17 Fourier Series

CHAPTER

"Fourier series arise naturally while analyzing many physical phenomena like electrical oscillations, vibrating mechanical systems, longitudinal oscillations in crystals, etc. Many functions including some discontinuous periodic functions of practical interest, which do not find Taylor series representation, can be expanded in a Fourier series and, as such, Fourier series are more universal than Taylor series. They are very powerful tools in solving certain ordinary and partial differential equations. Modern-day applications of Fourier series include areas like data compression, filtering and signal analysis, CAT scans, sattelite communication, etc."

17.1 THE FOURIER SERIES OF A FUNCTION

A Fourier series is a representation of a periodic function as a series of cosine and/or sine terms. Before studying Fourier series we need to consider periodic functions.

17.1.1 Periodic Functions

A function $f(x)$ is called a 'periodic function', if there is some positive number T such that for every x in the domain of f

$$f(x+T) = f(x); \qquad \text{...(17.1)}$$

and the number $T > 0$, with this property, is called a 'period' of $f(x)$.

For example, $\sin x$ is periodic with period 2π, since $\sin(x + 2\pi) = \sin x$ for all x. The function $\tan x$ is periodic with period π, for $\tan(x + \pi) = \tan x$ for all x.

Examples of *non-periodic* functions are x, e^x, etc.

Further, we note that if f is periodic with period T, it is necessarily periodic with period $2T$, $3T$, $4T$ ... as well, since $f(x + 2T) = f((x + T) + T) = f(x + T) = f(x)$. Of all these possible periods, the smallest one, if it exists, is called the *'fundamental period'* of $f(x)$. For example, for $\cos x$ and $\sin x$ the fundamental period is 2π while for $\cos 2x$ and $\sin 2x$ it is π.

The function $f(x)$ = constant is periodic and every $T > 0$ is a period. Thus there is no smallest period so, $f(x)$ = constant does not have a fundamental period.

Further, if $f(x)$ and $g(x)$ are periodic with fundamental period T, then the function $h(x) = af(x) + bg(x)$, a, b being constants is also periodic with period T.

Now we define the Fourier series representation of a function $f(x)$.

17.1.2 Fourier Series

If a function $f(x)$ is periodic with period 2π and is integrable over $-\pi < x < \pi$, then the Fourier series representation of $f(x)$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \quad \text{...(17.2)}$$

where the coefficients $a_0, a_1, a_2, \ldots; b_0, b_1, b_2, \ldots$, called the Fourier coefficients, are determined by the function $f(x)$.

To determine the coefficients we need the following results which follow from the orthogonality property of the trigonometric functions $\cos x$, $\sin x$, $\cos 2x$, $\sin 2x$ ... $\cos nx$, $\sin nx$. However, the results can be proved otherwise also, for all integral values of m and n.

1. $$\int_{-\pi}^{\pi} \sin nx\,dx = \int_{-\pi}^{\pi} \cos nx\,dx = 0 \quad \text{...(17.3)}$$

2. $$\int_{-\pi}^{\pi} \sin mx \sin nx\,dx = \begin{cases} 0, & \text{if } m \neq n \\ \pi, & \text{if } m = n \end{cases} \quad \text{...(17.4)}$$

3. $$\int_{-\pi}^{\pi} \cos mx \cos nx\,dx = \begin{cases} 0, & \text{if } m \neq n \\ \pi, & \text{if } m = n \end{cases} \quad \text{...(17.5)}$$

4. $$\int_{-\pi}^{\pi} \sin mx \cos nx\,dx = 0, \text{ for all } m \text{ and } n. \quad \text{...(17.6)}$$

In fact, the results from (17.3) to (17.6) hold for the interval of integration $(\alpha, \alpha + 2\pi)$, for an arbitrary α. In particular, for $\alpha = -\pi$, the interval becomes $(-\pi, \pi)$ and for $\alpha = 0$ the interval becomes $(0, 2\pi)$.

Next, we determine the coefficients a_0, a_n and b_n.

Determination of the constant term a_0. Integrating both sides of (17.2) from $-\pi$ to π and assuming that term by term integration is possible, we obtain

$$\int_{-\pi}^{\pi} f(x)dx = a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos nx\,dx + b_n \int_{-\pi}^{\pi} \sin nx\,dx \right). \quad \text{...(17.7)}$$

Using (17.3), and integrating the first term on the right side of (17.7), we obtain

$$\boxed{a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx.}$$

Determination of the coefficients a_n. Multiplying (17.2) by cos mx for any fixed positive integer m and integrating on both sides from $-\pi$ to π; assuming that term by term integration is possible, we obtain

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = a_0\int_{-\pi}^{\pi}\cos mx\,dx + \sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos nx\cos mx\,dx + b_n\int_{-\pi}^{\pi}\sin nx\cos mx\,dx\right). \qquad ...(17.8)$$

Using (17.3), (17.5) and (17.6) on the right side of (17.8), we obtain

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx.$$

Determination of the coefficients b_n. Similarly multiplying (17.2) by sin mx, for any fixed positive integer m, and integrating on both sides from $-\pi$ to π; assuming that term by term integration is possible, we obtain

$$\int_{-\pi}^{\pi} f(x)\sin mx = a_0\int_{-\pi}^{\pi}\sin mx\,dx + \sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos nx\sin mx\,dx + b_n\int_{-\pi}^{\pi}\sin nx\sin mx\,dx\right). \qquad ...(17.9)$$

Using (17.3), ((17.4) and (17.6) on the right side of (17.9), we obtain

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx.$$

The results:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \qquad ...(17.10)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \qquad ...(17.11)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \qquad ...(17.12)$$

for $n = 1, 2, ...$ are called the *Euler's formulae* for the Fourier coefficients a_n and b_n associated with the Fourier series representation of $f(x)$ given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \qquad \ldots(17.13)$$

17.1.3 Convergence and Sum of a Fourier Series

Suppose that $f(x)$ is any given periodic function of period 2π which is continuous or merely piecewise continuous over the interval of integration. Then we can compute the Fourier coefficients a_0, a_n, b_n of $f(x)$ and use them to form the Fourier series (17.13) of $f(x)$. We would expect that the series thus obtained converges to $f(x)$ over the domain of definition of f.

Various results are available that give sufficient conditions on f for the Fourier series of $f(x)$ to represent $f(x)$. However, one such result which covers the majority of periodic functions appearing in practical applications, is stated as follows.

Theorem 17.1: *If $f(x)$ is a periodic function with period 2π and if $f(x)$ and $f'(x)$ both are piecewise continuous in the interval $-\pi \le x \le \pi$, then the Fourier series of $f(x)$ is convergent. It converges to $f(x)$ at every point x at which $f(x)$ is continuous, and to the mean value $[f(x+) + f(x-)]/2$ at every point x at which $f(x)$ is discontinuous, where $f(x+)$ and $f(x-)$ are the right and left hand limits respectively.*

But we must note that the jumps at the points of discontinuity must be finite. Figure 17.1 shows the graph of a typical piecewise continuous function.

$$f(x) = \begin{cases} x^2, & 0 \le x \le 1 \\ x/2, & 1 < x \le 2. \\ 2x & 2 < x \le 3 \end{cases}$$

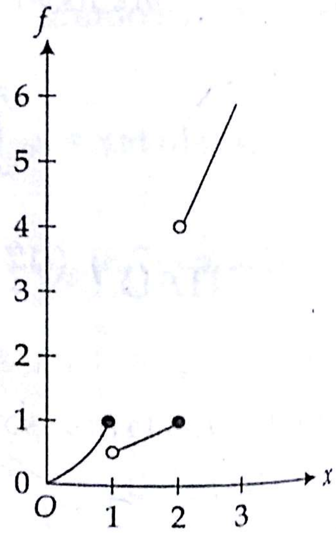

Fig. 17.1

At points of discontinuity (e.g. at $x = 1, 2$), the function $f(x)$ has 'finite' jumps.

An example of a simple function which is not piecewise continuous is

$$f(x) = \begin{cases} 0, & x = 0 \\ 1/x, & 0 < x \le 1 \end{cases}$$

since $\lim_{x \to 0^+} f(x) = \infty$ and so the jump at the discontinuity $x = 0$ is not finite and thus $f(x)$ is not piecewise continuous on $[0,1]$.

Example 17.1 (Saw-tooth wave): Find the Fourier series for the function $f(x) = x$, $-\pi < x < \pi$, when $f(x) = f(x + 2\pi)$.

