11 Second and Higher Order Linear Differential Equations

CHAPTER

"The theory of linear differential equations, particularly that of with constant coefficients, is quite comprehensive. There are standard methods for solving many practically important linear differential equations. Second order linear differential equations are important, since they occur more frequently while modelling the physical situations and the techniques developed to solve these equations can be extended to higher order equations as well."

11.1 BASIC CONCEPTS

The general linear differential equation of the nth order is of the form

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{dy^{n-1}}{dx^{n-1}} + a_2(x)\frac{dy^{n-2}}{dx^{n-2}} + \dots + a_n(x)y = f(x), \quad \text{...(11.1)}$$

where $a_0(x)$, $a_1(x)$, $a_2(x)$ ..., $a_n(x)$ and $f(x)$ are functions of x only, and $a_0(x) \neq 0$.

In case $a_0, a_1, ..., a_n$ are constants, the Eq. (11.1) is called linear differential equation with constant coefficients. The characteristic feature of a linear equation is that it is linear in the unknown function y and its derivatives.

If $f(x) = 0$, then equation is called *homogeneous equation*, otherwise it is called *non-homogeneous equation*. For example, a second order homogeneous equation with constant coefficients is of the form

$$a_0\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = 0, \ a_0 \neq 0,$$

while a non-homogeneous second order equation is of the form

$$a_0\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = f(x), \ a_0 \neq 0.$$

Of the higher order equations, the second order equations are the simplest one and have many applications in mechanics and electric circuit theory, for example

(a) The equation $L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E(t)$ represents the charge Q in an RLC-series circuit.

(b) The equation $m\frac{d^2x}{dt^2} + a\frac{dx}{dt} + kx = f(t)$ governs the motion of a mass m on a spring. Here x is the distance from a fixed point after t seconds, a is the damping factor, k is the spring stiffness and $f(t)$ is an external force.

(c) The equations

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0, \quad \text{and} \quad x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0,$$

respectively the Legendre and Bessel equations, are important equations in applied mathematics and physics.

(d) The non-linear equation $\frac{d^2y}{dx^2} = c\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$ represents the shape of a uniform flexible cable, or catenary, hanging under the action of its own weight, refer to Fig. 11.1.

Here $y(x)$ is the deflection and c is a constant depending upon the mass density of the cable.

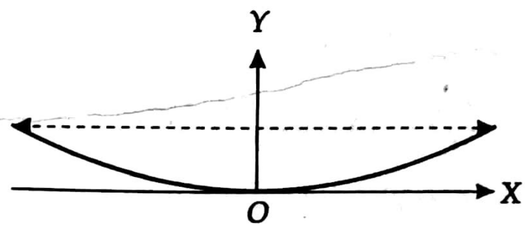

Fig. 11.1

(e) The equation, $EI\frac{d^4y}{dx^4} + ky = f(x)$, $k > 0$, occurs while studying the deflected shape $y(x)$ of a beam on an elastic foundation, under a load of $f(x)$ units per unit length, refer to Fig. 11.16 (p. 678).

In fact, there are numerous physical situations modelled by second and higher order differential equations.

11.2 SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS. LINEARLY INDEPENDENT AND DEPENDENT SOLUTIONS

In this section we shall discuss the solution of linear differential equations, particularly their linear dependence and independence. In what follows we shall assume that x varies on an interval I, may be the whole real line, which generally will not be mentioned explicitly.

11.2.1 Existence and Uniqueness of Solutions for Initial Value Problems

If $y = y_1(x)$ is a solution of the differential equation (11.1) on an interval I, then it must satisfy (11.1) indentically and hence $y_1(x)$ must be continuously differentiable $(n - 1)$ times and $y_1^{(n)}$ must be continuous on I.

We state the following result:

Theorem 11.1: *If the coefficients $a_0(x), a_1(x), ..., a_n(x)$ and $f(x)$ are continuous over an interval I, then there exists a unique solution to the initial value problem*

$$a_0y^{(n)} + a_1y^{(n-1)} + ... + a_{n-1}y^{(1)} + a_ny = f(x);\ y(x_0) = k_0,\ y^{(1)}(x_0) = k_1, ..., y^{(n-1)}(x_0) = k_{n-1}$$

where, $x_0 \in I$ and $k_0, k_1, ..., k_{n-1}$ are constants.

We must note the derivative $y^{(n)}(x_0)$ cannot be specified as an initial condition, because it is determined by the differential equation itself once the stated initial conditions have been given.

11.2.2 Superposition, or Linearity Principle

Theorem 11.2: *If $y_1(x)$ and $y_2(x)$ are two solutions of the linear homogeneous equation*

$$a_0y^{(n)} + a_1y^{(n-1)} + ... + a_{n-1}y^{(1)} + a_ny = 0 \quad \text{...(11.2)}$$

← homogeneous

then $y = c_1y_1(x) + c_2y_2(x)$, a linear combination of y_1 and y_2, where c_1, c_2 are two arbitrary constants, is also its solution.

Proof. Substituting for $y = c_1y_1(x) + c_2y_2(x)$ in left-hand side of (11.2), we obtain

$$a_0(c_1y_1 + c_2y_2)^{(n)} + a_1(c_1y_1 + c_2y_2)^{(n-1)} + ... + a_n(c_1y_1 + c_2y_2)$$

$$= c_1[a_0y_1^{(n)} + a_1y_1^{(n-1)} + ... + a_ny_1] + c_2[a_0y_2^{(n)} + a_1y_2^{(n-1)} + ... + a_ny_2] = c_1(0) + c_2(0) = 0,$$

since $y_1(x), y_2(x)$ are solutions of the linear homogeneous Eq. (11.2).

This result can be generalized to the case of more than two solutions.

Example 11.1: Show that $y_1 = e^x$, and $y_2 = e^{-2x}$ and their linear combination $c_1e^x + c_2e^{-2x}$ are solutions of the differential equation $y'' + y' - 2y = 0$.

Solution: For $y_1 = e^x$, we have $y_1' = e^x$, $y_1'' = e^x$, and thus $y'' + y' - 2y = e^x + e^x - 2e^x = 0$.

Hence, $y_1 = e^x$ is a solution of $y'' + y' - 2y = 0$.

Similarly, $y_2 = e^{-2x}$ is also a solution of $y'' + y' - 2y = 0$.

Next, $y = c_1e^x + c_2e^{-2x}$ gives $y' = c_1e^x - 2c_2e^{-2x}$ and $y'' = c_1e^x + 4c_2e^{-2x}$. Hence,

$$y'' + y' - 2y = (c_1e^x + 4c_2e^{-2x}) + (c_1e^x - 2c_2e^{-2x}) - 2(c_1e^x + c_2e^{-2x}) = c_1[0] + c_2[0] = 0.$$

Thus $y = c_1e^x + c_2e^{-2x}$ is also its solution.

We must note that *the linearity principle does not hold in case of non-homogeneous equations and non-linear equations.*

For example, $y_1 = 1 + \sin x$ and $y_2 = 1 + \cos x$ are solutions of the linear non-homogeneous differential equation $y'' + y = 1$ but $y_1 + y_2$, combination of y_1 and y_2 is not its solution. Similarly, $y_1 = x^2$ and $y_2 = 1$ are solutions of the non-linear differential equation $yy'' - xy' = 0$ but $y_1 + y_2$ is not a solution of this equation. We note that even $-y_1$, a simple constant multiple of y_1, is not a solution of this equation.

Next, if in $c_1y_1 + c_2y_2$, y_2 is simply a constant multiple of y_1, say $y_2 = ky_1$, then $c_1y_1 + c_2y_2 = c_1y_1 + c_2ky_1 = (c_1 + c_2k)y_1$ is just another constant multiple of y_1. In such a case y_2 does not provide any additional information. We now distinguish the two cases, first, when the two solutions are constant multiple of each other; and second, when the two solutions are not constant multiple of each other.

11.2.3 Linear Independent and Dependence

Functions $y_1(x), y_2(x), \ldots, y_n(x)$, $n > 1$ are called *linearly independent* on some interval I, where they are defined, if the equation

$$c_1y_1(x) + c_2y_2(x) + \ldots + c_ny_n(x) = 0 \qquad \ldots(11.3)$$

on I implies that $c_1 = c_2 = \ldots = c_n = 0$. These functions are said to be *linearly dependent* on I if the Eq. (11.3) also holds on I for some $c_1, c_2, \ldots, c_n$ not all zero. In such a case, one or more of y_i's can be expressed as a linear combination of the remaining functions. For example, if $c_1 \neq 0$, then

$$y_1(x) = -\frac{1}{c_1}[c_2y_2(x) + \ldots + c_ny_n(x)].$$

In particular, two functions y_1 and y_2 are linearly dependent on I, if there exists a constant $c \neq 0$ such that $y_1(x) = cy_2(x)$ on I; otherwise, $y_1(x)$ and $y_2(x)$ are linearly dependent on I. For example, $y_1 = \cos x$ and $y_2 = \sin x$ are two linearly independent solutions of the differential equation $y'' + y = 0$, since $y_2 = cy_1$, gives $\tan x = c$, for c, for all x, which is not true. But obviously, $y_1 = \cos x$ and $y_2 = 5\cos x$ are two linearly dependent solutions of this equation.

It is difficult to examine independence and dependence like this in case of more than two functions. A very systematic procedure to test the linear independence and dependence of a given set of functions is the application of Wronskian. Let $y_1, y_2, \ldots, y_n$ be the given functions, then the Wronskian of these functions, denoted by $W(y_1, y_2, \ldots, y_n)$, is defined as

$$W(y_1, y_2, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} = W(x).$$

Obviously the Wronskian of y_i's exists only if all the y_i's are differentiable $(n-1)$ times on I. We have the following result for testing the linear dependence or independence of the solutions of the linear homogeneous differential equation

$$a_0(x)\, y_{(x)}^{(n)} + a_1(x)\, y_{(x)}^{(n-1)} + \ldots + a_{n-1}(x)\, y'(x) + a_n(x)\, y(x) = 0,\ a_0(x) \neq 0 \qquad \ldots(11.4)$$

Theorem 11.3: If the coefficients $a_0(x), a_1(x), \ldots, a_n(x)$ of the differential Eq. (11.4) are continuous on I, then n solutions $y_1, y_2, \ldots, y_n$ of (11.4) are linearly dependent on I if and only if Wronskian $W(x) = 0$ for some $x = x_0$ on I. Further, if $W(x) = 0$ for $x = x_0$, then $W(x) \equiv 0$ on I; hence if there is some x_1 at which $W(x) \neq 0$, then $y_1, y_2, \ldots, y_n$ are linearly independent solutions on I.

Proof Let $y_1(x), y_2(x), \ldots, y_n(x)$ be linearly dependent on I. Then there exist constants $c_1, c_2, \ldots, c_n$, not all zero, such that for all x on I,

$$c_1y_1(x) + c_2y_2(x) + \ldots + c_ny_n(x) = 0 \quad \ldots(11.5)$$

Differentiating this successively $(n-1)$ times, we obtain

$$\left.\begin{aligned} c_1y'_1(x) + c_2y'_2(x) + \ldots + c_ny'_n(x) &= 0 \\ &\vdots \\ c_1y_{(x)}^{(n-1)}(x) + c_2y_{(x)}^{(n-1)} + \ldots + c_ny_n^{(n-1)}(x) &= 0 \end{aligned}\right\} \quad \ldots(11.6)$$

Equations (11.5) and (11.6) is a homogeneous linear system of algebraic equations with a non-trivial solution $c_1, c_2, \ldots, c_n$. Hence its coefficient determinant, which is the Wronskian $W(x) = W(y_1, y_2, \ldots, y_n)$, must be zero for every x on I. Conversely, let $W(x) = 0$ for some fixed $x_0 \in I$. Then the system of Eqs. (11.5) and (11.6) has a solution $c_1^*, c_2^*, \ldots, c_n^*$ not all zero by the result just proved. Hence $y^*(x) = c_1^*y_1(x) + c_2^*y_2(x) + \ldots + c_n^*y_n(x)$ is a solution of the linear homogeneous Eq. (11.4). By using the system of Eqs. (11.5) and (11.6), we find that $y^*(x)$ also satisfies the initial conditions $y^*(x_0) = 0$, $(y^*)'(x_0) = 0, \ldots, (y^*)_{(x_0)}^{(n-1)} = 0$. Thus $y^*(x)$ is the solution of initial value problem, and since the solution of the initial value problem is unique, refer to Theorem 11.1, thus $y^*(x) = y(x) = 0$ holds identically. That is, $c_1^* y_1(x) + c_2^* y(x) + \ldots + c_n^* y_n(x) = 0$, for all c_i's not zero and this implies the linear dependence of $y_1(x), y_2(x), \ldots, y_n(x)$. Thus $W(x_0) = 0$ for some $x_0 \in I$ implies $W(x) = 0$ for all $x \in I$.

From the results proved above, it follows obviously that $W(x) \neq 0$ at solve $x_1 \in I$ implies linear independence of the solutions $y_1, y_2, \ldots, y_n$ on I.

We have considered the solutions $y_1(x) = \cos x$, and $y_2(x) = \sin x$ of $y'' + y = 0$ for all x. In this case linear independence was obvious. Also the Wronskian of these solutions is

$$W(x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x \neq 0.$$

But it is not always obvious whether the two solutions are linearly independent or dependent on an interval. For example, consider the equation $y'' + xy = 0$. This equation can be solved by power series method to be discussed in Chapter 12; the two solutions are

$$y_1(x) = 1 - \frac{1}{6}x^3 + \frac{1}{180}x^6 - \frac{1}{12960}x^9 + \ldots$$

and,
$$y_2(x) = x - \frac{1}{12}x^4 + \frac{1}{504}x^7 - \frac{1}{45360}x^{10} + \ldots$$

Both $y_1(x)$ and $y_2(x)$ are convergent for all x on the real line. It is difficult to evaluate the Wronskian of these solutions at any non-zero x. Consider the Wronskian at $x = 0$, we have

$$W(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = y_1(0)y_2'(0) - y_1'(0)y_2(0) = (1)(1) - (0)(0) = 1 \neq 0$$

and this is sufficient to ensure that $y_1(x)$ and $y_2(x)$ as defined above, are linearly independent for all x on the entire line.

Fundamental solutions: a basis. The n linearly independent solutions $y_1(x), y_2(x) \ldots, y_n(x)$ are called the *fundamental solutions* of the homogeneous Eq. (11.2) on I. The set $\{y_1, y_2, \ldots, y_n\}$ of fundamental solutions forms a basis of the nth order linear homogeneous Eq. (11.2).

Now we are in a position to define the general solution of the homogeneous linear equation (11.2).

The general solution If $y_1(x), y_2(x), \ldots, y_n(x)$ are n linearly independent solutions of the nth order linear homogeneous equation

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_{n-1}(x)y^{(1)} + a_n(x) = 0$$

that is, if the set $\{y_1(x), y_2(x), \ldots, y_n(x)\}$ forms a basis of the nth order linear homogeneous equation, then the general solution of this equation is

$$y(x) = c_1y_1(x) + c_2y_2(x) + \ldots + c_ny_n(x) \qquad \ldots(11.7)$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Example 11.2: Show that the set $\{1, e^x, e^{-x}\}$ forms a basis of the differential equation $y''' - y' = 0$, but $e^x, e^{-x}, \cosh x$ is not so.

Solution: It is easy to verify that each of the functions $1, e^x, e^{-x}$ satisfy the differential equation $y''' - y' = 0$. Their Wronskian is

$$W(x) = W(1, e^x, e^{-x}) = \begin{vmatrix} 1 & e^x & e^{-x} \\ 0 & e^x & -e^{-x} \\ 0 & e^x & e^{-x} \end{vmatrix} = 2 \neq 0.$$

Thus the set $\{1, e^x, e^{-x}\}$ is a fundamental set and forms a basis of the differential equation $y''' - y' = 0$. Also we can verify that e^x, e^{-x} and $\cosh x$ are solutions of the differential equation $y''' - y' = 0$. But

$$W(x) = W(e^x, e^{-x}, \cosh x) = \begin{vmatrix} e^x & e^{-x} & \cosh x \\ e^x & -e^{-x} & \sinh x \\ e^x & e^{-x} & \cosh x \end{vmatrix} = 0,$$

the first and third rows being equal.

Hence, the set $\{e^x, e^{-x}, \cosh x\}$ does not form a basis. In fact, we note the $\cosh x$ is a linear combination of e^x and e^{-x}.

Example 11.3: Show that the set of functions $\{x, 1/x\}$ forms a basis of the equation $x^2y'' + xy' - y = 0, 1 \leq x < \infty$. Obtain a particular solution when $y(1) = 1, y'(1) = 2$.

Solution: We have $y_1 = x, y_1' = 1, y_1'' = 0$, and hence $x^2y_1'' + xy_1' - y_1 = x - x = 0$. Thus, $y_1 = x$ is a solution of $x^2y'' + xy' - y = 0$. Similarly, we can verify that $y_2 = 1/x$ is also a solution of the given homogeneous equation.

The Wronskian of y_1, y_2 is, $W(y_1, y_2) = \begin{vmatrix} x & 1/x \\ 1 & -1/x^2 \end{vmatrix} = -\dfrac{2}{x} \neq 0$. Therefore the set $\{x, 1/x\}$ forms a basis of the equation, and hence the general solution is $y(x) = c_1x + \dfrac{c_2}{x}$. It gives, $y'(x) = c_1 - \dfrac{c_2}{x^2}$.

Using the initial values, $y(1) = 1$, $y'(1) = 2$, we get respectively $c_1 + c_2 = 1$, $c_1 - c_2 = 2$; thus $c_1 = 3/2$, $c_2 = -1/2$. Hence the particular solution is $y(x) = \frac{1}{2}\left(3x - \frac{1}{x}\right)$.

11.3 FINDING SECOND LINEARLY INDEPENDENT SOLUTION FROM A KNOWN SOLUTION: REDUCTION OF ORDER

Suppose we know one solution $y_1(x)$ of the homogeneous linear second order differential equation.

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0,\ a_0(x) \neq 0 \qquad ...(11.8)$$

and we need to find the second linealy independent solution $y_2(x)$. The same can be obtained by the method of reduction of order which involves a second solution of the form

$$\boxed{y_2(x) = u(x)y_1(x),} \qquad ...(11.9)$$

where the function $u(x) \neq$ constant is to be determined. From (11.9), we have

$$y'_2 = u'y_1 + uy'_1, \text{ and } \quad y''_2 = u''y_1 + 2u'y'_1 + uy''_1.$$

Substituting for y_2, y'_2, y''_2 in (11.8) and rearranging the terms, we obtain

$$a_0(x)y_1u'' + [2a_0(x)y'_1 + a_1(x)y_1]u' + [a_0(x)y''_1 + a_1(x)y'_1 + a_2(x)y_1]u = 0 \qquad ...(11.10)$$

Using the fact that y_1 is a solution of Eq. (11.8), (11.10) reduces to

$$a_0(x)y_1u'' + [2a_0(x)y'_1 + a_1(x)y_1]u' = 0 \qquad ...(11.11)$$

Let $v = u'$, Eq. (11.11) becomes $a_0(x)y_1v' + [2a_0(x)y'_1 + a_1(x)y_1]v = 0$, a first order differential equation in v. Separating the variables, we obtain $\frac{dv}{v} = -\left[\frac{2a_0(x)y'_1 + a_1(x)y_1}{a_0(x)y_1}\right]dx$.

Integrating we get, $\ln v = -2\ln y_1 - \int \frac{a_1(x)}{a_0(x)}dx + \text{const.}$

or,
$$v = A_1\left[\frac{\exp\left\{-\int p(x)dx\right\}}{y_1^2}\right], \qquad ...(11.12)$$

where $p(x) = a_1(x)/a_0(x)$, and A_1 is a constant.

Since $v = u'$, integration of (11.12) gives,
$$u(x) = A_1\int\left[\frac{\exp\left\{-\int p(x)dx\right\}}{y_1^2}\right]dx + A_2, \qquad ...(11.13)$$

where A_2 is another arbitrary constant of integration. Without any loss of generality, we can have $A_1 = 1$ and $A_2 = 0$, and thus (11.13) gives $u(x) = \int\left[\frac{\exp\left\{-\int p(x)dx\right\}}{y_1^2}\right]dx$.

Therefore, from (11.9), the second linearly independent solution is

$$y_2(x) = y_1(x) \int \left[\frac{\exp\left\{-\int p(x)dx\right\}}{y_1^2} \right] dx \qquad \text{...(11.14)}$$

and hence the general solution of (11.8) is, $y(x) = c_1y_1(x) + c_2y_2(x)$, where c_1 and c_2 are two arbitrary constants.

Example 11.4: Given that $y_1(x) = e^{-3x}$ is a solution of $y'' + 6y' + 9y = 0$, find a second linearly independent solution and hence find the general solution of the equation.

Solution: Let the second linearly independent solution be $y_2(x) = u(x)y_1(x)$, where $y_1(x) = e^{-3x}$. Here $p(x) = a_1(x)/a_0(x) = 6$. Hence,

$$u(x) = \int \left[\frac{\exp\left\{-\int p(x)dx\right\}}{y_1^2} \right] dx = \int \left[\frac{\exp\left\{-\int 6dx\right\}}{e^{-6x}} \right] dx = \int dx = x.$$

Thus $y_2(x) = xe^{-3x}$ and hence, the general solution is $y(x) = c_1e^{-3x} + c_2xe^{-3x}$.

Example 11.5: Given that $y_1(x) = x^2$ is a solution of $y'' - (3/x)y' + (4/x^2)y = 0$, $x > 0$, find a second linealy independent solution and hence find the general solution of the equation.

Solution: Let the second linearly independent solution be $y_2(x) = u(x)y_1(x)$, where $y_1(x) = x^2$.

Here $\quad p(x) = \dfrac{a_1(x)}{a_0(x)} = -3/x$. Hence,

$$u(x) = \int \left[\frac{\exp\left\{-\int p(x)dx\right\}}{y_1^2} \right] dx = \int \left[\frac{\exp\left\{-\int \frac{-3}{x}dx\right\}}{x^4} \right] dx = (\ln x. + A)$$

Thus, $y_2(x) = x^2 \ln x$, and hence the general solution is $y = c_1x^2 + c_2x^2 \ln x$, where c_1 and c_2 are two arbitrary constants.

EXERCISE 11.1

Verify that each of the given function is a solution of the given differential equation. Verify if the set forms a basis or not. If so, find the general solution of the differential equation.

1. $y''' - 6y'' + 11y' - 6y = 0$, $\{e^x, e^{2x}, e^{3x}\}$ form.
2. $y''' - 6y'' + 9y' - 4y = 0$, $\{e^x, xe^x, (1-x)e^x\}$ ✗
3. $x^2y'' - xy' + y = 0$, $x > 0$, $\{x, x \ln x\}$ ✓
4. $y^{(4)} + 2y^{(2)} + y = 0$, $\{\cos x, \sin x, x\cos x, x\sin x\}$

5. $x^2y'' + 4xy' + 2y = 0,\ \left\{\frac{1}{x}, \frac{1}{x^2}\right\}$

6. $(1 + x^2)y'' + (1+ x)y' + y = 0,\ \{\cos[\ln(1 + x)], \sin[\ln(1 + x)]\}$

Verify that y_1 and y_2 are solutions of the given differential equation. Show that they form a basis. Also find the particular solution of the initial value problem.

7. $y'' - 4y = 0;\ y(0) = 1,\ y'(0) = 0,\ y_1 = \cosh(2x),\ y_2 = \sinh(2x)$

8. $y'' + 11y' + 24y = 0;\ y(0) = 1,\ y'(0) = 0,\ y_1 = e^{-3x},\ y_2 = e^{-8x}$

9. $y'' - \frac{7}{x}y' + \frac{16}{x^2}y = 0;\ y(1) = 2,\ y'(1) = 4,\ y_1 = x^4,\ y_2 = x^4 \ln x$

10. $x^2y'' + xy' - 4y = 0;\ y(1) = 2,\ y'(1) = 6,\ y_1 = x^2,\ y_2 = \frac{1}{x^2}$

11. $4x^2y'' - 3y = 0;\ y(1) = 3,\ y'(1) = 2.5,\ y_1 = x^{-1/2},\ y_2 = x^{3/2}$

Find the solution of the following differential equations by reducing the order, one solution is known in each case.

12. $y'' - y' - 6y = 0,\ y_1 = e^{-2x}$

13. $x^2y'' - xy + y = 0;\ x > 0,\ y_1 = x$

14. $x^2y'' - 5xy' + 9y = 0,\ y_1 = x^3$

15. $x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0,\ y_1 = x^{-1/2}\cos x$

Reduce the following equations to first order and solve

16. $y'' = y'$

17. $yy'' = 2y'^2$

18. $xy'' = \sqrt{1 + y'^2}$

19. $y^{(4)}y^{(3)} = 1$

20. A small body moves on a straight line so that the product of its velocity and acceleration is constant, say $1\ \text{m}^2/\text{sec}^3$. If at $t = 0$ the body's distance from the origin is 2 meter and its velocity is 2 meter/sec; then what are the distance and velocity at $t = 6$ sec?

11.4 DIFFERENTIAL OPEREATOR D. SOLUTION OF CONSTANT COEFFICIENTS HOMOGENEOUS LINEAR EQUATIONS

In this section we introduce the differential operator D and discuss methods of finding solution of linear differential equations with constant coefficients.

11.4.1 Differential Operator D

By an operator we mean a transformation that transforms a function into another function. Let D denote the differentiation with respect to x, that is, $D \equiv \frac{d}{dx}$, and we write

$$Df(x) = Df = f' = \frac{df}{dx}.$$

Thus D transforms $f(x)$ into its derivative $f'(x)$. For example, $D(x^3) = 3x^2$, $D(\sin x) = \cos x$. Also D is a linear operator, that is, $D(af + bg) = aDf + bDg$, where a and b are constants.

Further applying D twice, we have $D(Df) = D(f') = f''$, and we simply write $D^2f = f''$, $D^3f = f'''$ etc., where f is sufficiently differentiable.

We define $D^0 = 1$, and thus $D^0f = 1f = f$.

The homogeneous linear differential equation $a_0\dfrac{d^2y}{dx^2} + a_1\dfrac{dy}{dx} + a_2y = 0$, can be written as

$$a_0D^2y + a_1Dy + a_2y = 0, \text{ or } (a_0D^2 + a_1D + a_2)y = 0$$

or, $\quad F(D)y = 0, \text{ where } F(D) \equiv a_0D^2 + a_1D + a_2$

is the *second order differential operator*.

Similarly, linear differential equation of the nth order,

$$a_0\frac{d^ny}{dx^2} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_{n-1}\frac{dy}{dx}\, a_ny = f(x)$$

can be expressed as $F(D)y = f(x)$,

where $F(D) \equiv a_0D^n + a_1D^{n-1} + \dots + a_{n-1}D + a_n$, is the *nth order differential operator*.

When a_i, $i = 0, 1, \dots n$, are constants, the differential operator $F(D)$ can be factorized. For example, $(D^2 + 3D + 2) = (D + 1)(D + 2)$.

We must note that when $a(x)$ is a function of x, then $D[a(x)f] \neq a(x)Df$. In that case $D[a(x)f] = a(x)f' + a'(x)f$. We shall apply operator methods to solve linear differential equations with constant coefficients only. Extension of these methods to variable-coefficient equations is comparatively difficult and will not be considered here.

11.4.2 Solution of the Constant Coefficients Homogeneous Linear Equation of Second Order

Consider the second order homogeneous linear equation

$$ay'' + by' + cy = 0, \qquad \text{...(11.15)}$$

where $a, b, c, a \neq 0$ are constants.

