4

CHAPTER

Differentiation and Its Applications

Calculus is a major area in mathematics with applications in science, engineering, medicine and finance. The concept of derivative, measuring the rate of charge, is at the core of differential calculus. Applications of derivative include computations involving velocity and acceleration, errors and approximations, power series representation of a function, finding tangents, normals and asymptotes to a curve, determining curvature, etc. The interesting mean value theorems and their consequences extend the scope of differential calculus further.

4.1 SINGLE AND HIGHER ORDER DERIVATIVES

In this section, first we discuss the concept of derivative of a function and afterward, we consider successive differentiation of $f(x)$ resulting in higher order derivatives.

4.1.1 Derivative of a Function

Let $y = f(x)$ be a real-valued function defined on an interval I and let x_0 be a point in I. Then $f(x)$ is said to be differentiable at $x = x_0$, if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}, \text{ or } \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

exists and is finite. The derivative of $f(x)$ at $x = x_0$ is denoted by $f'(x_0)$ or $\left.\frac{df}{dx}\right|_{x=x_0}$, or $Df(x_0)$.

If $f(x)$ is differentiable at every point of the interval (a, b), then $f(x)$ is said to be differentiable in (a, b). If the interval $[a, b]$ is closed, then we say that $f(x)$ is differentiable at every point of $[a, b]$, if $f(x)$ is differentiable in (a, b) and right hand derivative of $f(x)$ exists at 'a' and left hand derivative of $f(x)$ exists at 'b'. Geometrically, *the derivative of $f(x)$ at a given point $P(x_0, y_0)$ gives the slope of the tangent line to the curve $y = f(x)$ at that point P, that is, $\tan \theta = f'(x_0)$*, as shown in Fig. 4.1. Also if a function $y = f(x)$ is differentiable at a point P, it is necessarily continuous there. The converse, however is not true, for example, $f(x) = |x|$ is continuous at $x = 0$, but not differentiable.

If f and g are two differentiable functions, then the following properties are satisfied:

(a) $(cf')(x) = cf'(x)$, c is any constant

(b) $(f \pm g)'(x) = f'(x) \pm g'(x)$

(c) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$

(d) $\left(\dfrac{f}{g}\right)'(x) = \dfrac{g(x)f'(x) - g'(x)f(x)}{[g(x)]^2}, g(x) \neq 0$

(e) If $h(x) = g[f(x)]$, then $h'(x) = g'[f(x)]f'(x)$

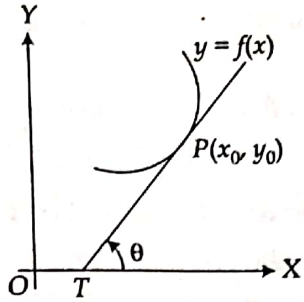

Fig. 4.1

Example 4.1: (i) Investigate the function $f(x) = |\ln x|$ for differentiability at the point $x = 1$.

(ii) Find the derivative of the function $y = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

Solution: (i) At $x = 1$, $\dfrac{\Delta y}{\Delta x} = \dfrac{f(1+\Delta x) - f(1)}{\Delta x} = \dfrac{|\ln(1+\Delta x)|}{\Delta x}$

$$= \begin{cases} \dfrac{\ln(1+\Delta x)}{\Delta x} & \text{at } \Delta x > 0 \\ \dfrac{-\ln(1+\Delta x)}{\Delta x} & \text{at } \Delta x < 0 \end{cases}$$

Hence, $\lim\limits_{\Delta x \to 0^+} \dfrac{\Delta y}{\Delta x} = +1$ and $\lim\limits_{\Delta x \to 0^-} \dfrac{\Delta y}{\Delta x} = -1.$

Since, the left hand and right hand derivatives are different at the point $x = 1$. Hence $f(x)$ is not differentiable at $x = 1$.

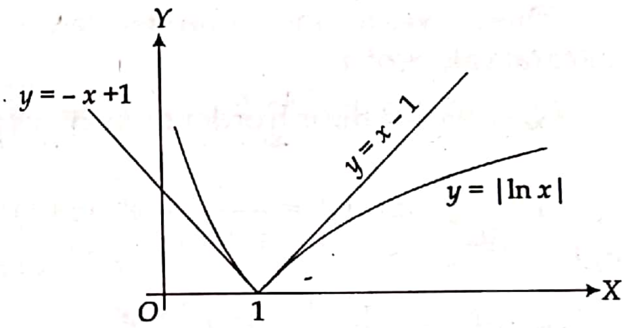

Fig. 4.2

(ii) We have, $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-\left(\dfrac{2x}{1+x^2}\right)^2}} \cdot \dfrac{2(1+x^2) - 4x^2}{(1+x^2)^2} = \dfrac{2(1-x^2)}{|1-x^2|(1+x^2)} = \begin{cases} \dfrac{2}{1+x^2} & \text{at } |x| < 1 \\ -\dfrac{2}{1+x^2} & \text{at } |x| > 1 \end{cases}$

At $|x| = 1$ the derivative does not exist.

4.1.2 Successive Differentiation: Higher Order Derivatives

The derivative of $f(x)$ at any point x, if it exists, is again a function of x say $f'(x) = g(x)$. If the function $g(x)$ is itself differentiable, then the second derivative of $f(x)$ is defined as $f''(x) = g'(x)$. It is also denoted by $f^{(2)}(x)$, or $\dfrac{d^2 f}{dx^2}$, or $D^2 f(x)$. The nth order derivative of $f(x)$ is defined as $f^{(n)}(x) = \dfrac{d}{dx}\left[f^{(n-1)}_{(x)}\right]$.

The nth Derivative of the Product of Two Functions

The nth order derivative of the product of two functions f and g is obtained by *Leibnitz rule*, stated as follows.

Leibnitz's Rule: *If f and g are functions of x possessing derivatives of the nth order, then*

$$(fg)^{(n)} = f^{(n)}g + C_1^n f^{(n-1)}g^{(1)} + C_2^n f^{(n-2)}g^{(2)} + \dots + C_n^n fg^{(n)} \qquad \dots(4.1)$$

Proof. The result (4.1) will be proved by induction.

Since, we have $(fg)^{(1)} = f^{(1)}g + fg^{(1)}$, thus, the result is true for $n = 1$.

Let it be true for $n = k$, that is, let

$$(fg)^{(k)} = f^{(k)}g + C_1^k f^{(k-1)}g^{(1)} + C_2^k f^{(k-2)}g^{(2)} + \dots + C_k^k fg^{(k)}.$$

Differentiating it both sides w.r.t. x, gives

$$(fg)^{(k+1)} = (f^{(k+1)}g + f^{(k)}g^{(1)}) + C_1^k (f^{(k)}g^{(1)} + f^{(k-1)}g^{(2)}) + C_2^k(f^{(k-1)}g^{(2)} + f^{(k-1)}g^{(3)}) + \dots + C_k^k(f^{(1)}g^{(k)} + fg^{(k+1)})$$

$$= f^{(k+1)}g + (C_0^k + C_1^k)f^{(k)}g^{(1)} + (C_1^k + C_2^k)f^{(k-1)}g^{(2)} + \dots + C_k^k fg^{(k+1)}$$

$$= f^{(k+1)}g + C_1^{k+1}f^{(k)}g^{(1)} + C_2^{k+1}f^{(k-1)}g^{(2)} + \dots + C_{k+1}^{k+1}fg^{(k+1)}$$

Thus, the result is true for $n = k + 1$ also.

This proves the induction step and since the result is true for $n = 1$, hence it is true for all positive integral values of n.

Next, we list the nth order derivatives of some frequently used functions.

1. $\dfrac{d^n}{dx^n}(ax+b)^m = \dfrac{m!}{(m-n)!}a^n(ax+b)^{m-n}, \quad n \le m, \quad m > 0$

2. $\dfrac{d^n}{dx^n}\left(\dfrac{1}{ax+b}\right) = \dfrac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}$

3. $\dfrac{d^n}{dx^n}\ln(ax+b) = \dfrac{(-1)^{n-1}(n-1)!\, a^n}{(ax+b)^n}$

4. $\dfrac{d^n}{dx^n}(a^{mx}) = m^n(\ln a)^n a^{mx}$

5. $\dfrac{d^n}{dx^n}\sin(ax+b) = a^n \sin\left(ax+b+n.\dfrac{\pi}{2}\right)$

6. $\dfrac{d^n}{dx^n}\cos(ax+b) = a^n \cos\left(ax+b+n.\dfrac{\pi}{2}\right)$

7. $\dfrac{d^n}{dx^n}[e^{ax}\sin(bx+c)] = (a^2+b^2)^{n/2}e^{ax}\sin\left(bx+c+n\tan^{-1}\dfrac{b}{a}\right)$

8. $\frac{d^n}{dx^n}[e^{ax}\cos(bx+c)] \quad = (a^2+b^2)^{n/2} e^{ax} \cos\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$

Example 4.2: Find the nth derivatives of the following functions

(i) $\frac{x^4}{(x-1)(x-2)}$ (ii) $e^{ax}\sin(bx+c)$

Solution: (i) Let $y = \frac{x^4}{(x-1)(x-2)}$. Resolving into partial fractions, we have

$$y = x^2 + 3x + 7 - \frac{1}{x-1} + \frac{16}{x-2}$$

Differentiating it n (> 2) times, we obtain

$$y^{(n)} = (-1n)\, n!\left[\frac{16}{(x-2)^{n+1}} - \frac{1}{(x-1)^{n+1}}\right]$$

(ii) Consider, $y = e^{ax}\sin(bx+c)$. We have

$$y^{(1)} = e^{ax}[a\sin(bx+c) + b\cos(bx+c)]$$

Substituting $a = r\cos\phi$, $b = r\sin\phi$, and thus, $r = \sqrt{a^2+b^2}$ and $\phi = \tan^{-1}\frac{b}{a}$, and simplifying we obtain

$$y^{(1)} = (a^2+b^2)^{1/2} e^{ax}\sin\left(bx+c+\tan^{-1}\frac{b}{c}\right)$$

Similarly, $$y^{(2)} = (a^2+b^2)^{2/2} e^{ax}\sin\left(bx+c+2\tan^{-1}\frac{b}{c}\right)$$

and, in general $$y^{(n)} = (a^2+b^2)^{n/2} e^{ax}\sin\left(bx+c+n\tan^{-1}\frac{b}{c}\right)$$

Example 4.3: Find the nth order derivative of the function $y = \frac{3x+2}{x^2-2x-5}$ at the point $x = 0$.

Solution: Rewriting the given function as

$$y(x^2-2x+5) = 3x+2. \qquad \text{...(4.2)}$$

Differentiating this n times using Leibnitz rule, we obtain

$$y^{(n)}(x^2-2x+5) + ny^{(n-1)}(2x-2) + \frac{n(n-1)}{2}y^{(n-2)}(2) = 0.$$

For $x = 0$, we have $5y^{(n)}(0) - 2ny^{n-1}(0) + n(n-1)y^{n-2}(0) = 0$. Hence,

$$y^{(n)}(0) = \frac{2}{5}ny^{n-1}(0) - \frac{n(n-1)}{5}y^{n-2}(0), \quad n \geq 2. \quad \text{...(4.3)}$$

Also from (4.2), $y(0) = 2/5$, and $y'(x) = \dfrac{-3x^2 - 4x + 19}{(x^2 - 2x + 5)^2}$, which gives $y'(0) = \dfrac{19}{25}$.

For $n = 2, 3, \ldots$ in (4.3), the values of the derivatives of higher orders can be obtained at $x = 0$.

Example 4.4: If $y = a\cos(\ln x) + b\sin(\ln x)$, show that $x^2y^{(2)} + xy^{(1)} + y = 0$, and

$$x^2y^{(n+2)} + (2n+1)xy^{(n+1)} + (n^2+1)y^{(n)} = 0.$$

Solution: We have, $y = a\cos(\ln x) + b\sin(\ln x)$. Differentiating w.r.t. x,

$$y^{(1)} = -a\sin(\ln x).\frac{1}{x} + b\cos(\ln x).\frac{1}{x}, \text{ or } xy^{(1)} = -a\sin(\ln x) + b\cos(\ln x).$$

Again differentiating w.r.t. x, we get, $xy^{(2)} + y^{(1)} = -a\cos(\ln x)\dfrac{1}{x} - b\sin(\ln x)\dfrac{1}{x}$

or,
$$x^2y^{(2)} + xy^{(1)} + y = 0. \quad \text{...(4.4)}$$

Differentiating (4.4) n times using Leibnitz's rule, we have

$$x^2y^{(n+2)} + 2nxy^{(n+1)} + n(n-1)y^{(n)} + xy^{(n+1)} + ny^{(n)} + y^{(n)} = 0.$$

or,
$$x^2y^{(n+2)} + (2n+1)xy^{(n+1)} + (n^2+1)y^{(n)} = 0.$$

EXERCISE 4.1

1. Show that function $y = 3|x| + 1$ is not differentiable at $x = 0$.
2. Find the derivative of $f(x) = x|x|, -1 \leq x \leq 1$.
3. Find the derivatives of the nth order of the following functions:

   (a) $y = \ln(x^2 + x - 2)$ (b) $y = \dfrac{ax+b}{cx+d}$ (c) $\dfrac{1}{1+x+x^2}$

4. If $y = (\sin^{-1}x)^2$, prove that $(1-x^2)y^{(n+2)} - (2n+1)xy^{(n+1)} - n^2y^{(n)} = 0$.
5. If $y = x^{n-1}\ln x$, prove that $y^{(n)} = [(n-1)!/x]$.
6. If $y^{1/m} + y^{-1/m} = 2x$, prove that $(x^2-1)y^{(n+2)} + (2n+1)xy^{(n+1)} + (n^2-m^2)y^{(n)} = 0$.
7. If $y = \left[x + \sqrt{(1+x^2)}\right]^m$, find $y^{(n)}(0)$.
8. If $y = \tan^{-1}x$, prove that $(1+x^2)\,y^{(n+2)} + 2(n+1)\,xy^{(n+1)} + n(n+1)y^{(n)} = 0$. Find $y^{(n)}(0)$.
9. If $x = \sin t$, $y = \cos pt$ prove that $(1-x^2)y^{(2)} - xy^{(1)} + p^2y = 0$. Hence prove that $(1-x^2)y^{(n+2)} - (2n+1)xy^{(n+1)} - (n^2-p^2)y^{(n)} = 0$.
10. If $y^{(n)} = \dfrac{d^n}{dx^n}(x^n\ln x)$, then $y^{(n)} = ny^{(n-1)} + (n-1)!$ and hence show that

$$y^{(n)} = n!\left(\ln x + 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right).$$

4.2 ERRORS AND APPROXIMATIONS

Let $y = f(x)$ be a real valued differentiable function then the increment Δy in the function $y = f(x)$, corresponding to the increment Δx in x can be expressed as

$$\Delta y = f(x + \Delta x) - f(x) = f'(x)\Delta x + \alpha\Delta x,$$

where α is an infinitesimal small quantity dependent on Δx and tends to zero as $\Delta x \to 0$. The principal linear part of this increment $f'(x)\Delta x$ is called the *differential* and is denoted by $df(x)$ or dy.

In the limiting form, the differential is also written as $df(x) = f'(x)dx$. Hence an *approximation* to $f(x + \Delta x)$ can be written as

$$f(x + \Delta x) \approx f(x) + f'(x)dx. \qquad \text{...(4.5)}$$

As we move from x to a nearby point $x + \Delta x$, we define $df = f'dx$, $\frac{df}{f}$, and $\frac{df}{f} \times 100$ as *absolute error*, *relative error* and *percentage error*, respectively.

Example 4.5: About how accurately should we measure the radius r of a sphere to calculate the surface area $S = 4\pi r^2$ within 1% of its true value?

Solution: We require, $|\Delta S| \le \frac{S}{100} = \frac{4\pi r^2}{100}$ and here, $S = 4\pi r^2$, thus $dS = 8\pi r dr$.

Replacing ΔS in this inequality with dS, we have

$$|8\pi r dr| \le \frac{4\pi r^2}{100}, \text{ or } \quad |dr| \le \frac{r}{200}.$$

Thus r should be measured no more than 0.5% of the true value.

Example 4.6: Using the concept of differential, find the approximate value of the function

$$f(x) = \sqrt[5]{\frac{2-x}{2+x}} \text{ at } x = 0.15.$$

Solution: We have

$$f(x + \Delta x) \approx f(x) + f'(x)dx \qquad \text{...(4.6)}$$

Here, $f(x) = \sqrt[5]{\frac{2-x}{2+x}}$. It gives $f'(x) = -\frac{4}{5}\left(\frac{2+x}{2-x}\right)^{4/5} \frac{1}{(2+x)^2}$.

At $x = 0$, $f(0) = 1$ and $f'(0) = -\frac{1}{5}$.

Further taking $x = 0$ and $\Delta x = 0.15$ in (4.6) and approximating dx with Δx, we obtain

$$f(0.15) \approx 1 - \frac{0.15}{5} = 0.97.$$

Example 4.7: The volume V of a fluid flowing through a small pipe in a unit of time at a fixed pressure is a constant times the fourth power of the pipe's radius r. How will a 10% increase in r affect V?

Solution: We have $V = kr^4$, where k is a constant. Thus $dV = 4kr^3 dr$, and hence, $\frac{dV}{V} = 4\frac{dr}{r}$.

A 10% increase in r means $\frac{dr}{r} = \frac{1}{10}$, which gives, $\frac{dV}{V} = 4 \times \frac{1}{10} = \frac{2}{5}$.

Thus a 10% increase in r will produce a 40% increase in V.

EXERCISE 4.2

1. The radius of a sphere is increasing at a variable rate and is equal to 1 cm/sec, when the radius is 3 cm. Find the rate of change in volume at this time.
2. If there is a possible error of 0.02 cm in the measurement of the diameter of a sphere, then find the possible percentage error in its volume, when the radius is 10 cm.
3. A vessel is in the form of an inverted cone of semivertical angle 45°. Water is poured into this vessel at the rate of 100 cc per second. Find the rate of rise in the water level when it is 2 cm deep.
4. Use the concept of differential to find an approximate value of
$$y = 3(4.02)^2 - 2(4.02)^{3/2} + 8(4.02)^{-1/2}.$$
5. All faces of a copper cube with 5 cm sides were uniformly ground down. As a result the weight of the cube was reduced by 0.96 gm. Knowing the specific density of copper as 8, find the reduction in the cube size, that is, the amount by which its side was reduced.

4.3 TANGENTS AND NORMALS

The '*equation of the tangent*' to the curve $y = f(x)$ at a point $P(x_0, y_0)$, as shown in Fig. 4.3 is

$$y - y_0 = f'(x_0)\,(x - x_0) \qquad \text{...(4.7)}$$

where $f'(x_0) = \tan\psi$ is the slope of the tangent line PT, ψ is the angle which this line makes with the positive direction of the x-axis.

A straight line PN passing through the point of contact $P(x_0, y_0)$ and perpendicular to the tangent line PT is called the *normal to the curve* $y = f(x)$ at $P(x_0, y_0)$. Its equation is given by

$$y - y_0 = -\frac{1}{f'(x_0)}(x - x_0)\ f'(x_0) \neq 0. \qquad ...(4.8)$$

The angle of intersection of two curves is the angle between the tangents to the curves at their point of intersection.

If m_1 and m_2 are the slopes of the tangents to the curves $y = f(x)$ and $y = g(x)$ at their point of intersection, say (x_0, y_0), then the angle of intersection 'θ' between these two curves at (x_0, y_0) is

$$\tan\theta = \frac{|m_1 - m_2|}{1 + m_1 m_2} \qquad ...(4.9)$$

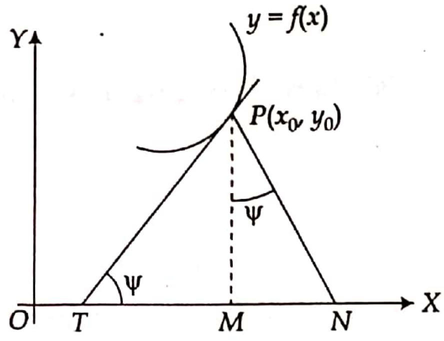

Fig. 4.3

In case $m_1 m_2 = -1$, $\theta = 90°$, that is, the two curves cut orthogonally.

Segments of tangent, normal, subtangent and subnormal

Draw the ordinate PM. Then PT and PN are called the *segment of the tangent* and *segment of the normal* respectively; and TM, MN are called the *sub-tangent* and *sub-normal* respectively.

From Fig. 4.3, for any arbitrary point $P(x, y)$ we have

(a) *Length of tangent segment:* $TP = MP \operatorname{cosec}\psi = |y|\sqrt{1 + (dx/dy)^2}$ ...(4.10)

(b) *Length of normal segment:* $NP = MP\sec\psi = |y|\sqrt{1 + (dy/dx)^2}$ ...(4.11)

(c) *Length of subtangent:* $TM = MP\cot\psi = \left|y\frac{dx}{dy}\right|$ ...(4.12)

(d) *Length of subnormal:* $MN = MP\tan\psi = \left|y\frac{dy}{dx}\right|$. ...(4.13)

Example 4.8: Find the equations of the tangent and normal at the point 'θ' to the curve $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$. Also find the length of tangent segment, normal segment, and that of the subtangent and the subnormal.

Solution: We have,

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{2\sin\theta/2\cos\theta/2}{2\cos^2\theta/2} = \tan\frac{\theta}{2}$$

Hence, the equations of the tangent and that of the normal at point θ respectively are:

$$y - a(1 - \cos\theta) = \tan\frac{\theta}{2}(x - a(\theta + \sin\theta))$$

and,
$$y - a(1 - \cos\theta) = -\cot\frac{\theta}{2}(x - a(\theta + \sin\theta))$$

(a) Length of tangent segment $= |y|\sqrt{1+\left(\frac{dx}{dy}\right)^2} = |a(1-\cos\theta)|\sqrt{1+\cot^2\frac{\theta}{2}}$

$$= \left|2a\sin^2\frac{\theta}{2}\operatorname{cosec}\frac{\theta}{2}\right| = 2\left|a\sin\frac{\theta}{2}\right|.$$

(b) Length of normal segment $= |y|\sqrt{1+\left(\frac{dy}{dx}\right)^2} = |a(1-\cos\theta)\sec\theta/2|$

$$= 2|a\sin\theta/2\tan\theta/2|.$$

(c) Subtangent $= \left|y\frac{dx}{dy}\right| = \left|a(1-\cos\theta)\cot\frac{\theta}{2}\right| = a|\sin\theta|.$

(d) Subnormal $= \left|y\frac{dy}{dx}\right| = \left|a(1-\cos\theta)\tan\frac{\theta}{2}\right| = \left|2a\sin^2\frac{\theta}{2}\tan\frac{\theta}{2}\right|.$

Example 4.9: Show that the condition for the line $x\cos\alpha + y\sin\alpha = p$ to touch the curve $(x/a)^m + (y/b)^m = 1$ is $(a\cos\alpha)^{m/(m-1)} + (b\sin\alpha)^{m/(m-1)} = p^{m/m-1}$

Solution: Equation of the curve is

$$(x/a)^m + (y/b)^m = 1. \qquad \text{...(4.14)}$$

Differentiating (4.14) w.r.t. x, we have

$$\frac{mx^{m-1}}{a^m} + \frac{my^{m-1}}{b^m}\frac{dy}{dx} = 0, \text{ which gives, } \frac{dy}{dx} = -\left(\frac{b}{a}\right)^m\left(\frac{x}{y}\right)^{m-1}.$$

Therefore, the equation of the tangent at point $P(x_0, y_0)$ to the curve (4.14) is

$$y - y_0 = -\left(\frac{b}{a}\right)^m\left(\frac{x_0}{y_0}\right)^{m-1}(x - x_0)$$

or,
$$\frac{x_0^{m-1}}{a^m}x + \frac{y_0^{m-1}}{b^m}y = \frac{x_0^m}{a^m} + \frac{y_0^m}{b^m} = 1. \qquad \text{...(4.15)}$$

If the given line touches the curve (4.14) at (x_0, y_0) then (4.15) must be of the form

$$x\cos\alpha + y\sin\alpha = p \qquad \text{...(4.16)}$$

Comparing the corresponding coefficients of (4.15) and (4.16), we get

$$\frac{x_0^{m-1}}{a^m\cos\alpha} = \frac{y_0^{m-1}}{b^m\sin\alpha} = \frac{1}{p}$$

or, $$\left(\frac{x_0}{a}\right)^{m-1} = \frac{a\cos\alpha}{p}, \quad \left(\frac{y_0}{b}\right)^{m-1} = \frac{b\sin\alpha}{p}$$

or, $$\left(\frac{a\cos\alpha}{p}\right)^{\frac{m}{m-1}} + \left(\frac{b\sin\alpha}{p}\right)^{\frac{m}{m-1}} = \left(\frac{x_0}{a}\right)^m + \left(\frac{y_0}{b}\right)^m = 1.$$

Hence, $(a\cos\alpha)^{m/(m-1)} + (b\sin\alpha)^{m/(m-1)} = p^{m/(m-1)}$, the desired condition.

Example 4.10: Find the condition that the curves $ax^2 + by^2 = 1$ and $a'x^2 + b'y^2 = 1$ cut orthogonally.

Solution: Given curves are

$$ax^2 + by^2 = 1, \quad \text{...(4.17)}$$

and $$a'x^2 + b'y^2 = 1 \quad \text{...(4.18)}$$

Let $P(h, k)$ be a point of intersection of these curves, thus

$$ah^2 + bk^2 = 1 \quad \text{and} \quad a'h^2 + b'k^2 = 1,$$

which give $$\frac{h^2}{-b+b'} = \frac{k^2}{-a'+a} = \frac{1}{ab'-a'b}$$

or, $$h^2 = (b'-b)/(ab'-a'b), \quad k^2 = (a-a')/(ab'-a'b) \quad \text{...(4.19)}$$

From (4.17), we have

$$2axdx + 2by\frac{dy}{dx} = 0, \text{ or } \frac{dy}{dx} = -ax/by.$$

Similarly from (4.18), we have $\frac{dy}{dx} = -a'x/b'y$.

Thus, m = slope of the tangent to (4.17) at $P(h, k) = -\frac{ah}{bk}$,

m' = slope of the tangent to (4.18) at $P(h, k) = -\frac{a'h}{b'k}$.

The curves (4.17) and (4.18) cut orthogonally, if $mm' = -1$,

or if, $$\left(-\frac{ah}{bk}\right)\left(-\frac{a'h}{b'k}\right) = -1, \text{ or if, } aa'h^2 + bb'k^2 = 0$$

or if, $$\frac{aa'(b'-b)}{(ab'-a'b)} + \frac{bb'(a-a')}{(ab'-a'b)} = 0 \quad \text{(using 4.19)}$$

or if, $\frac{a-a'}{aa'} = \frac{b-b'}{bb'}$, or if, $\frac{1}{a} - \frac{1}{a'} = \frac{1}{b} - \frac{1}{b'}$, the required condition.

Example 4.11: In the catenary $y = \cosh(x/c)$, prove that the length of the portion of the normal intercepted between the curve and the axis of x is y^2/c.

Solution: The equation of the curve is $y = c\cosh x/c$. Differentiating with respect to x, we obtain

$$\frac{dy}{dx} = \sinh\frac{x}{c}.$$

The length of the normal intercepted between the curve and the axis of x, refer to Eq. (4.11), is

$$|y|\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \left|c\cosh\frac{x}{c}\right|\sqrt{1+\sinh^2 x/c} = c\left|\cosh\frac{x}{c}\right|^2 = y^2/c.$$

4.4 DERIVATIVE OF ARC LENGTH

Before finding the derivative of arc length, we derive an important result concerning the *limit of the ratio of the arc to the chord for any two points P and Q on a curve* to be used to find the derivative. *We prove that,*

$$\lim_{Q \to P}\frac{\text{arc } PQ}{\text{chord } PQ} = 1. \qquad ...(4.20)$$

Let $P(x, y)$ and $Q(x + \Delta x, y + \Delta y)$ be any two points on a curve $y = f(x)$ such that the arc PQ, throughout its length, is concave to the chord PQ as shown in Fig. 4.4. Let QS be the perpendicular from Q to the tangent to the curve at P. We have

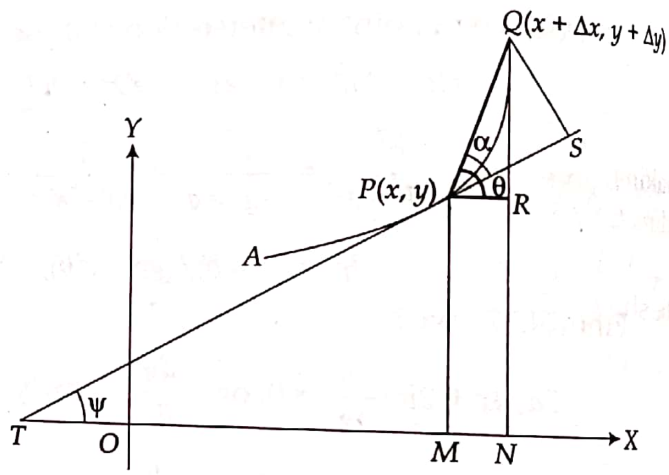

Fig. 4.4

$$\text{Chord } PQ < \text{arc } PQ < PS + SQ.$$

Dividing throughout by PQ, we obtain

$$1 < \frac{\text{arc } PQ}{PQ} < \frac{PS + SQ}{PQ}$$

or,
$$1 < \frac{\text{arc } PQ}{PQ} < \cos\alpha + \sin\alpha, \quad \alpha = \underline{|QPS} \qquad ...(4.21)$$

Let $Q \to P$ so that the chord PQ tends to the tangent PS in its limiting position and $\alpha \to 0$. Thus, from (4.21), we obtain (4.20) since $\cos\alpha + \sin\alpha \to 1$ as $\alpha \to 0$.

4.4.1 Length of Arc as a Function

With reference to a fixed point A, as shown in Fig. 4.4, let $\overset{\frown}{AP} = s$, and $\overset{\frown}{AQ} = s + \Delta s$.

We note that s is a function of x for the curve $y = f(x)$.

For the curve in parametric form: $x = f(t)$, $y = g(t)$, s is a function of the parameter t; and, s is a function of θ when the curve is in polar form $r = f(\theta)$.

Next we derive the derivative of s in all the three cases.

(A) Cartesian form: For the curve $y = f(x)$, from the right angled ΔPQR, refer to Fig. 4.4, we have

$$(PQ)^2 = (PR)^2 + (RQ)^2 = (\Delta x)^2 + (\Delta y)^2.$$

In the limiting position arc PQ = chord PQ, the above equation may be written as

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2$$

Dividing throughout by $(\Delta x)^2$ and considering the limit, $\Delta x \to 0$ (that is, $Q \to P$), we obtain

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2.$$

Thus, we have derivative of arc in cartesian form as

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \qquad \text{...(4.22)}$$

taking the positive sign as per the convention that for the curve $y = f(x)$, s is measured positively in the direction of increasing x.

If the equation of the curve is of the form $x = f(y)$, then s is considered as a function of y and it can be shown that

$$\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}. \qquad \text{...(4.23)}$$

(B) Parametric form: If the curve is of parametric form $x = f(t)$, $y = g(t)$, then the derivative of arc is

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}, \qquad \text{...(4.24)}$$

where we measure s positively in the increasing direction of t.

The result can be derived from (4.22) or directly also.

Also in Fig. 4.4, assuming $\underline{|RPQ} = \theta$, it is observed that as $Q \to P$, the angle θ tends to coincide with ψ. The angle made by the tangent to the curve at P with the positive direction of the x-axis.

From ΔPQR, $\sin\theta = \dfrac{RQ}{PQ} = \dfrac{\Delta y}{\Delta s}$, $\cos\theta = \dfrac{PR}{PQ} = \dfrac{\Delta x}{\Delta s}$, $\tan\theta = \dfrac{RQ}{PR} = \dfrac{\Delta y}{\Delta x}$.

Hence, as $Q \to P$, we obtain

$$\sin\psi = \frac{dy}{ds}, \quad \cos\psi = \frac{dx}{ds}, \quad \tan\psi = \frac{dy}{dx}. \qquad \text{...(4.25)}$$

We will be using these results later on.

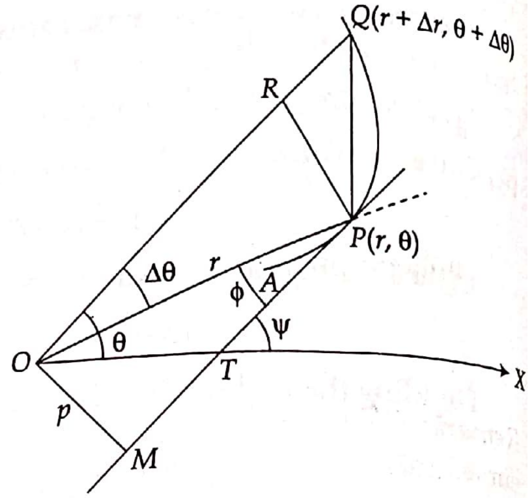

Fig. 4.5

(C) Polar form: Let s denote the arc length of any point $P(r, \theta)$ from some fixed point A on the curve $r = f(\theta)$ as shown in Fig. 4.5. Take a point $Q(r + \Delta r, \theta + \Delta\theta)$ near to P on the curve and let arc $AQ = s + \Delta s$, so that arc $PQ = \Delta s$.

If PR is perpendicular to OQ, then

$$PR = r \sin \Delta\theta. \qquad ...(4.26)$$

Also,
$$RQ = OQ - OR = r + \Delta r - r\cos\Delta\theta$$
$$= r(1 - \cos\Delta\theta) + \Delta r. \qquad ...(4.27)$$

In the limiting position as $Q \to P, \Delta\theta \to 0$, so we can take, $\sin\Delta\theta \approx \Delta\theta$ and $\cos\Delta\theta \approx 1$, and hence from (4.26) and (4.27), we have respectively

$$PR \approx r\Delta\theta \text{ and } \quad RQ \approx \Delta r. \qquad ...(4.28)$$

Next, from the rt. angled ΔPQR, refer Fig. 4.5, we have

$$(PQ)^2 = (PR)^2 + (RQ)^2.$$

In the limiting position, arc PQ = chord PQ, the above equation using (4.28) may be written as

$$(\Delta s)^2 = (\Delta r)^2 + (r\Delta\theta)^2,$$

which, as $Q \to P$, gives the *derivative of arc length in polar form* as

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \qquad ...(4.29)$$

and,
$$\frac{ds}{dr} = \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2} \qquad ...(4.30)$$

here s is to be measured in the positive direction of θ, or r, as the case may be.

The formula (4.29) is used when the equation is of the form $r = f(\theta)$, while (4.30) is used when equation is of the form $\theta = f(r)$.

4.4.2 Angle Between Tangent and Radius Vector

From the right angled triangle PQR, refer to Fig. 4.5, we have

$$\tan \angle RQP = \frac{RP}{QR} = \frac{r\sin\Delta\theta}{r(1-\cos\Delta\theta) + \Delta r}, \qquad \text{(from (4.26) and (4.27))}$$

$$= \frac{r\sin\Delta\theta}{2r\sin^2\frac{\Delta\theta}{2} + \Delta r}. \qquad ...(4.31)$$

When $Q \to P$ the angle $\angle OQP = (|\underline{RQP})$ tends to ϕ, the angle between the positive direction of the tangent and the radius vector at $P(r, \theta)$, and also $\Delta\theta \to 0$, thus (4.31) gives

$$\tan\phi = \lim_{\Delta\theta\to 0}\frac{r\sin\Delta\theta}{2r\sin^2\frac{\Delta\theta}{2}+\Delta r} = \lim_{\Delta\theta\to 0}\frac{r(\sin\Delta\theta/\Delta\theta)}{r\left(\sin\frac{\Delta\theta}{2}\right)\left(\sin\frac{\Delta\theta}{2}\Big/\frac{\Delta\theta}{2}\right)+\frac{\Delta r}{\Delta\theta}}$$

$$= r\frac{d\theta}{dr}, \text{ using } \lim_{\theta\to 0} = \frac{\sin\theta}{\theta} = 1. \qquad \text{...(4.32a)}$$

Similarly, we can show that

$$\sin\phi = r\frac{d\theta}{ds} \text{ and } \cos\phi = \frac{dr}{ds} \qquad \text{...(4.32b)}$$

Remark: The result (4.32a) is useful in finding the angle of intersection of two curves when the curves are in polar form, since, *if ϕ_1 and ϕ_2 are the angles between the common radius vector and the tangents to the two curves at their point of intersection, then the angle of intersection of these curves is $\phi_1 - \phi_2$.*

4.4.3 Pedal Equation of a Curve

If $OM = p$ is the length of the perpendicular from the pole O to the tangent PT, as shown in Fig. 4.5 then (p, r) are called the *pedal co-ordinates of the point P*.

In right angled triangle MPO, we have

$$\sin\phi = \frac{p}{r}, \text{ or } p = r\sin\phi \qquad \text{...(4.33)}$$

From (4.33), $\frac{1}{p^2} = \frac{1}{r^2\sin^2\phi} = \frac{1}{r^2}[1+\cot^2\phi] = \frac{1}{r^2}\left[1+\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right]$ (using (4.32a))

or,
$$\frac{1}{p^2} = \frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2, \qquad \text{...(4.34)}$$

an expression for the *length of the perpendicular from pole on the tangent,* to the curve $r = f(\theta)$ at the point $P(r, \theta)$.

The equation obtained by eliminating θ between (4.34) and the curve $r = f(\theta)$ is called the *pedal equation of the curve.*

4.4.4 Polar subtangent and Polar subnormal. Polar tangent and Polar normal

With reference to Fig. 4.6, let NOT be a straight line through the pole O and perpendicular to the radius vector OP. Let the tangent at P meets this line at T and the normal at P meets the line at N. Then OT and ON at called the *polar subtangent* and *polar subnormal* respectively.

Let $OM \perp PT$ it is obvious that $\underline{|PNO} = \underline{|MOT} = \phi$, the angle between the radius vector and the tangent at $P(r, \theta)$. Thus,

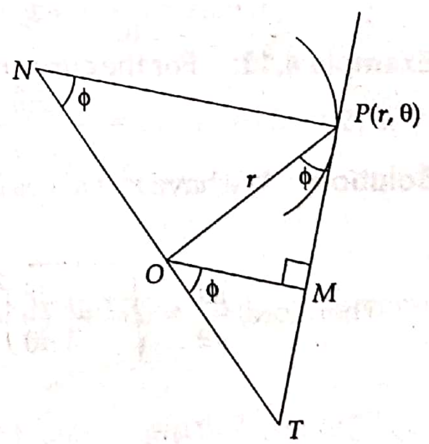

Fig. 4.6

Polar subtangent; $OT = r\tan\phi = r.r\dfrac{d\theta}{dr} = r^2\dfrac{d\theta}{dr}$; ...(4.35)

Polar subnormal; $ON = r\cot\phi = r.\dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{dr}{d\theta}$; ...(4.36)

Polar tangent; $PT = r\sqrt{1+\tan^2\phi} = r\sqrt{1+r^2\left(\dfrac{d\theta}{dr}\right)^2}$ ...(4.37)

Polar normal; $PN = r\sqrt{1+\cot^2\phi} = \sqrt{r^2+\left(\dfrac{dr}{d\theta}\right)^2}$ ...(4.38)

Example 4.12: Find $\dfrac{ds}{dx}, \dfrac{ds}{dy}$ for the curve $y = a\ln\sec\dfrac{x}{a}$ and also prove that $x = \psi$, where ψ is the angle which the tangent at an arbitrary point $P(x, y)$ makes with the positive direction of the x-axis.

Solution: Equation of the curve is, $y = a\ln\sec\dfrac{x}{a}$.

This gives, $\dfrac{dy}{dx} = \tan\dfrac{x}{a}$ ...(4.39)

Thus, $\dfrac{ds}{dx} = \sqrt{1+\left(\dfrac{dy}{dx}\right)^2} = \sqrt{1+\tan^2\dfrac{x}{a}} = \sec\dfrac{x}{a}$. Similarly, $\dfrac{ds}{dy} = \operatorname{cosec}\dfrac{x}{a}$.

Also, $\dfrac{dy}{dx} = \tan\psi$ ...(4.40)

From (4.39) and (4.40), we obtain, $\dfrac{x}{a} = \psi$, or $x = a\psi$.

