10. $f(x, y) = \begin{cases} \dfrac{\sin^{-1}(x+2y)}{\tan^{-1}(2x+4y)}, & (x,y) \neq (0,0) \\ 1/2, & (x,y) = (0,0) \end{cases}$

11. $f(x, y) = \begin{cases} \dfrac{2xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

12. $f(x, y) = \begin{cases} \tan^{-1}\left(\dfrac{|x|+|y|}{x^2+y^2}\right), & (x,y) \neq (0,0) \\ \pi/2, & (x,y) = (0,0) \end{cases}$

Assign a suitable value for $f(0, 0)$ such that $f(x, y)$ is continuous at the point (0, 0)

13. $f(x, y) = \ln\left(\dfrac{3x^2 - x^2y^2 + 3y^2}{x^2+y^2}\right)$

14. $f(x, y) = \dfrac{2xy^2}{x^2+y^2}$

15. Discuss the continuity of the function

$$f(x, y, z) = \begin{cases} \dfrac{xyz}{x^2+y^2+z^2}, & (x, y\ z) \neq (0,0,0) \\ 0, & (x, y\ z) = (0,0,0) \end{cases}$$

at (0, 0, 0).

Using the $\delta - \epsilon$ definition show that

16. $f(x, y, z) = x + y - z$ is continuous at every point.
17. $f(x, y, z) = x^2 + y^2 + z^2$ is continuous at the origin.
18. $f(x, y) = (x + y)/(2 + \cos x)$ is continuous everywhere by taking $\epsilon = 0.02$.
19. $f(x, y) = \tan^2 x + \tan^2 y + \tan^2 z$ is continuous everywhere by taking $\epsilon = 0.03$.
20. Does knowing that $|\cos(1/y)| \leq 1$ tell anything about $\lim\limits_{(x,y)\to(0,0)} x \cos \dfrac{1}{y}$? Give reasons for your answer.

5.2 PARTIAL DERIVATIVES

When we keep all but one of the independent variables of a function constant and differentiate the function with respect to that one variable, we get a partial derivative.

Consider a function $z = f(x, y)$ of two independent variables x and y. Let (x_0, y_0) be a point in the domain of $f(x, y)$. The partial derivative of f with respect to x at the point (x_0, y_0) is the limit

$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x},$$

provided the limit exists. In case it exists, it is denoted by $\left(\dfrac{\partial z}{\partial x}\right)_{(x_0, y_0)}$, *or* $\left(\dfrac{\partial f}{\partial x}\right)_{(x_0, y_0)}$, *or* $f_x(x_0, y_0)$.

In fact the partial derivative of f with respect to x at the point (x_0, y_0) is the ordinary derivative of $f(x, y_0)$ with respect to x at the point $x = x_0$.

Similarly, *the partial derivative of $f(x, y)$ with respect to y at the point (x_0, y_0) is the limit* $\lim\limits_{\Delta y \to 0} \dfrac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$*, provided the limit exists, and is denoted by* $\left(\dfrac{\partial z}{\partial y}\right)_{(x_0, y_0)}$*, or* $\left(\dfrac{\partial f}{\partial y}\right)_{(x_0, y_0)}$ or $f_y(x_0, y_0)$. Also the partial derivative of f with respect to y at the point (x_0, y_0) is the ordinary derivative of $f(x_0, y)$ with respect to y at the point $y = y_0$.

For example, if $z = e^{ax} \sin by$, then $\dfrac{\partial z}{\partial x} = ae^{ax} \sin by$, and $\dfrac{\partial z}{\partial y} = be^{ax} \cos by$.

In case we consider a function $u = f(x, y, z)$ of three independent variables, then we have three partial derivatives of first order denoted by $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}$ and $\dfrac{\partial u}{\partial z}$. Here $\dfrac{\partial u}{\partial x}$ is obtained by differentiating u with respect to x, treating both y and z as constants; and

$$\left.\frac{\partial u}{\partial x}\right|_{(x_0, y_0, z_0)} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0, z_0) - f(x_0, y_0, z_0)}{\Delta x} \text{ etc.}$$

For example, if $f(x, y, z) = x^2 + y^2 + z^2 + xye^x$, then

$$f_x = 2x + y\,(1 + x)e^x,\ f_y = 2y + xe^x \text{ and } f_z = 2z.$$

Remark: A function $f(x, y)$ can have partial derivatives with respect to both x and y at a point without being continuous there. This is different from functions of a single variable, where the existence of a derivative implies continuity. For example, consider the function

$$f(x, y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$$

Here the limit of $f(x, y)$ as $(x, y) \to (0, 0)$ along the line $y = x$ is 0, but $f(0, 0) = 1$ and hence the function $f(x, y)$ is not continuous at $(0, 0)$ but

$$\left(\frac{\partial f}{\partial x}\right)_{(0,\,0)} = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{1 - 1}{\Delta x} = \lim_{\Delta x \to 0} (0) = 0, \text{ and}$$

similarly, $\left(\dfrac{\partial f}{\partial y}\right)_{(0,\,0)} = 0$. Thus both partial derivatives exist at $(0, 0)$.

Geometrical Interpretation of $\partial z/\partial x$ and $\partial z/\partial y$: The partial derivatives $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ of a function $z = f(x, y)$ have a very simple geometric interpretation. The function $z = f(x, y)$ represents a surface in space. The equation $y = y_0$ then represents a vertical plane intersecting this surface in a curve $z = f(x, y_0)$. The partial derivative $\dfrac{\partial z}{\partial x}$ at the point (x_0, y_0) is the slope of the tangent to the curve $z = f(x, y_0)$ at the point $P(x_0, y_0, f(x_0, y_0))$ as shown in Fig. 5.2.

Similarly, the partial derivative $\dfrac{\partial z}{\partial y}$ at (x_0, y_0) is the slope of the tangent to the curve $z = f(x_0, y)$ at the point $(x_0, y_0, f(x_0, y_0))$.

Partial derivatives of second and higher orders: The derivatives $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ are called partial derivatives of first order. By differentiating these derivatives once again we obtain the four partial derivatives of second order. These are

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}$$

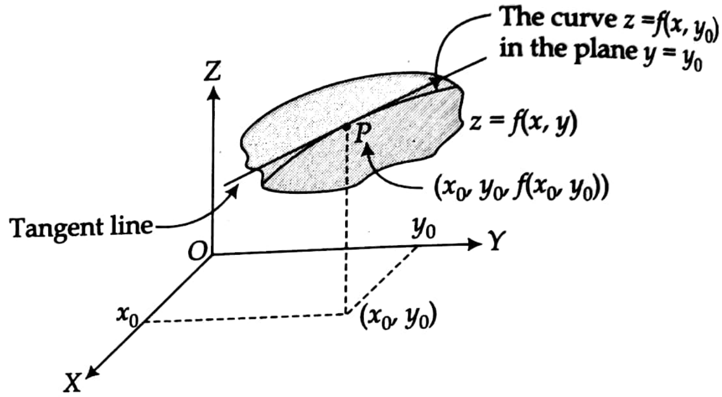

Fig. 5.2

It can be shown that if f_x, f_y and f_{xy} exist and are continuous, then the two mixed partial derivatives are equal and so the order of differentiation does not matter. In practical applications these conditions are satisfied and therefore we shall assume that the order of differentiation is immaterial.

By differentiating the partial derivatives of second order again with respect to x and y we obtain partial derivatives of the third and higher orders.

Example 5.4: If $u = x^2 \tan^{-1} \dfrac{y}{x} - y^2 \tan^{-1} \dfrac{x}{y}$, prove that $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{x^2 - y^2}{x^2 + y^2}$.

Solution: We have

$$\frac{\partial u}{\partial y} = x^2 \frac{1}{1 + \frac{y^2}{x^2}} \frac{1}{x} - 2y \tan^{-1} \frac{x}{y} + y^2 \frac{1}{1 + \frac{x^2}{y^2}} \frac{x}{y^2}$$

$$= \frac{x^3}{x^2 + y^2} + \frac{xy^2}{x^2 + y^2} - 2y \tan^{-1} \frac{x}{y}$$

$$= x - 2y \tan^{-1} \frac{x}{y}.$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = 1 - 2y\frac{1}{1+\frac{x^2}{y^2}}\frac{1}{y}$$

$$= 1 - \frac{2y^2}{y^2+x^2} = \frac{x^2-y^2}{x^2+y^2}.$$

Example 5.5: If $\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$, prove that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right).$$

Solution: Consider

$$\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1. \qquad ...(5.2)$$

Differentiating partially w.r.t. x, we have

$$\frac{2x}{a^2+u} - \frac{x^2}{(a^2+u)^2}\frac{\partial u}{\partial x} - \frac{y^2}{(b^2+u)^2}\frac{\partial u}{\partial x} - \frac{z^2}{(c^2+u)^2}\frac{\partial u}{\partial x} = 0$$

or,

$$\left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]\frac{\partial u}{\partial x} = \frac{2x}{a^2+u}$$

or,

$$\frac{\partial u}{\partial x} = \frac{2x}{a^2+u}\Big/\left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right] \qquad ...(5.3)$$

Similarly differentiating (5.2) partially w.r.t. y and z, we get respectively

$$\frac{\partial u}{\partial y} = \frac{2y}{a^2+u}\Big/\left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right] \qquad ...(5.4)$$

and,

$$\frac{\partial u}{\partial z} = \frac{2z}{a^2+u}\Big/\left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right] \qquad ...(5.5)$$

Squaring and adding (5.3), (5.4) and (5.5), we obtain

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = \frac{4\left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]}{\left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]^2}$$

$$= \frac{4}{\dfrac{x^2}{(a^2+u)^2} + \dfrac{y^2}{(b^2+u)^2} + \dfrac{z^2}{(c^2+u)^2}} \qquad \text{...(5.6)}$$

Also, from (5.3), (5.4) and (5.5), we have

$$2\left\{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right\} = \frac{4\left[\dfrac{x^2}{a^2+u} + \dfrac{y^2}{b^2+u} + \dfrac{z^2}{c^2+u}\right]}{\dfrac{x^2}{(a^2+u)^2} + \dfrac{y^2}{(b^2+u)^2} + \dfrac{z^2}{(c^2+u)^2}}$$

$$= 4\Bigg/\left(\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right), \qquad \text{...(5.7)}$$

using (5.2). From (5.6) and (5.7), we obtain the desired result.

Example 5.6: Find the value of n, so that the equation $v = r^n(3\cos^2\theta - 1)$ satisfies the relation

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right) = 0.$$

Solution: Consider $\quad v = r^n(3\cos^2\theta - 1). \qquad$ ...(5.8)

Differentiating it partially w.r.t. r, we obtain $\dfrac{\partial v}{\partial r} = nr^{n-1}(3\cos^2\theta - 1)$

or, $$r^2\frac{\partial v}{\partial r} = nr^{n+1}(3\cos^2\theta - 1).$$

Differentiating again partially w.r.t. r, we get

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) = n(n+1)r^n(3\cos^2\theta - 1). \qquad \text{...(5.9)}$$

Next, differentiating (5.8) w.r.t. θ, we obtain $\dfrac{\partial v}{\partial\theta} = -6r^n\cos\theta\sin\theta$

or, $$\sin\theta\frac{\partial v}{\partial\theta} = -6r^n\cos\theta\sin^2\theta. \qquad \text{...(5.10)}$$

Differentiating (5.10) w.r.t. θ, we get

$$\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right) = -6r^n[-\sin^3\theta + 2\sin\theta\cos^2\theta] = 6r^n\sin\theta[\sin^2\theta - 2\cos^2\theta]$$

or, $$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left\{\sin\theta\frac{\partial v}{\partial\theta}\right\} = 6r^n(\sin^2\theta - 2\cos^2\theta)$$

$$= -6r^n(3\cos^2\theta - 1). \qquad \text{...(5.11)}$$

Adding (5.9) and (5.11), we get

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial v}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial v}{\partial\theta}\right) = \{n(n+1)-6\}r^n\{3\cos^2\theta - 1\} \qquad ...(5.12)$$

The expression on the left side of (5.12) will be zero for all θ and r, provided

$$n(n+1) - 6 = 0, \text{ or } n^2 + n - 6 = 0 \text{ or } \quad n = 2, -3.$$

Example 5.7: If $x^x y^y z^z = c$, show that at $x = y = z$, $\dfrac{\partial^2 z}{\partial x \partial y} = -(x \ln ex)^{-1}$.

Solution: We have, $x^x y^y z^z = c$. Taking logarithm, we get

$$x \ln x + y \ln y + z \ln z = \ln c. \qquad ...(5.13)$$

Differentiating (5.13) w.r.t. x, we obtain $(\ln x + 1) + \dfrac{\partial z}{\partial x} \ln z + \dfrac{\partial z}{\partial x} = 0$,

which gives

$$\frac{\partial z}{\partial x} = -\frac{1+\ln x}{1+\ln z} = -\frac{\ln ex}{\ln ez}. \qquad ...(5.14)$$

Similarly, $$\frac{\partial z}{\partial y} = -\frac{\ln ey}{\ln ez}. \qquad ...(5.15)$$

Differentiating (5.15) w.r.t. x, we obtain

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\ln ey}{(\ln ez)^2}\frac{1}{ez} e \frac{\partial z}{\partial x} = \frac{\ln ey}{z(\ln ez)^2}\frac{\partial z}{\partial x} \qquad ...(5.16)$$

Using (5.14) in (5.15), we get

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{(\ln ex)(\ln ey)}{z(\ln ez)^3}. \qquad ...(5.17)$$

At $x = y = z$, (5.17) becomes $\dfrac{\partial^2 z}{\partial x \partial y} = -(x \ln ex)^{-1}$.

Imp.

Example 5.8: If $u = \ln(x^3 + y^3 + z^3 - 3xyz)$, prove that $\left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y} + \dfrac{\partial}{\partial z}\right)^2 u = -9(x + y + z)^{-2}$.

Solution: Consider, $u = \ln(x^3 + y^3 + z^3 - 3xyz)$. Differentiating w.r.t. x, we have

$$\frac{\partial u}{\partial x} = \frac{3(x^2 - yz)}{x^3 + y^3 + z^3 - 3xyz}.$$

Finding corresponding expressions for $\dfrac{\partial u}{\partial y}$ and $\dfrac{\partial u}{\partial z}$ and then adding, we obtain

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3(x^2+y^2+z^2-yz-zx-xy)}{x^3+y^3+z^3-3xyz}$$
$$=3/(x+y+z). \qquad ...(5.18)$$

Also, $$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right) \qquad \text{using (5.18)}$$

$$=-3\left[\frac{1}{(x+y+z)^2}+\frac{1}{(x+y+z)^2}+\frac{1}{(x+y+z)^2}\right]=-9/(x+y+z)^2.$$

Example 5.9: If (x, y) and (r, θ) are respectively cartesian and polar co-ordinates of a point P, find

$$\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial \theta}{\partial x}, \frac{\partial \theta}{\partial y}, \frac{\partial y}{\partial \theta}, \frac{\partial y}{\partial r}, \frac{\partial x}{\partial \theta} \text{ and } \frac{\partial x}{\partial r}.$$

Solution: The relation between the cartesian and polar co-ordinates is

$$x=r\cos\theta, \quad y=r\sin\theta. \qquad ...(5.19)$$

We have, $$\frac{\partial x}{\partial r}=\cos\theta, \quad \frac{\partial x}{\partial \theta}=-r\sin\theta \text{ and, } \frac{\partial y}{\partial r}=\sin\theta, \quad \frac{\partial y}{\partial \theta}=r\cos\theta.$$

Also from (5.19), we have

$$r^2=x^2+y^2 \qquad ...(5.20)$$

and, $$\tan\theta=y/x. \qquad ...(5.21)$$

From (5.20), we have $2r\dfrac{\partial r}{\partial x}=2x$, or $\dfrac{\partial r}{\partial x}=\dfrac{x}{r}=\cos\theta$. Similarly, $\dfrac{\partial r}{\partial y}=\dfrac{y}{r}=\sin\theta$.

Next differentiating (5.21) w.r.t. x, we obtain $\sec^2\theta\dfrac{\partial\theta}{\partial x}=-\dfrac{y}{x^2}$,

or, $$\frac{\partial\theta}{\partial x}=-\frac{y}{x^2}\cos^2\theta=-\frac{y}{x^2}\frac{x^2}{r^2}=-\frac{y}{r^2}=-\frac{\sin\theta}{r}.$$

Similarly, $$\frac{\partial\theta}{\partial y}=\frac{1}{x}\cos^2\theta=\frac{1}{x}\frac{x^2}{r^2}=\frac{x}{r^2}=\frac{\cos\theta}{r}.$$

Remark: In the above example, we have four variables x, y, r, θ connected by the two relations $x=r\cos\theta$ and $y=r\sin\theta$. To find $\dfrac{\partial r}{\partial x}$ we need a relation between r and x and such a relation will contain one more variable θ or y as given by $r=x\sec\theta$, $r^2=x^2+y^2$. We can find $\dfrac{\partial r}{\partial x}$ from any of these relations but there is no reason to suppose that the two values of $\frac{\partial r}{\partial x}$ as determined by these, where we regard θ and y respectively as constants, are equal. To avoid confusion we can denote first by $\left(\frac{\partial r}{\partial x}\right)_\theta$ to mean the partial derivative of r with respect to x, keeping θ constant and, second by $\left(\frac{\partial r}{\partial x}\right)_y$ to mean the partial derivative of r with respect to x, keeping y constant.

In general, when no indication is given regarding the variable to be kept constant, then $(\partial/\partial x)$ means $\left(\frac{\partial}{\partial x}\right)_y$ and $\left(\frac{\partial}{\partial y}\right)$ means $\left(\frac{\partial}{\partial y}\right)_x$; similarly $\frac{\partial}{\partial r}$ mean $\left(\frac{\partial}{\partial r}\right)_\theta$ and $\left(\frac{\partial}{\partial \theta}\right)$ means $\left(\frac{\partial}{\partial \theta}\right)_r$.

Example 5.10: If $u = f(r)$ and $x = r\cos\theta$, $y = r\sin\theta$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r}f'(r)$.

Solution: We have $u = f(r)$, and thus

$$\frac{\partial u}{\partial x} = f'(r)\frac{\partial r}{\partial x}, \quad \frac{\partial^2 u}{\partial x^2} = f''(r)\left(\frac{\partial r}{\partial x}\right)^2 + f'(r)\frac{\partial^2 r}{\partial x^2};$$

and

$$\frac{\partial u}{\partial y} = f'(r)\frac{\partial r}{\partial y}, \quad \frac{\partial^2 u}{\partial y^2} = f''(r)\left(\frac{\partial r}{\partial y}\right)^2 + f'(r)\frac{\partial^2 r}{\partial y^2}.$$

Here $\frac{\partial r}{\partial x}$ and $\frac{\partial r}{\partial y}$, etc. are to be evaluated from $x = r\cos\theta$, $y = r\sin\theta$.

Also, $r^2 = x^2 + y^2$, and thus, $\frac{\partial r}{\partial x} = \frac{x}{r}$, and $\frac{\partial^2 r}{\partial x^2} = \frac{r - x^2/r}{r^2} = \frac{y^2}{r^3}$.

Similarly, $$\frac{\partial r}{\partial y} = \frac{y}{r} \text{ and } \frac{\partial^2 r}{\partial y^2} = \frac{x^2}{r^3}.$$

Thus,
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r)\left\{\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right\} + f'(r)\left\{\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2}\right\}$$

$$= f''(r)\left[\frac{x^2}{r^2} + \frac{y^2}{r^2}\right] + f'(r)\left[\frac{x^2}{r^3} + \frac{y^2}{r^3}\right]$$

$$= f''(r) + \frac{1}{r}f'(r).$$

EXERCISE 5.2

1. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, if

   (i) $z = y \sin xy$ (ii) $z = x^2 + 3xy + y + 1$

   (iii) $x + y + z = \ln z$ (iv) $z = \int_x^y g(t)dt.$

2. Find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial y}$, if

   (i) $u = \sin^{-1} xyz$ (ii) $\tanh(x + 2y + 3z)$.

