DELTA Pg No.
Date / /

DEEPAK MISHRA

ROLL No → 66

BATCH → B-5

MATHS - ASSIGNMENT-4

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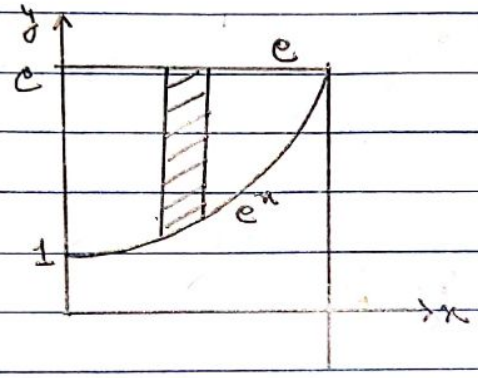
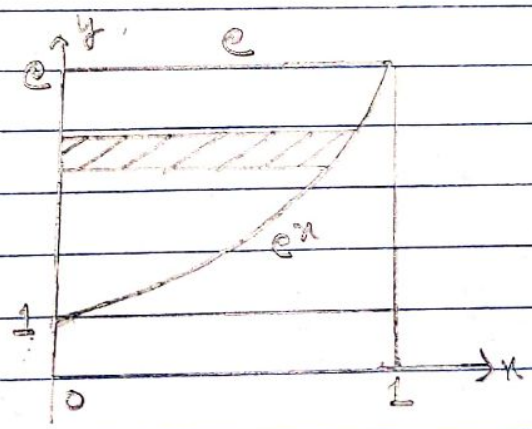
1. Evaluate the following integral by changing order of integration :

a) $\int_0^1 \int_{e^x}^{e} \frac{dx\,dy}{\log y}$.

→



changing order of integration :→



$$\int_1^e \int_0^{\ln y} \frac{dx\,dy}{\log y}$$

$$\rightarrow \int_1^e \frac{(\log y)\,dy}{\log y} \rightarrow \int_1^e dy \rightarrow [y]_1^e$$

$$\rightarrow (e-1).$$

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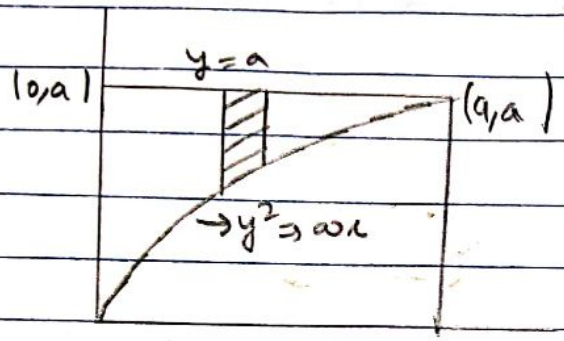
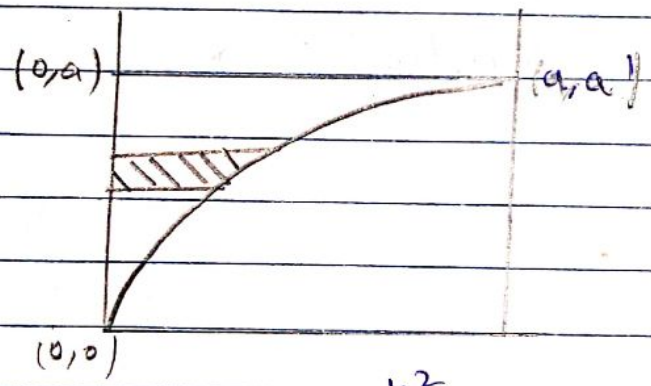
b) $\int_0^a \int_{\sqrt{ax}}^a \frac{y^2\,dy\,dx}{\sqrt{y^4-a^2x^2}}$



changing order of integration :→



Area ⇒ $\int_0^a \int_0^{\frac{y^2}{a}} \frac{y^2\,dy\,dx}{\sqrt{y^4-a^2x^2}}$

$\longrightarrow \int_0^a \left(\int_0^{y^2/a} \frac{1}{a} y^2 \frac{1}{\sqrt{(y^2/a)^2 - (x)^2}}\,dx \right) dy$

$\longrightarrow \int_0^a \frac{y^2}{a} \left[\sin^{-1}\left(\frac{xa}{y^2}\right)\Big]_0^{y^2/a} \right] dy$

$\longrightarrow \int_0^a \frac{y^2}{a} \times \frac{\pi}{2}\,dy \longrightarrow \frac{\pi}{2a} \times \left[\frac{y^3}{3}\right]_0^a$

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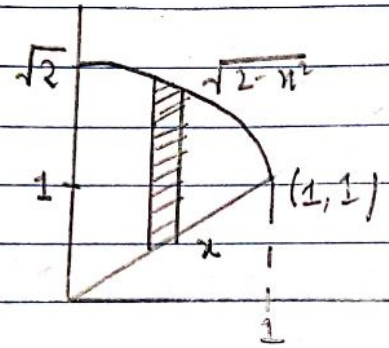
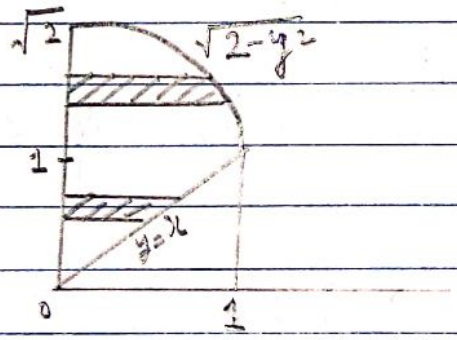
$\longrightarrow \dfrac{\pi a^2}{6}$.

