Maths Assignment -5

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DTU/2K16/B1/COE
BTech COE.

Q1 → $\vec{A} = x^2z\,\hat{i} + yz^3\,\hat{j} - 3xy\,\hat{k}$

$\phi = 2x^2 + yz$

a) $\vec{A}\cdot(\vec{\nabla}\phi)$

$$\vec{\nabla}\phi = \frac{\partial\phi}{\partial x}\cdot\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$= 4x\hat{i} + z\hat{j} + y\hat{k}$$

$$\vec{A}\,(\vec{\nabla}\phi) = (x^2z\,\hat{i} + yz^3\,\hat{j} - 3xy\,\hat{k})\cdot(4x\hat{i} + z\hat{j} + y\hat{k})$$

$$= 4x^3z + yz^4 - 3xy^2$$

b) $(\vec{A}\cdot\vec{\nabla})\phi$

$$= \left((x^2z\,\hat{i} + yz^3\,\hat{j} - 3xy\,\hat{k})\cdot\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\right)\cdot\phi$$

$$= (2xz + z^3 - 0)\,\phi$$

$$= (2xz + z^3)\cdot(2x^2 + yz)$$

$$= 4x^3z + 2xyz + 2x^2z^3 + yz^4$$

c) $(\vec{A}\times\vec{\nabla})\phi$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x^2z & yz^3 & -3xy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}\cdot\phi$$

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$$= \frac{\left[\hat{i}\,2x + \hat{j}\,2y + \hat{k}\,2z\right]}{2\sqrt{x^2+y^2+z^2}} \cdot \frac{1}{2x}$$

$$= \frac{\left(\hat{i}\,2x + \hat{j}\,2y + \hat{k}\,2z\right)}{2x^{2-1}}$$

$$= x^{2-1}\left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right)$$

$$= x^{2-1}\left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right)(x^n)$$

$$= \vec{\nabla}(x^n)$$

(a)

$$\overset{\text{Ans}}{=}$$

$$= \frac{x^2}{\frac{1}{-1}}$$

$$= \frac{x^2}{-1}$$

$$= \frac{x^2}{-\left(\sqrt{x^2+y^2+z^2}\right)^3}$$

$$= \frac{x^2}{-\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}$$

$$\left[\hat{i}\,z + \hat{j}\,y + \hat{k}\,x\right]$$

$$\left[\hat{i}\,zx + \hat{j}\,yx + \hat{k}\,xy\right]$$

$$\left[\hat{i}\,zx + \hat{j}\,yx + \hat{k}\,xy\right]\frac{1}{\sqrt{x^2+y^2+z^2}}$$

$$\frac{1}{r}(\vec{r})$$

$$\frac{1}{r}(\vec{r})$$

$$= \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right)$$

$$\vec{\nabla}\left(\frac{1}{r}\right) = \left(\frac{\partial}{\partial x}\,\frac{\partial}{\partial y}\right)$$

$$|\vec{r}| = r = \sqrt{x^2+y^2+z^2}$$

$$\text{Put} \quad x = x \quad y = y \quad z = z$$

$$\overset{\text{Ans}}{=}$$

$$= \hat{i}\left(6x^2y + 3y^2z - 6x^2 - 3y^2z\right)$$

$$= \hat{i}\left(6x^2y - 6x^2\right)$$

$$= \hat{i}\left(3x^2y - 3x^2\right)$$

$$= \hat{i}\left(3x^2y - 3x^2\right)$$

$$= \left[\hat{i}\left(3x^2y + 3x\right) - \hat{j}\left(x^2 + 3x\right) + \hat{k}\left(0\right)\right]$$

$$\phi$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)x^n = \frac{\partial^2}{\partial x^2}x^n + \frac{\partial^2}{\partial y^2}x^n + \frac{\partial^2}{\partial z^2}x^n$$

$$= \left(\frac{\partial}{\partial x}\right)\cdot\left(\frac{\partial}{\partial x}x^n\right) + \left(\frac{\partial}{\partial y}\right)\cdot\left(\frac{\partial}{\partial y}x^n\right) + \left(\frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial}{\partial z}x^n\right)$$

RHS

(d) $\vec{\nabla}^2(x^n) = n(n+1)\,x^{n-2}$.

$$= \frac{1}{x^2}\,\frac{\partial}{\partial x}\left(\frac{x^2+y^2+z^2}{x^2+y^2+z^2}\right)$$

$$= \frac{\partial}{\partial x}\left[n r^{n-1} \cdot \frac{2x}{2\sqrt{x^2+y^2+z^2}} \right] + \frac{\partial}{\partial y}\left[n r^{n-1} \cdot \frac{y}{\sqrt{x^2+y^2+z^2}} \right]$$

$$+ \frac{\partial}{\partial z}\left[n r^{n-1} \cdot \frac{z}{\sqrt{x^2+y^2+z^2}} \right].$$

$$= n\left[\frac{\partial}{\partial x}(r^{n-2} \cdot x) + \frac{\partial}{\partial y}(r^{n-2} \cdot y) + \frac{\partial}{\partial z}(r^{n-2} \cdot z) \right]$$

$$= n\left[r^{n-2} \cdot 1 + n(n-2) r^{n-3} \cdot \frac{x}{r} + r^{n-2} \cdot 1 + y(n-2) r^{n-3} \cdot \frac{y}{r} \right.$$

$$\left. + r^{n-2} + z(n-2) \cdot r^{n-3} \cdot \frac{z}{r} \right]$$

$$= n r^{n-3}\left[3r + (n-2)\left[\frac{x^2+y^2+z^2}{r} \right] \right.$$

$$= n r^{n-3}\left[3r + (n-2) r \right]$$

$$= n(n+1)(r^{n-2})$$

HP

Q4 a) $\vec{A} = 3y^4z^2\hat{i} + 4x^3z^2\hat{j} - 3x^2y^2\hat{k}$ is solenoidal.

$$\vec{\nabla}\cdot(\vec{A}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(3z^2y^4\hat{i} + 4x^3z^2\hat{j} - 3x^2y^2\hat{k}\right)$$

$$= \frac{\partial}{\partial x}\left[3z^2y^4\right] + \frac{\partial}{\partial y}\left[4x^3z^2\right] - \frac{\partial}{\partial z}\left[3x^2y^2\right]$$

$$= 0$$

Hence, $\vec{A}$ is solenoidal.

