Q.1 $f(x) = x^2 \quad , 0 \le x \le \pi$

$\quad = -x^2 \quad , -\pi \le x \le 0$

let the fourier series for $f(x)$ be

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} -x^2\,dx + \frac{1}{\pi}\int_{0}^{\pi} x^2\,dx$$

$$= -\frac{1}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{x^3}{3}\right]_{0}^{\pi}$$

$$= \frac{-1}{3\pi}\left[0 + \pi^3\right] + \frac{1}{3\pi}\left[\pi^3 + 0\right]$$

$$= 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} -x^2 \cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} x^2 \cos nx\,dx$$

$$= -\frac{1}{\pi}\left[x^2.\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} - \int_{-\pi}^{0} 2x\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} dx\right]$$

$$+ \frac{1}{\pi}\left[x^2.\left[\frac{\sin nx}{n}\right]_{0}^{\pi} - \int_{0}^{\pi} 2x\left[\frac{\sin nx}{n}\right]_{0}^{\pi} dx\right]$$

$$= -\frac{1}{\pi}\left[x^2[0] - \frac{2}{n}\left[\int_{-\pi}^{0} x \sin nx\,dx\right]\right] + \frac{1}{\pi}\left[x^2(0) - \frac{2}{n}\left[\int \ldots\right.\right.$$

$$= \frac{-1}{\pi}\left[-\frac{2}{n}\left[x\cdot\frac{-\cos nx}{n}\right]_{-\pi}^{0} + \int_{-\pi}^{0}\frac{\cos nx}{n}\right] + \frac{1}{\pi}\left[-\frac{2}{n}\left[x\cdot\frac{-\cos nx}{n}\right]_{0}^{\pi} + \int_{0}^{\pi}\frac{\cos nx}{n}\right]$$

$$= -\frac{1}{\pi}\left[+\frac{2}{n}\left[+\frac{\pi\cos n\pi}{n}\right] + \frac{1}{n}\times 0\right] + \frac{1}{\pi}\left(-\frac{2}{n}\cdot\left(\frac{\pi\cdot -\cos n\pi}{n}\right] + \frac{1}{n}\sin nx\Big]_0^{\pi}\right)$$

$$= -\frac{2\pi\cos n\pi}{\pi n^2} + \frac{2\pi\cos n\pi}{\pi n^2} = 0.$$

for $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

$$= \frac{-1}{\pi}\int_{-\pi}^{0} x^2\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} x^2\sin nx\,dx$$

$$= \frac{-1}{\pi}\left[x^2\cdot\frac{-\cos nx}{n} - \int 2x\cdot\frac{-\cos nx}{n}\,dx\right]_{-\pi}^{0} + \frac{1}{\pi}\left[x^2\cdot\frac{-\cos nx}{n} + \int 2x\frac{\cos nx\,dx}{n}\right]_0^{\pi}$$

$$= \frac{-1}{\pi}\left[-\frac{x^2\cos nx}{n} + \frac{2}{n}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]\right]_{-\pi}^{0} + \frac{1}{\pi}\left[-\frac{x^2\cos nx}{n} + \frac{2}{n}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]\right]_0^{\pi}$$

$$= \frac{-1}{\pi}\left[0 + \frac{2}{n}\left[0 + \frac{1}{n^2}\right] - \left[-\frac{\pi^2\cos n\pi}{n} + \frac{2}{n}\left[0 + \frac{\cos n\pi}{n^2}\right]\right]\right]$$
$$+ \frac{1}{\pi}\left[-\frac{\pi^2\cos n\pi}{n} + \frac{2}{n}\left[0 + \frac{\cos n\pi}{n^2}\right] - \left[0 + \frac{2}{n}\left[0 + \frac{1}{n^2}\right]\right]\right]$$

$$= -\frac{1}{\pi}\left[\frac{2}{n^3} + \frac{\pi^2\cos n\pi}{n} + \frac{2\cos n\pi}{n^3}\right]$$
$$+ \frac{1}{\pi}\left[-\frac{\pi^2\cos n\pi}{n} + \frac{2\cos n\pi}{n^3} - \frac{2}{n^3}\right]$$

$$= -\frac{2}{\pi n^3} - \frac{\pi \cos n\pi}{n} - \frac{2\cos n\pi}{\pi n^2} - \frac{\pi \cos n\pi}{n} + \frac{2\cos n\pi}{\pi n^2} - \frac{2}{\pi n^3}$$

