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1) <u>Case 1</u> A or B is singular.

If A or B is singular, det(A) or det(B) = 0

$\Rightarrow$ $\det(C) = \det(A)\det(B) = 0$

$\Rightarrow$ C is also singular.

For a singular matrix, the minimum singular value is 0.

So,

If A or B is singular $\Rightarrow$ $a_m = 0$ or $b_m = 0$

and $c_m = 0$.

$\Rightarrow$ $\min(a_1 b_m, a_m b_1) = 0$

$c_m = 0$

$a_m b_m = 0$

So, $\min(a_1 b_m, a_m b_1) \geq c_m \geq a_m b_m$ is trivially true

$0 \geq 0 \geq 0$.

$\Rightarrow$ $\max(a_1 b_m, a_m b_1) = 0$

(TP = To prove) $\Rightarrow$ TP $a_1 b_1 \geq c_1 \geq 0$ $\Rightarrow$ TP $a_1 b_1 \geq c_1$

since $c_1 \geq 0$ is trivially true.

<u>Proof</u> : $\|C\|_2 = \|AB\|_2$

Let $A = U_1 \Sigma_1 V_1^*$, $B = U_2 \Sigma_2 V_2^*$, $C = U_3 \Sigma_3 V_3^*$

Also, $\|UAV\|_2 = \|A\|$ where U, V are unitary.

$\Rightarrow$ $\|U_3 \Sigma_3 V_3^*\|_2 = \|U_1 \Sigma_1 V_1^* U_2 \Sigma_2 V_2^*\|_2$

$\Rightarrow$ $\|\Sigma_3\|_2 = \|\Sigma_1 V_1^* U_2 \Sigma_2\|_2$

$$\Rightarrow \quad \|\Sigma_3\|_2 \le \|\Sigma_1 V_1^*\| \, \|U_2 \Sigma_2\|$$

$$\Rightarrow \quad \|\Sigma_3\|_2 \le \|\Sigma_1\| \, \|U_2 \Sigma_2\|$$

$$\Rightarrow \quad c_1 \le a_1 b_1$$

Hence proved.

<u>Case 2</u> A and B are invertible.

i) TP $a_1 b_1 \ge c_1$ $\Rightarrow$ Same proof as in case 1

ii) TP $c_m \ge a_m b_m$

<u>Proof</u> : $C = AB \Rightarrow C^{-1} = B^{-1} A^{-1}$

$$\Rightarrow \|C^{-1}\|_2 = \|B^{-1} A^{-1}\|_2$$

$$\Rightarrow \|C^{-1}\|_2 \le \|B^{-1}\|_2 \|A^{-1}\|_2$$

$$\Rightarrow \frac{1}{c_m} \le \frac{1}{b_m} \times \frac{1}{a_m}$$

$$\Rightarrow a_m b_m \le c_m$$

Hence proved.

iii) TP $c_1 \ge \max(a_1 b_m, a_m b_1)$

<u>Part 1</u>

$$a_m \le \frac{\|Ax\|}{\|x\|} \quad \forall x$$

Let $x = By$ (B is invertible $\Rightarrow$ one-one)

$$\Rightarrow a_m \|x\| \le \|Ax\| \quad \Rightarrow \quad a_m \|By\| \le \|ABy\| \quad \forall y$$

$$\Rightarrow a_m \frac{\|By\|}{\|y\|} \le \frac{\|ABy\|}{\|y\|} \quad \forall y$$

$$\Rightarrow a_m b_1 \le c_1$$

Part 2

$$\frac{1}{c_1} \le \frac{\|(AB)^{-1} y\|}{\|y\|} \quad \forall y$$

$$\Rightarrow \frac{1}{c_1} \times \|y\| \le \|B^{-1}A^{-1} y\| \quad \forall y$$

$$\Rightarrow \frac{1}{c_1} \times \|Ax\| \le \|B^{-1} x\| \quad \forall x \qquad [\text{Let } y = Ax]$$

$$\Rightarrow \frac{1}{c_1} \times \frac{\|Ax\|}{\|x\|} \le \frac{\|B^{-1}x\|}{\|x\|} \quad \forall x$$

$$\Rightarrow \frac{1}{c_1} \times a_1 \le \frac{1}{b_m}$$

$$\Rightarrow c_1 \ge a_1 b_m.$$

Hence $c_1 \ge \max(a_1 b_m, a_m b_1)$

iv) TP $c_m \le \min(a_1 b_m, a_m b_1)$

Part 1

$$\frac{1}{b_1} \le \frac{\|B^{-1} x\|}{\|x\|} \quad \forall x$$

$$\Rightarrow \frac{1}{b_1} \times \|x\| \le \|B^{-1} x\| \quad \forall x$$

Let $x = A^{-1} y$.

