8 Vector Differential Calculus

Most of the problems in science and engineering deal with the analysis of forces, velocities and various other quantities which are vectors. These are not generally static but vary with position and time and are thus functions of one or more variables. Vector differential calculus extends the concept of differential calculus over a real line to these vector functions, enabling to analyze the problems over curves and surfaces in three dimensions and finds applications in fluid-flow, heat flow, solid mechanics, electrostatic and many other branches.

8.1 DIFFERENTIATION OF A VECTOR FUNCTION

Let $\vec{V}(t)$ be a vector, continuous and single valued function of a scalar variable t. The basic concepts of calculus such as convergence, continuity and differentiability can be defined for vector functions in a simple and natural way. Among these the concept of derivative is the most important one from applications point of view.

8.1.1 Convergence. Limit and Continuity

An infinite sequence of vectors $\vec{v}_n(t)$, $n = 1, 2, \ldots$ is said to converge if there exists a vector $\vec{v}(t)$ such that, $\lim_{n\to\infty} |\vec{v}_n(t) - \vec{v}(t)| = 0$. The vector $\vec{v}(t)$ is called the limit vector of that sequence, and we write $\lim_{n\to\infty} \vec{v}_n(t) = \vec{v}(t)$.

Assuming the co-ordinate system to be cartesian, this sequence of vectors converges to $\vec{v}(t)$ if, and only if the three sequences of components of the vectors converge to the corresponding components of the vector $\vec{v}(t)$.

Similarly, a vector function $\vec{v}(t)$ of a real variable t is said to have the limit $\vec{l}$ as t approaches t_0, if $\vec{v}(t)$ is defined in some neighbourhood of t_0, possibly except at t_0, and $\lim_{t\to t_0} |\vec{v}(t) - \vec{l}| = \vec{0}$, and then we write

$$\lim_{t\to t_0} |\vec{v}(t) - \vec{l}| = \vec{l}.$$

A vector function v(t) is said to be continuous at $t = t_0$, if it is defined in some neighbourhood of t_0 and $\lim_{t\to t_0} \vec{v}(t) = \vec{v}(t_0)$.

In case of cartesian co-ordinate system, we may write $\vec{v}(t)= v_1(t)\,\hat{i} + v_2(t)\,\hat{j} + v_3(t)\,\hat{k}$, where $\hat{i}, \hat{j}, \hat{k}$ have their usual meanings. Then continuity of $\vec{v}(t)$ at t_0 implies and is implied by the continuity of its three components at t_0.

Next, we define the derivative of a vector function.

8.1.2 Derivatives of a Vector Function

A vector function $\vec{v}(t)$ is said to be differentiable at a point t, if the limit $\lim_{\delta t\to t_0} \dfrac{\vec{v}(t+\delta t)-\vec{v}(\delta t)}{\delta t}$ *exists. If so, then this limit, denoted by* $\dfrac{d\vec{v}}{dt}$, *or* $\vec{v}\,'(t)$, *is called the derivative of the vector function* $\vec{v}(t)$.

If $\vec{v}(t) = v_1(t)\,\hat{i} + v_2(t)\,\hat{j} + v_3(t)\,\hat{k}$, then $\vec{v}(t)$ is differentiable at a point t if, and only if its three components are differentiable at t, and the derivative of $\vec{v}(t)$ is obtained by differentiating each component separately.

Since $\dfrac{d\vec{v}}{dt}$ is itself a vector function of t, its derivative, denoted by $\dfrac{d^2\vec{v}}{dt^2}$, is called the *second order derivative of $\vec{v}$ with respect to t*. Similarly, we can define higher order derivatives of $\vec{v}(t)$.

General rules of differentiation. If ϕ is a scalar function and $\vec{u}, \vec{v}, \vec{w}$ are vector functions of a scalar variable t, then we have the following general rules of differentiation of vector functions similar to those of ordinary differential calculus, provided the order of factors in vector product is maintained.

1. $\dfrac{d}{dt}(\vec{u}\pm\vec{v}) = \dfrac{d\vec{u}}{dt}\pm\dfrac{d\vec{v}}{dt}$

2. $\dfrac{d}{dt}(\vec{u}\cdot\vec{v}) = \vec{u}\cdot\dfrac{d\vec{v}}{dt}+\dfrac{d\vec{u}}{dt}\cdot\vec{v}$

3. $\dfrac{d}{dt}(\vec{u}\times\vec{v}) = \vec{u}\times\dfrac{d\vec{v}}{dt}+\dfrac{d\vec{u}}{dt}\times\vec{v}$

4. $\dfrac{d}{dt}(\phi\vec{u}) = \phi\dfrac{d\vec{u}}{dt}+\dfrac{d\phi}{dt}\vec{u}$

5. $\dfrac{d}{dt}[\vec{u}\;\vec{v}\;\vec{w}] = \left[\dfrac{d\vec{u}}{dt}\;\vec{v}\;\vec{w}\right]+\left[\vec{u}\;\dfrac{d\vec{v}}{dt}\;\vec{w}\right]+\left[\vec{u}\;\vec{v}\;\dfrac{d\vec{w}}{dt}\right]$

6. $\dfrac{d}{dt}[\vec{u}\times(\vec{v}\times\vec{w})] = \dfrac{d\vec{u}}{dt}\times(\vec{v}\times\vec{w})+\vec{u}\times\left(\dfrac{d\vec{v}}{dt}\times\vec{w}\right)+\vec{u}\times\left(\vec{v}\times\dfrac{d\vec{w}}{dt}\right)$

As an illustration, we prove **3.** By definition

$$\frac{d}{dt}(\vec{u}\times\vec{v}) = \lim_{\delta t\to 0}\frac{(\vec{u}+\delta\vec{u})\times(\vec{v}+\delta\vec{v})-(\vec{u}\times\vec{v})}{\delta t} = \lim_{\delta t\to 0}\frac{(\vec{u}\times\vec{v}+\vec{u}\times\delta\vec{v}+\delta\vec{u}\times\vec{v}+\delta\vec{u}\times\delta\vec{v})-(\vec{u}\times\vec{v})}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\vec{u} \times \delta\vec{v} + \delta\vec{u} \times \vec{v} + \delta\vec{u} \times \delta\vec{v}}{\delta t} = \lim_{\delta t \to 0}\left[\vec{u} \times \frac{\delta\vec{v}}{\delta t} + \frac{\delta\vec{u}}{\delta t} \times \vec{v} + \frac{\delta\vec{u}}{\delta t} \times \delta\vec{v}\right]$$

$$= \vec{u} \times \frac{d\vec{v}}{dt} + \frac{d\vec{u}}{dt} \times \vec{v}, \text{ since } \delta\vec{v} \to 0 \text{ as } \delta t \to 0.$$

We must note that since $\vec{u} \times \vec{v} \neq \vec{v} \times \vec{u}$, thus while evaluating $\frac{d}{dt}(\vec{u} \times \vec{v})$ the order of vectors $\vec{u}$ and $\vec{v}$ must be maintained. Similarly while evaluating $\frac{d}{dt}[\vec{u}\,\vec{v}\,\vec{w}]$, the cyclic order of vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ must be maintained.

Derivative of a vector function constant in magnitude, or direction only. A vector function changes if either its magnitude changes or its direction changes, or the direction and magnitude both change.

We find conditions under which a vector function will remain constant in magnitude, or in direction, or in both.

First, let $\vec{v}(t)$ be a vector with constant magnitude, say $|\vec{v}(t)| = c$. Then, we have

$$\vec{v} \cdot \vec{v} = |\vec{v}(t)|^2 = c^2 \qquad \text{...(8.1)}$$

Differentiating (8.1) w.r.t. t, we get $\frac{d\vec{v}}{dt} \cdot \vec{v} + v \cdot \frac{d\vec{v}}{dt} = 0.$

Since the dot product is commutative this gives

$$\vec{v} \cdot \frac{d\vec{v}}{dt} = 0. \qquad \text{...(8.2)}$$

Thus the derivative of a vector function $\vec{v}(t)$ of constant magnitude is either the zero vector or is perpendicular to $\vec{v}(t)$.

Next, let $\vec{v}(t)$ be a vector with constant direction and let $\vec{a}$ be a unit vector in that direction, then

$$\vec{v} = \phi\vec{a} \qquad \text{...(8.3)}$$

where $\phi = |\vec{v}|$.

From (8.3), we get $\frac{d\vec{v}}{dt} = \phi\frac{d\vec{a}}{dt} + \frac{d\phi}{dt}\vec{a}$, and thus

$$\vec{v} \times \frac{d\vec{v}}{dt} = \phi\vec{a} \times \left[\phi\frac{d\vec{a}}{dt} + \frac{d\phi}{dt}\vec{a}\right] = \phi^2\vec{a} \times \frac{d\vec{a}}{dt} \qquad \text{...(8.4)}$$

since $\vec{a} \times \vec{a} = 0$

Further since $\vec{a}$ is constant vector in magnitude as well as direction, thus $\frac{d\vec{a}}{dt} = 0$. Hence (8.4) becomes

$$\vec{v} \times \frac{d\vec{v}}{dt} = \vec{0} \quad \text{...(8.5)}$$

Thus the derivative of a vector function $\vec{v}(t)$ of constant direction is either the zero vector or is parallel to $\vec{v}(t)$.

In fact, if $\vec{v}(t)$ is a constant vector, then $\vec{v}(t + \Delta t) = \vec{v}(t)$, for all t which gives

$$\lim_{\Delta t \to 0} \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t} = \vec{0}. \text{ or } \frac{d\vec{v}}{dt} = \vec{0}. \quad \text{...(8.6)}$$

In particular, we have

$$\frac{d\vec{i}}{dt} = \frac{d\vec{j}}{dt} = \frac{d\vec{k}}{dt} = \vec{0}, \quad \text{...(8.7)}$$

where $\vec{i}$, $\vec{j}$ and $\vec{k}$ are the unit vectors along x-axis, y-axis and z-axis respectively.

8.1.3 Geometrical Interpretation of the Derivative of a Vector Function

Let $\vec{r}(t)$ be the position vector of a point P with respect to the origin of reference O. As t varies continuously P traces out a curve C as shown in Fig. 8.1. Thus a vector function $\vec{r}(t)$ represents a curve in space.

For example (i) the vector $\vec{r}(t) = a\cos t\vec{i} + b\sin t\hat{j}$ represents an ellipse in the xy-plane with center at the origin and principal axes in the direction of x and y axes, since from $\vec{r}(t)$, we have $x = a\cos t$, $y = b\sin t$, $z = 0$, which give

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad z = 0.$$

If $b = a$, then $\vec{r}(t)$ represents a circle or radius a.

(ii) The vector $\vec{r}(t) = at^2\hat{i} + 2at\hat{j}$ represents the parabola $y^2 = 4ax$, $z = 0$ in the xy-plane.

Let $\vec{r}$ and $\vec{r} + \delta\vec{r}$ be the position vectors of two neighbouring points P and Q on this curve C as shown in Fig. 8.2. Then,

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (\vec{r} + \delta\vec{r}) - \vec{r} = \delta\vec{r}.$$

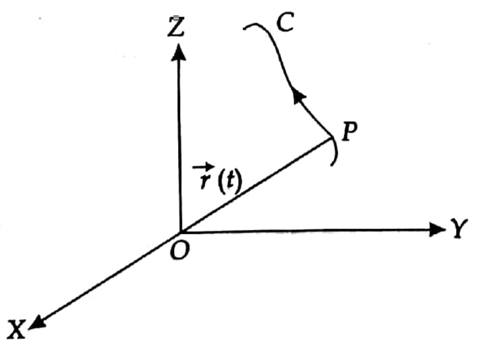

Fig. 8.1

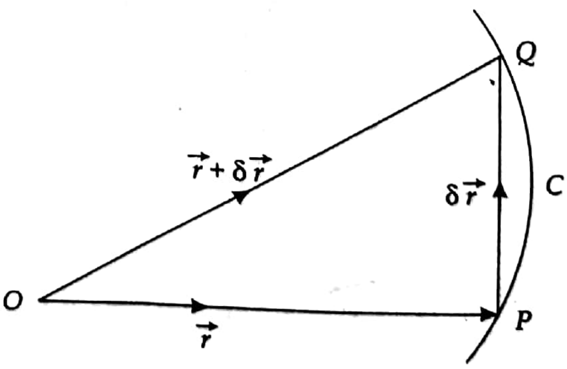

Fig. 8.2

Therefore $\frac{\delta\vec{r}}{\delta t}$ is directed along the chord PQ.

As $\delta t \to 0$, $Q \to P$, the chord PQ becomes the tangent to the curve at P. Thus $\lim_{\delta t\to 0} \frac{\delta\vec{r}}{\delta t} = \frac{\delta\vec{r}}{\delta t}$ is a vector along the tangent to the curve at P. If $\vec{r}'(t) \neq \vec{0}$, then $\vec{r}'(t)$ is called a *tangent vector to the curve C at P* because it has the direction of the tangent to C at point P. The corresponding unit vector is the *unit tangent vector* given by $\frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \hat{u}(t)$.

Both $\vec{r}'(t)$ and $\hat{u}(t)$ point in the direction of increasing t. Hence, their sense depends on the orientation of C.

Suppose that scalar parameter t is replaced by s, the arc length from any convenient point A on the curve up to the point P, and let $\widehat{AP} = s$, $\widehat{AQ} = s + \delta s$, so that, $\delta s = \widehat{PQ}$. In this case $\frac{d\vec{r}}{ds}$ will be a vector along the tangent at P and

$$\left|\frac{d\vec{r}}{ds}\right| = \lim_{\delta s\to 0}\left|\frac{d\vec{r}}{ds}\right| = \lim_{Q\to P}\frac{\text{chord } PQ}{\text{arc } PQ} = 1.$$

Thus $\frac{d\vec{r}}{ds}$ *is the unit vector along the tangent* at P.

8.2 VELOCITY AND ACCELERATION. TANGENTIAL AND NORMAL ACCELERATION

In this section we discuss the concepts of velocity and acceleration of a particle. The decomposition of acceleration along the tangent and normal is of great practical importance.

8.2.1 Velocity and Acceleration

If the scalar t denotes the time and $\vec{r}$ is the position vector of a moving particle P, then $\frac{d\vec{r}}{dt}$ represents the velocity vector $\vec{v}$ of the particle at P. Its direction is along the tangent at P. Further, $\frac{d^2\vec{r}}{dt}$ or $\frac{d\vec{v}}{dt}$ represents the *acceleration* $\vec{a}(t)$ of the particle at P. For example, the vector function

$$\vec{r}(t) = R\cos wt\,\hat{i} + R\sin wt\,\hat{j},\ w > 0 \qquad \text{...(8.8)}$$

represents a circle of radius R with centre at the origin in the xy-plane. It describes the motion of a particle P in the counterclockwise sense. The velocity of the particle at P is given by

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = -Rw\sin wt\,\hat{i} + Rw\cos wt\,\hat{j}.$$

Its magnitude

$$|\vec{v}(t)| = \sqrt{\frac{d\vec{r}}{dt}\cdot\frac{d\vec{r}}{dt}} = Rw \quad \text{...(8.9)}$$

is constant, and the direction is along the tangent to C.

The angular speed is, $\frac{Rw}{R} = w$, and the acceleration is

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = -Rw^2 \cos wt\,\hat{i} - Rw^2 \sin wt\,\hat{j} = -w^2\vec{r} \quad \text{...(8.10)}$$

We observe that the acceleration is of constant magnitude, $|\vec{a}(t)| = w^2|\vec{r}| = w^2R$ and is directed towards the origin. The acceleration $\vec{a}(t)$ is called the *centripetal acceleration* and results from the fact that the velocity vector is changing its direction at a constant rate.

8.2.2 Tangential and Normal Accerlerations

Next we study the decomposition of acceleration into a component in the direction of motion, called the *tangential component,* and a component perpendicular to it, called the *normal component.*

The acceleration $\vec{a}$ is the time rate of change of the velocity $\vec{v}$. As discussed in case of motion described by Eq. (8.8), we have $|\vec{v}|$ = constant, but $|\vec{a}| \neq 0$. Thus, *magnitude of acceleration is not always the rate change of* $|\vec{v}|$. *The reason is that, in general,* $\vec{a}$ *is not along the tangent to the path C.* In fact

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt} = \vec{u}(s)\frac{ds}{dt}, \quad \text{...(8.11)}$$

where $\vec{u}(s)$ is the unit tangent vector of C.

Further differentiating it again

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\vec{u}(s)\frac{ds}{dt}\right) = \frac{d\vec{u}}{ds}\left(\frac{ds}{dt}\right)^2 + \vec{u}(s)\frac{d^2s}{dt^2}. \quad \text{...(8.12)}$$

Now $\vec{u}(s)$ is along tangent to C and of constant length one, so $\frac{d\vec{u}}{ds}$ is perpendicular to $\vec{u}(s)$. Hence, the acceleration $\vec{a}(t)$ is composed of

(i) the tangential component $\vec{u}(s)\frac{d^2s}{dt^2}$, called the *tangential acceleration;* and

(ii) the normal component $\left(\frac{d\vec{u}}{ds}\right)\left(\frac{ds}{dt}\right)^2$, called the *normal acceleration.*

Thus we observe that *if, and only if the normal acceleration is zero* $|\vec{a}(t)|$ *equals the time rate of change of* $|\vec{v}(t)| = \frac{ds}{dt}$, because only then from (8.12), we have

$$|\vec{a}(t)| = |\vec{u}(s)|\left|\frac{d^2s}{dt^2}\right| = \left|\frac{d^2s}{dt^2}\right|. \quad \text{...(8.13)}$$

8.2.3 Relative Velocity and Acceleration

Let two particles P_1 and P_2 moving along the curves C_1 and C_2 have position vectors $\vec{r}_1$ and $\vec{r}_2$ at time t, with reference to the origin O. From Fig. 8.3,

$$\vec{r} = \overrightarrow{P_1P_2} = \vec{r}_2 - \vec{r}_1.$$

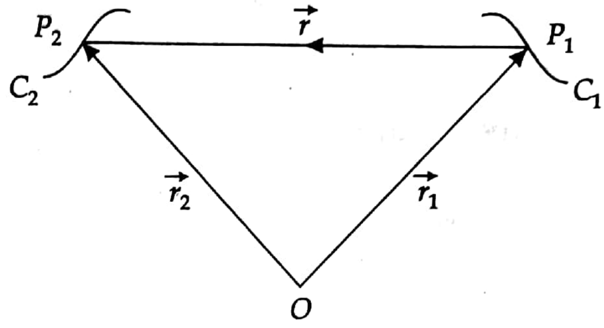

Fig. 8.3

Differentiating w.r.t. t, we get

$$\frac{d\vec{r}}{dt} = \frac{d\vec{r}_2}{dt} - \frac{d\vec{r}_1}{dt}. \qquad \text{...(8.14)}$$

This defines the relative velocity of P_2 w.r.t. P_1 and thus, *velocity of P_2 relative to P_1 = velocity of P_2 – velocity of P_1*.

Again differentiating (8.14) w.r.t. t, we get

$$\frac{d^2\vec{r}}{dt^2} = \frac{d^2\vec{r}_2}{dt^2} - \frac{d^2\vec{r}_1}{dt^2} \qquad \text{...(8.15)}$$

That is, *acceleration of P_2 relative to P_1 = acceleration of P_2 – acceleration of P_1*.

Example 8.1: Find $\dfrac{d\vec{w}}{dt}$ in each of the following cases:

(a) $\vec{w}(t) = (3t\hat{i} + 5t^2\hat{j} + 6\hat{k}) \cdot (t^2\hat{i} - 2t\hat{j} + t\hat{k})$ (b) $\vec{w}(t) = (t\hat{i} + e^t\hat{j} - t^2\hat{k}) \times (t^2\hat{i} + \hat{j} + t^3\hat{k})$

Solution: (a) Using $\dfrac{d}{dt}(\vec{u}\cdot\vec{v}) = \vec{u}\cdot\dfrac{d\vec{v}}{dt} + \dfrac{d\vec{u}}{dt}\cdot\vec{v}$, we have

$$\frac{d\vec{w}}{dt} = (3t\hat{i} + 5t^2\hat{j} + 6\hat{k}) \cdot \frac{d}{dt}(t^2\hat{i} - 2t\hat{j} + t\hat{k}) + \frac{d}{dt}(3t\hat{i} + 5t^2\hat{j} + 6\hat{k}) \cdot (t^2\hat{i} - 2t\hat{j} + t\hat{k})$$

$$= (3t\hat{i} + 5t^2\hat{j} + 6\hat{k}) \cdot (2t\hat{i} - 2\hat{j} + \hat{k}) + (3\hat{i} + 10t\hat{j}) \cdot (t^2\hat{i} - 2t\hat{j} + t\hat{k})$$

$$= 6t^2 - 10t^2 + 6 + 3t^2 - 20t^2 = 6 - 21t^2.$$

(b) Using $\dfrac{d}{dt}(\vec{u}\times\vec{v}) = \vec{u}\times\dfrac{d\vec{v}}{dt} + \dfrac{d\vec{u}}{dt}\times\vec{v}$, we have

$$\frac{d\vec{w}(t)}{dt} = (t\hat{i} + e^t\hat{j} - t^2\hat{k}) \times \frac{d}{dt}(t^2\hat{i} + \hat{j} + t^3\hat{k}) + \frac{d}{dt}(t\hat{i} + e^t\hat{j} - t^2\hat{k}) \times (t^2\hat{i} + \hat{j} + t^3\hat{k})$$

$$= (t\hat{i} + e^t\hat{j} - t^2\hat{k}) \times (2t\hat{i} + 3t^2\hat{k}) + (\hat{i} + e^t\hat{j} - 2t\hat{k}) \times (t^2\hat{i} + \hat{j} + t^3\hat{k})$$

$$= [3t^2e^t\hat{i} - 5t^3\hat{j} - 2te^t\hat{k}] + [(t^3e^t + 2t)\hat{i} - 3t^3\hat{j} + (1 - t^2e^t)\hat{k}]$$

$$= [t^2e^t(3 + t) + 2t]\hat{i} - 8t^3\hat{j} + [1 - te^t(2 + t)]\hat{k}.$$

Example 8.2: If $\vec{r}$ is a vector function of a scalar t and $\vec{a}$ is a constant vector, differentiate the following with respect to t

(a) $\dfrac{\vec{r}\times\vec{a}}{\vec{r}\cdot\vec{a}}$ Imp' (b) $\dfrac{\vec{r}+\vec{a}}{|\vec{r}|^2+|\vec{a}|^2}$.

Solution: (a) Let $\vec{R} = \dfrac{\vec{r}\times\vec{a}}{\vec{r}\cdot\vec{a}}$. Here $\vec{r}\cdot\vec{a}$ is a scalar function of t; also $\dfrac{d\vec{a}}{dt} = \vec{0}$, since $\vec{a}$ is a constant vector. Therefore,

$$\frac{d\vec{R}}{dt} = \frac{1}{\vec{r}\cdot\vec{a}}\frac{d}{dt}(\vec{r}\times\vec{a}) + \left\{\frac{d}{dt}\left(\frac{1}{\vec{r}\cdot\vec{a}}\right)\right\}(\vec{r}\times\vec{a}) = \frac{1}{\vec{r}\cdot\vec{a}}\left(\vec{r}\times\frac{d\vec{a}}{dt}+\frac{d\vec{r}}{dt}\times a\right) - \frac{\frac{d}{dt}(\vec{r}\cdot\vec{a})}{(\vec{r}\cdot\vec{a})^2}(\vec{r}\cdot\vec{a})$$

$$= \frac{\frac{d\vec{r}}{dt}\times\vec{a}}{\vec{r}\cdot\vec{a}} - \frac{\vec{r}\cdot\frac{d\vec{a}}{dt}+\frac{d\vec{r}}{dt}\cdot\vec{a}}{(\vec{r}\cdot\vec{a})^2}(\vec{r}\times\vec{a}) = \frac{\frac{d\vec{r}}{dt}\times\vec{a}}{\vec{r}\cdot\vec{a}} - \frac{\frac{d\vec{r}}{dt}\cdot\vec{a}}{(\vec{r}\cdot\vec{a})^2}(\vec{r}\times\vec{a}).$$

(b) Let $\vec{R} = \dfrac{\vec{r}+\vec{a}}{|\vec{r}|^2+|\vec{a}|^2}$. Here $|\,\vec{r}\,|^2$ is a scalar function of t and $|\,\vec{a}\,|^2$ is a scalar independent of t.

Therefore,

$$\frac{d\vec{R}}{dt} = \frac{1}{|\vec{r}|^2+|\vec{a}|^2}\frac{d}{dt}(\vec{r}+\vec{a}) + \left\{\frac{d}{dt}\left(\frac{1}{|\vec{r}|^2+|\vec{a}|^2}\right)\right\}(\vec{r}+\vec{a})$$

$$= \frac{1}{|\vec{r}|^2+|\vec{a}|^2}\frac{d\vec{r}}{dt} - \left\{\frac{\frac{d}{dt}(|\vec{r}|^2+|\vec{a}|^2)}{(|\vec{r}|^2+|\vec{a}|^2)^2}\right\}(\vec{r}+\vec{a}) = \frac{\frac{d\vec{r}}{dt}}{|\vec{r}|^2+|\vec{a}|^2} - \frac{2\vec{r}\cdot\frac{d\vec{r}}{dt}}{(|\vec{r}|^2+|\vec{a}|^2)^2}(\vec{r}+\vec{a}),$$

since, $\dfrac{d}{dt}|\vec{r}|^2 = \dfrac{d}{dt}(\vec{r}\cdot\vec{r}) = 2\vec{r}\cdot\dfrac{d\vec{r}}{dt}$ and $\dfrac{d}{dt}|\vec{a}|^2 = 0.$

Example 8.3: A particle moves along a curve whose parametric equations are $x = e^{-t}$, $y = 2\cos 3t$, $z = \sin 3t$. Find the velocity and acceleration at $t = 0$.

Solution: Let $\vec{r}(t)$ be the position vector of the particle at any time t, then

$$\vec{r}(t) = x\hat{i} + y\hat{j} + z\hat{k} = e^{-t}\hat{i} + 2\cos 3t\,\hat{j} + \sin 3t\,\hat{k}.$$

The velocity $\vec{u}(t)$ of the particle at time t is $\vec{u}(t) = \dfrac{d\vec{r}}{dt} = -e^{-t}\hat{i} - 6\sin 3t\,\hat{j} + 3\cos 3t\,\hat{k}$.

Thus the velocity at $t = 0$ is $\left(\dfrac{d\vec{r}}{dt}\right)_{t=0} = -\hat{i} + 3\hat{k}$.

Similarly the acceleration $\vec{a}(t)$ of the particle at time t is

$$\vec{a}(t) = \frac{d^2 r}{dt^2} = e^{-t}\hat{i} - 18\cos 3t\,\hat{j} - 9\sin 3t\,\vec{k}.$$

Thus the acceleration at $t = 0$ is $\left(\frac{d^2 r}{dt^2}\right)_{t=0} = \hat{i} - 18\hat{j}$.

Example 8.4: A particle moves along a curve whose parametric equations are $x = 3t^2$, $y = t^2 - 2t$, $z = t^3$, where the parameter t is time. Find its velocity and acceleration at $t = 2$.

Solution: Let $\vec{r}(t)$ be the position vector of the particle at time t, then

$$\vec{r}(t) = x\hat{i} + y\hat{j} + z\hat{k} = 3t^2\hat{i} + (t^2 - 2t)\hat{j} + t^3\hat{k}$$

The velocity $\vec{v}(t)$ of the particle at time t is $\vec{v}(t) = \frac{d\vec{r}}{dt} = 6t\hat{i} + (2t-2)\hat{j} + 3t^2\hat{k}$,

and the acceleration $\vec{a}(t)$ of the particle at time t is

$$\vec{a}(t) = \frac{d^2\vec{r}}{dt^2} = 6\hat{i} + 2\hat{j} + 6t\hat{k}$$

At $t = 2$, $\left(\frac{d\vec{r}}{dt}\right)_{t=2} = 12\hat{i} + 2\hat{j} + 12\hat{k}$, and $\left(\frac{d^2\vec{r}}{dt^2}\right)_{t=2} = 6\hat{i} + 2\hat{j} + 12\hat{k}$.

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Example 8.5: Find the angle between the tangents to the curve $\vec{r} = t^2\hat{i} + 2t\hat{j} - t^3\hat{k}$ at the points $t = \pm 1$.

Solution: A vector along the tangent at any point t to the given curve

$$\vec{r}(t) = t^2\hat{i} + 2t\hat{j} - t^3\hat{k} \text{ is } \frac{d\vec{r}}{dt} = 2t\hat{i} + 2\hat{j} - 3t^2\hat{k}$$

If $\vec{T_1}$ and $\vec{T_2}$ are the vectors along the tangents at $t = 1$ and $t = -1$ respectively, then

$$\vec{T_1} = 2\hat{i} + 2\hat{j} - 3\hat{k}, \text{ and } \vec{T_2} = -2\hat{i} + 2\hat{j} - 3\hat{k}.$$

If θ is the angle between $\vec{T_1}$ and $\vec{T_2}$, then

$$\cos\theta = \frac{\vec{T_1}\cdot\vec{T_2}}{|\vec{T_1}||\vec{T_2}|} = \frac{2(-2) + 2(2) + (-3)(-3)}{\sqrt{4+4+9}\cdot\sqrt{4+4+9}} = \frac{9}{17}$$

Therefore, $\theta = \cos^{-1}(9/17)$.

Example 8.6: For the curve $x = \cos t + t \sin t$, $y = \sin t - t \cos t$, find the tangential and the normal components of the acceleration at any time t.

