Finding recurrence relation :

Assignment - 3.

MA - 102, Applied Mathematics.

By

Dhruv Ramdev

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Q1 → Find the power series soln about origin of the equation

$$x^2 \frac{d^2y}{dx^2} + 6x\frac{dy}{dx} + (6+x^2)\, y = 0 .$$

A1 → Comparing the given eqn with

$$P_0(x)\, y'' + P_1(x)\, y' + P_2(x)\, y = 0$$

where $P_0(x)$, $P_1(x)$, $P_2(x)$ are polynomials of x

$\therefore \; P_0(x) = x^2$

When $P_0(x) = 0$

$\therefore$

$x^2 = 0$

$x = 0, 0$

Hence, zero is the singular point.

Let $y = \sum_{c=0}^{\infty} a_c x^{m+c}$ be the soln of given differential eqn.

$$\therefore \; \frac{dy}{dx} = \sum_{c=0}^{\infty} a_c (m+c)\, x^{m+c-1}$$

Similarly

$$\frac{d^2y}{dx^2} = \sum_{c=0}^{\infty} a_c (m+c)(m+c-1)\, x^{m+c-2}.$$

Putting the value of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in,

$$x^2\frac{d^2y}{dx^2} + 6x\frac{dy}{dx} + (6+x^2)\, y = 0,$$ we get.

$$\Rightarrow \quad x^2 \sum_{c=0}^{\infty} a_c (m+c)(m+c-1)\, x^{m+c-2} + 6x \sum_{c=0}^{\infty} a_c (m+c)\, x^{m+c-1}$$

$$+ (6+x^2) \sum_{c=0}^{\infty} a_c x^{m+c} = 0$$

$$\Rightarrow \quad \sum a_c (m+c)(m+c-1)\, x^{m+c} + \sum 6a_c (m+c)\, x^{m+c} + \sum 6a_c x^{m+c}$$

$$+ \sum a_c x^{m+c+2} = 0$$

$$\Rightarrow \quad \sum a_c x^{m+c} \left[(m+c)(m+c-1) + 6(m+c) + 6 \right]$$

$$+ \sum a_c x^{m+c+2} = 0$$

$$\Rightarrow \quad \sum a_c x^{m+c} \left[(m+c)^2 + 5(m+c) + 6 \right] + \sum a_c x^{m+c+2} = 0$$

Equating the coefficient of lowest power of x, i.e x^m to zero.

$$\therefore \quad a_0 \left[m^2 + 5m + 6 \right] = 0$$

Since $a_0 \neq 0$

$$\therefore \quad m^2 + 5m + 6 = 0$$

$$(m+2)(m+3) = 0$$

$$\boxed{m = -2, -3}$$

Indicial roots are distinct and differ by integer.

Equating the coeff of x^{m+1} to zero.

$$a_1 \left[(m+1)^2 + 5(m+1) + 6 \right] = 0.$$

$$a_1 \left[(m+3)(m+4) \right] = 0$$

For $m = -3$, a_1 is arbitory constant

For $m = -2$, a_1 is zero.

Finding recurrence relation.

$$\underbrace{\sum a_c x^{m+c}\left[(m+c)^2 + 5(m+c) + 6\right]}_{c = c+2} + \sum a_c (x^{m+c+2}) = 0$$

$$\therefore \sum x^{m+c+2}\left[a_{c+2}\left[(m+c+4)(m+c+5)\right] + a_c\right] = 0$$

equating coeffs to zero

$$a_{c+2} = \frac{-a_c}{(m+c+4)(m+c+5)}$$

$\therefore$

$$a_2 = \frac{-a_0}{(m+4)(m+5)} \qquad a_3 = \frac{-a_1}{(m+5)(m+6)}$$

$$a_4 = \frac{+a_0}{(m+4)(m+5)(m+6)(m+7)} \qquad a_5 = \frac{+a_1}{(m+5)(m+6)(m+7)(m+8)}$$

$\vdots \qquad \vdots$

For $m = -3$,

$$y_1(x) = x^m\left[a_0 + a_1 x + a_2 x^2 + a_3 x^3 \cdots\right]$$

$$= x^{-3}\left[a_0 + a_1 x + \frac{-a_0}{(-3+4)(-3+5)} x^2 - \frac{a_1 (x^3)}{(-3+5)(-3+6)} \cdots\right]$$

$$= \frac{1}{x^3}\left[a_0\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots\right] + a_1\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\right]\right]$$

The series corresponding to $m = -2$, is included in above series, and since it has two arbitary constants, it can be considerd complete.

soln of given differential eqn.

$\therefore$

$$y = a_0\left[x^{-3} - \frac{x^{-1}}{2!} + \frac{x^{1}}{4!} \cdots\right]$$

$$+ a_1\left[x^{-2} - \frac{x^{1}}{3!} + \frac{x^{3}}{5!} \cdots\right]$$

where a_0 and a_1 are arbitary constants.

Q2 Find the fourth degree polynomial approximation (a power series ~~approximation~~ about the origin) of the eqn of initial value problem. $\neq 0$.

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0 \quad , \quad y(0) = 2, \quad \frac{dy}{dx}(0) = 0$$

Given differential eqn is

$$y'' - y' = 0 \quad - (1)$$

$P_0(x) = 1$

Hence, $P_0(0) \neq 0$, hence, 0 is ordinary point.

Let, $y = \sum_{c=0}^{\infty} a_c x^c$.

