Infinite Series

7 Aug, 2k19

$\{x_n\} \rightarrow \{x_1, x_2, x_3 \ldots\ldots x_n\}$

$\sum_{i=1}^{\infty} a_i \rightarrow$ infinitely many pts/elements

Partial sum $= S_n \rightarrow$ sum of n terms of series

$S_n = a_1 + a_2 + a_3 + \ldots\ldots + a_n$

$S_n = a_2 + a_3 + \ldots\ldots a_{n+1}$

$S_n = a_2 + a_4 + \ldots\ldots a_{2n}$

If $\lim_{n \to \infty} S_n \rightarrow l$ finite and unique then series converges

An ∞ +ve term series is said to be <u>convergent</u> if its partial sum has finite limit.

ie $\lim_{n \to \infty} S_n \rightarrow l$ (finite and unique)

Limit

$\lim_{x \to a} f(x) \rightarrow l$

If

$\lim_{x \to a^-} f(x) \rightarrow l$ and $\lim_{x \to a^+} f(x) \rightarrow l$

* LHL and RHL must exist and be equal.

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Infinite series = $1+1+1+1 \ldots\ldots$

$$S_n = 1+1+1+\ldots\ldots 1_n$$

$$n \to \infty$$

Hence its a divergent series.

Infinite series $= 1-1+1-1+1-1 \ldots$

$S_n \Rightarrow 0, -1, 1$ depending on choosing the nth term.

Hence its a divergent series.

<u>Geometric series</u>

$$a + ar + ar^2 \ldots\ldots + ar^{n-1} + ar^n + \ldots$$

A geometric series is: ① convergent if $|r| < 1$

② divergent if $|r| \geq 1$

Ex: 1. $5 + \frac{5}{2} + \frac{5}{4} + \ldots$

$a = 5$ Hence convergent

$r = 1/2$

2. $1 + \frac{1}{2} + \frac{1}{2^2} \ldots$

$a = 1$ Hence convergent.

$r = 1/2$

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3. $\frac{5}{2} + \left(\frac{5}{2}\right)^2 + \left(\frac{5}{2}\right)^3 + \ldots\ldots$

$a = 5/2$ Hence divergent.

$r = 5/2$

→ <u>Theorem of convergence in Geometric Series</u>

<u>Proof</u>: <u>Case 1</u>

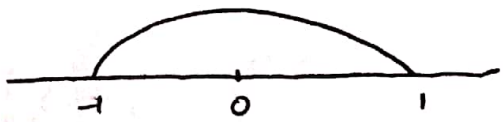
$|r| < 1$

$$S_n = \frac{a(1-r^n)}{(1-r)}$$

$$= \frac{a}{1-r} - \frac{a(r^n)}{1-r}$$

$$\lim_{n\to\infty} S_n = \frac{a}{1-r} - a \lim_{n\to\infty} \cancel{\frac{r^n}{1-r}} \to 0$$

$$= \frac{a}{1-r}$$

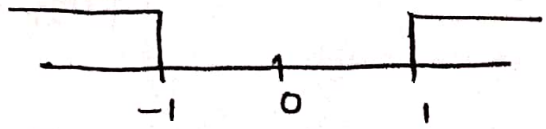


As $|r|<1$

finite and unique ∴ series converges.

<u>Case 2</u>

$|r| \geq 1$



$r=1$ $\lim_{n\to\infty} S_n = a+a+ae \ldots a_n$

$r=-1$ $S_n = a-a+a-a+\ldots$ → divergent

<u>$r>1$</u>

$$S_n = \frac{a(r^n-1)}{(r-1)}$$

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$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \cancel{\frac{ar^n}{r-1}} - \frac{a}{r-1} \longrightarrow \infty$$

∞

$\underline{r<-1}$

$$S_n = \frac{a(1-r^n)}{(1-r)}$$

$r = -k \quad k>0$

$$= \frac{a}{1-r} - \frac{a(-k)^n}{1-r}$$

$$= \frac{a}{1-r} + \frac{a}{1-r}(-1)^{n+1}\cancel{k^n} \to \infty$$

$\longrightarrow \cancel{\pm 0} \pm\infty$

Ex: 14. $\quad 1 - \frac{5}{2} + \frac{25}{4} - \frac{125}{8} \ldots$

$$1 + \left(-\frac{5}{2}\right) + \left(-\frac{5}{2}\right)^2 + \left(-\frac{5}{2}\right)^3 \ldots$$

$a = 1$

$r = -5/2 < -1$

Hence divergent

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Types of series

1.) +ve term series [Individual term +ve]

Comparison test

Let $\sum_{n=1}^{\infty} a_n$ be a +ve term series if each term of this series does not exceed the corresponding term of a convergent +ve term series $\sum_{n=1}^{\infty} b_n$ then the given series is convergent.

$$a_i \leq b_i \ \forall i$$

and Σb_i is convergent then Σa_i is convergent.