Solution: Let $\vec{r}(t)$ be the position vector of any point P on the curve at time t, then

$$\vec{r}(t) = (\cos t + t \sin t)\hat{i} + (\sin t - t\cos t)\hat{j}.$$

The velocity $\vec{v}(t)$ of particle P is

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = (-\sin t + \sin t + t\cos t)\hat{i} + (\cos t - \cos t + t \sin t)\hat{j} = (t\cos t)\hat{i} + (t \sin t)\hat{j}.$$

which is along the tangent to the curve at P.

The acceleration $\vec{a}(t)$ is

$$\vec{a}(t) = \frac{d^2\vec{r}}{dt^2} = (-t\sin t + \cos t)\hat{i} + (t\cos t + \sin t)\hat{j} \qquad ...(8.16)$$

Let

$$\vec{a}(t) = a_T\hat{T} + a_N\hat{N} \qquad ...(8.17)$$

where a_T and a_N are the tangential and normal components of the acceleration, respectively, and $\hat{T}$ and $\hat{N}$ respectively are unit vectors along the tangent and normal at P. From (8.17),

along velocity

$$\vec{a}(t)\cdot\hat{T} = a_T\hat{T}\cdot\hat{T} + a_N\hat{N}\cdot\hat{T} = a_T \qquad ...(8.18)$$

Also, $\hat{T} = \frac{\vec{T}}{|\vec{T}|} = \frac{(t\cos t)\hat{i} + (t\sin t)\hat{j}}{\sqrt{(t\cos t)^2 + (t\sin t)^2}} = \cos t\hat{i} + \sin t\hat{j}$.

Therefore, from (8.18), we obtain

$$a_T = \vec{a}\cdot\hat{T} = \{(-t\sin t + \cos t)\hat{i} + (t\cos t + \sin t)\hat{j}\}\cdot\{\cos t\hat{i} + \sin t\hat{j}\}$$

$$= \cos t(-t\sin t + \cos t) + \sin t(t\cos t + \sin t) = \cos^2 t + \sin^2 t = 1.$$

Next, from (8.17) $a_N\hat{N} = \vec{a}(t) - a_T\hat{T}$. Thus

$$a_N^2 = (a_N\hat{N})\cdot(a_N\hat{N}) = (\vec{a} - a_T\hat{T})\cdot(\vec{a} - a_T\hat{T}) = (\vec{a} - \hat{T})\cdot(\vec{a} - \hat{T}), \quad \text{since } a_T = 1$$

$$= \vec{a}\cdot\vec{a} - 2\vec{a}\cdot\hat{T} + \hat{T}\cdot\hat{T} = (-t\sin t + \cos t)^2 + (t\cos t + \sin t)^2 - 2a_T + 1$$

$$= t^2\sin^2 t + \cos^2 t - 2t\sin t\cos t + t^2\cos^2 t + \sin^2 t + 2t\sin t\cos t - 2 + 1$$

$$= t^2 + 1 - 1 = t^2.$$

Hence, $a_N = t$.

Thus, the tangential and normal components of the acceleration are 1 and t, respectively.

Example 8.7: Obtain the tangential and normal components of the acceleration of a particle which is at a point $P(x, y)$ on the curve $x = e^t\cos t$, $y = e^t \sin t$ at any time t.

Solution: Let $\vec{r}(t)$ be the position vector of any point P on the curve at time t, then $\vec{r}(t) = (e^t \cos t)\hat{i} + (e^t \sin t)\hat{j}$

The velocity $\vec{v}(t)$ of P at time t is $\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = e^t(-\sin t + \cos t)\hat{i} + e^t(\cos t + \sin t)\hat{j}$

$$= e^t(\cos t - \sin t)\hat{i} + e^t(\sin t + \cos t)\hat{j}$$

which is along the tangent to the curve at P.

The acceleration $\vec{a}(t)$ is

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = [e^t(\cos t - \sin t) + e^t(-\sin t - \cos t)]\hat{i} + [e^t(\sin t + \cos t) + e^t(\cos t - \sin t)]\hat{j}$$

$$= -2e^t \sin t\,\hat{i} + 2e^t \cos t\,\hat{j}. \qquad ...(8.19)$$

Let $\vec{a}(t) = a_T\hat{T} + a_N\hat{N}$ ...(8.20)

where a_T and a_N are the tangential and normal components of the acceleration respectively and $\hat{T}$, $\hat{N}$ are the unit vectors along the tangent and the normal at point P. Next, from (8.20)

$$\vec{a}\cdot\hat{T} = a_T\hat{T}\cdot\hat{T} + a_N\hat{N}\cdot\hat{T} = a_T. \qquad ...(8.21)$$

Also, $\hat{T} = \frac{\vec{T}}{|\vec{T}|} = \frac{e^t(\cos t - \sin t)\hat{i} + e^t(\sin t + \cos t)\hat{j}}{\sqrt{e^{2t}(\cos^2 t + \sin^2 t - 2\cos t \sin t + \sin^2 t + \cos^2 t + 2\cos t \sin t)}}$

$$= \frac{1}{\sqrt{2}}[(\cos t - \sin t)\hat{i} + (\sin t + \cos t)\hat{j}].$$

Therefore, from (8.21)

$$a_T = \vec{a}\cdot\hat{T} = [-2e^t \sin t\,\hat{i} + 2e^t \cos t\,\hat{j}]\cdot\frac{1}{\sqrt{2}}[(\cos t - \sin t)\hat{i} + (\sin t + \cos t)\hat{j}]$$

$$= -\sqrt{2}\,e^t(\sin t \cos t - \sin^2 t) + \sqrt{2}\,e^t(\cos t \sin t + \cos^2 t) = \sqrt{2}\,e^t.$$

Also from (8.20), $a_N\hat{N} = \vec{a} - a_T\hat{T}$. It gives

$$a_N^2 = (a_N\hat{N})\cdot(a_N\hat{N}) = (\vec{a} - a_T\hat{T})\cdot(\vec{a} - a_T\hat{T}) = \vec{a}\cdot\vec{a} - 2a_T\vec{a}\cdot\hat{T} + a_T^2\,\hat{T}\cdot\hat{T}$$

$$= 4e^{2t}(\sin^2 t + \cos^2 t) - 2(\sqrt{2}\,e^t)(\sqrt{2}\,e^t) + 2e^{2t}.\ \frac{1}{2}[(\cos t - \sin t)^2 + (\sin t + \cos t)^2]$$

$$= 4e^{2t} - 4e^{2t} + 2e^{2t} = 2e^{2t}.$$

Hence, $a_N = \sqrt{2}\,e^t$.

Thus both the tangential and normal components of the acceleration are equal to $\sqrt{2}\,e^t$.

Example 8.8: A person going eastwards with a velocity of 4 km per hour, observes that the wind appears to blow directly from the north. He doubles his speed and the wind appears to come from north-east. Find the actual velocity of the wind.

Solution: Let the actual velocity of the wind be $\vec{v}_w = x\hat{i} + y\hat{j}$, where $\hat{i}$ and $\hat{j}$ represent velocities of 1 km per hour towards the east and the north, respectively, as shown in Fig. 8.4.

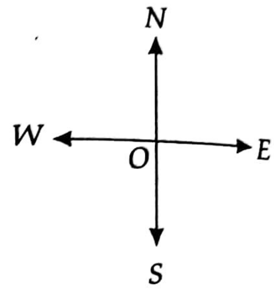

Fig. 8.4

If $\vec{v}_p$ be the velocity of the person, then $\vec{v}_p = 4\hat{i}$. Thus $\vec{v}_{wp}$ the velocity of the wind relative to that of the person is

$$\vec{v}_{wp} = \vec{v}_w - \vec{v}_p = (x\hat{i} + y\hat{j}) - 4\hat{i} = (x-4)\hat{i} + y\hat{j}$$

But it is given to be parallel to $-\hat{j}$ since it appears to blow from the north. Hence, $x = 4$.

When the velocity of the person becomes $8\hat{i}$, the velocity of the wind relative to the person is

$$\vec{v}_{wp} = (x\hat{i} + y\hat{j}) - 8\hat{i} = (x-8)\hat{i} + y\hat{j}.$$

But this is parallel to $-(\hat{i} + \hat{j})$, since it appears to come from north-east.

Therefore, $\dfrac{x-8}{y} = 1$, which by using $x = 4$ gives $y = -4$. Hence, the actual velocity of the wind is $\vec{v}_w = 4\hat{i} - 4\hat{j}$, that is, $4\sqrt{2}$ km per hour towards the south-east.

EXERCISE 8.1

In the following problems, find the indicated derivative using the differentiation rules assuming that all the given vector functions are differentiable.

1. $\vec{u}(t) = 5t^2\hat{i} + t\hat{j} + t^3\hat{k}$, $f(t) = \sin t$, find $[f(t)\vec{u}(t)]'$
2. $\vec{u}(t) = [\sin(2t)\hat{i} - \cos(2t)\hat{j} + t\hat{k}]$, $v(t) = [\cos(2t)\hat{i} - \sin(2t)\hat{j} + t^2\hat{k}]$, find $[u(t)\cdot v(t)]$
3. $\vec{u}(t) = (\cos wt)\hat{i} + (\sin wt)\hat{j}$, find $\vec{u}(t) \times \vec{u}'(t)$
4. $\vec{u}(t) = (1-t)\hat{i} + t^2\hat{j} + e^t\hat{k}$, $\vec{v}(t) = (1+t)\hat{i} + e^t\hat{j} + t\hat{k}$, find $[\vec{u}(t) \times \vec{v}(t)]$

If a and t are scalars, then find

5. $[t^2\vec{u}(t^2)]'$
6. $[\vec{u}(at) + \vec{v}(a/t)]'$
7. $[\vec{u}(t) \times \vec{u}''(t)]'$
8. $[\vec{u}(t)\cdot\vec{u}'(t) \times \vec{u}''(t)]'$

9. Verify the formula $\frac{d}{dt}(\vec{u}(t)\cdot\vec{v}(t)) = \vec{u}(t)\cdot\frac{d\vec{v}(t)}{dt} + \frac{d\vec{u}(t)}{dt}\cdot\vec{v}(t)$,

for $\vec{u}(t) = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$, and $\vec{v}(t) = \sin t\,\hat{i} - \cos t\,\hat{j}$.

10. Find the parametric equation of the tangent line to the curve $x = \sin t$, $y = \cos t$, $z = t$ at $t = \pi/4$.

11. Find the unit tangent vector at any point on the curve $x = 3\cos t$, $y = 3\sin t$, $z = 4t$.

12. Find the angle between the tangents to the curve $x = t$, $y = t^2$, $z = t^3$ at $t = \pm 1$.

13. A particle moves along the curve $x = e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$, where t is the time. Determine its velocity and acceleration vectors and also the magnitudes of velocity and acceleration at $t = 0$.

14. A particle moves on the curve $x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$, where t is the time. Find the components of velocity and acceleration at time $t = 1$ in the direction $\hat{i} - 3\hat{j} + 2\hat{k}$.

15. The position vector of a moving particle at a time t is $\vec{r} = t^2\hat{i} - t^3\hat{j} + t^4\hat{k}$. Find the tangential and normal components of its acceleration at $t = 1$.

16. The velocity of a boat relative to water is represented by $3\hat{i} + 4\hat{j}$ and that of water relative to earth is $\hat{i} - 3\hat{j}$. What is the velocity of the boat relative to the earth if $\hat{i}$ and $\hat{j}$ represent one km an hour east and north respectively?

17. A person travelling towards the north-east with a velocity of 6 km per hour finds that the wind appears to blow from the north, but when he doubles his speed it seems to come from a direction inclined at an angle $\tan^{-1} 2$ to the north of east. Show that the actual velocity of the wind is $3\sqrt{2}$ km. per hour towards the east.

8.3 SCALAR AND VECTOR FIELDS. GRADIENT OF A SCALAR FIELD. DIRECTIONAL DERIVATIVES.

In this section we discuss two kinds of functions: scalar and vector functions and their fields. In fact some of the vector fields can be obtained from the scalar fields by applying 'gradient'. This concept is of great practical advantage since comparatively it is easy to deal with a scalar fields, and because of this, gradient finds applications in engineering and physical sciences.

8.3.1 Scalar and Vector Point Functions

A variable quantity which depends for its value on its position only, that is, upon the co-ordinate of the points of a region, say (x, y, z) in space, is called a point function. There are two types of point functions, as explained below.

Scalar point function. *Let E be a region at each point P(x, y, z) of which a scalar $\phi = \phi(x, y, z)$ is specified, then we say that ϕ is a scalar point function, and the region E defined* so, is called a *scalar field*. The scalar point function does not depend upon the choice of co-ordinate system. It only depends on the point in the field. For example, the temperature distribution in a medium, the distribution of atmospheric pressure in space, density of a body are all examples of scalar point functions.

Vector point function. *Let E be a region at each point P(x, y, z) of which a vector* $\vec{v} = \vec{v}(x, y, z)$ *is specified, then we say that* $\vec{v}$ *is a vector point function and the region E defined so is* called a *vector field.* For example, the velocity of a moving fluid at any instant, the gravitation force, or electrical intensity are all examples of vector point functions.

Level surfaces. *Let $\phi(x, y, z)$ be single valued continuous scalar point function defined at every point P(x, y, z) of E. Then the surface $\phi(x, y, z) = c$, a constant, defines the equation of a surface, and is called the level surface of the function.* For example, if $\phi(x, y, z)$ represents temperature in a medium, then $\phi(x, y, z) = c$ represents a surface on which the temperature is a constant c. Such surfaces are called *isothermal surfaces*. Another example is of *equipotential surfaces*. Note that for different values of c, we get different level surfaces, no two of which intersect.

For example, the level surfaces of the scalar fields in space defined by the function $f(x, y, z) = z - \sqrt{x^2 + y^2}$ are given by $z - \sqrt{x^2 + y^2} = c$, or, $(x^2 + y^2) = (z - c)^2$ which are cones.

8.3.2 Gradient of a Scalar Field

Let $\phi(x, y, z)$ be a scalar point function defining a scalar field. To define the gradient of a scalar field, we first introduce a differential vector operator ∇, called 'del' or nabla. The differential vector operator ∇ in two and three dimensions is defined as

$$\nabla \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} \text{ and } \nabla \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

respectively.

The 'gradient' of a scalar field $\phi(x, y, z)$ denoted by $\nabla\phi$, or grad ϕ, is defined as

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

We observe that the operator ∇ operates on a scalar field and produces a vector field. Thus, *gradient of a scalar function is always a vector.*

8.3.3 Geometrical Interpretation of the Gradient

Let $\phi(P) = \phi(x, y, z)$ be a differentiable scalar point function. Consider the level surface through P at which the function has value ϕ and another level surface through a neighbouring point Q where the value is $\phi + \delta\phi$ as shown in Fig. 8.5. Let $\vec{r}$ and $\vec{r} + \delta\vec{r}$ be the position vectors of the points P and Q respectively with reference to the origin O, and thus, the vector here

$$\overrightarrow{PQ} = \delta\vec{r} = \hat{i}\,\delta x + \hat{j}\,\delta y + \hat{k}\,\delta z.$$

Consider, $\phi + \delta\phi = \phi(x + \delta x, y + \delta y, z + \delta z) = \phi(x, y, z) + \frac{\partial\phi}{\partial x}\delta x + \frac{\partial\phi}{\partial y}\delta y + \frac{\partial\phi}{\partial z}\delta z$
+ terms of second and higher orders, (by Taylor series).

Neglecting terms of second and higher order, we obtain

$$\delta\phi \approx \frac{\partial\phi}{\partial x}\delta x + \frac{\partial\phi}{\partial y}\delta y + \frac{\partial\phi}{\partial z}\delta z = \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)\cdot(\hat{i}\,\delta x + \hat{j}\,\delta y + \hat{k}\,\delta z) = \nabla\phi\cdot\delta\vec{r}$$

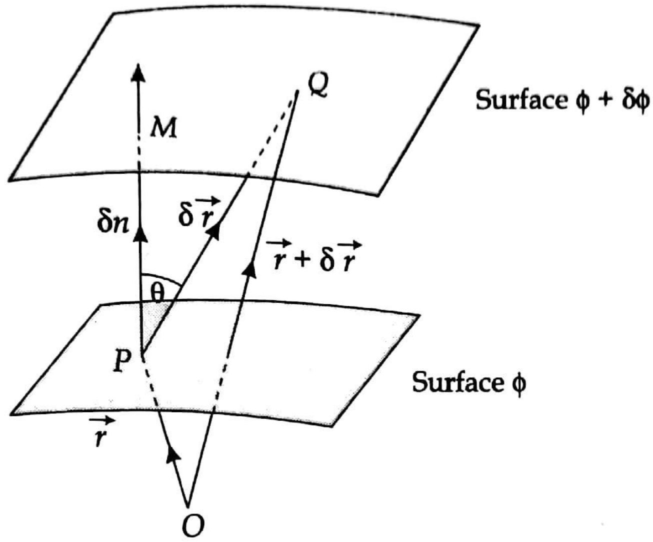

Fig. 8.5

Next, if Q approaches P, $\delta\phi \to 0$, then $\nabla\phi \cdot \delta\vec{r} = 0$. This means that $\nabla\phi$ is perpendicular to every $\vec{r}$ (whose direction is along the tangent to the level surface) lying on this surface. Thus *$\nabla\phi$ is normal o the level surface $\phi(x, y, z) = c$ at the point P*. Therefore, $\nabla\phi = |\nabla\phi|\,\hat{N}$, where $\hat{N}$ is a unit normal ector to this surface. If the perpendicular distance PM between the surfaces through P and Q be δn, hen the rate of change of ϕ along the normal PM to the surface through P is given by

$$\frac{\partial\phi}{\partial n} = \lim_{\delta n \to 0} \frac{\delta\phi}{\delta n} = \lim_{\delta n \to 0} \left(\Delta\phi \cdot \frac{\delta\vec{r}}{\delta n}\right) = |\nabla\phi| \lim_{\delta n \to 0} \frac{\hat{N}\cdot\delta\vec{r}}{\delta n} = |\nabla\phi|, \text{ since } \hat{N}\cdot\delta\vec{r} = |\delta\vec{r}|\cos\theta = \delta n.$$

Thus the magnitude of $\nabla\phi$ is $\dfrac{\partial\phi}{\partial n}$.

Hence the gradient of a scalar field ϕ is a vector normal to the level surface $\phi = c$ and has a magnitude equal to the rate of change of ϕ along the normal.

8.3.4 Directional Derivative

Let $\phi(x, y, z)$ be a differentiable scalar field, then $\dfrac{\partial\phi}{\partial x}, \dfrac{\partial\phi}{\partial y}$ and $\dfrac{\partial\phi}{\partial z}$ are the rates of change of ϕ in the directions of x, y and z axis respectively. Let $P_0(x_0, y_0, z_0)$ be any fixed point and $\hat{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ be any unit vector. Then the position vector of any point Q on the line passing through P_0 in the direction of $\hat{b}$, as shown in Fig. 8.6, is given by

$$\vec{r} = (x_0 + b_1 t)\,\hat{i} + (y_0 + b_2 t)\,\hat{j} + (z_0 + b_3 t)\,\hat{k}, \qquad \text{...(8.22)}$$

where t is the parameter.

Further, since $|\hat{b}| = 1$, the distance from P_0 to Q, that is $|P_0Q|$ is equal to t.

The directional derivative of ϕ at P_0 in the direction of $\hat{b}$, that is along P_0Q, is defined as

$$\frac{\partial \phi}{\partial t} = \lim_{t \to 0} \frac{\phi(Q) - \phi(P_0)}{t}.$$

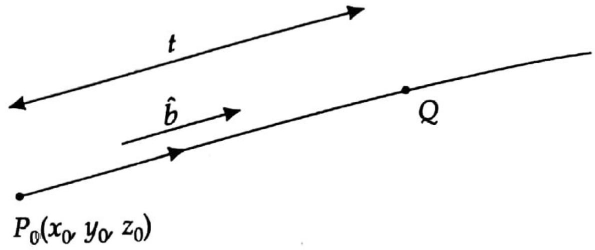

Fig. 8.6

By chain rule, $$\frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial x}\frac{dx}{dt} + \frac{\partial \phi}{\partial y}\frac{dy}{dt} + \frac{\partial \phi}{\partial z}\frac{dz}{dt}$$

$$= \left(i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z} \right) \cdot \left(i\frac{dx}{dt} + j\frac{dy}{dt} + k\frac{dz}{dt} \right)$$

$$\boxed{= \nabla \phi \cdot \frac{\partial \vec{r}}{\partial t}.}$$

From (8.22), $\frac{\partial \vec{r}}{\partial t} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k} = \hat{b}$. Therefore, *the directional derivative of ϕ in the direction of $\hat{b}$, denoted by $D_b(\phi)$, is*

$$D_b(\phi) = \nabla \phi \cdot \hat{b} = grad\ \phi \cdot \hat{b} \qquad \text{...(8.23)}$$

In general, *the directional derivative of ϕ in the direction of a vector $\vec{u}$, denoted by $D_u(\phi)$, is given by*

$$\boxed{D_u(\phi) = grad\ \phi \cdot \frac{\vec{u}}{|\vec{u}|}.}$$

Thus the directional derivative represents the rate of change of ϕ with respect to distance at any point $P(x, y, z)$ in the direction of unit vector $\hat{b}$ and is a scalar quantity.

We observe that the directional derivative of ϕ along the positive axes are given by $\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}$. For example, along x-axis it is

$$\text{grad } \phi \cdot i = \left(i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z} \right) \cdot i = \frac{\partial \phi}{\partial x},$$

and similarly for others axes.

Maximum rate change of a scalar function. We have,

$$D_b(\phi) = \nabla \phi \cdot \hat{b} = |\nabla \phi|\ |\hat{b}| \cos \theta = |\nabla \phi| \cos \theta,$$

where θ is the angle between the vectors $\nabla \phi$ and $\hat{b}$. Since $-1 \leq \cos \theta \leq 1$, so maximum value of $D_b(\phi)$ is $\nabla \phi$ at $\theta = 0$, when $\hat{b}$ is along $\nabla \phi$, that is $\hat{b} = \hat{N}$. This direction is the direction of the normal. So *the rate of change of ϕ at a point is maximum along the normal to the surface at that point.*

When $\theta = \pi$, then $D_b(\phi) = -|\nabla \phi|$, gives the minimum value of the rate of change of ϕ. It is along the direction opposite to that of $\nabla \phi$, that is along $-\hat{N}$.

In fact we may comment that the *vector $\nabla \phi$ points in the direction in which ϕ increases most rapidly and $-\nabla \phi$ points in the direction in which ϕ decreases most rapidly.*

8.3.5 Properties of Gradients

Let ϕ and ψ be any two scalar point functions. Then,

1. $\nabla(\phi + \psi) = \nabla\phi + \nabla\psi$
2. $\nabla(c_1\phi + c_2\psi) = c_1\nabla\phi + c_2\nabla\psi$, where c_1, c_2 are two arbitrary constants.
3. $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$
4. $\nabla\left(\dfrac{\phi}{\psi}\right) = \dfrac{\psi\Delta\phi - \phi\Delta\psi}{\psi^2}, \psi \neq 0.$

The above properties can be proved very easily, for example to prove **3**, we have, by definition

$$\nabla(\phi\psi) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(\phi\psi) = \hat{i}\frac{\partial(\phi\psi)}{\partial x} + \hat{j}\frac{\partial(\phi\psi)}{\partial y} + \hat{k}\frac{\partial(\phi\psi)}{\partial z}$$

$$= \hat{i}\left[\phi\frac{\partial\psi}{\partial x} + \psi\frac{\partial\phi}{\partial x}\right] + \hat{j}\left[\phi\frac{\partial\psi}{\partial y} + \psi\frac{\partial\phi}{\partial y}\right] + \hat{k}\left[\phi\frac{\partial\psi}{\partial z} + \psi\frac{\partial\phi}{\partial z}\right]$$

$$= \phi\left[\hat{i}\frac{\partial\psi}{\partial x} + \hat{j}\frac{\partial\psi}{\partial y} + \hat{k}\frac{\partial\psi}{\partial z}\right] + \psi\left[\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right] = \phi\nabla\psi + \psi\nabla\phi$$

Conservative vector field. *A vector field $\vec{F}$ is said to be conservative, if the vector function $\vec{F}$ can be expressed as the gradient of some scalar function ϕ, that is, $\vec{F} = \nabla\phi$.* In such a field the work done in moving a particle from a point A to a point B depends only on the position of points A and B and is independent of the path along which the particle is displaced from A to B. It is useful to mention here that every vector field is not conservative. We shall discuss this concept further in Section 9.2.2.

Example 8.9: Find grad ϕ at the point (1, 2, 1) when $\phi = \ln(x^2 + y^2 + z^2)$.

Solution: $$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\ln(x^2 + y^2 + z^2)$$

$$= \hat{i}\frac{\partial}{\partial x}\ln(x^2 + y^2 + z^2) + \hat{j}\frac{\partial}{\partial y}\ln(x^2 + y^2 + z^2) + \hat{k}\frac{\partial}{\partial z}\ln(x^2 + y^2 + z^2)$$

$$= \frac{2(x\hat{i} + y\hat{i} + z\hat{k})}{x^2 + y^2 + z^2} = \frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k} \text{ at } (1, 2, 1).$$

Example 8.10: Find a unit vector normal to the surface $x^3 + y^3 + 3xyz = 3$ at the point (1, 2, –1).

Solution: Since the gradient of ϕ is normal to the surface ϕ = constant, therefore, the unit vector normal to the surface at a point $P(x, y, z)$ is $\nabla\phi/|\nabla\phi|$. We have,

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^3 + y^3 + 3xyz)$$

$$= (3x^2 + 3yz)\hat{i} + (3y^2 + 3xz)\hat{j} + 3xy\hat{k} = 3(x^2 + yz)\hat{i} + 3(y^2 + xz)\hat{j} + 3xy\hat{k}.$$

Thus $\nabla\phi$ at $(1, 2, -1)$ is $-3\hat{i} + 9\hat{j} + 6\hat{k}$; also $|\nabla\phi|$ at $(1, 2, -1)$ is $3\sqrt{14}$.

Thus the unit vector normal to the given surface at $(1, 2, -1)$ is

$$\frac{\nabla\phi}{|\nabla\phi|} = \frac{-3\hat{i} + 9\hat{j} + 6\hat{k}}{3\sqrt{14}} = \frac{1}{\sqrt{14}}(-\hat{i} + 3\hat{j} + 2\hat{k}).$$

Example 8.11: Find the directional derivative of $\phi(x, y, z) = xy^2 + yz^3$ at the point $(2, -1, 1)$ in the direction of the vector $\hat{i} + 2\hat{j} + 2\hat{k}$.

Solution: We have, $\nabla\phi = \hat{i}\frac{\partial\phi}{\partial z} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} = y^2\hat{i} + (2xy + z^3)\hat{j} + 3yz^2\hat{k}$

Thus, $\nabla\phi$ at $(2, -1, 1) = \hat{i} - 3\hat{j} - 3\hat{k}$.

The directional derivative of ϕ is the component of $\nabla\phi$ at the given point in the direction of the given vector $\hat{i} + 2\hat{j} + 2\hat{k}$. Thus it is equal to

$$(\hat{i} - 3\hat{j} - 3\hat{k})\cdot\frac{i + 2\hat{j} + 2\hat{k}}{\sqrt{1+4+4}} = \frac{1-6-6}{3} = -\frac{11}{3}.$$

Example 8.12: Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$, and $z = x^2 + y^2 - 3$ at $(2, -1, 2)$.

Solution: Let $\phi(x, y, z) = x^2 + y^2 + z^2$ and $\psi(x, y, z) = x^2 + y^2 - z$

Therefore, $\nabla\phi = 2(x\hat{i} + y\hat{j} + z\hat{k})$, and $\nabla\psi = 2x\hat{i} + 2y\hat{j} - \hat{k}$. Thus

$$\nabla\phi \text{ at } (2, -1, 2) = 4\hat{i} - 2\hat{j} + 4\hat{k}, \text{ and } \nabla\psi \text{ at } (2, -1, 2) = 4\hat{i} - 2\hat{j} - \hat{k}.$$

Also, $|\nabla\phi|$ at $(2, -1, 2) = \sqrt{16+4+16} = 6$, and $|\nabla\psi|$ at $(2, -1, 2) = \sqrt{16+4+1} = \sqrt{21}$

Let $\hat{N}$ and $\hat{N}'$ be the unit vectors normal to the surfaces $\phi(x, y, z) = c_1$ and $\psi(x, y, z) = c_2$, respectively at $(2, -1, 2)$. Then,

$$\hat{N} = \frac{4\hat{i} - 2\hat{j} + 4\hat{k}}{6} = \frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}, \text{ and } \hat{N}' = \frac{4\hat{i} - 2\hat{j} - \hat{k}}{\sqrt{21}} = \frac{4}{\sqrt{21}}\hat{i} - \frac{2}{\sqrt{21}}\hat{j} - \frac{1}{\sqrt{21}}\hat{k}.$$

If θ is the angle between the two surfaces $\phi(x, y, z) = c_1$ and $\psi(x, y, z) = c_2$ at $(2, -1, 2)$, then

$$\cos\theta = \hat{N}\cdot\hat{N}' = \left(\frac{2}{3}\right)\left(\frac{4}{\sqrt{21}}\right) + \left(\frac{-1}{3}\right)\left(\frac{-2}{\sqrt{21}}\right) + \left(\frac{2}{3}\right)\left(\frac{-1}{\sqrt{21}}\right) = \frac{8+2-2}{3\sqrt{21}} = \frac{8}{3\sqrt{21}}.$$

Therefore, $\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$.

Imp.

Example 8.13: If $r^2 = x^2 + y^2 + z^2$, then show that $\nabla\phi(r) = \frac{\phi'(r)}{r}\vec{r}$ and hence, or otherwise, prove that

(a) $\nabla(r) = \hat{r}$

(b) $\nabla\left(\frac{1}{r}\right) = -\frac{1}{r^3}\vec{r}$

(c) $\nabla(r^n) = n\, r^{n-2}\vec{r}$

(d) $\nabla\left(\int r^n\, dr\right) = r^{n-1}\vec{r}$. Here $r = |\vec{r}|$.

