Some formulas:

(1) $$\int_{\alpha}^{\alpha+2\pi} \cos mx \cdot \cos nx \cdot dx = \frac{1}{2}\int_{\alpha}^{\alpha+2\pi} [\cos(m+n)x + \cos(m-n)x] dx$$

$$= \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_{\alpha}^{\alpha+2\pi} = 0.$$

(if $m \neq n$)

(2) $$\int_{\alpha}^{\alpha+2\pi} \cos^2 nx \, dx = \left[\frac{x}{2} + \frac{\sin 2nx}{4n}\right]_{\alpha}^{\alpha+2\pi} = \pi.$$ (if $n \neq 0$)

(3) $$\int_{\alpha}^{\alpha+2\pi} \sin mx \cdot \cos nx \cdot dx = -\frac{1}{2}\left[\frac{\cos(m-n)x}{m-n} + \frac{\cos(m+n)x}{m+n}\right]_{\alpha}^{\alpha+2\pi} = 0.$$

if ($m \neq n$)

(4) $$\int_{\alpha}^{\alpha+2\pi} \sin nx \cdot \cos nx \, dx = \left[\frac{\sin^2 nx}{2n}\right]_{\alpha}^{\alpha+2\pi} = 0$$ (if $n \neq 0$)

(5) $$\int_{\alpha}^{\alpha+2\pi} \sin mx \cdot \sin nx \, dx = \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_{\alpha}^{\alpha+2\pi} = 0$$ ($m \neq n$)

(6) $$\int_{\alpha}^{\alpha+2\pi} \sin^2 nx \, dx = \left[\frac{x}{2} - \frac{\sin 2nx}{4n}\right]_{\alpha}^{\alpha+2\pi} = \pi$$ ($n \neq 0$).

(7) $$\int_{\alpha}^{\alpha+2\pi} e^{ax} \cos bx \, dx = \left[\frac{1}{a^2+b^2} e^{ax}(a\cos bx + b\sin bx)\right]_{\alpha}^{\alpha+2\pi}$$

(8) $$\int_{\alpha}^{\alpha+2\pi} e^{ax} \cdot \sin bx \, dx = \left[\frac{1}{a^2+b^2} e^{ax}(a\sin bx - b\cos bx)\right]_{\alpha}^{\alpha+2\pi}$$

Unit-V (Fourier Series)

Fouries Series:- A series expansion of a function in terms of trigonometric functions $\cos mx$ and $\sin nx$ is called fouries series.

The fourier series for the function $f(x)$ in the interval $\alpha < x < \alpha + 2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx .$$

where

$$a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\,dx .$$

$$a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cdot \cos nx\,dx .$$

$$b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cdot \sin nx \cdot dx .$$

Example:- Obtain the fourier series for $f(x) = e^{-x}$ in the interval $0 < x < 2\pi$.

Solⁿ. Let

$$e^{-x} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx . \quad \text{---(i)}$$

Then

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\,dx$$

[Here $\alpha = 0$ then limit from $\alpha = 0$ to $\alpha + 2\pi = 2\pi$.]

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{-1}\right]_0^{2\pi} = -\frac{1}{\pi}\left[e^{-2\pi} - 1\right].$$

$$\boxed{a_0 = \frac{1 - e^{-2\pi}}{\pi}}$$

& $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot \cos nx \cdot dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cdot \cos nx \cdot dx.$

Now by formula (7),

$$= \frac{1}{\pi}\cdot\frac{1}{n^2+1}\left[e^{-x}\cdot(-\cos nx + n\sin nx)\right]_0^{2\pi}$$

$$= \frac{1}{\pi(n^2+1)}\left[e^{-2\pi}(-\cos n.2\pi + n\sin n.2\pi) - e^0(-\cos 0 + n.\sin 0)\right]$$

$$= \frac{1}{\pi(n^2+1)}\left[e^{-2\pi}(-1+0)+1\right] = \frac{1-e^{-2\pi}}{\pi(n^2+1)}$$

$$\therefore a_1 = \left(\frac{1-e^{-2\pi}}{\pi}\right)\cdot\frac{1}{2}, \quad a_2 = \left(\frac{1-e^{-2\pi}}{\pi}\right)\frac{1}{5} \text{ etc.}$$