Solution: The graph of the function $f(x)$ is shown in Fig. 17.2. It is periodic with period 2π. The Fourier coefficients are

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} x\, dx = 0$$

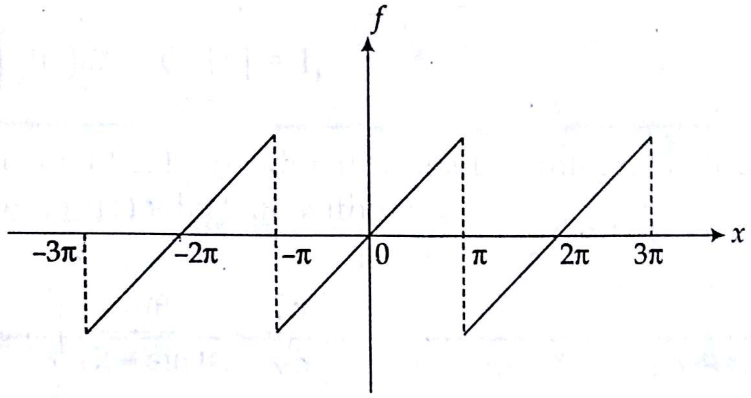

Fig. 17.2

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx = \frac{1}{\pi}\left[\left(\frac{x\sin nx}{n}\right)_{-\pi}^{\pi} - \frac{1}{n}\int_{-\pi}^{\pi}\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_{-\pi}^{\pi} = 0.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx = \frac{1}{\pi}\left[-\left(\frac{x\cos nx}{n}\right)_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi}\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\frac{x\cos nx}{n} + \frac{1}{n^2}\sin nx\right]_{-\pi}^{\pi} = -\frac{2}{n}\cos n\pi = (-1)^{n+1}\frac{2}{n}.$$

Hence Fourier series of $f(x)$ on $[-\pi, \pi]$ is

$$x = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin nx = 2\sin x - \sin 2x + \frac{2}{3}\sin 3x - \frac{1}{2}\sin 4x + \ldots$$

We note that the Fourier series converges to $f(x)$ at every point at which it is continuous. The function $f(x)$ has points of finite discontinuity at $x = -3\pi, -\pi, \pi, 3\pi \ldots$ The average of the extremes at each discontinuity is $\frac{1}{2}(\pi + (-\pi)) = 0$; and it can be verified by direct substitution that the above series converges to zero at $x = -3\pi, -\pi, \pi, 3\pi$, etc.

Example 17.2 (Rectangular wave): Find the Fourier series for the periodic function

$$f(x) = \begin{cases} 0, & -\pi < x \le 0 \\ 4, & 0 < x < \pi \end{cases}$$

Solution: The graph of the periodic function $f(x)$ is shown in Fig. 17.3

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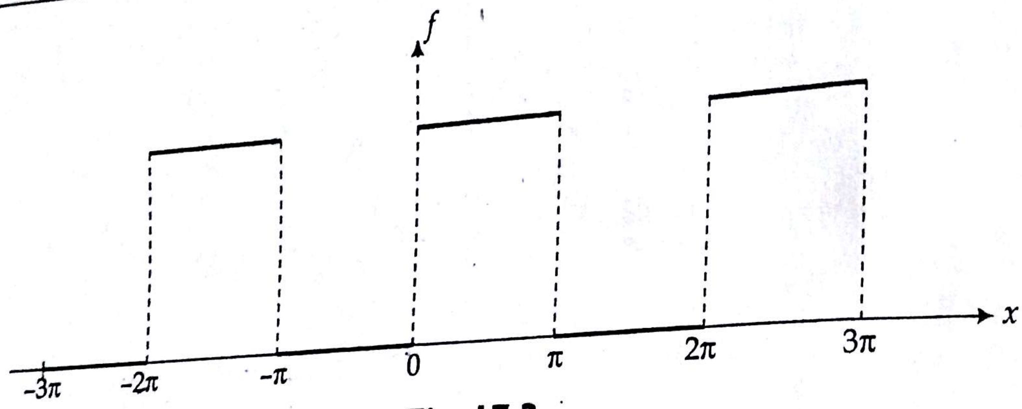

Fig. 17.3

It is periodic with period 2π. The Fourier coefficients are

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} 4dx\right] = 2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cos nx\,dx + \int_{0}^{\pi} 4\cos nx\,dx\right] = \frac{4}{n\pi}[\sin nx]_0^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\sin nx\,dx + \int_{0}^{\pi} 4\sin nx\,dx\right]$$

$$= \frac{-4}{n\pi}[\cos nx]_0^{\pi} = \frac{4}{n\pi}[1-(-1)^n]$$

Thus,
$$b_n = \begin{cases} \dfrac{8}{n\pi}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

Hence the Fourier series of $f(x)$ on $(-\pi, \pi)$ is

$$f(x) = 2 + \frac{8}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right) = 2 + \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)} \qquad \dots(17.14)$$

We note that at points of finite discontinuity of $f(x)$, that is, at $x = -2\pi, -\pi, 0, \pi, 2\pi$, etc., the series converges to 2, the mean value $\frac{1}{2}(0+4)$ and for all others x in the domain of definition of f, the series converges to $f(x)$.

To illustrate how this convergence to f is achieved, we plot some of the partial sums of the series (17.14). The partial sums are

$$S_1(x) = 2, \quad S_2(x) = 2 + \frac{8}{\pi} \sin x, \quad S_3(x) = 2 + \frac{8}{\pi} \sin x + \frac{8}{3\pi} \sin 3x,$$

$$S_4(x) = 2 + \frac{8}{\pi} \sin x + \frac{8}{3\pi} \sin 3x + \frac{8}{5\pi} \sin 5x$$

and so on.

Their graphs in Fig. 17.4 show that the series is convergent and has the sum $f(x)$ Also we note that at $x = -\pi, 0, \pi$, the points of discontinuity of $f(x)$, all partial sums have the value '2', the average of the values 0 and 4 of $f(x)$.

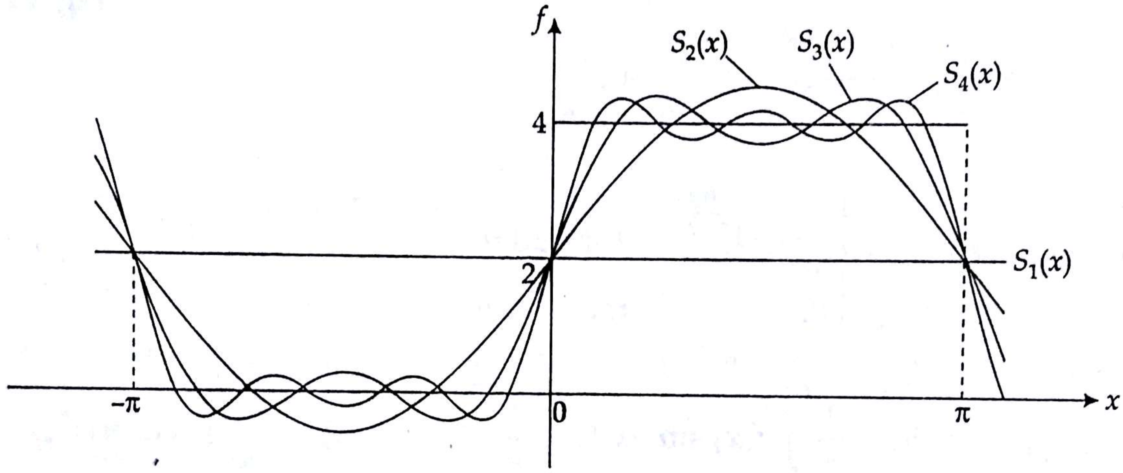

Fig. 17.4

The graph in Fig. 17.4 shows upto the first four partial sum approximations to the function $f(x)$ defined in Example (17.2). The Nth partial sum is given by

$$S_N(x) = a_0 + \sum_{n=1}^{N-1} (a_n \cos nx + b_n \sin nx)$$

and,
$$f(x) = \lim_{N \to \infty} S_N(x).$$

Remark: It should be noted that not every function has a Fourier expansion involving an infinite number of terms. For example, $f(x) = 1 + 2 \sin x \cos x$ when rewritten as $f(x) = 1 + \sin 2x$ is in fact its own Fourier series.