Solution: By definition

$$\nabla\phi(r) = \hat{i}\,\frac{\partial}{\partial x}\phi(r) + \hat{j}\,\frac{\partial}{\partial y}\phi(r) + \hat{k}\,\frac{\partial}{\partial z}\phi(r)$$

$$= \hat{i}\,\phi'(r)\frac{\partial r}{\partial x} + \hat{j}\,\phi'(r)\frac{\partial r}{\partial y} + \hat{k}\,\phi'(r)\frac{\partial r}{\partial z} = \phi'(r)\left(\hat{i}\,\frac{\partial r}{\partial x} + \hat{j}\,\frac{\partial r}{\partial y} + \hat{k}\,\frac{\partial r}{\partial z}\right)$$

From $r^2 = x^2 + y^2 + z^2$, we have, $2r\frac{\partial r}{\partial x} = 2x$, or $\frac{\partial r}{\partial x} = \frac{x}{r}$, etc.

Hence, $\nabla\phi(r) = \phi'(r)\left[\hat{i}\,\frac{x}{r} + \hat{j}\,\frac{y}{r} + \hat{k}\,\frac{z}{r}\right] = \frac{1}{r}\,\phi'(r)\vec{r}$.

(a) $\nabla(r) = \hat{i}\,\frac{\partial r}{\partial x} + \hat{j}\,\frac{\partial r}{\partial y} + \hat{k}\,\frac{\partial r}{\partial z} = \hat{i}\,\frac{x}{r} + \hat{j}\,\frac{y}{r} + \hat{k}\,\frac{z}{r} = \frac{\vec{r}}{r} = \hat{r}$.

(b) $\nabla\left(\frac{1}{r}\right) = \hat{i}\,\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \hat{j}\,\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \hat{k}\,\frac{\partial}{\partial z}\left(\frac{1}{r}\right) = \hat{i}\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial x} + \hat{j}\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial y} + \hat{k}\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial z}$

$$= -\frac{1}{r^2}\left[\hat{i}\,\frac{x}{r} + \hat{j}\,\frac{y}{r} + \hat{k}\,\frac{z}{r}\right] = -\frac{1}{r^3}\vec{r}.$$

(c) Take $\phi(r) = r^n$, then $\nabla\phi(r) = \frac{1}{r}\,\phi'(r)\vec{r}$, gives $\nabla(r^n) = \frac{1}{r}\,nr^{n-1}\vec{r} = nr^{n-2}\vec{r}$.

(d) Take $\phi(r) = \int r^n\, dr$, so that $\phi'(r) = r^n$, then

$$\nabla\phi(r) = \frac{1}{r}\,\phi'(r)\vec{r}, \text{ gives } \nabla\left(\int r^n dr\right) = \frac{1}{r}\,r^n\vec{r} = r^{n-1}\vec{r}.$$

Example 8.14: If $u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = xy + yz + zx$, show that ∇u, ∇v, ∇w are coplanar vectors.

Solution: By definition

$$\nabla u = \left(\hat{i}\,\frac{\partial u}{\partial x} + \hat{j}\,\frac{\partial u}{\partial y} + \hat{k}\,\frac{\partial u}{\partial z}\right) = \hat{i} + \hat{j} + \hat{k}$$

$$\nabla v = \left(\hat{i}\,\frac{\partial v}{\partial x} + \hat{j}\,\frac{\partial v}{\partial y} + \hat{k}\,\frac{\partial v}{\partial z}\right) = 2(x\hat{i} + y\hat{j} + z\hat{k})$$

$$\nabla w = \left(\hat{i}\,\frac{\partial w}{\partial x} + \hat{j}\,\frac{\partial w}{\partial y} + \hat{k}\,\frac{\partial w}{\partial z}\right) = \hat{i}\,(y+z) + \hat{j}\,(z+x) + \hat{k}\,(x+y)$$

The vectors ∇u, ∇v, ∇w, will be coplanar if their scalar triple product is zero, that is, if $\nabla u \cdot (\nabla v \times \nabla w) = 0$, or if

$$\begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & z+x & x+y \end{vmatrix} = 0 \quad \text{or, if} \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ x+y+z & x+y+z & x+y+z \end{vmatrix} = 0, \; [R_3 \to R_3 + \frac{1}{2} R_2]$$

$$\text{or, if } (x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix} = 0,$$

since the first and third rows in the determinant are the same. Thus, ∇u, ∇v, ∇w are coplanar.

Example 8.15: The temperature of a point in space is given by $T(x, y, z) = x^2 + y^2 - z$. A mosquito located at (1, 1, 2) desires to fly in such a direction that it will get warm as soon as possible. In what direction should it fly?

Solution: If $T = c$ is the level surface, then ∇T gives the direction of maximum rate of change we have $\nabla T = 2x\hat{i} + 2y\hat{j} - \hat{k}$, and ∇T at $(1, 1, 2) = 2\hat{i} + 2\hat{j} - \hat{k}$

It should move in the direction of the unit vector normal along ∇T, that is, along $(1/3)(2\hat{i} + 2\hat{j} - \hat{k})$.

Imp

Example 8.16: If f and $\vec{g}$ are respectively the scalar and vector point functions, prove that the components of the latter, normal and tangential to the surface $f = 0$, are

$$\frac{(\vec{g}\cdot\nabla f)\,\nabla f}{|\nabla f|^2} \quad \text{and} \quad \frac{\nabla f \times (\vec{g} \times \nabla f)}{|\nabla f|^2}.$$

Solution: We know that ∇f is normal to the level surface $f = 0$. Thus we are to find the components of $\vec{g}$ along and normal to ∇f.

Let O be the point of reference and F and G be two points such that $\nabla f = \overrightarrow{OF}$ and $\vec{g} = \overrightarrow{OG}$ as shown in Fig. 8.7, and let OM be the projection of $\vec{g}$ along ∇f.

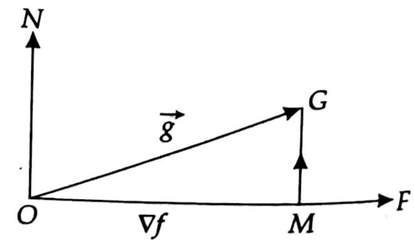

Fig. 8.7

The component of $\vec{g}$ along ∇f is = OM times the unit vector along $\nabla f = (\vec{g}\cdot\hat{\nabla}f)\,\hat{\nabla}f = \dfrac{(\vec{g}\cdot\nabla f)\,\nabla f}{|\nabla f|^2}$.

The component of $\vec{g}$ normal to ∇f is = $\overrightarrow{MG} = \overrightarrow{OG} - \overrightarrow{OM}$

$$= \vec{g} - \frac{(\vec{g}\cdot\nabla f)\,\nabla f}{|\nabla f|^2} = \frac{(\nabla f\cdot\nabla f)\vec{g} - (\vec{g}\cdot\nabla f)\,\nabla f}{|\nabla f|^2} = \frac{(\nabla f\times\vec{g})\times\nabla f}{|\nabla f|^2}.$$

Imp

Example 8.17: Determine the constant a such that at any point of intersection of the two spheres $(x-a)^2+y^2+z^2=3$, and $x^2+(y-1)^2+z^2=1$ their tangent planes are perpendicular to each other.

Solution: Let $\phi_1 = (x-a)^2+y^2+z^2$ and $\phi_2 = x^2+(y-1)^2+z^2$. We have

$$\nabla\phi_1 = \hat{i}\,\frac{\partial\phi_1}{\partial x} + \hat{j}\,\frac{\partial\phi_1}{\partial y} + \hat{k}\,\frac{\partial\phi_1}{\partial z} = 2(x-a)\,\hat{i} + 2y\,\hat{j} + 2z\,\hat{k}$$

$$\nabla\phi_2 = \hat{i}\,\frac{\partial\phi_2}{\partial x} + \hat{j}\,\frac{\partial\phi_2}{\partial y} + \hat{k}\,\frac{\partial\phi_2}{\partial z} = 2x\,\hat{i} + 2(y-1)\,\hat{j} + 2z\,\hat{k}$$

The vectors $\nabla\phi_1$ and $\nabla\phi_2$ are along the normals to the two spheres at a point (x, y, z) of their intersection. The tangent planes to the two spheres at a point of intersection will be perpendicular when their normals are perpendicular to each other and for that

$$[2(x-a)\,\hat{i} + 2y\,\hat{j} + 2z\,\hat{k}]\cdot[2x\,\hat{i} + 2(y-1) + 2z\,\hat{k}] = 0$$

or, $4x(x-a) + 4y(y-1) + 4z^2 = 0$

or, $$x^2+y^2+z^2-ax-y=0. \qquad ...(8.24)$$

Also at any point $P(x, y, z)$ of intersection of the given spheres, we have

$$x^2+y^2+z^2-2ax+a^2-3=0 \text{ and } x^2+y^2+z^2-2y=0.$$

Adding these two and dividing by 2, we obtain

$$x^2+y^2+z^2-ax-y+\frac{1}{2}a^2-\frac{3}{2}=0. \qquad ...(8.25)$$

Imp step.

From (8.24) and (8.25), we obtain $\dfrac{a^2}{2} - \dfrac{3}{2} = 0$, or $a = \sqrt{3}$.

Imp

Example 8.18: Show that the vector field defined by $\vec{F} = xyz(yz\,\hat{i} + xz\hat{j} + xy\,\hat{k})$ is conservative.

Solution: If the vector field defined by the given vector function $\vec{F}$ is conservative, then there exists some scalar function f such that $\vec{F} = \nabla f$. It gives

$$xyz(yz\,\hat{i} + xz\,\hat{j} + xy\,\hat{k}) = \hat{i}\,\frac{\partial f}{\partial x} + \hat{j}\,\frac{\partial f}{\partial y} + \hat{k}\,\frac{\partial f}{\partial z}$$

This implies $\dfrac{\partial f}{\partial x} = xy^2z^2$, $\dfrac{\partial f}{\partial y} = yx^2z^2$, $\dfrac{\partial f}{\partial z} = zx^2y^2$. Integrating $\dfrac{\partial f}{\partial x} = xy^2z^2$ w.r.t. x, we get

$$f(x, y, z) = \frac{1}{2}x^2y^2z^2 + g(y, z) \qquad ...(8.26)$$

Substituting for f in $\frac{\partial f}{\partial y} = yx^2z^2$, we get $x^2yz^2 + \frac{\partial g}{\partial y} = x^2yz^2$, or $\frac{\partial g}{\partial y} = 0$, that is, $g = g(z)$.

Therefore, from (8.26), we have

$$f(x, y, z) = \frac{1}{2}x^2y^2z^2 + g(z) \qquad ...(8.27)$$

Substituting for f in $\frac{\partial f}{\partial z} = zx^2y^2$, we get $x^2y^2z + \frac{\partial g}{\partial z} = zx^2y^2$, or $\frac{\partial g}{\partial z} = 0$, that is, $g = c$, a constant.

Hence from (8.27), $f(x, y, z) = \frac{1}{2}x^2y^2z^2 + c$

Thus there exists a scalar function $f(x, y, z)$ such that $\vec{F} = \nabla f$.

EXERCISE 8.2

1. Find the gradient of the following scalar fields.

   (a) $\phi(x, y) = y^2 - 4xy$ at $(1, 2)$ (b) $\phi(x, y, z) = x^2y^2 + xy^2 - z^2$ at $(3, 1, 1)$

2. Find a unit vector normal to the surface

   (a) $xy^3z = 2$ at $(-1, -1, 2)$ (b) $x^2y + 2xz = 4$ at $(2, -2, 3)$.

3. Show that the equation of the tangent plane to the surface $z = \sqrt{x^2 + y^2}$ at the point $(3, 4, 5)$ is $3x + 4y - 5z = 0$. $\vec{r} \cdot \hat{m} = 0$ $\vec{r} = (a - x)\hat{i} + (b - y)\hat{j}$ ...

4. Show that the angle of intersection of the spheres $x^2 + y^2 + z^2 = 29$ and $x^2 + y^2 + z^2 + 4x - 6y - 8z = 47$ at $(4, -3, 2)$ is $\theta = \cos^{-1}(19/29)$.

5. Show that the angle between the tangent planes to the surface $x \ln z = y^2 - 1$, $x^2y = 2 - z$ at the point $(1, 1, 1)$ is $\cos^{-1}(-1/\sqrt{30})$.

6. Find the directional derivative of the function

   (a) $f(x, y, z) = xy^2 + yz^3$ at the point $(2, -1, 1)$ in the direction of the vector $\hat{i} + 2\hat{j} + 2\hat{k}$.

   (b) $f(x, y, z) = 4xz^3 - 3x^2yz^2$ at the point $(2, -1, 2)$ along z-axis.

   (c) $f(x, y, z) = xy^2 + yz^3$ at the point $(2, -1, 1)$ in the direction of the normal to surface $x \ln z - y^2 + 4 = 0$ at $(-1, 2, 1)$.

   (d) $f(x, y) = x^2y^3 + xy$ at $(2, 1)$ in the direction of a unit vector which makes an angle of $\pi/3$ with x-axis.

7. In what direction from $(3, 1, -2)$ is the directional derivative of $\phi = x^2y^2z^4$ maximum and what is its magnitude?

8. Find the constants λ and μ so that the surfaces $\lambda x^2 - \mu yz = (\lambda + 2)x$ and $4x^2y + z^3 = 4$ may intersect orthogonally at the point $(1, -1, 2)$.

9. If $u = u(x, y, z, t)$, $x = x(t)$, $y = y(t)$, $z = z(t)$, show that

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \left(\frac{d\vec{r}}{dt}\cdot\nabla\right)u, \text{ where } \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}.$$

10. If $\vec{r}$ is the position vector of the point (x, y, z) and $\vec{a}$ and $\vec{b}$ are constant vectors, prove that

$$\vec{a}\cdot\nabla\left\{\vec{b}\cdot\nabla\left(\frac{1}{r}\right)\right\} = \frac{3(\vec{a}\cdot\vec{r})(\vec{b}\cdot\vec{r})}{r^5} - \frac{\vec{a}\cdot\vec{b}}{r^3}.$$

8.4 DIVERGENCE AND CURL OF A VECTOR FIELD

The concepts of divergence and curl of a vector fields, like that of gradient, are of wide applications in engineering and physics. We introduce these two in this section.

8.4.1 Divergence of a Vector Field

Let $\vec{F}(x, y, z) = F_1(x, y, z)\hat{i} + F_2(x, y, z)\hat{j} + F_3(x, y, z)\hat{k}$ be a differentiable vector point function. Then *the divergence of the vector field $\vec{F}$, denoted by div $\vec{F}$, is defined as*

$$\text{div}\,\vec{F} = \nabla\cdot\vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{z}\right)\cdot(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

We note that $\nabla\cdot\vec{F}$ is simply a notation and not a scalar product in the usual sense, since $\nabla\cdot\vec{F} \neq \vec{F}\cdot\nabla$. In fact, $\vec{F}\cdot\nabla = F_1\frac{\partial}{\partial x} + F_2\frac{\partial}{\partial y} + F_3\frac{\partial}{\partial z}$ is a scalar operator while $\nabla\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$ is a scalar. For example,

$$\nabla\cdot\vec{r} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot(x\hat{i} + y\hat{j} + z\hat{k}) = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3.$$

As another example, if $\vec{F} = 3xz^2\hat{i} + 2xy\hat{j} - y^2z^2\hat{k}$, then div $\vec{F} = \nabla\cdot\vec{F} = 3z^2 + 2x - 2y^2z$.

8.4.2 Physical Meaning of the Divergence

To give the physical interpretation to the divergence, consider the motion of a fluid in a region R having no source or sink in R, that is, there is no point in R at which the fluid is produced or disappears.

Let $\vec{v} = v_x(x, y, z)\hat{i} + v_y(x, y, z)\hat{j} + v_z(x, y, z)\hat{k}$ be the velocity of the fluid at a point $P(x, y, z)$. Consider a rectangular parallelopiped of sides δx, δy, δz in the fluid as shown in Fig. 8.8. Consider that the fluid is flowing in the positive direction of the y-axis.

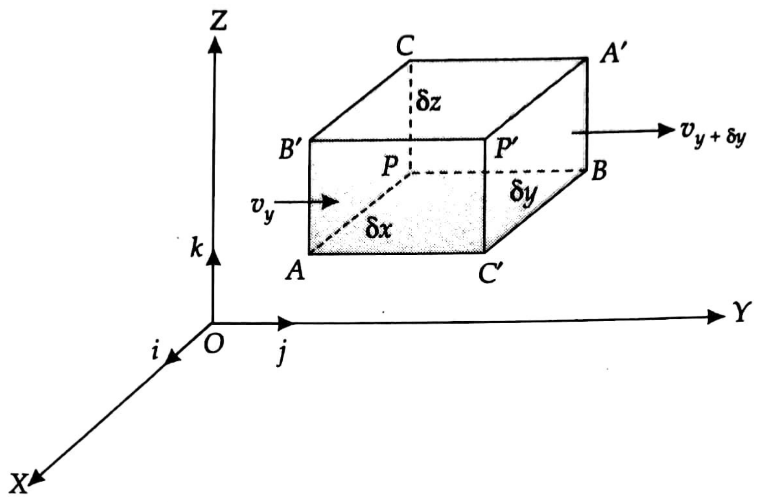

Fig. 8.8

The amount of fluid entering the face $PAB'C$ in a unit time $= v_y\, \delta z\, \delta x$.

The amount of fluid leaving the face $P'C'BA'$ in a unit time $= v_{y+\delta y}\, \delta z\, \delta x$.

Therefore the resultant fluid flow out of these two parallel faces is

$$= (v_{y+\delta y} - v_y)\, \delta z\, \delta x \approx \frac{\partial v_y}{\partial y}\, \delta x\, \delta y\, \delta z,$$

using Taylor series and neglecting the terms of second and higher orders

This is called the *flux* of the vector field $\vec{v}$ through the area $\delta z\, \delta x$.

Accounting for the resultant fluid flows across the other two pairs of faces, the total flux of $\vec{v}$ through the six faces is

$$\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) \delta x\, \delta y\, \delta z.$$

Dividing by the volume $\delta x\, \delta y\, \delta z$, the flux per unit volume is given by

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

which is equal to div $\vec{v}$. *Hence divergence gives a measure of the outward flux per unit volume of the flow at a point (x, y, z).*

Similarly, if $\vec{v}$ represents an electric flux, then div $\vec{v}$ is the amount of electric flux which diverges per unit volume in unit time. If $\vec{v}$ represents the heat flux, then div $\vec{v}$ is the rate at which heat is issuing from a point per unit volume. In general the divergence of a vector point function, representing any physical quantity gives at each point the rate per unit volume at which the physical quantity is issuing from that point. This justifies the name divergence of a vector point function.

If the fluid is incompressible, then the balance of outflow and inflow for a given volume element is zero at any time and hence, div $\vec{v} = 0$. This equation is known as the *condition of incompressibility*. Clearly the assumption that the flow has no source or sink in the region is essential for this argument.

If for a vector point function $\vec{v}$, div $\vec{v} = 0$ everywhere, then such a point function $\vec{v}$ is called a solenoidal vector function and the field represented by such a vector function is called the solenoidal field.

8.4.3 Curl of a Vector Field

The curl of a vector $\vec{F}$, denoted by curl $\vec{F}$, is defined as

$$curl\ \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & i & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}.$$

We note that $\nabla \times \vec{F}$ is simply a notation and is not a vector product in the usual sense, since $\boxed{\nabla \times \vec{F} \neq -\vec{F} \times \nabla.}$ Sometimes, curl $\vec{F}$ is also written as

$$\text{curl } \vec{F} = \sum \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i},$$

where summation is over the cyclic rotation of the unit vectors $\hat{i}$, $\hat{j}$, $\hat{k}$, the components F_1, F_2, F_3 and the independent variables x, y, z. For example, if

$$\vec{F} = yz\hat{i} + 3zx\hat{j} + z\hat{k}, \text{ then curl } \vec{F} = \begin{vmatrix} i & i & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & 3zx & z \end{vmatrix} = -3x\hat{i} + y\hat{j} + 2z\hat{k}.$$

Next we give a physical interpretation to curl of a vector function.

8.4.4. Physical Interpretation of the Curl

Suppose a rigid body rotates about a fixed axis through the origin O with a uniform angular velocity $\vec{w} = w_1\hat{i} + w_2\hat{j} + w_3\hat{k}$ and let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of any point $\vec{P}(x, y, z)$ on the body.

The linear velocity $\vec{v}$ of the point $P(x, y, z)$ is given by

$$\vec{v} = \vec{w} \times \vec{r} = \begin{vmatrix} i & j & k \\ w_1 & w_2 & w_3 \\ x & y & z \end{vmatrix} = (w_2 z - w_3 y)\hat{i} + (w_3 x - w_1 z)\hat{j} + (w_1 y - w_2 x)\hat{j}.$$

Therefore $\text{curl } \vec{v} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ w_2z - w_3y & w_3x - w_1z & w_1y - w_2x \end{vmatrix}$

$$= (w_1 + w_1)\hat{i} + (w_2 + w_2)\hat{j} + (w_3 + w_3)\hat{k} = 2(w_1\hat{i} + w_2\hat{j} + w_3\hat{k}) = 2w,$$

or, $$\vec{w} = \frac{1}{2}\text{curl } \vec{v}$$

Hence, the angular velocity of a uniformly rotating body is equal to one- half of the curl of the linear velocity.

Because of this interpretation, sometimes the word *rotation* is also used in place of curl. In fluid mechanics, if $\vec{v}$ is the velocity of a fluid, and curl $\vec{v} = \vec{0}$, then $\vec{v}$ is said to be *irrotational field* and the corresponding motion is said to be *irrotational* otherwise, *rotational.*

Example 8.19: Find div $\vec{F}$ and curl $\vec{F}$, when $\vec{F} = \text{grad } (x^3 + y^3 + z^3 - 3xyz)$.

Solution: Take $u = x^3 + y^3 + z^3 - 3xyz$. Then

$$\vec{F} = \text{grad } u = \hat{i}\frac{\partial u}{\partial x} + \hat{j}\frac{\partial u}{\partial y} + \hat{k}\frac{\partial u}{\partial z} = (3x^2 - 3yz)\hat{i} + (3y^2 - 3xz)\hat{j} + (3z^2 - 3xy)\hat{k}.$$

$$\text{div } \vec{F} = \nabla\cdot[(3x^2 - 3yz)\hat{i} + (3y^2 - 3xz)\hat{j} + (3z^2 - 3xy)\hat{k}]$$

$$= \frac{\partial}{\partial x}(3x^2 - 3yz) + \frac{\partial}{\partial y}(3y^2 - 3xz) + \frac{\partial}{\partial z}(3z^2 - 3xy) = 6x + 6y + 6z = 6(x + y + z).$$

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3(x^2 - yz) & 3(y^2 - xz) & 3(z^2 - xy) \end{vmatrix}$$

$$= \hat{i}\left[3\frac{\partial}{\partial y}(z^2 - xy) - 3\frac{\partial}{\partial z}(y^2 - xz)\right] + \hat{j}\left[3\frac{\partial}{\partial z}(x^2 - yz) - 3\frac{\partial}{\partial x}(z^2 - yx)\right]$$
$$+ \hat{k}\left[3\frac{\partial}{\partial x}(y^2 - zx) - 3\frac{\partial}{\partial y}(x^2 - zy)\right]$$

$$= \hat{i}(-3x + 3x) + \hat{j}(-3y + 3y) + \hat{k}(-3z + 3z) = \vec{0}.$$

Example 8.20: If $\vec{a}$ is a constant vector and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, show that curl $(\vec{a} \times \vec{r}) = 2\vec{a}$.

Solution: Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, where a_1, a_2, a_3 are constants. Then

$$\vec{a}\times\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = (a_2 z - a_3 y)\hat{i} + (a_3 x - a_1 z)\hat{j} + (a_1 y - a_2 x)\hat{k}$$

$$\text{Thus, curl }(\vec{a}\times\vec{r}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_2 z - a_3 y & a_3 x - a_1 z & a_1 y - a_2 x \end{vmatrix}$$

$$= \hat{i}\,(a_1 + a_1) + \hat{j}\,(a_2 + a_2) + \hat{k}\,(a_3 + a_3) = 2(a_1\hat{i} + a_2 j + a_3 k) = 2\vec{a}.$$

Example 8.21. Show that (a) $\nabla\cdot(r^n\vec{r}) = (n+3)r^n$, (b) $\nabla\cdot(\vec{a}\times\vec{r}) = 0$, where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{a}$ is a constant vector.

Solution: (a) $\nabla\cdot(r^n\vec{r}) = \left(\hat{i}\dfrac{\partial}{\partial x} + \hat{j}\dfrac{\partial}{\partial y} + \hat{k}\dfrac{\partial}{\partial z}\right)\cdot(r^n x\hat{i} + r^n y\hat{j} + r^n z\hat{k})$

$$= \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$$

$$= r^n + xnr^{n-1}\frac{\partial r}{\partial x} + r^n + ynr^{n-1}\frac{\partial r}{\partial y} + r^n + znr^{n-1}\frac{\partial r}{\partial z}$$

$$= 3r^n + nr^{n-1}\left(x\frac{\partial r}{\partial x} + y\frac{\partial r}{\partial y} + z\frac{\partial r}{\partial z}\right) = 3r^n + nr^{n-1}\left(x\frac{x}{r} + y\frac{y}{r} + z\frac{z}{r}\right)$$

$$= 3r^n + nr^n = (n+3)r^n.$$

(b) $$\nabla\cdot(\vec{a}\times\vec{r}) = \sum \hat{i}\,\frac{\partial}{\partial x}\cdot(\vec{a}\times\vec{r}) = \sum \hat{i}\cdot\frac{\partial}{\partial x}(\vec{a}\times\vec{r}) = \sum \hat{i}\cdot\left[\frac{\partial\vec{a}}{\partial x}\times\vec{r} + \vec{a}\times\frac{\partial\vec{r}}{\partial x}\right]$$

$$= 0 + \sum \hat{i}\cdot\left[\vec{a}\times\frac{\partial\vec{r}}{\partial x}\right] = \sum \hat{i}\cdot(\vec{a}\times\hat{i}) = 0.$$

EXERCISE 8.3

In the Problems (1-2), compute div $\vec{F}$, curl $\vec{F}$ and verify that div(curl $\vec{F}$) = 0.

1. $\vec{F} = xe^{-y}\hat{i} + 2ze^{-y}\hat{j} + xy^2\hat{k}$.

2. $\vec{F} = (x^2 - y^2)\hat{i} + 4xy\hat{j} + (x^2 - xy)\hat{k}$.

3. If $\vec{F} = (x + y + 1)\hat{i} + \hat{j} - (x + y)\hat{k}$, show that $\vec{F} \cdot \text{curl } \vec{F} = 0$.

4. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{a}$ is constant vector, show that

   (i) $\vec{r}$ is irrotational curl $\vec{r} = 0$ (ii) curl $(\vec{a} \times \vec{r}) = 2\vec{a}$.

5. Determine the values of m and n so that the vector point function

   $\vec{F} = (xyz)^m (x^n\hat{i} + y^n\hat{j} + z^n\hat{k})$ is irrotational.

6. Show that the vector field defined by $\vec{F} = e^{x+y-2z}(\hat{i} + \hat{j} + \hat{k})$ is solenoidal.

7. If $f = x^2 + y^2 + z^2$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ show that div$(f\vec{r}) = 5f$.

8. Show that the vector field $\vec{F} = 3x^2y^2z^4\hat{i} + 2x^3yz^4\hat{j} + 4x^3y^2z^3\hat{k}$ is irrotational and find a scalar function f such that $\vec{F}$ = grad f.

8.5 SOME VECTOR IDENTITIES

In this section we study two types of vector identities. First type results *when del is applied twice to a point function,* and the second type results *when del is applied to product of two point functions.* These identities contribute further in the development of the subject.

8.5.1 'Del' Applied Twice to a Point Function

Since ∇f and $\nabla \times \vec{F}$ are the vector point functions, so we can form their divergence and curl, while $\nabla \cdot \vec{F}$ is a scalar point function, only gradient can be formulated. This results in the following formulae.

1. div grad $f = \nabla \cdot \nabla f = \nabla^2 f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$

2. curl grad $f = \nabla \times \nabla f = \vec{0}$

3. div curl $\vec{F} = \nabla \cdot \nabla \times \vec{F} = 0$

4. curl curl $\vec{F} = \nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$ = grad div $\vec{F} - \nabla^2 \vec{F}$

5. grad div $\vec{F} = \nabla(\nabla \cdot \vec{F}) = \nabla \times (\nabla \times \vec{F}) + \nabla^2 \vec{F}$ = curl curl $\vec{F} + \nabla^2 \vec{F}$

We note that in **1** the operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is a scalar operator, called the *Laplacian operator*.

Next, we prove these results.