Now,

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot \sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx$$

$$= \frac{1}{\pi(n^2+1)}\left[e^{-x}(-\sin nx - n\cos nx)\right]_0^{2\pi}$$

$$= \frac{1}{\pi(n^2+1)}\left[e^{-2\pi}(-\sin n.2\pi - n.\cos n.2\pi) - e^{-0}(-\sin 0 - n\cos 0)\right]$$

$$b_n = \frac{1}{\pi(n^2+1)}\left[e^{-2\pi}(-n)+n\right] = \left(\frac{1-e^{-2\pi}}{\pi}\right)\frac{n}{n^2+1}$$

$$\therefore b_1 = \left(\frac{1-e^{-2\pi}}{\pi}\right)\frac{1}{2}, \quad b_2 = \left(\frac{1-e^{-2\pi}}{\pi}\right)\frac{2}{5} \text{ etc.}$$

Substituting the values of a_0, a_n, b_n in (i), we get

$$e^{-x} = \frac{1-e^{-2\pi}}{\pi}\left\{\frac{1}{2} + \left(\frac{1}{2}\cos x + \frac{1}{5}\cos 2x + \frac{1}{10}\cos 3x + \cdots\right) + \left(\frac{1}{2}\sin x + \frac{2}{5}\sin 2x + \cdots\right)\right\}$$

Example:- Expand $f(x) = \sqrt{(1-\cos x)}$, $0 < x < 2\pi$ in a fourier series. Hence evaluate

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots$$

Solⁿ:-

$$f(x) = \sqrt{(1-\cos x)} = \sqrt{2\sin^2\frac{x}{2}} = \sqrt{2}\sin\frac{x}{2}$$

Let $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \cdot \sin nx. \quad \text{——(i)}$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} \sqrt{2}\sin\frac{x}{2}\,dx = \frac{\sqrt{2}}{\pi}\left[-\cos\frac{x}{2}\cdot 2\right]_0^{2\pi}$$

$$= \frac{\sqrt{2}}{\pi}\left[-2\cos\frac{2\pi}{2} + 2\cdot\cos 0\right] = \frac{\sqrt{2}}{\pi}\left[-2\cdot -1 + 2\right]$$

$$= \frac{4\sqrt{2}}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \sqrt{2}\sin\frac{x}{2}\cos nx\cdot dx = \frac{\sqrt{2}}{2\pi}\int_0^{2\pi} 2\sin\frac{x}{2}\cdot\cos nx\,dx.$$

$$= \frac{\sqrt{2}}{2\pi}\left\{\int_0^{2\pi}\left[\sin\left(n+\frac{1}{2}\right)x - \sin\left(n-\frac{1}{2}\right)x\right]dx.\right.$$

$$= \frac{\sqrt{2}}{2\pi}\left[-\cos\left(n+\frac{1}{2}\right)x\cdot\frac{2}{2n+1} + \cos\left(n-\frac{1}{2}\right)x\cdot\frac{2}{2n-1}\right]_0^{2\pi}$$

$$= \frac{\sqrt{2}}{2\pi}\left[-\cos\left(n+\frac{1}{2}\right)\cdot 2\pi\cdot\left(\frac{2}{2n+1}\right) + \frac{2}{2n-1}\cdot\cos\left(n-\frac{1}{2}\right)\cdot 2\pi + \frac{2}{2n+1}\cdot\cos\left(n+\frac{1}{2}\right)\cdot 0\right.$$
$$\left. - \frac{2}{2n-1}\cos\left(n-\frac{1}{2}\right)\cdot 0\right].$$

$$\cancel{= \frac{\sqrt{2}}{2\pi}\left[-\frac{2}{2n+1}\cos(2n+1)\pi + \frac{2}{2n-1}\cos(2n-1)\pi + \frac{2}{2n+1} - \frac{2}{2n-1}\right]}.$$

$$= \frac{\sqrt{2}}{2\pi}\left[-\frac{2}{2n+1}\cos(2n+1)\pi + \frac{2}{2n-1}\cos(2n-1)\pi + \frac{2}{2n+1} - \frac{2}{2n-1}\right].$$

$$= \frac{\sqrt{2}}{2\pi}\left[\frac{2}{2n+1} - \frac{2}{2n-1} + \frac{2}{2n+1} - \frac{2}{2n-1}\right] = \frac{\sqrt{2}}{2\pi}\left[\frac{4}{2n+1} - \frac{4}{2n-1}\right]$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{2}{2n+1} - \frac{2}{2n+1}\right] = \frac{\sqrt{2}}{\pi}\left[\frac{4n-2-4n-2}{4n^2-1}\right] = \frac{-4\sqrt{2}}{\pi(4n^2-1)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \sqrt{2}\sin\frac{x}{2}\cdot\sin nx\,dx = \frac{\sqrt{2}}{2\pi}\int_0^{2\pi} 2\sin nx\cdot\sin\frac{x}{2}\cdot dx$$