Example 17.3 (Rectangular pulse): Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 0, & -\pi < x < -\pi/2 \\ 1, & -\pi/2 \le x \le \pi/2 \\ 0, & \pi/2 < x < \pi \end{cases}$$

Solution: The graph of the function $f(x)$ on the interval $(-\pi, \pi)$ is shown in Fig. 17.5. The function is periodic with period 2π. The Fourier coefficients are

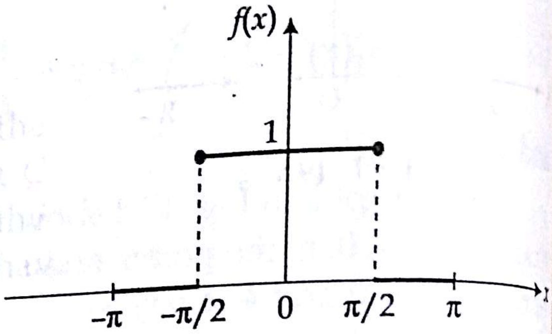

Fig. 17.5

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} dx = \frac{1}{2},$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos nx\,dx$$

$$= \frac{1}{\pi}\left(\frac{\sin nx}{n}\right)_{-\pi/2}^{\pi/2} = \frac{2}{n\pi}\sin\frac{n\pi}{2}$$

which gives,
$$a_n = \begin{cases} \dfrac{2}{n\pi}(-1)^{\frac{n-1}{2}} & \text{for odd } n \\ 0, & \text{for even } n \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \sin nx\,dx = \frac{1}{n\pi}[-\cos nx]_{-\pi/2}^{\pi/2} = 0.$$

Hence the Fourier series of $f(x)$ on $(-\pi, \pi)$ is

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left[\cos x - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - \frac{\cos 7x}{7} + \dots\right]. \quad \dots(17.15)$$

Figs. 17.6a and 17.6b show respectively that graphs of the first five and the first ten terms of the Fourier series expansion (17.15) in the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$. It can be observed that the graphs of $S_5(x)$ and $S_{10}(x)$ exhibit over and undershoots close to the discontinuities $x = -\pi/2, \pi/2$. This oscillatory behaviour of the partial sums $S_N(x)$ near a point of jump discontinuity continues even for large value N and is called the *Gibbs phenomenon*.

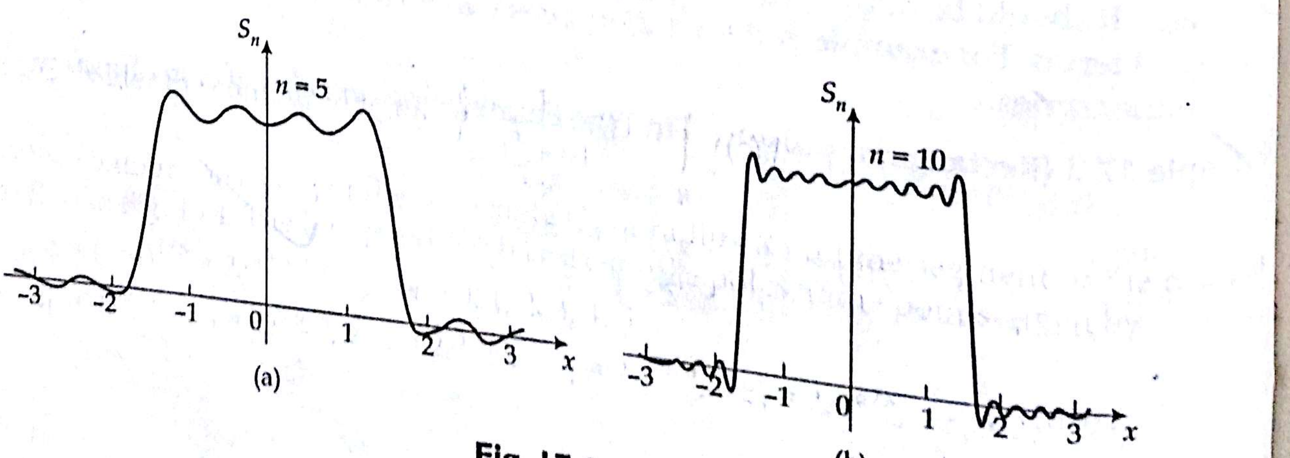

Fig. 17.6

Example 17.4: Find the Fourier series for the function $f(x) = e^{-x}$, $0 < x < 2\pi$ with $f(x + 2\pi) = f(x)$.

Solution: The function $f(x)$ is periodic with period 2π. The Fourier coefficients are

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx = \frac{1}{2\pi}\int_0^{2\pi} e^{-x}dx = -\frac{1}{2\pi}(e^{-x})_0^{2\pi} = \frac{1-e^{-2\pi}}{2\pi},$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx$$

$$= \frac{1}{\pi(n^2+1)}[e^{-x}(-\cos nx + n\sin nx)]_0^{2\pi} = \frac{1-e^{-2\pi}}{\pi(n^2+1)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx$$

$$= \frac{1}{\pi(n^2+1)}[e^{-x}(-\sin nx - n\cos nx)]_0^{2\pi} = \frac{n(1-e^{-2\pi})}{\pi(n^2+1)}.$$

Hence the Fourier series of $f(x)$ on $(0, 2\pi)$ is

$$e^{-x} = \frac{1-e^{-2\pi}}{\pi}\left\{\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{\cos nx}{n^2+1} + \frac{n\sin nx}{n^2+1}\right)\right\}.$$

17.2 FOURIER SERIES OF FUNCTIONS OF PERIOD $T = 2l$

So far we have considered the Fourier series expansion of functions with period 2π. In many applications, we need to find the Fourier series expansion of periodic functions with arbitrary period, say $2l$. The transition from period $T = 2l$ to period $T = 2\pi$ is quite simple and involves only a proportional change of scale.

Consider the periodic function $f(x)$ with period $2l$ defined in $(-l, l)$. To change the problem to period 2π, set

$$v = \frac{\pi x}{l}, \text{ which gives, } \quad x = \frac{lv}{\pi}.$$

Thus $x = \pm l$ corresponds to $v = \pm\pi$ and the function $f(x)$ of period $2l$ in $(-l, l)$ may be regarded as function $g(v)$ of period 2π in $(-\pi, \pi)$. Hence,

$$g(v) = a_0 + \sum_{n=1}^{\infty}(a_n\cos nv + b_n\sin nv) \quad \text{...(17.16)}$$

with coefficients

$$\left.\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(v)\,dv \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(v)\cos nv\,dv \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(v)\sin nv\,dv \end{aligned}\right\} \qquad \ldots(17.17)$$

Making the inverse substitutions, $v = \dfrac{\pi x}{l}$ and $g(v) = f(x)$ in (17.16) and (17.17), we obtain the Fourier series expansion of $f(x)$ in the interval $(-l, l)$ given by

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

with coefficients

$$\boxed{\left.\begin{aligned} a_0 &= \frac{1}{2l}\int_{-l}^{l} f(x)\,dx \\ a_n &= \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx \\ b_n &= \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx \end{aligned}\right\}} \qquad \ldots(17.18)$$

We may replace the interval of integration by any interval of length $T = 2l$, say by the interval $(0, 2l)$.

Example 17.5: Find the Fourier series for the function

$$f(x) = \begin{cases} x, & -1 < x \le 0 \\ x+2, & 0 < x < 1, \end{cases}$$

where $f(x) = f(x+2)$. From the series obtained deduce the sum of the series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$.

Solution: The graph of the periodic function $f(x)$ in the interval $(-1, 1)$ is shown in Fig. 17.7.

The function is periodic with period 2. The Fourier coefficients are

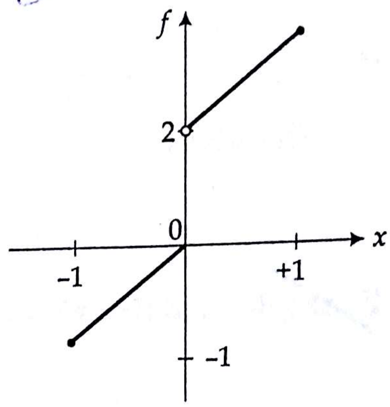

Fig. 17.7

$$a_0 = \frac{1}{2}\int_{-1}^{1} f(x)dx = \frac{1}{2}\left[\int_{-1}^{0} x\,dx + \int_{0}^{1}(x+2)dx\right] = 1,$$

$$a_n = \int_{-1}^{1} f(x)\cos n\pi x\,dx = \int_{-1}^{0} x\cos n\pi x\,dx + \int_{0}^{1}(x+2)\cos n\pi x\,dx$$

$$= 2\int_{0}^{1}\cos n\pi x\,dx = \frac{2}{n\pi}(\sin n\pi x)_0^1 = \frac{2}{n\pi}(\sin n\pi) = 0,$$

$$b_n = \int_{-1}^{1} f(x)\sin n\pi x\,dx = \int_{-1}^{0} x\sin n\pi x\,dx + \int_{0}^{1}(x+2)\sin n\pi x\,dx$$

$$= 2\int_{0}^{1} x\sin n\pi x\,dx + 2\int_{0}^{1}\sin n\pi x\,dx = 2\left[-\frac{x\cos n\pi x}{n\pi} - \frac{\sin n\pi x}{n^2\pi^2}\right]_0^1 - 2\left[\frac{\cos n\pi x}{n\pi}\right]_0^1$$

$$= -\frac{2\cos n\pi}{n\pi} - \frac{2\cos n\pi}{n\pi} + \frac{2}{n\pi} = \frac{2}{n\pi} - \frac{4}{n\pi}(-1)^n = \frac{2}{n\pi}[1-(-1)^n 2]$$