$$= \frac{-4}{\pi n^3} - \frac{2\pi \cos n\pi}{n}$$

$$= -\frac{2}{n}\left[\frac{2}{\pi n^2} - \pi(-1)^n\right]$$

So $$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$= \frac{1}{2}\times 0 + \sum_{n=1}^{\infty}\left(0 + \frac{2}{n}\left(\pi(-1)^n - \frac{2}{\pi n^2}\right)\sin nx\right)$$

$$= \sum_{n=1}^{\infty}\frac{2}{n}\left(\pi(-1)^n - \frac{2}{\pi n^2}\right)\sin nx$$

$$= 2\left(-\pi - \frac{2}{\pi}\right)\sin x + \frac{2}{2}\left(\pi - \frac{2}{\pi 4}\right)\sin 2x$$

$$+ \dots$$

Ans 27] An alternating current, after passing through a rectifier has the form

$$i = I_0 \sin x \qquad 0 \le x \le \pi$$
$$= 0 \qquad \pi \le x \le 2\pi$$

So let be the fourier series i be

$$i = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} I_0 \sin x \, dx = \frac{I_0}{\pi}\left[-\cos x\right]_0^{\pi}$$

$$= \frac{I_0}{\pi}(+2) = \frac{2I_0}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} I_0 \sin x \cdot \cos nx \, dx$$

$$= \frac{I_0}{\pi}\int_0^{\pi} \sin x \cdot \cos nx \, dx$$

$$= \frac{I_0}{2\pi}\int_0^{\pi} 2\sin x \cdot \cos nx \, dx$$

$$= \frac{I_0}{2\pi}\int_0^{\pi} \sin(n+1)x - \sin(n-1)x \, dx$$

$$= \frac{I_0}{2\pi}\left[\left[\frac{-\cos(n+1)x}{n+1}\right]_0^{\pi} - \left[-\frac{\cos(n-1)}{n-1}\right]_0^{\pi}\right]$$

$$= \frac{I_0}{2\pi}\left[\frac{-1}{n+1}\left((-1)^n - 1\right]\right. + \left(\frac{\cos(n-1)\pi}{n-1} - \frac{1}{n-1}\right)$$

$$= \frac{I_0}{2\pi}\left[\frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{I_0}{2\pi}\left(\frac{n-1 - n+1}{(n+1)(n-1)}\right)$$

$$= \frac{-I_0}{\pi(n+1)(n-1)}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} I_0 \sin x \sin nx \, dx$$

$$= \frac{I_0}{2\pi}\int_0^{\pi} 2\sin x \sin nx \, dx$$

$$= \frac{I_0}{2\pi}\int_0^{\pi} \cos(n-1)x - \cos(n+1)x \, dx$$

$$= \frac{I_0}{2\pi}\left[\left[\frac{\sin(n-1)x}{n-1}\right]_0^{\pi} - \left[\frac{\sin(n+1)x}{n+1}\right]_0^{\pi} \right]$$

$$= \frac{I_0}{2\pi}\left[\frac{1}{n-1}[0-0] - \frac{1}{n+1}[0] \right] \quad \text{where } n \neq 1$$

$$= 0.$$

$$b_1 = \frac{I_0}{\pi}\int_0^{\pi} \sin^2 x \, dx$$

$$= \frac{I_0}{\pi}\int_0^{\pi} \left(\frac{1-\cos 2x}{2}\right) dx$$

$$= \frac{I_0}{2\pi}\int_0^{\pi} 1-\cos 2x \, dx$$

$$= \frac{I_0}{2\pi}\left[x - \frac{\sin 2x}{2} \right]_0^{\pi}$$

$$= \frac{I_0}{2\pi}\left[\pi - 0 \right] = \frac{I_0}{2}$$

$$a_1 = \frac{I_0}{\pi}\int_0^{\pi} \sin x \cos x \, dx$$

$$= \frac{I_0}{2\pi}\int_0^{\pi} \sin 2x \, dx$$

$$= \frac{I_0}{2\pi}\left[-\frac{\cos 2x}{2} \right]_0^{\pi}$$

$$= \frac{-I_0}{4\pi}\left[-1-1 \right] = \frac{I_0}{2\pi}.$$

$$i = \frac{1}{2}a_0 + a_1\cos\varphi + b_1\sin\varphi + \sum_{n=2}^{\infty}(a_n\cos nx + b_n \ldots$$

$$= \frac{I_0}{\pi} + \frac{I_0}{2\pi}\cos\varphi + \frac{I_0}{2}\sin\varphi + \sum_{n=2}^{\infty} -\frac{I_0\cos\ldots}{\pi(n+1)(n\ldots} \quad a_n = \frac{1}{\pi}\ldots$$

Ans 3] Convergence of fourier Series : if a $f(x)$ is a periodic function with period 2π and if $f(x)$ and $f'(x)$ both are piecewise continous in the interval $-\pi \le x \le \pi$ then fourier series of $f(x)$ is convergent it converges to $f(x)$ at every point x at which $f(x)$ is continous and mean value of $[f(x+) + f(x-)]/2$ at every point x at which $f(x)$ is discontinous, where $f(x+)$ and $f(x-)$ are right & left limits respectively.

