Fourier Series

17 April 2020 12:20

Dirichlet

$c < x < c+2\pi$ $f(x)$ is a periodic fn.

with period 2π

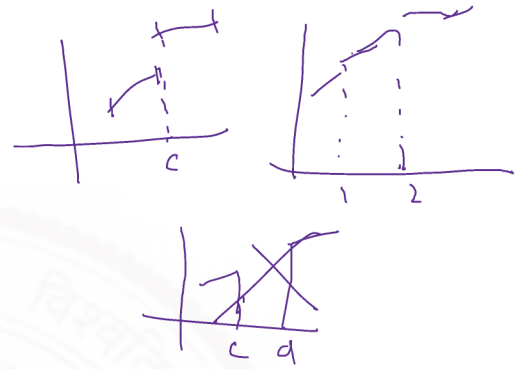
$\sqrt{x} = \sqrt{4} = \begin{cases} -2 \\ 2 \end{cases}$

(1) Single valued

(2) finite no. of disc.

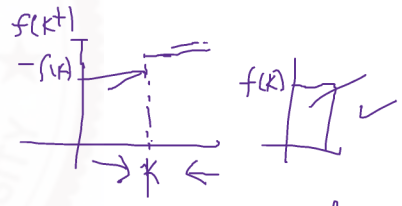
p. cont

(3) $\int_c^{c+2\pi} |f(z)|\, dz$ exist



Fourier Series:

Let $f(x)$ be periodic with period 2π and satisfies the Dirichlet condn. then we can expand $f(x)$ as follows:



$x=k$ is pt of cont then

R.H.S $\to f(k)$

$x=k$ is pt of disc.

R.H.S $\to \dfrac{f(k^+)+f(k^-)}{2}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where a_0, a_n & b_n are called Fourier coeff.

given by

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(z)\, dz$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(z) \cos nx\, dx$$

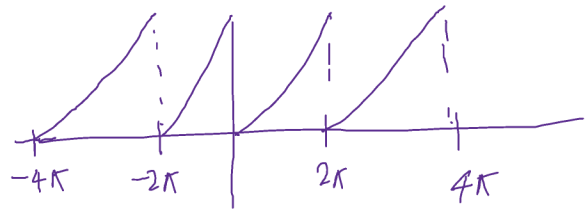
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(z) \sin nx\, dx$$

H.W.

H.W. $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(z)\sin nx\, dx$

Problems:

1. $f(x) = x \quad 0 < x < 2\pi$



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} = \frac{\cancel{4}^{2}\pi^{\cancel{2}}}{\cancel{2}\,\cancel{\pi}} = 2\pi$$

$$\therefore \boxed{a_0 = 2\pi}$$

$$\left[\int u\,dv = uv - \int v\,du\right.$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\, d\left(\frac{\sin nx}{n}\right)$$

$$= \cancel{x\,\frac{\sin nx}{n}\Big|_0^{2\pi}} - \int_0^{2\pi}\frac{\sin nx}{n}\,dx$$

$$= + \frac{\cos nx}{n^2}\Big|_0^{2\pi}$$

$$= \frac{1}{n^2}(1-1) = 0$$

$$\therefore \boxed{a_n = 0}$$

$$b_n = \frac{1}{\quad}\int^{2\pi} f(x)\sin nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\; d\left(-\frac{\cos nx}{n}\right)$$

$$= \frac{1}{\pi}\left\{ -x\frac{\cos nx}{n}\Big|_0^{2\pi} + \int_0^{2\pi}\cancel{\frac{\cos nx}{n}}\,dx \right\}$$

$$= \frac{1}{\cancel{\pi}}\left(-\frac{2\cancel{\pi}}{n}\right) = -\frac{2}{n}$$

$$\therefore \boxed{b_n = -\frac{2}{n}}$$

$\therefore$ The Fourier series of the given fn $f(x) = x$ is

$$x = \frac{\cancel{2}\pi}{\cancel{2}} + \sum_{n=1}^{\infty}\left[\cancel{(0)\cos nx} - \frac{2}{n}\sin nx\right]$$

$$\therefore x = \pi - 2\sum_{n=1}^{\infty}\frac{\sin nx}{n} \;//$$

$$\therefore x = \pi - 2\left(\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots\right)$$

$[c, c+2\pi]$ Arbitrary int $[-c, c]$ Harmonic series

F.S — General ✓ — Even odd ✓ — half range series ✓

① $\boxed{y = f(x)}$ empirical relation ✓

②

③

x	x_0	x_1			x_n
y	y_0	y_1			y_n

3

x	x_0	x_1			x_n
y	y_0	y_1			y_n

2. Given that $f(x) = x + x^2$ for $x \in (-\pi, \pi)$ then find the F.S expn of $f(x)$ and Deduce $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