Thus
$$b_n = \begin{cases} \dfrac{6}{n\pi}, & \text{for odd } n \\ \dfrac{-2}{n\pi}, & \text{for even } n \end{cases}$$

Hence the Fourier series of $f(x)$ on $(-1, 1)$ is

$$f(x) = 1 + \frac{2}{\pi}\left[3\sin \pi x - \frac{1}{2}\sin 2\pi x + \frac{1}{3}(3\sin 3\pi x) - \frac{1}{4}\sin 4\pi x + \frac{1}{5}(3\sin 5\pi x) - \frac{1}{6}\sin 6\pi x + - \ldots\right].$$

Further, for $x = 1/2$, $f(x) = x + 2 = 1/2 + 2 = 5/2$.
Setting $x = 1/2$ on both sides of the series above, we obtain

$$\frac{5}{2} = 1 + \frac{2}{\pi}\left[3\sin\frac{\pi}{2} - \frac{1}{2}\sin\pi + \sin\frac{3\pi}{2} - \frac{1}{4}\sin 2\pi + \frac{3}{5}\sin\frac{5\pi}{2} - \frac{1}{6}\sin 3\pi + \frac{3}{7}\sin\frac{7\pi}{2} - \ldots\right]$$

$$= 1 + \frac{2}{\pi}\left[3 - 1 + \frac{3}{5} - \frac{3}{7} + \ldots\right]$$

$\frac{5}{2} = 1 + \frac{6}{\pi}(-)$

or,
$$\frac{3\pi}{4} = 3 - 1 + \frac{3}{5} - \frac{3}{7} + \ldots,$$

This gives
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots.$$

Example 17.6 (Half-wave rectifires): Find the Fourier series of the periodic function

$$f(x) = \begin{cases} 0, & -l < x \le 0 \\ E\sin wx, & 0 < x < l \end{cases}$$

with period $T = 2l = \dfrac{2\pi}{w}$.

Solution: The graph of the periodic function $f(x)$ with period $2\pi/w$ is shown in Fig. 17.8.

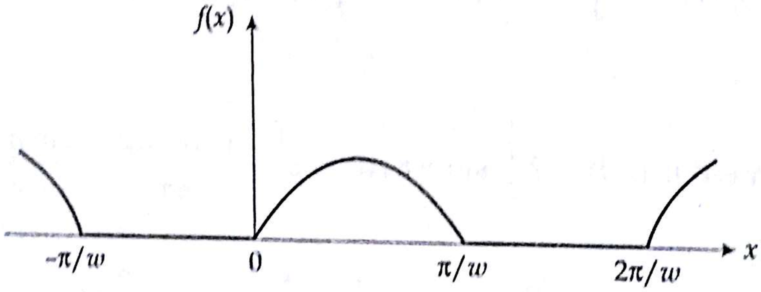

Fig. 17.8

The Fourier coefficients are

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)dx = \frac{w}{2\pi}\left[\int_{-\pi/w}^{0} 0\,dx + \int_{0}^{\pi/w} E\sin wx\,dx\right] = \frac{w}{2\pi}\left[-\frac{E\cos wx}{w}\right]_0^{\pi/w} = \frac{w}{2\pi}\left(\frac{2E}{w}\right) = \frac{E}{\pi}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx = \frac{w}{\pi}\left[\int_{-\pi/w}^{0} 0\cos nwx\,dx + \int_{0}^{\pi/w} E\sin wx\cos nwx\,dx\right]$$

$$= \frac{wE}{2\pi}\int_{0}^{\pi/w}[\sin(1+n)wx + \sin(1-n)wx]dx = \frac{wE}{2\pi}\left[\frac{-\cos(1+n)wx}{(1+n)w} - \frac{\cos(1-n)wx}{(1-n)w}\right]_0^{\pi/w}, n \ne 1$$

$$= \frac{E}{2\pi}\left[\left(\frac{-\cos(1+n)\pi + 1}{1+n}\right) + \left(\frac{-\cos(1-n)\pi + 1}{1-n}\right)\right] = \frac{E}{2\pi}\left[\frac{-(-1)^{1+n} + 1}{1+n} + \frac{-(-1)^{1-n} + 1}{1-n}\right]$$

$(-1)^{n+1}$

$$= \begin{cases} 0, & \text{if } n \text{ is odd} \quad (\text{except } n = 1) \\ \dfrac{-2E}{(n-1)(n+1)\pi}, & \text{if } n \text{ is even} \end{cases}$$

For $n = 1$, $a_1 = \dfrac{w}{\pi}\displaystyle\int_0^{\pi/w} E \sin wx \cos wx\, dx = \dfrac{wE}{2\pi}\int_0^{\pi/w} \sin 2wx\, dx = \dfrac{-E}{2\pi}\left[\dfrac{\cos 2wx}{2}\right]_0^{\pi/w} = 0.$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} dx = \frac{w}{\pi}\left[\int_{-\pi/w}^{0} 0.\sin nwx\, dx + \int_0^{\pi/w} E \sin wx \sin nwx\, dx\right]$$

$$= \frac{wE}{2\pi}\int_0^{\pi/w} [\cos(1-n)wx - \cos(1+n)wx]dx$$

$$= \frac{wE}{2\pi}\left[\frac{\sin(1-n)wx}{(1-n)w} - \frac{\sin(1+n)wx}{(1+n)w}\right]_0^{\pi/w}, \quad n \neq 1$$

$$= \frac{E}{2\pi}(0) = 0.$$

For $\quad n = 1$, $b_1 = \dfrac{wE}{\pi}\displaystyle\int_0^{\pi/w} \sin wx \sin wx\, dx = \dfrac{wE}{\pi}\int_0^{\pi/w} \sin^2 wx\, dx$

$$= \frac{wE}{2\pi}\int_0^{\pi/w} (1 - \cos 2wx)dx = \frac{wE}{2\pi}\left[x - \frac{\sin 2wx}{2w}\right]_0^{\pi/w} = \frac{wE}{2\pi}\left(\frac{\pi}{w}\right) = \frac{E}{2}.$$

Hence Fourier series of $f(x)$ on $\left(\dfrac{-\pi}{w}, \dfrac{\pi}{w}\right)$ is

$$f(x) = \frac{E}{\pi} + \frac{E}{2}\sin wx - \frac{2E}{\pi}\left[\frac{1}{1.3}\cos 2wx + \frac{1}{3.5}\cos 4wx + \ldots\right].$$

$n \neq l$

Example 17.7: Obtain the Fourier series for the periodic function

$$f(x) = e^{-x}, \quad -l < x < l, \text{ where } f(x + 2l) = f(x).$$

Solution: The function $f(x)$ is periodic with period $2l$. The Fourier coefficients are

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)dx = \frac{1}{2l}\int_{-l}^{l} e^{-x}dx = \frac{1}{2l}[-e^{-x}]_{-l}^{l} = \frac{1}{2l}(e^l - e^{-l}) = \frac{\sinh l}{l},$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx = \frac{1}{l}\int_{-l}^{l} e^{-x}\cos\frac{n\pi x}{l}dx$$

$$= \frac{1}{l}\left[\frac{e^{-x}}{1+\left(\frac{n\pi}{l}\right)^2}\left(-\cos\frac{n\pi x}{l}+\frac{n\pi}{l}\sin\frac{n\pi x}{l}\right)\right]_{-l}^{l}$$

$$= \frac{l}{l^2+n^2\pi^2}[-e^{-l}\cos n\pi + e^{l}\cos n\pi]$$

$$= \frac{2l\cos n\pi}{l^2+n^2\pi^2}\left(\frac{e^{l}-e^{-l}}{2}\right) = \frac{2(-1)^n l}{l^2+n^2\pi^2}\sinh l$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx = \frac{1}{l}\int_{-l}^{l} e^{-x}\sin\frac{n\pi x}{l}dx$$

$$= \frac{1}{l}\left[\frac{e^{-x}}{1+\left(\frac{n\pi}{l}\right)^2}\left(-\sin\frac{n\pi x}{l}-\frac{n\pi}{l}\cos\frac{n\pi x}{l}\right)\right]_{-l}^{l}$$

$$= -\frac{l}{l^2+n^2\pi^2}\left[\frac{n\pi}{l}(e^{-l}-e^{l})\cos n\pi\right]$$

$$= \frac{2n\pi\cos n\pi}{l^2+n^2\pi^2}\left(\frac{e^{l}-e^{-l}}{2}\right) = \frac{2(-1)^n n\pi}{l^2+n^2\pi^2}\sinh l$$