Ans 4] $f(x) = x\sin x \qquad 0 \le x < 2\pi$

let the fourier series be

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\,dx$$

$$= \frac{1}{\pi}\Big[-x\cos x\Big]_0^{2\pi} - \frac{1}{\pi}\int_0^{2\pi} 1(-\cos x)\,dx$$

$$= -2.$$

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$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left(\sin(n+1)x - \sin(n-1)x\right)dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\sin(n+1)x\,dx - \frac{1}{2\pi}\int_0^{2\pi} x\sin(n-1)x\,dx$$

$$= \frac{1}{2\pi}\left[x\left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\}\right]_0^{2\pi}$$

$$-\frac{1}{2\pi}\int_0^{2\pi} 1\left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\}dx \qquad \boxed{n\neq 1}$$

$$= \frac{2\pi}{2\pi}\left\{-\frac{\cos(n+1)2\pi}{n+1} + \frac{\cos(n-1)2\pi}{n-1}\right\}$$

$$-\cancel{2}\frac{1}{2\pi}\left[\left\{-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{2\pi} \qquad (n\neq$$

$$= -\frac{1}{n+1} + \frac{1}{n-1} = \frac{2}{n^2-1} \qquad \boxed{n\neq 1}$$

here a_1 is calculated separately

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos x\,dx = \frac{1}{2\pi}\int_0^{2\pi} x\sin 2x\,dx$$

$$= \frac{1}{2\pi}\left[-\frac{x\cos 2x}{2}\right]_0^{2\pi} - \frac{1}{2\pi}\int_0^{2\pi} -\frac{1}{2}\cos 2x\,dx$$

$$= -\frac{1}{2} + 0.$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin nx \, dx$$

$$= \frac{1}{2\pi}\int_0^{\pi} x\{\cos(n-1)x - \cos(n+1)x\}\, dx$$

$$= \frac{1}{2\pi}\left[x \left\{ \frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1} \right\}\right]_0^{2\pi}$$

$$- \frac{1}{2\pi}\int_0^{2\pi} 1 \left\{ \frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1} \right\} dx$$

$$= 0 - \frac{1}{2\pi}\left[-\frac{\cos(n-1)x}{(n-1)^2} + \frac{\cos(n+1)x}{(n+1)^2} \right]_0^{2\pi}$$

$$= 0 \qquad (n \neq 1)$$

So b_1 is calculated seperately

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin^2 x \, dx$$

$$= \pi.$$

$$f(x) = -1 - \frac{1}{2}\cos x + \pi \sin x + 2\sum_{n=2}^{\infty} \frac{1}{n^2-1}\cos nx$$

Q5] To prove $x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} (-1)^n \frac{\cos nx}{n^2}$; $-\pi < x < \pi$

Hence show that

$\sum \frac{1}{n^2} = \frac{\pi^2}{6}$ (ii) $\sum \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

• $x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n \frac{\cos nx}{n^2} \quad -\pi < x < \pi$

$(n \neq 1)$

Let fourier series be

$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx)$

~~functi~~ Since it is even function. So $b_n = 0$

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx$

$= \frac{2}{\pi}\int_{0}^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2\pi^2}{3}$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\,dx$

$= \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos nx\,dx$

$= \frac{2}{\pi}\left[x^2 \cdot \frac{\sin nx}{n} - \int 2x\frac{\sin nx}{n}\,dx\right]_0^{\pi}$

$= \frac{2}{\pi}\left[x^2\frac{\sin nx}{n} - \frac{2}{n}\left[x - \frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}\right.$

$= \frac{2}{\pi}\left[0 - \frac{2}{n}\left[\frac{-\pi.(-1)^n}{n} + 0\right]\right]$

$= \frac{4(-1)^n}{n^2}$

So $f(x) = \frac{1}{\cancel{2}} \times \frac{\cancel{2}\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx$

$= \frac{\pi^3}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$ Hence Proved

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$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} (-1)^n \frac{\cos nx}{n^2}$$

put $x = \pi$

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{n^2}$$

$$\frac{2\pi^2}{3 \times 4} = \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2}$$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}} \quad \text{—— (i)}$$