$$= \frac{1}{\sqrt{2}\pi}\int_0^{2\pi}\left[\cos\left(n-\frac{1}{2}\right)x - \cos\left(n+\frac{1}{2}\right)x\right]dx$$

$$= \frac{1}{\sqrt{2}\pi}\left[\frac{2}{2n-1}\cdot\sin\left(\frac{2n-1}{2}\right)x - \frac{2}{2n+1}\sin\left(\frac{2n+1}{2}\right)x\right]_0^{2\pi}$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{1}{2n-1}\cdot\sin(2n-1)\pi - \frac{1}{2n+1}\sin(2n+1)\cdot\pi - \frac{2}{2n-1}\cdot\sin 0 + \frac{2}{2n+1}\cdot\sin 0\right].$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{1}{2n-1}\cdot 0 - \frac{1}{2n+1}\cdot 0 - 0 + 0\right] = 0.$$

Substituting the values of a's & b's in (1)

$$\sqrt{(1-\cos x)} = \frac{2\sqrt{2}}{\pi} - \sum_{n=1}^{\infty}\frac{4\sqrt{2}}{(4n^2-1)\pi}\cos nx.$$

when $x=0$, we have

$$0 = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)}$$

i.e. $-\frac{2\sqrt{2}}{\pi} = \frac{-4\sqrt{2}}{\pi}\left(\frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \cdots\right)$

$$\frac{1}{2} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots$$

Any function $f(x)$ can be developed as a fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ where a_0, a_n, b_n are constants, provided:

(i) $f(x)$ is periodic, single-valued & finite;

(ii) $f(x)$ has a finite number of discontinuties in any one period;

functions having Points of discontinuity

if in the interval $(\alpha, \alpha+2\pi)$ $f(x)$ is defined by

$$f(x) = \begin{cases} \phi(x), & \alpha < x < c \\ \psi(x), & c < x < \alpha + 2\pi \end{cases}$$

i.e. c is the pt. of discontinuity, then

$$a_0 = \frac{1}{\pi}\left[\int_{\alpha}^{c} \phi(x)\,dx + \int_{c}^{\alpha+2\pi} \psi(x)\,dx\right].$$

$$a_n = \frac{1}{\pi}\left[\int_{\alpha}^{c} \phi(x)\cos nx\,dx + \int_{c}^{\alpha+2\pi} \psi(x)\cdot\cos nx\,dx\right]$$

$$b_n = \frac{1}{\pi}\left[\int_{\alpha}^{c} \phi(x)\sin nx\,dx + \int_{c}^{\alpha+2\pi} \psi(x)\sin nx\,dx\right].$$

At ~~a~~ point of finite discontinuity $x = c$, both the limit on the left [i.e. $f(c-0)$] and the limit on the right [i.e. $f(c+0)$] exist & are different. At such a pt, fouries series gives the value of $f(x)$ as the arthmetic mean of these two limits.

i.e. at $x = c$, $f(x) = \frac{1}{2}\left[f(c-0) + f(c+0)\right]$.

Example: Find the fourier series expansion for $f(x)$, if

$$f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi. \end{cases}$$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

Sol: Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi)\,dx + \int_0^{\pi} x\,dx\right] = \frac{1}{\pi}\left[(-\pi x)_{-\pi}^{0} + \left(\frac{x^2}{2}\right)_0^{\pi}\right].$$

$$= \frac{1}{\pi}\left[-\pi^2 + \frac{\pi^2}{2}\right] = \frac{1}{\pi}\left[-\frac{\pi^2}{2}\right] = -\frac{\pi}{2}.$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi).\cos nx\,dx + \int_0^{\pi} x.\cos nx\,dx.\right.$$

$$= \frac{1}{\pi}\left[-\pi.\left(\frac{\sin nx}{n}\right)_{-\pi}^{0} + \left(\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right)_0^{\pi}\right].$$

$$= \frac{1}{\pi}\left[-\pi.\frac{\sin n(-\pi)}{n} + \pi.\frac{\sin n.0}{n} + \frac{\pi\sin n\pi}{n} + \frac{\cos n\pi}{n^2} - 0 - \frac{\cos 0}{n^2}\right]$$

$$= \frac{1}{\pi}\left[0 + 0 + 0 + \frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{1}{\pi n^2}\left[\cos n\pi - 1\right].$$

$$\therefore\ a_n = \frac{1}{\pi n^2}\left[(-1)^n - 1\right] \qquad \left[\because \cos n\pi = (-1)^n\right]$$

$$a_1 = \frac{1}{\pi}[-1-1] = -\frac{2}{\pi},\ a_2 = 0,\ a_3 = \frac{-2}{\pi.3^2},\ a_4 = 0 \text{ etc.}$$