Hence the Fourier series of $f(x)$ on $(-l, l)$ is

$$e^{-x} = \frac{\sinh l}{l} + 2\sinh l\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{l^2+n^2\pi^2}\left(l\cos\frac{n\pi x}{l}+n\pi\sin\frac{n\pi x}{l}\right)\right)$$

$$= \sinh l\left[\frac{1}{l} - 2l\left(\frac{1}{l^2+\pi^2}\cos\frac{\pi x}{l} - \frac{1}{l^2+2^2\pi^2}\cos\frac{2\pi x}{l} + \frac{1}{l^2+3^3\pi^2}\cos\frac{3\pi x}{l} - \dots\right)\right.$$
$$\left. - 2\pi\left(\frac{1}{l^2+\pi^2}\sin\frac{\pi x}{l} - \frac{2}{l^2+2^2\pi^2}\sin\frac{2\pi x}{l} + \frac{3}{l^2+3^2\pi^2}\sin\frac{3\pi x}{l} - \dots\right)\right]$$

EXERCISE 17.1

1. Obtain the Fourier series to represent

$$f(x) = \frac{1}{4}(\pi - x)^2, \quad 0 < x < 2\pi, \; f(x + 2\pi) = f(x).$$

2. Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1, & -\pi < x \le 0 \\ -1, & 0 < x < \pi \end{cases}, \quad f(x + 2\pi) = f(x).$$

3. Find the Fourier series expansion of the function

$$f(x) = x \sin x, \quad 0 < x < 2\pi, \; f(x + 2\pi) = f(x).$$

4. Find the Fourier series to represent $f(x) = x - x^2$, $-\pi < x < \pi$. Hence show that

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}.$$

5. Obtain the Fourier series for the function $f(x) = x^2$, $-\pi < x < \pi$. Hence show that

(i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \frac{\pi^2}{6}$ (ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}$

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots = \frac{\pi^2}{8}$

6. Let f be the periodic function shown in the Fig. 17.9, each segment of which is a semicircle of radius π. Show that its Fourier series expansion is

$$f(x) = \frac{\pi^2}{4} + \frac{\pi^2}{2} \sum_{n=1}^{\infty} [J_0(n\pi) + J_2(n\pi)] \cos nx,$$

where J_0 and J_2 have their usual meanings.

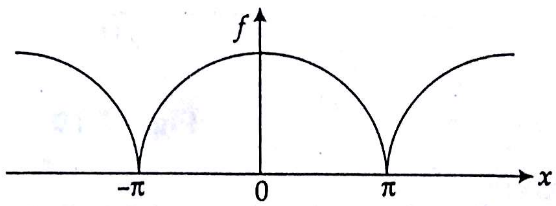

Fig. 17.9

7. Find the Fourier series expansion of the function

$$f(x) = e^{-4x}, \quad -2 \le x \le 2, \; f(x + 4) = f(x).$$

8. Find the Fourier series of the *periodic square wave* given by

$$f(x) = \begin{cases} 0, & -2 < x \le -1 \\ k, & -1 < x < 1 \\ 0, & 1 \le x < 2 \end{cases} \quad f(x + 4) = f(x).$$

9. Find the Fourier series of the periodic function

$$f(x) = \pi \sin \pi x, \quad 0 < x < 1, \; f(x + 1) = f(x).$$

10. Prove that in the range $-\pi < x < \pi$,

$$\cosh ax = \frac{2a^2}{\pi} \sinh a\pi \left(\frac{1}{2a^2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + a^2} \cos nx \right).$$

17.3 FOURIER SERIES EXPANSIONS OF EVEN AND ODD FUNCTIONS

We can save some work in computing the Fourier coefficients, if a function is even or odd. Before proceeding further, first we discuss the concept of even and odd functions and a few of their properties from calculus.

17.3.1 Even and Odd Functions

A function $f(x)$ is an even function on $[-l, l]$, if $f(-x) = f(x)$, for $-l \le x \le l$.

For example, $y = x^2, x^4, \cos x, e^{-|x|}$ are even functions of x on any interval $[-l, l]$. The graph of such a function is symmetric with respect to the y-axis as shown in Fig. 17.10 for $y = x^2$.

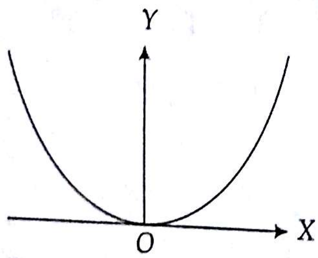

Fig. 17.10

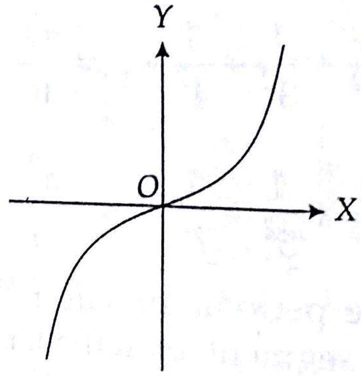

Fig. 17.11

A function $g(x)$ is an odd function on $[-l, l]$, if $g(-x) = -g(x)$, for $-l \le x \le l$.

For example, $y = x, x^3, \sin x$ are odd functions of x on any interval $[-l, l]$. The graph of such a function is symmetric with respect to the orgin as shown in Fig. 17.11 for $y = x^3$.

A function may neither be an even function nor an odd function. For example, functions $y = x + x^2, e^x$ are not even nor odd on any interval $[-l, l]$.

Further, the product of two even functions or two odd functions is an even function and the product of an even function with an odd function is an odd function. Also we have following results from calculus

1. $$\int_{-l}^{l} f(x)dx = 2\int_{0}^{l} f(x)dx,$$

if f is an even function of x on $[-l, l]$, and ...(17.19)

2.
$$\int_{-l}^{l} f(x)dx = 0, \qquad \text{...(17.20)}$$

if f is an odd function on $[-l, l]$.

17.3.2 Fourier Series Expansions of Even and Odd Functions

We have already observed in Example (17.1) with $f(x) = x$, which is an odd function of x on $[-\pi, \pi]$, that the cosine coefficients were all zeros, since $x \cos nx$ is an odd function of x and thus the Fourier expansion of $f(x) = x$ consists only sine terms. We have the following results for the Fourier series expansion of even and odd functions:

Theorem 17.2: *If $f(x)$ is an even function on $[-l, l]$ of period $2l$, then*

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} dx \qquad \text{...(17.21)}$$

with coefficients

$$a_0 = \frac{1}{l}\int_0^l f(x)dx \text{ and } a_n = \frac{2}{l}\int_0^l f(x) \cos \frac{n\pi x}{l} dx. \qquad \text{...(17.22)}$$

Theorem 17.3: *If $f(x)$ is an odd function on $[-l, l]$ of period $2l$, then*

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} dx, \qquad \text{...(17.23)}$$

with coefficients

$$b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l} dx. \qquad \text{...(17.24)}$$

These results follow easily from the applications of (17.19) and (17.20) to the Euler's formulae for the Fourier coefficients given by (17.18).

The series in (17.21) is called the *Fourier cosine series* and the series in (17.23) is called the *Fourier sine series.*

When the period is 2π, then (17.21) reduces to

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx, \qquad \text{...(17.25)}$$

with coefficients

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)dx \text{ and } a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\, dx; \qquad \text{...(17.26)}$$

and (17.23) reduces to

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \qquad ...(17.27)$$

with coefficients

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot \sin nx\, dx. \qquad ...(17.28)$$

Example 17.8: Find the Fourier series of $f(x) = x^2$ on $(-1, 1)$, when $f(x+2) = f(x)$.

Solution: Since the function $f(x) = x^2$ is an even function of x on $[-1, 1]$, thus its Fourier series expansion consists only of constant term and cosine terms; also $f(x)$ is periodic with period 2. The Fourier coefficients are

$$a_0 = \frac{1}{l}\int_0^{l} f(x)dx = \int_0^1 x^2 dx = \left|\frac{x^3}{3}\right|_0^1 = \frac{1}{3},$$

$$a_n = \frac{2}{l}\int_0^{l} f(x)\cos\frac{n\pi x}{l}dx = 2\int_0^1 x^2\cos n\pi x\, dx = 2\left[\left[x^2\frac{\sin n\pi x}{n\pi}\right]_0^1 - 2\int_0^1 x\frac{\sin n\pi x}{n\pi}dx\right]$$

$$= -\frac{4}{n\pi}\left[-x\frac{\cos n\pi x}{n\pi} + \frac{\sin n\pi x}{n^2\pi^2}\right]_0^1 = \frac{4}{n^2\pi^2}\cos n\pi = \frac{4(-1)^n}{n^2\pi^2}.$$

Hence, the Fourier series for $f(x)$ on $(-1, 1)$ is

$$x^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi x.$$

Example 17.9: Find the Fourier series expansion of the function $f(x) = \sin ax$, $-\pi < x < \pi$, where a is not an integer.