$x = 0$

$$\frac{+\pi^2}{12} = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \quad \text{—— (ii)}$$

adding both, we get

$$2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) = \frac{\pi^2}{12} + \frac{\pi^2}{6}$$

$$2 \cdot \sum \frac{1}{(2n+1)^2} = \frac{\pi^2}{4}$$

$$\sum \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

Ans 6] $f(x) = |\sin x|$ for $-\pi < x < \pi$

Since it is even function

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx)$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi/2} \sin x\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} -\sin$$

$$= \frac{2}{\pi}(1-0) - \frac{2}{\pi}(0-1) = \frac{4}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} f$$

$$= \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx \cos nx$$

$$= \frac{2}{\pi}\int_0^{\pi} |\sin x| \cos nx \, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \sin x \cdot \cos nx \, dx + \int_{\pi/2}^{\pi} -\sin x \cos nx \, dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2} 2\sin x \cdot \cos nx \, dx - \int_{\pi/2}^{\pi} 2\sin x \cos nx \, dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2} \sin(n+1)x - \sin(n-1)x \, dx - \int_{\pi/2}^{\pi} \sin(n+1)x - \sin(n-1)x \, dx\right]$$

$$= \frac{1}{\pi}\left[\left[-\frac{\cos(n+1)x}{n+1}\right]_0^{\pi/2} + \left[\frac{\cos(n-1)}{n-1}\right]_0^{\pi/2}\right] + \left[\frac{\cos(n+1)x}{n+1}\right]_{\pi/2}^{\pi} - \left[\frac{\cos(n-1)x}{n-1}\right]_{\pi/2}$$

$$= \frac{1}{\pi}\left[-\left[\frac{\cos(n+1)\pi/2}{n+1} - \frac{1}{n+1}\right] + \frac{\cos(n-1)\pi/2 - \frac{1}{n-1}}{n-1} + \frac{\cos(n+1)\pi - \cos(n+1)\pi/2}{n+1} - \frac{\cos(n-1)\pi}{n-1} + \frac{\cos(n-1)\pi/2}{n-1}\right]$$

$$= \frac{1}{\pi}\left[+\frac{2\cos(n+1)\pi/2}{n+1} + \frac{2\cos(n-1)\pi/2}{n-1}\right]$$

$$= \frac{2}{\pi}\left(\frac{2\sin\frac{n\pi}{2}}{n^2-1}\right)$$

$$= \frac{4\sin n\pi/2}{\pi n^2-1}$$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} f(x)\cos x \, dx \quad = \frac{2}{\pi}\int_0^{\pi/2} \cos^2 x \, dx + \int_{\pi/2}^{\pi} \cos^2 x$$

$$= 0.$$

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$$f(x) = |\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{2} \frac{\cos n\pi/2 \cos nx}{n^2-1}$$

Ans 7] Half range cosine series for $f(x) = e^x$

$$a_n = \frac{2}{\pi}\int_0^{\pi} e^x \cdot \cos \frac{n\pi x}{\pi}\, dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} e^x \cos nx \, dx$$

$$= \frac{2}{\pi}\left[\frac{e^x}{1^2+n^2}(\cos nx + n\sin nx)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{e^{\pi}}{n^2+1}(\cos n\pi + n\sin n\pi) - \frac{e^0}{1+n^2}(\cos 0 + n\cdot\sin 0)\right]$$

$$= \frac{2}{\pi}\left[\frac{e^{\pi}}{n^2+1}((-1)^n) - \frac{1}{1+n^2}\right]$$

$$= \frac{2}{\pi(n^2+1)}\left((-1)^n e^{\pi} - 1\right)$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} e^x \, dx$$

$$= \frac{2}{\pi}\left[e^x\right]_0^{\pi}$$

$$= \frac{2(e^{\pi}-1)}{\pi}$$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \frac{e^{\pi}-1}{\pi}$$

Ans 8] f(

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$$= \frac{e^{\pi}-1}{\pi} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1-(-1)^n e^{\pi}}{1+n^2} \cos nx.$$

Ans 8] $f(x) = -a \;, \; -c < x < 0$

$\qquad\qquad a \;, \; 0 < x < c$

Since the function is odd function so

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi} \int_{-c}^{c} f(x) = \frac{1}{\pi} \int_{-c}^{0} -a \, dx + \frac{1}{\pi} \int_{0}^{c} a \, dx$$

$$= -\frac{a}{\pi} [0 + c] + \frac{a}{\pi} [c + 0]$$

$$= 0$$

$$b_n = \frac{2a}{\pi} \int_{0}^{c} \sin nx \, dx$$

$$= \frac{2a}{\pi} \left[-\frac{\cos nx}{n} \right]_0^c$$

$$= \frac{2a}{n\pi} [-\cos nc + 1]$$

$$= \frac{2a}{n\pi} [1 - \cos nc]$$

$$f(x) = \sum_{n=1}^{\infty} \frac{2a}{n\pi} (1 - \cos nc) \sin nx$$

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(9) $-\pi < x < \pi$

to prove that

$$x\sin x = 1 - \frac{1}{2}\cos x - \frac{2\cos 2x}{1\cdot 3} + \frac{2\cdot\cos 3x}{2\cdot 4} + \dots$$

