Q1 Find the fourier series of the following function :-

$$f(x) = \begin{cases} x^2 & , 0 \le x \le \pi \\ -x^2 & , -\pi \le x \le 0 . \end{cases}$$

The fourier series of the given function can be written as

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right)$$

where

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx \quad ; \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\,dx .$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\,dx .$$

The given function is an odd function ,

since

$$-f(x) = f(-x) \quad , \quad \left[\text{from definition above of } f(x) \right]$$

$\therefore \quad a_0 = a_n = 0$

and

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) . \sin(nx)\,dx .$$

$$= \frac{2}{\pi} \left[\int_0^{\pi} x^2 . \sin(nx)\,dx \right]$$

$$= \frac{2}{\pi} \left[x^2 . \left(-\frac{\cos nx}{n} \right) \Big|_0^{\pi} + \int_0^{\pi} 2x \left(\frac{+\cos nx}{n} \right) dx \right]$$

$$= \frac{2}{\pi} \left[\frac{-\pi^2}{n} . \cos(n\pi) + \left[2x . \frac{\sin nx}{n^2} \Big|_0^{\pi} - \int_0^{\pi} \frac{2 \sin n\pi}{n^2} \right] \right]$$

$$= \frac{2}{\pi} \left[\frac{(-1)^n (-1) \pi^2}{n} + \frac{2\pi}{n^2} \sin n\pi + \frac{2}{n^3} (-1)^n - \frac{2}{n^3} \right]$$

$$= \frac{2}{\pi} \left[(-1)^n \left[\frac{2}{n^3} - \frac{\pi^2}{n} \right] + 0 - \frac{2}{n^3} \right]$$

$$b_n = \frac{2}{\pi}\left[(-1)^n\left(\frac{2}{n^3} - \frac{\pi^2}{n}\right) - \frac{2}{n^3}\right]$$

$$\therefore\ b_1 = \frac{2}{\pi}\left[-2 + \pi^2 - 2\right] = 2\left(\pi - \frac{4}{\pi}\right)$$

$$b_2 = \frac{2}{\pi}\left[\frac{\cancel{1}}{4} - \frac{\pi^2}{2} - \frac{\cancel{1}}{4}\right] = -\pi$$

$$b_3 = \frac{2}{\pi}\left[\frac{-2}{27} + \frac{\pi^2}{3} - \frac{2}{27}\right] = \frac{2}{3}\left(\pi - \frac{4}{9\pi}\right)$$

$$b_4 = \frac{2}{\pi}\left[\frac{1}{32} - \frac{\pi^2}{4} - \frac{1}{32}\right] = \frac{-\pi}{2}$$

$$\vdots$$

$$\therefore\ f(x) = \sum_{n=1}^{\infty} b_n \sin(nx) = 2\left(\pi - \frac{4}{\pi}\right)\sin x - \pi\sin 2x + \frac{2}{3}\left(\pi - \frac{4}{9\pi}\right)\sin 3x$$
$$- \frac{\pi}{2}\sin 4x \ldots$$

Q2. An alternating current, after passing through a rectifier has this form

$$i = \begin{cases} I_0 \sin x, & 0 \le x \le \pi \\ 0, & \pi \le x \le 2\pi \end{cases}$$

where I_0 is maximum current and period is 2π.
Express i as a fourier series.

The fourier series for given alternating current is as follows

$$i = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos(nx) + b_n \sin(nx)\right)$$

where $a_0 = \frac{1}{2\pi}\int_0^{2\pi} i\, dx$ $\qquad a_n = \frac{1}{\pi}\int_0^{2\pi} i(x)\cdot\cos(nx)\, dx$.