Now

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi)\sin nx\,dx + \int_0^{\pi} x.\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi\cos nx}{n}\right)_{-\pi}^{0} + \left(-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right)_0^{\pi}\right].$$

$$= \frac{1}{\pi}\left[\frac{\pi.\cos n0}{n} - \frac{\pi.\cos n.(-\pi)}{n} - \frac{\pi\cos n\pi}{n} + \frac{\sin n\pi}{n^2} + \frac{0.\cos n.0}{n} - \frac{\sin n.0}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n} - \frac{\pi}{n}\cos n\pi - \frac{\pi}{n}\cos n\pi\right].$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n} - \frac{2\pi}{n}\cos n\pi\right] = \frac{1}{\pi}\cdot\frac{\pi}{n}\left[1 - 2\cos n\pi\right]$$

$$b_n = \frac{1}{n}\left[1-2\cos n\pi\right] = \frac{1}{n}\left[1-2(-1)^n\right]$$

$\therefore\ b_1 = 3,\ b_2 = -\frac{1}{2},\ b_3 = 1,\ b_4 = -\frac{1}{4}$ etc.

Substituting the values of a's & b's in (i), we get

$$f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right) + 3\sin x - \frac{\sin 2x}{2} + \frac{3\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots \quad \text{---(ii)}$$

Put $x=0$ in (ii), we obtain $f(0) = -\frac{\pi}{4} - \frac{2}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$

Now $f(x)$ is discontinuous at $x=0$.

$f(0-0) = -\pi$ & $f(0+0) = 0$

$\therefore\ f(0) = \frac{1}{2}\left[f(0-0) + f(0+0)\right] = -\frac{\pi}{2}$

Hence,
$$-\frac{\pi}{2} = -\frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right].$$

If $f(x)$ is defined in interval $(\alpha, \alpha+2c)$, then fourier expansion of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c}.$$

where
$$a_0 = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\,dx.$$

$$a_n = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\cdot\cos\frac{n\pi x}{c}\cdot dx.$$

$$b_n = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\cdot \sin\frac{n\pi x}{c}\,.dx\,.$$

In short:-

If $f(x)$ is defined in interval $(-l, l)$ then fouries expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\,.$$

where $$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx\,.$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,.dx\,.$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\cdot\sin\frac{n\pi x}{l}\,.dx\,.$$

Holds for both interval $[\alpha, \alpha+2\pi]$ & $[\alpha, \alpha+2c]$.

Example:- Expand $f(x) = e^{-x}$ as a fourier series in $(-l, l)$.

Solⁿ:- series is of the form

$$e^{-x} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\,. \qquad \text{—— (i)}$$

Then $$a_0 = \frac{1}{l}\int_{-l}^{l} e^{-x}.dx = \frac{1}{l}\left[-e^{-x}\right]_{-l}^{l} = \frac{1}{l}\left[e^{l} - e^{-l}\right] = \frac{2\sinh l}{l}$$

& $$a_n = \frac{1}{l}\int_{-l}^{l} e^{x}.\cos\frac{n\pi x}{l}\,dx\,.$$

$$= \frac{1}{l}\left[\frac{e^{-x}}{1+\left(\frac{n\pi}{l}\right)^2}\left(-\cos\frac{n\pi x}{l} + \frac{n\pi}{l}\sin\frac{n\pi x}{l}\right)\right]_{-l}^{l}$$

$$= \frac{1}{l}\left[\frac{e^{-l}}{1+(n\pi/l)^2}\left(-\cos\frac{n\pi l}{l} + \frac{n\pi}{l}\sin\frac{n\pi l}{l}\right) - \frac{e^{l}}{1+\left(\frac{n\pi}{l}\right)^2}\left(-\cos\frac{n\pi(-l)}{l} + \right.\right.$$

