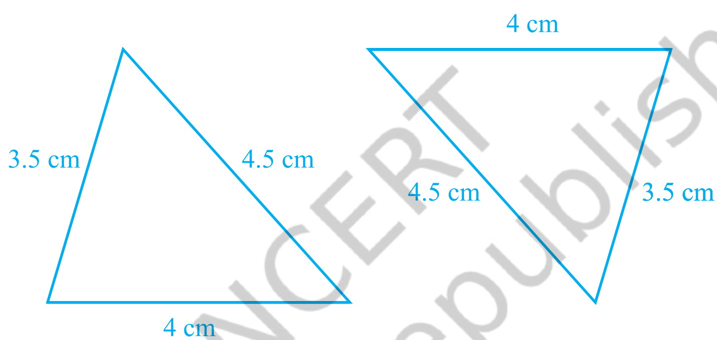

Fig. 7.35

Repeat this activity with some more triangles. We arrive at another rule for congruence.

Theorem 7.4 (SSS congruence rule) : *If three sides of one triangle are equal to the three sides of another triangle, then the two triangles are congruent.*

This theorem can be proved using a suitable construction.

You have already seen that in the SAS congruence rule, the pair of equal angles has to be the included angle between the pairs of corresponding pair of equal sides and if this is not so, the two triangles may not be congruent.

Perform this activity:

Construct two right angled triangles with hypotenuse equal to 5 cm and one side equal to 4 cm each (see Fig. 7.36).

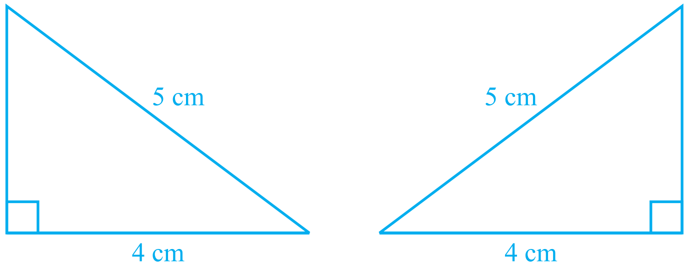

Fig. 7.36

Cut them out and place one triangle over the other with equal side placed on each other. Turn the triangles, if necessary. What do you observe?

The two triangles cover each other completely and so they are congruent. Repeat this activity with other pairs of right triangles. What do you observe?

You will find that two right triangles are congruent if one pair of sides and the hypotenuse are equal. You have verified this in earlier classes.

Note that, the right angle is ***not*** the included angle in this case.

So, you arrive at the following congruence rule:

Theorem 7.5 (RHS congruence rule) : *If in two right triangles the hypotenuse and one side of one triangle are equal to the hypotenuse and one side of the other triangle, then the two triangles are congruent.*

Note that RHS stands for **Right angle - Hypotenuse - Side.**

Let us now take some examples.

Example 7 : AB is a line-segment. P and Q are points on opposite sides of AB such that each of them is equidistant from the points A and B (see Fig. 7.37). Show that the line PQ is the perpendicular bisector of AB.

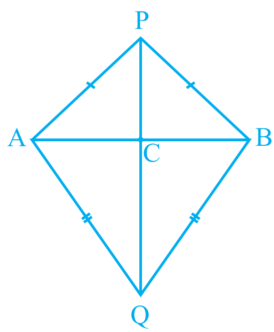

Fig. 7.37

Solution : You are given that PA = PB and QA = QB and you are to show that PQ $\perp$ AB and PQ bisects AB. Let PQ intersect AB at C.

Can you think of two congruent triangles in this figure?

Let us take Δ PAQ and Δ PBQ.

In these triangles,

$$AP = BP \quad \text{(Given)}$$

$$AQ = BQ \quad \text{(Given)}$$

$$PQ = PQ \quad \text{(Common)}$$

So, $\Delta\ PAQ \cong \Delta\ PBQ$ (SSS rule)

Therefore, $\angle\ APQ = \angle\ BPQ$ (CPCT).

Now let us consider Δ PAC and Δ PBC.

You have : $AP = BP$ (Given)

$\angle\ APC = \angle\ BPC$ ($\angle\ APQ = \angle\ BPQ$ proved above)

$PC = PC$ (Common)

So, $\Delta\ PAC \cong \Delta\ PBC$ (SAS rule)

Therefore, $AC = BC$ (CPCT) (1)

and $\angle\ ACP = \angle\ BCP$ (CPCT)

Also, $\angle\ ACP + \angle\ BCP = 180°$ (Linear pair)

So, $2\angle\ ACP = 180°$

or, $\angle\ ACP = 90°$ (2)

From (1) and (2), you can easily conclude that PQ is the perpendicular bisector of AB.

[Note that, without showing the congruence of Δ PAQ and Δ PBQ, you cannot show that Δ PAC $\cong$ Δ PBC even though AP = BP (Given)

$PC = PC$ (Common)

and $\angle\ PAC = \angle\ PBC$ (Angles opposite to equal sides in ΔAPB)

It is because these results give us SSA rule which is not always valid or true for congruence of triangles. Also the angle is not included between the equal pairs of sides.]

Let us take some more examples.

Example 8 : P is a point equidistant from two lines l and m intersecting at point A (see Fig. 7.38). Show that the line AP bisects the angle between them.

Solution : You are given that lines l and m intersect each other at A. Let PB $\perp$ l, PC $\perp$ m. It is given that PB = PC.

You are to show that $\angle$ PAB = $\angle$ PAC.

Let us consider Δ PAB and Δ PAC. In these two triangles,

$$PB = PC \qquad \text{(Given)}$$

$$\angle PBA = \angle PCA = 90^\circ \qquad \text{(Given)}$$

$$PA = PA \qquad \text{(Common)}$$

So, $\Delta PAB \cong \Delta PAC$ (RHS rule)

So, $\angle PAB = \angle PAC$ (CPCT)

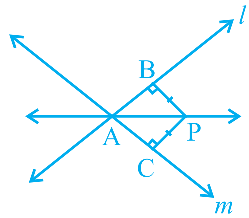

Fig. 7.38

Note that this result is the converse of the result proved in Q.5 of Exercise 7.1.

EXERCISE 7.3

1. Δ ABC and Δ DBC are two isosceles triangles on the same base BC and vertices A and D are on the same side of BC (see Fig. 7.39). If AD is extended to intersect BC at P, show that

(i) $\Delta ABD \cong \Delta ACD$

(ii) $\Delta ABP \cong \Delta ACP$

(iii) AP bisects $\angle$ A as well as $\angle$ D.

(iv) AP is the perpendicular bisector of BC.

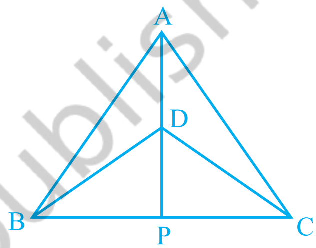

Fig. 7.39

2. AD is an altitude of an isosceles triangle ABC in which AB = AC. Show that

(i) AD bisects BC (ii) AD bisects $\angle$ A.

3. Two sides AB and BC and median AM of one triangle ABC are respectively equal to sides PQ and QR and median PN of Δ PQR (see Fig. 7.40). Show that:

(i) $\Delta ABM \cong \Delta PQN$

(ii) $\Delta ABC \cong \Delta PQR$

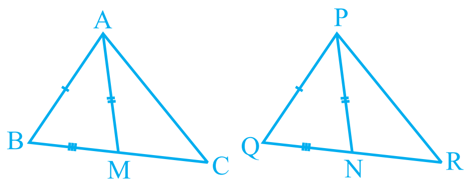

Fig. 7.40

4. BE and CF are two equal altitudes of a triangle ABC. Using RHS congruence rule, prove that the triangle ABC is isosceles.

5. ABC is an isosceles triangle with AB = AC. Draw AP $\perp$ BC to show that $\angle B = \angle C$.

7.6 Summary

In this chapter, you have studied the following points :

1. Two figures are congruent, if they are of the same shape and of the same size.
2. Two circles of the same radii are congruent.
3. Two squares of the same sides are congruent.
4. If two triangles ABC and PQR are congruent under the correspondence $A \leftrightarrow P$, $B \leftrightarrow Q$ and $C \leftrightarrow R$, then symbolically, it is expressed as $\Delta ABC \cong \Delta PQR$.
5. If two sides and the included angle of one triangle are equal to two sides and the included angle of the other triangle, then the two triangles are congruent (SAS Congruence Rule).
6. If two angles and the included side of one triangle are equal to two angles and the included side of the other triangle, then the two triangles are congruent (ASA Congruence Rule).
7. If two angles and one side of one triangle are equal to two angles and the corresponding side of the other triangle, then the two triangles are congruent (AAS Congruence Rule).
8. Angles opposite to equal sides of a triangle are equal.
9. Sides opposite to equal angles of a triangle are equal.
10. Each angle of an equilateral triangle is of 60°.
11. If three sides of one triangle are equal to three sides of the other triangle, then the two triangles are congruent (SSS Congruence Rule).
12. If in two right triangles, hypotenuse and one side of a triangle are equal to the hypotenuse and one side of other triangle, then the two triangles are congruent (RHS Congruence Rule).

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CHAPTER 8

QUADRILATERALS

8.1 Properties of a Parallelogram

You have already studied quadrilaterals and their types in Class VIII. A quadrilateral has four sides, four angles and four vertices. A parallelogram is a quadrilateral in which both pairs of opposite sides are parallel.

Let us perform an activity.

Cut out a parallelogram from a sheet of paper and cut it along a diagonal (see Fig. 8.1). You obtain two triangles. What can you say about these triangles?

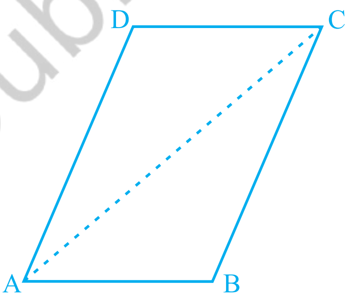

Fig. 8.1

Place one triangle over the other. Turn one around, if necessary. What do you observe?

Observe that the two triangles are congruent to each other.

Repeat this activity with some more parallelograms. Each time you will observe that each diagonal divides the parallelogram into two congruent triangles.

Let us now prove this result.

Theorem 8.1 : *A diagonal of a parallelogram divides it into two congruent triangles.*

Proof : Let ABCD be a parallelogram and AC be a diagonal (see Fig. 8.2). Observe that the diagonal AC divides parallelogram ABCD into two triangles, namely, Δ ABC and Δ CDA. We need to prove that these triangles are congruent.

In Δ ABC and Δ CDA, note that BC || AD and AC is a transversal.

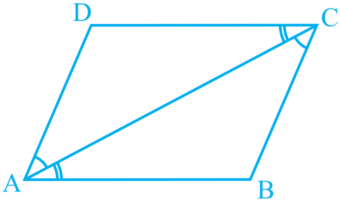

Fig. 8.2

So, $\angle$ BCA = $\angle$ DAC (Pair of alternate angles)

Also, AB || DC and AC is a transversal.

So, $\angle$ BAC = $\angle$ DCA (Pair of alternate angles)

and AC = CA (Common)

So, Δ ABC $\cong$ Δ CDA (ASA rule)

or, diagonal AC divides parallelogram ABCD into two congruent triangles ABC and CDA.

Now, measure the opposite sides of parallelogram ABCD. What do you observe?

You will find that AB = DC and AD = BC.

This is another property of a parallelogram stated below:

Theorem 8.2 : *In a parallelogram, opposite sides are equal.*

You have already proved that a diagonal divides the parallelogram into two congruent triangles; so what can you say about the corresponding parts say, the corresponding sides? They are equal.

So, AB = DC and AD = BC

Now what is the converse of this result? You already know that whatever is given in a theorem, the same is to be proved in the converse and whatever is proved in the theorem it is given in the converse. Thus, Theorem 8.2 can be stated as given below :

If a quadrilateral is a parallelogram, then each pair of its opposite sides is equal. So its converse is :

Theorem 8.3 : *If each pair of opposite sides of a quadrilateral is equal, then it is a parallelogram.*

Can you reason out why?

Let sides AB and CD of the quadrilateral ABCD be equal and also AD = BC (see Fig. 8.3). Draw diagonal AC.

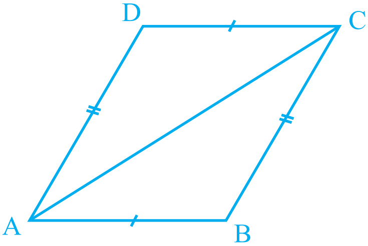

Fig. 8.3

Clearly, Δ ABC $\cong$ Δ CDA (Why?)

So, $\angle$ BAC = $\angle$ DCA

and $\angle$ BCA = $\angle$ DAC (Why?)

Can you now say that ABCD is a parallelogram? Why?

You have just seen that in a parallelogram each pair of opposite sides is equal and conversely if each pair of opposite sides of a quadrilateral is equal, then it is a parallelogram. Can we conclude the same result for the pairs of opposite angles?

Draw a parallelogram and measure its angles. What do you observe?

Each pair of opposite angles is equal.

Repeat this with some more parallelograms. We arrive at yet another result as given below.

Theorem 8.4 : *In a parallelogram, opposite angles are equal.*

Now, is the converse of this result also true? Yes. Using the angle sum property of a quadrilateral and the results of parallel lines intersected by a transversal, we can see that the converse is also true. So, we have the following theorem :

Theorem 8.5 : *If in a quadrilateral, each pair of opposite angles is equal, then it is a parallelogram.*

There is yet another property of a parallelogram. Let us study the same. Draw a parallelogram ABCD and draw both its diagonals intersecting at the point O (see Fig. 8.4).

Measure the lengths of OA, OB, OC and OD.

What do you observe? You will observe that

OA = OC and OB = OD.

or, O is the mid-point of both the diagonals.

Repeat this activity with some more parallelograms.

Each time you will find that O is the mid-point of both the diagonals.

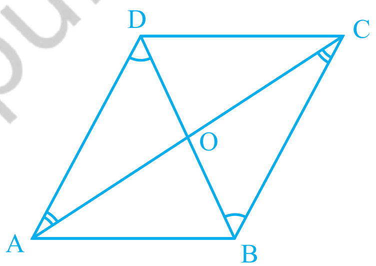

Fig. 8.4

So, we have the following theorem :

Theorem 8.6 : *The diagonals of a parallelogram bisect each other.*

Now, what would happen, if in a quadrilateral the diagonals bisect each other? Will it be aparallelogram? Indeed this is true.

This result is the converse of the result of Theorem 8.6. It is given below:

Theorem 8.7 : *If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.*

You can reason out this result as follows:

Note that in Fig. 8.5, it is given that OA = OC and OB = OD.

So, Δ AOB $\cong$ Δ COD (Why?)

Therefore, $\angle$ ABO = $\angle$ CDO (Why?)

From this, we get AB || CD

Similarly, BC || AD

Therefore ABCD is a parallelogram.

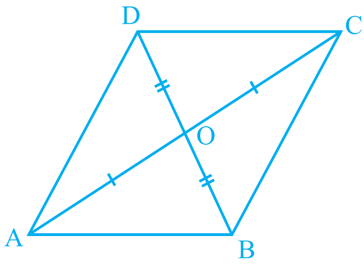

Fig. 8.5

Let us now take some examples.

Example 1 : Show that each angle of a rectangle is a right angle.

Solution : Let us recall what a rectangle is.

A rectangle is a parallelogram in which one angle is a right angle.

Let ABCD be a rectangle in which $\angle$ A = 90°.

We have to show that $\angle$ B = $\angle$ C = $\angle$ D = 90°

We have, AD || BC and AB is a transversal (see Fig. 8.6).

So, $\angle$ A + $\angle$ B = 180° (Interior angles on the same side of the transversal)

Fig. 8.6

But, $\angle$ A = 90°

So, $\angle$ B = 180° – $\angle$ A = 180° – 90° = 90°

Now, $\angle$ C = $\angle$ A and $\angle$ D = $\angle$ B

(Opposite angles of the parallellogram)

So, $\angle$ C = 90° and $\angle$ D = 90°.

Therefore, each of the angles of a rectangle is a right angle.

Example 2 : Show that the diagonals of a rhombus are perpendicular to each other.

Solution : Consider the rhombus ABCD (see Fig. 8.7).

You know that AB = BC = CD = DA (Why?)