$$b_n = \frac{1}{\pi}\int_0^{2\pi} i(x)\cdot\sin(nx)\, dx$$

Since

$i(x) = 0$ in $[\pi, 2\pi] = 0$ therefore a_0, b_n, a_n reduces to

$$a_0 = \frac{1}{2\pi}\int_0^{\pi} I_0 \sin x \, dx + 0$$

$$= \frac{I_0}{2\pi}\left[-\cos x\right]_0^{\pi}$$

$$= \frac{I_0}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} I_0 \sin x \cos(nx) \, dx + 0$$

$$= \frac{I_0}{\pi}\int_0^{\pi}\left[\sin(n+1)x - \sin(n-1)x\right] dx$$

$$= \frac{I_0}{\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi} \qquad \boxed{n \neq 1}$$

$$= \frac{I_0}{\pi}\left[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n}}{n-1} + \left(+\frac{1}{n+1} - \frac{1}{n-1}\right)\right]$$

$$= \frac{I_0}{\pi}\left[-\frac{1}{n+1}\left[(-1)^{n+1} - 1\right] + \frac{1}{n-1}\left[1 - (-1)^{n-1}\right]\right]$$

$\therefore$ for $n = 1$

$$a_1 = \frac{1}{\pi}\int_0^{\pi} I_0 \sin x \cos x \, dx$$

$$= \frac{I_0}{2\pi}\int_0^{\pi} \sin 2x \, dx = \frac{I_0}{2\pi}\left[-1 + 1\right] = 0$$

$a_1 = 0$

$a_2 = -\frac{2}{3}$

$a_3 = 0$

$a_4 = -\frac{2}{15}$

$a_5 = 0$

$a_6 = -\frac{2}{35}$

$\vdots$

$$b_n = \frac{1}{\pi}\int_0^\pi i(x)\, \sin(nx)\, dx$$

$$= \frac{I_0}{\pi}\int_0^\pi \sin x \, \sin(nx)\, dx$$

$$= \frac{I_0}{2\pi}\int_0^\pi \left[\cos(n-1)x - \cos(n+1)x\right] dx$$

$$= \frac{I_0}{2\pi}\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]_0^\pi \qquad n \neq 1$$

$$= \frac{I_0}{2\pi}\left[0\right]$$

$$b_n = 0 \qquad n \neq 1$$

$\therefore$ At $n = 1$

$$b_1 = \frac{I_0}{\pi}\int_0^\pi \sin^2 x \, dx$$

$$= \frac{I_0}{\pi}\int_0^\pi \left[\frac{1-\cos 2x}{2}\right] dx$$

$$= \frac{I_0}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_0^\pi$$

$$b_1 = \frac{I_0}{2\pi}\left[\pi\right]$$

$$= \frac{I_0}{2}.$$

$$\therefore\ i = \frac{I_0}{\pi} + \frac{I_0}{2}\sin x - \frac{I_0}{\pi}\left[\frac{2}{1\cdot 3}\cos 2x + \frac{2}{3\cdot 5}\cos 4x \ \ldots\right]$$

Q4 Find the fourier series to represent $f(x) = x\sin x$ for $0 < x < 2\pi$

The fourier series for given function can be written as :-

$$f(x) = x\sin x = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

where

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} x\sin x \, dx \qquad -(1)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x\sin x \cos(nx)\, dx \qquad -(2)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x\sin x \sin(nx)\, dx \qquad -(3)$$

Solving a_0,

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} x\sin x \, dx$$

$$= \frac{1}{2\pi}\left[x\left(\frac{-\cos x}{1}\right)\Big|_0^{2\pi} + \underbrace{\int_0^{2\pi} \cos x \, dx}_{0} \right]$$

$$= \frac{1}{2\pi}\left[-2\pi \cdot 1 \right].$$

$$= -1$$

Solving a_n =, $\frac{1}{\pi}\int_0^{2\pi} x\sin x \cos(nx)\, dx.$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\sin(n+1)x - \sin(n-1)x\right] dx$$

$$= \frac{1}{2\pi}\left[\frac{x(-\cos(n+1)x)}{n+1}\Big|_0^{2\pi} + \int_0^{2\pi} \frac{\cos(n+1)x}{n+1}\, dx \right. \qquad -(4)$$

$$\left. - \left[\frac{x(-\cos(n-1)x)}{n-1}\Big|_0^{2\pi} + \int_0^{2\pi} \frac{\cos(n-1)x}{n-1}\, dx \right] \right]$$