3. If $z = \sin^{-1}(x/y)$, prove that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0.$

4. If $u = \frac{1}{\sqrt{x^2+y^2+z^2}}$, show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$

5. If $u = \ln \frac{x^2+y^2}{xy}$, verify that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}.$

6. Show that $z = f(x - cy) + g(x + cy)$ satisfies the equation $\frac{\partial^2 z}{\partial y^2} = c^2 \frac{\partial^2 z}{\partial x^2}.$

7. Prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, if $u = \tan^{-1}\left[\frac{2xy}{x^2 - y^2}\right].$

8. If $f(x, y) = (1 - 2xy + y^2)^{-1/2}$, show that $\frac{\partial}{\partial x}\left[(1-r^2)\frac{\partial f}{\partial x}\right] + \frac{\partial}{\partial y}\left[y^2 \frac{\partial f}{\partial y}\right] = 0.$

9. Let $r^2 = x^2 + y^2 + z^2$ and $u = r^m$, prove that $u_{xx} + u_{yy} + u_{zz} = m(m+1)r^{m-2}$.

10. If $u = \ln(x^2 + y^2 + z^2)$, prove that $(x^2 + y^2 + z^2)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 1.$

11. If $u = e^{x-at}\cos(x - at)$, show that $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}.$

12. Given that $u = x(x^2 + y^2 + z^2)^{-3/2}$, show that $xu_x + yu_y + zu_z = -2u.$

13. Show that $v = (r^n + r^{-n}) \sin n\theta$ satisfies Laplace equation in polar co-ordinates, that is,

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0.$$

14. If $w = \sin^{-1} u$, $u = (x^2 + y^2 + z^2) / (x + y + z)$, then $x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} + z\dfrac{\partial w}{\partial z} = \tan w$.

15. If $z = \ln (u^2 + v)$, $u = e^{x+y^2}$, $v = x + y^2$, then $2y\dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y} = 0$.

16. If $x = r\cos\theta$, $y = r\sin\theta$, prove that $\dfrac{\partial^2 r}{\partial x^2} + \dfrac{\partial^2 r}{\partial y^2} = \dfrac{1}{r}\left[\left(\dfrac{\partial r}{\partial x}\right)^2 + \left(\dfrac{\partial r}{\partial y}\right)^2\right]$.

17. If $u = f(r)$, where $r = \sqrt{x^2 + y^2 + z^2}$, show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = f''(r) + \dfrac{2}{r} f'(r)$.

18. If $z = \ln (e^x + e^y)$, show that $rt - s^2 = 0$, where $r = \dfrac{\partial^2 z}{\partial x^2}$, $t = \dfrac{\partial^2 z}{\partial y^2}$ and $s = \dfrac{\partial^2 z}{\partial x \partial y}$.

19. If $z = y + f(u)$, $u = \dfrac{x}{y}$, then show that $u\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 1$.

20. The equation $\dfrac{\partial u}{\partial t} = \mu\dfrac{\partial^2 u}{\partial x^2}$ refers to the conduction of heat along a bar without radiation.

Show that if $u = Ae^{-gx}\sin(nt - gx)$, where A, g, n are positive constants, then $g = \sqrt{(n/2\mu)}$.

5.3 TOTAL DIFFERENTIAL AND APPROXIMATION

We define the concept of total differential of a function of two or more variables and then apply it to estimate errors in calculations.

5.3.1 Total Differential of a Function

Consider a function $z = f(x, y)$ defined throughout in some domain D. Let $P(x, y)$ and $Q(x + \Delta x, y + \Delta y)$ be two neighbouring points in D, so that Δx, Δy are the changes in the independent variables and let Δz be the corresponding change in z. Then

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y) \qquad ..(5.22)$$

is called the *total increment* in z corresponding to the increments Δx in x and Δy in y.

Subtracting and adding $f(x + \Delta x, y)$ in (5.22), we have

$$\Delta z = \{f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y)\} + \{f(x + \Delta x, y) - f(x, y)\} \qquad ...(5.23)$$

Applying Lagrange's mean value theorem, refer to Eq. (3.50), under the assumption that $f(x, y)$ satisfies the desired conditions in D, we have

$$\Delta z = \Delta y f_y(x + \Delta x, y + \theta_1 \Delta y) + \Delta x\, f_x(x + \theta_2 \Delta x, y),\ 0 < \theta_1, \theta_2 < 1 \qquad ...(5.24)$$

Writing $f_x(x + \theta_2\Delta x, y) - f_x(x, y) = \epsilon_1$, and $f_y(x + \Delta x, y + \theta_1 \Delta y) - f_y(x, y) = \epsilon_2$, and substituting in (5.24) we obtain

$$\Delta z = \{f_x(x, y)\Delta x + f_y(x, y)\Delta y\} + (\epsilon_1 \Delta x + \epsilon_2 \Delta y), \quad ...(5.25)$$

where $\epsilon_1 = \epsilon_1(\Delta x, \Delta y)$ and $\epsilon_2 = \epsilon_2(\Delta x, \Delta y)$ are infinitesimal small such that $\epsilon_1 \to 0, \epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

Thus the change Δz in z given by (5.25) consists of two parts.

The part $f_x \Delta x + f_y \Delta y$ in (5.25) which is linear in Δx and Δy is called the *total differential* or simply the *differential* of z and is denoted by dz or df, thus

$$dz = f_x \Delta x + f_y \Delta y \quad ...(5.26)$$

In the limiting case (5.25) becomes

$$\boxed{dz = f_x dx + f_y dy} \quad ...(5.27)$$

We note that the differential dz of the dependent variable z is not the same as the change Δz; it is the *principal part* of the increment Δz.

Similarly, for a function of more than two independent variables say $u = f(x, y, z)$, the *total differential* is

$$\boxed{du = f_x dx + f_y dy + f_z dz} \quad ...(5.28)$$

and so on.

Example 5.11: Find the total differential of the following functions.

(i) $z = \sin^{-1}(x/y)$ (ii) $\ln(x^2 + y^2 + z^2)$

Solution: (i) Here $f(x, y) = \sin^{-1}(x/y)$, which gives

$$f_x = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\left(\frac{1}{y}\right) = \frac{1}{\sqrt{y^2 - x^2}}; f_y = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\left(-\frac{x}{y^2}\right) = -\frac{x}{y\sqrt{y^2-x^2}}.$$

Therefore, the total differential is

$$dz = f_x dx + f_y dy = \frac{dx}{\sqrt{y^2-x^2}} - \frac{xdy}{y\sqrt{y^2-x^2}} = \frac{ydx - xdy}{y\sqrt{y^2-x^2}}.$$

(ii) Here $f(x, y, z) = \ln(x^2 + y^2 + z^2)$, which gives

$$f_x = \frac{2x}{x^2+y^2+z^2}, \quad f_y = \frac{2y}{x^2+y^2+z^2}, \quad f_z = \frac{2z}{x^2+y^2+z^2}.$$

Therefore, the total differential is $dz = f_x dx + f_y dy + f_z dz = \dfrac{2(xdx + ydy + zdz)}{x^2+y^2+z^2}$.

5.3.2 Approximation by Total Differentials

From above we observe that the approximate change dz in z corresponding to the small changes dx in x and dy in y is $dz = \dfrac{\partial z}{\partial x}dx + \dfrac{\partial z}{\partial y}dy$, where the partial derivatives are evaluated at the given point (x, y). This result has applications in estimating errors in calculations.

Since, the partial derivatives and errors in arguments can be both positive and negative. Thus, the *maximum absolute error* is given by $|df| = |f_x|\,|\Delta x| + |fy|\Delta y$. The ratio $|df|/|f|$ is defined as the *maximum relative error* and $(|df|/|f|) \times 100$ as the *maximum percentage error.* These concepts can be extended to functions of more than two variables also.

An important result useful in calculated approximate value of a function is obtained from (5.25) by replacing Δz with $f(x + \Delta x, y + \Delta y) - f(x, y)$ and rewriting it as

$$f(x + \Delta x, y + \Delta y) = f(x, y) + f_x(x, y)\,\Delta x + f_y(x, y)\,\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

and it gives

$$f(x + \Delta x, y + \Delta y) \approx f(x, y) + f_x(x, y)\Delta x + f_y(x, y)\Delta y. \qquad \text{...(5.29)}$$

Example 5.12: Find the percentage error in the computed area of an ellipse when an error of 2% is made in measuring its major and minor axes.

Solution: If a, b, A are the semi-major axis, semi-minor axis and area of an ellipse respectively, then $A = \pi ab$. It gives $dA = \pi b da + \pi a db$.

It is given that $da = 0.02a$, $db = 0.02b$. Therefore, $|dA| = 0.04\pi ab = 0.04A$, and hence,

percentage error $= \left|\dfrac{dA}{A}\right| \times 100 = 4\%.$

Example 5.13: Suppose that the variables r and h change from the initial values of $r_0 = 1.0$, $h_0 = 5$ by the amount $dr = 0.03$ and $dh = -0.1$. Estimate the resulting absolute, relative and percentage changes in the volume of the right circular cone of initial radius r_0 and height h_0.

Solution: The volume of a right circular cone with radius r and height h is $v = \dfrac{1}{3}\pi r^2 h$.

Hence, the absolute change, $dv = \dfrac{1}{3}\pi[2rhdr + r^2 dh]$

$$= \frac{1}{3}\pi[2(1)(5)(0.03) + (1)^2(-0.1)] = \frac{1}{3}\pi[0.3 - 0.1] = \frac{0.2\pi}{3}.$$

$$\text{Relative change} = \frac{dv}{v(r_0, h_0)} = \frac{0.2\pi}{3} \Big/ \frac{1}{3}\pi r_0^2 h_0 = \frac{0.2\pi}{\pi(1)^2 5} = 0.04.$$

$$\text{Percentage change} = \frac{dv}{v(r_0, h_0)} \times 100 = 0.04 \times 100 = 4\%$$

Example 5.14: The volume $v = \pi r^2 h$ of a right circular cylinder is to be calculated from measured values of r and h. Suppose that r is measured with an error of no more than 2% and h with an error of no more than 0.5%. Estimate the resulting possible percentage error in the calculation of v.

Solution: Here $\left|\dfrac{dr}{r} \times 100\right| \le 2$ and $\left|\dfrac{dh}{h} \times 100\right| \le 0.5$. Also $\dfrac{dv}{v} = \dfrac{2\pi rhdr + \pi r^2 dh}{\pi r^2 h} = \dfrac{2dr}{r} + \dfrac{dh}{h}.$

Thus, $$\left|\frac{dv}{v} \times 100\right| \le 2\left|\frac{dr}{r} \times 100\right| + \left|\frac{dh}{h} \times 100\right| = 2 \times 2 + 0.5 = 4.5.$$

Thus, the resulting possible error in the calculation of v is 4.5%

Example 5.15: Find a reasonable square about the point $(r, h) = (5, 12)$ in which the value of $v = \pi r^2 h$ will not vary by more than ± 0.1.

Solution: We have $dv = 2\pi rh\,dr + \pi r^2 dh$.

About the point (5, 12), $dv = 2\pi(5)(12)dr + \pi(5)^2 dh = 120\pi dr + 25\pi dh$

Since the region to be restricted is a square, thus $dr = dh$ and, therefore,

$$dv = 120\pi\, dr + 25\pi\, dr = 145\pi\, dr.$$

Now $|dv| \le 0.1$ implies $|145\pi dr| \le 0.1$, or $|dr| \le \dfrac{0.1}{145\pi} \approx 2.1 \times 10^{-4}$.

Thus, the required square is given by

$$|r - 5| \le 2.1 \times 10^{-4}, \quad |h - 12| \le 2.1 \times 10^{-4}.$$

Example 5.16: Using differentials, find an approximate value of $f(x, y) = x^y$ at (2.1, 3.2).

Solution: Take $(x, y) = (2, 3)$, $\Delta x = 0.1$, $\Delta y = 0.2$.

Here $f(x, y) = x^y$, thus $f_x(x, y) = yx^{y-1}$ and $f_y(x, y) = x^y \ln x$.

We have, $f(x + \Delta x, y + \Delta y) \approx f(x, y) + f_x(x, y)\,\Delta x + f_y(x, y)\Delta y$

Thus,
$$f(2.1, 3.2) \approx f(2, 3) + f_x(2, 3)(0.1) + f_y(2, 3)(0.2)$$
$$= 8 + 12\,(0.1) + 8 \ln 2\,(0.2) = 10.309.$$

EXERCISE 5.3

1. Suppose T is to be measured from the formula $T = x(e^y + e^{-y})$ where x and y are found to be 2 and ln 2 with maximum possible error of $|dx| = 0.1$ and $|dy| = 0.02$. Estimate the maximum possible error in the computed value of T.
2. Give a reasonable square about (1, 1) over which the value of $f(x, y) = x^3 y^4$ will not vary by more than ± 0.1.
3. When an x ohm and y ohm resistors are in parallel, by about what percentage their resultant resistance R will change if x increases from 20 to 20.1 ohms and y decreases from 25 to 24.9 ohms?
4. The power consumed in an electric resistor is given by $P = E^2/R$ watts. If $E = 80$ volts and $R = 5$ ohms, by how much the power consumption will change if E is increased by 3 volts and R is decreased by 0.1 ohms.
5. At a distance of 50 meters from the foot of a tower the elevation of its top is 30°. If the possible errors in measuring the distance and elevation are 2 cm and 0.05 degrees, find the approximate error in calculating the height.
6. If the sides of a plane triangle ABC vary in such a way that its circum-radius remains constant, prove that

$$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0.$$

7. If the radius r and the altitude h of a cone are measured with an absolute error of 1% in each measurement, then find the approximate percentage change in the lateral area of the cone if the measured values are $r = 3ft$ and $h = 4ft$.
8. A balloon in the form of right circular cylinder of radius 1.5 m and length 4.0 m is surmounted by hemispherical ends. If the radius is increased by 0.01 m, and the length by 0.05 m, find the percentage change in the volume of the balloon.
9. Using differentials, obtain the approximate values of the following:

   (a) $(4.05)^{1/2}(7.97)^{1/3}$ (b) $\frac{1}{\sqrt{1.05}}+\frac{1}{\sqrt{3.97}}+\frac{1}{\sqrt{9.01}}$

10. Using differentials, obtain the approximate values of the following.

    (a) $\cos 44° \sin 32°$ (b) $\sin 26° \cos 57° \tan 48°$.

5.4 THE CHAIN RULE: DIFFERENTIATION OF COMPOSITE AND IMPLICIT FUNCTIONS

In this section we study the differentiation of composite and implicit functions by the application of chain rule to functions of two or more variables.

5.4.1 Derivative of a Composite Function

Let $z = f(x, y)$ be a function of two independent variables x and y where x and y themselves are functions of an independent variable t, say $x = \phi(t)$ and $y = \psi(t)$. Then $z = f[\phi(t), \psi(t)]$ is said to define a *composite function of the independent variable t*.

Again if $x = \phi(u, v)$ and $y = \psi(u, v)$, so that x, y are functions of the variables u and v, then $z = f[\phi(u, v), \psi(u, v)]$ is said to define a *composite function of u and v*.

Now, if we wish to know the rate at which f changes with respect to t, we need to differentiate this composite function with respect to t, provided the derivative exists. Sometime we can do this by substituting the formula for $\phi(t)$ and $\psi(t)$ into the formula for f and then differentiating directly with respect to t, which may be a little bit cumbersome and so we prefer to apply *chain rule* given as follows.

Let Δx, Δy and Δz be the increments respectively in x, y and z corresponding to the increment Δt in t. Then from (5.25)

$$\Delta z = \left(\frac{\partial z}{\partial x}\right)\Delta x + \left(\frac{\partial z}{\partial y}\right)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y, \quad ...(5.30)$$

where $\epsilon_1 = \epsilon_1(\Delta x, \Delta y)$ and $\epsilon_2 = \epsilon_2(\Delta x, \Delta y) \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

Dividing both sides of (5.30) by Δt and letting $\Delta t \to 0$, and hence, Δx, Δy, Δz and ϵ_1, ϵ_2 approach zero, we get the chain rule as

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \quad ...(5.31)$$

In case z is a composite function of two independent variables u and v, then the chain rule gives

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \qquad ...(5.32)$$

and,
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \qquad ...(5.32a)$$

The rule can easily be extended to functions of more than two independent variables.

Example 5.17: Find dz/dt when $z = xy^2 + x^2y,\ x = at^2,\ y = 2at$ by chain rule and verify by direct substitution.

Solution: We have, $\dfrac{\partial z}{\partial x} = y^2 + 2xy,\ \dfrac{\partial z}{\partial y} = 2xy + x^2,\ \dfrac{dx}{dt} = 2at$, and $\dfrac{dy}{dt} = 2a.$

By chain rule, $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt}$. Substituting for $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}, \dfrac{dx}{dt}, \dfrac{dy}{dt}$, we obtain

$$\frac{dz}{dt} = (y^2 + 2xy)(2at) + (2xy + x^2)(2a)$$
$$= (4a^2t^2 + 4a^2t^3)(2at) + (4a^2t^3 + a^2t^4)2a = 2a^3t^3(8 + 5t).$$

Also, $z = x^2y + xy^2 = 2a^3t^5 + 4a^3t^4$

Therefore, $\dfrac{dz}{dt} = 10a^3t^4 + 16a^3t^3 = 2a^3t^3(8 + 5t)$, hence the verification.

V. Imp.

Example 5.18: If $z = f(x, y)$ where $x = r\cos\theta$ and $y = r\sin\theta$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial^2 z}{\partial x^2}$.

Solution: We have $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}(y/x)$, and hence

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta, \quad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1 + y^2/x^2}(-y/x^2) = -\frac{y}{x^2 + y^2} = -\frac{y}{r^2} = -\frac{\sin\theta}{r}$$

and,
$$\frac{\partial \theta}{\partial y} = \frac{1}{1 + y^2/x^2}\cdot\frac{1}{x} = \frac{x}{r^2} = \frac{\cos\theta}{r}.$$

We have,
$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial x}$$
$$= \frac{\partial z}{\partial r}\cos\theta - \frac{\sin\theta}{r}\frac{\partial z}{\partial \theta} = \cos\theta\frac{\partial z}{\partial r} - \frac{\sin\theta}{r}\frac{\partial z}{\partial \theta} \qquad ...(5.33)$$
$$= \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)z. \qquad ...(5.34)$$

From (5.34),
$$\frac{\partial}{\partial x} \equiv \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \qquad ...(5.35)$$

Thus $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial x}\right)$

$$= \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial z}{\partial r} - \frac{\sin\theta}{r}\frac{\partial z}{\partial\theta}\right) \text{ using (5.33) and (5.35).}$$

$$= \cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial z}{\partial r} - \frac{\sin\theta}{r}\frac{\partial z}{\partial\theta}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial z}{\partial r} - \frac{\sin\theta}{r}\frac{\partial z}{\partial\theta}\right)$$

$$= \cos\theta\left[\cos\theta\frac{\partial^2 z}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial z}{\partial\theta} - \frac{\sin\theta}{r}\frac{\partial^2 z}{\partial r\partial\theta}\right]$$

$$-\frac{\sin\theta}{r}\left[-\sin\theta\frac{\partial z}{\partial r} + \cos\theta\frac{\partial^2 z}{\partial\theta\,\partial r} - \frac{\cos\theta}{r}\frac{\partial z}{\partial\theta} - \frac{\sin\theta}{r}\frac{\partial^2 z}{\partial\theta^2}\right]$$

$$= \cos^2\theta\frac{\partial^2 z}{\partial r^2} + \frac{2\cos\theta\sin\theta}{r^2}\frac{\partial z}{\partial\theta} - \frac{2\cos\theta\sin\theta}{r}\frac{\partial^2 z}{\partial r\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial z}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 z}{\partial\theta^2}$$

Example 5.19: If $f = f(y - z, z - x, x - y)$, prove that $\dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y} + \dfrac{\partial f}{\partial z} = 0$.