Sol: Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x + x^2\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\cancel{\frac{\pi^2}{2}} + \frac{\pi^3}{3} - \cancel{\frac{\pi^2}{2}} + \frac{\pi^3}{3}\right]$$

$$\boxed{a_0 = \frac{2\pi^2}{3}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$$

$$\int_0^{\pi} x^2\, d\left(\frac{\sin nx}{n}\right)$$

$$= \frac{2}{\pi}\left\{\cancel{x^2 \frac{\sin nx}{n}\Big|_0^{\pi}} + \int_0^{\pi} 2x\,\frac{\sin nx}{n}\,dx\right\}$$

$$= +\frac{4}{\pi}\left[-x\,\frac{\cos nx}{n^2}\Big|_0^{\pi} - \int_0^{\pi} \frac{\cos nx}{n^2}\,dx\right]$$

$$= \frac{4}{\pi}\left\{\frac{\pi(-1)^n}{n^2} - \cancel{\frac{\sin nx}{n^3}\Big|_0^{\pi}}\right\}$$

$$= \frac{4}{\pi}\left\{ \frac{\pi(-1)}{n^2} - \frac{\sin n\pi}{n^3}\Big|_0 \right\}$$

$$\boxed{a_n = \frac{4(-1)^n}{n^2}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx$$

$$= \frac{2}{\pi}\left[-x\frac{\cos nx}{n}\Big|_0^{\pi} + \int_0^{\pi}\frac{\cos nx}{n}\,dx\right]$$

$$= \frac{2}{\pi}\left[-\frac{\pi(-1)^n}{n} + \cancel{\frac{\sin nx}{n^2}\Big|_0^{\pi}}\right]$$

$$\boxed{b_n = -\frac{2(-1)^n}{n}}$$

$$\therefore\ x+x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n\cos nx}{n^2} - 2\sum_{n=1}^{\infty}\frac{(-1)^n\sin nx}{n} \quad -(1)$$

At $x=\pi$

$$\pi+\pi^2 = \frac{\pi^2}{3} + 4\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right) \quad -(2)$$

At $x=-\pi$

$$-\pi+\pi^2 = \frac{\pi^2}{3} + 4\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right) \quad -(3)$$

(2)+(3) gives

$$2\pi^2 = \frac{2\pi^2}{3} + 8\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right)$$

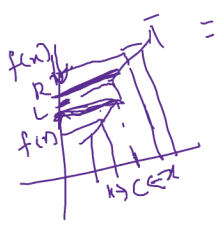
$$= 1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots = \sum\frac{1}{n^2}$$

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2}$$

Functions defined in two or more sub-range :

(3) If $f(x) = \begin{cases} -\pi & -\pi < x \le 0 \\ x & 0 < x < \pi \end{cases}$ then find the F.S. of $f(x)$

(2) S.T. $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$



$x = x_0$

$f(x_0) = R.H.S$

$x = x_0$

$$\frac{f(x-) + f(x+)}{2} = \frac{x-\pi}{2}$$

Let $\frac{x-\pi}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi^2 + \frac{x^2}{2}\Big|_0^{\pi}\right]$$

$$-\pi\,[x]_{-\pi}^{0} = -\pi\,[0+\pi] = -\pi^2$$

$$= \frac{1}{\pi}\left[-\frac{\pi^2}{2}\right] = -\frac{\pi}{2}$$

$$\therefore \boxed{a_0 = -\frac{\pi}{2}}$$

$$b_n = \frac{1}{n}(1 - 2\cos n\pi)$$

$$a_n = \frac{1}{\pi n^2}(\cos n\pi - 1)$$

$$\frac{x-\pi}{2} = \frac{-\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \cdots\right)$$
$$+ 3\sin x - \frac{\sin 2x}{2} + \frac{3\sin 3x}{3} - \cdots$$

At $x = 0$

$$\frac{-\pi}{2} = \frac{-\pi}{4} - \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

Dt: 22.04.20

Even and odd

If $f(-x) = f(x)$ then f is even

Ex $\cos x$, x^2, $|x|$ ...

If $f(-x) = -f(x)$ then f is odd

Ex: $\sin x$, x, x^3, ...

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f \text{ is even} \\ 0 & \text{if } f \text{ is odd} \end{cases}$$

If f is a even fn. $[-\pi, \pi]$ then its F.S. is given by

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx}$$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

If f is an odd fn in $[-\pi, \pi]$ then F.S. of $f(x)$ is

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin nx}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$\int_0 f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$\cancel{b_n = 0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

Ex: $f(x) = x^2 \quad x \in [-\pi, \pi]$

Hence find $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots \infty = ?$

Sol: $f(x) = x^2$ is an even fn.

$$\therefore \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$$= \frac{2}{\pi}\int_0^{\pi} x^2\,dx$$

$$= \frac{2}{\pi}\left.\frac{x^3}{3}\right|_0^{\pi}$$

$$\boxed{a_0 = \frac{2}{3}\pi^2}$$

$$\int u\,dv = uv - \int v\,du$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \cancel{f(x)}\,x^2 \cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} \underbrace{x^2}_{u}\, d\underbrace{\left(\frac{\sin nx}{n}\right)}_{v}$$

$$= \frac{2}{\pi}\left[\cancel{x^2 \left.\frac{\sin nx}{n}\right|_0^{\pi}} - \int_0^{\pi} 2x\,\frac{\sin nx}{n}\,dx\right]$$

$$\int x\, d\left(\frac{+\cos nx}{n}\right)$$

$$= \frac{4}{n\pi}\left[x\left(\frac{+\cos nx}{n}\right)\Big|_0^{\pi} - \int^{\pi} \frac{\cos nx}{n}\,dx\right]$$

$$\frac{2}{\pi}\left[x\left(\frac{+\cos nx}{n}\right)\Big|_0 - \int_0 \frac{\cos nx}{n}\,dx\right]$$

$$= \frac{4}{n\pi}\left[\frac{\pi}{n}(-1)^n - \cancel{\frac{\sin nx}{n^2}\Big|_0^{\pi}}\right]$$

$$\boxed{a_n = \frac{4}{n^2}(-1)^n}$$

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

Take $x=0$

$$0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots \infty = \frac{\pi^2}{12}$$