Example 4.13: For the curve $r^2 = a^2\cos 2\theta$, find $\dfrac{ds}{d\theta}$ and $\dfrac{ds}{dr}$.

Solution: We have $r^2 = a^2\cos 2\theta$, which gives $2r\dfrac{dr}{d\theta} = -2a^2\sin 2\theta$, or $\dfrac{dr}{d\theta} = -\dfrac{a^2\sin 2\theta}{r}$.

Therefore, $\dfrac{ds}{d\theta} = \sqrt{r^2+\left(\dfrac{dr}{d\theta}\right)^2} = \sqrt{r^2+\dfrac{a^4\sin^2 2\theta}{r^2}} = \sqrt{a^2\cos 2\theta+\dfrac{a^4\sin^2 2\theta}{a^2\cos 2\theta}}$

$$= a\sqrt{\frac{\cos^2 2\theta+\sin^2 2\theta}{\cos 2\theta}} = \frac{a}{\sqrt{\cos 2\theta}} = \frac{a^2}{r}.$$

Also $$\frac{ds}{dr} = \sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2} = \sqrt{1+r^2\,\frac{r^2}{a^4\sin^2 2\theta}}$$

$$= \sqrt{1+\frac{a^4\cos^2 2\theta}{a^4\sin^2 2\theta}} = \operatorname{cosec} 2\theta = \frac{a^2}{\sqrt{a^2-r^2}}.$$

Example 4.14: Find the pedal equation of the parabola $y^2 = 4a(x+a)$.

Solution: Differentiating the given equation $y^2 = 4a(x+a)$ with respect to x, we have

$$2y\frac{dy}{dx} = 4a, \text{ or } \frac{dy}{dx} = \frac{2a}{y}.$$

Thus, the equation of the tangent to the given curve at the point (x, y) is

$$Y - y = \frac{2a}{y}(X-x), \text{ or } \frac{2a}{y}X - Y - \left(\frac{2ax}{y} - y\right) = 0$$

If p be the length of the perpendicular from the origin to this tangent, then

$$p = \frac{\left|\frac{2ax}{y} - y\right|}{\sqrt{\left(\frac{2a}{y}\right)^2 + 1}} = \frac{|2ax - y^2|}{\sqrt{y^2+4a^2}} = \frac{|2ax - 4a(x+a)|}{\sqrt{4a(x+a)+4a^2}} \text{ or, } p = \sqrt{a(x+2a)} \quad ...(4.41)$$

Also, $r^2 = x^2 + y^2 = x^2 + 4a(x+a) = (x+2a)^2$ or, $r = |x+2a|$ ...(4.42)

Eliminating x between (4.41) and (4.42), we obtain $p^2 = ar$ as the required pedal equation.

Example 4.15: Find the polar subtangent and polar subnormal to the curve $r = a(1-\cos\theta)$. Also find the pedal equation of this curve.

Solution: The equation of the given curve is $r = a(1-\cos\theta)$, which gives $\frac{dr}{d\theta} = a\sin\theta$.

Hence, polar subtangent $= r^2\frac{d\theta}{dr} = \frac{a^2(1-\cos\theta)^2}{a\sin\theta} = a\frac{4.\sin^4\theta/2}{2\sin\theta/2\cos\theta/2} = 2a\sin^3\theta/2\sec\theta/2.$

Also, polar subnormal $= \frac{dr}{d\theta} = a\sin\theta.$

Next, if ϕ is the angle between the radius vector and the tangent at a point $P(r,\theta)$ to the given curve $r = a(1-\cos\theta)$, then

$\tan\phi = r\frac{d\theta}{dr} = \frac{a(1-\cos\theta)}{a\sin\theta} = \tan\theta/2.$ Thus, $\phi = \theta/2$, and hence $p = r\sin\phi = r\sin\theta/2.$

Also, $r = a(1 - \cos\theta) = 2a\sin^2\theta/2$. Eliminating θ, between these two, we obtain $r = 2a\frac{p^2}{r^2}$, or $r^3 = 2ap^2$, as the required pedal equation of the curve.

Example 4.16: Find the angle of intersection of the curves $r = \sin\theta + \cos\theta$ and $r = 2\sin\theta$.

Solution: Let $P(r, \theta)$ be a point of intersections of the two curves. Consider the first curve,

$$r = \sin\theta + \cos\theta \qquad ...(4.43)$$

Taking logarithm this gives, $\ln r = \ln(\sin\theta + \cos\theta)$, and hence

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\cos\theta - \sin\theta}{\sin\theta + \cos\theta} = \frac{1 - \tan\theta}{1 + \tan\theta}.$$

Thus, $\tan\phi_1 = r\frac{d\theta}{dr} = \frac{1 + \tan\theta}{1 - \tan\theta} = \tan\left(\frac{\pi}{4} + \theta\right)$, which implies $\phi_1 = \frac{\pi}{4} + \theta$, where ϕ_1 is the angle between the tangent and the radius vector to the curve (4.43) at $P(r, \theta)$.

Similarly, for the second curve

$$r = 2\sin\theta \qquad ...(4.44)$$

we have, $\ln r = \ln(2\sin\theta) = \ln 2 + \ln\sin\theta$ and hence we obtain

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\cos\theta}{\sin\theta}. \text{ Thus, } \tan\phi_2 = r\frac{d\theta}{dr} = \tan\theta,$$

which implies $\phi_2 = \theta$ where ϕ_2 is the angle between the tangent and the radius vector at the same point $P(r, \theta)$ on the curve (4.44). Therefore, the angle of intersection between the two curves is

$$\phi_1 - \phi_2 = \left(\frac{\pi}{4} + \theta\right) - \theta = \frac{\pi}{4}$$

EXERCISE 4.3

1. If the tangent to the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at any point on it meets the x-axis at P and y-axis at Q, then show that $OP + OQ = a$, O being the origin.
2. If the normal to the curve $x^{2/3} + y^{2/3} = a^{2/3}$ makes an angle ϕ with the x-axis, then show that its equation is $y\cos\phi - x\sin\phi = a\cos 2\phi$.
3. Show that the parabolas $y^2 = 4ax$ and $2x^2 = ay$ intersect at an angle $\tan^{-1}(3/5)$.
4. Show that the curves $\frac{x^2}{a^2 + \alpha} + \frac{y^2}{b^2 + \alpha} = 1$ and $\frac{x^2}{a^2 + \beta} + \frac{y^2}{b^2 + \beta} = 1$ intersect orthogonally irrespective of the values of α and β.
5. Show that in the exponential curve $y = be^{x/a}$, the subtangent is the constant length and that the subnormal varies as the square of the ordinate.

6. For the curve $x = a(\cos t + \ln \tan t/2), y = a \sin t$, prove that the portion of the tangent between the curve and x-axis is constant. Also find its subtangent.
7. Prove that for the curve $ky^2 = (x + a)^3$, the square of the subtangent is proportional to the subnormal.
8. Show that the curves given by $r = \dfrac{a}{1+\cos\theta}$ and $r = \dfrac{b}{1-\cos\theta}$ intersect orthogonally.
9. Find the angle of intersection of the curves given by $r = a$, and $r = 2a\cos\alpha$.
10. Prove that the pedal equation of the asteroid $x = a\cos^3 t, y = a\sin^3 t$ is $r^2 = a^2 - 3p^2$.
11. Find the pedal equation of the following curves:
    (a) $r^2 = a^2 \sin^2\theta$
    (b) $r = a\sec t,\quad \theta = \tan t - t$
12. Find $\dfrac{ds}{dx}$ for the curve $3ay^2 = x(x-a)^2$.
13. For the ellipse $x = a\cos t, y = b\sin t$, prove that $\dfrac{ds}{dt} = a(1 - e^2\cos^2 t)^{1/2}$.
14. Show that for the hyperbolical spiral $r\theta = a$, $\dfrac{ds}{dr} = \dfrac{\sqrt{r^2+a^2}}{r}$.
15. Show that for any pedal curve $p = f(r)$, $\dfrac{ds}{dr} = \dfrac{r}{\sqrt{r^2-p^2}}$.
16. Show that the pedal equation of the curve $a^2(x^2+y^2) = x^2y^2$ is $\dfrac{1}{p^2} + \dfrac{3}{r^3} = \dfrac{1}{a^2}$.

4.5 MEAN VALUE THEOREMS

Mean value theorems play a very important role in the study of differential calculus. Of these, Rolle's theorem is the most fundamental one.

Theorem 4.1: **(Rolle's Theorem)** *If a function $f(x)$ is such that (i) it is continuous in the closed interval $[a, b]$, (ii) derivable in the open interval (a, b), and (iii) $f(a) = f(b)$, then there exists at least one value 'c' of x lying within (a, b) such that $f'(c) = 0$.*

Geometrically, *if the graph of a function is continuous curve from A to B, has a unique tangent at every point between A and B, and the ordinates of its extremities A, B are equal, then there exists at least one point P of the curve other than A and B, the tangent at which is parallel to x-axis*, a result quite evident from Figs. 4.7a, b, & c.

Proof. When $f(x)$ satisfies the above conditions, two cases arise. Either $f(x) = 0$ identically, or $f(x) \neq 0$ at least at some points in $[a, b]$.

If $f(x) = 0$ for all $x \in [a, b]$, then $f'(x) = 0$ at all x, and hence the theorem is proved trivially.

If $f(x) \neq 0$ at least at some points in $[a, b]$, then $f(x)$ being continuous in $[a, b]$, it must be bounded. Let

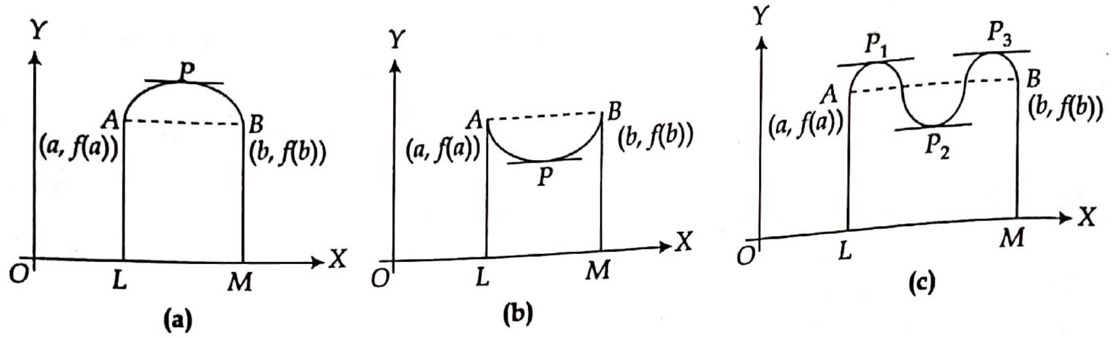

Fig. 4.7

M and m be the upper and lower bounds respectively. Thus, for some $x_1, x_2 \in [a, b]$, we have $f(x_1) = M$ and $f(x_2) = m$.

If $M = m$ then $f(x)$ = constant and hence $f'(x) = 0$; the theorem is proved.

However, if $M \neq m$, then at least one of M or m must be different from $f(a)$ and $f(b)$. Let for some c, $M = f(c) \neq f(a)$ or $f(b)$. Then $c \in (a, b)$ and we prove that $f'(c) = 0$.

Now $f(c) \geq f(c+h)$ for values of h both positive or negative. Then

$$\frac{f(c+h)-f(c)}{h} \leq 0, \text{ for } h>0 \qquad \ldots(4.45)$$

and,

$$\frac{f(c+h)-f(c)}{h} \geq 0, \text{ for } h<0 \qquad \ldots(4.46)$$

Since, f is differentiable in (a, b), from (4.45) and (4.46) as $h \to 0$, we have $f'(c) \leq 0$ and $f'(c) \geq 0$. Hence, $f'(c) = 0$ for some $c \in (a, b)$.

Similarly, we can prove in case $m = f(d)$ for some $d \in (a, b)$.

This completes the proof.

An other form of the Rolle's theorem is obtained by taking $b = a + h, h > 0$. In this case *if $f(x)$ is continuous in $[a, a+h]$, derivable in $(a, a+h)$ and $f(a) = f(a+h)$, then there exists at least one number $\theta \in (0,1)$ such that $f'(a+\theta h) = 0, 0 < \theta < 1$.*

The conclusion of Rolle's Theorem may not hold good for a function which does not satisfy any of these conditions.

For example, consider the function $f(x) = |x|$ in the interval $[-1, 1]$. Clearly $f(x)$ is continuous in the interval $[-1, 1]$, and $f(-1) = f(1)$. Its derivative $f'(x)$ is 1 for $0 < x \leq 1$ and is -1 for $-1 \leq x < 0$; and $f'(x)$ does not exist for $x = 0$. Thus, $f'(x)$ vanishes no where in the interval $(-1, 1)$, and hence the Rolle's theorem fails.

The failure is explained by the fact that $f(x)$ is not derivable in $(-1, 1)$. Geometrically, $y = |x|$ does not have a unique tangent at the origin as shown in Fig. 4.8.

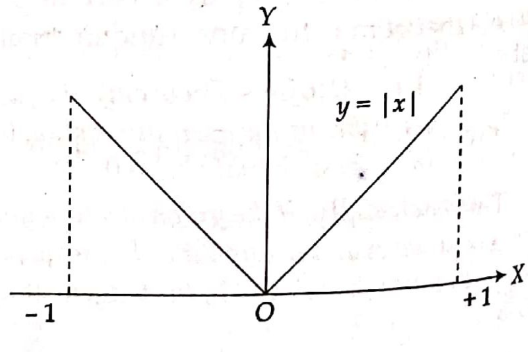

Fig. 4.8

Theorem 4.2: **(*Lagrange's Mean Value Theorem*)** *If a function $f(x)$ is such that (i) it is continuous in the closed interval $[a, b]$, (ii) derivable in the open interval (a, b), then there exists at least one value 'c' of x lying in (a, b) such that*

$$\frac{f(b)-f(a)}{b-a} = f'(c). \qquad ...(4.47)$$

Geometrically, if the graph of a function is a continuous curve from A to B and has a unique tangent at every point between A and B, then there exists at least one point P on the curve such that the tangent at P is parallel to the chord AB joining its extremities, a result quite evident from Figs. 4.9a & b.

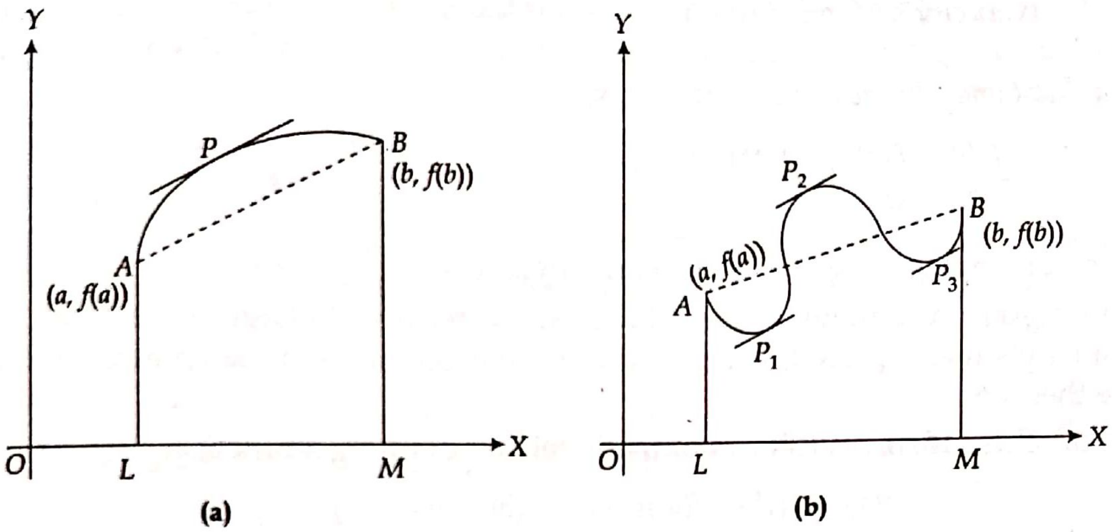

Fig. 4.9

Proof. Define a function

$$\phi(x) = f(x) + Ax \qquad ...(4.48)$$

where A is a constant such that $\phi(a) = \phi(b)$. Using this gives

$$A = -\frac{f(b)-f(a)}{(b-a)} \qquad ...(4.49)$$

From the form of $\phi(x)$, obviously it is continuous in $[a, b]$, differentiable in (a, b) and since $\phi(a) = \phi(b)$, thus $\phi(x)$ satisfies all the conditions of Rolle's Theorem and hence there exists at least one $c \in (a, b)$ such that $\phi'(c) = 0$, which gives from (4.48)

$$f'(c) + A = 0, \text{ or } f'(c) = -A, \text{ that is } f'(c) = \frac{f(b)-f(a)}{b-a} \quad \text{using (4.49)}$$

This completes the proof.

Another form of the Lagrange's mean value theorem is obtained by taking $b = a + h, h > 0$. In this case, *if $f(x)$ is continuous in $[a, a + h]$ and derivable in $(a, a + h)$, then there exists at least one number θ*, $0 < \theta < 1$, such that

$$f(a+h) = f(a) + hf'(a+\theta h), \qquad ...(4.50)$$

A few important results that follow obviously from the Lagrange's mean value theorem are:

1. *If the derivative of a function vanishes for all values of x in an interval, then the function must be a constant.*

2. *If two functions $f(x)$ and $g(x)$ have the same derivative for every value of x in an interval, then they differ only by a constant.*
3. *A function whose derivative is positive for every value of x in an interval is a monotonically increasing function of x in that interval.*
4. *A function whose derivative is negative for every value of x in an interval is a monotonically decreasing function of x in that interval.*

Theorem 4.3: **(*Cauchy's Mean Value Theorem*)** *If two functions $f(x)$ and $g(x)$ are (i) continuous in the closed interval (a, b), (ii) derivable in the open interval (a, b), and $g'(x) \neq 0$ for any value of x in (a, b), then there exists at least one c in the open interval (a, b) such that*

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}. \qquad \ldots(4.51)$$

Here we note that $g(b) - g(a)$ can't be equal to zero.

Clearly for $g(x) = x$, Cauchy mean value theorem reduces to Lagrange's mean value theorem. Also the Cauchy's mean value theorem has the same geometrical interpretation as of Lagrange's mean value theorem.

The proof of this theorem follows immediately by defining a function

$$\phi(x) = \{f(b) - f(a)\}g(x) - \{g(b) - g(a)\}f(x)$$

The function $\phi(x)$ satisfies all the conditions of Rolle's theorem; the result (4.51) follows from $\phi'(c) = 0$, for some $c \in (a, b)$.

Example 4.17: Prove that the equation $3x^5 + 15x - 8 = 0$ has only one real root.

Solution: The existence of at least one real root follows from the fact that the polynomial $f(x) = 3x^5 + 15x - 8$ is of an odd power.

To prove its uniqueness let us suppose that there exists two roots a and b, say $a < b$. Then in the interval $[a, b]$, the function $f(x) = 3x^5 + 15x - 8$ satisfies all the conditions of Rolle's theorem, hence there exists at least one $c \in (a, b)$ such that $f'(c) = 0$.

But $f'(x) = 15(x^4 + 1) > 0$. This contradiction proves that $f(x) = 3x^5 + 15x - 8$ has only one real root.

Example 4.18: Show that $\frac{h}{1+h^2} < \tan^{-1} h < h$, when $h \neq 0$ and $h > 0$.

Solution: Take $f(x) = \tan^{-1}x$, $0 \le x \le h$. The function $f(x)$ satisfies the conditions of Lagrange's mean value theorem in $[0, h]$. Thus,

$$\frac{\tan^{-1} h - \tan^{-1} 0}{h-0} = \frac{1}{1+k^2}, \text{ for some } k \in (0, h), \quad \text{or} \quad \tan^{-1} h = \frac{h}{1+k^2}.$$

Now $0 < k < h \Rightarrow 0 < k^2 < h^2$, which gives

$$1 < 1 + k^2 < 1 + h^2$$

or, $h > \frac{h}{1+k^2} > \frac{h}{1+h^2}$, $(h > 0)$, or $h > \tan^{-1} h > \frac{h}{1+h^2}$.

Example 4.19: Using the mean value theorem show that $3+\frac{1}{28}<\sqrt[3]{28}<3+\frac{1}{27}$.

Solution: If a function $f(x)$ satisfies all the conditions of Lagranges MVT in $[x, x+h]$, then

$$f(x+h)=f(x)+hf'(x+\theta h), \quad 0<\theta<1.$$

Take $f(x)=\sqrt[3]{x}=x^{1/3}$, so that $f'(x)=\frac{1}{3x^{2/3}}$. Set $x=27$, $h=1$, we get

$$f(x+h)=3+\frac{1}{3(27+\theta)^{2/3}}, \quad 0<\theta<1 \qquad \text{...(4.52)}$$

For $\theta=0$, the right side of (4.52), is

$$3+\frac{1}{3(27)^{2/3}}=3+\frac{1}{27} \qquad \text{...(4.53)}$$

and, for $\theta=1$, it is

$$3+\frac{1}{3(28)^{2/3}}=3+\frac{1}{(27)^{1/3}(28)^{2/3}}>3+\frac{1}{28} \qquad \text{...(4.54)}$$

Since $0<\theta<1$, from (4.52), (4.53) and (4.54), we get $3+\frac{1}{28}<\sqrt[3]{28}<3+\frac{1}{27}$.

Example 4.20: If $1<a<b$, then show that there exists a point c in (a, b) such that

$$\frac{\ln b-\ln a}{b-a}=\frac{b+a}{2c^2}.$$

Solution: Take $f(x)=\ln x$ and $g(x)=x^2$. Functions $f(x)$ and $g(x)$ satisfies the conditions of Cauchy's mean value theorem in the interval $[a, b]$. Hence, for some c in (a, b), we have

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}, \text{ or } \frac{\ln b-\ln a}{b^2-a^2}=\frac{1/c}{2c}=\frac{1}{2c^2}, \text{ or } \frac{\ln b-\ln a}{b-a}=\frac{b+a}{2c^2}.$$

Example 4.21: Let C be a curve defined parametrically as $x=a\cos^3\theta$, $y=a\sin^3\theta$, $0\le\theta\le\pi/2$. Determine a point P on C, where the tangent to C is parallel to the chord joining the point $(a, 0)$ and $(0, a)$.

Solution: Take $x=a\cos^3\theta=f(\theta)$ and $y=a\sin^3\theta=g(\theta)$. Clearly $f(\theta)$ and $g(\theta)$ satisfy the conditions of Cauchy's mean value theorem over the interval $[0, \pi/2]$.

Thus, using the Cauchy's mean value theorem, at some point θ, the slope of tangent is equal to the slope of the chord joining the points corresponding to $\theta=0$ and $\theta=\pi/2$, that is, the points $(a, 0)$ and $(0, a)$. Thus,

$$\frac{g'(0)}{f'(0)}=\frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta}=\frac{a-0}{0-a}, \text{ or } -\tan\theta=-1, \text{ or } \theta=\pi/4.$$

Therefore, the required point is $\left(a/2\sqrt{2},\ a/2\sqrt{2}\right)$.

Example 4.22: Show that $x/(1+x) < \ln(1+x) < x$, for $x > 0$.

Solution: Consider

$$f(x) = \ln(1+x) - \frac{x}{1+x}, \qquad \text{...(4.55)}$$

It gives
$$f'(x) = \frac{1}{1+x} - \frac{1}{(1+x)^2} = \frac{x}{(1+x)^2}.$$

Thus, $f'(x) > 0$, for $x > 0$ and $f'(0) = 0$. Hence, $f(x)$ is monotonically increasing in the interval $[0, \infty]$. Also $f(0) = 0$, therefore, $f(x) > f(0) = 0$ for $x > 0$. Thus, (4.55) gives

$$\ln(1+x) > x/(1+x), \text{ for } x > 0 \qquad \text{...(4.56)}$$

Similarly by taking $g(x) = x - \ln(1+x)$ we can prove that $g(x)$ is monotonically increasing in the interval $[0, \infty]$ and also $g(0) = 0$. Therefore,

$$x > \ln(1+x), \text{ for } x > 0 \qquad \text{...(4.57)}$$

From (4.56) and (4.57), we get the desired result.

EXERCISE 4.4

1. On the curve $y = x^3$ find the point at which the tangent line is parallel to the chord through the points $A(-1, -1)$ and $B(2, 8)$.
2. Without solving, prove that the equation $x^4 - 4x - 1 = 0$ has two different real roots.
3. Prove that all roots of the derivative of the given polynomial $f(x) = (x+1)(x-1)(x-2)(x-3)$ are real.
4. Calculate approximately $\sqrt[5]{245}$ by using Lagrange's mean value theorem.
5. Determine the root of the equation $x^3 + 5x - 10 = 0$ which lies in (1, 2) correct to two decimal places using mean value theorem.
6. Show that for all x, $\sin x$ lies between $x - \dfrac{x^3}{6}$ and $x - \dfrac{x^3}{6} + \dfrac{x^5}{120}$.
7. Show that $\dfrac{\tan x}{x} > \dfrac{x}{\sin x}$, if $0 < x < \dfrac{\pi}{2}$.
8. Determine the intervals in which the function $(x^4 + 6x^3 + 17x^2 + 32x + 32)e^{-x}$ is increasing or decreasing.
9. Show that for all $x > 0$, $1 - x < e^{-x} < 1 - x + \dfrac{x^2}{2}$.
10. A twice differentiable function f is such that $f(a) = f(b) = 0$ and $f(c) > 0$ for $0 < c < b$. Prove that there is at least one value ξ, $a < \xi < b$ for which $f''(\xi) < 0$.

11. Prove the inequality $\tan^{-1}x_2 - \tan^{-1}x_1 < x_2 - x_1$

12. Using Rolle's Theorem prove that the derivative $f'(x)$ of the function

$$f(x) = \begin{cases} x\sin\dfrac{\pi}{x}, & x > 0 \\ 0, & x = 0 \end{cases}$$

vanishes on an infinite set of points of the interval (0, 1).

4.6 TAYLOR'S AND MACLAURIN'S THEOREMS AND SERIES

A useful technique in the analysis of real valued functions is the approximation of continuous functions by polynomial. Taylor's and Maclaurin's theorems are important tools which provide such an approximation of the real valued functions. These theorems are regarded as *'generalized mean value theorems'* in the sense that mean value theorems relate the value of the function and its first order derivative, whereas, Taylor's and Maclaurin's theorems generalize this relation to higher order derivatives.

4.6.1 Taylor's Theorem with Lagrange's form of Remainder

Theorem 4.4: **(*Taylor's Theorem*)** *If a function $f(x)$ is such that*

(a) *$f(x), f'(x), f''(x), \ldots, f^{(n-1)}(x)$ are continuous in the closed interval $[a, a+h]$*

(b) *$f^{(n)}(x)$ exists in the open interval $(a, a+h)$, then there exists at least one number θ, between 0 and 1, such that*

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n}{n!}f^{(n)}(a+\theta h),\ 0 < \theta < 1 \quad \ldots(4.58)$$

Proof. We define a function $\phi(x)$ involving $f(x)$ and its derivatives $f'(x), f''(x), \ldots, f^{(n)}(x)$, designed so as to satisfy the conditions for Rolle's theorem. Let

$$\phi(x) = f(x) + (a+h-x)f'(x) + \frac{(a+h-x)^2}{2!}f''(x) + \ldots + \frac{(a+h-x)^{n-1}}{(n-1)!}f^{(n-1)}(x) + \frac{(a+h-x)^n}{n!}A, \quad \ldots(4.59)$$

where A is a constant to be determined such that

$$\phi(a) = \phi(a+h). \quad \ldots(4.60)$$

From (4.59) and (4.60), we obtain

$$f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n}{n!}A = f(a+h) \quad \ldots(4.61)$$

The functions $f(x), f'(x), f''(x), \ldots, f^{(n-1)}(x)$ are continuous in the closed interval $[a, a+h]$ and are derivable in the open interval $(a, a+h)$; also $(a+h-x)$, $\dfrac{(a+h-x)^2}{2!}, \ldots, \dfrac{(a+h-x)^n}{n!}$ and A are continuous in the closed interval $[a, a+h]$ and are derivable in the open interval $(a, a+h)$, therefore, $\phi(x)$ as defined in (4.59) is continuous in $[a, a+h]$, derivable in $(a, a+h)$ and also $\phi(a+h) = \phi(a)$.

Thus, $\phi(x)$ satisfies all the conditions for Rolle's theorem. There exists, therefore, at least one number θ between 0 and 1 such that $\phi'(a+\theta h) = 0$. From (4.59),

$$\phi'(x) = f'(x) - f'(x) + (a+h-x) f''(x) - (a+h-x) f''(x)$$

$$+ \frac{(a+h-x)^2}{2!} f'''(x) - \frac{(a+h-x)^2}{2!} f'''(x) + \dots + \frac{(a+h-x)^{n-1}}{(n-1)!} f^{(n)}(x) - \frac{(a+h-x)^{n-1}}{(n-1)!} A$$

$$= \frac{(a+h-x)^{n-1}}{(n-1)!} [f^{(n)}(x) - A], \text{ other terms cancelling in pairs.}$$

Therefore $\phi'(a+\theta h) = 0$ gives, $0 = \phi'(a+\theta h) = \dfrac{(h-\theta h)^{n-1}}{(n-1)!} [f^{(n)}(a+\theta h) - A]$ which gives

$$A = f^{(n)}(a+\theta h), \text{ since } \quad (1-\theta) \neq 0, \text{ for } \quad 0 < \theta < 1.$$

Substituting this value of A in (4.61), we get (4.58).

The $(n+1)$th term $\dfrac{h^n}{n!} f^{(n)}(a+\theta h)$ in (4.58) is called the Lagrange's form of remainder after n terms in the Taylors expansion of $f(a+h)$ and is denoted by $R_n(x)$.

Taking $n = 1$ in (4.58), we obtain

$$f(a+h) = f(a) + hf'(a+\theta h), \quad 0 < \theta < 1,$$

which is Lagrange's mean value theorem, refer to (4.50)

Thus LMVT is only a paticular case of Taylor's theorem with Lagrange's form of the remainder.

Remark: Another form of Taylor's expansion is with *Cauchy's form of remainder*. In this case the remainder after n terms is

$$R_n = \frac{h^n}{(n-1)!} (1-\theta)^{n-1} f^{(n)}(a+\theta h), \quad 0 < \theta < 1. \qquad \dots(4.62)$$

4.6.2 Maclaurin's Theorem with Lagrange's Form of Remainder

Theorem 4.5: **(*Maclaurin's Theorem*)** *If a function $f(x)$ is such that*

(a) $f(x), f'(x), f''(x), \dots, f^{(n-1)}(x)$ are continuous in the closed interval $[0, x]$.

(b) $f^{(n)}(x)$ exists in the open interval $(0, x)$, then there exists at least one number θ between 0 and 1 such that

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + \frac{x^n}{n!} f^{(n)}(\theta x). \qquad \dots(4.63)$$

This is obtained from Taylor's theorem by considering the interval $[0, x]$ instead of $[a, a+h]$ and changing a to 0 and h to x in (4.58).

4.6.3 Taylor's Infinite Series

Rewrite (4.58) as $f(a+h) = S_n + R_n$, where

$$S_n = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a),$$

and,
$$R_n = \frac{h^n}{n!}f^{(n)}(a+\theta h), \quad 0<\theta<1.$$

Suppose that $R_n \to 0$, as $n \to \infty$, then $\lim_{n\to\infty} S_n = f(a+h)$, so that the series

$$f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \dots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \dots$$

converges and its sum is equal to $f(a+h)$. Thus, we obtain that

(i) *if $f(x)$ possesses derivatives of every order in the interval $[a, a+h]$, and*

(ii) *the remainder $\frac{h^n}{n!}f^{(n)}(a+\theta h)$ tends to zero as n tends to infinity, then*

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots + \frac{h^n}{n!}f^{(n)}(a) + \dots \qquad \text{...(4.64)}$$

This series is known as *Taylor's infinite series.*

4.6.4 Maclaurin's Infinite Series

Set 0 for a and x for h in theTaylor's infinite series (4.64), we obtain that

(i) *if $f(x)$ possesses derivatives of every order in the interval $[0, x]$, and*

(ii) *the remainder $\frac{x^n}{n!}f^n(\theta x)$ tends to zero as n tends to infinity, then*

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots + \frac{x^n}{n!}f^{(n)}(0) + \dots \qquad \text{...(4.65)}$$

This series is known as *Maclaurin's series* for the expansion of function $f(x)$ in powers of x.

Remark: In order to find out if any given function can be expanded as an infinite Taylor series (or, Maclaurin series) or not, it is necessary to examine the behaviour of R_n as n tends to infinity, and for this we need to obtain the general expression for the nth derivative of the function. However, a formal expansion of a function as a power series is obtained under the assumption that as $n \to \infty$, $R_n \to 0$.

Example 4.23: Obtain the fourth degree polynomial approximation to $f(x) = e^{2x}$ about $x = 0$. Find the maximum error when $0 \le x \le 0.5$.

Solution: The Maclaurin's theorem with Lagrange's form of remainder after the fourth degree term is

$$f(x) = f(0) + xf^{(1)}(0) + \frac{x^2}{2!}f^{(2)}(0) + \frac{x^3}{3!}f^{(3)}(0) + \frac{x^4}{4!}f^{(4)}(0) + \frac{x^5}{5!}f^{(5)}(\theta x), 0 < \theta < 1.$$

For $f(x) = e^{2x}$, we have $f^{(r)}(x) = 2^r e^{2x}$, and therefore, $f^{(r)}(0) = 2^r$.

Also, $$f^{(5)}(\theta x) = 2^5 e^{2\theta x} = 32 e^{2\theta x}$$

Therefore, $$f(x) = e^{2x} \approx 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \frac{16x^4}{4!} + \dots$$

$$\approx 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4.$$

The error term is given by $R_5(x) = \frac{x^5}{5!}f^{(5)}(\theta x) = \frac{32}{5!}x^5 e^{2\theta x}, \quad 0 < \theta < 1$

$$= \frac{32}{5!}x^5 e^{2c}, \quad 0 < c < x, \text{ taking } \theta x = c.$$

Thus, $$|R_5(x)| \le \frac{32}{120}\left[\max_{0 \le x \le 0.5} . x^5\right]\left[\max_{0 < c < 0.5} e^{2c}\right] \le \frac{32}{120}(0.5)^5(e^{2(0.5)}) = \frac{e}{120}.$$

Thus maximum error is $e/120$, for $0 < x < 0.5$.

Example 4.24: Prove by Taylor's series that

$$\tan^{-1}(x+h) = \tan^{-1}x + h\sin\alpha\frac{\sin\alpha}{1} - (h\sin\alpha)^2\frac{\sin 2\alpha}{2} + (h\sin\alpha)^3\frac{\sin 3\alpha}{3} - \dots,$$

where $x = \cot\alpha$.

Solution: Here $x = \cot\alpha$, which gives $\alpha = \cot^{-1}x$. Thus,

$$\frac{d\alpha}{dx} = \frac{-1}{1+x^2} = \frac{-1}{1+\cot^2\alpha} = -\sin^2\alpha.$$

Consider $f(x) = \tan^{-1}x$, therefore,

$$f'(x) = \frac{1}{1+x^2} = \frac{1}{1+\cot^2\alpha} = \frac{1}{\operatorname{cosec}^2\alpha} = \sin^2\alpha$$

$$f''(x) = 2\sin\alpha\cos\alpha\frac{d\alpha}{dx} = \sin 2\alpha\,(-\sin^2\alpha) = -\sin^2\alpha\sin 2\alpha$$

$$f'''(x) = -[2\sin\alpha\cos\alpha.\sin 2\alpha + 2\cos 2\alpha\sin^2\alpha]\frac{d\alpha}{dx}$$

$$= -2\sin\alpha[\cos\alpha\sin 2\alpha + \cos 2\alpha\sin\alpha](-\sin^2\alpha)$$

$$= 2\sin^3\alpha\sin 3\alpha.$$

By Taylor's series, $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + ...$ Substituting for $f(x), f'(x), f''(x), ...,$ we obtain

$$\tan^{-1}(x+h) = \tan^{-1}x + h\sin^2\alpha + \frac{h^2}{2!}(-\sin^2\alpha\sin 2\alpha) + \frac{h^3}{3!}(2\sin^3\alpha\sin 3\alpha) - ...$$

or, $$\tan^{-1}(x+h) = \tan^{-1}x + h\sin\alpha\frac{\sin\alpha}{1} - (h\sin\alpha)^2\frac{\sin 2\alpha}{2} + (h\sin\alpha)^3\frac{\sin 3\alpha}{3} - ...$$

the required expansion.

Example 4.25: Expand $\ln\sin x$ in powers of $(x-3)$.

Solution: Let $f(x) = \ln\sin x$.

Therefore, $f'(x) = \frac{\cos x}{\sin x} = \cot x,\qquad f'(3) = \cot 3$

$f''(x) = -\operatorname{cosec}^2 x,\qquad f''(3) = -\operatorname{cosec}^2 3$

$f'''(x) = 2\operatorname{cosec}^2 x\cot x,\qquad f'''(3) = 2\operatorname{cosec}^2 3\cot 3$ etc.

write $f(x) = f(3 + \overline{x-3}) = f(3+h)$, where $h = x-3$.

By Taylor's series $f(x) = f(3+h) = f(3) + hf'(3) + \frac{h^2}{2!}f''(3) + ...$

Substituting for $f(x), f(3), f'(3), f''(3)$ etc. we obtain

$$\ln\sin x = \ln\sin 3 + (x-3)\cot 3 - \frac{(x-3)^2}{2}\operatorname{cosec}^2 3 + \frac{(x-3)^3}{3}\operatorname{cosec}^2 3\cot 3 + ...$$

Example 4.26: Apply Taylor's series to calculate the value of $f(11/10)$, where

$$f(x) = x^3 + 3x^2 + 15x - 10.$$

Solution: By Taylor's series, $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + ...$

Put $x = 1,\quad h = 1/10$, this gives

$$f\left(\frac{11}{10}\right) = f(1) + \frac{1}{10}f'(1) + \left(\frac{1}{10}\right)^2\frac{f''(1)}{2!} + \left(\frac{1}{10}\right)^3\frac{f'''(1)}{3!} + ... \qquad ...(4.66)$$

Here, $f(x) = x^3 + 3x^2 + 15x - 10, \quad f(1) = 9$

$f'(x) = 3x^2 + 6x + 15, \quad f'(1) = 24$

$f''(x) = 6x + 6, \quad f''(1) = 12$

$f'''(x) = 6. \quad f'''(1) = 6$ etc.

Substituting for $f(1), f'(1), f''(1)$, etc. in (4.66), we obtain

$$f\left(\frac{11}{10}\right) = 9 + \frac{1}{10}(24) + \frac{1}{100}\left(\frac{12}{2}\right) + \frac{1}{1000}\left(\frac{6}{6}\right) = 9 + 2.4 + 0.06 + 0.0001 = 11.461.$$

Example 4.27: Calculate the approximate value of $\sqrt{10}$ to four decimal places using Taylor's series.

Solution: Consider $f(x) = \sqrt{x}$. By Taylor's series $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots$

Put $x = 9, h = 1;$ we obtain

$$f(10) = f(9) + f'(9) + \frac{f''(9)}{2!} + \frac{f'''(9)}{3!} + \ldots \qquad \ldots(4.67)$$

Now, $f(x) = \sqrt{x}, \quad f(9) = 3$

$f'(x) = \frac{1}{2\sqrt{x}}, \quad f'(9) = \frac{1}{6}$

$f''(x) = -\frac{1}{4x\sqrt{x}}, \quad f''(9) = -\frac{1}{108}$

$f'''(x) = \frac{3}{8x^2\sqrt{x}}, \quad f'''(9) = \frac{1}{648}$ etc.

Substituting for $f(9), f'(9), f''(9)$, etc. in (4.67), we obtain

$$\sqrt{10} = 3 + \frac{1}{6} - \frac{1}{2(108)} + \frac{1}{6(648)} - \ldots = 3 + 0.16666 - 0.00463 + 0.00025 + \ldots = 3.1623$$

Example 4.28: Expand $\tan\left(x + \frac{\pi}{4}\right)$ as far as the term x^4, and evaluate tan 44° to four significant digits.

