* Mathematics Assignment
Infinite Series.

Dhruw Ramder.

Q1 Test the convergence of series :-

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x+1)^n}{2^n n^2}$$

A1

$$u_n = \frac{(-1)^n (x+1)^n}{2^n (n^2)} \quad , \quad u_{n+1} = \frac{(-1)^n (x+1)^{n+1}}{2^{n+1} (n+1)^2}$$

By Absolute limit test

$$\lim_{n\to\infty} \frac{|u_n|}{|u_{n+1}|} = \lim_{n\to\infty} \frac{|x+1|^n}{2^n (n^2)} \cdot \frac{2^{n+1} (n+1)^2}{|x+1|^{n+1}}$$

$$= \lim_{n\to\infty} \frac{2}{|x+1|} \left(1+\frac{1}{n}\right)^2$$

$$= \frac{2}{|x+1|}$$

For convergence

$$\frac{2}{|x+1|} > 1$$

$$|x+1| < 2$$

$x+1 < 2$ $\quad$ $x+1 > -2$

$\boxed{x < 1}$ $\quad$ $\boxed{x > -3}$

For divergence

$$\frac{2}{|x+1|} < 1$$

$$|x+1| > 2$$

$x+1 > 2$ $\quad$ $x+1 < -2$

$\boxed{x > 1}$ $\quad$ $\boxed{x < -3}$

For $x = 1$

$$u_n = \frac{(-1)^n (2^{\cancel{n}})}{2^{\cancel{n}} (n^2)}$$

$$u_n = \frac{(-1)^n}{n^2}$$

$|u_n|$ is convergent

u_n converges for

$\boxed{x = 1}$

For $x = 3$

$$u_n = \frac{(-1)^n (-2)^n}{2^n (n^2)}$$

$$u_n = 1/n^2$$

u_n is convergent for $\boxed{x = -3}$

$\therefore$ Given series is

convergent for $-3 \le x \le 1$

divergent for $x > 1$ and $x < -3$

Scanned by CamScanner

Q2 Test the convergence of series

$$\sum_{n=1}^{\infty} \frac{(n-1)^{n-1}}{(n)^n}$$

A2 $u_n = \frac{(n-1)^{n-1}}{n^n}$ $\qquad u_{n+1} = \frac{(n)^n}{(n+1)^{n+1}}$

$$\frac{u_n}{u_{n+1}} = \frac{(n-1)^{n-1}}{n^n} \cdot \frac{(n+1)^{n+1}}{(n)^n}$$

$$= \left(\frac{n-1}{n}\right)^{n-1} \cdot \left(\frac{n+1}{n}\right)^{n+1} = \left(1-\frac{1}{n}\right)^{n-1}\left(1+\frac{1}{n}\right)^{n+1}$$

By log's test

$$= \lim_{n\to\infty} n \log_e\left(\frac{u_n}{u_{n+1}}\right)$$

$$= \lim_{n\to\infty} n \ln\left[\left(1-\frac{1}{n}\right)^{n-1}\left(1+\frac{1}{n}\right)^{n+1}\right]$$

$$= \lim_{n\to\infty} n\left[(n-1)\ln\left(1-\frac{1}{n}\right) + (n+1)\ln\left(1+\frac{1}{n}\right)\right]$$

$$= \lim_{n\to\infty} n\left[(n-1)\left[-\frac{1}{n}+\frac{1}{2n^2}-\frac{1}{3n^3}\cdots\right] + (n+1)\left[\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}\cdots\right]\right]$$

$$= \lim_{n\to\infty} n\left[n\left(-\frac{1}{n}+\frac{1}{2n^2}-\frac{1}{3n^3}\cdots+\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}\right) + 2\left[\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}\cdots\right]\right]$$

$$= \lim_{n\to\infty}\left[\left[-1+\frac{1}{2}-\frac{1}{3n}\cdots+1-\frac{1}{2}+\frac{1}{3n}\cdots\right] + 2\left[1-\frac{1}{2n}+\frac{1}{3n^2}\cdots\right]\right]$$

$$= [0 + 2]$$

$$= 2$$

$\therefore$ Given series is convergent.