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$$\frac{n\pi}{l}\sin\frac{n\pi(-l)}{l}\Big]$$

$$=\frac{1}{l}\left[\frac{e^{-l}}{1+\left(\frac{n\pi}{l}\right)^2}\left(-\cos n\pi+\frac{n\pi}{l}\sin n\pi\right)-\frac{e^{l}}{1+\left(\frac{n\pi}{l}\right)^2}\left(-\cos\frac{n\pi l}{l}-\frac{n\pi}{l}\sin\frac{n\pi l}{l}\right)\right]$$

$$=\frac{1}{l}\left[\frac{e^{-l}}{1+\left(\frac{n\pi}{l}\right)^2}\left(-(-1)^n+0\right)-\frac{e^{l}}{1+\left(\frac{n\pi}{l}\right)^2}\left(-(-1)^n-0\right)\right]$$

$$=\frac{1}{l}\left[-\frac{e^{-l}}{1+\left(\frac{n\pi}{l}\right)^2}\cdot(-1)^n+\frac{e^{l}}{1+\left(\frac{n\pi}{l}\right)^2}(-1)^n\right].$$

$$=\frac{(-1)^n}{l\left(1+\left(\frac{n\pi}{l}\right)^2\right)}\left[e^{l}-e^{-l}\right]=\frac{(-1)^n\cdot 2l^2\cdot\sinh l}{l\left[l^2+(n\pi)^2\right]}$$

$$=\frac{(-1)^n\cdot 2l\sinh l}{l^2+(n\pi)^2}.$$

$\therefore\ a_1=\dfrac{-2l\sinh l}{l^2+\pi^2}$, $a_2=\dfrac{2l\sinh l}{l^2+2^2\pi^2}$, $a_3=\dfrac{-2l\sinh l}{l^2+3^2\pi^2}$ etc.

$$b_n=\frac{1}{l}\int_{-l}^{l}e^{-x}\sin\frac{n\pi x}{l}\,.dx.$$

$$=\frac{1}{l}\left[\frac{e^{-x}}{1+\left(\frac{n\pi}{l}\right)^2}\left(-\sin\frac{n\pi x}{l}-\frac{n\pi}{l}\cos\frac{n\pi x}{l}\right)\right]_{-l}^{l}$$

$$b_n=\frac{2n\pi(-1)^n\sinh l}{l^2+(n\pi)^2}.$$

$\therefore\ b_1=\dfrac{-2\pi\sinh l}{l^2+\pi^2}$, $b_2=\dfrac{4\pi\sinh l}{l^2+2^2\pi^2}$, $b_3=\dfrac{-6\pi\sinh l}{l^2+3^2\pi^2}$ etc.

Substituting the values of a's & b's in (i)

$$e^{-x} = \sinh l \left\{ \frac{1}{l} - 2l\left(\frac{1}{l^2+\pi^2} \cos\frac{\pi x}{l} - \frac{1}{l^2+2^2\pi^2} \cos\frac{2\pi x}{l} + \frac{1}{l^2+3^2\pi^2} \cos\frac{3\pi x}{l} \right.\right.$$

$$\left.\left. - \cdots \right) - 2\pi\left(\frac{1}{l^2+\pi^2} \sin\frac{n\pi}{l} - \frac{2}{l^2+2^2\pi^2} \sin\frac{2\pi x}{l} + \cdots \right) \right\}$$

Even & odd functions

A fu^n $f(x)$ is said to be even if $f(-x) = f(x)$

eg:- $\cos x$, $\sec x$, x^2.

$f(x) = \cos x$

$f(-x) = \cos(-x) = \cos x$,

$\Rightarrow f(x) = f(-x) \Rightarrow$ even fu^n.

A fu^n $f(x)$ is said to be odd if $f(-x) = -f(x)$.

eg:- $\sin x$, x^3.

Note:-
$$\int_{-c}^{c} f(x)\,dx = \begin{cases} 2\int_0^c f(x)\,dx, & \text{when } f(x) \text{ is even } f^n. \\ 0, & \text{when } f(x) \text{ is odd } fu^n. \end{cases}$$

Expansions of Even or odd periodic fu^n:-

A periodic fu^n $f(x)$ defined in $(-c, c)$ can be represented by the fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c}.$$

where $a_0 = \frac{1}{c}\int_{-c}^{c} f(x)\,dx$, $a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}\,dx$

$$b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}\,dx.$$

Case I :- when $f(x)$ is even fu^n.

$$a_0 = \frac{1}{c}\int_{-c}^{c} f(x)\,dx = \frac{2}{c}\int_{0}^{c} f(x)\,dx.$$

since $f(x)\cdot\cos\frac{n\pi x}{c}$ is also even fu^n,

$$\therefore\ a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cdot\cos\frac{n\pi x}{c}\,dx = \frac{2}{c}\int_{0}^{c} f(x)\cos\frac{n\pi x}{c}\,dx.$$