(c) $\displaystyle\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$



changing order of integral.



$\longrightarrow \displaystyle\int_0^1 \int_0^y \left(\frac{x\,dx}{\sqrt{x^2+y^2}}\right) dy + \int_1^{\sqrt{2}} \int_0^{\sqrt{2-y^2}} \left(\frac{x\,dx}{\sqrt{x^2+y^2}}\right) dy$

$\longrightarrow \displaystyle\int_0^1 \left[\sqrt{x^2+y^2}\right]_0^y dy + \int_1^{\sqrt{2}} \left[\sqrt{x^2+y^2}\right]_0^{\sqrt{2-y^2}} dy$

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$\rightarrow \int_0^1 (\sqrt{2}-1)y\,dy + \int_1^{\sqrt{2}} (\sqrt{2}-y)\,dy$

$\rightarrow \left[(\sqrt{2}-1)\frac{y^2}{2}\right]_0^1 + \left[\sqrt{2}y - \frac{y^2}{2}\right]_1^{\sqrt{2}}$

$\rightarrow \frac{(\sqrt{2}-1)1}{2} + \left(\sqrt{2}\times\sqrt{2} - \frac{2}{2}\right) - \left(\sqrt{2} - \frac{1}{2}\right)$

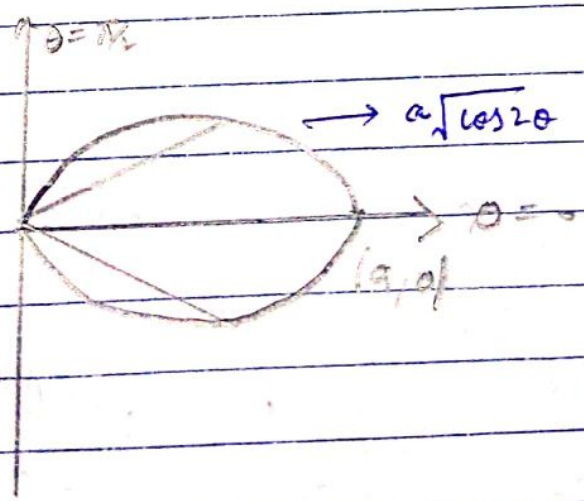
$\rightarrow \frac{\sqrt{2}-1}{2} + 1 + \frac{1}{2} - \sqrt{2}$

$\rightarrow \frac{1}{\sqrt{2}} + 1 - \sqrt{2} \rightarrow \frac{1-2}{\sqrt{2}} + 1 \rightarrow 1 - \frac{1}{\sqrt{2}}$.

(2) Evaluate $\iint \frac{r\,dr\,d\theta}{\sqrt{a^2+r^2}}$ over one loop of lemniscate $r^2 = a^2\cos 2\theta$.

$\rightarrow$ Region $= \left\{(r,\theta),\ 0 \le r \le a\sqrt{\cos 2\theta},\ -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}\right\}$.

$\rightarrow$



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$\rightarrow \quad \cos 2\theta \Rightarrow 0.$

$2\theta \Rightarrow \frac{\pi}{2}, -\frac{\pi}{2}$

$\theta \Rightarrow \frac{\pi}{4}, -\frac{\pi}{4}.$

$$\therefore \text{Area} = \int_{-\pi/4}^{\pi/4} \int_0^{a\sqrt{\cos 2\theta}} \left(\frac{r\,dr}{\sqrt{r^2+a^2}} \right) d\theta$$

$$\rightarrow \int_{-\pi/4}^{\pi/4} \frac{1}{2} \int_0^{a\sqrt{\cos 2\theta}} \frac{2r\,dr}{(\sqrt{r^2+a^2})} d\theta$$

$$\rightarrow \int_{-\pi/4}^{\pi/4} \left[\sqrt{r^2+a^2} \right]_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$\rightarrow \int_{-\pi/4}^{\pi/4} \left[\sqrt{a^2(\cos 2\theta + 1)} - a \right] d\theta.$$

$$\rightarrow \int_{-\pi/4}^{\pi/4} \sqrt{2}\, a\, |\cos\theta|\, d\theta - a \int_{-\pi/4}^{\pi/4} d\theta.$$

$$\rightarrow \sqrt{2}\, a \times \left(\sin\theta \right)_{-\pi/4}^{\pi/4} - \frac{a\pi}{2}.$$

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$\rightarrow \sqrt{2}a \times \left(\frac{1}{\sqrt{2}}\right) - \frac{a\pi}{2}$

$\rightarrow 2a - \frac{a\pi}{2}.$

$\rightarrow 2a\left(1 - \frac{\pi}{4}\right).$

(3) Evaluate the given double integral by converting to polar coordinates.