Solution: Let $f(x) = \tan x$. By Taylor's series expansion

$$f\left(x + \frac{\pi}{4}\right) = f\left(\frac{\pi}{4}\right) + xf'\left(\frac{\pi}{4}\right) + \frac{x^2}{2!}f''\left(\frac{\pi}{4}\right) + \frac{x^3}{3!}f'''\left(\frac{\pi}{4}\right) + \frac{x^4}{4!}f^{iv}\left(\frac{\pi}{4}\right) + \ldots \qquad \ldots(4.68)$$

Now,
$$f(x) = \tan x$$
$$f'(x) = \sec^2 x = 1 + \tan^2 x$$
$$f''(x) = 2\sec^2 x \tan x = 2[\tan x + \tan^3 x]$$
$$f'''(x) = 2[\sec^2 x + 3\tan^2 x \sec^2 x] = 2[1 + 4\tan^2 x + 3\tan^4 x]$$
$$f^{iv}(x) = 2[8\tan x \sec^2 x + 12\tan^3 x \sec^2 x] = 16\tan x + 40\tan^3 x + 24\tan^5 x.$$

Therefore, $f\left(\frac{\pi}{4}\right) = 1,\ f'\left(\frac{\pi}{4}\right) = 2,\ f''\left(\frac{\pi}{4}\right) = 4,\ f'''\left(\frac{\pi}{4}\right) = 16,\ f^{iv}\left(\frac{\pi}{4}\right) = 80.$

Substituting in (4.68), we obtain

$$\tan\left(x + \frac{\pi}{4}\right) = 1 + 2x + \frac{4x^2}{2!} + \frac{16x^3}{3!} + \frac{80x^4}{4!} + \ldots$$

$$= 1 + 2x + 2x^2 + \frac{8}{3}x^3 + \frac{10}{3}x^4 + \ldots \qquad \ldots(4.69)$$

To evaluate tan 44°, take $x = -1° = -\frac{\pi}{180} = -0.017453$ radians. Substituting in (4.69), and simplifying, we obtain $\tan 44° \simeq 0.9657$, up to four significant digits.

Example 4.29: Show that $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$ and from this derive the expansion for $\cos x$.

Solution: Here $f(x) = \sin x$, therefore

$$f'(x) = \cos x,\quad f''(x) = -\sin x,\quad f'''(x) = -\cos x, f^{iv}(x) = \sin x,\quad f^{v}(x) = \cos x, \text{ etc.}$$

Maclaurin's series expansion is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \frac{x^5}{5!}f^{v}(0) + \ldots$$

Substituting for $f(x), f(0), f''(0), \ldots$ we get $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$

To obtain the expansion for $\cos x$ differentiating it term by term, we obtain $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$

Example 4.30: Imp Show that, $\sin^{-1} x = x + \frac{1^2}{3!}x^3 + \frac{1^2.3^2}{5!}x^5 + \frac{1^2.3^2.5^2}{7!}x^7 + \ldots$

Hence, find the value for π correct upto three decimal places.

Solution: Consider $\quad y(x) = \sin^{-1} x$. It gives $\qquad \ldots(4.70)$

$$y_1(x) = \frac{1}{\sqrt{1-x^2}} \quad ...(4.71)$$

or, $$y_1^2(1-x^2) = 1$$

Differentiating w. r. t. x, we obtain

$$2y_1y_2(1-x^2) - 2y_1^2x = 0$$

$$y_2(1-x^2) - y_1x = 0 \quad ...(4.72)$$

Differentiating (4.72) n times using Leibnitz rule, we have

$$[y_{n+2}(1-x^2) + C_1^n y_{n+1}(-2x) + C_2^n y_n(-2)] - [y_{n+1}\,x + C_1^n y_n] = 0$$

or, $$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0 \quad ...(4.73)$$

Substituting $x = 0$ in (4.70), (4.71), (4.72) and (4.73), we obtain respectively

$$y(0) = 0,\ y_1(0) = 1,\ y_2(0) = 0, \text{ and } y_{n+2}(0) = n^2y_n(0) \quad ...(4.74)$$

From (4.74), $y_3(0) = 1^2y_1(0) = 1^2$, $y_4(0) = 2^2y_2(0) = 0$

$y_5(0) = 3^2y_3(0) = 1^2.3^2$, $y_6(0) = 4^2y_4(0) = 0$, $y_7(0) = 5^2y_5(0) = 1^2.3^2.5^2$, and so on.

Substituting these in the Maclaurin's series

$$y(x) = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \ldots$$

we obtain, $$\sin^{-1} x = x + \frac{1^2}{3!}x^3 + \frac{1^2.3^2}{5!}x^5 + \frac{1^2.3^2.5^2}{7!}x^7 + \ldots \quad ...(4.75)$$

the required expansion.

Since, the series (4.75) is convergent for $-1 < x < 1$, to find the value for π, put $x = \frac{1}{2}$ in (4.75), we have

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{6}\left(\frac{1}{2}\right)^3 + \frac{3}{40}\left(\frac{1}{2}\right)^5 + \frac{5}{112}\left(\frac{1}{2}\right)^7 + \ldots$$

or, $$\pi = 3 + \frac{1}{8} + \frac{9}{640} + \frac{30}{14336} + \ldots = 3.141 \text{ approximately.}$$

Example 4.31: Prove that if terms of x^3 and higher orders are neglected, then

$$\tan^{-1}[(1+x)\tan\alpha] = \alpha + x\sin\alpha\cos\alpha - x^2\sin^3\alpha\cos\alpha.$$

Solution: Let $f(x) = \tan^{-1}[(1+x)\tan\alpha]$

It gives $f'(x) = \dfrac{\tan\alpha}{1+(1+x)^2\tan^2\alpha}$, $f''(x) = -\tan\alpha[1+(1+x)^2\tan^2\alpha]^{-2}[2(1+x)\tan^2\alpha]$.

Therefore, $f(0) = \tan^{-1}(\tan\alpha) = \alpha$, $f'(0) = \dfrac{\tan\alpha}{1+\tan^2\alpha} = \dfrac{\tan\alpha}{\sec^2\alpha} = \sin\alpha\cos\alpha$

$$f''(0) = -\tan\alpha[1+\tan^2\alpha]^{-2}(2\tan^2\alpha) = -\frac{2\tan^3\alpha}{\sec^4\alpha} = -2\sin^3\alpha\cos\alpha.$$

Maclaurin's series expansion, neglecting the terms of x^3 and higher order, is

$f(x) \approx f(0) + xf'(0) + \dfrac{x^2}{2!}f''(0)$. Substituting for $f(x), f(0), f'(0), f''(0)$, we obtain

$$\tan^{-1}[(1+x)\tan\alpha] = \alpha + x\sin\alpha\cos\alpha - x^2\sin^3\alpha\cos\alpha.$$

Example 4.32: Expand $y = \sin[\ln(x^2+2x+1)]$ in powers of x using Maclaurin's series up to x^5.

Solution: Let $$y(x) = \sin[\ln(x^2+2x+1)]. \quad ...(4.76)$$

It gives $$y_1(x) = \cos[\ln(x^2+2x+1)].\frac{2x+2}{x^2+2x+1}$$

$$= \frac{2}{x+1}\cos[\ln(x^2+2x+1)] \quad ...(4.77)$$

or, $$(x+1)y_1 = 2\cos[\ln(x^2+2x+1)].$$

Differentiating it again w.r.t. x, we get

$$(x+1)y_2 + y_1 = -2\sin[\ln(x^2+2x+1)].\frac{2(x+1)}{(x+1)^2} = -\frac{4y}{x+1}$$

or, $$(x+1)^2y_2 + (x+1)y_1 + 4y = 0. \quad ...(4.78)$$

Differentiating (4.78) n times using Leibnitz's rule, we get

$$[(x+1)^2y_{n+2} + C_1^n 2(x+1)y_{n+1} + C_2^n(2)y_n] + [(x+1)y_{n+1} + C_1^n(1)y_n] + 4y_n = 0$$

or, $$(x+1)^2y_{n+2} + (2n+1)(x+1)y_{n+1} + (n^2+4)y_n = 0. \quad ...(4.79)$$

Substituting $x = 0$, in (4.76), (4.77), (4.78) and (4.79), we obtain

$y(0) = 0$, $y_1(0) = 2$, $y_2(0) = -2$, and $y_{n+2}(0) + (2n+1)y_{n+1}(0) + (n^2+4)y_n(0) = 0$

or, $$y_{n+2}(0) = -(2n+1)y_{n+1}(0) - (n^2+4)y_n(0) \quad ...(4.80)$$

Set $n = 1, 2, 3, \ldots$ in (4.80), we get

$$y_3(0) = -3y_2(0) - 5y_1(0) = -4,\ y_4(0) = -5y_3(0) - 8y_2(0) = 36$$

$$y_5(0) = -7y_4(0) - 13y_3(0) = -200, \text{ and so on.}$$

Substituting these values in Maclaurin's series expansion,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots,$$

we obtain, $\sin[\ln(x^2+2x+1)] = 2x - x^2 - \frac{2}{3}x^3 + \frac{3}{2}x^4 - \frac{5}{3}x^5 + \ldots$

as the required expansion.

EXERCISE 4.5

1. Using Taylor's series prove that

(i) $\cos\left(x+\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\left(1 - x - \frac{x^2}{2!} + \frac{x^3}{3!} - \ldots\right)$

(ii) $\ln\sin(x+h) = \ln\sin x + h\cot x - \frac{h^2}{2}\operatorname{cosec}^2 x + \frac{h^3}{3}\operatorname{cosec}^2 x\cot x.$

(iii) $a^{x+h} = a^x\left[1 + h\ln a + \frac{h^2}{2!}(\ln a)^2 + \frac{h^3}{3!}(\ln a)^3 + \ldots\right]$

(iv) $\ln(x+h) = \ln x + \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \ldots$

(v) $\tan^{-1}(x+h) = \tan^{-1}x + \frac{h}{1+x^2} - \frac{xh^2}{(1+x^2)^2} + \ldots$

(vi) $\tan(x+h) = \tan x + h\sec^2 x + h^2\sec^2 x\tan x + \frac{h^3}{3}\sec^2 x(1+3\tan^2 x) + \ldots$

(viii) $\sec^{-1}(x+h) = \sec^{-1}x + \frac{h}{x\sqrt{x^2-1}} - \frac{h^2}{2!}\cdot\frac{2x^2-1}{x^2(x^2-1)^{3/2}} + \ldots,$

2. Using Taylor's series expand

(i) $\cos x$ in powers of $\left(x-\frac{\pi}{4}\right)$ up to 4 terms.

(ii) e^x in powers of $(x-2)$

(iii) $\sin x$ in powers of $\left(x-\frac{\pi}{2}\right)$. Hence, find the value of $\sin 91°$ correct to 4 decimal places.

(iv) $\tan^{-1}x$ in powers of $(x-1)$

(v) $2x^3+7x^2+x-1$ in powers of $(x-2)$.

3. Compute to four decimal places, the value of $\cos 32°$ using Taylor's series.

4. Given $\log_{10} 4 = 0.6021$, calculate approximately $\log_{10} 404$.

5. If $f(x) = x^3 + 2x^2 - 5x + 11$, calculate the value of $f\left(\frac{9}{10}\right)$ by the application of Taylor's series.

6. Calculate the approximate value of $\sqrt{17}$ correct to four decimal places using Taylor's series.

7. Prove that: (i) $f(ax) = f(x) + (a-1)xf'(x) + \frac{(a-1)^2x^2}{2!}f''(x) + \ldots$

(ii) $f\left(\frac{x^2}{1+x}\right) = f(x) - \frac{x}{1+x}f'(x) + \left(\frac{x}{1+x}\right)^2 \frac{f''(x)}{2!} - \left(\frac{x}{1+x}\right)^3 \frac{f'''(x)}{3!} + \ldots$

8. Show that the error terms in the Taylor's polynomial expansion of the function $f(x) = \sin x$ about the point $x = \frac{\pi}{4}$ tends to zero as $n \to \infty$ for any real x.

9. Expand the following functions using Maclaurin's series in powers of x.
(i) e^x (ii) $\cos x$ (iii) $\tan x$ (iv) $\sinh x$ (v) $\tan^{-1} x$ (vi) $\ln(1 + x)$ (viii) $\sec x$ (viii) $\tan^{-1}(1 + x)$ (ix) a^{x+h}

10. Prove that

(i) $e^x \sec x = 1 + x + \frac{x^2}{2} + \frac{2x^3}{3} + \ldots$ (ii) $\ln\cosh x = \frac{1}{2}x^2 - \frac{x^4}{12} + \frac{x^6}{45} - \ldots$

(iii) $\cos^2 x = 1 - x^2 + \frac{x^4}{3} - \ldots$ (iv) $\frac{x}{\sin x} = 1 + \frac{x^2}{6} + \frac{7x^4}{360} + \ldots$

(v) $\frac{e^x}{\cos x} = 1 + x + \frac{2x^2}{2!} + \frac{4x^3}{3!} + \ldots$ (vi) $e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \ldots$

11. Prove that

(i) $\ln\tan\left(x + \frac{\pi}{4}\right) = 2x + \frac{4}{3}x^3 + \frac{4}{2}x^5 + \ldots$ (ii) $\ln(1 + \sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \ldots$

(iii) $\ln\sec x = \frac{1}{2}x^2 + \frac{1}{12}x^4 + \frac{1}{45}x^6 + \ldots$ (iv) $\ln(\sec x + \tan x) = x + \frac{x^3}{6} + \frac{x^5}{24} + \ldots$

12. By forming a differential equation, prove that

(i) $(\sin^{-1} x)^2 = 2\frac{x^2}{2!} + 2.2^2\frac{x^4}{4!} + 2.2^2 4^2\frac{x^6}{6!} + \ldots$

(ii) $e^{m\tan^{-1} x} = 1 + mx + \frac{m^2}{2!}x^2 + \frac{m(m^2-2)}{3!}x^3 + \frac{m^2(m^2-8)}{4!}x^4 + \ldots$

(iii) $\ln\left[x+\sqrt{1+x^2}\right] = x - \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} - \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{x^7}{7} + \ldots$

(iv) $\left(x+\sqrt{1+x^2}\right)^n = 1 + nx + \frac{n^2x^2}{2!} + \frac{n^2(n^2-1^2)}{3!}x^3 + \frac{n^2(n^2-2^2)}{4!}x^4 + \ldots$

(v) $\frac{\ln(1+x)}{1+x} = x - \frac{3}{2}x^2 + \frac{11}{6}x^3 - \frac{25}{12}x^4 + \ldots$

(vi) $\sin(2\sin^{-1}x) = 2x - x^3 - \frac{x^5}{4} - \ldots$ (vii) $\frac{\sin^{-1}x}{\sqrt{1-x^2}} = x + \frac{2}{3}x^3 + \frac{8}{15}x^5 + \ldots$

13. Using Maclaurin's series show that, $e^{x\cos\alpha}\cos(x\sin\alpha) = 1 + x\cos\alpha + \frac{x^2}{2!}\cos 2\alpha + \ldots$

Also deduce that $\cos x = 1 - \frac{x^2}{2!} + \ldots$

14. Apply Maclaurin's series to obtain the expansion of the function $e^{ax}\sin bx$ in an infinite series of power of x giving the general term.

15. Obtain by Maclaurin's theorem, the first four terms of the expansion of $e^{x\cos x}$. Using this show that $\lim\limits_{x\to 0}\frac{e^x - e^{x\cos x}}{x - \sin x} = 3.$

16. Find the minimum number of terms that must be retained in the Maclaurin's series expansion of the function $\sin x\cos x$ in the interval $[0, 1]$, such that $|\text{Error}| < 0.0005$.

4.6.5 Use of Some Standard Series

Sometimes it is cumbersome to find the successive derivatives of a function. In such cases the use of Maclaurin's expansion of some standard functions is advisable. The expansions for some standard functions are listed below:

1. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$

2. $a^x = 1 + x(\ln a) + \frac{x^2}{2!}(\ln a)^2 + \frac{x^3}{3!}(\ln a)^3 + \ldots$

3. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

4. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

5. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots, |x|<1$

6. $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots$

7. $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$

8. $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots$ 9. $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots$

10. $(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \ldots \; (|x|<1).$

Next we give some examples illustrating the use of some of these series.

Example 4.33: Prove that $\ln(1+x+x^2+x^3+x^4) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} - \frac{4}{5}x^5 + \frac{x^6}{6} + \ldots$

Solution: We have, $1+x+x^2+x^3+x^4 = \frac{1-x^5}{1-x}$. Therefore,

$$\ln(1+x+x^2+x^3+x^4) = \ln(1-x^5) - \ln(1-x)$$

$$= \left[-x^5 - \frac{(x^5)^2}{2} - \frac{(x^5)^3}{3} - \ldots\right] - \left[-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \frac{x^6}{6}\right]$$

$$= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} - \frac{4}{5}x^5 + \frac{x^6}{6} + \ldots, \quad |x|<1,$$

the required expansion.

Example 4.34: Prove that $(1+x)^x = 1 + x^2 - \frac{x^3}{2} + \frac{5}{6}x^4 - \frac{3}{4}x^5 + \frac{33}{40}x^6 + \ldots$

Solution: We have $(1+x)^x = e^{\ln(1+x)^x} = e^{x\ln(1+x)} = e^t$, where $t = x\ln(1+x)$. Consider

$$t = x\ln(1+x) = x\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots\right) = x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \frac{x^6}{5} - \ldots$$

Therefore, $(1+x)^x = e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots$

$$= 1 + \left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \frac{x^6}{5} - \ldots\right) + \frac{1}{2!}\left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \frac{x^6}{5} - \ldots\right)^2$$

$$+ \frac{1}{3!}\left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \frac{x^6}{5} - \ldots\right)^3 + \ldots$$

$$= 1 + x^2 - \frac{x^3}{2} + x^4\left(\frac{1}{3} + \frac{1}{2}\right) + x^5\left(-\frac{1}{4} - \frac{1}{2}\right) + x^6\left(\frac{1}{5} + \frac{1}{8} + \frac{1}{6} + \frac{1}{3}\right) + \ldots$$

$$= 1 + x^2 - \frac{x^3}{2} + \frac{5}{6}x^4 - \frac{3}{4}x^5 + \frac{33}{40}x^6 + \ldots$$

is the required expansion.

Example 4.35: Expand ln [1 – ln (1 – x)] in powers of x by Maclaurin's series up to the term of x^3 and deduce the expansion of ln [1 + ln (1 + x)].

Solution: Let $f(x) = \ln[1 - \ln(1-x)]$. We have, $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots$

Therefore, $f(x) = \ln[1 - \ln(1-x)]$

$$= \ln\left[1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\right] = \ln(1+t), \text{ where } t = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots$$

Further $\ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} + \ldots$ Therefore,

$$\ln[1 - \ln(1-x)] = \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\right) - \frac{1}{2}\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\right)^2 + \frac{1}{3}\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\right)^3 + \ldots$$

$$= x + \frac{1}{6}x^3 + \ldots, \qquad \ldots(4.81)$$

on simplification.

To get the expansion of ln [1 + ln (1 + x)], replace x by $\frac{x}{1+x}$ in (4.81). The left side of (4.81) becomes

$$\ln[1 - \ln(1-x)] = \ln\left[1 - \ln\left(1 - \frac{x}{1+x}\right)\right] = \ln\left[1 - \ln\left(\frac{1}{1+x}\right)\right] = \ln[1 + \ln(1+x)].$$

Therefore, (4.81) becomes

$$\ln[1 + \ln(1+x)] = x(1+x)^{-1} + \frac{1}{6}x^3(1+x)^{-3} + \ldots$$

$$= x\left[1 - x + x^2 + x^3 - \ldots\right] +$$

$$\frac{1}{6}x^3\left[1 - 3x + \frac{(-3)(-4)}{2!}x^2 + \frac{(-3)(-4)(-5)}{3!}x^3 + \ldots\right]$$

$$= x - x^2 + x^3\left(1 + \frac{1}{6}\right) + \ldots = x - x^2 + \frac{7}{6}x^3 + \ldots$$

Example 4.36: Expand $\cos^{-1}\left(\dfrac{x-x^{-1}}{x+x^{-1}}\right)$ in ascending powers of x.

Solution: Put $x = \cot\theta$, then

$$\cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right) = \cos^{-1}\left(\frac{\cot\theta-\tan\theta}{\cot\theta+\tan\theta}\right) = \cos^{-1}\left(\frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta+\sin^2\theta}\right) = \cos^{-1}(\cos 2\theta)$$

$$= 2\theta = 2\cot^{-1}x = 2\left(\frac{\pi}{2}-\tan^{-1}x\right) = \pi - 2\tan^{-1}x.$$

Also, $\tan^{-1}x = x - \dfrac{x^3}{3}+\dfrac{x^5}{5}-\dfrac{x^7}{7}+\ldots$ Therefore,

$$\cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right) = \pi - 2\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots\right) = \pi - 2x + \frac{2x^3}{3}-\frac{2x^5}{5}+\frac{2x^7}{7}-\ldots$$

Example 4.37: Find the first four terms in the expansion of $\ln(1+\tan x)$.

Solution: Write $\ln(1+\tan x) = \ln(1+t)$, where $t = \tan x$. We have,

$$\ln(1+t) = t - \frac{t^2}{2}+\frac{t^3}{3}-\frac{t^4}{4}+\ldots \text{ and, } t = \tan x = x + \frac{x^3}{3}+\frac{2x^5}{15}+\ldots$$

Therefore, $\ln(1+\tan x) = \left(x+\dfrac{x^3}{3}+\dfrac{2x^5}{15}+\ldots\right)-\dfrac{1}{2}\left(x+\dfrac{x^3}{3}+\dfrac{2x^5}{15}+\ldots\right)^2$

$$+\frac{1}{3}\left(x+\frac{x^3}{3}+\frac{2x^5}{15}+\ldots\right)^3-\frac{1}{4}\left(x+\frac{x^3}{3}+\frac{2x^5}{15}+\ldots\right)^4+\ldots$$

$$= x - \frac{1}{2}x^2 + x^3\left(\frac{1}{3}+\frac{1}{3}\right)+x^4\left(-\frac{1}{3}-\frac{1}{4}\right)+\ldots$$

$$= x - \frac{1}{2}x^2 + \frac{2}{3}x^3 - \frac{7}{12}x^4 + \ldots$$

is the required expansion.

Example 4.38: Show that $\tan^{-1}\dfrac{\sqrt{1+x^2}-1}{x} = \dfrac{1}{2}\left(x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\dfrac{x^7}{7}+\ldots\right)$.

Solution: Put $x = \tan\theta$, we obtain

$$\tan^{-1}\frac{\sqrt{1+x^2}-1}{x} = \tan^{-1}\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta} = \tan^{-1}\frac{\sec\theta - 1}{\tan\theta}$$

$$= \tan^{-1}\frac{1-\cos\theta}{\sin\theta} = \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x$$

The requisite result follows from the expression, $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$

4.7 INDETERMINATE FORMS

If functions $f(x)$ and $g(x)$ are both zero at $x = a$, then $\lim_{x\to a}(f(x)/g(x))$ cannot be found by substitution $x = a$. The substitution produces 0/0, a meaningless expression, known as an *indeterminate form*. This does not imply that $\lim_{x\to a}(f(x)/g(x))$ does not exist. In fact in many cases it has a finite value, as we have already noticed that the determination of the differential coefficient dy/dx is itself equivalent to finding the limit of a fraction $\Delta y/\Delta x$ which assumes an indeterminate form as $\Delta x \to 0$.

4.7.1 L' Hospital's Rule

We will assume that $f(x)$ and $g(x)$ possess continuous derivatives of every order that appear in the process of finding the limit in a certain interval enclosing $x = a$.

Since $f(a) = 0 = g(a)$, we can write

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)} = \lim_{x\to a}\frac{[f(x)-f(a)]/(x-a)}{[g(x)-g(a)]/(x-a)}$$

$$= \lim_{x\to a}\frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)} \quad \ldots(4.82)$$

provided the limit on the right exists. This rule is known as *L' Hospital's rule.*

Suppose now that $f'(a) = 0 = g'(a)$. Then we repeat the application of the L' Hospital's rule on $f'(x)/g'(x)$ and obtain

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} = \lim_{x\to a}\frac{f''(x)}{g''(x)} = \frac{f''(a)}{g''(a)} \quad \ldots(4.83)$$

provided the limits exist.

The application of this rule can be repeated as long as the indeterminate form is evaluated.

L' Hospital's rule also applies to quotients that lead to the indeterminate form ∞/∞. If $f(x)$ and $g(x)$ both approach infinity as $x \to a$, then it can be shown that

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

provided the limit on the right exists. Here, a may itself, be either finite or infinite.

The L' Hospital's rule is designed to use when the ratio is of the form $0/0$ or ∞/∞. But we can sometimes handle the indeterminate forms $0.\infty$ and $\infty-\infty$ by using algebra to get $0/0$ or ∞/∞. Limits that lead to the indeterminate forms 1^∞, 0^0 and ∞^0 can sometimes be handled by taking logarithms first. We use this rule to find the limit of the logarithm and then exponentiate to find the original limit.

We must note that since the functions of the form 0^∞, $\infty.\infty$, $\infty+\infty$, ∞^∞, $\infty^{-\infty}$ are not of indeterminate form and hence L' Hospital's rule is not applicable in these cases.

Example 4.39: Evaluate $\lim\limits_{x\to 1}\dfrac{x^x - x}{x-1-\ln x}$.

Solution: The function is of the form $0/0$, using L' Hospital rule we have

$$\lim_{x\to 1}\frac{x^x - x}{x-1-\ln x} = \lim_{x\to 1}\frac{x^x(1+\ln x)-1}{1-1/x} = \lim_{x\to 1}\frac{x^x(1+\ln x)^2 + x^x(1/x)}{1/x^2} = \frac{2}{1} = 2.$$

Example 4.40: Find the values of a and b in order that $\lim\limits_{x\to 0}\dfrac{x(1+a\cos x)-b\sin x}{x^3} = 1.$

Solution: The function is of the form $0/0$, using L' Hospital's rule, we have

$$\lim_{x\to 0}\frac{x(1+a\cos x)-b\sin x}{x^3} = \lim_{x\to 0}\frac{1+a\cos x - ax\sin x - b\cos x}{3x^2}.$$

The denominator being 0 for $x=0$, the function will tend to a finite limit if the numerator is also zero for $x=0$ and for this

$$1+a-b=0 \qquad \text{...(4.84)}$$

Suppose that (4.84) is satisfied, we have

$$\lim_{x\to 0}\frac{1+a\cos x - ax\sin x - b\cos x}{3x^2} = \lim_{x\to 0}\frac{-2a\sin x - ax\cos x + b\sin x}{6x}$$

$$= \lim_{x\to 0}\frac{-3a\cos x + ax\sin x + b\cos x}{6} = \frac{b-3a}{6}$$

As given, $\dfrac{b-3a}{6} = 1$, or $b-3a=6$ ...(4.85)

Solving (4.84) and (4.85), we obtain $a=-5/2$, $b=-3/2$.

Example 4.41: Find $\lim\limits_{x\to 0}\left(\dfrac{1}{\sin x}-\dfrac{1}{x}\right)$.

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Solution: If $x \to 0^+$, then $\lim_{x\to 0^+}\left(\frac{1}{\sin x}-\frac{1}{x}\right) = \infty - \infty$ and if, $x \to 0^-$, then $\lim_{x\to 0^-}\left(\frac{1}{\sin x}-\frac{1}{x}\right) = -\infty+\infty$, in both cases the form is indeterminate.

In case we combine the fraction, then it is of the form 0/0. Applying L' Hospital's rule, we have

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right) = \lim_{x\to 0}\frac{x-\sin x}{x\sin x} = \lim_{x\to 0}\frac{1-\cos x}{\sin x + x\cos x} = \lim_{x\to 0}\frac{\sin x}{2\cos x - x\sin x} = \frac{0}{2} = 0.$$

Example 4.42: Find $\lim_{x\to 0}(\cos x)^{1/x^2}$.

Solution: Consider $f(x) = (\cos x)^{1/x^2}$. If $x \to 0$, then $f(x)$ is of the form 1^∞, which is indeterminate.

Taking logarithm both sides we obtain, $\ln f(x) = \dfrac{\ln(\cos x)}{x^2}$

It is of the form 0/0. Applying L' Hospital's rule we have

$$\lim_{x\to 0}\ln f(x) = \lim_{x\to 0}\frac{\ln(\cos x)}{x^2} = \lim_{x\to 0}\frac{-\tan x}{2x} = -\frac{1}{2}, \text{ since } \lim_{x\to 0}\frac{\tan x}{x} = 1.$$

Thus, $\lim_{x\to 0}(\ln f(x)) = -\frac{1}{2}$, or $\lim_{x\to 0} f(x) = e^{-1/2}$, that is, $\lim_{x\to 0}(\cos x)^{1/x^2} = e^{-1/2}$

Example 4.43: Evaluate $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2}$.

Solution: Consider

$$\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2} = \lim_{x\to 0}\left[\frac{x+\frac{x^3}{3}+\frac{2}{15}x^5+\dots}{x}\right]^{1/x^2}$$

$$= \lim_{x\to 0}\left[1+\frac{x^2}{3}+\frac{2}{15}x^4+\dots\right]^{1/x^2}$$

$$= \lim_{x\to 0}[1+x^2 f(x)]^{1/x^2}, \text{ where } f(x) = \frac{1}{3}+\frac{2}{5}x^2+\dots$$

$$= \lim_{x\to 0}\left[\left(1+x^2 f(x)\right)^{\frac{1}{x^2 f(x)}}\right]^{f(x)}$$

$$= \lim_{x\to 0} e^{f(x)}, \text{ since } \lim_{x\to 0}(1+x)^{1/x} = e \text{ and } x^2 f(x) \to 0 \text{ as } x \to 0,$$

$$= e^{f(0)} = e^{1/3}.$$

EXERCISE 4.6

Evaluate the following limits

1. $\lim_{x\to 0} \frac{1-\cos x}{x+x^2}$

2. $\lim_{x\to 0} \frac{xe^x - \ln(1+x)}{x^2}$

3. $\lim_{x\to 0} \frac{e^x - e^{-x} - 2\ln(1+x)}{x \sin x}$

4. $\lim_{x\to 0} \left\{\frac{1}{x} - \frac{1}{x^2}\ln(1+x)\right\}$

5. If the limit of $\frac{\sin 2x + a\sin x}{x^3}$ as x tends to zero be finite, find the value of a and the limit.

Evaluate the following limits.

6. $\lim_{x\to\infty} x\tan\left(\frac{1}{x}\right)$

7. $\lim_{x\to\infty} \int_x^{2x} \frac{1}{t}dt$

8. $\lim_{x\to a} \frac{\ln(x-a)}{\ln(e^x - e^a)}$

9. $\lim_{x\to 0} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right)$

10. $\lim_{x\to 0} (\cot x)^{\sin 2x}$

11. $\lim_{x\to 0} \frac{(1+x)^{1/x} - e}{x}$

12. $\lim_{x\to 0} \frac{1+\sin x - \cos x + \ln(1-x)}{x\tan^2 x}$

13. $\lim_{x\to\infty} \sqrt{\frac{x+\sin x}{x-\cos^2 x}}$

14. If $f(x) = e^{-1/x^2}$, $x \neq 0$ and $f(0) = 0$, show that the derivative of every order of $f(x)$ vanishes for $x = 0$, i.e., $f^n(0) = 0$ for all n.

15. Discuss the continuity of $f(x)$ at the origin when $f(x) = x \ln \sin x$ for $x \neq 0$ and $f(0) = 0$.

4.8 EXTREME VALUES OF A FUNCTION

Here we shall be interested in determining those values of a continuous function defined over an interval $[a, b]$ which are the greatest or the least in their immediate neighbourhood, technically known as the *local maximum* and the *local minimum* values, or the *extreme values*. The knowledge of these values is helpful in studying the behaviour of a system over an interval and is of great practical importance in engineering and science.

4.8.1 Maxima and Minima

Let c be any interior point of the domain of definition of a function $f(x)$. We say that $f(c)$ is a *maxima* of the function $f(x)$, if there exists some interval $(c-h, c+h)$, $h > 0$, around c such that $f(c) > f(x)$, for all $x \in (c-h, c+h)$ other than c itself. The interval $(c-h, c+h)$ is called a *neighbourhood (nbd.)* of c.

On the other hand, $f(c)$ is a *minima* of the function $f(x)$, if $f(c) < f(x)$, for all $x \in (c-h, c+h)$ other than c itself.

The points at which maximum or minimum values of a function exist are called the *critical points*, or the *stationary points*.

The values of the function at these points are called the *extreme values*.

Thus, if $f(c)$ is an extreme value of a function $f(x)$, then $f(c+h) - f(c)$ keeps the same sign for values of h, sufficiently small numerically. Further it should be clear that a maximum value may not be the greatest and a minimum value may not be the least of all the values of the function in any finite interval. In fact a function can have several maximum and minimum values and a minimum value can even be greater than the maximum value as shown in Fig. 4.10.

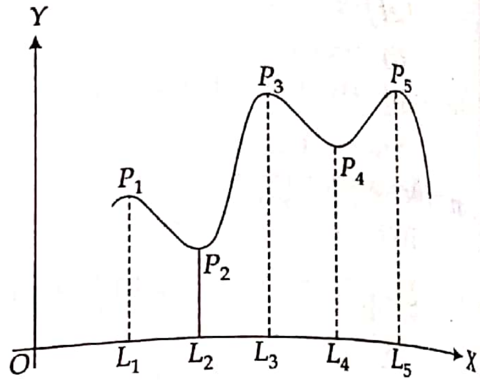


Fig. 4.10

For the function $y = f(x)$ the points P_1, P_3, P_5 are the points of maxima, while the points P_2, P_4 are the points of minima, but the ordinate L_4P_4 at P_4 is greater than that of L_1P_1 at P_1.

4.8.2 A Necessary Condition for Extreme Values

A necessary condition for $f(c)$ to be an extreme value of $f(x)$ is that $f'(c) = 0$.

To prove it, consider $f(x)$ to be maximum at $x = c$, thus $f(c) \geq f(c+h)$, for all h close to zero.

Also, $f'(c) = \lim_{h\to 0} \dfrac{f(c+h) - f(c)}{h}$. If $h > 0$, then $\dfrac{f(c+h)-f(c)}{h} \leq 0$ and thus $\lim_{h\to 0^+} \dfrac{f(c+h)-f(c)}{h} \leq 0$.

Similarly if $h < 0$, then $\lim_{h\to 0^-} \dfrac{f(c+h)-f(c)}{h} \geq 0$. Since $f'(c)$ exists, thus both of these inequalities will hold simultaneously if, and only if, $f'(c) = 0$.

The case when $f(x)$ is minimum at $x = c$ follows on the same line.

Geometrically interpreted, the necessary condition for extreme values means that the tangent to a curve at a point where the ordinate is maximum or minimum is parallel to x-axis.

Further we note that $f'(c) = 0$ *is only a necessary but not sufficient condition for $f(x)$ to be an extreme value.* For example, consider the function $f(x) = x^3$ at $x = 0$. For $x > 0, f(x)$ is positive and is, therefore, greater than $f(0)$ which is zero; and for $x < 0, f(x)$ is negative and is, therefore, less than $f(0)$. Thus, $f(0)$ is not an extreme value even though $f'(0) = 0$.

Stationary value: A function $f(x)$ is said to be stationary for $x = c$, if the derivative $f'(x)$ vanishes for $x = c$, and c is said to be a *stationary* value of $f(x)$.

In fact, *a maximum or a minimum value is always a stationary value but a stationary value may neither be a maximum nor a minimum value.*

4.8.3 A Sufficient Criteria for Extreme Values

A function $f(x)$ has a maximum value at $x = c$ if, and only if the sign of $f'(x)$ changes from positive to negative as x passes through c and $f(x)$ has a minimum value at $x = c$, if, and only if $f'(x)$ changes sign from negative to positive as x passes through c.

The second and higher order derivatives test for extreme values. The extreme values of a function can sometimes be found more conveniently by the use of derivatives of the second and higher orders. The result is as follows:

Let $f(x)$ be differentiable at $x=c$, and $f'(c)=0$. If $f''(x)$ exists and is continuous in a nbd. of c, then

(i) $f(x)$ has a maximum value at $x=c$, if $f''(c)<0$,

(ii) $f(x)$ has a minimum value at $x=c$, if $f''(c)>0$.

When $f''(c)=0$, further investigation is needed to decide whether $x=c$ is a point of maxima, or minima, or neither a maxima nor a minima.

In this situation the following is the general criteria:

Let $f'(c)=f''(c)=\ldots=f^{(n-1)}(c)=0$ but $f^{(n)}(c)\neq 0$. Then $x=c$ is

(i) a point of minima of $f(x)$, if $f^{(n)}(c)>0$ and n is even;

(ii) a point of maxima of $f(x)$, if $f^{(n)}(c)<0$ and n is even;

(iii) neither a point of maxima nor a minima of $f(x)$, if n is odd.

4.8.4 Extreme Values of a Function Represented Parametrically

Let a function $y=f(x)$ be represented parametrically as $x=\phi(t)$, $y=\psi(t)$, where the functions $\phi(t)$ and $\psi(t)$ have derivatives both of the first and second orders within a certain interval of the argument t, and $\phi'(t)\neq 0$ in that interval.

Further, let, at $t=t_0$, $\psi'(t_0)=0$. Then

(i) if $\psi''(t_0)<0$, the function $y=f(x)$ has a maximum at $x=c=\phi(t_0)$,

(ii) if $\psi''(t_0)>0$, the function $y=f(x)$ has a minimum at $x=c=\phi(t_0)$

(iii) if $\psi''(t_0)=0$, the existence of maximum or minimum remains open.

The points at which $\phi'(t)=0$ require a special investigation.

4.8.5 Concavity and Points of Inflexion

Consider a curve $y=f(x)$. Draw the tangents at the points P_1, P_2, and P_3 as shown in Fig. 4.11. The curve is *concave downward* (or, *convex upward*) at the point P_1 and is *concave upward* (or, *convex downward*) at the point P_3.

At the point P_2, however there is a change in the direction of bending of the curve from concavity downwards to concavity upwards. Such a point is called a *point of inflexion* of the curve. At a point of inflexion, the curve changes from concavity downwards to concavity upwards or vice versa. Infact, it crosses the tangent at the point of inflexion.

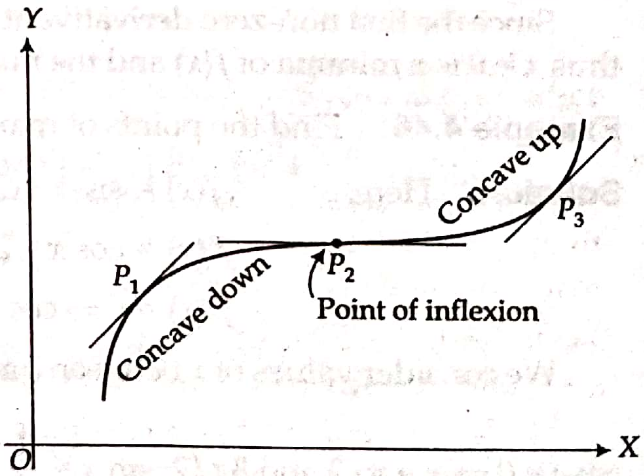


Fig. 4.11

The second order derivative test for concavity

Let $y=f(x)$ be twice differentiable on an interval I. Then

(i) if $f''(x)>0$ on I, the graph of $y=f(x)$ is concave upward on I,

(ii) if $f''(x)<0$ on I, the graph of $y=f(x)$ is concave downward on I,

(iii) a point where $f''(x)$ changes sign from positive to negative or viceversa is a point of inflexion.

We observe that at the point of inflexion $\frac{d^2y}{dx^2} = 0$, but $\frac{d^3y}{dx^3} \neq 0$.

Example 4.44: Find the points of maxima and minima values of $x^4 - 8x^3 + 22x^2 - 24x + 1$.

Solution: Let $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 1$. This gives

$$f'(x) = 4x^3 - 24x^2 + 44x - 24 = 4(x^3 - 6x^2 + 11x - 6) = 4(x-1)(x-2)(x-3).$$

Thus $f'(x) = 0$, for $x = 1, 2, 3$; and hence these are the possible extreme values.