$\therefore$ $y' = \frac{dy}{dx} = \sum_{c=0}^{\infty} a_c (c)(x^{c-1})$

, $y'' = \frac{d^2y}{dx^2} = \sum_{c=0}^{\infty} a_c (c)(c-1) x^{c-2}$

$\therefore$ Putting these value in eqn (1), we get.

$$\Rightarrow \sum_{c=0}^{\infty} a_c\, c(c-1)\, x^{c-2} - \sum_{c=0}^{\infty} a_c\, c\, x^{c-1} = 0.$$

Finding re-currence relation.

$$\Rightarrow \sum_{c=0}^{\infty} \underbrace{a_c\, c(c-1)\, x^{c-2}}_{c=c+1} - \sum a_c\, c(x^{c-1}) = 0$$

$$\Rightarrow \sum x^{c-1}\left[a_{c+1}(c+1)(c) - a_c c \right] = 0$$

equating coeff to zero.

$$a_{c+1}(c+1)(c) = a_c(c)$$

$$a_{c+1} = \frac{a_c}{c+1}, \quad \text{when} \quad c \neq 0.$$

$\Rightarrow$ Comparing the coeff for lowest power, x^{-2}, when $c=0$

$$a_0(0)(0-1) = 0$$

a_0 = arbitary.

Comparing the coeff for x^{-1}, when $c=0$.

$$a_1[1][1-1] - a_0 . 0 = 0$$

$$a_1(\text{zero}) = 0$$

a_1 is arbitary.

Now, $a_{c+1} = \frac{a_c}{c+1}$, $c = 1, 2, 3 \ldots$

$c=1$

$$a_2 = \frac{a_1}{2} = \frac{a_1}{2!}$$

$$a_3 = \frac{a_2}{3} = \frac{a_1}{3!} \qquad a_4 = \frac{a_1}{4!} \ldots$$

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Therefore

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \cdots$$

$$y = a_0 + a_1 \left[x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \cdots \right] \quad - (3)$$

Since, it has two arbitrary constant, it is complete solution of given differential eqn.

It is given that, $y(0) = 2$

At $x = 0$

$$y(0) = 2 = a_0 + a_1 [0 + 0 + 0 \cdots]$$

$$\therefore \boxed{a_0 = 2}$$

Also $y_1(0) = 0$,

$$\therefore y_1 = a_1 [1 + x + x^2 + x^3 \cdots]$$

$$y_1(0) = 0 = a_1 [1 + 0 + 0 \cdots]$$

$$\boxed{a_1 = 0}$$

Putting these values in (3)

$$\therefore \boxed{y = 2}$$

Q3 Find the general soln (using power series) of

$$x^2 \frac{d^2y}{dx^2} + 4x\frac{dy}{dx} - 3y = 0.$$

$$x^2 y'' + 4xy' - 3y = 0. \quad -(1)$$

Comparing the given differential eqn with.

$$P_0(x) y'' + P_1(x) y' + P_2(x) y = 0$$

$\therefore$ $P_0(x) = x^2$

At $x = 0$,

$P_0(0) = 0$.

0 is a singular point.

let $y = \sum_{c=0}^{\infty} a_c x^{m+c}$.

$\therefore$ $y_1 = \sum_{c=0}^{\infty} a_c (m+c) x^{m+c-1}$

$y_2 = \sum_{c=0}^{\infty} a_c (m+c)(m+c-1) x^{m+c-2}$

Substituting back these values in (1).

$$\Rightarrow x^2 \sum_{c=0}^{\infty} a_c (m+c)(m+c-1) x^{m+c-2} + 4x \sum_{c=0}^{\infty} a_c (m+c) x^{m+c-1} - 3\sum_{c=0}^{\infty} a_c x^{m+c} = 0$$

$$\Rightarrow \sum a_c x^{m+c} \left[(m+c)(m+c-1) + 4(m+c) - 3 \right] = 0$$

$$\Rightarrow \sum a_c x^{m+c} \left[(m+c)^2 + 3(m+c) - 3 \right] = 0$$

Equating the coeffient of lowest power of x, i.e x^m to zero

$$a_0 \left[m^2 + 3m - 3 \right] = 0$$

Since, $a_0 \neq 0$.

$$\therefore \quad m^2 + 3m - 3 = 0$$

$$\therefore \quad m = \frac{-3 \pm \sqrt{9+12}}{2}.$$

$$\therefore \quad m_1 = \frac{-3+\sqrt{21}}{2}, \quad m_2 = \frac{-3-\sqrt{21}}{2}.$$

$\therefore$ Finding recurrence relation.

$$a_c\left[(m+c)^2 + 3(m+c) - 3\right] = 0, \quad \text{for } c = 1, 2, 3$$

c = 1

$$a_1\left[m^2 + 5m + 1\right] = 0$$

At $m = m_1$ or $m = m_2$

$$m^2 + 5m + 1 \neq 0.$$

$$\therefore \quad a_1 = 0$$

c = 2

$$a_2\left[m^2 + 7m + 7\right] = 0$$

At $m_0 = m_1$ or $m = m_2$

$$m^2 + 7m + 7 \neq 0$$

$$a_2 = 0$$

Similarly $a_3 = a_4 = a_5 \cdots = 0$.