$\rightarrow \int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\, dx\, dy.$

$\rightarrow$ changing to polar coordinates :-

$\rightarrow \int_0^{\pi/2}\int_0^{\infty} (e^{-r^2})\, r\, dr\, d\theta.$

$\rightarrow \frac{-1}{2}\int_0^{\pi/2} d\theta \int_0^{\infty} (-2re^{-r^2})\, dr.$

$\rightarrow -\frac{1}{2} \times \frac{\pi}{2} \times \left[e^{-r^2}\right]_0^{\infty}$

$\rightarrow -\frac{\pi}{4} \times [0 - 1] \rightarrow \frac{\pi}{4}.$

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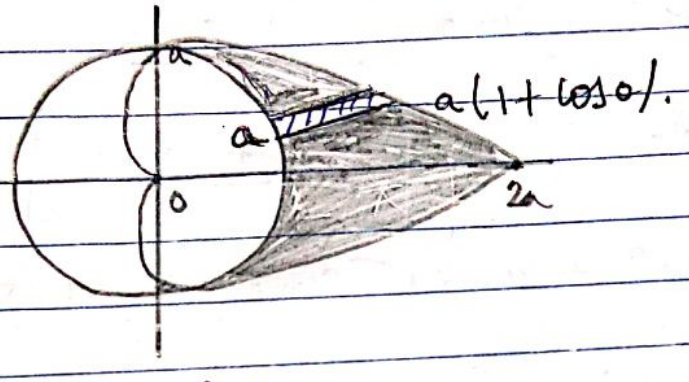
(4) Find by double integen area lying inside cardioid $r = a(1+\cos\theta)$ & outside circle $r = a$.

→



→ Intersection point

$$a = a(1+\cos\theta).$$

$$\boxed{\cos\theta = 0.}$$

$$\boxed{\theta = \pm\frac{\pi}{2}, -\frac{\pi}{2}.}$$

Area

$$\rightarrow 2\int_0^{\pi/2}\int_a^{a(1+\cos\theta)} r\,dr\,d\theta.$$

$$\rightarrow \cancel{2}\int_0^{\pi/2}\left[\frac{r^2}{\cancel{2}}\right]_a^{a(1+\cos\theta)} d\theta.$$

$$\rightarrow a^2\int_0^{\pi/2}(1+\cos\theta)^2-(1)^2$$

$$\rightarrow a^2\int_0^{\pi/2}(2+\cos\theta)\cos\theta\,d\theta$$

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$$\longrightarrow \int_0^{\pi/2} a^2 (2\cos\theta)\, d\theta + a^2 \int_0^{\pi/2} \cos^2\theta \, d\theta .$$

$$\longrightarrow 2a^2 \times \left[\sin\theta\right]_0^{\pi/2} + \frac{a^2}{2} \times \int_0^{\pi/2} \left[\cos 2\theta + 1\right]$$

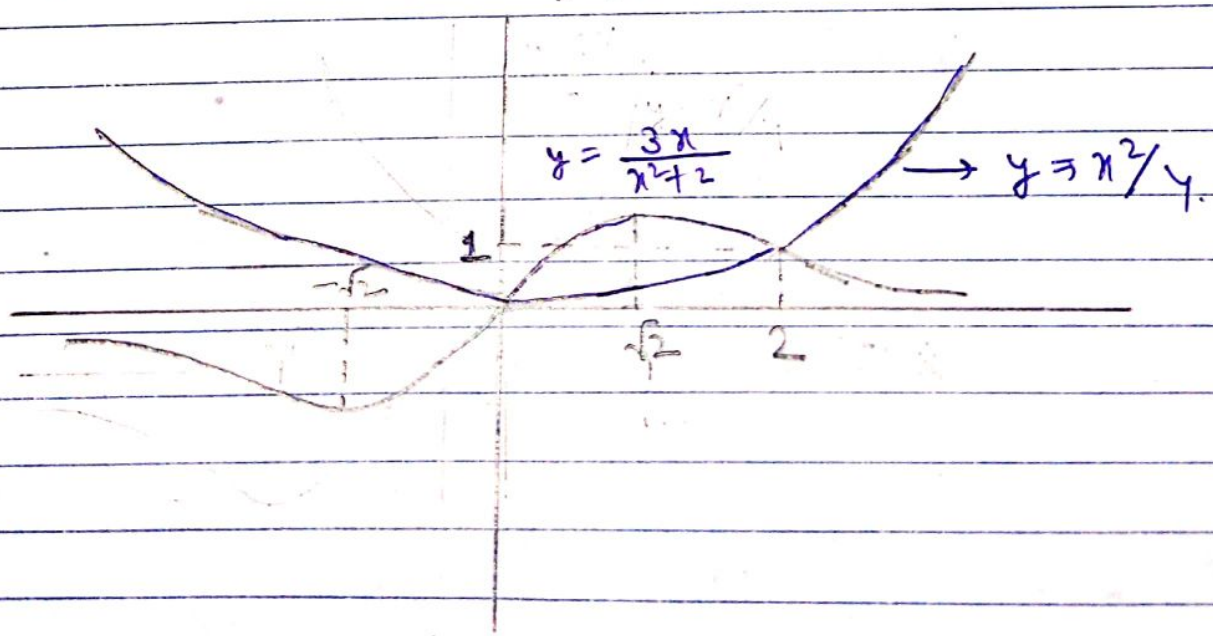
$$\longrightarrow 2a^2 + \frac{a^2}{2}\left[\frac{\sin 2\theta}{2} + \theta\right]_0^{\pi/2}$$

$$\longrightarrow 2a^2 + \frac{a^2}{2} \times \left[\frac{\sin 2\pi - \sin 0}{2} + \frac{\pi}{2}\right]$$

$$\longrightarrow \frac{\pi a^2}{4} + 2a^2$$

$$\longrightarrow a^2\left(\frac{\pi + 8}{4}\right).$$

(5) Find, by double integration, area enclosed by the curves $y = \dfrac{3x}{x^2+2}$ and $4y = x^2$.