$$\Rightarrow \frac{1}{b_1} \times \frac{\|A^{-1} y\|}{\|y\|} \le \frac{\|(AB)^{-1} y\|}{\|y\|} \quad \forall y.$$

$$\Rightarrow \frac{1}{b_1} \times \frac{1}{a_m} \le \frac{1}{c_m}$$

$$\Rightarrow c_m \le a_m b_1$$

Part 2

$$c_m \leq \frac{\|ABy\|}{\|y\|} \quad \forall y$$

$$\Rightarrow c_m\|y\| \leq \|ABy\| \quad \forall y$$

Let $y = B^{-1}x$,

$$\Rightarrow c_m \frac{\|B^{-1}x\|}{\|x\|} \leq \frac{\|Ax\|}{\|x\|} \quad \forall x$$

$$\Rightarrow c_m \times \frac{1}{b_m} \leq a_1$$

$$\Rightarrow c_m \leq a_1 b_m$$

Hence $c_1 \leq \max(a_1 b_m, a_m b_1)$ //

2) Let $\{v \in \mathbb{C}^m : u^* v = 0 \text{ for all } u \in S\} = T'$

TP $T' \subseteq T$ and $T \subseteq T'$

Part 1 $[T' \subseteq T$ i.e $\vec{x} \in T' \rightarrow \vec{x} \in T]$

Let $\vec{x} \in T'$. Then $\vec{x} \in \mathbb{C}^m$

$\Rightarrow \exists\, s_1 \in S,\ t_1 \in T$ s.t $\vec{x} = \vec{s_1} + \vec{t_1}$

We know that $\vec{x}^* \vec{U} = 0, \quad \forall \vec{U} \in S$

$\Rightarrow (\vec{s_1} + \vec{t_1})^* \vec{U} = 0 \quad \forall \vec{U} \in S$

but $\vec{t_1}^* \vec{U} = 0$ bcoz $\vec{U} \in S$ and $\vec{t_1} \in T$ and $S \perp T$.

$\Rightarrow \vec{s_1}^* \vec{U} = 0 \quad \forall \vec{U} \in S$

$\Rightarrow \vec{s_1} = 0$ //

$\Rightarrow \vec{x} = \vec{t_1}$

$\Rightarrow \vec{x} \in T$ // Proved.

Part 2 $[T \subseteq T'$ i.e $\vec{x} \in T \rightarrow \vec{x} \in T']$

Let $\vec{x} \in T$.

We know that $S \perp T \Rightarrow \vec{x}^* \vec{U} = 0 \quad \forall \vec{U} \in S$

$\Rightarrow \vec{x} \in T'$ // Proved.

Hence $T = T'$ //

3) a) $\|x_i\|_2 = \|y_i\|_2 \Rightarrow x_i^* x_i = y_i^* y_i \quad \forall i$ — ①

$\|x_i - x_j\|_2 = \|y_i - y_j\|_2 \Rightarrow (x_i - x_j)^*(x_i - x_j) = (y_i - y_j)^*(y_i - y_j)$

$$\Rightarrow x_i^* x_i - x_j^* x_i - x_i^* x_j + x_j^* x_j$$
$$= y_i^* y_i - y_j^* y_i - y_i^* y_j$$

But since $x_i^* x_i = y_i^* y_i$

$$\Rightarrow \cancel{x_i^* x_i} - x_j^* x_i - x_i^* x_j + \cancel{x_j^* x_j}$$
$$= \cancel{y_i^* y_i} - y_j^* y_i - y_i^* y_j + \cancel{y_j^* y_j}$$