Now, in Δ AOD and Δ COD,

OA = OC (Diagonals of a parallelogram bisect each other)

OD = OD (Common)

AD = CD

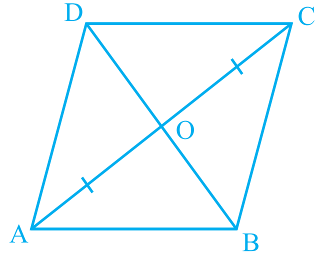

Fig. 8.7

Therefore, Δ AOD $\cong$ Δ COD

(SSS congruence rule)

This gives, $\angle$ AOD = $\angle$ COD (CPCT)

But, $\angle$ AOD + $\angle$ COD = 180° (Linear pair)

So, 2$\angle$ AOD = 180°

or, $\angle$ AOD = 90°

So, the diagonals of a rhombus are perpendicular to each other.

Example 3 : ABC is an isosceles triangle in which AB = AC. AD bisects exterior angle PAC and CD || AB (see Fig. 8.8). Show that

(i) $\angle$ DAC = $\angle$ BCA and (ii) ABCD is a parallelogram.

Solution : (i) Δ ABC is isosceles in which AB = AC (Given)

So, $\angle$ ABC = $\angle$ ACB (Angles opposite to equal sides)

Also, $\angle$ PAC = $\angle$ ABC + $\angle$ ACB

(Exterior angle of a triangle)

or, $\angle$ PAC = 2$\angle$ ACB (1)

Now, AD bisects $\angle$ PAC.

So, $\angle$ PAC = 2$\angle$ DAC (2)

Therefore,

2$\angle$ DAC = 2$\angle$ ACB [From (1) and (2)]

or, $\angle$ DAC = $\angle$ ACB

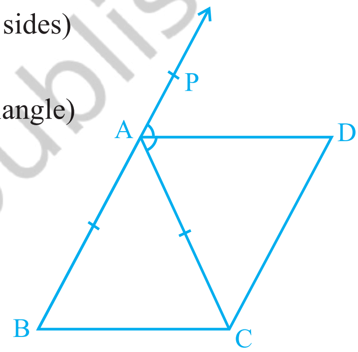

Fig. 8.8

(ii) Now, these equal angles form a pair of alternate angles when line segments BC and AD are intersected by a transversal AC.

So, BC || AD

Also, BA || CD (Given)

Now, both pairs of opposite sides of quadrilateral ABCD are parallel.

So, ABCD is a parallelogram.

Example 4 : Two parallel lines l and m are intersected by a transversal p (see Fig. 8.9). Show that the quadrilateral formed by the bisectors of interior angles is a rectangle.

Solution : It is given that PS || QR and transversal p intersects them at points A and C respectively.

The bisectors of $\angle$ PAC and $\angle$ ACQ intersect at B and bisectors of $\angle$ ACR and $\angle$ SAC intersect at D.

We are to show that quadrilateral ABCD is a rectangle.

Now, $\angle$ PAC = $\angle$ ACR

(Alternate angles as l || m and p is a transversal)

So, $\frac{1}{2}\angle \text{PAC} = \frac{1}{2}\angle \text{ACR}$

i.e., $\angle$ BAC = $\angle$ ACD

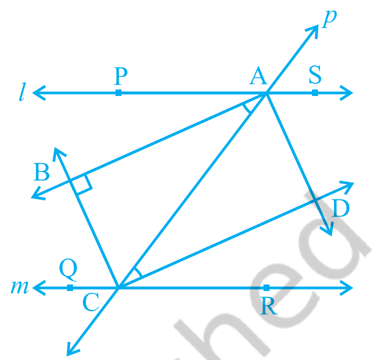

Fig. 8.9

These form a pair of alternate angles for lines AB and DC with AC as transversal and they are equal also.

So, AB || DC

Similarly, BC || AD (Considering $\angle$ ACB and $\angle$ CAD)

Therefore, quadrilateral ABCD is a parallelogram.

Also, $\angle$ PAC + $\angle$ CAS = 180° (Linear pair)

So, $\frac{1}{2}\angle \text{PAC} + \frac{1}{2}\angle \text{CAS} = \frac{1}{2} \times 180° = 90°$

or, $\angle$ BAC + $\angle$ CAD = 90°

or, $\angle$ BAD = 90°

So, ABCD is a parallelogram in which one angle is 90°.

Therefore, ABCD is a rectangle.

Example 5 : Show that the bisectors of angles of a parallelogram form a rectangle.

Solution : Let P, Q, R and S be the points of intersection of the bisectors of $\angle$ A and $\angle$ B, $\angle$ B and $\angle$ C, $\angle$ C and $\angle$ D, and $\angle$ D and $\angle$ A respectively of parallelogram ABCD (see Fig. 8.10).

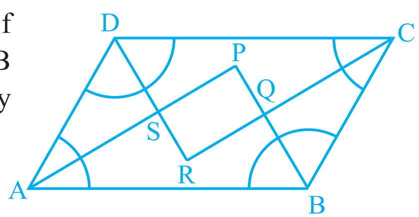

Fig. 8.10

In Δ ASD, what do you observe?

Since DS bisects $\angle$ D and AS bisects $\angle$ A, therefore,

$$\angle \text{DAS} + \angle \text{ADS} = \frac{1}{2}\angle \text{A} + \frac{1}{2}\angle \text{D}$$

$$= \frac{1}{2}(\angle \text{A} + \angle \text{D})$$

$$= \frac{1}{2} \times 180° \quad (\angle \text{A and } \angle \text{D are interior angles on the same side of the transversal})$$

$$= 90°$$

Also, $\angle \text{DAS} + \angle \text{ADS} + \angle \text{DSA} = 180°$ (Angle sum property of a triangle)

or, $90° + \angle \text{DSA} = 180°$

or, $\angle \text{DSA} = 90°$

So, $\angle \text{PSR} = 90°$ (Being vertically opposite to $\angle$ DSA)

Similarly, it can be shown that $\angle$ APB = 90° or $\angle$ SPQ = 90° (as it was shown for $\angle$ DSA). Similarly, $\angle$ PQR = 90° and $\angle$ SRQ = 90°.

So, PQRS is a quadrilateral in which all angles are right angles.

Can we conclude that it is a rectangle? Let us examine. We have shown that $\angle$ PSR = $\angle$ PQR = 90° and $\angle$ SPQ = $\angle$ SRQ = 90°. So both pairs of opposite angles are equal.

Therefore, PQRS is a parallelogram in which one angle (in fact all angles) is 90° and so, PQRS is a rectangle.

EXERCISE 8.1

1. If the diagonals of a parallelogram are equal, then show that it is a rectangle.
2. Show that the diagonals of a square are equal and bisect each other at right angles.
3. Diagonal AC of a parallelogram ABCD bisects $\angle$ A (see Fig. 8.11). Show that
   (i) it bisects $\angle$ C also,
   (ii) ABCD is a rhombus.

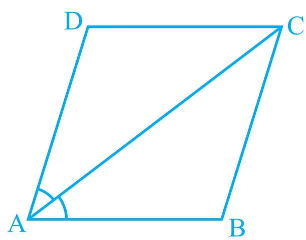

Fig. 8.11

4. ABCD is a rectangle in which diagonal AC bisects $\angle$ A as well as $\angle$ C. Show that: (i) ABCD is a square (ii) diagonal BD bisects $\angle$ B as well as $\angle$ D.

5. In parallelogram ABCD, two points P and Q are taken on diagonal BD such that DP = BQ (see Fig. 8.12). Show that:

   (i) $\Delta\, APD \cong \Delta\, CQB$

   (ii) $AP = CQ$

   (iii) $\Delta\, AQB \cong \Delta\, CPD$

   (iv) $AQ = CP$

   (v) APCQ is a parallelogram

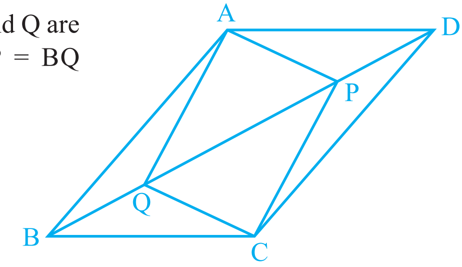

Fig. 8.12

6. ABCD is a parallelogram and AP and CQ are perpendiculars from vertices A and C on diagonal BD (see Fig. 8.13). Show that

   (i) $\Delta\, APB \cong \Delta\, CQD$

   (ii) $AP = CQ$

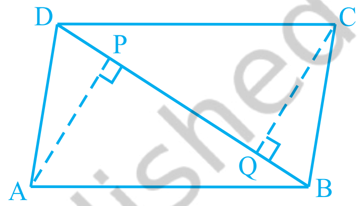

Fig. 8.13

7. ABCD is a trapezium in which AB || CD and AD = BC (see Fig. 8.14). Show that

   (i) $\angle A = \angle B$

   (ii) $\angle C = \angle D$

   (iii) $\Delta\, ABC \cong \Delta\, BAD$

   (iv) diagonal AC = diagonal BD

   [*Hint* : Extend AB and draw a line through C parallel to DA intersecting AB produced at E.]

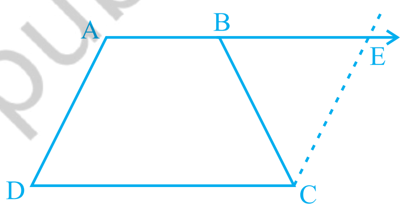

Fig. 8.14

8.2 The Mid-point Theorem

You have studied many properties of a triangle as well as a quadrilateral. Now let us study yet another result which is related to the mid-point of sides of a triangle. Perform the following activity.

Draw a triangle and mark the mid-points E and F of two sides of the triangle. Join the points E and F (see Fig. 8.15).

Measure EF and BC. Measure $\angle$ AEF and $\angle$ ABC.

What do you observe? You will find that :

$$EF = \frac{1}{2}\, BC \text{ and } \angle AEF = \angle ABC$$

so, EF || BC

Repeat this activity with some more triangles.

So, you arrive at the following theorem:

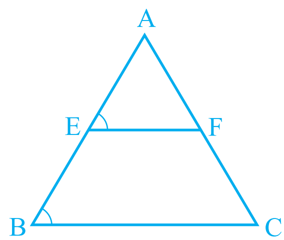

Fig. 8.15

Theorem 8.8 : *The line segment joining the mid-points of two sides of a triangle is parallel to the third side.*

You can prove this theorem using the following clue:

Observe Fig 8.16 in which E and F are mid-points of AB and AC respectively and CD || BA.

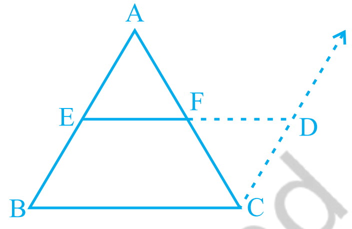

Fig. 8.16

Δ AEF $\cong$ Δ CDF (ASA Rule)

So, EF = DF and BE = AE = DC (Why?)

Therefore, BCDE is a parallelogram. (Why?)

This gives EF || BC.

In this case, also note that EF = $\frac{1}{2}$ ED = $\frac{1}{2}$ BC.

Can you state the converse of Theorem 8.8? Is the converse true?

You will see that converse of the above theorem is also true which is stated as below:

Theorem 8.9 : *The line drawn through the mid-point of one side of a triangle, parallel to another side bisects the third side.*

In Fig 8.17, observe that E is the mid-point of AB, line *l* is passsing through E and is parallel to BC and CM || BA.

Prove that AF = CF by using the congruence of Δ AEF and Δ CDF.

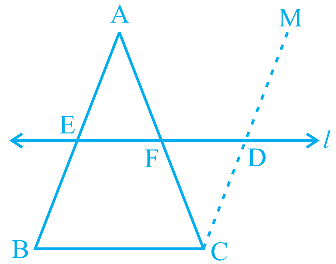

Fig. 8.17

Example 6 : In Δ ABC, D, E and F are respectively the mid-points of sides AB, BC and CA (see Fig. 8.18). Show that Δ ABC is divided into four congruent triangles by joining D, E and F.

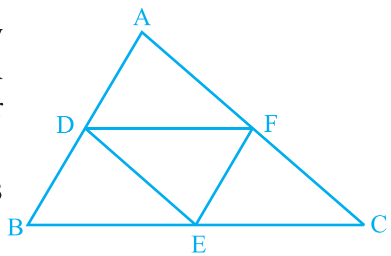

Fig. 8.18

Solution : As D and E are mid-points of sides AB and BC of the triangle ABC, by Theorem 8.8,

DE || AC

Similarly, DF || BC and EF || AB

Therefore ADEF, BDFE and DFCE are all parallelograms.

Now DE is a diagonal of the parallelogram BDFE,

therefore, Δ BDE $\cong$ Δ FED

Similarly Δ DAF $\cong$ Δ FED

and Δ EFC $\cong$ Δ FED

So, all the four triangles are congruent.

Example 7 : l, m and n are three parallel lines intersected by transversals p and q such that l, m and n cut off equal intercepts AB and BC on p (see Fig. 8.19). Show that l, m and n cut off equal intercepts DE and EF on q also.

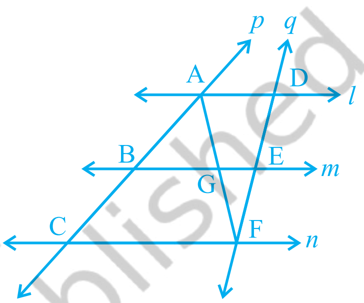

Fig. 8.19

Solution : We are given that AB = BC and have to prove that DE = EF.

Let us join A to F intersecting m at G..

The trapezium ACFD is divided into two triangles; namely Δ ACF and Δ AFD.

In Δ ACF, it is given that B is the mid-point of AC (AB = BC)

and BG || CF (since $m \parallel n$).

So, G is the mid-point of AF (by using Theorem 8.9)

Now, in Δ AFD, we can apply the same argument as G is the mid-point of AF, GE || AD and so by Theorem 8.9, E is the mid-point of DF,

i.e., DE = EF.

In other words, l, m and n cut off equal intercepts on q also.

EXERCISE 8.2

1. ABCD is a quadrilateral in which P, Q, R and S are mid-points of the sides AB, BC, CD and DA (see Fig 8.20). AC is a diagonal. Show that :

   (i) SR || AC and SR = $\frac{1}{2}$ AC

   (ii) PQ = SR

   (iii) PQRS is a parallelogram.

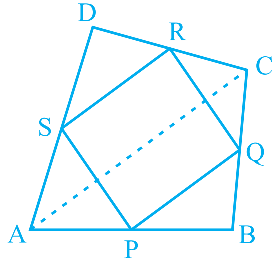

Fig. 8.20

2. ABCD is a rhombus and P, Q, R and S are the mid-points of the sides AB, BC, CD and DA respectively. Show that the quadrilateral PQRS is a rectangle.

3. ABCD is a rectangle and P, Q, R and S are mid-points of the sides AB, BC, CD and DA respectively. Show that the quadrilateral PQRS is a rhombus.

4. ABCD is a trapezium in which AB || DC, BD is a diagonal and E is the mid-point of AD. A line is drawn through E parallel to AB intersecting BC at F (see Fig. 8.21). Show that F is the mid-point of BC.

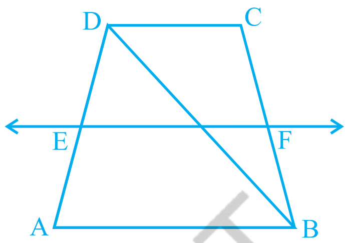

Fig. 8.21

5. In a parallelogram ABCD, E and F are the mid-points of sides AB and CD respectively (see Fig. 8.22). Show that the line segments AF and EC trisect the diagonal BD.

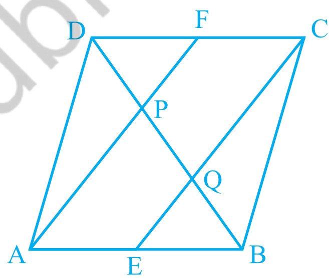

Fig. 8.22

6. ABC is a triangle right angled at C. A line through the mid-point M of hypotenuse AB and parallel to BC intersects AC at D. Show that

(i) D is the mid-point of AC (ii) MD $\perp$ AC

(iii) CM = MA = $\frac{1}{2}$AB

8.3 Summary

In this chapter, you have studied the following points :

1. A diagonal of a parallelogram divides it into two congruent triangles.
2. In a parallelogram,
   (i) opposite sides are equal (ii) opposite angles are equal
   (iii) diagonals bisect each other
3. Diagonals of a rectangle bisect each other and are equal and vice-versa.
4. Diagonals of a rhombus bisect each other at right angles and vice-versa.
5. Diagonals of a square bisect each other at right angles and are equal, and vice-versa.
6. The line-segment joining the mid-points of any two sides of a triangle is parallel to the third side and is half of it.
7. A line through the mid-point of a side of a triangle parallel to another side bisects the third side.