Hence,

$$y_0(x) = a_0 x^m$$

$$\therefore \quad y_1 = y\big|_{m=m_1} = a_0 x^{(3+\sqrt{21})/2}$$

$$y_2 = y\big|_{m=m} = a_0 x^{(-3-\sqrt{21})/2}.$$

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... distinct and does not differ by

The complete soln

$$y = c_1' y_1 + c_2' y_2$$

$$= c_1 a_0 x^{(-3+\sqrt{21})/2} + c_2' a_0 x^{(-3-\sqrt{21})/2}$$

$$= c_1 x^{(-3+\sqrt{21})/2} + c_2 x^{(-3-\sqrt{21})/2}$$

$$\cancel{= x^{-3}\left[c_1 + c_2\right]}$$

$$= x^{-3}\left[c_1 x^{\sqrt{21}/2} + \frac{c_2}{x^{\sqrt{21}/2}}\right]$$

$$= x^{(-3-\sqrt{21}/2)}\left[c_1 x^{\sqrt{21}} + c_2\right]$$

Q4 Find the series solution about the indicated point of the following differential eqn by Frobenius method.

$$2(1-x)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0, \quad x = 1$$

Ans 4

$$2(1-x)y'' - xy' + y = 0$$

$P_0(x) = 2(1-x)$

At $x=1$

$P_0(1) = 2(1-1) = 0$

Hence, it is a singular point.

$\therefore$ Let $y = \sum_{c=0}^{\infty} a_c (x - x_0)^{m+c}$

where $x = x_0$

$$\therefore \quad y = \sum_{c=0}^{\infty} a_c (x-1)^{m+c}$$

$$y' = \sum_{c=0}^{\infty} a_c (m+c)(x-1)^{m+c-1} \qquad -(3)$$

$$y'' = \sum_{c=0}^{\infty} a_c (m+c-1)(m+c)(x-1)^{m+c-2} \qquad -(4)$$

From (1).

$$\Rightarrow \quad 2(1-x) y'' - x y' + y = 0$$

$$\Rightarrow \quad -2(x-1) y'' - (x-1) y' - y' + y = 0 .$$

$$\Rightarrow \quad 2(x-1) y'' + (x-1) y' + y' - y = 0 . \qquad -(5)$$

Putting value of y, y', y'' from (2), (3), (4) in (5), we get.

$$\Rightarrow 2(x-1) \sum_{c=0}^{\infty} a_c (m+c)(m+c-1)(x-1)^{m+c-2} + (x-1) \sum_{c=0}^{\infty} a_c (m+c)(x-1)^{m+c-1}$$

$$+ \sum_{c=0}^{\infty} a_c (m+c)(x-1)^{m+c-1} - \sum_{c=0}^{\infty} a_c (x-1)^{m+c} = 0$$

$$\Rightarrow \sum a_c (x-1)^{m+c-1} \left[2(m+c)(m+c-1) + (m+c) \right]$$

$$+ \sum a_c (x-1)^{m+c} \left[(m+c) - 1 \right] = 0$$

$$\Rightarrow \sum a_c (x-1)^{m+c-1} \left[(m+c)(2m+2c-1) \right]$$

$$+ \sum a_c (x-1)^{m+c} \left[m+c-1 \right] = 0 .$$

Comparing the coeff of lowest power of $(x-1)$, $(x-1)^{m-1}$

$$a_0 \left[m(2m-1) \right] = 0$$

since $a_0 \neq 0$, $\boxed{m = 0, \quad m = 1/2}$

Initial roots are distinct and does not differ by integer.

Finding recurrence ~~sol~~ relation

$$\Rightarrow \quad \sum (x-1)^{m+c} \left[a_{c+1} (m+c+1)(2m+2c+1) + (m+c-1) a_c \right] = 0$$

Comparing coefficient to zero.

$$\Rightarrow \quad a_{c+1} = \frac{-a_c (m+c-1)}{(m+c+1)(2m+2c+1)}$$

$\therefore$ At $c=0$.

$$a_1 = -\frac{a_0 (m-1)}{(m+1)(2m+1)}$$

$$a_2 = -\frac{a_1 (m)}{(m+2)(2m+3)}$$

$$= \frac{a_0 (m)(m-1)}{(m+1)(m+2)(2m+1)(2m+3)}$$

Similarly

$$a_3 = -\frac{a_0 (m-1)(m)(m+1)}{(m+1)(m+2)(m+3)(2m+1)(2m+3)(2m+5)}$$

$\vdots$

At $m=0$

$a_1 = a_0$

$a_2 = 0, \quad a_3 = 0, \quad a_4 = 0 \ldots\ldots$

At $m=0$

$$a_1 = \frac{a_0 [\,1/2\,]}{3/2 \cdot 2} = \frac{a_0}{3!} \; ; \quad a_2 = \frac{-a_0 [\,1/2\,][\,1/2\,]}{5/2 \cdot 3/2 \cdot 4 \cdot 2} = -\frac{a_0}{5!}$$

$$a_3 = \frac{+a_0\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(+\frac{1}{2}\right)}{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot 4\cdot 2\cdot 6} = \frac{3a_0}{7!}$$

$$y = (x-1)^m\left[a_0 + a_1(x-1) + a_2(x-1)^2 \cdots\cdots\right]$$

$$y_1 = y\big|_{m=0} = (x-1)^0\left[a_0 + a_0(x-1)\right]$$

$$y_2 = y\big|_{m=\frac{1}{2}} = (x-1)^{\frac{1}{2}}\left[a_0\right]\left[1 + \frac{(x-1)}{3!} - \frac{(x-1)^2}{5!} + \frac{(x-1)^3}{7!}\cdot 3 \cdots\right]$$

$$\therefore \quad y = c_1' y_1 + c_2' y_2$$

$$= c_1' a_0\left[\cancel{1} + x - 1\right]$$

$$+ c_2' a_0\left[x-1\right]^{\frac{1}{2}}\left[1 + \frac{(x-1)}{3!} - \frac{(x-1)^2}{5!} + \frac{3(x-1)^3}{7!} \cdots\right]$$

$$y = c_1 x + c_2\left[x-1\right]^{\frac{1}{2}}\left[1 + \frac{(x-1)}{3!} - \frac{(x-1)^2}{5!} + \frac{3}{7!}(x-1)^3 \cdots\right].$$