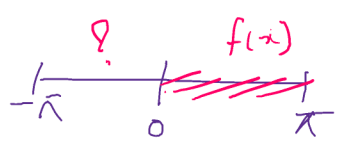
H.W. Find F.S.E of $f(x) = x^3$ $x \in [-\pi, \pi]$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x^3 \sin nx\,dx = 2(-1)^n\left[-\frac{\pi^2}{n} + \frac{6}{n^3}\right]$$

Half range Series:

$f(x)$ $[0, \pi]$



$f(x)$ — Sine, Cosine

f odd

Sine series

∞

Cosine series

...

Sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$$

Cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

Ex: Find the F.S.S. of

$$f(x) = e^{ax} \qquad 0 < x < \pi$$

Sol: Let $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$

$$= \frac{2}{\pi}\int_0^{\pi} e^{ax}\sin nx\, dx$$

$$\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$$

$$= \frac{2}{\pi}\left\{\frac{e^{ax}}{a^2+n^2}\left[a\sin nx - n\cos nx\right]\right\}_0^{\pi}$$

$$= \frac{2}{\pi}\left\{\frac{e^{a\pi}}{a^2+n^2}\left(\cancel{a\sin n\pi} - n\cos n\pi\right) + \frac{n}{a^2+n^2}\right]$$

$$b_n = \frac{2n}{(a^2+n^2)\pi}\left[1 - (-1)^n e^{a\pi}\right]$$

$$b_1 = \frac{2(1+e^{a\pi})}{\quad} \qquad b_2 = \frac{2.2(1-e^{a\pi})}{\quad}$$

$$b_1 = \frac{2(1+e^{a\pi})}{(1+a^2)\pi} \qquad b_2 = \frac{2.2(1-e^{a\pi})}{(4+a^2)\pi}$$

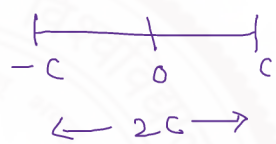
$$f(x) = e^{ax} = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots$$

$$= \frac{2}{\pi}\left[\frac{1+e^{a\pi}}{1+a^2}\sin x + \frac{2(1-e^{a\pi})}{(4+a^2)}\sin 2x + \cdots\right]$$

F.S — Gen, Even, H.R } Arbitrary interval, change of interval

$(0, 2\pi)$, $(-\pi, \pi)$, $(c, c+2\pi)$

Let $f(x)$ be periodic in $(-c, c)$



2c interval for the variable x

" " $\frac{x}{2c}$

" $\frac{2\pi x}{2c} = \frac{\pi x}{c} = z \Rightarrow x = \frac{cz}{\pi}$

2π

$\boxed{f(x)}$ having period 2c is transformed into

$f\left(\frac{cz}{\pi}\right) = F(z)$ is periodic with period 2π $\qquad 2\pi = 2c$

put $z = \frac{\pi x}{c}$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} F(z)\,dz = \frac{1}{\pi}\int_0^{2\pi} f\left(\frac{cz}{\pi}\right)dz$$

$dz = \frac{\pi}{c}\,dx$

$x = 0$

$x = 2c$

$$\boxed{a_0 = \frac{1}{c}\int_0^{2c} f(x)\,dx}$$

$n(z)$ $\qquad (-c, c)$

$$a_n = \frac{1}{c}\int_x^{2c} f(x)\cos\frac{n(\pi x)}{c}\,dx$$

$$b_n = \frac{1}{c}\int_0^{2c} f(x)\sin\frac{n\pi x}{c}\,dx$$

$$F(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos nz + b_n\sin nz\right) \qquad z\in[0,2\pi]$$

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{c}\right) + b_n\sin\left(\frac{n\pi x}{c}\right)\right)}$$

$$x\in(0,2c)$$

If f is even in $[-c,c]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{c}\right)$$

where

$$a_0 = \frac{2}{c}\int_0^{c} f(x)\,dx$$

$$a_n = \frac{2}{c}\int_0^{c} f(x)\cos\left(\frac{n\pi x}{c}\right)dx$$

If f is an odd fn $[-c,c]$

Then F.S. expn. of f is given by

$$f(x) = \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{c}\right)$$

where

$$b_n = \frac{2}{c}\int_0^{c} f(x)\sin\left(\frac{n\pi x}{c}\right)dx$$

<u>Half range</u> f is define $[0,c]$

Cosine Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{c}\right)$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos\left(\frac{n\pi x}{c}\right)dx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

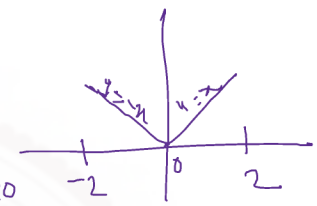
Sine Series.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{c}\right)$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin\left(\frac{n\pi x}{c}\right)dx$$