Ex: $\sum_{n=0}^{\infty} \frac{1}{n!}$. Test for convergence/divergence.

$$\Sigma a_n = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} \cdots$$

comparing with $\sum_{n=0}^{\infty} 1/2^n = \Sigma b_n$

$1 = 1$

$\frac{1}{2!} = \frac{1}{2}$

$\frac{1}{3!} < \frac{1}{2}$

$\vdots \qquad \vdots$

Since b_n is convergent a_n is also convergent.

* We can ignore some terms like we ignored 1.

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- If each term of submission a_n exceeds the corresponding term of a divergent series $\sum_{n=1}^{\infty} b_n$ then $\sum a_n$ diverges.

Ex:

$$\sum a_n = \sum_{n=0}^{\infty} n! \rightarrow 0! + 1! + 2! + 3! \ldots$$

$$\sum b_n = \sum_{n=0}^{\infty} n \rightarrow 0 + 1 + 2 + 3 + \ldots$$

$0! > 0$
$1! = 1$
$2! > 2$
$3! > 3$
$\vdots \quad \vdots$

since

$b_i \leq a_i$

hence

a_n is divergent because b_n is divergent (prove).

<u>P series test</u> (Most Powerful test)

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} \ldots \frac{1}{n^p}$$

Fix p,

$$\sum \frac{1}{n^1}, \sum \frac{1}{n^2}, \sum \frac{1}{n^3}, \sum \frac{1}{n^{3/2}}, \sum \frac{1}{\sqrt{n}}$$

$\downarrow$ div, $\downarrow$ con, $\downarrow$ con, $\downarrow$ con, $\downarrow$ div

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Ex 1 (a n) $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2} \leq \frac{1}{n^2}$

2. $\sum_{n=1}^{\infty} \frac{\sin(n)}{n}$ $\qquad \frac{\sin(n)}{n} \leq \frac{1}{n}$ ✗

3.) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

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- If $p \leq 1$ series is divergent and if $p > 1$ series is convergent

- If we remove or add some (finite) terms in an ∞ series its convergence or divergence is not affected.

Now,

$$S_n = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} \cdots \frac{1}{n^p}$$

Proof

i) $p > 1$

$$S_n = \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) \cdots \frac{1}{n^p}$$

Now, as

$$\frac{1}{2^2} > \frac{1}{3^2} \quad \text{and} \quad \frac{1}{4^2} > \frac{1}{5^2}$$

we can

$$S_n < \frac{1}{1^p} + \underbrace{\left(\frac{1}{2^p} + \frac{1}{2^p}\right)}_{\substack{2 \\ grp}} + \underbrace{\left(\frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}\right)}_{\substack{4 \\ grp}} \cdots \underbrace{\quad}_{\substack{8 \\ grp}}$$

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$$= \frac{1}{1^p} + \frac{2}{2^p} + \frac{4}{4^p} \cdots$$

$$= \frac{1}{1^p} + \frac{1}{2^{p-1}} + \left(\frac{1}{2^{p-1}}\right)^2 \cdots$$

this is a GP with common ratio $r = \frac{1}{2^{p-1}}$

S_n converges when $r = \frac{1}{2^{p-1}} < 1$

$$2^{p-1} > 1 = 2^0$$

$$\boxed{p > 1}$$

ii) <u>p=1</u>

$$\sum \frac{1}{n^p} = \sum \frac{1}{n} \quad 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \cdots$$

$$1 + \frac{1}{2} = 1 + \frac{1}{2}$$

$$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$$

and so on.

$$\sum \frac{1}{n} > 1 + \underbrace{\frac{1}{2} + \frac{1}{2} + \frac{1}{2} \cdots}_{\text{divergent series.}}$$

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How,

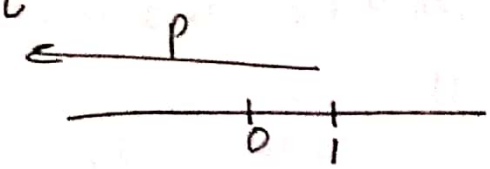
take n terms

- we are neglecting 1.

$$S_n = n \times \frac{1}{2} \longrightarrow \infty \quad \text{when } n \to \infty$$

By comparison test $\sum \frac{1}{n}$ diverges

iii) $\underline{p < 1}$



$$\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} \cdots$$

$$\frac{1}{1^p} = 1 \qquad \frac{1}{3^p} > \frac{1}{3}$$

$$\frac{1}{2^p} > \frac{1}{2} \qquad \frac{1}{4^p} > \frac{1}{4}$$

$$\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} \cdots \cdots > \underbrace{1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \cdots}_{\text{diverges}}$$

so by comparison

$\sum \frac{1}{n^p}$ diverges.