Scanned by CamScanner

Q3 Discuss the convergence of series

$$1 + \frac{x}{2} + \frac{2!}{3^2}x^2 + \frac{3!}{4^3}x^3 + \cdots \qquad \boxed{x>0}$$

A3

$$u_n = \frac{x^{n-1}}{n^{n-1}} \cdot (n-1)! \qquad u_{n+1} = \frac{x^n}{(n+1)^n} \cdot n!$$

$$\frac{u_n}{u_{n+1}} = \frac{1}{x} \cdot \frac{(n+1)^n}{n^{n-1}} \cdot \frac{1}{n}$$

$$\frac{u_n}{u_{n+1}} = \frac{1}{x} \cdot \left(1+\frac{1}{n}\right)^n$$

$$l = \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{1}{x}\left(1+\frac{1}{n}\right)^n$$

$$= \frac{1}{x}e$$

$\therefore$ For convergence of series

$$l > 1$$

$$\frac{e}{x} > 1$$

$$\boxed{x < e}$$

For divergence of series

$$l < 1$$

$$\frac{e}{x} < 1$$

$$\boxed{x > e}$$

$\boxed{\text{At } x = e}$

$$\frac{u_n}{u_{n+1}} = \frac{1}{e} \cdot \left(1+\frac{1}{n}\right)^n$$

$$\therefore \quad l_1 = \lim_{n\to\infty}\left[n \log_e\left(\frac{u_n}{u_{n+1}}\right)\right] = n \ln\left[\frac{1}{e} \cdot \left(1+\frac{1}{n}\right)^n\right]$$

$$= \lim_{n\to\infty}\left[n \ln\left(\frac{1}{e}\right) + n^2 \ln\left(1+\frac{1}{n}\right)\right]$$

$$= \lim_{n\to\infty}\left[-n + n^2\left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} \cdots\right]\right]$$

$$= \lim_{n\to\infty}\left[-x + x - \frac{1}{2} + \frac{1}{3n} \cdots\right]$$

$$= -\frac{1}{2}$$

$\therefore$ series is divergent for $\boxed{x = e}$

Scanned by CamScanner

Q4 $x^2(\ln 2)^q + x^3(\ln 3)^q + x^4[\ln(4)]^q \cdots$ $\boxed{x>0}$

Discuss the convergence of above series

A4 $u_n = x^n[\ln(n)]^q$ $\qquad u_{n+1} = x^{n+1}[\ln(n+1)]^q$

$$\frac{u_n}{u_{n+1}} = \frac{1}{x}\left[\frac{\ln(n)}{\ln(n+1)}\right]^q$$

$$l = \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \frac{1}{x}\lim_{n\to\infty}\frac{1}{x}\left[\frac{\ln(n)}{\ln(n+1)}\right]^q$$

by L Hospital Rule.

$$l = \lim_{n\to\infty}\frac{1}{x}\left[\frac{1/n}{1/n+1}\right]^q$$

$$= \lim_{n\to\infty}\frac{1}{x}\left[\frac{n+1}{n}\right]^q$$

$$= \lim_{n\to\infty}\frac{1}{x}\left[1+\frac{1}{n}\right]^q$$

$$= \frac{1}{x}[1]^q$$

$$l = \frac{1}{x}$$

$\therefore$ For convergence

$l>1$

$\frac{1}{x}>1$

$\boxed{x<1}$

For divergence

$l<1$

$\frac{1}{x}<1$

$\boxed{x>1}$

At $\boxed{x=1}$

$$u_n = [\ln(n)]^q$$

$$\lim_{n\to\infty} u_n = \infty$$

$\therefore$ At $x=1$, series is divergence.

$\therefore$ Series

convergence for $x<1$

divergence for $x\geq 1$

Scanned by CamScanner

of series :

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p} \quad \boxed{p>0}$$

$$\therefore u_n = \frac{1}{n(\log n)^p} \qquad \cancel{u_{n+1} = \frac{1}{(n+1)[\log(n+1)]^p}}$$

$$\cancel{\therefore \frac{u_n}{u_{n+1}} = \left(\frac{n+1}{n}\right)\left[\frac{\log(n+1)}{\log(n)}\right]^p}$$

$$\cancel{= \left[1+\frac{1}{n}\right]\left[\frac{\log(n+1)}{\log(n)}\right]^p}$$

By integral test

$$= \int_1^{\infty} \frac{dn}{(n+1)[\log(n+1)]^p}$$

Put $\log(n+1) = t$

$$\frac{1}{n+1}\, dn = dt$$

$$\therefore = \int_{\log 2}^{\infty} \frac{1}{t^p}$$

$$= \left[\frac{t^{-p+1}}{-p+1}\right]_{\log 2}^{\infty} \qquad \boxed{\text{for } p \neq 1}$$

For convergence

$-p+1 < 0$

$\boxed{p>1}$

For divergence

$-p+1 > 0$

$p<1$

At $\underline{p=1}$

$$= \left[\log(t)\right]_{\log 2}^{\infty}$$

$$= \infty$$

, divergent.