Solution: Let $u = y - z$, $v = z - x$, $w = x - y$, so that $f = f(u, v, w)$ is a composite function of x, y, z.

Therefore, $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial f}{\partial v}\dfrac{\partial v}{\partial x} + \dfrac{\partial f}{\partial w}\dfrac{\partial w}{\partial x} = \dfrac{\partial f}{\partial u}(0) + \dfrac{\partial f}{\partial v}(-1) + \dfrac{\partial f}{\partial w}(1)$

$$= -\frac{\partial f}{\partial v} + \frac{\partial f}{\partial w}. \qquad \text{...(5.36)}$$

Similarly, $$\frac{\partial f}{\partial y} = -\frac{\partial f}{\partial w} + \frac{\partial f}{\partial u} \qquad \text{...(5.37)}$$

and, $$\frac{\partial f}{\partial z} = -\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v} \qquad \text{...(5.38)}$$

Adding (5.36), (5.37) and (5.38) we get the desired result.

5.4.2 Derivative of an Implicit Function

The equation $f(x, y) = 0$ defines y implicitly as a function of x, say $y = h(x)$. Suppose the function $f(x, y)$ is differentiable, then applying chain rule to $w = f(x, y) = 0$, we obtain

$$0 = \frac{dw}{dx} = f_x\frac{dx}{dx} + f_y\frac{dy}{dx} = f_x + f_y\frac{dy}{dx},$$

or, $$\frac{dy}{dx} = -f_x/f_y, \text{ provided } f_y \neq 0. \qquad \text{...(5.39)}$$

Differentiating again with respect to x, regarding f_x and f_y as composite functions of x, we get

$$\frac{d^2 y}{dx^2} = -\frac{\left(f_{xx} + f_{xy}\dfrac{dy}{dx}\right)f_y - f_x\left(f_{xy} + f_{yy}\dfrac{dy}{dx}\right)}{(f_y)^2}$$

$$= -\frac{(f_{xx}f_y - f_{xy}f_x)f_y - f_x(f_{xy}f_y - f_{yy}\,f_x)}{(f_y)^3}, \qquad \text{using (5.39)}$$

$$\boxed{= -\frac{f_{xx}(f_y)^2 - 2f_{xy}f_x\,f_y + f_{yy}(f_x)^2}{(f_y)^3}.} \qquad \text{...(5.40)}$$

Example 5.20: If $y^3 - 3ax^2 + x^3 = 0$, then using partial differentiation prove that

$$\frac{d^2y}{dx^2} + \frac{2a^2x^2}{y^5} = 0.$$

Solution: The implicit function is $f(x, y) = y^3 - 3ax^2 + x^3$. This gives

$$f_x = -6ax + 3x^2, \quad f_y = 3y^2, \; f_{xy} = 0, \quad f_{xx} = -6a + 6x, \text{ and } f_{yy} = 6y.$$

Substituting these values in (5.40), we obtain

$$\frac{d^2y}{dx^2} = -\frac{6(x-a)(9y^4) + 6y\,[9x^2(x-2a)^2]}{27y^6} = -\frac{2[(x-a)y^3 - x^2(x-2a)^2]}{y^5} \qquad \text{....(5.41)}$$

Now $y^3 - 3ax^2 + x^3 = 0 \Rightarrow y^3 = x^2(3a - x)$. Using this in (5.41) and simplifying, we obtain

$$\frac{d^2y}{dx^2} = \frac{-2a^2x^2}{y^5}, \text{ or } \frac{d^2y}{dx^2} + \frac{2a^2x^2}{y^5} = 0.$$

Example 5.21: Given that $F(x, y, z) = 0$, prove that $\left(\frac{\partial y}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial x}{\partial z}\right) = -1.$

Solution: Since, $F(x, y, z) = 0$, we have

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz = 0. \qquad \text{...(5.42)}$$

If z is kept constant, then $dz = 0$, from (5.42) we obtain

$$\left(\frac{dy}{dx}\right)_z = \frac{\partial y}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial y} \qquad \text{...(5.43)}$$

Similarly, if x is kept constant, then $dx = 0$, we get

$$\frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z} \qquad \text{...(5.44)}$$

and, if y is kept constant, then $dy = 0$, we get

$$\frac{\partial x}{\partial z} = -\frac{\partial F/\partial z}{\partial F/\partial x} \qquad \text{...(5.45)}$$

Multiplying (5.43), (5.44) and (5.45), we get

$$\left(\frac{\partial y}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial x}{\partial z}\right) = -\frac{\partial F/\partial x}{\partial F/\partial y} \times \frac{\partial F/\partial y}{\partial F/\partial z} \times \frac{\partial F/\partial z}{\partial F/\partial x} = -1.$$

Example 5.22: Find $\partial w/\partial x$, if $w = x^2 + y^2 + z^2$, and $z^3 - xy + yz + y^3 = 1$, when x and y are independent variables.

Solution: It is not convenient to eliminate z from the two equations. We differentiate both w and z implicitly w.r.t. x treating x and y as independent variables. Thus,

$$\frac{\partial w}{\partial x} = 2x + 2z\frac{\partial z}{\partial x} \qquad \text{...(5.46)}$$

and,

$$3z^2\frac{\partial z}{\partial x} - y + y\frac{\partial z}{\partial x} = 0. \qquad \text{...(5.47)}$$

Solving (5.47) for $\frac{\partial z}{\partial x}$, we have $\frac{\partial z}{\partial x} = \frac{y}{y+3z^2}$.

Substituting this in (5.46), we obtain $\frac{\partial w}{\partial x} = 2x + \frac{2yz}{y+3z^2}$.

EXERCISE 5.4

1. Find $\frac{df}{dt}$ for the following functions:

   (a) $f = x^2 + y^2,\quad x = (t^2-1)/t,\quad y = t/(t^2+1)$ at $t = 1$

   (b) $f = e^{2x+3y}\cos 4z,\quad x = \ln t,\quad y = \ln(t^2+1),\quad z = t.$

   (c) $f = z\ln y + y\ln z + xyz,\quad x = \sin t,\quad y = t^2+1,\quad z = \cos^{-1} t$ at $t = 1$

2. If $z = f(x, y)$, $x = e^{2u} + e^{-2v}$, $y = e^{-2u} + e^{2v}$, then show that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = 2\left[x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}\right]$.

3. If $u = f(r, s, t)$, $r = x/y$, $s = y/z$, $t = z/x$, then show using chain rule that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

4. If $u = f(x^2 + 2yz, y^2 + 2zx)$ then prove that $(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z} = 0.$

5. Given that $z^3 + xy - y^2 z = 6$, obtain the expressions for $\partial y/\partial x$, $\partial z/\partial x$ in terms of x, y, z and find their values at the point (0, 1, 2).

6. Given that $u = f(x^2 + y^2 + z^2)$, where $x = r\cos\theta\sin\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\phi$, find $\dfrac{\partial u}{\partial\theta}$, and $\dfrac{\partial u}{\partial\phi}$.

7. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$, if $\cos xy + \cos yz + \cos zx = 1$.

8. Find $\dfrac{dy}{dx}$, if $x^y + y^x = a$, a is any constant and $x, y > 0$.

9. If $x + y = 2e^{\theta}\cos\phi$ and $x - y = 2ie^{\theta}\sin\phi$; prove that $\dfrac{\partial^2 v}{\partial\theta^2} + \dfrac{\partial^2 v}{\partial\phi^2} = 4xy\dfrac{\partial^2 v}{\partial x\partial y}$.

10. If v is a function of r only where $r^2 = \sum_{i=1}^{n} x_i^2$, then show that

$$\sum_{i=1}^{n}\frac{\partial^2 v}{\partial x_i^2} = \frac{d^2 v}{dr^2} + \frac{n-1}{r}\frac{dv}{dr}.$$

11. If $x = r(\sec\theta + \tan\theta)$, and $y = r(\sec\theta - \tan\theta)$, then show that

$$4\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = \left(\frac{\partial u}{\partial r}\right)^2 - \frac{\cos^2\theta}{r^2}\left(\frac{\partial u}{\partial\theta}\right)^2.$$

12. If $x = r^2\cosh\theta$, $y = r^2\sinh\theta$, then prove that

$$\frac{1}{2}r^2\left\{\frac{\partial^2 u}{\partial r^2}\right\}\left\{\frac{\partial^2 u}{\partial\theta^2}\right\} = (2x^2 - y^2)\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + (2y^2 - x^2)\frac{\partial^2 u}{\partial y^2}.$$

13. If $u = x^2 + y^2$, $v = 2xy$ and $f(x, y) = \phi(u, v)$, then show that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4(x^2 + y^2)\left[\frac{\partial^2\phi}{\partial u^2} + \frac{\partial^2\phi}{\partial v^2}\right].$$

14. If $u = f(x, y, z)$ and $x = r\cos\theta\sin\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\phi$, then show that

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2 = \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial\phi}\right)^2 + \frac{1}{r^2\sin^2\phi}\left(\frac{\partial f}{\partial\theta}\right)^2.$$

15. Find $\left(\dfrac{\partial w}{\partial y}\right)_x$ and $\left(\dfrac{\partial w}{\partial y}\right)_z$ at the point $(w, x, y, z) = (4, 2, 1, -1)$, if

$$w = x^2y^2 + yz - z^3 \text{ and } x^2 + y^2 + z^2 = 6$$

5.5 JACOBIANS

The study of Jacobians is useful in connection with the transformations of variables applied while studying problems in partial differentiation and multiple integrals.

5.5.1 Definition and Properties

If $u = u(x, y)$ and $v = v(x, y)$ are functions of two independent variables x and y, then the determinant $\begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix}$ is defined as the Jacobian of u, v with respect to x and y and is denoted by $J\left(\dfrac{u,v}{x,y}\right)$ or $\dfrac{\partial(u,v)}{\partial(x,y)}$.

In the case of three functions $u = u(x, y, z)$, $v = v(x, y, z)$, $w = w(x, y, z)$ of independent variables x, y and z, the jacobian of u, v, w with respect to x, y, z is defined as

$$J\left(\frac{u,v,w}{x,y,z}\right) = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix}.$$

Similarly, we can define Jacobians for four or more variables.

Properties of Jacobians: Following are some of the properties satisfied by Jacobians.

1. $$\frac{\partial(u,v)}{\partial(x,y)} \frac{\partial(x,y)}{\partial(u,v)} = 1.$$

Consider $u = u(x, y)$ and $v = v(x, y)$, then

$$\frac{\partial u}{\partial u} = 1 = \frac{\partial u}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial u}{\partial v} = 0 = \frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v}$$

$$\frac{\partial v}{\partial u} = 0 = \frac{\partial v}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial v}{\partial v} = 1 = \frac{\partial v}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial v}.$$

Therefore, $$\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial u} & \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial v} \\ \dfrac{\partial v}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial v}{\partial y}\dfrac{\partial y}{\partial u} & \dfrac{\partial v}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial v}{\partial y}\dfrac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

2. *If u, v are functions of x, y and x, y are themselves functions of s, t, then*

$$\frac{\partial(u,v)}{\partial(x,y)}\cdot\frac{\partial(x,y)}{\partial(s,t)} = \frac{\partial(u,v)}{\partial(s,t)} \quad \textit{(The chain rule).}$$

Consider
$$\frac{\partial(u,v)}{\partial(x,y)}\cdot\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}\begin{vmatrix}\frac{\partial x}{\partial s} & \frac{\partial x}{\partial t}\\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t}\end{vmatrix} = \begin{vmatrix}\frac{\partial u}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial s} & \frac{\partial u}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial t}\\ \frac{\partial v}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial s} & \frac{\partial v}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial t}\end{vmatrix}$$

$$= \begin{vmatrix}\frac{\partial u}{\partial s} & \frac{\partial u}{\partial t}\\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t}\end{vmatrix} = \frac{\partial(u,v)}{\partial(s,t)}.$$

3. *The necessary and sufficient condition for the functions of two independent variables x, y, say u(x, y) and v(x, y) to be functionally dependent is* $\frac{\partial(u,v)}{\partial(x,y)} = 0.$

We prove only the necessary part, let us assume that u, v are functionally dependent then there exists a functional relation of the type $F(u, v) = 0$.

Differentiating this with respect to x and y, we obtain respectively

$$\frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} = 0 \qquad \text{...(5.48)}$$

and,
$$\frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y} = 0 \qquad \text{...(5.49)}$$

Eliminating $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$ from (5.48) and (5.49), we obtain

$$\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix} = 0, \text{ that is, } \quad \frac{\partial(u,v)}{\partial(x,y)} = 0.$$

4. *If the functions u, v, w of the variables x, y, z are defined by the relations u = u(x), v = v(x, y) and w = w(x, y, z), then*

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\frac{\partial w}{\partial z}.$$

We observe that $\quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial z} = 0$, and also, $\frac{\partial v}{\partial z} = 0.$

Thus, $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x} & 0 & 0 \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & 0 \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\frac{\partial w}{\partial z}.$$

This result can be easily extended to more than three functions and the corresponding number of independent variables.

5. *(Jacobian of implicit functions).* *If* $f(u, v, w; x, y, z) = 0$, $g(u, v, w; x, y, z) = 0$, and $h(u, v, w; x, y, z) = 0$, then

$$\frac{\partial(f,g,h)}{\partial(u,v,w)}\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\partial(f,g,h)}{\partial(x,y,z)}. \qquad ...(5.50)$$

From $f(u, v, w; x, y, z) = 0$, we have

$$\left.\begin{aligned} \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} &= 0 \\ \frac{\partial f}{\partial y} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y} &= 0 \\ \frac{\partial f}{\partial z} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial z} &= 0 \end{aligned}\right\}, \text{ or } \left.\begin{aligned} \sum \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} &= \frac{-\partial f}{\partial x} \\ \sum \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} &= \frac{-\partial f}{fy} \\ \sum \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} &= \frac{-\partial f}{\partial z} \end{aligned}\right\}, \qquad ...(5.51)$$

where the summation is over u, v and w only.

We can obtain similar sets of equations corresponding to the functions.

$g(u, v, w, x, y, z) = 0$, and $h(u, v, w, x, y, z) = 0$

Hence, $$\frac{\partial(f,g,h)}{\partial(u,v,w)}\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \sum \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} & \sum \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} & \sum \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} \\ \sum \frac{\partial g}{\partial u}\frac{\partial u}{\partial x} & \sum \frac{\partial g}{\partial u}\frac{\partial u}{\partial y} & \sum \frac{\partial g}{\partial u}\frac{\partial u}{\partial z} \\ \sum \frac{\partial h}{\partial u}\frac{\partial u}{\partial x} & \sum \frac{\partial h}{\partial u}\frac{\partial u}{\partial y} & \sum \frac{\partial h}{\partial u}\frac{\partial u}{\partial z} \end{vmatrix},$$

where summation is over u, v and w only.

Using (5.51) and similar equations, we have

$$\frac{\partial(f,g,h)}{\partial(u,v,w)}\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{-\partial f}{\partial x} & \frac{-\partial f}{\partial y} & \frac{-\partial f}{\partial z} \\ \frac{-\partial g}{\partial x} & \frac{-\partial g}{\partial y} & \frac{-\partial g}{\partial z} \\ \frac{-\partial h}{\partial x} & \frac{-\partial h}{\partial y} & \frac{-\partial h}{\partial z} \end{vmatrix} = (-1)^3 \frac{\partial(f,g,h)}{\partial(x,y,z)}.$$

Example 5.23: (i) If $x = r\cos\theta$, $y = r\sin\theta$, prove that $\frac{\partial(x,y)}{\partial(r,\theta)} = r.$

(ii) If $x = r\sin\phi\cos\theta$, $y = r\sin\phi\sin\theta$, $z = r\cos\phi$, prove that $\frac{\partial(x,y,z)}{\partial(r,\phi,\theta)} = r^2\sin\phi.$

Solution: (i) We have

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r.$$

$$\text{(ii)} \quad \frac{\partial(x,y,z)}{\partial(r,\phi,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & r\cos\phi\cos\theta & -r\sin\phi\sin\theta \\ \sin\phi\sin\theta & r\cos\phi\sin\theta & r\sin\phi\cos\theta \\ \cos\phi & -r\sin\phi & 0 \end{vmatrix}$$

$$\begin{aligned} &= \cos\phi(r^2\cos\phi\sin\phi\cos^2\theta + r^2\cos\phi\sin\phi\sin^2\theta) \\ &\quad + r\sin\phi[r\sin^2\phi\cos^2\theta + r\sin^2\phi\sin^2\theta] \\ &= r^2\cos^2\phi\sin\phi + r^2\sin^3\phi = r^2\sin\phi(\cos^2\phi + \sin^2\phi) = r^2\sin\phi. \end{aligned}$$

Example 5.24: For $x = u$, $y = u\tan v$, $z = w$, verify $\frac{\partial(x,y,z)}{\partial(u,v,w)}\cdot\frac{\partial(u,v,w)}{\partial(x,y,z)} = 1.$

Solution: $$J_1 = \frac{\partial(x,y\,z)}{\partial(u,v\,w)} = \begin{vmatrix} 1 & 0 & 0 \\ \tan v & u\sec^2 v & 0 \\ 0 & 0 & 1 \end{vmatrix} = u\sec^2 v.$$

Solving for u, v, w in terms of x, y and z, we have $u = x$, $v = \tan^{-1}(y/x)$, $w = z$. Thus,

$$J_2 = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \dfrac{1}{\;} & 0 & 0 \\ \dfrac{-y}{x^2+y^2} & \dfrac{x}{x^2+y^2} & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \frac{x}{x^2+y^2} = \frac{1}{x[1+(y/x)^2]} = \frac{1}{u[1+\tan^2 v]} = \frac{1}{u \sec^2 v}.$$

Hence, $J_1 J_2 = 1$.

Example 5.25: If $u = \sqrt{yz}$, $v = \sqrt{zx}$, $w = \sqrt{xy}$ and $x = r\sin\phi\cos\theta$, $y = r\sin\phi\sin\theta$, $z = r\cos\phi$, find

$$\frac{\partial(u,v,w)}{\partial(r,\phi,\theta)}.$$

Solution: By definition,

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 0 & \dfrac{1}{2}\sqrt{\dfrac{z}{y}} & \dfrac{1}{2}\sqrt{\dfrac{y}{z}} \\ \dfrac{1}{2}\sqrt{\dfrac{z}{x}} & 0 & \dfrac{1}{2}\sqrt{\dfrac{x}{z}} \\ \dfrac{1}{2}\sqrt{\dfrac{y}{x}} & \dfrac{1}{2}\sqrt{\dfrac{x}{y}} & 0 \end{vmatrix} = \frac{1}{8}\left[\frac{\sqrt{xyz}}{\sqrt{xyz}} + \frac{\sqrt{xyz}}{\sqrt{xyz}}\right] = \frac{1}{4}$$

Also, $\dfrac{\partial(x,y,z)}{\partial(r,\phi,\theta)} = r^2\sin\phi$, refer to Example 5.23 (ii).

By chain rule, $\dfrac{\partial(u,v,w)}{\partial(r,\phi,\theta)} = \dfrac{\partial(u,v,w)}{\partial(x,y,z)} \cdot \dfrac{\partial(x,y,z)}{\partial(r,\phi,\theta)} = \dfrac{1}{4}r^2\sin\phi.$

Example 5.26: Show that $u = \dfrac{x+y}{1-xy}$ and $v = \tan^{-1}x + \tan^{-1}y$, $xy \neq 1$ are functionally dependent. Also find the relationship between u and v.