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Comparison test II

suppose $a_n > 0$ and $b_n > 0$ $\forall n \geq N$

1. If $\lim_{n \to \infty} \frac{a_n}{b_n} = c$ (finite) then Σa_n and Σb_n converge and diverge together
2. If $\lim_{n \to \infty} \frac{a_n}{b_n} = 0$ and Σb_n converges then Σa_n converges
3. If $\lim_{n \to \infty} \frac{a_n}{b_n} \to \infty$ and Σb_n diverges then Σa_n diverges.

Ques

1.) Check nature

1) $\sum_{n=2}^{\infty} \frac{2n}{n^2 - n + 1}$

2) $\sum_{n=2}^{\infty} \frac{8n^3 + 100n^2 + 1000}{2n^6 - n + 5}$

3) $\frac{3}{4} + \frac{5}{9} + \frac{7}{16} + \frac{9}{25} \cdots \underbrace{\sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}}$ $+ \cdots$

will not be given in exam.

4) $1 + \frac{1}{3} + \frac{1}{7} + \frac{1}{15} \cdots \frac{1}{2^n - 1} \cdots$

5) $\sum_{n=2}^{\infty} \frac{1 + n \ln(n)}{n^2 + 5}$

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Ans.

1.) $\sum_{n=2}^{\infty} \frac{2n}{n^2-n+1} = a_n$

Removing constants,

$\frac{n}{n^2-n} = \frac{1}{n-1}$

Removing const again,

$= \frac{1}{n}$

Now,

choose $\sum \frac{1}{n} = \sum b_n$ dvergent by p series test.

Now

$\lim_{n\to\infty} \frac{\frac{2n}{n^2-n+1}}{1/n}$ $\Rightarrow \frac{2n}{1-\frac{1}{n}+\frac{1}{n^2}}$ dividing by n^2

when $n \to \infty$,

this tends to 2 → finite

so series is divergent.

2) $\sum_{n=2}^{\infty} \frac{8n^3+100n^2+1000}{2n^6-n+5}$

Removing const

$\frac{n^3+n^2}{2n^6-n}$

simplifying up and removing const,

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$= \frac{1}{n^3} \rightarrow b_n$ converges by p series test.

$$\frac{a_n}{b_n} = \frac{\frac{8n^3+100n^2+1000}{2n^6-n+5}}{1/n^3}$$

$$= \frac{8n^6+100n^5+1000n^3}{2n^6-n^5+5}$$

dividing by n^6 and putting the $\lim_{n\to\infty}$

$= \frac{8}{2} = 4 \rightarrow$ finite

series converges.

3) $\sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2} = a_n$

removing const,

$\frac{n}{(n+1)^2} \rightarrow \frac{n}{n^2} \rightarrow \frac{1}{n} \rightarrow$ diverges by p series test

$\downarrow$

b_n

$$\frac{a_n}{b_n} = \frac{(2n+1)n}{(n+1)^2} = \frac{2n^2+n}{n^2+2n+1}$$

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$$\frac{2+\frac{1}{n}}{1+\frac{2}{n}+\frac{1}{n^2}} \quad \text{putting } \lim_{n\to\infty}$$

$= 2$ finite hence series diverges.

4) $\sum \frac{1}{2^n-1}$

removing const.

$\frac{1}{2^n} \to b_n$ by GS test is convergent

$$\frac{\frac{1}{2^n-1}}{\frac{1}{2^n}} \to \frac{2^n}{2^n-1} \to 2 \qquad \frac{1}{1-\frac{1}{2^n}} \to 1$$

hence series converges.

5) $\sum_{n=2}^{\infty} \frac{1+n\ln(n)}{n^2+5}$

removing constants,

$\frac{\ln(n)}{n} \to b_n$

$\frac{\ln(n)}{n} > \frac{1}{n} \quad \forall\, n \geq 3$

Σb_n diverges by comparison test.

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$$\frac{1+n(\log(n))}{n^2}$$

$$\sum b_n \Rightarrow \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{n+n^2\ln(n)}{n^2+5} \longrightarrow \infty$$

by pt ③ of comparison test II

- In exam, no one will ask how you reached at b_n, so skip its calculation, just write let b_n be ...