Hence, series converges of $p>1$

diverges for $0<p\leq 1$

Scanned by CamScanner

Q6 Test for absolute convergence of series

$$\frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} + \dots$$

$$|u_n| = \frac{|\sin nx|}{n^3}$$

let us assume $V_n = \frac{1}{n^2}$

$\therefore$ since $|\sin nx| \leq 1$

$$\therefore \frac{|\sin nx|}{n^3} < \frac{1}{n^2} \quad [\text{for } n > 1]$$

$\therefore$ V_n is convergent, therefore by comparision test $|u_n|$ is also convergent.

Hence, given series is absolutely convergent.

Q7 a) Discuss the converge of series

$$\sum_{n=2}^{\infty} \frac{(-1)^{n-1} x^n}{n(n-1)} \qquad 0 < x < 1$$

$$|u_n| = \frac{x^n}{n(n-1)}$$

$$|u_{n+1}| = \frac{x^{n+1}}{(n+1)(n)}$$

$$\therefore \frac{|u_n|}{|u_{n+1}|} = \frac{1}{x} \frac{(n+1)}{(n-1)}$$

$$l = \lim_{n \to \infty} \frac{|u_n|}{|u_{n+1}|} = \lim_{n \to \infty} \frac{1}{x} \frac{(1 + 1/n)}{1 - 1/n}$$

$$= \frac{1}{x}$$

since $0 < x < 1$, $\boxed{\frac{1}{x} > 1}$, hence series is convergent.

Scanned by CamScanner

$$\sum \frac{\sqrt{n}\cos(n\pi)}{\ln(n)}$$

$\cos(n\pi) = +1$ for n is odd
$\quad\quad\quad -1$ for n is even.

$$\therefore u_n = \frac{\sqrt{n}\,(-1)^{n+1}}{\ln(n)}$$

$$\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \frac{\sqrt{n}}{\ln(n)}$$

By L. Hospital Rule

$$= \lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{2} \neq 0$$

$\therefore$ Series is not convergent.

Q8 Explain the absolute & conditional convergence. Find the range of the value of 'x' for which series is ~~con~~ absolutely convergent.

$$\frac{x}{2\cdot 5} - \frac{x^2}{3\cdot 5^2} + \frac{x^3}{4\cdot 5^3} - \frac{x^4}{5\cdot 5^4} \cdots$$

- An infinite series of number is said to absolutely convergent, if the sum of the absolute value of the summation is finite.

$\sum_{n=0}^{\infty} a_n$ is said to converge absolutely if

$$\sum_{n=0}^{\infty} |a_n| = L \quad \text{for some real number } L.$$

- A infinite series is said to be conditional convergent if it converges but it is not absolutely convergent.

$\therefore$ $\sum_{n=0}^{\infty} a_n$ is said to converge conditionally if $\sum a_n$ converges, but $\sum |a_n|$ does not

Scanned by CamScanner

$$u_n = \frac{(-1)^{n+1}(x^n)}{(n+1)(5^n)}$$

$$|u_n| = \frac{|x|^n}{(n+1)(5^n)} \qquad |u_{n+1}| = \frac{|x|^{n+1}}{(n+2)(5^{n+1})}$$

$$l = \lim_{n\to\infty} \frac{|u_n|}{|u_{n+1}|} = \lim_{n\to\infty} \frac{1}{|x|} \cdot \frac{5}{n+1}(n+2)$$

$$= \lim_{n\to\infty} \frac{5}{|x|} \frac{(1+2/n)}{(1+1/n)}$$

$$= \frac{5}{|x|}$$

For convergence

$$l > 1$$

$$\frac{5}{|x|} > 1$$

$$\boxed{|x| < 5}$$

$\therefore$ Series is absolutely convergent for $|x| < 5$

Q9 Test following series for convergence or divergence.

$$\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{1}{n^2+n+1}\right)$$

$$\therefore u_n = \tan^{-1}\left(\frac{1}{1+(n)(n+1)}\right)$$

$$= \tan^{-1}\left(\frac{n+1-n}{1+(n)(n+1)}\right)$$

$$= \tan^{-1}(n+1) - \tan^{-1}(n)$$

$\therefore$ A series is convergent if $\sum u_n$, $n\to\infty$ is finite.