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$$\int_0^2 \int_{x^2/4}^{\frac{3x}{x^2+2}} (dy)\,dx.$$

$$\rightarrow \int_0^2 [y]_{x^2/4}^{3x/x^2+2}\, dx.$$

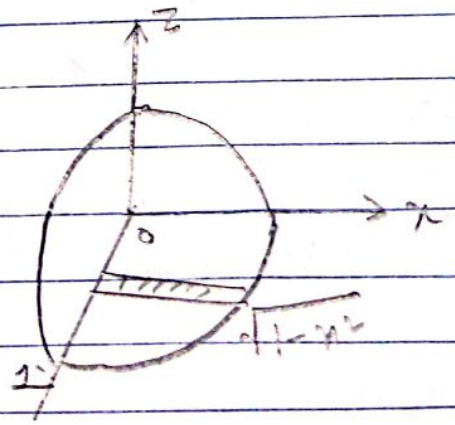
$$\rightarrow \int_0^2 \left(\frac{3x}{x^2+2} - \frac{x^2}{4}\right) dx$$

$$\rightarrow \frac{3}{2} \ln|x^2+2|_0^2 - \frac{1}{12} \times [8-0]$$

$$\rightarrow \frac{3}{2} (\ln 6 - \ln 2) - \frac{8}{12}$$

$$\rightarrow \frac{3}{2} \ln 3 - \frac{2}{3}.$$

(6) Evaluate $\rightarrow \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-(x^2+y^2)}} (xyz)\, dx\, dy\, dz.$



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changing to spherical coordinates :→
$x \to r\sin\phi\cos\theta$, $y \to r\sin\phi\sin\theta$ & $z \to r\cos\phi$.

$$\to \int_0^1 \int_0^{\pi/2} \int_0^{\pi/2} \left[(r)\sin\phi\cos\theta\right]\left[r\sin\phi\sin\theta\right] \times \left(r\cos\phi\right) r^2 \sin\phi \, dr \, d\theta \, d\phi .$$

$$\to \int_0^1 \int_0^{\pi/2} \int_0^{\pi/2} (r^5) \times \left(\sin^3\phi \, d\phi \cos\phi\right) \times \left(\sin\theta\cos\theta \, d\theta\right).$$

$$\to \int_0^1 r^5 \times \int_0^{\pi/2} \sin^3\phi \, d\phi \cos\phi \times \int_0^{\pi/2} \frac{\sin 2\theta}{2} \, d\theta .$$

$$\to \frac{1}{6} \times \frac{\left[\sin^4\theta\right]_0^{\pi/2}}{4} \times \frac{-\left[\cos 2\theta\right]_0^{\pi/2}}{4}$$

$$\to \frac{1}{6} \times \frac{1}{4} \times \frac{-\left[-1-1\right]}{4} \to \frac{1}{6} \times \frac{1}{4} \times \frac{1}{2} .$$

$$\to \frac{1}{48} .$$

(7) Evaluate the given integral by converting into spherical coordinates.

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{1} \frac{dz \, dy \, dx}{\sqrt{x^2+y^2+z^2}} .$$

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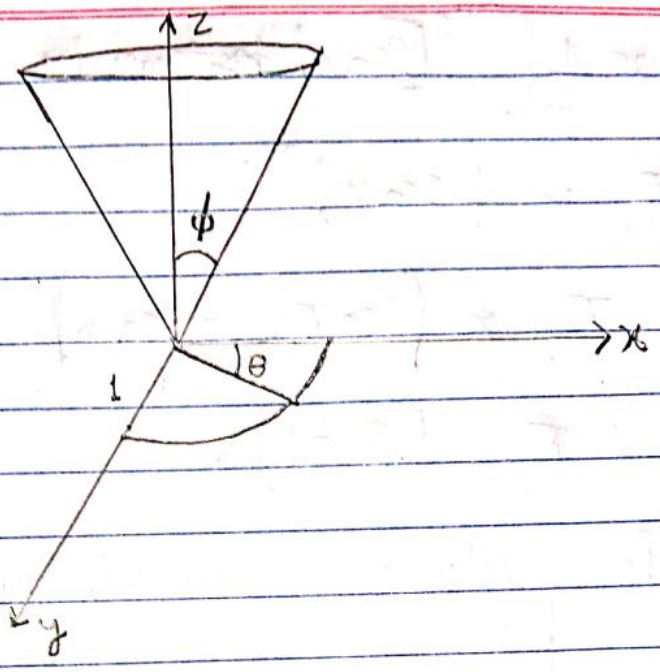

Let $x \to r\sin\phi\cos\theta$, $y \to r\sin\phi\sin\theta$, $z \to r\cos\phi$.