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D Alembert Ratio test / Higher ratio test

Let $\sum u_n$ be a +ve term series

$$\cancel{\lim_{n\to\infty} \frac{a_n}{a_{n+1}}}$$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = l$$ then the series

1) converges if $l > 1$
2) diverges if $l < 1$
3) test fails at $l = 1$

Ques

1. $\sum_{n=0}^{\infty} \frac{2^n+5}{3^n}$ 2) $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$ 3) $\sum_{n=1}^{\infty} \frac{4^n (n!)^2}{(2n)!}$

Ans

1) $\sum_{n=0}^{\infty} \frac{2^n+5}{3^n} \to a_n$

$$\frac{a_n}{a_{n+1}} = \frac{\frac{2^n+5}{3^n}}{\frac{2^{n+1}+5}{3^{n+1}}} = \frac{(2^n+5)(3^{n+1})}{(2^{n+1}+5)\,3^n}$$

$$= \frac{3(2^n+5)}{2^{n+1}+5}$$

$$= \frac{3/2 + \cancel{\frac{5}{2^{n+1}}}}{1 + \cancel{\frac{5}{2^{n+1}}}} = 3/2$$

$l > 1$

hence converges

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2) $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} \rightarrow a_n$

$$\frac{a_n}{a_{n+1}} = \frac{(2n)!}{(n!)^2} \times \frac{((n+1)!)^2}{(2(n+1))!}$$

$$= (n+1)^2 \times \frac{(2n)!}{(2(n+1))!}$$

$$= (n+1)^{\cancel{2}} \times \frac{1}{2\cancel{(n+1)}}$$

$$= \frac{n+1}{2} = 1/2$$

$$= \frac{(2n)!}{(2n+2)!} \times \frac{(n+1)!\,(n+1)!}{n!\,n!}$$

$$= \frac{(2n)!}{(2n+2)(2n+1)(2n!)} \times \frac{(n+1)(n!) \times (n+1)n!}{n!\,n!}$$

$$= \frac{(n+1)(n+1)}{(2n+2)(2n+1)}$$

dividing by n,

$\longrightarrow$ 1/4 $\rightarrow$ diverges.

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D'Alembert Ratio test (contd.)

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$$\lim_{n \to \infty} \frac{U_{n+1}}{U_n} = k$$

$k < 1$ converges

$k > 1$ diverges

$k = 1$ test fails.

Proof

1) When $k < 1$

then we can always find a +ve no.

$k < \lambda < 1$.

where λ is a fixed +ve no.

$$\frac{U_{n+1}}{U_n} = k < \lambda$$

$$\frac{U_{n+1}}{U_n} < \lambda$$

then

$$\frac{U_2}{U_1} < \lambda \qquad \frac{U_3}{U_2} < \lambda$$

$$\sum U_n = U_1 + U_2 + U_3 + \ldots\ldots$$

$$= U_1 \left[1 + \frac{U_2}{U_1} + \frac{U_3}{U_1} + \frac{U_4}{U_1} \ldots \right]$$

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$$= U_1\left[1+\frac{U_2}{U_1}+\frac{U_3}{U_2}\cdot\frac{U_2}{U_1}+\frac{U_4}{U_3}\cdot\frac{U_3}{U_2}\cdot\frac{U_2}{U_1}+\cdots\right]$$

$$\leq U_1\left[1+\lambda+\lambda^2+\lambda^3\cdots\right]$$

$$= \frac{U_1}{1-\lambda} = \text{a finite term}$$

Hence the series is convergent.

2) <u>When $k>1$</u>

~~ΣU_n~~. then by limit $\frac{U_{n+1}}{U_n} > 1$

$$\frac{U_2}{U_1} > 1 \qquad \frac{U_3}{U_2} > 1 \cdots$$

$$\Sigma U_n = U_1+U_2+U_3\cdots$$

$$= U_1\left[1+\frac{U_2}{U_1}+\frac{U_3}{U_1}+\frac{U_4}{U_1}\cdots\right]$$

$$= U_1\left[1+\frac{U_2}{U_1}+\frac{U_3}{U_2}\cdot\frac{U_2}{U_1}+\frac{U_4}{U_3}\cdot\frac{U_3}{U_2}\cdot\frac{U_2}{U_1}\cdots\right]$$

$$> U_1\left[1+1+1+1\cdots\right]$$

$$= nU_1$$

Hence series is divergent.