In the operator notion, Eq. (11.15) can be written as

$$(aD^2 + bD + c)y = 0. \qquad \text{...(11.16)}$$

The form of the Eq. (11.16) requires that constant multiples of derivatives of $y(x)$ must sum to zero. Since the derivative of an exponential function $e^{\lambda x}$ is a constant multiple of $e^{\lambda x}$, therefore, we consider $y = e^{\lambda x}$ for solution. Substituting in (11.16), we get $(a\lambda^2 + b\lambda + c)e^{\lambda x} = 0$. Since $e^{\lambda x} \neq 0$, we obtain

$$a\lambda^2 + b\lambda + c = 0, \qquad \text{...(11.17)}$$

an algebraic equation in λ.

It is called *the characteristic equation or the auxiliary equation* of the Eq. (11.15) and the roots of this equation $\lambda_1 = \dfrac{-b+\sqrt{b^2-4ac}}{2a}$, $\lambda_2 = \dfrac{-b-\sqrt{b^2-4ac}}{2a}$ are called the characteristic roots of the Eq. (11.15). Thus the functions $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$ are the solutions of the differential equation (11.15). Depending on the sign of the discriminant $b^2 - 4ac$, the following three cases arise

Case I: *$b^2 - 4ac > 0$, the roots are real and distinct, that is, $\lambda_1 \neq \lambda_2$.*

Case II: *$b^2 - 4ac = 0$, the roots are real and equal, that is, $\lambda_1 = \lambda_2$.*

Case III: *$b^2 - 4ac < 0$, the roots are complex conjugate.*

To find the complete solution in each case, we proceed as follows:

Case I: Real and distinct roots. In this case $y_1 = e^{\lambda_1 x}$ and $y_2 = e^{\lambda_2 x}$ form two linearly independent solutions of the differential Eq. (11.15) on any interval, since $y_1/y_2 = e^{(\lambda_1-\lambda_2)x} \neq$ constant for $\lambda_1 \neq \lambda_2$. The general solution of Eq. (11.5) is

$$\boxed{y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x},}$$ — $\lambda_1 \neq \lambda_2$, i.e $D > 0$

where c_1 and c_2 are two arbitrary constants.

Case II: Real and equal roots. In case the discriminant $b^2 - 4ac = 0$, the characteristic equation has the repeated root $\lambda = \lambda_1 = \lambda_2 = -b/2a$, so $y_1 = e^{-bx/2a}$ is one solution of the Eq. (11.15).

To obtain a second linearly independent solution y_2, we use the method of reduction of order as discussed in Section 11.3. Set $y_2 = uy_1$ in Eq. (11.15), we obtain

$$a(uy_1)'' + b(uy_1)' + cuy_1 = 0$$

or, $$a(u''y_1 + 2u'y_1' + uy_1'') + b(u'y_1 + uy_1') + cuy_1 = 0.$$

Collecting terms in u'', u' and u, we obtain

$$ay_1u'' + (2ay_1' + by_1)u' + (ay_1'' + by_1' + cy_1)u = 0. \quad \text{...(11.18)}$$

Since y_1 is a solution of Eq. (11.15), therefore, $ay_1'' + by_1' + cy_1 = 0$.

Also, $2ay_1' = 2a\left(-\dfrac{b}{2a}e^{-\frac{b}{2a}x}\right) = -be^{-\frac{b}{2a}x} = -by_1$, which gives, $(2ay_1' + by_1) = 0$.

Substituting these in (11.18), we obtain $au''y_1 = 0$, or $u'' = 0$, since $a \neq 0$ and $y_1 \neq 0$. It gives $u = Ax + B$. We can simply take $u = x$, and thus making $y_2 = xy_1$ as the second linearly independent solution of Eq. (11.15), for $y_2/y_1 = x$ is not a constant. Thus, in the case of real double root of the characteristic equation, the general solution of Eq. (11.15) is

$$\boxed{y = (c_1 + c_2x)e^{\lambda x}, \quad \lambda = -b/2a,}$$ $D = 0$, equal roots

where c_1 and c_2 are two arbitrary constants.

Case III: Complex conjugate roots. In case $b^2 - 4ac < 0$, then the roots of the Eq.(11.17) are complex conjugates, say $p \pm iq$, where $p = -\frac{b}{2a}$ and $q = \sqrt{4ac - b^2}/2a$ are reals. Thus the two solutions of the Eq. (11.15) are $y_1 = e^{(p+iq)x}$, and $y_2 = e^{(p-iq)x}$.

These two solutions are linearly independent, since $q \neq 0$ and thus

$$\frac{y_1}{y_2} = e^{(p+iq)x}/e^{(p-iq)x} = e^{2iqx} \neq \text{constant}.$$

Hence the general solution of the equation is

$$y = Ay_1 + By_2 = Ae^{(p+iq)x} + Be^{(p-iq)x} = (Ae^{iqx} + Be^{-iqx})e^{px}$$
$$= [A(\cos px + i\sin px) + B(\cos qx - i\sin qx)]e^{px} \quad ...(11.19)$$

by Euler's formula. Rewriting (11.19), we obtain

$$\boxed{y(x) = (c_1 \cos qx + c_2 \sin qx)e^{px},} \quad ...(11.20)$$

as the general solution, where $c_1 = A + B$ and $c_2 = i(A - B)$ are constants.

Hence, we have the following results to remember:

For the differential equation $ay'' + by' + cy = 0$, the characteristic equation is $a\lambda^2 + b\lambda + c = 0$. The three cases are:

I $b^2 - 4ac > 0$: *The general solution is $y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$, where*

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

II $b^2 - 4ac = 0$: *The general solution is $y(x) = (c_1 + c_2x)e^{\lambda x}$, where $\lambda = -b/2a$.*

III $b^2 - 4ac < 0$: *The general solution is $y(x) = (c_1 \cos qx + c_2 \sin qx)e^{px}$, where*

$$p = -\frac{b}{2a} \text{ and } q = \sqrt{4ac - b^2}/2a.$$

We note that the characteristic Eq. (11.17) for the Eq. (11.15) can be written directly replacing simply D by λ in the Eq. (11.15).

Example 11.6: Find the general solution of the differential equation $2y'' + 5y' - 3y = 0$.

Solution: The differential equation is $(2D^2 + 5D - 3)y = 0$.

The corresponding characteristic equation is given by

$$2\lambda^2 + 5\lambda - 3 = 0, \text{ or } (2\lambda - 1)(\lambda + 3) = 0.$$

It gives $\lambda = 1/2, -3$ as two distinct real roots. Hence, the general solution of the given equation is $y(x) = c_1e^{x/2} + c_2e^{-3x}$, where c_1 and c_2 are arbitrary constants.

Example 11.7: Find the general solution and solve the initial value differential equation

$$y'' + 4y' + 4y = 0; \; y(0) = 3 \text{ and } y'(0) = 1.$$

Solution: The differential equation is $(D^2 + 4D + 4)y = 0$.

The corresponding characteristic equation is given by

$$\lambda^2 + 4\lambda + 4 = 0, \text{ or } (\lambda + 2)^2 = 0,$$

which gives $\lambda = -2, -2$ as repeated root. Hence the general solution is

$$y = (c_1 + c_2 x)e^{-2x}. \qquad ...(11.21)$$

Using the initial conditions $y(0) = 3$, and $y'(0) = 1$ in (11.21), we get $c_1 = 3$ and $c_2 = 7$. So the solution of the given initial value problem is $y(x) = (3 + 7x)e^{-2x}$.

Example 11.8: Solve the initial value problem $y'' + 6y' + 13y = 0; y(0) = 3, y'(0) = 7$.

Solution: The differential equation is $(D^2 + 6D + 13)y = 0$.

The corresponding characteristic equation is given by $\lambda^2 + 6\lambda + 13 = 0$, which gives $\lambda = -3 \pm 2i$, as the two characteristic roots. Hence the general solution is

$$y(x) = e^{-3x}(c_1 \cos 2x + c_2 \sin 2x), \qquad ...(11.22)$$

where c_1 and c_2 are arbitrary constants.

Using the initial condition $y(0) = 3$ in (11.22) gives $c_1 = 3$. Also from (11.22),

$$y'(x) = e^{-3x}[(2c_2 - 3c_1) \cos 2x - (2c_1 + 3c_2) \sin 2x].$$

Using $y'(0) = 7$, we obtain $2c_2 - 3c_1 = 7$, which gives $c_2 = 8$, since $c_1 = 3$.

Substituting for c_1 and c_2 in (11.22), the solution of the initial value problem becomes $y = e^{-3x}(3 \cos 2x + 8 \sin 2x)$.

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Example 11.9: Solve the boundary value problem $y'' + w^2y = 0$, $y(0) = 0$ and $y(e) = 0$.

Solution: The differential equation is

$$(D^2 + w^2)y = 0, \; y(0) = 0 \text{ and } y(e) = 0 \qquad ...(11.23)$$

The characteristic equation is $\lambda^2 + w^2 = 0$, which gives, $\lambda = \pm iw$, as two complex conjugate roots. Therefore the general solution is

$$y(x) = c_1 \cos wx + c_2 \sin wx \qquad ...(11.24)$$

Using the boundary conditions $y(0) = 0$, $y(e) = 0$ in Eq. (11.24), we obtain

$$c_1 = 0, \text{ and } c_2 \sin(we) = 0$$

In case $c_2 = 0$, the solution is trivial one, $y(x) = 0$.

For $c_2 \neq 0$, we get $\sin we = 0$, which gives $w = \dfrac{n\pi}{e}$, $n = 0, \pm 1, \pm 2, ...$ Therefore, the solution is

$y_n(x) = B_n\left[\sin\dfrac{n\pi x}{e}\right]$, $n = 0, \pm 1, \pm 2, ...$, where B_i's are arbitrary constants.

Since Eq. (11.23) is homogeneous, using the superposition principle, the general solution is given by

$$y(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{e}\right), \quad ...(11.25)$$

where B_i's are arbitrary constants.

Example 11.10: Solve the differential equation $9y'' - 24y' + 16y = 0$ by factorizing the differential operator and reducing it into first order equation.

Solution: The differential equation is $(9D^2 - 24D + 16)y = 0$

or, $$(3D - 4)(3D - 4)y = 0. \quad ...(11.26)$$

Set $(3D - 4)y = u$, the Eq. (11.26) becomes $(3D - 4)u = 0$, which gives $u = ce^{4x/3}$ as its solution. Therefore,

$$(3D - 4)y = ce^{4x/3}, \text{ or } \left(D - \frac{4}{3}\right)y = \frac{c}{3}e^{4x/3},$$

which is a linear first order equation. The integrating factor is $e^{-4x/3}$, hence the solution is

$$ye^{-4x/3} = \int \frac{c}{3}dx + c_1 = \frac{cx}{3} + c_1, \text{ or } y = (c_1 + c_2x)e^{4x/3}; \quad c_2 = c/3,$$

where c_1, c_2 are two arbitrary constants.

11.4.3 Solution of the Constant Coefficients Homogeneous Linear Equations of Higher Order

The method discussed in case of second order equations can be extended in a natural way to find the solution of the higher order homogeneous linear differential equations with constant coefficients. The characteristic equation for the homogeneous linear differential equation of the nth order

$$(a_0D^n + a_1D^{n-1} + ... + a_{n-1}D + a_n)y = 0 \quad ...(11.27)$$

is written as

$$a_0\lambda^n + a_1\lambda^{n-1} + ... + a_{n-1}\lambda + a_n = 0. \quad ...(11.28)$$

This is a polynomial of degree n and hence has n roots, say $\lambda_1, \lambda_2, ..., \lambda_n$. All the roots may be real and distinct, all or some of the roots may be equal, all or some of the roots may be in complex conjugate pairs. We discuss the following cases.

Distinct real roots. *If all the n roots $\lambda_1, \lambda_2, ..., \lambda_n$ of Eq. (11.28) are real and distinct, then the n solutions $y_1(x) = e^{\lambda_1 x}$, $y_2(x) = e^{\lambda_2 x}$, ..., $y_n(x) = e^{\lambda_n x}$ form a set of n linearly independent solutions of the differential equation (11.27), (The Wronskian of these being non-zero, for $\lambda_1 \neq \lambda_2 \neq ... \neq \lambda_n$), and hence the general solution of Eq. (11.28) is*

$$y(x) = c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x} + ... + c_ne^{\lambda_n x},$$

where $c_1, c_2, ..., c_n$ are n arbitrary constants.

Multiple real roots. If a real double root occurs, say $\lambda_1 = \lambda_2 = \lambda$, then corresponding to this root, as discussed in case of homogeneous equation of second order with constant coefficient, we take $y_1 = e^{\lambda x}$ and $y_2 = xe^{\lambda x}$ as two linearly independent solutions.

If a triple root occurs, say $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, then corresponding to this root, three linearly independent solutions are $y_1 = e^{\lambda x}$, $y_2 = xe^{\lambda x}$, and $y_3 = x^2 e^{\lambda x}$.

In general, if λ is a root of multiplicity k, then the corresponding k linearly independent solutions are $y_1 = e^{\lambda x}$, $y_2 = xe^{\lambda x}$, ..., $y_k = x^{k-1} e^{\lambda x}$

The linear independece of $y_1, y_2, ..., y_k$ can be verified by finding the Wronskian and proving it to be non-zero over any open interval.

Complex roots. The coefficients in the characteristic equation (11.28) being real, the complex roots will occur only in conjugate pairs, say $p \pm iq$. The corresponding linearly independent solutions are given by $e^{px} \cos qx$ and $e^{px} \sin qx$.

In case the characteristic equation has k simple conjugate pairs of complex roots given by $q_r \pm ip_r$; $r = 1, 2, ..., k$, then the corresponding 2k linearly independent solutions are $e^{p_r x} \cos q_r x$, $e^{p_r x} \sin q_r x$, $r = 1, 2, ..., k$.

In case of multiple complex roots, say $p \pm iq$ is a pair of complex conjugate roots of multiplicity 2, then the corresponding four linearly independent solutions are: $e^{px} \cos qx$, $e^{px} \sin qx$, $xe^{px} \cos qx$ and $xe^{px} \sin qx$.

This can be extended in case roots of multiplicity of higher order occur.

Example 11.11: Find the general solution of the differential equation $y''' - 8y' + 8y = 0$.

Solution: The differential equation is $(D^3 - 8D + 8)y = 0$. The characteristic equation is

$$\lambda^3 - 8\lambda + 8 = 0 \qquad ...(11.29)$$

By inspection one root of Eq. (11.29) is $\lambda = 2$. The equation can be expressed as $(\lambda - 2)(\lambda^2 + 2\lambda - 4) = 0$. The quadratic equation $\lambda^2 + 2\lambda - 4 = 0$ has the roots $\lambda = -1 \pm \sqrt{5}$.

Hence the three roots of the characteristic Eq. (11.29) are: 2, $-1 + \sqrt{5}$ and $-1 - \sqrt{5}$. Since, the roots are distinct and real, therefore, the general solution is

$$y(x) = c_1 e^{2x} + c_2 e^{(-1+\sqrt{5})x} + c_3 e^{(-1-\sqrt{5})x} = c_1 e^{2x} + e^{-x}[c_2 e^{\sqrt{5}x} + c_3 e^{-\sqrt{5}x}],$$

where c_1, c_2 and c_3 are arbitrary constants.

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Example 11.12: Find the general solution of the equation $y'''' + ky = 0$ arising in studying the deflected shape $y(x)$ of a beam on an elastic foundation, where $k > 0$ is a constant. ← V. Imp

De Moivre's Theorem

Solution: The differential equation is $(D^4 + k)y = 0$.

The corresponding characteristic equation is $\lambda^4 + k = 0$. Its roots are given by

$$\lambda = (-k)^{1/4} = k^{1/4}\left[\cos\frac{(2m+1)\pi}{4} + i \sin\frac{(2m+1)\pi}{4}\right], m = 0, 1, 2, 3.$$

Simplifying, the roots are $\lambda = \frac{k^{1/4}}{\sqrt{2}}(1 \pm i)$, $\frac{k^{1/4}}{\sqrt{2}}(-1 \pm i)$, two pairs of simple complex conjugates. Hence the general solution is

De'moivre theorem

$(\cos\theta + i\sin\theta)^n$

$\left(\cos\frac{2m\pi+\theta}{n}\right) + i\sin\left(\frac{2m\pi+\theta}{n}\right)$

$$y(x) = e^{\frac{k^{1/4}}{\sqrt{2}}x}\left(c_1 \cos \frac{k^{1/4}}{\sqrt{2}}x + c_2 \sin \frac{k^{1/4}}{\sqrt{2}}x\right) + e^{-\frac{k^{1/4}}{\sqrt{2}}x}\left(c_3 \cos \frac{k^{1/4}}{\sqrt{2}}x + c_4 \sin \frac{k^{1/4}}{\sqrt{2}}x\right),$$

where c_1, c_2, c_3, c_4 are arbitrary constants.

Example 11.13: Find the general solution of a homogeneous equation with the characteristic equation $\lambda^3(\lambda + 4)^2 (\lambda^2 + 2\lambda + 5)^2 = 0$.

Solution: In the characteristic equation $\lambda^3(\lambda + 4)^2 (\lambda^2 + 2\lambda + 5)^2 = 0$, the root $\lambda = 0$ occurs with multiplicity three, the root $\lambda = -4$ occurs with multiplicity two, and the pairs of complex conjugate roots $\lambda = -1 \pm 2i$, occur with multiplicity two.

Hence, the general solution of the differential equation with the given characteristic equation is

$$y(x) = c_1 + c_2x + c_3x^2 + (c_4 + c_5x)e^{-4x} + e^{-x}[(c_6 + c_7x) \cos 2x + (c_8 + c_9x) \sin 2x],$$

where c_i's, $i = 1, 2, \ldots, 9$ are arbitrary constants.

Example 11.14: Find the non-trivial solution of the boundary value problem $y'''' - k^4y = 0$, $y(0) = y''(0) = y(l) = y''(l) = 0$, where $k > 0$ is a constant.

Solution: The differential equation is $(D^4 - k^4)y = 0$. The corresponding characteristic equation is $\lambda^4 - k^4 = 0$. Its roots are: $\lambda = \pm k, \pm ik$, and hence the general solution is

$$y(x) = c_1'e^{kx} + c_2'e^{-kx} + c_3 \cos kx + c_4 \sin kx.$$

Since $e^{kx} = \cosh kx + \sinh kx$, and $e^{-kx} = \cosh kx - \sinh kx$, the general solution can be expressed as

$$y(x) = c_1 \cosh kx + c_2 \sinh kx + c_3 \cos kx + c_4 \sin kx, \qquad \text{...(11.30)}$$

where c_i's are constants.

Using the initial condition $y(0) = 0$ in (11.30), we get $c_1 + c_3 = 0$.

Also from (11.30), $y''(x) = k^2(c_1 \cosh kx + c_2 \sinh kx - c_3 \cos kx - c_4 \sin kx)$.

Using the initial condition $y''(0) = 0$, we get $k^2(c_1 - c_3) = 0$, which gives $c_1 - c_3 = 0$, since $k \neq 0$. Solving $c_1 + c_3 = 0$, $c_1 - c_3 = 0$, we obtain, $c_1 = c_3 = 0$, and hence (11.30) becomes

$$y(x) = c_2 \sinh kx + c_4 \sin kx \qquad \text{...(11.31)}$$

Next, from (11.31), we have

$$y''(x) = k^2(c_2 \sinh kx - c_4 \sin kx) \qquad \text{...(11.32)}$$

Using $y(l) = 0$ in (11.31) and $y''(l) = 0$ in (11.32) we obtain respectively

$$c_2 \sinh kl + c_4 \sin kl = 0, \qquad \text{...(11.33)}$$

$$\text{and, } c_2 \sinh kl - c_4 \sin kl = 0. \qquad \text{...(11.34)}$$

Adding these two, we get, $c_2 \sinh kl = 0$, which gives $c_2 = 0$, since $\sinh kl \neq 0$.

Using $c_2 = 0$ in (11.33), we obtain $c_4 \sin kl = 0$. Since we are interested in non-trivial solution, taking $c_4 \neq 0$. Hence $\sin kl = 0$, which gives $k = \frac{n\pi}{l}$, $n = 1, 2, \ldots$ Thus the solutions are

$$y_n(x) = b_n \sin \frac{n\pi x}{l}, n = 1, 2, \ldots$$

Since the given equation is homogeneous, using the superposition principle, the solution is given by $y(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$ where b_i's are arbitrary constants.

EXERCISE 11.2

Find the general solution of

1. $y'' + y' - 2y = 0$
2. $y'' + 2y' + 4y = 0$
3. $4y'' + 4y' + y = 0$
4. $y''' + y'' + 4y' + 4y = 0$
5. $y'''' + 4y = 0$
6. $y'''' + 8y'' + 16y = 0$

Solve the following initial value problems

7. $2y'' + 5y' - 3y = 0$; $y(0) = 4$, $y'(0) = 9$
8. $4y'' + 20y' + 125y = 0$; $y(0) = 3$, $y'(0) = 2.5$
9. $y'' - 6y' + 9y = 0$; $y(0) = 2$, $y'(0) = 0$
10. $y''' + 2y'' + 4y' = 0$; $y(0) = 0$; $y'(0) = 1$, $y''(0) = 0$
11. $y''' + y'' - 4y = 0$; $y(0) = 1$; $y'(0) = 1$, $y''(0) = 0$
12. $y^{iv} - y'' - 2y = 0$; $y(0) = 1$; $y'(0) = 0$, $y''(0) = 0$, $y'''(0) = 0$

Solve the following boundary value problems

13. $y'' + 2y' + 2y = 0$; $y(0) = 1$, $y(\pi/2) = 0$
14. $y'' + 36y = 0$; $y(0) = 2$, $y(1/6) = 1/e$
15. $y'' + 2y' + 2y = 0$; $y(0) = 1$, $y(\pi/2) = e^{-\pi/2}$
16. $y''' + \pi^2 y' = 0$; $y(0) = 0$, $y(1) = 0$, $y'(0) + y'(1) = 0$
17. $y^{iv} + 13y'' + 36y = 0$; $y(0) = 0$, $y''(0) = 0$, $y(\pi/2) = -1$, $y'(\pi/2) = -4$
18. $y^{iv} + 4y''' + 8y'' + 8y' + 4y = 0$; $y(0) = 1$, $y'(0) = 0$, $y''(0) = -2$, $y'''(0) = 2$
19. If the roots of a characteristic equation are 4, 4, 4, i and $-i$, then find the original differential equation and also its general solution.

Solve the following differential equations by factorizing the differential operator and reducing it into first order equations

20. $(D^2 + 4D + 4)y = 0$
21. $(4D^2 + 8D + 3)y = 0$
22. $(D^3 + 3D^2 - 4)y = 0$
23. Find non-trivial solution of the boundary value problem $y'' + k^2 y = 0$; $y(0) = y(\pi) = 0$.
24. If $k > 0$, then show that the general solution of $y^{(iv)} - k^4 y = 0$ can be expressed as $y = c_1 \cos kx + c_2 \sin kx + c_3 \cosh kx + c_4 \sinh kx$.

De moivre theorem $\left[\cos\left(\frac{2m\pi+\theta}{m}\right)+i\sin\left(\frac{2m\pi+\theta}{m}\right)\right]$

$r^n(\cos\theta + i\sin m\theta)^m$

11.5 SOLUTION OF CONSTANT COEFFICIENTS NON-HOMOGENEOUS LINEAR EQUATIONS.

In this section we discuss the method to find the solution of non-homogeneous linear differential equations of the form

$$a_0y^{(n)} + a_1y^{(n-1)} + \ldots + a_{n-1}y' + a_ny = f(x),\ a_0 \neq 0 \qquad \ldots(11.35)$$

when the general solution of the corresponding homogeneous equation

$$a_0y^{(n)} + a_1y^{(n-1)} + \ldots + a_{n-1}y' + a_ny = 0,\ a_0 \neq 0 \qquad \ldots(11.36)$$

is known.

11.5.1 The General Solution of the Non-homogeneous Equation

We have the following theorem:

Theorem 11.4: *If $\{y_1(x), y_2(x), \ldots, y_n(x)\}$ is a fundamental set of solution and $y_c(x) = c_1y_1(x) + c_2y_2(x) + \ldots + c_ny_n(x)$ is the general solution of the homogeneous linear Eq. (11.36), and further, if $y_p(x)$ is any particular solution of the non-homogeneous Eq. (11.35), then the general solution of the non-homogeneous Eq. (11.35) is given by*

$$y(x) = c_1y_1(x) + c_2y_2(x) + \ldots + c_ny_n(x) + y_p(x) \qquad \ldots(11.37)$$

Proof. Since $y(x)$ and $y_p(x)$ both are solutions of the non-homogeneous linear equation (11.35), thus

$$a_0[y^{(n)} - y_p^{(n)}] + a_1[y^{(n-1)} - y_p^{(n-1)}] + \ldots + a_{n-1}[y' - y_p'] + a_n[y - y_p]$$
$$= (a_0y^{(n)} + a_1y^{(n-1)} + \ldots + a_{n-1}y' + a_ny) - (a_0y_p^{(n)} + a_1y_p^{(n-1)} + \ldots + a_{n-1}y'_p + a_ny_p)$$
$$= f(x) - f(x) = 0$$

Therefore $y - y_p$ is a solution of the homogeneous Eq.(11.36). Further $\{y_1, y_2, \ldots, y_n\}$ forms a fundamental set of solutions for this homogeneous equation, and thus there are constants $c_1, c_2, \ldots, c_n$ such that

$$y(x) - y_p(x) = c_1y_1(x) + c_2y_2(x) + \ldots + c_ny_n(x)$$

that is, $$y(x) = c_1y_1(x) + c_2y_2(x) + \ldots + c_ny_n(x) + y_p(x).$$

The above result suggests the following strategy to find the general solution of the non-homogeneous linear differential equation with constant coefficients.

1. *Find the general solution of the corresponding homogeneous equation. This solution is called the complementary function and is denoted by $y_c(x)$.*
2. *Find a particular solution, a solution not containing any arbitrary constant, of the non-homogeneous equation; this solution is called the particular integral and is denoted by $y_p(x)$.*

The general solution of the non-homogeneous equation is then given as $y(x) = y_c(x) + y_p(x)$, and it contains all possible solutions on the interval.

The methods for finding $y_c(x)$ have already been discussed in the preceding section. Here, we discuss methods for finding the particular integral $y_p(x)$ of the non-homogeneous equation.

11.5.2 The Operator Method for Finding Particular Integral

The D-operator method is a concise method for finding a particular integral of a linear homogeneous equation with constant coefficients.

The general linear homogeneous equation with constant coefficients can be expressed as

$F(D)y = f(x)$, where $F(D) = a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \dots + a_{n-1}D + a_n$,

with a_i's, $i = 0, 1, 2, \dots, n$, $a_0 \neq 0$, all being constant and $f(x)$ a function of x only.

We define $\frac{1}{F(D)} f(x)$, *or* $[F(D)]^{-1}(x)$ *as a that function of x, without any arbitrary constant, which when operated by the polynomial operator F(D) gives f(x), that is,* $F(D)\,[F(D)]^{-1}f(x) = f(x)$.

Thus, $[F(D)]^{-1}f(x)$ *satisfies the linear differential equation* $F(D)y = f(x)$ *and is, therefore, its particular integral* $y_p(x)$. *Hence,* $y_p(x) = [F(D)]^{-1}f(x)$.

Obviously $F(D)$ and $[F(D)]^{-1}$ respresent inverse operators.

In particular, if $F(D) = D$, then $[F(D)]^{-1}f(x) = \frac{1}{D}f(x) = \int f(x)dx$, for

$$y = \frac{1}{D}f(x) \Rightarrow Dy = D\left(\frac{1}{D}f(x)\right) = f(x), \text{ or } \frac{dy}{dx} = f(x) \Rightarrow y = \int f(x)dx;$$

no constant is being added since we are looking for a particular integral.