Solution: To show that u and v are functionally dependent, we prove that $\dfrac{\partial(u,v)}{\partial(x,y)} = 0$.

We have, $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{1+y^2}{(1-xy)^2} & \dfrac{1+x^2}{(1-xy)^2} \\ \dfrac{1}{1+x^2} & \dfrac{1}{1+y^2} \end{vmatrix} = \dfrac{1}{(1-xy)^2} - \dfrac{1}{(1-xy^2)} = 0.$

To find the relation between u and v, consider

$v = \tan^{-1} x + \tan^{-1} y = \tan^{-1} \dfrac{x+y}{1-xy}$, which gives $\tan v = \dfrac{x+y}{1-xy}$, or $u = \tan v$.

Example 5.27: If the three roots of the equation in λ given by $(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$ are u, v and w, then prove that

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = -2\frac{(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}$$

Solution: The given equation $(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$ can be rewritten as

$$3\lambda^3 - 3(x+y+z)\lambda^2 + 3(x^2+y^2+z^2)\lambda - (x^3+y^3+z^3) = 0.$$

Since u, v, w are the three roots of this equation, therefore,

$$u + v + w = x + y + z,\ uv + vw + wu = x^2 + y^2 + z^2, \text{ and } uvw = \frac{1}{3}(x^3 + y^3 + z^3).$$

We define, $f(u, v, w; x, y, z) = \Sigma u - \Sigma x = 0$, $g(u, v, w; x, y, z) = \Sigma uv - \Sigma x^2 = 0$

and, $$h(u, v, w; x, y, z) = uvw - \frac{1}{3}\Sigma x^3 = 0.$$

Here, u, v, w are the implicit functions of the three independent variables x, y and z, and thus from (5.50)

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = (-1)^3 \frac{\partial(f, g, h)}{\partial(x, y, z)} \Big/ \frac{\partial(f, g, h)}{\partial(u, v, w)}. \qquad ...(5.52)$$

Now, $$\frac{\partial(f, g, h)}{\partial(x, y, z)} = \begin{vmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} & \dfrac{\partial g}{\partial z} \\ \dfrac{\partial h}{\partial x} & \dfrac{\partial h}{\partial y} & \dfrac{\partial h}{\partial z} \end{vmatrix} = \begin{vmatrix} -1 & -1 & -1 \\ -2x & -2y & -2z \\ -x^2 & -y^2 & -z^2 \end{vmatrix}$$

$$= -2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = -2\begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ x^2 & y^2-x^2 & z^2-x^2 \end{vmatrix}$$

$$= -2[(y-x)(z^2-x^2) - (z-x)(y^2-x^2)]$$

$$= -2(y-x)(z-x)[z + x - y - x] = -2(x-y)(y-z)(z-x) \qquad ...(5.53)$$

Also, $\dfrac{\partial(f,g,h)}{\partial(u,v,w)} = \begin{vmatrix} \dfrac{\partial f}{\partial u} & \dfrac{\partial f}{\partial v} & \dfrac{\partial f}{\partial w} \\ \dfrac{\partial g}{\partial u} & \dfrac{\partial g}{\partial v} & \dfrac{\partial g}{\partial w} \\ \dfrac{\partial h}{\partial u} & \dfrac{\partial h}{\partial v} & \dfrac{\partial h}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ (v+w) & (w+u) & (u+v) \\ vw & wu & uv \end{vmatrix}$

$$= \begin{vmatrix} 1 & 0 & 0 \\ v+w & u-v & u-w \\ vw & w(u-v) & v(u-w) \end{vmatrix} = (u-v)v(u-w) - w(u-w)(u-v)$$

$$= (u-v)(u-w)(v-w) = -(u-v)(v-w)(w-u) \qquad \text{...(5.54)}$$

Substituting from (5.53) and (5.54) in (5.52), we get

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = -2\frac{(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}.$$

5.5.2 Applications in Change of Variables

Jacobians are applied in connection with the change of variables in multiple integrals to be discussed in Chapter 7. Here we discuss applications in connection with the change of variables in partial differentiation.

Suppose that $f(x, y)$ is a function of two independent variables x, y and x, y are functions of two new independent variables u, v given by $x = \phi(u, v)$, $y = \psi(u, v)$. By chain rule

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \qquad \text{...(5.55)}$$

and,

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} \qquad \text{...(5.56)}$$

Suppose we want to determine $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$ in terms of $\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial v}$. Solving (5.55) and (5.56) by Cramer's rule, we get

$$\frac{\partial f/\partial x}{\dfrac{\partial f}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial f}{\partial v}\dfrac{\partial y}{\partial u}} = \frac{\partial f/\partial y}{\dfrac{\partial f}{\partial v}\dfrac{\partial x}{\partial u} - \dfrac{\partial f}{\partial u}\dfrac{\partial x}{\partial v}} = \frac{1}{\dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial x}{\partial v}\dfrac{\partial y}{\partial u}}$$

Hence,

$$\frac{\partial f}{\partial x} = \frac{1}{J}\left[\frac{\partial(f,y)}{\partial(u,v)}\right] \text{ and } \frac{\partial f}{\partial y} = -\frac{1}{J}\left[\frac{\partial(f,x)}{\partial(u,v)}\right], \qquad \text{...(5.57)}$$

where $J = \dfrac{\partial(x,y)}{\partial(u,v)}$ is the Jacobian of the variables of transformation.

Similarly, if $f(x, y, z)$ is a function of three independent variables x, y, z and x, y, z are functions of three new independent variables say u, v, w, then it can be shown that

$$\frac{\partial f}{\partial x} = \frac{1}{J}\left[\frac{\partial(f, y, z)}{\partial(u, v, w)}\right], \frac{\partial f}{\partial y} = -\frac{1}{J}\left[\frac{\partial(f, x, z)}{\partial(u, v, w)}\right] \text{ and } \frac{\partial f}{\partial z} = \frac{1}{J}\left[\frac{\partial(f, x, y)}{\partial(u, v, w)}\right], \quad ...(5.58)$$

where $J = \dfrac{\partial(x, y, z)}{\partial(u, v, w)}$ is the Jacobian of the variables of transformation.

Example 5.28: If $u = f(x, y)$ and $x = r\cos\theta$, $y = r\sin\theta$, then prove that

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2.$$

Solution: We have $x = r\cos\theta$, $y = r\sin\theta$, thus

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r. \quad ...(5.59)$$

Also, $$\frac{\partial(f, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \theta} \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos\theta\frac{\partial f}{\partial r} - \sin\theta\frac{\partial f}{\partial \theta} \quad ...(5.60)$$

and, $$\frac{\partial(f, x)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \theta} \\ \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \theta} \\ \cos\theta & -r\sin\theta \end{vmatrix} = -r\sin\theta\frac{\partial f}{\partial r} - \cos\theta\frac{\partial f}{\partial \theta}. \quad ...(5.61)$$

Hence using (5.57), $$\frac{\partial f}{\partial x} = \frac{1}{J}\frac{\partial(f, y)}{\partial(r, \theta)} = \cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial \theta} \quad ...(5.62)$$

and, $$\frac{\partial f}{\partial y} = -\frac{1}{J}\frac{\partial(f, x)}{\partial(r, \theta)} = \sin\theta\frac{\partial f}{\partial r} + \frac{\cos\theta}{r}\frac{\partial f}{\partial \theta} \quad ...(5.63)$$

Squaring and adding (5.62) and (5.63) and rearranging, we obtain

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = (\cos^2\theta + \sin^2\theta)\left(\frac{\partial f}{\partial r}\right)^2 + \frac{(\cos^2\theta + \sin^2\theta)}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2.$$

Example 5.29: If $x = u + v + w$, $y = vw + wu + uv$, $z = uvw$ and F is a function of x, y, z, then show that

$$u\frac{\partial F}{\partial u} + v\frac{\partial F}{\partial v} + w\frac{\partial F}{\partial w} = x\frac{\partial F}{\partial x} + 2y\frac{\partial F}{\partial y} + 3z\frac{\partial F}{\partial z}.$$

Solution: We have $x = u + v + w,\ y = vw + wu + uv, \quad z = uvw.$ ...(5.64)

Thus,
$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ v+w & w+u & u+v \\ vw & wu & uv \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ v+w & u-v & u-w \\ vw & w(u-v) & v(u-w) \end{vmatrix} = -(u-v)(v-w)(w-u).$$

Also,
$$\frac{\partial(F,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} & \frac{\partial F}{\partial w} \\ v+w & w+u & u+v \\ vw & wu & uv \end{vmatrix} = u^2(v-w)\frac{\partial F}{\partial u} + v^2(w-u)\frac{\partial F}{\partial v} + w^2(u-v)\frac{\partial F}{\partial w},$$

$$\frac{\partial(F,x,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} & \frac{\partial F}{\partial w} \\ 1 & 1 & 1 \\ vw & wu & uv \end{vmatrix} = u(v-w)\frac{\partial F}{\partial u} + v(w-u)\frac{\partial F}{\partial v} + w(u-v)\frac{\partial F}{\partial w}$$

and,
$$\frac{\partial(F,x,y)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} & \frac{\partial F}{\partial w} \\ 1 & 1 & 1 \\ v+w & w+u & u+v \end{vmatrix} = (v-w)\frac{\partial F}{\partial u} + (w-u)\frac{\partial F}{\partial v} + (u-v)\frac{\partial F}{\partial w}.$$

Using (5.58), we obtain

$$\frac{\partial F}{\partial x} = -\frac{u^2(v-w)\frac{\partial F}{\partial u} + v^2(w-u)\frac{\partial F}{\partial v} + w^2(u-v)\frac{\partial F}{\partial w}}{(u-v)(v-w)(w-u)} \qquad ...(5.65)$$

$$\frac{\partial F}{\partial y} = \frac{u(v-w)\frac{\partial F}{\partial u} + v(w-u)\frac{\partial F}{\partial v} + w(u-v)\frac{\partial F}{\partial w}}{(u-v)(v-w)(w-u)} \qquad ...(5.66)$$

$$\frac{\partial F}{\partial z} = -\frac{(v-w)\frac{\partial F}{\partial u} + (w-u)\frac{\partial F}{\partial v} + (u-v)\frac{\partial F}{\partial w}}{(u-v)(v-w)(w-u)} \qquad ...(5.67)$$

Multiplying (5.65) by x, (5.66) by $2y$ and (5.67) by $3z$ and adding; also using values for x, y and z from (5.64), we obtain

$$x\frac{\partial F}{\partial x} + 2y\frac{\partial F}{\partial y} + 3z\frac{\partial F}{\partial z} = (u+v+w) \times [\text{right side of (5.65)}]$$

$$+ 2(vw + wu + uv) \times [\text{right side of (5.66)}] + 3\,uvw \times [\text{right side of (5.67)}] \quad \text{...(5.68)}$$

The coefficient of $\frac{\partial F}{\partial u}$ on the right side of (5.68) is

$$= \frac{-(u+v+w)(u^2)(v-w) + 2(vw+wu+uv)\,u\,(v-w) - 3uvw\,(v-w)}{(u-v)(v-w)(w-u)}$$

$$= \frac{[-(u+v+w)u + 2(vw+wu+uv) - 3vw]u}{(u-v)(w-u)} = \frac{[-u^2+wu+uv-vw]u}{(u-v)(w-u)} = u.$$

Similarly, we can show that coefficients of $\frac{\partial F}{\partial v}$ and $\frac{\partial F}{\partial w}$ on the right side of (5.68) are v and w respectively. Hence (5.68) simplifies to

$$x\frac{\partial F}{\partial x} + 2y\frac{\partial F}{\partial y} + 3z\frac{\partial z}{\partial z} = u\frac{\partial F}{\partial u} + v\frac{\partial F}{\partial v} + w\frac{\partial z}{\partial w}, \text{ the desired result.}$$

EXERCISE 5.5

1. If $x = r\cos\theta$, $y = r\sin\theta$, show that $\dfrac{\partial(r,\theta)}{\partial(x,y)} = \dfrac{1}{r}$.

2. If $x = e^u \cos v$, $y = e^u \sin v$, show that $\dfrac{\partial(x,y)}{\partial(u,v)}\dfrac{\partial(u,v)}{\partial(x,y)} = 1$.

3. If $u = x^2 - 2y^2$, $v = 2x^2 - y^2$ and $x = r\cos\theta$, $y = r\sin\theta$, show that
$$\frac{\partial(u,v)}{\partial(r,\theta)} = 6r^3 \sin 2\theta.$$

4. If $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, show that $\dfrac{\partial(r,\theta,z)}{\partial(x,y,z)} = \dfrac{1}{r}$.

5. If $F(x, y, u, v) = 0$ and $G(x, y, u, v) = 0$, prove that $\dfrac{\partial u}{\partial x}\dfrac{\partial y}{\partial u} + \dfrac{\partial v}{\partial x}\dfrac{\partial y}{\partial v} = 0$.

6. If $u = x + y + z$, $uv = y + z$, $uvw = z$, show that $\partial(x, y, z)/\partial(u, v, w) = u^2 v$.

7. If $u^3 = xyz$, $\dfrac{1}{v} = \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}$, $w^2 = x^2 + y^2 + z^2$, prove that
$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{-v(y-z)(z-x)(x-y)(x+y+z)}{3u^2 w(yz+zx+xy)}$$

8. If $u^3 + v^3 + w^3 = x + y + z$, $u^2 + v^2 + w^2 = x^3 + y^3 + z^3$, $u + v + w = x^2 + y^2 + z^2$, then prove that

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \frac{(y - z)(z - x)(x - y)}{(u - v)(v - w)(w - u)}.$$

9. If $u = x^2 + y^2 + z^2$, $v = x + y + z$, $w = xy + yz + zx$, show that the Jacobian $\dfrac{\partial(u, v, w)}{\partial(x, y, z)}$ vanishes identically. Also find the relation between u, v and w.

10. Show that the functions $u = x + y - z$, $v = x - y + z$, $w = x^2 + y^2 + z^2 - 2yz$ are not independent of one another. Also find the relation between them.

11. Show that the functions $u = \sin^{-1} x + \sin^{-1} y$, and $v = x\sqrt{1 - y^2} + y\sqrt{1 - x^2}$ are functionally dependent. Also find the relationship.

5.6 HOMOGENEOUS FUNCTIONS

A function $f(x, y)$ is said to be a homogeneous function of order n, if the degree of each of its terms in x and y is equal to n. For example,

$$f(x, y) = a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \ldots + \ldots a_{n-1} x y^{n-1} + a_n y^n$$

is a homogeneous function of order n.

This definition of homogeneity is applicable only to polynomial functions. To enlarge the concept of homogeneity we say that z *is a homogeneous function of order n in x and y, if it can be expressed in the form $x^n f(y/x)$.* In addition to the polynomial functions, this brings transcendental functions also within its scope. For example, $x^n \sin(y/x)$ is a homogeneous function of degree n, while

$$\frac{\sqrt{y} + \sqrt{x}}{y + x} = x^{-1/2} \frac{(1 + \sqrt{y/x})}{(1 + y/x)} = x^{-1/2} f(y/x)$$

is a homogeneous function of degree $-1/2$.

To cover some other functions of the form $f(x, y) = x^n f(y/x) + y^n g(x/y)$, we say that *$F(x, y)$ is a homogeneous function of order n in x and y if it is expressible in the form $F(\lambda x, \lambda y) = \lambda^n F(x, y)$.*

Similarly, *a function $F(x, y, z)$ of three variables is said to be homogeneous function of order n in x, y and z if it can be expressible in the form*

$F(\lambda x, \lambda y, \lambda z) = \lambda^n F(x, y, z)$, or $F(x, y, z) = x^n f(y/x, z/x)$ or $y^n g(x/y, z/y)$, or $z^n h(x/z, y/z)$.

We have the following important result concerning homogeneous functions.

If u is a homogeneous function of order n in x and y, then $\partial u/\partial x$ and $\partial u/\partial y$ both are homogeneous functions of order $(n - 1)$ in x and y.

To prove it, consider $u = x^n f(y/x)$. This gives

$$\frac{\partial u}{\partial x} = n x^{n-1} f(y/x) + x^n f'(y/x)(-y/x^2) = x^{n-1}\left[n f(y/x) - \frac{y}{x} f'(y/x)\right] = x^{n-1} g(y/x),$$

a homogeneous function of order n, where $g(y/x) = \left[nf(y/x) - \frac{y}{x}f'(y/x)\right]$.

Similarly, we can prove for $\partial u/\partial y$.

Next we consider an important theorem on homogeneous functions.

Theorem 5.1: (Euler's Theorem on Homogeneous Functions) *If $f(x, y)$ is a homogeneous function of order n in x and y, then*

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf. \qquad ...(5.69)$$

Proof. Since $f(x, y)$ is a homogeneous function of order n in x and y, we can write

$$f(x, y) = x^n g(y/x). \qquad ...(5.70)$$

Differentiating this partially with respect to x and y separately, we obtain respectively

$$\frac{\partial f}{\partial x} = nx^{n-1}g(y/x) + x^n g'(y/x)(-y/x^2)$$

$$= nx^{n-1}g(y/x) - yx^{n-2}g'(y/x). \qquad ...(5.71)$$

and,
$$\frac{\partial f}{\partial y} = x^n g'(y/x)\frac{1}{x} = x^{n-1}g'(y/x). \qquad ...(5.72)$$

Multiplying (5.71) by x and (5.72) by y, we obtain

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nx^n g(y/x) - yx^{n-1}g'(y/x) + yx^{n-1}g'(y/x) = nx^n g(y/x) = nf, \quad \text{using (5.70)}.$$

Another result on homogeneous functions which follows from Euler's theorem is given as follows.

Theorem 5.2: *If f is a homogeneous function of x, y of order n, then*

$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x\partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f \qquad ...(5.73)$$

Proof. Differentiating (5.69) partially with respect to x and y separately, we get

$$\frac{\partial f}{\partial x} + x\frac{\partial^2 f}{\partial x^2} + y\frac{\partial^2 f}{\partial x\partial y} = n\frac{\partial f}{\partial x} \qquad ...(5.74)$$

and,
$$x\frac{\partial^2 f}{\partial y\partial x} + \frac{\partial f}{\partial y} + y\frac{\partial^2 f}{\partial y^2} = n\frac{\partial f}{\partial y} \qquad ...(5.75)$$

Multiplying (5.74) and (5.75) by x and y respectively, adding and rearranging the terms and assuming $\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x}$, we obtain

$$x^2\frac{\partial^2 f}{\partial x^2}+2xy\frac{\partial^2 f}{\partial x\partial y}+y^2\frac{\partial^2 f}{\partial y^2}=n\left[x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right]-\left[x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right]=n(nf)-nf=n(n-1)f, \text{ using (5.69)}$$

Euler's theorem on homogeneous functions can be extended to functions of three or more independent variables. For example, if f is a homogeneous function of three independent variables x, y, and z, then

$$f=x^n\phi\left(\frac{y}{x},\frac{z}{x}\right)=x^n\phi(u,v), \quad \text{...(5.76)}$$

where $u=y/x$ and $v=z/x$.

We have,

$$f_x=nx^{n-1}\phi+x^n\left[\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial\phi}{\partial v}\frac{\partial v}{\partial x}\right]$$

$$=n\,x^{n-1}\phi-y\,x^{n-2}\frac{\partial\phi}{\partial u}-zx^{n-2}\frac{\partial\phi}{\partial v}, \quad \text{...(5.77)}$$

$$f_y=x^n\left[\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial\phi}{\partial v}\frac{\partial v}{\partial y}\right]=x^{n-1}\frac{\partial\phi}{\partial u}, \quad \text{...(5.78)}$$

and,

$$f_z=x^n\left[\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial z}+\frac{\partial\phi}{\partial v}\cdot\frac{\partial v}{\partial z}\right]=x^{n-1}\frac{\partial\phi}{\partial v}. \quad \text{...(5.79)}$$

Multiplying (5.77), (5.78), and (5.79) by x, y and z respectively and adding, then using (5.76), we obtain

$$xf_x+yf_y+zf_z=nf. \quad \text{...(5.80)}$$

This is Euler's theorem on homogeneous functions for three variables.