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3) <u>When $K=1$</u>

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = 1$$

Let us suppose $U_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{n}{n+1} \rightarrow \frac{1}{1+1/n} = 1$$

hence series is divergent

But if $U_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1$$

But $\Sigma \frac{1}{n^2}$ is convergent (p series)

so test fails.

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Ques: Test convergence of the series

$$\frac{1}{3}x + \frac{1.2}{3.5}x^2 + \frac{1.2.3}{3.5.7}x^3 \ldots$$

$$a_n = \frac{1.2.3 \ldots n}{3.5.7 \ldots (2n+1)} x^n$$

$$\frac{a_n}{a_{n+1}} = \frac{1.2.3 \ldots n}{3.5.7 \ldots (2n+1)} \cancel{x^n} \times \frac{3.5.7 \ldots (2(n+1)+1)}{1.2.3 \ldots (n+1)\, \cancel{x^{n+1}}\, x}$$

$$= \frac{(2n+3)}{x(n+1)}$$

$$= \frac{(2+3/n)}{x(1+1/n)} \quad \text{at } n \to \infty$$

$$= \frac{2}{x} > 1 \Rightarrow x < 2$$

diverges $x > 2$

Test fails at $x = 2$.

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Raabe's Test

If Σa_n be a +ve term series and

$$\lim_{n\to\infty} n\left[\frac{a_n}{a_{n+1}} - 1\right] \to l$$

then

1) series converges at $l > 1$
2) series diverges at $l < 1$
3) Test fails at $l = 1$

- you can continue this test after ratio tes because you'll have the ratio which will be used here.

Let us use raabes test in previous ques.

$$\frac{a_n}{a_{n+1}} = \frac{2n+3}{(n+1)2} \to x=2 \text{ bcz test fails at } 2.$$

$$\lim_{n\to\infty} n\left[\frac{a_n}{a_{n+1}} - 1\right]$$

$$\lim_{n\to\infty} \frac{n}{2n+2}$$ (substituting and simplifying)

$$\downarrow$$

$$1/2 < 1$$

Result: series cgs for $x<2$
series dgs for $x \geq 2$

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Ques: 1) $1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} \cdots$

2) $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} \cdots$

3) $\frac{x}{1+x} + \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3}$

Ans 1) $a_n = \frac{n^2}{n!}$

$$\frac{a_n}{a_{n+1}} = \frac{n^2}{n!} \times \frac{(n+1)!}{(n+1)^2}$$

$$= \frac{(n+1)\,n^2}{(n+1)^2}$$

$$= \frac{n^2}{n+1} = \frac{1}{1+1/n} \quad n \to \infty$$

$= 1 \quad \infty > 1$ hence converges.

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Ans 2) $a_n = \dfrac{1}{n(n+1)(n+2)}$

$$\frac{a_n}{a_{n+1}} = \frac{1}{n\cancel{(n+1)}\cancel{(n+2)}} \times \frac{\cancel{(n+1)}\cancel{(n+2)}(n+3)}{1}$$

$$= \frac{n+3}{n}$$

$$= 1 + 3/n \rightarrow n \to \infty$$

$$= 1$$

Test fails using rabbe's test.

$$\lim_{n\to\infty} n\left[\frac{n+3}{n} - 1\right]$$

$$n+3-n$$

$$= 3 \quad \text{converges.}$$

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Ans 3) $a_n = \dfrac{x^n}{1+x^n}$

$$\frac{a_n}{a_{n+1}} = \frac{\cancel{x^n}}{1+x^n} \times \frac{1+x^{n+1}}{\cancel{x^{n+1}}\, x}$$

$$= \frac{1+x^{n+1}}{(1+x^n)\,x}$$

$$= \frac{1+x^{n+1}}{x+x^{n+1}}$$

$$= \frac{\cancel{x^{n+1}}}{\cancel{x^{n+1}}} \frac{[1/x^{n+1}+1]}{[1/x^n+1]}$$

$$= 1$$

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If $0 < x < 1$

then $x^n \to 0$ as $n \to \infty$

$$\frac{1}{x} \lim_{n\to\infty} \left(\frac{1+x^{n+1}}{1+x^n} \right) \longrightarrow \frac{1}{x} \quad \text{for } x < 1$$

$1/x > 1$

series converges for $0 < x < 1$

- <u>Necessary condn for converges</u>

If $\sum a_n$ cgs then $\lim_{n\to\infty} a_n \longrightarrow 0$

<u>Proof</u>

$\sum a_n$ cgs

$\lim_{n\to\infty} S_n \longrightarrow l$

$\lim_{n\to\infty} S_{n-1} \longrightarrow l$

$$\Rightarrow \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} \to l - l = 0$$

$$(a_1 + a_2 \cdots a_n) - (a_1 + a_2 \cdots a_{n-1}) \longrightarrow 0$$

<u>case $x > 1$</u>

$$\lim_{n\to\infty} u_n = \frac{x^n}{1+x^n} = \lim_{n\to\infty} \frac{1}{1+\left(\frac{1}{x}\right)^n}$$

diverges $x > 1$.