$\boxed{n \neq 1}$

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for $n \neq 1$

$$= \frac{1}{2\pi}\left[\frac{2\pi}{n+1} + \frac{2\pi}{n-1}\right]$$

$$= \frac{1}{n-1} - \frac{1}{n+1}$$

$$a_n = \frac{2}{n^2-1}$$

For $n=1$,

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos x \, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x \sin 2x \, dx .$$

$$= \frac{1}{4\pi}\left[-x\cos 2x\Big|_0^{2\pi} + \underbrace{\int_0^{2\pi} \cos 2x \, dx}_{\text{zero}}\right]$$

$$= \frac{-1}{4\pi} \cdot 2\pi$$

$$a_1 = -1/2.$$

Solving b_n,

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin(nx) \, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\cos(n-1)x - \cos(n+1)x\right] dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\cos(n-1)x \, dx - \frac{1}{2\pi}\int_0^{2\pi} x\cos(n+1)x \, dx$$

$$= \frac{1}{2\pi}\left[\frac{x\sin(n-1)x}{(n-1)}\Big|_0^{2\pi} - \int_0^{2\pi}\frac{\sin(n-1)x\,dx}{n-1}\right] - \frac{1}{2\pi}\left[\frac{x\sin(n+1)x}{n+1}\Big|_0^{2\pi} - \int_0^{2\pi}\frac{\sin(n+1)x\,dx}{n+1}\right]$$

$$= \frac{1}{2\pi}[0] = 0.$$

$\boxed{n \neq 1}$

For $n = 1$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin^2 x \, dx.$$

$$= \frac{1}{2\pi}\left[\int_0^{2\pi}(x - x\cos 2x)\, dx\right]$$

$$= \frac{1}{2\pi}\left[\frac{(2\pi)^2}{2} - \frac{x\sin 2x}{2}\Big|_0^{2\pi} + \int_0^{2\pi}\frac{\sin 2x}{2}\, dx\right]$$

$$= \frac{1}{2\pi}\left[2\pi^2 - 0 + 0\right]$$

$$= \pi.$$

$\therefore$ $a_0 = -1$ $\qquad$ $b_1 = \pi$

$a_1 = -1/2$ $\qquad$ $b_2 = b_3 = b_4 \dots b_n = 0.$

$a_2 = \frac{2}{2^2-1}$

$a_3 = \frac{2}{3^3-1}$

$$\therefore \quad x\sin x = -1 + \pi\sin x - \tfrac{1}{2}\cos x + 2\left[\frac{\cos 2x}{2^2-1} + \frac{\cos 3x}{3^2-1} + \frac{\cos 4x}{4^2-1} \dots\right]$$

Q5 Prove that $x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2}$, $-\pi < x < \pi$

Hence show that

i) $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$ $\qquad$ ii) $\sum \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

The fourier series for given function can be given as follows:-

$$f(x) = x^2 = a_0 + \sum_{n=1}^{n=\infty}\left(a_n\cos(nx) + b_n\sin(nx)\right)$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \; ; \; b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx.$$

Since $f(x) = x^2$ is an even function

$\therefore b_n = 0$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} x^2 \, dx \quad ; \quad a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos(nx)\, dx$$

$$a_0 = \frac{1}{\pi} \left.\frac{x^3}{3}\right]_0^{\pi}$$

$$a_0 = \pi^2/3.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos(nx)\, dx$$

$$= \frac{2}{\pi}\left[\left.\frac{x^2 \sin(nx)}{n}\right|_0^{\pi} - 2\int_0^{\pi} \frac{x\sin(nx)}{n}\, dx \right]$$

$$= \frac{2}{\pi}\left[0 + 2\left[\left. + \frac{x\cos(nx)}{n^2}\right]_0^{\pi} - \cancel{\phi}\, 2\underbrace{\int_0^{\pi} \frac{\cos(nx)}{n^2}\, dx}_{\text{zero}} \right.\right.$$

$$= \frac{4}{\cancel{\pi}}(-1)^n \cdot \frac{\cancel{\pi}}{n^2}$$

$$= \frac{4}{n^2}(-1)^n.$$

$\therefore$

$$f(x) = x^2 = \frac{\pi^3}{3} + 4\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n^2}\cos(nx) \right]$$

$$x^2 = \frac{\pi^3}{3} + 4\sum_{n=1}^{\infty}\left[\frac{(-1)^n \cos(nx)}{n^2} \right].$$

i) Put $x = \pi$

$$\therefore \pi^2 = \frac{\pi^3}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{n^2}$$

$$\therefore \frac{2\cancel{2}\pi^3}{3} = \cancel{4}\,2\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\therefore \quad \frac{\pi^2}{6} = \sum \frac{1}{n^2}$$