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b) $\vec{V} = (2x+y)\hat{i} - (3y+2z)\hat{j} + (x+az)\hat{k}$ is solenoidal

Hence $\vec{\nabla}\cdot\vec{V} = 0$

$$\therefore \frac{\partial}{\partial x}(2x+y) - \frac{\partial}{\partial y}(3y+2z) + \frac{\partial}{\partial z}(x+az) = 0$$

$$2 - 3 + a = 0$$

$$\boxed{a = 1}$$

Q5 → $\vec{V} = (y\sin z - \sin x)\hat{i} + (x\sin z + 2yz)\hat{j} + (xy\cos z + y^2)\hat{k}$

To know irrotational or not, curl is to found out

$$\text{curl } \vec{V} = \vec{\nabla}\times\vec{V}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y\sin z - \sin x & x\sin z + 2yz & xy\cos z + y^2 \end{vmatrix}$$

$$= \hat{i}\left[x\cos z + 2y - x\cos z - 2y\right] - \hat{j}\left[y\cos z - y\cos z\right] + \hat{k}\left[\sin z - \sin z\right].$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k}$$

Hence, vector is irrotational.

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To find potential function.

$$\vec{F} = \vec{\nabla} f.$$

$$(y\sin z - \sin x)\,\hat{i} + (x\sin z + 2yz)\,\hat{j} + (xy\cos z + y^2)\,\hat{k} = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

By equating coefficients.

$$\frac{\partial f}{\partial x} = y\sin z - \sin x. \quad -(1) \qquad \frac{\partial f}{\partial z} = xy\cos z + y^2. \quad -(3)$$

$$\frac{\partial f}{\partial y} = x\sin z + 2yz. \quad (2)$$

Integrating (1) wrt x

$$f(x,y,z) = xy\sin z + \cos x. + f_1(y,z). \quad -(4)$$

Now differentiating wrt y

$$\frac{\partial f}{\partial y} = x\sin z + \frac{\partial f_1}{\partial y}.$$

$$\boxed{\frac{\partial f_1}{\partial y} = 2yz} \quad -(5).$$

Integrating it wrt y.

$$f_1(y,z) = y^2 z + f_2(z).$$

$$f(x,y,z) = xy\sin z + \cos x + y^2 z + f_2 z.$$

$$\frac{\partial f}{\partial z} = xy\cos z + y^2 + \frac{\partial f_2}{\partial z}.$$

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$$\therefore \frac{\partial f_2}{\partial z} = 0 \quad \boxed{\text{From } 3}$$

$$\boxed{f_2 = c}$$

Hence,

Velocity potential $f(x,y,z) = xy \sin z + \cos x + y^2 z$

Q6 $\vec{u}$ & $\vec{r}$ have continous second order derivates

a) $\text{curl}(\text{grad } u) = 0$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial^2 u}{\partial y \partial z} - \frac{\partial^2}{\partial z \partial y}\right] - \hat{j}\left[\frac{\partial^2 u}{\partial x \partial z} - \frac{\partial^2 u}{\partial z \partial x}\right] + k\left[\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y \partial x}\right]$$

Since, u is continous at second order derivative

Hence $\frac{\partial^2 u}{\partial y \partial z} = \frac{\partial^2 u}{\partial z \partial y}$ or so on.

Hence $\text{curl}(\text{grad } u) = 0$.

b) $\text{div}(\text{curl } \vec{r}) = 0$

Let $\vec{r} = r_1 \hat{i} + r_2 \hat{j} + r_3 \hat{k}$.

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$$\therefore \operatorname{div}(\operatorname{curl}\vec{r}) = \operatorname{div}\left(\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r_1 & r_2 & r_3 \end{vmatrix}\right)$$

$$= \operatorname{div}\left(\hat{i}\left(\frac{\partial r_3}{\partial y} - \frac{\partial r_2}{\partial z}\right) - \hat{j}\left(\frac{\partial r_3}{\partial x} - \frac{\partial r_1}{\partial z}\right) + \hat{k}\left(\frac{\partial r_2}{\partial x} - \frac{\partial r_1}{\partial y}\right)\right)$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(\hat{i}\left(\frac{\partial r_3}{\partial y} - \frac{\partial r_2}{\partial z}\right) - \hat{j}\left(\frac{\partial r_3}{\partial x} - \frac{\partial r_1}{\partial z}\right) + \hat{k}\left(\frac{\partial r_2}{\partial x} - \frac{\partial r_1}{\partial y}\right)\right)$$

$$= \frac{\partial^2 r_3}{\partial x \partial y} - \frac{\partial^2 r_2}{\partial x \partial z} - \frac{\partial^2 r_3}{\partial y \partial x} + \frac{\partial^2 r_1}{\partial y \partial z} + \frac{\partial^2 r_2}{\partial z \partial x} - \frac{\partial^2 r_1}{\partial z \partial y}.$$

$$= 0$$

$$\left[\because \frac{\partial^2 r_3}{\partial x \partial y} = \frac{\partial^2 r_3}{\partial y \partial x}, \text{ Continuity of } \vec{r}\right].$$

Q7 $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

a). $\vec{\nabla} \cdot \left[r \vec{\nabla}\left(\frac{1}{r^3}\right)\right] = 3r^{-4}$