$$\sum_{n=1}^{\infty} u_n = \left[\tan^{-1}(n+1) - \tan^{-1}(n)\right] + \left[\tan^{-1}(n) - \tan^{-1}(n-1)\right] \cdots$$

$$\cdots \left[\tan^{-1}(3) - \tan^{-1}(2)\right] +$$

$$\left[\tan^{-1}(2) - \tan^{-1}(1)\right]$$

Scanned by CamScanner

$$= \tan^{-1}(n+1) - \tan^{-1}(1)$$

$$= \tan^{-1}(\infty) - \tan(1)$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}, \quad \text{which is finite}$$

Hence, given series is convergent.

Q10 Test the convergence of series

$$\sum_{n=1}^{\infty} \left(\sqrt{n^2+1} - n\right) x^{2n} \qquad \boxed{x > 0}$$

$$\therefore \quad u_n = \left(\sqrt{n^2+1} - n\right) x^{2n}$$

$$= \left(\sqrt{n^2+1} - n\right)\frac{\left(\sqrt{n^2+1} + n\right)}{\sqrt{n^2+1} + n} x^{2n}$$

$$= \frac{1}{\sqrt{n^2+1} + n}\left(x^{2n}\right)$$

$$u_{n+1} = \frac{x^{2n+2}}{\sqrt{(n+1)^2+1} + n+1}$$

$$l = \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{1}{x^2}\left[\frac{\sqrt{(n+1)^2+1} + n+1}{\sqrt{n^2+1} + n}\right].$$

$$= \frac{1}{x^2}\left[\frac{n+1}{n}\right]\left[\frac{\sqrt{1+(1/n+1)^2} + 1}{\sqrt{1+1/n^2} + 1}\right]$$

$$= \frac{1}{n^2}\left[1+\frac{1}{n}\right]\left[\frac{\sqrt{1+\left(\frac{1}{n+1}\right)^2} + 1}{\sqrt{1+\frac{1}{n^2}} + 1}\right]$$

$$= \frac{1}{x^2}$$

Given series is convergent for $x^2 < 1$

divergent for $x^2 > 1$

For $x^2 = 1 \Rightarrow x = 1$

$$u_n = \left(\sqrt{n^2+1} + n\right)^{-1}$$

$\therefore$ By raabe's test

$$\cancel{l = \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right)}$$

$$\cancel{= \lim_{n\to\infty} n\left[\frac{\sqrt{n^2+1} + (n)}{\sqrt{(n+1)^2+1} + (n+1)} - 1\right]}$$

$$\cancel{= \lim_{n\to\infty} n\left[\frac{\sqrt{n^2+1} - \sqrt{(n+1)^2+1} + 1}{\sqrt{(n+1)^2+1} - (n+1)}\right].}$$

$$\cancel{l = \lim_{n\to\infty} n\left[\frac{\sqrt{(n+1)^2+1} + n+1 - \sqrt{n^2+1} - n}{\sqrt{n^2+1} + n}\right]}$$

$$u_n = \left(\frac{1}{n + \sqrt{n^2+1}}\right)$$

$$v_n = \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{\cancel{n}\left(1 + \sqrt{1 + 1/n^2}\right)} \cdot \cancel{n}$$

$$= \frac{1}{1+1} = \frac{1}{2}$$

and v_n is ~~convergent~~ divergent

$\therefore$ u_n is ~~convergent~~ divergent for $\boxed{x = 1}$.

$\therefore$ Given series converges for $x^2 < 1$

and diverges for $x^2 \geq 1$

Scanned by CamScanner

1. Test the following series for convergence or divergence

a) $\sum \left[\frac{1}{n(1+\frac{1}{n})} \right]$

$$u_n = \frac{1}{n(1+1/n)} = \frac{1}{n+1}.$$

$$v_n = \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n}{n+1}$$

$$= \lim_{n\to\infty} \frac{1}{1+1/n} = 1.$$

$\therefore$ And we know that v_n is ~~convergent~~ divergent

$\therefore$ by comparison test,

u_n is divergent.