But since $x_i^* x_j = x_j^* x_i$ and $y_i^* y_j = y_j^* y_i$
since $x_i, x_j, y_i, y_j \in \mathbb{R}^m$

$$\Rightarrow \cancel{2} x_i^* x_j = \cancel{2} y_i^* y_j$$

② $\Rightarrow x_i^* x_j = y_i^* y_j \quad \forall i, j,\ i \neq j$

① and ② imply that $X^* X = Y^* Y$ //

Let $X = \hat{Q}_1 \hat{R}_1$ and $Y = \hat{Q}_2 \hat{R}_2$ be their reduced QR factorizations (with positive diagonal elements in both $\hat{R}_1$ and $\hat{R}_2$)

$$X^* X = Y^* Y$$
$$\Rightarrow \hat{R}_1^* \hat{Q}_1^* \hat{Q}_1 \hat{R}_1 = \hat{R}_2^* \hat{Q}_2^* \hat{Q}_2 \hat{R}_2$$
$$\Rightarrow \hat{R}_1^* \hat{R}_1 = \hat{R}_2^* \hat{R}_2$$

[$\hat{Q}_1^* \hat{Q}_1 = I$ and $\hat{Q}_2^* \hat{Q}_2 = I$]

Then, there exists a matrix $D \in \mathbb{C}^{n \times n}$ s.t.

$$\hat{R}_1^* \hat{R}_1 = D = \hat{R}_2^* \hat{R}_2$$

- $D = \hat{R}_1^* \hat{R}_1$ is a Cholesky factorization with positive diagonal elements in $\hat{R}_1$
- similarly for $D = \hat{R}_2^* \hat{R}_2$

But we know that ~~for a given matrix there~~ Cholesky factorization for a matrix is unique.

So, $\hat{R}_1 = \hat{R}_2$ // Hence proved.

b) Let $X = \hat{Q}_1 \hat{R}$, $Y = \hat{Q}_2 \hat{R}$

$$\Rightarrow \hat{Q}_1^* X = \hat{Q}_1^* \hat{Q}_1 \hat{R} \quad \text{and} \quad \hat{Q}_2^* Y = \hat{Q}_2^* \hat{Q}_2 \hat{R}$$

$$\Rightarrow \hat{Q}_1^* X = \hat{R} = \hat{Q}_2^* Y$$

$$\Rightarrow \hat{Q}_2 \hat{Q}_1^* X = \hat{Q}_2 \hat{Q}_2^* Y \quad \Rightarrow \quad \boxed{\hat{Q}_2 \hat{Q}_1^* X = Y}$$

$$\Rightarrow \hat{Q}_2 \hat{Q}_1^* x_i = y_i \quad \forall i$$

$$\boxed{Q = \hat{Q}_2 \hat{Q}_1^*}$$

Algo

- Compute reduced QR factorization of $X = \hat{Q}_1 \hat{R}_1$
- Compute reduced QR factorization of $Y = \hat{Q}_2 \hat{R}_2$
- Return $\hat{Q}_2 \hat{Q}_1^*$

4) a) $FA = A - 2\dfrac{vv^*}{v^*v} \times A$

$= A - \dfrac{2}{v^*v} \times v \times (v^*A)$

$= A - v\left[\dfrac{2}{v^*v}(v^*A)\right]$

Hence,

$w = \left[\dfrac{-2}{\underbrace{v^*v}_{\text{scalar } \in \mathbb{R}}}(v^*A)\right]^* = \left[\dfrac{-2}{v^*v}(A^*v)\right]$ // $\rightarrow$ an $n\times 1$ vector

Note: v^*A is compatible since $\dim(v^*) = 1\times \underline{\underline{m}}$
$\dim(A) = \underline{\underline{m}}\times n$

As a result, $\dim(w) = n\times 1 \rightarrow$ vector.

i) The dominating operation is the matrix multiplication FA. [Computing F takes $\sim 3m^2$ flops].

↳ FA has $m\times n$ entries.
$\dim(F) = m\times m$, $\dim(A) = m\times n$.

For each entry in FA, we need to do
$= m$ multiplications $+ m-1$ additions
$= 2m-1$ flops

For all entries $= (2m-1)\times m\times n$
$= 2m^2n - mn$
$\sim 2m^2n$ flops //

$\boxed{\text{Ans: } 2m^2n \text{ flops}}$

ii) Computing w => computing A^*v ~ $(2m-1)n$ flops

computing $\frac{-2}{v^*v}$ ~ $2m$ flops

Scalar multiplication of $\frac{-2}{v^*v}$ with A^*v ~ n flops

~ $2mn$ flops

Computing vw^* => mn multiplications

~ mn flops

Computing $A+vw^*$ => mn additions

~ mn flops

Total ~ $\underline{\underline{4mn}}$ flops

b) Given $\dfrac{\|\tilde{v}-v\|_2}{\|v\|_2} = O(\epsilon_m)$

TP $\|\tilde{F}-F\|_2 = O(\epsilon_m)$

We know that r.f.e $\le (K(v) + O(1))$ r.b.e.