& hence show that

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \dots$$

let the fourier series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} -x\sin x + \frac{1}{\pi}\int_{0}^{\pi} x\sin x$$

$$= \frac{1}{\pi}\left[x - \cos x + \sin x\right]_{-\pi}^{0} + \frac{1}{\pi}\left(x - \cos x - \sin x\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left(0 - 1 - 0 - (-\pi + 1)\right] + \frac{1}{\pi}\left(\pi \mp 1 - (0+1)\right]$$

$$= \frac{1}{\cancel{\pi}}\left(\cancel{-1} + \cancel{\pi} + \cancel{1}\right) + \frac{1}{\cancel{\pi}}(\pi)$$

$$= 2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} x\sin x\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\sin x\cos nx\,dx$$

$$= \frac{-2\cos n\pi}{n^2 - 1} \quad n \neq 1$$

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$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin x \cos x \, dx$$

$$= -\frac{1}{2}.$$

So $f(x) = 1 - \frac{1}{2}\cos x - \sum_{n=2}^{\infty} \frac{2(-1)^n \cos nx}{n^2-1}$

$$f(x) = 1 - \frac{1}{2}\cos x - \frac{2\cos 2x}{1.3} + \frac{2\cos 3x}{3.4} - \cdots$$

So $x = \pi/4$

$$\frac{\pi}{4}\sin\frac{\pi}{4} = 1 - \frac{1}{2} - \frac{1}{1.3} + \frac{1}{3.5} - \frac{1}{5.7}$$

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1.3} + \frac{1}{3.5} + \cdots$$

Hence Proved

A 10] $f(x) = 2x - x^2, \quad 0 < x < 3$

here period $= 2l = 3$ So $l = \frac{3}{2}$

let $f(x) = 2x - x^2 = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cdot \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$$

$$= \frac{2}{3}\int_0^3 (2x - x^2)\,dx$$

$$= \frac{2}{3}\left[x^2 - \frac{x^3}{3}\right]_0^3$$

$$= 0$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{3}\int_0^{3}(2x-x^2)\cos\frac{n\pi x 2}{3}\,dx$$

$$= \frac{2}{3}\left[(2x-x^2)\cdot\frac{3}{2n\pi}\sin\left(\frac{2n\pi x}{3}\right)\right]_0^3 - \frac{2}{3}\int_0^3 (2-2x)\cdot\frac{3}{2n\pi}\sin$$

$$= -\frac{2\cdot 3}{n^2\pi^2}\cos 2n\pi - \frac{3}{n^2\pi^2}\cos 0 + \frac{3}{n^2\pi^2}\left[\sin\left(\frac{2n\pi x}{3}\right)\cdot\frac{3}{2n\pi}\right]$$

$$= \frac{-6}{n^2\pi^2} - \frac{3}{n^2\pi^2} = -\frac{9}{n^2\pi^2}$$

$3 - \frac{9}{4} =$

$\frac{3}{4}$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{3}\int_0^3 (2x-x^2)\sin\frac{2\pi n x}{3}\,dx$$

$$= \frac{2}{3}\left[-(2x-x^2)\cos\frac{2n\pi x}{3}\times\frac{3}{2n\pi}\right]_0^3 - \frac{2}{3}\int_0^3 -(2-2x)\cos\frac{2n\pi x}{3}\cdot\frac{3}{2n\pi}$$

$$= \frac{1}{n\pi}\left[3\cos 2n\pi + 0\right] + \frac{2}{n\pi}\left[(1-x)\left(\sin\frac{2n\pi x}{3}\right)\left(\frac{3}{2n\pi}\right)\right]_0^3$$

$$- \frac{2}{n\pi}\int_0^3 (-1)\sin\frac{2n\pi x}{3}\Big)\left(\frac{3}{2n\pi}\right)dx$$

$$= \frac{3}{n\pi} - \frac{3}{n^2\pi^2}\left[\cos\left(\frac{2n\pi x}{3}\right)\left(\frac{3}{2n\pi}\right)\right]_0^3$$