Now, $x < 1 \Rightarrow f'(x) < 0, 1 < x < 2 \Rightarrow f'(x) > 0, 2 < x < 3 \Rightarrow f'(x) < 0, x > 3 \Rightarrow f'(x) > 0$.

Since, $f'(x)$ changes sign from negative to positive as x passes through 1 and 3, thus $x = 1$ and $x = 3$ are the points of minima and $f(1) = -8$ and $f(3) = -8$ are the corresponding minimum values.

Again $f'(x)$ changes sign from positive to negative as x passes through 2, and therefore $x = 2$ is the point of maxima and the corresponding maximum value is $f(2) = -7$.

Example 4.45: Investigate the nature of the extreme value of the function $f(x) = \cosh x + \cos x$ at the point $x = 0$.

Solution: Since the function $f(x) = \cosh x + \cos x$ is an even function of x, thus $x = 0$ is an extreme value of $f(x)$.

Here,

$$f'(x) = \sinh x - \sin x; \quad f'(0) = 0$$
$$f''(x) = \cosh x - \cos x; \quad f''(0) = 0$$
$$f'''(x) = \sinh x + \sin x; \quad f'''(0) = 0$$
$$f''''(x) = \cosh x + \cos x; \quad f''''(0) = 2 > 0.$$

Since the first non-zero derivative at the point $x = 0$ is a derivative of an even order, and is positive, thus $x = 0$ is a minima of $f(x)$ and the minimum value is $f(0) = 2$.

Example 4.46: Find the points of maxima and minima values of the function $\sin x + \cos 2x$.

Solution: Here,

$$f(x) = \sin x + \cos 2x$$
$$f'(x) = \cos x - 2\sin 2x = \cos x - 4\sin x \cos x$$
$$f'(x) = 0 \Rightarrow \cos x = 0, \text{ or } \sin x = 1/4.$$

We consider values of x between 0 and 2π only, for the given function is periodic with period 2π.

$\cos x = 0 \Rightarrow x = \pi/2$ and $3\pi/2$, $\sin x = \frac{1}{4} \Rightarrow x = \sin^{-1}\frac{1}{4}$, and $\pi - \sin^{-1}\frac{1}{4}$.

Now, $$f''(x) = -\sin x - 4\cos 2x.$$

At $$x = \pi/2, \quad f''(\pi/2) = 3 > 0$$
$$x = 3\pi/2, \quad f''(3\pi/2) = 5 > 0$$

At $x = \sin^{-1}\frac{1}{4}$ and $\pi - \sin^{-1}\frac{1}{4}$,

$$f''(x) = -\sin x - 4\cos 2x = -\sin x - 4(1 - 2\sin^2 x) = -\frac{1}{4} - 4\,(1 - 2/16) = -15/4 < 0.$$

Therefore $x = \pi/2, 3\pi/2$ are the points of minima and $x = \sin^{-1}\frac{1}{4}, \pi - \sin^{-1}\frac{1}{4}$ are the points of maxima of $f(x)$. The corresponding minimum values are $f(\pi/2) = 0$ and $f(3\pi/2) = -2$ and the corresponding maximum values are 9/8 and 9/8.

Example 4.47: 20 metre of wire is available for fencing off a flower-bed which should have the form of a circular sector. What must be the radius of the circle bed if we wish the flower-bed of the greatest possible surface area?

Solution: Let x be the radius of the circle and y be the length of the arc, as shown in Fig. 4.12. Then

$$20 = 2x + y. \text{ This gives}$$

$$y = 2(10 - x).$$

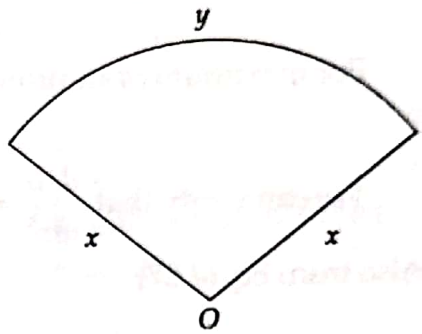


Fig. 4.12

The area of the circular sector $S = \frac{1}{2}xy = x(10 - x), 0 \le x \le 10.$

Here $\frac{dS}{dx} = 0$, gives $x = 5$ and $\frac{d^2S}{dx^2} = -2 < 0$. Hence $x = 5$ gives the maximum surface area.

Example 4.48: Assuming that the fuel burnt per hour in driving a motor boat varies as the cube of its velocity, show that the most economical speed when going against a current of c miles per hour is $3c/2$ miles per hour.

Solution: Let the velocity of the boat be v miles per hour so that its relative velocity is $v - c$ when going against the current. Thus, the time required to cover a distance of d miles $= \frac{d}{v-c}$ hours.

The fuel burnt per hour $= kv^3$, where k is a constant.

If y be the amount of the fuel burnt to cover a distance of d miles, then $y = kd\frac{v^3}{v-c}$.

Therefore, $\frac{dy}{dv} = kd\,\frac{v^2(2v-3c)}{(v-c)^2}$, and $\frac{dy}{dv} = 0$, gives $v = 0, 3c/2$. Also, $\frac{dy}{dv}$ changes sign from negative to positive as v passes through $3c/2$. Hence minima exists at $v = 3c/2$. Since, $v = 0$ is not desirable, thus, $v = 3c/2$ gives the desired least value of y and is the most economical speed.

Example 4.49: Find the diameter and height of a cylinder of maximum volume which can be cut from a sphere of radius 12 cm.

Solution: A cylinder of radius r and height h is shown enclosed in a sphere of radius 12 cm. in Fig. 4.13.

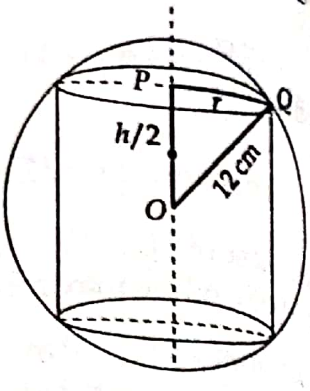


Fig. 4.13

Volume of cylinder $v = \pi r^2 h$. ...(4.86)

Also from ΔOPQ, we have, $r^2 + \left(\dfrac{h}{2}\right)^2 = 144$. ...(4.87)

From (4.86) and (4.87), we obtain

$$v = 144\pi h - \frac{\pi h^3}{4}, \text{ which gives } \frac{dv}{dh} = 144\pi - \frac{3\pi h^2}{4}.$$

For maximum or minimum value $\dfrac{dv}{dh} = 0$. Hence, $h = \dfrac{\sqrt{(144)(4)}}{3} = 13.86$ cm.

We can verify that $\dfrac{d^2v}{dh^2} = -\dfrac{6\pi h}{4}$ is negative for $h = 13.86$. Thus $h = 13.86$ gives a maximum value.

Also from Eq. (4.87)

$$r^2 = 144 - \frac{h^2}{4} = 144 - \frac{(13.86)^2}{4}, \text{ or, } r = 9.80 \text{ cm.}$$

Hence, diameter of cylinder $= 2r = 2(9.80) = 19.60$ cm.

Example 4.50: Find the points of inflexion for the curve $y = \dfrac{x+1}{x^2+1}$.

Solution: The equation of the curve is $y = \dfrac{x+1}{x^2+1}$.

The first and the second derivatives are; $y' = \dfrac{-x^2-2x+1}{(x^2+1)^2}$, $y'' = \dfrac{2x^3+6x^2-6x-2}{(x^2+1)^3}$

Now, $y'' = 0$ implies that $x^3 + 3x^2 - 3x - 1 = 0$. This gives $x = -2-\sqrt{3},\ -2+\sqrt{3},\ 1$.

Sign of y'' depends upon the sign of the numerator

$$2x^3 + 6x^2 - 6x - 2 = 2(x+2+\sqrt{3})(x+2-\sqrt{3})(x-1)$$

We have the following table:

Interval	Sign of y''	Conclusion
$-\infty < x < -2-\sqrt{3}$	$-$ve	concave downward
$-2-\sqrt{3} < x < -2+\sqrt{3}$	$+$ve	concave upward
$-2+\sqrt{3} < x < 1$	$-$ve	concave downward
$1 < x < \infty$	$+$ve	concave upward

Hence, $x = -2 \pm \sqrt{3}, 1$ corresponds to the points of inflexion and the corresponding points of inflexion are

$$\left(-2 \pm \sqrt{3}, \frac{1 \pm \sqrt{3}}{4}\right), \text{ and } (1, 1).$$

EXERCISE 4.7

1. Using the first derivative, find the points of maxima and minima of the function $f(x) = x(x+1)^3 (x-3)^2$.
2. Find the extreme value (s) of the function $f(x) = \sqrt{e^{x^2} - 1}$.
3. Investigate $f(x) = x^4 e^{-x^2}$ for its extreme values.
4. For the function $x = \phi(t) = t^5 - 5t^3 - 20t + 7, \quad y = \psi(t) = 4t^3 - 3t^2 - 18t + 3$, find the extreme values.
5. Find the points of maxima and minima of the function $f(x) = 2 \sin x + \cos 2x$.
6. Find the points of maxima and minima of the function $f(x) = \dfrac{40}{3x^4 + 8x^3 - 18x^2 + 60}$.
7. Find the points of maxima and minima of the function $f(x) = \sin x(1 + \cos x), \quad 0 \le x \le 2\pi$.
8. It is required to construct an open cylindrical reservoir of capacity V_0. The thickness of the material is d. What dimensions, that is, the base radius and height should the reservoir have so as to ensure the least possible expenditure on the material?
9. Show that the radius of the right circular cylinder of greatest curved surface which can be inscribed in a given cone is half that of the cone.
10. Find the semi-vertical angle of a right circular cone of maximum volume and a given surface area.
11. Prove that the least parameter of an isosceles triangle in which a circle of radius r can be inscribed is $6r\sqrt{3}$.
12. A tree trunk L feet long is in the shape of a frustum of a cone the radii of its ends being a and b feet $(a > b)$. It is required to cut from it a beam of uniform square section. Prove that the beam of the greatest volume that can be cut is $aL/3(a-b)$ feet long.
13. Show that the curve $y = \dfrac{x+1}{x^2+1}$ has three points of inflexion lying in a straight line.
14. Show that the points of inflexion of the curve $y = x \sin x$ lie on the curve $y^2(4 + x^2) = 4x^2$.

4.9 CURVATURE AND EVOLUTE

In many practical problems we are concerned with the comparison of bending of two curves or even with the bending of a curve at its different points. For example the problem may be of interest while laying the rail tracks or designing the highways. As another example, in case of parabolic path, bending is different at its different points being maximum at its vertex. The *curvature gives a numerical measure of the sharpness of the bending of a curve.*

4.9.1 Definition and Measure of Curvature

Let A be a fixed point on the curve from which the arc length is measured and let arc lengths of the points P and Q on the curve be s and $s + \delta s$, respectively so that arc $PQ = \delta s$ as shown in Fig. 4.14. Let ψ, $\psi + \delta\psi$ be the angles which the positive directions of the tangents at P and Q make with some fixed line. Thus $\delta\psi$ is the angle through which the tangent turns as a point moves along the curve from P to Q through a distance δs.

The quantity $\delta\psi$ is called the *total curvature* and the ratio $\delta\psi/\delta s$ is called the *average curvature* of the arc PQ.

The curvature of the curve at P is defined as $\lim\limits_{Q\to P} \dfrac{\delta\psi}{\delta s} = \lim\limits_{\delta s\to 0} \dfrac{\delta\psi}{\delta s} = \dfrac{d\psi}{ds}$.

Generally it is denoted by Greek letter k (kappa), thus

$$k = \frac{d\psi}{ds}. \qquad \text{...(4.88)}$$

Curvature of a circle: *The curvature of a circle is constant.*

Consider a circle with radius r and centre O. Let P, Q be any two points on the circle and let arc $PQ = \delta s$, as shown in Fig. 4.15.

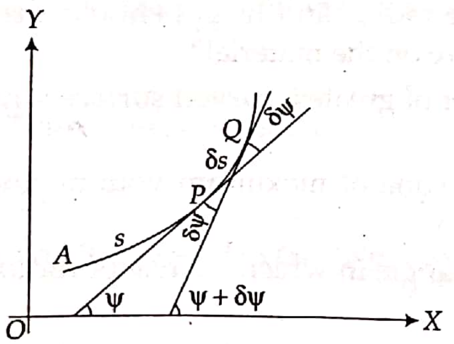


Fig. 4.14

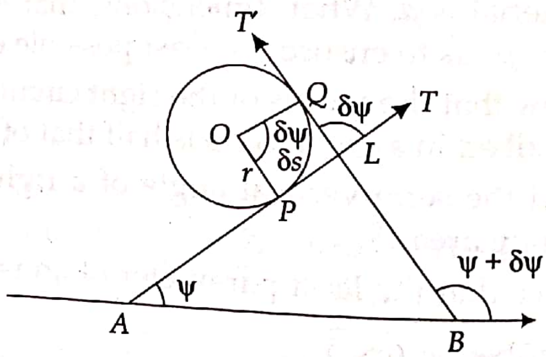


Fig. 4.15

Let L be the point where the tangents PT, QT' at P and Q meet. We have $\angle POQ = \angle TLT' = \delta\psi$.

Also in case of a circle, $\dfrac{\delta s}{r} = \delta\psi$, or $\dfrac{\delta\psi}{\delta s} = \dfrac{1}{r}$.

Taking limit $Q \to P$, we obtain

$$\lim_{\delta s\to 0} \frac{\delta\psi}{\delta s} = \frac{1}{r}, \text{ or } \frac{d\psi}{ds} = \frac{1}{r}, \qquad \text{...(4.89)}$$

which is a constant, being the reciprocal of its radius.

We observe that as the radius r increases, the curvature $1/r$ decreases. When r tends to infinity, the arc of the circle approximates a section of a straight line and the curvature tends to zero. Hence, *the curvature of a straight line is zero at any of its point*. The same is expected intuitively also, since a line has no bending from one point to another point.

4.9.2 Radius of Curvature

The reciprocal of the curvature of a curve at any point, in case it is non-zero, is called the radius of curvature at that point. Generally it is denoted by ρ, thus

$$\rho = \frac{ds}{d\psi}. \qquad \text{...(4.90)}$$

We observe that the radius of curvature of a circle at any point is constant and is equal to its radius, refer to (4.89), and also the radius of curvature of a straight line is infinity.

The expression (4.90) for radius of curvature is suitable only for curves with equation of the form $s = \phi(\psi)$, called the *intrinsic form*. We transform (4.90) suitable to cartesian, parametric, and polar curves.

4.9.3 Radius of Curvature for Cartesian Curves

(a) *Explicit equation:* $y = f(x)$

Let ψ be the angle which the tangent at any point $P(x, y)$ makes with the positive direction of the x-axis. Then, $\tan\psi = \dfrac{dy}{dx}$.

Differentiating w.r.t. s, we get, $\sec^2\psi \dfrac{d\psi}{ds} = \dfrac{d^2y}{dx^2}\dfrac{dx}{ds}$, which gives

$$\frac{ds}{d\psi} = \frac{(1+\tan^2\psi)\dfrac{ds}{dx}}{\dfrac{d^2y}{dx^2}}$$

Also, $\dfrac{ds}{dx} = \sqrt{1+\left(\dfrac{dy}{dx}\right)^2}$, refer to (4.22). Therefore,

$$\rho = \frac{ds}{d\psi} = \frac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}} = \frac{(1+y_1^2)^{3/2}}{y_2} \qquad \text{...(4.91)}$$

provided $y_2 \neq 0$.

The radius of curvature ρ is positive or negative according as $\frac{d^2y}{dx^2}$ is positive or negative, that is, as the curve is concave upwards or downwards, refer to Section 4.8.5. At a point of inflexion, that is, at a point where $\frac{d^2y}{dx^2} = 0$, curvature is zero.

Remark: In case the tangent to the curve at the point under consideration is parallel to y-axis, then dy/dx tends to infinity and thus (4.91) does not hold good. In this case $dx/dy = 0$, so we use the formula

$$\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{d^2x}{dy^2}}, \text{ provided } \frac{d^2x}{dy^2} \neq 0. \qquad \text{...(4.92)}$$

Since, the value of ρ is independent of the choice of the axes, (4.92) is obtained simply by interchanging x and y in (4.91).

(b) *Implicit equation:* $f(x, y) = 0$.

Here, $\frac{dy}{dx} = -\frac{f_x}{f_y}$, provided $f_y \neq 0$, and $\frac{d^2y}{dx^2} = -\frac{f_{xx}(f_y)^2 - 2f_xf_yf_{xy} + f_{yy}(f_x)^2}{(f_y)^3}$

Substituting these values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (4.91) and simplifying we obtain

$$\rho = \frac{[(f_x)^2+(f_y)^2]^{3/2}}{f_{xx}(f_y)^2 - 2f_xf_yf_{xy} + f_{yy}(f_x)^2} \qquad \text{...(4.93)}$$

(c) *Parametric equations:* $x = f(t), \ y = g(t)$

Here, $\frac{dy}{dx} = \frac{g'(t)}{f'(t)}$, provided $f'(t) \neq 0$, and $\frac{d^2y}{dx^2} = \frac{f'(t)g''(t) - g'(t)f''(t)}{[f'(t)]^2} \frac{1}{f'(t)}$.

Substituting these values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (4.91), we get

$$\rho = \frac{\left[(f'(t))^2 + (g'(t))^2\right]}{f'(t)g''(t) - g'(t)f''(t)}. \qquad \text{...(4.94)}$$

Example 4.51: Find the radius of curvature at any point (s, ψ) on the curve

$$s = a \ln \cot(\pi/4 - \psi/2) + \frac{a \sin\psi}{\cos^2\psi}$$

Solution: The curve is

$$s = a \ln \cot(\pi/4 - \psi/2) + \frac{a \sin\psi}{\cos^2\psi}.$$

The radius of curvature is

$$\rho = \frac{ds}{d\psi} = a\frac{\frac{1}{2}\operatorname{cosec}^2\left(\frac{\pi}{4}-\frac{\psi}{2}\right)}{\cot(\pi/4-\psi/2)} + a\frac{\cos^2\psi\cos\psi - \sin\psi(-2\sin\psi\cos\psi)}{\cos^4\psi}$$

$$= a\left[\frac{1}{\sin\left(\frac{\pi}{2}-\psi\right)} + \frac{\cos^2\psi + 2\sin^2\psi}{\cos^3\psi}\right] = a\left[\frac{1}{\cos\psi} + \frac{1+\sin^2\psi}{\cos^3\psi}\right] = 2a\sec^3\psi.$$

Example 4.52: Find the radius of curvature at any point (x, y) of the rectangular hyperbola $xy = c^2$.

Solution: The curve is $y = c^2/x$.

It gives, $\dfrac{dy}{dx} = -c^2/x^2$ and $\dfrac{d^2y}{dx^2} = \dfrac{2c^2}{x^3}$. Therefore,

$$\rho = \left[\frac{1+\left(\frac{dy}{dx}\right)^2}{d^2y/dx^2}\right]^{3/2} = \frac{\left(1+\frac{c^4}{x^4}\right)^{3/2}}{2c^2/x^3} = \frac{(x^4+c^4)^{3/2}}{2x^3c^2} = \frac{(x^2+y^2)^{3/2}}{2c^2}.$$

Imp.

Example 4.53: Find the radius of curvature at any point on the curve

$$x = c\ln\left[s+\sqrt{s^2+c}\right], \quad y = \sqrt{s^2+c}.$$

Solution: The curve is $x = c\ln\left[s+\sqrt{s^2+c}\right]$, $y = \sqrt{s^2+c}$.

Differentiating w.r.t. s, we obtain $\dfrac{dx}{ds} = \dfrac{c}{\sqrt{s^2+c^2}}$, $\dfrac{dy}{ds} = \dfrac{s}{\sqrt{s^2+c^2}}$.

Therefore, $\dfrac{dy}{dx} = \dfrac{dy}{ds}\Big/\dfrac{dx}{ds} = s/c$, which gives $\tan\psi = s/c$, or $s = c\tan\psi$,

where ψ is the angle which the tangent at $P(x, y)$ to the given curve makes with the positive direction of x-axis. Therefore, the radius of curvature

$$\rho = \frac{ds}{d\psi} = c\sec^2\psi = c(1+\tan^2\psi) = c\left(1+\frac{s^2}{c^2}\right) = \frac{s^2+c^2}{c}.$$

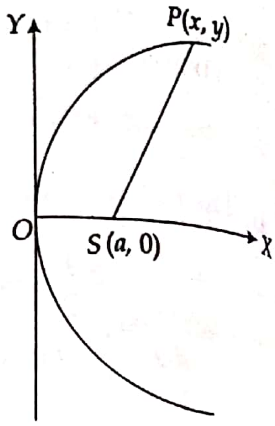


Fig. 4.16

Imp *

Example 4.54: If ρ is the radius of curvature at any point P on the parabola $y^2 = 4ax$ and S is its focus, then show that ρ^2 varies as $(SP)^3$; also show that the radius of curvature at the vertex is equal to the length of the semi-latus rectum.

Solution: The parabola is $y^2 = 4ax$ with $S(a, 0)$ as its focus and $P(x, y)$ any point on it as shown in Fig. 4.16.

We have, $2yy_1 = 4a$, or $y_1 = \frac{2a}{y}$. It gives $y_2 = -\frac{2a}{y^2}y_1 = -\frac{2a}{y^2}\frac{2a}{y} = -\frac{4a^2}{y^3}$.

The radius of curvature is

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{\left(1+\frac{4a^2}{y^2}\right)^{3/2}}{-\frac{4a^2}{y^3}} = -\frac{(y^2+4a^2)^{3/2}}{4a^2}$$

$$= \frac{-(4ax+4a^2)^{3/2}}{4a^2} = -\frac{2}{\sqrt{a}}(x+a)^{3/2}. \qquad \text{...(4.95)}$$

Also, $SP = \sqrt{(x-a)^2+y^2} = \sqrt{(x-a)^2+4ax} = |x+a|$. From (4.95), we have

$$\rho^2 = \frac{4}{a}(x+a)^3 = \frac{4}{a}(SP)^3,$$

which implies that $\rho^2 \alpha (SP)^3$.

Next, to find ρ at vertex ~~(0, 0) we note that~~ $y_1 \to \infty$ as $y \to 0$. Therefore, we apply (4.92) to find ρ, given by $\rho = \frac{\left[1+(dx/dy)^2\right]^{3/2}}{d^2x/dy^2} = \frac{1}{d^2x/dy^2}$.

$\frac{dy}{dx} = \infty$

$\frac{dx}{dy} = 0$

Now $\frac{dx}{dy} = \frac{y}{2a}$ gives $\frac{d^2x}{dy^2} = \frac{1}{2a}$ and, therefore, $\rho = 2a$, the length of the semi-latus rectum of the parabola $y^2 = 4ax$.

Example 4.55: Find the radius of curvature at any point θ of the cycloid

$$x = a(\theta + \sin\theta), \quad y = a(1-\cos\theta).$$

Solution: The curve is $x = a(\theta+\sin\theta), y = a(1-\cos\theta)$. It gives $\frac{dx}{d\theta} = a(1+\cos\theta)$, $\frac{dy}{d\theta} = a\sin\theta$.

Therefore, $$\frac{dy}{dx} = \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\theta/2} = \tan\theta/2.$$

and, $$\frac{d^2y}{dx^2} = \frac{1}{2}\sec^2\frac{\theta}{2}\frac{d\theta}{dx} = \frac{1}{2}\sec^2\frac{\theta}{2}\frac{1}{2a\cos^2\frac{\theta}{2}} = \frac{1}{4a}\frac{1}{\cos^4\frac{\theta}{2}}.$$

The radius of curvature $$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left(1+\tan^2\frac{\theta}{2}\right)^{3/2}}{\frac{1}{4a\cos^4\frac{\theta}{2}}} = 4a\cos\frac{\theta}{2}.$$

Example 4.56: Show that the curvature at the point $(3a/2, 3a/2)$ on the folium $x^3 + y^3 = 3axy$ is $-8\sqrt{2}/3a$.

Solution: The curve is $x^3 + y^3 = 3axy$.

Differentiating w.r.t. x and cancelling 3 from both sides, we obtain

$$x^2 + y^2\frac{dy}{dx} = ay + ax\frac{dy}{dx} \qquad \text{...(4.96)}$$

or, $$\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}. \text{ Therefore, } \left[\frac{dy}{dx}\right]_{\left(\frac{3}{2}a,\frac{3}{2}a\right)} = -1.$$

Again differentiating (4.96) w.r.t. x, we get

$$2x + 2y\left(\frac{dy}{dx}\right)^2 + y^2\frac{d^2y}{dx^2} = a\frac{dy}{dx} + a\frac{dy}{dx} + ax\frac{d^2y}{dx^2}$$

Substituting $x = y = \frac{3a}{2}$ and $\frac{dy}{dx} = -1$ in this, we get $\left[\frac{d^2y}{dx^2}\right]_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = -\frac{32}{3a}$.

Hence, the curvature at $$\left(\frac{3a}{2},\frac{3a}{2}\right) = \frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}} = \frac{-8\sqrt{2}}{3a}.$$

Example 4.57: Imp. If ρ_1 and ρ_2 be the radii of curvature at the extremities of two conjugate semi diameters of the ellipse $x^2/a^2 + y^2/b^2 = 1$, prove that $\left(\rho_1^{2/3} + \rho_2^{2/3}\right)(ab)^{2/3} = a^2 + b^2$.

Solution: Let P and Q be the extremities of the conjugate semi-diameters of the ellipse $x^2/a^2 + y^2/b^2 = 1$, as shown in Fig. 4.17.

Now, if P is $(a\cos\theta, b\sin\theta)$, then Q is $\left[a\cos\left(\theta+\frac{\pi}{2}\right), b\sin\left(\theta+\frac{\pi}{2}\right)\right]$.

The parametric equations of the ellipse are $x = a\cos\theta,\ \ y = b\sin\theta$.

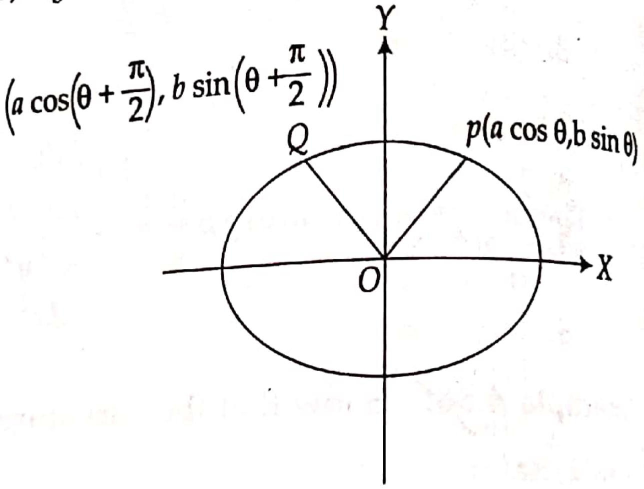


Fig. 4.17

Thus, $\quad \dfrac{dx}{d\theta} = -a\sin\theta, \quad \dfrac{dy}{d\theta} = b\cos\theta$

and, $\quad \dfrac{d^2x}{d\theta^2} = -a\cos\theta, \quad \dfrac{d^2y}{d\theta^2} = -b\sin\theta.$

Therefore, ρ at θ

$$= \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''}$$

$$= \frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}{ab\sin^2\theta + ab\cos^2\theta}$$

That is,

$$\rho_1 = \frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}{ab}. \qquad \text{...(4.97)}$$

The ρ at $\left(\theta + \frac{\pi}{2}\right)$, that is, ρ_2 is obtained by replacing θ with $\theta + \frac{\pi}{2}$ in (4.97) and, therefore,

$$\rho_2 = \frac{(a^2\cos^2\theta + b^2\sin^2\theta)^{3/2}}{ab} \qquad \text{...(4.98)}$$

From (4.97) and (4.98), we have

$$\rho_1^{2/3} + \rho_2^{2/3} = \frac{a^2\sin^2\theta + b^2\cos^2\theta}{(ab)^{2/3}} + \frac{a^2\cos^2\theta + b^2\sin^2\theta}{(ab)^{2/3}} = \frac{a^2 + b^2}{(ab)^{2/3}}$$

or, $(\rho_1^{2/3} + \rho_2^{2/3})(ab)^{2/3} = a^2 + b^2$.

EXERCISE 4.8

1. Find the radius of curvature at any point of the following curves

(i) $s = 4a\sin\psi$ (ii) $s = 4a\sin\frac{1}{3}\psi$

(iii) $s = a\ln(\tan\psi + \sec\psi) + a\tan\psi\sec\psi$

2. Find the radius of curvature for the curves

(i) $\sqrt{x}+\sqrt{y} = \sqrt{a}$ at $\left(\frac{a}{4}, \frac{a}{4}\right)$ (ii) $y = 4 \sin x - \sin 2x$ at $x = \frac{\pi}{2}$

(iii) $y^2 = \frac{a^2(a-x)}{x}$ at $(a, 0)$

3. Show that for the parabola $(y - y_1)^2 = 4a(x - x_1)$, ρ^2 varies as $(SP)^3$, where ρ is the radius of curvature at any point P of the parabola, S is the focus of the parabola and (x_1, y_1) are the co-ordinates of vertex of the parabola.

4. Find the radius of curvature of the curve $y = e^x$ at the point where it crosses the y-axis.

5. Prove that the radius of curvature at any point of the asteroid $x^{2/3} + y^{2/3} = a^{2/3}$ is three times the length of the perpendicular from the origin to the tangent at that point.

6. If ρ_1 and ρ_2 be the radii of curvature at the extremities of a focal chord of a parabola whose latus rectum is $2l$, then prove that $\rho_1^{-2/3} + \rho_2^{-2/3} = l^{-2/3}$.

7. Show that for the curve $x = a \cos \theta(1 + \cos \theta)$, $y = a \sin \theta\ (1 + \cos \theta)$, the radius of curvature at the point $\theta = -\pi/4$ is a.

8. Show that the radius of curvature of the curve given by $x^2y = a\left(x^2 + a^2/\sqrt{5}\right)$ is least for the point $x = a$ and its least value is $9a/10$.

9. Show that $3\sqrt{3}/2$ is the least value of $|\rho|$ for $y = \ln x$.

10. Find the radius of curvature at any point for the following curves:

(i) $x = a\cos^3\theta,\quad y = a\sin^3\theta$ (ii) $x = 3a\cos t - a\cos 3t,\ y = 3a\sin t - a\sin 3t$

(iii) $x = a\sin 2\theta(1 + \cos 2\theta),\quad y = a\cos^2\theta(1 - \cos 2\theta)$

11. Find the radius of curvature at the origin of the two branches of the curve given by $x = 1 - t^2, y = t - t^3$.

12. Show that the radius of curvature at an end of the major axis of the ellipse $x^2/a^2 + y^2/b^2 = 1$ is equal to the semi-latus rectum of the ellipse.

13. Prove that the radius of curvature at the point $(-2a, 2a)$ on the curve $x^2y = a(x^2 + y^2)$ is, $-2a$.

14. For the curve $y = ax/(a + x)$, if ρ is the radius of curvature at any point (x, y), then show that $(2\rho/a)^{2/3} = (x/y)^2 + (y/x)^2$.

15. Show that the radius of curvature at any point of the curve

$$x = a\left[\ln \tan\frac{\theta}{2} + \cos\theta\right],\ y = a\sin\theta,$$

where θ is the parameter and a is a constant, varies inversely as the length of the normal intercepted between the point on the curve and the x-axis.

16. Show that in the curve, $x = \frac{3}{2}a(\sinh t\cosh t + t),\quad y = a\cosh^2 t$, if the normal at $P(x, y)$ meets the axis of x in Q, the radius of curvature at P is equal to $3PQ$.

4.9.4 Radius of Curvature at (0, 0). Newtonian Method

If a curve passes through the origin and the axis of x is tangent there, then

$$\rho \text{ at } (0,0) = \lim_{x\to 0}\left(\frac{x^2}{2y}\right). \quad \text{...(4.99)}$$

Since x-axis is tangent at (0, 0) so $\frac{dy}{dx}$ at (0, 0) is zero, thus from (4.91), $\rho = 1/y_2$ at (0, 0).

Consider $\lim_{x\to 0}\left(\frac{x^2}{2y}\right) = \lim_{x\to 0}\left(\frac{2x}{2y_1}\right) = \lim_{x\to 0}\left(\frac{1}{y_2}\right) = 1/y_2$ at (0, 0). This proves (4.99).

Similarly, *if a curve passes through the origin and the axis of y is tangent there, then*

$$\rho \text{ at } (0,0) = \lim_{y\to 0}\left(\frac{y^2}{2x}\right) \quad \text{...(4.100)}$$

When the curve passes through the origin and neither of the axes is tangent there, then the radius of curvature at (0, 0) can be found by the method of expansion as follows.

Let the curve $y = f(x)$ pass through (0, 0). Then $f(0) = 0$, and by Maclaurin's expansion

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots = px + \frac{1}{2}qx^2 + \ldots$$

where $p = f'(0)$ and $q = f''(0)$.

Thus, $$\rho \text{ at } (0,0) = \frac{(1+p^2)^{3/2}}{q}. \quad \text{...(4.101)}$$

Remarks:

1. To find p and q in numerical problems, we substitute $y = px + \frac{1}{2}qx^2 + \ldots$ in the given equation and equate the coefficients of like powers of x on the both sides.
2. The equation (s) of the tangent at the origin is found by equating to zero the terms of the lowest degree in the equation of the curve.

Example 4.58: Find the radius of curvature at the origin for the following curves

(a) $x^3 + y^3 - 2x^2 + 6y = 0$

(b) $2x^4 + 4x^3y + xy^2 + 6y^3 - 3x^2 - 2xy + y^2 - 4x = 0.$

Solution: (a) The curve $x^3 + y^3 - 2x^2 + 6y = 0$ passes through (0, 0) and the equation of the tangent there is $y = 0$, the x-axis. Hence the radius of curvature ρ at (0, 0) is $\lim_{x\to 0}(x^2/2y)$.

Dividing each term in the equation by $2y$, we obtain $x\left(\dfrac{x^2}{2y}\right)+\dfrac{1}{2}y^2-2\left(\dfrac{x^2}{2y}\right)+3=0$.

Taking limit as $x \to 0$, (also $y \to 0$) and using $\rho = \lim\limits_{x\to 0}(x^2/2y)$, we obtain $\rho = 3/2$.

(b) The curve $2x^4+4x^3y+xy^2+6y^3-3x^2-2xy+y^2-4x=0$ passes through $(0, 0)$, and the equation of the tangent at origin is $x=0$, the y-axis. Hence, the radius of curvature ρ at $(0, 0)$ is $\lim\limits_{y\to 0}(y^2/2x)$.

Dividing each term in the equation by $2x$, we obtain

$$x^3+2x^2y+\frac{y^2}{2}+6y\left(\frac{y^2}{2x}\right)-\frac{3}{2}x-y+\frac{y^2}{2x}-2=0.$$

Taking limit as $x \to 0$, (also $y \to 0$) and using $\rho = \lim(y^2/2x)$, we obtain $\rho = 2$.

Example 4.59: Show that the radii of curvature at the origin for the curve $x^3+y^3=3axy$ are each equal to $3a/2$.

Solution: The curve $x^3+y^3=3axy$ passes through $(0, 0)$ and the tangents at the origin are obtained by equating to zero the lowest degree term, $xy=0$. It gives $x=0, y=0$. Thus, x-axis and y-axis both are tangents at $(0, 0)$.

Therefore, ρ at $(0, 0)$ is $\lim\limits_{x\to 0}\dfrac{x^2}{2y}=\rho_1$ (say), or $\lim\limits_{y\to 0}\dfrac{y^2}{2x}=\rho_2$ (say).

Dividing each term in the equation by $2xy$, we have

$$\frac{x^2}{2y}+\frac{y^2}{2x}=\frac{3a}{2} \text{ or, } \frac{x^2}{2y}+\frac{1}{4}xy\frac{2y}{x^2}=\frac{3a}{2}.$$

Taking limit as $x \to 0$ and $y \to 0$, we get $\rho_1 + 0\dfrac{1}{\rho_1}=\dfrac{3a}{2}$, or $\rho_1=\dfrac{3a}{2}$.

Similarly, we can show that $\rho_2=\dfrac{3a}{2}$.

Example 4.60: Find the radii of curvature at the origin for the curve

$$y^2-3xy-4x^2+x^3+x^4y+y^3=0.$$

Solution: The curve passes through the origin but clearly neither of the axes is tangent at $(0, 0)$.

To find the curvature at $(0, 0)$ put $y=px+\dfrac{1}{2!}qx^2+\ldots$ in the given equation, we obtain

$$\left(px+\frac{1}{2!}qx^2+\ldots\right)^2-3x\left(px+\frac{1}{2!}qx^2+\ldots\right)-4x^2+x^3+x^4\left(px+\frac{1}{2!}qx^2+\ldots\right)+\left(px+\frac{1}{2!}qx^2+\ldots\right)^3=0$$

Simplifying, we obtain

$$(p^2-3p-4)x^2+\left(pq-\frac{3}{2}q+1\right)x^3+\ldots=0. \quad \ldots(4.102)$$

Equating to zero the coefficients of x^2 and x^3 on both sides of (4.102), we obtain respectively

$$p^2-3p-4=0, \text{ and } pq-\frac{3}{2}q+1=0$$

Solving these equations for p and q, we get $p=-1, q=2/5$, and $p=4, q=-2/5$ as the two solutions.

Also the radius of curvature ρ at $(0,0)$ is $\rho=\dfrac{(1+p^2)^{3/2}}{q}$, refer to (4.101).

For, $p=1,\ q=2/5,\ \rho=5/\sqrt{2}$, and for, $p=4,\ q=-2/5,\ \rho=\dfrac{-85\sqrt{17}}{2}$.

Thus, radii of curvature at origin are $5\sqrt{2}$ and $\dfrac{-85\sqrt{17}}{2}$.

Example 4.61: Obtain the radii of curvature for the curve $a(y^2-x^2)=x^3$ at the origin.

Solution: The given curve is $a(y^2-x^2)=x^3$. It gives

$$y=\pm x\left(1+\frac{x}{a}\right)^{\frac{1}{2}}=\pm x\left[1+\frac{1}{2}\left(\frac{x}{a}\right)+\ldots\right]=\pm\left[x+\frac{1}{2a}x^2+\ldots\right],$$

which is of the form $y=px+\dfrac{1}{2!}qx^2+\ldots$. Thus, we get $p=\pm 1,\quad q=\pm\dfrac{1}{a}$

Hence the radius of curvature ρ at $(0,0)$ for $p=1, q=\dfrac{1}{a}$ is $\rho_1=\dfrac{(1+p^2)^{3/2}}{q}=2\sqrt{2}a$;

and for $p=-1,\quad q=-\dfrac{1}{a}$ it is $\rho_2=\dfrac{(1+p^2)^{3/2}}{q}=-2\sqrt{2}a$.

Thus the radii of curvature at $(0,0)$ are $\pm 2\sqrt{2}a$.

EXERCISE 4.9

1. Find the radius of curvature at the origin for the curves
   (i) $y^4+x^3+a(x^2+y^2)-a^2y=0$ (ii) $2x^4+3y^4+4x^2y+xy-y^2+2x=0$
2. Find the radius of curvature at the origin for the curves
   (i) $y-x=x^2+2xy+y^2$ (ii) $y=6x+5x^2+x^3$

3. Find the radius of curvature at the origin for the curve $x = t - \frac{1}{3}t^3, y = t^2$.
4. Show that the radii of curvature of the curve $y^2 = x^2(a+x)/(a-x)$ at the origin are $\pm a\sqrt{2}$.
5. Find the radius of curvature at the origin for the cycloid $x = a(\theta + \sin\theta),\quad y = a(1-\cos\theta)$ using Newtonian method.

4.9.5 Radius of Curvature for Polar Curves: $r = f(\theta)$

To find the radius of curvature for polar curves we need to express $ds/d\psi$ in terms of r and its derivatives with respect to θ.

Let the tangent at any point $P(r, \theta)$ to the curve $r = f(\theta)$ make an angle ψ with the initial line OX and angle ϕ with the radius vector OP as shown in Fig. 4.18.