$$\Rightarrow \frac{\|\tilde{F}-F\|_2}{\|F\|_2} \le (K(v) + O(1)) \frac{\|\tilde{v}-v\|_2}{\|v\|_2}$$

(1) $$\Rightarrow \|\tilde{F}-F\|_2 \le \|F\|_2 (K(v) + O(1)) O(\epsilon_m)$$

To find: $K(v)$. [Assume $\|\cdot\| = \|\cdot\|_2$

$$K(v) = \sup_{\delta v} \frac{\|\delta F\|}{\|F\|} \Big/ \frac{\|\delta v\|}{\|v\|}$$

$$= \frac{\|v\|}{\|F\|} \sup_{\delta v} \frac{\|\delta F\|}{\|\delta v\|}$$

$$= \frac{\|v\|}{\|F\|} \sup_{\delta v} \frac{\left\| \left(I - \frac{2(v+\delta v)(v+\delta v)^*}{(v+\delta v)^*(v+\delta v)}\right) - \left(I - \frac{2vv^*}{v^*v}\right)\right\|}{\|\delta v\|}$$

$$= \frac{2\|v\|}{\|F\|} \sup_{\delta v} \frac{\left\| \frac{(v+\delta v)(v+\delta v)^*}{\|v+\delta v\|^2} - \frac{vv^*}{\|v\|^2}\right\|}{\|\delta v\|}$$

(Approximating $\|v+\delta v\| \to \|v\|$)

$$= \frac{2\|v\|}{\|F\|} \sup_{\delta v} \left\| \frac{(v+\delta v)(v+\delta v)^*}{\|v\|^2} - \frac{vv^*}{\|v\|^2}\right\| \Big/ \|\delta v\|$$

$$= 2 \times \frac{\cancel{\|v\|}}{\|F\|} \times \frac{1}{\|v\|^{\cancel{2}}} \sup_{\delta v} \frac{\|(v+\delta v)(v+\delta v)^* - vv^*\|}{\|\delta v\|}$$

$$= \frac{2}{\|v\|\|F\|} \sup_{\delta v} \frac{\| \delta v . v^* + v \delta v^* + \delta v \delta v^* \|}{\| \delta v \|}$$

Ignoring $\delta v \delta v^*$ because it is negligible,

$$= \frac{2}{\|v\|\|F\|} \sup_{\delta v} \frac{\| \delta v . v^* + v \delta v^* \|}{\| \delta v \|}$$

$$\leq \frac{2}{\|v\|\|F\|} \sup_{\delta v} \frac{\| \delta v . v^* \| + \| v \delta v^* \|}{\| \delta v \|}$$

[Note: $\| A \|_2 = \| A^* \|_2$ (can be proved using SVD)]

$$\leq \frac{2}{\|v\|\|F\|} \sup_{\delta v} \frac{\cancel{\| \delta v \|} \| v \|}{\cancel{\| \delta v \|}} + \frac{\| v \| \cancel{\| \delta v \|}}{\cancel{\| \delta v \|}}$$

$$\leq \frac{2}{\cancel{\|v\|}\|F\|} \times 2 \cancel{\|v\|}$$

$$K(v) \leq \frac{4}{\|F\|}$$

Substituting in (1),

$$\Rightarrow \| \tilde{F} - F \|_2 \leq \|F\| \left(\frac{4}{\|F\|} + o(1) \right) o(\epsilon_m)$$

$$\Rightarrow \| \tilde{F} - F \|_2 \leq \left[4 + \|F\| o(1) \right] o(\epsilon_m)$$