$$= \frac{3}{n\pi} + 0 = \frac{3}{n\pi}$$

Substituting these values of a_n & b_n

$$2x - x^2 = 0 + \sum_{n=1}^{\infty}\left[\frac{-9}{n^2\pi^2}\cos\left(\frac{2n\pi x}{3}\right) + \frac{3}{n\pi}\sin\left(\frac{2n\pi x}{3}\right)\right]$$

at $x = \frac{3}{2}$

$$3-\frac{9}{4} = \sum_{n=1}^{\infty}\left(-\frac{9}{n^2\pi^2}\cos n\pi + \frac{3}{n\pi}\sin n\pi\right)$$

$$\frac{3}{4} = -\frac{9}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = \frac{9}{\pi^2}\left[\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right]$$

$$\frac{\pi^2}{12} = \left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right) - 2\left(\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}\right)$$

$$\frac{\pi^2}{12} = \frac{1}{2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\right)$$

$$\frac{\pi^2}{6} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots$$

Hence Proved

(11) obtain half range sine series of $f(x) = lx - x^2$ in $(0, l)$ and show that

$$\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\cdots = \frac{\pi^3}{32}$$

let the required sine series be $lx - x^2 = \sum_{1}^{\infty} b_n \sin\frac{n\pi x}{l}$

$$b_n = \frac{2}{l}\int_0^l (lx-x^2)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[(lx-x^2)\left(-\cos\frac{n\pi x}{l}\cdot\frac{l}{n\pi}\right)\right]_0^l - \frac{2}{l}\int_0^l (l-2x)\left(-\cos\frac{n\pi x}{l}\cdot\frac{l}{n\pi}\right)$$

$$= 0 + \frac{2}{n\pi}\left((l-2x)\sin\frac{n\pi x}{l}\cdot\frac{l}{n\pi}\right]_0^l - \frac{2}{n\pi}\int_0^l(-2)\sin\frac{n\pi x}{l}\cdot\frac{l}{n\pi}$$

$$= \frac{4l}{n^2\pi^2}\left[-\cos\frac{n\pi x}{l}\cdot\frac{l}{n\pi}\right]_0^l = \frac{4l^2}{n^3\pi^3}\left[-\cos n\pi + 1\right]$$

$b_n = 0$ when n is even

$= \frac{8l^2}{n^3\pi^3}$ where n is odd

Hence the required fourier half range sine series is

$$lx - x^2 = \frac{8l^2}{\pi^3}\left[\frac{1}{1^3}\sin\frac{\pi x}{l} + \frac{1}{3^3}\sin\frac{3\pi x}{l} + \frac{1}{5^3}\sin\frac{5\pi x}{l} + \cdots\right]$$

put $x = \frac{l}{2}$

$$\frac{l^2}{2} - \frac{l^2}{4} = \frac{8l^2}{\pi^3}\left[\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots\right]$$

$$\frac{\cancel{l^2}}{4} = \frac{8\cancel{l^2}}{\pi^3}\left[\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots\right]$$

$$\frac{\pi^3}{32} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots$$

FOURIER TRANSFORM

Q.1] Do fourier sine and cosine transform of e^x exist? explain

Ans] The function whose fourier transform is to be obtained, it must satisfy Dirichlet's Conditions and since e^x is not a bounded function and $\int_{-\infty}^{\infty} |e^x|\,dx$ does not exist hence fourier sine and fourier cosine transform of e^x do not exist.

2] $f(x) = \frac{1}{1+x^2}$

Cosine transform:

$$F_c\{f(x)\} = \int_0^\infty \frac{\cos \lambda x}{1+x^2}\, dx$$

Consider $g(x) = e^{-x}$

then

$$F_c\{g(x)\} = \int_0^\infty e^{-x}.\cos \lambda x\, dx$$

$$= \left[\frac{e^{-x}}{1+\lambda^2}.(-1.\cos\lambda x + \lambda \sin \lambda x)\right]_0^\infty$$

$$= \frac{1}{1+\lambda^2}$$

Taking inverse Fourier Cosine transform

$$g(x) = \frac{2}{\pi}\int_0^\infty \frac{1}{1+\lambda^2} \cos \lambda x\, dx$$

So $\displaystyle\int_0^\infty \frac{\cos \lambda x}{1+x^2}\, dx = F_c\{f(x)\} = \frac{\pi}{2} e^{-\lambda}$. Ans