Ex: A periodic fn of period 4 is defined as

$$f(x) = |x| \qquad -2 < x < 2$$



Find its F.S. expn. $f(x) = \begin{cases} -x & -2 < x < 0 \\ x & 0 < x < 2 \end{cases}$

Sol: Since $|x|$ is an even fn. Given $2c = 4$

$\boxed{c = 2}$

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}$

where $a_0 = \frac{\cancel{2}}{\cancel{2}}\int_0^2 f(x)\,dx = \int_0^2 x\,dx = \frac{x^2}{2}\Big|_0^2 = 2$

$\boxed{a_0 = 2}$

$$a_n = \frac{\cancel{2}}{\cancel{2}}\int_0^2 x\cos\left(\frac{n\pi x}{2}\right)dx = \int_0^2 x\, d\left(\frac{\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}\right)$$

$$= \frac{2}{n\pi}\left[\cancel{x\sin\left(\frac{n\pi x}{2}\right)\Big|_0^2} - \int_0^2 \sin\left(\frac{n\pi x}{2}\right)dx\right]$$

$$= +\frac{2}{n\pi}\left.\frac{\cos\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}\right|_0^2$$

$$a_n = \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$|x| = 1 + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\left[(-1)^n - 1\right]}{n^2}\cos\left(\frac{n\pi x}{2}\right) \quad //$$

Fourier Transform

28 April 2020 13:58

Integral transform

$$I[f(x)] = F(S) = \int_a^b K(S,x)\, f(x)\, dx$$

e^{-st}

Def: A transform of the form:

(1) $F[f(t)] = F(S) = \int_{-\infty}^{\infty} e^{ist} f(t)\, dt$ — Fourier transform of $f(t)$

(2) $F^{-1}[F(S)] = f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(S)\, e^{-ist}\, dS$ Inv. Four. trans. of $F(S)$

Note: Fourier Integral:

F.T.

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} \underbrace{\left[\int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt\right]}_{F(S)} ds$$

Fourier Sine transform

$f(x) \Rightarrow F(S)$
$\rightarrow \bar{F}(S)$

$$F_S(f(x)) = F_S(S) = \int_0^{\infty} f(x) \sin sx\, dx$$

$$F_S^{-1}[F_S(S)] = f(x) = \frac{2}{\pi}\int_0^{\infty} F_S(S) \sin sx\, ds$$

F. cosine trans:

$$F_c[f(x)] = F_c(S) = \int_0^{\infty} f(x) \cos sx\, dx$$

$$F_c^{-1}[F_c(S)] = f(x) = \frac{2}{\pi}\int_0^{\infty} f(x) \cos sx\, ds$$

✓ F.T F.S.T . F.C.T

~~Finite F.T~~

~~Finite F.T~~

F.T , F.S.T , F.C.T

1) <u>Linearity</u>: Let $F[f(x)] = F(S)$ & $F[g(x)] = G(S)$

then $F[\alpha f(x) + \beta g(x)] = \alpha F(S) + \beta G(S)$

$$= \alpha F[f(x)] + \beta F[g(x)]$$

<u>pr</u>: $L.H.S = \int_{-\infty}^{\infty} e^{iSx} [\alpha f(x) + \beta g(x)] dx$

$$= \alpha \int_{-\infty}^{\infty} e^{iSx} f(x) dx + \beta \int_{-\infty}^{\infty} e^{iSx} g(x) dx$$

$$= \alpha F(S) + \beta G(S)$$

$$= R.H.S.$$

2) Change of Scale: Let $F[f(x)] = F(S)$ then

$$F[f(ax)] = \frac{1}{a} F\left(\frac{S}{a}\right) \quad a \neq 0$$

<u>proof</u>: $L.H.S = \int_{-\infty}^{\infty} e^{iSx} f(ax) dx$

put $ax = t$

$x = \frac{t}{a}$

$dx = \frac{dt}{a}$

$$= \frac{1}{a} \int_{-\infty}^{\infty} e^{i\left(\frac{S}{a}\right) t} f(t) dt$$

$$= \frac{1}{a} F\left(\frac{S}{a}\right)$$

$= R.H.S$

3) <u>Shifting</u> : Let $F[f(x)] = F(s)$ then

$$F[f(x-a)] = e^{isa} F(s)$$

put $x-a = t$
$x = t+a$

<u>proof</u>:

$$L.H.S = \int_{-\infty}^{\infty} e^{isx} f(x-a)\, dx$$

$$= \int_{-\infty}^{\infty} e^{is(t+a)} f(t)\, dx$$

$$= e^{isa} \underbrace{\int_{-\infty}^{\infty} e^{ist} f(t)\, dt}$$

$$= e^{isa} F(s)$$

$$= R.H.S$$

<u>Modulation theorem</u>: Let $F[f(x)] = F(s)$ then

$$F[f(x)\cos ax] = \frac{1}{2}[F(s+a) + F(s-a)]$$

<u>proof</u>:

$$L.H.S = \int_{-\infty}^{\infty} e^{isx} f(x) \cos ax\, dx$$

$$= \int_{-\infty}^{\infty} e^{isx} \left(\frac{e^{iax} + e^{-iax}}{2} \right) f(x)\, dx$$