Again since $f(x)\cdot\sin\frac{n\pi x}{c}$ is odd fu^n

$$\therefore\ b_n = \frac{1}{c}\int_{-c}^{c} f(x)\cdot\sin\frac{n\pi x}{c}\,dx = 0.$$

Hence, fourier expansion contains only cosine terms &

$$a_0 = \frac{2}{c}\int_{0}^{c} f(x)\,dx.$$

$$a_n = \frac{2}{c}\int_{0}^{c} f(x)\cdot\cos\frac{n\pi x}{c}\,dx.$$

* Use this formula for even fu^n fourier series expansion

Case II :- Illy, when $f(x)$ is odd fu^n; then its fourier expansion contains only sine terms &

$$a_0 = 0,\ a_n = 0$$

$$\&\ b_n = \frac{2}{c}\int_{0}^{c} f(x)\cdot\sin\frac{n\pi x}{c}\,dx.$$

Example :- Express $f(x) = \frac{x}{2}$ as a fourier series in interval $-\pi < x < \pi$.

Sol^n :- Since, $f(-x) = \frac{-x}{2} = -f(x)$

$\therefore$ $f(x)$ is odd fu^n & ~~Hen~~

hence $f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot \sin nx \cdot dx$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{x}{2}\cdot \sin nx\, dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$\cancel{= \frac{1}{\pi}\left[\pi\left(-\frac{\cos n\pi}{n}\right) + 0\left(\frac{1}{n}\cdot \cos n\cdot 0\right) + \left(\frac{\sin n\pi}{n^2} - \frac{\sin n\cdot 0}{n^2}\right)\right]}$$

$$\cancel{= \frac{1}{\pi}\left[\pi\left(\frac{-(-1)^n}{n} + \frac{1}{n}\right)\right] = \frac{1-(-1)^n}{n}}$$

$\therefore$ ~~$b_1 =$~~

$$= \frac{1}{\pi}\left[\pi\left(-\frac{\cos n\pi}{n} + \frac{\sin n\pi}{n^2}\right) + 0\left(\frac{\cos n\cdot 0}{n} - \frac{\sin n\cdot 0}{n^2}\right)\right].$$

$$= \frac{1}{\pi}\left[\pi\cdot\left(-\frac{\cos n\pi}{n}\right)\right] = -\frac{\cos n\pi}{n}$$

$\therefore\ b_1 = 1,\ b_2 = \frac{-1}{2},\ b_3 = \frac{1}{3},\ b_4 = \frac{-1}{4}$ etc.

Hence the series is $\frac{x}{2} = \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \cdots$

Example:- find a fourier series ~~to represent x^2 in the interval~~ for the fun^n $f(x)$ given by

$$f(x) = \begin{cases} 1 + \frac{2x}{\pi} & ; \quad -\pi \le x \le 0 \\ 1 - \frac{2x}{\pi} & ; \quad 0 \le x \le \pi. \end{cases}$$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

Sol^n:- Since $f(-x) = 1 - \frac{2x}{\pi}$ in $(-\pi, 0) = f(x)$ in $(0, \pi)$.

& $f(-x) = 1 + \frac{2x}{\pi}$ in $(0, \pi)$. $= f(x)$ in $(-\pi, 0)$.

$\therefore$ $f(x)$ is an even fu^n in $(-\pi, \pi)$.

$\therefore$ $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$.

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)dx$

$$= \frac{2}{\pi}\left(x - \frac{2x^2}{\pi.2}\right)_0^{\pi} = \frac{2}{\pi}\left(\pi - \frac{\pi^2}{\pi}\right) = 0.$$

& $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)\cos nx\,dx$

$$= \frac{2}{\pi}\left[\left(1 - \frac{2x}{\pi}\right)\frac{\sin nx}{n} - \left(-\frac{2}{\pi}\right)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(1 - \frac{2\pi}{\pi}\right)\frac{\sin n\pi}{n} - \frac{2}{\pi}\left(\frac{\cos n\pi}{n^2}\right) - \left(1 - 0\right)\frac{\sin n\cdot 0}{n} + \frac{2}{\pi}\frac{\cos n\cdot 0}{n^2}\right]$$

$$= \frac{2}{\pi}\left[-\frac{\sin n\pi}{n} - \frac{2}{\pi}\frac{\cos n\pi}{n^2} + \frac{2}{\pi}\cdot\frac{1}{n^2}\right].$$

$$= \frac{2}{\pi}\left[0 - \frac{2}{\pi}\frac{(-1)^n}{n^2} + \frac{2}{\pi}\cdot\frac{1}{n^2}\right] = \frac{4}{\pi n^2}\left[1 - (-1)^n\right].$$

$\therefore$ $a_1 = \frac{8}{\pi^2}$, $a_3 = \frac{8}{3^2\cdot\pi^2}$, $a_5 = \frac{8}{5^2\pi^2}$ etc.