ϕ can ~~goes~~ from $(0 \text{ to } \frac{\pi}{4})$, θ from $(0 \text{ to } \frac{\pi}{2})$

and r : 0 to $\sec\phi$,

$$\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{\sec\phi} \frac{(r^2\sin\phi)\,d\theta\,d\phi\,dr}{r}$$

$$\longrightarrow \int_0^{\pi/2} d\theta \times \int_0^{\pi/4}\int_0^{\sec\phi} (r\sin\phi)\,dr\,d\phi$$

$$\longrightarrow \left[\frac{\pi}{2}\right]_0^{\pi/2} \times \int_0^{\pi/4} \frac{\sin\phi}{2} \times [\sec^2\phi]\,d\phi .$$

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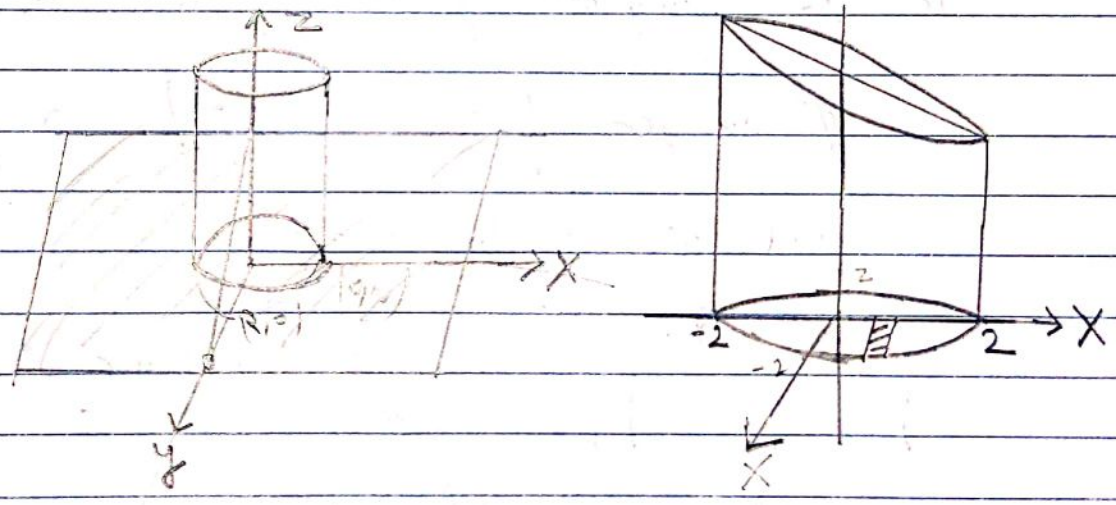
$$\rightarrow \frac{\pi}{2} \times \frac{1}{2}\int_0^{\pi/4} (\sec x)\, \tan x\, dx.$$

$$\rightarrow \frac{\pi}{4} \times \left| \sec x \right|_0^{\pi/4}$$

$$\rightarrow \frac{\pi}{4} \times \left[\sec \pi/4 - \sec 0 \right]$$

$$\rightarrow \frac{\pi}{4} \times (\sqrt{2} - 1).$$

(8) Find the volume bounded by cylinder $x^2+y^2=4$ and the plane $y+z=4$ and $z=0$.



Volume $\rightarrow$

$$\iint z\, dx\, dy.$$

$$V \rightarrow 2\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}} (z)\, dy\, dx$$

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changing z in terms of y.

$$\longrightarrow 2\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}} (4-y)\, dy\, dx.$$

<u>Changing order in circular form :</u> $\longrightarrow$

$$\longrightarrow 2\left[\int_{-\pi/2}^{\pi/2}\int_{0}^{2} (4-r\sin\theta)\, r\, d\theta\, dr\right.$$

$$\longrightarrow 2\left[\int_{-\pi/2}^{\pi/2} d\theta \int_{0}^{2} (4r)\, dr - \int_{-\pi/2}^{\pi/2} \sin\theta\, d\theta \times \int_{0}^{2} r^2\, dr\right.$$

$$\longrightarrow 2\times(\pi)\times 2\times(2)^2 + \left[\cos\left(\frac{\pi}{2}\right) - \cos\frac{\pi}{2}\right]\times\left(\frac{2^3}{3}\right)$$

$$\longrightarrow 16\pi - 0 \longrightarrow 16\pi.$$

Q9. using triple integration, find the value of ellipsoid :$\longrightarrow$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \Rightarrow 1.$$

$\longrightarrow$ volume :$\longrightarrow$

$$\int_0^a\int_0^{a\sqrt{1-y^2/b^2}}\int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\, dy\, dx. \longrightarrow \int_0^a\int_0^{a\sqrt{1-\frac{y^2}{b^2}}}\int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\, dy\, dx$$

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→ Let $\frac{x}{a} = p$, $\frac{y}{b} = q$, $\frac{z}{c} = r$.

For Jacobian transformation :→
$dx\,dy\,dz$ to $dp\,dq\,dr$.