Similarly, $\frac{1}{D-a}f(x) = e^{ax}\int f(x)e^{-ax}dx$, for

$$y = \frac{1}{D-a}f(x) \Rightarrow (D-a)y = (D-a)\left(\frac{1}{D-a}f(x)\right) = f(x) \Rightarrow \frac{dy}{dx} - ay = f(x),$$

which is a linear differential equation of order one and degree one, with solution as $y = e^{ax}\int f(x)e^{-ax}dx$, and hence,

$$\frac{1}{D-a}f(x) = e^{ax}\int f(x)e^{-ax}dx. \qquad \text{...(11.38)}$$

The procedure for finding the particular integral by this method depends upon the form of $f(x)$ and is described below for some specific cases.

Case I: $f(x) = e^{ax}$, *where a is a constant*. When $f(x) = e^{ax}$, we have,

$De^{ax} = ae^{ax}$, $D^2e^{ax} = a^2e^{ax}$, ..., $D^ne^{ax} = a^ne^{ax}$, and thus

$$\begin{aligned} F(D)e^{ax} &= (a_0D^n + a_1D^{n-1} + \dots + a_{n-1}D + a_0)e^{ax} \\ &= a_0D^ne^{ax} + a_1D^{n-1}e^{ax} + \dots + a_{n-1}De^{ax} + a_0e^{ax} \\ &= a_0a^ne^{ax} + a_1a^{n-1}e^{ax} + \dots + a_{n-1}ae^{ax} + a_0e^{ax} \end{aligned}$$

$= (a_0a^n + a_1a^{n-1} + \dots + a_{n-1}a + a_0)e^{ax}$, which gives

$$F(D)e^{ax} = F(a)e^{ax} \qquad \text{...(11.39)}$$

Operating on both sides of (11.39) by $[F(D)]^{-1}$, we have

$$[F(D)]^{-1}F(D)e^{ax} = [F(D)]^{-1}F(a)e^{ax}, \text{ or } e^{ax} = F(a)[F(D)]^{-1}e^{ax}.$$

Dividing by $F(a)$, we get

$$y_p(x) = F(D)^{-1}e^{ax} = \frac{1}{F(a)}e^{ax}, \text{ provided } F(a) \neq 0 \qquad \text{...(11.40)}$$

In case $F(a) = 0$, then $(D - a)$ is a factor of $F(D)$, and say $F(D) = (D - a)\,G(D)$, where $G(a) \neq 0$. Then

$$[F(D)]^{-1} e^{ax} = [(D-a)\,G(D)]^{-1}e^{ax} = \frac{1}{D-a}[G(D)]^{-1}e^{ax} = \frac{1}{D-a}\,\frac{1}{G(a)}e^{ax} = \frac{1}{G(a)}\,\frac{1}{(D-a)}e^{ax}$$

$$= \frac{1}{G(a)}\,e^{ax}\int e^{ax}\,e^{-ax}dx, \text{ using (11.38)}$$

$$= x\,\frac{1}{G(a)}e^{ax} = x\,\frac{1}{F'(a)}e^{ax}, \text{ provided } F'(a) \neq 0, \qquad \text{...(11.41)}$$

for $F(D) = (D - a)G(D)$ implies $F'(D) = G(D) + (D - a)G'(D)$ which in turn gives $F'(a) = G(a)$.

In case $F'(a) = 0$, then re-applying the result, to get

$$\boxed{[F(D)]^{-1}e^{ax} = x^2\frac{1}{F''(a)}e^{ax},} \qquad \text{...(11.42)}$$

provided, $F''(a) \neq 0$, and so on.

Example 11.15: Find the general solution of the differential equation

$$(D^2 - 13D + 12)y = 3e^{-2x}.$$

Solution: The corresponding homogeneous equation is

$$(D^2 - 13D + 12)y = 0. \qquad \text{...(11.43)}$$

To find the complementary function, consider the characteristic equation corresponding to (11.43), which is $\lambda^2 - 13\lambda + 12 = 0$. It has roots $\lambda = 1, 12$.

Thus the complementary function is $y_c(x) = c_1e^x + c_2e^{12x}$, where c_1, c_2 are arbitrary constants.

The particular integral is

$$y_p(x) = (D^2 - 13D + 12)^{-1}(3e^{-2x}) = 3\frac{1}{(-2)^2 - 13(-2) + 12}e^{-2x} = \frac{1}{14}e^{-2x}.$$

Hence the general solution of the given differential equation is $y = c_1e^x + c_2e^{12x} + \frac{1}{14}e^{-2x}$, where c_1 and c_2 are two arbitrary constants.

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Example 11.16: Find the general solution of the differential equation

$$(D^4 + 5D^3 + 6D^2 - 4D - 8)y = e^{-2x} + 2e^{-x} + 3e^x - 3.$$

Solution: The corresponding homogeneous equation is

$$(D^4 + 5D^3 + 6D^2 - 4D - 8)y = 0 \qquad \text{...(11.44)}$$

To find the complementary function, consider the characteristic equation corresponding to (11.44), which is $\lambda^4 + 5\lambda^3 + 6\lambda^2 - 4\lambda - 8 = 0$. It has roots $\lambda = -2, -2, -2, 1$. Thus the complementary function $y_c(x)$ is $y_c(x) = (c_1 + c_2x + c_3x^2)e^{-2x} + c_4e^x$, where c_1, c_2, c_3, c_4 are arbitrary constants.
The particular integral is

$$\begin{aligned}
y_p(x) &= [(D+2)^3 (D-1)]^{-1}(e^{-2x} + 2e^{-x} + 3e^x - 3) \\
&= [(D+2)^{-3}(D-1)^{-1}]e^{-2x} + 2[(D+2)^{-3}(D-1)^{-1}]e^{-x} \\
&\qquad + 3[(D+2)^{-3}(D-1)^{-1}]e^x - 3[(D+2)^{-3}(D-1)^{-1}]e^0 \\
&= -\frac{1}{3}(D+2)^{-3}e^{-2x} + 2\left(-\frac{1}{2}\right)e^{-x} + 3\,\frac{1}{27}(D-1)^{-1}e^x - 3\left(-\frac{1}{8}\right) \\
&= -\frac{1}{3}\,\frac{x^3}{3!}e^{-2x} - e^{-x} + \frac{1}{9}xe^x + \frac{3}{8} = -\frac{x^3e^{-2x}}{18} - e^{-x} + \frac{1}{9}xe^x + \frac{3}{8}.
\end{aligned}$$

Hence the general solution is

$$y(x) = (c_1 + c_2x + c_3x^2)e^{-2x} + c_4e^x - \frac{x^3}{18}e^{-2x} - e^{-x} + \frac{1}{9}xe^x + \frac{3}{8}$$

or,
$$y(x) = \left(c_1 + c_2x + c_3x^2 - \frac{x^3}{18}\right)e^{-2x} - e^{-x} + \left(c_4 + \frac{1}{9}x\right)e^x + \frac{3}{8}.$$

Case II: *$f(x) = \sin(ax + b)$, or $\cos(ax + b)$ where a and b are constants.*

When $f(x) = \sin(ax + b)$, we have

$D^2 \Rightarrow -a^2$

$D\sin(ax + b) = a\cos(ax + b)$, $D^2\sin(ax + b) = (-a^2)\sin(ax + b)$

$D^3\sin(ax + b) = -a^3\cos(ax + b)$, $D^4\sin(ax + b) = (-a^2)^2\sin(ax + b)$

Thus, for even n, say $n = 2m$, $D^n\sin(ax + b) = (-a^2)^m\sin(ax + b)$.

Hence, in case $F(D)$ contains only even powers of D,

$$F(D^2)\sin(ax + b) = F(-a^2)\sin(ax + b).$$

Operating both sides by $[F(D^2)]^{-1}$ and dividing by $F(-a^2)$, the particular integral is

$$y_p(x) = [F(D^2)]^{-1}\sin(ax + b) = \frac{1}{F(-a^2)}\sin(ax + b), \qquad \text{...(11.45)}$$

provided $F(-a^2) \neq 0$.

Similarly for the case $f(x) = \cos(ax + b)$, the particular integral is

$$[F(D^2)]^{-1}\cos(ax + b) = \frac{1}{F(-a^2)}\cos(ax + b), \qquad \text{...(11.46)}$$

provided $F(-a^2) \neq 0$.

When $F(D)$ contains odd powers of D also, to find the particular integral, we proceed tentatively on similar lines as described above, however, the exact procedure will be illustrated in the examples to follow.

In case $F(-a^2) = 0$, we write $\cos(ax + b) = Re(e^{i(ax+b)})$ and $\sin(ax + b) = Im.(e^{i(ax+b)})$ and apply the formulae for the *Case I, when* $f(x) = e^{ax}$, as follows,

$$y_p(x) = \frac{1}{F(D)} \cos(ax + b) = Re.\frac{1}{F(D)} e^{i(ax+b)} = Re.\frac{1}{F(ia)} e^{i(ax+b)} \qquad \text{...(11.47)}$$

provided $F(ia) \neq 0$, and so on. Similarly,

$$\frac{1}{F(D)} \sin(ax + b) = Im.\frac{1}{F(D)} e^{i(ax+b)} = Im.\frac{1}{F(ia)} e^{i(ax+b)}, \qquad \text{...(11.48)}$$

provided $F(ia) \neq 0$, and so on.

Example 11.17: Find the particular integral of the equation $(D^2 + 1)y = \cos(2x - 1)$.

Solution: The particular integral is

$$y_p(x) = (D^2 + 1)^{-1} \cos(2x - 1) = \frac{1}{-4+1} \cos(2x - 1) = -\frac{1}{3} \cos(2x - 1).$$

Example 11.18: Find the general solution of the differential equation $(D^3 + D^2 - D - 1)y = \sin(2x - 3)$.

Solution: Characteristic equation of the corresponding homogeneous equation is $\lambda^3 + \lambda^2 - \lambda - 1 = 0$. Its roots are $\lambda = -1, -1, 1$. Thus the complementary function is $y_c(x) = (c_1 + c_2x)e^{-x} + c_3e^x$.

The particular integral is

$$y_p(x) = \frac{1}{D^3 + D^2 - D - 1} \sin(2x - 3)$$

$$= \frac{1}{-2^2D - 2^2 - D - 1} \sin(2x - 3), \text{ (replacing } D^2 = -2^2)$$

$$= -\frac{1}{5(D+1)} \sin(2x - 3) = \frac{-(D-1)}{5(D^2 - 1)} \sin(2x - 3)$$

$$= \frac{1}{25}(D - 1) \sin(2x - 3), \text{ (replacing } D^2 = -2^2)$$

$$= \frac{1}{25}[2\cos(2x - 3) - \sin(2x - 3)].$$

Therefore the general solution is

$$y(x) = (c_1 + c_2 x)e^{-x} + c_3 e^x + \frac{2}{25}\cos(2x-3) - \frac{1}{25}\sin(2x-3)$$

where c_1, c_2, c_3 are arbitrary constants.

Example 11.19: Find the particular integral of the equation $(D^2 + 4)y = \sin 2x$.

Solution: The particular integral is, $y_p(x) = \dfrac{1}{D^2+4}\sin 2x$.

Here, $F(-a)^2 = -a^2 + 4 = 0$ at $a = 2$, therefore, writing $y_p(x)$ as

$$y_p(x) = Im.\frac{1}{D^2+4}e^{i2x} = Im.\frac{x}{2D}e^{2ix} = Im.\frac{x}{4i}e^{2ix}$$

$$= Im.\frac{x}{4i}(\cos 2x + i\sin 2x) = -Im.\frac{x}{4}(i\cos 2x - \sin 2x) = -\frac{x}{4}\cos 2x.$$

Alternatively,
$$y_p(x) = \frac{1}{D^2+4}\sin 2x,$$

Here, $F(-a^2) = 0$ proceeding on similar lines as in Case I when $f(x) = e^{ax}$ and $F(a) = 0$, we have

$$y_p(x) = \frac{x}{2D}\sin 2x = \frac{x}{2}\int \sin 2x\,dx = -\frac{x}{4}\cos 2x.$$

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Example 11.20: Solve the differential equation $(D^2 - 4D + 3)y = \sin 3x\cos 2x$.

Solution: Characteristic equation of the corresponding homogeneous equation is $\lambda^2 - 4\lambda + 3 = 0$. Its roots are $\lambda = 1, 3$. The complementary function is thus $y_c(x) = c_1e^x + c_2e^{3x}$, where c_1, c_2 are arbitrary constants.

The particular integral is

$$y_p(x) = \frac{1}{D^2-4D+3}(\sin 3x\cos 2x) = \frac{1}{2}\frac{1}{D^2-4D+3}\sin 5x + \frac{1}{2}\frac{1}{D^2-4D+3}\sin x$$

$$= \frac{1}{2}\frac{1}{-25-4D+3}\sin 5x + \frac{1}{2}\frac{1}{-1-4D+3}\sin x$$

$$= -\frac{1}{4}\frac{11-2D}{121-4D^2}\sin 5x + \frac{1}{4}\frac{1+2D}{1-4D^2}\sin x = -\frac{1}{884}(11-2D)\sin 5x + \frac{1}{20}(1+2D)\sin x$$

$$= \frac{1}{884}(10\cos 5x - 11\sin x) + \frac{1}{20}(\sin x + 2\cos x).$$

Hence the general solution is

$$y_c(x) = c_1e^x + c_2e^{3x} + \frac{1}{884}(10\cos 5x - 11\sin 5x) + \frac{1}{20}(\sin x + 2\cos x).$$

Example 11.21: Solve the differential equation $(D^2 + 2D + 1)y = \cosh x - \cos^2 x$.

Solution: The complementary function is $y_c(x) = (c_1 + c_2 x)e^{-x}$. The particular integral is

$$y_p(x) = \frac{1}{(D^2+2D+1)}[\cosh x - \cos^2 x]$$

$$= \frac{1}{2}\frac{1}{(D^2+2D+1)}(e^x + e^{-x}) - \frac{1}{2}\frac{1}{(D^2+2D+1)}(1 + \cos 2x)$$

$$= \frac{1}{2(D^2+2D+1)}e^x + \frac{1}{2(D^2+2D+1)}e^{-x} - \frac{1}{2(D^2+2D+1)}e^0 - \frac{1}{2(D^2+2D+1)}\cos 2x$$

$$= \frac{e^x}{8} + \frac{x^2e^{-x}}{4} - \frac{1}{2} - \frac{1}{2(2D-3)}\cos 2x = \frac{e^x}{8} + \frac{x^2e^{-x}}{4} - \frac{1}{2} - \frac{(2D+3)}{2(4D^2-9)}\cos 2x$$

$$= \frac{e^x}{8} + \frac{x^2e^{-x}}{4} - \frac{1}{2} + \frac{1}{50}(2D+3)\cos 2x = \frac{e^x}{8} + \frac{x^2e^{-x}}{4} - \frac{4}{50}\sin 2x + \frac{3}{50}\cos 2x - \frac{1}{2}.$$

The general solution is

$$y(x) = (c_1 + c_2x)\,e^{-x} + \frac{e^x}{8} + \frac{x^2e^{-x}}{4} - \frac{4}{50}\sin 2x + \frac{3}{50}\cos 2x - \frac{1}{2}.$$

Case III: *$f(x) = x^m$, $m > 0$ is an integer.* When $f(x) = x^m$, the particular integral is

$$y_p(x) = [F(D)]^{-1}x^m. \qquad ...(11.49)$$

Symbolically, we expand the operator $[F(D)]^{-1}$ as an infinite series in ascending powers of D and operate on x^m. We need not to consider terms with power $m + 1$ and higher, since $(m + 1)$th and higher order derivatives of x^m are zeros.

Example 11.22: Find the particular integral of the differential equation
$(D^2 + 2D + 1)y = 2x + x^3$.

Solution: The particular integral is

$$y_p(x) = \frac{1}{D^2+2D+1}(2x + x^3) = (1 + D)^{-2}(2x + x^3)$$

$$= (1 - 2D + 3D^2 - 4D^3 + ...)(2x + x^3)$$

$$= 2x + x^3 - 2(2 + 3x^3) + 3(6x) - 24 = x^3 - 6x^2 + 20x - 28.$$

Case IV: *$f(x) = e^{ax}g(x)$, $g(x)$ being some function of x.*

In case $f(x) = e^{ax}g(x)$,

$$Df(x) = D[e^{ax}g(x)] = e^{ax}Dg(x) + ae^{ax}g(x) = e^{ax}(D + a)g(x)$$

$$D^2f(x) = D^2[e^{ax}g(x)] = D[e^{ax}Dg(x) + ae^{ax}g(x)]$$

$$= e^{ax}D^2g(x) + 2ae^{ax}Dg(x) + a^2e^{ax}g(x) = e^{ax}(D + a)^2g(x).$$

In general, $D^n(e^{ax}g(x)) = e^{ax}(D + a)^ng(x)$.

Therefore, $F(D)e^{ax}g(x) = e^{ax}F(D + a)g(x)$ and hence the particular integral is

$$y_p(x) = [F(D)]^{-1}e^{ax}g(x) = e^{ax}[F(D + a)]^{-1}g(x) \qquad \text{...(11.50)}$$

and $[F(D + a)]^{-1}g(x)$ can be evaluated for a specific value of $g(x)$.

Example 11.23: Solve the differential equation $(D^2 - 4)y = x \sinh x$.

Solution: The complementary function is $y_c(x) = c_1e^{2x} + c_2e^{-2x}$. The particular integral is

$$y_p(x) = \frac{1}{D^2-4}\, x \sinh x = \frac{1}{2}\frac{1}{D^2-4}\, x(e^x - e^{-x})$$

$$= \frac{1}{2}\left[\frac{1}{D^2-4}xe^x - \frac{1}{D^2-4}xe^{-x}\right] = \frac{1}{2}\left[e^x\frac{1}{(D+1)^2-4}x - e^{-x}\frac{1}{(D-1)^2-4}\right]$$

$$= \frac{1}{2}\left[e^x\frac{1}{D^2+2D-3}x - e^{-x}\frac{1}{D^2-2D-3}x\right]$$

$$= \frac{1}{2}\left[\frac{e^x}{-3}\left\{1-\left(\frac{2D}{3}+\frac{D^2}{3}\right)\right\}^{-1}x - \frac{e^{-x}}{-3}\left\{1+\left(\frac{2D}{3}-\frac{D^2}{3}\right)\right\}^{-1}x\right]$$

$$= -\frac{1}{6}\left[e^x\left\{1+\left(\frac{2D}{3}+\frac{D^2}{3}\right)+\ldots\right\}x - e^{-x}\left\{1-\left(\frac{2D}{3}-\frac{D^2}{3}\right)+\ldots\right\}x\right]$$

$$= -\frac{1}{6}\left[e^x\left(x+\frac{2}{3}\right) - e^{-x}\left(x+\frac{2}{3}\right)\right] = -\frac{x}{3}\left(\frac{e^x - e^{-x}}{2}\right) - \frac{2}{9}\left(\frac{e^x+e^{-x}}{2}\right) = -\frac{x}{3}\sinh x - \frac{2}{9}\cosh x.$$

Hence the general solution is $y(x) = c_1e^{2x} + c_2e^{-2x} - \frac{x}{3}\sinh x - \frac{2}{9}\cosh x$.

Example 11.24: Solve the differential equation $(D^2 + 4)y = x \sin x$.

Solution: The complementary function is $y_c(x) = c_1 \cos 2x + c_2 \sin 2x$. Particular integral is

$$y_p(x) = \frac{1}{D^2+4}\, x \sin x = Im\,\frac{1}{D^2+4}\, xe^{ix} = Im\, e^{ix}\frac{1}{(D+i)^2+4}\, x = Im\, e^{ix}\frac{1}{D^2+2iD+3}\, x$$

$$= Im\,\frac{e^{ix}}{3}\left[1+\frac{2iD+D^2}{3}\right]^{-1}x = Im\,\frac{e^{ix}}{3}\left[1-\frac{2iD+D^2}{3}+\ldots\right]x = Im\,\frac{e^{ix}}{3}\left[x-\frac{2i}{3}\right]$$

$$= \frac{1}{3}Im\,(\cos x + i \sin x)\left(x-\frac{2i}{3}\right) = \frac{x}{3}\sin x - \frac{2}{9}\cos x.$$

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Hence, the general solution is $y(x) = c_1 \cos 2x + c_2 \sin 2x + \frac{x}{3}\sin x - \frac{2}{9}\cos x$.

So far have discussed the procedure to find particular integral using operator method for some specific forms of $f(x)$. In general, if $f(x)$ is any function of x, then we can proceed to find the particular integral as follow. By definition $y_p(x) = \frac{1}{F(D)}f(x)$.

If $F(D) = a_0(D - m_1)(D - m_2) \dots (D - m_n)$, resolving $\frac{1}{F(D)}$ into partial fractions, we have

$$\frac{1}{F(D)} = \frac{c_1}{D-m_1} + \frac{c_2}{D-m_2} + \dots \frac{c_n}{D-m_n} = \sum_{i=1}^{n}\frac{c_i}{D-m_i}.$$

Thus,
$$y_p(x) = \frac{1}{F(D)}f(x) = \sum_{i=1}^{n}\frac{c_i}{D-m_i}f(x)$$

$$= \sum_{i=1}^{n}\left(c_i e^{m_i x}\int f(x)e^{-m_i x}dx\right), \qquad \dots(11.51)$$

using (11.38). In case of repeated roots the result can be modified accordingly.

Example 11.25: Solve the differential equation $(D^2 + 3D + 2)y = e^{e^x}$.

Solution: The compliementary function is $y_c(x) = c_1e^{-x} + c_2e^{-2x}$, where c_1, c_2 are arbitrary constants.

The particular integral is

$$y_p(x) = \frac{1}{(D+1)(D+2)}e^{e^x} = \left(\frac{1}{D+1} - \frac{1}{D+2}\right)e^{e^x}$$

$$= \frac{1}{D+1}e^{e^x} - \frac{1}{D+2}e^{e^x} = e^{-x}\int e^{e^x}e^x dx - e^{-2x}\int e^{e^x}e^{2x}dx. \qquad \dots(11.52)$$

Substitute $e^x = t$, (11.52) gives

$$y_p(x) = \frac{1}{t}\int e^t dt - \frac{1}{t^2}\int te^t dt = \frac{e^t}{t} - \frac{1}{t^2}(te^t - e^t) = \frac{e^t}{t} - \frac{e^t}{t} - \frac{e^t}{t^2} = \frac{e^t}{t^2} = \frac{e^{e^x}}{e^{2x}} = e^{(e^x - 2x)}.$$

Hence, the general solution is $y(x) = c_1e^{-x} + c_2e^{-2x} + e^{(e^x - 2x)}$.

Example 11.26: Solve the differential equation $(D^2 + a^2)\,y = \tan ax$.

Solution: The complementary function is $y_c(x) = c_1 \cos ax + c_2 \sin ax$. The particular integral is

$$y_p(x) = \frac{1}{D^2+a^2}\tan ax = \frac{1}{(D+ai)(D-ai)}\tan ax$$

$$= \frac{1}{2ai}\left[\frac{1}{D-ai} - \frac{1}{D+ai}\right]\tan ax$$

$$= \frac{1}{2ai}\frac{1}{D-ai}\tan ax - \frac{1}{2ai}\frac{1}{D+ai}\tan ax. \quad ...(11.53)$$

We have, $\frac{1}{D-ai}\tan ax = e^{aix}\int e^{-aix}\tan ax\,dx$ using (11.38)

$$= e^{aix}\int [\cos ax - i\sin ax]\tan ax\,dx = e^{aix}\int\left[\sin ax - i\frac{(1-\cos^2 ax)}{\cos ax}\right]dx$$

$$= e^{aix}\int (\sin ax + i\cos ax - i\sec ax)dx$$

$$= e^{aix}\left[-\frac{\cos ax}{a} + \frac{i\sin ax}{a} - \frac{i}{a}\ln|\sec ax + \tan ax|\right] \quad ...(11.54)$$

Replacing i by $-i$ in (11.54), we have

$$\frac{1}{D+ai}\tan ax = e^{-aix}\left[-\frac{1}{a}\cos ax - \frac{i}{a}\sin ax + \frac{i}{a}\ln|\sec ax + \tan ax|\right]. \quad ...(11.55)$$

Using (11.54) and (11.55) in (11.53), we have

$$y_p(x) = \frac{1}{2a^2 i}\left[-(e^{aix} - e^{-aix})\cos ax + i(e^{aix} + e^{-aix})\sin ax - i(e^{aix} + e^{-aix})\ln|\sec ax + \tan ax|\right]$$

$$= \frac{1}{2a^2 i}\left[-2i\cos ax\sin ax + 2i\sin ax\cos ax - 2i\cos ax\ln|\sec ax + \tan ax|\right]$$

$$= -\frac{1}{a^2}\cos ax\ln|\sec ax + \tan ax|.$$

Hence the general solution is $y(x) = c_1\cos ax + c_2\sin ax - \frac{1}{a^2}\cos ax\ln|\sec ax + \tan ax|$.

EXERCISE 11.3

Find the general solution of the following equations by the operator method

1. $(2D^2 + 5D - 3)y = 6$ e^{0x}
2. $(D^2 - D - 6)y = 2e^x$
3. $(2D^2 - D - 3)y = 5e^{3/2x}$
4. $(D^3 + 5D)y = \sinh 2x$

5. $(D^2 - 4)y = x^2e^{3x}$.
6. $(D^2 - 2D + 1)y = xe^x \cos x$
7. $(D^2 + D)y = (1 + e^x)^{-1}$
8. $(D^2 + D)y = x^2 + 2x + 4$
9. $(D^2 + a^2)y = \sec ax$ $\quad \frac{1}{(D+ai)(D-ai)}$
10. $(D^4 - 1)y = \cos x \cosh x$
11. $(D^2 - 4)y = \cosh (2x - 1) + 3^x$
12. $(D^2 + 5D + 6)y = e^{-2x} \sec^2 x\, (1 + 2 \tan x)$

Solve the following equations

13. $(D^2 + n^2)y = k \sin px$; k, n and p are constants, $n^2 \neq p^2$; $y(0) = y'(0) = 0$.
14. $(D^2 - 7D + 10)y = e^{2x} + 20$; $y(0) = 0$, $y'(0) = -1/3$.
15. $(2D^2 - D - 6)\, y = 6e^x \cos x$; $y(0) = -21/29$, $y'(0) = -194/29$
16. $(D^2 + D) = 2 + 2x + x^2$; $y(0) = 8$, $y'(0) = -1$.

11.6 EQUATIONS REDUCIBLE TO LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

In this section we shall study two such forms of linear differential equations with variable coefficients which can be reduced to linear differential equations with constant coefficients by suitable substitutions.

11.6.1 Cauchy's Homogeneous Linear Equation

An equation of the form

$$a_0x^n \frac{d^ny}{dx^n} + a_1x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \dots + a_{n-1}x \frac{dy}{dx} + a_ny = f(x), \qquad \text{...(11.56)}$$

where $a_0, a_1, \dots, a_n$, $a_0 \neq 0$ are constants and $f(x)$ is a function of x only, is called Cauchy's homogeneous linear equation. This can be reduced to linear differential equation with constant coefficients if we substitute $x = e^t$, or $t = \ln x$. Then, if $D \equiv \frac{d}{dt}$, we have

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{1}{x}, \text{ which gives, } x\frac{dy}{dx} = \frac{dy}{dt} = Dy.$$

Next, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2}$, which gives, $x^2\frac{d^2y}{dx^2} = D(D-1)y$.

Similarly, $x^3\frac{d^3y}{dx^2} = D(D-1)(D-2)y$, and so on.

Substituting these in (11.56) and simplifying, we get a linear equation with constant coefficients. Solving the equation obtained, we get the solution in terms of t. Substituting $t = \ln x$, we obtain the solution of the given equation.