1. $$\nabla^2 f = \nabla \cdot \nabla f = \nabla \cdot \left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z} \right)$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) \cdot \left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z} \right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

2. $$\nabla \times \nabla f = \nabla \times \left(i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} \right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial^2 f}{\partial x\, \partial z} - \frac{\partial^2 f}{\partial y\, \partial z} \right) + \hat{j}\left(\frac{\partial^2 f}{\partial z\, \partial x} - \frac{\partial^2 f}{\partial z\, \partial x} \right) + \hat{k}\left(\frac{\partial^2 f}{\partial x\, \partial y} - \frac{\partial^2 f}{\partial x\, \partial y} \right) = \vec{0}$$

3. $$\nabla \cdot \nabla \times \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) \cdot \left[\hat{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) + \hat{j}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) + k\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \right]$$

$$= \left(\frac{\partial^2 F_3}{\partial x\, \partial y} - \frac{\partial^2 F_2}{\partial x\, \partial z} \right) + \left(\frac{\partial^2 F_1}{\partial y\, \partial z} - \frac{\partial^2 F_3}{\partial y\, \partial x} \right) + \left(\frac{\partial^2 F_2}{\partial z\, \partial x} - \frac{\partial^2 F_1}{\partial z\, \partial y} \right) = 0$$

4. $$\nabla \times (\nabla \times \vec{F}) = \left(\sum i\frac{\partial}{\partial x} \right) \times \left[\sum i\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \right] = \sum i\left[\frac{\partial}{\partial y}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) - \frac{\partial}{\partial z}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \right]$$

$$= \sum i\left[\frac{\partial^2 F_2}{\partial y\, \partial x} - \frac{\partial^2 F_1}{\partial y} - \frac{\partial^2 F_1}{\partial z} + \frac{\partial^2 F_3}{\partial z\, \partial x} \right] = \sum i\left[\left(\frac{\partial^2 F_2}{\partial y\, \partial x} + \frac{\partial^2 F_3}{\partial z\, \partial x} \right) - \left(\frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2} \right) \right]$$

$$= \sum i\left[\frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) - \left(\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2} \right) \right]$$

$$= \left(\sum i\frac{\partial}{\partial x} \right)(\nabla \cdot \vec{F}) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right)\left(\sum i\, F_1 \right) = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$$

Thus, $\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$

Also this implies $\nabla(\nabla \cdot \vec{F}) = \nabla \times (\nabla \times \vec{F}) + \nabla^2 \vec{F}$, which is **5.**

Remark. The interpretation of ∇ as a vector operator gives the justification of the results and also helps to remember the above formulae as follows.

1. $\nabla \cdot \nabla f = \nabla^2 f;$ here $\nabla \cdot \nabla = \nabla^2$

2. $\nabla \times \nabla f = \vec{0};$ here $\nabla \times \nabla = \vec{0}$

3. $\nabla \cdot \nabla \times \vec{F} = 0;$ here$[\nabla, \nabla, \vec{F}] = 0$, being the scalar tripple product with two vectors equal.

4. & 5. $\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla \cdot \nabla \vec{F} = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$; here expanding $\nabla \times (\nabla \times \vec{F})$ as a vector tripple product.

In continuation to the above noted results, next we consider the operation of ∇ to the product of two point functions.

8.5.2 'Del' Applied to the Product of Two Point Functions

Consider two scalar point functions f and g and two vector point functions $\vec{F}$ and $\vec{G}$. The possible forms of the product are: fg, $\vec{F} \cdot \vec{G}$, the scalar products, and $f\vec{G}$, $\vec{F} \times \vec{G}$, the vector products. When del is applied to these products, we arrive at the following formulae:

6. $\nabla(fg) = f\nabla g + g\nabla f$

7. $\nabla \cdot (f\vec{G}) = \nabla f \cdot \vec{G} + f\nabla \cdot \vec{G}$

8. $\nabla \times (f\vec{G}) = \nabla f \times \vec{G} + f\nabla \times \vec{G}$

9. $\nabla(\vec{F} \cdot \vec{G}) = (F \cdot \nabla)\vec{G} + (G \cdot \nabla)\vec{F} + \vec{F} \times (\nabla \times \vec{G}) + \vec{G} \times (\nabla \times \vec{F})$

10. $\nabla \cdot (\vec{F} \times \vec{G}) = \vec{G} \cdot (\nabla \times \vec{F}) - \vec{F} \cdot (\nabla \times \vec{G})$

*11. $\nabla \times (\vec{F} \times \vec{G}) = \vec{F}(\nabla \cdot \vec{G}) - \vec{G}(\nabla \cdot \vec{F}) + (\vec{G} \cdot \nabla)\vec{F} - (\vec{F} \cdot \nabla)\vec{G}$

We note that in result **9,** the term

$$(\vec{F} \cdot \nabla)\vec{G} = \left(\hat{i}\,\frac{\partial f}{\partial x} + \hat{j}\,\frac{\partial f}{\partial y} + \hat{k}\,\frac{\partial f}{\partial z}\right) = F_1 \frac{\partial \vec{G}}{\partial x} + F_2 \frac{\partial \vec{G}}{\partial y} + F_3 \frac{\partial \vec{G}}{\partial z}$$

where $\vec{G} = G_1\hat{i} + G_2\hat{j} + G_3\hat{k}$; and in result **11,** the term

$$\vec{F}(\nabla \cdot \vec{G}) = \vec{F}\frac{\partial G_1}{\partial x} + \vec{F}\frac{\partial G_2}{\partial y} + \vec{F}\frac{\partial G_3}{\partial z}, \text{ where } \vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}.$$

These results can also be proved using the results on vector differentiation. The result **6** is quite obvious one.

7. $$\nabla\cdot(f\vec{G}) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot(\hat{i}\,fG_1+\hat{j}\,fG_2+\hat{k}\,fG_3) = \frac{\partial}{\partial x}(fG_1)+\frac{\partial}{\partial y}(fG_2)+\frac{\partial}{\partial z}(fG_3)$$

$$= f\left(\frac{\partial G_1}{\partial x}+\frac{\partial G_2}{\partial y}+\frac{\partial G_3}{\partial z}\right)+\frac{\partial f}{\partial x}G_1+\frac{\partial f}{\partial y}G_2+\frac{\partial f}{\partial z}G_3 = f(\nabla\cdot\vec{G})+(\nabla f)\cdot\vec{G}.$$

8. $$\nabla\times(f\vec{G}) = \left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\times(f\vec{G}) = \sum\vec{i}\times\frac{\partial}{\partial x}(f\vec{G})$$

$$= \sum\vec{i}\times\left\{\frac{\partial f}{\partial x}\vec{G}+f\frac{\partial\vec{G}}{\partial x}\right\} = \sum\frac{\partial f}{\partial x}\vec{i}\times\vec{G}+f\sum\vec{i}\times\frac{\partial\vec{G}}{\partial x} = \nabla f\times\vec{G}+f(\nabla\times\vec{G}).$$

9. $$\nabla(\vec{F}\cdot\vec{G}) = \sum\hat{i}\frac{\partial}{\partial x}(\vec{F}\cdot\vec{G}) = \sum\hat{i}\left[\vec{F}\cdot\frac{\partial\vec{G}}{\partial x}+\frac{\partial\vec{F}}{\partial x}\cdot\vec{G}\right]$$

$$= \sum\left(\vec{F}\cdot\frac{\partial\vec{G}}{\partial x}\right)\hat{i}+\sum\left(\frac{\partial\vec{F}}{\partial x}\cdot\vec{G}\right)i. \qquad \text{...(8.28)}$$

Also, $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$, therefore, $(\vec{a}\cdot\vec{b})\vec{c} = (\vec{a}\cdot\vec{c})\vec{b}-\vec{a}\times(\vec{b}\times\vec{c})$

Thus, $$\left(\vec{F}\cdot\frac{\partial\vec{G}}{\partial x}\right)\hat{i} = (\vec{F}\cdot\hat{i})\frac{\partial\vec{G}}{\partial x}-\vec{F}\times\left(\frac{\partial\vec{G}}{\partial x}\times i\right) = (\vec{F}\cdot\hat{i})\frac{\partial\vec{G}}{\partial x}+\vec{F}\times\left(i\times\frac{\partial\vec{G}}{\partial x}\right)$$

and therefore, $$\sum\left(\vec{F}\cdot\frac{\partial\vec{G}}{\partial x}\right)\hat{i} = \left(\vec{F}\cdot\sum i\frac{\partial}{\partial x}\right)\vec{G}+\vec{F}\times\sum\left(i\times\frac{\partial\vec{G}}{\partial x}\right) = (\vec{F}\cdot\nabla)\vec{G}+\vec{F}\times(\nabla\times\vec{G})$$

Interchanging $\vec{F}$ and $\vec{G}$, we have

$$\sum\left(\vec{G}\cdot\frac{\partial\vec{F}}{\partial x}\right)\hat{i} = (\vec{G}\cdot\nabla)\vec{F}+\vec{G}\times(\nabla\times\vec{F})$$

Substituting the values of $\sum\left(\vec{F}\cdot\frac{\partial\vec{G}}{\partial x}\right)\hat{i}$ and $\sum\left(\vec{G}\cdot\frac{\partial\vec{F}}{\partial x}\right)\hat{i}$ in (8.28), we get the desired result.

10. $$\nabla\cdot(\vec{F}\times\vec{G}) = \sum\vec{i}\cdot\frac{\partial}{\partial x}(\vec{F}\times\vec{G}) = \sum\vec{i}\cdot\left(\frac{\partial\vec{F}}{\partial x}\times\vec{G}+\vec{F}\times\frac{\partial\vec{G}}{\partial x}\right)$$

$$= \sum\vec{i}\cdot\left(\frac{\partial\vec{F}}{\partial x}\times\vec{G}\right)-\sum i\left(\frac{\partial\vec{G}}{\partial x}\times\vec{F}\right) = \sum\left(\vec{i}\times\frac{\partial\vec{F}}{\partial x}\right)\cdot\vec{G}-\sum\left(i\times\frac{\partial\vec{G}}{\partial x}\right)\cdot\vec{F}$$

$$= (\nabla\times\vec{F})\cdot\vec{G}-(\nabla\times\vec{G}).\vec{F} = \vec{G}.(\nabla\times\vec{F})-\vec{F}\cdot(\nabla\times\vec{G}).$$

11. $\nabla \times (\vec{F} \times \vec{G}) = \sum \vec{i} \times \left[\frac{\partial}{\partial x}(\vec{F} \times \vec{G})\right] = \sum \vec{i} \times \left[\frac{\partial \vec{F}}{\partial x} \times \vec{G} + \vec{F} \times \frac{\partial \vec{G}}{\partial x}\right]$

$$= \sum \vec{i} \times \left(\frac{\partial \vec{F}}{\partial x} \times \vec{G}\right) + \sum \vec{i} \times \left(\vec{F} \times \frac{\partial \vec{G}}{\partial x}\right). \quad \text{...(8.29)}$$

Now $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$. Using this, we have

$$\sum \vec{i} \times \left(\frac{\partial \vec{F}}{\partial x} \times \vec{G}\right) = \sum (\vec{i} \cdot \vec{G}) \frac{\partial \vec{F}}{\partial x} - \sum \left(\vec{i} \cdot \frac{\partial \vec{F}}{\partial x}\right)\vec{G} = \sum (\vec{G} \cdot \vec{i}) \frac{\partial \vec{F}}{\partial x} - \sum \left(\vec{i} \cdot \frac{\partial \vec{F}}{\partial x}\right)\vec{G}$$

$$= \vec{G} \cdot \sum \left(i \frac{\partial}{\partial x}\right)\vec{F} - \left(\sum \vec{i} \cdot \frac{\partial \vec{F}}{\partial x}\right)\vec{G} = (\vec{G} \cdot \nabla)\vec{F} - (\nabla \cdot \vec{F})\vec{G}.$$

Similarly,

$$\sum \vec{i} \times \left(\vec{F} \times \frac{\partial \vec{G}}{\partial x}\right) = \sum \left(\vec{i} \cdot \frac{\partial \vec{G}}{\partial x}\right)\vec{F} - \sum (\vec{i} \cdot \vec{F}) \frac{\partial \vec{G}}{\partial x} = (\nabla \cdot \vec{G})\vec{F} - \left(\vec{F} \cdot \sum i \frac{\partial}{\partial x}\right)\vec{G} = (\nabla \cdot \vec{G})\vec{F} - (\vec{F} \cdot \nabla)\vec{G}.$$

Thus from (8.29), we have

$\nabla \times (\vec{F} \times \vec{G}) = (\vec{G} \cdot \nabla)\vec{F} - (\nabla \cdot \vec{F})\vec{G} + (\nabla \cdot \vec{G})\vec{F} - (\vec{F} \cdot \nabla)\vec{G}$, which is **11.**

Example 8.22: Prove that (a) $\nabla^2(r^n) = n(n+1)r^{n-2}$ (b) $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$.

Solution: (a) $\nabla^2 r^n = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) r^n = \sum \frac{\partial^2}{\partial x^2}(r^n) = \sum \frac{\partial}{\partial x}\left(\frac{\partial r^n}{\partial x}\right)$

$$= \sum \frac{\partial}{\partial x}\left(nr^{n-1} \frac{\partial r}{\partial x}\right) = \sum \frac{\partial}{\partial x}\left(nr^{n-1} \frac{x}{r}\right) = \sum \frac{\partial}{\partial x}(nr^{n-2}x)$$

$$= \sum n\left[r^{n-2} + x(n-2)\, r^{n-3} \frac{\partial r}{\partial x}\right] = \sum n[r^{n-2} + (n-2)r^{n-4}x^2]$$

$$= n[3r^{n-2} + (n-2)r^{n-4}(x^2 + y^2 + z^2)] = n[3r^{n-2} + (n-2)r^{n-2}] = n(n+1)r^{n-2}$$

(b) $\nabla^2 f(r) = \nabla \cdot (\nabla f(r)) = \text{div}\,(\text{grad}\, f(r)) = \text{div}\,\{f'(r)\, \text{grad}\, r\}$

$$= \text{div}\left\{f'(r) \frac{\vec{r}}{r}\right\} = \text{div}\left\{\frac{f'(r)}{r} \vec{r}\right\} = \frac{f'(r)}{r} \text{div}\, \vec{r} + \vec{r} \cdot \text{grad}\left(\frac{f'(r)}{r}\right)$$

$$= \frac{3f'(r)}{r} + \vec{r} \cdot \left[\frac{1}{r} \text{grad}\, f'(r) + f'(r)\, \text{grad}\, \frac{1}{r}\right] = \frac{3f'(r)}{r} + \vec{r} \cdot \left\{\frac{1}{r} f''(r) \frac{\vec{r}}{r} + f'(r)\left(\frac{-\vec{r}}{r^3}\right)\right\}$$

$$= \frac{3}{r} f'(r) + \frac{1}{r^2} f''(r)\,\vec{r}\cdot\vec{r} - \frac{1}{r^3} f'(r)\,\vec{r}\cdot\vec{r}$$

$$= \frac{3}{r} f'(r) + f''(r) - \frac{f'(r)}{r} = f''(r) + \frac{2}{r} f'(r).$$

Example 8.23: If $f = (x^2 + y^2 + z^2)^{-n}$, find div grad f and determine n if div grad $f = 0$.

Solution: We have, $f = r^{-2n}$. Thus grad f = grad $r^{-2n} = (-2n)\, r^{-2n-2}\vec{r}$, and

$$\text{div grad } f = \nabla\cdot(-2n\, r^{-2n-2}\vec{r}) = (-2n)\,[r^{-2n-2}\,\nabla\cdot\vec{r} + \vec{r}\cdot\nabla r^{-2n-2}]$$

$$= (-2n)\,[3r^{-2n-2} + \vec{r}\cdot(-2n-2)\, r^{-2n-4}\vec{r}]$$

$$= (-2n)\,[3r^{-2n-2} + (-2n-2) r^{-2n-2}]$$

$$= (-2n)\,(-2n+1) r^{-2n-2} = (2n)\,(2n-1) r^{-2n-2}.$$

div grad $f = 0$, gives $n = 0$, or $1/2$.

V.Imp.

Example 8.24: Prove that curl curl curl curl $\vec{F} = \nabla^4\vec{F}$, where $\vec{F}$ is solenoidal field.

Solution: Since the vector field $\vec{F}$ is solenoidal, therefore, div $\vec{F} = 0$. Thus

curl curl $\vec{F}$ = grad div $\vec{F} - \nabla^2\vec{F} = -\nabla^2\vec{F} = \vec{G}$, say.

Consider, curl curl $\vec{G}$ = grad div $\vec{G} - \nabla^2\vec{G}$

Now div $\vec{G} = \nabla\cdot(-\nabla^2\vec{F}) = -\nabla^2(\nabla\cdot\vec{F}) = 0$. Therefore, curl curl $\vec{G} = -\nabla^2\vec{G}$, and thus

curl curl curl curl $\vec{F} = -\nabla^2(-\nabla^2\vec{F}) = \nabla^4\vec{F}$. Imp.

Imp

Example 8.25: If $\vec{a}$ and $\vec{b}$ are constant vectors, show that

(a) $\nabla\left[\frac{\vec{a}\cdot\vec{r}}{r^n}\right] = \frac{\vec{a}}{r^n} - n\frac{(\vec{a}\cdot\vec{r})}{r^{n+2}}\vec{r}$ Imp SS (b) $\nabla\times\{\vec{a}\times(\vec{b}\times\vec{r})\} = \vec{a}\times\vec{b}$.

Solution:

(a) $$\nabla\left[\frac{\vec{a}\cdot\vec{r}}{r^n}\right] = \nabla\left[\frac{1}{r^n}\,\vec{a}\cdot\vec{r}\right] = \frac{1}{r^n}\nabla(\vec{a}\cdot\vec{r}) + (\vec{a}\cdot\vec{r})\nabla\left(\frac{1}{r^n}\right)$$

$$= \frac{1}{r^n}\,\vec{a} + (\vec{a}\cdot\vec{r})\,(-n) r^{-n-2}\vec{r}, \text{ since } \nabla(\vec{a}\cdot\vec{r}) = \vec{a}.$$

$$= \frac{\vec{a}}{r^n} - n\frac{(\vec{a}\cdot\vec{r})}{r^{n+2}}\vec{r}$$

(b) $$\nabla\times(\vec{a}\times(\vec{b}\times\vec{r})) = \nabla\times[(\vec{a}\cdot\vec{r})\vec{b} - (\vec{a}\cdot\vec{b})\vec{r}] = \nabla\times(\vec{a}\cdot\vec{r})\vec{b} - \nabla\times(\vec{a}\cdot\vec{b})\vec{r}$$

$$= \{\nabla(\vec{a}\cdot\vec{r})\times\vec{b} + (\vec{a}\cdot\vec{r})\nabla\times\vec{b}\} - \{\nabla(\vec{a}\cdot\vec{b})\times\vec{r} + (\vec{a}\cdot\vec{b})\nabla\times\vec{r}\}$$

$$= \vec{a}\times\vec{b}, \text{ since } \nabla(\vec{a}\cdot\vec{r}) = \vec{a},\ \nabla\times\vec{r} = \vec{0},\ \nabla\times\vec{b} = \vec{0}, \text{ and } \nabla(\vec{a}\cdot\vec{b}) = 0.$$

Example 8.26: If r and $\vec{r}$ have their usual meanings and $\vec{a}$ is a constant vector, prove that

$$\nabla \times \left(\frac{\vec{a} \times \vec{r}}{r^n}\right) = \frac{2-n}{r^n}\vec{a} + \frac{n(\vec{a}\cdot\vec{r})\vec{r}}{r^{n+2}}.$$

Solution:

$$\text{L.H.S.} = \nabla \times [r^{-n}(\vec{a} \times \vec{r})]$$

$$= r^{-n}[\nabla \times (\vec{a} \times \vec{r})] + \nabla r^{-n} \times (\vec{a} \times \vec{r}) \qquad \text{...(8.30)}$$

We have, $\nabla \times (\vec{a} \times \vec{r}) = \vec{a}(\nabla\cdot\vec{r}) - \vec{r}(\nabla\cdot\vec{a}) + (\vec{r}\cdot\nabla)\vec{a} - (\vec{a}\cdot\nabla)\vec{r}$ ...(8.31)

Since $\vec{a}$ is a constant vector, therefore, $\nabla\cdot\vec{a}$ and $(\vec{r}\cdot\nabla)\vec{a}$ are zeros; also $\nabla\cdot\vec{r} = 3$ and $(\vec{a}\cdot\nabla)\vec{r} = \vec{a}$, thus (8.31) becomes

$$\nabla \times (\vec{a} \times \vec{r}) = 3\vec{a} - \vec{a} = 2\vec{a} \qquad \text{...(8.32)}$$

Also $\nabla r^{-n} = -nr^{-(n+2)}\vec{r}$ ...(8.33)

Using (8.32) and (8.33) in (8.30), we have

$$\text{L.H.S.} = \frac{2\vec{a}}{r^n} - \frac{n}{r^{n+2}}[\vec{r} \times (\vec{a} \times \vec{r})] = \frac{2\vec{a}}{r^n} - \frac{n}{r^{n+2}}[(\vec{r}\cdot\vec{r})\vec{a} - (\vec{a}\cdot\vec{r})\vec{r}]$$

$$= \frac{2\vec{a}}{r^n} - \frac{n}{r^{n+2}}[r^2\vec{a} - (\vec{a}\cdot\vec{r})\vec{r}] = \frac{2-n}{r^n}\vec{a} + \frac{n(\vec{a}\cdot\vec{r})\vec{r}}{r^{n+2}} = \text{R.H.S.}$$

Example 8.27: Prove that $\nabla\cdot\left(\frac{f(r)}{r}\vec{r}\right) = \frac{1}{r^2}\frac{d}{dr}(r^2 f(r))$

Solution:

$$\nabla\cdot\left(\frac{f(r)}{r}\vec{r}\right) = \nabla\cdot\left[\frac{f(r)}{r}(x\hat{i} + y\hat{j} + z\hat{k})\right] = \frac{\partial}{\partial x}\left[\frac{f(r)}{r}x\right] + \frac{\partial}{\partial y}\left[\frac{f(r)}{r}y\right] + \frac{\partial}{\partial z}\left[\frac{f(r)}{r}z\right]$$

Now,

$$\frac{\partial}{\partial x}\left[\frac{f(r)}{r}x\right] = \frac{f(r)}{r} + x\frac{d}{dr}\left\{\frac{f(r)}{r}\right\}\frac{\partial r}{\partial x} = \frac{f(r)}{r} + x\left\{\frac{f'(r)}{r} - \frac{f(r)}{r^2}\right\}\frac{x}{r}$$

$$= \frac{f(r)}{r} + \frac{x^2}{r^2}f'(r) - \frac{x^2}{r^3}f(r). \text{ Similarly,}$$

$$\frac{\partial}{\partial y}\left[\frac{f(r)}{r}y\right] = \frac{f(r)}{r} + \frac{y^2}{r^2}f'(r) - \frac{y^2}{r^3}f(r), \text{ and } \frac{\partial}{\partial z}\left[\frac{f(r)}{r}z\right] = \frac{f(r)}{r} + \frac{z^2}{r^2}f'(r) - \frac{z^2}{r^3}f(r)$$

Thus,

$$\nabla\cdot\left(\frac{f(r)}{r}\vec{r}\right) = \frac{3}{r}f(r) + \frac{1}{r^2}(x^2 + y^2 + z^2)f'(r) - \frac{1}{r^3}(x^2 + y^2 + z^2)f(r)$$

$$= \frac{2}{r}f(r) + f'(r) = \frac{1}{r^2}[2rf(r) + r^2 f'(r)] = \frac{1}{r^2}\frac{d}{dr}(r^2 f(r)).$$

Example 8.28: If $\vec{A}$ and $\vec{B}$ are irrotational, prove that $\vec{A} \times \vec{B}$ is solenoidal.

Solution: Since $\vec{A}$ and $\vec{B}$ are irrotational, therefore, curl $\vec{A} = \vec{0}$, and curl $\vec{B} = \vec{0}$

We have, $\quad \text{div}\,(\vec{A} \times \vec{B}) = \vec{B} \cdot \text{curl}\vec{A} - \vec{A} \cdot \text{curl}\vec{B} = \vec{B} \cdot \vec{0} - \vec{A} \cdot \vec{0} = 0$

Thus, $\vec{A} \times \vec{B}$ is solenoidal.

EXERCISE 8.4

1. If $u = x^2 + y^2 + z^2$ and $\vec{v} = x\hat{i} + y\hat{j} + z\hat{k}$, show that div $(u\vec{v}) = 5u$.
2. If $u\vec{F} = \nabla v$, where u and v are scalars and $\vec{F}$ is a vector, show that $\vec{F} \cdot \text{curl}\ \vec{F} = 0$
3. Find the directional derivative of $\nabla \cdot (\nabla \phi)$ at the point $(1, -2, 1)$ in the direction of the normal to the surface $xy^2z = 3x + z^2$, where $\phi = 2x^3y^2z^4$.
4. If $\vec{r}_1$ and $\vec{r}_2$ be the vectors joining the fixed points (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively to a variable point (x, y, z) prove that

   (a) div $(\vec{r}_1 \times \vec{r}_2) = 0$ (b) grad$(\vec{r}_1 \cdot \vec{r}_2) = \vec{r}_1 + \vec{r}_2$ (c) curl $(\vec{r}_1 \times \vec{r}_2) = 2(\vec{r}_1 - \vec{r}_2)$.
5. Show that the vector $\nabla\phi \times \nabla\psi$ is solenoidal.
6. Show that curl $\left[\hat{k} \times \text{grad}\,\frac{1}{r}\right] + \text{grad}\left[\hat{k} \cdot \text{grad}\,\frac{1}{r}\right] = \vec{0}$, where r is the distance of a point (x, y, z) from the origin and $\hat{k}$ is a unit vector in the direction of z-axis.
7. Let $f(x, y, z)$ be a solution of the Laplace equation $\nabla^2 f = 0$, show that ∇f is a vector which is both irrotational and solenoidal.
8. Let $f(x, y, z)$ be a solution of the Poisson equation $\nabla^2 f = c$, where c is a constant. If $\vec{v} = \nabla f$, show that curl $\vec{v} = 0$, but div $\vec{v} \neq 0$.
9. Show that (a) $\nabla \cdot (f\nabla g) - \nabla \cdot (g\nabla f) = f\nabla^2 g - g\nabla^2 f$ (b) $\nabla \cdot [(f\nabla g) \times (g\nabla f)] = 0$.
10. If $\nabla \cdot \vec{e} = 0, \nabla \cdot \vec{h} = 0, \nabla \times \vec{e} = -\frac{1}{c}\frac{\partial \vec{h}}{\partial t}$, and $\nabla \times \vec{h} = \frac{1}{c}\frac{\partial \vec{h}}{\partial t}$, then show that $\vec{e}$ and $\vec{h}$ satisfy the wave equation $\frac{\partial^2 f}{\partial t^2} = c^2\nabla^2 \vec{f}$, where c is a constant, and $\vec{f}$ is a vector function.

ANSWERS

Exercise 8.1 (p. 494)

1. $(5t^2 \cos t + 10t \sin t)\,\hat{i} + (t \cos t + \sin t)\,\hat{j} + (t^3 \cos t + 3t^2 \sin t)\,\hat{k}$
2. $4 \cos 4t + 3t^2$
3. $w\hat{k}$

4. $(3t^2 - 2e^{2t})\hat{i} - [(1-2t) - (2+t)e^t]\hat{j} - [te^t + t(2+3t)]\hat{k}$

5. $2t\vec{u}(t^2) + 2t^3\,\vec{u}'(t^2)$ 6. $a\vec{u}'(at) - (a/t^2)\vec{v}'(a/t)$

7. $\vec{u}(t) \times \vec{u}'''(t) + \vec{u}'(t) \times \vec{u}''(t)$ 8. $\vec{u}(t) \cdot \vec{u}'(t) \times \vec{u}'''(t)$

10. $x(t) = (1+t)/\sqrt{2}$, $y(t) = (1-t)/\sqrt{2}$, $z(t) = (\pi/4) + t$

11. $(-3\sin t\,\hat{i} + 3\cos t\,\hat{j} + 4\hat{k})/5$ 12. $\cos^{-1}(3/7)$

13. $\sqrt{37}$, $5\sqrt{13}$ 14. $8\sqrt{14}/7$, $-\sqrt{14}/7$

15. $70/\sqrt{29}$, $\sqrt{(436/29)}$

16. $\sqrt{17}$ m.p.h in the direction $\tan^{-1}(0.25)$ north of east

Exercise 8.2 (p. 504)

1. (a) $-8\hat{i}$ (b) $7\hat{i} + 24\hat{j} - 2\hat{k}$

2. (a) $-(\hat{i} + 3\hat{j} - \hat{k})/\sqrt{11}$ (b) $(-\hat{i} + 2\hat{j} + 2\hat{k})/3$

6. (a) $-\dfrac{11}{3}$ (b) 144 (c) $15/\sqrt{17}$ (d) $(15 + 14\sqrt{3})/2$

7. $(4\hat{i} + 3\hat{j} - 12\hat{k})/13, 1$ 8. $\lambda = 2.5, \mu = 1.$

10. $(xx_0/a^2) + (yy_0/b^2) + (zz_0/c^2) = 1$

Exercise 8.3 (p. 510)

5. $m = 0, n = 1$

Exercise 8.4 (p. 517)

3. $1724/\sqrt{21}$

9

CHAPTER

Vector Integral Calculus

Vector integral calculus, similar to vector differential calculus, extends the concept of integration to vector functions, enabling to generalize the idea of definite integration to curves and surfaces in three dimensions. This helps in better understanding of the physical interpretations of divergence and curl, and has applications in solid mechanics, fluid flow and heat flow problems.

9.1 INTEGRATION OF VECTOR FUNCTIONS

The integration of vector functions is defined as the reverse process of differentiation. Let $\vec{f}(t)$ and $\vec{F}(t)$ be two vector functions of a scalar variable t such that $\frac{d}{dt}\vec{F}(t) = \vec{f}(t)$, then $\vec{F}(t)$ is called the integral of $\vec{f}(t)$ with respect to t and, since $\frac{d}{dt}\left(\vec{F}(t)+\vec{c}\right)=\frac{d}{dt}\vec{F}(t)$, we write

$$\int \vec{f}(t)dt = \vec{F}(t) + \vec{c}, \qquad ...(9.1)$$

where $\vec{c}$ is any arbitrary constant vector independent of t.