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CHAPTER 9

CIRCLES

9.1 Angle Subtended by a Chord at a Point

You have already studied about circles and its parts in Class VI.

Take a line segment PQ and a point R not on the line containing PQ. Join PR and QR (see Fig. 9.1). Then $\angle$ PRQ is called the angle subtended by the line segment PQ at the point R. What are angles POQ, PRQ and PSQ called in Fig. 9.2? $\angle$ POQ is the angle subtended by the chord PQ at the centre O, $\angle$ PRQ and $\angle$ PSQ are respectively the angles subtended by PQ at points R and S on the major and minor arcs PQ.

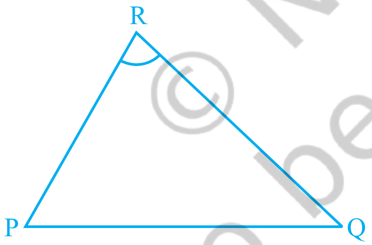

Fig. 9.1

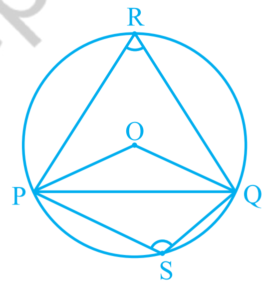

Fig. 9.2

Let us examine the relationship between the size of the chord and the angle subtended by it at the centre. You may see by drawing different chords of a circle and angles subtended by them at the centre that the longer is the chord, the bigger will be the angle subtended by it at the centre. What will happen if you take two equal chords of a circle? Will the angles subtended at the centre be the same or not?

Draw two or more equal chords of a circle and measure the angles subtended by them at the centre (see Fig.9.3). You will find that the angles subtended by them at the centre are equal. Let us give a proof of this fact.

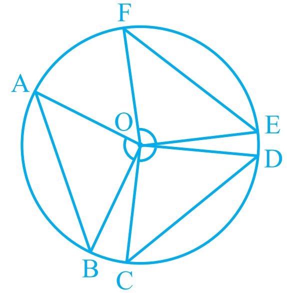

Fig. 9.3

Theorem 9.1 : *Equal chords of a circle subtend equal angles at the centre.*

Proof : You are given two equal chords AB and CD of a circle with centre O (see Fig.9.4). You want to prove that $\angle$ AOB = $\angle$ COD.

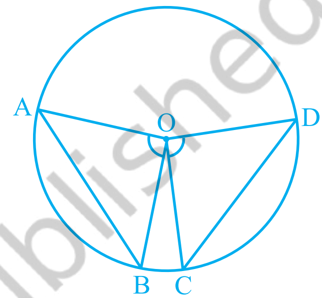

Fig. 9.4

In triangles AOB and COD,

OA = OC (Radii of a circle)

OB = OD (Radii of a circle)

AB = CD (Given)

Therefore, Δ AOB $\cong$ Δ COD (SSS rule)

This gives $\angle$ AOB = $\angle$ COD

(Corresponding parts of congruent triangles)

Remark : For convenience, the abbreviation CPCT will be used in place of 'Corresponding parts of congruent triangles', because we use this very frequently as you will see.

Now if two chords of a circle subtend equal angles at the centre, what can you say about the chords? Are they equal or not? Let us examine this by the following activity:

Take a tracing paper and trace a circle on it. Cut it along the circle to get a disc. At its centre O, draw an angle AOB where A, B are points on the circle. Make another angle POQ at the centre equal to $\angle$AOB. Cut the disc along AB and PQ (see Fig. 9.5). You will get two segments ACB and PRQ of the circle. If you put one on the other, what do you observe? They cover each other, i.e., they are congruent. So AB = PQ.

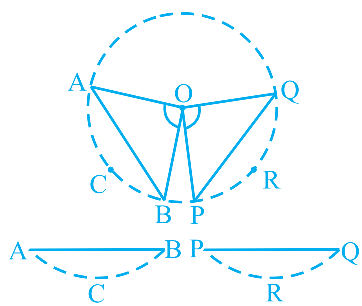

Fig. 9.5

Though you have seen it for this particular case, try it out for other equal angles too. The chords will all turn out to be equal because of the following theorem:

Theorem 9.2 : *If the angles subtended by the chords of a circle at the centre are equal, then the chords are equal.*

The above theorem is the converse of the Theorem 9.1. Note that in Fig. 9.4, if you take $\angle$ AOB = $\angle$ COD, then

$$\Delta \text{ AOB} \cong \Delta \text{ COD (Why?)}$$

Can you now see that AB = CD?

EXERCISE 9.1

1. Recall that two circles are congruent if they have the same radii. Prove that equal chords of congruent circles subtend equal angles at their centres.
2. Prove that if chords of congruent circles subtend equal angles at their centres, then the chords are equal.

9.2 Perpendicular from the Centre to a Chord

Activity : Draw a circle on a tracing paper. Let O be its centre. Draw a chord AB. Fold the paper along a line through O so that a portion of the chord falls on the other. Let the crease cut AB at the point M. Then, $\angle$ OMA = $\angle$ OMB = 90° or OM is perpendicular to AB. Does the point B coincide with A (see Fig.9.6)?

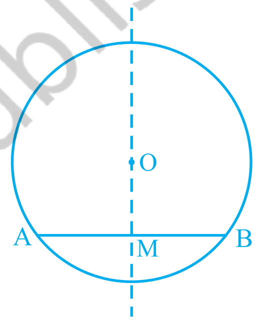

Fig. 9.6

Yes it will. So MA = MB.

Give a proof yourself by joining OA and OB and proving the right triangles OMA and OMB to be congruent. This example is a particular instance of the following result:

Theorem 9.3 : *The perpendicular from the centre of a circle to a chord bisects the chord.*

What is the converse of this theorem? To write this, first let us be clear what is assumed in Theorem 9.3 and what is proved. Given that the perpendicular from the centre of a circle to a chord is drawn and to prove that it bisects the chord. Thus in the converse, what the hypothesis is 'if a line from the centre bisects a chord of a circle' and what is to be proved is 'the line is perpendicular to the chord'. So the converse is:

Theorem 9.4 : *The line drawn through the centre of a circle to bisect a chord is perpendicular to the chord.*

Is this true? Try it for few cases and see. You will see that it is true for these cases. See if it is true, in general, by doing the following exercise. We will write the stages and you give the reasons.

Let AB be a chord of a circle with centre O and O is joined to the mid-point M of AB. You have to prove that OM $\perp$ AB. Join OA and OB (see Fig. 9.7). In triangles OAM and OBM,

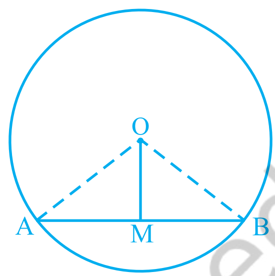

Fig. 9.7

OA = OB (Why ?)

AM = BM (Why ?)

OM = OM (Common)

Therefore, $\Delta OAM \cong \Delta OBM$ (How ?)

This gives $\angle OMA = \angle OMB = 90°$ (Why ?)

9.3 Equal Chords and their Distances from the Centre

Let AB be a line and P be a point. Since there are infinite numbers of points on a line, if you join these points to P, you will get infinitely many line segments PL_1, PL_2, PM, PL_3, PL_4, etc. Which of these is the distance of AB from P? You may think a while and get the answer. Out of these line segments, the perpendicular from P to AB, namely PM in Fig. 9.8, will be the least. In Mathematics, we define this least length PM to be **the distance of AB from P**. So you may say that:

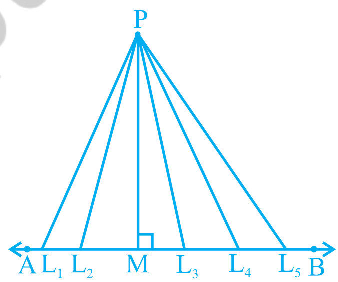

Fig. 9.8

The length of the perpendicular from a point to a line is the distance of the line from the point.

Note that if the point lies on the line, the distance of the line from the point is zero.

A circle can have infinitely many chords. You may observe by drawing chords of a circle that longer chord is nearer to the centre than the smaller chord. You may observe it by drawing several chords of a circle of different lengths and measuring their distances from the centre. What is the distance of the diameter, which is the longest chord from the centre? Since the centre lies on it, the distance is zero. Do you think that there is some relationship between the length of chords and their distances from the centre? Let us see if this is so.

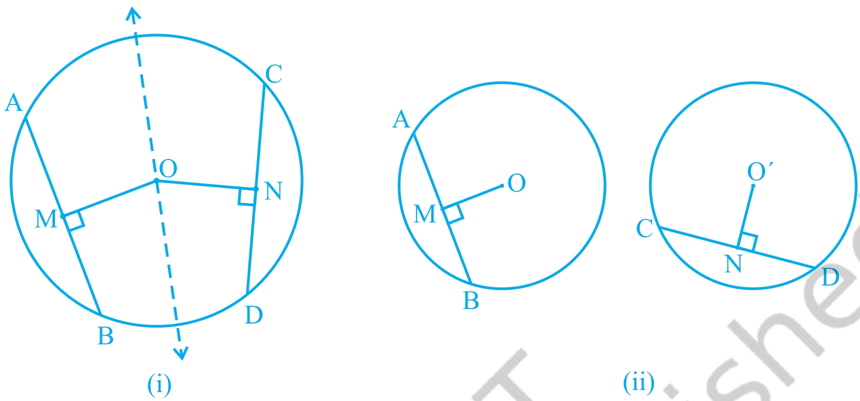

Fig. 9.9

Activity : Draw a circle of any radius on a tracing paper. Draw two equal chords AB and CD of it and also the perpendiculars OM and ON on them from the centre O. Fold the figure so that D falls on B and C falls on A [see Fig.9.9 (i)]. You may observe that O lies on the crease and N falls on M. Therefore, OM = ON. Repeat the activity by drawing congruent circles with centres O and O′ and taking equal chords AB and CD one on each. Draw perpendiculars OM and O′N on them [see Fig. 9.9(ii)]. Cut one circular disc and put it on the other so that AB coincides with CD. Then you will find that O coincides with O′ and M coincides with N. In this way you verified the following:

Theorem 9.5 : *Equal chords of a circle* (*or of congruent circles*) *are equidistant from the centre* (*or centres*).

Next, it will be seen whether the converse of this theorem is true or not. For this, draw a circle with centre O. From the centre O, draw two line segments OL and OM of equal length and lying inside the circle [see Fig. 9.10(i)]. Then draw chords PQ and RS of the circle perpendicular to OL and OM respectively [see Fig 9.10(ii)]. Measure the lengths of PQ and RS. Are these different? No, both are equal. Repeat the activity for more equal line segments and drawing the chords perpendicular to them. This verifies the converse of the Theorem 9.5 which is stated as follows:

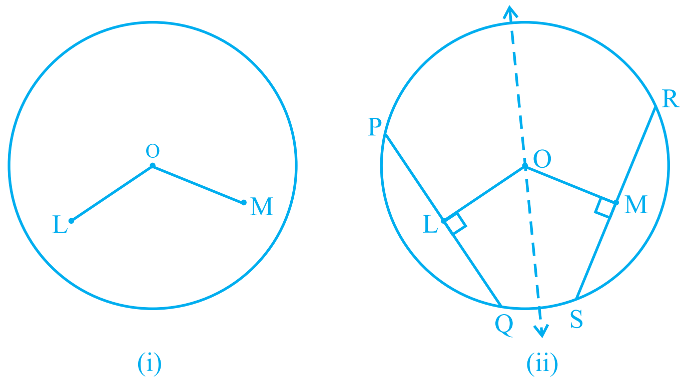

Fig. 9.10

Theorem 9.6 : *Chords equidistant from the centre of a circle are equal in length.*

We now take an example to illustrate the use of the above results:

Example 1 : If two intersecting chords of a circle make equal angles with the diameter passing through their point of intersection, prove that the chords are equal.

Solution : Given that AB and CD are two chords of a circle, with centre O intersecting at a point E. PQ is a diameter through E, such that $\angle$ AEQ = $\angle$ DEQ (see Fig.9.11). You have to prove that AB = CD. Draw perpendiculars OL and OM on chords AB and CD, respectively. Now

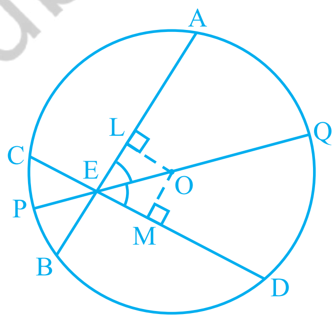

Fig. 9.11

$\angle$ LOE = 180° – 90° – $\angle$ LEO = 90° – $\angle$ LEO
(Angle sum property of a triangle)

= 90° – $\angle$ AEQ = 90° – $\angle$ DEQ

= 90° – $\angle$ MEO = $\angle$ MOE

In triangles OLE and OME,

	$\angle$ LEO = $\angle$ MEO	(Why ?)
	$\angle$ LOE = $\angle$ MOE	(Proved above)
	EO = EO	(Common)
Therefore,	Δ OLE $\cong$ Δ OME	(Why ?)
This gives	OL = OM	(CPCT)
So,	AB = CD	(Why ?)

EXERCISE 9.2

1. Two circles of radii 5 cm and 3 cm intersect at two points and the distance between their centres is 4 cm. Find the length of the common chord.
2. If two equal chords of a circle intersect within the circle, prove that the segments of one chord are equal to corresponding segments of the other chord.
3. If two equal chords of a circle intersect within the circle, prove that the line joining the point of intersection to the centre makes equal angles with the chords.
4. If a line intersects two concentric circles (circles with the same centre) with centre O at A, B, C and D, prove that AB = CD (see Fig. 9.12).

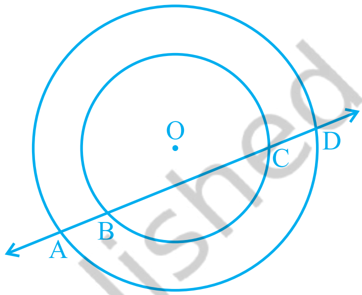

Fig. 9.12

5. Three girls Reshma, Salma and Mandip are playing a game by standing on a circle of radius 5m drawn in a park. Reshma throws a ball to Salma, Salma to Mandip, Mandip to Reshma. If the distance between Reshma and Salma and between Salma and Mandip is 6m each, what is the distance between Reshma and Mandip?
6. A circular park of radius 20m is situated in a colony. Three boys Ankur, Syed and David are sitting at equal distance on its boundary each having a toy telephone in his hands to talk each other. Find the length of the string of each phone.

9.4 Angle Subtended by an Arc of a Circle

You have seen that the end points of a chord other than diameter of a circle cuts it into two arcs – one major and other minor. If you take two equal chords, what can you say about the size of arcs? Is one arc made by first chord equal to the corresponding arc made by another chord? In fact, they are more than just equal in length. They are congruent in the sense that if one arc is put on the other, without bending or twisting, one superimposes the other completely.

You can verify this fact by cutting the arc, corresponding to the chord CD from the circle along CD and put it on the corresponding arc made by equal chord AB. You will find that the arc CD superimpose the arc AB completely (see Fig. 9.13). This shows that equal chords make congruent arcs and conversely congruent arcs make equal chords of a circle. You can state it as follows:

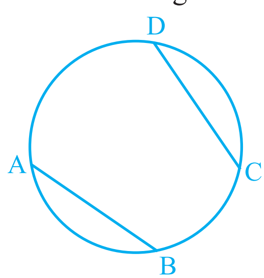

Fig. 9.13

If two chords of a circle are equal, then their corresponding arcs are congruent and conversely, if two arcs are congruent, then their corresponding chords are equal.

Also the angle subtended by an arc at the centre is defined to be angle subtended by the corresponding chord at the centre in the sense that the minor arc subtends the angle and the major arc subtends the reflex angle. Therefore, in Fig 9.14, the angle subtended by the minor arc PQ at O is ∠POQ and the angle subtended by the major arc PQ at O is reflex angle POQ.