HP

ii) Put $x = 0$

$$-\frac{\pi^2}{3} = 4 \sum \frac{(-1)^n}{n^2}$$

$$-\frac{\pi^2}{12} = -\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} \cdots$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \cdots$$

Adding it to $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \cdots$

we get

$$\frac{\pi^2}{4} = 2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots\right.$$

$$\frac{\pi^2}{8} = \sum \frac{1}{(2n-1)^2}$$

HP

Q6 Obtain a fourier series to represent the function $f(x) = |\sin x|$ for $-\pi < x < \pi$.

$$f(x) = \begin{cases} \sin x, & x \geq 0 \\ -\sin x, & x < 0. \end{cases}$$

The fourier series for given function ~~said~~ is :-

$$f(x) = |\sin x| = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where
$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx.$$

Since $f(x) = f(-x)$, therefore $f(x)$ is an even function.

fourier series reduces to,

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx)) \quad ; \quad b_n = 0$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} \sin x\,dx$$

$$= \frac{1}{\pi} -\cos x \Big|_0^{\pi} = \frac{2}{\pi}.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin(x)\cos(nx)\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi} \sin(n+1)x - \sin(n-1)x\right]dx$$

$$= \frac{1}{2}\times\frac{2}{\pi}\left[\frac{\cos(n+1)x}{-(n+1)} + \frac{\cos(n-1)x}{(n-1)}\Big|_0^{\pi}\right] \qquad \boxed{n \neq 1}$$

$$= \frac{1}{2}\times\frac{2}{\pi}\left[\frac{(-1)^{n+1}-1}{-(n+1)} + \frac{(-1)^{n-1}-1}{(n-1)}\right] \qquad (-1)^{n+1} = (-1)^2(-1)^{n-1}$$

$$= \frac{1}{2}\times\frac{2}{\pi}\left[(-1)^{n-1}-1\right]\left[\frac{1}{(n-1)} - \frac{1}{n+1}\right]$$

$$= \frac{1}{2}\times\frac{2}{\pi}\left[(-1)^{n-1}-1\right]\frac{2}{(n^2-1)}$$

$$= \frac{2}{\pi}\left[\frac{(-1)^{n-1}-1}{(n^2-1)}\right].$$

For odd value of 'n'

$$(-1)^{n-1} = 1$$

$\therefore$

$$a_n = 0, \quad [\text{for odd } n]$$

And

$$a_n = -\frac{4}{\pi(n^2-1)} \quad \text{for even } n$$

$\boxed{\text{For } n=1}$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \frac{\sin 2x}{2}\,dx$$

$$= \frac{1}{\pi}\frac{\cos 2x}{-2}\Big|_0^{\pi} dx = 0$$

Hence the fourier series becomes.

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{1}{3}\cos 2x + \frac{1}{15}\cos 4x + \frac{1}{35}\cos 6x \ldots\right]$$

Q7. Find the half range cosine series for $f(x) = e^x$

$0 \le x < \pi$.

Assuming the function $f(x)$ in the interval $[-\pi, 0)$, such that it becomes even function in the interval $[-\pi, \pi]$, therefore half range fourier series will be

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

where
$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$$
$$a_n = 2 \cdot \frac{1}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx.$$

Solving a_0,
$$a_0 = \frac{1}{\pi}\int_0^{\pi} e^x\,dx = \frac{1}{\pi}\left[e^{\pi} - 1\right]$$

Solving a_n
$$a_n = \frac{2}{\pi}\int_0^{\pi} e^x \cos(nx)\,dx$$

Consider
$$I = \int e^x \cos(nx)\,dx$$
$$= \cos nx\, e^x + n\int e^x \sin nx\,dx$$
$$I = \cos(nx)e^x + n\Big[\sin nx\, e^x - n\underbrace{\int \cos(nx)e^x\,dx}_{I}\Big]$$
$$I = -n^2 I + e^x\left[\cos(nx) + n\sin(nx)\right]$$
$$I = \frac{e^x}{1+n^2}\left[\cos(nx) + n\sin(nx)\right].$$

$$\therefore\ a_n = \frac{2}{\pi}\left[\frac{e^x}{1+n^2}\left[\cos(nx) + n\sin(nx)\right]\right]_0^{\pi}$$
$$= \frac{2}{\pi}\,\frac{1}{1+n^2}\left[e^{\pi}(-1)^n - 1\right]$$
$$a_n = \frac{2}{\pi}\left[\frac{e^{\pi}(-1)^n - 1}{1+n^2}\right]$$