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Ques. Check convergence

1) $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

2) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \sin\left(\frac{1}{n}\right)$

3) $\left(\frac{1}{3}\right)^2 + \left(\frac{1.4}{3.6}\right)^2 + \left(\frac{1.4.7}{3.6.9}\right)^2 \cdots$

Ans, $\cancel{\sin\left(\frac{1}{n}\right)}$ $a_n = \sin\left(\frac{1}{n}\right)$

Rabbes test

$$n\left[\frac{\sin(1/n)}{\sin(1/n+1)} - 1\right]$$

$$\frac{\sin 1/n - \sin(1/n+1)}{1/n \quad \sin(1/n+1)}$$

Ans 1 Using comparison test.

$\sum b_n = \sum \frac{1}{n} \rightarrow$ dgt by p series test

$\lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = 1 \rightarrow$ finite / unique

therefore $\sin(1/n)$ diverges.

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Cauchy's n^{th} root test

Let $\sum a_n$ be a +ve term series
and $\lim_{n\to\infty} (a_n)^{1/n} \longrightarrow l$

* Try to prove it.

then $\sum a_n$

- cgs if $l<1$
- dgs if $l>1$
- test fails at $l=1$

Ques. 1) $a_n = \begin{cases} \frac{n}{2^n} & n = \text{odd} \\ 1/2^n & n = \text{even} \end{cases}$

2) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

3) $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$

4) $1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} \cdots$

5) $\sum (1+1/n)^{n^2}$

6) $\frac{1}{2}x + \left(\frac{2}{3}\right)^4 x^2 + \left(\frac{3}{4}\right)^9 x^3 + \cdots$

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Ans 4) $a_n = \frac{1}{n^n}$

Using nth root method

$$a_n = \frac{1}{n^n}$$

$$(a_n)^{1/n} = \frac{1}{n} \qquad n \to \infty$$

$$\to 0 \qquad \text{convergent.}$$

Ans 5) $a_n = (1 + 1/n)^{n^2}$

$$= (1 + 1/n)^n \qquad [1^\infty \text{ form}]$$

$$= e > 1 \text{ dgs.}$$

Ans 2) $a_n = \frac{n^2}{2^n}$

$$(a_n)^{1/n} = \left(\frac{n^2}{2^n}\right)^{1/n}$$

$$= \frac{n^{2/n}}{2} \longrightarrow 1/2 < 1$$

converges

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Practice ques:

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1) $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}$

2) $\frac{1}{2}x + \left(\frac{2}{3}\right)^4 x^2 + \left(\frac{3}{4}\right)^9 x^3 \ldots$

3) $\sum \frac{1}{\sqrt{n}} \tan\left(\frac{1}{n}\right)$

4) $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$

5) $\frac{1}{1\cdot 3} + \frac{2}{3\cdot 5} + \frac{3}{5\cdot 7} + \frac{4}{7\cdot 9} \ldots$

6) $\frac{1}{\sqrt{2}} + \frac{2}{\sqrt{5}} + \frac{4}{\sqrt{17}} + \frac{8}{\sqrt{65}} \ldots$

7) $\sum_{n=0}^{\infty} \frac{1}{n!}$

8) $\frac{1}{1+2} + \frac{2}{1+2^2} + \frac{3}{1+2^3} \ldots$

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Q: $1 + \frac{x}{2} + \frac{2!}{3^2}x^2 + \frac{3!}{4^3}x^3 + \frac{4!}{5^2}x^4$

$$a_n = \frac{n!\,x^n}{(n+1)^n}$$

$$\frac{a_n}{a_{n+1}} = \frac{n!\,x^n}{(n+1)^n} \times \frac{(n+2)^{n+1}}{(n+1)!\,x^{n+1}}$$

$$= \lim_{n\to\infty}\left(\frac{n+2}{n+1}\right)^{n+1} \cdot \frac{1}{x}$$

$$= \left(\frac{1+2/n}{1+1/n}\right)^n \cdot \lim_{n\to\infty}\left(\frac{1+2/n}{1+1/n}\right) \to 1$$

$$= \frac{1}{x}\cdot\frac{e^2}{e} = \frac{e}{x}$$

Ratio test

$\frac{e}{x} > 1$ cgs

$x < e$ cgs

$x > e$ dgs

$\underbrace{x = e}$ test fails

indication that logrithmic test is application.