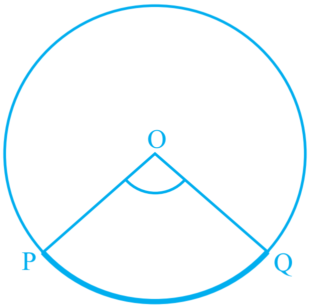

Fig. 9.14

In view of the property above and Theorem 9.1, the following result is true:

Congruent arcs (or equal arcs) of a circle subtend equal angles at the centre.

Therefore, the angle subtended by a chord of a circle at its centre is equal to the angle subtended by the corresponding (minor) arc at the centre. The following theorem gives the relationship between the angles subtended by an arc at the centre and at a point on the circle.

Theorem 9.7 : *The angle subtended by an arc at the centre is double the angle subtended by it at any point on the remaining part of the circle.*

Proof : Given an arc PQ of a circle subtending angles POQ at the centre O and PAQ at a point A on the remaining part of the circle. We need to prove that ∠ POQ = 2 ∠ PAQ.

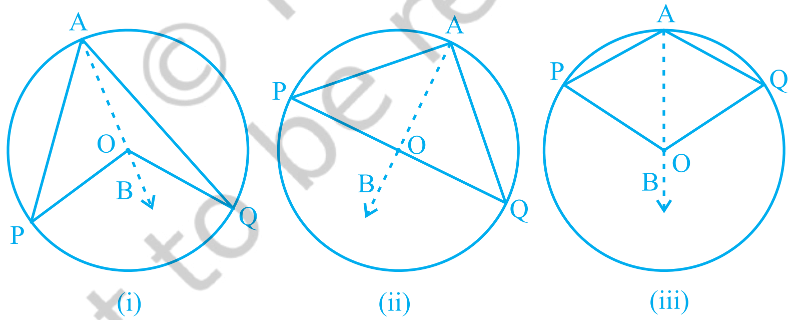

Fig. 9.15

Consider the three different cases as given in Fig. 9.15. In (i), arc PQ is minor; in (ii), arc PQ is a semicircle and in (iii), arc PQ is major.

Let us begin by joining AO and extending it to a point B.

In all the cases,

$$\angle BOQ = \angle OAQ + \angle AQO$$

because an exterior angle of a triangle is equal to the sum of the two interior opposite angles.

Also in Δ OAQ,

$$OA = OQ \quad \text{(Radii of a circle)}$$

Therefore, $\angle OAQ = \angle OQA$ (Theorem 7.5)

This gives $\angle BOQ = 2 \angle OAQ$ (1)

Similarly, $\angle BOP = 2 \angle OAP$ (2)

From (1) and (2), $\angle BOP + \angle BOQ = 2(\angle OAP + \angle OAQ)$

This is the same as $\angle POQ = 2 \angle PAQ$ (3)

For the case (iii), where PQ is the major arc, (3) is replaced by

$$\text{reflex angle } POQ = 2 \angle PAQ$$

Remark : Suppose we join points P and Q and form a chord PQ in the above figures. Then $\angle PAQ$ is also called the angle formed in the segment PAQP.

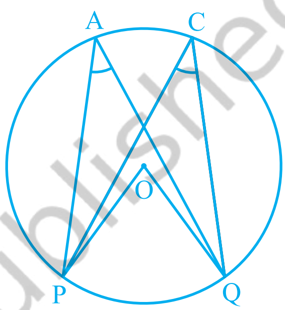

Fig. 9.16

In Theorem 9.7, A can be any point on the remaining part of the circle. So if you take any other point C on the remaining part of the circle (see Fig. 9.16), you have

$$\angle POQ = 2 \angle PCQ = 2 \angle PAQ$$

Therefore, $\angle PCQ = \angle PAQ.$

This proves the following:

Theorem 9.8 : *Angles in the same segment of a circle are equal.*

Again let us discuss the case (ii) of Theorem 10.8 separately. Here $\angle PAQ$ is an angle in the segment, which is a semicircle. Also, $\angle PAQ = \frac{1}{2} \angle POQ = \frac{1}{2} \times 180° = 90°$.

If you take any other point C on the semicircle, again you get that

$$\angle PCQ = 90°$$

Therefore, you find another property of the circle as:

Angle in a semicircle is a right angle.

The converse of Theorem 9.8 is also true. It can be stated as:

Theorem 9.9 : *If a line segment joining two points subtends equal angles at two other points lying on the same side of the line containing the line segment, the four points lie on a circle (i.e. they are concyclic).*

You can see the truth of this result as follows:

In Fig. 9.17, AB is a line segment, which subtends equal angles at two points C and D. That is

$$\angle ACB = \angle ADB$$

To show that the points A, B, C and D lie on a circle let us draw a circle through the points A, C and B. Suppose it does not pass through the point D. Then it will intersect AD (or extended AD) at a point, say E (or E′).

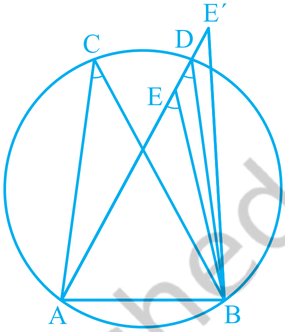

Fig. 9.17

If points A, C, E and B lie on a circle,

$$\angle ACB = \angle AEB \qquad \text{(Why?)}$$

But it is given that $\angle ACB = \angle ADB$.

Therefore, $\angle AEB = \angle ADB$.

This is not possible unless E coincides with D. (Why?)

Similarly, E′ should also coincide with D.

9.5 Cyclic Quadrilaterals

A quadrilateral ABCD is called *cyclic* if all the four vertices of it lie on a circle (see Fig 9.18). You will find a peculiar property in such quadrilaterals. Draw several cyclic quadrilaterals of different sides and name each of these as ABCD. (This can be done by drawing several circles of different radii and taking four points on each of them.) Measure the opposite angles and write your observations in the following table.

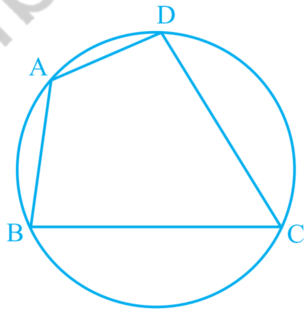

Fig. 9.18

S.No. of Quadrilateral	$\angle A$	$\angle B$	$\angle C$	$\angle D$	$\angle A + \angle C$	$\angle B + \angle D$
1.						
2.						
3.						
4.						
5.						
6.						

What do you infer from the table?

You find that $\angle A + \angle C = 180°$ and $\angle B + \angle D = 180°$, neglecting the error in measurements. This verifies the following:

Theorem 9.10 : *The sum of either pair of opposite angles of a cyclic quadrilateral is* 180º.

In fact, the converse of this theorem, which is stated below is also true.

Theorem 9.11 : *If the sum of a pair of opposite angles of a quadrilateral is* 180º, *the quadrilateral is cyclic.*

You can see the truth of this theorem by following a method similar to the method adopted for Theorem 9.9.

Example 2 : In Fig. 9.19, AB is a diameter of the circle, CD is a chord equal to the radius of the circle. AC and BD when extended intersect at a point E. Prove that $\angle$ AEB = 60°.

Solution : Join OC, OD and BC.

Triangle ODC is equilateral (Why?)

Therefore, $\angle \text{COD} = 60°$

Now, $\angle \text{CBD} = \frac{1}{2} \angle \text{COD}$ (Theorem 9.7)

This gives $\angle \text{CBD} = 30°$

Again, $\angle \text{ACB} = 90°$ (Why ?)

So, $\angle \text{BCE} = 180° - \angle \text{ACB} = 90°$

Which gives $\angle \text{CEB} = 90° - 30° = 60°$, i.e., $\angle \text{AEB} = 60°$

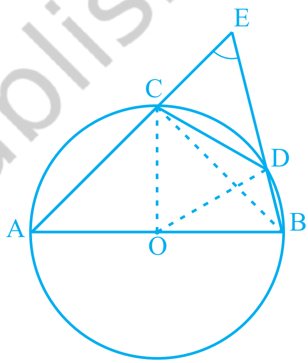

Fig. 9.19

Example 3 : In Fig 9.20, ABCD is a cyclic quadrilateral in which AC and BD are its diagonals. If $\angle$ DBC = 55° and $\angle$ BAC = 45°, find $\angle$ BCD.

Solution : $\angle \text{CAD} = \angle \text{DBC} = 55°$
(Angles in the same segment)

Therefore, $\angle \text{DAB} = \angle \text{CAD} + \angle \text{BAC}$
$= 55° + 45° = 100°$

But $\angle \text{DAB} + \angle \text{BCD} = 180°$
(Opposite angles of a cyclic quadrilateral)

So, $\angle \text{BCD} = 180° - 100° = 80°$

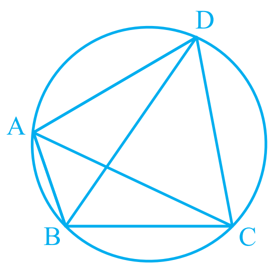

Fig. 9.20

Example 4 : Two circles intersect at two points A and B. AD and AC are diameters to the two circles (see Fig. 9.21). Prove that B lies on the line segment DC.

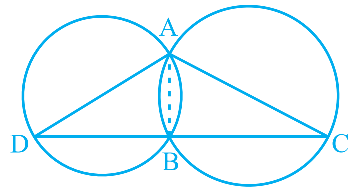

Fig. 9.21

Solution : Join AB.

$$\angle ABD = 90^\circ \quad \text{(Angle in a semicircle)}$$

$$\angle ABC = 90^\circ \quad \text{(Angle in a semicircle)}$$

So, $\angle ABD + \angle ABC = 90^\circ + 90^\circ = 180^\circ$

Therefore, DBC is a line. That is B lies on the line segment DC.

Example 5 : Prove that the quadrilateral formed (if possible) by the internal angle bisectors of any quadrilateral is cyclic.

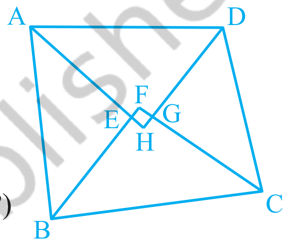

Fig. 9.22

Solution : In Fig. 9.22, ABCD is a quadrilateral in which the angle bisectors AH, BF, CF and DH of internal angles A, B, C and D respectively form a quadrilateral EFGH.

Now, $\angle FEH = \angle AEB = 180^\circ - \angle EAB - \angle EBA$ (Why ?)

$$= 180^\circ - \frac{1}{2}(\angle A + \angle B)$$

and $\angle FGH = \angle CGD = 180^\circ - \angle GCD - \angle GDC$ (Why ?)

$$= 180^\circ - \frac{1}{2}(\angle C + \angle D)$$

Therefore, $\angle FEH + \angle FGH = 180^\circ - \frac{1}{2}(\angle A + \angle B) + 180^\circ - \frac{1}{2}(\angle C + \angle D)$

$$= 360^\circ - \frac{1}{2}(\angle A + \angle B + \angle C + \angle D) = 360^\circ - \frac{1}{2} \times 360^\circ$$

$$= 360^\circ - 180^\circ = 180^\circ$$

Therefore, by Theorem 9.11, the quadrilateral EFGH is cyclic.

EXERCISE 9.3

1. In Fig. 9.23, A,B and C are three points on a circle with centre O such that $\angle BOC = 30^\circ$ and $\angle AOB = 60^\circ$. If D is a point on the circle other than the arc ABC, find $\angle ADC$.

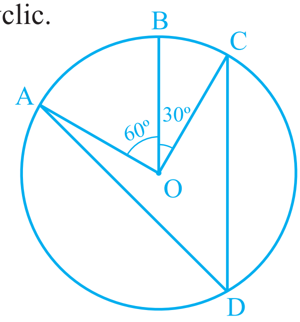

Fig. 9.23

2. A chord of a circle is equal to the radius of the circle. Find the angle subtended by the chord at a point on the minor arc and also at a point on the major arc.

3. In Fig. 9.24, ∠ PQR = 100°, where P, Q and R are points on a circle with centre O. Find ∠ OPR.

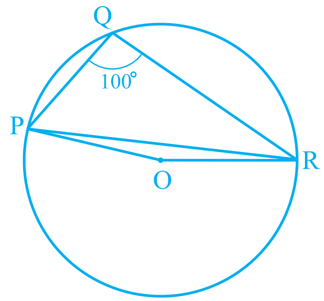

Fig. 9.24

4. In Fig. 9.25, ∠ ABC = 69°, ∠ ACB = 31°, find ∠ BDC.

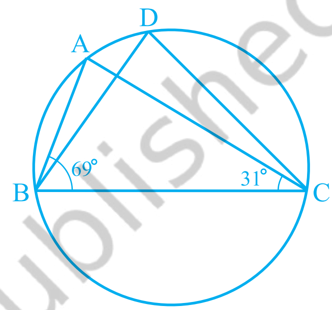

Fig. 9.25

5. In Fig. 9.26, A, B, C and D are four points on a circle. AC and BD intersect at a point E such that ∠ BEC = 130° and ∠ ECD = 20°. Find ∠ BAC.

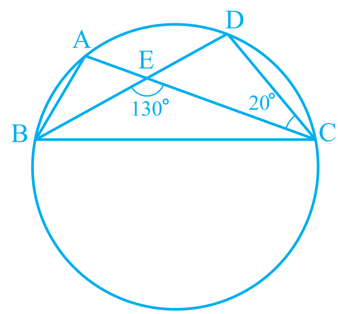

Fig. 9.26

6. ABCD is a cyclic quadrilateral whose diagonals intersect at a point E. If ∠ DBC = 70°, ∠ BAC is 30°, find ∠ BCD. Further, if AB = BC, find ∠ ECD.

7. If diagonals of a cyclic quadrilateral are diameters of the circle through the vertices of the quadrilateral, prove that it is a rectangle.

8. If the non-parallel sides of a trapezium are equal, prove that it is cyclic.

9. Two circles intersect at two points B and C. Through B, two line segments ABD and PBQ are drawn to intersect the circles at A, D and P, Q respectively (see Fig. 9.27). Prove that $\angle ACP = \angle QCD$.

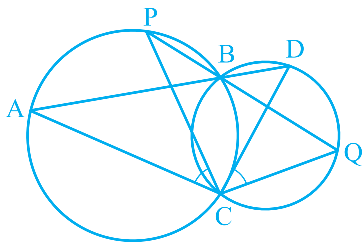

Fig. 9.27

10. If circles are drawn taking two sides of a triangle as diameters, prove that the point of intersection of these circles lie on the third side.

11. ABC and ADC are two right triangles with common hypotenuse AC. Prove that $\angle CAD = \angle CBD$.

12. Prove that a cyclic parallelogram is a rectangle.

9.6 Summary

In this chapter, you have studied the following points:

1. A circle is the collection of all points in a plane, which are equidistant from a fixed point in the plane.

2. Equal chords of a circle (or of congruent circles) subtend equal angles at the centre.

3. If the angles subtended by two chords of a circle (or of congruent circles) at the centre (corresponding centres) are equal, the chords are equal.

4. The perpendicular from the centre of a circle to a chord bisects the chord.

5. The line drawn through the centre of a circle to bisect a chord is perpendicular to the chord.

6. Equal chords of a circle (or of congruent circles) are equidistant from the centre (or corresponding centres).

7. Chords equidistant from the centre (or corresponding centres) of a circle (or of congruent circles) are equal.

8. If two arcs of a circle are congruent, then their corresponding chords are equal and conversely if two chords of a circle are equal, then their corresponding arcs (minor, major) are congruent.

9. Congruent arcs of a circle subtend equal angles at the centre.

10. The angle subtended by an arc at the centre is double the angle subtended by it at any point on the remaining part of the circle.

11. Angles in the same segment of a circle are equal.

12. Angle in a semicircle is a right angle.

13. If a line segment joining two points subtends equal angles at two other points lying on the same side of the line containing the line segment, the four points lie on a circle.

14. The sum of either pair of opposite angles of a cyclic quadrilateral is 180^0.

15. If sum of a pair of opposite angles of a quadrilateral is 180^0, the quadrilateral is cyclic.

CHAPTER 10

HERON'S FORMULA

10.1 Area of a Triangle — by Heron's Formula

We know that the area of triangle when its height is given, is $\frac{1}{2}$ × base × height. Now suppose that we know the lengths of the sides of a scalene triangle and not the height. Can you still find its area? For instance, you have a triangular park whose sides are 40 m, 32 m, and 24 m. How will you calculate its area? Definitely if you want to apply the formula, you will have to calculate its height. But we do not have a clue to calculate the height. Try doing so. If you are not able to get it, then go to the next section.