We have, $\psi = \theta + \phi$ and, therefore,

$$\frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds}$$

$$= \frac{d\theta}{ds} + \frac{d\phi}{d\theta}\frac{d\theta}{ds} = \frac{d\theta}{ds}\left(1 + \frac{d\phi}{d\theta}\right). \qquad ...(4.103)$$

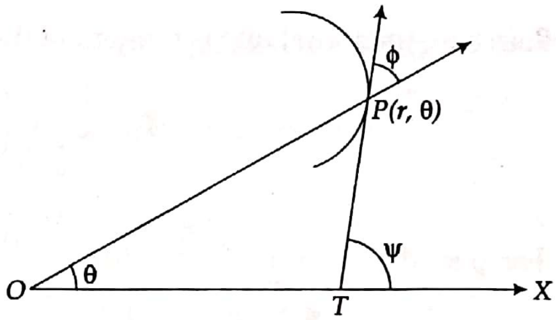


Fig. 4.18

Also, $\tan\phi = r\frac{d\theta}{dr} = \frac{r}{\frac{dr}{d\theta}}$, refer to (4.32a). Differentiating w.r.t. θ, we have

$$\sec^2\phi\frac{d\phi}{d\theta} = \frac{\frac{dr}{d\theta}\cdot\frac{dr}{d\theta} - r\cdot\frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2} \text{ or, } \frac{d\phi}{d\theta} = \frac{\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{(1+\tan^2\phi)\left(\frac{dr}{d\theta}\right)^2} = \frac{\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2 + r^2}. \qquad ...(4.104)$$

For the curve $r = f(\theta)$, refer to (4.29), we have

$$\frac{ds}{d\theta} = \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}. \qquad ...(4.105)$$

From (4.103), (4.104) and (4.105), we obtain

$$\frac{d\psi}{ds} = \frac{1}{\sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}}\left[1 + \frac{\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2 + r^2}\right] = \frac{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{\left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right]^{3/2}} = \frac{r^2 + 2r_1^2 - rr_2}{(r^2 + r_1^2)^{3/2}}.$$

Therefore the radius of curvature ρ is given by

$$\rho = \frac{ds}{d\psi} = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}, \quad \text{...(4.106)}$$

where $r_1 = \dfrac{dr}{d\theta}$ and $r_2 = \dfrac{d^2 r}{d\theta^2}$.

Remarks: **1.** In case the curve is of the form $u = f(\theta)$, where $u = 1/r$, then $r_1 = -u_1/u^2$, and $r_2 = (2u_1^2 - uu_2)/u^3$. Substituting in (4.106) and simplifying, we obtain

$$\rho = \frac{\left(u^2 + u_1^2\right)^{3/2}}{u^3\left(u + u_2\right)} \quad \text{...(4.107a)}$$

2. In case the initial line is tangent at the pole, then the radius of curvature ρ at $(0, 0)$ is

$$\rho = \frac{1}{2} \lim_{\substack{r \to 0 \\ \theta \to 0}} \left(\frac{dr}{d\theta}\right).$$

For ρ at $(0, 0) = \lim\limits_{\substack{x \to 0 \\ y \to 0}} \left(\dfrac{x^2}{2y}\right) = \lim\limits_{\substack{r \to 0 \\ \theta \to 0}} \dfrac{r^2 \cos^2 \theta}{2r \sin \theta} = \lim\limits_{\substack{r \to 0 \\ \theta \to 0}} \left[\dfrac{r}{2\theta} \cdot \dfrac{\theta}{\sin \theta} \cos^2 \theta\right]$

$$= \frac{1}{2} \lim_{\substack{r \to 0 \\ \theta \to 0}} \left[\frac{r}{\theta}\right] = \frac{1}{2} \lim_{\substack{r \to 0 \\ \theta \to 0}} \left(\frac{dr}{d\theta}\right) \quad \text{...(4.107b)}$$

4.9.6 Radius of Curvature for Pedal Curves: $p = f(r)$

Let the tangent at any point P to the curve make an angle ψ with the initial line OX and let $OM = p$ be the length of the perpendicular from the origin to the tangent line to the curve at P, as shown in Fig. 4.19. Then (p, r) are the pedal co-ordinates of the point P.

From Fig. 4.19, $\psi = \theta + \phi$. It gives

$$\frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds}. \quad \text{...(4.108)}$$

Also from Fig. 4.19, $p = r \sin \phi$. Differentiating this w.r.t. r, we have

$$\frac{dp}{dr} = \sin \phi + r \cos \phi \frac{d\phi}{dr}$$

$$= \sin \phi + r \cos \phi \frac{d\phi}{ds} \frac{ds}{dr} \quad \text{...(4.109)}$$

Further $\sin \phi = r\dfrac{d\theta}{ds}$, and $\cos \phi = \dfrac{dr}{ds}$, refer to (4.32b)

Using in (4.109), we have

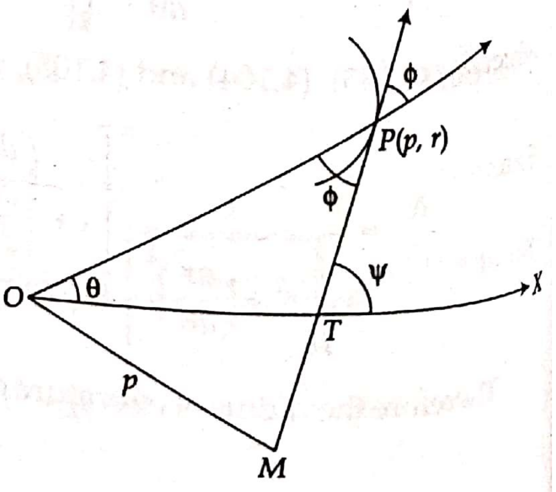


Fig. 4.19

$$\frac{dp}{dr} = r\frac{d\theta}{ds} + r\frac{dr}{ds}\frac{ds}{dr}\frac{d\phi}{ds}$$

$$= r\left(\frac{d\theta}{ds} + \frac{d\phi}{ds}\right) = r\frac{d\psi}{ds}, \quad \text{using (4.108)}$$

which gives

$$\frac{ds}{d\psi} = r\frac{dr}{dp},$$

or,

$$\rho = r\frac{dr}{dp} \qquad ...(4.110)$$

as the desired result.

Remark: A relation between p and ψ, holding for every point of a curve, is called the *tangential polar equation*, and for the tangential polar curve $p = f(\psi)$ the radius of curvature is given by $\rho = p + \frac{d^2p}{d\psi^2}$.

Example 4.62: Find the radius of curvature at the point (r, θ) of the curve $r = a(1 - \cos\theta)$ and show that ρ^2 varies as r.

Solution: The equation is $r = a(1 - \cos\theta)$, hence $r_1 = a\sin\theta$ and $r_2 = a\cos\theta$. Therefore

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{2r_1^2 + r^2 - rr_2} = \frac{(a^2(1-\cos\theta)^2 + a^2\sin^2\theta)^{3/2}}{2a^2\sin^2\theta + a^2(1-\cos\theta)^2 - a^2(1-\cos\theta)\cos\theta}$$

$$= \frac{a[1 - 2\cos\theta + \cos^2\theta + \sin^2\theta]^{3/2}}{2\sin^2\theta + 1 + 2\cos^2\theta - 3\cos\theta}$$

$$= \frac{a[2(1-\cos\theta)]^{3/2}}{3(1-\cos\theta)} = \frac{2a\sqrt{2}}{3}\sqrt{1-\cos\theta} = \frac{4a}{3}\sin\frac{\theta}{2}.$$

It gives,

$$\rho^2 = \frac{16a^2}{9}\sin^2\theta/2 = \frac{8a}{9}[a(1-\cos\theta)] = \frac{8a}{9}r,$$

which implies that ρ^2 varies as r.

Example 4.63: For the curve $r^n = a^n\cos n\theta$, prove that radius of curvature is $\frac{a^n}{(n+1)r^{n-1}}$.

Solution: The curve is $r^n = a^n\cos n\theta$. Taking logarithm, we obtain

$$n\ln r = n\ln a + \ln\cos n\theta$$

Differentiating w.r.t. θ, $\frac{n}{r}\frac{dr}{d\theta} = -\frac{n\sin n\theta}{\cos n\theta}$. It gives $r_1 = -r\tan n\theta$.

Also, $r_2 = \dfrac{d^2r}{d\theta^2} = -rn\sec^2 n\theta - \tan n\theta \dfrac{dr}{d\theta} = -rn\sec^2 n\theta + r\tan^2 n\theta.$

Thus, $\rho = \dfrac{(r^2 + r_1^2)^{3/2}}{2r_1^2 + r^2 - rr_2} = \dfrac{(r^2 + r^2\tan^2 n\theta)^{3/2}}{2r^2\tan^2 n\theta + r^2 + nr^2\sec^2 n\theta - r^2\tan^2 n\theta}$

$$= \frac{r\sec^3 n\theta}{\tan^2 n\theta + 1 + n\sec^2 n\theta} = \frac{r\sec^3 n\theta}{(n+1)\sec^2 n\theta}$$

$$= \frac{r\sec n\theta}{(n+1)} = \frac{r^n \sec n\theta}{(n+1)r^{n-1}} = \frac{a^n\cos n\theta\,.\sec n\theta}{(n+1)r^{n-1}} = \frac{a^n}{(n+1)r^{n-1}}.$$

Example 4.64: Show that at the point of intersection of the curves $r = a\theta$ and $r\theta = a$, the curvatures are in the ratio 3:1, $(0 < \theta < 2\pi)$.

Solution: The points of intersection of the curves $r = a\theta$ and $r\theta = a$ are given by $a\theta^2 = a$, or $\theta = \pm 1$. For the curve $r = a\theta$, we have $r_1 = a$, $r_2 = 0$. If ρ_1 is the radius of curvature for this curve at $\theta = \pm 1$, then

$$\rho_1 = \left[\frac{(r^2 + r_1^2)^{3/2}}{2r_1^2 + r^2 - rr_2}\right] = \frac{(a^2\theta^2 + a^2)^{3/2}}{2a^2 + a^2\theta^2} = \frac{a(1+\theta^2)^{3/2}}{2+\theta^2} = \frac{a(2\sqrt{2})}{3}.$$

Next, for the curve $r\theta = a$, we have, $r_1 = -a/\theta^2$ and $r_2 = 2a/\theta^3$. If ρ_2 is the radius of curvature for this curve at $\theta = \pm 1$, then

$$\rho_2 = \left[\frac{\left(\dfrac{a^2}{\theta^2} + \dfrac{a^2}{\theta^4}\right)^{3/2}}{2\dfrac{a^2}{\theta^4} + \dfrac{a^2}{\theta^2} - \dfrac{2a^2}{\theta^4}}\right] = \frac{a(1+\theta^2)^{3/2}}{\theta^4} = 2a\sqrt{2}.$$

Thus, $$\frac{\rho_2}{\rho_1} = \frac{3}{1} \quad \text{or,} \quad \rho_2 : \rho_1 = 3:1.$$

Example 4.65: Find the radius of curvature at the point (p, r) on the ellipse $\dfrac{1}{p^2} = \dfrac{1}{a^2} + \dfrac{1}{b^2} - \dfrac{r^2}{a^2b^2}$.

Solution: The given curve is $\dfrac{1}{p^2} = \dfrac{1}{a^2} + \dfrac{1}{b^2} - \dfrac{r^2}{a^2b^2}$. Differentiating w.r.t. r, we have

$$-\frac{2}{p^3}\frac{dp}{dr} = -\frac{2r}{a^2b^2}. \text{ It gives } r\frac{dr}{dp} = \frac{a^2b^2}{p^3}, \quad \text{or} \quad \rho = \frac{a^2b^2}{p^3}$$

as the required radius of curvature.

Example 4.66: Find the radius of curvature of the curve $p = a \sin\psi \cos\psi$.

Solution: The curve is, $p = a \sin\psi \cos\psi = \frac{a}{2}\sin 2\psi$. Differentiating we obtain

$$\frac{dp}{d\psi} = a\cos 2\psi, \quad \frac{d^2p}{d\psi^2} = -2a\sin 2\psi.$$

Thus the radius of curvature is

$$\rho = p + \frac{d^2p}{d\psi^2} = \frac{a}{2}\sin 2\psi - 2a\sin 2\psi = -\frac{3a}{2}\sin 2\psi = 3p \text{ (numerically)}.$$

Example 4.67: Find the radius of curvature of the curve $r = a\sin n\theta$ at the pole.

Solution: Equation of the curve is $r = a\sin n\theta$. This gives $\frac{dr}{d\theta} = na\cos n\theta$.

Therefore, $\tan\phi = r\frac{d\theta}{dr} = \frac{a\sin n\theta}{na\cos n\theta} = \frac{1}{n}\tan n\theta.$

Further when $\theta = 0$, we have $r = 0$ and also $\tan\phi = 0$, which gives, $\phi = 0$.
Thus, the curve passes through the pole and also initial line is the tangent to the curve at pole.

By Newton's method, ρ at pole $= \lim\limits_{\substack{r\to 0\\ \theta\to 0}}\left(\frac{1}{2}\frac{dr}{d\theta}\right)$, refer to (4.107)

$$= \lim_{\substack{r\to 0\\ \theta\to 0}}\left(\frac{1}{2}.na\cos n\theta\right) = \frac{na}{2}.$$

Example 4.68: Find the radius of curvature at any point (r, θ) on the curve $\frac{l}{r} = 1 + e\cos\theta$.

Solution: The curve is $\frac{l}{r} = 1 + e\cos\theta$, or $u = \frac{1}{l}(1 + e\cos\theta)$, where $u = \frac{1}{r}$.

It gives $u_1 = \frac{du}{d\theta} = \frac{-e\sin\theta}{l}$, $\quad u_2 = \frac{d^2u}{d\theta^2} = \frac{-e\cos\theta}{l}$

Thus, the radius of curvature, refer to (4.107a), is

$$\rho = \frac{\left(u^2 + u_1^2\right)^{3/2}}{u^3(u + u_2)} = \frac{\left[\frac{(1+e\cos\theta)^2}{l^2} + \frac{e^2\sin^2\theta}{l^2}\right]^{3/2}}{\left(\frac{1+e\cos\theta}{l}\right)^3\left[\frac{1+e\cos\theta}{l} - \frac{e\cos\theta}{l}\right]}$$

$$= \frac{(1+e^2+2e\cos\theta)^{3/2}}{l^3} \cdot \frac{l^4}{(1+e\cos\theta)^3} = \frac{l(1+e^2+2e\cos\theta)^{3/2}}{(1+e\cos\theta)^3}.$$

EXERCISE 4.10

1. Find the radius of curvature at the point (r, θ) of the following curves:

   (i) $\sqrt{r}\cos\left(\frac{\theta}{2}\right) = \sqrt{a}$ (ii) $r^2 = a^2\cos 2\theta$

2. Find the radius of curvature at any point (r, θ) of the following curves:

   (i) $\frac{2a}{r} = 1 + \cos\theta$ (ii) $r = ae^{\theta\cot\alpha}$

3. If ρ_1, ρ_2 be the radii of curvature at the extremities of any chord of the cardioid $r = a(1+\cos\theta)$ which passes through the pole, then show that

$$\rho_1^2 + \rho_2^2 = 16a^2/9.$$

4. Show that for the curve $r = a(1+\cos\theta)$, the radius of curvature $\rho = \frac{4a}{3}\cos\frac{\theta}{2}$. Also find the radius of curvature where the tangent is parallel to the initial line.
5. Prove that in the curve $r^2 = a^2\sin 2\theta$ the curvature varies as the radius vector.
6. A line is drawn through the origin meeting the cardioids $r = a(1-\cos\theta)$ in the points P, Q and the normals at P, Q meet in C. Show that the radii of curvature at P and Q are proportional to PC and QC.
7. Find the radius of curvature at the pole to the curve $r = a(1-\cos\theta)$ by Newtonian method.
8. Find the radius of curvature at any point (p, r) on the following curves:

   (i) $p^2 = ar$ (ii) $2ap^2 = r^3$

   (iii) $pa^2 = r^3$ (iv) $p^2 + a^2 = r^2$

   (v) $p^2(a^2+b^2-r^2) = a^2b^2$ (vi) $\frac{1}{p^2} = \frac{A}{r^2} + B.$

9. Find the radius of curvature for the ellipse $p^2 = a^2\cos^2\psi + b^2\sin^2\psi$.
10. For any curve $r = f(\theta)$, prove that $\frac{r}{\rho} = \sin\phi\left(1+\frac{d\phi}{d\theta}\right)$.

4.9.7 Centre of Curvature. Circle of Curvature.

The centre of curvature for any point P of a curve is the point which lies on the positive direction of the normal at P and is at a distance ρ, the radius of curvature at P, from it.

The positive direction of the normal is obtained by rotating the positive direction of the tangent through $\pi/2$ in the anti-clockwise direction, where the positive direction of the tangent to $y = f(x)$ is the one in which x increases.

Co-ordinates of the centre of curvature: Let $C(X, Y)$ be the centre of curvature corresponding to any point $P(x, y)$ on the curve and ρ be the radius of curvature at the point P, then $PC = \rho$, as shown in Fig. 4.20.

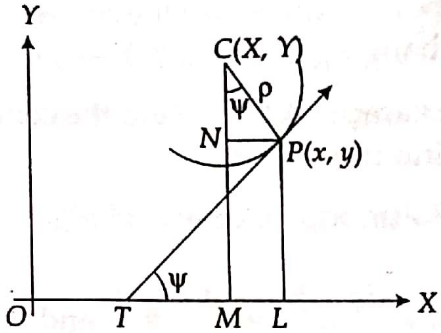


Fig. 4.20

Let the tangent PT make an angle ψ with the positive direction of the x-axis. Draw PL and CM perpendicular on x-axis and draw PN perpendicular on CM.

Consider $$\angle NCP = \frac{\pi}{2} - \angle NPC$$

$$= \frac{\pi}{2} - \left(\frac{\pi}{2} - \angle XTP\right) = \angle XTP = \psi$$

Thus, $$X = OM = OL - ML = OL - NP$$

$$= OL - PC \sin\psi = x - \rho \sin\psi. \quad \text{...(4.111)}$$

$$Y = MC = MN + NC = LP + NC$$

$$= LP + PC \cos\psi = y + \rho\cos\psi. \quad \text{...(4.112)}$$

Further, $\tan\psi = \dfrac{dy}{dx} = y_1$ gives $\sin\psi = \dfrac{y_1}{\sqrt{1+y_1^2}}$ and $\cos\psi = \dfrac{1}{\sqrt{1+y_1^2}}$.

Also, $$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}.$$

Using these in (4.111) and (4.112), we obtain

$$X = x - \frac{y_1(1+y_1^2)}{y_2}, \quad Y = y + \frac{(1+y_1^2)}{y_2} \quad \text{...(4.113)}$$

as the co-ordinates of the centre of curvature.

Circle of curvature: *The circle of curvature of any point P of a curve is the circle whose centre is at the centre of curvature and whose radius is ρ, the radius of curvature at P.*

If ρ is the radius of curvature and (X, Y) the centre of curvature at the given point P, then the equation of the circle of curvature of P is given by

$$(x - X)^2 + (y - Y)^2 = \rho^2. \quad \text{...(4.114)}$$

Obviously the circle of curvature will touch the curve at P and its curvature will be the same as that of the curve at the point P.

4.9.8 Evolute

The locus of the centres of curvature of a curve is called its evolute and the curve is said to be involute of its evolute.

To find evolute of a curve we eliminate the parameters x and y between

$$X = x - \frac{y_1(1+y_1^2)}{y_2} \quad \text{and} \quad Y = y + \frac{1+y_1^2}{y_2},$$

the co-ordinates of the centre of curvature, and then the required equation of the evolute is obtained by changing $X \to x$ and $Y \to y$ in the relation obtained between X and Y.

Example 4.69: Find the centre of curvature of the parabola $x = at^2, y = 2at$ at the point t and hence find its evolute.

Solution: Curve is $x = at^2$, $y = 2at$. Therefore,

$$y_1 = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{2a}{2at} = \frac{1}{t} \text{ and, } y_2 = \frac{d}{dx}\left(\frac{1}{t}\right) = -\frac{1}{t^2}\frac{dt}{dx} = -\frac{1}{t^2}\frac{1}{2at} = -\frac{1}{2at^3}.$$

Thus the co-ordinates of the centre of curvature (X, Y) are

$$X = at^2 - \frac{\frac{1}{t}\left(1+\frac{1}{t^2}\right)}{-\frac{1}{2at^3}} = at^2 + 2a(1+t^2) = a(2+3t^2)$$

$$Y = 2at + \frac{\left(1+\frac{1}{t^2}\right)}{-\frac{1}{2at^3}} = 2at - 2at(1+t^2) = -2at^3$$

Eliminating t between X and Y, we obtain

$$(X-2a)^3 = (3a)^3\left(-\frac{Y}{2a}\right)^2 \quad \text{or,} \quad 4(X-2a)^3 = 27a\,Y^2.$$

Changing X to x and Y to y, the required equation of the evolute is $4(X-2a)^3 = 27a\,Y^2$.

Example 4.70: Show that the evolute of the tractrix $x = c\cos t + c\ln\tan\frac{t}{2}, y = c\sin t$ is the catenary $y = c\cosh\frac{x}{c}$.

Solution: Equation of the curve is $x = c\cos t + c\ln\tan\frac{t}{2}$, $y = c\sin t$.

It gives $\frac{dx}{dt} = -c\sin t + \frac{c}{\tan\frac{t}{2}}\frac{1}{2}\sec^2\frac{t}{2} = \frac{c\cos^2 t}{\sin t}$, $\frac{dy}{dt} = c\cos t$.

Therefore, $$y_1 = \frac{dy}{dt}\Big/\frac{dx}{dt} = c\cos t\,\frac{\sin t}{c\cos^2 t} = \tan t.$$

$$y_2 = \frac{d}{dx}(\tan t) = \sec^2 t\,\frac{dt}{dx} = \sec^2 t\,\frac{\sin t}{c\cos^2 t} = \frac{\sin t}{c\cos^4 t}.$$

If (X, Y) is the centre of curvature at any point on the curve, then

$$X = x - \frac{y_1(1+y_1^2)}{y_2} = c\cos t + c\ln\tan\frac{t}{2} - \frac{\tan t(1+\tan^2 t)}{\dfrac{\sin t}{c\cos^4 t}}$$

$$= c\cos t + c\ln\tan\frac{t}{2} - c\cos t = c\ln\tan\frac{t}{2} \qquad \text{...(4.115)}$$

$$Y = y + \frac{1+y_1^2}{y_2} = c\sin t + \frac{1+\tan^2 t}{\dfrac{\sin t}{c\cos^4 t}} = c\sin t + \frac{c\cos^2 t}{\sin t} = \frac{c}{\sin t}. \qquad \text{...(4.116)}$$

For evolute we eliminate t between X and Y.

From (4.115) $\quad \ln\tan\frac{t}{2} = \frac{X}{c}$, or $\quad \tan\frac{t}{2} = e^{\frac{X}{c}}$ ...(4.117)

From (4.116) $\quad \frac{Y}{c} = \frac{1}{\sin t} = \frac{1+\tan^2\frac{t}{2}}{2\tan\frac{t}{2}} = \frac{1}{2}\left[\frac{1}{\tan\frac{t}{2}} + \tan\frac{t}{2}\right] = \frac{1}{2}\left[e^{-\frac{X}{c}} + e^{\frac{X}{c}}\right]$, using (4.117)

$$= \cosh\frac{X}{c}$$

or,
$$Y = c\cosh\frac{X}{c}.$$

Changing X to x and Y to y, we obtain, $y = c\cosh\frac{x}{c}$, as the equation of the evolute.

Example 4.71: Find the circle of curvature at the point $t = \frac{\pi}{2}$ on the ellipse $x = a\cos t;\ y = b\sin t$.

Solution: The curve is $x = a\cos t$, $y = b\sin t$. It gives $\frac{dx}{dt} = -a\sin t$, $\frac{dy}{dt} = b\cos t$. Therefore,

$$y_1 = \frac{dy}{dt}\Big/\frac{dx}{dt} = -\frac{b\cos t}{a\sin t} = -\frac{b}{a}\cot t.$$

$$y_2 = \frac{d}{dx}\left(-\frac{b}{a}\cot t\right) = \frac{b}{a}\operatorname{cosec}^2 t\,\frac{dt}{dx} = \frac{b}{a}\operatorname{cosec}^2 t\left(-\frac{\operatorname{cosec} t}{a}\right) = -\frac{b}{a^2}\operatorname{cosec}^3 t.$$

At $\quad t = \pi/2;\ (x, y) = (0, b),\ y_1 = 0,\ y_2 = -b/a^2$, therefore

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = -\frac{a^2}{b}$$

and, the centre of curvature (X, Y) is $\left(x-\dfrac{y_1(1+y_1^2)}{y_2},\ y+\dfrac{1+y_1^2}{y_2}\right)=\left(0, \dfrac{b^2-a^2}{b}\right)$.

Hence the circle of curvature at $t=\pi/2$ is $x^2+\left(y-\dfrac{b^2-a^2}{b}\right)^2=\dfrac{a^4}{b^2}$.

Example 4.72: Find the equation of the evolute of the curve $x^{2/3}+y^{2/3}=a^{2/3}$.

Solution: The parametric equations of the curve are $x=a\cos^3\theta,\quad y=a\sin^3\theta$.

We obtain, $\dfrac{dx}{d\theta}=-3a\cos^2\theta\sin\theta,\quad \dfrac{dy}{d\theta}=3a\sin^2\theta\cos\theta.$

Therefore, $\dfrac{dy}{dx}=\dfrac{dy}{d\theta}\Big/\dfrac{dx}{d\theta}=-\dfrac{3a\sin^2\theta\cos\theta}{3a\cos^2\theta\sin\theta}=-\tan\theta.$

$$\frac{d^2y}{dx^2}=-\frac{d}{dx}(\tan\theta)=-\sec^2\theta\,\frac{d\theta}{dx}=-\frac{1}{\cos^2\theta}\,\frac{1}{(-3a\cos^2\theta\sin\theta)}=\frac{1}{3a\sin\theta\cos^4\theta}.$$

If (X, Y) is the centre of curvature at point θ, then

$$X=x-\frac{y_1(1+y_1^2)}{y_2}=a\cos^3\theta-\frac{-\tan\theta(1+\tan^2\theta)}{\dfrac{1}{3a\sin\theta\cos^4\theta}}=a\cos\theta(\cos^2\theta+3\sin^2\theta)$$

and $$Y=y+\frac{1+y_1^2}{y_2}=a\sin^3\theta+\frac{1+\tan^2\theta}{\dfrac{1}{3a\sin\theta\cos^4\theta}}=a\sin\theta(\sin^2\theta+3\cos^2\theta).$$

To find the equation of the evolute, we eliminate θ between X and Y.

We have, $X+Y=a(\cos^3\theta+\sin^3\theta)+3a\sin\theta\cos\theta(\sin\theta+\cos\theta)=a[\cos\theta+\sin\theta]^3$

or, $$(X+Y)^{2/3}=a^{2/3}[\cos\theta+\sin\theta]^2 \qquad ...(4.118)$$

Similarly, $$(X-Y)^{2/3}=a^{2/3}[\cos\theta-\sin\theta]^2 \qquad ...(4.119)$$

From (4.118) and (4.119), we have

$$(X+Y)^{2/3}+(X-Y)^{2/3}=a^{2/3}[(\cos\theta+\sin\theta)^2+(\cos\theta-\sin\theta)^2]=2a^{2/3}.$$

Changing X to x and Y to y the equation of evolute is $(x+y)^{2/3}+(x-y)^{2/3}=2a^{2/3}$.

EXERCISE 4.11

1. Find the co-ordinates of the centre of curvature at the point (x, y) on the parabola $y^2=4ax$. Also find the equation of its evolute.

2. Find the co-ordinates of the centre of curvature of the curve $a^2y = x^3$.
3. Show that the evolute of ellipse $x = a\cos\theta$, $y = b\sin\theta$ is $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$.
4. Prove that the centres of curvature at points of a cycloid lie on another cycloid.
5. Show that the parabolas $y = -x^2 + x + 1$, $x = -y^2 + y + 1$ have the same circle of curvature at the point $(1, 1)$.
6. Show that $\left(x - \frac{3}{4}a\right)^2 + \left(y - \frac{3}{4}a\right)^2 = \frac{1}{2}a^2$ is the circle of curvature of the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at the point $\left(\frac{a}{4}, \frac{a}{4}\right)$.
7. Find the radius of curvature and the centre of curvature for the curve $y = \tan x$ at the point when $x = \pi/4$.
8. Show that the evolute of the curve $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ is $x^2 + y^2 = a^2$.
9. Find the circle of curvature at the origin for the curve $x + y = ax^2 + by^2 + cx^3$.
10. Show that the circle of curvature at the origin of the parabola $y = mx + \frac{x^2}{a}$, is $(x^2 + y^2) = a(1 + m^2)(y - mx)$.

4.10 ENVELOPES

Consider the following one parameter family of straight lines

$$y = mx + a/m \qquad ...(4.120)$$

Here m is parameter and a is some constant.

For different values of m, (4.120) represents different straight lines. All these straight lines thus obtained constitute a family whose members touch the parabola $y^2 = 4ax$. In fact $y^2 = 4ax$ is the locus of the point of intersection two consecutive members of the family (4.120). Consider two members of the family

$$y = mx + \frac{a}{m} \qquad ...(4.121)$$

$$y = (m + \delta m)x + \frac{a}{m + \delta m}. \qquad ...(4.122)$$

The point of intersection of these lines is $P\left(\frac{a}{m(m + \delta m)}, \frac{a(2m + \delta m)}{m(m + \delta m)}\right)$.

Keeping m fixed and making $\delta m \to 0$, then the point of intersection P tends to the point

$$Q\left(\frac{a}{m^2}, \frac{2a}{m}\right). \qquad ...(4.123)$$

The point Q is the limiting position of the point of intersection of the two lines (4.121) and (4.122), when the latter tends to coincide with the former, and it lies on the line (4.121). A similar point will lie on every line of the family (4.120) and the locus of such points is obtained by eliminating m between x and y given by

$$x = \frac{a}{m^2}, \quad y = \frac{2a}{m} \qquad ...(4.124)$$

which gives the parabola $y^2 = 4ax$.

The *locus* obtained such is called the *envelope* of the given family.

Thus $y^2 = 4ax$ is the envelope of the one-parameter family (4.20) of straight lines as shown in Fig. 4.21.

As another example, the circle $x^2 + y^2 = p^2$ is the envelope of the family of straight lines $x \cos \alpha + y \sin \alpha = p$. All members of this family touch the circle $x^2 + y^2 = p^2$.

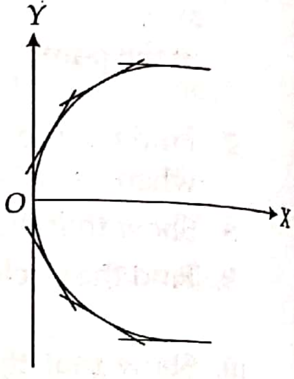


Fig. 4.21

A formal definition is given as follows.

The envelope of a one-parameter family of curves given by $f(x, y, \alpha) = 0$ is the locus of the limiting position of the points of intersection of any two consecutive members of the family when one tends to coincide with the other which is fixed.

The envelope of the one-parameter family of curves $f(x, y, \alpha) = 0$ is obtained by eliminating α between

$$f(x, y, \alpha) = 0, \quad \text{and} \quad \frac{\partial f}{\partial \alpha}(x, y, \alpha) = 0. \qquad ...(4.125)$$

In case of two-parameters family of curves $f(x, y, \alpha, \beta) = 0$; α, β being two independent parameters, envelope is obtained by eliminating α and β from

$$f(x, y, \alpha, \beta) = 0, \quad \frac{\partial f}{\partial \alpha}(x, y, \alpha, \beta) = 0 \text{ and } \frac{\partial f}{\partial \beta}(x, y, \alpha, \beta) = 0. \qquad ...(4.126)$$

Another result of interest is

Evolute of a curve is the envelope of the normals to that curve.

In the examples to follow, we will learn that a family of curves may have no envelope, or one envelope, or more than one envelope.

Example 4.73: Find the envelope of the family of curves given by

(a) $y = mx + \frac{a}{m}$ (b) $x \cos \alpha + y \sin \alpha = p$

α, m being paramers and a, p are constants.

Solution: (a) The family of curves is

$$f(x, y, m) = y - mx - \frac{a}{m} = 0. \qquad ...(4.127)$$

Differentiating with respect to m, we have

$$\frac{\partial f}{\partial m} = -x + \frac{a}{m^2} = 0. \qquad ...(4.128)$$

From (4.128) $m^2 = a/x$. ...(4.129)

Also from (4.127) $y^2 = m^2x^2 + \frac{a^2}{m^2} + 2ax$ ...(4.130)

Substituting for m^2 from (4.129) in (4.130), we obtain $y^2 = 4ax$ as the envelope.

(b) The family of curves is

$$f(x, y, \alpha) = x\cos\alpha + y\sin\alpha - p = 0. \quad ...(4.131)$$

It gives $$\frac{\partial f}{\partial \alpha} = -x\sin\alpha + y\cos\alpha = 0. \quad ...(4.132)$$

From (4.132), we have $\tan\alpha = y/x$, which gives

$$\sin\alpha = \frac{\pm y}{\sqrt{x^2+y^2}}, \text{ and } \cos\alpha = \frac{\pm x}{\sqrt{x^2+y^2}}. \quad ...(4.133)$$

Substituting for $\sin\alpha$ and $\cos\alpha$ from (4.133) in (4.131), we obtain

$$\pm\frac{(x^2+y^2)}{\sqrt{x^2+y^2}} = p, \text{ or} \quad ...(4.134)$$

Squaring and simplifying (4.134), we obtain $x^2 + y^2 = p^2$ as the envelope.

Example 4.74: Find the envelope of the family

$$x^2(x-a) + (x+a)(y-m)^2 = 0,$$

where m is a parameter and a is some constant.

Solution: The family of curves is

$$x^2(x-a) + (x+a)(y-m)^2 = 0. \quad ...(4.135)$$

Differentiating it partially w.r.t. m, we obtain

$$-2(x+a)(y-m) = 0 \quad ...(4.136)$$

Eliminating m between (4.135) and (4.136), we obtain

$$x^2(x-a) = 0 \quad ...(4.137)$$

as the envelope.

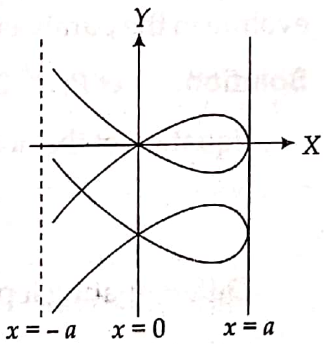


Fig. 4.22

From (4.137) it is clear that the envelope consists of two lines as $x = 0$ and $x = a$, as shown in Fig. 4.22.

Example 4.75: Check the family of circles $x^2 + (y-b)^2 = b^2$ for its envelope.

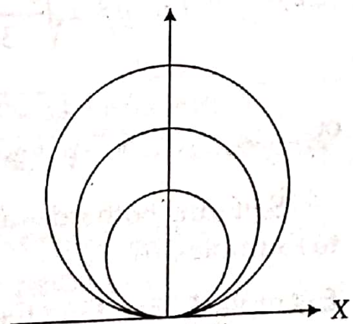


Fig. 4.23

Solution: The family of circles is

$$x^2 + (y-b)^2 - b^2 = 0 \quad ...(4.138)$$

Differentiating partially w.r.t. b, we obtain

$$2(y-b)(-1) - 2b = 0, \text{ or } y = 0 \quad ....(4.139)$$

Eliminating b between the two equations we obtain $x^2 = 0$, or $x = 0$. Taking it along with (4.139), it gives (0, 0) as the envelope or simply, *no envelope* as shown in Fig. 4.23.

Example 4.76: Find the envelope of the family of ellipses $x^2/a^2 + y^2/b^2 = 1$, where the two parameters a, b are connected by the relation $a + b = c$.

Solution: The family can be expressed as

$$\frac{x^2}{a^2} + \frac{y^2}{(c-a)^2} = 1. \qquad ...(4.140)$$

Differentiating partially w.r.t. a, we obtain

$$-x^2/a^3 + \frac{y^2}{(c-a)^3} = 0, \quad \text{or} \quad \frac{c-a}{a} = \frac{y^{2/3}}{x^{2/3}}. \qquad ...(4.141)$$

From (4.141), we have

$$a = cx^{2/3}/(x^{2/3} + y^{2/3}) \quad \text{and,} \quad c - a = cy^{2/3}/(x^{2/3} + y^{2/3})$$

Substituting these in (4.140), we obtain

$$x^{2/3}(x^{2/3} + y^{2/3})^2 + y^{2/3}(x^{2/3} + y^{2/3})^2 = c^2, \quad \text{or} \quad x^{2/3} + y^{2/3} = c^{2/3}$$

as the required envelope.

Example 4.77: Using the result that the evolute of a curve is the envelope of its normals, find the evolute to the parabola $y^2 = 4ax$.

Solution: Let $P(at^2, 2at)$ be any point on the parabola $y^2 = 4ax$.

Equation of the normal at t is

$$y + xt = 2at + at^3. \qquad ...(4.142)$$

Differentiating it partially w.r.t. t, we obtain $x = 2a + 3a\,t^2$, which gives $t = \sqrt{\dfrac{x-2a}{3a}}$

Substituting for t in (4.142), we obtain

$$y + x\sqrt{\frac{x-2a}{3a}} = 2a\sqrt{\frac{x-2a}{3a}} + a\left(\frac{x-2a}{3a}\right)^{3/2}$$

or,
$$y = \sqrt{\frac{x-2a}{3a}}\left[2a - x - a\left(\frac{x-2a}{3a}\right)\right] = \sqrt{\frac{x-2a}{3a}}\left(\frac{4a}{3} - \frac{2x}{3}\right).$$

Squaring both sides and simplifying, we get $27ay^2 = 4(x-2a)^3$ as the required envelope, also refer to Example 4.69.

Example 4.78: Find the envelope of straight lines drawn at right angles to the radii vectors of the cardiod $r = a(1 + \cos\theta)$ through their extremities.

Solution: Let P be any point on the cardiod $r = a(1 + \cos\theta)$. If α be its vectorial angle, then the radius vector $OP = a(1 + \cos\alpha)$.

The equation of a line through P at right angle to the radius vector OP is

$$r\cos(\theta - \alpha) = a(1 + \cos\alpha) \quad \ldots(4.143)$$

here α is the parameter, since for different α the lines are different.

Differentiating w.r.t. α, we obtain,

$$r\sin(\theta - \alpha) = -a\sin\alpha$$

or,
$$r\sin\theta\cos\alpha - (r\cos\theta - a)\sin\alpha = 0, \text{ which gives } \tan\alpha = r\sin\theta/(r\cos\theta - a).$$

Hence, $\sin\alpha = \dfrac{r\sin\theta}{\sqrt{r^2 + a^2 - 2ar\cos\theta}}$, and $\cos\alpha = \dfrac{r\cos\theta - a}{\sqrt{r^2 + a^2 - 2ar\cos\theta}}$

Also from (4.143), we have $(r\cos\theta - a)\cos\alpha + r\sin\theta\sin\alpha = a$

Substituting for $\sin\alpha$ and $\cos\alpha$, we obtain

$$\frac{(r\cos\theta - a)^2 + r^2\sin^2\theta}{\sqrt{r^2 + a^2 - 2ar\cos\theta}} = a, \text{ or } \frac{r^2 + a^2 - 2a^2\cos\theta}{\sqrt{r^2 + a^2 - 2ar\cos\theta}} = a$$

or,
$$\sqrt{r^2 + a^2 - 2ar\cos\theta} = a, \text{ or } r^2 + a^2 - 2ar\cos\theta = a^2, \text{ or } r = 2a\cos\theta$$

as the required envelope.

EXERCISE 4.12

1. Find the envelope of the following family of lines
   (i) $y = mx \pm \sqrt{a^2m^2 + b^2}$, m being the parameter.
   (ii) $y = mx - 2am - am^3$, m being the parameter.
   (iii) $y = mx + am^3$, m being the parameter.
   (iv) $x\cos\alpha - y\sin\alpha = c$, α being the parameter.
   (v) $x\tan\alpha + y\sec\alpha = c$, α being the parameter
   (vi) $x\cos^n\alpha + y\sin^n\alpha = c$, α being the parameter
2. Find the envelope of the family of circles $(x - \alpha)^2 + y^2 = 2\alpha$, where α is the parameter.
3. Find the envelope of the family of semi-cubical parabolas $y^2 - (x + \alpha)^3 = 0$; α being the parameter.
4. Find the envelope of a system of concentric and co-axial ellipses of constant area A.
5. Find the envelope of a family of lines $x/a + y/b = 1$, where the parameters a and b are connected by the relation $a^n + b^n = c^n$.
6. Circles are described on the double ordinates of $y^2 = 4ax$ as diameter. Show that the envelope is $y^2 = 4a(x + a)$.