Solution: Since $\sin ax$ is an odd function of x on $[-\pi, \pi]$, thus its Fourier series expansion consists of only sine terms. The Fourier coefficients are

$$b_n = \frac{2}{l}\int_0^{l} f(x)\sin\frac{n\pi x}{l}dx = \frac{2}{\pi}\int_0^{\pi}\sin ax \sin nx\, dx = \frac{1}{\pi}\int_0^{\pi}[\cos(n-a)x - \cos(n+a)x]dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(n-a)x}{n-a} - \frac{\sin(n+a)x}{(n+a)}\right]_0^{\pi} = \frac{1}{\pi}\left[\frac{\sin(n-a)\pi}{(n-a)} - \frac{\sin(n+a)\pi}{n+a}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n(-\sin a\pi)}{n-a} - \frac{(-1)^n \sin a\pi}{n+a}\right] = (-1)^{n+1}\frac{2n \sin a\pi}{\pi(n^2-a^2)}.$$

Hence the Fourier series for $f(x)$ on $(-\pi, \pi)$ is

$$\sin ax = \frac{2 \sin a\pi}{\pi} \sum_{n=1}^{\infty} \frac{n(-1)^{n+1}}{n^2-a^2} \sin nx.$$

Example 17.10: Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1+2x/\pi, & -\pi < x < 0 \\ 1-2x/\pi, & 0 \le x < \pi \end{cases} \qquad f(x+2\pi) = f(x).$$

Also deduce that $$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots = \frac{\pi^2}{8}.$$

Solution: The graph of the function $f(x)$ is shown in Fig. 17.12. The graph is symmetrical about y-axis. Hence, $f(x)$ is an even function of x over $(-\pi, \pi)$ with period 2π. Thus its Fourier expansion consists of only constant term and cosine terms. The Fourier coefficients are

$$a_0 = \frac{1}{l}\int_0^l f(x)dx = \frac{1}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)dx = \frac{1}{\pi}\left[x-\frac{x^2}{\pi}\right]_0^{\pi} = 0,$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx = \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx\,dx$$

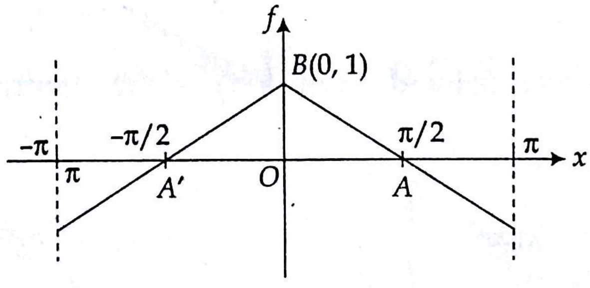

Fig. 17.12

$$= \frac{2}{\pi}\left[\left[\left(1-\frac{2x}{\pi}\right)\frac{\sin nx}{n}\right]_0^{\pi} + \int_0^{\pi}\frac{2}{\pi}\frac{\sin nx}{n}dx\right]$$

$$= \frac{4}{\pi^2}\left[\frac{-\cos nx}{n^2}\right]_0^{\pi} = \frac{4}{n^2\pi^2}(1-\cos n\pi) = \frac{4}{n^2\pi^2}(1-(-1)^n).$$

$$\text{Thus, } a_n = \begin{cases} \dfrac{8}{n^2\pi^2}, & \text{for odd } n \\ 0, & \text{for even } n \end{cases}.$$

Hence the Fourier series of $f(x)$ on $(-\pi, \pi)$ is

$$f(x) = \frac{8}{\pi^2}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right].$$

At $x = 0$, $f(0) = 1$. Setting $x = 0$ in (17.29), we obtain

$$1 = \frac{8}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right), \quad \text{or} \quad \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}.$$

17.4 FOURIER HALF-RANGE COSINE AND SINE SERIES

We have seen that in case $f(x)$ is defined on $-l \le x \le l$ we can write its Fourier series, and the coefficients of the series are determined by the function and the interval. Let us suppose that a function $f(x)$ of period $2l$ is specified only on a half-range interval $0 \le x \le l$. In such a case, we have a choice to extend the definition of the function to the interval $-l \le x \le l$ in a suitable manner, even or odd, to find its Fourier cosine or sine expansion respectively and then restricting the Fourier series representation of the extended function to the original half-range interval $0 \le x \le l$.

17.4.1 The Fourier Cosine Series on $0 \le x \le l$

Let a function $f(x)$ specified on the interval $0 \le x \le l$ is extended to the interval $-l \le x \le l$ as an even function $g(x)$ of x, given by

$$g(x) = \begin{cases} f(-x), & -l \le x \le 0 \\ f(x), & 0 \le x \le l \end{cases}$$

which coincides with $f(x)$ on the interval $0 \le x \le l$, refer Fig. 17.13a and 17.13b, then

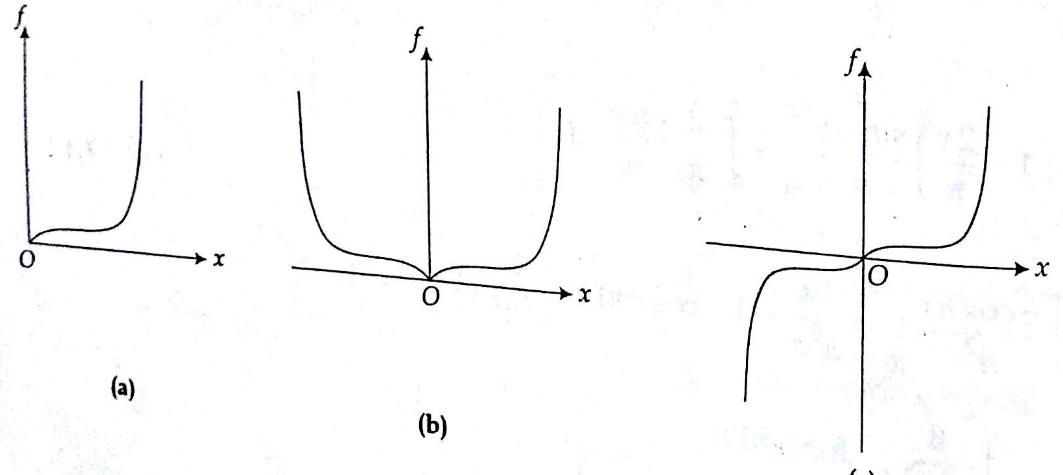

Fig. 17.13

the Fourier series representation of $f(x)$ on the interval $0 \le x \le l$ is the cosine series given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}, \qquad \dots(17.30)$$

where $\quad a_0 = \frac{1}{l}\int_0^l f(x)dx$ and $a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx.$

17.4.2 The Fourier Sine Series on $0 \le x \le l$

If $f(x)$ specified on the interval $0 \le x \le l$, is extended to the interval $-l \le x \le l$ as an odd function $g(x)$ of x, given by

$$g(x) = \begin{cases} -f(-x), & -l \le x \le 0 \\ f(x), & 0 \le x \le l \end{cases}$$

which coincides with $f(x)$ on the interval $0 \le x \le l$, refer Figs. 17.13a and 17.13c, then the Fourier series representation of $f(x)$ on the interval $0 \le x \le l$ is the sine series given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}, \qquad \text{...(17.31)}$$

where $\quad b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx.$

The expansions (17.30) and (17.31) are respectively referred to as *half-range cosine series* and *half-range sine series* expansions of $f(x)$.

Example 17.11: Find the Fourier sine and cosine series expansions of $f(x) = x$ for $0 \le x \le \pi$.

Solution: The sine series representation of $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx; \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx.$$

Substituting for $f(x)$, we have

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx = \frac{2}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi} = -\frac{2}{n}\cos n\pi = (-1)^{n+1}\frac{2}{n}.$$

Hence, the required sine series expansion of $f(x) = x$ is

$$x = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}, \quad 0 \le x \le \pi.$$

Next, the cosine series representation of $f(x)$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx; \quad a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)dx \text{ and } a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx.$$

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Substituting for $f(x)$, we have

$$a_0 = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{1}{\pi}\left(\frac{x^2}{2}\right)_0^{\pi} = \pi/2, \text{ and}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx = \frac{2}{\pi}\left[x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi} = \frac{2}{\pi n^2}[(-1)^n - 1].$$

Thus,
$$a_n = \begin{cases} -\dfrac{4}{\pi n^2}, & \text{when } n \text{ is odd.} \\ 0, & \text{when } n \text{ is even.} \end{cases}$$

Hence the cosine series expansion of $f(x) = x$ is

$$x = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}, \quad 0 \le x \le \pi.$$

Example 17.12: Write the sine series expansion of

$$f(x) = \begin{cases} 1, & 0 < x \le \pi/2 \\ 2, & \pi/2 < x < \pi \end{cases}$$

on $[0, \pi]$ and also discuss its convergence.