Example 5.30: If $u=\tan^{-1}\left(\dfrac{x^3+y^3}{x-y}\right)$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=\sin 2u$.

Solution: Here $u(x,y)=\tan^{-1}\left(\dfrac{x^3+y^3}{x-y}\right)$ is not a homogeneous function, however,

$$z=\tan u=\frac{x^3+y^3}{x-y}=x^2\frac{1+(y/x)^3}{1-(y/x)},$$

is a homogeneous function of order two in x and y. Using Euler's theorem, we obtain

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2z. \quad \text{...(5.81)}$$

Here $\dfrac{\partial z}{\partial x}=\sec^2 u\,\dfrac{\partial u}{\partial x};\quad \dfrac{\partial z}{\partial y}=\sec^2 u\,\dfrac{\partial u}{\partial y}.$

Substituting in (5.81), we get $\sec^2 u\left\{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right\}=2\tan u$

or, $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{2\sin u}{\cos u}\cos^2 u=\sin 2u$$

Example 5.31: If $u(x, y)=\sin^{-1}\frac{x}{y}+\tan^{-1}\frac{y}{x}$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0.$

Solution: We have $u(\lambda x, \lambda y)=u(x, y)$, thus $u(x, y)$ is a homogeneous function of degree zero.

Therefore by Euler's theorem $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0.u=0.$

Example 5.32: If $u=\sin^{-1}\frac{x+2y+3z}{x^8+y^8+z^8}$, then find the value of $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}$.

Solution: Here $u(x, y, z)=\sin^{-1}\frac{x+2y+3z}{x^8+y^8+z^8}$ is not a homogeneous function. However,

$$f=\sin u=\frac{x+2y+3z}{x^8+y^8+z^8}=x^{-7}\frac{1+2(y/x)+3(z/x)}{1+(y/x)^8+(z/x)^8}$$

is a homogeneous function of order -7 in x, y, z. Hence, by Euler's theorem

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=(-7)f \qquad \text{...(5.82)}$$

But $\frac{\partial f}{\partial x}=\cos u\frac{\partial u}{\partial x}$, $\frac{\partial f}{\partial y}=\cos u\frac{\partial u}{\partial y}$, and $\frac{\partial f}{\partial z}=\cos u\frac{\partial u}{\partial z}$.

Substituting in (5.82), we get

$$x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y}+z\cos u\frac{\partial u}{\partial z}=-7\sin u$$

or, $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=-7\tan u.$$

Example 5.33: If $z=x^m f(y/x)+x^n g(y/x)$, then show that

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}+mnz=(m+n-1)\left(x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right). \qquad \text{...(5.83)}$$

Solution: Let $z=x^m f(y/x)+x^n g(y/x)=z_1+z_2$, say.

The function $z_1=x^m f(y/x)$, is a homogeneous function of order m in x and y, thus

$$x\frac{\partial z_1}{\partial x}+y\frac{\partial z_1}{\partial y}=mz_1 \qquad \text{...(5.84)}$$

and, $$x^2\frac{\partial^2 z_1}{\partial x^2}+2xy\frac{\partial^2 z_1}{\partial x\partial y}+y^2\frac{\partial^2 z_1}{\partial y^2}=m(m-1)z_1. \qquad ...(5.85)$$

Similarly, for $z_2 = x^n g(y/x)$, we have

$$x\frac{\partial z_2}{\partial x}+y\frac{\partial z_2}{\partial y}=nz_2 \qquad ...(5.86)$$

and, $$x^2\frac{\partial^2 z_2}{\partial x^2}+2xy\frac{\partial^2 z_2}{\partial x\partial y}+y^2\frac{\partial^2 z_2}{\partial y^2}=n(n-1)z_2. \qquad ...(5.87)$$

Adding (5.84) and (5.86) and using $z = z_1, + z_2$, we obtain

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=mz_1+nz_2. \qquad ...(5.88)$$

Similarly from (5.85) and (5.87), we obtain

$$x^2\frac{\partial z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=m(m-1)z_1+n(n-1)z_2 \qquad ...(5.89)$$

Thus left side of (5.83), using (5.89) becomes

$$\begin{aligned}
&= m(m-1)z_1+n(n-1)z_2+mnz\\
&= m\,(m-1)\,z_1+mnz_1+n(n-1)z_2+mnz_2\\
&= (m+n-1)mz_1+(m+n-1)nz_2\\
&= (m+n-1)\,(mz_1+nz_2),
\end{aligned}$$

using (5.88) it is the same as the right side of (5.83).

EXERCISE 5.6

1. Verify Euler's theorem for
   (a) $z = ax^2 + 2hxy + by^2$
   (b) $z = (x^2 + xy + y^2)^{-1}$
   (c) $z = \sin^{-1}\dfrac{x}{y} + \tan^{-1}\dfrac{y}{x}$
   (d) $z = x^n \ln\dfrac{y}{x}$
   (e) $z = x^2(x^2 - y^2)^3/(x^2 + y^2)^3$.
2. If u is a homogeneous function of order n in x, y, then prove that
   (a) $x\dfrac{\partial^2 u}{\partial x^2}+y\dfrac{\partial^2 u}{\partial y\partial y}=(n-1)\dfrac{\partial u}{\partial x}$
   (b) $x\dfrac{\partial^2 u}{\partial x\partial y}+y\dfrac{\partial^2 u}{\partial y^2}=(n-1)\dfrac{\partial u}{\partial y}$
3. If $z = xy\,f(x/y)$, then show that $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=2z$.

4. If $u = \cos^{-1} \dfrac{x+y}{\sqrt{x}+\sqrt{y}}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = -\dfrac{1}{2}\cot u$.

5. If $u = \sin^{-1} \dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$.

6. If $u = \sin^{-1}(x/y) + \tan^{-1}(y/x)$, prove that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 0$.

7. If $u = \sin(y/x) + x\sin^{-1}(y/x)$, prove that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial z^2} = 0$.

8. If $u = f(r, s, t)$, $r = x/y$, $s = y/z$, $t = z/x$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0$.

9. If $u = \tan^{-1}\dfrac{x^3+y^3}{x-y}$, prove that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 2\cos 3u \sin u$.

10. If $u = \tan^{-1}\dfrac{y^2}{x}$, prove that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = -\sin 2u \sin^2 u$.

5.7 TAYLOR'S EXPANSION. APPROXIMATION AND ERROR ESTIMATION

We have already studied the Taylor's expansion for functions in one variable in Chapter 4. Here we extend this to the functions of more than one variable.

5.7.1 Taylor's Expansion for Two Variables

Theorem 5.3: (Taylor's Expansion) If $f(x, y)$ and its partial derivatives up to order $(n + 1)$ are continuous throughout the domain D centered at a point (x_0, y_0), then throughout D

$$f(x_0 + h, y_0 + k) = f(x_0, y_0) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f(x_0, y_0) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0, y_0) + \dots + $$

$$+ \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x_0, y_0) + R_n \qquad \text{...(5.90)}$$

where R_n is the remainder term given by

$$R_n = \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(x_0 + \theta h, y_0 + \theta k), \quad 0 < \theta < 1. \qquad \text{...(5.91)}$$

Proof. To prove this, let $x = x_0 + \lambda h$ and $y = y_0 + \lambda k$, where $0 < \lambda < 1$, and let

$$F(\lambda) = f(x_0 + \lambda h, y_0 + \lambda k).$$

By chain rule, $F'(\lambda) = \dfrac{dF}{d\lambda} = \dfrac{\partial f}{\partial x}\dfrac{dx}{d\lambda} + \dfrac{\partial f}{\partial y}\dfrac{dy}{d\lambda} = h\dfrac{\partial f}{\partial x} + k\dfrac{\partial f}{\partial y} = \left(h\dfrac{\partial}{\partial x} + k\dfrac{\partial}{\partial y}\right)f(x, y).$

Similarly, $F''(\lambda) = \dfrac{d^2F}{d\lambda^2} = \left(h\dfrac{\partial}{\partial x} + k\dfrac{\partial}{\partial y}\right)^2 f(x, y)$

....

$$F^{(n)}(\lambda) = \frac{d^nF}{d\lambda^{n-1}} = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x, y), \text{ and } F^{(n+1)}(\lambda) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(x, y).$$

Set $\lambda = 0$ in $F(\lambda), F'(\lambda), F''(\lambda), \ldots, F^n(\lambda)$ and $\lambda = \theta$ in $F^{(n+1)}(\lambda)$, we obtain

$$\left.\begin{aligned}
F(0) &= f(x_0, y_0) \\
F'(0) &= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x_0, y_0) \\
F''(0) &= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0, y_0) \\
&\;\;\vdots \\
F^{(n)}(0) &= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x_0, y_0), \\
\text{and,}\quad F^{(n+1)}(\theta) &= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(x_0 + \theta h, y_0 + \theta k).
\end{aligned}\right\} \quad \ldots(5.92)$$

By Taylor's theorem for a function of one variable, refer to (4.58), we have expansion of $F(1)$ as

$$F(1) = F(0) + F'(0) + \frac{1}{2!}F''(0) + \ldots + \frac{1}{n!}F^{(n)}(0) + \frac{1}{(n+1)!}F^{(n+1)}(\theta), \; 0 < \theta < 1 \quad \ldots(5.93)$$

Hence, using (5.92) in (5.93), we obtain

$$f(x_0 + h, y_0 + k) = f(x_0, y_0) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x_0, y_0) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0, y_0) + \ldots$$

$$+ \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x_0, y_0) + \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(x_0 + \theta h,\; y_0 + \theta k), \; 0 < \theta < 1.$$

which is the Taylor's expansion for a function of two variables. The result can be extended to functions of three or more variables on the similar lines.

5.7.2 Taylor's Expansion for $f(x, y)$ at the Origin: Maclaurin's Expansion

If $(x_0, y_0) = (0, 0)$, then we treat h and k as independent variables, replacing h by x and k by y in (5.90), we obtain Taylor's expansion for $f(x, y)$ at $(0, 0)$, also called the Maclaurin's expansion, given by

$$f(x,y) = f(0,0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f(0,0) + \frac{1}{2!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(0,0) + \ldots$$

$$+ \frac{1}{n!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^n f(0,0) + \frac{1}{(n+1)!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^{n+1} f(\theta x, \theta y), 0 < \theta < 1. \quad \ldots(5.94)$$

When $R_n \to 0$ as $n \to \infty$, from (5.90) and (5.94), we get respectively, the Taylor's series and Maclaurin's series, respectively as

$$f(x_0 + h, y_0 + k) = f(x_0, y_0) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x_0, y_0) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0, y_0)$$

$$+ \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f(x_0, y_0) + \ldots \quad \ldots(5.95)$$

$$f(x,y) = f(0,0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f(0,0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(0,0) + \frac{1}{3!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^3 f(0,0) + \ldots \quad \ldots(5.96)$$

The results for the Taylor's expansion can be extended to more than two variables on the similar line.

5.7.3 Approximation and Error Estimation

Taylor's formula (5.90) provides polynomial approximation to two variables functions. The first $n+1$ terms give the polynomial of degree n and the last term gives approximation error.

For example for $n = 1$, the linear approximation of $f(x, y)$ about the point (x_0, y_0) is

$$f(x,y) = f(x_0 + \overline{x - x_0}, y_0 + \overline{y - y_0}) \approx f(x_0, y_0) + (x - x_0) f_x(x_0, y_0) + (y - y_0) f_y(x_0, y_0) \quad \ldots(5.97)$$

with error term, $R_1 = \dfrac{1}{2!}[(x - x_0)^2 f_{xx} + 2(x - x_0)(y - y_0) f_{xy} + (y - y_0)^2 f_{yy}]$ ...(5.98)

Here, f_{xx}, f_{xy} and f_{yy} are to be evaluated at the point $[x_0 + \theta(x - x_0), y_0 + \theta(y - y_0)]$, $0 < \theta < 1$.

Since θ is unknown we cannot evaluate R_1 exactly, however, it is possible to find an upper bound to R_1 in a given rectangular region $R = \{(x, y) : |x - x_0| < \delta_1, \ |y - y_0| < \delta_2\}$. From (5.98)

$$|R_1| \le \frac{1}{2}[|x - x_0|^2 |f_{xx}| + 2|x - x_0|\,|y - y_0|\,|f_{xy}| + |y - y_0|^2 |f_{yy}|]$$

If $M = max\{|f_{xx}|, |f_{xy}|, |f_{yy}|\}$, for all $(x, y) \in R$, then

$$|R_1| \le \frac{M}{2}[|x - x_0|^2 + 2|x - x_0|\,|y - y_0| + |y - y_0|^2]$$

$$= \frac{M}{2}[|x-x_0| + |y-y_0|]^2 \le \frac{M}{2}[\delta_1 + \delta_2]^2, \text{ for all } (x, y) \in R.$$

Thus we have the following result:

If $M = \max\{|f_{xx}|, |f_{xy}|, |f_{yy}|\}$, then the maximum absolute error $|R_1|$ in the linear approximation of $f(x, y)$ in the rectangular region $R = \{(x, y) : |x - x_0| < \delta_1, |y - y_0| < \delta_2\}$ about the point (x_0, y_0), is

$$\frac{M}{2}[\delta_1 + \delta_2]^2. \qquad ...(5.99)$$

Similarly, it can be shown that *the maximum absolute error $|R_2|$ in the quadratic approximation of $f(x, y)$ in the rectangular region $R = \{(x, y) : |x - x_0| < \delta_1, |y - y_0| < \delta_2\}$ about the point (x_0, y_0), is*

$$\frac{M}{6}(\delta_1 + \delta_2)^3, \qquad ...(5.100)$$

where $M = \max. \{|f_{xxx}|, |f_{xxy}|, |f_{xyy}|, |f_{yyy}|\}$.

We note that the *quadratic approximation of $f(x, y)$* about (x_0, y_0) is given by

$$f(x, y) \approx f(x_0, y_0) + (x - x_0)f_x + (y - y_0)f_y + \frac{1}{2!}[(x - x_0)^2 f_{xx} + 2(x - x_0)(y - y_0)f_{xy} + (y - y_0)^2 f_{yy}].$$

Example 5.34: Find the quadratic Taylor series polynomial approximation to the function $f(x, y) = \sin x \sin y$ about the origin. Obtain the maximum absolute error if $|x| \le 0.1$ and $|y| \le 0.1$.

Solution: We have

$$\begin{aligned}
f(x, y) &= \sin x \sin y, & f(0, 0) &= 0\\
f_x(x, y) &= \cos x \sin y, & f_x(0, 0) &= 0\\
f_y(x, y) &= \sin x \cos y, & f_y(0, 0) &= 0\\
f_{xx}(x, y) &= -\sin x \sin y, & f_{xx}(0, 0) &= 0\\
f_{xy}(x, y) &= \cos x \cos y, & f_{xy}(0, 0) &= 1\\
f_{yy}(x, y) &= -\sin x \sin y, & f_{yy}(0, 0) &= 0\\
f_{xxx}(x, y) &= -\cos x \sin y, & f_{xxy}(x, y) &= -\sin x \cos y,\\
f_{xyy}(x, y) &= -\cos x \sin y, & f_{yyy}(x, y) &= -\sin x \cos y.
\end{aligned}$$

The quadratic approximation about (0, 0) is given by

$$f(x, y) \approx f(0, 0) + \{xf_x(0, 0) + yf_y(0, 0)\} + \frac{1}{2!}\{x^2 f_{xx}(0, 0) + 2xyf_{xy}(0, 0) + y^2 f_{yy}(0, 0)\}$$

Substituting for $f(0, 0), f_x(0, 0), f_y(0, 0)$, etc., we obtain $\sin x \sin y \approx xy$.

The maximum absolute error in the quadratic approximation, refer to (5.100), is given by

$$|R_2| \le \frac{M}{6}[|x| + |y|]^3 \le \frac{M}{6}[0.1 + 0.1]^3 = 0.00133\, M,$$

where $M = \max. \{|f_{xxx}|, |f_{xxy}|, |f_{xyy}|\ |f_{yyy}|\}$ for all $(x, y) \in R: |x| \le 0.1$ and $|y| \le 0.1$.

Here the derivatives are to be evaluated at point $(\theta x, \theta y), 0<\theta<1$.

Since the third order derivatives in this case, as derived above, are products of sines and cosines thus they can never exceed unity and hence M can at the most be equal to 1; and thus $|R_2| \leq 0.00133$.

Example 5.35: Find the linear and quadratic Taylor series polynomial approximation to the function $f(x, y) = x^2y + 3y - 2$ about the point $(-1, 2)$ and obtain the maximum absolute error in the region $|x + 1| < 0.1$ and $|y - 2| < 0.1$.

Solution: We have,

$$\begin{aligned}
f(x, y) &= x^2y + 3y - 2, & f(-1, 2) &= 6\\
f_x &= 2xy & f_x(-1, 2) &= -4\\
f_y &= x^2 + 3 & f_y(-1, 2) &= 4\\
f_{xx} &= 2y & f_{xx}(-1, 2) &= 4\\
f_{xy} &= 2x & f_{xy}(-1, 2) &= -2\\
f_{yy} &= 0 & f_{yy}(-1, 2) &= 0\\
f_{xxx} &= 0 & f_{xxx}(-1, 2) &= 0\\
f_{xxy} &= 2 & f_{xxy}(-1, 2) &= 2\\
f_{xyy} &= 0 & f_{xyy}(-1, 2) &= 0\\
f_{yyx} &= 0 & f_{yyx}(-1, 2) &= 0\\
f_{yyy} &= 0 & f_{yyy}(-1, 2) &= 0.
\end{aligned}$$

The linear approximation of $f(x, y)$ about $(-1, 2)$ is

$$\begin{aligned}
f(x, y) &\approx f(-1, 2) + [(x + 1)f_x(-1, 2) + (y - 2)f_y(-1, 2)]\\
&= 6 + (x + 1)(-4) + (y - 2)(4) = 6 - 4(x + 1) + 4(y - 2).
\end{aligned}$$

The maximum absolute error in the linear approximation, refer to (5.99), is

$$|R_1| \leq \frac{M}{2}[|x+1| + |y-2|]^2 \leq \frac{M}{2}[0.1 + 0.1]^2 = 0.02M$$

where $M = \max\{|f_{xx}|, |f_{xy}|, |f_{yy}|\}$, for all $(x, y) \in R: |x+1| \leq 0.1$, and $|y-2| \leq 0.1$.

$$\begin{aligned}
\text{Now,}\quad \max|f_{xx}| &= \max|2y| = 2\max|y| = 2\max|(y-2)+2|\\
&\leq 2\max[|y-2| + 2] \leq 2(0.1 + 2) = 4.2\\
\max|f_{xy}| &= \max|2x| = 2\max|x| = 2\max|(x+1) - 1|\\
&\leq 2\max[|x+1| + 1] \leq 2[0.1 + 1] = 2.2\\
\text{and,}\quad \max|f_{yy}| &= 0.
\end{aligned}$$

Hence, $M = 4.2$, and thus, $|R_1| \leq 0.02\,(4.2) = .084$

Next, the quadratic approximation about $(-1, 2)$ is

$$f(x, y) \approx f(-1, 2) + [(x + 1)f_x(-1, 2) + (y - 2)f_y(-1, 2)] + \frac{1}{2!}[(x + 1)^2 f_{xx}(-1, 2)$$

$$+ 2(x+1)(y-2)f_{xy}(-1,2) + (y-2)^2 f_{yy}(-1,2)]$$

$$= 6 + (x+1)(-4) + (y-2)4 + \frac{1}{2}[4(x+1)^2 + 2(x+1)(y-2)(-2)]$$

$$= 6 - 4(x+1) + 4(y-2) + 2(x+1)^2 - 2(x+1)(y-2).$$

The maximum absolute error in the quadratic approximation, refer (5.100), is

$$|R_2| \le \frac{M}{6}[|x+1| + |y-2|]^3 \le \frac{M}{6}[0.1+0.1]^3 = \frac{.008M}{6} \approx .00133\text{ M},$$

where $M = \max\{|f_{xxx}|, |f_{xxy}|, |f_{xyy}|, |f_{yyy}|\}$, for all $(x, y) \in R: |x+1| \le 0.1$, and $|y-2| \le 0.1$.