$\therefore \Rightarrow \vec{\nabla}\left(\frac{1}{r^3}\right).$

$$\Rightarrow \frac{-3}{r^4} \cdot \left[\frac{\partial r}{\partial x} + \frac{\partial r}{\partial y} + \frac{\partial r}{\partial z}\right]$$

$$\Rightarrow \frac{-3}{r^4}\left[\frac{x+y+z}{r}\right] = -3\left[\frac{x+y+z}{r^5}\right].$$

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$$\vec{\nabla}\left[x\,\vec{\nabla}\left(\frac{1}{r^3}\right)\right] = \vec{\nabla}\left[x \cdot \frac{-3(x+y+z)}{r^4}\right]$$

$$= -3\,\vec{\nabla}\left[\frac{x+y+z}{r^4}\right]$$

$$= -3\left[\frac{\partial}{\partial x}\left(\frac{x+y+z}{r^4}\right) + \frac{\partial}{\partial y}\left[\frac{x+y+z}{r^4}\right] + \frac{\partial}{\partial z}\left[\frac{x+y+z}{r^4}\right]\right]$$

$$= -3\left[\frac{r^4 - 4r^3 \cdot \frac{x}{r} \cdot x + r^4 - 4r^3 \frac{y}{r} \cdot y + r^4 - 4r^3 \cdot \frac{z}{r} \cdot z}{r^8}\right]$$

$$= -3\left[3r^4 - 4r^2\left[x^2+y^2+z^2\right]\right] / r^8$$

$$= 3\left[\frac{4r^4 - 3r^4}{r^8}\right]$$

$$= \frac{3}{r^4} = 3r^{-4}.$$

HB

b). $\text{curl}\left[r^2 \cdot \vec{r}\right]$.

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^2 x & r^2 y & r^2 z \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(r^2 z) - \frac{\partial}{\partial z}(r^2 y)\right]$$

$$- \hat{j}\left[\frac{\partial}{\partial x}(r^2 z) - \frac{\partial}{\partial z}(r^2 x)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(r^2 y) - \frac{\partial}{\partial y}(r^2 x)\right].$$

$$= \hat{i}\left[2yz - 2zy\right]$$

$$- \hat{j}\left[2xz - 2zx\right] + \hat{k}\left[2xy - 2yx\right].$$

$$= 0.$$

HB

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$$= \frac{r^2 \cdot 1 - x\left(2x \cdot \frac{x}{r}\right) + r^2 - y\left(2y \cdot \frac{y}{r}\right) + r^2 - z\left(2z \cdot \frac{z}{r}\right)}{r^4}$$

$$= \frac{3r^2 - 2r^2}{r^4}$$

$$= \frac{1}{r^2} \quad \text{HR}$$

e) $\vec{\nabla} \cdot \left(\frac{\vec{r}}{r}\right)$

LHS $\vec{\nabla} \cdot \left(\frac{\vec{r}}{r}\right)$

$$= \frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r}\right).$$

$$= \frac{r \cdot 1 - \frac{x}{r} \cdot x + r - \frac{y}{r} \cdot y + r - \frac{z}{r} \cdot z}{r^2}$$

$$= \frac{3r^2 - x^2 - y^2 - z^2}{r^3}$$

$$= \frac{2r^2}{r^3} = \frac{2}{r}.$$

Q8 $(x-1)^2 + y^2 + (z+2)^2 = 9$, at pt $(3,1,-4)$

Outward normal :→ $\vec{\nabla} \cdot \phi$, where $\phi \Rightarrow (x-1)^2 + y^2 + (z+2)^2 - 9 = 0$

$$\left[\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right]$$

$$= \hat{i}\left[2(x-1)\right] + \hat{j}\left[2y\right] + \hat{k}\,(2)(z+2)$$

At $(3,1,-4)$.

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Outward normal = $4\hat{i} + 2\hat{j} - 4\hat{k}$.

Q9 Directional derivative

$\phi = x^2 - y^2 + 2z^2$, at point $P(1,2,3)$

in direction of $Q(5,0,4)$.

$$\therefore \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= 2x\hat{i} + (-2y)\hat{j} + (4z)\hat{k}.$$

$$\therefore \hat{r} = \frac{\vec{r}}{|r|} = \frac{(5-1)\hat{i} + (0-2)\hat{j} + (4-3)\hat{k}}{\sqrt{4^2 + 2^2 + 1^2}}$$

$$= \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{21}}$$

$$\therefore \underset{\text{At}(1,2,3)}{dd} = \left(2\hat{i} + (-4\hat{j}) + 12\hat{k}\right) \cdot \frac{(4\hat{i} - 2\hat{j} + \hat{k})}{\sqrt{21}}$$

$$= \frac{8 + 8 + 12}{\sqrt{21}} = \frac{28}{\sqrt{21}}.$$

Q10 $\phi = 4e^{2x-y+z}$,

$$\nabla\phi = \hat{i}\left(\frac{\partial\phi}{\partial x}\right) + \hat{j}\left(\frac{\partial\phi}{\partial y}\right) + \hat{k}\left(\frac{\partial\phi}{\partial z}\right)$$

$$= \phi\left[2\hat{i} - \hat{j} + \hat{k}\right].$$

At $(1,1,-1)$.

$\phi = 4e^0 = 4$

Hence $\nabla\phi = 8\hat{i} - 4\hat{j} + 4\hat{k}$.