Heron was born in about 10AD possibly in Alexandria in Egypt. He worked in applied mathematics. His works on mathematical and physical subjects are so numerous and varied that he is considered to be an encyclopedic writer in these fields. His geometrical works deal largely with problems on mensuration written in three books. Book I deals with the area of squares, rectangles, triangles, trapezoids (trapezia), various other specialised quadrilaterals, the regular polygons, circles, surfaces of cylinders, cones, spheres etc. In this book, Heron has derived the famous formula for the area of a triangle in terms of its three sides.

Heron (10 C.E. – 75 C.E.)

Fig. 10.1

The formula given by Heron about the area of a triangle, is also known as *Hero's formula*. It is stated as:

$$\textbf{Area of a triangle} = \sqrt{s(s-a)(s-b)(s-c)} \qquad \text{(I)}$$

where a, b and c are the sides of the triangle, and s = *semi-perimeter, i.e., half the perimeter of the triangle* $= \dfrac{a+b+c}{2}$,

This formula is helpful where it is not possible to find the height of the triangle easily. Let us apply it to calculate the area of the triangular park ABC, mentioned above (see Fig. 10.2).

Let us take $a = 40$ m, $b = 24$ m, $c = 32$ m,

so that we have $s = \dfrac{40 + 24 + 32}{2}$ m = 48 m.

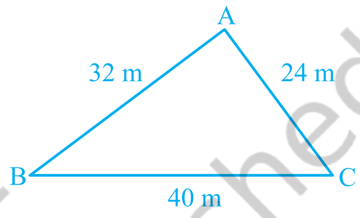

Fig. 10.2

$s - a = (48 - 40)$ m = 8 m,

$s - b = (48 - 24)$ m = 24 m,

$s - c = (48 - 32)$ m = 16 m.

Therefore, area of the park ABC

$= \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{48 \times 8 \times 24 \times 16}\ \text{m}^2 = 384\ \text{m}^2$

We see that $32^2 + 24^2 = 1024 + 576 = 1600 = 40^2$. Therefore, the sides of the park make a right triangle. The largest side, i.e., BC which is 40 m will be the hypotenuse and the angle between the sides AB and AC will be 90°.

We can check that the area of the park is $\dfrac{1}{2} \times 32 \times 24\ \text{m}^2 = 384\ \text{m}^2$.

We find that the area we have got is the same as we found by using Heron's formula.

Now using Heron's formula, you verify this fact by finding the areas of other triangles discussed earlier *viz.*,

(i) equilateral triangle with side 10 cm.

(ii) isosceles triangle with unequal side as 8 cm and each equal side as 5 cm.

You will see that

For (i), we have $s = \dfrac{10+10+10}{2}$ cm = 15 cm.

Area of triangle $= \sqrt{15(15-10)\,(15-10)\,(15-10)}\ \text{cm}^2$

$$= \sqrt{15 \times 5 \times 5 \times 5}\ \text{cm}^2 = 25\sqrt{3}\ \text{cm}^2$$

For (ii), we have $s = \dfrac{8+5+5}{2}$ cm = 9 cm

Area of triangle = $\sqrt{9(9-8)\,(9-5)\,(9-5)}\ \text{cm}^2 = \sqrt{9 \times 1 \times 4 \times 4}\ \text{cm}^2 = 12\ \text{cm}^2$.

Let us now solve some more examples:

Example 1 : Find the area of a triangle, two sides of which are 8 cm and 11 cm and the perimeter is 32 cm (see Fig. 10.3).

Solution : Here we have perimeter of the triangle = 32 cm, a = 8 cm and b = 11 cm.

Third side c = 32 cm – (8 + 11) cm = 13 cm

So, $2s = 32$, i.e., $s = 16$ cm,

$s - a = (16 - 8)$ cm = 8 cm,

$s - b = (16 - 11)$ cm = 5 cm,

$s - c = (16 - 13)$ cm = 3 cm.

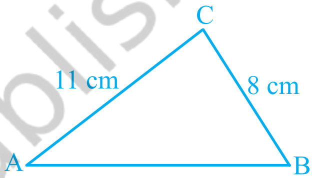

Fig. 10.3

Therefore, area of the triangle $= \sqrt{s(s-a)\,(s-b)\,(s-c)}$

$$= \sqrt{16 \times 8 \times 5 \times 3}\ \text{cm}^2 = 8\sqrt{30}\ \text{cm}^2$$

Example 2 : A triangular park ABC has sides 120m, 80m and 50m (see Fig. 10.4). A gardener *Dhania* has to put a fence all around it and also plant grass inside. How much area does she need to plant? Find the cost of fencing it with barbed wire at the rate of ₹20 per metre leaving a space 3m wide for a gate on one side.

Solution : For finding area of the park, we have

$2s$ = 50 m + 80 m + 120 m = 250 m.

i.e., s = 125 m

Now, $s - a = (125 - 120)$ m = 5 m,

$s - b = (125 - 80)$ m = 45 m,

$s - c = (125 - 50)$ m = 75 m.

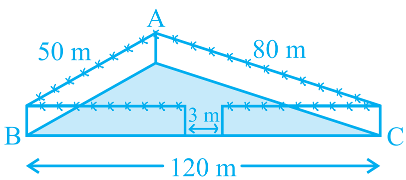

Fig. 10.4

Therefore, area of the park = $\sqrt{s(s-a)(s-b)(s-c)}$

$$= \sqrt{125 \times 5 \times 45 \times 75} \text{ m}^2$$

$$= 375\sqrt{15} \text{ m}^2$$

Also, perimeter of the park = AB + BC + CA = 250 m

Therefore, length of the wire needed for fencing = 250 m – 3 m (to be left for gate)

= 247 m

And so the cost of fencing = ₹20 × 247 = ₹4940

Example 3 : The sides of a triangular plot are in the ratio of 3 : 5 : 7 and its perimeter is 300 m. Find its area.

Solution : Suppose that the sides, in metres, are $3x$, $5x$ and $7x$ (see Fig. 10.5).

Then, we know that $3x + 5x + 7x = 300$ (perimeter of the triangle)

Therefore, $15x = 300$, which gives $x = 20$.

So the sides of the triangle are 3 × 20 m, 5 × 20 m and 7 × 20 m

i.e., 60 m, 100 m and 140 m.

Can you now find the area [Using Heron's formula]?

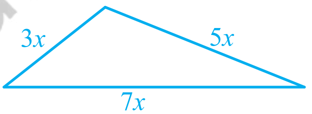

Fig. 10.5

We have $s = \dfrac{60 + 100 + 140}{2}$ m = 150 m,

and area will be $\sqrt{150(150-60)(150-100)(150-140)}$ m²

$$= \sqrt{150 \times 90 \times 50 \times 10} \text{ m}^2$$

$$= 1500\sqrt{3} \text{ m}^2$$

EXERCISE 10.1

1. A traffic signal board, indicating 'SCHOOL AHEAD', is an equilateral triangle with side 'a'. Find the area of the signal board, using Heron's formula. If its perimeter is 180 cm, what will be the area of the signal board?

2. The triangular side walls of a flyover have been used for advertisements. The sides of the walls are 122 m, 22 m and 120 m (see Fig. 10.6). The advertisements yield an earning of ₹ 5000 per m^2 per year. A company hired one of its walls for 3 months. How much rent did it pay?

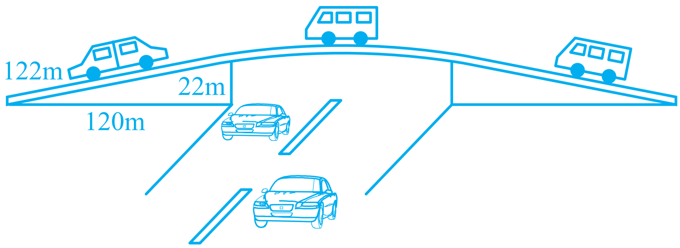

Fig. 10.6

3. There is a slide in a park. One of its side walls has been painted in some colour with a message "KEEP THE PARK GREEN AND CLEAN" (see Fig. 10.7). If the sides of the wall are 15 m, 11 m and 6 m, find the area painted in colour.

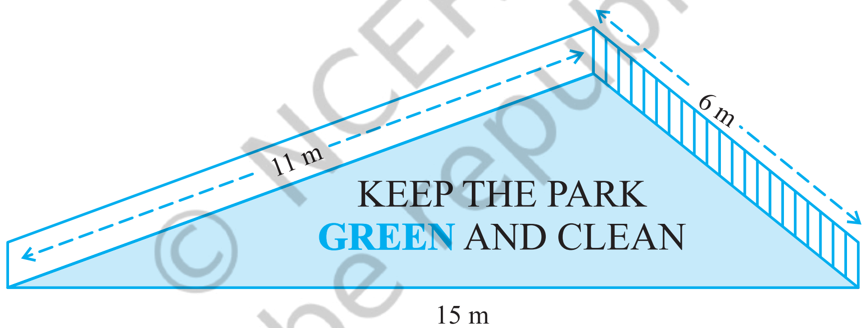

Fig. 10.7

4. Find the area of a triangle two sides of which are 18cm and 10cm and the perimeter is 42cm.

5. Sides of a triangle are in the ratio of 12 : 17 : 25 and its perimeter is 540cm. Find its area.

6. An isosceles triangle has perimeter 30 cm and each of the equal sides is 12 cm. Find the area of the triangle.

10.2 Summary

In this chapter, you have studied the following points :

1. Area of a triangle with its sides as a, b and c is calculated by using Heron's formula, stated as

$$\text{Area of triangle} = \sqrt{s(s-a)(s-b)(s-c)}$$

where
$$s = \frac{a+b+c}{2}$$

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CHAPTER 11

SURFACE AREAS AND VOLUMES

11.1 Surface Area of a Right Circular Cone

We have already studied the surface areas of cube, cuboid and cylinder. We will now study the surface area of cone.

So far, we have been generating solids by stacking up congruent figures. Incidentally, such figures are called *prisms*. Now let us look at another kind of solid which is not a prism (These kinds of solids are called *pyramids*.). Let us see how we can generate them.

Activity : Cut out a right-angled triangle ABC right angled at B. Paste a long thick string along one of the perpendicular sides say AB of the triangle [see Fig. 11.1(a)]. Hold the string with your hands on either sides of the triangle and rotate the triangle about the string a number of times. What happens? Do you recognize the shape that the triangle is forming as it rotates around the string [see Fig. 11.1(b)]? Does it remind you of the time you had eaten an ice-cream heaped into a container of that shape [see Fig. 11.1 (c) and (d)]?

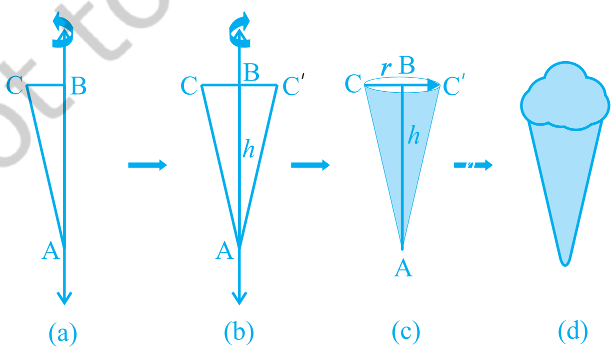

Fig. 11.1

This is called a *right circular cone*. In Fig. 11.1(c) of the right circular cone, the point A is called the vertex, AB is called the height, BC is called the *radius* and AC is called the slant height of the cone. Here B will be the centre of circular base of the cone. The height, radius and slant height of the cone are usually denoted by h, r and l respectively. Once again, let us see what kind of cone we can *not* call a right circular cone. Here, you are (see Fig. 11.2)! What you see in these figures are not right circular cones; because in (a), the line joining its vertex to the centre of its base is not at right angle to the base, and in (b) the base is not circular.

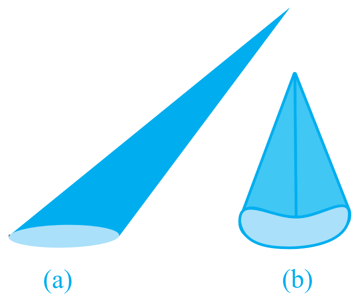

Fig. 11.2

As in the case of cylinder, since we will be studying only about right circular cones, remember that by 'cone' in this chapter, we shall mean a 'right circular cone.'

Activity : (i) Cut out a neatly made paper cone that does not have any overlapped paper, straight along its side, and opening it out, to see the shape of paper that forms the surface of the cone. (The line along which you cut the cone is the *slant height* of the cone which is represented by l). It looks like a part of a round cake.

(ii) If you now bring the sides marked A and B at the tips together, you can see that the curved portion of Fig. 11.3 (c) will form the circular base of the cone.

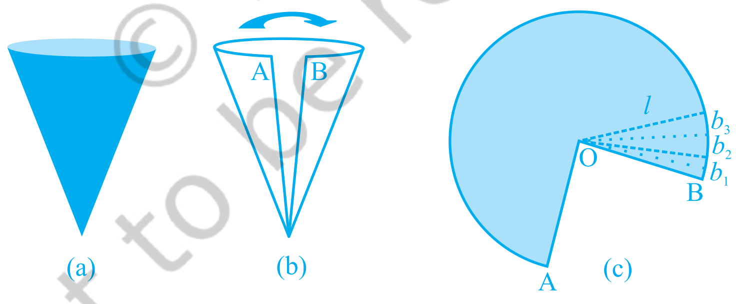

Fig. 11.3

(iii) If the paper like the one in Fig. 11.3 (c) is now cut into hundreds of little pieces, along the lines drawn from the point O, each cut portion is almost a small triangle, whose height is the slant height l of the cone.

(iv) Now the area of each triangle = $\frac{1}{2}$ × base of each triangle × l.

So, area of the entire piece of paper

$= \text{sum of the areas of all the triangles}$

$$= \frac{1}{2}b_1 l + \frac{1}{2}b_2 l + \frac{1}{2}b_3 l + \cdots = \frac{1}{2} l \left(b_1 + b_2 + b_3 + \cdots\right)$$

$$= \frac{1}{2} \times l \times \text{length of entire curved boundary of Fig. 11.3(c)}$$

(as $b_1 + b_2 + b_3 + \ldots$ makes up the curved portion of the figure)

But the curved portion of the figure makes up the perimeter of the base of the cone and the circumference of the base of the cone = $2\pi r$, where r is the base radius of the cone.

So, $\boxed{\textbf{Curved Surface Area of a Cone} = \dfrac{1}{2} \times l \times 2\pi r = \pi r l}$

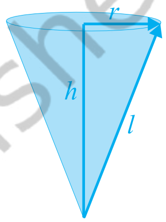

Fig. 11.4

where r is its base radius and l its slant height.

Note that $l^2 = r^2 + h^2$ (as can be seen from Fig. 11.4), by applying Pythagoras Theorem. Here h is the *height* of the cone.

Therefore, $l = \sqrt{r^2 + h^2}$

Now if the base of the cone is to be closed, then a circular piece of paper of radius r is also required whose area is πr^2.

So, $\boxed{\textbf{Total Surface Area of a Cone} = \pi r l + \pi r^2 = \pi r(l + r)}$

Example 1 : Find the curved surface area of a right circular cone whose slant height is 10 cm and base radius is 7 cm.

Solution : Curved surface area $= \pi r l$

$$= \frac{22}{7} \times 7 \times 10 \text{ cm}^2$$

$$= 220 \text{ cm}^2$$

Example 2 : The height of a cone is 16 cm and its base radius is 12 cm. Find the curved surface area and the total surface area of the cone (Use $\pi = 3.14$).

Solution : Here, $h = 16$ cm and $r = 12$ cm.

So, from $l^2 = h^2 + r^2$, we have

$$l = \sqrt{16^2 + 12^2} \text{ cm} = 20 \text{ cm}$$

So, curved surface area = $\pi r l$

$= 3.14 \times 12 \times 20 \text{ cm}^2$

$= 753.6 \text{ cm}^2$

Further, total surface area $= \pi r l + \pi r^2$

$= (753.6 + 3.14 \times 12 \times 12) \text{ cm}^2$

$= (753.6 + 452.16) \text{ cm}^2$

$= 1205.76 \text{ cm}^2$

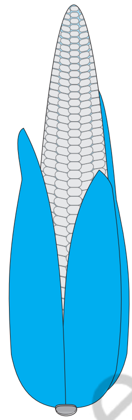

Fig. 11.5

Example 3 : A corn cob (see Fig. 11.5), shaped somewhat like a cone, has the radius of its broadest end as 2.1 cm and length (height) as 20 cm. If each 1 cm^2 of the surface of the cob carries an average of four grains, find how many grains you would find on the entire cob.