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial p} & \frac{\partial x}{\partial q} & \frac{\partial x}{\partial r} \\ \frac{\partial y}{\partial p} & \frac{\partial y}{\partial q} & \frac{\partial y}{\partial r} \\ \frac{\partial z}{\partial p} & \frac{\partial z}{\partial q} & \frac{\partial z}{\partial r} \end{vmatrix} \rightarrow \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$$

$$\boxed{|J| = abc}$$

$$\therefore \quad \boxed{p^2 + q^2 + r^2 = 1}$$

$p^2 + q^2 + r^2 = 1$ is an equation of sphere.

$$dV = \int_0^1 \int_0^{\sqrt{1-p^2}} \int_0^{\sqrt{1-p^2-q^2}} dp\,dq\,dr\,|J|.$$

$$dV = |J| \int_0^1 \int_0^{\sqrt{1-p^2}} \int_0^{\sqrt{1-p^2-q^2}} dp\,dq\,dr.$$

$$\rightarrow abc \int_0^1 \int_0^{\sqrt{1-p^2}} \int_0^{\sqrt{1-p^2-q^2}} dp\,dq\,dr$$

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solving $\int_0^1 \int_0^{\sqrt{1-p^2}} \int_0^{\sqrt{1-(p^2+q^2)}} (dp\,dq\,dr)$ by spherical coordinates.

$\rightarrow 8 \int_0^1 \int_0^{\pi/2} \int_0^{\pi/2} (r^2) \sin\phi \, dr\, d\theta\, d\phi$

where
$p \rightarrow r\sin\phi\cos\theta$
$q \rightarrow r\sin\phi\sin\theta$
$q \rightarrow r\cos\phi$

$\rightarrow 8 \times \left[\frac{r^3}{3}\right]_0^1 \times \left[\sin\phi\right]_0^{\pi/2} \times \left[d\theta\right]_0^{\pi/2}$

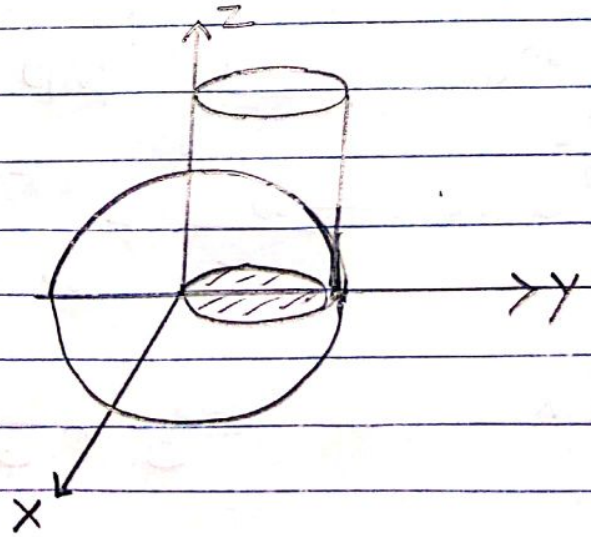
$\rightarrow \frac{4\pi}{3}$

volume $\rightarrow \frac{4\pi}{3} \times |J|$

$\rightarrow \frac{4\pi}{3} (abc)$.

$\rightarrow \frac{4\pi}{3} (abc)$ unit^3.

10. Find the volume of the portion of the sphere $x^2+y^2+z^2=a^2$ lying inside the cylinder $x^2+a^2 \Rightarrow ay$.



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Volume $\longrightarrow \iint z \, dx \, dy$

changing to polar coordinate $\Rightarrow$

$x^2+y^2 \rightarrow ay$

$r^2 = ar\sin\theta$.

$\boxed{r = a\sin\theta.}$

and θ from $(0$ to $\pi.)$

$z \rightarrow \sqrt{a^2-(x^2+y^2)} \rightarrow \sqrt{a^2-r^2}$

Volume $\rightarrow$ $2\int_0^{\pi}\left[\int_0^{a\sin\theta}\left(\sqrt{a^2-r^2}\right) r \, dr\right] d\theta$

$\rightarrow 2 \times \int_0^{\pi}\left(-\frac{1}{3}\left[\sqrt{a^2-r^2}\right]^3\right)_0^{a\sin\theta} d\theta$

$\rightarrow -\frac{2}{3}\int_0^{\pi}\left(a^3|\cos|^3\theta - a^3\right) d\theta.$

$\rightarrow \frac{2}{3}\int_0^{\pi} a^3 \, d\theta - \frac{2a^3}{3}\int_0^{\pi} |\cos|^3\theta \, d\theta.$

$\rightarrow \frac{2}{3}\int_0^{\pi} a^3 \, d\theta - \frac{2a^3}{3} \times 2\int_0^{\pi/2} \cos^3\theta \, d\theta$

$\rightarrow \frac{2}{3}a^3\pi. - \frac{4a^3}{3} \times \frac{2}{3}$

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$\longrightarrow \quad 2a^3\left(\frac{3\pi-4}{9}\right).$

11. Prove the following :→

(a) $\beta(m, \frac{1}{2}) = 2^{2m-1}\beta(m,m)$ where β represents the Beta function.

$\longrightarrow \beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

put $x = \sin^2\theta$.

$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta.$

If $n = \frac{1}{2}$.

$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\, d\theta. \quad \ldots\ldots (i)$

Putting $\underline{n = m}$. :→

$\beta(m,m) = 2\int_0^{\pi/2} (\sin\theta\cos\theta)^{2m-1}\,d\theta.$

$\longrightarrow \frac{2}{2^{2m-1}}\int_0^{\pi/2} (\sin 2\theta)^{2m-1}\,d\theta.$

$\longrightarrow \frac{\cancel{2}}{\cancel{2}\times 2^{2m-1}}\int_0^{\pi/2} (\sin t)^{2m-1}\,dt$ putting $2\theta = t$.