$$= \frac{1}{2} \int^{\infty} [e^{i(s+a)x} + e^{i(s-a)x}] f(x)\, dx$$

$$= \frac{1}{2}\int_{-\infty} \left[e^{i(s+a)x} + e^{i(s-a)x} \right] f(x)\,dx$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} e^{i(s+a)x} f(x)\,dx + \int_{-\infty}^{\infty} e^{i(s-a)x} f(x)\,dx \right]$$

$$= \frac{1}{2}\left[F(s+a) + F(s-a) \right]$$

$$= R.H.S.$$

Cor: 1) $F_s[f(x)\cos ax] = \frac{1}{2}\left[F_s(s+a) + F_s(s-a)\right]$

2) $F_c[f(x)\sin ax] = \frac{1}{2}\left[F_s(s+a) - F_s(s-a)\right]$

3) $F_s[f(x)\sin ax] = \frac{1}{2}\left[F_c(s-a) - F_s(s+a)\right]$

problems:

Ex: Find the F.T. of $f(x) = \begin{cases} 1 & \text{for } |x|<1 \\ 0 & \text{for } |x|>1 \end{cases}$

Hence evaluate $\int_0^{\infty} \frac{\sin x}{x}\,dx$.

Sol: $F(s) = F[f(x)] = \int_{-\infty}^{\infty} e^{isx} f(x)\,dx$

$$= \int_{-1}^{1} e^{isx}\,dx = \int_{-1}^{1} dx = 2 \quad \text{if } s=0$$

$$\sin s = \frac{e^{is} - e^{-is}}{2i}$$

$$= \left.\frac{e^{isx}}{is}\right|_{-1}^{1} = \frac{e^{is} - e^{-is}}{is} = \frac{2\sin s}{s}, \; s \neq 0$$

$$\therefore \quad F(s) = \begin{cases} 2\dfrac{\sin s}{s}, & s \neq 0 \\ 2 & s = 0 \end{cases}$$

Taking inverse transformation, we have

$-1 < x < 1$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin s}{s}\, e^{-isx}\, ds = \begin{cases} 1 & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}$$

Let $x = 0 \Rightarrow f(x) = 1$ then

$$\frac{2}{\pi}\int_0^{\infty} \frac{\sin s}{s}\, ds = 1$$

$$\therefore \int_0^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2} \quad //$$

Ex:

2. Find F.T. of

$$f(x) = \begin{cases} 1 - x^2 & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}$$

Hence $\displaystyle\int_0^{\infty} \frac{x\cos x - \sin x}{x^3} \cos\frac{x}{2}\, dx$.

Sol:

$$F[f(x)] = \int_{-1}^{1} e^{isx}(1 - x^2)\, dx \qquad \int u\,dv = uv - \int v\,du$$

$$= uv - u'v_1 + u''v_2$$

$$= \int_{-1}^{1} (1 - x^2)\, d\left(\frac{e^{isx}}{is}\right)$$

$$= \cancel{(1 - x^2)\frac{e^{isx}}{\ }}\Big|^{1} - (-2x)\frac{e^{isx}}{\ }\Big|^{1} + (-2)\frac{e^{isx}}{\ }\Big|^{1}$$

$$= \cancel{(1-x^2)\frac{e^{isx}}{is}}\Big|_{-1}^{1} - (-2x)\frac{e^{isx}}{(is)^2}\Big|_{-1}^{1} + (-2)\frac{e^{isx}}{(is)^3}\Big|_{-1}^{1}$$

$$= \frac{-4}{s^2}\left[\frac{(e^{is}+e^{-is})}{2}\right] + \frac{4}{s^3}\frac{(e^{is}-e^{is})}{2i}$$

$$= -\frac{4}{s^2}\cos s + \frac{4}{s^3}\sin s$$

$$F[f(x)] = F(s) = -\frac{4}{s^3}[s\cos s - \sin s] \quad //$$

Using F.I.T, we have.

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds \qquad 1-\left(\tfrac{1}{2}\right)^2 = 1-\tfrac{1}{4} = \tfrac{3}{4}$$

$$= \frac{4}{2\pi}\int_{-\infty}^{\infty}\frac{s\cos s - \sin s}{s^3}\, e^{-isx}\, ds = \begin{cases} 1-x^2 & |x|\le 1 \\ 0 & |x|>1 \end{cases}$$

Take $x = \frac{1}{2}$

$$\Rightarrow \frac{-\cancel{4}^{\,2}}{\cancel{2}\pi}\int_{-\infty}^{\infty}\frac{s\cos s - \sin s}{s^3}\, e^{-\frac{is}{2}}\, ds = \frac{3}{4}$$

Comparing real part on both sides, we get

$$\Rightarrow \therefore \int_{0}^{\infty}\frac{x\cos x - \sin x}{x^3}\cos\frac{x}{2}\, dx = -\frac{3\pi}{16} \quad //$$

$\mathrm{Re}\, e^{i\theta} = \cos\theta + i\sin\theta$

3. (a) $F[e^{-a^2x^2}]$, $a>0$ Hence evaluate $e^{-\frac{x^2}{2}}$ (self reciprocal w.r.t F.T)