7. Show that the envelope of a circle whose centre lies on the parabola $y^2 = 4ax$ and which passes through its vertex is the cissoid $y^2(2a + x) + x^3 = 0$.
8. Considering that the evolute of a curve is the envelope of its normals, show that the evolute of the (i) ellipse $x^2/a^2 + y^2/b^2 = 1$ is $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$ (ii) hyperbola $x^2/a^2 - y^2/b^2 = 1$ is $(ax)^{2/3} - (by)^{2/3} = (a^2 + b^2)^{2/3}$

4.11 ASYMPTOTES

The concept of asymptotes is associated with those curves which extend to infinity like parabola and hyperbola. For example, the two infinite branches of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ are $y = \pm \frac{b}{a}\sqrt{x^2 - a^2}$, since here as $x \to \pm\infty$, y also tends to $\pm\infty$. On the other hand, in case of a circle, say $x^2 + y^2 = a^2$, x and y both can take only finite values.

A point $P(x, y)$ on an infinite branch of a curve is said to tend to ∞ along the curve when either x, or y, or x and y both tend to ∞ or $-\infty$, as P moves along the infinite branch of the curve.

Next we define asymptote:

A straight line at a finite distance from the origin is said to be an 'asymptote' to an infinite branch of a curve if the perpendicular distance of a point P on the curve from the straight line approaches zero as the point P moves to infinity along the curve.

In other words, *asymptote is the limiting position of the tangent line to the curve as the point of contact recedes to infinity along the infinite branch of the curve.*

Asymptotes parallel to x-axis are called *horizontal asymptotes* and those parallel to y-axis are called *'vertical asymptotes'*. Asymptotes which are not parallel to any of the axes are called *oblique asymptotes*. In some typical cases asymptotes are curves also.

4.11.1 Asymptotes Parallel to Axes

For the curve $y = f(x)$: First we find the asymptotes parallel to y-axis. If $x = k$ is an asymptote for the curve $y = f(x)$, then, by definition of an asymptote, $|x - k|$, the distance of the line $x = k$ from a point $P(x, y)$ on the curve, must tend to zero as $P \to \infty$ (in this case $y \to \infty$) along the curve. Thus, *to find asymptotes parallel to y-axis we find from the given equation of the curve the definite values $k_1, k_2, \ldots$ to which x tends as y tends to ∞*, or $-\infty$ *and, then $x = k_1, x = k_2, \ldots$ are equations of the asymptotes.*

Similarly, *to find asymptotes parallel to x-axis we find from the given equation of the curve the definite values $l_1, l_2, \ldots$ to which y tends as x tends to ∞, or $-\infty$ and, then $y = l_1, y = l_2, \ldots$ are equations of the asymptotes.*

For the curve $f(x, y) = 0$: Let the given equation $f(x, y) = 0$, after arranging in descending powers of y, be

$$\phi_0(x)\, y^n + \phi_1(x) y^{n-1} + \phi_2(x) y^{n-2} + \ldots + \phi_n(x) = 0 \qquad \ldots(4.144)$$

where $\phi_i(x), i = 0, 1, 2, \ldots, n$ are polynomials in x.

Dividing (4.144) throughout by y^n, we obtain

$$\phi_0(x) + \phi_1(x)\frac{1}{y} + \phi_2(x)\frac{1}{y^2} + \ldots + \phi_n(x)\frac{1}{y^n} = 0 \qquad \ldots(4.145)$$

To find asymptotes $x = k$, parallel to y-axis, as above, we find from (4.145) the definite values $k_1, k_2, \ldots$ to which x tends as $y \to \pm\infty$. These are given by $\phi_0(x) = 0$. Thus *to find asymptotes parallel to y-axis for the algebraic curve f(x, y) = 0 equate to zero the coefficient of the highest power of y, present in the equation of the curve. Resolve it into real linear factors. Then these factors equated to zero give the equation of the asymptotes. If the coefficient of the highest power of y is either a constant or not resolvable into real linear factors, then there are no asymptotes parallel to the y-axis.*

Similarly, *to find the asymptotes parallel to x-axis for the algebraic curve f(x, y) = 0, equate to zero the coefficient of the highest power of x, present in the equation of the curve. Resolve it into real linear factors. Then these factors equated to zero give the equation of the asymptotes. If the coefficient of the highest power of x is either a constant or not resolvable into real linear factors, then there are no asymptotes parallel to the x-axis.*

Example 4.79: Find the asymptotes parallel to the axes for the curves:

(i) $x^2y^2 = a^2(x^2 + y^2)$ (ii) $\dfrac{a^2}{x^2} - \dfrac{b^2}{y^2} = 1.$

Solution: (i) The equation of the curve is $x^2y^2 - a^2(x^2 + y^2) = 0$.

To find asymptotes parallel to x-axis equate to zero the coefficient of the highest power of x, we have

$$(y^2 - a^2) = 0, \text{ or } \quad (y - a)(y + a) = 0, \text{ or } \quad y = \pm a,$$

is the required asymptotes parallel to x-axis.

Next, to find asymptotes parallel to y-axis equate to zero the coefficient of the highest power of y, we have

$$(x^2 - a^2) = 0, \text{ or } \quad (x - a)(x + a) = 0, \text{ or } \quad x = \pm a,$$

is the required asymptotes parallel to y-axis.

(ii) The equation of the curve is $a^2/x^2 - b^2/y^2 = 1$ or, $x^2y^2 - a^2y^2 + b^2x^2 = 0$.

Equating to zero the coefficient of the highest power of x we have $(y^2 + b^2) = 0$. Since it can't be resolved into real linear factors. Hence there is no asymptote parallel to x-axis.

Proceeding on the same lines as in (i) above, the asymptotes parallel to y-axis are $x = \pm a$.

Example 4.80: Find the asymptotes parallel to axes to the curves

(i) $y = e^x$ (ii) $y = \sec x$.

Solution: (i) Equation of the curve is $y = e^x$.

Here, $\lim\limits_{x\to\infty} y = \lim\limits_{x\to\infty} e^x \to \infty$ and $\lim\limits_{x\to-\infty} y = \lim\limits_{x\to-\infty} e^x = 0$.

Therefore, $y = 0$ is an asymptote parallel to x-axis.

Further $\quad y = e^x$ gives $x = \ln y$

Here, $\lim\limits_{y\to\infty} x = \lim\limits_{y\to\infty} \ln y \to \infty$, and $\lim\limits_{y\to-\infty} x = \lim\limits_{y\to-\infty} \ln y$, which does not exist.

Therefore, there is no asymptote parallel to y-axis.

(ii) Equation of the curve is $y = \sec x$.

Here $\lim\limits_{x\to\pm\infty} y = \lim\limits_{x\to\pm\infty} \sec x$ does not tend to a unique finite value. Therefore there is no asymptote parallel to x-axis.

However, $y \to \pm\infty$ gives $\cos x \to 0$, which implies $x = (2n+1)\pi/2$, for integral values of n. Thus, $x = (2n+1)\pi/2$ are asymptotes parallel to the y-axis.

EXERCISE 4.13

Find the asymptotes parallel to the axes for the following curves

1. $x(y^2 - x^2) = a(x^2 + y^2)$
2. $x^2y^2 - y^2 - 2 = 0$
3. $x^2y^2 - a^2(x^2 + y^2) - a^3(x + y) + a^4 = 0$
4. $\dfrac{a^2}{x^2} + \dfrac{b^2}{y^2} = 1$
5. $y = \sin x$
6. $y = \operatorname{cosec} x$.

4.11.2 Oblique Asymptotes

For the curve $y = f(x)$. Let $y = mx + c$ be the equation of the asymptote. If d is the distance of $y = mx + c$ from any point $P(x, y)$ on an infinite branch of the curve, then

$$d = \frac{|y - mx - c|}{\sqrt{1+m^2}}. \text{ This gives}$$

$$y - mx = c \pm d\sqrt{1+m^2} \qquad \text{...(4.146)}$$

or,

$$\frac{y}{x} = m + \frac{c}{x} \pm \frac{d}{x}\sqrt{1+m^2}. \qquad \text{...(4.147)}$$

Making the point $P(x, y) \to \infty$ along the curve, (4.147) gives $\lim\limits_{x\to\infty}\left(\dfrac{y}{x}\right) = m$, since d, by definition of the asymptote, tends to zero, as $P \to \infty$.

Also as $P \to \infty$, (4.146) gives $\lim\limits_{x\to\infty}(y - mx) = c$.

Hence, *if $y = mx + c$ is an oblique asymptote to the curve $y = f(x)$, then*

$$m = \lim_{x\to\infty}(y/x) \text{ and } \quad c = \lim_{x\to\infty}(y - mx).$$

For the curve $f(x, y) = 0$. Write the equation of the curve in the form

$$x^n\phi_n(y/x) + x^{n-1}\phi_{n-1}(y/x) + x^{n-2}\phi_{n-2}(y/x) + \ldots = 0, \qquad \text{...(4.148)}$$

where $\phi_r(y/x)$ is a polynomial of degree r in (y/x).

To find the point of intersection of the line $y = mx + c$ with (4.148), put $y/x = m + c/x$, and expand each of $\phi_r\left(m + \dfrac{c}{x}\right)$ by Taylor's series, we obtain

$$x^n\left[\phi_n(m)+\frac{c}{x}\phi_n'(m)+\frac{c^2}{2!\,x^2}\phi_n''(m)+\ldots\right]+x^{n-1}\left[\phi_{n-1}(m)+\frac{c}{x}\phi_{n-1}'(m)+\frac{c^2}{2!\,x^2}\phi_{n-1}''(m)\right]+\ldots=0$$

$$\phi_n(m)x^n+[c\phi_n'(m)+\phi_{n-1}(m)]x^{n-1}+\left[\frac{c^2}{2!}\phi_n''(m)+c\phi_{n-1}'(m)+\phi_{n-2}(m)\right]x^{n-2}+\ldots=0.$$

Dividing throughout by x^n, we obtain

$$\phi_n(m)+[c\phi_n'(m)+\phi_{n-1}(m)].\frac{1}{x}+\left[\frac{c^2}{2!}\phi_n''(m)+c\,\phi_{n-1}'(m)+\ \phi_{n-2}(m)\right]\frac{1}{x^2}+\ldots=0 \qquad \ldots(4.149)$$

Also from (4.148)

$$\phi_n(y/x)+\frac{1}{x}\phi_{n-1}(y/x)+\frac{1}{x^2}\phi_{n-2}(y/x)+\ldots=0 \qquad \ldots(4.150)$$

If $y=mx+c$ is the asymptote to the given curve, then $\lim\limits_{x\to\infty}(y/x)=m$ and hence from (4.150), we obtain

$$\phi_n(m)=0. \qquad \ldots(4.151)$$

The real roots of the Eq. (4.151) give the slopes of the asymptotes.

Substituting $\phi_n(m)=0$ in (4.149) and multiplying throughout by x and taking $x\to\infty$ we get

$$c\phi_n'(m)+\phi_{n-1}(m)=0,\text{ or }\quad c=\frac{-\phi_{n-1}(m)}{\phi_n'(m)},\text{ provided }\quad \phi_n'(m)\neq 0. \qquad \ldots(4.152)$$

Thus, if m_1, m_2, ... are the distinct values of m as obtained from (4.151) and c_1, c_2, ... are corresponding values of c given by (4.152), then the equations of the asymptotes are: $y_1=m_1x+c_1$, $y=m_2x+c_2$..., provided $\phi_n'(m)\neq 0$.

If $\phi_n'(m)=0$ but $\phi_{n-1}(m)\neq 0$, then from (4.152) values of c are infinite and thus there is no asymptote in this case.

But if $\phi_n'(m)=0$ and $\phi_{n-1}(m)=0$, then $c\phi_n'(m)+\phi_{n-1}(m)=0$ becomes an identity. Now since $\phi_n'(m)=0$. Thus $\phi_n(m)$ has repeated values of m. Proceeding on the similar lines, corresponding to the repeated values of m we get two values of c say c_1 and c_2 from

$$\frac{c^2}{2!}\phi_n''(m)+c\,\phi_{n-1}''(m)+\phi_{n-2}(m)=0 \qquad \ldots(4.153)$$

provided $\phi_n''(m)\neq 0$; and so on.

Working rules to find the oblique asymptotes.

1. *To obtain $\phi_n(m)$ put $x=1$, $y=m$ in the highest degree terms. Equate it to zero and solve for m, and let m_1, m_2, ... be its real roots.*
2. *To obtain $\phi_{n-1}(m)$, put $x=1$, $y=m$ in the next lower degree terms. Similarly for $\phi_{n-2}(m)$, etc.*

3. *Find the values $c_1, c_2, \ldots$ corresponding to values $m_1, m_2, \ldots$ by using the relation* $c = \dfrac{\phi_{n-1}(m)}{\phi_n'(m)}$, *provided* $\phi_n'(m) \neq 0$, *the corresponding asymptotes are:* $y = m_1 x + c_1, \quad y = m_2 x + c_2, \ldots$
4. *If $\phi_n'(m) = 0$ for some value of m, but $\phi_{n-1}(m) \neq 0$, then corresponding to that value of m there is no asymptote.*
5. *If $\phi_n'(m) = 0 = \phi_{n-1}(m)$ for some value of m, that is, two roots of $\phi_n(m) = 0$ are equal, then the values of c are obtained from the equation,*

$$\frac{c^2}{2!}\phi_n''(m) + c\phi_{n-1}'(m) + \phi_{n-2}(m) = 0$$

This gives two values of c, provided $\phi_n''(m) \neq 0$ and the two parallel asymptotes, corresponding to this value of m, are obtained. However, if $\phi_n''(m) = 0$, we proceed further on similar lines as above.

Remark: Before finding the oblique asymptotes, find the asymptotes parallel to the axes, if any. If the number of asymptotes parallel to the axes is less than the degree of the equation of the curve only then proceed to find for the oblique asymptotes since *the total number of asymptotes of a curve cannot exceed the degree of the equation of the curve.*

Example 4.81: Find the oblique asymptotes, if any, to the curves

(i) $y = e^x$ (ii) $y = \sec x$.

Solution: (i) Let $y = mx + c$ be an oblique asymptote, then $m = \lim\limits_{x\to\infty} \dfrac{y}{x} = \lim\limits_{x\to\infty} \dfrac{e^x}{x} \to \infty$.

Therefore there exists no finite value of m hence the curve has no oblique asymptote.

(ii) In this case, $$m = \lim_{x\to\infty} \frac{y}{x} = \lim_{x\to\infty} \frac{\sec x}{x} = \lim_{x\to\infty} \frac{1}{x\cos x} = 0.$$

The asymptote corresponding to $m = 0$, if any, will be parallel to x-axis, which does not exist refer to Example 4.80(ii). Hence, the curve does not have any oblique asymptote.

Example 4.82: Find all the asymptotes to the parabola $y^2 - 4ax = 0$.

Solution: Since the coefficient of the highest degree terms both in x and y are constants, so the curve has no asymptotes parallel to any of the axes.

To find the oblique asymptotes put $x = 1, y = m$ in the highest degree term, and the next lower degree term, we obtain

$$\phi_2(m) = m^2, \quad \phi_1(m) = -4a$$

Now, $\phi_2(m) = 0$ gives $m^2 = 0$, or $m = 0, 0$

Also, $\phi_2'(m) = 2m$, thus $\phi_2'(m) = 0$, at $m = 0$

but $\phi_1(m) \neq 0$. Hence no oblique asymptote. Otherwise also there can't be any oblique asymptote corresponding to $m = 0$. Thus, the curve $y^2 - 4ax = 0$ has no asymptote.

Example 4.83: Find the asymptotes of the curve $y^3 - 6xy^2 + 11x^2y - 6x^3 + x + y = 0$.

Solution: Since the coefficients of y^3 and that of x^3, the highest degree terms in x and y are constants, therefore, the curve has no asymptote parallel to any of the axes.

To find the oblique asymptotes, put $x = 1$, $y = m$ in the highest degree terms and next lower degree terms, we get

$$\phi_3(m) = m^3 - 6m^2 + 11m - 6,\ \phi_2(m) = 0 \text{ and } \phi_1(m) = 1 + m.$$

Now, $\phi_3(m) = 0$, gives $m^3 - 6m^2 + 11m - 6 = 0$, or $(m-1)(m-2)(m-3) = 0$, or $m = 1, 2, 3$.

Next, c is given by $$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{0}{3m^2 - 12m + 11} = 0,$$

as $\phi_2(m) = 0$ for $m = 1, 2, 3$ and $\phi_3'(m) \neq 0$.

Therefore, the three asymptotes of the curve are $y = x$, $y = 2x$, and $y = 3x$.

Example 4.84: Find all the asymptotes of the curve

$$x^3 + 4x^2y + 4xy^2 + 5x^2 + 15xy + 10y^2 - 2y + 1 = 0.$$

Solution: Since, the coefficient of x^3, the highest degree term in x is constant, thus, no asymptote parallel to x-axis.

Next equating to zero the coefficient of y^2, the highest degree term in y, gives $4x + 10 = 0$, or $2x + 5 = 0$, gives the equation of the asymptote parallel to y-axis.

To find oblique asymptotes, put $x = 1$, $y = m$ in the third degree terms and the next lower degree terms, we have

$$\phi_3(m) = 1 + 4m + 4m^2,\quad \phi_2(m) = 5 + 15m + 10m^2,\ \phi_1(m) = -2m.$$

Now, $\phi_3(m) = 0$, gives $4m^2 + 4m + 1 = 0$, or $(2m+1)^2 = 0$, which gives, $m = -\frac{1}{2}, -\frac{1}{2}$.

Also, $\phi_3'(m) = 4 + 8m = 4(1 + 2m)$.

Next, $$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{5 + 15m + 10m^2}{4(1+2m)} = \frac{0}{0} \text{ at } m = -\frac{1}{2}.$$

Therefore, c is given by

$$\frac{c^2}{2!}\phi_3''(m) + c\,\phi_2'(m) + \phi_1(m) = 0. \qquad ...(4.154)$$

We have, $\phi_3''(m) = 8$, $\phi_2'(m) = 15 + 20m$ and $\phi_1(m) = -2m$

At $m = -\frac{1}{2}$, $\phi_3''(m) = 8$, $\phi_2'(m) = 5$ and $\phi_1(m) = 1$

Substituting in (4.154), gives

$$\frac{c^2}{2!}(8) + c(5) + 1 = 0, \text{ or } 4c^2 + 5c + 1 = 0, \text{ or } (4c+1)(c+1) = 0 \text{ or, } c = -1,\ -\frac{1}{4}$$

Thus the oblique asymptotes are: $y = -\frac{1}{2}x - 1, \quad y = -\frac{1}{2}x - \frac{1}{4}$.

or,
$$x + 2y + 2 = 0, \quad 2x + 4y + 1 = 0$$

Hence, the asymptotes are: $2x + 5 = 0, \quad x + 2y + 2 = 0, \quad 2x + 4y + 1 = 0.$

Example 4.85: Find the asymptotes to the curve
$$y^4 - 2xy^3 + 2x^3y - x^4 - 3x^3 + 3x^2y + 3xy^2 - 3y^3 - 2x^2 + 2y^2 - 1 = 0.$$

Solution: Since, the coefficients of y^4 and x^4, the highest degree terms in x and y, are constants, so there are no asymptotes parallel to any of the axes.

To find oblique asymptotes, put $x = 1$and $y = m$ in the highest degree terms and the subsequent lower degree terms, we have

$$\phi_4(m) = m^4 - 2m^3 + 2m - 1,\ \phi_3(m) = -3m^3 + 3m^2 + 3m - 3,\ \phi_2(m) = 2(m^2 - 1), \text{ and } \phi_1(m) = 0.$$

$\phi_4(m) = 0$ gives $m^4 - 2m^3 + 2m - 1 = 0$. By inspection $m = 1$ is a root of this equation. Using synthetic division

1	1	−2	0	2	−1
		1	−1	−1	1
	1	−1	−1	1	0

The depressed equation is $m^3 - m^2 - m + 1 = 0$

Again $m = 1$ is its root. Using synthetic division,

1	1	−1	−1	1
		1	0	−1
	1	0	−1	0

The depressed equation is $m^2 - 1 = 0$ which gives $m = \pm 1$

Hence roots are: $m = 1, 1, 1$ and -1.

Also, $\phi_4'(m) = 4m^3 - 6m^2 + 2, \quad \phi_4''(m) = 12m^2 - 12m \quad \phi_4'''(m) = 24m - 12;$

$\phi_3'(m) = -9m^2 + 6m + 3, \quad \phi_3''(m) = -18m + 6$, and $\phi_2'(m) = 4m.$

Value of c of the asymptote with slope $m = -1$ is given by

$$c = \frac{-\phi_3(m)}{\phi_4'(m)} = -\frac{-3m^3 + 3m^2 + 3m - 3}{4m^3 - 6m^2 + 2} = -\frac{3 + 3 - 3 - 3}{-4 - 6 + 2} = 0.$$

Hence, equation of the asymptote is $y = -x$.

The values of c for the three parallel asymptotes corresponding to $m = 1$ are given by

$$\frac{c^3}{3!}\phi_4'''(m) + \frac{c^2}{2!}\phi_3''(m) + c\phi_2'(m) + \phi_1(m) = 0$$

or, $$\frac{c^3}{6}(24m-12)+\frac{c^2}{2}(-18m+6)+c(4m)+0=0$$

or, $$c[(4m-2)c^2+(-9m+3)c+4m]=0$$

Put $m=1$, we get $2c(c^2-3c+2)=0$, or $c(c-1)(c-2)=0$, or $c=0,1,2$.

Thus the three asymptotes for $m=1$ are $y=x$, $y=x+1$, and $y=x+2$.

Therefore, the four asymptotes are $y=-x$, $y=x$, $y=x+1$ and $y=x+2$.

EXERCISE 4.14

Find the asymptotes of the following curves:

1. $x^3-2y^3+xy(2x-y)+y(x-1)+1=0$
2. $x^3+2x^2y-xy^2-2y^3+4y^2+2xy+y-1=0$
3. $y^3-xy^2-x^2y+x^3+x^2-y^2-1=0$
4. $y^3-2xy^2-x^2y+2x^3+3y^2-7xy+2x^2+2y+2x+1=0$
5. $x^2(x-y)^2-a^2(x^2+y^2)=0$
6. $x^2(x-y)^2+a(x^2-y^2)-a^2xy=0$
7. $xy(x^2-y^2)(x^2-4y^2)+xy(x^2-y^2)+x^2+y^2-7=0$
8. $(2x-3y+1)^2(x+y)=8x-2y+9$
9. $(x+y)(x^4+y^4)-a(x^4+a^4)=0$
10. $(a_1x+b_1y+c_1)(a_2x+b_2x+c_2)+c_3=0$
11. Show that the asymptotes of the curve $x^2y^2=a^2(x^2+y^2)$ form a square of side $2a$.
12. Show that the asymptotes of the curve $x^2y^2-a^2(x^2+y^2)-a^3(x+y)+a^4=0$ form a square, through two of whose angular points the curve passes.
13. Find the asymptotes of the curve $x^2y^2-4(x-y)^2+2y-3=0$ and show that they form a square.

4.11.3 Some Further Results on Asymptotes

We state a few more results on asymptotes which are of vital importance.

1. *If the equation of a curve of degree n can be put in the form $F_n+F_{n-2}=0$, where F_n consists of n non-repeated linear factors and F_{n-2} is of degree $(n-2)$ at the most, then every linear factor of F_n equated to zero will give an asymptote, provided no two linear factors differ by a constant.*

For example, the equation of the curve $x^2y-xy^2+xy-y^2+x+y=0$ ...(4.155)

is of the form $F_n+F_{n-2}=0$, with $F_n=x^2y-xy^2+xy-y^2$, and $F_{n-2}=x+y$.

$F_n=0$ gives $x^2y-xy^2+xy-y^2=0$, or $y(x+1)(x-y)=0$.

Therefore, the asymptotes of the curve (4.155) are $y=0$, $x+1=0$, $x-y=0$.

This method of finding the asymptotes is called the *method of inspection*.

2. *Any asymptote of an algebraic curve of the nth degree cuts the curve in $(n-2)$ points. Thus, if a curve of degree n has n asymptotes then they all intersect the curve in $n(n-2)$ points.*

Further, *if the curve can be put in the form $F_n + F_{n-2} = 0$, then these $n(n-2)$ points of intersection, of the curve $F_n + F_{n-2} = 0$ and its asymptotes $F_n = 0$, lie on the curve $F_{n-2} = 0$.*

For example, for the curve (4.155) of degree 3, the three asymptotes cut it in $3(3-2) = 3$ points which lie on a curve of degree $(3-2) = 1$, that is, on a straight line. In this case it is $x + y = 0$; and it can be verified easily.

Example 4.86: Show that the points of intersection of the curve $xy(x^2 - y^2) + a^2y^2 + b^2x^2 - a^2b^2 = 0$ with its asymptotes lie on an ellipse.

Solution: The equation of the curve is

$$xy(x^2 - y^2) + a^2y^2 + b^2x^2 - a^2b^2 = 0 \qquad \text{...(4.156)}$$

Equating the coefficient of the highest power of x to zero, we get $y = 0$ as the asymptote parallel to x-axis.

Equating the coefficient of the highest power of y to zero, we get $x = 0$ as the asymptote parallel to y-axis.

To find the oblique asymptotes, put $x = 1$ and $y = m$ in the terms of degree 4 and the next lower degree terms, we have

$$\phi_4(m) = m(1 - m^2), \quad \phi_3(m) = 0, \quad \phi_2(m) = a^2m^2 + b^2.$$

Now, $\quad \phi_4(m) = 0$, or $\quad m(1 - m^2) = 0$, or $\quad m = 0, \pm 1$. Also, $\phi_4'(m) = 1 - 3m^2$.

The value of c is given by $c = \dfrac{\phi_3(m)}{\phi_4'(m)} = 0$, since $\phi_3(m) = 0$ and $\phi_4'(m) \neq 0$, for $m = 0, \pm 1$. Therefore, the asymptotes are: $y = 0,\ y = x,\ y = -x$.

Thus, the four asymptotes of the curve (4.156) are

$$x = 0, \quad y = 0, \quad x + y = 0, \text{ and } \quad x - y = 0.$$

These asymptotes cut the curve in $4(4-2) = 8$ points.

To find the curve on which these eight points will lie, first we find the joint equation of the asymptotes, which is

$$xy(x + y)(x - y) = 0, \text{ or } \quad xy(x^2 - y^2) = 0. \qquad \text{...(4.157)}$$

Subtracting (4.157) from (4.156), we obtain

$$a^2y^2 + b^2x^2 - a^2b^2 = 0, \text{ or } \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

an ellipse, as the curve on which points of intersection lie.

Alternatively, the equation of the given curve is

$$xy(x^2 - y^2) + a^2y^2 + b^2x^2 - a^2b^2 = 0$$

It is of the form $F_n + F_{n-2} = 0$, where $\quad F_n = xy(x^2 - y^2)$, and $\quad F_{n-2} = a^2y^2 + b^2x^2 - a^2b^2$.

$F_n = 0$ gives $xy(x^2 - y^2) = 0$, or $x = 0$, $y = 0$, $x - y = 0$, and $x + y = 0$
as the asymptotes to the given curve.

The eight points of intersection of these four asymptotes with the given curve lie on the conic $F_{n-2} = 0$, that is, on $a^2y^2 + b^2x^2 - a^2b^2 = 0$, or $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$
which is an ellipse.

Example 4.87: Find the equation of a quartic which has $x = 0$, $y = 0$, $y = x$, and $y = -x$ as its asymptotes and passes through a point (a, b) and it cuts its asymptotes again in eight points that lie on a circle $x^2 + y^2 = a^2$.

Solution: The joint equation of the asymptotes is

$$F_4 = xy(y - x)(y + x) = 0 \qquad ...(4.158)$$

Therefore the equation of the quartic is of the form

$$F_4 + F_2 = 0, \qquad ..(4.159)$$

where F_2 is of at the most of degree 2.

Also the curve of points of intersection of (4.158) and (4.159) is $x^2 + y^2 = a^2$, therefore (4.159) will be of the form

$$xy(y - x)(y + x) + k(x^2 + y^2 - a^2) = 0, \qquad ...(4.160)$$

where k is a constant.

Since it passes through the point (a, b), thus

$$ab(b - a)(b + a) + k(a^2 + b^2 - a^2) = 0, \text{ or } k = \frac{a(a^2 - b^2)}{b}.$$

Substituting the value of k in (4.160), the equation of the required quartic is

$$xy(y - x)(y + x) + \frac{a(a^2 - b^2)}{b}(x^2 + y^2 - a^2) = 0$$

or, $$bxy(y^2 - x^2) + a(a^2 - b^2)(x^2 + y^2 - a^2) = 0$$

Example 4.88: Find the equation of a cubic which has the same asymptotes as the curve $x^3 - 6x^2y + 11xy^2 - 6y^3 + 4x + 5y + 7 = 0$ and which passes through the points $(0, 0)$, $(0, 2)$ and $(2, 0)$.

Solution: The equation of the given curve is

$$x^3 - 6x^2y + 11xy^2 - 6y^3 + 4x + 5y + 7 = 0$$

or, $$(x - y)(x - 2y)(x - 3y) + (4x + 5y + 7) = 0$$

which is of the form $F_3 + F_1 = 0$.

Thus the asymptotes are given by $F_3 = 0$.

or, $$(x - y)(x - 2y)(x - 3y) = 0$$

or, $$x - y = 0, \quad x - 2y = 0, \text{ and } \quad x - 3y = 0.$$

Now the equation of cubic which has three asymptotes given by $F_3 = 0$ is $F_3 + F_1 = 0$, where F_1 is at most of degree 1. So in this case cubic will be of the form

$$(x - y)(x - 2y)(x - 3y) + ax + by + c = 0 \qquad ...(4.161)$$

where a, b, c are arbitrary constants.

Since, this cubic passes through (0, 0), (0, 2) and (2, 0), therefore, these points must satisfy (4.161), which give $a = -4$, $b = 24$ and $c = 0$. Substituting for a, b and c in (4.161), we obtain

$$(x - y)(x - 2y)(x - 3y) - 4x + 24y = 0$$

or, $$x^3 - 6x^2y + 11xy^2 - 6y^3 - 4x + 24y = 0$$

as the required cubic.

EXERCISE 4.15

1. Show that the points of intersection of the curve
$$2y^3 - 2x^2y - 4xy^2 + 4x^3 - 14xy + 6y^2 + 4x^2 + 6y + 1 = 0$$
and its asymptotes lie on the straight line $8x + 2y + 1 = 0$.
2. Find the asymptotes of the curve $4(x^4 + y^4) - 17x^2y^2 - 4x(4y^2 - x^2) + 2(x^2 - 2) = 0$ and show that they pass through the points of intersection of the curve with the ellipse $x^2 + 4y^2 = 4$.
3. Show that asymptotes of the curve $xy(x^2 - y^2) + 9x^2 + 25y^2 = 114$ cut the curve in eight points which lie on an ellipse of eccentricity 4/5.
4. Show that the asymptotes of the curve
$$(x^2 - y^2)(y^2 - 4x^2) + 6x^3 - 5x^2y - 3xy^2 + 2y^3 - x^2 + 3xy - 1 = 0$$
cut the curve again in eight points which lie on a circle of radius unity.
5. Find the equation of the cubic which has the same asymptotes as the curve $x^3 - 6x^2y + 11xy^2 - 6y^3 + x + y + 1 = 0$ and which touches the axis of y at the origin and passes through the point (3, 2).
6. Find the equation of the cubic which has the same asymptotes as the curve $x^3 - 6x^2y + 11xy^2 - 6y^3 + x + y + 1 = 0$ and which passes through the points (0, 0), (1, 0) and (0, 1).
7. Find the equation of the conic which has the same asymptotes as the conic $3x^2 - 2xy - 5y^2 + 7x - 9y = 0$ and passes through the point (2, 2).

4.11.4 Asymptotes of Polar Curves

To find the asymptotes of polar curves we have the following result:

If θ_1 is the root of the equation $f(\theta) = 0$, then equation of the asymptote of the curve $1/r = f(\theta)$ is $r \sin(\theta - \theta_1) = 1/f'(\theta_1)$, provided $f'(\theta_1) \neq 0$.

Working rules to find the asymptotes of polar curves:

1. *Write the equation of the curve in the form $(1/r) = f(\theta)$ and change all the T-ratios, if any, into $\sin\theta$ and $\cos\theta$.*
2. *Solve the equation $f(\theta) = 0$. Let the roots be $\theta = \theta_1, \theta_2$ etc.*

3. *Find $f'(\theta)$ and calculate $f'(\theta)$ for $\theta = \theta_1, \theta_2$, etc.*
4. *The asymptote corresponding to $\theta = \theta_i$ is $r \sin(\theta - \theta_i) = 1/f'(\theta_i)$ provided $f'(\theta_i) \neq 0, i = 1, 2,$ etc.*

Example 4.89: Find the asymptotes of the curve $r\theta = a$.

Solution: The equation of the curve is $\frac{1}{r} = \frac{\theta}{a} = f(\theta)$, say.

Here, $f(\theta) = 0$ gives, $\frac{\theta}{a} = 0$, or $\theta = 0$.

Also, $f'(\theta) = \frac{1}{a}$, therefore $f'(0) = \frac{1}{a}$.

Thus equation of the asymptote is

$$r \sin(\theta - 0) = a, \text{ that is } a = r\sin\theta.$$

Example 4.90: Prove that the curve $r = \frac{a}{1-\cos\theta}$ has no asymptotes.

Solution: The equation of the curve is

$$\frac{1}{r} = \frac{1-\cos\theta}{a} = f(\theta), \quad \text{say.}$$

Here, $f(\theta) = 0$, gives $\frac{1-\cos\theta}{a} = 0$,

or, $\cos\theta = 1$, or $\theta = 2n\pi$,

where n is an integer

Further $f'(\theta) = \frac{\sin\theta}{a}$, gives $f'(2n\pi) = \frac{\sin(2n\pi)}{a} = 0$.

The equation of the asymptote corresponding to $\theta = \theta_i$ is $r\sin(\theta - \theta_i) = 1/f'(\theta_i)$ and since for the given curve $f'(\theta_i) = 0$ at $\theta_1 = 2n\pi$, so the curve has no asymptote.

Example 4.91: Find the asymptotes of the curve $r = a\sec\theta + b\tan\theta$.

Solution: The equation of the curves is $\frac{1}{r} = \frac{1}{a\sec\theta + b\tan\theta} = \frac{\cos\theta}{a + b\sin\theta} = f(\theta)$, say.

Here, $f(\theta) = 0$ gives $\cos\theta = 0$, that is, $\theta = (2n+1)\frac{\pi}{2}$, where n is an integer.

Also, $f'(\theta) = \frac{-\sin\theta(a + b\sin\theta) - b\cos\theta\cos\theta}{(a+b\sin\theta)^2}$. Futher

$$\sin(2n+1)\frac{\pi}{2} = \sin\left(\frac{\pi}{2} + n\pi\right) = \cos n\pi = (-1)^n, \text{ and } \cos(2n+1)\frac{\pi}{2} = \cos\left(\frac{\pi}{2} + n\pi\right) = -\sin n\pi = 0.$$

Therefore, $f'\left((2n+1)\frac{\pi}{2}\right) = \frac{(-1)^{n+1}}{[a+b(-1)^n]}$.

The equation of the asymptote corresponding to root $\theta = \theta_1$ is, $r\sin(\theta - \theta_i) = \frac{1}{f'(\theta)}$; it becomes

$$r\sin\left(\theta - (2n+1)\frac{\pi}{2}\right) = \frac{a+(-1)^n b}{(-1)^{n+1}}, \text{ or } -r\sin\left(n\pi + \frac{\pi}{2} - \theta\right) = \frac{a+(-1)^n b}{(-1)^{n+1}}, \text{ or } (-1)^{n+1} r\cos\theta = \frac{a+(-1)^n b}{(-1)^{n+1}}$$

or, $r\cos\theta = a + (-1)^n b$.

If n is odd, asymptote is $r\cos\theta = a - b$. If n is even, asymptote is $r\cos\theta = a + b$.

Hence, the curve has two asymptotes, $r\cos\theta = a \pm b$.

Circular asymptotes: *Let the equation of the curve be $r = f(\theta)$. If $\lim_{\theta\to\infty} f(\theta) = a$, then the circle $r = a$ is called the circular asymptote of the curve $r = r(\theta)$.*

Example 4.92: Find the circular asymptote of the curves

(i) $r = \frac{a\theta}{\theta - 1}$ (ii) $r(\theta + \sin\theta) = 2\theta + \cos\theta$.

Solution: (i) The equation of the given curve is $r = \frac{a\theta}{\theta - 1} = f(\theta)$, say.

Consider
$$\lim_{\theta\to\infty} f(\theta) = \lim_{\theta\to\infty} \frac{a\theta}{\theta - 1} = \lim_{\theta\to\infty} \frac{a}{1 - \frac{1}{\theta}} = a.$$

Hence the required circular asymptote is $r = a$

(ii) The equation of the given curve is $r = \frac{2\theta + \cos\theta}{\theta + \sin\theta} = f(\theta)$, say.

Here,
$$\lim_{\theta\to\infty} f(\theta) = \lim_{\theta\to\infty} \frac{2\theta + \cos\theta}{\theta + \sin\theta} = \lim_{\theta\to\infty} \frac{2 + \frac{\cos\theta}{\theta}}{1 + \frac{\sin\theta}{\theta}} = \frac{2+0}{1+0} = 2.$$

Hence, the required circular asymptote is $r = 2$.

EXERCISE 4.16

Find the asymptotes of the polar curves

1. $\frac{2}{r} = 1 + 2\sin\theta$
2. $r = 4(\sec\theta + \tan\theta)$
3. $r\theta\cos\theta = a\cos 2\theta$
4. $r = a\,\text{cosec}\,\theta + b$

5. $r \ln \theta = a$

6. $r = \dfrac{3a \sin\theta \cos\theta}{\sin^3\theta + \cos^3\theta}$

7. Show that the curve $r^n \sinh \theta = a^n$ has no asymptotes.

Find the circular asymptotes of the curves **(8-10)**

8. $r(e^\theta - 1) = a(e^\theta + 1)$

9. $r(2\theta^2 + \theta + 1) = 3\theta^2 + 2\theta + 1$

10. $r \cos\theta = a \sin\theta$.

4.12 CURVE TRACING

In applications such as computing the areas, length, volumes of solids of revolution, and surfaces of solids of revolution, it is very useful to know the shape of the curve represented by the given equation. The knowledge of curve tracing helps us to obtain the rough shape of the curve without actual plotting a large number of points on it. We shall study tracing of curves in cartesian, polar and parametric form.

4.12.1 Tracing of Cartesian Curves

While tracing curve of cartesian form we need to observe the following characteristics of the curve from the equation given.

1. Symmetry: (a) If only even powers of y occur in the equation, then the curve is *symmetrical about x-axis*. In this case the equation of the curve remains unchanged if y is changed to $-y$. For example, the curve $y^2 = 4ax$ is symmetrical about x-axis, refer to Fig. 4.24a.

(b) If only even powers of x occur in the equation, then the curve is *symmetrical about y-axis*. In this case the equation of the curve remains unchanged if x is changed to $-x$. For example, the curve $x^2 = 4ay$ is symmetrical about y-axis, refer to Fig. 4.24b.

(c) If the equation of the curve contains even powers in both of x and y, then the curve is *symmetrical about both the axes*. For example, the circle $x^2 + y^2 = a^2$, refer to Fig. 4.24c.

(d) If the equation of the curve remains unchanged when both x and y are changed to $-x$ and $-y$ respectively, then the curve is *symmetrical in opposite quadrants*. This is also called the *symmetry about the origin*. For example, the curve $xy = c^2$ is symmetrical in opposite quadrants, refer to Fig. 4.24d. We note that symmetry about both the axes implies symmetry about origin also but the converse is not true.