b) $\sum \frac{n^p}{(1+n)^q}$

$$u_n = \frac{n^p}{(1+n)^q} \qquad u_{n+1} = \frac{(n+1)^p}{(n+2)^q}$$

$$= \frac{n^{p-q} \cdot n^q}{(1+n)^q}$$

$$u_n = \frac{n^{p-q}}{\left(\frac{1}{n}+1\right)^q}$$

$$\frac{u_n}{u_{n+1}} = \frac{n^p}{(n+1)^q} \cdot \frac{(n+2)^q}{(n+1)^p}$$

$$= \left(1+\frac{1}{n}\right)^{-p} \left(\frac{1+2/n}{1+1/n}\right)^q$$

$$l = \lim_{n\to\infty} n \log\left(\frac{u_n}{u_{n+1}}\right)$$

$$= \lim_{n\to\infty} n\left[-p\log\left(1+\frac{1}{n}\right) + q\left[\log\left(1+\frac{2}{n}\right)\right] - q\left[\log\left(1+\frac{1}{n}\right)\right]\right]$$

$$= \lim_{n\to\infty} -p + 2q - q$$

$$\boxed{l = q - p}$$

Scanned by CamScanner

For convergence

$l > 1$

$q - p > 1$

$q > 1+p$

For divergence

$l < 1$

$q - p < 1$

$q < 1+p$

At $l = 1$, $q = 1+p$

$$u_n = \frac{n^p}{(1+n)^{p+1}}, \quad \text{let us assume } v_n = \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n^{p+1}}{(n+1)^{p+1}} = \lim_{n\to\infty} \frac{1}{(1+1/n)^{p+1}} = 1$$

$\therefore$ u_n is divergent at $q = 1+p$

$\therefore$ Series converges for $q > 1+p$

diverges for $q \leq 1+p$

c)

$$\sum \frac{(-1)^{n-1} \cos^2 nx}{n\sqrt{n}}.$$

$$|u_n| = \frac{\cos^2(nx)}{n\sqrt{n}} = \frac{\cos^2(nx)}{n^{3/2}}$$

We know that $\cos^2(nx)$ ranges from 0 to 1.

$\therefore$ $|u_n|$ will always be less than or equal to $\frac{1}{n^{3/2}}$

$$\therefore \quad u_n = \frac{\cos^2(nx)}{n^{3/2}}$$

$$v_n = \frac{1}{n^{3/2}},$$

$\therefore$ for all x,

$$u_n \leq v_n$$

We know that v_n is convergent.

$\therefore$ u_n is also convergent.

Scanned by CamScanner

Q12 Test the convergence of series

$$\frac{2^2}{3\cdot 4} + \frac{2^2\cdot 4^2}{3\cdot 4\cdot 5\cdot 6} + \frac{2^2\cdot 4^2\cdot 6^2}{3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8} + \cdots$$

$$u_n = \frac{2^2\cdot 4^2 \cdots (2n)^2}{3\cdot 4\cdot 5\cdot 6 \cdots (2n+2)}$$

$$u_{n+1} = \frac{2^2\cdot 4^2 \cdots (2n)^2 (2n+2)^2}{3\cdot 4\cdot 5\cdot 6 \cdots (2n+2)(2n+3)(2n+4)}$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{(2n+3)(2n+4)}{(2n+2)^2}$$

$$\frac{u_n}{u_{n+1}} = \frac{4n^2 + 14n + 12}{4n^2 + 8n + 4}.$$

$$l = \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right)$$

$$= \lim_{n\to\infty} n\left[\frac{+6n + 8}{4n^2 + 8n + 4}\right]$$

$$= \frac{6}{4} = \frac{3}{2} > 1$$

$\therefore$ Hence, series is convergent.

Q13 Investigate the convergence of series.

$$= \frac{1}{3}x + \frac{1}{3}\cdot\frac{2}{5}x^2 + \frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}x^3 \cdots$$

$$u_n = \frac{1\cdot 2\cdot 3 \cdots (n)}{3\cdot 5\cdot 7 \cdots (2n+1)} \left|(x^n)\right|$$

$$u_{n+1} = \frac{1\cdot 2\cdot 3 \cdots n\cdot n+1}{3\cdot 5\cdot 7 \cdots (2n+1)(2n+3)} \left|x^{n+1}\right|$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{2n+3}{n+1}\,\frac{1}{|x|}$$

$$l = \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{2}{|x|}.$$

Scanned by CamScanner

For convergence

$$\frac{2}{|x|} > 1$$

$$|x| < 2$$

For divergence.

$$\frac{2}{|x|} < 1$$

$$|x| > 2$$

At $|x| = 2$

$$\frac{u_n}{u_{n+1}} = \frac{2n+3}{2n+2}$$

$\therefore$ By roobe's test

$$l_1 = \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right)$$

$$= \lim_{n\to\infty} n\left(\frac{2n+3}{2n+2} - 1\right)$$

$$= \lim_{n\to\infty} n\left(\frac{1}{2n+2}\right)$$

$$= \frac{1}{2}.$$

$\therefore$ Series is divergent for $|x| = 2$

$\therefore$ Series converges for $|x| < 2$

diverges for $|x| \geq 2$

i.e converges for $x \in (-2, 2)$

diverges for $x \in (-\infty, -2] \cup [2, \infty)$.