3. a) $F[e \quad]$, $a > 0$ Hence evaluate e (w.r.t F.T)

b) $F[e^{-2(x-3)^2}]$ c) $e^{-x^2}\cos 3x$

Sol: a) $F[e^{-a^2x^2}] = \int_{-\infty}^{\infty} e^{isx} e^{-a^2x^2}\, dx$ $\quad \because \int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$

$$= \int_{-\infty}^{\infty} e^{-a^2(x - is/2a^2)^2} \cdot e^{-s^2/4a^2}\, dx$$

$-a^2\left[x^2 - 2x\frac{is}{2a^2} - \frac{s^2}{4a^4}\right]$

$-a^2x^2 + isx + \frac{s^2}{4a^2} - \frac{s^2}{4a^2}$

put $a\left(x - \frac{is}{2a^2}\right) = t \Rightarrow dx = \frac{dt}{a}$

$$= \int_{-\infty}^{\infty} e^{-t^2} \cdot e^{-s^2/4a^2} \frac{dt}{a}$$

$F[e^{-a^2x^2}] = \frac{\sqrt{\pi}}{a} e^{-s^2/4a^2}$ Take $a = \frac{1}{\sqrt{2}}$ then

$\Rightarrow F[e^{-x^2/2}] = \sqrt{2\pi}\, e^{-\frac{s^2}{2}}$ Self reciprocal w.r.t F.T

b) $F[e^{-2(x-3)^2}]$

$e^{-2x^2} = e^{-\frac{(2x)^2}{2}} = f(2x)$ where $f(x) = e^{-\frac{x^2}{2}}$

$F(s) = \sqrt{2\pi}\, e^{-\frac{s^2}{2}}$

by change of Scale prop.

$$F[f(2x)] = \frac{1}{2} F\left(\frac{s}{2}\right)$$

$\therefore F[e^{-2x^2}] = \frac{1}{2}\sqrt{2\pi}\, e^{-\left(\frac{s}{2}\right)^2}$

$$\therefore F\left[\underbrace{e^{-2x^2}}_{f(x)}\right] = \frac{1}{2}\sqrt{2\pi}\, e^{-\frac{\left(\frac{s}{2}\right)^2}{2}}$$

$$= \sqrt{\frac{\pi}{2}}\, e^{-\frac{s^2}{8}} = F(s)$$

Shifting prop: $F[f(x-3)] = e^{i3s} F(s)$ — $F[f(x)]$

$$\therefore F\left[e^{-2(x-3)^2}\right] = e^{3is}\sqrt{\frac{\pi}{2}}\, e^{-\frac{s^2}{8}}$$

$$= \sqrt{\frac{\pi}{2}}\, e^{(3is - s^2/8)}$$

c) $F\left[\underbrace{e^{-x^2}}_{f(x)} \cos \underbrace{3}_{a} x\right]$ W.K.T $F(s) = F\left[\underbrace{e^{-x^2/2}}\right] = \sqrt{2\pi}\, e^{-s^2/2}$

Recall the modulation formula:

$$F[f(x)\cos ax] = \frac{1}{2}\left[F(s+a) + F(s-a)\right]$$

$F(f(\sqrt{2}x)) = \frac{1}{\sqrt{2}} F\left(\frac{s}{\sqrt{2}}\right)$

$F\left[e^{-x^2}\right] = \frac{\sqrt{2\pi}}{\sqrt{2}} e^{-\frac{s^2}{4}}$

$= F(s)$

$$\therefore F\left[e^{-x^2}\cos 3x\right] = \frac{1}{2}\left[F(s+3) + F(s-3)\right]$$

$$= \frac{\sqrt{\pi}}{2}\left[e^{-\frac{(s+3)^2}{4}} + e^{-\frac{(s-3)^2}{4}}\right]$$

— o —

29.04.20

Find $F_c\left[e^{-x^2}\right]$

Sol: $F_c(s) = F_c\left[e^{-x^2}\right] = \int_0^\infty e^{-x^2}\cos sx\, dx = I$ —(1)

Diff. (1) w.r.to s

$$\frac{dI}{ds} = -\int_0^\infty x e^{-x^2} \sin sx \, dx$$

$$= \frac{1}{2}\int_0^\infty \sin sx \, (-2x e^{-x^2}) \, dx \quad \Big| \int u\,dv = uv - \int v\,du$$

$$= \frac{1}{2}\int_0^\infty \sin sx \, d(e^{-x^2})$$

$$= \frac{1}{2}\left[\cancel{\sin sx \, e^{-x^2}}\Big|_0^\infty - s\int_0^\infty e^{-x^2}\cos sx \, dx \right]$$

$$\frac{dI}{ds} = \frac{-s}{2}\underbrace{\int_0^\infty e^{-x^2}\cos sx \, dx}_{I} = -\frac{sI}{2}$$

$$\frac{dI}{I} = -\frac{s}{2}\,ds$$

$$\Rightarrow \log I = -\frac{s^2}{4} + \log c = \log(c\, e^{-s^2/4})$$

$$\Rightarrow I = c\, e^{-s^2/4}$$

$$\int_0^\infty e^{-x^2}\cos sx \, dx = c\, e^{-s^2/4} \qquad \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

put $s = 0$, $\int_0^\infty e^{-x^2}dx = c \Rightarrow c = \frac{\sqrt{\pi}}{2}$

$$F_c[e^{-x^2}] = \int_0^\infty e^{-x^2}\cos sx \, dx = \frac{\sqrt{\pi}}{2} e^{-s^2/4}$$