11.6.2 Legendre's Homogeneous Linear Equation

An equation of the form

$$a_0(ax+b)^n \frac{d^n y}{dx^n} + a_1(ax+b)^{n-1}\frac{d^n y}{dx^n} + \ldots + a_{n-1}(ax+b)\frac{dy}{dx} + a_n y = f(x), \qquad \ldots(11.57)$$

where $a_0, a_1, \ldots, a_n, a_0 \neq 0$ are constants and $f(x)$ is a function of x only, is called Legendre's homogeneous linear equation. It reduces to linear differential equation with constant coefficient if we substitute $ax + b = e^t$, or $t = \ln(ax + b)$. Proceeding on the similar lines as above, we obtain

$$(ax+b)\frac{dy}{dx} = aDy, \quad (ax+b)^2 \frac{d^2y}{dx^2} = a^2D(D-1)y,$$

$$(ax+b)^3 \frac{d^3y}{dx^3} = a^3D(D-1)(D-2)y, \text{ and so on; here } D \equiv \frac{d}{dt}.$$

Substituting these in Eq. (11.57), the resultant equation is linear with constant coefficients and thus can be solved accordingly.

Example 11.27: Solve the differential equation $x^3\dfrac{d^3y}{dx^3} + 3x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = x\ln x.$

Solution: Substituting $x = e^t$, the equation becomes

$$[D(D-1)(D-2) + 3D(D-1) + D + 1]y = te^t, \text{ where } D \equiv \frac{d}{dt}.$$

Simplifying it gives, $(D^3+1)y = te^t$. The characteristic equation is $\lambda^3 + 1 = 0$, which has the roots $\lambda = -1, \dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}$ and, therefore, the complimentary function is

$$y_c(t) = c_1e^{-t} + e^{t/2}\left(c_2\cos\frac{\sqrt{3}}{2}t + c_3\sin\frac{\sqrt{3}}{2}t\right).$$

The particular integral is

$$y_p(t) = \frac{1}{(D^3+1)}te^t = e^t\frac{1}{(D+1)^3+1}t = e^t\frac{1}{D^3+3D^2+3D+2}t$$

$$= \frac{e^t}{2}\left[1+\frac{3}{2}D+\frac{3}{2}D^2+\frac{1}{2}D^3\right]^{-1}t = \frac{e^t}{2}\left[1-\left(\frac{3}{2}D+\frac{3}{2}D^2+\frac{1}{2}D^3\right)+\ldots\right]t = \frac{e^t}{2}\left(t-\frac{3}{2}\right).$$

Therefore, the complete solution is

$$y(t) = c_1e^{-t} + e^{t/2}\left(c_2\cos\frac{\sqrt{3}}{2}t + c_3\sin\frac{\sqrt{3}}{2}t\right) + \frac{1}{2}e^t\left(t-\frac{3}{2}\right).$$

Substituting $t = \ln x$ gives

$$y = \frac{c_1}{x} + \sqrt{x}\left[c_2 \cos\left(\frac{\sqrt{3}}{2}\ln x\right) + c_3 \sin\left(\frac{\sqrt{3}}{2}\ln x\right)\right] + \frac{x}{2}\left[\ln x - \frac{3}{2}\right]$$

as the general solution for the given equation.

Example 11.28: Solve the differential equation $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + y = \ln x.\left[\sin(\ln x) + \dfrac{1}{x}\right]$

Solution: Substituting $x = e^t$, the equation becomes

$$[D(D-1) - 3D + 1]y = t \sin t + te^{-t}, \text{ where } D \equiv \frac{d}{dt}.$$

Simplifying it gives, $(D^2 - 4D + 1)y = te^{-t} + t\sin t$. The characteristic equation is $\lambda^2 - 4\lambda + 1 = 0$ which has roots $\lambda = 2 \pm \sqrt{3}$. Thus, the complementary function is $y_c(t) = (c_1e^{\sqrt{3}\,t} + c_2e^{-\sqrt{3}\,t})e^{2t}$. The particular integer is

$$y_p(t) = \frac{1}{D^2 - 4D + 1}te^{-t} + \frac{1}{(D^2 - 4D + 1)}t\sin t$$

$$= e^{-t}\frac{1}{(D-1)^2 - 4(D-1) + 1}t + Im.\frac{1}{D^2 - 4D + 1}te^{it}$$

$$= e^{-t}\frac{1}{D^2 - 6D + 6}t + Im.e^{it}\frac{1}{(D+i)^2 - 4(D+i) + 1}t$$

$$= \frac{e^{-t}}{6}\left\{1 - \left(D - \frac{D^2}{6}\right)\right\}^{-1} t + Im.e^{it}\frac{1}{D^2 + 2(i-2)D - 4i}t$$

$$= \frac{e^{-t}}{6}\left\{1 + \left(D - \frac{D^2}{6}\right) + \ldots\right\}t + Im.e^{it}\left[-4i\left\{1 - \frac{2(1+2i)D - iD^2}{4}\right\}^{-1} t\right]$$

$$= \frac{(t+1)e^{-t}}{6} + Im.e^{it}\left[-4i\left\{1 + \frac{2(1+2i)D - iD^2}{4} + \ldots\right\}t\right]$$

$$= \frac{(t+1)e^{-t}}{6} + Im.(\cos t + \sin t)\left[-4i\left\{t + \frac{1}{2}(1+2i)\right\}\right]$$

$$= \frac{(t+1)e^{-t}}{6} - 2(2t\cos t + \cos t - 2\sin t).$$

Therefore, the general solution is

$$y(t) = (c_1 e^{\sqrt{3}t} + c_2 e^{-\sqrt{3}t}) e^{2t} + \frac{(t+1)}{6} e^{-t} - 2(2t \cos t + \cos t - 2 \sin t).$$

Substituting $t = \ln x$, we obtain

$$y = \left(c_1 x^{\sqrt{3}} + \frac{c_2}{x^{\sqrt{3}}}\right) x^2 + \frac{(\ln x + 1)}{6x} - 2(2 \ln x \cos (\ln x) + \cos (\ln x) - 2 \sin (\ln x)]$$

$$= c_1 x^{2+\sqrt{3}} + c_2 x^{2-\sqrt{3}} + \frac{(1+\ln x)}{6x} - 2[(2 \ln x + 1) \cos (\ln x) - 2 \sin (\ln x)],$$

as the general solution of the given differential equation.

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Example 11.29: Solve the differential equation $(2x+3)^2 \frac{d^2y}{dx^2} - (2x+3)\frac{dy}{dx} - 12y = 6x.$

Solution: Substituting $2x + 3 = e^t$, the equation becomes

$$[4D(D-1) - 2D - 12]y = 3(e^t - 3), \quad D \equiv \frac{d}{dt}.$$

Simplifying it gives, $2(2D^2 - 3D - 6)y = 3e^t - 9$. The characteristic equation is, $2\lambda^2 - 3\lambda - 6 = 0$, with roots $\lambda = \frac{3}{4} \pm \frac{\sqrt{57}}{4}$. Thus the complementary function is $y_c(t) = \left(c_1 e^{\frac{\sqrt{57}}{4}t} + c_2 e^{-\frac{\sqrt{57}}{4}t}\right)(e^{3t/4})$. The particular integral is

$$y_p(t) = \frac{1}{2(2D^2 - 3D - 6)}(3e^t - 9) = \frac{3}{2}\left[\frac{1}{2D^2 - 3D - 6}e^t - 3\frac{1}{2D^2 - 3D - 6}e^0\right]$$

$$= \frac{3}{2}\left[\frac{e^t}{2-3-6} - 3.\frac{1}{-6}\right] = \frac{3}{2}\left[-\frac{1}{7}e^t + \frac{1}{2}\right] = -\frac{3}{14}e^t + \frac{3}{4}.$$

Hence the general solution is

$$y(t) = \left(c_1 e^{\frac{\sqrt{57}}{4}t} + c_2 e^{-\frac{\sqrt{57}}{4}t}\right) e^{3t/4} - \frac{3}{14}e^t + \frac{3}{4}.$$

Substituting $t = \ln (2x + 3)$, we obtain

$$y = \left[c_1(2x+3)^{\sqrt{57}/4} + c_2(2x+3)^{-\sqrt{57}/4}\right](2x+3)^{3/2} - \frac{3}{14} + (2x+3) + \frac{3}{4},$$

as the general solution for the given differential equation.

EXERCISE 11.4

Solve the differential equations:

1. $(x^2D^2 - xD + 1)y = \ln x$
2. $(x^2D^2 - xD - 3)y = x^2(\ln x)^2$
3. $\left(xD^2 - \dfrac{2}{x}\right)y = x + \dfrac{1}{x^2}$
4. $(x^2D^2 + 3xD + 1)y = \dfrac{1}{(1-x)^2}$
5. $(x^3D^3 + 3x^2D^2 + xD + 8)y = 65 \cos(\ln x)$
6. $(x^4D^4 + 2x^3D^3 + x^2D^2 - xD + 1)y = \ln x$
7. $[(1 + x)^2D^2 + (1 + x)D + 1]y = 4 \cos \ln(1 + x)$
8. $[(3x + 2)^2D^2 + 3(3x + 2)D - 36]y = 3x^2 + 4x + 1$
9. $(4x^2D^2 + 1)y = \ln x,\ x > 0;\ y(1) = 0,\ y(e) = 5$
10. $(x^2D^2 + 3xD + 10)y = 9x^2;\ y(1) = 5/2,\ y'(1) = 8$
11. The radial displacement u in a rotating disc at a distance r from the axis is given by $r^2\dfrac{d^2u}{dr^2} + r\dfrac{du}{dr} - u + kr^3 = 0$, where k is a constant. Solve the equation under the conditions, $u(0) = u(a) = 0$.

11.7 METHOD OF VARIATION OF PARAMETERS. METHOD OF UNDETERMINED COEFFICIENTS

So far we have discussed the operator method for finding the particular integral which is easily applicable in some specific forms of $f(x)$. In this section we discuss two general methods for finding the particular integral of a non-homogeneous differential equation, whenever the complementary function is known. The methods are:

I. *Method of variation of parameters*

II. *Method of undetermined coefficients.*

First method is applicable to both constant coefficients and variable-coefficients non-homogeneous differential equations, while the second is applicable only to constant coefficients one.

11.7.1 Method of Variation of Parameters

Consider the non-homogeneous differential equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = f(x),\ a_0(x) \neq 0. \quad ...(11.58)$$

If $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of the homogeneous equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0, \quad ...(11.59)$$

$\dfrac{f(x)}{a_0(x)} = g(x)$

then the complementary function is

$$y(x) = c_1y_1(x) + c_2y_2(x), \quad ...(11.60)$$

where c_1 and c_2 are two arbitrary constants.

The method of variation of parameters consists of finding a particular solution of the non-homogeneous equation by replacing the constants $c_1$and c_2 in (11.60) with functions of x, that is, we find functions $u(x)$ and $v(x)$ such that

$$y_p(x) = u(x)\, y_1(x) + v(x) y_2(x) \qquad \text{...(11.61)}$$

is a particular solution of Eq. (11.58). To determine $u(x)$ and $v(x)$ we need two equations. These are obtained as follows.

We compute

$$y'_p = uy'_1 + vy'_2 + u'y_1 + v'y_2 \qquad \text{...(11.62)}$$

To simplify this expression we find u and v such that

$$u'y_1 + v'y_2 = 0. \qquad \text{...(11.63)}$$

Thus (11.62) becomes

$$y'_p = uy'_1 + vy'_2 \qquad \text{...(11.64)}$$

Next we compute

$$y''_p = uy''_1 + vy''_2 + u'y'_1 + v'y'_2 \qquad \text{...(11.65)}$$

Substituting expressions for y_p, y'_p and y''_p from (11.61), (11.64) and (11.65) in Eq. (11.58), we obtain

$$a_0(x)\,[uy''_1 + vy''_2 + u'y'_1 + v'y'_2] + a_1(x)\,[uy'_1 + vy'_2] + a_2(x)\,[uy_1 + vy_2] = f(x)$$

Rearranging the terms to obtain

$$u[a_0(x)y''_1 + a_1(x)y'_1 + a_2(x)y_1] + v[a_0(x)y''_2 + a_1(x)\,y'_2 + a_2(x)y_2] + a_0(x)\,[u'y'_1 + v'y'_2] = f(x) \qquad \text{...(11.66)}$$

Using the fact that y_1 and y_2 are two solutions of the homogeneous Eq. (11.59), the Eq. (11.66) reduces to

$$a_0(x)\,(u'y'_1 + v'y'_2) = f(x)$$

or,

$$u'y'_1 + v'y'_2 = f(x)/a_0(x) = g(x), \text{ say.} \qquad \text{...(11.67)}$$

Solving Eqs. (11.63) and (11.67) for u' and v' we obtain

$$u'(x) = \frac{W_1(x)}{W(x)} \text{ and } v'(x) = \frac{W_2(x)}{W(x)}, \qquad \text{...(11.68)}$$

where $W(x) = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} \neq 0$; $W_1(x) = \begin{vmatrix} 0 & y_2 \\ g(x) & y'_2 \end{vmatrix}$, and $W_2(x) = \begin{vmatrix} y_1 & 0 \\ y'_1 & g(x) \end{vmatrix}$.

Here $W(x)$, the Wronskian of y_1, y_2 is non-zero, since y_1 and y_2 are two linearly independent solutions of the homogeneous Eq. (11.59).

Integrating (11.68), we obtain u and v to get y_p. The arbitrary constants of integrations, are taken zeros since our interest is to find y_p only. In fact, if we consider the constants of integration to be non-zeros, then $y_p(x)$ gives the general solution of the non-homogeneous differential equation. The method of variation of parameter is applicable to equations of higher order also.

For example, consider the third order equation

$$a_0(x)y''' + a_1(x)y'' + a_2(x)y' + a_3(x)y = f(x), \; a_0(x) \neq 0,$$

If y_1, y_2, y_3 are three linearly independent solutions of the associated homogeneous equation, then the complementary function is $y_c(x) = c_1y_1(x) + c_2y_2(x) + c_3y_3(x)$, where c_1, c_2, c_3 are three arbitrary constants. Consider

$$y_p(x) = u(x)y_1(x) + v(x)\, y_2(x) + w(x)y_3(x)$$

as the particular integral and proceed on the similar lines as above. The required conditions to determine $u(x)$, $v(x)$ and $w(x)$ are:

$$\left.\begin{aligned} u'y_1 + v'y_2 + w'y_3 &= 0 \\ u'y_1' + v'y_2' + w'y_3' &= 0 \\ u'y_1'' + v'y_2'' + w'y_3'' &= \frac{f(x)}{a_0(x)} = g(x) \end{aligned}\right\} \quad ...(11.69)$$

If $\quad W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} \neq 0$, is the Wronskian of y_1, y_2, y_3 and

$$W_1(x) = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ g(x) & y_2'' & y_3'' \end{vmatrix}, \quad W_2(x) = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & g(x) & y_3'' \end{vmatrix}, \quad W_3(x) = \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & g(x) \end{vmatrix},$$

then, $\quad u'(x) = \dfrac{W_1(x)}{W(x)}, \quad v'(x) = \dfrac{W_2(x)}{W(x)}$, and $\quad w'(x) = \dfrac{W_3(x)}{W(x)}$. $\quad ...(11.70)$

Hence, the particular integral is

$$y_p(x) = \left(\int \frac{W_1(x)}{W(x)}dx\right)y_1 + \left(\int \frac{W_2(x)}{W(x)}dx\right)y_2 + \left(\int \frac{W_3(x)}{W(x)}dx\right) y_3. \quad ...(11.71)$$

The result can be generalized to the equation of the nth order.

Example 11.30: Solve by the method of variation of parameters the differential equation $(D^2 + 4)y = \sec x$.

Solution: The characteristic equation of the associated homogeneous equation is

$$\lambda^2 + 4 = 0, \text{ with roots } \lambda = \pm 2i.$$

Thus, two independent solutions are $y_1(x) = \cos 2x$ and $y_2 = \sin 2x$.

The Wronskian of y_1 and y_2 is $W(x) = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2(\cos^2 2x + \sin^2 2x) = 2.$

Also, $\quad W_1(x) = \begin{vmatrix} 0 & \sin 2x \\ \sec x & 2\cos 2x \end{vmatrix} = -\sin 2x \sec x = -2\sin x,$

and, $$W_2(x) = \begin{vmatrix} \cos 2x & 0 \\ -2\sin 2x & \sec x \end{vmatrix} = \cos 2x \sec x.$$

Hence, $u'(x) = W_1(x)/W(x) = -\sin x$, which gives $u(x) = \cos x$.

Similarly, $v'(x) = \dfrac{w_2(x)}{w(x)} = \dfrac{1}{2}\cos 2x \sec x = \cos x - \dfrac{1}{2}\sec x$, which gives

$$v(x) = \sin x - \frac{1}{2}\ln|\sec x + \tan x|.$$

Thus the particular integral is

$$y_p(x) = u(x)y_1(x) + v(x)y_2(x) = \cos x \cos 2x + \left(\sin x - \frac{1}{2}\ln|\sec x + \tan x|\right)\sin 2x.$$

Therefore, the general solution is

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + \cos x \cos 2x + \left(\sin x - \frac{1}{2}\ln|\sec x + \tan x|\right)\sin 2x.$$

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Example 11.31: Solve by the method of variation of parameter the differential equation $(x^2D^2 - 3xD + 4)y = \ln x,\ x > 0.$

Solution: It is homogeneous differential equation of the Euler's form, we can see easily that the two linearly independent solutions of the associated homogeneous differential equation are

$$y_1 = x^2, \text{ and } y_2 = x^2 \ln x.$$

The Wronskian of y_1 and y_2 is

$$W(x) = \begin{vmatrix} x^2 & x^2 \ln x \\ 2x & 2x\ln x + x \end{vmatrix} = 2x^3 \ln x + x^3 - 2x^3 \ln x = x^3 \neq 0, \text{ since } x > 0.$$

Also here $g(x) = f(x)/a_0(x) = \ln x/x^2$; and hence

$$W_1(x) = \begin{vmatrix} 0 & x^2 \ln x \\ \ln x/x^2 & 2x\ln x + x \end{vmatrix} = -(\ln x)^2, \text{ and } W_2(x) = \begin{vmatrix} x^2 & 0 \\ 2x & \ln x/x^2 \end{vmatrix} = \ln x.$$

Thus, $u'(x) = -(\ln x)^2/x^3$, and $v'(x) = \ln x/x^3$. Integrating, we obtain

$$u(x) = \frac{1}{2}\frac{(\ln x)^2}{x^2} + \frac{1}{2}\frac{\ln x}{x^2} + \frac{1}{4x^2}, \text{ and } v(x) = -\frac{1}{2}\frac{\ln}{x^2} - \frac{1}{4x^2}.$$

Thus the particular integral is

$$\begin{aligned} y_p(x) &= u(x)y_1(x) + v(x)y_2(x) \\ &= \frac{1}{2}\left(\frac{(\ln x)^2}{x^2} + \frac{\ln x}{x^2} + \frac{1}{2x^2}\right)x^2 - \frac{1}{2}\left(\frac{\ln x}{x^2} + \frac{1}{2x^2}\right)x^2 \ln x = \frac{1}{4}\ln x + \frac{1}{4}. \end{aligned}$$

1) if of the form $(x^n D^n + x^{n-1} D^{n-1} - - - xD + c)y$ —
use $\frac{w_1}{w}$, $\frac{w_2}{w}$ - - -

if $(D^2 + \dots)y = f(x)$, use $u(x)A(x) + B(x)v(x)$ when $A(x) = -\int \frac{Rv}{w}$; $B(x) = \int \frac{Ru}{w}$

Therefore, the general solution is $y(x) = c_1x^2 + c_2x^2 \ln x + \dfrac{1}{4} \ln x + \dfrac{1}{4}$.

Example 11.32: Imp. It is given that $y_1 = \dfrac{1}{x}$, $y_2 = x$ and $y_3 = x^2$ are three linearly independent solutions of the associated homogeneous equation $x^3y''' + x^2y'' - 2xy' + 2y = \dfrac{2}{x}$, $x > 0$. Find a particular integral to this equation using the method of variation of parameters.

Solution: The Wronskian of y_1, y_2, y_3 is $W(x) = \begin{vmatrix} 1/x & x & x^2 \\ -1/x^2 & 1 & 2x \\ 2/x^3 & 0 & 2 \end{vmatrix} = \dfrac{6}{x}$, after simplification.

We have $g(x) = f(x)/a_0(x) = 2/x^4$.

Thus, $$W_1(x) = \begin{vmatrix} 0 & x & x^2 \\ 0 & 1 & 2x \\ 2/x^4 & 0 & 2 \end{vmatrix} = 2/x^2, \quad W_2(x) = \begin{vmatrix} 1/x & 0 & x^2 \\ -1/x^2 & 0 & 2x \\ 2/x^3 & 2/x^4 & 2 \end{vmatrix} = -6/x^4,$$

and, $$W_3(x) = \begin{vmatrix} 1/x & x & 0 \\ -1/x^2 & 1 & 0 \\ 2/x^3 & 0 & 2/x^4 \end{vmatrix} = 4/x^5.$$ This gives

$$u'(x) = \frac{W_1}{W} = \frac{1}{3x}, \quad v'(x) = \frac{W_2}{W} = \frac{1}{x^3}, \quad \text{and} \quad w'(x) = \frac{W_3}{W} = \frac{1}{3x^4}.$$ Hence,

$$u(x) = \frac{1}{3} \ln x, \quad v(x) = 1/2x^2, \text{ and } \quad w(x) = -\frac{2}{9x^3}$$

Thus, particular integral is

$$y_p(x) = u(x)y_1 + v(x)y_2 + w(x)y_3 = \frac{\ln x}{3x} + \frac{1}{2x} - \frac{2}{9x} = \frac{\ln x}{3x} + \frac{5}{18x}.$$

We note that the term $5/18x$ can be dropped from $y_p(x)$, since it will appear in the C.F. because of the solution $y_1(x) = 1/x$. Hence, $y_p(x)$ can be taken as $y_p = \ln x/3x$. The same can be verified by direct substitution in the given non-homogeneous equation.

11.7.2 Method of Undetermined Coefficients

This method of finding particular integral is applicable only to linear differential equations with constant coefficients. When the right hand side $f(x)$ is of a special form, say containing polynomials, exponentials, cosine and sine function, sum or product of these functions, then the form of $y_p(x)$ can be guessed. By substituting this in the differential equation, the undetermined constants in $y_p(x)$ are determined.

For example, if $f(x) = x^m$, then its derivatives contain the terms $x^m, x^{m-1}, \ldots, x, 1$ and hence $y_p(x)$ can be chosen as

$$y_p(x) = c_0x^m + c_1x^{m-1} + \ldots + c_{m-1}x + c_m,$$

where c_i's are constants.

Similarly, if $f(x) = e^{ax}\cos bx$, or $e^{ax}\sin bx$, then $y_p(x)$ can be chosen as

$$y_p(x) = e^{ax}(c_1\cos bx + c_2\sin bx)$$

However, if any term in the choice of the particular integral is also a solution of the associated homogeneous equation, then we multiply this term by x^m, where m is the multiplicity of the root. Also this method is applicable only if the repeated differentiation of $f(x)$, the right hand side of the non-homogeneous equation, produces only a finite number of linearly independent terms.

For example, in case $f(x) = x^2e^x$. The sequence consisting of this term and successive derivatives is

$$\{x^2e^x, x^2e^x + 2xe^x, x^2e^x + 4xe^x + 2e^x, \ldots\}$$

We observe that sequence consists of only three linearly independent functions x^2e^x, xe^x and e^x. However, in case of $f(x) = \ln x$, the sequence consists of

$$\left\{\ln x, \frac{1}{x}, -\frac{1}{x^2}, +\frac{2}{x^3}, \ldots\right\},$$

an infinite number of linearly independent terms. Thus method of undetermined coefficients cannot be applied in this case. Similarly this method fails in case of $f(x) = \tan x$ or $\sec x$.

Example 11.33: Solve by the method of undetermined coefficients the differential equation $(D^2 - 4)y = 8x^2 - 2x$.

Solution: The characteristic equation is $\lambda^2 - 4 = 0$. Its roots are $\lambda = \pm 2$. Hence, the complementary function is $y_c(x) = c_1e^{2x} + c_2e^{-2x}$.

Here $f(x) = x^2 - 2x$; assuming the particular integral as

$$y_p(x) = a_0x^2 + a_1x + a_2$$

where a_0, a_1 and a_2 are undetermined constants. It gives

$$y'_p(x) = 2a_0x + a_1, \text{ and } y_p''(x) = 2a_0.$$

Substituting these in the given equation to obtain

$$2a_0 - 4(a_0x^2 + a_1x + a_2) = 8x^2 - 2x$$

Rearranging it as $-4a_0x^2 - 4a_1x + (2a_0 - 4a_2) = 8x^2 - 2x$.

Comparing on both sides, we obtain $-4a_0 = 8$, $-4a_1 = -2$, and $2(a_0 - 2a_2) = 0$, which gives $a_0 = -2$, $a_1 = \frac{1}{2}$, $a_2 = -1$. Hence, the particular integral is, $y_p(x) = -2x^2 + \frac{1}{2}x - 1$.

Thus, the general solution is, $y(x) = c_1e^{2x} + c_2e^{-2x} - 2x^2 + \frac{1}{2}x - 1$.

Example 11.34: Solve $y''(x) + y(x) = \sin x$ by the method of undetermined coefficient.

Solution: It is easy to see that the complementary function is $y_c(x) = c_1 \cos x + c_2 \sin x$.

Here $f(x) = \sin x$, and as a normal guess the particular integral would have been, $a_1 \cos x + a_2 \sin x$, but since, $\cos x$ and $\sin x$ have already appeared in $y_c(x)$, we choose particular integral as

$$y_p(x) = x(a_1 \cos x + a_2 \sin x),$$

where, a_1, a_2 are undetermined constants. This gives

$$y_p'(x) = (a_1 + a_2 x)\cos x + (a_2 - a_1 x)\sin x, \text{ and } y_p''(x) = (2a_2 - a_1 x)\cos x - (2a_1 + a_2 x)\sin x.$$

Substituting these in the given equation we obtain

$$(2a_2 - a_1 x)\cos x - (2a_1 + a_2 x)\sin x + x(a_1 \cos x + a_2 \sin x) = \sin x.$$

Simplifying it gives, $2a_2 \cos x - 2a_1 \sin x = \sin x$. Comparing coefficients of $\sin x$ and $\cos x$ on both sides, we get $a_1 = (-1/2)$, $a_2 = 0$. Hence, the particular integral is $y_p(x) = (-1/2)x \cos x$.

Thus the general solution is $y(x) = c_1 \cos x + c_2 \sin x - (1/2)x \cos x$.

Example 11.35: Solve, $y''' - 5y'' + 6y' = x^2 + \sin x$, by the method of undetermined coefficients.

Solution: The characteristic equation is $\lambda^3 - 5\lambda + 6\lambda = 0$. It gives $\lambda_1 = 0$, $\lambda_2 = 2$ and $\lambda_3 = 3$. So the complementary function is $y_c(x) = c_1 + c_2 e^{2x} + c_3 e^{3x}$, where c_1, c_2 and c_3 are arbitrary constants.