This fourier series becomes

$$f(x) = e^x = \frac{e^\pi - 1}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{e^\pi(-1)^n - 1}{n^2+1}\right]\cos(nx)$$

Q8. Find the fourier series to represent $f(x)$, where

$$f(x) = \begin{cases} -a, & -c < x < 0 \\ a, & 0 < x < c \end{cases}$$

The given function function is periodic in range $(-c, c)$, converting it into an interval $(-\pi, \pi)$, we get the fourier transform as follow

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{c}\right) + b_n \sin\left(\frac{n\pi x}{c}\right)\right]$$

where

$$a_0 = \frac{1}{2c}\int_{-c}^{c} f(x)\,dx$$

$$a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\left(\frac{n\pi x}{c}\right)dx$$

$$b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\left(\frac{n\pi x}{c}\right)dx$$

Since, the given function is a odd function in range $(-c, c)$, ~~[illegible]~~ the fourier transform reduces to

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{c}\right)dx$$

where

$$b_n = \frac{2}{c}\int_{0}^{c} f(x)\sin\left(\frac{n\pi x}{c}\right)dx$$

Solving b_n, we get

$$b_n = \frac{2}{c}\int_{0}^{c} a\sin\left(\frac{n\pi x}{c}\right)dx$$

$$= \frac{2a}{c}\left.\frac{-\cos\left(\frac{n\pi x}{c}\right)}{\frac{n\pi}{c}}\right|_0^c$$

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$$b_n = \frac{2a}{n\pi}\left[1-(-1)^n\right]$$

for odd value of 'n' , $(-1)^n = -1$

$$b_n = \frac{4a}{n\pi}$$

for even value of 'n' ; $(-1)^n = 1$

$$b_n = 0$$

$\therefore$ $b_1 = \frac{4a}{\pi}$, $b_3 = \frac{4a}{3\pi}$ . . .

$b_2 = b_4 = b_6 \dots = 0$.

Hence fourier series become

$$f(x) = \frac{4a}{\pi}\left[\sin\left(\frac{\pi x}{c}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{c}\right) + \frac{1}{5}\sin\left(\frac{5\pi x}{c}\right) \dots\right].$$

Q9 If $-\pi < x < \pi$, prove that

$$x\sin x = 1 - \frac{1}{2}\cos x - \frac{2\cos 2x}{1\cdot 3} + \frac{2\cos 3x}{2\cdot 4} - \frac{2\cos 4x}{3\cdot 5} \dots$$

And Hence show that .

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} \dots$$

Ans 9

$$f(x) = x\sin x$$

$$f(-x) = -x \cdot -\sin x = x\sin x$$

$$= f(x)$$

Hence, f(x) is a even function.

Therefore, the fourier series for f(x) will be

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) \quad ; \; b_n = 0$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} x\sin x\,dx .$$

$$= \frac{1}{\pi}\left[-x\cos x + \sin x\right]_0^{\pi} = \frac{1}{\pi}[+\pi] = 1$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\sin x \cos(nx)\,dx$$

$$= \frac{2}{\pi}\cdot\frac{1}{2}\int_0^{\pi} x(\sin(n+1)x) - x\sin(n-1)x \,dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos(n+1)x}{n+1}\right) + \frac{\sin(n+1)x}{(n+1)^2} + \frac{x\cos(n-1)x}{n-1} - \frac{\sin(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{(-1)^{n+1}\pi}{n+1} + \frac{(-1)^{n-1}\pi}{n-1}\right] \qquad n \neq 1$$

$$= (-1)^{n+1}\frac{2}{n^2-1} \qquad n \neq 1$$

For $n = 1$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos x\,dx$$

$$= \frac{2}{\pi}\cdot\frac{1}{2}\int_0^{\pi} x\sin 2x\,dx = x\left[\frac{-\cos 2x}{2}\right]_0^{\pi} - \left[-\frac{\sin 2x}{4}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\cdot -\frac{\pi}{2} = -\frac{1}{2}.$$