Q5 Find the series soln about the indicated point of the following differential eqn by Frobenius method.

$$x(x-2)\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 3y = 0, \quad \boxed{x=2}$$

$P_0(x) = x(x-2)$

$P_0(x) = 0 = x(x-2)$

$x = 0, 2$

$\therefore$ 2 is a singular point

Let $y = \sum_{c=0}^{\infty} a_c (x-x_0)^{m+c}$ — ①

$\boxed{\text{where } x_0 = 2}$

Now, the given differential eqn is

$\Rightarrow \quad x(x-2)y'' + 4xy' + 3y = 0$

$\Rightarrow \quad (x-2+2)(x-2)y'' + 4(x-2)y' + 8y' + 3y = 0$

$\Rightarrow \quad (x-2)^2 y'' + 2(x-2)y'' + 4(x-2)y' + 8y' + 3y = 0$ — ②

Now,

$$y' = \sum_{c=0}^{\infty} (m+c) a_c (x-2)^{m+c-1}$$

$$y'' = \sum_{c=0}^{\infty} (m+c-1)(m+c) a_c (x-2)^{m+c-2}$$

Putting back in ②

$$\Rightarrow (x-2)^2 \sum_{c=0}^{\infty} (m+c-1)(m+c) a_c (x-2)^{m+c-2} + 2(x-2)\sum_{c=0}^{\infty} a_c (m+c-1)(m+c)(x-2)^{m+c-2}$$
$$+ 4(x-2)\sum_{c=0}^{\infty} a_c (m+c)(x-2)^{m+c-1} + 8\sum_{c=0}^{\infty} a_c (m+c)(x-2)^{m+c-1}$$
$$+ 3\sum_{c=0}^{\infty} a_c (x-2)^{m+c} = 0.$$

Simplyfying terms -

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$$\Rightarrow \sum_{r=0}^{\infty} a_r (x)^{m+r-1}\left[2(m+r-1)(m+r) + 8(m+r)\right]$$

$$+ \sum_{r=0}^{\infty} a_r x^{m+r}\left[(m+r-1)(m+r) + 4(m+r) + 3\right] = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} a_r x^{m+r-1}\left[(m+r)\left[2m+2r+6\right]\right]$$

$$+ \sum_{r=0}^{\infty} a_r x^{m+r}\left[(m+r)^2 + 3(m+r) + 3\right] = 0$$

Equating the coeff. of lowest power of 'x', i.e. x^{m-1} to zero

$$a_0[m][2m+6] = 0$$

since $a_0 \neq 0$

$\therefore m = 0$ or $2m+6 = 0$

$$\boxed{m = 0, -3}$$

Finding recurrence relation

$$a_{r+1}\left[(m+r+1)(2m+2r+8)\right]$$

$$+ a_r\left((m+r)^2 + 3(m+r) + 3\right] = 0$$

$$a_{r+1} = \frac{-a_r}{2}\left[\frac{(m+r)^2 + 3(m+r) + 3}{(m+r+4)(m+r+1)}\right]$$

For $r = 0$

$$a_1 = \frac{-a_0}{2}\left[\frac{m^2+3m+3}{(m+4)(m+1)}\right]$$

$$a_2 = \frac{-a_1}{2}\left[\frac{m^2+5m+7}{(m+5)(m+2)}\right]$$

$$= \frac{a_0}{4}\left[\frac{[m^2+5m+7][m^2+3m+3]}{(m+4)(m+5)(m+1)(m+2)}\right]$$

$\therefore$ Recurrence relation -

$$a_3 = -\frac{a_2}{2}\frac{[m^2+7m+13]}{(m+6)(m+3)}$$

$$= -\frac{a_0}{8}\frac{[m^2+7m+13][m^2+5m+7][m^2+3m+3]}{(m+6)(m+5)(m+4)(m+3)(m+2)(m+1)}$$

Similarly

$$a_4 = +\frac{a_0}{16}\left[\frac{[m^2+9m+21][m^2+7m+13][m^2+5m+7][m^2+3m+3]}{(m+6)(m+5)(m+4)(m+7)(m+4)(m+3)(m+2)(m+1)}\right]$$

$\vdots$

$$y = x^m[a_0]\left[1 - \frac{(m^2+3m+3)x}{2(m+1)(m+4)} + \frac{x^2(m^2+3m+3)(m^2+5m+7)}{2^2(m+1)(m+2)(m+4)(m+5)} - \frac{(m^2+3m+3)(m^2+5m+7)(m^2+7m+13)x^3}{2^3(m+1)(m+2)(m+3)(m+4)(m+5)(m+6)} + \cdots\right]$$

$$y_1(x) = y|_{m=0} = a_0\left[1 - \frac{3x}{2\cdot4} + \frac{3\cdot7}{2^2\cdot2\cdot4\cdot5}x^2 - \frac{3\cdot7\cdot13\,x^3}{2^3\cdot2\cdot4\cdot3\cdot5\cdot6}\right] \quad \text{— (A)}$$