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$$\hat{n} = \frac{(-3-1)\hat{i} + (5-1)\hat{j} + (6+1)\hat{k}}{\sqrt{4^2 + 4^2 + 7^2}}$$

$$= \frac{-4\hat{i} + 4\hat{j} + 7\hat{k}}{9}$$

$\therefore$ DD at $(1,1,-1)$ towards $(3,5,6)$

$$= \frac{(8\hat{i} - 4\hat{j} + 4\hat{k})(-4\hat{i} + 4\hat{j} + 7\hat{k})}{9}$$

$$= \frac{-32 - 16 + 28}{9}$$

$$= -20/9$$

Q11 $\vec{A} = (3x-2y)\hat{i} + (y+2z)\hat{j} - x^2\hat{k} \Rightarrow \int_C \vec{A}\cdot d\vec{r}$

Along C, $x = t,\ y = t^2,\ z = t^3$.

$$= \int_C \vec{A}\cdot d\vec{r}$$, by converting into scalar parametric form

$$= \int_C \left[A_1 \frac{dx}{dt} + A_2 \frac{dy}{dt} + A_3 \frac{dz}{dt}\right] dt$$

where $A = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$

$$= \int_C \left[(3x-2y)\frac{dx}{dt} + (y+2z)\frac{dy}{dt} + (-x^2)\frac{dz}{dt}\right] dt$$

Putting $x = t,\ y = t^2,\ z = t^3$. $\therefore$

$$= \int_0^1 (3t-2t^2).1 + (t^2+2t^3).2t - (t^2).3t^2 \ .\ dt.$$

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$$= \int_0^1 \left(3t - 2t^2 + 2t^3 + 4t^4 - 3t^4\right) dt.$$

$$= \int_0^1 \left(3t - 2t^2 + 2t^3 + t^4\right) dt.$$

$$= \left[\frac{3t^2}{2} - \frac{2t^3}{3} + \frac{2t^4}{4} + \frac{t^5}{5}\right]_0^1$$

$$= \frac{3}{2} - \frac{2}{3} + \frac{1}{2} + \frac{1}{5}$$

$$= \frac{45 - 20 + 15 + 6}{30}$$

$$= \frac{46}{30} = \frac{23}{15}.$$

Q12 → $\vec{F} = (5xy - 6x^2)\hat{i} + (2y - 4x)\hat{j}$, $y = x^3$ from $(1,1)$ to $(2,8)$

$$= \int_C \vec{A} \cdot \vec{dr}$$, by converting in scalar differential form -

$$= \int_C \left[A_1\, dx + A_2\, dy\right].$$

$$= \int_C (5xy - 6x^2)\, dx + (2y - 4x)\, dy.$$

Putting $y = x^3$ $\therefore dy = 3x^2\, dx$. $\therefore x \Rightarrow 1$ to 2.

$$= \int_0^2 (5x^4 - 6x^2)\, dx + (2x^3 - 4x)\, 3x^2\, dx.$$

$$= \int_0^2 \left(5x^4 + 6x^5 - 6x^2 - 12x^3\right) dx$$

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$$= \left[x^5 + x^6 - 2x^2 - 3x^4\right]_1^2$$

$$\Rightarrow 32-1+64-1-2(8-1)-3(16-1)$$

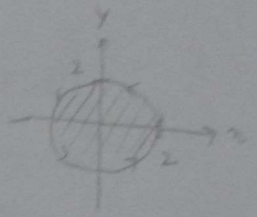
$$\Rightarrow 31+63-14-45$$

$$\Rightarrow \underline{\underline{35}}$$

$$\begin{array}{r} 94 \\ 59 \\ \hline 35 \end{array}$$

Q13 $A = \int_C (x^2+y^2)\,dx + 3xy^2\,dy$

$C: x^2+y^2=4.$



By Green thm

$$A = \iint_R \frac{\partial}{\partial x}\left[3xy^2\right] - \frac{\partial}{\partial y}\left[x^2+y^2\right] dx\,dy$$

$$= \iint_R (3y^2 - 2y)\,dx\,dy.$$

Put converting into polar.

$$= \int_0^{2\pi}\int_0^2 (3r^2\sin^2\theta - 2r\sin\theta)\,r\,dr\,d\theta$$

$$\Rightarrow \int_0^{2\pi}\left[\frac{3r^4}{4}\sin^2\theta - \frac{2r^3}{3}\sin\theta\right]_0^2 d\theta$$

$$\Rightarrow \int_0^{2\pi}\left[12\sin^2\theta - \frac{16}{3}\sin\theta\right] d\theta$$

$\sin^2\theta = \left[\frac{1-\cos 2\theta}{2}\right]$

$\frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]$

$$\Rightarrow \left[\frac{12}{2}\left[\theta - \frac{\sin 2\theta}{2}\right] + \frac{16}{3}\cos\theta\right]_0^{2\pi}$$

$$= \boxed{12\pi} \text{ squints.}$$

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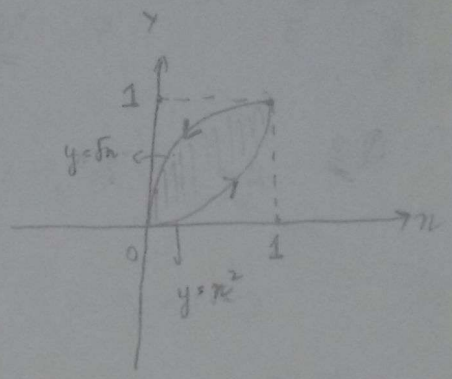
Q14 $A = \int_C (3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy.$

$C \rightarrow$ bounded by $y = \sqrt{x}$, $y = x^2$



According to green's theorem

$$A = \underbrace{\int_C (3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy}_{LHS} = \underbrace{\iint_R \left[\frac{\partial}{\partial x}(4x - 6xy) - \frac{\partial}{\partial y}(3x^2 - 8y^2)\right] dx\, dy}_{RHS}$$