Here $f(x) = x^2 + \sin x$. As a normal guess the form of particular integral corresponding to the term x^2 in $f(x)$ is $a_0 x^2 + a_1 x + a_2$, but since a constant term is already there in $y_c(x)$, so we modify this to as $a_0 x^3 + a_1 x^2 + a_2 x$. Next, the form corresponding to the term $\sin x$ in $f(x)$ is $a_3 \sin x + a_4 \cos x$. Combining these two, we choose $y_p(x)$ as

$$y_p(x) = a_0 x^3 + a_1 x^2 + a_2 x + a_3 \sin x + a_4 \cos x$$

where a_i's are undetermined coefficients to be determined. It gives

$$y_p'(x) = 3a_0 x^2 + 2a_1 x + a_2 + a_3 \cos x - a_4 \sin x$$

$$y_p''(x) = 6a_0 x + 2a_1 - a_3 \sin x - a_4 \cos x, \text{ and } y_p'''(x) = 6a_0 - a_3 \cos x + a_4 \sin x.$$

Substituting these in the given differential equation, we obtain

$$(6a_0 - a_3 \cos x + a_4 \sin x) - 5(6\, a_0 x + 2a_1 - a_3 \sin x - a_4 \cos x)$$
$$+ 6(3a_0 x^2 + 2a_1 x + a_2 + a_3 \cos x - a_4 \sin x) = x^2 + \sin x.$$

Rewriting it as

$$2(3a_0 - 5a_1 + 3a_2) - 6(5a_0 - 2a_1)\, x + 18a_1\, x^2 + 5(a_3 + a_4)\cos x + 5(a_3 - a_4)\sin x = x^2 + \sin x.$$

Comparing the coefficients of like terms on both sides, we obtain

$$3a_0 - 5a_1 + 3a_2 = 0, \quad 5a_0 - 2a_1 = 0, \quad 18a_1 = 1, \quad a_3 + a_4 = 0, \quad \text{and } 5(a_3 - a_4) = 1.$$

Solving for a_0, a_1, a_2, a_3 and a_4, we have $a_0 = 1/45$, $a_1 = 1/18$, $a_2 = 19/270$, $a_3 = 1/10$ and $a_4 = -1/10$.

Thus the particular integral is $y_p(x) = \dfrac{1}{45}x^3 + \dfrac{1}{18}x^2 + \dfrac{19}{270}x + \dfrac{1}{10}(\sin x - \cos x)$.

Hence the general solution is

$$y(x) = c_1 + c_2 e^{2x} + c_3 e^{3x} + \frac{1}{45}x^3 + \frac{1}{18}x^2 + \frac{19}{270}x + \frac{1}{10}(\sin x - \cos x).$$

Remark. Though the method of undetermined coefficients is applicable only to linear differential equations with constant coefficients, yet, whenever the differential equation with variable coefficients is reducible to a constant coefficients one, this method can be applied, provided the repeated differentiation of the right hand side of the reduced equation generate only finite number of linearly independent functions.

Example 11.36: Solve $x^2y'' - 5xy' + 8y = 2\ln x$ using the method of undetermined coefficients.

Solution: It is Euler's homogeneous equation. Applying the transformation $x = e^t$, the equation reduces to

$$y''(t) - 6y'(t) + 8y(t) = 2t \qquad \ldots(11.72)$$

a linear differential equation with constant coefficients.

Its complementary function is $y_c(t) = c_1e^{2t} + c_2e^{4t}$.

Also we can find easily by applying the method of undermined coefficients, that its particular integral as $y_p(t) = t/4 + 3/16$.

Hence the general solution for Eq. (11.72) is $y(t) = c_1e^{2t} + c_2e^{4t} + t/4 + 3/16$.

Using $t = \ln x$, the general solution for the given equation is

$$y(x) = c_1x^2 + c_2x^4 + (\ln x)/4 + 3/16.$$

EXERCISE 11.5

For the following equations find the general solution by the method of variation of parameter.

1. $y'' + 2y' + y = xe^x$
2. $y'' + y = \operatorname{cosec} x,\ x \neq n\pi$
3. $y'' + 16y = 32\sec 2x,\ x \neq \left(n + \dfrac{1}{2}\right)\dfrac{\pi}{2}$
4. $y'' + 3y' + 2y = 3/(1 + e^x)$
5. $y'' + y = \sec^2 x$
6. $y'' + 4y' + 5y = xe^{-2x}\cos x$
7. $y''' - 6y'' + 11y' - 6y = e^{-x}$
8. $y''' + 4y' = \sec 2x$
9. $x^2y'' + xy' - 4y = x^2\ln x$
10. $x^2y'' - 2xy' + 2y = x^3 + x$

For the following equations verify that the functions $y_1(x), y_2(x)$ are linearly independent solutions of the associated homogeneous equation. Using these find a particular integral and general solution of the given equation

11. $x^2y'' + xy' - y = x,\ x \neq 0;\ y_1 = x,\ y_2 = 1/x$
12. $y'' + 4y' + 8y = 16e^{-2x}\operatorname{cosec}^2 2x,\ x \neq \dfrac{n\pi}{2};\ y_1 = e^{-2x}\cos 2x,\ y_2 = e^{-2x}\sin 2x$
13. $x^2y'' + 3xy' - 3y = \sqrt{x};\ y_1 = x,\ y_2 = 1/x^3$
14. $(1 - x^2)y'' - cy' + 4y = x;\ y_1 = 2x^2 - 1,\ y_2 = x\sqrt{x^2 - 1}$
15. $y''' + y'' - y' - y = x;\ y_1 = e^x,\ y_2 = e^{-x},\ y_3 = xe^{-x}$

For the following equations find the general solution by the method of undetermined coefficients.

16. $y'' + 2y' + 4y = 2x^2 + 3e^{-x}$

17. $y'' + 5y' + 6y = 4e^{-x} + 5\sin x$

18. $y'' + y' - 12y = e^{3x}$

19. $y'' - y = e^{3x}\cos 2x - e^{2x}\sin 3x$

20. $y''' - 6y'' + 12y' - 8y = 12e^{2x} + 27e^{-x}$

21. $x^2y'' + xy' + 4y = \sin(2\ln x)$

22. $x^2y'' + 3xy' + y = 9x^2 + 8x + 5$

11.8 SIMULTANEOUS LINEAR DIFFERENTIAL EQUATIONS

So far we have considered the case where there is a single dependent variable such as the current $I(t)$ in a circuit at the time t. However, many problems involve two or more dependent variables, but only one independent variable. For example, in a chemical reaction two substances with specific concentrations at an instant t, react to form a third substance with some concentration. Since the concentrations are interrelated the governing differential equations are coupled and lead to system of simultaneous differential equations. As another example, governing of various currents in an electric circuit, comprising two or more loops, leads to a system of simultaneous linear differential equations.

We shall consider the solution of a system of two linear first order equations in two dependent variables x and y and one independent variable t with constant coefficients only. For example, the equations

$$\frac{dx}{dt} - \frac{dy}{dt} - y = e^{-t}, \quad \frac{dy}{dt} + x - y = e^{2t}$$

represent a system of linear constant coefficients first order differential equations. These can be expressed in operator form as

$$Dx - (D+1)y = e^{-t}, \quad x + (D-1)y = e^{2t}; \; D \equiv \frac{d}{dt}$$

The method of solving these simultaneous equations consists of eliminating one of the variables x or y, solving the resulting differential equation and substituting the value of that variable in one of the two equations to get the value of the second variable. The complete solution consists of the two equations giving expressions for x and y in term of the independent variable t. This method can be extended to a system of simultaneous equations with more than two dependent variables and one independent variable.

Example 11.37: Solve the system of differential equations

$$\frac{dx}{dt} - 2x + y = 4 - t^2, \quad \frac{dy}{dt} + x - 2y = 1$$

Solution: The equations can be expressed as

$$(D-2)x + y = 4 - t^2 \qquad \text{...(11.73)}$$

$$x + (D-2)y = 1 \qquad \text{...(11.74)}$$

Operating (11.73) by $(D - 2)$, and then subtracting (11.74) from the resultant equation, we obtain

$$(D-2)^2x - x = (D-2)(4-t^2) - 1$$

or,
$$(D^2 - 4D + 3)x = 2t^2 - 2t - 9. \qquad \text{...(11.75)}$$

Solving Eq. (11.75) by operator method, the general solution is

$$x(t) = c_1e^{3t} + c_2e^{t} + \frac{2}{3}t^2 + \frac{10}{9}t - \frac{53}{27}, \qquad \text{...(11.76)}$$

where c_1 and c_2 are two arbitrary constants.

To find $y(t)$, from (11.73) we have $y(t) = -(D - 2)\, x(t) + 4 - t^2$.

Using (11.76) in it, gives

$$y(t) = -c_1e^{3t} + c_2e^{t} + \frac{1}{3}t^2 + \frac{8}{9}t - \frac{28}{27}. \qquad \text{...(11.77)}$$

The Eqs. (11.76) and (11.77) form the general solution for the given system of differential equations.

Example 11.38: Solve the system of differential equations

$$(3D + 1)x + 3Dy = 3t + 1 \qquad \text{...(11.78)}$$

$$(D - 3)x + Dy = 2t \qquad \text{...(11.79)}$$

Solution: To eliminate y, multiply Eq. (11.79) by 3 and subtract from Eq. (11.78) and simplify we obtain

$$x(t) = (1 - 3t)/10 \qquad \text{...(11.80)}$$

To find $y(t)$ from Eq. (11.79) we have

$$Dy = 2t - (D - 3)x = 2t - \frac{1}{10}(D - 3)\,(1 - 3t), \quad \text{using (11.80)}$$

$$= \frac{1}{10}(11t + 6), \text{ a linear first order equation in } y.$$

Integrating, we obtain

$$y(t) = \frac{11}{20}t^2 + \frac{3}{5}t + c, \qquad \text{...(11.81)}$$

where c is an arbitrary constant.

The Eqs. (11.80) and (11.81) form the general solution for the given system of differential equations.

Example 11.39: Solve the system of differential equations

$$(D^2 - 3)x - 4y = 0 \qquad \text{...(11.82)}$$

$$x + (D^2 + 1)y = 0 \qquad \text{...(11.83)}$$

Solution: To eliminate y, multiply Eq. (11.82) by $(D^2 + 1)$ and Eq. (11.83) by 4 and add, we get

$$[(D^2 + 1)\,(D^2 - 3) + 4]x = 0$$

or,
$$(D^4 - 2D^2 + 1)x = 0 \qquad \text{...(11.84)}$$

The general solution of Eq. (11.84) is

$$x = (c_1 + c_2 t)e^{t} + (c_3 + c_4 t)e^{-t}, \quad \text{...(11.85)}$$

where c_1, c_2, c_3, and c_4 are arbitrary constants.

To find y, from (11.82) we have $y = \frac{1}{4}(D^2 - 3)x$.

Substituting for x from (11.85) and simplifying, we obtain

$$y = \frac{1}{2}(c_2 - c_1 - c_2 t)e^{t} - \frac{1}{4}(c_2 + c_4 + c_4 t)e^{-t}. \quad \text{...(11.86)}$$

Eqs. (11.85) and (11.86) form the general solution for the given system of differential equations.

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Example 11.40: The small oscillations of a certain system with two degrees of freedom is given by

$$(D^2 + 2)x - y = 0 \quad \text{...(11.87)}$$

$$-x + (D^2 + 2)y = 0 \quad \text{...(11.88)}$$

Find x and y as a function of t.

Solution: To eliminate y, multiply (11.87) by $(D^2 + 2)$ and add to (11.88), we obtain

$$(D^4 + 4D^2 + 3)x = 0 \quad \text{...(11.89)}$$

Similarly eliminating x from (11.87) and (11.88), we get

$$(D^4 + 4D^2 + 3)y = 0 \quad \text{...(11.90)}$$

The general solution for Eqs. (11.89) and (11.90) are respectively

$$x(t) = c_1 \cos t + c_2 \sin t + c_3 \cos \sqrt{3}\, t + c_4 \sin \sqrt{3}\, t \quad \text{...(11.91)}$$

and, $$y(t) = c'_1 \cos t + c'_2 \sin t + c'_3 \cos \sqrt{3}\, t + c'_4 \sin \sqrt{3}\, t. \quad \text{...(11.92)}$$

To find any relation among the constants c_i's and c'_i's, we use the fact that (11.91) and (11.92) are solutions of the given simultaneous equations. Substituting these in, say Eq. (11.87), we obtain

$$(c_1 - c'_1)\cos t + (c_2 - c'_2)\sin t - (c_3 + c'_3)\cos \sqrt{3}\, t - (c_4 + c'_4)\sin \sqrt{3}\, t = 0, \text{ for all } t$$

which gives, $c_1 = c'_1$, $c_2 = c'_2$, $c_3 = -c'_3$ and $c_4 = -c'_4$.

Hence (11.92) becomes

$$y(t) = c_1 \cos t + c_2 \sin t - c_3 \cos \sqrt{3}\, t - c_4 \sin \sqrt{3}\, t \quad \text{...(11.93)}$$

The Eqs. (11.91) and (11.93) form the general solution for the given system of equations.

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Example 11.41: Solve the system of differential equations

$$Dx = y + z, \quad Dy = x + z, \quad \text{and} \quad Dz = x + y.$$

Solution: Equations are

$$Dx = y + z \quad \text{...(11.94)}$$

$$Dy = x + z \quad \text{...(11.95)}$$

$$Dz = x + y \qquad ...(11.96)$$

Operating (11.94) with D and substituting for Dy and Dz respectively from (11.95) and (11.96), we obtain $(D^2 - D - 2)x = 0$, which is a second order linear homogeneous equation with solution

$$x(t) = c_1e^{-t} + c_2e^{2t}, \qquad ...(11.97)$$

where c_1 and c_2 are two arbitrary constants.

Next, operating (11.95) with D and substituting for Dz from (11.96) in it, we obtain $(D^2 - 1)y = (D + 1)x$, or $(D^2 - 1)y = 3\ c_2\ e^{2t}$, using (11.97). It is a second order non-homogeneous linear equation with constant coefficients. Its general solution is

$$y(t) = c_2e^{2t} + c_3e^{-t} + c_4e^{t}. \qquad ...(11.98)$$

We note that while solving these linear differential equations, the four constants have already appeared, which is not desirable. To rectify this, from (11.94) and (11.95) we have

$$Dx - Dy = y - x. \qquad ...(11.99)$$

Substituting from (11.97) and (11.98) in (11.99) and simplifying, we get $-c_4e^t = c_4e^t$, which gives, $c_4 = 0$, since $e^t \neq 0$. Hence, (11.98) becomes

$$y(t) = c_2e^{2t} + c_3e^{-t} \qquad ...(11.100)$$

To find z, from (11.94), we have $z = Dx - y$.

Substituting for x and y from (11.97) and (11.100) respectively and simplifying we get

$$z(t) = -(c_1 + c_3)e^{-t} + c_2e^{2t} \qquad ...(11.101)$$

The Eqs. (11.97), (11.100) and (11.101) form the solution of the given system of equations.

Remark. We have been solving system of differential equations like that of system of linear algebraic equations. As in case of system of algebraic equations, we have system of differential equations with no solution, that is, an inconsistent system, or a system with infinite number of solutions, a redundant system. For example, the system of equations

$$2Dx + Dy = 1, \quad 4Dx + 2Dy = 4$$

has no solution, since the process of elimination of variable x or y leads to $2 = 4$, which is not possible. But if we modify the system to

$$2Dx + Dy = 2, \quad 4Dx + 2Dy = 4$$

then it has infinite solutions. Indeed then the second equation is merely twice the first and thus can be discarded, leaving the single equation $2Dx + Dy = 2$ in two unknowns $x(t)$ and $y(t)$. Choosing one of these arbitrarily and solving for the second, so there are infinitely many linearly independent solutions.

EXERCISE 11.6

Solve the following system of differential equations

1. $(D + 2)x + 3y = 0,\ 3x + (D + 2)y = 2e^{2t}$ 2. $(D + 2)x - y = 1 + e^{-t},\ x + (D + 2)y = 3$
3. $2(D - 2)x + (3D + 5)y = 3t + 2,\ (D - 2)x + (D + 1)y = t$
4. $(D + a)x - ay = 0,\ ax + (D + a)y = 0,\ a \neq 0$

5. $(D-2)x+12y=0,\ 3x-(2D+8)y=0;\ x(0)=0$ and $y(0)=1$
6. $(D^2-3)x-y=0,\ 2x+Dy=0$
7. $D^2x=y,\ D^2y=x$
8. $(D^2+3)x-2y=0,\ (D^2-3)x+(D^2+5)y=0;\ x(0)=y(0)=0,\ Dx(0)=3$ and $Dy(0)=2$
9. $D^2x+y=\sin t,\ x+D^2y=\cos t$
10. $Dx=2y,\ Dy=2x,\ Dz=2x.$

11.9 MODELLING SIMPLE HARMONIC MOTION

A simple harmonic motion (S.H.M.) is a periodic motion in which the acceleration of a particle is proportional to its displacement from a fixed point called the centre and is always directed towards it.

Let x be the displacement of the particle P at any time t from the fixed point O as shown in Fig. 11.2, then by definition of S.H.M.

$$\frac{d^2x}{dt^2}=-\mu^2x, \qquad \text{...(11.102)}$$

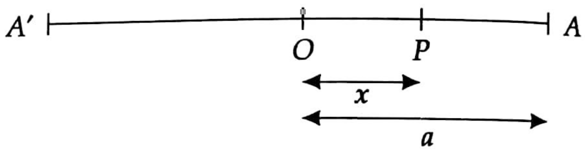

Fig. 11.2

μ^2 is the constant of proportionality negative sign is taken since the force acting on the particle is directed towards the fixed point, a direction of decreasing x.

Rewriting Eq. (11.102) as $(D^2+\mu^2)x=0$, which is a linear differential equation of second order with constant coefficients with complete solution as

$$x=c_1\cos\mu t+c_2\sin\mu t. \qquad \text{...(11.103)}$$

In case the particle starts from rest at A, where $OA=a$, then

$$x(0)=a, \text{ and } x'(0)=0. \qquad \text{...(11.104)}$$

Using (11.104) in (11.103) gives, $c_1=a$ and $c_2=0$ and hence, Eq. (11.103) becomes

$$x=a\cos\mu t, \text{ which gives, } \frac{dx}{dt}=-a\mu\sin\mu t=-\mu\sqrt{(a^2-x^2)}.$$

These respectively are the expressions for the *displacement* and the *velocity* of particle P at any time t. The maximum displacement, from the centre a is called the *amplitude* and the time of complete oscillation, $\frac{2\pi}{\mu}$ is called the *periodic time* and $\frac{1}{(\text{Periodic time})}=\frac{\mu}{2\pi}$ is called the *frequency*, the number of oscillations per second.

Also we note that the general solution (11.103), can be expressed as

$$x=a\cos(\mu t-\alpha), \text{ where } \alpha=\tan^{-1}(c_2/c_1)$$

The quantity α is called the *starting phase or the epoch of the motion* and the quantity $(\mu t-\alpha)$ is called the *argument of the motion* at time t.

Example 11.42: A particle is moving linearly with the speed v given by the relation $v^2=a+2bx-cx^2$, where x is the displacement of the particle from a fixed point on the path, and a, b, c are constants, with $c>0$. Show that the motion is simple harmonic and find its period and amplitude.

Solution: The velocity v of the particle is given by

$$v^2 = a + 2bx - cx^2 \qquad \text{...(11.105)}$$

Differentiating both sides w.r.t. x, we get, $2v\dfrac{dv}{dx} = 2b - 2cx$.

or, $$\frac{d^2x}{dt^2} = b - cx = -c\left(x - \frac{b}{c}\right) \qquad \text{...(11.106)}$$

Since $c > 0$, the Eq. (11.106) represents a S.H.M. directed towards the point $x = b/c$ and period $\dfrac{2\pi}{\mu} = \dfrac{2\pi}{\sqrt{c}}$. To find amplitude put $v = 0$ in (11.105), we obtain, $x = \dfrac{b \pm \sqrt{b^2 + ac}}{c}$. Thus the distances of two positions of instantaneous rest from the fixed point O, $\left(x = \dfrac{b}{c}\right)$, are

$$OA = \frac{b + \sqrt{b^2 + ac}}{c}, \text{ and } OA' = \frac{b - \sqrt{b^2 + ac}}{c}.$$

Hence the amplitude of the motion is $\left|\dfrac{b \pm \sqrt{b^2 + ac}}{c} - \dfrac{b}{c}\right| = \dfrac{\sqrt{b^2 + ac}}{c}$

Example 11.43: A particle of mass m executes simple harmonic motion in the line joining the points A and A' on a smooth table and is connected with these points by elastic strings. If T is tension in equilibrium and l, l' are the extensions of the strings beyond their natural lengths, find the periodic time.

Solution: Let O be the position of equilibrium of the particle, so that $OA = a + l$ and $A'O = a' + l'$ where a and a' are the natural lengths of the strings, refer Fig. 11.3. The tension T in the position of equilibrium is given by

$$T = \lambda l/a = \lambda' l'/a'.$$

Let P be the position of the particle at instant t during its motion such that $OP = x$ and T_1, T_2 be the tensions in the two portions, then

$$T_1 = \lambda\frac{l + x}{a} \text{ and } T_2 = \lambda'\frac{l' - x}{a'}.$$

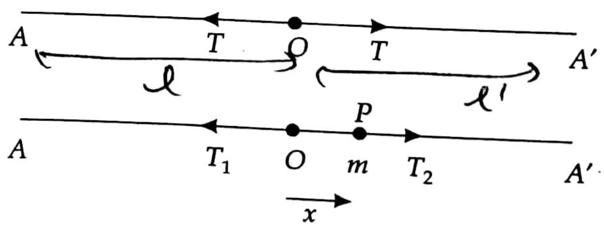

Fig. 11.3

Hence the equation of motion is

$$m\frac{d^2x}{dt^2} = T_2 - T_1 = \lambda'\frac{l' - x}{a'} - \lambda\frac{l + x}{a} = \left(\frac{\lambda' l'}{a'} - \frac{\lambda l}{a}\right) - \left(\frac{\lambda'}{a'} + \frac{\lambda}{a}\right)x$$

$$= (T - T) - T\left(\frac{1}{l'} + \frac{1}{l}\right)x = -\frac{T(l + l')}{ll'}x \text{ or,}$$

$\frac{d^2x}{dt^2} = -\frac{T(l+l')}{mll'}x = -\mu x$. The periodic time is given by $T = \frac{2\pi}{\sqrt{\mu}} = 2\pi\sqrt{\frac{mll'}{(l+l')T}}$.

EXERCISE 11.7

1. A horizontal shelf with a body of mass m placed on it is moving up and down in a simple harmonic motion of period 1 sec. Find its greatest amplitude so that the body placed on it is not thrown off.
2. Find the time required for a particle in simple harmonic motion with amplitude 20 cm and period 4 seconds, in passing between two points which are at distances 15 cm and 5 cm from the origin.
3. An elastic string of natural length $2a$ and modulus λ is stretched between two points A and B, distant $4a$ apart on a smooth horizontal table. A particle of mass m is attached to the middle of the string. Show that it can vibrate in line AB with period $\sqrt{(2am/\lambda)}\ \pi$.
4. If x_1, x_2, x_3 are the positions of a particle at the end of Ist, 2nd, 3rd second of its motion in S.H.M., then show that the time period is
$$2\pi \Big/ \cos^{-1}\left(\frac{x_1 + x_2}{x_3}\right).$$

11.10 MODELLING MASS-SPRING SYSTEM: FREE AND FORCED OSCILLATIONS

Consider a spring with unstretched length l and spring modulus k, a measure of the stiffness of the spring. The spring is suspended vertically from a fixed support. Let a body of mass m is attached at the lower end of the spring, (assuming m to be large so that the mass of the spring is neglected), which stretches the spring by d units over its natural length before coming to rest in its equilibrium position. Next, let us suppose that the body is then displaced vertically (up, or down) by distance y_0 units and is released possibly with an initial velocity. We want to study the motion of this spring-mass system, refer to Fig. 11.4.

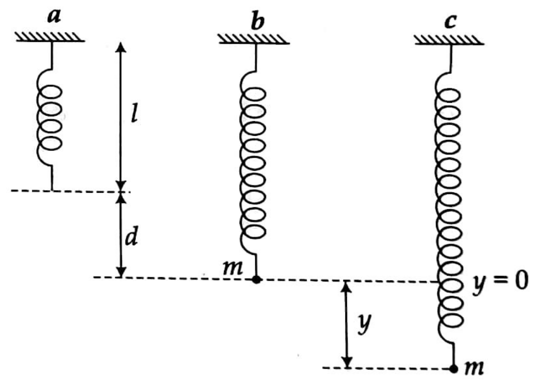

Fig. 11.4 (a) Unstretched; (b) Static equilibrium; (c) System in motion

11.10.1 The Spring Equation

Let $y(t)$ be the displacement of the object at time t from the equilibrium position, say $y = 0$ and select the downward direction to be positive. Consider the forces acting on the body of mass m at time t.

The force due to gravity which pulls it downward is of magnitude mg.

The force the spring exerts on the body at time t, due to Hooke's law, is of magnitude ky. At equilibrium position the force of the spring is of magnitude $-kd$, sign is negative since the force acts upward. If the body is displaced downward by a distance y from the equilibrium position, then an additional force $-ky$ is exerted on it at time t.

Thus, the total force on the body due to gravity and the spring is

$$mg - kd - ky.$$

At equilibrium position ($y = 0$) this force is zero, and hence $mg = kd$. Thus the net force acting on the object is therefore just $F_1 = -ky$, an upward force. It is a *restoring force* and has the tendency to restore the system, that is, pull the body back to the equilibrium position $y = 0$.

Next, every system is subjected to some damping or retarding forces, which may be air resistance or the viscosity of the medium if the body is suspended in some fluid such as oil, etc. In case these forces are not negligible then we need to take the corresponding damping into account. Experiments show that *the magnitude of the damping forces at any instant t is proportional to the velocity $y'(t)$, and direction is opposite to the instantaneous motion.* Thus, the *damping force* is $F_2 = -cy'$, where $c > 0$ is some constant, called the *damping constant.*

Also there may be a driving force of magnitude $F_3 = f(t)$ acting on the body, and then the total external force on the body has the magnitude

$$F_1 + F_2 + F_3 = -ky - cy' + f(t). \qquad \text{...(11.107)}$$

Using Newton's second law of motion, *the equation of the mass-spring system* is,

$$my'' = -ky - cy' + f(t)$$

$$\text{or, } my'' + cy' + ky = f(t). \qquad \text{...(11.108)}$$

This is called *the spring equation.*

In the absence of the external force, that is $f(t) = 0$, the equation (11.108) becomes homogeneous one given by $my'' + cy' + ky = 0$, and the motion of the mass-spring system are called the *free motions.*

In the presence of the external force the motions of the spring-mass system are called the *forced motion*. The external force $f(t)$ is called the *input*, or *the deriving force*; and the corresponding solution is called an *output* or a *response* of the system to the deriving force.

Next we analyze the motion described by solutions of the spring Eq. (11.108) under various conditions.

11.10.2 Mass-Spring System: Free Motions

In case of free motions the deriving force $f(t) = 0$, hence the spring equation is

$$my'' + cy' + ky = 0. \qquad \text{...(11.109)}$$

We study the following cases of practical interest.

A ***Undamped system*** In case the damping forces are negligible, (at least it can be so when the velocity is small during the initial phase of the motion), then $c = 0$, the Eq. (11.109) becomes

$$my'' + ky = 0. \qquad \text{...(11.110)}$$

The general solution of (11.110) is

$$y(t) = c_1 \cos wt + c_2 \sin wt,\ w = \sqrt{k/m} \qquad \text{...(11.111).}$$

In case the body is first pulled to a point at a distance a units from the position of static equilibrium and is released, then $y(0) = a$ and $y'(0) = 0$.

Using these initial conditions in (11.111) gives $c_1 = a$ and $c_2 = 0$, and hence (11.111) becomes $y(t) = a \cos wt$. The motion is simple harmonic motion with period $\dfrac{2\pi}{w} = 2\pi\sqrt{\dfrac{m}{k}}$ with amplitude a as shown in Fig. 11.5. The curve touches the line $y = \pm a$ when wt is an integral multiple of π.