$\vec{F}(t)$ is called the *indefinite integral* of $\vec{f}(t)$. The constant vector $\vec{c}$ is called the constant of integration and is determined on the basis of the initial conditions given.

The *definite integral* of $\vec{f}(t)$ between the limits $t = a$ and $t = b$ is given by

$$\int_a^b \vec{f}(t)dt = [\vec{F}(t)]_a^b = \vec{F}(b) - \vec{F}(a). \qquad ...(9.2)$$

As in case of differentiation of vectors, in order to integrate a vector function, we integrate its components, that is, if $\vec{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$, then

$$\int f(t)dt = \hat{i}\int f_1(t)dt + \hat{j}\int f_2(t)dt + \hat{k}\int f_3(t)dt.$$

By considering the derivatives of suitable vector functions we obtain some standard results for integration of vector functions. For example

1. $\frac{d}{dt}(\vec{r}\,.\vec{s}\,) = \frac{d\vec{r}}{dt}.\vec{s} + \vec{r}.\frac{d\vec{s}}{dt}$, implies $\int\left(\frac{dr}{dt}.\vec{s} + \vec{r}.\frac{d\vec{s}}{dt}\right)dt = \vec{r}\,.\vec{s}\, + c$, where c is a scalar independent of t.

2. $\frac{d}{dt}(\vec{r}^{\,2}) = 2\vec{r}\,.\frac{d\vec{r}}{dt}$, implies $\int\left(\vec{r}.\frac{d\vec{r}}{dt}\right)dt = \frac{1}{2}\vec{r}^{\,2} + c$, where c is a scalar independent of t.

3. $\frac{d}{dt}\left(\vec{r}\times\frac{d\vec{r}}{dt}\right) = \frac{d\vec{r}}{dt}\times\frac{d\vec{r}}{dt} + \vec{r}\times\frac{d^2\vec{r}}{dt^2} = \vec{r}\times\frac{d^2\vec{r}}{dt^2}$, implies $\int\left(\vec{r}\times\frac{d^2\vec{r}}{dt^2}\right)dt = \vec{r}\times\frac{d\vec{r}}{dt} + \vec{c}$,

where $\vec{c}$ is a vector independent of t.

4. If $\vec{a}$ is a constant vector, then

$$\frac{d}{dt}(\vec{a}\times\vec{r}\,) = \frac{d\vec{a}}{dt}\times\vec{r} + \vec{a}\times\frac{d\vec{r}}{dt} = \vec{a}\times\frac{d\vec{r}}{dt}, \text{ implies} \int\left(\vec{a}\times\frac{d\vec{r}}{dt}\right)dt = \vec{a}\times\vec{r} + \vec{c}\,,$$

where $\vec{c}$ is a vector independent of t.

5. If r and $\hat{r}$ have their usual meanings, then

$\frac{d}{dt}(\hat{r}) = \frac{d}{dt}\left(\frac{1}{r}\vec{r}\right) = \frac{1}{r}\frac{d\vec{r}}{dt} - \frac{1}{r^2}\frac{dr}{dt}\vec{r}$, implies $\int\left[\frac{1}{r}\frac{d\vec{r}}{dt} - \frac{1}{r^2}\frac{dr}{dt}\vec{r}\right]dt = \hat{r} + \vec{c}$, where $\vec{c}$ is a vector independent of t.

Example 9.1: The acceleration of a particle at time t is given by

$$\vec{a}(t) = 18\cos 3t\,\hat{i} - 8\sin 2t\hat{j} + 6t\hat{k}.$$

If the velocity vector $\vec{v}$ and displacement $\vec{r}$ are zero at $t = 0$, find $\vec{v}$ and $\vec{r}$ at any time t.

Solution: The acceleration is $\vec{a}(t) = \frac{d^2\vec{r}}{dt^2} = 18\cos 3t\hat{i} - 8\sin 2t\hat{j} + 6t\hat{k}$

Integrating w.r.t. t, we have

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = 18\frac{\sin 3t}{3}\hat{i} + 8\frac{\cos 2t}{2}\hat{j} + 6\frac{t^2}{2}\hat{k} + \vec{c} = 6\sin 3t\hat{i} + 4\cos 2t\hat{j} + 3t^2\hat{k} + \vec{c}\,.$$

At $t = 0$, $v = \vec{0}$, therefore, $\vec{0} = 4\hat{j} + \vec{c}$, or $\vec{c} = -4\hat{j}$, thus

$$v(t) = \frac{d\vec{r}}{dt} = 6\sin 3t\hat{i} + 4(\cos 2t - 1)\hat{j} + 3t^2\hat{k}.$$

Integrating again w.r.t. t, we have $\vec{r}(t) = -2\cos 3t\,\hat{i} + (2\sin 2t - 4t)\hat{j} + t^3\hat{k} + \vec{c}$

At $t = 0$, $\vec{r} = \vec{0}$, therefore, $\vec{0} = -2\hat{i} + \vec{c}$, or $\vec{c} = 2\hat{i}$, thus

$$\vec{r} = 2(1 - \cos 3t)\hat{i} + 2(\sin 2t - 2t)\hat{j} + t^3\hat{k}.$$

Example 9.2: Evaluate $\displaystyle\int_1^2 (\vec{a}\,.\,\vec{b}\times\vec{c})\,dt$, where

$$\vec{a} = t\hat{i} - 3\hat{j} + 2t\hat{k},\quad \vec{b} = \hat{i} - 2\hat{j} + 2\hat{k},\quad \vec{c} = 3\hat{i} + t\hat{j} - \hat{k}.$$

Solution: We have,

$$\vec{a}.\vec{b}\times\vec{c} = [\vec{a}\ \vec{b}\ \vec{c}\,] = \begin{vmatrix} t & -3 & 2t \\ 1 & -2 & 2 \\ 3 & t & -1 \end{vmatrix} = t(2 - 2t) + 3(-1 - 6) + 2t(t + 6) = 7(2t - 3).$$

$$\text{Therefore, } \int_1^2 (\vec{a}\,.\,\vec{b}\times\vec{c})\,dt = 7\int_1^2 (2t - 3)\,dt = 7\left[t^2 - 3t\right]_1^2 = 7(-2 + 2) = 0.$$

Example 9.3: If $\vec{r}(t) = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$, then prove $\displaystyle\int_1^2 \left[\vec{r}\times\frac{d^2\vec{r}}{dt^2}\right]dt = -14\hat{i} + 75\hat{j} - 15\hat{k}$.

Solution: We have, $\dfrac{d}{dt}\left[\vec{r}\times\dfrac{d\vec{r}}{dt}\right] = \dfrac{d\vec{r}}{dt}\times\dfrac{d\vec{r}}{dt} + \vec{r}\times\dfrac{d^2\vec{r}}{dt^2} = \vec{r}\times\dfrac{d^2\vec{r}}{dt^2}$, therefore,

$$\int_1^2 \left[\vec{r}\times\frac{d^2\vec{r}}{dt^2}\right]dt = \left[\vec{r}\times\frac{d\vec{r}}{dt}\right]_1^2.$$

$$\text{Also, } \vec{r}\times\frac{d\vec{r}}{dt} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5t^2 & t & -t^3 \\ 10t & 1 & -3t^2 \end{vmatrix} = -2t^3\hat{i} + 5t^4\hat{j} - 5t^2\hat{k}\text{. Thus}$$

$$\int_1^2 \left[\vec{r}\times\frac{d^2\vec{r}}{dt^2}\right]dt = \left[-2t^3\hat{i} + 5t^4\hat{j} - 5t^2\hat{k}\right]_1^2 = (-16 + 2)\hat{i} + (80 - 5)\hat{j} - (20 - 5)\hat{k} = -14\hat{i} + 75\hat{j} - 15\hat{k}.$$

Imp -

Example 9.4: If $r(t) = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$, then evaluate $\displaystyle\int_0^1 \left(\vec{r}.\frac{d\vec{r}}{dt}\right)dt$.

Solution: We have, $\frac{d}{dt}(\vec{r}^2) = 2\vec{r}\cdot\frac{d\vec{r}}{dt}$, therefore, $\int_0^1 \left(\vec{r}\cdot\frac{d\vec{r}}{dt}\right)dt = \frac{1}{2}\left[\vec{r}^2\right]_0^1$

Also, $\vec{r}^2 = \vec{r}\cdot\vec{r} = (5t^2\hat{i} + t\hat{j} - t^3\hat{k})\cdot(5t^2\hat{i} + t\hat{j} - t^3\hat{k}) = 25t^4 + t^2 + t^6.$

Thus, $\int_0^1 \left(\vec{r}\cdot\frac{d\vec{r}}{dt}\right)dt = \frac{1}{2}[25t^4 + t^2 + t^6]_0^1 = \frac{1}{2}[25 + 1 + 1] = \frac{27}{2}.$

Example 9.5: If $\frac{d^2\vec{r}}{dt^2} = 6t\hat{i} - 12t^2\hat{j} + 4\cos t\hat{k}$, then find $\vec{r}$ given that at $t = 0$,

$$\frac{d\vec{r}}{dt} = -\hat{i} - 3\hat{k} \text{ and } \vec{r} = 2\hat{i} + \hat{j}.$$

Solution: We have, $\frac{d^2\vec{r}}{dt^2} = 6t\hat{i} - 12t^2\hat{j} + 4\cos t\hat{k}$. Integrating this w.r.t t,

$\frac{d\vec{r}}{dt} = 3t^2\hat{i} - 4t^3\hat{j} + 4\sin t\hat{k} + \vec{c}$, where $\vec{c}$ is a vector independent of t.

At $t = 0$, $\frac{d\vec{r}}{dt} = -\hat{i} - 3\hat{k}$, therefore, $-\hat{i} - 3\hat{k} = \vec{c}$, and thus

$$\frac{d\vec{r}}{dt} = (3t^2 - 1)\hat{i} - 4t^3\hat{j} + (4\sin t - 3)\hat{k}.$$

Integrating w.r.t. t, $\vec{r}(t) = (t^3 - t)\hat{i} - t^4\hat{j} - (4\cos t + 3t)\hat{k} + \vec{c}$, where $\vec{c}$ is a vector independent of t.

At $t = 0$, $\vec{r} = 2\hat{i} + \hat{j}$, this gives $2\hat{i} + \hat{j} = -4\hat{k} + \vec{c}$, or $\vec{c} = 2\hat{i} + \hat{j} + 4\hat{k}$.

Thus, $\vec{r}(t) = (t^3 - t + 2)\hat{i} - (t^4 - 1)\hat{j} - (4\cos t + 3t - 4)\hat{k}$.

EXERCISE 9.1

1. Given $\vec{r}(t) = (5t^2 - 3t)\hat{i} + 6t^3\hat{j} - 7t\hat{k}$, evaluate $\int_2^4 \vec{r}(t)dt$.

2. If $\vec{r} = t\hat{i} - t^2\hat{j} + (t-1)\hat{k}$ and $\vec{s} = 2t^2\hat{i} + 6t\hat{k}$, evaluate

(a) $\int_0^2 \vec{r}\cdot\vec{s}dt$ (b) $\int_0^2 \vec{r}\times\vec{s}dt$

3. Given that $\vec{r}(t) = 2\hat{i} - \hat{j} + 2\hat{k}$ when $t = 2$ and $\vec{r}(t) = 4\hat{i} - 2\hat{j} + 3\hat{k}$, when $t = 3$, show that $\int_2^3 \left(\vec{r} \cdot \frac{d\vec{r}}{dt}\right) dt = 10.$

4. Find the value of $\vec{r}$ satisfying $\frac{d^2\vec{r}}{dt^2} = 6t\hat{i} - 24t^2\hat{j} + 4\sin t\,\hat{k}$, given that $\vec{r} = 2\hat{i} + \hat{j}$ and $\frac{d\vec{r}}{dt} = -\hat{i} - 3\hat{k}$ at $t = 0$.

9.2 THE LINE INTEGRAL. INDEPENDENCE OF PATH

The concept of line integral is a generalization of the concept of the definite integral $\int_a^b f(x)dx$ in integral calculus. In definite integral we integrate the integrand $f(x)$ from $x = a$, along the x-axis, to $x = b$. In line integral we shall integrate the given function along a curve C in the plane or in the space.

A curve C in space can be represented by a vector function

$$\vec{r}(t) = [x(t), y(t), z(t)] = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k},\ a \le t \le b,$$

where, x, y, z are cartesian co-ordinates. This is called a *parametric representation* of the curve C and t is called the *parameter* of the representation. To each value of t, there corresponds a point P on C, as shown in Fig. 9.1a.

We call C *the path of integration*. With A: $\vec{r}(a)$, its initial point, and B: $\vec{r}(b)$, its terminal point, the curve C is now *oriented*. The direction from A to B in which t increases is taken as the positive direction on C. The direction is indicated by an arrow. If the points A and B coincide, as in Fig. 9.1b, then C is called a *closed path*. Further, C is called a *smooth curve* if it has a unique tangent at each of its points whose direction varies continuously as we move along C. Mathematically, it means that $\vec{r}(t)$ is differentiable and the derivative $\frac{d\vec{r}}{dt}$ is continuous and different from the zero vector at every point of C.

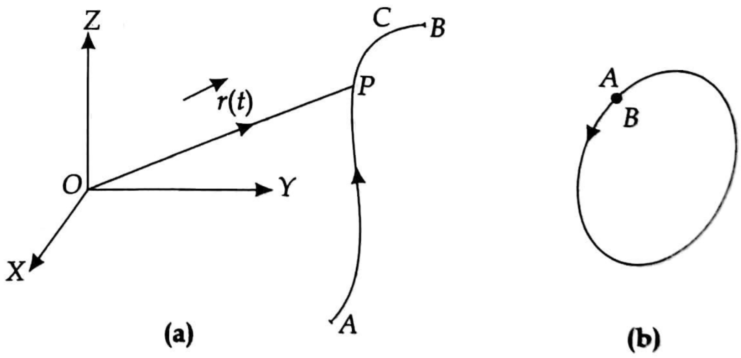

Fig. 9.1

We shall assume *every path of integration of a line integral to be piecewise smooth, that is, consisting of finitely many smooth curves.*

Now, we define line integral.

9.2.1 Line Integral of $\vec{F}$ Over C

A line integral of a vector function $\vec{F}(\vec{r})$ over a curve C is defined by

$$\int_C \vec{F}(\vec{r}).d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\,dt \qquad ...(9.3)$$

If $\vec{F}(\vec{r}) = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ and $d\vec{r} = dx\,\hat{i} + dy\hat{j} + dz\hat{k}$, then

$$\int_C \vec{F}(\vec{r}).d\vec{r} = \int_C (f_1dx + f_2dy + f_3dz) = \int_a^b \left(f_1\frac{dx}{dt} + f_2\frac{dy}{dt} + f_3\frac{dz}{dt}\right)dt \qquad ...(9.4)$$

Here, f_1, f_2 and f_3 are functions of x, y, z, which in turn depend upon parameter t, $a \le t \le b$. When the path of integration C is a closed curve, then instead of $\int_C$ we write $\oint_C$. We note that line integral (9.4) is a scalar, not a vector. Two other types of line integrals are

$$\int_C \vec{F} \times d\vec{r} \quad \text{and} \quad \int_C f d\vec{r}$$

which are both vectors.

The line integral $\int_C \vec{F}(\vec{r}).d\vec{r}$ arises naturally in mechanics. If $\vec{F}$ represents the force acting on a particle moving along an arc AB, then the work done during the small displacement $\delta\vec{r}$ is $\vec{F}.\delta\vec{r}$. Thus *the total work done by* $\vec{F}$ *during the displacement from A to B* is given by the line integral $\int_A^B \vec{F}.d\vec{r}$.

Similarly, if $\vec{F}$ represents the velocity of a fluid particle, then the line integral $\int_C \vec{F}.d\vec{r}$ is called *the circulation of* $\vec{F}$ *around the curve C*; and *when the circulation of* $\vec{F}$ *around every closed curve C in a region E vanishes, then* $\vec{F}$ *is said to be irrotational in E.*

Example 9.6: If $\vec{F} = (5xy - 6x^2)\hat{i} + (2y - 4x)\hat{j}$, evaluate $\int_C \vec{F}.d\vec{r}$ along the curve $C: y = x^3$ in the xy-plane from the point (1, 1) to (2, 8).

Solution: Since the particle moves in the xy-plane, therefore, $d\vec{r} = dx\hat{i} + dy\hat{j}$. Thus, we have

$$\int_C \vec{F}.d\vec{r} = \int_{C:y=x^3} [(5xy - 6x^2)dx + (2y - 4x)dy]$$

Substituting $y = x^3$, where x goes from 1 to 2, we have

$$\int_C \vec{F}.d\vec{r} = \int_1^2 [(5x^4 - 6x^2)dx + (2x^3 - 4x)\cdot 3x^2dx] = \int_1^2 (5x^4 - 6x^2 + 6x^5 - 12x^3)dx = \left[x^5 - 2x^3 + x^6 - 3x^4\right]_1^2$$

$$= 32 - (-3) = 35.$$

Example 9.7: Using the line integral compute the work done by the force $\vec{F} = (2y+3)\hat{i} + xz\hat{j} + (yz-x)\hat{k}$ when it moves a particle from the point (0, 0, 0) to the point (2, 1, 1) along the curve $x = 2t^2$, $y = t$, $z = t^3$.

Solution: Since the particle moves in the space, therefore, $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{j}$. The work W done by the force $\vec{F}$ in moving a particle from the point $A(0, 0, 0)$ to the point $B(2, 1,1)$, is given by

$$W = \int_A^B \vec{F}.d\vec{r} = \int_A^B [(2y+3)dx + xzdy + (yz-x)dz]$$

Substituting $x = 2t^2$, $y = t$ and $z = t^3$, where t goes from 0 to 1, we have

$$W = \int_0^1 [(2t+3)4t\,dt + 2t^5dt + (t^4 - 2t^2)3t^2dt] = \int_0^1 (8t^2 + 12t + 2t^5 + 3t^6 - 6t^4)dt$$

$$= \left[\frac{8t^3}{3} + 6t^2 + \frac{t^6}{3} + \frac{3}{7}t^7 - \frac{6}{5}t^5\right]_0^1 = \frac{8}{3} + 6 + \frac{1}{3} + \frac{3}{7} - \frac{6}{5} = 8\frac{8}{35} \text{ units.}$$

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Example 9.8: A vector field is given by $\vec{F} = (\sin y)\hat{i} + x(1+\cos y)\hat{j}$. Evaluate the line integral over a circular path given by $x^2 + y^2 = a^2$, $z = 0$

Solution: The parametric equation of the circular path $x^2 + y^2 = a^2$, $z = 0$ are:

$x = a\cos t$, $y = a\sin t$, $z = 0$, $0 \le t \le 2\pi$. Also $d\vec{R} = dx\hat{i} + dy\hat{j}$. Therefore the line integral is

$$\oint_C \vec{F}.d\vec{R} = \oint_C [\sin y\,dx + x(1+\cos y)dy] = \oint_C d(x\sin y) + \oint_C xdy$$

Substituting $x = a\cos t$, $y = a\sin t$, we have

$$\oint_C \vec{F}.d\vec{R} = \int_0^{2\pi} d[a\cos t\sin(a\sin t)] + \int_0^{2\pi} a\cos t\, a\cos t\,dt$$

$$= [a\cos t\sin(a\sin t)]_0^{2\pi} + \frac{a^2}{2}\int_0^{2\pi}(1+\cos 2t)dt = 0 + \frac{a^2}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi} = \pi a^2.$$

 Anti-clockwise : +ve

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Example 9.9: Compute the line integral $\int_C (y^2dx - x^2dy)$ about the triangle whose vertices are (1, 0), (0, 1) and (−1, 0).

Solution: Here the path is *ABCD* as shown in Fig. 9.2

Equation of line *AB* is, $x + y = 1$, of *BC* is, $-x + y = 1$, and of *CA* is, $y = 0$.

We have,

$$\int_{ABCA} (y^2dx - x^2dy) = \int_{AB} (y^2dx - x^2dy) + \int_{BC} (y^2dx - x^2dy) + \int_{CA} (y^2dx - x^2dy) \qquad ...(9.5)$$

To evaluate $\int_{AB} (y^2dx - x^2dy)$, put $y = 1 - x$, where x goes from 1 to 0. Thus

$$\int_{AB} (y^2dx - x^2dy) = \int_1^0 [(1-x)^2dx - x^2(-dx)] = \int_1^0 (1 + x^2 - 2x + x^2)dx$$

$$= \int_1^0 (1 - 2x + 2x^2)dx = \left[x - x^2 + \frac{2x^3}{3}\right]_1^0 = -\frac{2}{3}.$$

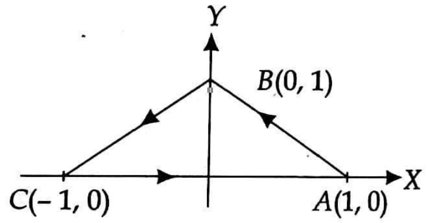

Fig. 9.2

Similarly, $\int_{BC} (y^2dx - x^2dy) = \int_0^{-1} [(1+x)^2dx - x^2dx] = \int_0^{-1} (1 + x^2 + 2x - x^2)dx$

$$= \int_0^{-1} (1 + 2x)\,dx = \left[x + x^2\right]_0^{-1} = 0.$$

and, $\int_{CA} (y^2dx - x^2dy) = \int_{-1}^{-1} 0.dx = 0$. Substituting these values in (9.5), we obtain

$$\int_{ABCA} (y^2dx - x^2dy) = -\frac{2}{3} + 0 + 0 = -\frac{2}{3}.$$

9.2.2 Independence of Path. Conservative Vector Field

An important question of interest is: *Does the value of a line integral $\int_C \vec{F}.d\vec{r}$ change if we integrate from the same initial point A to the same terminal point B but along another path?* The answer is *yes*, in general. Consider the following example.

xample 9.10: A vector field is given by $\vec{F} = 5z\hat{i} + xy\hat{j} + x^2z\hat{k}$. Evaluate the line integral $\int \vec{F}.d\vec{r}$ long the two different paths C_1 and C_2 with the same initial point A (0, 0, 0) and the same terminal oint $B(1, 1, 1)$, as given below:

(a) C_1: the straight-line segment, $x = y = z = t, \quad 0 \le t \le 1$.

(b) C_2: the parabolic arc, $x = y = t, \quad z = t^2, 0 \le t \le 1$.

Solution: We have $\vec{F} = 5z\hat{i} + xy\hat{j} + x^2z\hat{k}$. The line integral from the initial point $A(0, 0, 0)$ to the terminal point $B(1, 1, 1)$ is

$$\int_A^B \vec{F}.d\vec{r} = \int_A^B (5z\,dx + xy\,dy + x^2z\,dz) \qquad ...(9.6)$$

(a) Along the path C_1: $x = y = z = t, \quad 0 \le t \le 1$. Thus, (9.6) becomes

$$\int_A^B \vec{F}.d\vec{r} = \int_0^1 (5t\,dt + t^2dt + t^3dt) = \int_0^1 (5t + t^2 + t^3)dt$$

$$= \left[\frac{5t^2}{2} + \frac{t^3}{3} + \frac{t^4}{4}\right]_0^1 = \frac{37}{12}.$$

(b) Along the path C_2: $x = y = t, \quad z = t^2, \quad 0 \le t \le 1$. Thus (9.6) becomes

$$\int_A^B \vec{F}.d\vec{r} = \int_0^1 (5t^2dt + t^2dt + 2t^5dt) = \int_0^1 (6t^2 + 2t^5)dt = \left[2t^3 + \frac{t^6}{3}\right]_0^1 = \frac{7}{3}.$$

We observe that the two results are different, although the end points are the same. This shows that, *the value of a line integral, in general, depends not only on $\vec{F}$ and on the endpoints A, B, of the path but also on the path along which we integrate from the initial point A to the final point B.*

Next we find condition which ensures that the value of the line integral is independent of the path, and this aspect is of great physical applications. For instance, in mechanics, independence of path may mean that we have to do the same amount of work regardless of the path to the mountain top, be it short and steep, or long and smooth. But we must note that not all forces are of this type. We have the following result.

Theorem 9.1: (Independence of Path) *A line integral $\int_C \vec{F}.d\vec{r}$ is independent of path in domain D if, and only if $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ with continuous F_1, F_2, F_3 in D is the gradient of some function f in D, that is, if $\vec{F} = grad\, f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$.*

Proof. Let $\vec{F}$ = grad f for some function f in D. Consider C to be any path in D from point A to point B, both A and B being arbitrary, given by $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$, where t, $a \le t \le b$, is the parameter. Consider

$$\int_C \vec{F}.d\vec{r} = \int_A^B \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right)\cdot(dx\hat{i} + dy\hat{j} + dz\hat{k}) = \int_A^B \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\right)$$

$$= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right)dt = \int_a^b \frac{df}{dt}dt = \left[f[x(t), y(t), z(t)]\right]_{t=a}^{t=b} = f(B) - f(A).$$

Thus the value of the integral is simply the difference of the value of f at the two endpoints of C and is therefore independent of the path C.

To prove the converse part, suppose that the line integral is independent of the path in the domain D. Choose a fixed point $A(x_0, y_0, z_0)$ in D and an arbitrary point $B(x, y, z,)$ in D, and define function $f(x, y, z)$ as

$$f(x, y, z) = k + \int_A^B (F_1dx' + F_2dy' + F_3dz'), \qquad \text{...(9.7)}$$

where k is a constant and C' is any path from A to the arbitrary point B in D, as shown in Fig. 9.3.

Since, the line integral is independent of the path and further A is a fixed point, thus (9.7) depends only on the co-ordinates x, y, z of the point B and hence defines a function $f(x, y, z)$.

Next, because of independence of path, we may integrate along the path A to $B_1(x_1, y, z)$ and then parallel to the x-axis along B_1B as shown in the Fig. 9.3, then

$$f(x, y, z) = k + \int_A^{B_1} (F_1dx' + F_2dy' + F_3dz')$$

$$+ \int_{B_1}^{B} (F_1dx' + F_2dy' + F_3dz') \qquad \text{...(9.8)}$$

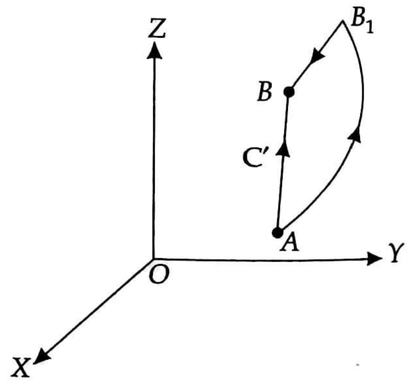

Fig. 9.3

The first integral on the right of (9.8) does not depend on x since $A(x_0, y_0, z_0)$ is fixed and $B_1(x_1, y, z)$ also does not depend on x, and the second integral does not depend on y and z, since the path B_1B is parallel to x-axis, therefore $dy' = dz' = 0$, and hence the second integral on the right of (9.8) can be written as $\int_{x_1}^{x} F_1(x', y, z)dx'$.

Taking partial derivative with respect to x on both sides of (9.8) gives

$$\frac{\partial f}{\partial x} = F_1. \qquad ...(9.9)$$

Similarly, choosing suitable paths of integration, we obtain

$$\frac{\partial f}{\partial y} = F_2, \quad \text{and} \quad \frac{\partial f}{\partial z} = F_3.$$

Thus, $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k} = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$ = grad f, and this completes the proof.

We have seen that if the line integral $\int_C \vec{F}.d\vec{r}$ is independent of path, then F = grad f. In such a situation we say that the vector field $\vec{F}$ is a *gradient field* and the function f is called the *potential function for* $\vec{F}$, and such a force field where the work done by the force $\vec{F}$ in moving a particle is independent of the path and depends only on the endpoints is called a *conservative* field. We must note that if $\vec{F}$ is a conservative force field, then the work done along any simple closed path is zero, and further, to verify that the field $\vec{F}$ is conservative, we need to show that curl $\vec{F} = 0$.

Example 9.11: Show that the work done by the force $\vec{F} = 2x\hat{i} + 2y\hat{j} + 4z\hat{k}$ in moving a particle from the point $A(0, 0, 0)$ to the point $B(2, 2, 2)$ is independent of the path and find its value also.

Solution: Here, $\vec{F} = 2x\hat{i} + 2y\hat{j} + 4z\hat{k}$. We can check very easily that curl $\vec{F} = \mathbf{0}$. Hence $\vec{F}$ *is* a conservative field, and say $\vec{F}$ = grad f, for some scalar function f. Thus we have

$$2x\hat{i} + 2y\hat{j} + 4z\hat{k} = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}.$$

This gives, $\frac{\partial f}{\partial x} = 2x, \frac{\partial f}{\partial y} = 2y, \frac{\partial f}{\partial z} = 4z.$

Integrating these we obtain, $f = x^2 + \phi_1(y, z), f = y^2 + \phi_2(z, x)$, and $f = 2z^2 + \phi_3(x, y)$, hence f is given by $f = x^2 + y^2 + 2z^2$.

Now since $\vec{F}$ is a conservative field, thus the work done by $\vec{F}$ in moving a particle from $A(0, 0, 0)$ to $B(2, 2, 2)$ is independent of the path, and is given by

$$\int_A^B \vec{F}.d\vec{r} = \int_{A(0,0,0)}^{B(2,2,2)} (2x\,dx + 2y\,dy + 4z\,dz) = \int_{A(0,0,0)}^{B(2,2,2)} d(x^2 + y^2 + 2z^2) = \left[x^2 + y^2 + 2z^2\right]_{A(0,0,0)}^{B(2,2,2)} = 16.$$

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Example 9.12: Show that $\int_C \frac{x\,dx + y\,dy}{\sqrt{x^2 + y^2}}$ is independent of any path of integration which does not pass through the origin.

Solution: Let $\int_C \frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}} = \int_C \vec{F}.d\vec{r}$, where $\vec{F} = \frac{x\hat{i}}{\sqrt{x^2+y^2}} + \frac{y\hat{j}}{\sqrt{x^2+y^2}}$, and $d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$.