Now

$$y_2(x) = \frac{dy}{dm}\Big|_{m=0}$$

$$= y(x)\log x + x^m a_0\left[\frac{-1}{2}\left[\frac{(m+1)(m+4)(2m+3) - (m^2+3m+3)(2m+5)}{(m+1)^2(m+4)^2}\right]\right.$$

$$+\left[\frac{\left((m+1)(m+2)(m+4)(m+5)\left\{(2m+5)(m^2+5m+7)+(2m+5)(m^2+3m+3)\right\} - (m^2+3m+3)(m^2+5m+7)\left\{(m+2)(m+4)(m+5)+(m+1)(m+4)(m+5)+(m+1)(m+2)(m+5)+(m+1)(m+2)(m+4)\right\}\right)}{2^2(m+1)^2(m+2)^2(m+4)^2(m+5)^2}\right]$$

$\cdots$

[illegible]

$$y_2 = \frac{\partial y}{\partial m}\Big|_{m=0}$$

$$= y_1 \ln(x) + x_0\left[\frac{2x}{2\cdot 1^2\cdot 4^2}x - \frac{1}{2^2}\cdot\frac{4x^{[illegible]}}{1^2\cdot 2^2\cdot 4^2\cdot [illegible]} \cdots \right]$$

— (B)

General soln ⇒ $y(x) = c_1 y_1 + c_2 y_2$

where

y_1 is represented by (A)

y_2 is represented by (B)

Q6 Find the linearly independent soln to the eqn

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-1)y = 0$$

$P_0(x) = x^2$

$P_0(x) = 0 = x^2 \Rightarrow x = 0, 0$

0 is singular point.

Let $y = \sum_{i=0}^{\infty} x^{m+i}$

$$x^2y'' + xy' + (x^2-1)y = 0$$

Putting value of y, y', y'' in above eqn

$$\Rightarrow x^2 \sum_{i=0}^{\infty} a_i(m+i)(m+i-1)x^{m+i-2} + x\sum_{i=0}^{\infty} a_i(m+i)x^{m+i-1} + (x^2-1)\sum_{i=0}^{\infty} a_i x^{m+i} = 0$$

$$\Rightarrow \sum_{i=0}^{\infty} a_i x^{m+i}\left[(m+i)^2 - 1\right] + \sum_{i=0}^{\infty} a_i x^{m+i+2} = 0$$

Comparing coefficient of lowest power to zero, i.e x^m

$$a_0(m^2-1) = 0$$

Since $a_0 \neq 0$

$$\boxed{m = \pm 1}$$

Comparing coefficient of next power x^{m+1}

$$a_1\left[(m+1)^2 - 1\right] = 0$$

$$a_1\left[m^2 + 2m\right] = 0$$

$\therefore a_1 = 0$, $\because m(m+2) \neq 0$, from $m = 1$ or $m = -1$

$\therefore$ Recurrence relation -

$$\Rightarrow \quad a_{c+2}\left[(m+c+2)^2-1\right] = -a_c$$

$$\Rightarrow \quad a_{c+2} = \frac{-a_c}{(m+c+2)^2-1}$$

$\therefore$ At $\boxed{c=0}$

$$\cancel{a_{c+2} = \frac{-a_c}{(c+1)^2-1}}$$

$$a_2 = \frac{-a_0}{(m+2)^2-1}$$

$$= \frac{-a_0}{(m+2+1)(m+2-1)} = \frac{-a_0}{(m+3)(m+1)}$$

$$a_3 = \frac{-a_1}{(m+3)^2-1}$$

Since $a_1 = 0 \quad \therefore \quad a_3 = 0$

$$a_4 = \frac{-a_2}{(m+4)^2-1}$$

$$= \frac{a_0}{(m+5)(m+3)(m+3)(m+1)}$$

$$\therefore \quad y = x^m\left[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \dots\right]$$

$$= x^m\left[a_0\right]\left[1 - \frac{x^2}{(m+3)(m+1)} + \frac{x^4}{(m+5)(m+3)(m+3)(m+1)} \dots\right]$$

Let $b_0 = a_0(m+1)$

$$\therefore \quad y = x^m b_0\left[(m+1) - \frac{x^2}{(m+3)} + \frac{x^4}{(m+5)(m+3)^2} \dots\right]$$

$$y_1 = y\big|_{m=-1} = \frac{b_0}{x}\left[-\frac{x^2}{2} + \frac{x^4}{16} - \frac{x^6}{384} \dots\right] \Rightarrow \text{(A)}$$

$$y_2 = \left.\frac{\partial y}{\partial m}\right|_{m=-1}$$

$$= y_1 \log x + x^m b_0 \left[1 + \frac{x^2 (1)}{(m+3)^2} - \frac{x^4 \left[(m+3)^2 + 2(m+3)(m+5) \right]}{(m+3)^4 (m+5)^4} \right]$$

$$= y_1 \log x + x^{-1} b_0 \left[1 + \frac{x^2}{4} - \frac{x^4 [20]}{4^2 \cdot 16} \cdots \right]$$

$$= y_1 \log x + \frac{b_0}{x} \left[1 + \frac{x^2}{4} - \frac{20 x^4}{(16)^2} \cdots \right] \Rightarrow B$$

$$\therefore \quad y = C_1 y_1 + C_2 y_2$$

where y_1 is represented by A

y_2 is represented by B

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Q7 Express the given polynomials in term of Legendre's polynomial.

$$(x^3 + 4x^2 + 3x + 1)$$

Ans Let $f(x) = \sum_{n=0}^{\infty} C_n P_n(x)$, where C_n are Legendre coefficients given by

$$C_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx, \quad n = 0, 1, 2, ---$$

For given $f(x)$.