(The $o(1)$ term tends to 0 as $\epsilon_m \to 0$). So,

$$\Rightarrow \| \tilde{F} - F \|_2 = o(\epsilon_m)$$

Let $\|\cdot\| = \|\cdot\|_2$

Part 2

Let E be a $m \times n$ matrix with ϵ_{ij} terms s.t. $|\epsilon_{ij}| \le \epsilon_m$ and $\|E\|_2 = O(\epsilon_m)$. Then

$$fl(\tilde{F}A) = \tilde{F}A(I+E)$$

Let $\tilde{F} = F + \Delta F$ with $\|\Delta F\| = O(\epsilon_m)$

Then,

$$\begin{aligned}
\tilde{F}A(I+E) &= (F+\Delta F)A(I+E) \\
&= (F+\Delta F)A + (F+\Delta F)AE \\
&= FA + \Delta FA + FAE + \Delta FAE.
\end{aligned}$$

$$\Rightarrow F\delta A = \Delta FA + FAE + \Delta FAE$$

Since F is unitary $FF^* = F^*F = I$. and $\|F\| = \|F^*\| = 1$

$$\Rightarrow \delta A = F^*\Delta FA + AE + F^*\Delta FAE$$

$$\begin{aligned}
\Rightarrow \frac{\|\delta A\|}{\|A\|} &= \frac{\|F^*\Delta FA + AE + F^*\Delta FAE\|}{\|A\|} \\
&\le \frac{\cancel{\|F^*\|}\|\Delta F\|\cancel{\|A\|}}{\cancel{\|A\|}} + \frac{\cancel{\|A\|}\|E\|}{\cancel{\|A\|}} + \frac{\cancel{\|F^*\|}\|\Delta F\|\cancel{\|A\|}\|E\|}{\cancel{\|A\|}} \\
&\le \|\Delta F\| + \|E\| + \|\Delta F\|\|E\| \\
&\le O(\epsilon_m) + O(\epsilon_m) + O(\epsilon_m^2)
\end{aligned}$$

Hence,

$$\frac{\|\delta A\|}{\|A\|} = O(\epsilon_m) \;//$$

//

5) a) The $\vec{y} \in range(A)$ which minimizes $\|b-y\|_2$ is the ~~which~~ $\vec{y}$ s.t $(\vec{b}-\vec{y}) \perp range(A)$,

$$\Rightarrow A^*(b-y) = 0$$

$$\Rightarrow \boxed{A^* y = A^* b}$$

$$\Rightarrow (U\Sigma V^*)^* y = (U\Sigma V^*)^* b$$

$$\Rightarrow V\Sigma^* U^* y = V\Sigma^* U^* b$$

$$\Rightarrow \Sigma^* U^* y = \Sigma^* U^* b \quad \left[\begin{array}{c} V \text{ is invertible} \\ VV^* = I = V^*V \end{array}\right]$$

$$\Rightarrow \Sigma[U^* y - U^* b] = 0$$

Σ is not a zero matrix. There has to be atleast one singular value for any matrix.

$$\Rightarrow U^* y = U^* b$$

$$\Rightarrow \boxed{y = UU^* b}$$

Ans: $\boxed{y = UU^* b}$

b) The Ax which minimizes $\|Ax - b\|_2$ is,

$$Ax = UU^* b$$

$$\Rightarrow \quad U \Sigma V^* x = UU^* b$$

$$\Rightarrow \quad \Sigma V^* x = U^* b \qquad [U^* U = UU^* = I]$$

Let $\vec{z} = V^* x$,

$$\Rightarrow \quad \Sigma z = U^* b$$

$$\Rightarrow \quad \begin{bmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_r & & \\ & & & 0 & \\ & & & & \ddots \; 0 \\ \hline & & 0 & & \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} u_1^* b \\ \vdots \\ u_n^* b \end{bmatrix}$$

where σ_i = singular values
z_i = entries of $\vec{z}$
u_i^* = rows of U^*

$$\Rightarrow \quad \sigma_1 z_1 = u_1^* b \qquad \Rightarrow \qquad z_1 = \frac{u_1^* b}{\sigma_1}$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$\sigma_r z_r = u_r^* b \qquad\qquad z_r = \frac{u_r^* b}{\sigma_r}$$

$$\Rightarrow \quad \boxed{\begin{array}{ll} z_i = \dfrac{u_i^* b}{\sigma_i}, & i = 1 \cdots r \\ z_i = \text{arbitrary}, & i = r+1 \cdots n \end{array}} \quad - \text{①}$$

Since $\vec{z} = V^* \vec{x} \Rightarrow \vec{x} = V\vec{z}$ $[V^*V = VV^* = I]$

Hence,

$$\boxed{\vec{x} = V\vec{z}}$$

The above $\vec{x}$ with the entries of $\vec{z}$ given in ① minimizes $\|b - Ax\|_2$