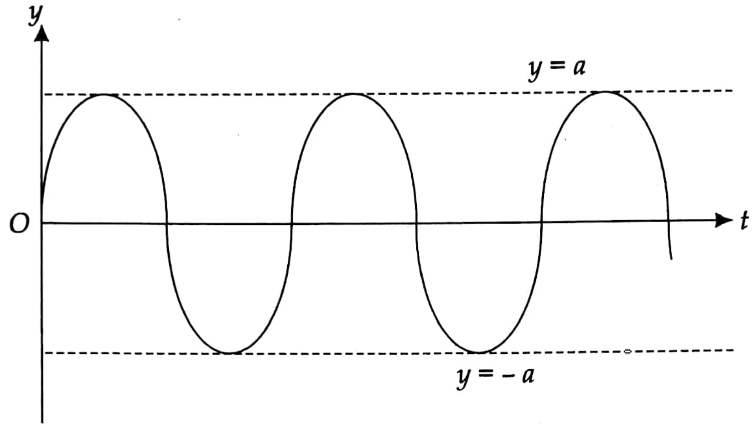

Fig. 11.5

B ***Damped system*** If the damping forces are not negligible, then $c \neq 0$ and the spring equation is

$$my'' + cy' + ky = 0. \qquad ...(11.112)$$

The corresponding characteristic equations is $\lambda^2 + \dfrac{c}{m}\lambda + \dfrac{k}{m} = 0$, with roots

$$\lambda_1 = -\alpha + \beta, \text{ and } \lambda_2 = -\alpha - \beta, \text{ where } \alpha = \frac{c}{2m} \text{ and } \beta = \frac{1}{2m}\sqrt{c^2 - 4mk}.$$

As is evident, the form of the solution of (11.112) will depend on the mass m, the amount of damping and the stiffness of the spring. We have the following three cases:

Case I: $c^2 > 4mk$: Two distinct real roots λ_1, λ_2 *(over damping),*

Case II: $c^2 = 4mk$: Two equal and real roots *(critical damping),*

Case III: $c^2 < 4mk$: Complex conjugate roots *(under damping).*

Case I: Over damping. If the damping constant c is so large that $c^2 > 4mk$, then λ_1 and λ_2 are two real and distinct roots and the general solution of the equation (11.112) is

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}. \qquad ...(11.113)$$

Obviously $\lambda_2 = -\alpha - \beta$ is negative, and also

$$\lambda_1 = -\alpha + \beta = -\frac{c}{2m} + \frac{\sqrt{c^2 - 4mk}}{2m} < -\frac{c}{2m} + \frac{c}{2m} = 0.$$

Since both λ_1 and λ_2 are negative, therefore, the terms in (11.113) tend to be zero as t approaches infinity. Thus the body does not oscillate and after a sufficiently long time the mass will be at rest at its static equilibrium position $y = 0$, a case of over damping.

Case II: Critical damping. If $c^2 = 4mk$, then $\beta = 0$ and the two roots of the equation (11.112) are

$\lambda_1 = \lambda_2 = -\dfrac{c}{2m} = -\alpha$. The general solution is

$$y(t) = (c_1 + c_2 t)e^{-\alpha t}. \qquad ...(11.114)$$

Here also $y(t) \to 0$ as $t \to \infty$, as in the case of over damping. This case marks the boundary between the over damped behaviour discussed above and the oscillatory behaviour to be discussed next.

Case III: Under damping. If the damping coefficient is so small that $c^2 < 4mk$, then the roots of the Eq. (11.114) are complex conjugate, say $\lambda_1 = -\alpha + i\beta^\bullet$ and $\lambda_2 = -\alpha - i\beta^\bullet$, where

$\alpha = \dfrac{c}{2m}$, and $\beta^\bullet = \dfrac{1}{2m}\sqrt{4km - c^2}$. The general solution of Eq. (11.112) is

$$y(t) = e^{-\alpha t}(c_1 \cos \beta^\bullet t + c_2 \sin \beta^\bullet t). \qquad ...(11.115)$$

Since $\alpha > 0$, thus $y(t) \to 0$ as $t \to \infty$. However, the motion is now oscillatory because of the sine and cosine terms in the solution. But it is not the periodic one because of the exponential factors which causes the amplitude of the oscillations to decay to zero as t becomes sufficiently large. The Eq. (11.115) can be expressed as

$$y(t) = ce^{-\alpha t}\cos(\beta^\bullet t - \theta), \text{ where } c = \pm\sqrt{c_1^2 + c_2^2}, \text{ and } \theta = \tan^{-1}\frac{c_2}{c_1}.$$

Thus the solution curve lies between $\pm ce^{-\alpha t}$, touching these curves when $(\beta^\bullet t - \theta)$ is an integral multiple of π, as shown in Fig. 11.6.

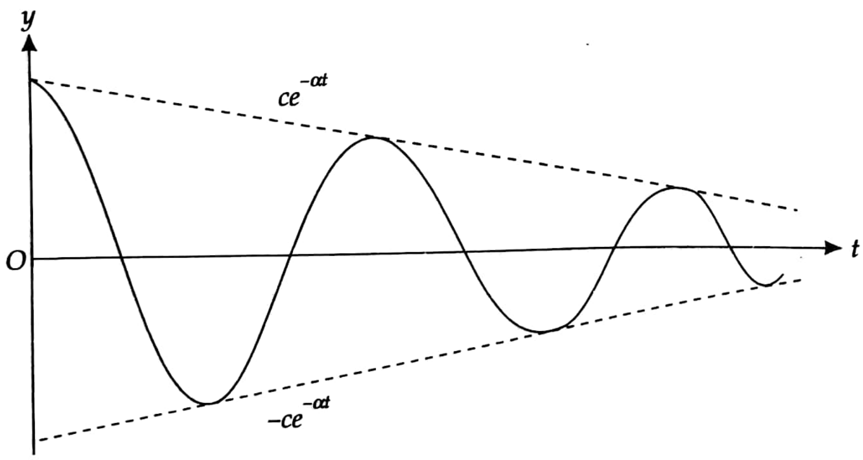

Fig. 11.6

Example 11.44: A weight of about 89.0 Newton stretches a spring by 10.0 cm. How many cycles per second will this mass-spring system execute? What will be its equation of motion in case the weight is pulled down by 15.00 cm from its position of static equilibrium and then released, ignore the damping forces. How does the motion will change if the system has damping given by
(i) $c = 200.0$ kg/sec (ii) $c = 179.81$ kg/sec (iii) $c = 100.0$ kg/sec?

Solution: If k is the coefficient of stiffness for the spring, then using Hookes' law, we have

$89.0 = 0.1\,k$, (10.0 cm = 0.1 meter), hence $k = 890$ N/meter. Also mass,

$m = w/g = 89.0/9.80 = 9.082$ kg. Thus frequency, $\dfrac{w}{2\pi} = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{m}} = \dfrac{1}{2\pi}\sqrt{\dfrac{890}{9.082}} = 1.576.$

If $y(t)$ denotes the displacement of the body at any instant from its position of static equilibrium, then ignoring the damped forces, motion is described by the initial value problem

$$y'' + w^2y = 0,\ y(0) = 0.15,\ y'(0) = 0;\ w = \sqrt{k/m} = 9.899. \quad \text{...(11.116)}$$

We can check that the solution of (11.116) is $y(t) = 0.1500 \cos 9.899t$.

In case the system has damping, then the spring equation is $my'' + cy' + ky = 0$.

(i) When $c = 200.0$, the spring equation for $m = 9.082$ and $k = 890$ becomes

$$9.082y'' + 200.0y' + 890.0y = 0, \quad \text{...(11.117)}$$

The characteristic equation has the roots $\lambda = -11.01 \pm 4.82 = -6.190, -15.83$.

The general solution is

$$y(t) = c_1e^{-6.19t} + c_2e^{-15.83t} \quad \text{...(11.118)}$$

Using the initial conditions $y(0) = 0.15$, and $y'(0) = 0$, Eq. (11.118) becomes

$$y(t) = 0.2463e^{-6.190t} - 0.0963e^{-15.83t}.$$

Thus, $y(t)$ tends to zero as $t \to \infty$. It is the case of *over damping*.

(ii) When $c = 179.81$, then $c^2 = 4mk$ and hence the characteristic equation has the double root $\lambda = -9.899$. Therefore the solution is

$$y(t) = (c_1 + c_2t)e^{-9.899t}. \quad \text{...(11.119)}$$

Using the initial conditions $y(0) = 0.15$, and $y'(0) = 0$, (11.119) becomes

$$y(t) = (0.150 + 1.48t)e^{-9.899t}.$$

In this case also $y(t) \to 0$ as $t \to \infty$. It is the case of *critical damping*

(iii) When $c = 100$, the roots of the characteristic equation are $\lambda = -5.506 \pm 8.227i$, the complex conjugate. The general solution is

$$y(t) = e^{-5.506t}(c_1 \cos 8.227t + c_2 \sin 8.227\ t). \quad \text{...(11.1120)}$$

Using the initial conditions $y(0) = 0.15$ and $y'(0) = 0$, (11.120) becomes

$y(t) = e^{-5.506t}(0.1500 \cos 8.227t + 0.1004 \sin 8.227t) = 0.1805e^{-5.506t} \cos (8.227t - 0.981).$

It is a case of *damped oscillation* with frequency, $w/2\pi = \dfrac{8.227}{2\pi} = 1.309.$

11.10.3 Mass-Spring System: Forced Motions

Now suppose that an external deriving force of magnitude $f(t)$ acts on the body. Different forces will cause different kinds of motions. Of practical interest are periodic deriving force of the type $f(t) = A \cos wt$, where $A > 0$ and $w > 0$ are constants. Then the spring equation (11.108) becomes

$$my'' + cy' + ky = A \cos wt. \qquad ...(11.121)$$

A general solution of the non-homogeneous Eq. (11.121) is the sum of the complementary function $y_c(t)$, a solution of the corresponding homogeneous equation given by

$$y_c(t) = \begin{cases} e^{-\frac{ct}{2m}}\left[c_1 \cos\sqrt{w_0^2 - \left(\frac{c}{m}\right)^2}\, t\right], c^2 < 4mk \text{ (underdamped)} \\ e^{-\frac{ct}{2m}}[c_1 + c_2 t], c^2 = 4mk \text{ (critically damped)} \\ e^{-\frac{ct}{2m}}\left[c_1 \cosh\sqrt{\left(\frac{c}{2m}\right)^2 - w_0^2}\, t\right], c^2 > 4mk \text{ (overdamped)} \end{cases} \qquad ...(11.122)$$

here $w_0 = \sqrt{k/m}$; and a particular solution $y_p(t)$ of the Eq. (11.121), which we derive next.

To determine $y_p(t)$ we use the method of undetermined coefficients and hence try a solution of the from

$$y_p(t) = a \cos wt + b \sin wt, \qquad ...(11.123)$$

where a and b are the coefficients to be determined.

Substituting this in (11.121) and comparing the coefficients of cos wt and sin wt on both sides, we have $(k - mw^2)a + wcb = A$, and $-wca + (k - mw^2)b = 0$. Solving these for a and b, we obtain

$$a = \frac{A(k - mw^2)}{(k - mw^2)^2 + w^2c^2} \text{ and } b = \frac{Awc}{(k - mw^2)^2 + w^2c^2}, \text{ provided } (k - mw^2)^2 + w^2c^2 \neq 0.$$

Set $w_0 = \sqrt{k/m}$. Then particular solution is

$$y_p(t) = \frac{mA(w_0^2 - w^2)}{m^2(w_0^2 - w^2)^2 + w^2c^2} \cos wt + \frac{Awc}{m^2(w_0^2 - w^2)^2 + w^2c^2} \sin wt, \qquad ...(11.124)$$

provided $w \neq w_0$, or $c \neq 0$.

We shall now discuss the behaviour of the spring mass system with this deriving force, distinguishing between the two cases $c = 0$, the undamped system, and $c > 0$, the damped system.

A. Damped forced motion

The particular solution (11.124) can be expressed in the form

$$y_p = B\cos(wt - \theta), \qquad \text{...(11.125)}$$

where the amplitude B and angle θ are given by

$$B = \frac{A}{\sqrt{m^2(w_0^2 - w^2)^2 + w^2c^2}}, \text{ and } \theta = \tan^{-1}\frac{wc}{m(w_0^2 - w^2)}, \quad 0 < \theta < \pi.$$

We observe that $y_c(t)$ part of the solution tends to zero as t goes to infinity because of the exp.$(-ct/2m)$ factor as long as $c > 0$, no matter how small it is. Practically it is zero after a sufficient long time. Thus, $y_c(t)$ is the *transient part* of the solution. The part $y_p(t)$ is the *steady-state part* since $y(t) \to B\cos(wt - \theta)$ as $t \to \infty$. Hence, after a sufficiently long time the output corresponding to a purely sinusoidal input will practically be a harmonic oscillation whose frequency is that of the input, and this is what happens in almost all cases since damped forces are never zero.

B. Undamped forced motion (resonance)

In the absence of the damping forces, $c = 0$. The spring equation becomes

$$my'' + ky = A\cos wt. \qquad \text{...(11.126)}$$

The general solution of this equation is

$$y(t) = c_1\cos w_0t + c_2\sin w_0t + \frac{A\cos wt}{m(w_0^2 - w^2)}, \qquad \text{...(11.127)}$$

where $w_0 = \sqrt{k/m}$ is called the *natural frequency* of the mass-spring system, and w is the *input frequency* to the system. The general solution (11.127) assumes that $w \neq w_0$.

The output represents a superposition of two harmonic oscillations with frequency $w_0/2\pi$ of the system and the frequency $w/2\pi$ of the input.

Consider the case when the input frequency matches the natural frequency of the system, that is, $w = w_0$ then the spring equation becomes

$$my'' + ky = A\cos w_0t, \qquad \text{...(11.128)}$$

and (11.127) is no longer its solution.

The complementary function of (11.128) is $y_c(t) = c_1\cos w_0t + c_2\sin w_0t$.

To find particular solution we use method of undetermined coefficients. Since the right side of Eq. (11.128) contains a term appearing in the complementary function, we attempt a function of the form

$$y_p(t) = at\cos w_0t + bt\sin w_0t. \qquad \text{...(11.129)}$$

Substituting (11.129) in (11.128) and comparing the coefficients of $\sin w_0t$ and $\cos w_0t$ on both sides, we get $a = 0$, and $b = A/2mw_0$, and thus, (11.129) becomes $y_p(t) = \dfrac{A}{2mw_0} t \sin w_0 t$, and hence the general solution of Eq. (11.128) is

$$y(t) = c_1 \cos w_0 t + c_2 \sin w_0 t + \frac{A}{2mw_0} t \sin w_0 t. \quad ...(11.130)$$

In this special case the response of $y_p(t)$ is not harmonic oscillation but t times a harmonic function which causes the magnitude to tend to infinity as $t \to \infty$, as shown in Fig. 11.7.

We observe that $y_p(t)$ becomes larger and larger. In practice this means that systems with very little damping may undergo large vibrations. This phenomena is called *resonance*.

Resonance is sometimes desirable and sometimes undesirable. For example, we may wish to amplify a given input when we tune a radio circuit to a desired broadcast frequency. But we may wish to suppress inputs from a bumpy road to an automobile. Practically we can never be equal to w_0. It may therefore be interesting to observe the case when w approaches to w_0.

Let the initial conditions be $y(0) = 0$ and $y'(0) = 0$. Using these conditions in (11.127) to evaluate c_1 and c_2, we obtain

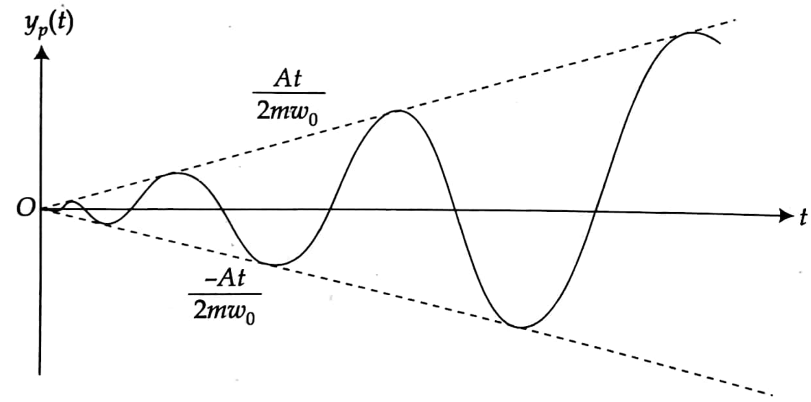

Fig. 11.7

$$y(t) = \frac{A}{m(w_0^2 - w^2)} (\cos wt - \cos w_0 t).$$

$$= \frac{2A}{m(w_0^2 - w^2)} \sin\left(\frac{w_0 + w}{2} t\right) \sin\left(\frac{w_0 - w}{2} t\right). \quad ...(11.131)$$

Since the difference $(w_0 - w)$ is small, so the period of the second sinusoid in (11. 131) is large. This results in a periodic variation of amplitude in the outcome $y(t)$ depending on the relative sizes of $w_0 + w$ and $w_0 - w$, as shown in Fig. 11.8. This periodic variation is called a *'beat'* and this is what which interests musicians while tuning their instruments.

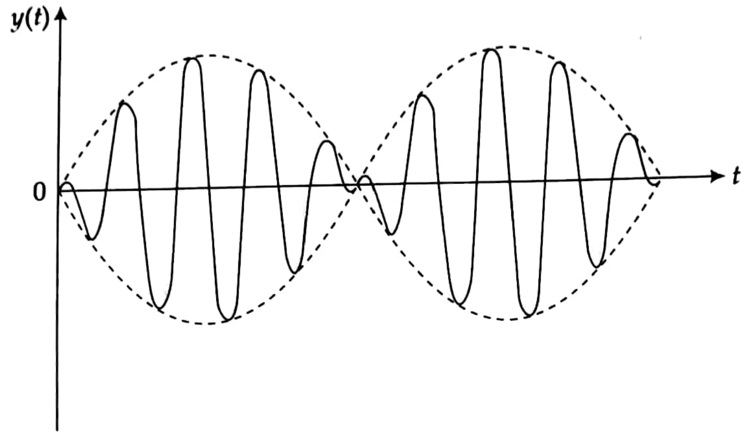

Fig. 11.8

Example 11.45: An 8 lb weight is placed at one end of a spring suspended from the ceiling. The weight is raised to 5 inches above the static equilibrium position and is released. Assuming the spring constant 12 lb./ft, find the equation of motion, displacement $y(t)$, amplitude, period and maximun velocity.

Solution: If $y(t)$ is the displacement of the mass with respect to its position of static equilibrium, then the equation of motion is

$$m\frac{d^2y}{dt^2} + ky = 0, \qquad \text{...(11.132)}$$

where m is the mass suspended and k is the spring constant.

Here, $m = \frac{8}{32} = \frac{1}{4}$, $k = 12$. The solution of Eq. (11.132) is

$$y(t) = c_1 \cos wt + c_2 \sin wt \qquad \text{...(11.133)}$$

where $w = \sqrt{k/m} = \sqrt{48} = 4\sqrt{3}$.

Using the initial conditions, $y(0) = -5/12$ and $y'(0) = 0$, Eq. (11.133) becomes

$$y(t) = -\frac{5}{12}\cos 4\sqrt{3}\,t = \frac{5}{12}\sin\left(4\sqrt{3}t - \frac{\pi}{2}\right).$$

Thus amplitude $A = \frac{5}{12}$ ft, period $T = \frac{2\pi}{w} = \frac{2\pi}{4\sqrt{3}} = \frac{\pi\sqrt{3}}{6}$ sec., and the velocity

$y'(t) = \frac{5\sqrt{3}}{3}\cos\left(4\sqrt{3}t - \frac{\pi}{2}\right)$; the maximum velocity is $\frac{5\sqrt{3}}{3}$ ft/sec.

Example 11.46: A weight of 980 gm is suspended at the lower end of a spring which is fixed at its upper end. The weight is pulled down 1/4 cm below its static equilibrium position and then released. In case the resistance (in gm. wt) to the motion of the weight is 1/10 of the velocity of the weight in cm/sec, write the equation of motion, displacement $y(t)$ and also the time it takes the damping factor to drop to 1/10 of its initial value. Assume that spring constant is 20 gm/cm.

Solution: If $y(t)$ is the displacement of the body at any time t from its position of static equilibrium, then the equation of motion is $my'' + cy' + ky = 0$.

Here, $m = \dfrac{980}{g} = \dfrac{980}{980} = 1$ gm, $c = \dfrac{1}{10}$, $k = 20$ gm/cm. Thus the equation becomes

$$10y'' + y' + 200y = 0. \quad \text{...(11.134)}$$

Solution of (11.134) is $y(t) = e^{-0.05t}\,[c_1 \cos (4.5)t + c_2 \sin (4.5)t]$.

Using the initial conditions $y(0) = \dfrac{1}{4}$, $y'(0) = 0$, we obtain

$$y(t) = e^{-0.05t}\,[0.25 \cos (4.5)t + 0.003 \sin (4.5)t]$$

$$= 0.25e^{-0.05t} \cos (4.5t - \theta), \text{ where } \theta = \tan^{-1}(0.012),\ 0 < \theta < \pi/2. \quad \text{...(11.135)}$$

Since $\cos (4.5t - \theta)$ lies between ± 1, thus the displacement $y(t)$ lies between the curves $y = \pm\, 0.25\, e^{-0.05t}$. Also the damping factor in (11.135) is $0.25e^{-0.05t}$. At $t = 0$ initial value is 0.25. If t is the time the damping factor reduces to its (1/10)th of its value, then

$$0.25e^{-0.05t} = 0.025, \text{ which gives, } t = 20 \ln 10 = 46 \text{ sec.}$$

Example 11.47: A weight of 16 lb. is suspended from a spring with spring constant 5 lb/ft. The system is subjected to an external force 24 sin 10 t and a damping force equal to 5 times the velocity of the weight. Find the displacement of the weight at any time t, if initially the weight is at rest at its equilibrium position. Describe the transient and steady state solutions.

Solution: If $y(t)$ is the displacement of the body at any time t from its position of static equilibrium, then the equation of motion is $my'' + cy' + ky = f(t)$.

Here, $m = w/g = 16/32 = (1/2)$ lb, $c = 5$, $k = 5$ lb/ft. and $f(t) = 24 \sin 10t$.

Substituting these values, equation of motion becomes

$$y'' + 10y' + 10y = 48 \sin 10t. \quad \text{...(11.136)}$$

The complementary function is $y_c(t) = c_1e^{-1.13t} + c_2e^{-8.87t}$.

The particular integral is $y_p(x) = \dfrac{1}{D^2 + 10D + 10}\, 48 \sin 10t = -\dfrac{240}{905} \cos 10t - \dfrac{216}{905} \sin 10t$

$$= -(0.265 \cos 10t + 0.238 \sin 10t).$$

Thus the complete solution of Eq (11.136) is

$$y(t) = c_1e^{-1.13t} + c_2e^{-8.87t} - (0.265 \cos 10t + 0.238 \sin 10t).$$

Using the initial conditions $y(0) = y'(0) = 0$, it gives $c_1 = 0.663$ and $c_2 = -0.398$. Thus

$$y_c(t) = 0.663e^{-1.13t} - 0.398e^{-8.87t} \qquad ...(11.137)$$

Further the particular integral $y_p(t)$ can be expressed as

$$y_p(t) = 0.356 \sin (10t + 3.982). \qquad ...(11.138)$$

Thus, the displacement of the weight at any time t is $y(t) = y_c(t) + y_p(t)$ where $y_c(t)$ and $y_p(t)$ are given by (11.137) and (11.138) respectively. We observe that $y_c(t)$ represents the transient solution and tends to zero as $t \to \infty$, while $y_p(t)$ represents the steady state solution and is a harmonic oscillation with amplitude 0.356 and time period $2\pi/10 = \pi/5$ sec. Since $y_c(t) \to 0$ as $t \to \infty$, thus after a sufficiently long time, the output becomes the sinusoidal wave $0.356 \sin (10t + 3.982)$.

Example 11.48: A weight of 6 lb is suspended from a spring with spring constant 12 lb./ft. and an external force $3 \cos 8t$ acts on the weight. Describe the motion under the assumption of no damping force when initially the body is at rest in its position of static equilibrium.

Solution: Let $y(t)$ be displacement of the body at time t, then motion is described by the equation

$$\frac{6}{32}y'' + 12y = 3 \cos 8t \text{ or, } y'' + 64y = 16 \cos 8t \qquad ...(11.139)$$

with initial conditions $y(0) = y'(0) = 0$.

The C.F. of (11.139) is $y_c(t) = c_1 \cos 8t + c_2 \sin 8t$.

To find the particular integral we use the method of undetermined coefficients and let $y_p(t)$ be

$$y_p(t) = at \cos 8t + bt \sin 8t, \qquad ...(11.140)$$

where a and b are coefficients to be determined. Substituting (11.140) in (11.139), and comparing the coefficients of the corresponding terms, we get $a = 0$, $b = 1$, thus $y_p(t) = t \sin 8t$. Hence the complete solution is

$$y(t) = c_1 \cos 8t + c_2 \sin 8t + t \sin 8t.$$

Using the initial conditions $y(0) = y'(0) = 0$, we obtain $c_1 = c_2 = 0$, and hence the displacement is given by $y(t) = t \sin 8t$.

Thus the displacement is t times the harmonic oscillation $\sin 8t$ which causes the magnitude to tend to infinity as $t \to \infty$, as shown in Fig. 11.9, and hence, the resonance occurs.

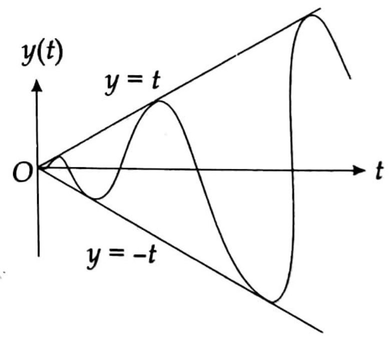

Fig. 11.9

EXERCISE 11.8

1. Show that the frequency of a harmonic oscillation of a body on a spring is $\left(\sqrt{g/l}\right)/2\pi$, where l is the elongation.
2. A body weight 10 kg is suspended from a spring with spring modulus 200 kg/metre. The body is pulled down to 0.20 metres below its position of static equilibrium and then released. Find the displacement of the body from its position of equilibrium at any time t, the maximum velocity and the period of oscillation.
3. A light elastic string of natural length l has one extremity fixed at a point A and other end attached to a stone, the weight of which in equilibrium extends the string to a depth l_1. Show that if the stone be dropped from rest at A, it will come to instantaneous rest at a depth $\sqrt{(l_1^2 - l^2)}$ below the equilibrium position.
4. A 2 lb. weight suspended from one end of a spring stretches it to (1/2)ft. A velocity of 5 ft/sec^2 upward is imparted to the weight at its position of equilibrium. In case the damping force is c, $(c > 0)$, times the velocity of the weight at time t, find the displacment of the weight at any time t. Find the amplitude, period and maximum velocity of the motion. Find the values of c for which the system is damped, overdamped or underdamped.
5. A mass m_1 is attached to a spring and allowed to vibrate with undamped motion having period p. At some later time, a second mass m_2 is instantaneously fused with m_1. Prove that the new object having mass $(m_1 + m_2)$ exhibits simple harmonic motion with period $p/\sqrt{1+m_2/m_1}$.
6. Let $y(t)$ be the solution of $y'' + w_0^2 y = (A/m)\cos wt$, with $y(0) = y'(0) = 0$. Assuming that $w \neq w_0$, find $\lim_{w \to w_0} y(t)$. How does this limit compare with the solution of $y'' + w_0^2 y = (A/m)\cos(w_0 t)$, with $y(0) = y'(0) = 0$?
7. Consider damped force motion governed by $my'' + cy' + ky = A\cos wt$. Show that the maximum amplitude of the steady-state solution is achieved, if $w = \dfrac{1}{m\sqrt{2}}\sqrt{2km - c^2}$.
8. Show that the damped forced motion of a mass m on a spring with forcing function A $\cos wt$ is always bounded in magnitude.
9. A body weighting 16 lb. is suspended by a spring in a fluid whose resistance in lb. wt. is twice the velocity of the body in ft./sec. A pull of 25 lb. wt would stretch the spring by 3 inches. The body is drawn 3 inches below the equilibrium position in the position and then released. Find the period of oscillation and the time required for the damping force to be reduced to (1/10)th of its initial value.
10. A body weighing 4 lb. hangs at rest at the lower end of a spring producing an extension of 1 ft. The upper end of the spring is subjected to a driving force $f(t) = \sin 4t$. If the body is subjected to a damping force (1/4) times its velocity in ft/sec, find the expression for the displacement of the body at time t, when t is large and explain it.