Solving for LHS

$$LHS = \int_C (3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy$$

$$= \int_0^1 (3x^2 - \underbrace{8x^4)\, dx}_{I} + (4x^2 - 6x^3)\, 2x\, dx$$

$$+ \int_1^0 (3x^2 - 8x)\, dx + \underbrace{(4\sqrt{x} - 6x^{3/2})}_{I} \frac{1}{2} x^{-1/2}\, dx$$

$$I = \int_0^1 (3x^2 - 20x^4 + 8x^3)\, dx$$

$$= \left[x^3 - 4x^5 + 2x^4\right]_0^1 = -1$$

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$$II = \int_0^1 \left[(3x^2 - 8x) + \left(4 - \frac{6x}{2}\right) \right] dx$$

$$= -\int_0^1 \left(3x^2 - 11x + \frac{4}{2}\right) dx$$

$$= -\left[\left(x^3 - \frac{11x^2}{2} + 2x\right) \right]_0^1$$

$$= -\left[1 - \frac{11}{2} + 2 \right]$$

$$= \frac{5}{2}$$

$$\therefore \quad LHS = -1 + \frac{5}{2} = -\frac{3}{2} = A$$

Solving for RHS

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} \left[\frac{\partial}{\partial x}(4y - 6xy) - \frac{\partial}{\partial y}(3x^2 - 8y^2) \right] dy\, dx$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} \left[-6y + 8 \times 2y \right] dy\, dx$$

$$= \int_0^1 \left[-3y^2 + 8y^2 \right]_{x^2}^{\sqrt{x}} dx$$

$$\int_0^1 5(x - x^4)\, dx$$

$$= -\left[\frac{5x^2}{2} - \frac{5x^5}{5} \right]_0^1$$

$$= -\left[\frac{5}{2} - 1 \right]$$

$$= -\frac{3}{2} = LHS = A$$

Hence, Proved

Q15 →

$$W = \oint_C (2x - y - z)\hat{i} + (x + y - z^2)\hat{j} + (3x - 2y + 4z)\hat{k} \cdot (d\vec{r})$$

By converting into parametric scalar differential

$C \Rightarrow x^2 + y^2 = 9$

$x = 3\cos\theta$

$y = 3\sin\theta$

$z = 0$

$$W = \oint_C \left[(2x - y - z)\frac{dx}{d\theta} + (x + y - z^2)\frac{dy}{d\theta} + (3x - 2y + 4z)\frac{dz}{d\theta} \right] d\theta$$

$$= \oint_C (6\cos\theta - 3\sin\theta)(3(-\sin\theta)) + 3(\cos\theta + \sin\theta)(3\cos\theta)\, d\theta$$

$$= \int_0^{2\pi} 9\left[-2\cos\theta\sin\theta + \sin^2\theta + \cos^2\theta + \cos\theta\sin\theta \right] d\theta$$

$$= \int_0^{2\pi} 9\left[-\cos\theta\sin\theta + 1 \right]$$

$$= 9\int_0^{2\pi} \left[1 - \frac{\sin 2\theta}{2} \right]$$

$$= 9\left(2\pi + \frac{1}{2}\left[\cos 2\theta\right]_0^{2\pi} \right)$$

$$\Rightarrow 9(2\pi - 0)$$

$$= 18\pi$$

// curl ≠ 0, hence no potential func, dont write further

Finding potential func,

$\frac{\partial f}{\partial x} = 2x - y - z$ $\quad \frac{\partial f}{\partial y} = x + y - z^2$, $\quad \frac{\partial f}{\partial z} = 3x - 2y + 4z$

$f(x, y, z) = x^2 - xy - xz + f_1(y, z)$ — (1)

$\frac{\partial f}{\partial y} = -x + \frac{\partial f_1}{\partial y}$

$$x+y-z^2 = -x + \frac{\partial f_1}{\partial y}$$

$$\frac{\partial f_1}{\partial y} = 2x+y-z^2$$

$$f_1 = 2xy + \frac{y^2}{2} - yz^2 + f_2(z)$$

$$\therefore f(x,y,z) = x^2 + \frac{y^2}{2} + xy - xz - yz^2 + f_2(z)$$

$$\frac{\partial f}{\partial z} = 3x - 2y + 4z = -x - 2yz + \frac{df_2}{dz}$$

$$\therefore \frac{df_2}{dz} = 4x + 2yz - 2y + 4z$$

$$f_2 = 4xz + yz^2 - 2yz + 2z^2$$

$$\therefore f = x^2 + \frac{y^2}{2} + xy - \cancel{xz} - \cancel{yz^2} + \overset{3}{\cancel{4}}xz + yz^2 - 2yz + 2z^2$$

$$= x^2 + \frac{y^2}{2} + 2z^2 + xy + 3xz - 2yz$$

Q16 $\vec{F} = (2xy+z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$.

$$\text{Curl of } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy+z^3 & x^2 & 3xz^2 \end{vmatrix}.$$

$$= \hat{i}\left[0-0\right] - \hat{j}\left[3z^2-3z^2\right] + \hat{k}\left[2x-2x\right]$$

$$= \vec{0}$$

Hence, $\vec{F}$ is conservative.

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For scalar potential

$$\frac{\partial f}{\partial x} = 2xy + z^3$$

$$\frac{\partial f}{\partial y} = x^2$$

$$\frac{\partial f}{\partial z} = 3xz^2$$

integrate wrt x

$$f(x,y,z) = x^2 y + xz^3 + f_1(y,z)$$

differentiate wrt y

$$\frac{\partial f}{\partial y} = x^2 + \frac{\partial f_1}{\partial y}$$

$$\boxed{\frac{\partial f_1}{\partial y} = 0}$$

$$f_1 = f_2(z)$$

$$\therefore f(x,y,z) = x^2 y + xz^3 + f_2(z)$$

$$\therefore \frac{\partial f}{\partial z} = 3xz^2 + \frac{\partial f_2}{\partial z}$$

$$\frac{\partial f_2}{\partial z} = 0$$

$$\boxed{f = x^2 y + xz^3}$$

Q17 $\iint \vec{A} \cdot \hat{n} \, dS$. $\quad A = y\hat{i} + 2x\hat{j} - z\hat{k}$.

$$\vec{n} = \nabla(2x+y)$$

$$\vec{n} = (2\hat{i} + \hat{j}) \quad \therefore \hat{n} = \frac{2\hat{i} + \hat{j}}{\sqrt{5}}.$$