& $a_2 = a_4 = a_6 = \cdots = 0$.

Thus substituting the values of a's in (i)

$$f(x) = \frac{8}{\pi^2}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right].$$

Half Range Series:-

(1) Half Range fourier sine series:-

$$f(x)=\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{c} \qquad 0<x<c.$$

where $b_n=\frac{2}{c}\int_0^c f(x).\sin\left(\frac{n\pi x}{c}\right).dx.$

(2) Half Range fourier cosine series

$$f(x)=a_0+\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{c}\right) \quad , 0<x<c$$

$$a_0=\frac{1}{c}\int_0^c f(x)dx.$$

$$a_n=\frac{2}{c}\int_0^c f(x).\cos\left(\frac{n\pi x}{c}\right).dx.$$

Example:- Find half Range cosine series of
$f(x)=x, \quad 0<x<2.$

Sol^n:-

$$f(x)=a_0+\sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right).$$

$$a_0=\frac{1}{2}\int_0^2 f(x)dx=\frac{1}{2}\int_0^2 x\,dx=\frac{1}{2}\left(\frac{x^2}{2}\right)_0^2=\frac{1}{2}\left(\frac{4}{2}-0\right)$$

$$=\frac{1}{2}$$

$$a_n=\frac{2}{2}\int_0^2 x.\cos\left(\frac{n\pi x}{2}\right)dx.$$

$$a_n=x\int_0^2 \cos\frac{n\pi x}{2}.dx-\int_0^2 1.\int_0^2 \cos\frac{n\pi x}{2}.dx\,.dx.$$

$$=\left[x.\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}+\frac{\cos\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2}\right]_0^2$$

$$=2.\frac{2}{n\pi}.\sin\frac{n\pi.2}{2}+\frac{4}{n^2\pi^2}\cos\frac{n\pi.2}{2}-0.\frac{2}{n\pi}\sin\frac{n\pi.0}{2}-\left(\frac{2}{n\pi}\right)^2.\cos 0$$

$$=\frac{4}{n\pi}.0+\frac{4}{n^2\pi^2}(-1)^n-\frac{4}{n^2\pi^2}.1$$

$$a_n = \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right].$$

$$\therefore\ a_1 = \frac{-8}{\pi^2},\quad a_2 = 0,\quad a_3 = \frac{-8}{3^2\pi^2},\quad a_4 = 0.$$

Hence,

$$f(x) = 1 - \frac{8}{\pi^2}\left[\frac{\cos \pi x/2}{1^2} + \frac{\cos 3\pi x/2}{3^2} + \cdots\right].$$

<u>Example:</u> Find Half Range sine series of

$$f(x) = x,\quad 0 < x < 2.$$

<u>Sol^n:-</u>

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$$

where

$$b_n = \frac{2}{2}\int_0^2 x \sin\left(\frac{n\pi x}{2}\right) dx$$

$$= \left[-\frac{x\cos\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}} + \frac{\sin\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2}\right]_0^2$$

$$= \left[-2\times\frac{2}{n\pi}\cos\left(\frac{n\pi\cdot 2}{2}\right) + \frac{4}{n^2\pi^2}\cdot\sin\left(\frac{n\pi\cdot 2}{2}\right) + 0\cdot\frac{2}{n\pi}\cos\frac{n\pi}{2}\cdot 0 - \frac{4}{n^2\pi^2}\cdot\sin\left(\frac{n\pi}{2}\cdot 0\right)\right]$$

$$= \frac{-4}{n\pi}(-1)^n + 0 + \cancel{\frac{4}{n\pi}}\,0$$

$$\cancel{b_n} = \cancel{\frac{-4}{n\pi}\left((-1)^n - 1\right)}\quad b_n = \frac{-4(-1)^n}{n\pi}$$

$$\therefore\ b_1 = \frac{4}{\pi},\quad b_2 = \frac{-4}{2\pi},\quad b_3 = \frac{4}{3\pi},\quad b_4 = \frac{-4}{4\pi}\ \text{etc.}$$

Hence

$$f(x) = \frac{4}{\pi}\left(\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} - \frac{1}{4}\sin\frac{4\pi x}{2} + \cdots\right)$$

Complex form of fourier Series :-

complex form of fourier series is

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x/c}$$

where $C_n = \frac{1}{2c}\int_{-c}^{c} f(x) e^{-in\pi x/c} \cdot dx.$

Example :- Find the Complex form of the fourier series of $f(x) = e^{-x}$ in $-1 \le x \le 1$.