11.11 MODELLING R-L-C ELECTRICAL CIRCUIT: ANALOGY WITH MASS-SPRING SYSTEM

In Chapter 10, we have considered the applications of the linear first order differential equations to an R-L series circuit and an R-C series circuit. If a circuit contains a resistance R, inductance L, capacitance C, an electromotive force $E(t)$, and if, $I(t)$ is the current flowing in the circuit at any time t, then the voltage drop around the circuit, refer Fig. 10.10, by Kirchoff's law, satisfies

$$LI'(t) + RI(t) + \frac{1}{C}\int I dt = E(t). \qquad ...(11.141)$$

If $Q(t)$ is the charge, and since $Q'(i) = I(t)$, the Eq. (11.141) can be written as

$$LQ''(t) + RQ'(t) + \frac{1}{C}Q(t) = E(t). \qquad ...(11.142)$$

If L, R, and C are constants, then (11.142) is a second order linear differential equation of the type which we have already solved for various choices of $E(t)$.

11.11.1 An Analogy with Mass-Spring System

It is interesting to observe that Eq. (11.142) is of exactly the same form as the Eq. (11.108) for the displacement $y(t)$ of an object of mass m units suspended to a spring with spring modulus k, which is

$$my'' + cy' + ky = f(t). \qquad ...(11.143)$$

Here c is the damping force and $f(t)$ is the driving force to which the mass-spring system is subjected.

This provides an appropriate example of the important mathematical fact that entirely different physical systems may lead to the same mathematical model and thus can be solved by the same methods. This analogy between mechanical and electrical systems facilitates us to construct an electric circuit whose current will give the exact values of the displacement in the mechanical system when suitable scale factors are introduced. This may be of practical importance due to the fact that electric circuits are far easy to assemble and observe as compared to the mechanical systems.

The forms of the two Eqs. (11.142) and (11.143) suggest the following analogy between the two systems:

Displacement function	$y(t)$	$\leftrightarrow$ charge $Q(t)$
Velocity	$y'(t)$	$\leftrightarrow$ current $I(t)$
Driving force	$f(t)$	$\leftrightarrow$ e.m.f. $E(t)$
Mass	m	$\leftrightarrow$ inductance L
Damping force	c	$\leftrightarrow$ resistance R
Spring modulus	k	$\leftrightarrow$ reciprocal of the capacitance $1/C$,

Further the Eq. (11.141) can be written as

$$L\,I''(t) + RI'(t) + \frac{1}{C}I = \frac{dE}{dt}. \qquad ...(11.144)$$

This equation is used more often since in most practical problems we need to find current $I(t)$ instead of the charge $Q(t)$. Further an RLC-circuit reduces to an RL-circuit in the absence of a capacitor and to an RC-circuit in the absence of an inductor. Generally the e.m.f. is of the form of harmonic oscillations $E_0 \cos wt$ or $E_0 \sin wt$.

Next we consider the following different cases.

11.11.2 Free Oscillations in an Electric Circuit

LC-Circuit. Consider an electrical circuit containing a condenser of capacity C and an inductance L as shown in Fig. 11.10. In case no e.m.f. is employed in the circuit, then Eq. (11.142) gives

$$LQ''(t) + \frac{1}{C}Q(t) = 0 \text{ or, } Q''(t) + w^2 Q(t) = 0 \qquad ...(11.145)$$

where $w^2 = \dfrac{1}{LC}$.

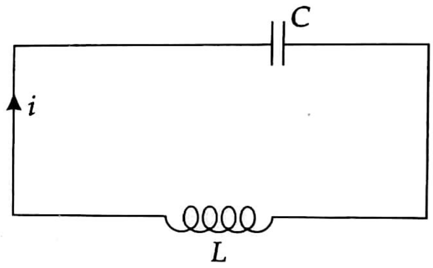

Fig. 11.10

This equation is similar to the Eq. (11.110) in case of mass-spring system, simply replacing the displacement $y(t)$ by the charge $Q(t)$. It represents free electrical oscillations. Thus, the discharging of a condenser through an inductance L is the same as the motion of an object of mass m suspended at the end of a spring.

LCR-circuit. Next consider the discharge of a condenser C through an inductance L and the resistance R as shown in Fig. 11.11.

In case no e.m.f. is employed, then Eq. (11.142) gives

$$LQ''(t) + RQ'(t) + \frac{1}{C}Q(t) = 0. \qquad ...(11.146)$$

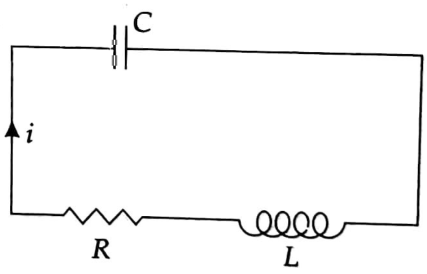

Fig. 11.11

The equation is similar to the Eq. (11.112) of free motions of a mass-spring system when the system is damped and hence has the same solution as for the object of mass m suspended at the end of a spring with spring modulus k and damping c.

Similarly, in case of *LC–circuit with e.m.f.* $E_0 \cos wt$, the equation giving the charge $Q(t)$ at time t is

$$LQ''(t) + \frac{1}{C}Q(t) = E_0 \cos wt. \qquad ...(11.147)$$

It is the same as Eq. (11.126), a case of undamped forced motion of mass-spring system and, therefore, has the same solution as for the motion of a mass m suspended at the end of a spring with a driving force $E_0 \cos wt$ acting on it.

Also in the case of *LCR-circuit with e.m.f.* $E_0 \cos wt$, equation giving the charge $Q(t)$ at time t

$$LQ''(t) + RQ'(t) + \frac{1}{C} Q(t) = E_0 \cos wt \qquad ...(11.148)$$

is the same as Eq. (11.121), a case of damped forced motion as of mass-spring system and hence can be solved accordingly.

Example 11.49: For the circuit as shown in Fig. 11.12, find the charge $Q(t)$ and the current $I(t)$ at time t, if at time $t = 0$ the current is zero and the charge on the capacitor is $1/1000$ coulomb.

Solution: If $Q(t)$ is the charge on the capacitor for $t > 0$, then by Kirchoff's law,

$$10\, Q'' + 120Q' + 1000Q = 17 \sin 2t. \qquad ...(11.149)$$

The C. F. is $Q_c(t) = e^{-6t}[c_1 \cos 8t + c_2 \sin 8t]$.

The P.I. is,

$$Q_p(t) = \frac{1}{10D^2 + 120D + 1000}(17 \sin 2t) = 17\frac{1}{120D + 960}\sin 2t$$

$$= -\frac{1}{480}(D - 8) \sin 2t = -\frac{1}{240}(\cos 2t - 4 \sin 2t)$$

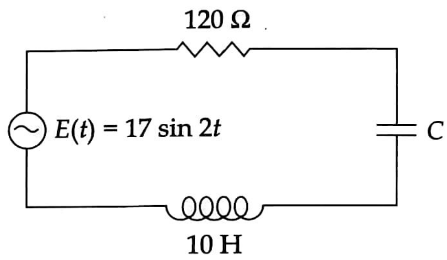

Fig. 11.12

Hence the solution of Eq. (11.149) is

$$Q(t) = e^{-6t}(c_1 \cos 8t + c_2 \sin 8t) - \frac{1}{240}(\cos 2t - 4 \sin 2t).$$

Using the initial conditions, $Q(0) = \frac{1}{2000}$ and $Q'(0) = 0$, we obtain $c_1 = \frac{7}{1500}$, $c_2 = \frac{-1}{1500}$, and hence the charge is

$$Q(t) = \frac{1}{1500} e^{-6t}[7 \cos 8t - \sin 8t] - \frac{1}{240}[\cos 2t - 4 \sin 2t]. \qquad ...(11.150)$$

The current, $I(t) = Q'(t) = -\frac{1}{30} e^{-6t}[\cos (8t) + \sin (8t)] + \frac{1}{120}[4 \cos 2t + \sin 2t]. \qquad ...(11.151)$

We observe that the current as given by (11.151) is a sum of a transient part, $-\frac{1}{30} e^{-6t}[\cos (8t) + \sin (8t)]$, which decays to zero as t increase, and a steady-part, $\frac{1}{120}[4 \cos 2t + \sin 2t]$, which is periodic.

Thus, after a long time, the output will be harmonic oscillation given by the steady part.

Example 11.50: Determine the current $I(t)$ in an RLC-circuit with e.m.f. $E(t) = E_0 \sin wt$, in case the circuit is tuned to resonance so that $w^2 = 1/LC$ and R/L is so small that second and higher order terms can be rejected. Assuming that at $t = 0$, $I(0) = I'(0) = 0$.

Solution: The differential equation giving the current in an RLC-circuit with e.m.f. $E_0 \sin wt$ is

$$L\frac{d^2I}{dt^2}+R\frac{dI}{dt}+\frac{1}{C}I=\frac{dE}{dt}=wE_0\cos wt. \qquad ...(11.152)$$

The roots of the auxiliary equation $L\lambda^2+R\lambda+\frac{1}{C}=0$, are

$$\lambda=-\frac{R}{2L}\pm\frac{1}{2L}\sqrt{R^2-\frac{4L}{C}}=-\frac{R}{2L}\pm\sqrt{\frac{R^2}{4L^2}-\frac{1}{LC}}\approx-\frac{R}{2L}\pm i\frac{1}{\sqrt{LC}}=-\frac{R}{2L}\pm iw.$$

Hence, the C.F. is $I_c(t)=e^{-\frac{R}{2L}t}(c_1\cos wt+c_2\sin wt)\simeq\left(1-\frac{R}{2L}t\right)(c_1\cos wt+c_2\sin wt)$,

neglecting terms of second and higher order in $\frac{R}{L}$.

The particular integral is

$$I_p(t)=\frac{1}{LD^2+RD+\frac{1}{C}}wE_0\cos wt=\frac{wE_0}{-Lw^2+RD+\frac{1}{C}}\cos wt$$

$$=\frac{wE_0}{RD}\cos wt,\text{ since }w^2=\frac{1}{LC}$$

$$=\frac{wE_0}{R}\int\cos wt\,dt=\frac{E_0}{R}\sin wt.$$

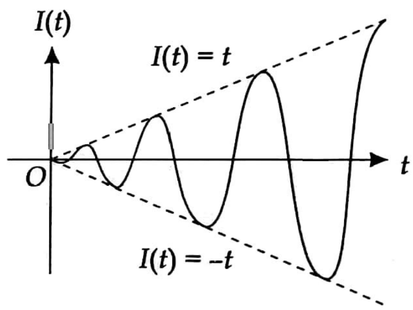

Fig. 11.13

Hence the general solution of Eq. (11.152) is

$$I(t)=\left(1-\frac{R}{2L}t\right)(c_1\cos wt+c_2\sin wt)+\frac{E_0}{R}\sin wt. \qquad ...(11.153)$$

The initial conditions $I(0)=0$, and $I'(0)=0$ gives $c_1=0$, and $c_2=-\frac{E_0}{R}$. Substituting in (11.153), we obtain $I(t)=\frac{E_0}{2L}t\sin wt$, which is t times the harmonic oscillation $\frac{E_0}{2L}\sin wt$, which causes the current to increase indefinitely as t increases as shown in Fig. 11.13 and hence resonance occurs.

EXERCISE 11.9

1. Show that the frequency of free vibrations in a closed electrical circuit with inductance L and capacitance C in series is $\frac{30}{\pi\sqrt{LC}}$ per minute.

2. An e.m.f. $E_0 \sin wt$ is applied at $t = 0$ to a circuit containing a capacitance C and inductance L. If $w^2 = 1/LC$ and initially the current I and the charge Q are zero, then show that the current at time t is $(E_0 t/2L) \sin wt$.
3. A circuit consists of an inductance of 0.05 H, a resistance of 5 Ω and a condensor of 4×10^{-4} C. If $Q(0) = I(0) = 0$, find $Q(t)$ and $I(t)$ when, (a) there is a constant e.m.f. of 110 V, (b) there is an alternating e.m.f. of 200 cos 100t; also find the study state solution in this case.
4. Find the current $I(t)$ in the LC circuit, assuming zero initial current and charge, with the following data:
   (a) $L = 10$ H, $C = 0.1$ F, $E = 10t$ V
   (b) $L = 2$ H, $C = 0.005$ F, $E = 220 \sin 4t$ V
5. What RLC circuit with $L = 1$ H is the analog of the mass-spring system with mass 2 kg, damping constant 20 kg/sec, spring constant 58 kg/sec^2, and driving force 110 cos 5t N?
6. Find the current in the RLC circuit, assuming zero initial current and capacitor charge, with the following data:
   (a) $R = 400\ \Omega$, $L = 0.12$ H, $C = 0.04$ F, $E(t) = 120 \sin 2t$ V
   (b) $R = 450\ \Omega$, $L = 0.95$ H, $C = 0.007$ F, $E(t) = e^{-t} \sin^2 3t$ V

11.12 MODELLING: BENDING OF ELASTIC BEAMS

In this section we consider an application of fourth-order differential equation in case of bending of an elastic beam such as that of wood or iron girder in a building or a bridge.

Consider a beam B of uniform elastic material with length L and of constant, say rectangular cross-section. Let the beam B is subjected to a uniform load in a vertical plane through the x-axis, the axis of symmetry, and as a result the beam is bent and its axis is curved into the elastic curve $y = y(x)$, as shown in Fig. 11.13.

Using the theory of elasticity it can be shown that the bending moment $M(x)$ is proportional to the curvature of the curve $y = y(x)$, that is,

$$M(x) = EI\frac{d^2y/dx^2}{[1+(dy/dx)^2]^{3/2}}, \qquad ...(11.154)$$

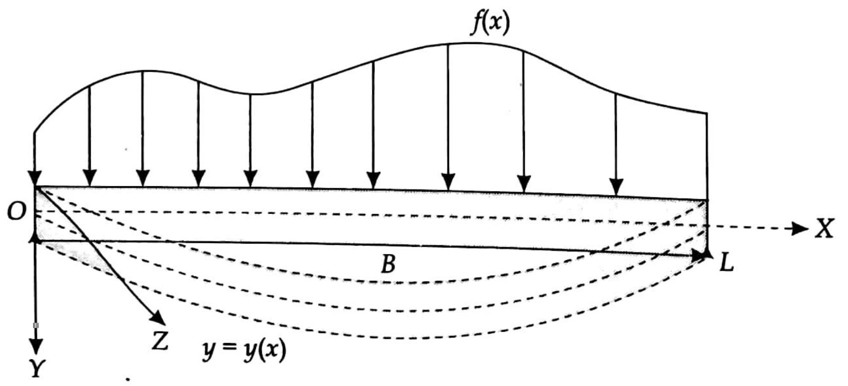

Fig. 11.14

where EI is the constant of proportionality called the *flexural rigidity of the beam*. Here E is the Young's modulus of the beam material and I is the moment of inertia of cross-section about the z-axis.

If $f(x)$ is the load per unit length acting along the beam creating a total load of $\int_a^b f(x)\,dx$ on the segment $a \le x \le b$, then it can be shown that

$$\frac{d^2M}{dx^2} = f(x), \qquad \text{...(11.155)}$$

and hence, from (11.154)

$$\frac{d^2}{dx^2}\left\{\frac{EI\, d^2y/dx^2}{[1+(dy/dx)^2]^{3/2}}\right\} = f(x). \qquad \text{...(11.156)}$$

In case of small bending the term $\left(\dfrac{dy}{dx}\right)^2$ can be neglected, and hence, (11.155) reduces to

$$EI\frac{d^4y}{dx^4} = f(x). \qquad \text{...(11.157)}$$

It is the *equation of deflection for the uniform elastic beam* under the load density of $f(x)$ units. Also in case of small bending, from (11.154), the *bending moment* is

$$M(x) = EI\frac{d^2y}{dx^2}. \qquad \text{...(11.158)}$$

Differentiating, we get

$$\frac{dM}{dx} = EI\frac{d^3y}{dx^3}, \qquad \text{...(11.159)}$$

as the *shear force*, and differentiating it again, we get

$$\frac{d^2M}{dx^2} = EI\frac{d^4y}{dx^4}, \qquad \text{...(11.160)}$$

as the *intensity of loading* of the beam

Next we mention a few important supports and the corresponding boundary conditions:

(a) *Simply supported ends*

In this case there is no displacement and no bending moment at the points $x = 0$ and $x = L$.

Thus, $y(0) = y(L) = 0$, and $y''(0) = y''(L) = 0$.

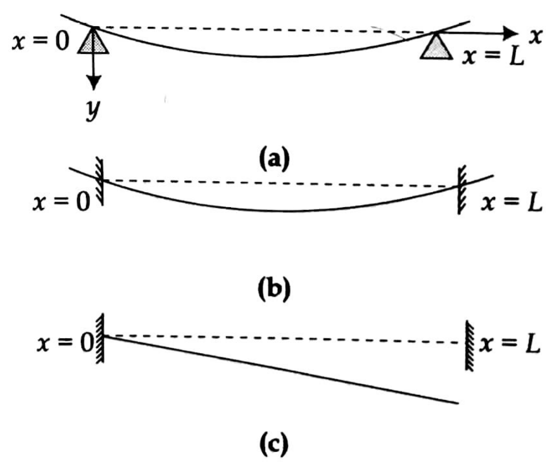

Fig. 11.15

(b) *Clamped at both ends*
In this case there is no deflection and no slope at the points $x = 0$ and $x = L$. Thus,

$$y(0) = y(L) = 0 \quad \text{and} \quad y'(0) = y'(L) = 0.$$

(c) *Clamped at $x = 0$ and free at $x = L$ (Cantilever)*
In this case there is displacement and no slope at the point $x = 0$ and no bending moment and no shear force at the point $x = L$. Thus,
$y(0) = y'(0) = 0$, and $y''(L) = y'''(L) = 0$.

As another example of the applications of higher order equations, the differential equation

$$EI\,\frac{d^4y}{dx^4} + ky = f(x) \qquad ...(11.161)$$

governs the deflection $y(x)$ of a beam that rests upon an elastic foundation under a load density of $f(x)$ units, as shown in Fig. 11.16.

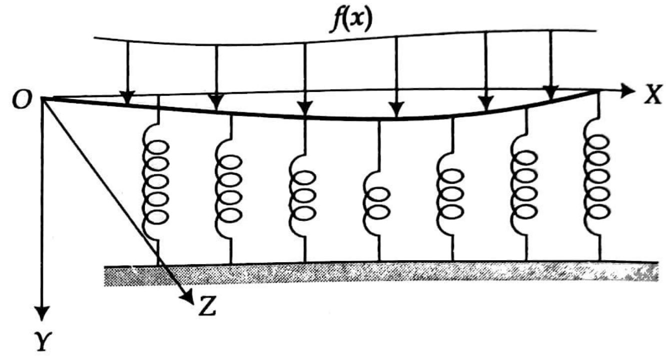

Fig. 11.16

Here, E is the Young's modulus of the beam material, I is moment of inertia of cross-section of the beam about the z-axis, and k is the *spring modulus* of the elastic foundation, called the *foundation modulus*.

Example 11.51: An elastic beam of length π, simply supported at two ends, is uniformly loaded with the load density $f(x) = c \sin x$, $0 \le x \le \pi$, $c > 0$ is a constant. Find an expression for the deflection of the beam and also show that the maximum deflection occurs in the middle of the beam.

Solution: Let E be the Young's modulus of the material of the beam and I be the moment of inertia of the cross-section about the z-axis and let the downward deflection be taken as positive, as shown in Fig. 11.17.

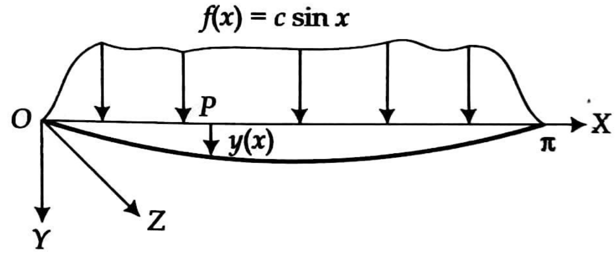

Fig. 11.17

If $y(x)$ is the deflection at a point P, x distance from the origin, then we have

$$EI\frac{d^4y}{dx^4} = c \sin x. \qquad ...(11.162)$$

Since there is no deflection or bending moment at the points $x = 0, \pi$, thus $y(0) = y(\pi) = 0$, $y''(0) = y''(\pi) = 0$.

Rewriting (11.162) as, $y^{iv}(x) = k \sin x$, where $k = \dfrac{c}{EI}$.

Integrating twice, $y''(x) = -k \sin x + c_1 x + c_2$

Applying $y''(0) = y''(\pi) = 0$, we obtain $c_1 = c_2 = 0$. Thus, $y''(x) = -k \sin x$.

Again integrating twice, $y(x) = k \sin x + c_3 x + c_4$.

Applying $y(0) = y(\pi) = 0$, we obtain $c_3 = c_4 = 0$. Thus, $y(x) = k \sin x$.

Obviously the maximum deflection occurs at the point $x = \pi/2$, and it is $y_{\max} = k = \dfrac{c}{EI}$.

Example 11.52: The deflection y of a beam of length L with one end built in and other end subjected to the end thrust P satisfies the equation $\dfrac{d^2y}{dx^2} + a^2y = \dfrac{a^2R}{P}(L-x)$.

Find the equation of the deflection curve, when y is the deflection of the beam at a distance x from the built-in-end, as shown in Fig. 11.18.

Solution: Since the end at $x = 0$ is built in, thus $y(0) = y'(0) = 0$

Hence, the initial value problem is

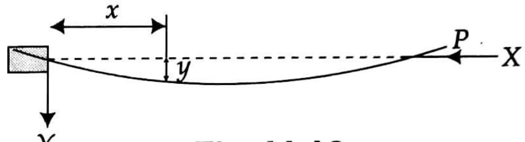

Fig. 11.18

$$(D^2 + a^2)y = \frac{a^2R}{P}(L-x),\ y(0) = y'(0) = 0. \qquad \text{...(11.163)}$$

Its complementary function is, $y_c = c_1 \cos ax + c_2 \sin ax$, and particular integral is,

$$y_p = \frac{1}{D^2 + a^2}\frac{a^2R}{P}(L-x) = \frac{R}{P}\left[1 + \frac{D^2}{a^2}\right]^{-1}(l-x)$$

$$= \frac{R}{P}\left[1 - \frac{D^2}{a^2} + \ldots\right](l-x) = \frac{R}{P}(l-x).$$

Thus, the general solution is

$$y = c_1 \cos ax + c_2 \sin ax + \frac{R}{P}(l-x). \qquad \text{...(11.164)}$$

Using the initial conditions, $y(0) = y'(0) = 0$, we obtain, $c_1 = \dfrac{-Rl}{P}$ and $c_2 = \dfrac{R}{aP}$. Hence, (11.164) gives, $y = \dfrac{R}{P}\left[\dfrac{1}{a}\sin ax - l\cos ax + l - x\right]$, as the desired deflection curve.

EXERCISE 11.10

1. An elastic beam of length L simply supported at two ends is uniformly loaded with the load density f_0 = const. Find an expression for the deflection of the beam and show that the maximum deflection is in the middle at $x = L/2$ and it is equal to $5f_0L^4/(16.24\,EI)$, where E and I have usual meanings.
2. A cantilever beam of length L and weighing f_0 lb. per unit length is subjected to a horizontal compressive force P applied at the free end. With origin at the free end and y-axis upwards, differential equation governing the deflection of the beam is $EI\dfrac{d^2y}{dx^2} + Py = -\dfrac{f_0x^2}{2}$.

   Find the maximum deflection of the beam.

3. The shape of a strut of length L subjected to an end thrust P and lateral load density f_0 units, when the ends are built-in, is governed by $EI\dfrac{d^2y}{dx^2} + Py = \dfrac{f_0x^2}{2} - \dfrac{f_0Lx}{2} + M, y(0) = y'(0) = 0,$ where M is the moment at a fixed end. Find y in terms of x.

4. A long column of length L fixed at one end and hinged at the other end, is under the action of axial load P. If a force F is applied laterally at the hinge to prevent lateral movement, then the equation of deflection is given by $\dfrac{d^2y}{dx^2} + w^2y = \dfrac{Ew^2}{P}(L - x)$. Find the equation of the deflection curve.

11.13 MODELLING: APPLICATIONS OF SIMULTANEOUS LINEAR DIFFERENTIAL EQUATIONS

So far we have considered the modelling in the situations where there is a single dependent variable such as the current $I(t)$ in a circuit, the time t is the independent variable. However, many problems, in engineering and science, involve two or more dependent variables of the single independent variable.

For example, an RL-circuit comprising two loops as shown in Fig. 11.19, using Kirchhoff's law, leads to following two simultaneous differential equations in two variables $I_1(t)$ and $I_2(t)$, given by

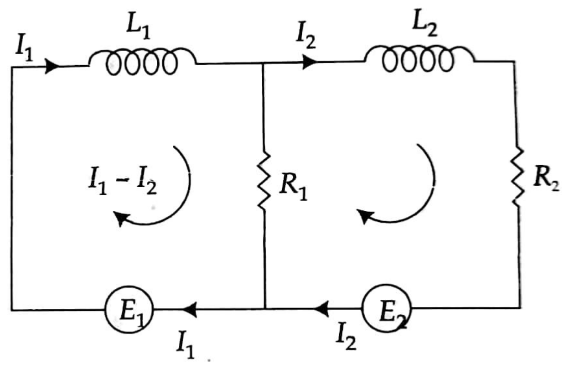

Fig. 11.19

$$L_1I_1' + R_1I_1 - R_1I_2 = E_1,\ L_2I_2' - R_1I_1 + (R_1 + R_2)\ I_2 = E_2$$

As another example, in the combustion of the fossil fuels there are many interacting chemical species whose generation and demise, as a function of time, are governed by a large set of differential equations. The mathematical formulation of such problems results into a system of simultaneous differential equations equal in number to the number of dependent variables. In the examples to follow, we shall elaborate the applications of simultaneous differential equations in modelling of mixing problems, mass-spring system and induction coils, etc.

Example 11.53: Initially a tank A contains 100 liters of brine and 50 kg of salt and another tank B contains 50 litres of fresh water. Fresh water runs in tank A, brine runs out from tank A in tank B and is taken out from tank B at a uniform rate of 2 litres per minute. If both the tanks A and B are kept stirring uniformly, find the amounts of salt present in tanks at time t.

Solution: Let x and y be the amounts of salt present in kg. in tanks A and B respectively, at time t. Under the given rates the volumes of brine solution in tanks A and B remains unchanged.

Thus for tank A, $\dfrac{dx}{dt} = -\dfrac{2x}{100} = -\dfrac{x}{50}$, which gives

$$(50D + 1)x = 0, \quad D = \frac{d}{dt} \qquad \text{...(11.165)}$$

For tank B, $\frac{dy}{dt} = \frac{2x}{100} - \frac{2y}{50} = \frac{x}{50} - \frac{y}{25}$, which gives

$$-x + (50D + 2)y = 0. \qquad \text{...(11.166)}$$

Solving (11.165) and (11.166), we obtain $x = c_1 e^{-t/50}$, and $y = c_1 e^{-t/50} + c_2 e^{-t/25}$.