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$$\therefore \quad ds = \frac{dx\,dy}{|\hat{N}\cdot\hat{k}|}$$

$$= \frac{dx\,dy}{2}\cdot\sqrt{5}.$$

$$\vec{A}\cdot\hat{N} = (y\hat{i} + 2x\hat{j} - z\hat{k})\left(\frac{2\hat{i}+\hat{j}}{\sqrt{5}}\right)$$

$$= \frac{2y+2x}{\sqrt{5}} \Rightarrow \frac{y+6}{\sqrt{5}}.$$

$$\int_0^6\int_0^4 \frac{y+6}{\sqrt{5}}\cdot\frac{dx\,dy}{2}\sqrt{5}$$

$$\frac{1}{2}\int_0^6\int_0^4 (y+6)\,dx\,dy$$

$$\frac{1}{2}\cdot 4\left[\frac{y^2}{2}+6y\right]_0^6$$

$$= \frac{1}{2}\cdot 4\left[18+36\right]$$

$$= \underline{\underline{108}}$$

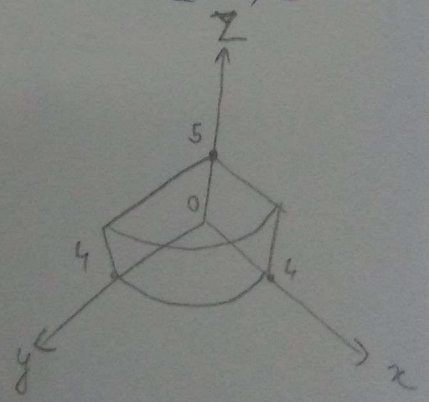
Q18 $\iint \vec{A}\cdot\hat{n}\,dS.$ where $\vec{A} = z\hat{i} + x\hat{j} - 3y^2z\hat{k}$

S $x^2+y^2=16$, in first octant

$z=0, z=5.$

$$\hat{N} = \nabla(x^2+y^2)$$

$$= \frac{2x\hat{i}+2y\hat{j}}{2\sqrt{x^2+y^2}} = \frac{x\hat{i}+\hat{j}y}{4}$$



Projection of S on yz plane ∴ rectangle with sides 5, 4.

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$$\therefore \; dS = \frac{dy\,dz}{|\hat{N}\cdot\hat{i}|} = \frac{dy\,dz}{\pi/4}$$

$$= \frac{4\,dy\,dz}{\pi}$$

$$\vec{F}\cdot\hat{N} = \frac{1}{4}\left[z\hat{i} + x\hat{j} - 3y^2z\hat{k}\right]\cdot\left[x\hat{i} + y\hat{j}\right]$$

$$= \frac{1}{4}\left[xz + xy\right]$$

$$= \iint_S \vec{A}\cdot\hat{n}\cdot ds$$

$$= \iint_S \frac{1}{4}\left[\cancel{x}z + \cancel{x}y\right]\cdot 4\,\frac{dy\,dz}{\cancel{x}}$$

$$= \int_0^4\int_0^5 (z+y)\,dz\,dy$$

$$= \int_0^4 \left(\frac{z^2}{2} + yz\right)_0^5 dy$$

$$= \int_0^4 \left[\frac{25}{2} + 5y\right] dy$$

$$= \frac{25}{\cancel{2}}\cdot\overset{2}{\cancel{4}} + 5\frac{(4)^2}{2}$$

$$= 50 + 40 = \boxed{90}$$

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Q17 $\iint \vec{F}\cdot\hat{n}\, dS$, where

$$\vec{F} = 2xy\,\hat{i} + yz^2\,\hat{j} - xz\,\hat{k}$$

$\therefore$ By Gauss divergence theorem

$$\iint \vec{F}\cdot\hat{n}\, dS = \iiint_V \nabla\cdot\vec{F}\, dV$$

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(yz^2) + \frac{\partial}{\partial z}(-xz)$$

$$= (2y + z^2 - x)$$

Then

$$= \int_0^3\int_0^1\int_0^2 (2y + z^2 - x)\, dx\, dy\, dz$$

$$= \int_0^3\int_0^1 \left[2xy + z^2x - \frac{x^2}{2}\right]_0^2 dy\, dz$$

$$= \int_0^3\int_0^1 \left[4y + 2z^2 - 2\right] dy\, dz$$

$$= \int_0^3 \left[2y^2 + 2z^2y - 2y\right]_0^1 dz$$

$$= \int_0^3 \left[2 + 2z^2 - 2\right] dz$$

$$= \frac{2}{3}\left[z^3\right]_0^3 = \frac{2}{3}\times 27 = 18$$

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R.H.S.

$$= \iiint_V \nabla \cdot F \, dv$$

$$\nabla \cdot F = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2)$$

$$= 2x + 2y + 2z$$

Now,

$$\iiint \nabla \cdot F \, dv = \int_0^1 \int_0^1 \int_0^1 (2x + 2y + 2z) \, dx \, dy \, dz$$

$$= 2 \int_0^1 \int_0^1 \left[\frac{x^2}{2} + xy + xz \right]_0^1 dy \, dz$$

$$= 2 \int_0^1 \int_0^1 \left(\frac{1}{2} + y + z \right) dy \, dz$$

$$= 2 \int_0^1 \left[\frac{y}{2} + \frac{y^2}{2} + yz \right]_0^1 dz$$

$$= 2 \int_0^1 \left[\frac{1}{2} + \frac{1}{2} + z \right] dz$$

$$= 2 \int_0^1 (1 + z) \, dz = 2 \left(z + \frac{z^2}{2} \right)_0^1$$

$$= 2 \left(1 + \frac{1}{2} \right) = 3$$

L.H.S.