Sol^n :-

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x}.$$

where $C_n = \frac{1}{2}\int_{-1}^{1} e^{-x} \cdot e^{-in\pi x} \cdot dx = \frac{1}{2}\int_{-1}^{1} e^{-(1+in\pi)x} \cdot dx$

$$= \frac{1}{2}\left[\frac{e^{-(1+in\pi)x}}{-(1+in\pi)}\right]_{-1}^{1} = \frac{e^{1+in\pi} - e^{-(1+in\pi)}}{2(1+in\pi)}$$

$$= \frac{e(\cos n\pi + i\sin n\pi) - e^{-1}(\cos n\pi - i\sin n\pi)}{2(1+in\pi)}$$

$$= \frac{(e-e^{-1})(-1)^n}{2} \cdot \frac{1}{(1+in\pi)} = \frac{(e-e^{-1})(-1)^n}{2} \cdot \frac{1-in\pi}{(1+in\pi)(1-in\pi)}$$

$$= \frac{(e-e^{-1})(-1)^n}{2} \cdot \frac{(1-in\pi)}{1+n^2\pi^2} = \frac{(-1)^n(1-in\pi)\sinh 1}{1+n^2\pi^2}$$

Hence, $e^{-x} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n(1-in\pi)\sinh 1 \cdot e^{in\pi x}}{1+n^2\pi^2}$

Harmonic Analysis:-

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos n\theta + b_n \sin n\theta \right)$$

$$a_0 = \frac{2}{N} \sum y.$$

$$a_n = \frac{2}{N} \sum y \cos n\theta.$$

$$b_n = \frac{2}{N} \sum y \sin n\theta.$$

Note

1) Find fourier series upto first Harmonic.

$$f(x) = a_0 + a_1 \cos\theta + b_1 \sin\theta.$$

(2) Amplitude of first Harmonic

$$A_1 = \sqrt{a_1^2 + b_1^2}$$

Ques:- Obtain ~~const~~ the first three coefficients in the fourier cosine series for y, where y is given in the following table:

x:	0	1	2	3	4	5
y:	4	8	15	7	6	2

Sol^n:- Taking the interval as $60°$ (because total ~~number~~ terms ~~of values for x~~ $= 6$, then $\frac{360°}{6} = 60°$), we have

$\theta =$	$0°$	$60°$	$120°$	$180°$	$240°$	$300°$
$x =$	0	1	2	3	4	5
$y =$	4	8	15	7	6	2

∴ fourier cosine series in the intervals $(0, 2\pi)$ is

$$y = \frac{a_0}{2} + a_1 \cos\theta + a_2 \cos 2\theta + a_3 \cos 3\theta + \ldots$$

θ°	$\cos\theta$	$\cos 2\theta$	$\cos 3\theta$	y	$y\cos\theta$	$y\cos 2\theta$	$y\cos 3\theta$
0°	$\cos 0 = 1$	1	1	4	4	4	4
60°	$1/2$	$-1/2$	-1	8	4	-4	-8
120°	$-1/2$	$-1/2$	1	15	-7.5	-7.5	15
180°	-1	1	-1	7	-7	7	-7
240°	$-1/2$	$-1/2$	1	6	-3	-3	6
300°	$1/2$	$-1/2$	-1	2	1	-1	-2
			$\Sigma =$	42	-8.5	-4.5	8

Hence, $\cancel{a_0 = \frac{1}{N}\sum y = \frac{1}{42}}$

$$a_0 = \frac{2}{N}\sum y = \frac{2}{6} \cdot 42 = 14.$$

$$\cancel{a_n = \frac{2}{6}\sum y\cos n\theta = \frac{2}{6} \cdot -8.5 = -2.8}.$$

$$\cancel{b_n = \frac{2}{6}\sum y \sin n\theta = \frac{1}{3}}.$$

$$a_1 = \frac{2}{6} \cdot y\cos\theta = \frac{1}{3} \cdot (-8.5) = -2.8$$

$$a_2 = \frac{2}{6} \cdot y\cos 2\theta = \frac{1}{3} \cdot (-4.5) = -1.5$$

$$a_3 = \frac{2}{6} \cdot y\cos 3\theta = \frac{1}{3} \times 8 = 2.7.$$

Hence $y = 7 + (-2.8)\cos\theta + (-1.5)\cos 2\theta + (2.7)\cos 3\theta + \cdots$

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