Solution : Since the grains of corn are found only on the curved surface of the corn cob, we would need to know the curved surface area of the corn cob to find the total number of grains on it. In this question, we are given the height of the cone, so we need to find its slant height.

Here, $l = \sqrt{r^2 + h^2} = \sqrt{(2.1)^2 + 20^2}$ cm

$= \sqrt{404.41}$ cm = 20.11 cm

Therefore, the curved surface area of the corn cob = $\pi r l$

$$= \frac{22}{7} \times 2.1 \times 20.11 \text{ cm}^2 = 132.726 \text{ cm}^2 = 132.73 \text{ cm}^2 \text{ (approx.)}$$

Number of grains of corn on 1 cm^2 of the surface of the corn cob = 4

Therefore, number of grains on the entire curved surface of the cob

$= 132.73 \times 4 = 530.92 = 531$ (approx.)

So, there would be approximately 531 grains of corn on the cob.

EXERCISE 11.1

Assume $\pi = \frac{22}{7}$, unless stated otherwise.

1. Diameter of the base of a cone is 10.5 cm and its slant height is 10 cm. Find its curved surface area.
2. Find the total surface area of a cone, if its slant height is 21 m and diameter of its base is 24 m.

3. Curved surface area of a cone is 308 cm^2 and its slant height is 14 cm. Find (i) radius of the base and (ii) total surface area of the cone.

4. A conical tent is 10 m high and the radius of its base is 24 m. Find

(i) slant height of the tent.

(ii) cost of the canvas required to make the tent, if the cost of 1 m^2 canvas is ₹ 70.

5. What length of tarpaulin 3 m wide will be required to make conical tent of height 8 m and base radius 6 m? Assume that the extra length of material that will be required for stitching margins and wastage in cutting is approximately 20 cm (Use $\pi = 3.14$).

6. The slant height and base diameter of a conical tomb are 25 m and 14 m respectively. Find the cost of white-washing its curved surface at the rate of ₹ 210 per 100 m^2.

7. A joker's cap is in the form of a right circular cone of base radius 7 cm and height 24 cm. Find the area of the sheet required to make 10 such caps.

8. A bus stop is barricaded from the remaining part of the road, by using 50 hollow cones made of recycled cardboard. Each cone has a base diameter of 40 cm and height 1 m. If the outer side of each of the cones is to be painted and the cost of painting is ₹ 12 per m^2, what will be the cost of painting all these cones? (Use $\pi = 3.14$ and take $\sqrt{1.04} = 1.02$)

11.2 Surface Area of a Sphere

What is a sphere? Is it the same as a circle? Can you draw a circle on a paper? Yes, you can, because a circle is a plane closed figure whose every point lies at a constant distance (called **radius**) from a fixed point, which is called the **centre** of the circle. Now if you paste a string along a diameter of a circular disc and rotate it as you had rotated the triangle in the previous section, you see a new solid (see Fig 11.6). What does it resemble? A ball? Yes. It is called a **sphere**.

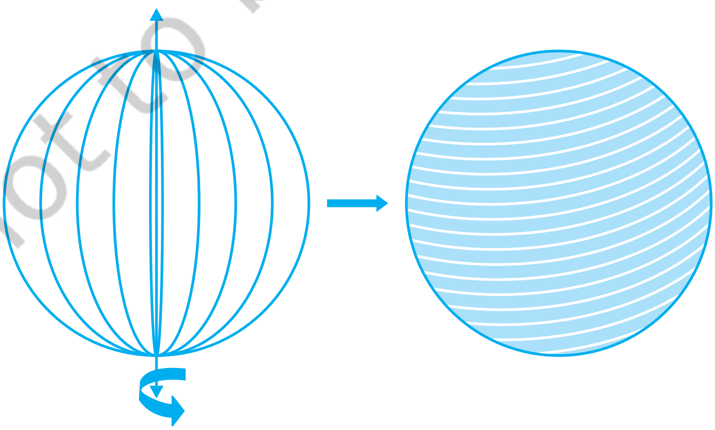

Fig. 11.6

Can you guess what happens to the centre of the circle, when it forms a sphere on rotation? Of course, it becomes the centre of the sphere. So, *a sphere is a three dimensional figure (solid figure), which is made up of all points in the space, which lie at a constant distance called the radius, from a fixed point called the centre of the sphere.*

Note : A sphere is like the surface of a ball. The word *solid sphere* is used for the solid whose surface is a sphere.

Activity : Have you ever played with a top or have you at least watched someone play with one? You must be aware of how a string is wound around it. Now, let us take a rubber ball and drive a nail into it. Taking support of the nail, let us wind a string around the ball. When you have reached the 'fullest' part of the ball, use pins to keep the string in place, and continue to wind the string around the remaining part of the ball, till you have completely covered the ball [see Fig. 11.7(a)]. Mark the starting and finishing points on the string, and slowly unwind the string from the surface of the ball.

Now, ask your teacher to help you in measuring the diameter of the ball, from which you easily get its radius. Then on a sheet of paper, draw four circles with radius equal to the radius of the ball. Start filling the circles one by one, with the string you had wound around the ball [see Fig. 11.7(b)].

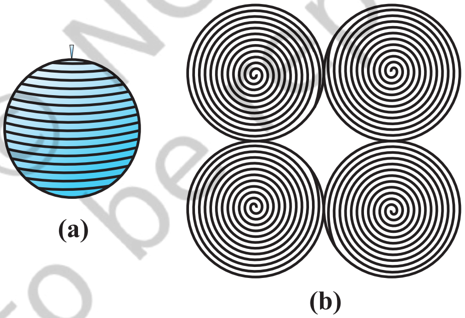

Fig. 11.7

What have you achieved in all this?

The string, which had completely covered the surface area of the sphere, has been used to completely fill the regions of four circles, all of the same radius as of the sphere.

So, what does that mean? This suggests that the surface area of a sphere of radius r

= 4 times the area of a circle of radius $r = 4 \times (\pi r^2)$

So, $\boxed{\textbf{Surface Area of a Sphere} = 4\pi r^2}$

where r is the radius of the sphere.

How many faces do you see in the surface of a sphere? There is only one, which is curved.

Now, let us take a solid sphere, and slice it exactly 'through the middle' with a plane that passes through its centre. What happens to the sphere?

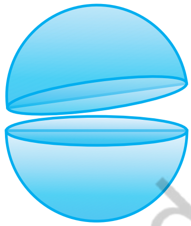

Fig. 11.8

Yes, it gets divided into two equal parts (see Fig. 11.8)! What will each half be called? It is called a **hemisphere**. (Because 'hemi' also means 'half')

And what about the surface of a hemisphere? How many faces does it have?

Two! There is a curved face and a flat face (base).

The curved surface area of a hemisphere is half the surface area of the sphere, which is $\frac{1}{2}$ of $4\pi r^2$.

Therefore, $\boxed{\textbf{Curved Surface Area of a Hemisphere} = 2\pi r^2}$

where r is the radius of the sphere of which the hemisphere is a part.

Now taking the two faces of a hemisphere, its surface area $2\pi r^2 + \pi r^2$

So, $\boxed{\textbf{Total Surface Area of a Hemisphere} = 3\pi r^2}$

Example 4 : Find the surface area of a sphere of radius 7 cm.

Solution : The surface area of a sphere of radius 7 cm would be

$$4\pi r^2 = 4 \times \frac{22}{7} \times 7 \times 7 \text{ cm}^2 = 616 \text{ cm}^2$$

Example 5 : Find (i) the curved surface area and (ii) the total surface area of a hemisphere of radius 21 cm.

Solution : The curved surface area of a hemisphere of radius 21 cm would be

$$= 2\pi r^2 = 2 \times \frac{22}{7} \times 21 \times 21 \text{ cm}^2 = 2772 \text{ cm}^2$$

(ii) the total surface area of the hemisphere would be

$$3\pi r^2 = 3 \times \frac{22}{7} \times 21 \times 21 \text{ cm}^2 = 4158 \text{ cm}^2$$

Example 6 : The hollow sphere, in which the circus motorcyclist performs his stunts, has a diameter of 7 m. Find the area available to the motorcyclist for riding.

Solution : Diameter of the sphere = 7 m. Therefore, radius is 3.5 m. So, the riding space available for the motorcyclist is the surface area of the 'sphere' which is given by

$$4\pi r^2 = 4 \times \frac{22}{7} \times 3.5 \times 3.5 \text{ m}^2$$
$$= 154 \text{ m}^2$$

Example 7 : A hemispherical dome of a building needs to be painted (see Fig. 11.9). If the circumference of the base of the dome is 17.6 m, find the cost of painting it, given the cost of painting is ₹ 5 per 100 cm^2.

Solution : Since only the rounded surface of the dome is to be painted, we would need to find the curved surface area of the hemisphere to know the extent of painting that needs to be done. Now, circumference of the dome = 17.6 m. Therefore, 17.6 = $2\pi r$.

So, the radius of the dome = $17.6 \times \frac{7}{2 \times 22}$ m = 2.8 m

The curved surface area of the dome = $2\pi r^2$

$$= 2 \times \frac{22}{7} \times 2.8 \times 2.8 \text{ m}^2$$
$$= 49.28 \text{ m}^2$$

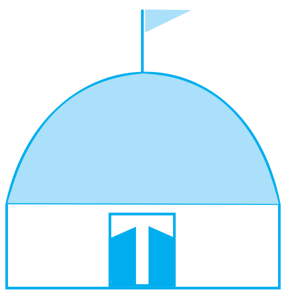

Fig. 11.9

Now, cost of painting 100 cm^2 is ₹ 5.

So, cost of painting 1 m^2 = ₹ 500

Therefore, cost of painting the whole dome

= ₹ 500 × 49.28

= ₹ 24640

EXERCISE 11.2

Assume $\pi = \frac{22}{7}$, unless stated otherwise.

1. Find the surface area of a sphere of radius:

(i) 10.5 cm (ii) 5.6 cm (iii) 14 cm

2. Find the surface area of a sphere of diameter:
(i) 14 cm (ii) 21 cm (iii) 3.5 m

3. Find the total surface area of a hemisphere of radius 10 cm. (Use $\pi = 3.14$)

4. The radius of a spherical balloon increases from 7 cm to 14 cm as air is being pumped into it. Find the ratio of surface areas of the balloon in the two cases.

5. A hemispherical bowl made of brass has inner diameter 10.5 cm. Find the cost of tin-plating it on the inside at the rate of ₹ 16 per 100 cm^2.

6. Find the radius of a sphere whose surface area is 154 cm^2.

7. The diameter of the moon is approximately one fourth of the diameter of the earth. Find the ratio of their surface areas.

8. A hemispherical bowl is made of steel, 0.25 cm thick. The inner radius of the bowl is 5 cm. Find the outer curved surface area of the bowl.

9. A right circular cylinder just encloses a sphere of radius r (see Fig. 11.10). Find
(i) surface area of the sphere,
(ii) curved surface area of the cylinder,
(iii) ratio of the areas obtained in (i) and (ii).

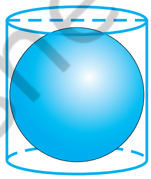

Fig. 11.10

11.3 Volume of a Right Circular Cone

In earlier classes we have studied the volumes of cube, cuboid and cylinder
In Fig 11.11, can you see that there is a right circular cylinder and a right circular cone of the same base radius and the same height?

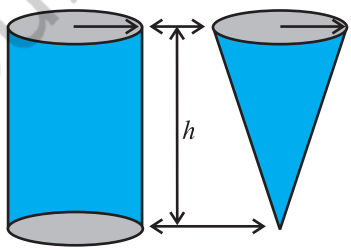

Fig. 11.11

Activity : Try to make a hollow cylinder and a hollow cone like this with the same base radius and the same height (see Fig. 11.11). Then, we can try out an experiment that will help us, to see practically what the volume of a right circular cone would be!

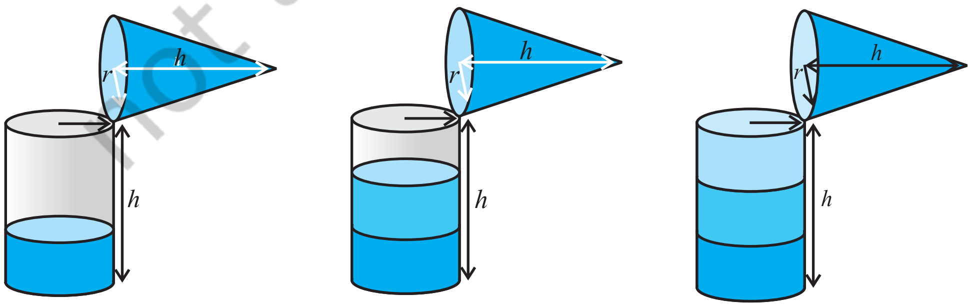

Fig. 11.12

So, let us start like this.

Fill the cone up to the brim with sand once, and empty it into the cylinder. We find that it fills up only a part of the cylinder [see Fig. 11.12(a)].

When we fill up the cone again to the brim, and empty it into the cylinder, we see that the cylinder is still not full [see Fig. 11.12(b)].

When the cone is filled up for the third time, and emptied into the cylinder, it can be seen that the cylinder is also full to the brim [see Fig. 11.12(c)].

With this, we can safely come to the conclusion that three times the volume of a cone, makes up the volume of a cylinder, which has the same base radius and the same height as the cone, which means that the volume of the cone is one-third the volume of the cylinder.

So, $\boxed{\textbf{Volume of a Cone} = \frac{1}{3}\pi r^2 h}$

where r is the base radius and h is the height of the cone.

Example 8 : The height and the slant height of a cone are 21 cm and 28 cm respectively. Find the volume of the cone.

Solution : From $l^2 = r^2 + h^2$, we have

$$r = \sqrt{l^2 - h^2} = \sqrt{28^2 - 21^2} \text{ cm} = 7\sqrt{7} \text{ cm}$$

So, volume of the cone $= \frac{1}{3}\pi r^2 h = \frac{1}{3} \times \frac{22}{7} \times 7\sqrt{7} \times 7\sqrt{7} \times 21 \text{ cm}^3$

$= 7546 \text{ cm}^3$

Example 9 : Monica has a piece of canvas whose area is 551 m^2. She uses it to have a conical tent made, with a base radius of 7 m. Assuming that all the stitching margins and the wastage incurred while cutting, amounts to approximately 1 m^2, find the volume of the tent that can be made with it.

Solution : Since the area of the canvas = 551 m^2 and area of the canvas lost in wastage is 1 m^2, therefore the area of canvas available for making the tent is (551 – 1) m^2 = 550 m^2.

Now, the surface area of the tent = 550 m^2 and the required base radius of the conical tent = 7 m

Note that a tent has only a curved surface (the floor of a tent is not covered by canvas!!).

Therefore, curved surface area of tent = 550 m^2.

That is, $\pi r l = 550$

or, $\frac{22}{7} \times 7 \times l = 550$

or, $l = 3\frac{550}{22}$ m = 25 m

Now, $l^2 = r^2 + h^2$

Therefore, $h = \sqrt{l^2 - r^2} = \sqrt{25^2 - 7^2}$ m $= \sqrt{625 - 49}$ m $= \sqrt{576}$ m

$= 24$ m

So, the volume of the conical tent $= \frac{1}{3}\pi r^2 h = \frac{1}{3} \times \frac{22}{7} \times 7 \times 7 \times 24 \text{ m}^3 = 1232 \text{ m}^3$.

EXERCISE 11.3

Assume $\pi = \frac{22}{7}$, unless stated otherwise.

1. Find the volume of the right circular cone with
(i) radius 6 cm, height 7 cm (ii) radius 3.5 cm, height 12 cm

2. Find the capacity in litres of a conical vessel with
(i) radius 7 cm, slant height 25 cm (ii) height 12 cm, slant height 13 cm

3. The height of a cone is 15 cm. If its volume is 1570 cm^3, find the radius of the base. (Use $\pi = 3.14$)

4. If the volume of a right circular cone of height 9 cm is 48π cm^3, find the diameter of its base.

5. A conical pit of top diameter 3.5 m is 12 m deep. What is its capacity in kilolitres?

6. The volume of a right circular cone is 9856 cm^3. If the diameter of the base is 28 cm, find
(i) height of the cone (ii) slant height of the cone
(iii) curved surface area of the cone

7. A right triangle ABC with sides 5 cm, 12 cm and 13 cm is revolved about the side 12 cm. Find the volume of the solid so obtained.