Q.3] Fourier Sine transform of $f(x) = e^{-x}$ $(x>0)$

Show that $\displaystyle\int_0^\infty \frac{x \sin mx}{1+x^2}\, dx$

$$F_s(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x) \sin \lambda x\, dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x} \sin \lambda x\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-x}}{1+\lambda^2}(-1.\sin\lambda x - \lambda \cos \lambda x)\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\ \frac{\lambda}{1+\lambda^2}$$

Now taking the inverse of fourier sine transform

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f_s(\lambda) \sin \lambda x \, d\lambda$$

$$e^{-x} = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}}\ \frac{\lambda}{\lambda^2+1} \sin \lambda x \, d\lambda$$

$$e^{-x} = \frac{2}{\pi} \int_0^\infty \frac{\lambda \sin \lambda x \, d\lambda}{1+\lambda^2}$$

Replacing x by m

$$e^{-m} = \frac{2}{\pi} \int_0^\infty \frac{\lambda}{1+\lambda^2} \sin m\lambda \, d\lambda$$

$$e^{-m} = \frac{2}{\pi} \int_0^\infty \frac{x}{1+x^2} \sin mx \, dx$$

So $$\int_0^\infty \frac{x \sin mx}{1+x^2} dx = \frac{\pi}{2} e^{-m} \quad (m>0)$$

Ans 47] find fourier transform of the function $f(x) = e^{-ax^2}$ and then find of $e^{-x^2/2}$

by using Complex fourier transform

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x) \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax^2} . e^{isx} \, dx$$

$$= \frac{1}{\ } \int^{\infty} e^{-(ax^2 - isx)} \, dx$$

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$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\left(x\sqrt{a} - \frac{is}{2\sqrt{a}}\right)^2 - \left(\frac{is}{2\sqrt{a}}\right)^2\right]} dx$$

$$= \frac{e^{-\frac{s^2}{4a}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(x\sqrt{a} - \frac{is}{2\sqrt{a}}\right)^2} dx$$

let $x\sqrt{a} - \frac{is}{2\sqrt{a}} = t$

$$= \frac{e^{-s^2/4a}}{\sqrt{2\pi a}} \int_{-\infty}^{\infty} e^{-t^2} dt$$

$$= \frac{e^{-s^2/4a}}{\sqrt{2\pi a}} \sqrt{\pi}$$

$$F(f(x)) = \frac{1}{\sqrt{2a}} e^{-s^2/4a}$$

taking $a - 1/2$ we get $F(e^{-x^2/2}) = e^{-s^2/2}$

(5) To prove

$$\int_0^{\infty} \frac{\sin \pi\lambda \sin \lambda x}{1-\lambda^2} d\lambda = \frac{\pi}{2} \sin x \quad , \quad 0 < x < \pi$$
$$= 0 \quad , \quad x > \pi$$

let us calculate the fourier sine transform

$$f(x) = \pi/2 \sin x \quad ; \quad 0 < x < \pi$$
$$= 0, \quad x > \pi$$

$$f_s(f(x)) = \int_0^{\pi} \frac{\pi}{2} \sin x . \sin \lambda x \, dx + \int_{\pi}^{\infty} 0 . \sin \lambda x \, dx$$

$$= \frac{\pi}{4} \int_0^{\pi} [\cos(\lambda-1)x - \cos(\lambda+1)x] \, dx$$

$$= \frac{\pi}{4} \left[\frac{\sin(\lambda-1)x}{\lambda-1} - \frac{\sin(\lambda+1)x}{\lambda+1} \right]_0^{\pi}.$$

$$= \frac{\pi}{4}\left[\frac{\sin(\lambda-1)\pi}{\lambda-1} - \frac{\sin(\lambda+1)\pi}{\lambda+1}\right]$$

$$= \frac{\pi}{4}\left[-\frac{\sin\pi\lambda}{\lambda-1} + \frac{\sin\pi\lambda}{\lambda+1}\right] = -\frac{\pi}{4}\sin\pi\lambda\left(\frac{2}{\lambda^2-1}\right) = \frac{\pi\sin\pi\lambda}{2(1-\lambda^2)}$$

inverting both sides

$$\frac{2}{\pi}\int_0^\infty \frac{\pi\sin\pi\lambda}{2(1-\lambda^2)}\sin\lambda x\, d\lambda = \frac{\pi}{2}\sin x.$$

6.] $\int_0^\infty f(x)\cos sx\, dx = 1-s, \quad 0 \le s \le 1$
$= 0, \quad s > 1$

$$F_c(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\, dx$$

$$= \begin{cases} \sqrt{\frac{2}{\pi}}(1-s) & \text{if } 0 \le s \le 1 \\ 0 & \text{if } s > 1 \end{cases}$$