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Logrithmic test

Let Σa_n be a +ve term series then

$$\lim_{n\to\infty} n\log\left(\frac{a_n}{a_{n+1}}\right) \to l$$

1) converges for $l > 1$
2) diverges for $l < 1$
3) test fails at $l = 1$.

continuing prv. ques

$$n\log\left(\frac{a_n}{a_{n+1}}\right) = n\log\left[\frac{(n+2)^{n+1}}{(n+1)^{n+1}} \cdot \underbrace{\frac{1}{x}}_{=e}\right]$$

$$= n\log\left[\frac{(1+2/n)^{n+1}}{(1+1/n)^{n+1}} \cdot \frac{1}{e}\right]$$

$$= n\left[(n+1)\log(1+2/n) - (n+1)\log(1+1/n) - \log e\right]$$

$$= n\left[(n+1)\left[\frac{2}{n} + \frac{1}{2}\left(\frac{2}{n}\right)^2 + \cdots\right]\right] - (n+1)\left[\frac{1}{n} - \frac{1}{2}\left(\frac{1}{n}\right)^2 \cdots\right] - 1\Big]$$

$\longrightarrow -\frac{1}{2} < 1$ hence diverges

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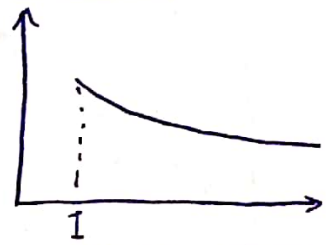
* Integral test [proof important]

Let f be a cont. +ve and ↓ function on the interval $[1,\infty)$ and $a_n = f(n)$

then the series $\sum_{n=1}^{\infty} a_n$ is convergent iff the integral

$$\int_1^{\infty} f(x)\,dx \text{ is convergent.}$$

general graph



ie

1) If $\int_1^{\infty} f(x)\,dx$ cgs then $\sum a_n$ is converges

2) If $\int_1^{\infty} f(x)\,dx$ dvgs then $\sum a_n$ is divergent.

Q: $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$

$$f(x) = \frac{\ln(x)}{x} \qquad x > 0$$

$$f'(x) = \frac{df}{dx} = \frac{1-\ln x}{x^2} < 0$$

$$1 - \ln x < 0$$

$$\ln x > 1$$

$$\boxed{x > e}$$

$$\int_e^{\infty} \frac{\ln x}{x}$$

$\ln x = t$

$x\,dt = dx$

$$= \int_e^{\infty} t\,dt = \infty - \frac{e^2}{2}$$

$$= \infty$$

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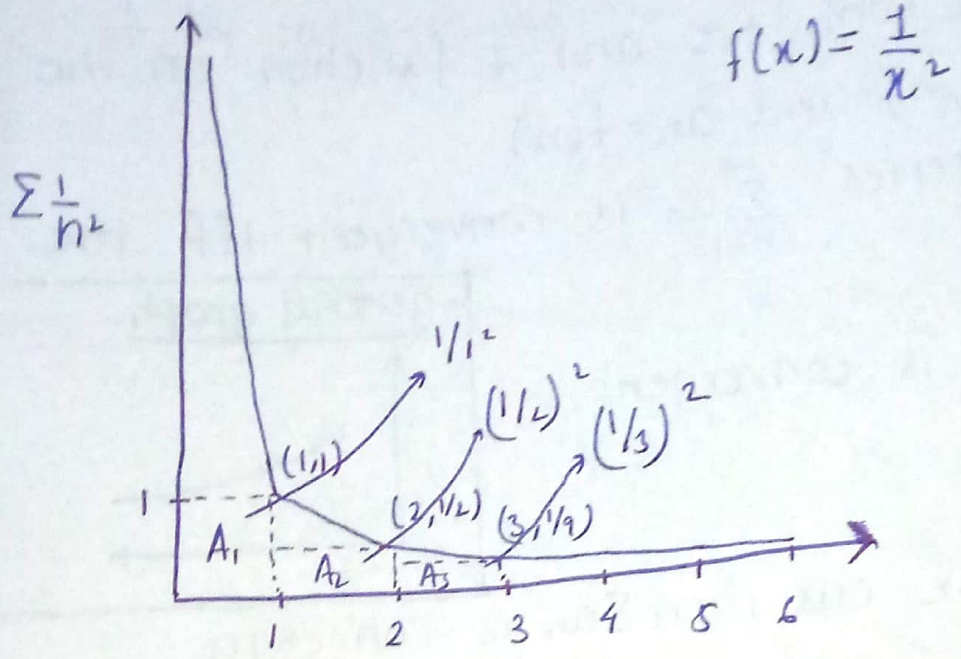