Thus $M = \max\{0, 2, 0, 0\} = 2$, and hence, $|R_2| \le (0.00133)(2) = 0.00266$.

Example 5.36: ss Expand $f(x, y) = e^x \ln(1+y)$ in Taylor series about the origin up to the terms of degree three.

Solution: We have

$$f(x,y) = e^x \ln(1+y), \qquad f(0,0) = 0$$

$$f_x = e^x \ln(1+y) \qquad f_x(0,0) = 0$$

$$f_y = \frac{e^x}{1+y} \qquad f_y(0,0) = 1$$

$$f_{xx} = e^x \ln(1+y) \qquad f_{xx}(0,0) = 0$$

$$f_{xy} = \frac{e^x}{1+y} \qquad f_{xy}(0,0) = 1$$

$$f_{yy} = -\frac{e^x}{(1+y)^2} \qquad f_{yy}(0,0) = -1$$

$$f_{xxx} = e^x \ln(1+y) \qquad f_{xxx}(0,0) = 0$$

$$f_{xxy} = \frac{e^x}{1+y} \qquad f_{xxy}(0,0) = 1$$

$$f_{xyy} = -\frac{e^x}{(1+y)^2} \qquad f_{xyy}(0) = -1$$

$$f_{yyy} = \frac{2e^x}{(1+y)^3} \qquad f_{yyy}(0) = 2.$$

Taylor's series expansion of $f(x, y)$ about $(0, 0)$ up to the terms of degree 3 is

$$f(x,y) \approx f(0,0) + [xf_x(0,0) + yf_y(0,0)] + \frac{1}{2}[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)]$$

$$+ \frac{1}{6}[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)].$$

Substituting the values for $f(0, 0), f_x(0, 0), f_y(0, 0)$ etc. we get

$$e^x \ln(1+y) = 0 + [x(0) + y(1)] + \frac{1}{2}[x^2(0) + 2xy(1) + y^2(-1)] + \frac{1}{6}[x^3(0) + 3x^2y(1) + 3xy^2(-1) + y^3(2)]$$

or $$e^x \ln(1+y) = y + xy + \frac{1}{2}x^2y - \frac{1}{2}xy^2 + \frac{1}{3}y^3,$$

as the required expansion.

Example 5.37: Expand $f(x, y) = \tan^{-1} xy$ in powers of $(x - 1)$ and $(y - 1)$ up to second degree terms. Hence compute $f(1.1, 0.8)$.

Solution: We have

$$f(x,y) = \tan^{-1} xy \qquad f(1,1) = \pi/4 \approx 0.7854$$

$$f_x = \frac{y}{1+x^2y^2} \qquad f_x(1,1) = \frac{1}{2};$$

$$f_y = \frac{x}{1+x^2y^2} \qquad f_y(1,1) = \frac{1}{2};$$

$$f_{xx} = \frac{-2xy^3}{(1+x^2y^2)^2} \qquad f_{xx}(1,1) = -\frac{1}{2};$$

$$f_{xy} = \frac{1-x^2y^2}{(1+x^2y^2)^2} \qquad f_{xy}(1,1) = 0;$$

$$f_{yy} = \frac{-2x^3y}{(1+x^2y^2)^2} \qquad f_{yy}(1,1) = -\frac{1}{2}.$$

Taylor's series expansion of $f(x, y)$ about $(1, 1)$ up to terms of degree 2 is

$$f(x,y) \approx f(1,1) + [(x-1)f_x(1,1) + (y-1)f_y(1,1)]$$

$$+ \frac{1}{2}[(x-1)^2 f_{xx}(1,1) + 2(x-1)(y-1)f_{xy}(1,1) + (y-1)^2 f_{yy}(1,1)].$$

Substituting the values for $f(1, 1), f_x(1, 1), f_y(1, 1)$ etc. we get

$$\tan^{-1} xy \approx 0.7854 + \left[(x-1)\left(\frac{1}{2}\right) + (y-1)\left(\frac{1}{2}\right)\right]$$

$$+ \frac{1}{2}\left[(x-1)^2\left(-\frac{1}{2}\right) + 2(x-1)(y-1)(0) + (y-1)^2\left(-\frac{1}{2}\right)\right]$$

or,
$$\tan^{-1} xy \approx 0.7854 + \frac{1}{2}(x-1) + \frac{1}{2}(y-1) - \frac{1}{4}(x-1)^2 - \frac{1}{4}(y-1)^2.$$

To compute the value of $\tan^{-1} xy$ at (1.1, 0.8), put $x = 1.1$ and $y = 0.8$, we obtain

$$f(1.1, 0.8) = 0.7854 + \frac{1}{2}(0.1) + \frac{1}{2}(-0.2) - \frac{1}{4}(0.1)^2 - \frac{1}{4}(-0.2)^2 = 0.7229.$$

Example 5.38: Evaluate $\ln[(1.03)^{1/3} + (0.98)^{1/4} - 1]$ approximately using linear Taylor's series approximation.

Solution: Let $f(x, y) = \ln[x^{1/3} + y^{1/4} - 1]$. Take $x_0 = 1$, $y_0 = 1$, $h = 0.03$ and $k = -0.02$.

We have,
$$f_x = \frac{\frac{1}{3}x^{-2/3}}{x^{1/3} + y^{1/4} - 1}, \text{ and } f_y = \frac{\frac{1}{4}y^{-3/4}}{x^{1/3} + y^{1/4} - 1}$$

Thus, $f(1, 1) = 0$, $f_x(1, 1) = 1/3$, and $f_y(1, 1) = 1/4$.

Linear Taylor's series approximation is $f(x_0 + h, y_0 + k) \approx f(x_0, y_0) + [f_x(x_0, y_0) + f_y(x_0, y_0)]$.

Substituting the values, we get

$$\ln[(1.03)^{1/3} + (0.98)^{1/4} - 1] \approx 0 + (.03)(1/3) - (.02)(1/4) = .01 - .005 = .005 \text{ (approx.)}$$

EXERCISE 5.7

1. Express $x^2 + 3y^2 - 9x - 9y + 26$ in powers of $(x - 2)$ and $(y - 2)$ using the Taylor's series expansion.
2. Obtain the quadratic Taylor's series polynomial approximation to the function $f(x, y) = 2x^3 + 3y^3 - 4x^2y$ about the point (1, 2) and obtain the maximum absolute error in the region $|x - 1| < 0.01$, $|y - 2| < 0.1$.
3. Using Taylor's series find a quadratic approximation of $\cos x \cos y$ at the origin and also estimate the error in the approximation, if $|x| \le 0.1$ and $|y| \le 0.1$.
4. Find the quadratic Taylor's series approximation of $e^x \cos y$ about the point $(1, \pi/4)$.
5. Find the cubic Maclaurin's approximation of $e^{ax} \sin by$.
6. Expand $\dfrac{(x+h)(y+k)}{(x+h)+(y+k)}$ in powers of h and k up to the second degree terms.
7. If $x^2 - xy + y^2$ is to be approximated by a linear Taylor's series polynomial about the point (2, 3), then find a square with centre at (2, 3) such that the error of approximation is less than or equal to 0.1 in magnitude for all points within this square.

8. Find the cubic Taylor's series polynomial approximation of $f(x, y) = \tan^{-1}(y/x)$ and hence compute $f(1.1, 0.9)$ approximately.

9. Expand $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ using Taylor's series up to first degree terms about the point (2, 2, 1). Obtain the maximum error in the region $|x - 2| < 0.1$, $|y - 2| < 0.1$, $|z - 1| < 0.1$.

10. Expand $f(x, y, z) = e^z \sin(x + y)$ in Taylor's series up to second order terms about the point (0, 0, 0). Obtain the maximum error in the region $|x| \le 0.1$, $|y| \le 0.1$, $|z| \le 0.1$.

5.8 EXTREME VALUES OF FUNCTIONS OF TWO VARIABLES

Similar to the case of function of a single variable as discussed in Section 4.8, here we discuss the extreme values of a function $z = f(x, y)$ of two independent variables x and y.

5.8.1 Local Maxima and Local Minima. Saddle Point

A function $z = f(x, y)$ is said to have a *local maxima* value at a point $P(a, b)$, if for all positive or negative small values of h and k

$$f(a + h, b + k) - f(a, b) < 0.$$

Similarly, the function $z = f(x, y)$ is said to have a *local minima* at a point $P(a, b)$, if for all positive or negative small values of h and k

$$f(a + h, b + k) - f(a, b) > 0.$$

Thus, if $\Delta f = f(a + h, b + k) - f(a, b)$ is of the same sign for all positive or negative small values of h and k, then point (a, b) is the point of local maxima, if $\Delta f < 0$, or of local minima, if $\Delta f > 0$, and $f(a, b)$ is the corresponding extreme, maximum or minimum value. In Fig. 5.3, $f(a, b)$ is the maximum value of $f(x, y)$ in the neighbourhood (*nbd.*) of (a, b), and P is the point $P(a, b, f(a, b))$.

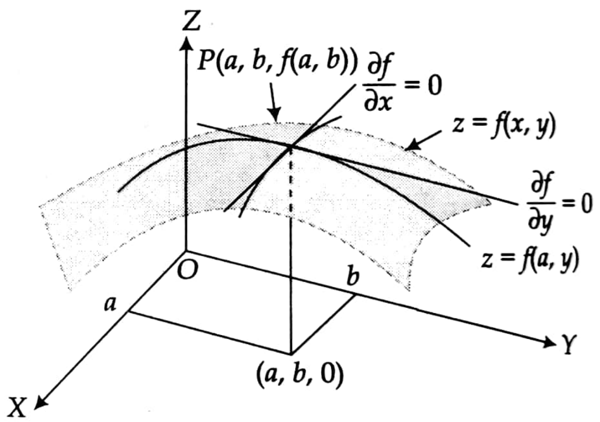

Fig. 5.3

In case the sign of Δf does not remain constant in the nbd. of (a, b), there will be neither a maxima nor a minima at (a, b), and then the point is said to be a *saddle point*.

5.8.2 Necessary Conditions For $f(x, y)$ to be Maximum or Minimum

We have seen that for $f(x, y)$ to be a maximum or minimum value at an arbitrary point (x, y), $\Delta f = f(x + h, y + k) - f(x, y)$ must keep the same sign for arbitrary small values of h and k. Using Taylor's expansion,

$$\Delta f = f(x + h, y + k) - f(x, y) = (hf_x + kf_y) + \frac{1}{2!}[h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}] + \ldots \quad \ldots(5.101)$$

For small values of h and k, the second and higher order terms in h and k are still smaller and hence may be neglected, and thus sign of Δf depends on the sign of $(hf_x + kf_y)$, which changes with

h and k. Hence the necessary condition for $f(x, y)$ to be a maximum or minimum at an arbitrary point (x, y) is $h f_x + k f_y = 0$, for arbitrary small values of h and k which is feasible, if $f_x = 0, f_y = 0$.

Hence, the necessary conditions for $f(x, y)$ to have a maximum or a minimum at a point (x, y) are

$$f_x = 0, \quad f_y = 0. \qquad \text{...(5.102)}$$

A point satisfying the conditions in (5.102), is called a *stationary point or a critical point.*

The conditions in (5.102) are only the necessary and not sufficient one for a point (x, y) to be a point of maxima or minima. For example, consider the function $f(x, y) = y^2 - x^2$.

Here, $\quad f_x = -2x = 0, \Rightarrow \; x = 0$, and $\; f_y = 2y = 0 \;\Rightarrow\; y = 0$.

Therefore local maximum or minimum can occur only at the origin (0, 0). However, along the positive x–axis f has the value $f(x, 0) = -x^2 < 0$ and along the positive y–axis f has the value $f(0, y) = y^2 > 0$. Therefore, every neighbourhood centred at (0, 0) in the xy–plane contains points where the function is positive and points where it is negative, Thus (0, 0) is neither a point of maxima nor of minima for $f(x, y) = y^2 - x^2$.

5.8.3 Sufficient Conditions for $f(x, y)$ to be Maximum or Minimum

Suppose that a function $f(x, y)$ is continuous and possess first and second derivatives at a critical point (x, y), then $f_x = 0, f_y = 0$ at (x, y), and thus

$$\Delta f = f(x + h, y + k) - f(x, y) \approx \frac{1}{2} [h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}] + \ldots$$

$$\approx \frac{1}{2f_{xx}} (h^2 f_{xx}^2 + 2hk f_{xy} f_{xx} + k^2 f_{xx} f_{yy})$$

$$\approx \frac{1}{2f_{xx}} [(hf_{xx} + k f_{xy})^2 + k^2(f_{xx}f_{yy} - f_{xy}^2)]. \qquad \text{...(5.103)}$$

Since $(hf_{xx} + kf_{xy})^2 > 0$, the sufficient condition for expression $[(hf_{xx} + kf_{xy})^2 + k^2(f_{xx}f_{yy} - f_{xy}^2)]$ to be positive is that $f_{xx} f_{yy} - f_{xy}^2 > 0$. Hence, if $f_{xx} f_{yy} - f_{xy}^2 > 0$, then $\Delta f < 0$, if $f_{xx} < 0$, and $\Delta f > 0$, if $f_{xx} > 0$.

Therefore, a sufficient condition for the critical point $P(x, y)$ to be a point of local maxima is

$$f_{xx}f_{yy} - f_{xy}^2 > 0 \text{ and } f_{xx} < 0 \qquad \text{...(5.104)}$$

and, to be a point of local minima is

$$f_{xx}f_{yy} - f_{xy}^2 > 0 \text{ and } f_{xx} > 0. \qquad \text{...(5.105)}$$

If $f_{xx}f_{yy} - f_{xy}^2 < 0$, then there will be neither a maxima nor a minima and point is called *saddle point*; and, if $f_{xx}f_{yy} - f_{xy}^2 = 0$, then the test is inconclusive.

The expression $f_{xx}f_{yy} - f_{xy}^2$ is called the *discriminant* of f. It is easy to remember the discriminant in the determinant form as

$$f_{xx} f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}.$$

Example 5.39: Find the maximum and minimum values of the function $x^3 + y^3 - 3axy$, $a > 0$.

Solution: We have, $f(x, y) = x^3 + y^3 - 3axy$. Here $f_x = 3x^2 - 3ay$ and $\; f_y = 3y^2 - 3ax$.

For critical points $f_x = 0$ and $f_y = 0$. This gives $x^2 = ay$ and $y^2 = ax$.

Therefore, critical points are (0, 0) and (a, a). Also, $f_{xx} = 6x$, $f_{xy} = -3a$, and $f_{yy} = 6y$.

At (0, 0), $f_{xx}f_{yy} - f_{xy}^2 = 0 - 9a^2 = -9a^2 < 0$, therefore at (0, 0), $f(x, y)$ is neither maximum nor minimum.

At (a, a), $f_{xx}f_{yy} - f_{xy}^2 = (6a)(6a) - 9a^2 = 27a^2 > 0$ and also $f_{xx} = 6a > 0$ and hence (a, a) is a point of minima and the minimum value of the function $f(x, y)$ is $a^3 + a^3 - 3a^3 = -a^3$.

Example 5.40: Find the absolute maximum and minimum values of the function

$$f(x, y) = 2 + 2x + 2y - x^2 - y^2$$

on the triangular plate in the first quadrant bounded by $x = 0$, $y = 0$, $y = 9 - x$.

Solution: The function $f(x, y)$ can have maximum or minimum values at the critical points inside the triangle or on its boundary $OABO$, refer to Fig. 5.4.

We have $f(x, y) = 2 + 2x + 2y - x^2 - y^2$ which gives

$$f_x = 2 - 2x, \quad \text{and} \quad f_y = 2 - 2y.$$

Now $f_x = 0$, $f_y = 0$ gives (1, 1) as the critical point.

Further, $f_{xx} = -2$, $f_{xy} = 0$, $f_{yy} = -2$.

At (1, 1), $f_{xx}f_{yy} - f_{xy}^2 = (-2)(-2) - 0 = 4 > 0$ and also $f_{xx} = -2 < 0$, and hence (1, 1) is a point of maxima and the maximum value of $f(x, y)$ at (1, 1) is $f(1, 1) = 4$.

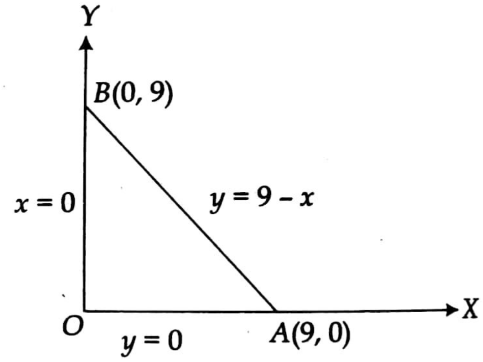

Fig. 5.4

On the boundary line OA, $y = 0$, and so the function

$$f(x, y) = f(x, 0) = g(x) = 2 + 2x - x^2,$$

which is a function of single variable on $0 \le x \le 9$.

Now, $dg/dx = 0$ gives $2 - 2x = 0$, or $x = 1$; also $\dfrac{d^2g}{dx^2} = -2 < 0$.

Therefore at $x = 1$, the function has a maxima and the maximum value is $g(1) = 3$.

Also at the end points $O(0, 0)$ and $A(9, 0)$ we have

$$f(0, 0) = g(0) = 2, \quad f(9, 0) = g(9) = 2 + 18 - 81 = -61$$

Next, on the boundary line OB, $x = 0$, and so the function

$$f(x, y) = f(0, y) = h(y) = 2 + 2y - y^2, \quad 0 \le y \le 9$$

As obtained above we have $f(0, 1) = h(1) = 3$, $f(0, 0) = h(0) = 2$, and $f(0, 9) = h(9) = -61$.

Next along the line AB, $y = 9 - x$ and thus

$$f(x, y) = f(x, 9 - x) = \phi(x) = 2 + 2x + 2(9 - x) - x^2 - (9 - x)^2$$
$$= -61 + 18x - 2x^2, \quad 0 < x < 9.$$

Here, $\dfrac{d\phi}{dx} = 0$ gives $18 - 4x = 0$, or $x = 9/2$, also, $\dfrac{d^2\phi}{dx^2} = -4 < 0$.

Therefore at $x = 9/2$, the function has a maxima and the maximum value is

$$f\left(\frac{9}{2},\frac{9}{2}\right) = 2 + 2\left(\frac{9}{2}\right) + 2\left(\frac{9}{2}\right) - \frac{81}{4} - \frac{81}{4} = -\frac{41}{2}.$$

We have already evaluated $f(x, y)$ at the boundary points $A(9, 0)$ and $B(0, 9)$.

Therefore, the absolute maximum value is 4, which $f(x, y)$ assumes at $(1, 1)$ and the absolute minimum value is -61 which $f(x, y)$ assumes at $(0, 9)$ and $(9, 0)$.

Example 5.41: Find the points of maxima and minima of $x^3y^2(1 - x - y)$.