Consider
$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} \qquad ...(9.10)$$
$$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} \qquad ...(9.11)$$

Integrating (9.10) w.r.t x, we have
$$f = \sqrt{x^2+y^2} + k(y), \qquad ...(9.12)$$
where $k(y)$ is a constant depending upon y only. Differentiating (9.12) partially w.r.t. y and comparing with (9.11), we obtain $k'(y) = 0$, or $k(y) = A$, a constant.

Thus $f(x, y, z) = \sqrt{x^2+y^2} + A$. Hence $\vec{F}$ can be expressed as $\vec{F} = \text{grad} f$. Thus the vector field $\vec{F}$ is conservative and hence the given integral is independent of the path of integration which does not pass through the origin.

Example 9.13: If C is a simple closed curve in the xy-plane not enclosing the origin show that $\int_C \vec{F}.d\vec{r} = 0$, where $\vec{F} = \frac{y\hat{i} - x\hat{j}}{x^2+y^2}$.

Solution: Here,
$$\vec{F} = \frac{y}{x^2+y^2}\hat{i} - \frac{x}{x^2+y^2}\hat{j}.$$

Consider
$$\frac{\partial f}{\partial x} = \frac{y}{x^2+y^2}, \qquad ...(9.13)$$
$$\frac{\partial f}{\partial y} = -\frac{x}{x^2+y^2}. \qquad ...(9.14)$$

Integrating (9.13) w.r.t. x, we have
$$f = \tan^{-1}\frac{x}{y} + k(y). \qquad ...(9.15)$$

Differentiating (9.15) w.r.t. y, we have
$$\frac{\partial f}{\partial y} = -\frac{x}{x^2+y^2} + k'(y).$$

Comparing it with (9.14), we obtain $k'(y) = 0$, that is, $k(y) = A$, a constant.

Thus $f(x, y, z) = \tan^{-1}\frac{x}{y} + A$. Hence, $\vec{F}$ can be expressed as $\vec{F}$ = grad f. Thus, the vector field $\vec{F}$ is conservative and hence the given integral is independent of the path of integration and depends only on the endpoints. Here, since C is a simple closed curve then endpoints coincide with each other and hence $\int_C \vec{F}.d\vec{r} = 0$.

EXERCISE 9.2

1. If $\vec{F} = 3xy\hat{i} - y^2\hat{j}$, evaluate $\int_C \vec{F}.d\vec{r}$, where C is the curve in the xy-plane $y = 2x^2$ from(0, 0) to (1, 2).
2. Evaluate the line integral $\int_C [(x^2 + xy)dx + (x^2 + y^2)dy]$, where C is the square formed by the lines $y = \pm 1$ and $x = \pm 1$.
3. Find the work done by the force $\vec{F} = x\hat{i} - z\hat{j} + 2y\hat{k}$ in the displacement along the closed path C consisting of the segments C_1, C_2 and C_3, where $C_1: 0 \le x \le 1, y = x, z = 0$; $C_2: 0 \le z \le 1, x = 1, y = 1$; $C_3: 1 \ge x \ge 0, \quad y = z = x$.
4. Find the work done in moving a particle in the force field $\vec{F} = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}$ along, (a) the straight line from (0, 0, 0) to (2, 1, 3). (b) the curve defined by $x^2 = 4y$, $3x^3 = 8z$ from $x = 0$ to $x = 2$.
5. If $\vec{F} = 2y\hat{i} - z\hat{j} + x\hat{k}$, evaluate $\int_C \vec{f} \times d\vec{r}$ along the curve $x = \cos t$, $y = \sin t$, $z = 2\cos t$ from $t = 0$ to $t = \pi/2$.
6. Show that the line integral $\int_C (3x^2dx + 2yzdy + y^2dz)$ is independent of the path in any domain in space and find its value if C has the initial point $A(0, 1, 2)$ and terminal point $B(1 - 1, 7)$.
7. If $\vec{F} = y\hat{i} + x\hat{j} + xyz^2\hat{k}$, evaluate $\int_C \vec{F}.d\vec{r}$, where C is the circle $x^2 + y^2 - 2y = 2$, $z = 1$ going around once in the anti-clockwise direction.
8. Show that the line integral $\int_A^B [(1 - \sin x \sin y)dx + (1 + \cos x \cos y)dy]$ is independent of path of integration, and also evaluate it from $P(\pi/4, \pi/4)$ to $Q(\pi/2, 0)$.
9. Find whether the vector field $\vec{F} = \cosh(x + y)(\hat{i} + \hat{j})$ is conservative. If it is so, find the potential function.

9.3 SURFACE AND SURFACE INTEGRALS

After discussing line integrals in the preceding section, we turn to surface integrals here, in which we integrate over surface in space. We will refer *surface*, also for a *portion of a surface*, for example, a portion of a sphere, or of a cylinder, etc.

A surface S in the xyz-space is represented by $z = g(x, y)$ or $f(x, y, z) = 0$.

For example, $z = \sqrt{a^2 - x^2 - y^2}$, or $x^2 + y^2 + z^2 - a^2 = 0, z \geq 0$, represents a semi-sphere of radius a with centre at (0, 0, 0).

Since the surfaces are two-dimensional, so to represent a surface parametrically we need two parameters say, u and v. Thus, a parametric representation of a surface S in space is of the form

$$\vec{R}(u, v) = x(u, v)\hat{i} + y(u, v)\hat{j} + z(u, v)\hat{k},$$

where u, v belong to some region D in the uv-plane.

Further the continuous functions $u = u(t)$ and $v = v(t)$ of a real parameter t represent a curve C on the surface S.

For example, a parametric representation of the sphere $x^2 + y^2 + z^2 = a^2$ is

$$\vec{R}(u, v) = a \cos v \cos u\hat{i} + a \cos v \sin u\hat{j} + a \sin v\hat{k},$$

where $0 \leq u \leq 2\pi, -\frac{\pi}{2} \leq v \leq \frac{\pi}{2}$.

Another parametric representation of the sphere is

$$\vec{R}(u, v) = a \cos u \sin v\hat{i} + a \sin u \sin v\hat{j} + a \cos v\hat{k},$$

where $0 \leq u \leq 2\pi, \quad 0 \leq v \leq \pi$.

9.3.1 Surface Normal Vector

To define a surface integral, we need the concept of *surface normal vector.*

A normal vector of a surface S at a point P is a vector perpendicular to the tangent plane of S at P.

Since a surface S is given by $\vec{R} = \vec{R}(u, v), u = u(t), v = v(t)$, differentiating w.r.t. t, we get

$$\frac{d\vec{R}}{dt} = \frac{\partial \vec{R}}{\partial u}\frac{du}{dt} + \frac{\partial \vec{R}}{\partial v}\frac{dv}{dt}. \qquad \text{...(9.16)}$$

The vectors $\frac{\partial \vec{R}}{\partial u}$ and $\frac{\partial \vec{R}}{\partial v}$ are tangential to S at P, we assume that these are linearly independent and so determine the tangent plane of S at P. Then their cross-product gives a normal vector $\vec{N}$ of S at P, that is, $\vec{N} = \frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v}$, and so, the corresponding unit normal vector $\hat{N}$ of S at P, is given by

$$\hat{N} = \frac{\frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v}}{\left|\frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v}\right|} \qquad ...(9.17)$$

Also we know that if the surface S is represented by $f(x, y, z) = 0$, then $\hat{N} = \frac{\text{grad } f}{|\text{grad } f|}$ is the unit outward normal to S.

A surface S is called a *smooth surface* if its surface normal vector depends continuously on the points of S. Further, S is called *piecewise smooth* if it consists of finitely many smooth surfaces. For example, a sphere is a smooth surface and a cube is a piecewise smooth surface.

9.3.2 The Surface Integral

Consider a piecewise-smooth surface S given by a parametric representation $\vec{R}(u, v) = x(u, v)\hat{i} + y(u, v)\hat{j} + z(u, v)\hat{k}$ and let $\hat{N}$ be the unit outward normal vector at a point P to S. For a given vector function $\vec{F}(\vec{R})$, ~~we define the *surface integral*~~ over S by

$$\iint_S \vec{F}.\vec{dS} = \iint_S \vec{F}.\hat{N}dS \qquad ...(9.18)$$

We note that integrand in (9.18) is a scalar being the dot product $\vec{F}.\hat{N}$, the normal component of $\vec{F}$.

Next, if $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ and $\hat{N} = \cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k}$, where α, β, γ are the angles between $\hat{N}$ and the positive directions of the co-ordinate axes, then

$$\iint_S \vec{F}.\hat{N}dS = \iint_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)dS \qquad ...(9.19)$$

Two other types of surface integrals are $\iint_S \vec{F} \times \vec{dS}$, and $\iint_S f\vec{dS}$, which are both vectors.

The surface integrals $\iint_S \vec{F}.\vec{dS}$ arise naturally in flow problems. If $\vec{F} = \rho\vec{v}$, where ρ is the density of the fluid and $\vec{v}$ is the velocity of flow, then the surface integral gives *the total outward flux across the surface* S. When the flux of $\vec{F}$ across every closed surface S in a region D vanishes, then $\vec{F}$ is said to be a *solenoidal vector point function in E.*

It may be noted that $\vec{F}$ may well be taken for gravitational force, electric force, magnetic force, etc.

9.3.3 Evaluation of the Surface Integral

To evaluate surface integrals it is, in general, convenient to express them as double integrals taken over the orthogonal projection of S on one of the co-ordinate planes, say in the xy-plane, as shown in Fig. 9.4.

Let $\gamma(<\pi/2)$ be the angle which the unit normal vector $\hat{N}$ makes with the positive direction of the z-axis, then

$$|\hat{N}.\hat{k}| = |\hat{N}|\,|\hat{k}|\cos\gamma = \cos\gamma$$

Also $\delta x\,\delta y$ = projection of ΔS on the xy-plane = $\Delta S\cos\gamma$.

Thus, $$\Delta S = \frac{\delta x\delta y}{\cos\gamma} = \frac{\delta x\delta y}{|\hat{N}.\hat{k}|}.$$

Hence, $$\iint_S \vec{F}.\hat{N}dS = \iint \vec{F}.\hat{N}\frac{dxdy}{|\hat{N}.\hat{k}|}, \quad \text{...(9.20)}$$

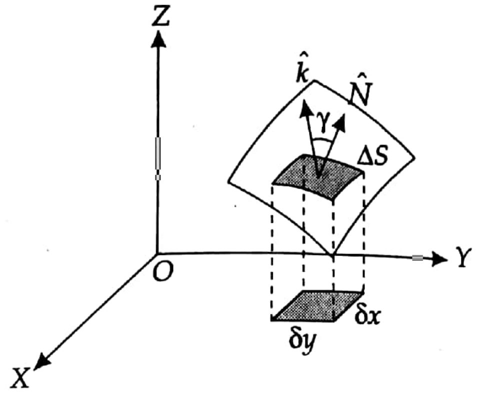

Fig. 9.4

where the integration on the right hand side of (9.20) is to be taken over the orthogonal projection of S on the xy-plane.

Another way of evaluating the surface integral is given as follows:

$$\iint_S \vec{F}.\hat{N}dS = \iint_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)dS = \iint_S (F_1dy\,dz + F_2dz\,dx + F_3dx\,dy) \quad \text{...(9.21)}$$

using $\Delta S = \dfrac{\delta x\delta y}{\cos\gamma}$, etc.

Example 9.14: Evaluate $\iint_S \vec{F}.\vec{dS}$, where $\vec{F} = x\hat{i} + (z^2 - zx)\hat{j} - xy\hat{k}$ and S is the triangular surface with vertices (2, 0, 0), (0, 2, 0), and (0, 0, 4).

Solution: Equation of the triangular surface S is $\dfrac{x}{2}+\dfrac{y}{2}+\dfrac{z}{4}=1$, or $2x + 2y + z = 4$.

A vector normal to the surface S is $\nabla(2x + 2y + z) = 2\hat{i} + 2\hat{j} + \hat{k}$.

Therefore $\hat{N}$, the unit vector normal to surface $S = \dfrac{2\hat{i}+2\hat{j}+\hat{k}}{\sqrt{4+4+1}} = \dfrac{2}{3}\hat{i}+\dfrac{2}{3}\hat{j}+\dfrac{1}{3}\hat{k}$.

Also $\hat{k}.\hat{N} = \hat{k}.\left(\dfrac{2}{3}\hat{i}+\dfrac{2}{3}\hat{j}+\dfrac{1}{3}\hat{k}\right) = \dfrac{1}{3}$. Thus, $\iint_S \vec{F}.\vec{dS} = \iint_E \vec{F}.\hat{N}\dfrac{dxdy}{|\hat{k}.\hat{N}|}$,

where E is the projection of S on the xy-plane which is a triangle OAB bounded by x-axis, y-axis and the line $x + y = 2$, as shown in Fig. 9.5.

Consider $\vec{F}.\hat{N} = [x\hat{i} + (z^2 - zx)\hat{j} - xy\hat{k}].\left(\frac{2}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}\right)$

$$= \frac{2}{3}x + \frac{2}{3}(z^2 - zx) - \frac{1}{3}xy$$

$$= \frac{2}{3}x + \frac{2}{3}[(4 - 2x - 2y)^2 - (4 - 2x - 2y)x] - \frac{1}{3}xy$$

(on the plane, $2x + 2y + z = 4$)

$$= \frac{1}{3}[2x + 2(4 - 2x - 2y)(4 - 3x - 2y) - xy]$$

$$= \frac{1}{3}[32 - 38x - 32y + 19xy + 12x^2 + 8y^2].$$ Thus

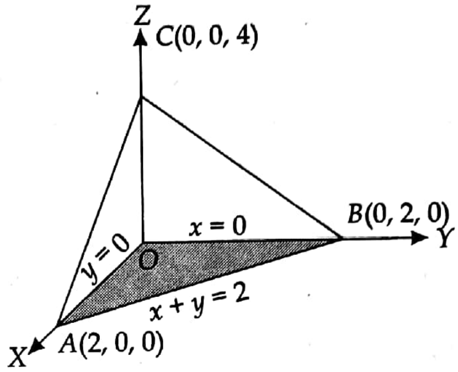

Fig. 9.5

$$\iint_S \vec{F}.\hat{N}dS = \iint_E \vec{F}.\hat{N}\frac{dxdy}{|\hat{k}.\hat{N}|}$$

$$= \int_0^2\int_0^{2-x} (32 - 38x - 32y + 19xy + 12x^2 + 8y^2)dxdy$$

$$= \int_0^2 \left[32y - 38xy - 16y^2 + \frac{19}{2}xy^2 + 12x^2y + \frac{8}{3}y^3\right]_0^{2-x}$$

$$= \int_0^2 \left[32(2-x) - 38x(2-x) - 16(2-x)^2 + \frac{19}{2}x(2-x)^2 + 12x^2(2-x) + \frac{8}{3}(2-x)^3\right]dx$$

$$= \int_0^2 \left(\frac{64}{3} - 38x + 24x^2 - \frac{31x^3}{6}\right)dx$$

$$= \left[\frac{64}{3}x - 19x^2 + 8x^3 - \frac{31}{24}x^4\right]_0^2$$

$$= \frac{128}{3} - 76 + 64 - \frac{62}{3} = 10.$$

Example 9.15: If $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$, evaluate $\iint_S \vec{F}.\hat{N}dS$, where S is the surface of the cube bounded by the planes $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

Solution: A cube is a piecewise smooth surface consisting of six smooth surfaces as shown in Fig. 9.6. Therefore the given integral has to be calculated over the six faces of cube. For the face $AC'PB'$, $x = 1$, $\hat{N} = \hat{i}$ and $dS = dydz$, thus

$$\iint_{AC'PB'} \vec{F}.\hat{N}dS = \int_0^1\int_0^1 (4xz\hat{i} - y^2\hat{j} + yz\hat{k}).\hat{i}\, dydz$$

$$= 4\int_0^1\int_0^1 zdydz = 4\left(\int_0^1 zdz\right)\int_0^1 dy = 2.$$

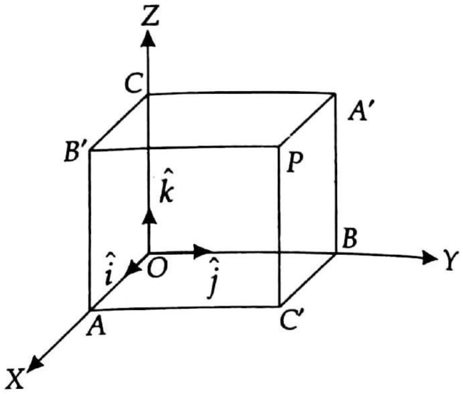

Fig. 9.6

For the face $OBA'C$, $x = 0$, $\hat{N} = -\hat{i}$ and $dS = dydz$, thus

$$\iint_{OBA'C} \vec{F}.\hat{N}dS = \int_0^1\int_0^1 ([4(0)z\hat{i} - y^2\hat{j} + yz\hat{k}].(-\hat{i})dydz = \int_0^1\int_0^1 0dydz = 0.$$

For the face $BC'PA'$, $y = 1$, $\hat{N} = \hat{j}$ and $dS = dxdz$, thus

$$\iint_{BCPA'} \vec{F}.\hat{N}dS = \int_0^1\int_0^1 [4xz\hat{i} - 1^2\hat{j} + 1z\hat{k}].\hat{j}dxdz = -\int_0^1\int_0^1 dxdz = -1.$$

For the face $OAB'C$, $y = 0$, $\hat{N} = -\hat{j}$ and $dS = dxdz$, thus

$$\iint_{OAB'C} \vec{F}.\hat{N}dS = \int_0^1\int_0^1 [4xz\hat{i} - (0)\hat{j} + (0)z\hat{k}](-\hat{j})dxdz = 0.$$

For the face $CB'PA'$, $z = 1$, $\hat{N} = \hat{k}$ and $dS = dxdy$, thus

$$\iint_{CB'PA'} \vec{F}.\hat{N}dS = \int_0^1\int_0^1 [4x(1)\hat{i} - y^2\hat{j} + y(1)\hat{k}].\hat{k}\,dxdy = \int_0^1\int_0^1 ydxdy = \frac{1}{2}.$$

For the face $OAC'B$, $z = 0$, $\hat{N} = -\hat{k}$ and $ds = dx\,dy$, thus

$$\iint_{OAC'B} \vec{F}.\hat{N}dS = \int_0^1\int_0^1 [4x(0)\hat{i} - y^2\hat{j} + y(0)\hat{k}].(-\hat{k})dxdy = 0.$$

Adding all these, $\iint_S \vec{F}.\hat{N}dS = 2 + 0 - 1 + 0 + \frac{1}{2} + 0 = 3/2.$

Example 9.16: Evaluate $\iint_S \vec{F}.\hat{N}dS$, where $\vec{F} = z^2\hat{i} + xy\hat{j} - y^2\hat{k}$ and S is the portion of the surface of the cylinder $x^2 + y^2 = 36$, $0 \le z \le 4$ included in the first octant.

Solution: Equation of surface S is $x^2 + y^2 - 36 = 0$. A vector normal to this surface is $\nabla(x^2 + y^2 - 36) = 2x\hat{i} + 2y\hat{j}$. Therefore $\hat{N}$, a unit vector normal to a point (x, y, z) of S is

$$\hat{N} = \frac{2x\hat{i} + 2y\hat{j}}{2\sqrt{x^2 + y^2}} = \frac{1}{6}(x\hat{i} + y\hat{j}), \text{ since } x^2 + y^2 = 36 \text{ on } S.$$

As shown in Fig. 9.7, the projection of S on xy-plane cannot be considered. We consider the projection of S on the yz-plane, which is a rectangle with sides of lengths 6 and 4.

We have $dS = \dfrac{dydz}{|\hat{N}.\hat{i}|} = \dfrac{dydz}{x/6}$, and

$$\vec{F}.\hat{N} = \frac{1}{6}(z^2\hat{i} + xy\hat{j} - y^2\hat{k})\cdot(x\hat{i} + y\hat{j}) = \frac{1}{6}(z^2 + y^2)x.$$

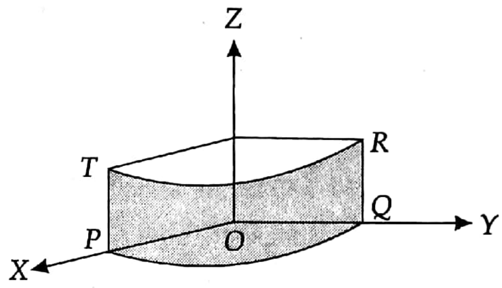

Fig. 9.7

Thus, $$\iint_S \vec{F}.\hat{N}dS = \int_{z=0}^{4}\left[\int_{y=0}^{6}(y^2 + z^2)dy\right]dz = \int_0^4 \left(\frac{y^3}{3} + yz^2\right)_0^6 dz$$

$$= \int_0^4 (72 + 6z^2)dz = \left[72z + 2z^3\right]_0^4 = 416.$$

Example 9.17: Evaluate $\iint_S \vec{F}.\hat{N}dS$, where $\vec{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$ and S is the portion of the surface of the sphere $x^2 + y^2 + z^2 = 1$ in the first octant.

Solution: Equation of the surface S is $x^2 + y^2 + z^2 - 1 = 0$. A vector normal to S is $\nabla(x^2 + y^2 + z^2 - 1) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$. Therefore $\hat{N}$, a unit vector normal to any point (x, y, z) of S is

$$\hat{N} = \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}} = x\hat{i} + y\hat{j} + z\hat{k},$$

since $x^2 + y^2 + z^2 = 1$ on S.

The projection of S on the xy-plane is the quadrant E of the circle $x^2 + y^2 = 1$ bounded by the lines $x = 0$ and $y = 0$. We have,

$$dS = \frac{dxdy}{|\hat{N}.\hat{k}|} = \frac{dxdy}{z}, \text{ and } \vec{F}.\hat{N} = (yz\hat{i} + zx\hat{j} + xy\hat{k})\cdot(x\hat{i} + y\hat{j} + z\hat{k}) = 3xyz.$$

Thus $$\iint_S \vec{F}.\hat{N}dS = 3\int_{y=0}^{1}\int_{x=0}^{\sqrt{1-y^2}} xy\,dx\,dy = 3\int_0^1 y\left[\frac{x^2}{2}\right]_0^{\sqrt{1-y^2}} dy$$

$$= \frac{3}{2}\int_0^1 y(1 - y^2)dy = \frac{3}{2}\left[\frac{y^2}{2} - \frac{y^4}{4}\right]_0^1 = \frac{3}{2}\left[\frac{1}{2} - \frac{1}{4}\right] = \frac{3}{8}.$$

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Example 9.18: If $\vec{r}$ denotes the position vector of any point (x, y, z) measured from the origin, then evaluate $\iint_S \frac{\vec{r}}{|\vec{r}|^3}.d\vec{S}$, where S is the surface of the sphere of radius a with centre at the origin.

Solution: Equation of the surface S is $x^2 + y^2 + z^2 - a^2 = 0$. A vector normal to S is $\nabla(x^2 + y^2 + z^2 - a^2)$ $= 2x\hat{i} + 2y\hat{j} + 2z\hat{j}$. Therefore, $\hat{N}$ a unit vector normal to a point (x, y, z) of S, is

$$\hat{N} = \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{1}{a}(x\hat{i} + y\hat{j} + z\hat{k}), \text{ since } x^2 + y^2 + z^2 = a^2 \text{ on } S.$$

Also $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, therefore, $|\vec{r}| = \sqrt{x^2 + y^2 + z^2} = a$, since $x^2 + y^2 + z^2 = a^2$ on S.

Let $\vec{F} = \frac{\vec{r}}{|\vec{r}|^3} = \frac{\vec{r}}{a^3} = \frac{1}{a^3}(x\hat{i} + y\hat{j} + z\hat{k})$. Then

$$\vec{F}.\hat{N} = \frac{1}{a^3}(x\hat{i} + y\hat{j} + z\hat{k}).\frac{1}{a}(x\hat{i} + y\hat{j} + z\hat{k}) = \frac{x^2 + y^2 + z^2}{a^4} = \frac{a^2}{a^4} = \frac{1}{a^2}.$$

Thus, $\iint_S \vec{F}.d\vec{S} = \frac{1}{a^2}\iint_s ds = \frac{1}{a^2}.4\pi a^2 = 4\pi$, since surface area of the sphere is $S = 4\pi a^2$.

EXERCISE 9.3

1. Evaluate $\iint_S \vec{F}.\hat{N}\,dS$, where $\vec{F} = 2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k}$ and S is the closed surface of the region in the first octant bounded by the cylinder $y^2 + z^2 = 9$ and the planes $x = 0, x = 2, y = 0$ and $z = 0$.

2. Evaluate $\iint_S \vec{F}.\hat{N}\,dS$, where $\vec{F} = 6z\hat{i} - 4\hat{j} + y\hat{k}$ and S is the portion of the plane $2x + 3y + 6z = 12$ in the first octant.

3. Evaluate $\iint_S \vec{F}.\hat{N}\,dS$, where $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ and S is the surface bounding the region $x^2 + y^2 = 4, z = 0, z = 3$.

4. If $\vec{r}$ is the position vector of any point (x, y, z) measured from the origin, then evaluate $\iint_S \vec{r}.d\vec{S}$ where S is that part of the surface of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, lying above the plane $z = 0$.

5. If $\vec{F} = y\hat{i} + (x-2xz)\hat{j} - xy\hat{k}$ and S is the surface of the sphere $x^2+y^2+z^2=a^2, 0 \le x, y, z \le a$, evaluate $\iint_S \vec{F}.\hat{N}\,dS$, where $\hat{N}$ is a unit vector along the outward normal to the surface.

6. If $\vec{F} = y\hat{i} + (x-2xz)\hat{j} - xy\hat{k}$, evaluate $\iint_S (\nabla \times \vec{F})\cdot\hat{N}\,dS$, where S is the surface of the sphere $x^2+y^2+z^2=a^2$ above the xy-plane.

9.4 GREEN'S THEOREM IN THE PLANE

Green's theorem provides a relationship between a double integral over a region E in a plane and the line integral over the closed curve C bounding E. This is of practical interest because it sometimes helps to make the evaluation of an integral easier. The theorem is stated as follows.

Theorem 9.2: (Green's Theorem) *Let E be a plane region in the xy-plane bounded by a closed curve C. If $f(x, y), g(x, y), \frac{\partial f}{\partial y}$ and $\frac{\partial g}{\partial x}$ are continuous on E, then*

$$\iint_E \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)dxdy = \oint_C [f(x, y)dx + g(x, y)dy], \qquad \text{...(9.22)}$$

the integration being carried in the counter-clockwise direction of C.

Proof. We prove Green's theorem for a special region E bounded by a closed curve C which is cut by any line parallel to the axes at the most in two points.

Let E be represented by $u_1(x) \le y \le v_1(x), a \le x \le b$, as shown in Fig. 9.8a.

$$\iint_E \frac{\partial f}{\partial y}dxdy = \int_a^b \left[\int_{u_1(x)}^{v_1(x)} \frac{\partial f}{\partial y}dy\right]dx = \int_a^b [f(x,y)]_{u_1(x)}^{v_1(x)}dx$$

$$= \int_a^b [f(x, v_1(x)) - f(x, u_1(x))]dx = -\int_b^a f(x, v_1(x))dx - \int_a^b f(x, u_1(x))dx$$

Since $y = v_1(x)$ represents the curve C^{**} and $y = u_1(x)$ represents the curve C^{*}, thus

$$\iint_E \frac{\partial f}{\partial y}dxdy = -\int_{C^{**}} f(x, y)dx - \int_{C^{*}} f(x, y)dx$$

$$= -\oint_C f(x, y)dx. \qquad \text{...(9.23)}$$

Similarly, it can be shown that for the region E represented by E: $u_2(y) \le x \le u_2(y), c \le y \le d$, refer Fig. 9.8b, we have

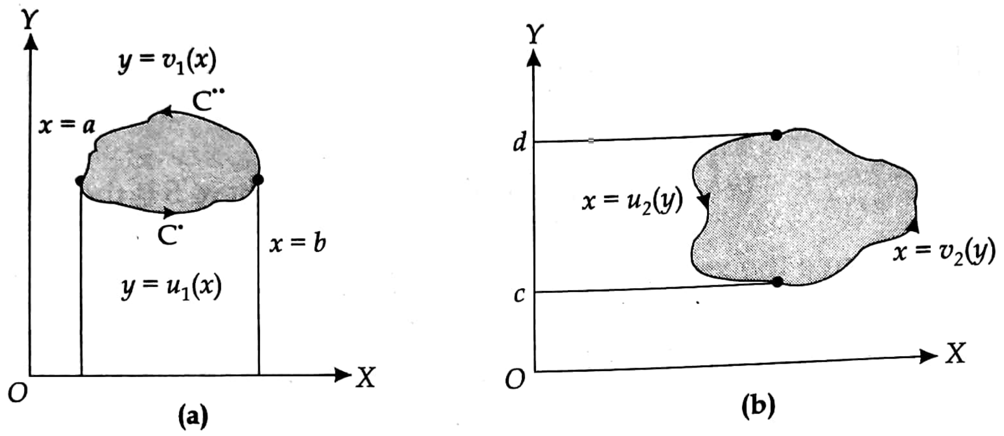

Fig. 9.8

$$\iint_E \frac{\partial g}{\partial x}\,dxdy = \oint_C g(x, y)dy. \quad ...(9.24)$$

(9.23) and (9.24) together give (9.22); this proves Green's theorem for the special regions.