Taking Inverse Fourier Cosine transform

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c(s)\cos sx\, ds$$

$$= \sqrt{\frac{2}{\pi}}\left[\int_0^1 \sqrt{\frac{2}{\pi}}(1-s)\cos sx\, dx + 0\right]$$

$$= \frac{2}{\pi}\left[(1-s)\frac{\sin sx}{x}\right]_0^1 - \frac{2}{\pi}\int_0^1 (-1)\frac{\sin sx}{x}\, ds$$

$$= 0 + \frac{2}{\pi x}\left[-\frac{\cos sx}{x}\right]_0^1$$

$$= \frac{2}{\pi x^2}(1-\cos x)$$

$= \frac{\pi \sin \pi \lambda}{2(1-\lambda^2)}$

In this given integral equation we set $s = 0$

$$\int_0^\infty f(x)\,dx = 1$$

from ① we have $\int_0^\infty f(x)\,dx = \int_0^\infty \frac{2}{\pi x^2}(1-\cos x)\,dx = 1$

or $\int_0^\infty \frac{1-\cos x}{x^2}\,dx = \frac{\pi}{2}$ or $\int_0^\infty \frac{2\sin^2 x/2}{x}\,dx = \pi/2$

putting $x = 2t$

$$\int_0^\infty \frac{2\sin^2 t \cdot 2dt}{4t^2} = \frac{\pi}{2}$$

So $$\int_0^\infty \frac{\sin^2 t\,dt}{t^2} = \pi/2$$

⑦ prove that

$$F[f(x)e^{-ax}] = F(s-a)$$

if $F(s)$ is the Fourier transform of $f(x)$

by definition $F(e^{-axi} f(x))$

$$= \int_{-\infty}^{\infty} f(x) \cdot e^{-iax} e^{isx}\,dx$$

$$= \int_{-\infty}^{\infty} f(x) \cdot e^{ix(s-a)}$$

$$= F(s-a)$$

Hence Proved.

Ans 6] The Convolution of two functions of $f(x)$ and $g(x)$ is defined by

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du$$

If $F[f(x) * g(x)] = F(f(x))\, F(g(x)) = F(\lambda)\, G(\lambda)$

Proof : By definition $F(\lambda) = \int_{-\infty}^{\infty} f(x) e^{i\lambda x} dx$

and $G(\lambda) = \int_{-\infty}^{\infty} g(x) e^{i\lambda x} dx$

therefore $F[f(x) * g(x)] = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(u)\, g(x-u)\, du \right] e^{i\lambda x} dx$

On changing the order of integration

$$F(f*g) = \int_{-\infty}^{\infty} f(u) \left[\int_{-\infty}^{\infty} g(x-u)\, e^{i\lambda x} dx \right] du$$

in the inner integral putting $x-u=t$; $x = t+u$, $dx = dt$

$$F(f*g) = \int_{-\infty}^{\infty} f(u) \left[\int_{-\infty}^{\infty} g(t)\, e^{i\lambda(t+u)} dt \right] du$$

$$= \int_{-\infty}^{\infty} f(u) \left[\int_{-\infty}^{\infty} g(t)\, e^{i\lambda t} dt \right] e^{i\lambda u} du$$

$$= \int_{-\infty}^{\infty} f(u)\, e^{i\lambda u} du \int_{-\infty}^{\infty} g(t) \cdot e^{i\lambda t} dt$$

$$= F(\lambda)\, G(\lambda)$$

Since $f(x) * g(x) = g(x) * f(x)$

Hence proved

Ans 9] $f''(x) - f(x) = 3e^{-2x}$, $0 < x < \infty$

$f(\infty) = $ Bounded

applying Sine transform to and using the linearity property

$$\hat{f}_s[f''(x)] - \hat{f}_s[f(x)] = 3\hat{f}_s[e^{-2x}]$$

or, $-\omega^2 F_s(\omega) + \omega\sqrt{\frac{2}{\pi}} f(0) - F_s(\omega) = 3\sqrt{\frac{2}{\pi}} \cdot \frac{\omega}{\omega^2+4}$

using $f(0) = x_0$

$$F_s(\omega) = \sqrt{\frac{2}{\pi}} \frac{\omega x_0}{\omega^2+1} - 3\sqrt{\frac{2}{\pi}} \frac{\omega}{(\omega^2+4)(\omega^2+1)} = \sqrt{\frac{2}{\pi}}\left[(x_0-1)\frac{\omega}{\omega^2+1} + \frac{\omega}{\omega^2+4}\right]$$

Taking inverse Fourier Sine Transform & using the linearity property of inverse transform

$$f(x) = (x_0 - 1)e^{-x} + e^{-2x}.$$