(e) If the equation of the curve remains unchanged when x and y are interchanged, then the curve is *symmetrical about the line* $y = x$. For example, the curve $x^3 + y^3 = 3axy$ is symmetric about the line $y = x$, refer to Fig. 4.24e.

2. *Passing through the origin; tangents at the origin:* (a) If there is no constant term in the equation of the curve, then the curve passes through the origin. For example, the curve $y^2 = 4ax$ passes through $(0, 0)$.

(b) If the curve passes through the origin, to find the equation (s) of the tangent (s) to the curve at the origin, equate to zero the lowest degree terms in the equation of the curve. For example, $x = 0$, that is, y-axis is tangent to the curve $y^2 = 4ax$ at $(0, 0)$, refer to Fig. 4.24a.

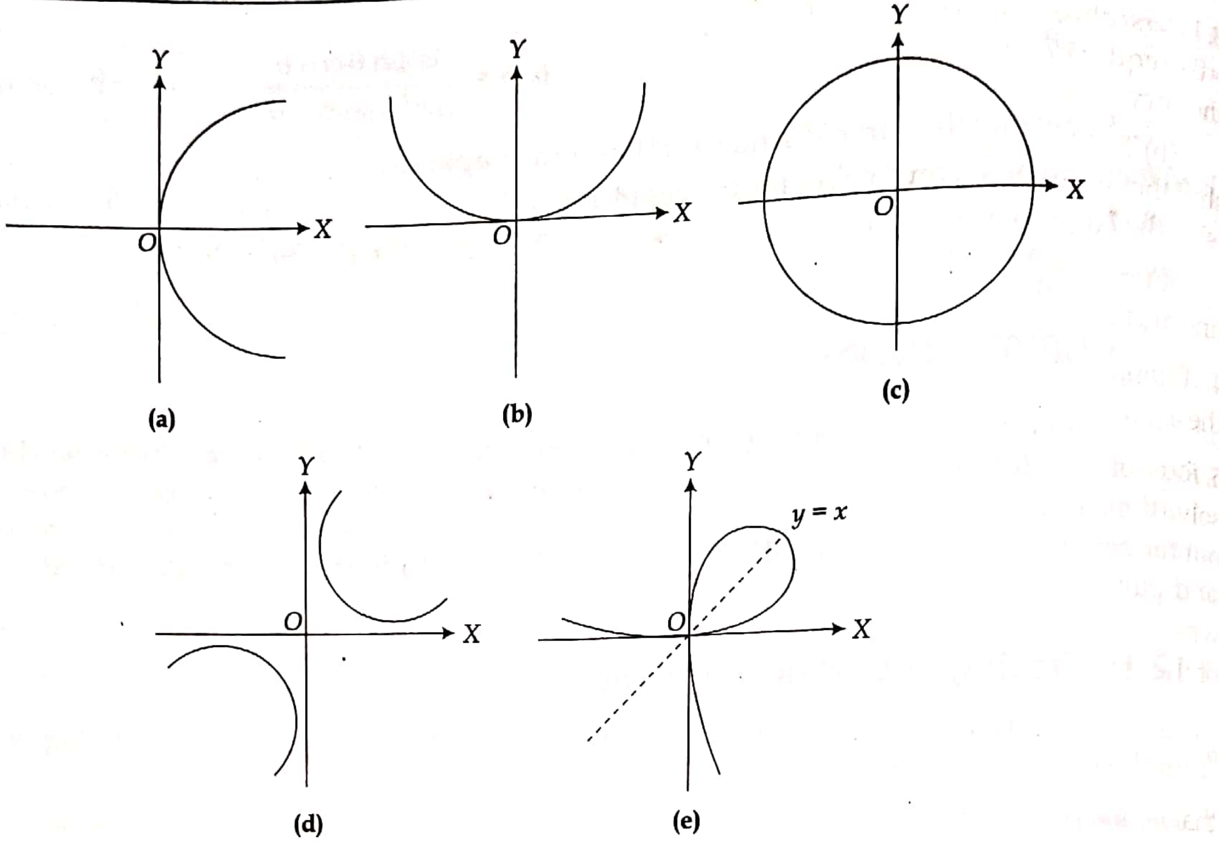


Fig. 4.24

(c) If there is only one tangent, find whether the curve lies below or above the tangent in the neighbourhood of the origin.

(d) If there are two or more tangents to the curve at the origin then the origin is a *multiple point*. In particular, if there are two tangents, the origin is called a *double point*. Further, the origin is called *a node, a cusp* or an *isolated point* according as the two tangents there are real and distinct, real and coincident, or imaginary. For example, in the curve $y^2(a - x) = x^2(a + x)$ as shown in Fig. 4.25a, the origin is a node; in the curve $y^2 - x^3 = 0$ as shown in Fig. 4.25b the origin is a cusp; and in the curve $x^2y^2 = a^2(x^2 + y^2)$ the origin is an isolated point as shown in Fig. 4.34, (refer to p. 277).

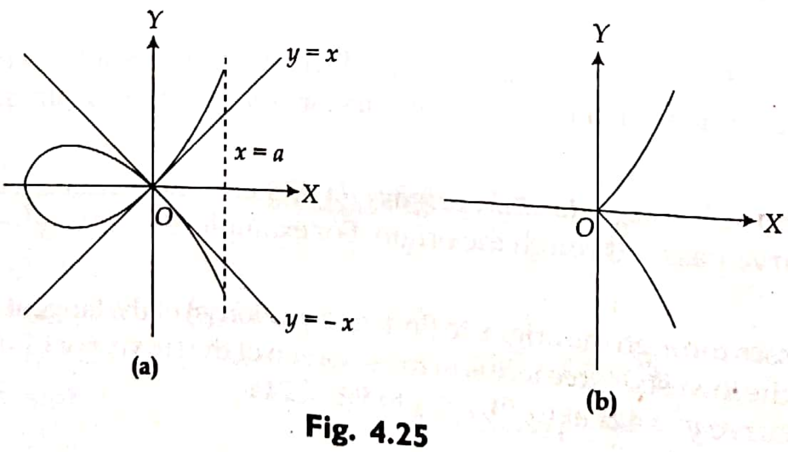


Fig. 4.25

3. ***Intersection with the co-ordinate axes:*** (a) To find the points where the curve cuts the x-axis, put $y = 0$ in equation of the curve and solve the resulting equation for x. Similarly, to find the points where the curve cuts the y-axis, put $x = 0$ in the equation of the curve and solve the resulting equation for y.

(b) To find the tangents to the curve at its point of intersection with the co-ordinate axes, we first shift the origin to this point and then by equating to zero the lowest degree term, we find the equation (s) of the tangent (s).

(c) If $y = x$, or $y = -x$ is a line of symmetry, then find the points of intersection of the curve and the line and also the tangents at that point.

4. ***Asymptotes:*** Find the equations of all the asymptotes parallel to the axes and also of oblique asymptotes as discussed in Section 4.11.

5. ***Regions in which the curve does not exist:*** To obtain such a region, solve the given equation for one variable in terms of the other and find out the set of values of one variable which make the other imaginary or undefined. For example in the curve $y^2(1 - x) = x^3$, y is imaginary when $x < 0$ and when $x > 1$ hence the curve does not exist on the right of the line $x = 1$ and on the left of y-axis as shown in Fig. 4.26.

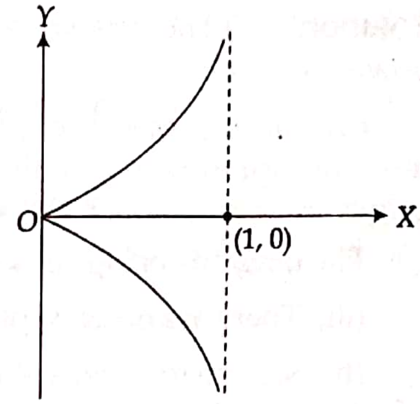


Fig. 4.26

6. ***The sign of*** $\frac{dy}{dx}$: Find the points on the curve where $\frac{dy}{dx} = 0$, or ∞, that is, the points where the tangents are parallel to x-axis or perpendicular to the x-axis. Also find the intervals where $\frac{dy}{dx}$ remains positive throughout, or negative throughout.

Find the points of maxima and minima of the curve and the points of inflexion, if any, by finding the value of $\frac{d^2y}{dx^2}$ at the points where $\frac{dy}{dx} = 0$. Also find the interval where $\frac{d^2y}{dx^2} > 0$, so the curve is concave upward there and interval where $\frac{d^2y}{dx^2} < 0$, so the curve is concave downward there, refer to Section 4.8.5.

7. Find the co-ordinates of some special points if necessary, which help in tracing out the curve. Sometimes it helps to convert the equation from the cartesian to polar form.

Taking all these observations in consideration we can draw the approximate shape of the curve as worked out in examples to follow.

Example 4.93: Trace the cissoid $y^2(2a - x) = x^3$.

Solution: (i) The curve is symmetrical about the x-axis, since only even powers of y occur.

(ii) The curve passes through the origin, since there is no constant term in its equation. Equating to zero the lowest degree terms, the tangents at the origin are $y = 0$, $y = 0$. Tangent being coincident, therefore origin is cusp.

(iii) The curve has an asymptote $x = 2a$, obtained by equating to zero the coefficient of y^2.

(iv) Also the curve meets the axes at (0, 0) only.

Further from the equation of the curve $y^2 = x^3/(2a - x)$.

When x is $-ve$, y^2 is $-ve$, so that no portion of the curve lies to the left of the y-axis. Also when $x > 2a$, y^2 is again $-ve$, so that no portion of the curve lies to the right of the line $3a = 2a$. The approximate shape of the curve is shown in Fig. 4.27.

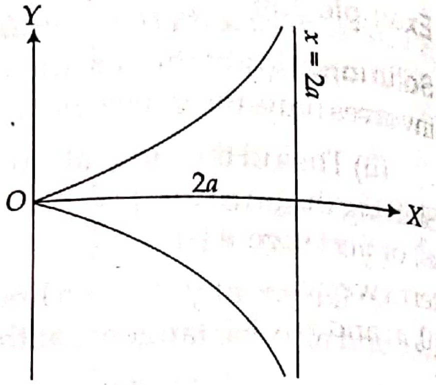


Fig. 4.27

Example 4.94: Trace the semi-cubical parabola $y^2 = x^3$.

Solution: (i) The curve is symmetrical about x-axis, since only even power of y occurs.

(ii) Curve passes through the origin, since there is no constant term. By equating to zero the lowest degree terms the tangent at the origin are $y^2 = 0$, which gives, $y = 0, y = 0$.

The tangents being coincident, there is a cusp at the origin.

(iii) There are no asymptotes parallel to either axis.

(iv) Solving for y, we get $y = \pm(x)^{3/2}$. When x is negative, the value of y becomes imaginary. Hence, the curve does not lie on the left of y-axis.

As x increases from 0 to ∞, y also increases from 0 to ∞. The approximate shape of the curve is shown in Fig. 4.28.

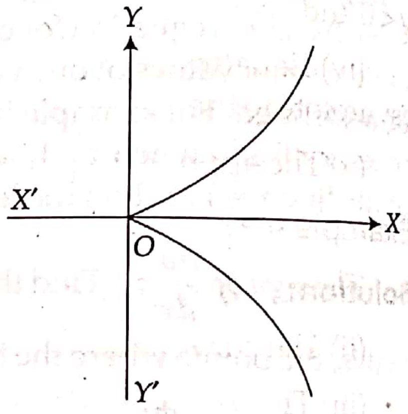


Fig. 4.28

Example 4.95: Trace the folium of Descartes $x^3 + y^3 = 3axy$.

Solution: (i) The curve is symmetrical about the line $y = x$, since the equation remains unchanged when x and y are interchanged.

(ii) It passes through the origin and tangents at the origin are given by $xy = 0$, that is $x = 0, y = 0$, thus origin is a node.

(iii) It has no asymptote parallel to the axes, since the coefficient of the highest powers of x and y in the equation are constant.

To find oblique asymptotes put $y = m, x = 1$ in third degree terms, we obtain

$$\phi_3(m) = 1 + m^3, \quad \phi_3(m) = 0 \text{ gives } \quad m = -1. \text{ Also } \phi_2(m) = -3\,am.$$

Therefore, $$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\left(\frac{-3am}{3m^2}\right) = \frac{a}{m} = -a, \text{ when } m = -1.$$

Hence $y = -x - a$ is an asymptote.

(iv) It meets the axes at the origin only.

Also when $y = x$, $2x^3 = 3ax^2$ which gives $x = 0$ or $3a/2$.

Thus the curve crosses the line $y = x$ at $(3a/2, 3a/2)$.

The approximate shape of the curve is shown in Fig. 4.29.

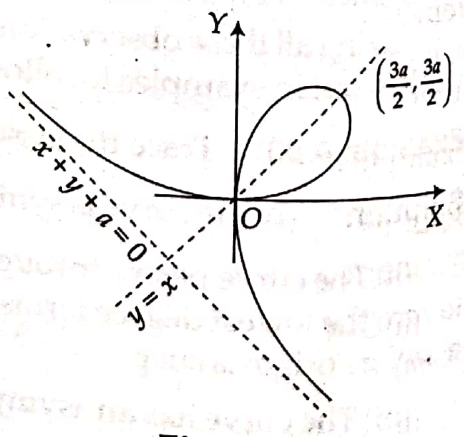


Fig. 4.29

Example 4.96: Trace the curve $y(x^2 + a^2) = a^3$.

Solution: (i) The curve is symmetrical about y-axis, since it involves only even power in x.

(ii) The curve does not meet x-axis but meets y-axis at $(0, a)$. Shifting the origin to $(0, a)$ the equation becomes $(y + a)(x^2 + a^2) = a^3$, or $y(x^2 + a^2) + ax^2 = 0$ and equating to zero the lowest degree term we get $y = 0$, as the equation of the tangent at the new origin $(0, a)$ and w.r.t. $O(0, 0)$ it is $y = a$.

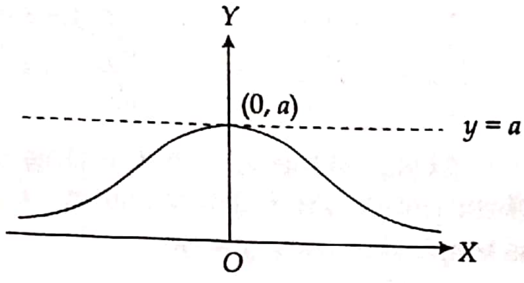


Fig. 4.30

(iii) Solving the given equation for x, we get $x = \pm a\sqrt{\dfrac{a-y}{y}}$ which shows that x is imaginary when $y < 0$ and also when $y > a$. Thus there is no curve below x-axis and above the line $y = a$.

(iv) Equating to zero the coefficient of the highest degree term in x we get $y = 0$, thus x-axis is asymptote to the curve.

(v) The approximate shape of the curve is shown in Fig. 4.30.

Example 4.97: Trace the curve $y(x^2 - 1) = x^2 + 1$.

Solution: (i) Since there are only even powers in x, so the curve is symmetrical about y-axis.

(ii) The curve does not pass through the origin.

(iii) The y-axis cuts the curve at $(0, -1)$. Shifting the origin to $(0, -1)$, the equation of the curve becomes

$$(y - 1)(x^2 - 1) = x^2 + 1 \text{ or, } yx^2 - 2 - 2x^2 = 0.$$

Now equating to zero, the lowest degree terms, we get the equation of the tangent at the new origin $(0, -1)$ as $y = 0$ and *w.r.t.* old origin as $y = -1$.

(iv) Equating to zero, the coefficients of the highest degree terms in x and y respectively, we get the asymptotes parallel to the axes as $y - 1 = 0$ and $x^2 - 1 = 0$ i.e., $x = 1, x = -1$.

(v) Rewriting as $y = [(x^2 + 1)/(x^2 - 1)]$. As x varies from 0 to 1, y varies from -1 to ∞ and as x varies from 1 to ∞, y varies from ∞ to 1. The approximate shape of the curve is shown in Fig. 4.31.

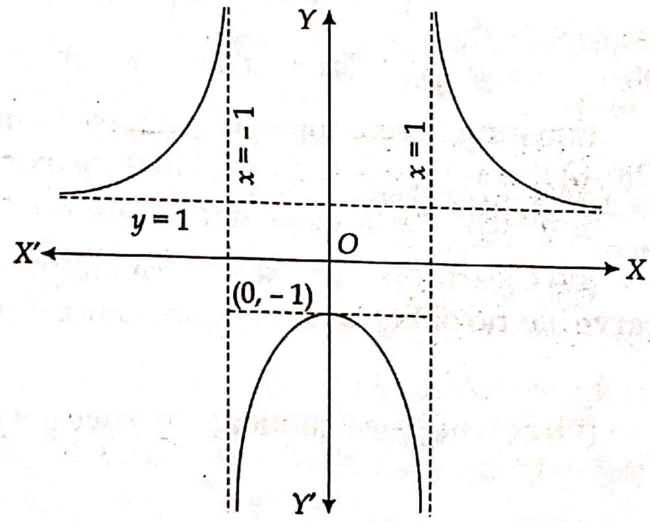


Fig. 4.31

Example 4.98: Trace the asteroid $x^{2/3} + y^{2/3} = a^{2/3}$.

Solution: (i) Writing the equation of the curve as $(x^{1/3})^2 + (y^{1/3})^2 = a^{2/3}$. It is clear that the curve is symmetrical about x-axis and y-axis both.

(ii) The curve does not pass through the origin but meets the x-axis at $(\pm a, 0)$ and the y-axis at $(0, \pm a)$.

(iii) Differentiating the given equation, we get

$$\frac{dy}{dx} = -\left(\frac{y}{x}\right)^{1/3}$$

At $(\pm a, 0)$ the slope of the tangent is 0 hence x-axis is the tangent at these points. At $(0, \pm a)$ the slope of the tangent is infinity, hence y-axis is tangent at these points.

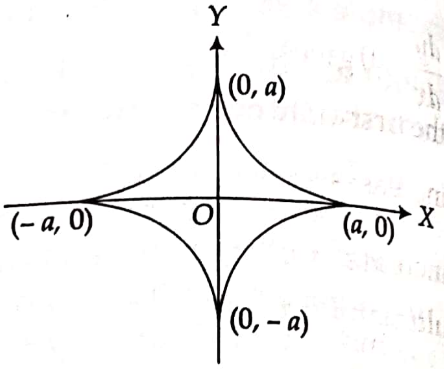


Fig. 4.32

(iv) Writing the equation in parametric form as $x = a\cos^3 t$, $y = a\sin^3 t$ which shows that $-a \le x \le a$ and $-a \le y \le a$. Therefore no point of the curve lies outside the square $x = \pm a, y \pm a$.

(v) The approximate shape of the curve is shown in Fig. 4.32.

Example 4.99: Trace the Lemniscate of Bernoulli $y^2(a^2 + x^2) = x^2(a^2 - x^2)$.

Solution: (i) Since the equation contains even powers both in x and y, the curve is symmetrical about both the axes.

(ii) The equation does not contain any constant term, therefore the curve passes through the origin. Equating to zero the lowest degree term; the tangents at the origin are given by $y^2 = x^2 \Rightarrow y = \pm x$, which being real and distinct, therefore origin is a node.

(iii) The curve meets the x-axis at $O(0, 0)$, $A(a, 0)$ and $A'(-a, 0)$.

Shifting the origin to the point $A(a, 0)$, the equation becomes

$$y^2[a^2 + (x + a)^2] = (x + a)^2[a^2 - (x + a)^2]$$

or,
$$y^2(x^2 + 2ax + 2a^2) = -(x + a)^2(x^2 + 2ax)$$

Equating to zero the lowest degree terms we get the tangent at the new origin $(a, 0)$ as $x = 0$, w.r.t. $O(0, 0)$ it is $x = a$, a line parallel to the y-axis.

Similarly, the tangent at the point $A'(-a, 0)$ is $x = -a$, again a line parallel to y-axis.

(iv) Clearly the curve has no asymptotes parallel to the axes. Also we can show very easily that the curve has no oblique asymptotes, (since $\phi_4(m) = 1 + m^2$ and $\phi_4(m) = 0$ gives imaginary values of m).

(vi) Solving the equations for y, we get $y = \pm x\sqrt{\dfrac{a^2 - x^2}{a^2 + x^2}}$, y is real only for $-a \le x \le a$.

Therefore the curve lies between the lines $x = \pm a$.

In the first quadrant $y = x\sqrt{\dfrac{a^2 - x^2}{a^2 + x^2}}$, therefore

$$\frac{dy}{dx} = \left(\frac{a^2 - x^2}{a^2 + x^2}\right)^{\frac{1}{2}}\left[\frac{a^4 - x^4 - 2a^2x^2}{(a^2 - x^2)(a^2 + x^2)}\right] = \frac{a^4 - x^4 - 2a^2x^2}{(a^2 - x^2)^{1/2}(a^2 + x^2)^{3/2}}.$$

$\frac{dy}{dx}=0$ gives $a^4-x^4-2a^2x^2=0$, or $x=a\sqrt{\sqrt{2}-1}$. Thus, in the first quadrant when $x=0, y=0$; as x increases, y also increases and maxima reaches at $x=a\sqrt{\sqrt{2}-1}$; as x increases further from $a\sqrt{\sqrt{2}-1}$ to a, y decreases and ultimately becomes zero when $x=a$. The maximum value of y at $x=a\sqrt{\sqrt{2}-1}$ is $a(\sqrt{2}-1)$.

The approximate shape of the curve is shown in Fig. 4.33.

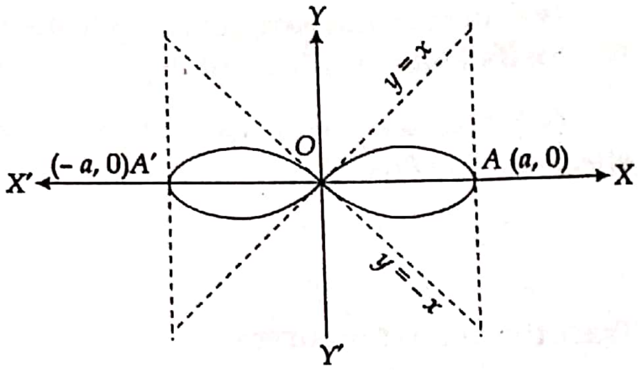


Fig. 4.33

Example 4.100: Trace the curve $x^2y^2=a^2(x^2+y^2)$.

Solution: (i) The curve is symmetrical about x-axis and y-axis both. Also it is symmetrical about the line $y=x$. Curve is symmetrical in opposite quadrants also.

(ii) The point (0, 0) lies on the curve and the tangents at the origin are given by $x^2+y^2=0$. But these are the imaginary, so the origin is an isolated point.

(iii) Equating to zero the coefficients of highest powers of x and y respectively, we get the asymptotes as $x=\pm a$ and $y=\pm a$.

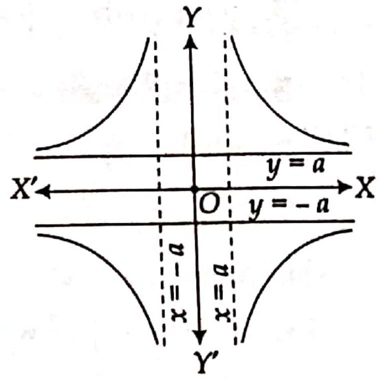


Fig. 4.34

(iv) Solving for y, we have $y^2=\frac{a^2x^2}{(x^2-a^2)}$ which gives $y=\pm\frac{ax}{\sqrt{(x^2-a^2)}}$.

When $0<x<a$, y is imaginary. It shows that the curve does not exist in the region bounded by the lines $x=0$ and $x=a$. As x varies from a to ∞, y varies from ∞ to a. The approximate shape of the curve is shown in Fig. 4.34.

Example 4.101: Trace strophoid $y^2(x+a)=x^2(3a-x)$.

Solution: (i) The curve is symmetrical about x–axis.

(ii) The curve passes through the origin. Equating to zero the lowest degree term, we get $ay^2=3ax^2$ or $y=\pm x\sqrt{3}$ as two tangents to the curve at the origin.

(iii) The curve meets y-axis only at the origin while it meets x-axis at $(3a, 0)$ also. Shifting the origin to this point the equation of the curve becomes

$$y^2(x+4a)=-(x+3a)^2x.$$

Equating to zero the lowest degree term we get $x=0$ as the tangent at the new origin $(3a, 0)$ and w.r.t. $O(0,0)$, it is $x=3a$. Thus, at $(3a, 0)$ tangent is $x=3a$.

(iv) y is imaginary for $x<-a$ and for $x>3a$, therefore, there is no portion of the curve on the left of the line $x=-a$ and on the right of the line $x=3a$.

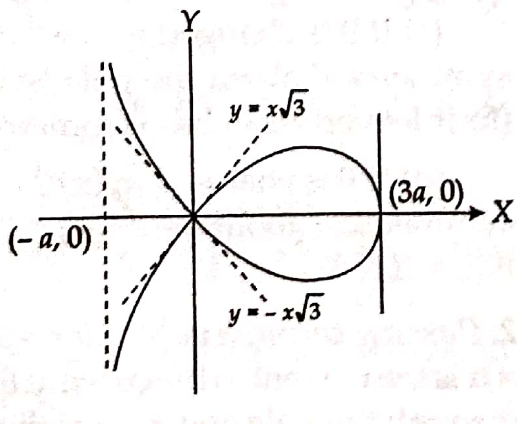


Fig. 4.35

(v) There is no asymptote parallel to x-axis, and for asymptote parallel to y-axis, we equate to zero the coefficient of the highest degree term in y so as to get $x + a = 0$ as the asymptote parallel to y-axis.

(vi) Also we can check that y is maximum at $x = a\sqrt{3}$. The approximate shape of the curve is shown in the Fig. 4.35.

EXERCISE 4.17

Trace the following curves

1. $y^2 a = x^3 + y^2 x, \quad a > 0$ (cissoid)
2. $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$ (astroid)
3. $y = c\cosh\frac{x}{c}$ (catenary)
4. $x^2y^2 = a^2(y^2 - x^2)$
5. $x^3 + y^3 = a^2x$
6. $4ay^2 = x(x - 2a)^2, \quad a > 0$
7. $x(x^2 + y^2) = a(x^2 - y^2)$ (strophoid)
8. $y^2x = a^2(x - a), \quad a > 0$
9. $(x^2 - a^2)(y^2 - b^2) = a^2b^2, \quad a, b > 0$
10. $xy^2 + (x + a)(x + 2a) = 0, \quad a > 0$
11. $y = x^2/1 - x^2$
12. $y^3 = x(a^2 - x^2)$
13. $a^2/x^2 - b^2/y^2 = 1$
14. $y^2(x^2 + y^2) + a^2(x^2 - y^2) = 0$
15. $y^2(a + x) = (a - x)^3$.

4.12.2 Tracing of Polar Curves

Similar to the curves in cartesian co-ordinates, we need to observe the following characteristics of the polar curves from the equation given.

1. *Symmetry:* (a) If θ is changed to $-\theta$ and the equation of the curve remains unchanged, then the curve is symmetrical about the initial line $\theta = 0$, (that is x-axis). For example, cardiod $r = a(1 + \cos\theta)$ is symmetrical about the initial line, refer to Fig. 4.36a.

(b) If θ is changed to $(\pi - \theta)$ and the equation of the curve remains unchanged, then the curve is symmetrical about the line $\theta = \pi/2$ (that is, y-axis). For example, the cardiod $r = a(1 + \sin\theta)$ is symmetrical about the line $\theta = \pi/2$, refer to Fig. 4.36b.

(c) If θ is changed to $\pi + \theta$ and the equation of the curve remains unchanged, then the curve is symmetrical about the pole (symmetry in opposite quadrant). For example, the curve $r = a\sin 2\theta$, (four leaved rose) has symmetry about the pole, refer to Fig. 4.36c.

(d) If θ is changed to $(\pi/2 - \theta)$ and the equation of the curve remains unchanged, then curve is symmetrical about the line $\theta = \pi/4$, (that is, the line $y = x$). For example, the curve $r = a\sin 2\theta$ refer to Fig. 4.36c.

2. *Passing through pole:* If $r = 0$ for some $\theta = \alpha$, then the curve passes through the pole and the line $\theta = \alpha$ is the tangent to the curve at the pole. For example, at $\theta = \pi$, $r = a(1 + \cos\theta) = 0$. Thus the curve passes through the pole and $\theta = \pi$ is the tangent there as shown in Fig. 4.36a.

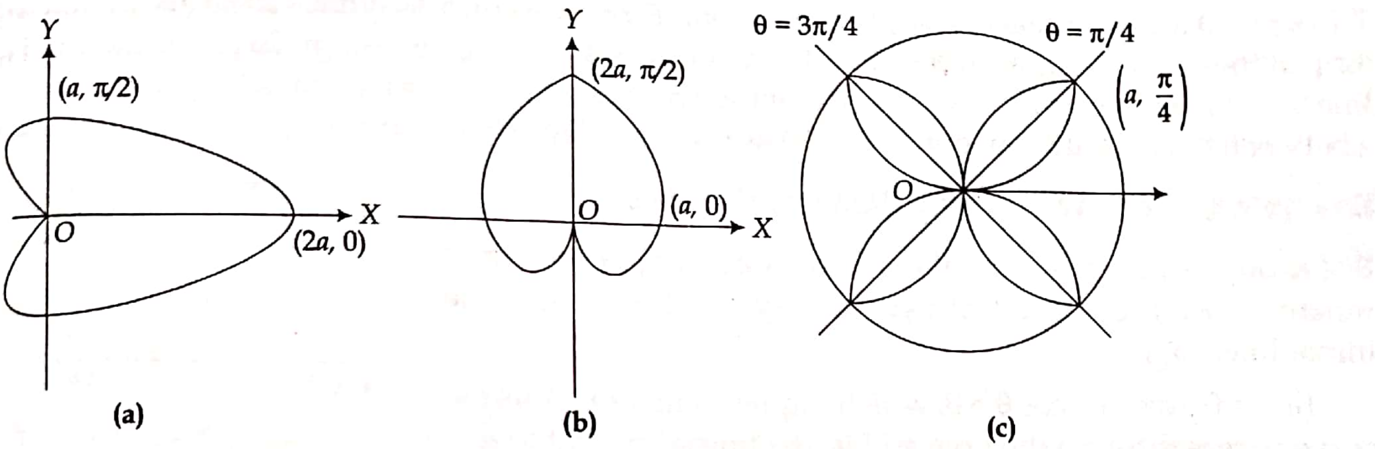


Fig. 4.36

3. *Direction of the tangent:* Find $\tan\phi = r\dfrac{d\theta}{dr}$, where ϕ is the angle between the tangent to the curve at a point $P(r, \theta)$ and its radius vector. It gives direction of the tangent at the point $P(r, \theta)$.

The points of specific interest are the points where the tangent concides with the radius vector or is perpendicular to it, that is, where $\tan\phi$ is zero or infinity.

For example, for the Lemniscates $r^2 = a^2 \sin 2\theta$, we have $2r\dfrac{dr}{d\theta} = 2a^2\cos 2\theta$, which gives $\tan\phi = r\dfrac{d\theta}{dr} = \dfrac{r^2}{a^2\cos 2\theta} = \tan 2\theta$, thus $\phi = 2\theta$. So when $\theta = \pi/4$, $\phi = \dfrac{\pi}{2}$. Thus, tangent to the curve at $\left(a, \dfrac{\pi}{4}\right)$ is perpendicular to the line $\theta = \pi/4$, refer to Fig. 4.37.

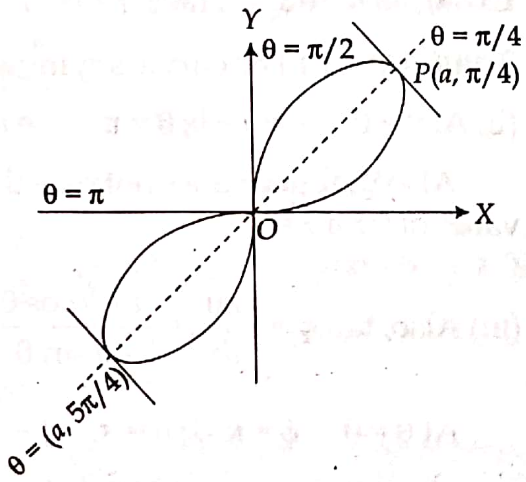


Fig. 4.37

4. *Asymptotes:* If $\dfrac{1}{r}$ becomes zero for $\theta = \theta_1$, then equation of the asymptote to the curve $\dfrac{1}{r} = f(\theta)$ at $\theta = \theta_1$ is $r\sin(\theta - \theta_1) = 1/f'(\theta_1)$. provided $f'(\theta_1) \neq 0$. For example, the spiral $r\theta = a$ has $r\sin\theta = a$ (that is, $y = a$) as asymptote, refer to Example 4.89.

5. *Interaction with axes:* Determine the points where the curve meets the lines $\theta = 0, \theta = \pi/4, \theta = \pi/2, \theta = \pi$ and $\theta = 3\pi/2$, etc.

6. *Region:* Determine the limits, if any, to which r and θ are confined. Find the greatest value of r (numerically), so as to find whether the curve lies within a circle or not. For example, $r = a\sin 2\theta$ lies within the circle $r = a$, refer to Fig. 4.36c.

Also find those values of θ for which r is imaginary. For example, $r^2 = a^2\sin 2\theta$ does not lie between the lines $\theta = \pi/2$ and $\theta = \pi$, refer to Fig. 4.37.

7. Loop: If a curve meets a line at points A and B and the curve is symmetrical about that line, then a loop of the curve exists between A and B. For example, the curve $r = a\sin 2\theta$ is symmetrical about the line $\theta = \pi/4$ and meets it at the pole and at the point $(a, \pi/4)$. Hence it forms a loop about the line $\theta = \pi/4$ between the pole and the point $(a, \pi/4)$ as shown in Fig. 4.36c.

Example 4.102: Trace the cardiod $r = a(1 - \cos\theta)$.

Solution: (i) By changing θ to $-\theta$, the equation of the curve remains unchanged hence the curve is symmetrical about the initial line.

(ii) $r = 0$ gives $1 - \cos\theta = 0$, which implies that $\theta = 0$, thus the curve passes through the pole and initial line is tangent to it.

(iii) As θ increase from 0 to π, r increases from 0 to $2a$.

(iv) Also $\tan\phi = r\dfrac{d\theta}{dr} = \tan(\theta/2)$, which gives $\phi = \theta/2$. At $\theta = \pi$, $\phi = \pi/2$. So the tangent at $(2a, \pi)$ is perpendicular to the initial line.

The approximate shape of the curve is shown in Fig. 4.38.

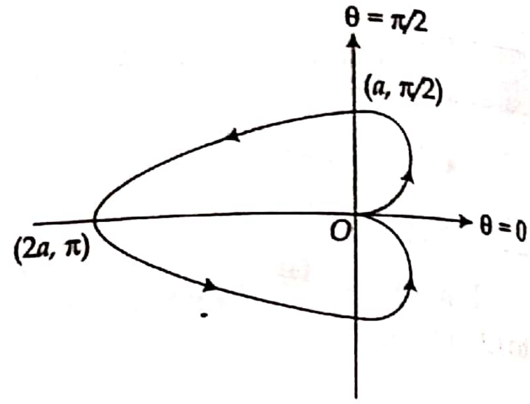


Fig. 4.38

Example 4.103: Trace $r = a + b\cos\theta$; $a > b$ (Lamicon).

Solution: (i) The curve is symmetrical about the initial line.

(ii) At $\theta = 0$, $r = a + b$; $\theta = \pi$, $r = a - b$

Also the curve does not pass through pole as $r = 0$ gives $\theta = \cos^{-1}(-a/b)$, $a > b$; thus there is no real value of θ for $r = 0$.

(iii) Also, $\tan\phi = r\dfrac{d\theta}{dr} = \dfrac{a + b\cos\theta}{-b\sin\theta}$.

At $\theta = 0$, $\phi = \pi/2$; $\theta = \pi$, $\phi = \pi/2$

thus the tangents to the curve at $(a + b, 0)$ and $(a - b, \pi)$ are perpendicular to the initial line. The shape is shown in Fig. 4.39.

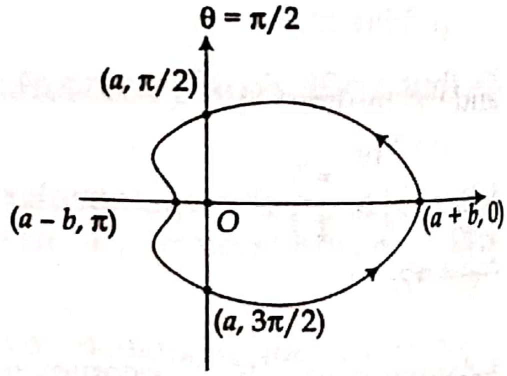


Fig. 4.39

Example 4.104: Trace the curve $r = a\sin 3\theta$, (three leaved rose)

Solution: (i) The curve is symmetrical about the line $\theta = \pi/2$, since changing $\theta \to \pi - \theta$ does not change the equation.

(ii) The curve wholly lies within the circle $r = a$, since $r = a\sin 3\theta$ gives $|r| < a$; and obviously it has no asymptotes.

(iii) Also $\tan\phi = r\dfrac{d\theta}{dr} = \dfrac{a\sin 3\theta}{3a\cos 3\theta} = \dfrac{1}{3}\tan 3\theta$

Thus, $\phi = 0$, when $\theta = 0, \quad \pi/3, \ldots$

$\phi = \pi/2$, when $\theta = \pi/6, \pi/2, \ldots$

Consider the following table for $0 \le \theta \le \pi/2$.

θ	r	Portion
0 to $\pi/6$	0 to a	O to A
$\pi/6$ to $\pi/3$	a to 0	A to O
$\pi/3$ to $\pi/2$	0 to $-a$	O to B

For, from $\pi/2$ to π, portions of the curve from B to O, O to C and C to O the curve is traced by symmetry about the line $\theta = \pi/2$.

Hence, the curve consists of three loops as shown in Fig. 4.40.

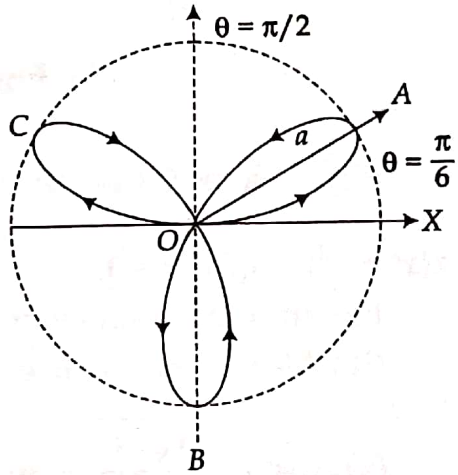


Fig. 4.40

Example 4.105: Trace the Leminiscate of Bernoulli $r^2 = a^2 \cos 2\theta$.

Solution: (i) The curve is symmetric about initial line, the line $\theta = \pi/2$ and about the pole.

(ii) The curve lies wholly within the circle $r = a$ since $r^2 \le a^2$. Also no portion of the curve lies between the lines $\theta = \pi/4$ and $\theta = 3\pi/4$, since r^2 is negative for $\pi/4 < \theta < \dfrac{3\pi}{4}$.

(iii) Further $\tan\phi = r\dfrac{d\theta}{dr} = -\cot 2\theta = \tan\left(\dfrac{\pi}{2} + 2\theta\right)$, that is, $\phi = \dfrac{\pi}{2} + 2\theta$. Thus $\phi = 0$, when $\theta = -\pi/4$ and the tangent at A is perpendicular to the initial line.

(b) The variations of r and θ and the portion traced is given as:

θ	r	Portion
0 to $\pi/4$	a to 0	ABO
$3\pi/4$ to π	0 to a	OCD

As θ increases from π to 2π, the portion can be traced by symmetry about the initial line. The approximate shape of the curve is shown in Fig. 4.41.

Example 4.106: Trace the curve $r = a(\cos\theta + \sec\theta)$.

Solution: (i) Changing θ to $-\theta$ does not change the equation, so the curve is symmetric about the initial line.

(ii) $r = 0$, gives $\cos^2\theta = -1$, hence the curve does not pass through pole for any real θ.