2. $F_s[e^{-|x|}] = ?$ Hence evaluate

$$\int_0^\infty \frac{x \sin mx}{1+x^2}\,dx \quad \text{for } m>0$$

Sol: $|x| = \begin{cases} x & x>0 \\ -x & x<0 \end{cases}$ $\quad |x| = x$ for $x \in (0,\infty)$

Now $F_s[e^{-|x|}] = \int_0^\infty e^{-x} \sin sx\,dx$

$$= \frac{e^{-x}}{1+s^2}\left[-\sin sx - s\cos sx\right]\Big|_0^\infty$$

$$F_s[e^{-|x|}] = \frac{s}{1+s^2} = F_s(s)$$

Using inversion formula, we have

$$e^{-x} = f(x) = F_s^{-1}[F_s(s)] = \frac{2}{\pi}\int_0^\infty \frac{s}{1+s^2}\sin sx\,ds$$

s by x $\quad \boxed{x=m}$

$$\therefore \int_0^\infty \frac{x \sin mx}{1+x^2} = \frac{\pi}{2}e^{-m} \quad m>0$$

3. Find $F_c[f(x)]$ where $f(x) = \begin{cases} x & 0<x<1 \\ 2-x & 1<x<2 \\ 0 & x>2 \end{cases}$

Sol: $F_c[f(x)] = \int_0^1 f(x)\cos sx\,dx + \int_1^2 f(x)\cos sx\,dx + \cancel{\int_2^\infty f(x)\cos sx\,dx}$

$$= \int_0^1 x\cos sx\,dx + \int_1^2 (2-x)\cos sx\,dx$$

$$= \frac{2\cos s}{s^2} - \frac{\cos 2s}{s^2} - \frac{1}{s^2} = \frac{1}{s^2}\left[2\cos s - \cos 2s - 1\right] \quad //$$

4. Find $F_s\left[e^{-ax}/x\right] = I$

Sol:

$$I = \int_0^\infty \frac{e^{-ax}}{x} \sin sx \, dx = F_s(s) \Rightarrow F_s(0) = 0$$

$$\frac{d F_s(s)}{ds} = \int_0^\infty \frac{\cancel{x} e^{-ax} \cos sx}{\cancel{x}} dx = \int_0^\infty e^{-ax} \cos sx \, dx$$

$$F_s(s) = \int \frac{a}{s^2 + a^2} ds + c$$

$$= \tan^{-1}\frac{s}{a} + c$$

$$\therefore F_s(0) = 0 \Rightarrow c = 0$$

$$\therefore F_s(s) = \tan^{-1}\frac{s}{a} \quad //$$

5. Find the $F_c\left[\frac{1}{1+x^2}\right]$

Hence derive $F_s\left[\frac{x}{1+x^2}\right]$

Sol:

$$F_c[f(x)] = \int_0^\infty \frac{\cos sx}{1+x^2} dx = I$$

$$F_s\left[\frac{x}{1+x^2}\right]$$

$$\frac{dI}{ds} = -\int_0^\infty \frac{\left[(1+x^2)-1\right] \sin sx}{x(1+x^2)} dx = -\int_0^\infty \left(\frac{x}{1+x^2}\right) \sin sx \, dx.$$

$$= -\int_0^\infty \frac{\sin sx}{x} dx + \int_0^\infty \frac{\sin sx}{x(1+x^2)} dx \quad F_s\left[\frac{x}{1+x^2}\right]$$

$$= -\int_0 \frac{\sin sx}{x}\,dx + \int_0 \frac{\sin sx}{x(1+x^2)}\,dx \quad F_s[\ 1+x^2\ ]$$

$$\frac{dI}{ds} = -\frac{\pi}{2} + \int_0^\infty \frac{\sin sx}{x(1+x^2)}\,dx$$

$$\frac{d^2I}{ds^2} = \int_0^\infty \frac{\cancel{x}\cos sx}{\cancel{x}(1+x^2)}\,dx = I$$

$$(D^2-1)\,I = 0$$

$$m^2-1=0$$

$$m = \pm 1$$

$$\therefore \quad \frac{d^2I}{ds^2} - I = 0$$

$$I = A e^{s} + B e^{-s}$$

$$\frac{dI}{ds} = A e^{s} - B e^{-s}$$

put $s=0$ $\quad A + B = \int_0^\infty \frac{dx}{1+x^2} = \tan^{-1}x \Big|_0^\infty = \frac{\pi}{2}$

$$A - B = -\frac{\pi}{2}$$

$$\Rightarrow \quad A = 0, \quad B = \frac{\pi}{2}$$

$$I = F_c\left[\frac{1}{1+x^2}\right] = \frac{\pi}{2} e^{-s}$$

$$F_s\left[\frac{x}{1+x^2}\right] = -\frac{dI}{ds} = -\left(-\frac{\pi}{2}e^{-s}\right)$$

$$= \frac{\pi}{2} e^{-s}$$

Ex:

$F_c[\ x^{n-1}\ ] = ?$ $\quad F_c[\ x^{n-1}\ ] = ?$ $\quad n > 0.$

6. $F_s[x^{n-1}] = ?$ $F_c[x^{n-1}] = ?$ $n > 0$.

$$F_s = F_s[x^{n-1}] = \int_0^\infty x^{n-1} \sin sx \, dx$$

$$F_c = F_c[x^{n-1}] = \int_0^\infty x^{n-1} \cos sx \, dx$$

put $isx = -t$
$dx = \frac{-dt}{is}$

$$\underline{F_c} + i\,\underline{F_s} = \int_0^\infty e^{isx} x^{n-1} \, dx$$

$$= \int_0^\infty e^{-t} \left(\frac{-t}{is}\right)^{n-1} \frac{-dt}{is}$$

$$= \left(\frac{-1}{si}\right)^n \int_0^\infty e^{-t} t^{n-1} \, dt$$

$$= \left(\frac{i}{s}\right)^n \Gamma(n)$$

$$= \left(\cos n\pi/2 + i \sin n\pi/2\right) \Gamma(n)/s^n$$

$$\boxed{L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}}$$

Eq. real & Imm. parts, we have

$$\boxed{\begin{aligned} F_c[x^{n-1}] &= \frac{\Gamma(n)}{s^n} \cos \frac{n\pi}{2} \\ F_s[x^{n-1}] &= \frac{\Gamma(n)}{s^n} \sin \frac{n\pi}{2} \end{aligned}}$$

Ex: S.T i) $F_s[x f(x)] = -\frac{d}{ds}[F_c(s)]$

ii) $F_c[x f(x)] = \frac{d}{ds}[F_s(s)]$

i) $\frac{d}{ds} F_c(s) = \int_0^\infty f(x)\left(\frac{\partial \cos sx}{\partial s}\right) dx$ ii) Similarly

$$= -\int_0^\infty \{x f(x)\} \sin sx \, dx$$

$$= -F_s[x f(x)]$$

$$\therefore F_s[x f(x)] = -\frac{d}{ds} F_c(s)$$

Find $F_s[x e^{-ax}]$ & $F_c[x e^{-ax}]$

$$F_s(s) = F_s[e^{-ax}] = \int_0^\infty e^{-ax} \sin sx \, dx = \frac{s}{s^2+a^2}$$

$$\text{||ly} \quad F_c(s) = \frac{a}{s^2+a^2}$$

$$F_s(x e^{-ax}) = -\frac{d}{ds} F_c(e^{-ax})$$

$$= -\frac{d}{ds}\left(\frac{a}{a^2+s^2}\right)$$

$$= \frac{2as}{(a^2+s^2)^2} \quad ||$$

$$F_c(x e^{-ax}) = \frac{d}{ds} F_s(s)$$

$$= \frac{d}{ds}\left(\frac{s}{s^2+a^2}\right)$$

$$= \frac{a^2 - s^2}{(a^2+s^2)^2} \quad //$$

Solve the integral eqn:

$$F_c[f(\theta)] = \int_0^\infty f(\theta) \cos\alpha\theta \, d\theta = \begin{cases} 1-\alpha & 0 \le \alpha \le 1 \\ 0 & \alpha > 1 \end{cases}$$

$= F_c(\alpha)$

Hence evaluate $\int_0^\infty \frac{\sin^2 t}{t^2} \, dt$

Sol:

$$F_c(\alpha) = \int_0^\infty f(\theta) \cos\alpha\theta \, d\theta$$

$$F_c(\alpha) = \begin{cases} 1-\alpha & 0 \le \alpha \le 1 \\ 0 & \alpha > 1 \end{cases}$$

Using inversion formula

$$f(\theta) = \frac{2}{\pi} \int_0^\infty F_c(\alpha) \cos\alpha\theta \, d\alpha$$

$$= \frac{2}{\pi} \int_0^1 (1-\alpha) \cos\alpha\theta \, d\alpha$$

$$\therefore f(\theta) = \frac{2(1-\cos\theta)}{\pi\theta^2} \quad //$$

Now $F_c(\alpha) = \int_0^\infty f(\theta) \cos\alpha\theta \, d\theta$

$$\left.\begin{matrix} 0 \le \alpha \le 1 & 1-\alpha \\ \alpha > 1 & 0 \end{matrix}\right\} = \frac{2}{\pi} \int_0^\infty \frac{1-\cos\theta}{\theta^2} \cos\alpha\theta \, d\theta$$

$$\alpha \to 0 \qquad 1 = \frac{2}{\pi} \int_0^\infty \frac{2\sin^2\theta/2}{\theta^2} \, d\theta$$

put $\theta/2 = t$

$d\theta = 2dt$

$\alpha = \pi$ $\quad I = \frac{2}{\pi} \int_0^{\infty}$ $\quad \theta$

put $\theta/2 = t$

$d\theta = 2dt$

$$= \frac{2}{\pi} \int_0^{\infty} \frac{\cancel{2} \sin^2 t}{\cancel{4}_2 t^2} \cancel{2} dt$$

$$\therefore \int_0^{\infty} \frac{\sin^2 t}{t^2} dt = \frac{\pi}{2} \,//$$