$\therefore\ a_1 = -1/2$, $a_2 = -2/3$, $a_3 = 2/8$, $a_4 = -2/15 \ldots$

Hence fourier series becomes

$$x\sin x = 1 - \frac{\cos x}{2} - \frac{2}{3}\cos 2x + \frac{2}{8}\cos 3x \ldots$$

At $x = \pi/2$.

$$\pi/2 = 1 - 0 - 2/3(-1) + \frac{2}{8}(0) - \frac{2}{3\cdot 5}(1)$$

$$\frac{\pi}{2\cdot 2} = 1/2 + \frac{1}{3} - \frac{1}{15} \ldots$$

$$\frac{\pi}{4} = 1/2 + \frac{1}{3} - \frac{1}{15} + \frac{1}{35} - \frac{1}{63} \ldots$$

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Q10 Find the fourier series for the func $f(x) = 2x - x^2$, $0 < x < 3$, and deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

The fourier series for given function can be given as

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right)$$

where

$$a_n = \frac{1}{c} \int_0^3 f(x) \cos\left(\frac{n\pi x}{c}\right) dx$$

$$b_n = \frac{1}{c} \int_0^3 f(x) \sin\left(\frac{n\pi x}{c}\right) dx$$

$$a_0 = \frac{1}{2c} \int_0^3 f(x)\, dx$$

where $f(x) = 2x - x^2$ and c is half interval size i.e $3/2$ $\therefore$ $c = 3/2$

$$\therefore \quad a_0 = \frac{1}{3} \int_0^3 (2x - x^2)\, dx = \frac{1}{3} \left[x^2 - \frac{x^3}{3} \right]_0^3 = 0$$

$$a_n = \frac{2}{3} \int_0^3 (2x - x^2) \cos\left(\frac{2\pi x n}{3}\right) dx$$

$$= \frac{2}{3} \left[(2x - x^2) \left(\frac{3}{2n\pi}\right) \sin\left(\frac{2\pi x n}{3}\right) \Big|_0^3 - \int_0^3 (2 - 2x)\left(\frac{3}{2n\pi}\right) \sin\left(\frac{2xn\pi}{3}\right) dx \right.$$

$$= \frac{2}{3} \left[0 - \frac{3}{n\pi} \left[\int_0^3 \sin\left(\frac{2nx\pi}{3}\right) \cdot (1 - x) \right] \right.$$

$$= -\frac{2}{n\pi} \left[\frac{3}{2n\pi} \left[-\cos\left(\frac{2nx\pi}{3}\right) (1 - x) \Big|_0^3 + \underbrace{\int_0^3 -\cos\left(\frac{2xn\pi}{3}\right) dx}_{\text{zero}} \right. \right.$$

$$= -\frac{3}{n^2\pi^2} \left[2 - (-1) \right]$$

$$= -\frac{9}{n^2\pi^2}$$

$$b_n = \frac{1}{3/2} \int_0^3 (2x - x^2) \sin\left(\frac{2n\pi x}{3}\right) dx$$

$$= \frac{2}{3} \left[-(2x-x^2)\left(\frac{3}{2n\pi}\right)\cos\left(\frac{2n\pi x}{3}\right) + \frac{(1-x)\left(\frac{3}{n\pi}\right)\left(\frac{3}{2n\pi}\right)}{} \sin\left(\frac{2n\pi x}{3}\right) + (-1)\left(\frac{3}{n\pi}\right)\left(\frac{3}{2n\pi}\right)\cos\left(\frac{2n\pi x}{3}\right) \right]_0^3$$

$$= \frac{3}{n\pi}.$$

Fourier series become :⇒

$$f(x) = 0 + \sum_{n=1}^{\infty} \left(\frac{-9}{n^2\pi^2} \cos\left(\frac{2n\pi x}{3}\right) + \frac{3}{n\pi} \sin\left(\frac{2n\pi x}{3}\right) \right]$$