Solution: We have $f(x, y) = x^3y^2(1 - x - y)$, hence

$$f_x = 3x^2y^2 - 4x^3y^2 - 3x^2y^3, \text{ and } f_y = 2x^3y - 2x^4y - 3x^3y^2.$$

For critical points $f_x = 0$, and $f_y = 0$, which give respectively

$$x^2y^2(3 - 4x - 3y) = 0, \text{ and } x^3y(2 - 2x - 3y) = 0.$$

Solving for x and y, the critical points are $\left(\frac{1}{2},\frac{1}{3}\right)$ and $(0, 0)$.

Also, $f_{xx} = 6xy^2 - 12x^2y^2 - 6xy^3$, $f_{xy} = 6x^2y - 8x^3y - 9x^2y^2$, $f_{yy} = 2x^3 - 2x^4 - 6x^3y$.

At $\left(\frac{1}{2},\frac{1}{3}\right)$, $f_{xx}f_{yy} - f_{xy}^2 = \left(\frac{1}{3} - \frac{1}{3} - \frac{1}{9}\right)\left(\frac{1}{4} - \frac{1}{8} - \frac{1}{4}\right) - \left(\frac{1}{2} - \frac{1}{3} - \frac{1}{4}\right)^2 = \frac{1}{144} > 0.$

Also, at $\left(\frac{1}{2},\frac{1}{3}\right)$, $f_{xx} = \left(\frac{1}{3} - \frac{1}{3} - \frac{1}{9}\right) = -\frac{1}{9} < 0$, therefore $\left(\frac{1}{2},\frac{1}{3}\right)$ is a point of maxima.

Next, at $(0, 0)$, $f_{xx}f_{yy} - f_{xy}^2 = 0$ and thus further investigation is needed.

For points along the line $y = x$, $f(x, y) = x^5(1 - 2x)$ which is positive for x slightly greater than zero (say, $x = .01$) and negative for x slightly less than zero (say, $x = -.01$) and thus $f(x, y)$ changes sign around the point $(0, 0)$ and hence $(0, 0)$ is neither a point of maxima nor of minima.

Example 5.42: Find the points on the surface $z^2 = xy + 1$ nearest to the origin.

Solution: Let (x, y, z) be an arbitrary point on the surface

$$z^2 = xy + 1 \qquad \text{...(5. 106)}$$

and let d be the distance of this point from the origin, then

$$d = \sqrt{x^2 + y^2 + z^2}$$

or,

$$d^2 = x^2 + y^2 + z^2 = x^2 + y^2 + xy + 1, \quad \text{using (5.106).}$$

Consider $f(x, y) = x^2 + y^2 + xy + 1$, then $f_x = 2x + y$, $f_y = 2y + x$.

For critical points $f_x = 0$, and $f_y = 0$, which give respectively $2x + y = 0$, and $2y + x = 0$.

Therefore critical point is $(0, 0)$.

Further $f_{xx} = 2$, $f_{xy} = 1$, $f_{yy} = 2$, and thus $f_{xx}f_{yy} - f_{xy}^2 = 4 - 1 = 3 > 0$, and also $f_{xx} = 2 > 0$, therefore, $f(x, y)$ has minima at $(0, 0)$. Substituting, $x = 0$, $y = 0$ in (5.106), we get $z = \pm 1$.

Hence the points on $z^2 = xy + 1$ nearest to the origin are $(0, 0, \pm 1)$.

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Example 5.43: Find the dimensions of a rectangular parallelopiped of maximum volume with edges parallel to co-ordinate axes which can be inscribed in the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

Solution: If $P(x, y, z)$ be the co-ordinates of one of the vertex of the required rectangular parallelopiped in the positive octant inscribed in the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \qquad \text{...(5.107)}$$

then edges being parallel to the axes are of the lengths $2x$, $2y$ and $2z$ respectively.

Thus the volume of the rectangular parallelopiped is

$$V = (2x)(2y)(2z) = 8xyz. \qquad \text{...(5.108)}$$

We need to find the maxima of (5.108) subject to (5.107), we have

$$V^2 = 64x^2y^2z^2 = 64x^2y^2c^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right), \text{ using (5.107)}$$

$$= 64c^2x^2y^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) = f(x, y), \text{ say.}$$

Consider $\qquad f(x, y) = 64c^2\left(x^2y^2 - \frac{x^4y^2}{a^2} - \frac{x^2y^4}{b^2}\right)$, then

$$f_x = 64c^2\left(2xy^2 - \frac{4x^3y^2}{a^2} - \frac{2xy^4}{b^2}\right), f_y = 64c^2\left(2x^2y - \frac{2x^4y}{a^2} - \frac{4x^2y^3}{b^2}\right).$$

For critical points $f_x = 0$ and $f_y = 0$, which give respectively

$$xy^2\left(1 - \frac{2x^2}{a^2} - \frac{y^2}{b^2}\right) = 0 \qquad \text{...(5.109)}$$

and,

$$x^2y\left(1 - \frac{x^2}{a^2} - \frac{2y^2}{b^2}\right) = 0. \qquad \text{...(5.110)}$$

Solving (5.109) and (5.110) and considering points with only positive values of abscissa and ordinate. The critical point obtained is $x = \frac{a}{\sqrt{3}}, \quad y = \frac{b}{\sqrt{3}}$. Also,

$$f_{xx} = 64c^2\left(2y^2 - \frac{12x^2y^2}{a^2} - \frac{2y^4}{b^2}\right), f_{xy} = 64c^2\left(4xy - \frac{8x^3y}{a^2} - \frac{8xy^3}{b^2}\right), f_{yy} = 64c^2\left(2x^2 - \frac{2x^4}{a^2} - \frac{12x^2y^2}{b^2}\right).$$

At $\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}\right)$,

$$f_{xx}f_{yy} - f_{xy}^2 = \left[64c^2\left(\frac{2b^2}{3} - \frac{12a^2b^2}{9a^2} - \frac{2b^4}{9b^2}\right)\right]\left[64c^2\left(\frac{2a^2}{3} - \frac{2a^4}{9a^2} - \frac{12a^2b^2}{9b^2}\right)\right]$$

$$- \left[64c^2\left(\frac{4ab}{3} - \frac{8a^3b}{9a^2} - \frac{8ab^3}{9b^2}\right)\right]^2$$

$$= (64)^2\left(\frac{4}{3}\right)^2 c^4a^2b^2\left[\left(\frac{2}{3}\right)^2 - \left(\frac{1}{3}\right)^2\right] > 0,$$

and, $$f_{xx} = 64c^2\left(\frac{2}{3}b^2 - \frac{4}{3}b^2 - \frac{2}{9}b^2\right) = -\frac{512}{9}b^2c^2 < 0.$$

Hence $f(x, y)$ is maximum at $x = \frac{a}{\sqrt{3}}, y = \frac{b}{\sqrt{3}}$.

Also at $x = \frac{a}{\sqrt{3}}$ and $y = \frac{b}{\sqrt{3}}, z = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} = \frac{c}{\sqrt{3}}$.

Thus dimensions of the desired rectangular parallelopiped are $\frac{2a}{\sqrt{3}}, \frac{2b}{\sqrt{3}}$ and $\frac{2c}{\sqrt{3}}$, and the corresponding volume is $V = 8xyz = \frac{8abc}{3\sqrt{3}}$.

5.9 CONSTRAINED EXTREME VALUES: LAGRANGE'S METHOD

Sometimes we need to find extreme values of a function which may be subject to some constraints. In such cases all variables in the function are not independent but are connected by some given relations. In general, we try to convert the given function to the one having the least number of independent variables using the given relation and then find the extreme values.

However, Lagrange's multiplier method is a powerful tool of finding the extreme values in case of constrained functions. The method was developed by Lagrange in 1755 to solve max-min problems in geometry.

5.9.1 Lagrange's Method

Suppose we need to find the extremum of the function $f(x, y, z)$ under the condition

$$\phi(x, y, z) = 0. \qquad ...(5.111)$$

We construct an auxiliary function of the form

$$F(x, y, z, \lambda) = f(x, y, z) + \lambda\phi(x, y, z), \quad ...(5.112)$$

where λ is an undetermined parameter called the *Lagrange multiplier*.

To determine the stationary points of F, the necessary conditions obtained from (5.112) are

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0 \quad \text{and} \quad \frac{\partial F}{\partial z} = 0,$$

which give respectively the equations

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0, \quad \frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0, \text{ and } \quad \frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0. \quad ...(5.113)$$

The Eqs. (5.111) and (5.113) give the values of x, y, z and λ for a maximum or minimum. The fact that whether the point is of maxima or minima is further established from the physical considerations of the problem.

In case the problem is subject to more than one constraint, say two, then two undetermined parameters are introduced and their values are determined.

Example 5.44: Find the maximum and minimum distances of the point $A(3, 4, 12)$ from the sphere $x^2 + y^2 + z^2 = 1$.

Solution: Let $P(x, y, z)$ be any point on the sphere $x^2 + y^2 + z^2 = 1$.

The distance of the given point $A(3, 4, 12)$ from (x, y, z) is

$$AP = \sqrt{(x-3)^2 + (y-4)^2 + (z-12)^2}.$$

We find the extreme values of the square of this distance, that is, of

$$(AP)^2 = (x-3)^2 + (y-4)^2 + (z-12)^2 = f(x, y, z), \quad \text{(say)} \quad ...(5.114)$$

subject to the condition

$$\phi(x, y, z) = x^2 + y^2 + z^2 - 1 = 0. \quad ...(5.115)$$

Consider the auxiliary function

$$F(x, y, z, \lambda) = (x-3)^2 + (y-4)^2 + (z-12)^2 + \lambda(x^2 + y^2 + z^2 - 1),$$

where λ is Lagrange's constant.

For extreme values

$$\frac{\partial F}{\partial x} = 2(x-3) + 2\lambda x = 0, \quad ...(5.116)$$

$$\frac{\partial F}{\partial y} = 2(y-4) + 2\lambda y = 0, \quad ...(5.117)$$

and,
$$\frac{\partial F}{\partial z} = 2(z-12) + 2\lambda z = 0. \quad ...(5.118)$$

Multiplying (5.116) by x, (5.117) by y, (5.118) by z and adding, we get

$$2(x^2 + y^2 + z^2) - 2(3x + 4y + 12z) + 2\lambda(x^2 + y^2 + z^2) = 0. \qquad ...(5.119)$$

Using (5.115), and simplifying (5.119) becomes

$$3x + 4y + 12z = 1 + \lambda. \qquad ...(5.120)$$

Next, from (5.116), (5.117) and (5.118), we have respectively

$$x = \frac{3}{1+\lambda}, \quad y = \frac{4}{1+\lambda}, \text{ and } \quad z = \frac{12}{1+\lambda}. \qquad ...(5.121)$$

Substituting these values of x, y, z in (5.120), we have

$$\frac{9}{1+\lambda} + \frac{16}{1+\lambda} + \frac{144}{1+\lambda} = 1 + \lambda,$$

which gives $(1 + \lambda)^2 = 169$, and hence $\lambda = 12$, or -14.

From (5.121) for $\lambda = 12$, the point is $P\left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right)$, and for $\lambda = -14$, the point is $Q\left(-\frac{3}{13}, -\frac{4}{13}, -\frac{12}{13}\right)$. Thus, $AP = \sqrt{\left(3 - \frac{3}{13}\right)^2 + \left(4 - \frac{4}{13}\right)^2 + \left(12 - \frac{12}{13}\right)^2} = 12,$

and,
$$AQ = \sqrt{\left(3 + \frac{3}{13}\right)^2 + \left(4 + \frac{4}{13}\right)^2 + \left(12 + \frac{12}{13}\right)^2} = 14.$$

Therefore, the minimum and maximum distances of the point $A(3, 4, 12)$ from the sphere $x^2 + y^2 + z^2 = 1$ are 12 and 14, respectively.

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Example 5.45: Find the shortest distance between the line $y = 10 - 2x$ and the ellipse $x^2/4 + y^2/9 = 1$.

Solution: Let (x, y) be a point on the ellipse $(x^2/4) + (y^2/9) - 1 = 0$ and, (u, v) be a point on the line $2x + y - 10 = 0$. The shortest distance between the line and the ellipse is the square root of the minimum value of

$$f(x, y, u, v) = (x - u)^2 + (y - v)^2 \qquad ...(5.122)$$

subject to the conditions

$$\phi_1(x, y) = \frac{x^2}{4} + \frac{y^2}{9} - 1 = 0, \qquad ...(5.123)$$

and,
$$\phi_2(u, v) = 2u + v - 10 = 0. \qquad ...(5.124)$$

Consider the auxiliary function

$$F(x, y, u, v, \lambda_1, \lambda_2) = (x - u)^2 + (y - v)^2 + \lambda_1\left(\frac{x^2}{4} + \frac{y^2}{9} - 1\right) + \lambda_2(2u + v - 10) \qquad ...(5.125)$$

where λ_1 and λ_2 are the Lagrange constants.

For extreme values, from (5.125)

$$\frac{\partial F}{\partial x} = 2(x-u) + \frac{2\lambda_1 x}{4} = 0, \text{ or } \lambda_1 x = 4(u-x), \quad \text{...(5.126)}$$

$$\frac{\partial F}{\partial y} = 2(y-u) + \frac{2\lambda_1 y}{9} = 0, \text{ or } \lambda_1 y = 9(v-y), \quad \text{...(5.127)}$$

$$\frac{\partial F}{\partial u} = -2(x-u) + 2\lambda_2 = 0, \text{ or } \lambda_2 = x-u, \quad \text{...(5.128)}$$

and,
$$\frac{\partial F}{\partial v} = -2(y-v) + \lambda_2 = 0, \text{ or } \lambda_2 = 2(y-v). \quad \text{...(5.129)}$$

From (5.126) and (5.127)

$$4(u-x)y = 9(v-y)x, \quad \text{...(5.130)}$$

and from (5.128) and (5.129)

$$(x-u) = 2(y-v). \quad \text{...(5.131)}$$

Next from (5.130) and (5.131), we obtain $8y = 9x$, or $y = \frac{9}{8}x$. Substituting in (5.123), we get $\frac{x^2}{4} + \frac{9x^2}{64} = 1$, which gives $x^2 = \frac{64}{25}$, or $x = \pm\frac{8}{5}$ and thus $y = \pm 9/5$.

Substituting $x = 8/5$ and $y = 9/5$ in (5.131) we obtain $u = 2v - 2$. Substituting this in (5.124), we get $u = 18/5$ and $v = 14/5$.

Hence an extremum corresponds to $(x, y) = (8/5, 9/5)$ and $(u, v) = (18/5, 14/5)$.

The distance between these two points is

$$\sqrt{\left(\frac{8}{5}-\frac{18}{5}\right)^2 + \left(\frac{9}{5}-\frac{14}{5}\right)^2} = \sqrt{4+1} = \sqrt{5}.$$

Similarly corresponding to $(x, y) = (-8/5, -9/5)$ we obtain $(u, v) = (22/5, 6/5)$ and the corresponding distance between these two points is $3\sqrt{5}$.

Thus the shortest distance between the line and the ellipse is $\sqrt{5}$.

Example 5.46: The plane $x + y + z = 1$ cuts the cylinder $x^2 + y^2 = 1$ in an ellipse. Find the point on the ellipse that lie closest to and farthest from the origin.

Solution: Let (x, y, z) be a point on the desired ellipse in which the plane $x + y + z = 1$ cuts the cylinder $x^2 + y^2 = 1$. We find the extreme values of

$$f(x, y, z) = x^2 + y^2 + z^2, \quad \text{...(5.132)}$$

the square of the distance of (x, y, z) from the origin subject to the constraints

$$\phi_1(x, y, z) = x^2 + y^2 - 1 = 0, \quad \text{...(5.133)}$$

and, $$\phi_2(x, y, z) = x + y + z - 1 = 0. \qquad ...(5.134)$$

Consider the auxiliary function

$$F(x, y, z, \lambda_1, \lambda_2) = x^2 + y^2 + z^2 + \lambda_1(x^2 + y^2 - 1) + \lambda_2(x + y + z - 1) \qquad ...(5.135)$$

where λ_1, λ_2, are Lagrange constants.

For extreme values, from (5.135), we have

$$\frac{\partial F}{\partial x} = 2x + 2\lambda_1 x + \lambda_2 = 0, \qquad ...(5.136)$$

$$\frac{\partial F}{\partial y} = 2y + 2\lambda_1 y + \lambda_2 = 0 \qquad ...(5.137)$$

$$\frac{\partial F}{\partial z} = 2z + \lambda_2 = 0 \qquad ...(5.138)$$

From (5.136) and (5.138), we obtain

$$(1 + \lambda_1)x = z. \qquad ...(5.139)$$

From (5.137) and (5.138), we obtain

$$(1 + \lambda_1)y = z \qquad ...(5.140)$$

The equations (5.139) and (5.140) are satisfied simultaneously if either $\lambda_1 = -1$ and $z = 0$, or, $\lambda_1 \neq -1$ and $x = y = \dfrac{z}{1+\lambda_1}$.

For $z = 0$ solving (5.133) and (5.134) simultaneously, the points are $P_1(1, 0, 0)$ and $P_2(0, 1, 0)$. Next, for $x = y$, from (5.133) and (5.134), we get $2x^2 = 1$ and $z = 1 - 2x$. Solving these, we get $x = \pm\dfrac{\sqrt{2}}{2}$, $z = 1 \mp \sqrt{2}$. The corresponding points on the ellipse are

$$Q_1\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1 - \sqrt{2}\right) \text{ and } Q_2\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 1 + \sqrt{2}\right).$$

The distance of $P_1(1, 0, 0)$ from origin is 1 and of $P_2(0, 1, 0)$ from the origin is 1.

Also the distance of Q_1, from the origin is $\sqrt{4 - 2\sqrt{2}}$, and of Q_2 from the origin is $\sqrt{4 + 2\sqrt{2}}$.

Hence $P_1(1, 0, 0)$ and $P_2(0, 1, 0)$ are the closest points to the origin and $Q_2\left(-\dfrac{\sqrt{2}}{2}, -\dfrac{\sqrt{2}}{2}, 1 + \sqrt{2}\right)$ is the farthest point to the origin lying on the desired ellipse.

EXERCISE 5.8

1. Find the maxima and minima of the functions

   (a) $x^3 + y^3 - 12x - 3y + 16$ (b) $x^3y + xy^3 + 2a^2xy - 3axy^2 - ax^3, \quad a > 0$

(c) $xy - x^2 - y^2 - 2x - 2y + 4$ (d) $x^4 + y^4 - 2x^2 + 4xy - 2y^2$

2. In a triangle ABC find the maximum value of $\cos A \cos B \cos C$.
3. Divide 36 into three parts such that the continued product of the first with the square of the second and the cube of the third is maximum.
4. If $xyz = 8$, find the values of x, y, z for which $5xyz/(x + 2y + 4z)$ is maximum.
5. Given the perimeter 20, determine the triangle of maximum area.
6. Given $x + y + z = a$, find the maximum value of $x^m y^n z^p$.
7. Find the point on the surface $y = x^2 + z^2$ nearest to the point $(3, 4, -6)$.
8. Find the maximum value of xyz subject to the condition $9x^2 + 36y^2 + 4z^2 = 36$. What is the geometrical interpretation of this problem?
9. Prove that of all rectangular parallelopiped of the same volume, the cube has the least surface.
10. Find the dimensions of the closed rectangular box with maximum volume that can be inscribed in the unit sphere.
11. A space probe in the shape of the ellipsoid $4x^2 + y^2 + 4z^2 = 16$ enters the earth's atmosphere and its surface begins to heat. After one hour, the temperature at the point (x, y, z) on the probe's surface is $T(x, y, z) = 8x^2 + 4yz - 16z + 600$. Find the hottest point on the probe's surface.
12. Find the point closest to the origin on the line of intersection of the planes $y + 2z = 12$ and $x + y = 6$.
13. Find the extreme values of the function $f(x, y, z) = xy + z^2$ on the circle in which the plane $y - x = 0$ intersects the sphere $x^2 + y^2 + z^2 = 4$.
14. Find the smallest and the largest distance between the points P and Q such that P lies on the plane $x + y + z = 2a$ and Q lies on the sphere $x^2 + y^2 + z^2 = a^2$, where a is any constant.
15. Using Lagrange's multipliers, show that the maximum and minimum values of $u = \dfrac{x^2}{a^4} + \dfrac{y^2}{b^4} + \dfrac{z^2}{c^4}$, where $lx + my + nz = 0$ and $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ are given by

$$\frac{l^2a^4}{a^2u - 1} + \frac{m^2b^4}{b^2u - 1} + \frac{n^2c^4}{c^2u - 1} = 0.$$

5.10 DIFFERENTIATION UNDER THE INTEGRAL SIGN: THE LEIBNITZ'S RULE

In applications sometimes we come across integrals of the form

$$\phi(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f(x, \alpha)dx \qquad \text{...(5.141)}$$

where α is a parameter and the integrand $f(x, \alpha)$ is such that it is not easily integrable. Leibnitz's rule gives us a procedure to evaluate these integrals by differentiating $\phi(\alpha)$ with respect to the parameter α to obtain $\phi'(\alpha)$ and then obtain the desired integral by integrating $\phi'(\alpha)$ with respect to α. The rule is stated as follows.