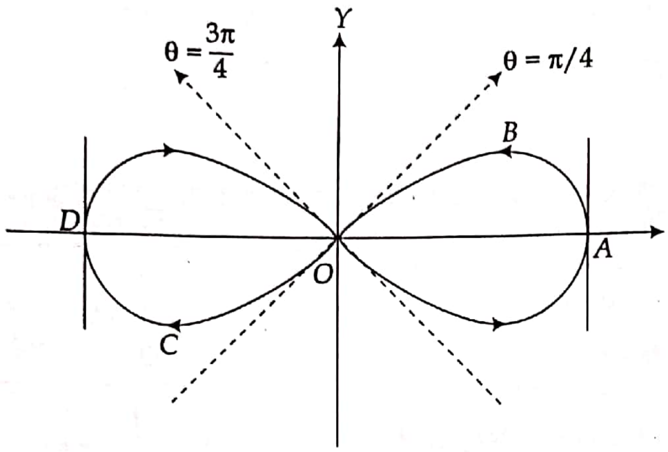


Fig. 4.41

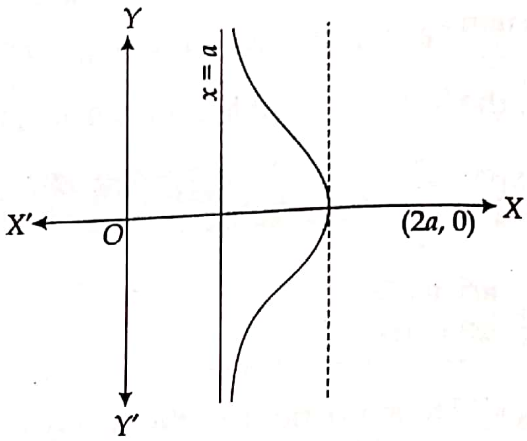


Fig. 4.42

(iii) $r = a(\cos\theta + \sec\theta)$ gives $r^2 = a\left[r\cos\theta + \frac{r^2}{r\cos\theta}\right]$; changing to cartesian co-ordinates it gives $x(x^2+y^2) = a(2x^2+y^2)$.

Equating the coefficient of highest power of y to zero gives $x = a$ as asymptote to the curve.

(iv) Also the curve meets the x-axis at $(2a, 0)$, the line $x = 2a$ is tangent at $(2a, 0)$.

(v) Also $y = \frac{x\sqrt{2a-x}}{\sqrt{x-a}}$ gives that y is defined only for $a < x \le 2a$ so the curve lies between the lines $x = a$ and $x = 2a$.

The approximate shape of the curve is shown in Fig. 4.42.

Example 4.107: Trace the equiangular spiral $r = ae^{b\theta}$ where $a, b > 0$ are constants.

Solution: (i) There is no symmetry in this curve.

(ii) At $\theta = 0$, we have $r = a$ when θ increases indefinitely r also increases indefinitely.

(iii) Consider $\tan\phi = r\frac{d\theta}{dr} = \frac{ae^{b\theta}}{abe^{b\theta}} = \frac{1}{b}$, a constant.

Thus, $\phi = \tan^{-1}\frac{1}{b} = \alpha$, say.

Hence the radius vector at any point of the curve always makes a constant angle with the tangent at that point.

The approximate shape of the curve is shown in Fig. 4.43.

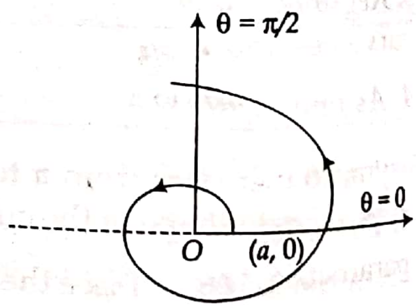


Fig. 4.43

Example 4.108: Trace the curve $r^2\cos 2\theta = a^2$.

Solution: (i) Since changing θ to $-\theta$ and θ to $\pi - \theta$, the equation of the curve remains unchanged, so the curve is symmetric about the initial line and about the line $\theta = \pi/2$.

(ii) Changing r to $-r$, the equation remains unchanged so the curve is symmetric about the pole.

(iii) Since $r^2 \cos 2\theta = a^2$ gives, $r^2 \geq a^2$, thus the curve is only in the region defined by $r \geq a$.

(iv) We have $r^2 \cos 2\theta = r^2(\cos^2\theta - \sin^2\theta) = x^2 - y^2$; thus the curve is $x^2 - y^2 = a^2$, a hyperbola.

Thus, $y = \pm x$, which give $\theta = \pm\pi/4$, are its asymptotes.

The approximate shape of the curve is shown in Fig. 4.44.

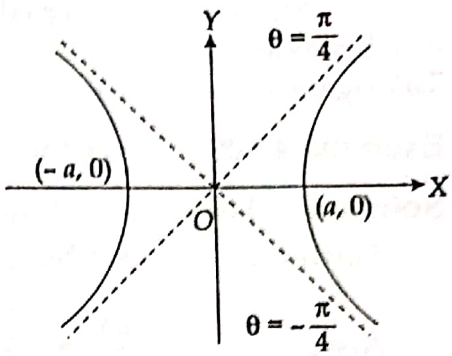


Fig. 4.44

EXERCISE 4.18

Trace the following curves

1. $r = a(1 + \sin\theta)$ (cardiod)
2. $r = 1 + \sqrt{2}\cos\theta$ (cardiod)
3. $r = ae^{\theta \cot\alpha}$, $a > 0$ (equiangular spiral)
4. $r\cos\theta = a\sin^2\theta$, $a > 0$
5. $r = a\cos 2\theta$ (four-leaved rose)
6. $r = a\cos 3\theta$ (three-leaved rose)
7. $r = a\sin 4\theta$ (eight-leaved rose)
8. $r = a\sin 5\theta$ (five-leaved rose)

4.12.3 Tracing of Parametric Curves

In case of the curve $x = f(t)$, $y = g(t)$, we try to eliminate the parameter t to get the curve in cartesian form since it is comparatively easy to plot the curve in cartesian form. However, to plot the curve in parametric form itself, we account for the following characteristics.

1. ***Origin:*** If for any value of t, $x = 0$ and $y = 0$, then the curve passes through the origin.

2. ***Intercept with the axes:*** Find the values of t for which $f(t) = 0$, then find $y = g(t)$ for those values of t, the curve meets the y-axis at these points.

Similarly, find the point of intersection of the curve with the x-axis.

3. ***Regions:*** Find the greatest and the least values of x and y which give the region in which the curve lies, that is find the limits of the parameter 't' beyond which x or y cannot lie.

4. ***Asymptotes:*** In case there is some $t = t_1$ such that $\lim_{t \to t_1} x = \infty$ and $\lim_{t \to t_1} y = \infty$, then $t = t_1$ is an asymptote.

5. ***Derivatives:*** In general, it is difficult to check the various symmetries for the curve in parametric form so finding derivatives is very essential. Find dx/dt and dy/dt and note the values of t for which x, y are increasing or decreasing functions of t. Find $\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt}$. Determine the points where tangent is parallel or perpendicular to the x-axis, that is the points where $\frac{dy}{dx} = 0$ or ∞.

Also find $\frac{d^2y}{dx^2}$ and check for the values of parameter t for which $\frac{d^2y}{dx^2} = 0$, $\frac{d^2y}{dx^2} > 0$, or $\frac{d^2y}{dx^2} < 0$ to know about the points of inflexion, concavity or convexity of the curve.

To plot the curve, assign the parameter t different values; find the corresponding values of x, y and also behaviour of dy/dx. We get different points on the curve and slope of the tangents at these points. Taking into consideration the characteristics mentioned above join these points to draw the curve.

Example 4.109: Trace the asteroid $x = a\cos^3 t, y = a\sin^3 t$.

Solution: The curve is symmetric about x-axis, since $t \to -t$ does not change x.

Further $|x|, |y| \le a$, hence the curve lies within the square bounded by the lines $x = \pm a$ and $y = \pm a$.

Also,
$$\frac{dx}{dt} = -3a\cos^2 t \sin t,$$
$$\frac{dy}{dt} = 3a\sin^2 t \cos t.$$

Hence, $\frac{dy}{dx} = -\tan t$. Therefore, $\frac{dy}{dx} = 0$ at $t = 0$ and $\frac{dy}{dx} \to \infty$ at $t = \pi/2, 3\pi/2$.

Following table gives the value of $t, x, y, dy/dx$ and the corresponding portion traced.

t	x	y	dy/dx	Portion
0 to $\pi/2$	+ve, decreases a to 0	+ve, increases 0 to a	0 to ∞ through –ve	A to B
$\pi/2$ to π	–ve, decreases 0 to $-a$	+ve, decreases a to 0	∞ to 0 through +ve	B to A

Since, curve is symmetric about initial line, portion from π to 2π is the image of 0 to π in the initial line. For $t > 2\pi$, the curve repeats. An approximate shape is shown in Fig. 4.45.

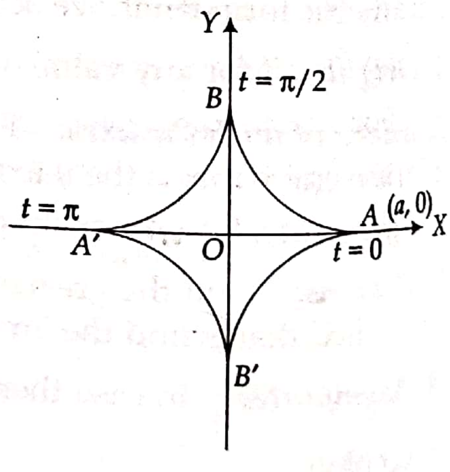


Fig. 4.45

Example 4.110: Trace the cycloid $x = a(\theta + \sin\theta), y = a(1 - \cos\theta)$.

Solution: (i) Since changing θ to $-\theta$ curve remains unchanged, thus the curve is symmetrical about y-axis, so it is sufficient to consider the curve only for $\theta > 0$.

(ii) The greatest and the least value for y are $2a$ and zero. Hence, the curve lies between the lines $y = 2a$ and $y = 0$.

(iii) The curve passes through the origin at $\theta = 0$. It meets the y-axis only at $(0, 0)$ while it meets x-axis where $1 - \cos\theta = 0$, that is, at $(0, 0), (2a\pi, 0), (4a\pi, 0)$ etc.

(iv) $\frac{dx}{d\theta} = a(1 + \cos\theta)$, $\frac{dy}{d\theta} = a\sin\theta$, thus $dy/dx = \tan\theta/2$.

Hence, $dy/dx = 0$, at $\theta = 0. 2\pi, 4\pi$, etc.

$= \infty$ at $\theta = \pi, 3\pi, 5\pi$, etc.

Tangent is parallel to y-axis at $x = a\pi, 3a\pi, 5a\pi$ etc. Following table gives the values of $\theta, x, y, dy/dx$ and the corresponding portion traced.

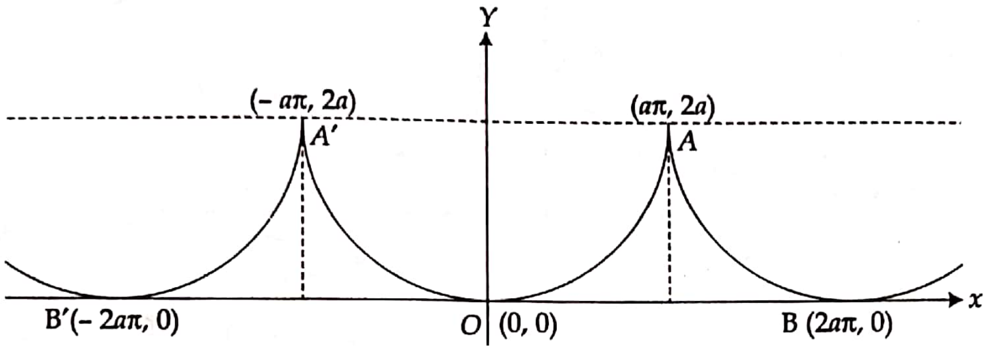


Fig. 4.46

θ	x	y	dy/dx	Portion
0 to π	Inc. 0 to $a\pi$	Inc. 0 to 2a	0 to ∞	O to A
π to 2π	Inc. $a\pi$ to $2a\pi$	dec. 2a to 0	∞ to 0	A to B

For $-2\pi \le \theta \le 0$, the curve is image of the portion $0 \le \theta \le 2\pi$ in the y-axis. $A'OA$ is one cycloid corresponding to $-\pi \le \theta \le \pi$. The curve extends to ∞ on both the sides as shown in Fig. 4.46.

Example 4.111: Trace the tractrix $x = a(\cos t + \ln|\tan(t/2)|)$, $y = a \sin t$.

Solution: Rewriting the equation as $x = a\left[\cos t + \frac{1}{2}\ln\tan^2\frac{t}{2}\right]$, $y = a\sin t$.

(i) Since t changes to $-t$, x remains unchanged so the curve is symmetrical about x-axis.

(ii) The origin does not lie on the curve.

For $t = \pm\pi/2$, we get $x = 0$ and $y = \pm a$. Hence, the curve meets the y-axis at points $(0, \pm a)$

(iii) $|y| < a$, hence curve lies within the lines $y = a$ and $y = -a$.

(iv) As $t \to 0$, we have $x \to \infty$ and $y \to 0$, therefore x-axis is an asymptote to the curve.

(v) Consider $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = a\cos t \bigg/ \frac{a\cos^2 t}{\sin t} = \tan t.$

Hence, $\frac{dy}{dx} = 0$, when $t = 0$

$= \infty$, when $t = \pm\pi/2$,

that is, at $(0, \pm a)$, so y-axis is tangent there.

Following table gives the values of t, x, y and dy/dx and the corresponding portion of the curve

t	x	y	dy/dx	Portion of the curve
0 to $\pi/2$	$-\infty$ to 0	Inc. 0 to a	0 to ∞	$-\infty$ to A
$\pi/2$ to π	Inc. 0 to ∞	dec. a to 0	∞ to 0	A to $+\infty$

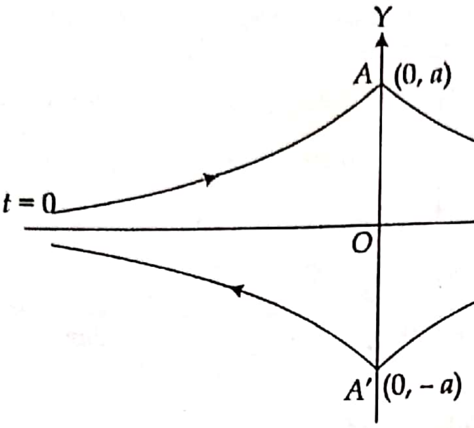


Fig. 4.47

Since it is symmetric about x-axis, curve for the portion π to 2π is the image in x-axis of the portion $0 \le x \le \pi$. The shape is shown in Fig. 4.47.

EXERCISE 4.19

Trace the following curve

1. $x = a \sec t, \quad y = b \tan t$
2. $x = a \cos^3 t, \quad y = b \sin^3 t$
3. $x = a(t - \sin t), \quad y = a(1 - \cos t), \quad 0 \le t \le 2\pi$
4. $x = a(t - \sin t), \quad y = a(1 + \cos t), \quad 0 \le t \le 2\pi$
5. $x = a(t + \sin t), \quad y = a\,(1 + \cos t)$
6. $x = a \sin 2t(1 + \cos t), \quad y = a \cos 2t(1 - \cos 2t).$

ANSWERS

Exercise 4.1 (p. 188)

2. $f'(x) = 2|x|$

3. (a) $(-1)^{n-1}\,(n-1)!\left[\dfrac{1}{(x-1)^n} + \dfrac{1}{(x+2)^n}\right]$ (b) $(-1)^n \dfrac{n!\,c^{n-1}}{(cx+d)^{n+1}}(bc - ad)$

7. $y_n(0) = \begin{cases} \{m^2 - (n-2)^2\}\ \{m^2 - (n-4)^2\}\ \ldots & (m^2) \\ \{m^2 - (n-2)^2\}\ \{m^2 - (n-4)^2\}\ \ldots & (m^2 - 1^2)(m) \end{cases}$

Exercise 4.2 (p. 190)

1. 36π 2. ± 0.3 3. $25/\pi$ 4. 36.35 5. 0.0016 cm.

Exercise 4.3 (p. 200)

6. $a\cos t$ 9. $\pi/3$ 11. (a) $r^3 = a^2 p$ (b) $p^2 + a^2 = r^2$ 12. $\dfrac{(3x+a)}{2\sqrt{3ax}}$.

Exercise 4.4 (p. 206)

1. $(1, 1)$ 4. 3.0049 5. 1.43
8. Increasing in $[-1, 1]$; decreasing in $(-\infty, -1)$, and $(1, \infty)$.

Exercise 4.5 (p. 216)

2. (i) $\cos x = \dfrac{1}{\sqrt{2}} - \left(x - \dfrac{\pi}{4}\right)\dfrac{1}{\sqrt{2}} - \dfrac{\left(x - \frac{\pi}{4}\right)^2 \frac{1}{\sqrt{2}}}{2!} + \dfrac{\left(x - \frac{\pi}{4}\right)^3 \frac{1}{\sqrt{2}}}{3!} + \ldots$

(ii) $e^2\left[1 + (x-2) + \dfrac{(x-2)^2}{2!} + \dfrac{(x-2)^3}{3!} + \ldots\right]$

(iii) $1 - \dfrac{1}{2!}\left(x - \dfrac{\pi}{2}\right)^2 + \dfrac{1}{4!}\left(x - \dfrac{\pi}{2}\right)^4 - \ldots$, 0.9998.

(iv) $\dfrac{\pi}{4} + \dfrac{1}{2}(x-1) - \dfrac{1}{4}(x-1)^2 + \dfrac{1}{12}(x-1)^3 + \ldots$

(v) $45 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3$

3. 0.8482 4. 2.6121 5. 8.849 6. 4.123

9. (i) $1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots$ (ii) $1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \ldots$

(iii) $x + \dfrac{x^3}{3} + \dfrac{2x^5}{15} + \ldots$ (iv) $x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \dfrac{x^7}{7!} + \ldots$

(v) $x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7}$ (vi) $x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots$

(vii) $1 + \dfrac{x^2}{2!} + \dfrac{5x^4}{4!} + \ldots$ (viii) $\dfrac{\pi}{4} + \dfrac{x}{2} - \dfrac{x^2}{4} + \dfrac{x^3}{12} - \ldots$

(ix) $a^h\left[1 + x(\ln a) + \dfrac{x^2}{2!}(\ln a)^2 + \dfrac{x^3}{3!}(\ln a)^3 + \ldots\right]$ 16. 6

Exercise 4.6 (p. 225)

1. 0 2. 3/2 3. 1 4. 1/2 5. $a = 2$; limit -1
6. 1 7. ln 8. 1 9. $-\dfrac{1}{3}$ 10. 1
11. $-\dfrac{e}{2}$ 12. $-\dfrac{1}{2}$ 13. 1 15. continuous at the origin.

Exercise 4.7 (p. 231)

1. Minima at $(3-\sqrt{17})/4$ and 3; maxima at $(3+\sqrt{17})/4$. Neither maxima nor minima at -1.
2. Minimum value 0 at $x = 0$.
3. Maximum $4/e^2$ at $x = \pm\sqrt{2}$; minimum 0 at $x = 0$.
4. Maximum 14 at $t = -1$, minimum -17.25 at $t = 3/2$.
5. Minima at $\pi/2, 3\pi/2$; maxima at $\pi/6, 5\pi/6$.
6. Minima at 0; maxima at -3 and 1.
7. Minima at $5\pi/3$; maxima at $\pi/3$.
8. Height = radius = $3\sqrt{v_0/\pi}$. 10. $\sin^{-1} 1/3$.
13. Pts. are $\left(-2-\sqrt{3}, \dfrac{-\sqrt{3}-1}{4}\right)$, $\left(-2+\sqrt{3}, \dfrac{1+\sqrt{3}}{4}\right)$, $(1, 1)$.

Exercise 4.8 (p. 238)

1. (i) $a \cos\psi$ (ii) $\dfrac{4}{3} a \cos \dfrac{1}{3}\psi$ (iii) $2a\sec^3\psi$ 2. (i) $\dfrac{a}{\sqrt{2}}$ (ii) $\dfrac{5\sqrt{5}}{4}$
(iii) $\dfrac{a}{2}$ 4. $2\sqrt{2}$ 10. (i) $3a \sin\theta\cos\theta$ (ii) $3a\sin t$
(iii) $4a\cos^3\theta$ 11. $2\sqrt{2}$

Exercise 4.9 (p. 242)

1. (i) $a/2$, (ii) 1 2. (i) $\dfrac{1}{2\sqrt{2}}$, (ii) $\dfrac{37\sqrt{37}}{10}$
3. $\dfrac{1}{2}$ 5. $4a$.

Exercise 4.10 (p. 248)

1. (i) $2r^{3/2}/a^{1/2}$ (ii) $\dfrac{a^2}{3r}$
2. (i) $\dfrac{2r^{3/2}}{\sqrt{a}}$ (ii) $r \operatorname{cosec} \alpha$
4. $\dfrac{2\sqrt{3}}{3}a$
7. 0
8. (i) $\dfrac{2r^{3/2}}{\sqrt{a}}$ (ii) $\dfrac{2}{3}\sqrt{2ar}$ (iii) $\dfrac{a^2}{3r}$ (iv) $\sqrt{r^2 - a^2}$ (v) a^2b^2/p^3 (vi) r^4/Ap^3
9. a^2b^2/p^3.

Exercise 4.11 (p. 252)

1. $\left(3x + 2a_1 - 2a^{-\frac{1}{2}} x^{3/2}\right)$, $4(x - 2a)^3 = 27\, ay^2$
2. $\left[\dfrac{x}{2}\left(1 - \dfrac{9x^4}{a^4}\right), \left(\dfrac{5x^3}{2a^2} + \dfrac{a^2}{6x}\right)\right]$
7. $\left(\dfrac{\pi}{3} - \dfrac{3}{2}, \dfrac{7}{4}\right)$
9. $(a + b)(x^2 + y^2) = 2x + 2y$

Exercise 4.12 (p. 257)

1. (i) $x^2/a^2 + y^2/b^2 = 1$ (ii) $27ay^2 = 4(x - 2a)^3$ (iii) $8x^3 + 27ay^2 = 0$ (iv) $x^2 - y^2 = c^2$ (v) $y^2 = a^2 + x^2$ (vi) $x^{2/2-n} + y^{2/2-n} = c^{2/2-n}$
2. $y^2 = 2x + 1$
3. $y = 0$
4. $2xy = A$.

Exercise 4.13 (p. 260)

1. $x = 0$
2. $y = 0,\quad x = \pm 1$
3. $x = \pm a,\quad y = \pm a$
4. $x = \pm a,\quad y = \pm b,$
5. No asymptote
6. $x = n\pi$.

Exercise 4.14 (p. 265)

1. $y = x + 1/6,\quad y = -x - 1/2,\quad y = -x/2 + 1/3$
2. $y = x + 1,\quad y = -x + 1,\quad x + 2y = 0$
3. $y + x = 0,\quad y - x = 0,\quad y = x + 1$
4. $y - x = 0,\quad y = -\dfrac{1}{2}x + \dfrac{1}{2},\quad y = -\dfrac{1}{2}x - \dfrac{1}{2}$
5. $x + a = 0,\quad x - a = 0,\quad x - y + \sqrt{2}a = 0,\quad x - y - \sqrt{2}a = 0$
6. $x = \pm a,\quad y = x + a,\quad y = x - a$
7. $x = 0,\quad y = 0,\quad x - y = 0,\quad x + y = 0,\quad x - 2y = 0,\quad x + 2y = 0$
8. $y + x = 0,\quad 3y - 2x - 3 = 0,\quad 3y - 2x + 1 = 0$
9. $y = -x + \dfrac{1}{2}a$
10. $a_1x + b_1y + c_1 = 0,\quad a_2x + b_2y + c_2 = 0.$

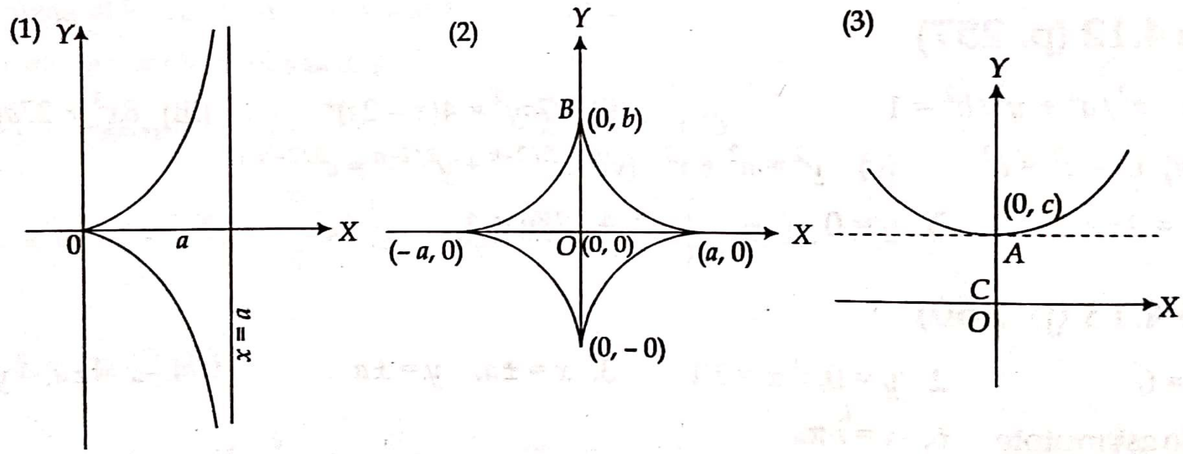
Exercise 4.15 (p. 268)

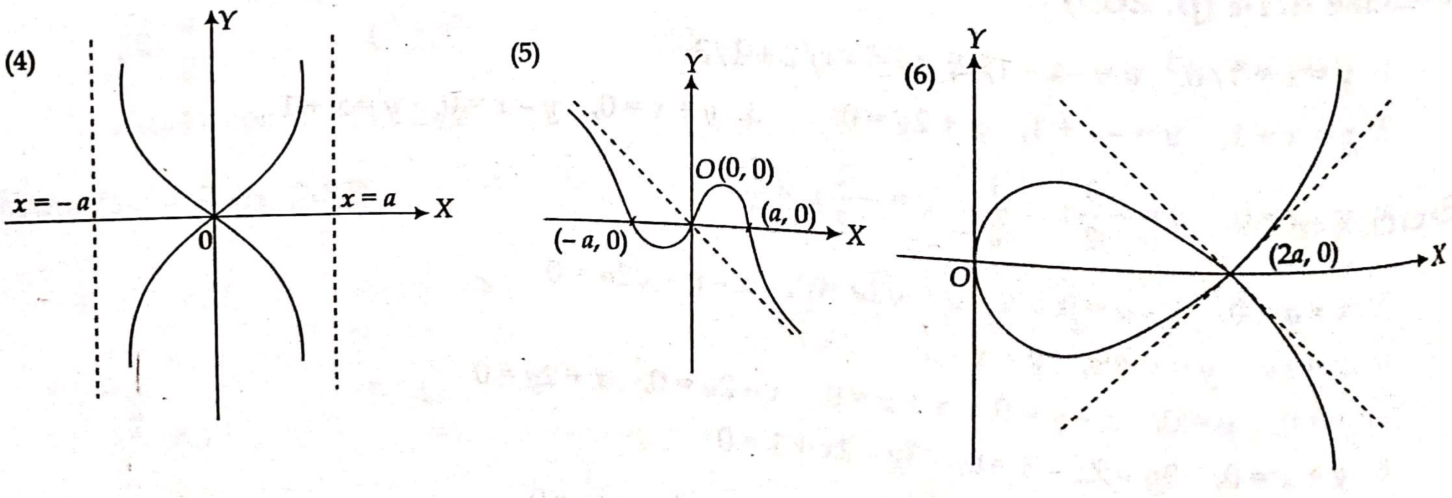
2. $x-2y=0, x+2y=0, \quad 2x-y+1=0, \quad 2x+y+1=0$
5. $x^3-6x^2y+11xy^2-6y^3-x=0$
6. $x^3-6x^2y+11xy^2-6y^3-x+6y=0$
7. $3x^2-2xy-5y^2+7x-9y-16=0.$

Exercise 4.16 (p. 270)

1. $r\sin\left(\theta\pm\frac{\pi}{6}\right)=2\sqrt{3}$
2. $r\cos\theta=8$
3. $r\sin\theta=a; \quad r\cos\theta=\frac{2a}{(2k+1)\pi}$
4. $r\sin\theta=a$
5. $r\sin(\theta-1)=a$
6. $r(\sin\theta+\cos\theta)+a=0$
8. $r=a$
9. $r=3/2$
10. $r=n\pi, n$ integer.

Exercise 4.17 (p. 278)





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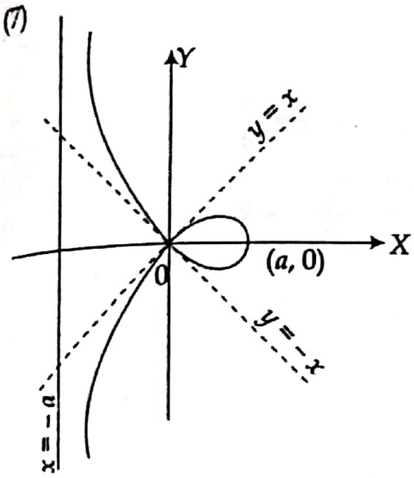

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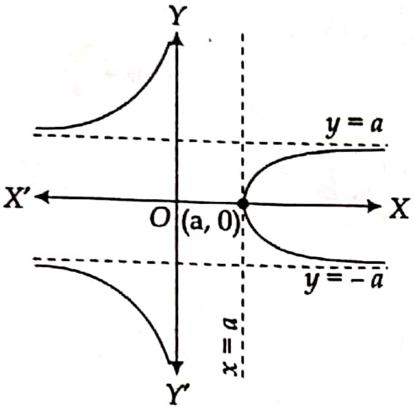

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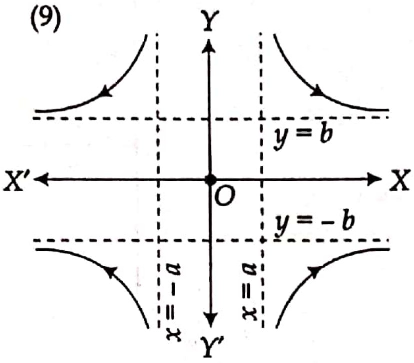

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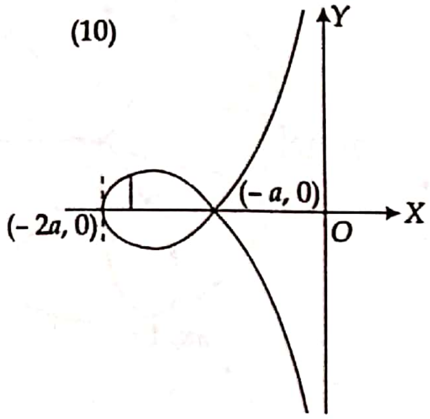

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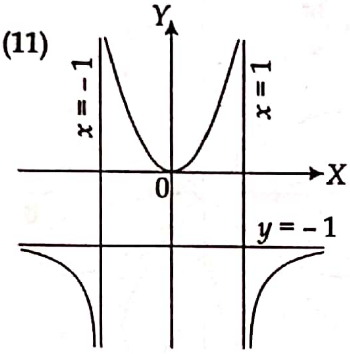

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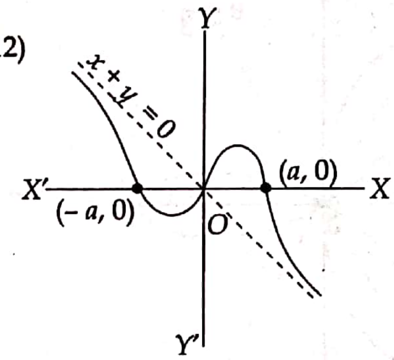

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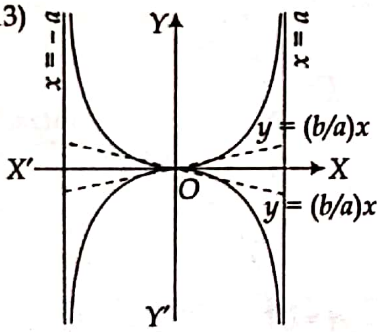

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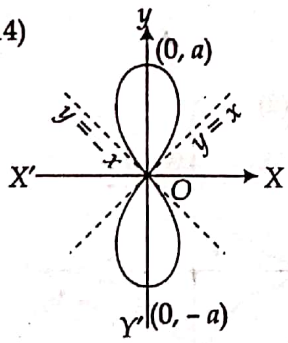

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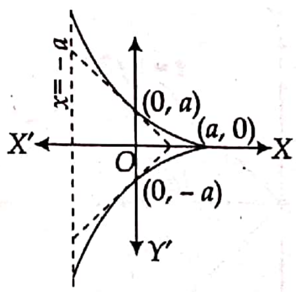
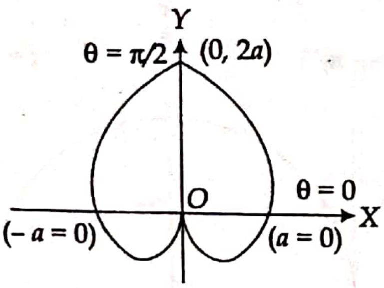


Exercise 4.18 (p. 283)

(1)



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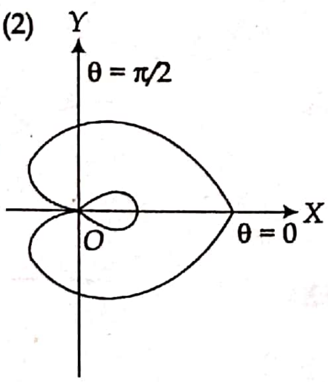

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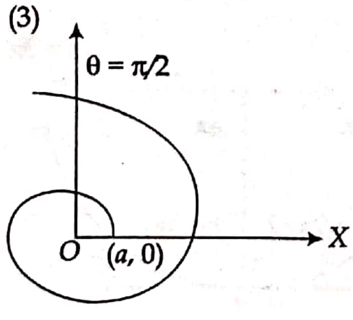

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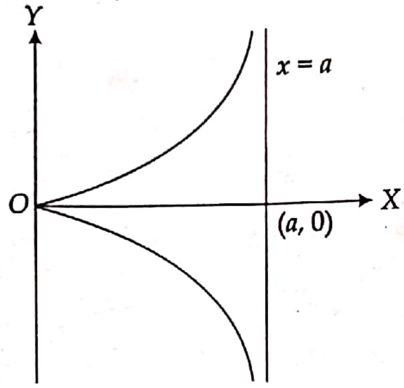

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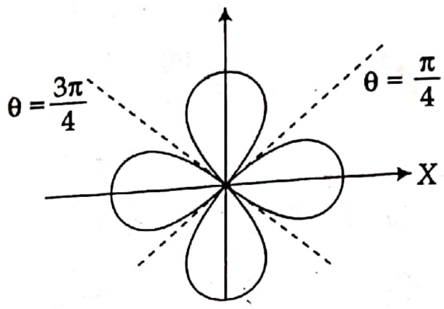

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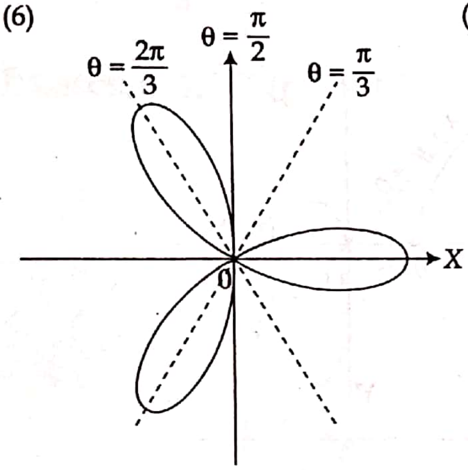

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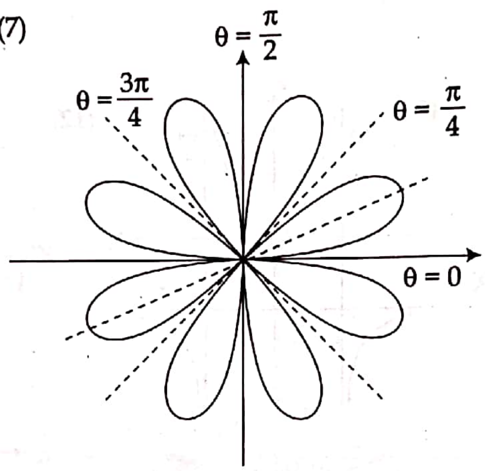

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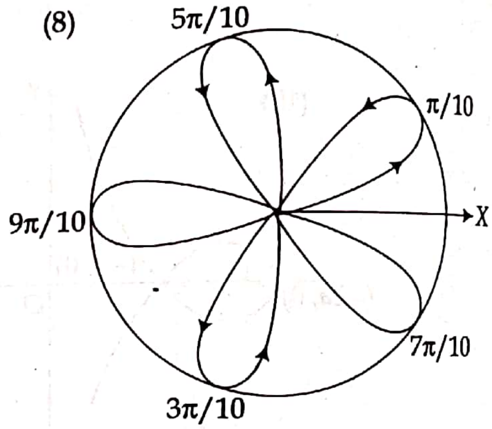
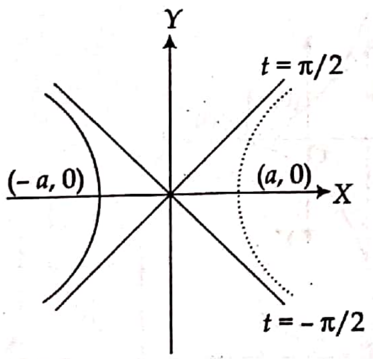


Exercise 4.19 (p. 286)

(1)



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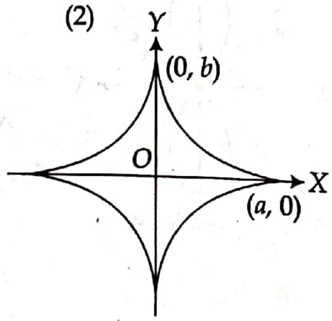

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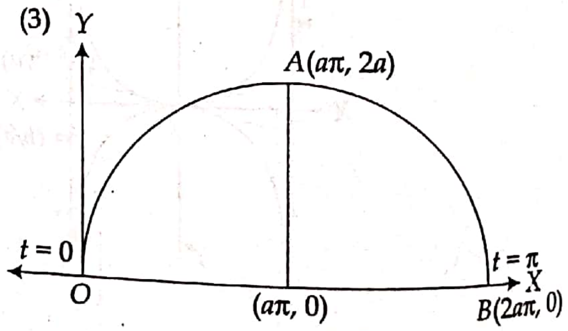

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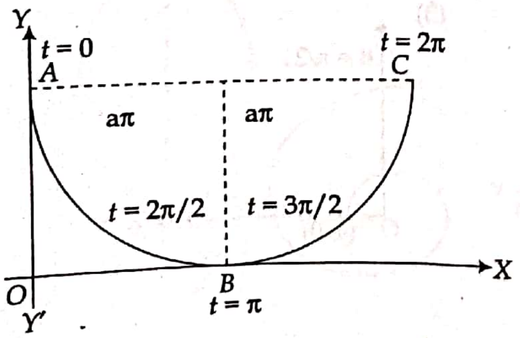

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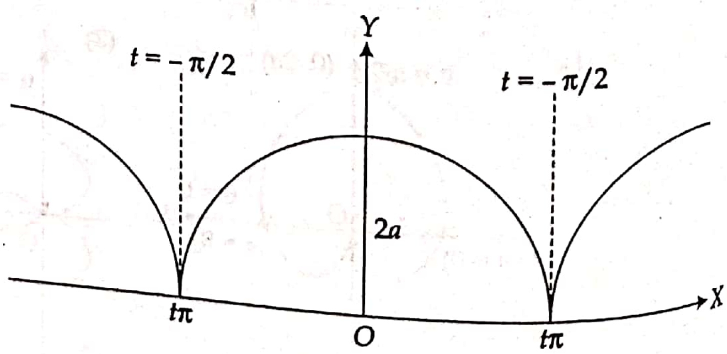

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