Solution: The sine series expansion is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \text{ with } b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

Substituting for $f(x)$, we have

$$b_n = \frac{2}{\pi}\left[\int_0^{\pi/2}\sin nx\,dx + \int_{\pi/2}^{\pi} 2\sin nx\,dx\right] = \frac{2}{\pi}\left[\left[-\frac{\cos nx}{n}\right]_0^{\pi/2} + 2\left[-\frac{\cos nx}{n}\right]_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{n\pi}\left[\left(1-\cos\frac{n\pi}{2}\right) - 2\left(\cos n\pi - \cos\frac{n\pi}{2}\right)\right] = \frac{2}{n\pi}\left[\cos\frac{n\pi}{2} + 1 - 2(-1)^n\right].$$

Hence, the sine series expansion of $f(x)$ is

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\cos\frac{n\pi}{2} + 1 - 2(-1)^n\right)\sin nx.$$

The series converges to 0, for $x = 0$; to 1, for $0 < x < \pi/2$; to $\frac{1}{2}(1+2) = 3/2$, for $x = \pi/2$; and to 2, for $\pi/2 < x < \pi$, and again to 0 for $x = \pi$.

EXERCISE 17.2

1. Expand the function $f(x) = x^4$ on $[-1, 1]$ as a Fourier series.
2. Expand the function $f(x) = x \sin x$ as a Fourier series in the interval $[-\pi, \pi]$. Also deduce that

$$\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \frac{1}{7.9} + \ldots = \frac{1}{4}(\pi - 2).$$

3. Expand the function $f(x) = |x|, -\pi < x < \pi$ as Fourier series and hence deduce that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots = \frac{\pi^2}{8}.$$

4. Expand $f(x) = |\cos x|, -\pi < x < \pi$ as a Fourier series.

5. Expand $f(x) = \begin{cases} -1/2, & -\pi < x < 0 \\ 1/2, & 0 < x < \pi \end{cases}$ $\quad f(x + 2\pi) = f(x)$

   as a Fourier series.

$\frac{1}{(2n-1)^{n+1}}$ $n>1$

6. Expand $f(x) = \begin{cases} -x+1, & -\pi \le x \le 0 \\ x+1, & 0 \le x \le \pi \end{cases}$ $\quad f(x + 2\pi) = f(x)$

$\frac{1}{1^2} + 1$

   as a Fourier series. Also deduce the value of $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$

7. Obtain the Fourier series expansion of $f(x) = 4 - x^2, -2 \le x \le 2$. Also show that

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$$

8. Find the Fourier cosine and sine series of $f(x) = 1, 0 \le x \le 2$.
9. Find the Fourier cosine series of the function

$$f(x) = \begin{cases} x^2, & 0 \le x < 2 \\ 4, & 2 \le x \le 4. \end{cases}$$

10. Find the Fourier sine series of the function

$$f(x) = \begin{cases} x, & 0 \le x < \pi/2 \\ \pi - x, & \pi/2 \le x \le \pi. \end{cases}$$

11. Expand $\sin\left(\frac{\pi x}{l}\right)$ in half-range cosine series in the interval $[0, l]$.

12. Find the Fourier cosine series for the function

$$f(x) = \begin{cases} \sin x, & 0 \le x \le \pi/2 \\ 0, & \pi/2 < x \le \pi. \end{cases}$$

17.5 INTEGRATION AND DIFFERENTIATION OF FOURIER SERIES. THE PARSEVAL'S FORMULA

First we discuss the termwise integration and differentiation of the Fourier series of a function $f(x)$ and then using the concept of termwise integration, we derive the Parseval's formula.

17.5.1 Termwise Integration and Differentiation of Fourier Series

Let $f(x)$ be piecewise continuous on $[-l, l]$ with Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right).$$

Then for any x, $-l \le x \le l$,

$$\int_{-l}^{x} f(t)dt = a_0(x + l) + \frac{l}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[a_n \sin\frac{n\pi x}{l} - b_n\left(\cos\frac{n\pi x}{l} - (-1)^n\right)\right]. \quad ...(17.32)$$

Note that the expression on the right side of (17.32) is exactly what we get by integrating the Fourier series term by term from $-l$ to x. This holds even if the Fourier series does not converge to $f(x)$ at a particular value of x.

Example 17.13: Use the Fourier series representation of $f(x) = x$, $-\pi < x < \pi$ to find the Fourier series representation for x^2 over $-\pi < x < \pi$.

Solution: The Fourier series representation of the function $f(x) = x$ over $[-\pi, \pi]$, refer Example 17.1 is

$$x = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin nx. \quad ...(17.33)$$

Integrating (17.33) term by term over the interval $(-\pi, x)$ for any x in $-\pi < x < \pi$, we obtain

$$\int_{-\pi}^{x} x\,dx = \sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\int_{-\pi}^{x}\sin nx\,dx,$$

or,

$$\frac{1}{2}(x^2 - \pi^2) = \sum_{n=1}^{\infty}\frac{2}{n^2}(-1)^n[\cos nx - \cos n\pi],$$

or,

$$x^2 - \pi^2 = 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}[\cos nx - (-1)^n],$$

$$= 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx - 4\sum_{n=1}^{\infty} \frac{1}{n^2}. \quad \ldots(17.34)$$

Using the result $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, (refer, Problem 5 Exercise 17.1), (17.34) becomes

$$x^2 - \pi^2 = 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx - \frac{2\pi^2}{3},$$

or,
$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx,$$

the Fourier series representation of x^2 over $-\pi \le x \le \pi$.

We have seen that termwise integration of Fourier series of a function $f(x)$ leads to some meaningful results. But same is not always true in case of termwise differentiation,

Consider again the Fourier series expansion of x over the interval $-\pi \le x \le \pi$. It is

$$x = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin nx. \quad \ldots(17.35)$$

The series converges to x for $-\pi < x < \pi$.

Differentiating w.r.t. x for $-\pi < x < \pi$, we obtain

$$1 = \sum_{n=1}^{\infty} 2(-1)^{n+1} \cos nx, \quad \ldots(17.36)$$

which is absurd, since the right side of (17.36) does not even converge over $-\pi < x < \pi$.

Thus, in this case termwise derivative of Fourier series is not related to the derivative of $f(x)$. However, for the validity of termwise differentiation of Fourier series, the series should be uniformly convergent over the given interval which is not true in case of the right side of (17.35).

Example 17.14: Use the Fourier series representation of $f(x) = x^2$, $-\pi < x < \pi$ to find the Fourier series representation for x over $-\pi < x < \pi$.

Solution: The Fourier series representation of the function $f(x) = x^2$ over $[-\pi, \pi]$, refer Example (17.13), is

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx. \quad \ldots(17.37)$$

The series on the right side of (17.37) is uniformly convergent over $-\pi < x < \pi$, thus termwise differentiation of (17.37) is admissible. Then for $-\pi < x < \pi$

$$f(x) = 2x = 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$$

or,
$$x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx,$$

which is the Fourier series representation of x over $-\pi < x < \pi$.

17.5.2 The Parseval's Formula

We state the following result:

Theorem 17.4 (Parseval's formula): *If the Fourier series for $f(x)$ converges uniformly on $(-l, l)$, then*

$$a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{1}{2l}\int_{-l}^{l} [f(x)]^2 dx, \qquad \text{...(17.38)}$$

where a_0, a_n, b_n, $n = 1, 2, \ldots$ are the Fourier coefficients of f on $(-l, l)$.

Proof. The Fourier series for $f(x)$ on $(-l, l)$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right).$$

Since the series is uniformly convergent over $(-l, l)$, multiplying both sides of it by $f(x)$ and integrating termwise from $-l$ to l, we get

$$\int_{-l}^{l} [f(x)]^2 dx = a_0\int_{-l}^{l} f(x)dx + \sum_{n=1}^{\infty}\left(a_n\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx + b_n\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx\right)$$

$$= l\left[2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right], \quad \text{using (17.18)}$$

or,
$$a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{2l}\int_{-l}^{l} [f(x)]^2 dx, \text{ which is (17.38).}$$

In case the interval is $(0, 2l)$, then the Parseval formula corresponding to (17.38), is

$$a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{2l}\int_{0}^{2l} [f(x)]^2 dx. \qquad \text{...(17.39)}$$

On the similar lines we can prove the following two results corresponding to half-range expansion of $f(x)$ over the interval $[0, l]$.