$$C_n = \frac{2n+1}{2} \int_{-1}^{1} (x^3+4x^2+3x+1) P_n(x)\, dx$$

For $n=0$,

$$P_n(x) = 1.$$

$$\therefore C_0 = \frac{1}{2} \int_{-1}^{1} (x^3+4x^2+3x+1)\, dx$$

$$= \frac{1}{2} \left[\frac{x^4}{4} + \frac{4x^3}{3} + \frac{3x^2}{2} + x \right]_{-1}^{1}$$

$$= \frac{1}{2} \left[\frac{8}{3} + 2 \right]$$

$$= 4/3 + 1 = 7/3$$

For $n=1$,

$$P_1(x) = x$$

$$\therefore C_1 = \frac{3}{2} \int_{-1}^{1} (x^4 + 4x^3 + 3x^2 + x)\, dx.$$

$$= \frac{3}{2} \left[\frac{x^5}{5} + x^4 + x^3 + \frac{x^2}{2} \right]_{-1}^{1} dx$$

$$= \frac{3}{2} \left[\frac{2}{5} + 2 \right] = \frac{18}{5}.$$

For $x = 2$,

$$P_2(x) = \left(\frac{3x^2 - 1}{2}\right)$$

$$C_2 = \frac{5}{2}\int_{-1}^{1} (x^3 + 4x^2 + 3x + 1)\left(\frac{3x^2 - 1}{2}\right) dx$$

$$= \frac{5}{4}\int_{-1}^{1} (3)\left[x^5 + 4x^4 + 3x^3 + x^2\right] - \left[x^3 + 4x^2 + 3x + 1\right] dx$$

$$= \frac{5}{4}\left[3\left[\frac{x^6}{6} + \frac{4}{5}x^5 + \frac{3x^4}{4} + \frac{x^3}{3}\right]_{-1}^{1} - \left[\frac{x^4}{4} + \frac{4x^3}{3} + \frac{3x^2}{2} + x\right]_{-1}^{1} \right]$$

$$= \frac{5}{4}\left[3\left(\frac{8}{5} + \frac{2}{3}\right) - \left(\frac{14}{3}\right)\right.$$

$$= \frac{5}{4}\left[\frac{72 + 30 - 70}{15}\right.$$

$$= \frac{\cancel{5}}{\cancel{4}} \times \frac{\cancel{32}\,8}{\cancel{15}\,3} = \frac{8}{3}$$

For $x = 3$

$$P_3(x) = \frac{5x^3 - 3x}{2}$$

$$C_3 = \frac{7}{2}\int_{-1}^{1} (x^3 + 4x^2 + 3x + 1)\left(\frac{5x^3 - 3x}{2}\right) dx$$

$$= \frac{7}{4}\int_{-1}^{1} 5\left(x^6 + 4x^5 + 3x^4 + x^3\right) - 3\left(x^4 + 4x^3 + 3x^2 + x\right) dx$$

$$= \frac{7}{4}\left[5\left(\frac{x^7}{7} + \frac{4x^6}{6} + \frac{3x^5}{5} + \frac{x^4}{4}\right) - 3\left(\frac{x^5}{5} + \frac{4x^4}{4} + \frac{3x^3}{3} + x\right)\right]_{-1}^{1}$$

$$= \frac{7}{4}\left[5\left(\frac{2}{7} + \frac{3\times 2}{5}\right) - 3\left(\frac{2}{5} + 2\right)\right]$$

$$= \frac{7}{4}\left[\frac{50 + 210 - 42 - 210}{35} \right]$$

$$= \frac{\cancel{7}}{\cancel{4}} \times \frac{\cancel{8}\,2}{\cancel{35}\,5} = \frac{2}{5}$$

The above eqns are derived from legendre's polynomial as shown above, now multiplying ① by P_n and ② by P_m and ~~adding~~ subtracting:

$$[n(n+1) + m(m+1)]\, P_n(x) \cdot P_m(x) = \frac{d}{dx}\left[(1-x^2)\frac{dP_m}{dx}\right] P_n - \frac{d}{dx}\left[(1-x^2)\frac{dP_n}{dx}\right] P_m$$

Let $C = n(n+1) - m(m+1) = n^2 - m^2 + (n-m)$

$= (n-m)(m+n+1)$

When, $m \neq n$, $C \neq 0$.

Now integrating both sides in $[-1, 1]$.

$$\therefore \quad C\int_{-1}^{1} P_n(x) \cdot P_m(x)\,dx = \int_{-1}^{1} \frac{d}{dx}\left[(1-x^2)\frac{dP_m}{dx}\right] P_n\,dx - \int_{-1}^{1} \frac{d}{dx}\left[(1-x^2)\frac{dP_n}{dx}\right] \cdot P_m\,dx$$

$$= I_1 - I_2$$

Consider.

$$I_1 = \int_{-1}^{1} \underbrace{\frac{d}{dx}\left[(1-x^2)\frac{dP_m}{dx}\right]}_{\mathrm{II}} \underbrace{P_n}_{\mathrm{I}}\,dx.$$

$$= \left\{ P_n(1-x^2)\frac{dP_m}{dx} \right\}_{-1}^{1} - \int_{-1}^{1} (1-x^2)\frac{dP_m}{dx} \cdot \frac{dP_n}{dx} \cdot dx$$

$$= 0 - \int_{-1}^{1} (1-x^2)\frac{dP_m}{dx} \cdot \frac{dP_n}{dx} \cdot dx.$$

Similarly

$$I_2 = \int_{-1}^{1} \frac{d}{dx}\left[(1-x^2)\frac{dP_n}{dx}\right] P_m\,dx$$

$\therefore \quad f(x) = x^3 + 4x^2 + 3x + 1$

$$= c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x) + c_3 P_3(x)$$

$$= \frac{7}{3} P_0(x) + \frac{18}{5} P_1(x) + \frac{8}{3} P_2(x) + \frac{2}{5} P_3(x)$$

$\therefore$ Given polynomial is represented in term of legendre's polynomials.