The Green's theorem can be extended to a region E that itself is not a special region but can be subdivided into finitely many special regions E_1, E_2 etc. Such that the boundary of each is cut at the most in two points by any line parallel to either axis, refer to Figs. 9.9 (a) & (b). We apply the theorem to each subregion and then add the results; the left-hand members add up to the integral over E while the right-hand side members add up to the line integral over C plus integrals over the curves introduced for subdividing E. These additional integrals over the common boundaries cancel each other, for each is covered twice but in opposite directions and we are left with remaining line integrals which combine to form the line integral over the external curve C.

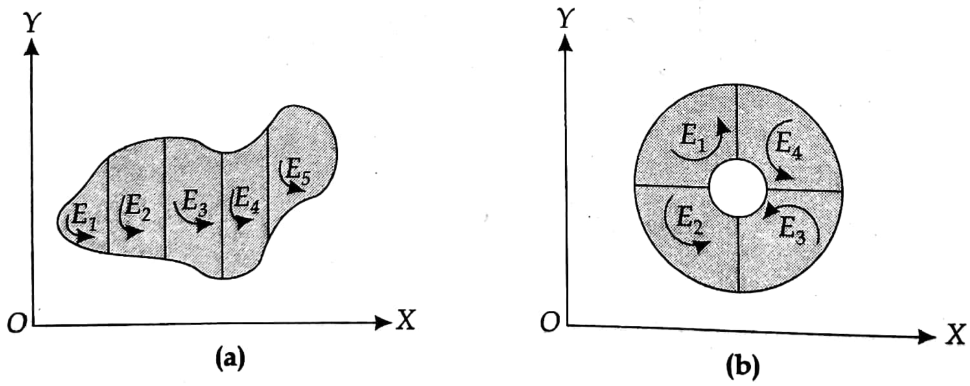

Fig. 9.9

9.4.1 Area of Plane Curves Using Green's Theorem

Green's theorem can be applied to find area of a plane region as a line integral over its boundary, e.g., for $f = 0$ and $g = x$, we have from (9.22) $\iint_E dx\,dy = \oint_C x\,dy$, and then, for $f = -y$, $g = 0$, again (9.22) gives $\iint\limits_E dx\,dy = -\oint\limits_C y\,dx$. Hence the area A of the region E is

$$\boxed{A = \iint\limits_E dx\,dy = \frac{1}{2}\oint\limits_C (x\,dy - y\,dx),} \qquad \text{...(9.25)}$$

which has been expressed in terms of a line integral over the boundary.

For an ellipse, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, we have $x = a\cos\theta$, $y = b\sin\theta$, and thus $dx = -a\sin\theta\,d\theta$, $dy = b\cos\theta\,d\theta$. Using (9.25) the area

$$A = \frac{1}{2}\int_0^{2\pi} [a\cos\theta)(b\cos\theta) + (b\sin\theta)(a\sin\theta)]d\theta = \frac{1}{2}ab\int_0^{2\pi} d\theta = \pi ab.$$

In terms of polar co-ordinates, for $x = r\cos\theta$, $y = r\sin\theta$, we have $dx = \cos\theta\,dr - r\sin\theta\,d\theta$ and $dy = \sin\theta\,dr + r\cos\theta\,d\theta$. Substituting in (9.25), we get

$$\boxed{A = \frac{1}{2}\oint\limits_C r^2 d\theta,} \qquad \text{...(9.26)}$$

a familiar result from calculus, used to find the area enclosed by polar curves.

Example ~~9.19~~: Verify Green's theorem for $\oint\limits_C [(3x^2 - 8y^2)dx + (4y - 6xy)dy]$ where C is the boundary of the region bounded by $x = 0$, $y = 0$ and $x + y = 1$.

Solution: By Green's theorem $\int\limits_C (f dx + g dy) = \iint\limits_E \left(\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y}\right) dx\,dy$, where E is the plane region enclosed by the closed curve C.

Here $f = 3x^2 - 8y^2$ and $g = 4y - 6xy$, and

$$\oint\limits_C (f dx + g dy) = \int\limits_{C_1} (f dx + g dy) + \int\limits_{C_2} (f dx + g dy) + \int\limits_{C_3} (f dx + g dy),$$

as shown in Fig. 9.10.

Along C_1, $y = 0$ and x varies from 0 to 1, thus $\int\limits_{C_1} (f dx + g dy) = 3\int_0^1 x^2 dx = 1$

Along C_2, $y = 1 - x$ and x varies from 1 to 0, thus

$$\int\limits_{C_2} (f dx + g dy) = \int_1^0 [\{3x^2 - 8(1-x)^2\}dx + \{4(1-x) - 6x(1-x)(-dx)\}$$

$$= \int_0^1 (12 - 26x + 11x^2)dx = \left[12x - 13x^2 + \frac{11x^3}{3}\right]_0^1 = \frac{8}{3}.$$

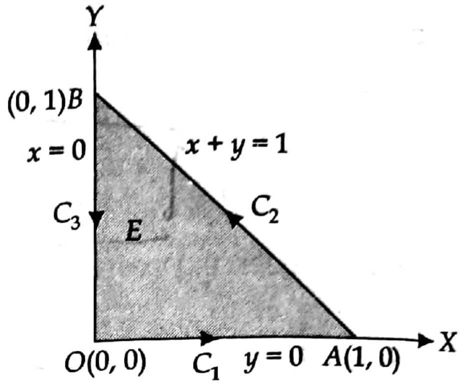

Fig. 9.10

Along C_3, $x = 0$ and y varies form 1 to 0, thus

$$\int_{C_3} (fdx + gdy) = 4\int_1^0 ydy = 4\left[\frac{y^2}{2}\right]_1^0 = -2.$$

Therefore, $$\int_C (fdx + gdy) = 1 + \frac{8}{3} - 2 = \frac{5}{3}. \qquad ...(9.27)$$

Now, $$\iint_E \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)dx\,dy = \iint_E \left[\frac{\partial}{\partial x}(4y - 6xy) - \frac{\partial}{\partial y}(3x^2 - 8y^2)\right]dx\,dy$$

$$= \int_0^1\int_0^{1-y} (-6y + 16y)dx\,dy = 10\int_0^1 (1 - y)y\,dy = 5/3. \qquad ...(9.28)$$

Green's theorem is verified from the equality of (9.27) and (9.28).

Example 9.20: Apply Green's theorem to evaluate

$$\int_C [\sin y\,dx + x(1 + \cos y)dy], \text{ where } C \text{ is the closed path given by } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Solution: By Green's theorem the given line integral is equal to

$$\iint_E \left[\frac{\partial}{\partial x}x(1 + \cos y) - \frac{\partial}{\partial y}(\sin y)\right]dx\,dy,$$

where E is the region enclosed by C. Thus,

$$\int_C [\sin y\,dx + x(1 + \cos y)dy] = \iint_E (1 + \cos y - \cos y)dx\,dy = \iint_E dx\,dy = \pi ab,$$

the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Example 9.21: Find the work done by the force $\vec{F} = (x^2 - y^3)\hat{i} + (x + y)\hat{j}$ in moving a particle along the closed path C containing the curve $x + y = 0$, $x^2 + y^2 = 16$, and $y = x$ in the first and the fourth quadrants.

Solution: The work done by the force $\vec{F}$ is given by

$$w = \oint_C \vec{F}.d\vec{r} = \oint_C [(x^2 - y^3)dx + (x + y)dy].$$

The closed path C bounding the region E is shown in Fig. 9.11. Using the Green's theorem, we obtain

$$\oint_C [(x^2 - y^3)dx + (x + y)dy] = \iint_E (1 + 3y^2)dx\,dy.$$

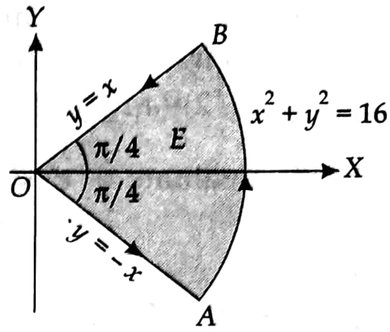

Fig. 9.11

Applying polar coordinates, the region E is given by

$$x = r\cos\theta,\quad y = r\sin\theta,\quad 0 \le r \le 4,\quad -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}.$$

Thus, $$\iint_E (1 + 3y^2)dx\,dy = \int_{-\pi/4}^{\pi/4}\left[\int_0^4 (1 + 3r^2\sin^2\theta)r\,dr\right]d\theta = \int_{-\pi/4}^{\pi/4}\left[\frac{r^2}{2} + \frac{3r^4}{4}\sin^2\theta\right]_0^4 d\theta$$

$$= \int_{-\pi/4}^{\pi/4} (8 + 192\sin^2\theta)d\theta = 16\int_0^{\pi/4} (1 + 24\sin^2\theta)d\theta$$

$$= 16\int_0^{\pi/4} (13 - 12\cos 2\theta)d\theta = 16\left[13\theta - 6\sin 2\theta\right]_0^{\pi/4} = 52\pi - 96.$$

EXERCISE 9.4

1. Verify Green's theorem for $\oint_C [(xy + y^2)dx + x^2dy]$, where C is bounded by $y = x$ and $y = x^2$.

2. Verify Green's theorem for $f(x, y) = e^{-x}\sin y$, $g(x, y) = e^{-x}\cos y$ and C is the square with vertices at $(0, 0)$, $(\pi/2, 0)$, $(\pi/2, \pi/2)$ and $(0, \pi/2)$.

3. Apply Green's theorem to evaluate $\int_C [y(1 - \sin x)dx + \cos x\,dy]$, where C is the plane triangle enclosed by the lines $y = 0$, $x = \pi/2$ and $y = (2/\pi)x$.

4. Evalute $\oint_C [(x^2 + y^2)dx + (y + 2x)dy]$, where C is the boundary of the region in the first quadrant bounded by the curves $y^2 = x$ and $x^2 = y$.

5. Using Green's theorem, prove that the area enclosed by a plane curve is $\frac{1}{2}\oint_C (x\,dy - y\,dx)$. Hence find the area of a hypocycloid $x = a\cos^3\theta$, $y = a\sin^3\theta$, $a > 0$, $0 \le \theta \le 2\pi$.

6. Show that the area of a polygon with vertices at (a_1, b_1), (a_2, b_2), ..., (a_n, b_n) taken in the anticlockwise direction is
$(1/2)[(a_1b_2 - a_2b_1) + (a_2b_3 - a_3b_2) + ... + (a_{n-1}b_n - a_nb_{n-1}) + (a_nb_1 - a_1b_n)]$.

7. Evaluate $\oint_C e^x(\sin y\,dx + \cos y\,dy)$, where C is the ellipse $4(x+1)^2 + 9(y-3)^2 = 36$.

9.5 STOKES' THEOREM: A GENERALIZATION OF GREEN'S THEOREM

We have seen the importance of the Green's theorem, next we consider another important theorem, the *Stokes' theorem,* which transforms line integrals into surface integrals, and conversely. Stokes' theorem generalizes Green's theorem in the sense that latter becomes a special case of the former.

The theorem is stated as follows.

Theorem 9.3: (Stokes' Theorem) *Let S be a piecewise smooth open surface bounded by a piecewise smooth simple closed curve C. If $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be any continuously differentiable vector point function, then*

$$\boxed{\oint_C \vec{F}.d\vec{r} = \iint_S \text{curl } \vec{F}.\hat{N}\,dS,} \qquad ...(9.29)$$

where $\hat{N} = \cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k}$ is a unit outward normal at any point of S, and C is traversed in positive direction.

Proof: Writing $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$, the Eq. (9.29) in components form is

$$\oint_C (F_1dx + F_2dy + F_3dz) = \iint_S \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\cos\gamma\right]dS. \quad ...(9.30)$$

We prove the result for a surface S that can be represented simultaneously in the forms

(a) $z = g(x, y)$, (b) $x = h(y, z)$,
(c) $y = k(z, x)$

where g, h, k are continuous functions and have continuous first order partial derivatives.

First we prove that

$$\oint_C F_1dx = \iint_S \left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right)dS. \qquad ...(9.31)$$

Consider the case when the equation of the surface S is written in the form $z = g(x, y)$ and the projection of S on the xy-plane is the region E and the projection of C on the xy-plane is the curve C' enclosing the region E as shown in Fig. 9.12.

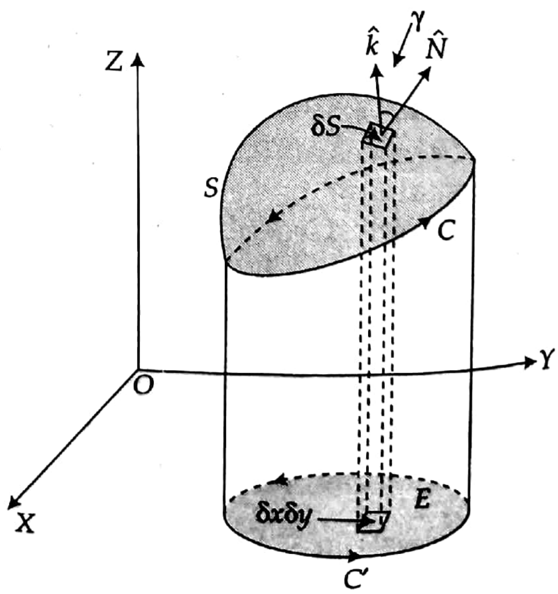

Fig. 9.12

In this case, we have

$$\oint_C F_1(x, y, z)dx = \oint_{C'} F_1[x, y, g(x, y)]dx$$

$$= \oint_{C'} (F_1[x, y, g(x, y)]dx + 0dy)$$

$$= -\iint_E \frac{\partial}{\partial y} F_1(x, y, g)dx\,dy.$$

using Green's theorem in plane.

Thus $$\int_C F_1 dx = -\iint_E \left[\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial g}{\partial y}\right] dx\,dy. \quad ...(9.32)$$

Next, the direction ratios of the normal to surface $z = g(x, y)$ are : $\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, -1$, and hence

$$\frac{\cos\alpha}{-\frac{\partial g}{\partial x}} = \frac{\cos\beta}{-\frac{\partial g}{\partial y}} = \frac{\cos\gamma}{1}, \text{ which gives, } \frac{\partial g}{\partial y} = -\frac{\cos\beta}{\cos\gamma}.$$

Also $dx\,dy$, the projection of dS on the xy-plane is $\cos\gamma dS$, and hence, (9.32) becomes

$$\oint_C F_1 dx = -\iint_S \left[\frac{\partial F_1}{\partial y} - \frac{\partial F_1}{\partial z}\frac{\cos\beta}{\cos\gamma}\right]\cos\gamma dS = \iint_S \left[\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right] dS,$$

This proves (9.31).

Similarly, we can prove the corresponding expressions for F_2 and F_3 by assuming the representations $x = h(y, z)$ and $y = k(z, x)$ respectively for the surface S. Adding all these, we get the required result (9.30) and hence (9.29).

9.5.1 Green's Theorem as a Special Case of Stokes' Theorem

Let $\vec{F} = F_1\hat{i} + F_2\hat{j}$ be a vector function which is continuously differentiable in a domain in the xy-plane containing region S bounded by a closed curve C. Then

$$\oint_C \vec{F}.d\vec{r} = \oint_C (F_1\hat{i} + F_2\hat{j})\cdot(dx\hat{i} + dy\hat{j}) = \oint_C (F_1 dx + F_2 dy).$$

$$\text{Also, curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & 0 \end{vmatrix} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}.$$

Here $\hat{N} = \hat{k}$, therefore, curl $\vec{F}.\hat{k} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$ and hence Stokes' theorem, refer to Eq. (9.29), takes the form $\oint_C (F_1 dx + F_2 dy) = \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$, which is Green's theorem in the plane, refer to Eq. (9.22).

Example 9.22: Verify Stokes' theorem for $\vec{F} = (x^2 - y^2)\hat{i} + 2xy\hat{j}$ in the rectangular region in the xy-plane given by (0, 0), $(a, 0)$, $(0, b)$ and (a, b).

Solution: Let $OACB$ be the given rectangle as shown in the Fig. 9.13.

Here, curl $\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix} = 4y\hat{k}$

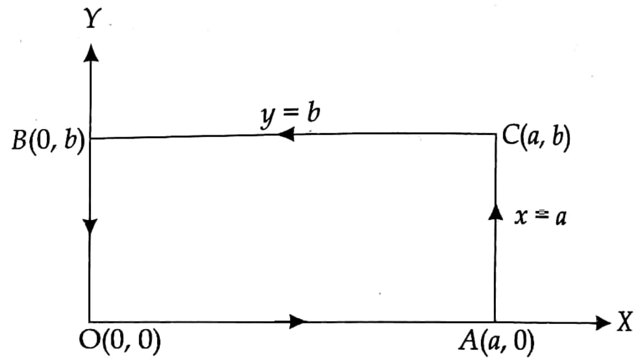

Fig. 9.13

and, curl $\vec{F}.d\vec{S} = (4y\hat{k})\cdot(\hat{k}\,dx\,dy) = 4y\,dx\,dy$. Therefore,

$$\iint_s \text{curl } \vec{F}.d\vec{S} = \int_0^a\int_0^b 4y\,dy\,dx = 2ab^2 \qquad ...(9.33)$$

We have, $\oint_{OACBO} \vec{F}.d\vec{r} = \int_{OA+AC+CB+BO} \vec{F}.d\vec{r}$, and

$$\vec{F}.d\vec{r} = [(x^2 - y^2)\hat{i} + 2xy\hat{j}]\cdot[\hat{i}\,dx + \hat{j}\,dy + \hat{j}\,dz] = (x^2 - y^2)dx + 2xy\,dy.$$

Thus, $\int_{OA} \vec{F}.d\vec{r} = \int_0^a x^2 dx = \frac{1}{3}a^3$, $\int_{AC} \vec{F}.d\vec{r} = \int_0^b 2ay\,dy = ab^2$

$\int_{CB} \vec{F}.d\vec{r} = -\int_0^a (x^2 - b^2)dx = ab^2 - \frac{1}{3}a^3$, and $\int_{BO} \vec{F}.d\vec{r} = 0.$

Therefore, $$\int_{OACBO} \vec{F}.d\vec{r} = \frac{1}{3}a^3 + ab^2 + ab^2 - \frac{1}{3}a^3 = 2ab^2. \qquad ...(9.34)$$

From the equality of (9.33) and (9.34), Stokes' theorem is verified.

Example 9.23: Verify Stokes' theorem for the field $\vec{F} = (2x - y)\,\hat{i} - yz^2\,\hat{j} - y^2 z\,\hat{k}$ over the upper half surface of $x^2 + y^2 + z^2 = 1$ bounded by its projection on the xy-plane.

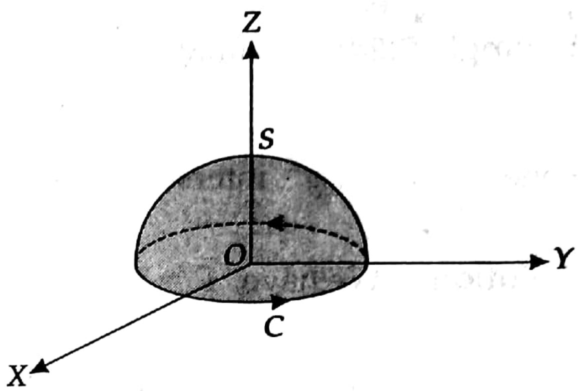

Fig. 9.14

Solution: The bounding curve C of the surface S, $x^2 + y^2 + z^2 = 1$, as shown in Fig. 9.14, in the xy-plane is $x^2 + y^2 = 1$, $z = 0$.

The equation of the curve in parametric form is

$$x = \cos\theta, \quad y = \sin\theta, \quad z = 0, \quad 0 \le \theta \le 2\pi.$$

Also
$$\oint_C \vec{F}.d\vec{r} = \oint_C [(2x - y)dx - yz^2 dy - y^2 z dz]$$

$$= \oint_C (2x - y)dx, \quad (z = 0)$$

$$= \int_0^{2\pi} (2\cos\theta - \sin\theta)(-\sin\theta)d\theta = \int_0^{2\pi} (-\sin 2\theta + \sin^2\theta)\,d\theta \qquad \text{...(9.35)}$$

$$= \int_0^{2\pi}\left[-\sin 2\theta + \frac{1}{2}(1 - \cos 2\theta)\right]d\theta = \left[\frac{\cos 2\theta}{2} + \frac{\theta}{2} - \frac{\sin 2\theta}{2}\right]_0^{2\pi} = \pi.$$

Next, curl $\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix} = (-2yz + 2yz)\,\hat{i} + (0 - 0)\,\hat{j} + (0 + 1)\,\hat{k} = \hat{k}.$

Therefore, curl $\vec{F}.\hat{N} = \hat{k}.\hat{N}$, where $\hat{N}$ is the unit outward normal to the surface S.

If E is the projection of S on the xy-plane then $dxdy = |\hat{N}.\hat{k}|dS$. Thus,

no need to find $\hat{N}$

$$\iint_S \text{curl } \vec{F}.\hat{N}\,dS = \iint_E \hat{k}.\hat{N}\frac{dxdy}{|\hat{k}.\hat{N}|} = \iint_E dxdy = 4\int_0^1\int_0^{\sqrt{1-x^2}} dydx$$

$$= 4\int_0^1 \sqrt{1-x^2}\,dx = 4\left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}x\right]_0^1 = 4\frac{1}{2}\frac{\pi}{2} = \pi. \qquad \text{...(9.36)}$$

From the equality of (9.35) and (9.36), Stokes' theorem is verified.

Example 9.24: Evaluate $\oint_C (2y^3dx + x^3dy + zdz)$, using Stokes' theorem where C is the trace of the cone $z = \sqrt{x^2+y^2}$ intersected by the plane $z = 4$ and S is the surface of the cone below $z = 4$.

Solution: We have, $\vec{F} = 2y^3\hat{i} + x^3\hat{j} + z\hat{k}$. Thus,

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2y^3 & x^3 & z \end{vmatrix} = (0)\hat{i} - (0)\hat{j} + (3x^2 - 6y^2)\hat{k}.$$

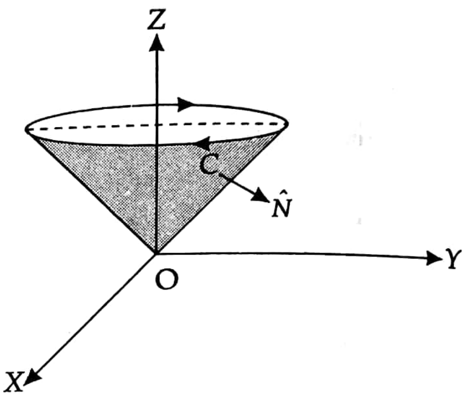

Fig. 9.15

The outward normal to the surface S points towards the downward direction as shown in the Fig. 9.15, and hence the direction of C is taken in the clockwise direction.

Let $f(x, y, z) = \sqrt{x^2+y^2} - z$ be the equation of the surface S, then the unit outward normal

$\hat{N} = \dfrac{\text{grad } f}{|\text{grad } f|}$. We have, grad $f = \dfrac{x\hat{i} + y\hat{j}}{\sqrt{x^2+y^2}} - \hat{k} = \dfrac{x\hat{i} + y\hat{j} - z\hat{k}}{z}$, and thus

$$\hat{N} = \frac{(x\hat{i} + y\hat{j} - z\hat{k})/z}{\sqrt{(\underbrace{x^2+y^2}_{=z^2} + z^2)/z^2}} = \frac{x\hat{i} + y\hat{j} - z\hat{k}}{\sqrt{2}z}, \text{ using } x^2 + y^2 = z^2.$$

By Stokes' theorem

$$\oint_C (2y^3dx + x^3dy + zdz) = \iint_S \text{curl } \vec{F}.\hat{N}\, dS, \text{ where } \vec{F} = 2y^3\hat{i} + x^3\hat{j} + z\hat{k}.$$

$$\text{Now, curl } \vec{F}.\hat{N} = (3x^2 - 6y^2)\hat{k}.\frac{x\hat{i} + y\hat{j} - z\hat{k}}{\sqrt{2}z} = -\frac{3x^2 - 6y^2}{\sqrt{2}}.$$

Also, $dxdy = (\hat{N}.\hat{k})ds = -\dfrac{1}{\sqrt{2}}dS$, or $dS = -\sqrt{2}\,dxdy$.

Therefore $\iint_S \text{curl } \vec{F}.\hat{N}\, dS = \iint_E (3x^2 - 6y^2)dxdy$, where E is the region $x^2 + y^2 = 16$.

Substituting $x = r\cos\theta$, $y = r\sin\theta$, we obtain

$$\iint_S \text{curl } \vec{F}.\hat{N}\, dS = \iint_E (3x^2 - 6y^2)dxdy = \int_0^4 \int_{2\pi}^0 (3\cos^2\theta - 6\sin^2\theta)r^3 drd\theta$$

* upper l = 0
lower l = 2π
cw dirn

$$= \frac{3}{2}\int_{0}^{4}\int_{2\pi}^{0} [(1 + \cos 2\theta) - 2(1 - \cos 2\theta)]r^3 dr d\theta$$

$$= \frac{3}{2}\int_{0}^{4}\int_{2\pi}^{0} [3\cos 2\theta - 1]r^3 dr d\theta = \frac{3}{2}\left[\frac{r^4}{4}\right]_0^4 \left[\frac{3\sin 2\theta}{2} - \theta\right]_{2\pi}^{0} = 192\pi.$$

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Example 9.25: Evaluate $\iint_S \nabla \times \vec{F} \cdot d\vec{S}$ over the surface of the paraboloid $z = 1 - x^2 - y^2, z \geq 0$, where $\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$.

Solution: By Stokes' theorem $\iint_S \nabla \times \vec{F} \cdot d\vec{S} = \oint_C \vec{F} . d\vec{r}$, where C is the closed curve binding the surface S, $z = 1 - x^2 - y^2, z \geq 0$; the curve C is given by $x^2 + y^2 = 1, z = 0$. We have $\oint_C \vec{F} . d\vec{r} = \oint_C (ydx + zdy + xdz)$. Substituting $x = \cos\theta$, $y = \sin\theta$, $z = 0$, $0 \leq \theta \leq 2\pi$, we have

$$\oint_C \vec{F} . d\vec{r} = -\int_0^{2\pi} \sin\theta \sin\theta d\theta = -\frac{1}{2}\int_0^{2\pi} (1 - \cos 2\theta) d\theta = -\pi.$$

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Example 9.26: Show that $\oint_C \vec{r} . d\vec{r} = 0$, independently of the origin of $\vec{r}$, the position vector of a point $P(x, y, z)$.

Solution: If S is the open surface enclosed by the closed curve C, then by Stokes' theorem

$$\oint_C \vec{r} . d\vec{r} = \iint_S \text{curl } \vec{r} . d\vec{S} = \iint_S \vec{0} . d\vec{S} = 0$$

Example 9.27: If S be the surface of the sphere $x^2 + y^2 + z^2 = 9$, prove that $\int_S \text{curl } \vec{F} . d\vec{S} = 0$.

Solution: Cut open the surface of the sphere $x^2 + y^2 + z^2 = 9$, by any plane and let S_1 and S_2 denote its upper and lower portions and let C be the common curve binding both these portions. Then

$$\iint_S \text{curl } \vec{F} . d\vec{S} = \iint_{S_1} \text{curl } \vec{F} . d\vec{S} + \iint_{S_2} \text{curl } \vec{F} . d\vec{S} = \oint_C \vec{F} . d\vec{r} - \oint_C \vec{F} . d\vec{r} = 0,$$

using Stokes' theorem; the second integral is negative because it is taken in a direction opposite to that of the first.

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Example 9.28: If ϕ is a scalar point function, using Stokes' theorem prove that curl (grad ϕ) $= \vec{0}$.

Solution: Let $\vec{F}$ be a vector point function such that $\vec{F}$ = grad ϕ, where ϕ is the given scalar point function. By Stokes' theorem $\oint_C \vec{F}.d\vec{r} = \int_S \text{curl } \vec{F}.d\vec{S}$, where S is the open surface bounded by the closed curve C.

Consider, $\vec{F}.d\vec{r}$ = grad ϕ. $d\vec{r}$

$$= \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)\cdot(dx\hat{i} + dy\hat{j} + dz\hat{k}) = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = d\phi.$$

Also C being a closed curve, we have $\oint_C d\phi = 0$, that is, $\oint_C \vec{F}.d\vec{r} = 0$, therefore,

$\int_S \text{curl grad } \phi\cdot d\vec{S} = 0$. Thus curl grad $\phi = \vec{0}$.

Example 9.29: Prove that $\int_C \vec{a}\times\vec{r}.d\vec{r} = 2\vec{a}.\iint_S d\vec{S}$, $\vec{a}$ being any constant vector and $\vec{r}$ being the position vector of a point $P(x, y, z)$.

Solution: If S be the open surface enclosed by the closed curve C, then by Stokes' theorem

$$\int_C \vec{a}\times\vec{r}.d\vec{r} = \iint_S \text{curl }(\vec{a}\times\vec{r}).d\vec{S}$$

Also, $\nabla\times(\vec{a}\times\vec{r}) = \vec{a}(\nabla\cdot\vec{r}) - \vec{r}(\nabla\cdot\vec{a}) + (\vec{r}\cdot\nabla)\vec{a} - (\vec{a}\cdot\nabla)\vec{r}$

We have, $\nabla\cdot\vec{r} = 3$, $\nabla\cdot\vec{a} = 0$, $(\vec{r}\cdot\nabla)\vec{a} = 0$ and $(\vec{a}\cdot\nabla)\vec{r} = \vec{a}$.

Therefore, curl $(\vec{a}\times\vec{r}) = \nabla\times(\vec{a}\times\vec{r}) = 3\vec{a} - \vec{a} = 2\vec{a}$.

Hence, $$\int_C \vec{a}\times\vec{r}.d\vec{r} = 2\iint_S \vec{a}\cdot d\vec{S} = 2\vec{a}\cdot\iint_S d\vec{S}.$$

EXERCISE 9.5

1. Show using Stokes' theorem that in an irrotational field $\vec{F}$, the circulation of $\vec{F}$ along every closed surface is zero.