For each surface of cube, $\hat{n}$ is different, thus we separately for each surface

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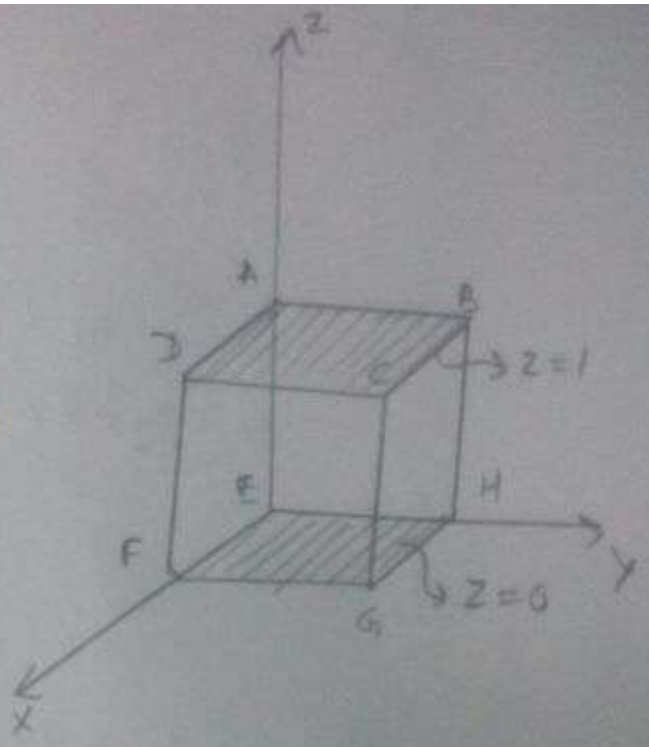
1) <u>For ABCD</u>

$\hat{n} = \hat{k}$

$\vec{F} \cdot \hat{n} = \vec{F} \cdot \hat{k} = z^2$

$ds = \dfrac{dx\,dy}{|\hat{n} \cdot \hat{k}|} = dx\,dy$



$\iint z^2\,dx\,dy \quad (z=1 \text{ for } ABCD)$

$= \int_0^1\int_0^1 dx\,dy = (1)(1) = 1$

2) <u>For EFGH</u>

$\hat{n} = -\hat{k}$ (normal outwards of face)

$\vec{F} \cdot \hat{n} = -z^2$

$\iint (-z^2)\,dx\,dy = 0 \quad (\text{as } z=0 \text{ for } EFGH)$

3) <u>AEFD</u>

$\hat{n} = -\hat{j}$

$\vec{F} \cdot \hat{n} = -y^2$

$\iint (-y^2)\,dx\,dz = 0 \quad (\text{as } y=0)$

4) <u>BCGH</u>

$\hat{n} = \hat{j}$

$\int_0^1\int_0^1 z^2\,dx\,dy = \int_0^1\int_0^1 dx\,dy = 1 \quad (\text{as } z=1)$

5) <u>For DCFG</u>

$\hat{n} = \hat{i}$

$\Rightarrow \int_0^1\int_0^1 x^2\,dy\,dz = \int_0^1\int_0^1 dy\,dz = 1 \quad (\text{as } x=1)$

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c) For ABEH,

$\hat{n} = -\hat{i}$

$$\Rightarrow \iint -x^2 \, dy\,dz = \iint (0) \, dy\,dz$$

$$= 0$$

Adding all, we get $= 1+0+1+0+1+0$

$= 3$

LHS = RHS

Hence Divergence theorem is verified.

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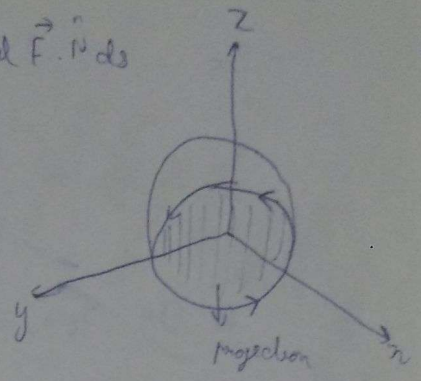
Q21 $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$. $\oint_C \vec{F}\cdot d\vec{r} = \iint_C \text{curl}\,\vec{F}\cdot\hat{N}\,ds$



$$\boxed{C \Rightarrow x^2 + y^2 = 1}$$

LHS

Put

$x = \cos\theta, \quad y = \sin\theta, \quad z = 0.$

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C x\,dx + y\,dy + z\,dz$$

$$\boxed{z = 0}$$

$$= \oint_C -\cos\theta\sin\theta\,d\theta + \sin\theta\cos\theta\,d\theta$$

$$= 0$$

RHS

$$\text{curl of } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \hat{i}(1-1) - \hat{j}(1-1) + \hat{k}(1-1)$$

$$= 0.$$

LHS = RHS

HP

Q22

$$\phi = xy^2 + yz^3 \quad \text{at } (1,-1,1).$$

Surface $(x^2+y^2+z^2 = 9)$ at $(1,2,2)$.

$$\vec{n} = \vec{\nabla} \cdot f$$

$$\vec{n} = (2x\hat{i} + 2y\hat{j} + 2z\hat{k})$$

$$\hat{n} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}}$$

$$\therefore \; dd \Rightarrow \nabla\phi \cdot \hat{n}$$

$$\Rightarrow \underbrace{\left[y^2\hat{i} + (2yx + z^3)\hat{j} + 3z^2y\hat{k}\right]}_{(1,-1,1)} \cdot \underbrace{\left[\frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}}\right]}_{(1,2,2)}$$

$$\Rightarrow (\hat{i} - \hat{j} + 3\hat{k})\left(\frac{\hat{i} + 2\hat{j} + 2\hat{k}}{3}\right)$$

$$\Rightarrow \frac{1-2-6}{3} = -\frac{7}{3}$$

Q24

For solenoidal flow,

$$\frac{\partial}{\partial x}\left[ax^2y + yz\right] + \frac{\partial}{\partial y}\left[xy^2 - az^2\right] + \frac{\partial}{\partial z}\left[2xyz - 2x^2y^2\right] = 0$$

$$2axy + 2xy + 2xy = 0$$

$$(a+2)(2xy) = 0$$

$$a+2 = 0$$

$$\boxed{a = -2}$$

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Q25 $\vec{F} = (x + 2y + az)\hat{i} + (bx - 3y - z)\hat{j} + (4x + cy + 2z)\hat{k}$