Q8 → State and prove orthogonality of Legendre's function of first kind of order 'n'.

Ans If n and m are non-negative integers, then

$$\int_{-1}^{1} P_m(x) \cdot P_n(x)\, dx = \begin{cases} 0, & m \neq n \\ \frac{2}{2n+1}, & m = n \end{cases}$$

This integral relationship is called orthogonality of legendre's polynomial on $[-1, 1]$.

Proof :-

Let $P_m(x)$ and $P_n(x)$ be two legendre's eqn of order 'm' and 'n' respectively and satisfying the following eqn respectively.

$$\frac{d}{dx}\left[(1-x)^2 \frac{dP_m}{dx}\right] = -m(m+1) P_m \quad -①$$

$$\frac{d}{dx}\left[(1-x)^2 \frac{dP_n}{dx}\right] = -n(n+1) P_n \quad -②$$

$$\left[\because \quad (1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0\right.$$

$$\left.\Rightarrow \frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] = -n(n+1)y.\right]$$

$$= -\int_{-1}^{1} (1-x^2) \frac{dP_n}{dx} \cdot \frac{dP_m}{dx} \cdot dx$$

$\therefore I_1 - I_2 = 0$

$$\Rightarrow c\int_{-1}^{1} P_n(x) \cdot P_m(x)\, dx = 0$$

$$\Rightarrow \int_{-1}^{1} P_n(x) \cdot P_m(x)\, dx = 0$$

$\therefore$ proved for $m \neq n$.

Now, when $n = m$,

We know that.

$$(1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n \cdot P_n(x).$$

Squaring both sides.

$$(1-2xt + t^2)^{-1} = \left[\sum_{n=0}^{\infty} t^n P_n(x)\right]^2 \quad - (4)$$

Integrating over $[-1,1]$.

$$\Rightarrow \int_{-1}^{1} \frac{dx}{1-2xt+t^2} = \left[\frac{\ln(1-2xt+t^2)}{-2t}\right]_{-1}^{1}$$

$$= \frac{-1}{2t}\left[\ln(1-2t+t^2) - \ln(1+2t+t^2)\right]$$

$$= \frac{+1}{t}\left[\ln(1+t) - \ln(1-t)\right]$$

$$= 2\left[1 + \frac{t^2}{3} + \frac{t^4}{5} + \cdots \frac{t^{2n}}{2n+1} \cdots\right]$$

RHS

$$\int_{-1}^{1} \left[\sum_{n=0}^{\infty} P_n(x)\, t^n \right]^2 dx$$

Using orthogonality property for $m \neq n$.

$$\int_{-1}^{1} \left[\sum_{n=0}^{\infty} P_n(x)\, t^n \right]^2 dx = \sum_{n=0}^{\infty} \left(\int_{-1}^{1} P_n^2(x)\, dx \right) t^{2n} \quad -⑥$$

from ⑤ and ⑥, equating coefficient of t^{2n}

$$\frac{2}{2n+1} = \int_{-1}^{1} P_n^2(x)\, dx \quad , \text{ for } n \geq 0.$$

Hence, orthogonality of legendre's polynomial is verified.

Q9 State and prove the orthogonality of Bessel's func^n.

Orthogonality of Bessel's function :- For $v \geq 0$, and $n = 1, 2, 3 \ldots$

$$\int_0^1 x J_v(j_n x)\, J_v(j_m x)\, dx = \begin{cases} 0 & \text{if } n \neq m \\ \frac{1}{2}\left[J_{v+1}[j_n]\right]^2 & , n = m \end{cases}$$

where j_n and j_m are zeros of $J_v(x)$

We know that $u(x) = J_v(j_n x)$ satisfies the bessel eqn

$$x^2 u'' + x u' + (j_n^2 x^2 - v^2) u = 0 \quad -①$$

and $v(x) = J_v(j_m x)$ the eqn

$$x^2 v'' + x v' + (j_m^2 x^2 - v^2) v = 0 \quad -②$$

Multiplying ① by v and ② by u, and subtracting, we get.

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$$\Rightarrow \quad x^2(u''v - uv'') + x(u'v - uv') = (j_m^2 - j_n^2)\, x^2 uv$$

dividing by x

$$\Rightarrow \quad x(u''v - uv'') + (u'v - uv') = (j_m^2 - j_n^2)\, xuv.$$

$$\left[x(u'v - uv') \right]' = (j_m^2 - j_n^2)\, xuv$$

integrating over $[0,1]$, we have

$$\int_0^1 \left[x(u'v - uv') \right]' dx = \int_0^1 (j_m^2 - j_n^2)\, xuv \, dx.$$

~~Division~~

Solving for ~~R~~LHS,

$$\int_0^1 \left[x(u'v - v'u) \right]' dx = \left[x(u'v - uv') \right]_0^1$$

$$= (u'v - uv')_{x=1}.$$

Further

$u = J_v(j_n x)$, gives $u' = j_n J_v'(j_n x)$

Similarly $v' = j_m J_v'(j_m x)$.

$\therefore$ LHS $\Rightarrow$

$$\left[u'v - uv' \right]_{x=1}$$

$$= j_n J_v'(j_n)\, J_v(j_m) - j_m J_v'(j_m)\, J_v(j_n)$$

$$= 0,$$

since j_m and j_n are zeroes of $J_v(x)$.