Using the initial conditions $x(0) = 50$ and $y(0) = 0$, we obtain $c_1 = 50$ and $c_2 = -50$. Thus,

$x = 50e^{-t/50}$, and $y = 50\,(e^{-t/50} - e^{-t/25})$ are the desired amounts.

Example 11.54: Two bodies each of mass 10 gm are suspended from two springs of same spring modulus $\frac{1}{10}$ gm/cm as shown in Fig. 11.20. After the system attains its static equilibrium, the lower mass is pulled 5 cm downwards and released. Discuss their motion assuming the mass of springs to be negligible.

Solution: Let $x(t)$ and $y(t)$ denote the displacement of the upper and lower masses at time t, from their respective positions of state equilibrium. Thus, elongation for the upper spring is x and for the lower spring is $(y - x)$; and hence the restoring force acting on the upper mass $= -kx + k(y - k) = k(y - 2x)$, and the restoring force for the lower mass $= -k(y - x)$, where k is the common spring modulus. The equations of motion are, therefore,

$$mx'' = k(y - 2x) \quad \text{and} \quad my'' = -k(y - x)$$

or,

$$(mD^2 + 2k)x - ky = 0, \quad \text{and} \quad -kx + (mD^2 + k)\,y = 0.$$

Here, $m = 10$ $k = \frac{1}{10}$, the equations become

$$(100\,D^2 + 2)x - y = 0 \qquad \text{...(11.167)}$$

$$-x + (100D^2 + 1)y = 0 \qquad \text{...(11.168)}$$

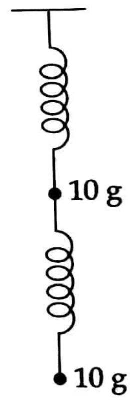

Fig. 11.20

Operating (11.167) by $(100D^2 + 1)$ and adding to (11.168) we obtain

$$(1000D^4 + 300D^2 + 1)\,x = 0.$$

Auxiliary equation is $1000\lambda^4 + 300\lambda^2 + 1 = 0$, which gives, $\lambda = \pm\, 0.162i, \pm\, 0.0618i$ as its roots, and hence

$$x(t) = c_1 \cos(0.162t) + c_2 \sin(0.162t) + c_3 \cos(0.0618t) + c_4 \sin(0.0618t). \qquad \text{...(11.169)}$$

Using this in (11.167) and simplifying, we obtain

$$y(t) = -0.624\,[c_1 \cos(0.162t) + c_2 \sin(0.162t)] + 1.618\,[c_3 \cos(0.0618t) + c_4 \sin(0.0618t)]. \qquad \text{...(11.170)}$$

Using the initial conditions $x(0) = y(0) = 5$ and $x'(0) = y'(0) = 0$, from (11.167) and (11.168), we obtain

$$c_1 + c_3 = 5, \; -0.624\,c_1 + 1.618\,c_3 = 5, \; 0.162c_2 + 0.0618c_4 = 0, \text{ and, } -0.101c_2 + .0999c_4 = 0.$$

Solving for c_1, c_2, c_3 and c_4, we have $c_1 = 1.378$, $c_2 = 0$, $c_3 = 3.622$, and $c_4 = 0$.

Hence, (11.167) and (11.168) give respectively

$x(t) = 1.378\cos(0.162t) + 3.622\cos(0.0618t)$ and $y(t) = -0.859\cos(0.162t) + 5.860\cos(0.0618t)$, as the desired displacements.

Thus the motion of the spring is a combination of two simple harmonic motions of period $2\pi/(0.162) = 38.8$ sec, and $2\pi/(0.0618) = 101.7$ sec.

Example 11.55: The two coils of an induction coil are identical with resistance R, inductance L, mutual inductance M, and a battery with e.m.f. E inserted in the primary coil. Determine the currents in the coils at any instant, assuming that initially there is no current in the either coil.

Solution: Let I_1 and I_2 be the currents flowing through the primary and the secondary coil at any instant as shown in Fig. 11.21.

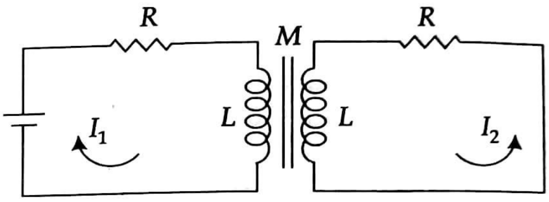

Fig. 11.21

Then using the Kirchhoff's law, we have

$$(R + LD)\, I_1 + MDI_2 = E \qquad \text{...(11.171)}$$

for the primary coil, and

$$(R + LD)\, I_2 + MDI_1 = 0 \qquad \text{...(11.172)}$$

for the secondary coil.

Eliminating I_2 from (11.171) and (11.172), we obtain

$$[(L^2 - M^2)\, D^2 + 2LRD + R^2]\, I_1 = RE. \qquad \text{...(11.173)}$$

The complementary function of (11.173) is

$$I_c = c_1 e^{-\frac{Rt}{L+M}} + c_2 e^{-\frac{Rt}{L-M}}$$

where, c_1 and c_2 are two arbitrary constants.

The particular integral of (11.173) is, $I_p = \dfrac{1}{(L^2 - M^2)D^2 + 2LRD + R^2} RE = \dfrac{E}{R}$.

Hence, the general solution for (11.173) is

$$I_1(t) = c_1 e^{-\frac{Rt}{L+M}} + c_2 e^{-\frac{Rt}{L-M}} + \frac{E}{R}. \qquad \text{...(11.174)}$$

From (11.172), $I_2 = -\dfrac{MD}{LD + R} I_1$. Substituting for I_1 from (11.174), we have

$$I_2 = -\frac{MD}{LD+R}\left(c_1 e^{-\frac{Rt}{L+M}}\right) - \frac{MD}{LD+R}\left(c_2 e^{-\frac{Rt}{L+M}}\right) - \frac{MD}{LD+R}\frac{E}{R}$$

$$= -\frac{c_1 M}{L\left(\frac{-R}{L+M}\right) + R} De^{-\frac{R}{L+M}t} - \frac{c_2 M}{L\left(\frac{-R}{L-M}\right) + R} De^{-\frac{R}{L-M}t} - \frac{M}{R} D\left(\frac{E}{R}\right)$$

$$= c_1 e^{-\frac{R}{L-M}} - c_2 e^{-\frac{R}{L-M}}. \qquad \text{...(11.175)}$$

Using the initial conditions $I_1(0) = I_2(0) = 0$ in (11.174) and (11.175), we obtain $c_1 + c_2 = -\frac{E}{R}$, $c_1 - c_2 = 0$, which gives $c_1 = c_2 = -E/2R$. Substituting in (11.174) and (11.175), we obtain

$$I_1(t) = \frac{E}{2R}\left[2 - e^{-\frac{Rt}{L+M}} - e^{-\frac{Rt}{L-M}}\right], \text{ and } I_2(t) = \frac{E}{2R}\left[e^{-\frac{Rt}{L-M}} - e^{-\frac{Rt}{L+M}}\right]$$

as the currents at any instant t.

EXERCISE 11.11

1. Initially a tank T_1 contains 400 litres of brine and 100 kg of salt and another tank T_2 contains 200 litres of fresh water. Brine from tank T_1 runs into tank T_2 at 12 litres per minute and from T_2 to T_1 at 8 litres per minute. If each tank is kept well stirred, find how much salt does tank T_1 contains after 50 minutes?
2. Initially a tank T_1 contains 100 gal. of pure water and another tank T_2 contains 100 gal. of water in which 150 lb. of fertilizer is dissolved. Liquid circulates through the tanks at a constant rate of 2 gal/min, and the mixture is kept uniform by stirring. Prove that the ultimately amount of fertilizer in each tank will be equal.
3. Find the currents $I_1(t)$ and $I_2(t)$ in the network shown in Fig. 11.22, assuming that all charges and currents are zero at $t = 0$.

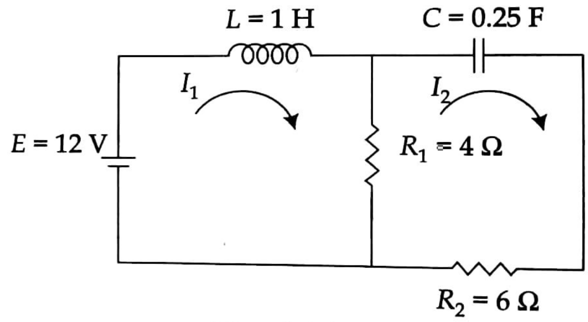

Fig. 11.22

4. The rate of increase of y with respect to x is $4z$ and that of z is $3y$. If y is 1000, and z is 500 when $x = 0$, find the value of x and y when z is 1000.
5. For the network as shown in Fig. 11.23, show that the currents I_1 and I_2 diminish numerically as t increases provided $L_1L_2 > M^2$.

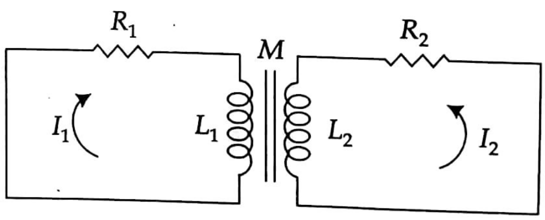

Fig. 11.23

6. The currents I_1 and I_2 in a mesh are given by the differential equations:

$$I'_1 - w_2 I_2 = a \cos pt, \; I'_2 + wI_1 = a \sin pt.$$

Find I_1 and I_2 assuming $I_1(0) = I_2(0) = 0$.

7. Under certain conditions the motion of an electron is given by the equations

$$m\frac{d^2x}{dt^2} + eH\frac{dy}{dt} = eE, \text{ and } m\frac{d^2y}{dt^2} - eH\frac{dx}{dt} = 0.$$

Find the path of the electron, if it started from rest at the origin.

8. Two particles each of mass m are suspended from two vertical springs of same stiffness k. After the system comes to rest the lower mass is pulled downward by 1 metre and then released. Show that the motion is combination of two simple harmonic motions.

9. A system consists of springs A, B, C and two objects D and E attached in a straight line on a frictionless horizontal table, with end of the springs A and B attached to fixed points P and Q as shown in Fig. 11.24. The system is set into vibration by holding the object D in place, moving the object E to the right through a distance $a > 0$ and then releasing the both. Assuming the masses of the two objects to be equal to m and spring coefficients of all the springs to be K, find the two equations of motion for the system, and show that system oscillations are combination of two simple harmonic oscillations.

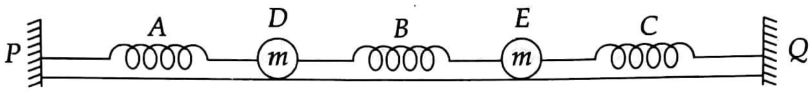

Fig. 11.24

10. Two substances with concentrations $x(t)$ and $y(t)$, react to form a third substance with concentration $z(t)$. The reaction is governed by the system $x' + \alpha x = 0$, $z' = \beta y$ and $x + y + z = \gamma$, where, α, β, γ are known positive constants. Solve for $x(t)$, $y(t)$, $z(t)$ subject to the initial conditions $z(0) = z'(0) = 0$ for the cases (i) $\alpha \neq \beta$ (ii) $\alpha = \beta$.

ANSWERS

Exercise 11.1 (p. 622)

1. $W(x) = 2e^{6x} \neq 0$, forms a basis $y = c_1 e + c_2 e^{2x} + c_3 e^{3x}$
2. $W(x) = 0$, not a basis
3. Forms a basis, $y = c_1 x + c_2 x \ln x$
4. Forms a basis, $y = (c_1 + c_2 x) \cos x + (c_2 + c_3 x) \sin x$
5. Forms a basis $y = c_1/x + c_2/x^2$
6. Forms a basis $y = c_1 \cos \ln (1 + x) + c_2 \sin \ln (1 + x)$
7. $y = \cos h\,(2x)$
8. $y = (12/5)e^{-3x} - (7/5)e^{-8x}$
9. $y = 2x^4 - 4x^4 \ln|x|$
10. $y = (1/2)(5x^2 - 1/x^2)$
11. $y = x^{-1/2} + 2x^{3/2}$
12. $y = c_1 e^{3x} + c_2 e^{-2x}$
13. $y = c_1 x + c_2 x \ln x$
14. $y = c_1 x^3 + c_2 x^3 \ln x$
15. $y = c_1 x^{-1/2} \cos x + c_2 x^{-1/2} \sin x$
16. $y = c_1 e^x + c_2$
17. $y = (c_1 x + c_2)^{-1}$
18. $y = \frac{1}{2}\left[\frac{c_1 x^2}{2} - \frac{1}{c_2} \ln x\right] + c_2$

19. $y = \frac{8\sqrt{2}}{105}(x + c_1)^{7/2} + \frac{1}{2}c_2 x^2 + c_3 x + c_4$

20. $y = \frac{1}{3}[(2t + 4)^{3/2} - 2]$, $y(6) = 62/3$, $y'(6) = 4$

Exercise 11.2 (p. 631)

1. $y = c_1 e^x + c_2 e^{-2x}$
2. $y = e^{-x}[c_1 \cos\sqrt{3}\,x + c_2 \sin\sqrt{3}\,x]$
3. $y = (c_1 + c_2 x)e^{-x/2}$
4. $y = c_1 e^{-x} + c_2 \cos 2x + c_3 \sin 2x$
5. $y = e^{-x}(c_1 \cos x + c_2 \sin x) + e^x(c_3 \cos x + c_4 \sin x)$
6. $y = (c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x$
7. $y = (1/6)e^{x/2} - 2e^{-3x}$
8. $y = e^{5x/2}(3\cos 5x + 2\sin 5x)$
9. $y = 2(1 - 3x)e^{3x}$
10. $y = \frac{1}{2} + e^{-x}\left(\frac{\sqrt{3}}{6}\sin\sqrt{3x} - \frac{1}{2}\cos\sqrt{3}\,x\right)$
11. $y = \frac{3}{4} + \left(\frac{1}{68}\right)\left[9\sqrt{17}\sinh\left(\frac{\sqrt{17}\,x}{2}\right) + 17\cosh\left(\frac{\sqrt{17}\,x}{2}\right)\right]e^{-x/2}$
12. $y = \left(\frac{1}{3}\right)\cosh(\sqrt{2}\,x) + \frac{2}{3}\cos x$
13. $y = e^{-x}\cos x$
14. $y = [(2e^2 - 1)e^{-6x} - e^{6x}]/(e^2 - 1)$
15. $y = \cos 5x + c\sin 5x$
16. $y = c\sin \pi x$
17. $y = 2\sin 2x + \sin 3x$
18. $y = (1 + x)e^{-x}\cos x$
19. $y^{(v)} - 12y^{(iv)} + 49y''' - 76y'' + 48y' - 64y = 0$
    $y = (c_1 + c_2 x + c_3 x^2)e^{4x} + c_4 \cos x + c_5 \sin x$
20. $y = (c_1 + c_2 x)e^{-2x}$
21. $y = c_1 e^{-x/2} + c_2 e^{-3x/2}$
22. $y = c_1 e^x + (c_2 + c_3 x)e^{-2x}$
23. $y(x) = \sum_{n=1} B_n \sin nx$

Exercise 11.3 (p. 641)

1. $y = c_1 e^{\frac{1}{2}x} + c_2 e^{-3x} - 2$
2. $y = c_1 e^{3x} + c_2 e^{-2x} - \frac{1}{3}e^x$
3. $y = c_1 e^{\frac{3}{2}x} + c_2 e^{-x} + xe^{\frac{3}{2}x}$
4. $y = c_1 + c_2\cos\sqrt{5}x + c_3\sin\sqrt{5}x + (1/18)\cos h\,2x$

5. $y = c_1e^{2x} + c_2e^{-2x} + (1/125)\,(25x^2 - 60x + 62)e^{3x}$
6. $y = (c_1 + c_2x)e^x + e^x(2\sin x - x\cos x)$
7. $y = c_1 + c_2e^{-x} + x - (1 + e^{-x})\ln(1 + e^x)$
8. $y = c_1 + c_2e^{-x} + 4x + x^3/3$
9. $y = (c_1\cos ax + c_2\sin ax) + (1/a)x\sin ax + (1/a^2)\cos ax\ln\cos ax$
10. $y = c_1e^x + c_2e^{-x} - (1/2)\,(x\sin x + \cos x) + xe^x(2x^2 - 3x + 9)/12$
11. $y = c_1e^{2x} + c_2e^{-2x} + (c_3\cos x + c_4\sin x) + x/4\sin h\,(2x - 1) + 3^x/[(\ln 3)^2 - 4]$
12. $y = (\tan x + c_1)\,e^{-2x} + c_2e^{-3x}$
13. $y = \dfrac{k}{n^2 - p^2}\left(\sin pt - \dfrac{p}{n}\sin nt\right)$
14. $y = (4/3)e^{5x} - (10/3)e^{2x} - (1/3)xe^{2x} + 2$
15. $y = 2e^{-3x/2} - 2e^{2x} + 3e^x(3\sin x - 7\cos x)/29$
16. $y = 3e^{-x} + 5 + 2x + (1/3)x^2$

Exercise 11.4 (p. 646)

1. $y = (c_1 + c_2\ln x)x + \ln x + 2$
2. $y = \dfrac{c_1}{x} + c_2x^3 - (14/27)x^2 - (4/9)x^2\ln x - x^2(\ln x)^{2/3}$
3. $y = c_1x^2 + c_2/x + (1/3)(x^2 - 1/x)\ln x$
4. $y = c_1/x + c_2(\ln x)/2 - \dfrac{\ln(1 - x)}{x}$
5. $y = c_1/x + \{c_2\cos(\ln x) + c_3\sin(\ln x)\}\,x + 5x + (10\ln x)/x$
6. $y = c_1x + c_2x\ln x + c_3\,x(\ln x)^2 + c_4\,x(\ln x)^3 + \ln x + 4$
7. $y = c_1\cos\{\ln(1 + x) - c_2\} + 2\ln(1 + x)\sin\ln(1 + x)$
8. $y = c_1\,(3x + 2)^2 + c_2\,(3x + 2)^{-2} + (1/108)\,[(3x + 2)^2\ln(3x + 2)]$
9. $y = 4(\ln x - 1)\sqrt{x} + \ln x + 4$
10. $y = 1/x[2\cos(3\ln x) + 3\sin(3\ln x)] + x^2/2$
11. $u = (kr/8)(a^2 - r^2)$

Exercise 11.5 (p. 653)

1. $y = c_1e^{-x} + c_2xe^{-x} + \dfrac{1}{6}x^3e^{-x}$
2. $y = c_1\cos x + c_2\sin x - x\cos x + \sin x\ln|\sin x|$
3. $y = c_1\cos 4x + c_2\sin 4x + 8\cos 2x - 4\sin 4x\ln|\sec 2x + \tan 2x|$
4. $y = c_1e^{-2x} + c_2e^{-x} + 3e^{-2x}\ln(1 + e^x) + 3e^{-x}\ln(1 + e^x)$
5. $y = c_1\cos x + c_2\sin x - 1 - \cos x + 2\tan h^{-1}\,[\sin x/(1 + \cos x)]\sin x$
6. $y = c_1\,e^{-2x}\cos x + c_2\,e^{-2x}\sin x + (1/4)\,xe^{-2x}\cos x + (1/4)\,x^2\,e^{-2x}\sin x$
7. $y = c_1e^x + c_2e^{2x} + c_3e^{3x} - (1/24)e^{-x}$
8. $y = c_1 + c_2\cos 2x + c_3\sin 2x - (1/4)x\cos 2x + (1/8)\sin 2x\ln|\cos 2x| + (1/8)\ln|\sec 2x + \tan 2x|$

9. $y = c_1 x^2 + c_2/x^2 + x^2\,[8(\ln x)^2 - 4\ln x + 1]$
10. $y = c_1 x + c_2 x^2 + x^{3/2} - x(1 + \ln x)$
11. $y = c_1 x + c_2/x + (x \ln x/2) - x/4$
12. $y = e^{-2x}(c_1 \cos 2x + c_2 \sin 2x) + 4e^{-2x} \cos 2x \ln |\text{cosec}\, 2x + \cot 2x| - 4e^{-2x}$
13. $y = c_1 x + c_2/x^3 - 4\sqrt{x}/7$
14. $y = c_1(2x^2 - 1) + c_2 x\,(x^2 - 1)^{1/2} + x/3$
15. $y = c_1 e^x + (c_2 + xc_3)e^{-x} + 1 - x$
16. $y = e^{-x}(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x) + 1x^2/2 - x/2 + e^{-x}$
17. $y = c_1 e^{-2x} + c_2 e^{-3x} + 2e^{-x} + (\sin x - \cos x)/2$
18. $y = c_1 e^{3x} + c_2 e^{-4x} + xe^{3x}/7$
19. $y = c_1 e^x + c_2 e^{-x} + e^{2x}(2\cos 3x + \sin 3x)/30 + e^{3x}(\cos 2x + 3\sin 2x)/40.$
20. $y = (c_1 x^2 + c_2 x + c_3)\, e^{2x} + 2x^3 e^{2x} - e^{-2x}$
21. $y = c_1 \cos(2\ln x) + c_2 \sin(2\ln x) - [\cos(2\ln x)\ln x]/4$
22. $y = c_1 x^{-1} + c_2 x^{-1}\ln x + x^2 + 2x + 5$

Exercise 11.6 (p. 657)

1. $x(t) = c_1 e^{-5t} + c_2 e^t - (6/7)e^{2t};\; y(t) = c_1 e^{-5t} + c_2 e^t + (8/7)e^{2t}$
2. $x(t) = c_1 e^{-2t}\cos t + c_2 e^{-2t}\sin t + 1 + (1/2)e^{-t}$
   $y(t) = -c_1 e^{-2t}\sin t + c_2 e^{-2t}\cos t + 1 - (1/2)e^{-t}$
3. $x(t) = c_2 e^{2t} - \dfrac{2}{5} c_1 e^{-3t} - \dfrac{t}{3} + \dfrac{5}{18};\; y(t) = c_1 e^{-3t} + \dfrac{t}{3} + \dfrac{5}{9}.$
4. $x(t) = e^{-at}(c_1 \cos at + c_2 \sin at);\; y(t) = e^{-at}(-c_1 \sin at + c_2 \cos at)$
5. $x(t) = -6e^{2t} + 6e^{-t};\quad y(t) = 4e^{2t} - 3e^{-t}$
6. $x(t) = (c_1 + c_2 t)e^t + c_3 e^{-2t};\; y(t) = 2(c_2 - c_1 - c_2 t)e^t + c_2 e^{-2t}$
7. $x(t) = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t;\; y(t) = c_1 e^t + c_2 e^{-t} - c_3 \cos t - c_4 \sin t$
8. $x(t) = (1/4)\,(11 \sin t + 1/3 \sin 3t);\; y(t) = (1/4)\,(11 \sin t - \sin t)$
9. $x(t) = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t - (t\cos t)/4 + (t \sin t)/4$
   $y(t) = -c_1 e^t - c_2 e^{-t} + c_3 \cos t + c_4 \sin t + (2 + t)\,(\sin t - \cos t)/4$
10. $x(t) = c_1 e^{2t} + c_2 e^{-t}\cos(\sqrt{3}\,t - c_3),\; y(t) = c_1 e^{2t} + c_2 e^{-t}\cos\left(\sqrt{3}t - c_3 + 2\pi/3\right)$

    $z(t) = c_1 e^{2t} + c_2 e^{-t}\cos\left(\sqrt{3}t - c_3 + 4\pi/3\right)$

Exercise 11.7 (p. 660)

1. $\dfrac{1}{4}m$
2. $\dfrac{2}{\pi}\left[\cos^{-1}\dfrac{3}{4} - coa^{-1}\dfrac{1}{4}\right]$

Exercise 11.8 (p. 671)

1. $0.2 \cos 14\,t$, 2.8 m/sec, 0.45 sec

4. $-5e^{-8ct}\left(\dfrac{1}{8\sqrt{1-c^2}}\right)\sin 8\sqrt{1-c^2}\,t$, $\dfrac{5}{8\sqrt{1-c^2}}e^{-8ct}$, $\dfrac{\pi}{4\sqrt{1-c^2}}$, $5e^{-8\sqrt{1-c^2}\,t}$,

   overdamped, damped, underdamped for $c^2 \gtreqless 1$.

6. $\dfrac{A}{2\ln w_0}\,t\sin(w_0\,t)$

9. 0.45 sec; 1.15 sec

10. $0.8(2\sin 4t - \cos 4t)$

Exercise 11.9 (p. 675)

3. (a) $Q(t) = -e^{-50t}\left(\dfrac{11}{250}\cos 50\sqrt{19}t + \dfrac{11\sqrt{19}}{4750}\sin 50\sqrt{19}\right) + \dfrac{11}{250}$,

   $I(t) = \dfrac{44}{\sqrt{19}}e^{-50t}\sin 50\sqrt{19}t$

   (b) $Q(t) = -e^{-50t}\left[\dfrac{16}{170}\cos wt + \dfrac{12\sqrt{19}}{1615}\sin wt\right] + \dfrac{4}{170}[4\cos 100\,t + \sin 100\,t]$

   $I(t) = -40e^{-50t} + \left(\dfrac{1}{17}\cos wt - \dfrac{410\sqrt{19}}{323}\sin wt\right) + \dfrac{40}{17}[\cos 100\,t - 4\sin 100\,t]$; $w = 50\sqrt{19}$

   The steady solutions:

   $Q(t) = \dfrac{4}{170}[4\cos 100\,t + \sin 100\,t]$, $I(t) = \dfrac{40}{17}[\cos 100\,t - 4\sin 100\,t]$

4. (a) $1 - \cos t$ (b) $\dfrac{110}{21}(\cos 4t - \cos 10t)$

5. $R = 10\,\Omega$, $C = 1/29$ F, $E = 11\sin 5t$

6. (a) $I(t) = .015\,e^{-0.0625t} - 5.4\times 10^{-7}\,e^{-3333.27t} + 0.015\cos 20\,t - 0.00043\sin 20\,t$

   (b) $I(t) = 0.001633e^{-t} + 0.00161e^{-0.3177t} + 0.0000\,23e^{-t}\cos 6t - 0.000183e^{-t}\sin 6t$

Exercise 11.10 (p. 679)

1. $y = \dfrac{f_0}{24EI}(x^4 - 2Lx^3 + L^3x)$

2. $y_{max} = \dfrac{f_0}{2Pw^2}\left[1 - \dfrac{L^2w^2}{2} - \sec wL + wl \tan wL\right]$

3. $y = \dfrac{f_0 l}{2Pw} \operatorname{cosec} \dfrac{wL}{2} \cos\left(wx - \dfrac{wL}{2}\right) - \dfrac{f_0 l}{2wP} \cot \dfrac{wL}{2} + \dfrac{f_0}{2P}(x^2 - Lx), w = P/EI$

4. $y = \dfrac{F}{P}(w \sin wx - L \cos wx + L - x)$

Exercise 11.11 (p. 683)

1. 34.32 kg
3. $I_1(t) = -8e^{-2t} + 5e^{-0.8t} + 3, \quad I_2(t) = -4e^{-2t} + 4e^{-0.8t}$
4. $x = 1408, \quad y = 1415$
6. $I_1 = \dfrac{a}{p+w} \sin pt, \quad I_2 = \dfrac{a}{p+w} \cos pt$
7. $x = \dfrac{E}{Hw}(1 - \cos wt), \quad y = \dfrac{E}{Hw}(wt - \sin wt), w = \dfrac{eH}{m}$
9. $m\dfrac{d^2x}{dt^2} + 2kx - ky = 0, \quad m\dfrac{d^2y}{dt^2} + 2\,ky - kx = 0$

   Motion is sum of two S.H.M. with period $2\pi\sqrt{\dfrac{m}{K}}$ and $\dfrac{2\pi}{\sqrt{3}}\sqrt{\dfrac{m}{K}}$.