For irrotational flow,

$$\text{curl} = 0$$

$$\begin{vmatrix} \hat{i} & \hat{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = 0$$

$$\hat{i}[c+1] - \hat{j}[4-a] + k[b-2] = 0$$

$c + 1 = 0$ $\quad$ $4 - a = 0$ $\quad$ $b - 2 = 0$

$\boxed{c = -1}$ $\quad$ $\boxed{4 = a}$ $\quad$ $\boxed{b = 2}$

$$F = \text{grad}\,\phi$$

$$\frac{\partial \phi}{\partial x} = x + 2y + 4z \qquad \frac{\partial \phi}{\partial y} = 2x - 3y - z$$

$$\frac{\partial \phi}{\partial z} = 4x - y + 2z$$

$$\phi(x,y,z) = \left(\frac{x^2}{2} + 2yx + 4zx\right) + \phi_1(y,z)$$

$$\frac{\partial \phi}{\partial y} = 2x + \frac{\partial \phi_1}{\partial y}$$

$$\frac{\partial \phi_1}{\partial y} = -3y - z$$

$$\phi_1 = -\frac{3y^2}{2} - zy + \phi_2(z)$$

$$\phi = \frac{x^2}{2} + 2yx + 4zx - \frac{3y^2}{2} - zy + f_2(z)$$

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$$\frac{\partial \phi}{\partial z} = 4x - y + \frac{\partial f_2}{\partial z}$$

$$\cancel{4x} - \cancel{y} + 2z = \cancel{4x} - \cancel{y} + \frac{\partial f_2}{\partial z}$$

$$\boxed{f_2 = z^2}$$

Hence

$$\phi = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy + 4zx - zy.$$

Q26 $\vec{F} = (y^2\cos x + z^3)\hat{i} + (2y\sin x - 4)\hat{j} + (3xz^2 + 2)\hat{k}$

$$\begin{vmatrix} \hat{i} & \hat{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2\cos x + z^3 & (2y\sin x - 4) & (3xz^2 + 2) \end{vmatrix}$$

$$\hat{i}\left[0 - 0\right] - \hat{j}\left[3z^2 - 3z^2\right] + \hat{k}\left[2y\cos x - 2y\cos x\right]$$

$$= \vec{0}$$

Hence, $\vec{F}$ is conservative.

$$\nabla\phi = \vec{F}$$

$$\therefore \frac{\partial \phi}{\partial x} = y^2\cos x + z^3, \quad \frac{\partial \phi}{\partial y} = (2y\sin x - 4)$$

$$\frac{\partial \phi}{\partial z} = (3xz^2 + 2).$$

$$\phi = y^2\sin x + xz^3 + \phi_1(y,z)$$

$$\frac{\partial \phi}{\partial y} = 2y\sin x + \frac{\partial \phi_1}{\partial y} \qquad \therefore \frac{\partial \phi_1}{\partial y} = -4$$

$$\boxed{\phi_1 = -4y}$$

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$$\phi = y^2 \sin x + xz^3 - 4y + \phi_2(z)$$

$$\frac{\partial \phi}{\partial z} = 3z^2 x + \frac{d\phi_2}{dz}$$

$$\therefore \frac{d\phi_2}{dz} = 2$$

$$\boxed{\phi_2 = 2z}$$

$$\therefore \boxed{\phi = y^2 \sin x + xz^3 - 4y + 2z}$$

$$\therefore W = -\phi(0, 1, -1) + \phi(\pi/2, -1, 2)$$

$$= +6 + 1 + \frac{\pi}{2} \cdot 8 + 4 + 4$$

$$\boxed{W = 4\pi + 15}$$

Q27 →

$$A = \int_C \vec{F} \cdot d\vec{r}$$

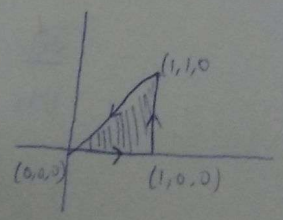
$$\vec{F} = y^2 \hat{i} + x^2 \hat{j} - (x+z) y^2 \hat{k}$$

By Stokes thm

$$A = \iint_S \nabla \times F \cdot n \, ds.$$

c is boundary.

(0,0,0), (1,0,0) or (1,1,0)



$$dS = dx\,dy, \quad n = \hat{k}$$

$$\frac{\partial f}{\partial x} = y^2, \quad \frac{\partial f}{\partial y} = x^2, \quad \frac{\partial f}{\partial z} = (x+z)y^2$$

$$f(x,y,z) = y^2 x + f_1(y,z)$$

$$x^2 + 2yx + \frac{\partial f_1}{\partial y}$$

$yz^3 j - 3xy k$, $\vec{B} =$

(b) $(\vec{A}.\vec{\nabla})\phi$

(b) $\vec{\nabla} r^n = nr^{n-2}\vec{r}$

Curl of F $= \begin{vmatrix} \hat{i} & \hat{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+2)(y^2) \end{vmatrix}$

$= \hat{i}\left[2y(-1)(x+2) - 0\right] - \hat{j}\left[-y^2 - 0\right]$

$+ \hat{k}\left[2x - 2y\right]$

$\hat{N} = \hat{k}$

$\therefore A = \iint_S (2x - 2y)\, dx\, dy.$

$= \int_0^1 \int_0^x (2x - 2y)\, dy\, dx$

$= \int_0^1 \left[2xy - y^2\right]_0^x$

$= \int_0^1 x^2\, dx$

$= \left[\frac{x^3}{3}\right]_0^1$

$= 1/3.$

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