$\longrightarrow \frac{1}{2^{2m-1}} \cdot 2\int_0^{\pi/2} \sin^{2m-1}\theta\, d\theta.$

$\longrightarrow \frac{1}{2^{2m-1}}\beta(m, \frac{1}{2})$

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$\rightarrow 2^{2m-1}\beta(m,m) \longrightarrow \beta(m,\frac{1}{2})$.

(b) $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2m-1}\theta \, d\theta$.

$\therefore \beta\left(m,\frac{1}{2}\right) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \, d\theta$.

putting $\underline{m=n}$.

$\therefore \beta(m,m) \longrightarrow 2\int_0^{\pi/2} (\sin\theta\cos\theta)^{2m-1} d\theta$

$\longrightarrow \frac{1}{2^{2m-2}} \int_0^{\pi/2} \sin^{2m-1} 2\theta \, d\theta$.

put $2\theta = t$.

$\longrightarrow \frac{1}{2^{2m-1}} \times \int_0^{\pi} \sin^{2m-1}\phi \, d\phi$

$\longrightarrow \frac{1}{2^{2m-1}} \times 2\int_0^{\pi/2} \sin^{2m-1}\phi \, d\phi$

$\therefore \boxed{\beta(m,m)\, 2^{2m-1} \longrightarrow 2\int_0^{\pi/2} \sin^{2m-1}\theta \, d\theta \rightarrow \beta\left(m,\frac{1}{2}\right).}$

$\frac{2^{2m-1} \times \cancel{\Gamma m}\, \Gamma m}{\Gamma 2m} \longrightarrow \frac{\cancel{\Gamma m}\, \Gamma\frac{1}{2}}{\Gamma m+\frac{1}{2}}$.

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$\Gamma m \, \Gamma m+\frac{1}{2} = \frac{\sqrt{\pi}\,\Gamma 2m}{2^{2m-1}}$ $\left(\Gamma\frac{1}{2} \rightarrow \sqrt{\pi}\right.$.

12. show that $\int_0^1 y^{q-1}\left(\log\frac{1}{y}\right)^{p-1} dy$.

$\rightarrow$ put $\log\frac{1}{y} = t$.

$I =$ $-\log y \rightarrow t$.

$-\frac{dy}{y} \rightarrow t$.

$-dy = e^{-t}\,dt$

$dy = -e^{-t}\,dt$.

$\therefore I = -\int_{\infty}^{0} e^{-(q-1)t}(t)^{p-1} e^{-t}\,dt$.

$\rightarrow -\int_{\infty}^{0} e^{-qt+t-1}(t)^{p-1}\,dt$.

$\rightarrow \int_0^{\infty} e^{-qt}(t)^{p-1}\,dt$.

put $qt = x$, $q\,dt = dx$.

$I = \int_0^{\infty} \frac{e^{-x}(x)^{p-1}}{q^{p-1}} \times \frac{dx}{q}$

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$$\rightarrow \frac{\int_0^\infty e^{-x} x^{p-1} dx}{q^{p-1+1}} \rightarrow \frac{\int_0^\infty e^{-x} x^{p-1}}{q^p}.$$

$$\rightarrow \frac{\Gamma p}{q^p}.$$

13. Show that $\int_a^b (x-a)^{m-1} (b-x)^{n-1} dx$,

where $p > 0$ and $q > 0$ and Γ represents the Gamma function.

$\rightarrow$ Let $x \Rightarrow a\cos^2\theta + b\sin^2\theta$.

$dx \rightarrow 2a\cos\theta(-\sin\theta) + 2b\sin\theta\cos\theta$.

$\rightarrow \underline{2(b-a)\sin\theta\cos\theta}$.

at $x = a$.

$a = a\cos^2\theta + b\sin^2\theta$.

$a - a\cos^2\theta = b\sin^2\theta$.

$(a-b)\sin^2\theta = 0$ $\quad (a \neq b)$

$\sin\theta = 0$.

similarly.

$b = a\cos^2\theta + b\sin^2\theta$

$b\cos^2\theta = a\cos^2\theta$.

$(b-a)\cos^2\theta = 0$

$\cos\theta \Rightarrow \frac{\pi}{2}$ $\quad (a \neq b)$

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$$\int_0^{\pi/2} (a\cos^2\theta - a + b\sin^2\theta)^{m-1} (b - a\cos^2\theta - b\sin^2\theta)^{n-1} \times 2(b-a)\sin\theta\cos\theta\, d\theta$$

$$\rightarrow \int_0^{\pi/2} (b-a)^{m-1} (\sin x)^{2m-2} \times (b-a)^{n-1} \times (\cos x)^{2m-2} \times 2\sin\theta\cos\theta \times (b-a).dx.$$

$$\rightarrow \int_0^{\pi/2} (b-a)^{m+n-1} \times 2(\sin x)^{2m-1} (\cos x)^{2m-1} dx.$$

$$\rightarrow (b-a)^{m+n-1}\, 2\int_0^{\pi/2} (\sin x)^{2m-1} (\cos x)^{2m-1} dx. \quad \ldots (i)$$

put $x = \sin^2\theta$ in Beta function

$$\beta(m,n) = 2\int_0^{\pi/2} (\sin x)^{2m-1} (\cos x)^{2n-1} dx. \quad \ldots (ii)$$

From eq (i) and (ii).

$$\boxed{\int_a^b (x-a)^{m-1} (b-x)^{n-1} dx \rightarrow (b-a)^{m+n-1} \beta(m,n).}$$

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14. Express the following in terms of Gamma functions :→

(a) $\int_0^\infty \frac{x^c}{c^x} dx$.