Scanned by CamScanner

Q17 Discuss the convergence of series.

$$\sum \frac{\sqrt{n}}{\sqrt{n^2+1}} x^n \quad (x>0)$$

$$u_n = \frac{\sqrt{n}}{\sqrt{n^2+1}} \cdot x^n$$

$$\frac{u_n}{u_{n+1}} = \sqrt{\frac{n}{n+1}} \sqrt{\frac{(n+1)^2+1}{n^2+1}} \cdot \frac{1}{x}.$$

$$= \sqrt{\frac{1}{1+1/n}} \sqrt{\frac{1+\frac{2}{n}+\frac{2}{n^2}}{1+1/n^2}} \cdot \frac{1}{x}$$

$$l = \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \sqrt{1}\sqrt{1} \cdot \frac{1}{x}$$

$$= \frac{1}{x}$$

Series is convergent for $0 < x < 1$

divergent for $x > 1$

At $x = 1$

$$u_n = \frac{\sqrt{n}}{\sqrt{n^2+1}} = \frac{\sqrt{n}}{n\sqrt{1+1/n^2}}$$

$$= \frac{1}{\sqrt{n}\sqrt{1+\frac{1}{n^2}}}$$

Let us assume $v_n = \frac{1}{\sqrt{n}}$

$$l_1 = \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\sqrt{1+1/n^2}} = 1$$

Since v_n is divergent

u_n will also be divergent for $\boxed{x=1}$

Given series converges for $0 < x < 1$

diverges for $x \geq 1$.

Scanned by CamScanner

Q15 Investigate the convergence of series.

$$\frac{x}{1+x} + \frac{x^2}{1+x^2} \cdots \qquad x > 0$$

$$u_n = \frac{x^n}{1+x^n}$$

$$\frac{u_n}{u_{n+1}} = \frac{1}{x}\left(\frac{1+x^{n+1}}{1+x^n}\right)$$

$$l = \lim_{n\to\infty} \frac{u_n}{u_{n+1}}$$

$$= \lim_{n\to\infty} \frac{1}{x}\left(\frac{1+x^{n+1}}{1+x^n}\right)$$

for $x < 1$

$$l = \frac{1}{x}$$

$\therefore \boxed{l > 1}$, hence series is convergent.

for $x > 1$

$$l = \lim_{n\to\infty} \frac{1}{x}\left(\frac{\cancel{x^{n+1}}}{\cancel{x^{n+1}}}\right)\frac{\left[\frac{1}{x^{n+1}}+1\right]}{\left[\frac{1}{x^{n+1}}+\frac{1}{x}\right]}$$

$$l = \lim_{n\to\infty} \frac{1}{x}\left[\frac{\frac{1}{x^{n+1}}+1}{\frac{1}{x^{n+1}}+\frac{1}{x}}\right].$$

$$l = 1$$

Apply roobe's test for $\boxed{x > 1}$

$$\therefore l_x = \lim_{n\to\infty} n\left[\frac{1}{x}\frac{(1+x^{n+1})}{1+x^n} - 1\right]$$

$$= \lim_{n\to\infty} n\left[\frac{1+x^{n+1} - x - x^{n+1}}{x+x^{n+1}}\right]$$

$$= \lim_{n\to\infty} \frac{(1-x)}{x}\frac{n}{(1+x^n)}$$

Scanned by CamScanner

by L Hospital Rule

$$= \lim_{n \to \infty} \left(\frac{1-x}{x}\right)\left(\frac{1}{n x^{n-1}}\right)$$

$$= 0$$

$\therefore$ divergent for $\boxed{x > 1}$

At $x = 1$

$$u_n = \frac{1}{2}$$

$\therefore$ which is divergent

Hence, given series converges for $0 < x < 1$

diverges for $x \geq 1$

Scanned by CamScanner