2. Verify Stokes' theorem for $\vec{F} = (x^2 + y^2)\hat{i} - 2xy\hat{j}$, taken around the rectangle bounded by the lines $x = \pm a$, $y = 0$, $y = b$.

3. Verify Stokes' theorem for the function $\vec{F} = x^2\hat{i} + xy\hat{j}$ integrated round the square of sides $x = a$, $y = a$, $x = 0$ and $y = 0$ in the plane $z = 0$.

4. Evaluate $\oint_C \vec{F}.d\vec{r}$ by Stokes' theorem, where $\vec{F} = y^2\hat{i} + x^2\hat{j} - (x+z)\hat{k}$ and C is the boundary of the triangle with vertices at (0, 0, 0), (1, 0, 0) and (1, 1, 0).

5. Evaluate by Stokes' theorem $\oint_C (yzdx + zxdy + xydz)$, where C is the curve $x^2 + y^2 = 1, z = y^2$.

6. Evaluate by Stokes' theorem $\oint_C (\sin zdx - \cos xdy + \sin ydz)$, where C is the boundary of the rectangle $0 \le x \le \pi, 0 \le y \le 1, z = 3$.

7. Apply Stokes' theorem to evaluate $\int_C [(x+y)dx + (2x-z)dy + (y+z)dz]$, where C is the boundary of the triangle with vertices (2, 0, 0), (0, 3,0) and (0, 0, 6).

8. Evaluate $\int_C \vec{F}.d\vec{r}$ where $\vec{F} = y\hat{i} + xz^3\hat{j} - zy^3\hat{k}$ and C is the circle $x^2 + y^2 = 4, z = 1.5$.

9. Evaluate $\oint_C \vec{F}.d\vec{r}$ using the Stokes' theorem, where $\vec{F} = 3y\hat{i} + 4z\hat{j} + 2x\hat{k}$ and C is the intersection of the sphere $x^2 + y^2 + z^2 = 16, x \ge 0$ and the cylinder $y^2 + z^2 = 4$.

10. Evaluate $\oint_C \vec{F}.d\vec{r}$ using the Stokes' theorem, where $\vec{F} = x\hat{i} + z\hat{j} + y\hat{k}$ and C is the boundary of the ellipsoid $y = \sqrt{144 - 36x^2 - 9z^2}/4$ in the plane $y = 0$.

11. Evaluate the integral $\iint_S (\nabla \times \vec{F})\cdot d\vec{S}$ by Stokes' theorem, where $\vec{F} = (x^2 - y^2)\hat{i} + (y^2 - x^2)\hat{j} + z\hat{k}$ and S is the portion of the surface $x^2 + y^2 - 2by + bz = 0$ whose boundary lies in the xy-plane; b being a constant.

9.6 VOLUME INTEGRAL GAUSS DIVERGENCE THEOREM

In this section we introduce volume integral. *Gauss divergence theorem* transforms surface integrals to volume integrals and conversely. The theorem is named so since it involves the divergence of a vector point function.

Volume Integral. To illustrate the concept of volume integral, consider a continuous vector function $\vec{F}(\vec{r})$ and a closed surface S enclosing the region (volume) E in space, for example, a solid cube, a ball, or the region between two concentric spheres. Subdivide the region E by planes parallel to the three co-ordinate planes into finite number of sub-regions $E_1, E_2, \ldots, E_n$. Let δV_i be the volume of the sub-region E_i enclosing an arbitrary point whose position vector is $\vec{r}_i$.

Consider the sum $\vec{V} = \sum_{i=1}^{n} \vec{F}(\vec{r}_i)\,\delta V_i$. The limit of this sum as $n \to \infty$ in such a way that $\delta V_i \to 0$, is called *the volume integral of* $\vec{F}(\vec{r})$ *over E and is* denoted by $\iiint_E \vec{F}(\vec{r})dV$. Under the assumption that $\vec{F}(\vec{r})$ is continuous in E and E is bounded by finitely many smooth surfaces, this limit is independent of the choice of subdivisions and the arbitrary position vector $\vec{r}_i$.

If $\vec{F}(\vec{r}) = F_1(x, y, z)\hat{i} + F_2(x, y, z)\hat{j} + F_3(x, y, z)\hat{k}$, then

$$\iiint_E \vec{F}\,dV = \hat{i}\iiint_E F_1(x,y,z)dxdydz + \hat{j}\iiint_E F_2(x,y,z)dxdydz + \hat{k}\iiint_E F_3(x,y,z)dxdydz \quad ...(9.37)$$

Next, we show that the triple integral of the divergence of a continuously differentiable vector function $\vec{F}(\vec{r})$ over a region E in space can be transformed into a surface integral of the normal component of $\vec{F}$ over the boundary surface S of E. This is executed by the Divergence theorem of Gauss, a three dimensional analog of Green's theorem in the plane.

Theorem 9.4: (Gauss Divergence Theorem) *Let E be a closed and bounded region in space whose boundary is a piecewise smooth oriented surface. Let $\vec{F}$ be a vector function which is continuous and has continuous first order partial derivatives in E,* then

$$\iiint_E \operatorname{div}\vec{F}\,dV = \iint_S \vec{F}\cdot\hat{N}\,dS, \quad ...(9.38)$$

where $\hat{N}$ is the outward unit normal vector of S.

Proof. Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ and α, β, γ be the angles which the outward unit normal vector $\hat{N}$ make with the positive direction of x, y, z axes respectively, then $\hat{N} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$. Thus, the cartesian equivalent of the divergence theorem is

$$\iiint_E \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)dxdydz = \iint_S (F_1\cos\alpha + F_3\cos\beta + F_3\cos\gamma)dS \quad ...(9.39)$$

To prove the divergence theorem it is sufficient to show that

$$\iiint_E \frac{\partial F_1}{\partial x}\,dxdydz = \iint_S F_1\cos\alpha dS \quad ...(9.40)$$

$$\iiint_E \frac{\partial F_2}{\partial y}\,dxdydz = \iint_S F_2\cos\beta dS \quad ...(9.41)$$

$$\iiint_E \frac{\partial F_3}{\partial z} dxdydz = \iint_S F_3 \cos \gamma dS \quad ...(9.42)$$

We prove (9.42) only, the remaining results can be proved on similar lines.

We prove it for a special region E which is bounded by a piecewise smooth orientable surface S that has the property that any straight line parallel to z-axis cuts it in two points only, as shown in Fig. 9.16.

Let R be the orthogonal projection of S in the xy-plane; and let the bottom surface be

$$S_1 : z = h(x, y), \quad (x, y) \in R;$$

the top surface be $\quad S_2 : z = g(x, y); \quad (x, y) \in R;$

and the side surface be $S_3 : h(x, y) \le z \le g(x, y); \quad (x, y) \in R.$

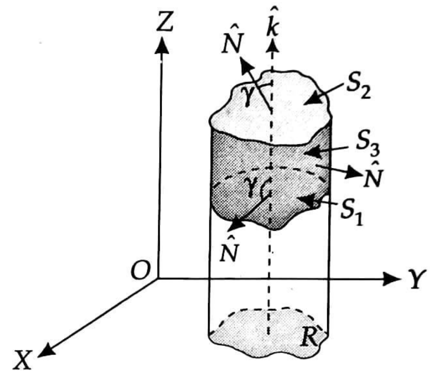

Fig. 9.16

Thus, $$\iiint_E \frac{\partial F_3}{\partial z} dx\,dy\,dz = \iint_R dxdy \left(\int_{h(x,y)}^{g(x,y)} \frac{\partial F_3}{\partial z} dz \right)$$

$$= \iint_R [F_3(x, y, g)dxdy - F_3(x, y, h)dx\,dy]$$

$$= \iint_R F_3(x, y, g)\,dx\,dy - \iint_R F_3(x, y, h)\,dx\,dy.$$

We show that this is equal to the R.H.S. of (9.42). On the lateral portion S_3 of S we have $\gamma = \pi/2$ and thus $\cos \gamma = 0$. Hence, this portion does not contribute to the surface integral in (9.42) refer to Fig. 9.16, and thus R.H.S. of (9.42) gives

$$\iint_S F_3 \cos \gamma dS = \iint_{S_1} F_3 \cos \gamma\, dS + \iint_{S_2} F_3 \cos \gamma dS \quad ...(9.43)$$

On S_1 of S, the normal $\hat{N}$ to S makes an obtuse angle γ with $\hat{k}$. Therefore $dxdy = -\cos \gamma ds$. Thus,

$$\iint_R F_3(x, y, h)dxdy = -\iint_{S_1} F_3 \cos \gamma dS.$$

Also on S_2 of S, the normal $\hat{N}$ to S makes an acute angle γ with $\hat{k}$, therefore, $dxdy = \cos \gamma dS$. Thus,

$$\iint_R F_3(x, y, g)dxdy = \iint_{S_2} F_3 \cos \gamma dS.$$

Therefore,

$$\iiint_E \frac{\partial F_3}{\partial z} dxdydz = \iint_{S_1} F_3 \cos \gamma dS + \iint_{S_2} F_3 \cos \gamma dS$$

which is the same as (9.43). This proves (9.42).

Similarly, by considering the projections on yz and zx-planes and proceeding on parallel lines we can prove (9.40) and (9.41).

This proves the divergence theorem for special region.

For any region E which can be subdivided into finitely many special regions by means of auxiliary surfaces, the theorem follows by adding the result for each part separately. The surface integrals over the auxiliary surfaces cancel in pairs, and the sum of the remaining surface integrals is the surface integral over the whole boundary S of E. The volume integrals over the parts of E add up to give the volume integral over E.

Example 9.30: Verify divergence theorem for $\vec{F} = x^2\hat{i} + z\hat{j} + yz\hat{k}$ taken over the cube bounded by $x = 0, x = 1; y = 0, y = 1; z = 0, z = 1$.

Solution: The Gauss divergence theorem is $\iiint\limits_E \text{div } \vec{F}\, dV = \iint\limits_S \vec{F}\cdot\hat{N}\, ds$.

Here $\vec{F} = x^2\hat{i} + z\hat{j} + yz\hat{k}$ gives div $\vec{F} = \dfrac{\partial}{\partial x}(x^2) + \dfrac{\partial}{\partial y}(z) + \dfrac{\partial}{\partial z}(yz) = 2x + y$.

$$\text{Thus,} \quad \iiint\limits_E \text{div } \vec{F}\, dv = \int_0^1\int_0^1\int_0^1 (2x+y)\,dxdydz = \int_0^1\int_0^1 \left[x^2 + xy\right]_0^1 dydz$$

$$= \int_0^1\int_0^1 (1+y)\,dydz = \int_0^1 \left(y + \frac{y^2}{2}\right)_0^1 dz = \frac{3}{2}\int_0^1 dz = \frac{3}{2}. \qquad \text{...(9.44)}$$

To evaluate the surface integral, we divide the piecewise smooth closed surface S of the cuboid into six smooth surfaces, as shown in Fig. 9.17, given by

S_1 : $AC'PB'$ $\qquad$ S_2 : $OBA'C$

S_3 : $BA'PC'$ $\qquad$ S_4 : $OAB'C$

S_5 : $CA'PB'$ $\qquad$ S_6 : $OBC'A$

$$\text{Thus,} \quad \iint\limits_S \vec{F}\cdot\hat{N}\, dS = \iint\limits_{(S_1+\ldots+S_6)} \vec{F}\cdot\hat{N}\, dS \qquad \text{...(9.45)}$$

On S_1, $x = 1$, we have $\hat{N} = \hat{i}$, thus $\vec{F}\cdot\hat{N} = x^2$, so that

$$\iint\limits_{S_1} \vec{F}\cdot\hat{N}\, dS = \int_0^1\int_0^1 x^2 dydz = \int_0^1\int_0^1 dydz = 1.$$

On S_2, $x = 0$, $\hat{N} = -\hat{i}$, thus $\vec{F}\cdot\hat{N} = -x^2$, so that

$$\iint\limits_{S_2} \vec{F}\cdot\hat{N}\, dS = -\int_0^1\int_0^1 x^2 dydz = \int_0^1\int_0^1 0.dydz = 0.$$

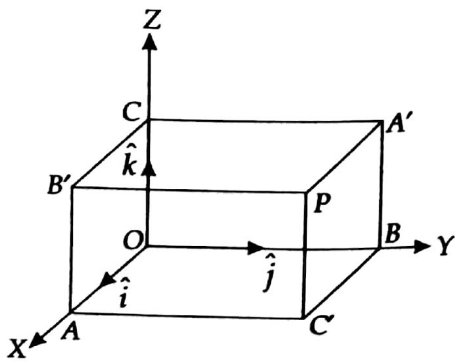

Fig. 9.17

On S_3, $y = 1$, $\hat{N} = \hat{j}$, thus $\vec{F}.\hat{N} = z$, so that

$$\iint_{S_3} \vec{F}.\hat{N}\, dS = \int_0^1\int_0^1 z dx dz = \int_0^1 \left(\frac{z^2}{2}\right)_0^1 dx = \int_0^1 \frac{dx}{2} = \frac{1}{2}.$$

On S_4, $y = 0$, $\hat{N} = -\hat{j}$, thus $\vec{F}.\hat{N} = -z$, so that

$$\iint_{S_4} \vec{F}.\hat{N}\, dS = -\int_0^1\int_0^1 z dx dz = -\int_0^1 \left(\frac{z^2}{2}\right)_0^1 dx = -\int_0^1 \frac{dx}{2} = -\frac{1}{2}.$$

On S_5, $z = 1$, $\hat{N} = \hat{k}$, thus $\vec{F}.\hat{N} = yz$, so that

$$\iint_{S_5} \vec{F}.\hat{N}\, dS = \int_0^1\int_0^1 yz dx dy = \int_0^1\int_0^1 y dx dy = \int_0^1 \left(\frac{y^2}{2}\right)_0^1 dx = \frac{1}{2}.$$

On S_6, $z = 0$, $\hat{N} = -\hat{k}$, thus $\vec{F}.\hat{N} = -yz$, so that

$$\iint_{S_6} \vec{F}.\hat{N}\, dS = -\int_0^1\int_0^1 yz dx dy = -\int_0^1\int_0^1 0.dx dy = 0.$$

Therefore from (9.45)

$$\iint_S \vec{F}.\hat{N}\, dS = 1 + 0 + \frac{1}{2} - \frac{1}{2} + \frac{1}{2} + 0 = 3/2. \qquad ...(9.46)$$

The equality of (9.44) and (9.46) verifies the divergence theorem.

Example 9.31: For any closed surface S, prove that

$$\iint_S [x(y-z)\hat{i} + y(z-x)\hat{j} + z(x-y)\hat{k}].d\vec{S} = 0.$$

Solution: If E be the volume enclosed by the closed surface S, then by divergence theorem

$$\iint_S \vec{F}.\hat{N}\, dS = \iiint_E \text{div}\,\vec{F}\, dv,$$ where the symbols have their usual meanings.

Here, $\vec{F} = x(y-z)\hat{i} + y(z-x)\hat{j} + z(x-y)\hat{k}$, gives $\text{div}\,\vec{F} = \frac{\partial}{\partial x}(y-z)x + \frac{\partial}{\partial y}(z-x)y + \frac{\partial}{\partial z}(x-y)z = 0.$

Therefore, $$\iiint_E \text{div}\,\vec{F}\, dv = 0.$$

This proves the result.

Example 9.32: Using divergence theorem, prove that

(a) $\iint_S \vec{r}.d\vec{S} = 3V$ (b) $\iint_S \nabla r^2.d\vec{S} = 6V$

where S is any closed surface enclosing a volume V, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r^2 = x^2 + y^2 + z^2$.

Solution: (*a*) By divergence theorem $\iint_S \vec{r}.d\vec{S} = \iiint_V \text{div}\ \vec{r}\ dV = 3\iiint_V dv = 3V.$

(b) Let $\vec{F} = \nabla r^2$, then by divergence theorem $\iint_S \nabla r^2.d\vec{S} = \iiint_V \text{div}\ \nabla r^2 dV.$

We have, $$\text{div}\ \nabla r^2 = \nabla\cdot\nabla r^2 = \nabla^2 r^2 = \frac{\partial^2}{\partial x^2}(x^2) + \frac{\partial^2}{\partial y^2}(y^2) + \frac{\partial^2}{\partial z^2}(z^2) = 6.$$

Thus, $$\iint_S \nabla r^2\cdot dS = \iiint_V 6dV = 6V.$$

This proves the result.

Example 9.33: Using divergence theorem evaluate $\iint_S \vec{F}.d\vec{S}$ where $\vec{F} = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$ and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$.

Solution: By divergence theorem $\iint_S \vec{F}.d\vec{S} = \iiint_E \text{div}\ \vec{F}dV$, where the symbols have their usual meanings.

We have $$\text{div}\ \vec{F} = \nabla\cdot(x^3\hat{i} + y^3\hat{j} + z^3\hat{k}) = 3(x^2 + y^2 + z^2) = 3a^2, \text{ thus}$$

$$\iint_S \vec{F}.d\vec{S} = 3a^2\iiint_E dV = 3a^2.\frac{4}{3}\pi a^3 = 4\pi a^5,$$

since the volume of $x^2 + y^2 + z^2 = a^2$ is $(4/3)\ \pi a^3$.

Example 9.34: Evaluate $\iint (xdydz + ydzdx + zdxdy)$ over the surface of a sphere of radius a.

Solution: We have $xdydz + ydzdx + zdxdy = (x\hat{i}\cdot\hat{N} + y\hat{j}\cdot\hat{N} + z\hat{k}\cdot\hat{N})dS = \vec{r}.d\vec{S}$.

By divergence theorem $\iint_S \vec{r}.d\vec{S} = \iiint_E \text{div}\ \vec{r}\ dV = 3\iiint_E dV = 3\frac{4}{3}\pi a^3 = 4\pi a^3.$

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Example 9.35: Verify divergence theorem for $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ taken over the region bounded by the cylinder $x^2 + y^2 = 4$, $z = 0$, $z = 3$.

Solution: We have, $\text{div}\,\vec{F} = \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2) = 4 - 4y + 2z$. Thus

$$\iiint_E \text{div}\,\vec{F}\,dv = \iiint_E (4 - 4y + 2z)dxdydz = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left(\int_0^3 (4 - 4y + 2z)dz\right)dydx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left[4z - 4yz + z^2\right]_0^3 dydx = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(21 - 12y)dydx$$

$$= 21\int_{-2}^{2}\left(\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy\right)dx, \quad \text{since} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 12y\,dy = 0$$

$$= 84\int_0^2 \sqrt{4-x^2}\,dx = 84\left[\frac{x\sqrt{4-x^2}}{2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2 = 84[2\sin^{-1}1] = 84\pi. \quad ...(9.47)$$

To evaluate the surface integral, we divide the piecewise smooth closed surface S of the cylinder into three smooth surfaces, as shown in Fig. 9.18.

S_1 : the circular base in the plane $z = 0$

S_2 : the circular top in the plane $z = 3$

S_3 : the curved surface of the cylinder given by $x^2 + y^2 = 4$; $0 \le z \le 3$.

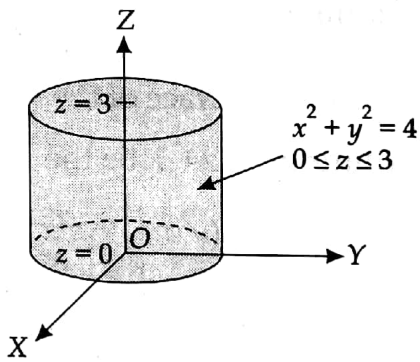

Fig. 9.18

We have, $$\iint_S \vec{F}.\hat{N}\,dS = \iint_{S_1+S_2+S_3} \vec{F}.\hat{N}\,dS \quad ...(9.48)$$

On S_1, $z = 0$, $\hat{N} = -\hat{k}$, thus $\vec{F}.\hat{N} = (4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}).(-\hat{k}) = -z^2 = 0$.

Therefore, $$\iint_{S_1} \vec{F}.\hat{N}\,dS = 0.$$

On S_2, $z = 3$, $\hat{N} = \hat{k}$, thus $\vec{F}.\hat{N} = (4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}).(\hat{k}) = z^2 = 9$.

Therefore, $$\iint_{S_2} \vec{F}.\hat{N}\,dS = 9\iint_{S_2} ds = 9\,(\text{area of circle } x^2 + y^2 = 4) = 36\pi.$$

On S_3, $x^2 + y^2 = 4$, $\hat{N} = \dfrac{\nabla(x^2+y^2)}{|\nabla(x^2+y^2)|} = \dfrac{2x\hat{i}+2y\hat{j}}{2\sqrt{x^2+y^2}} = \dfrac{x\hat{i}+y\hat{j}}{2}$,

thus $$\vec{F}.\hat{N} = (4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}).\frac{(x\hat{i}+y\hat{j})}{2} = 2x^2 - y^3.$$

Therefore, $$\iint_{S_3} \vec{F}.\hat{N}\,dS = \iint_{S_3} (2x^2 - y^3)dS.$$

Also on S_3, $x = 2\cos\theta$, $y = 2\sin\theta$ and $dS = 2d\theta dz$. Thus

$$\iint_{S_3} (2x^2 - y^3)dS = \int_{z=0}^{3}\int_{\theta=0}^{2\pi} (8\cos^2\theta - 8\sin^3\theta)\,2d\theta dz = 48\int_0^{2\pi} (\cos^2\theta - \sin^3\theta)d\theta$$

$$= 48\left[\frac{1}{2}\int_0^{2\pi}(1+\cos 2\theta)d\theta - \frac{1}{4}\int_0^{2\pi}(3\sin\theta - \sin 3\theta)d\theta\right] = 48 \times \frac{1}{2} \times 2\pi = 48\pi.$$

Thus (9.48) becomes

$$\iint_S \vec{F}.\hat{N}\,dS = 0 + 36\pi + 48\pi = 84\pi. \qquad ...(9.49)$$

The equality of (9.47) and (9.49) verifies the divergence theorem.

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Example 9.36: Use divergence theorem to evaluate $\iint_S (y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k}).d\vec{S}$, where S is the closed surface of the region bounded by the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$ and the plane $z = 0$.

Solution: Let V be the volume enclosed by the closed surface S, then by divergence theorem

$$\iint_S (y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k}).d\vec{S} = \iiint_V \text{div}\,(y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k})dV = 2\iiint_V zy^2 dV.$$

Using spherical polar co-ordinates, to find the volume enclosed by the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$, we substitute $x = r\sin\phi\cos\theta$, $y = r\sin\phi\sin\theta$, $z = r\cos\phi$, $dv = r^2\sin\phi\,dr\,d\theta\,d\phi$, where $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi/2$. Thus

$$2\iiint_V zy^2 dV = 2\int_0^{\pi/2}\int_0^{2\pi}\int_0^1 r\cos\phi . r^2\sin^2\phi\sin^2\theta\, r^2\sin\phi\,dr d\theta d\phi$$

$$= 2\int_0^{\pi/2}\int_0^{2\pi}\int_0^1 r^5\sin^3\phi\cos\phi\sin^2\theta\,dr d\theta d\phi$$

$$= \left(\int_0^{\pi/2} \sin^3 \phi \cos \phi \, d\phi \right) \left(\int_0^{2\pi} 2 \sin^2 \theta \, d\theta \right) \left(\int_0^1 r^5 dr \right)$$

$$= \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{2\pi} \left[\frac{\sin^4 \phi}{4} \right]_0^{\pi/2} \left[\frac{r^6}{6} \right]_0^1 = 2\pi \times \frac{1}{4} \times \frac{1}{6} = \pi/12.$$

Example 9.37: Use the divergence theorem to evaluate $\iint_S \vec{F} . \hat{N} \, dS$, where $\vec{F} = x^2 z \hat{i} + y \hat{j} - xz^2 \hat{k}$ and S is the boundary of the region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4y$.

Solution: Let V be volume enclosed by the closed surface S, then by divergence theorem

$$\iint_S \vec{F} . \hat{N} \, dS = \iiint_V \text{div } \vec{F} \, dV$$

$$\text{div } \vec{F} = \frac{\partial}{\partial x}(x^2 z) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(-xz^2) = 2xz + 1 - 2xz = 1. \text{ Thus}$$

$$\iint_S \vec{F} . \hat{N} \, dS = \iiint_E \text{div } \vec{F} \, dV = \int_{y=0}^{4} \int_{x=-\sqrt{4y-y^2}}^{\sqrt{4y-y^2}} \int_{z=x^2+y^2}^{4y} dz dx dy,$$

$x^2+(y-2)^2=4$

(projection of S on xy-plane is circle $x^2 + y^2 = 4y$)

$$= \int_{y=0}^{4} \int_{x=-\sqrt{4y-y^2}}^{\sqrt{4y-y^2}} (4y - x^2 - y^2) dx dy = 2 \int_0^4 \int_0^{\sqrt{4y-y^2}} [(4y - y^2) - x^2] dx dy$$

$$= 2 \int_0^4 \left[(4y - y^2)x - \frac{x^3}{3} \right]_0^{\sqrt{4y-y^2}} dy = 2 \int_0^4 [(4y - y^2)]^{3/2} - \frac{1}{3}(4y - y^2)^{3/2}] dy$$

$$= \frac{4}{3} \int_0^4 (4y - y^2)^{3/2} dy = \frac{4}{3} \int_0^4 [4 - (y - 2)^2]^{3/2} dy$$

$$= \frac{4}{3} \int_{-\pi/2}^{\pi/2} 16 \cos^4 t dt = \frac{128}{3} \int_0^{\pi/2} \cos^4 t dt, \qquad [(y - 2) = 2 \sin t]$$

$$= \frac{128}{3} \frac{3}{4} \frac{1}{2} \frac{\pi}{2} = 8\pi.$$

EXERCISE 9.6

1. Using Gauss divergence theorem show that if $\vec{F}$ defines a solenoidal field, then the flux of $\vec{F}$ around every closed surface is zero.
2. Verify divergence theorem for $\vec{F} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$ taken over the rectangular parallelopiped given by $0 \le x \le a, 0 \le y \le b, 0 \le z \le c$.
3. If $\vec{F} = xy^2\hat{i} + yz^2\hat{j} + zx^2\hat{k}$, evaluate $\iint_S \vec{F}.d\vec{S}$ over the sphere given by $x^2 + y^2 + z^2 = 1$.
4. Verify the divergence theorem for $\vec{F} = (2xy + z)\hat{i} + y^2\hat{j} - (x + 3y)\hat{k}$ when the surface S is that of the region bounded by the plane $2x + 2y + z = 6$ in the first octant.
5. Apply divergence theorem to evaluate $\iint (lx^2 + my^2 + nz^2)$ taken over the sphere $(x - a)^2 + (y - b)^2 + (z - c)^2 = \rho^2$; l, m, n being the direction cosines of the external normal to the sphere.
6. Evaluate $\iint_S \vec{F}.\hat{N}\,dS$ over the surface of the region above the xy-plane bounded by the cone $z^2 = x^2 + y^2$ and the plane $z = 4$, where $\vec{F} = 4xz\hat{i} + xyz^2\hat{j} + 3z\hat{k}$.
7. Evaluate $\iint_S (xdydz + ydzdx + zdxdy)$ using divergence theorem, where S is the surface of the sphere $(x - 2)^2 + (y - 2)^2 + (z - 2)^2 = 4$.
8. Evaluate $\iint_S (\nabla \times \vec{F}).\hat{N}\,dS$, where S is the surface of the cone $z = 2 - \sqrt{x^2 + y^2}$ above the xy-plane and $\vec{F} = (x - z)\hat{i} + (x^3 + yz)\hat{j} - 3xy^2\hat{k}$.
9. Show that $\iint_S (\vec{A}.\hat{N})\,dS = 0$, where $\vec{A}$ is a constant vector.
10. Show that $\iint_S r^n(\vec{r}.\hat{N})\,dS = (n + 3)\iiint_E r^n\,dv, n \neq 3$, where $\vec{r}$ is the position vector of the point $\rho(x, y, z)$ and $r = |\vec{r}|$.
11. Using divergence theorem, evaluate $\iint_S (x^3dydz + x^2ydzdx + x^2zdxdy)$, where S is the closed surface consisting of the cylinder $x^2 + y^2 = a^2$ and the circular discs $z = 0$, and $z = b$.

Exercise 9.1 (p. 522)

1. $75\frac{1}{3}\hat{i} + 360\hat{j} - 42\hat{k}$. 2. (a) 12 (b) $-24\hat{i} - \frac{40}{3}\hat{j} + \frac{64}{5}\hat{k}$.

Exercise 9.2 (p. 531)

1. $-7/6$
2. 0
3. $3/2$
4. (a) 16 (b) 16
5. $\left(2-\frac{\pi}{4}\right)\hat{i} - \left(\pi - \frac{1}{2}\right)\hat{j}$
6. 6
7. 0
8. $-1/2$
9. $\sinh(x+y)$

Exercise 9.3 (p. 538)

1. 180
2. 8
3. 84π
4. $2\pi abc$
5. $\frac{2}{3}a^3 - \frac{3}{8}a^4$
6. 0

Exercise 9.4 (p. 543)

1. $-\left(\frac{\pi}{4} + \frac{2}{\pi}\right)$
4. $\frac{11}{30}$
5. $\frac{3\pi a^2}{8}$
7. 0

Exercise 9.5 (p. 550)

2. $-4ab^2$
4. $\frac{1}{3}$
5. 0
6. 2
7. 21
8. $\frac{19}{2}\pi$
9. -16π
10. 0
11. $2\pi b^3$.

Exercise 9.6 (p. 560)

3. $\frac{4}{5}\pi$
5. $\frac{8\pi}{3}(a+b+c)\rho^3$
6. 320π
7. 32π
8. 12π
11. $\frac{5}{4}\pi a^4 b$.