Theorem 5.4: ***(Leibnitz's rule)*** *If $\phi(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f(x, \alpha)\, dx$, and $a(\alpha)$, $b(\alpha)$, $f(x, \alpha)$ are differentiable functions of α and $\frac{\partial f}{\partial \alpha}$ is continuous, then*

$$\frac{d\phi}{d\alpha} = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial f(x,\alpha)}{\partial \alpha} dx + f(b, \alpha)\frac{db}{d\alpha} - f(a, \alpha)\frac{da}{d\alpha}. \quad ...(5.142)$$

Proof. Let $\Delta\alpha$ be an increment in α and Δa, Δb, be the corresponding increments in $a(\alpha)$ and $b(\alpha)$. If $\Delta\phi$ is the resultant increment in $\phi(\alpha)$, then

$$\Delta\phi = \phi(\alpha + \Delta\alpha) - \phi(\alpha) = \int_{a+\Delta a}^{b+\Delta b} f(x, \alpha + \Delta\alpha)dx - \int_a^b f(x, \alpha)dx$$

$$= \int_{a+\Delta a}^{a} f(x, \alpha + \Delta\alpha)dx + \int_a^b f(x, \alpha + \Delta\alpha)dx + \int_b^{b+\Delta b} f(x, \alpha + \Delta\alpha)dx - \int_a^b f(x, \alpha)dx,$$

or, $$\Delta\phi = \int_{a+\Delta a}^{a} f(x, \alpha + \Delta\alpha)dx + \int_a^b [f(x, \alpha + \Delta\alpha) - f(x, \alpha)]dx + \int_b^{b+\Delta b} f(x, \alpha + \Delta\alpha)dx.$$

Dividing by $\Delta\alpha$, we obtain

$$\frac{\Delta\phi}{\Delta\alpha} = \int_{a+\Delta a}^{a} \frac{f(x, \alpha + \Delta\alpha)}{\Delta\alpha} dx + \int_a^b \frac{f(x, \alpha + \Delta\alpha) - f(x, \alpha)}{\Delta\alpha} dx + \int_b^{b+\Delta b} \frac{f(x, \alpha + \Delta\alpha)}{\Delta\alpha} dx. \quad ...(5.143)$$

Using the mean value theorem of integrals, refer to (6.8),

$$\int_{a+\Delta a}^{a} f(x, \alpha + \Delta\alpha)dx = -\Delta a f(\xi, \alpha + \Delta\alpha), \text{ for some } \xi,\ \alpha < \xi < a + \Delta a, \quad ...(5.144)$$

and $$\int_b^{b+\Delta b} f(x, \alpha + \Delta\alpha)dx = \Delta b f(\eta, \alpha + \Delta\alpha), \text{ for some } \eta,\ b < \eta < b + \Delta b. \quad ...(5.145)$$

Next using the Lagrange's mean value theorem,

$$f(x, \alpha + \Delta\alpha) - f(x, \alpha) = \Delta\alpha \frac{\partial f(x, \theta)}{\partial \alpha}, \text{ for some } \theta,\ \alpha < \theta < \alpha + \Delta\alpha. \quad ...(5.146)$$

Using (5.144), (5.145) and (5. 146) in (5.143), we obtain

$$\frac{\Delta\phi}{\Delta\alpha} = -f(\xi, \alpha + \Delta\alpha)\frac{\Delta a}{\Delta\alpha} + \int_a^b \frac{\partial f(x, \theta)}{\partial \alpha} dx + f(\eta, \alpha + \Delta\alpha)\frac{\Delta b}{\Delta\alpha}. \quad ...(5.147)$$

Taking limit as $\Delta\alpha \to 0$, and further we note that as $\Delta\alpha \to 0$, $\xi \to a$, $\eta \to b$ and $\theta \to \alpha$. Thus, (5.147) becomes

$$\frac{d\phi}{d\alpha} = \int_a^b \frac{\partial f(x,\alpha)}{\partial \alpha} dx + f(b,\alpha)\frac{db}{d\alpha} - f(a,\alpha)\frac{da}{d\alpha}.$$

In case $a(\alpha)$ and $b(\alpha)$ are independent of α, then Leibnitz's rule (5.142) becomes

$$\frac{d\phi}{d\alpha} = \int_a^b \frac{\partial f(x,\alpha)}{\partial \alpha} dx. \quad \text{...(5.148)}$$

In case the integrand f is independent of α, then (5.142) gives

$$\frac{d\phi}{d\alpha} = f(b)\frac{db}{d\alpha} - f(a)\frac{da}{d\alpha}. \quad \text{...(5.149)}$$

Example 5.47: Evaluate $\int_0^1 \frac{x^\alpha - 1}{\ln x} dx$, $\alpha \geq 0$ by applying differentiation under the integral sign.

Solution: Let $\phi(\alpha) = \int_0^1 \frac{x^\alpha - 1}{\ln x} dx$. ...(5.150)

Applying Leibnitz's rule, we obtain

$$\frac{d\phi}{d\alpha} = \int_0^1 \frac{\partial}{\partial \alpha}\left(\frac{x^\alpha - 1}{\ln x}\right) dx = \int_0^1 \frac{x^\alpha \ln x}{\ln x} dx = \int_0^1 x^\alpha dx = \frac{1}{\alpha+1}.$$

Integrating with respect to α, we get

$$\phi(\alpha) = \ln(\alpha+1) + c. \quad \text{...(5.151)}$$

From (5.150), $\phi(0) = 0$. Using in (5.151), we get $c = 0$. Thus, $\int_0^1 \frac{x^\alpha - 1}{\ln x} dx = \ln(\alpha + 1)$.

Example 5.48: Evaluate using the differentiation under the integral sign

$$\phi(\alpha) = \int_0^{\alpha^2} \tan^{-1}\left(\frac{x}{\alpha}\right) dx. \quad \text{...(5.152)}$$

Solution: Applying Leibnitz's rule, we obtain

$$\frac{d\phi}{d\alpha} = \int_0^{\alpha^2} \frac{\partial}{\partial x}\left(\tan^{-1}\left(\frac{x}{\alpha}\right)\right) dx + \tan^{-1}\left(\frac{\alpha^2}{\alpha}\right)\frac{d}{d\alpha}(\alpha^2) - \tan^{-1}\left(\frac{0}{\alpha}\right)\frac{d}{d\alpha}(0)$$

$$= \int_0^{\alpha^2} \frac{1}{1+\dfrac{x^2}{\alpha^2}}\left(\frac{-x}{\alpha^2}\right)dx + 2\alpha\tan^{-1}\alpha = -\int_0^{\alpha^2} \frac{x}{\alpha^2+x^2}\,dx + 2\alpha\tan^{-1}\alpha$$

$$= -\frac{1}{2}[\ln(\alpha^2+x^2)]_0^{\alpha^2} + 2\alpha\tan^{-1}\alpha = 2\alpha\tan^{-1}\alpha - \frac{1}{2}\ln(1+\alpha^2).$$

Integrating with respect to α, we get

$$\phi(\alpha) = \int 2\alpha\tan^{-1}\alpha\, d\alpha - \frac{1}{2}\int \ln(1+\alpha^2)d\alpha + c$$

$$= \left[\tan^{-1}\alpha.\alpha^2 - \int \frac{\alpha^2}{1+\alpha^2}d\alpha\right] - \frac{1}{2}\left[\ln(1+\alpha^2).\alpha - \int \frac{2\alpha^2}{1+\alpha^2}d\alpha\right] + c$$

$$= \alpha^2\tan^{-1}\alpha - \int \frac{\alpha^2}{1+\alpha^2}d\alpha - \frac{1}{2}\alpha\ln(1+\alpha^2) + \int \frac{\alpha^2}{1+\alpha^2}d\alpha + c$$

$$= \alpha^2\tan^{-1}\alpha - \frac{1}{2}\alpha\ln(1+\alpha^2) + c. \qquad \text{...(5.153)}$$

From (5.152), $\phi(0) = 0$. Using this in (5.153), we get $c = 0$. Thus,

$$\int_0^{\alpha^2} \tan^{-1}\left(\frac{x}{\alpha}\right)dx = \alpha^2\tan^{-1}\alpha - \frac{1}{2}\alpha\ln(1+\alpha^2).$$

Example 5.49: By successive differentiation of $\int_0^{\infty} e^{-\alpha^2x^2}dx = \frac{\sqrt{\pi}}{2\alpha}$, $(\alpha > 0)$, w.r.t. α, prove that

$$\int_0^{\infty} x^{2n}e^{-\alpha^2x^2}dx = \sqrt{\pi}\frac{(2n)!}{n!\,(2\alpha)^{2n+1}},$$ where n is a positive integer.

Solution: We have, $$\int_0^{\infty} e^{-\alpha^2x^2}dx = \frac{\sqrt{\pi}}{2\alpha}. \qquad \text{...(5.154)}$$

Differentiating both sides of (5.154) w. r. t. α and using Leibnitz's rule, we obtain

$$\int_0^{\infty} -2\alpha x^2 e^{-\alpha^2x^2}dx = \frac{\sqrt{\pi}}{2}\left(-\frac{1}{\alpha^2}\right)$$

or,
$$\int_0^{\infty} x^2 e^{-\alpha^2x^2}dx = \frac{\sqrt{\pi}}{2}\,\frac{1}{2\alpha^3}. \qquad \text{...(5.155)}$$

Again differentiating (5.155) w. r. t. α using Leibnitz's rule, and after simplification, we get

$$\int_0^\infty x^4 e^{-\alpha^2 x^2} dx = \frac{\sqrt{\pi}}{2} \frac{1.3}{2^2 \alpha^5}.$$

In general, we can write

$$\int_0^\infty x^{2n} e^{-\alpha^2 x^2} dx = \frac{\sqrt{\pi}}{2} \frac{1.3.5 \ldots (2n-1)}{2^n . \alpha^{2n+1}} = \frac{\sqrt{\pi}\,(2n)!}{n!\,(2\alpha)^{2n+1}}.$$

This proves the result.

EXERCISE 5.9

1. By differentiating under the integral sign, evaluate the integral $\int_0^\infty \frac{e^{-\alpha x} \sin x}{x} dx, \alpha > 0$, and

hence show that $\int_0^\infty \frac{\sin x}{x} dx = \int_0^\infty \frac{\sin \alpha x}{x} dx = \frac{\pi}{2}, \alpha > 0.$

2. Evaluate the integral $\int_0^\infty \frac{e^{-x} - e^{-\alpha x}}{x \sec x} dx, \alpha > 0.$

3. Evaluate the integral $\int_0^\infty \frac{\tan^{-1}(\alpha x)}{x(1+x^2)} dx, \alpha \geq 0$ and $\alpha \neq 1$.

4. Using the result $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$, evaluate $\int_0^\infty e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx.$

5. Show that $\int_0^\infty e^{-x}\left(\frac{1-\cos \alpha x}{x}\right) dx = \frac{1}{2} \ln (1+\alpha^2).$

Using the concept of differentiation under the integral sign, show that

6. $\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx = \ln \frac{b}{a}, a, b, > 0.$

7. $\int_0^1 \frac{x^a - x^b}{\ln x} dx = \ln [(a+1)/(b+1)], a, b, > -1.$

8. $\int_0^1 x^m (\ln x)^n dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$, $m > 1$ and $n > 0$ are integers.

9. $\int_0^{\pi/2} \ln\left(\frac{a + b\sin\theta}{a - b\sin\theta}\right)\frac{d\theta}{\sin\theta} = \pi \sin^{-1}\frac{a}{b}$, $a > b$.

10. $\int_0^{\pi/2} \ln(\alpha\cos^2\theta + \beta\sin^2\theta)\, d\theta = \pi \ln\left[\frac{1}{2}\left(\sqrt{\alpha} + \sqrt{\beta}\right)\right]$, $\alpha, \beta > 0$.

11. By successive differentiation of $\int_0^\infty \frac{dx}{x^2 + a^2} = \frac{\pi}{2a}$ w. r. t. a, show that

$$\int_0^\infty \frac{dx}{(x^2 + a^2)^{n+1}} = \frac{(2n)!\,\pi}{(n!)^2 (2a)^{2n+1}}$$

12. Evaluate $\int_0^\alpha \frac{\ln(1 + \alpha x)}{1 + x^2} dx$ by differentiating under the integral sign and hence show that

$$\int_0^1 \frac{\ln(1 + x)}{1 + x^2} dx = \frac{\pi}{8} \ln 2.$$

ANSWERS

Exercise 5.1 (p. 299)

1. 1
2. 1
3. 0
4. 4
5. 5
6. does not exist
7. 0
8. does not exist
9. discontinuous
8. continuous
11. discontinuous
12. continuous
15. continuous.

Exercise 5.2 (p. 308)

1. (i) $y^2 \cos xy$; $xy \cos xy + \sin xy$ (ii) $2x + 3y$; $3x + 1$

(iii) $\frac{z}{1 - z}$, $\frac{z}{1 - z}$ (iv) $-g(x)$; $g(y)$

2. (i) $\dfrac{yz}{\sqrt{1-x^2y^2z^2}}$, $\dfrac{xz}{\sqrt{1-x^2y^2z^2}}$, $\dfrac{xy}{\sqrt{1-x^2y^2z^2}}$

(ii) $\operatorname{sech}^2(x+2y+3z)$, $2\operatorname{sech}^2(x+2y+3z)$, $3\operatorname{sech}^2(x+2y+3z)$.

Exercise 5.3 (p. 312)

1. 0.31
2. $|x-1| \le 1/70$, $|y-1| \le 1/70$
3. 0.1
4. 121.6 watts
5. 7 cm.
7. 2%
8. 2.39
9. (a) 4.02 (b) 1.81
10. (a) $\dfrac{1}{2\sqrt{2}}\left[1+\dfrac{\pi}{180}(2\sqrt{3}+1)\right]$

(b) $\dfrac{1}{720}[180+\pi(6-\sqrt{3})]$.

Exercise 5.4 (p. 317)

1. (a) 0, (b) $2t(t^2-1)^2\{(4t^2+1)\cos 4t - 2t(t^2+1)\sin 4t\}$

(c) $(\pi/2 - 2/\pi)$

5. $y/(2yz-x)$, $y/(y^2-3z^2)$; $\dfrac{1}{4}$, $-1/11$
6. 0, 0
7. $-\dfrac{y(\sin xy)+z(\sin xz)}{y\sin(yz)+x(\sin xz)}$, $\dfrac{(x\sin xy)+z(\sin yz)}{y(\sin yz)+x(\sin xz)}$
8. $-(yx^{y-1}+y^x\ln y)/(xy^{x-1}+x^y\ln x)$
15. 5, 5.

Exercise 5.5 (p. 328)

9. $v^2 = u + 2w$
10. $uv = w$
11. $v = \sin u$.

Exercise 5.7 (p. 341)

1. $6-5(x-2)+3(y-2)+(x-2)^2+3(y-2)^2$
2. $18-10(x-1)+32(y-2)-2[(x-1)^2+4(x-1)(y-2)-9(y-2)^2]$; 0.004
3. $1-\dfrac{1}{2}x^2-\dfrac{1}{2}y^2$; 0.00134
4. $\dfrac{e}{\sqrt{2}}+\left[(x-1)\dfrac{e}{\sqrt{2}}+\left(y-\dfrac{\pi}{4}\right)\left(-\dfrac{e}{\sqrt{2}}\right)\right]$

$$+\frac{1}{2!}\left[(x-1)^2\frac{e}{\sqrt{2}}+2(x-1)\left(y-\frac{\pi}{4}\right)\left(-\frac{e}{\sqrt{2}}\right)+\left(y-\frac{\pi}{4}\right)^2\left(-\frac{e}{\sqrt{2}}\right)\right]$$

5. $(by + abxy) + \frac{1}{6}(3a^2bx^2y - b^3y^3)$

6. $\frac{xy}{x+y} + \frac{hy^2 + kx^2}{(x+y)^2} - \frac{h^2y^2 - 2hxy + k^2x^2}{(x+y)^3}$

7. $|x-2| \le 0.1581, \ |y-3| \le 0.1581$

8. $\frac{\pi}{4} - \frac{1}{2}[(x-1) - (y-1)] + \frac{1}{4}[(x-1)^2 - (y-1)^2]$

$- \frac{1}{12}[(x-1)^3 + 3(x-1)^2(y-1) - 3(x-1)(y-1)^2 - (y-1)^3]$; 0.6887

9. $3 + \frac{2}{3}[(x-2) + (y-2) + (z-1)]$; 0.017

10. $x + y + xz + yz$; 0.005.

Exercise 5.8 (p. 351)

1. (a) Max at $(-2, -1)$, min at $(2, 1)$

(b) Max at $\left(\frac{a}{2}, \frac{a}{2}\right)$, min at $\left(-\frac{a}{2}, \frac{a}{2}\right)$

(c) Max at $(-2, -2)$

(d) Min at $\left(\sqrt{2}, -\sqrt{2}\right)$ and $\left(-\sqrt{2}, \sqrt{2}\right)$

2. $\frac{1}{8}$

3. 6, 12, 18

4. 4, 2, 1

5. Equilateral triangle

6. $m^m n^n p^p a^{m+n+p}/(m+n+p)^{m+n+p}$

7. $(1, 5, -2)$

8. $\frac{2}{\sqrt{3}}$

10. $\frac{8}{3\sqrt{3}}$ cubic units

11. $\left(\pm\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}\right)$

12. $(2, 4, 4)$

13. Max is 4 at $(0, 0, \pm 2)$ and min is 2 at $\left(\pm\sqrt{2}, \pm\sqrt{2}, 0\right)$

14. $P\left(\frac{2a}{3}, \frac{2a}{3}, \frac{2a}{3}\right)$, $Q\left(\pm\frac{a}{\sqrt{3}}, \pm\frac{a}{\sqrt{3}}, \pm\frac{a}{\sqrt{3}}\right)$; $\frac{a}{\sqrt{3}}\sqrt{(7 - 4\sqrt{3})}$; $\frac{a}{\sqrt{3}}\sqrt{(7 + 4\sqrt{3})}$.

Exercise 5.9 (p. 356)

2. $\frac{1}{2}\ln\left(\frac{1+\alpha^2}{2}\right)$

3. $\frac{\pi}{2}\ln(1+\alpha)$

4. $\frac{\sqrt{\pi}}{2}e^{-2\alpha}$.