⇒ Graph ⇒

From graph, $f(x)$ is discontinuous at $x=3$,
$x=6$,

At $x=3$

$$f(3) = \frac{1}{2}\left[f(3^+) + f(3^-) \right]$$

$$= \frac{1}{2}\left[-3 + 0 \right] = -3/2.$$

$$\therefore -3/2 = -\frac{9}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} \cdots \right]$$

$$\frac{\pi^2}{6} = \sum \frac{1}{n^2}$$

H/P$_s$

Q 11. Obtain the half range sine series of $f(x) = lx - x^2$ in $(0, l)$ and hence show

$$\frac{1}{1^3} - \frac{1}{3^2} + \frac{1}{5^3} \cdots = \frac{\pi^3}{32}.$$

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Assuming the function $f(x)$ in the interval $(-l, 0)$ such that $f(x)$ become odd func in interval $(-l, l)$, Then half range fourier series becomes

$$f(x) = lx - x^2 = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n x \pi}{l}\right) \qquad \boxed{c = l}$$

where

$$b_n = \frac{2}{l}\int_0^l (lx - x^2)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left[(lx - x^2)\left(-\cos\left(\frac{n\pi x}{l}\right)\cdot\frac{l}{n\pi}\right)\Big|_0^l + \int_0^l \cos\left(\frac{n\pi x}{l}\right)\frac{l}{n\pi}(l - 2x)\,dx\right.$$

$$= \frac{2}{\cancel{l}}\left[0 + \frac{\cancel{l}}{n\pi}\int_0^l \cos\left(\frac{n\pi x}{l}\right)(l - 2x)\,dx\right.$$

$$= \frac{2}{n\pi}\left[\sin\left(\frac{n\pi x}{l}\right)\cdot\frac{l}{n\pi}\cdot l - 2x\Big|_0^l + \int_0^l \sin\left(\frac{n\pi x}{l}\right)\cdot\frac{l}{n\pi}\cdot(+2)\,dx\right.$$

$$= \frac{2}{n\pi}\cdot\frac{2l}{n\pi}\int_0^l \sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{4l}{n^2\pi^2}\, -\cos\left(\frac{n\pi x}{l}\right)\cdot\frac{l}{n\pi}\Big|_0^l$$

$$= \frac{4l^2}{n^3\pi^3}\left[1 - (-1)^n\right]$$

$\therefore$ Half range sine series for $f(x)$ is.

$$lx - x^2 = \sum_{n=1}^{\infty} \frac{4l^2}{n^3\pi^3}\left(1 - (-1)^n\right)\sin\left(\frac{n\pi x}{l}\right)$$

For even 'n', $1 - (-1)^n = 1 - 1 = 0$

$\therefore$ Series ~~~~ be

$$lx - x^2 = \sum_{n=1}^{\infty} \frac{4l^2 \cdot 2}{(2n-1)^3(\pi^3)} \sin\left(\frac{(2n-1)\pi x}{l}\right)$$

At $x = l/2$, continuous point

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$$\Rightarrow \quad \frac{l^4}{4} = \frac{8l^2}{\pi^3} \sum \frac{1}{(2n-1)^3} \sin\frac{(2n-1)\pi}{2}$$

$$\Rightarrow \quad \frac{\pi^3}{32} = \sum \frac{1}{(2n-1)^3} \sin\frac{(2n-1)\pi}{2}$$

$$\Rightarrow \quad \frac{\pi^3}{32} = \left[\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} \cdots\right]$$

Q3 Discuss the convergence of fourier series.

Let us consider an function $f(x)$ which is an period function of $2c$,

such that

$$f(x+2c) = f(x)$$

Containing finite number of discontinuity with piece wise continuity and finite number of maxima and minima in interval given, then at a point 'x' in interval the fourier series converges to

$$f(x) = \frac{1}{2}\left[f(x^+) + f(x^-)\right]$$

At a continous point,

$$f(x^+) = f(x^-) = f(x)$$

LHS $\rightarrow$ $f(x)$

RHS $\Rightarrow$ $\frac{1}{2}(f(x) + f(x))$

$= f(x)$.

At a discountinous point, the fourier series of converges to

$$f(x) = \frac{1}{2}\left[f(x^+) + f(x^-)\right].$$