8. If the triangle ABC in the Question 7 above is revolved about the side 5 cm, then find the volume of the solid so obtained. Find also the ratio of the volumes of the two solids obtained in Questions 7 and 8.

9. A heap of wheat is in the form of a cone whose diameter is 10.5 m and height is 3 m. Find its volume. The heap is to be covered by canvas to protect it from rain. Find the area of the canvas required.

11.4 Volume of a Sphere

Now, let us see how to go about measuring the volume of a sphere. First, take two or three spheres of different radii, and a container big enough to be able to put each of the spheres into it, one at a time. Also, take a large trough in which you can place the container. Then, fill the container up to the brim with water [see Fig. 11.13(a)].

Now, carefully place one of the spheres in the container. Some of the water from the container will over flow into the trough in which it is kept [see Fig. 11.13(b)]. Carefully pour out the water from the trough into a measuring cylinder (i.e., a graduated cylindrical jar) and measure the water over flowed [see Fig. 11.13(c)]. Suppose the radius of the immersed sphere is r (you can find the radius by measuring the diameter of the sphere). Then evaluate $\frac{4}{3}\pi r^3$. Do you find this value almost equal to the measure of the volume over flowed?

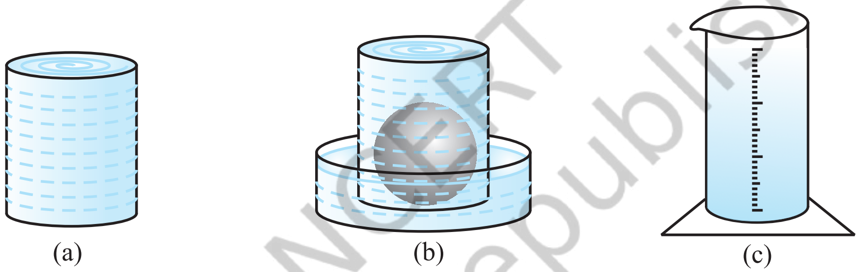

Fig. 11.13

Once again repeat the procedure done just now, with a different size of sphere. Find the radius R of this sphere and then calculate the value of $\frac{4}{3}\pi R^3$. Once again this value is nearly equal to the measure of the volume of the water displaced (over flowed) by the sphere. What does this tell us? We know that the volume of the sphere is the same as the measure of the volume of the water displaced by it. By doing this experiment repeatedly with spheres of varying radii, we are getting the same result, namely, the volume of a sphere is equal to $\frac{4}{3}\pi$ times the cube of its radius. This gives us the idea that

$$\boxed{\textbf{Volume of a Sphere} = \frac{4}{3}\pi r^3}$$

where r is the radius of the sphere.

Later, in higher classes it can be proved also. But at this stage, we will just take it as true.

Since a hemisphere is half of a sphere, can you guess what the volume of a hemisphere will be? Yes, it is $\frac{1}{2}$ of $\frac{4}{3}\pi r^3 = \frac{2}{3}\pi r^3$.

So, $\boxed{\textbf{Volume of a Hemisphere} = \frac{2}{3}\pi r^3}$

where r is the radius of the hemisphere.

Let us take some examples to illustrate the use of these formulae.

Example 10 : Find the volume of a sphere of radius 11.2 cm.

Solution : Required volume $= \frac{4}{3}\pi r^3$

$$= \frac{4}{3} \times \frac{22}{7} \times 11.2 \times 11.2 \times 11.2 \text{ cm}^3 = 5887.32 \text{ cm}^3$$

Example 11 : A shot-putt is a metallic sphere of radius 4.9 cm. If the density of the metal is 7.8 g per cm^3, find the mass of the shot-putt.

Solution : Since the shot-putt is a solid sphere made of metal and its mass is equal to the product of its volume and density, we need to find the volume of the sphere.

Now, volume of the sphere $= \frac{4}{3}\pi r^3$

$$= \frac{4}{3} \times \frac{22}{7} \times 4.9 \times 4.9 \times 4.9 \text{ cm}^3$$

$$= 493 \text{ cm}^3 \text{ (nearly)}$$

Further, mass of 1 cm^3 of metal is 7.8 g.

Therefore, mass of the shot-putt $= 7.8 \times 493$ g

$$= 3845.44 \text{ g} = 3.85 \text{ kg (nearly)}$$

Example 12 : A hemispherical bowl has a radius of 3.5 cm. What would be the volume of water it would contain?

Solution : The volume of water the bowl can contain

$$= \frac{2}{3}\pi r^3$$

$$= \frac{2}{3} \times \frac{22}{7} \times 3.5 \times 3.5 \times 3.5 \text{ cm}^3 = 89.8 \text{ cm}^3$$

EXERCISE 11.4

Assume $\pi = \frac{22}{7}$, unless stated otherwise.

1. Find the volume of a sphere whose radius is
   (i) 7 cm (ii) 0.63 m
2. Find the amount of water displaced by a solid spherical ball of diameter
   (i) 28 cm (ii) 0.21 m
3. The diameter of a metallic ball is 4.2 cm. What is the mass of the ball, if the density of the metal is 8.9 g per cm^3?
4. The diameter of the moon is approximately one-fourth of the diameter of the earth. What fraction of the volume of the earth is the volume of the moon?
5. How many litres of milk can a hemispherical bowl of diameter 10.5 cm hold?
6. A hemispherical tank is made up of an iron sheet 1 cm thick. If the inner radius is 1 m, then find the volume of the iron used to make the tank.
7. Find the volume of a sphere whose surface area is 154 cm^2.
8. A dome of a building is in the form of a hemisphere. From inside, it was white-washed at the cost of ₹ 4989.60. If the cost of white-washing is ₹ 20 per square metre, find the
   (i) inside surface area of the dome, (ii) volume of the air inside the dome.
9. Twenty seven solid iron spheres, each of radius r and surface area S are melted to form a sphere with surface area S′. Find the
   (i) radius r' of the new sphere, (ii) ratio of S and S′.
10. A capsule of medicine is in the shape of a sphere of diameter 3.5 mm. How much medicine (in mm^3) is needed to fill this capsule?

11.5 Summary

In this chapter, you have studied the following points:

1. **Curved surface area of a cone = πrl**
2. **Total surface area of a right circular cone = $\pi rl + \pi r^2$, i.e., $\pi r(l + r)$**
3. **Surface area of a sphere of radius $r = 4\pi r^2$**
4. **Curved surface area of a hemisphere = $2\pi r^2$**
5. **Total surface area of a hemisphere = $3\pi r^2$**
6. **Volume of a cone = $\frac{1}{3}\pi r^2 h$**
7. **Volume of a sphere of radius $r = \frac{4}{3}\pi r^3$**
8. **Volume of a hemisphere = $\frac{2}{3}\pi r^3$**

[Here, letters l, b, h, a, r, etc. have been used in their usual meaning, depending on the context.]

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CHAPTER 12

STATISTICS

12.1 Graphical Representation of Data

The representation of data by tables has already been discussed. Now let us turn our attention to another representation of data, i.e., the graphical representation. It is well said that one picture is better than a thousand words. Usually comparisons among the individual items are best shown by means of graphs. The representation then becomes easier to understand than the actual data. We shall study the following graphical representations in this section.

(A) Bar graphs

(B) Histograms of uniform width, and of varying widths

(C) Frequency polygons

(A) Bar Graphs

In earlier classes, you have already studied and constructed bar graphs. Here we shall discuss them through a more formal approach. Recall that a bar graph is a pictorial representation of data in which usually bars of uniform width are drawn with equal spacing between them on one axis (say, the x-axis), depicting the variable. The values of the variable are shown on the other axis (say, the y-axis) and the heights of the bars depend on the values of the variable.

Example 1 : In a particular section of Class IX, 40 students were asked about the months of their birth and the following graph was prepared for the data so obtained:

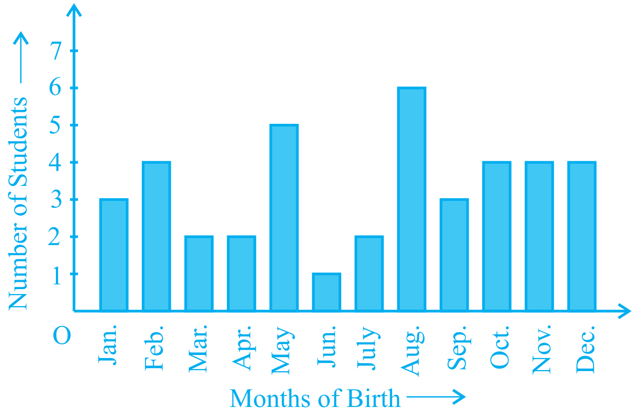

Fig. 12.1

Observe the bar graph given above and answer the following questions:

(i) How many students were born in the month of November?

(ii) In which month were the maximum number of students born?

Solution : Note that the variable here is the 'month of birth', and the value of the variable is the 'Number of students born'.

(i) 4 students were born in the month of November.

(ii) The Maximum number of students were born in the month of August.

Let us now recall how a bar graph is constructed by considering the following example.

Example 2 : A family with a monthly income of ₹ 20,000 had planned the following expenditures per month under various heads:

Table 12.1

Heads	Expenditure (in thousand rupees)
Grocery	4
Rent	5
Education of children	5
Medicine	2
Fuel	2
Entertainment	1
Miscellaneous	1

Draw a bar graph for the data above.

Solution : We draw the bar graph of this data in the following steps. Note that the unit in the second column is thousand rupees. So, '4' against 'grocery' means ₹4000.

1. We represent the Heads (variable) on the horizontal axis choosing any scale, since the width of the bar is not important. But for clarity, we take equal widths for all bars and maintain equal gaps in between. Let one Head be represented by one unit.
2. We represent the expenditure (value) on the vertical axis. Since the maximum expenditure is ₹5000, we can choose the scale as 1 unit = ₹1000.
3. To represent our first Head, i.e., grocery, we draw a rectangular bar with width 1 unit and height 4 units.
4. Similarly, other Heads are represented leaving a gap of 1 unit in between two consecutive bars.

The bar graph is drawn in Fig. 12.2.

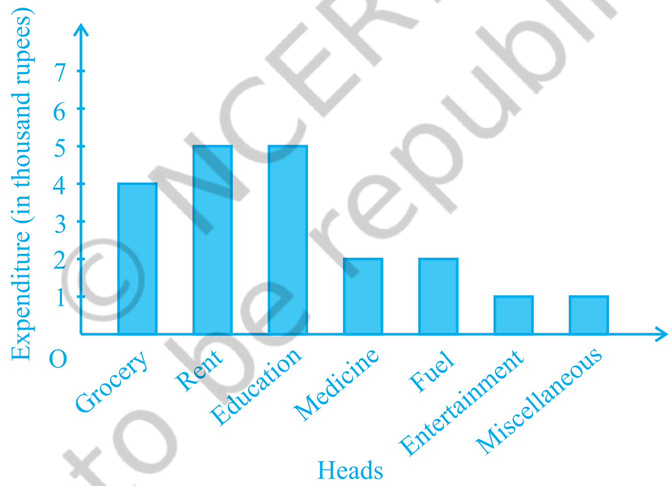

Fig. 12.2

Here, you can easily visualise the relative characteristics of the data at a glance, e.g., the expenditure on education is more than double that of medical expenses. Therefore, in some ways it serves as a better representation of data than the tabular form.

Activity 1 : Continuing with the same four groups of Activity 1, represent the data by suitable bar graphs.

Let us now see how a frequency distribution table for *continuous* class intervals can be represented graphically.

(B) Histogram

This is a form of representation like the bar graph, but it is used for continuous class intervals. For instance, consider the frequency distribution Table 12.2, representing the weights of 36 students of a class:

Table 12.2

Weights (in kg)	Number of students
30.5 - 35.5	9
35.5 - 40.5	6
40.5 - 45.5	15
45.5 - 50.5	3
50.5 - 55.5	1
55.5 - 60.5	2
Total	36

Let us represent the data given above graphically as follows:

(i) We represent the weights on the horizontal axis on a suitable scale. We can choose the scale as 1 cm = 5 kg. Also, since the first class interval is starting from 30.5 and not zero, we show it on the graph by marking a *kink* or a break on the axis.

(ii) We represent the number of students (frequency) on the vertical axis on a suitable scale. Since the maximum frequency is 15, we need to choose the scale to accomodate this maximum frequency.

(iii) We now draw rectangles (or rectangular bars) of width equal to the class-size and lengths according to the frequencies of the corresponding class intervals. For example, the rectangle for the class interval 30.5 - 35.5 will be of width 1 cm and length 4.5 cm.

(iv) In this way, we obtain the graph as shown in Fig. 12.3:

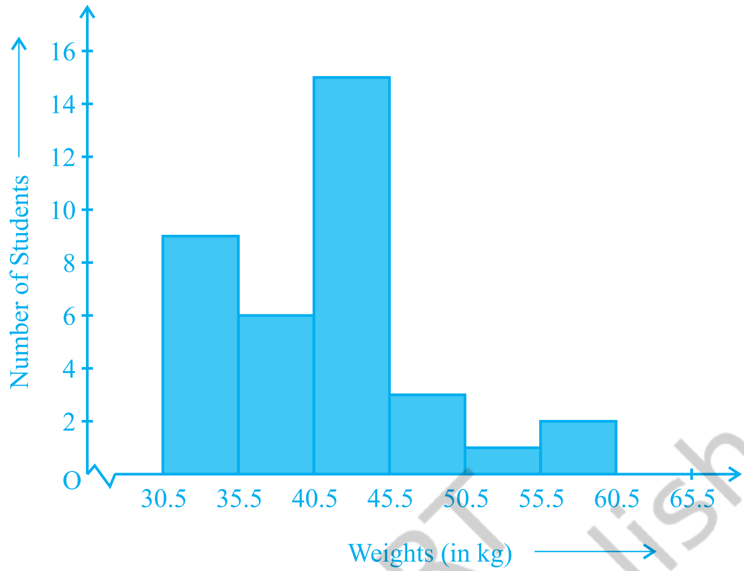

Fig. 12.3

Observe that since there are no gaps in between consecutive rectangles, the resultant graph appears like a solid figure. This is called a *histogram*, which is a graphical representation of a grouped frequency distribution with continuous classes. Also, unlike a bar graph, the width of the bar plays a significant role in its construction.

Here, in fact, areas of the rectangles erected are proportional to the corresponding frequencies. However, since the widths of the rectangles are all equal, the lengths of the rectangles are proportional to the frequencies. That is why, we draw the lengths according to (iii) above.

Now, consider a situation different from the one above.

Example 3 : A teacher wanted to analyse the performance of two sections of students in a mathematics test of 100 marks. Looking at their performances, she found that a few students got under 20 marks and a few got 70 marks or above. So she decided to group them into intervals of varying sizes as follows: 0 - 20, 20 - 30, . . ., 60 - 70, 70 - 100. Then she formed the following table:

Table 12.3

Marks	Number of students
0 - 20	7
20 - 30	10
30 - 40	10
40 - 50	20
50 - 60	20
60 - 70	15
70 - above	8
Total	90

A histogram for this table was prepared by a student as shown in Fig. 12.4.

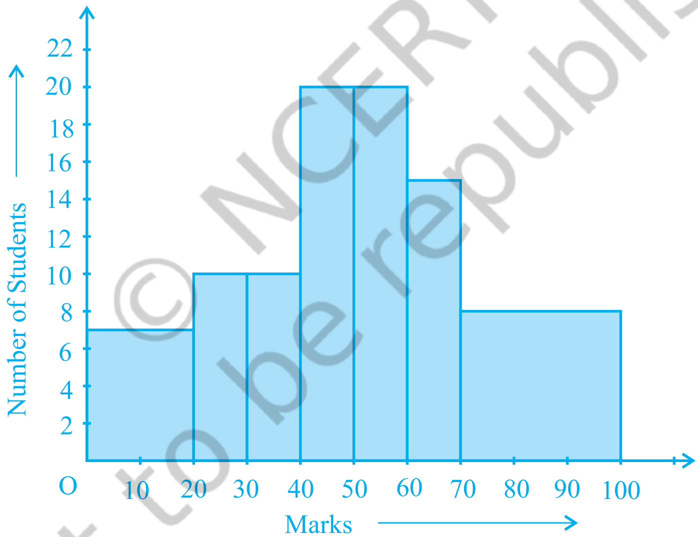

Fig. 12.4

Carefully examine this graphical representation. Do you think that it correctly represents the data? No, the graph is giving us a misleading picture. As we have mentioned earlier, the areas of the rectangles are proportional to the frequencies in a histogram. Earlier this problem did not arise, because the widths of all the rectangles were equal. But here, since the widths of the rectangles are varying, the histogram above does not give a correct picture. For example, it shows a greater frequency in the interval 70 - 100, than in 60 - 70, which is not the case.