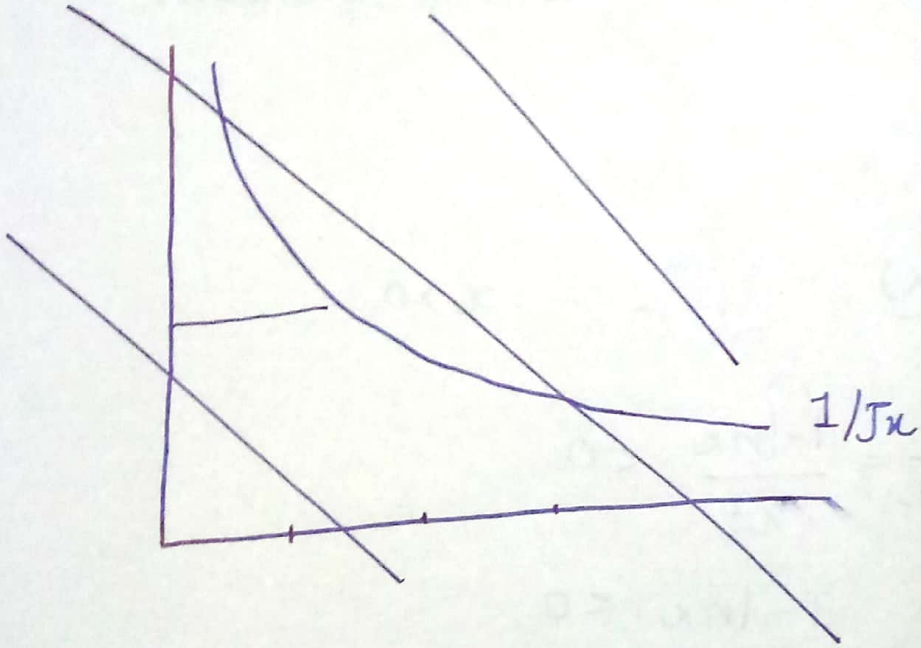

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$a_1 - a_2 + a_3 - a_4 + a_5 - a_6 \ldots \quad \sum (-1)^{n-1} a_n$

$= \sum_{n=1}^{\infty} (-1)^{n-1} a_n$

Alternating series

Test → Leibnitz test

- series cgs if

1) $|a_{n+1}| \leq |a_n|$

2) $\lim_{n \to \infty} a_n \to 0$

Q: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \ldots$

$\sum (-1)^{n-1} \frac{1}{n}$

$n+1 \geq n$

$\frac{1}{n+1} \leq \frac{1}{n}$

$a_{n+1} \leq a_n$

$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n} \to 0$

hence series cgs.

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Q: Check convergence

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\log(n+1)}$$

- Concept of absolute convergence

By ratio test to absolute series

$$\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{1}{|x|} \cdot \frac{\log(n+2)}{\log(n+1)}$$

$\rightarrow \frac{1}{|x|} > 1$ cgs

$|x| < 1$ cgs

$|x| > 1$ dvgs

$|x| = 1$ test fails.

- Absolute convergence

$|\Sigma(-1)^n a_n| \leq \Sigma |a_n|$ ie $|a_1 + a_2| \leq |a_1| + |a_2|$

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Proof of Liebnitz test

Let the sum of $2n$ number of terms be denoted by S_{2n}

$$S_{2n} = (a_1 - a_2) + (a_3 - a_4) + (a_5 - a_6) \dots (a_{2n-1} - a_{2n})$$

since $a_n > a_{n+1}$

$a_1 > a_2$

$a_3 > a_4$

$\vdots$

$a_{2n-1} > a_{2n}$

$S_{2n} > 0$

Now,

$$S_{2n+1} = U_1 - U_2 + U_3 - U_4 \dots U_{2n-1} - U_{2n} + U_{2n+1}$$

$$= U_1 - \underbrace{(U_2 - U_3)}_{>0} - \underbrace{(U_4 - U_5)}_{>0} \dots (U_{2n} - U_{2n+1})$$

$$S_{2n+1} \leq U_1$$

Also $S_{2n+1} = S_n + U_{2n+1}$

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} + \underbrace{\lim_{n\to\infty} U_{2n+1}}_{0 \text{ by given cond}^n}$$

$$\Rightarrow \lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n}$$

hence converges

$S_{2n} > 0$

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