→ $\int_0^\infty \frac{x^c}{e^{x\ln c}} dx$.

→ $\int_0^\infty x^c e^{-x\ln c} dx$.

→ let $x\ln c \Rightarrow t$.
$dx \rightarrow \frac{dt}{\ln c}$.

$\int_0^\infty \frac{t^c}{(\ln c)^c} \times \frac{e^{-t} dt}{\ln c}$

→ $\int_0^\infty \frac{e^{-t} (t)^{c+1-1} dt}{(\ln c)^{c+1}}$

→ $\int_0^\infty \frac{e^{-t} (t)^{(c+1)-1} dt}{(\ln c)^{c+1}}$

→ $\frac{\Gamma(c+1)}{(\ln c)^{c+1}}$.

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(b) $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta$.

$\rightarrow \int_0^{\pi/2} \frac{\sqrt{\sin\theta}}{\sqrt{\cos\theta}}\, d\theta$

$\rightarrow \int_0^{\pi/2} \sin\theta^{1/2} \cos\theta^{-1/2}\, d\theta$.

$\int_0^{\pi/2} |\sin\theta|^{2\times\frac{3}{4}-1} |\cos\theta|^{2\times\frac{1}{4}-1}\, d\theta$

$m = \frac{3}{4}, \quad n = \frac{1}{4}$.

In case, of Beta function :→

$\beta(m,n) = 2\int_0^{\pi/2} |\sin\theta|^{2m-1} |\cos\theta|^{2n-1}\, d\theta$

putting $m = \frac{3}{4}, n = \frac{1}{4}$.

$\frac{\Gamma\frac{3}{4}\,\Gamma\frac{1}{4}}{2\,\Gamma 1} \longrightarrow \int_0^{\pi/2} |\sin\theta|^{1/2} |\cos\theta|^{-1/2}$.

Hence,

$\int_0^{\pi/2} \sqrt{\tan x} \longrightarrow \frac{1}{2}\Gamma\frac{3}{4}\,\Gamma\frac{1}{4}$.

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15. Prove that

$$\int_0^1 \frac{x^2\,dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{dx}{\sqrt{1+x^4}}.$$

$\rightarrow$ let $x^2 = \sin\theta$ in 1st integral.

$$2x\,dx \rightarrow \cos\theta\,d\theta.$$

$$dx \rightarrow \frac{\cos\theta\,d\theta}{2\sqrt{\sin\theta}}.$$

$$\therefore I_1 \rightarrow \frac{1}{2}\int_0^{\pi/2} \sqrt{\sin\theta}\,d\theta$$

$$\rightarrow \frac{2}{4}\int_0^{\pi/2} (\sin\theta)^{2\times\frac{3}{4}-1} (\cos\theta)^{2\times\frac{1}{2}-1}\,d\theta$$

$$\rightarrow \frac{\beta\left(\frac{3}{4}, \frac{1}{2}\right)}{4}$$

$I_2 \rightarrow$ put $x^2 = \tan\theta$.

$$2x\,dx \rightarrow \sec^2\theta\,d\theta.$$

$$I_2 \rightarrow \int_0^{\pi/4} \frac{\sec^2\theta\,d\theta}{\sqrt{\tan\theta \times \sec\theta}}$$

$$\rightarrow \frac{1}{2}\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta\cos\theta}}.$$

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$$\rightarrow \frac{1}{\sqrt{2}} \int_0^{\pi/4} \frac{d\theta}{\sqrt{\sin 2\theta}}.$$

put $2\theta = t$.

$2d\theta = dt$.

$$\rightarrow \frac{1}{2\sqrt{2}} \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}}.$$

$$\rightarrow \frac{2}{4\sqrt{2}} \int_0^{\pi/2} (\sin)^{2\times\frac{1}{4}-1} (\cos\theta)^{\frac{1}{2}-1}$$

$$\rightarrow \frac{\beta\left(\frac{1}{4}, \frac{1}{2}\right)}{4\sqrt{2}}.$$

$$\rightarrow \frac{1}{16\sqrt{2}} \times \frac{\cancel{\Gamma\frac{3}{4}} \times \Gamma\frac{1}{2}}{\Gamma\frac{5}{4}} \times \frac{\Gamma\frac{1}{4}\,\Gamma\frac{1}{2}}{\cancel{\Gamma\frac{3}{4}}}$$

$$\rightarrow \frac{1}{16\sqrt{2}} \times \frac{\pi \times \cancel{\Gamma\frac{1}{4}}}{\frac{1}{4}\,\cancel{\Gamma\frac{1}{4}}}$$

$$\rightarrow \frac{\pi}{4\sqrt{2}}.$$

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