RHS $\Rightarrow$

$$(j_m^2 - j_n^2) \int_0^1 xuv \, dx = (j_m^2 - j_n^2) \int_0^1 x J_v(j_n x)\, J_v(j_m x)\, dx.$$

$\therefore$ Comparing both side.

$$\int_0^1 x J_v(j_n x)\, J_v(j_m x)\, dx = 0, \quad \text{for } j_n \neq j_m$$

$dV = 2 \dots$

In case, $j_n = j_m$, then considering j_n as a root of $J_v(x) = 0$ and j_m as variable approaching j_n, the LHS is equal to

$j_n J_v'(j_n) J_v(j_m)$ and thus;

$$\lim_{j_m \to j_n} \int_0^1 x J_v(j_n x) J_v(j_m x)\, dx = \lim_{j_m \to j_n} \frac{j_n J_v'(j_n) J_v(j_m)}{j_m^2 - j_n^2}$$

$$= \lim_{j_m \to j_n} \frac{j_n J_v'(j_n) J_v'(j_m)}{2 j_m}$$

$$= \frac{1}{2}\left[J_v'(j_n)\right]^2$$

$$= \frac{1}{2}\left[J_{v+1}(j_n)\right]^2$$

for $x = j_n$,

which proves that result

And orthogonility of bessel function is proved.

Q11 Prove that

$$P_n(x) = \frac{1}{n!\,2^n} \frac{d^n}{dx^n} (x^2-1)^n .$$

Hence evaluate

$P_{10}(x)$.

Ans We know that legendre differential eqn is

$$(1-x^2) y'' - 2xy' + (n)(n+1) y = 0 .$$

Let $v = (x^2-1)^n$

$$\frac{dV}{dx} = 2nx(x^2-1)^{n-1}$$

Multiply by $(1-x^2)$ on both sides

$$(1-x^2)\frac{dV}{dx} = -2nx(x^2-1)^n$$

$$\underset{\text{II}}{(1-x^2)}\,\underset{\text{I}}{V_1} + 2nxV = 0$$

differentiating $(n+1)$ times, using leibnitz rule, we get

$$(1-x^2)V_{n+2} + {}^{n+1}C_1 D(1-x^2)V_{n+1} + {}^{n+1}C_2 D^2(1-x^2)D^{n-1}V_1 + 0$$

$$+ 2nx D^{n+1}V + 2n\,{}^{n+1}C_1 Dx\,D^n V = 0$$

$$\Rightarrow (1-x^2)V_{n+2} + [(n+1)(-2x) + 2nx]V_{n+1} + n(n+1)V_n = 0$$

$$(1-x^2)\frac{d^2}{dx^2}V_n - 2x\frac{d}{dx}V_n + n(n+1)V_n = 0$$

Thus, V_n is a polynomial of degree of n constituting the soln of legendre's differential eqn and must be same then

$$P_n(x) = cV_n$$

$$= c\frac{d^n}{dx^n}(x^2-1)^n$$

$$= c\left[\frac{d^n}{dx^n}(x-1)^n(x+1)^n\right].$$

$$= c\left[n!(x+1)^n + \text{term with some power of } x-1\right].$$

$\therefore$ At $\boxed{x=1}$

$$P_n(1) = c(n!\,2^n)$$

$$c = \frac{1}{2^n\, n!}$$

$$P_n(x) = \frac{1}{2^n\, n!} V_n$$

$$= \frac{1}{2^n\, n!} \frac{d^n}{dx^n} (x^2-1)^n$$

Thus formula for $P_n(x)$ is proved.

Q 10

Ans $\int_0^{\pi/2} J_0(x\cos\theta)\cos\theta \, d\theta = \frac{\sin x}{x}$

We know that, for integer n,

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \cdot x^{2m+n}}{2^{2m+n} \cdot (m!) \cdot (m+n)!}$$

$$\therefore J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \cdot x^{2m}}{2^{2m+0} \cdot (m!)^2}$$

$$= 1 - \frac{x^2}{2^2 \cdot 1^2} + \frac{x^4}{2^4 \cdot (2!)^2} \cdots\cdots]$$

$$\therefore \cos\theta \cdot J_0(x\cos\theta) \cdot 1 = \cos\theta - \frac{x^2\cos^3\theta}{2^2} + \frac{x^4 \cdot \cos^5\theta}{2^4 \cdot (2!)^2} \cdots\cdots$$

$\therefore$ LHS becomes

$$\int_0^{\pi/2} \left(\cos\theta - \frac{x^2\cos^3\theta}{2^2} + \frac{x^4\cos^5\theta}{2^4 \cdot (2!)^2} \cdots\right) d\theta .$$

$$= \left[\sin\theta - \frac{x^2}{2^2}\left[\sin\theta - \frac{\sin^3\theta}{3}\right] + \frac{x^4}{2^4 (2!)^2}\left[\sin\theta + \frac{\sin^5\theta}{5} - \frac{2\sin^3\theta}{3}\right] \cdots\right]_0^{\pi/2}$$

$$= \left[1 - \frac{x^2}{2^2}\left[1 - \frac{1}{3}\right] + \frac{x^4}{2^4 (2!)^2}\left[1 + \frac{1}{5} - \frac{2}{3}\right] \cdots\cdots\right]$$

$$= 1 - \frac{x^2}{6} + \frac{x^4}{120} \cdots\cdots$$

$+ \ -3$

Multiplying & dividing by x

$$= \frac{1}{x}\Big[\underbrace{x - \frac{x^3}{6} + \frac{x^5}{120} \cdots\cdots}_{\text{expansion of } \sin x}\Big]$$

$$= \frac{1}{x}\sin x$$

$= RHS$

Hence proved