The $\vec{x}$ with min $\|x\|_2$ is

$$\vec{x} = V\vec{z} \quad \text{s.t.} \quad z_i = u_i^* b / \sigma_i, \; i = 1 \ldots r$$
$$z_i = 0, \; i = r+1 \ldots n$$

//

/

6) a) Let the coefficients of p be $\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}$

q be $\begin{bmatrix} b_0 \\ \vdots \\ b_n \end{bmatrix}$

$$(p,q) = \int_{-1}^{1} (\bar{a}_0 + \bar{a}_1 x + \cdots \bar{a}_n x^n)(b_0 + \cdots b_n x^n)$$

$$= \int_{-1}^{1} \sum_{i=0}^{n} \sum_{j=0}^{n} \bar{a}_i b_j x^{i+j}$$

$$= \left[\sum_{i=0}^{n} \sum_{j=0}^{n} \frac{\bar{a}_i b_j \, x^{i+j+1}}{i+j+1} \right]_{-1}^{1}$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{n} \bar{a}_i b_j \times \frac{2}{i+j+1} \times [(i+j+1) \bmod 2]$$

$\downarrow$
even degree powers cancel out

Let $G = \begin{bmatrix} | & | & & | \\ g_0 & g_1 & \cdots & g_n \\ | & | & & | \end{bmatrix}$

where g_i are the columns of G and $g_{ij} = (i,j)^{th}$ entry of G

$$[p]^* G [q] = [\bar{a}_0 \ \bar{a}_1 \ \cdots \ \bar{a}_n] \begin{bmatrix} g_0 & g_1 & \cdots & g_n \end{bmatrix} \begin{bmatrix} b_0 \\ \vdots \\ b_n \end{bmatrix}$$

$$= \begin{bmatrix} [p]^* g_0 & [p]^* g_1 & \cdots & [p]^* g_n \end{bmatrix} \begin{bmatrix} b_0 \\ \vdots \\ b_n \end{bmatrix}$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{n} \bar{a}_i b_j g_{ij}$$

$$[p]^* G [q] = (p, q)$$

$$\sum_{i=0}^{n} \sum_{j=0}^{n} \bar{a}_i b_j g_{ij} = \sum_{i=0}^{n} \sum_{j=0}^{n} \bar{a}_i b_j \times \left[\frac{2[(i+j+1) \bmod 2]}{i+j+1} \right]$$

By comparing the two, we see that

$$\boxed{g_{ij} = \frac{2[(i+j+1) \bmod 2]}{i+j+1}}$$

This describes the matrix G

Also, $G^* = G$ ~~bcoz~~ because

$$g_{ij} = \overline{g_{ji}} = \frac{2[(i+j+1) \bmod 2]}{i+j+1}$$ → real valued.

Hence G exists.

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b) ~~We Know that~~ $\cancel{P_{pq} = \vec{p}(\vec{p}^* q)}$ ~~and~~ $\cancel{P_{\perp p} = I - \vec{p}\vec{p}^*}$

~~Similarly, for polynomials (set~~ $\cancel{P_n}$ ~~is also a vector space),~~

$\cancel{P_p q = p(x)[(p,}$

b) Provjection of q onto the $\langle p \rangle$ $\Rightarrow$ $\dfrac{p(p^* q)}{(p^* p)} = p(x)\dfrac{(p,q)}{(p,p)}$

(Algebraic Defn) $\Rightarrow p(x)\dfrac{\left[\int_{-1}^{1} \overline{p(x)}\, q(x)\, dx\right]}{\left[\int_{-1}^{1} \overline{p(x)}\, p(x)\, dx\right]}$ //

Matrix form $\longrightarrow$ $\bar{p}\dfrac{(p,q)}{(p,p)} = [p]\cdot\dfrac{[p]^* G [q]}{[p]^* G [p]}$

$$= \left(\frac{[p][p]^* G}{[p]^* G [p]}\right)[q]$$

Matrix $= \dfrac{[p][p]^* G}{[p]^* G [p]}$

Similarly,

Provjection of q orthogonal to $\langle p \rangle = q(x) - p(x)\dfrac{\left[\int_{-1}^{1} \overline{p(x)}\, q(x)\, dx\right]}{\left[\int_{-1}^{1} \overline{p(x)}\, p(x)\, dx\right]}$

Matrix form $= I - \dfrac{[p][p]^* G}{[p]^* G [p]}$ //