So, we need to make certain modifications in the lengths of the rectangles so that the areas are again proportional to the frequencies.

The steps to be followed are as given below:

1. Select a class interval with the minimum class size. In the example above, the minimum class-size is 10.
2. The lengths of the rectangles are then modified to be proportionate to the class-size 10.

For instance, when the class-size is 20, the length of the rectangle is 7. So when the class-size is 10, the length of the rectangle will be $\frac{7}{20} \times 10 = 3.5$.

Similarly, proceeding in this manner, we get the following table:

Table 12.4

Marks	Frequency	Width of the class	Length of the rectangle
0 - 20	7	20	$\frac{7}{20} \times 10 = 3.5$
20 - 30	10	10	$\frac{10}{10} \times 10 = 10$
30 - 40	10	10	$\frac{10}{10} \times 10 = 10$
40 - 50	20	10	$\frac{20}{10} \times 10 = 20$
50 - 60	20	10	$\frac{20}{10} \times 10 = 20$
60 - 70	15	10	$\frac{15}{10} \times 10 = 15$
70 - 100	8	30	$\frac{8}{30} \times 10 = 2.67$

Since we have calculated these lengths for an interval of 10 marks in each case, we may call these lengths as "proportion of students per 10 marks interval".

So, the correct histogram with varying width is given in Fig. 12.5.

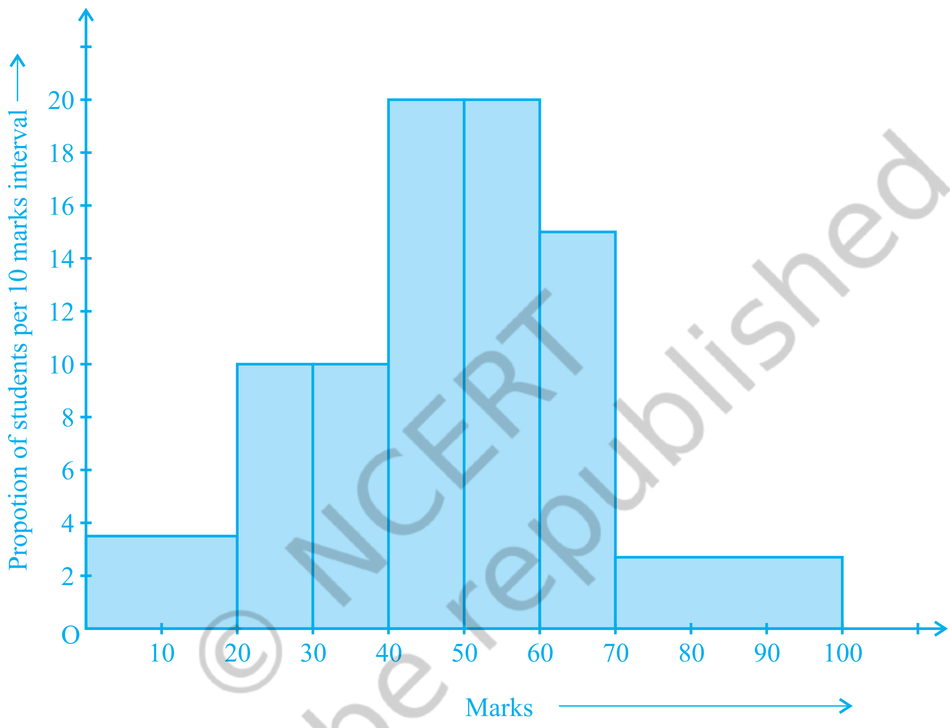

Fig. 12.5

(C) Frequency Polygon

There is yet another visual way of representing quantitative data and its frequencies. This is a polygon. To see what we mean, consider the histogram represented by Fig. 12.3. Let us join the mid-points of the upper sides of the adjacent rectangles of this histogram by means of line segments. Let us call these mid-points B, C, D, E, F and G. When joined by line segments, we obtain the figure BCDEFG (see Fig. 12.6). To complete the polygon, we assume that there is a class interval with frequency zero before 30.5 - 35.5, and one after 55.5 - 60.5, and their mid-points are A and H, respectively. ABCDEFGH is the frequency polygon corresponding to the data shown in Fig. 12.3. We have shown this in Fig. 12.6.

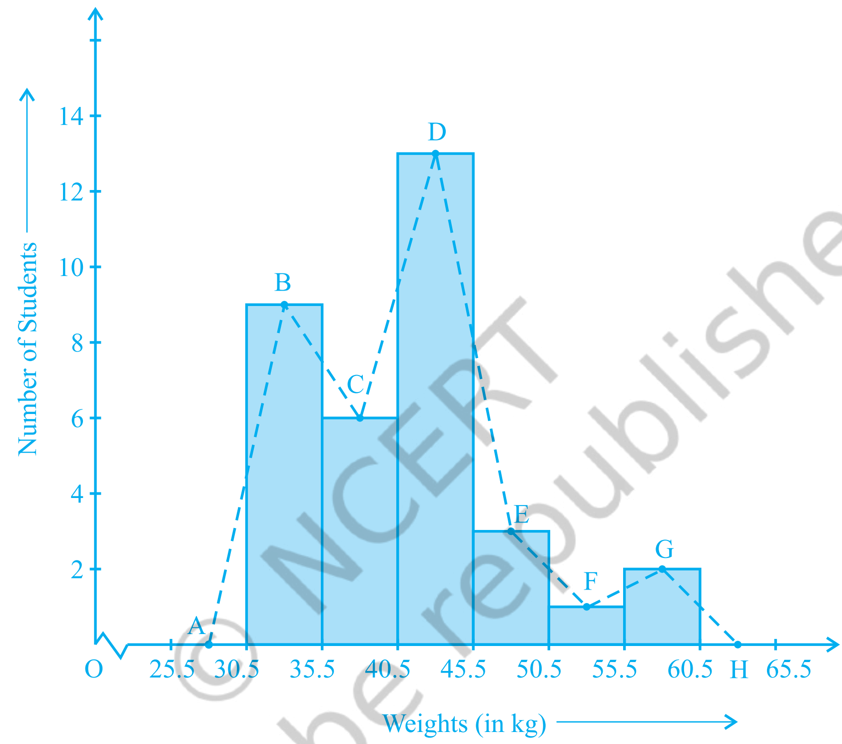

Fig. 12.6

Although, there exists no class preceding the lowest class and no class succeeding the highest class, addition of the two class intervals with zero frequency enables us to make the area of the frequency polygon the same as the area of the histogram. Why is this so? (**Hint :** Use the properties of congruent triangles.)

Now, the question arises: how do we complete the polygon when there is no class preceding the first class? Let us consider such a situation.

Example 4 : Consider the marks, out of 100, obtained by 51 students of a class in a test, given in Table 12.5.

Table 12.5

Marks	Number of students
0 - 10	5
10 - 20	10
20 - 30	4
30 - 40	6
40 - 50	7
50 - 60	3
60 - 70	2
70 - 80	2
80 - 90	3
90 - 100	9
Total	51

Draw a frequency polygon corresponding to this frequency distribution table.

Solution : Let us first draw a histogram for this data and mark the mid-points of the tops of the rectangles as B, C, D, E, F, G, H, I, J, K, respectively. Here, the first class is 0-10. So, to find the class preceeding 0-10, we extend the horizontal axis in the negative direction and find the mid-point of the imaginary class-interval (–10) - 0. The first end point, i.e., B is joined to this mid-point with zero frequency on the negative direction of the horizontal axis. The point where this line segment meets the vertical axis is marked as A. Let L be the mid-point of the class succeeding the last class of the given data. Then OABCDEFGHIJKL is the frequency polygon, which is shown in Fig. 12.7.

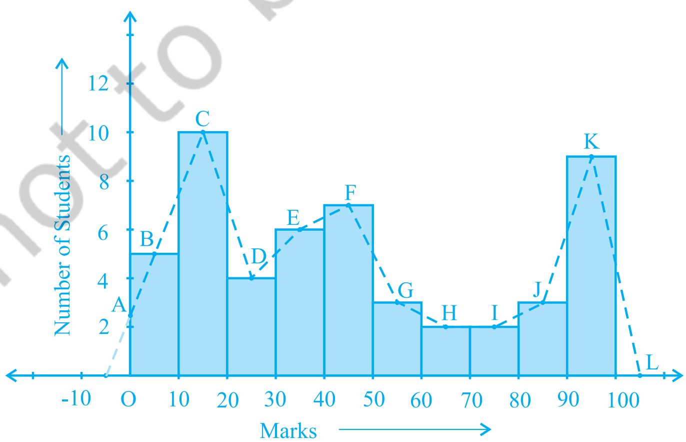

Fig. 12.7

Frequency polygons can also be drawn independently without drawing histograms. For this, we require the mid-points of the class-intervals used in the data. These mid-points of the class-intervals are called **class-marks**.

To find the class-mark of a class interval, we find the sum of the upper limit and lower limit of a class and divide it by 2. Thus,

$$\textbf{Class-mark} = \frac{\textbf{Upper limit + Lower limit}}{\textbf{2}}$$

Let us consider an example.

Example 5 : In a city, the weekly observations made in a study on the cost of living index are given in the following table:

Table 12.6

Cost of living index	Number of weeks
140 - 150	5
150 - 160	10
160 - 170	20
170 - 180	9
180 - 190	6
190 - 200	2
Total	52

Draw a frequency polygon for the data above (without constructing a histogram).

Solution : Since we want to draw a frequency polygon without a histogram, let us find the class-marks of the classes given above, that is of 140 - 150, 150 - 160,....

For 140 - 150, the upper limit = 150, and the lower limit = 140

So, the class-mark = $\frac{150+140}{2} = \frac{290}{2} = 145.$

Continuing in the same manner, we find the class-marks of the other classes as well. So, the new table obtained is as shown in the following table:

Table 12.7

Classes	Class-marks	Frequency
140 - 150	145	5
150 - 160	155	10
160 - 170	165	20
170 - 180	175	9
180 - 190	185	6
190 - 200	195	2
Total		52

We can now draw a frequency polygon by plotting the class-marks along the horizontal axis, the frequencies along the vertical-axis, and then plotting and joining the points B(145, 5), C(155, 10), D(165, 20), E(175, 9), F(185, 6) and G(195, 2) by line segments. We should not forget to plot the point corresponding to the class-mark of the class 130 - 140 (just before the lowest class 140 - 150) with zero frequency, that is, A(135, 0), and the point H (205, 0) occurs immediately after G(195, 2). So, the resultant frequency polygon will be ABCDEFGH (see Fig. 12.8).

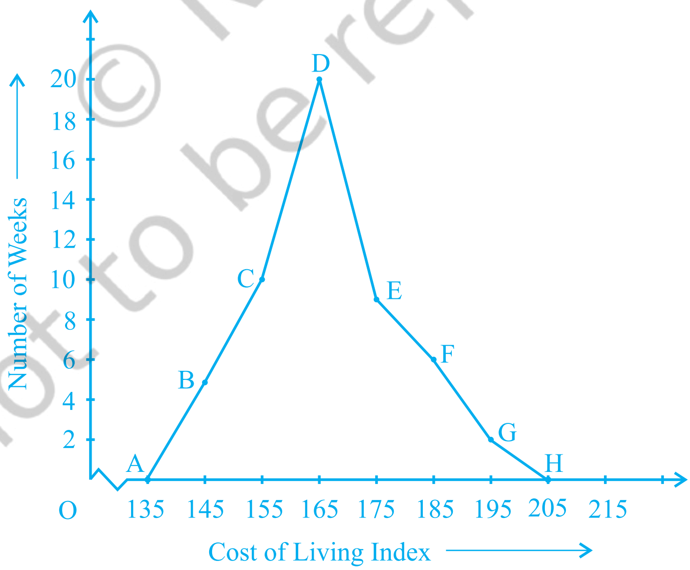

Fig. 12.8

Frequency polygons are used when the data is continuous and very large. It is very useful for comparing two different sets of data of the same nature, for example, comparing the performance of two different sections of the same class.

EXERCISE 12.1

1. A survey conducted by an organisation for the cause of illness and death among the women between the ages 15 - 44 (in years) worldwide, found the following figures (in %):

S.No.	Causes	Female fatality rate (%)
1.	Reproductive health conditions	31.8
2.	Neuropsychiatric conditions	25.4
3.	Injuries	12.4
4.	Cardiovascular conditions	4.3
5.	Respiratory conditions	4.1
6.	Other causes	22.0

(i) Represent the information given above graphically.

(ii) Which condition is the major cause of women's ill health and death worldwide?

(iii) Try to find out, with the help of your teacher, any two factors which play a major role in the cause in (ii) above being the major cause.

2. The following data on the number of girls (to the nearest ten) per thousand boys in different sections of Indian society is given below.

Section	Number of girls per thousand boys
Scheduled Caste (SC)	940
Scheduled Tribe (ST)	970
Non SC/ST	920
Backward districts	950
Non-backward districts	920
Rural	930
Urban	910

(i) Represent the information above by a bar graph.

(ii) In the classroom discuss what conclusions can be arrived at from the graph.

3. Given below are the seats won by different political parties in the polling outcome of a state assembly elections:

Political Party	A	B	C	D	E	F
Seats Won	75	55	37	29	10	37

(i) Draw a bar graph to represent the polling results.

(ii) Which political party won the maximum number of seats?

4. The length of 40 leaves of a plant are measured correct to one millimetre, and the obtained data is represented in the following table:

Length (in mm)	**Number of leaves**
118 - 126	3
127 - 135	5
136 - 144	9
145 - 153	12
154 - 162	5
163 - 171	4
172 - 180	2

(i) Draw a histogram to represent the given data. [Hint: First make the class intervals continuous]

(ii) Is there any other suitable graphical representation for the same data?

(iii) Is it correct to conclude that the maximum number of leaves are 153 mm long? Why?

5. The following table gives the life times of 400 neon lamps:

Life time (in hours)	**Number of lamps**
300 - 400	14
400 - 500	56
500 - 600	60
600 - 700	86
700 - 800	74
800 - 900	62
900 - 1000	48

(i) Represent the given information with the help of a histogram.

(ii) How many lamps have a life time of more than 700 hours?

6. The following table gives the distribution of students of two sections according to the marks obtained by them:

Section A		Section B	
Marks	**Frequency**	**Marks**	**Frequency**
0 - 10	3	0 - 10	5
10 - 20	9	10 - 20	19
20 - 30	17	20 - 30	15
30 - 40	12	30 - 40	10
40 - 50	9	40 - 50	1

Represent the marks of the students of both the sections on the same graph by two frequency polygons. From the two polygons compare the performance of the two sections.

7. The runs scored by two teams A and B on the first 60 balls in a cricket match are given below:

Number of balls	Team A	Team B
1 - 6	2	5
7 - 12	1	6
13 - 18	8	2
19 - 24	9	10
25 - 30	4	5
31 - 36	5	6
37 - 42	6	3
43 - 48	10	4
49 - 54	6	8
55 - 60	2	10

Represent the data of both the teams on the same graph by frequency polygons.

[**Hint :** First make the class intervals continuous.]

8. A random survey of the number of children of various age groups playing in a park was found as follows:

Age (in years)	Number of children
1 - 2	5
2 - 3	3
3 - 5	6
5 - 7	12
7 - 10	9
10 - 15	10
15 - 17	4

Draw a histogram to represent the data above.

9. 100 surnames were randomly picked up from a local telephone directory and a frequency distribution of the number of letters in the English alphabet in the surnames was found as follows:

Number of letters	Number of surnames
1 - 4	6
4 - 6	30
6 - 8	44
8 - 12	16
12 - 20	4

(i) Draw a histogram to depict the given information.

(ii) Write the class interval in which the maximum number of surnames lie.

12.2 Summary

In this chapter, you have studied the following points:

1. How data can be presented graphically in the form of bar graphs, histograms and frequency polygons.