Theorem 17.5: *If half-range cosine series of $f(x)$ converges uniformly to $f(x)$ over $(0, l)$, then*

$$a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 = \frac{1}{l}\int_0^l [f(x)]^2 dx, \qquad \text{...(17.40)}$$

where $$a_0 = \frac{1}{l}\int_0^l f(x)dx \text{ and } a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx.$$

Theorem 17.6: *If the half-range sine series of $f(x)$ converges uniformly to $f(x)$ over $(0, l)$, then*

$$\frac{1}{2}\sum_{n=1}^{\infty} b_n^2 = \frac{1}{l}\int_0^l [f(x)]^2 dx, \qquad \text{...(17.41)}$$

where $$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx.$$

The expression

$$[f(x)]_{rms} = \sqrt{\frac{1}{2l}\int_{-l}^{l}[f(x)]^2 dx} \qquad \text{...(17.42)}$$

is called the *root mean square (r.m.s.) value of the function $f(x)$ over the interval $(-l, l)$*. The *r.m.s.* value of a periodic function finds applications in engineering physics.

Example 17.15: Find the Fourier series expansion of the function $f(x) = |x|$ defined over the interval $(-2, 2)$. Using Parseval equality, prove that

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots = \frac{\pi^4}{96}.$$

Solution: Since $f(x)$ is an even function of $f(x)$ over the interval $(-2, 2)$, thus the Fourier series expansion of $f(x)$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}, \qquad \text{...(17.43)}$$

We have $$a_0 = \frac{1}{2}\int_0^2 x\,dx = \frac{1}{2}\left[\frac{x^2}{2}\right]_0^2 = 1$$

$$a_n = \frac{2}{2}\int_0^2 x\cos\frac{n\pi x}{2}dx = \int_0^2 x\cos\frac{n\pi x}{2}dx = \frac{4}{n^2\pi^2}(\cos n\pi - 1) = \frac{4}{n^2\pi^2}[(-1)^n - 1].$$

Hence, (17.43) becomes

$$f(x) = 1 + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos\frac{n\pi x}{2}.$$

Applying the Parseval equality

$$a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 = \frac{1}{l}\int_0^l [f(x)]^2 dx,$$

we obtain

$$1^2 + \frac{1}{2}\cdot\frac{16}{\pi^4}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]^2 = \frac{1}{2}\int_0^2 x^2 dx$$

or, $$1 + \frac{32}{\pi^4}\left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\right) = 4/3, \text{ which gives}$$

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots = \frac{\pi^4}{96}.$$

Example 17.16: Using the Fourier coefficients of $f(x) = \cos(x/2)$ on $(-\pi, \pi)$ prove that

$$\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)^2} = \frac{\pi^2 - 8}{16}.$$

Solution: The function $f(x) = \cos(x/2)$ is an even function of x on the interval $(-\pi, \pi)$. The Fourier coefficients are

$$a_0 = \frac{1}{\pi}\int_0^{\pi}\cos\frac{x}{2}dx = \frac{2}{\pi}\left[\sin\frac{x}{2}\right]_0^{\pi} = \frac{2}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}\cos\frac{x}{2}\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi}\left[\cos\left(n+\frac{1}{2}\right)x + \cos\left(n-\frac{1}{2}\right)x\right]dx$$

$$= \frac{2}{\pi}\left[\frac{\sin\left(n+\frac{1}{2}\right)x}{2n+1} + \frac{\sin\left(n-\frac{1}{2}\right)x}{2n-1}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{\sin\left(n+\frac{1}{2}\right)\pi}{2n+1} + \frac{\sin\left(n-\frac{1}{2}\right)\pi}{2n-1}\right]$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n}{2n+1} - \frac{(-1)^n}{2n-1}\right] = \frac{-4(-1)^n}{\pi(4n^2-1)}.$$

By Parseval formula

$$a_0^2 + \frac{1}{2}\sum a_n^2 = \frac{1}{l}\int_0^l [f(x)]^2 dx.$$

Substituting for a_0, a_n, l and $f(x)$, we get

$$\frac{4}{\pi^2} + \frac{16}{2\pi^2}\sum \frac{1}{(4n^2-1)^2} = \frac{1}{\pi}\int_0^\pi \cos^2(x/2)\,dx = \frac{1}{2\pi}\int_0^\pi (1+\cos x)\,dx = \frac{1}{2}$$

$$\text{or,}\quad \sum_{n=1}^{\infty} \frac{1}{(4n^2-1)^2} = \frac{\pi^2-8}{16}.$$

EXERCISE 17.3

1. If $f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$ has the Fourier series representation, $f(x) = \dfrac{4}{\pi}\sum_{n=1}^{\infty} \dfrac{\sin(2n-1)x}{2n-1}$,

   then find the Fourier series representation of $g(x) = \begin{cases} -x-\pi, & -\pi < x < 0 \\ x-\pi, & 0 < x < \pi. \end{cases}$

2. Find the Fourier series of $f(x) = \pi^2 - x^2$ for $-\pi < x < \pi$ and use it to find the Fourier series of x and $x(\pi^2 - x^2)$.

3. Let $f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ x, & 0 < x \le \pi \end{cases}$. Write the Fourier series of $f(x)$ on $[-\pi, \pi]$ and show that this series converges to $f(x)$ on $(-\pi, \pi)$, and can be integrated term by term, and thus obtain a trigonometric series expansion for $g(x) = \begin{cases} 0, & -\pi \le x < 0 \\ x^2/2, & 0 < x \le \pi. \end{cases}$

4. If $f(x) = x \sin x$, $-\pi < x < \pi$ with the Fourier series representation

   $$f(x) = \pi - \frac{1}{2}\pi\cos x + 2\pi\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^2-1}\cos nx,$$

   show that the series can be differentiated term by term and hence find the Fourier series expansion for $g(x) = x\cos x + \sin x$, for $-\pi < x < \pi$.

5. Given $f(x) = \begin{cases} \sin 2x, & -\pi \le x < -\pi/2 \\ 0, & -\pi/2 \le x \le \pi/2 \\ \sin 2x, & \pi/2 < x \le \pi \end{cases}$

Find the Fourier series expansion for $f'(x)$ by differentiating the Fourier expansion for $f(x)$.

6. Using the Fourier coefficients for the function $f(x) = \begin{cases} -1, & -\pi < x \le 0 \\ 1, & 0 < x < \pi \end{cases}$, show that

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}.$$

7. Using the Fourier coefficients for the function $y = x^2$ on $[-\pi, \pi]$, show that $\dfrac{\pi^4}{90} = \sum_{n=1}^{\infty} \dfrac{1}{n^4}$.

8. From the coefficients of the half-range cosine series of the function

$$f(x) = \begin{cases} \pi x, & 0 < x < 1 \\ \pi(2-x), & 1 < x < 2 \end{cases}, \quad \text{find } \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}.$$

17.6 COMPLEX FORM OF THE FOURIER SERIES

To simplify the calculations it is sometimes convenient to work in terms of complex numbers even when the parameters under reference are reals. In this context we study the complex form of the Fourier series of a real function $f(x)$. This form is of special interest in the study of electrical circuits.

17.6.1 Complex Fourier Series

Let $f(x)$ be a real periodic function of period $2l$ over the interval $(-l, l)$. Then

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right).$$

Since, $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$, therefore, this series can be expressed as

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left\{ \frac{a_n}{2}\left(e^{\frac{in\pi x}{l}} + e^{\frac{-in\pi x}{l}} \right) + \frac{b_n}{2i}\left(e^{\frac{in\pi x}{l}} - e^{\frac{-in\pi x}{l}} \right) \right\}$$

and after regrouping the terms, we have

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(\frac{a_n - ib_n}{2} \right) e^{\frac{in\pi x}{l}} + \sum_{n=1}^{\infty} \left(\frac{a_n + ib_n}{2} \right) e^{\frac{-in\pi x}{l}} \qquad \text{...(17.44)}$$

We define $c_0 = a_0$, $c_n = \dfrac{a_n - ib_n}{2}$ and $c_{-n} = \dfrac{a_n + ib_n}{2}$, for $n = 1, 2, \ldots$.

Clearly c_n and c_{-n} are complex conjugates. Using these, (17.44) becomes

$$f(x) = \lim_{k\to\infty} \sum_{n=-k}^{k} c_n e^{\frac{in\pi x}{l}}, \quad \text{for} \quad -l < x < l, \qquad ...(17.45)$$

where $$c_n = \frac{1}{2}\left[\frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx - \frac{i}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx\right]$$

$$= \frac{1}{2l}\int_{-l}^{l} f(x)\left(\cos\frac{n\pi x}{l} - i\sin\frac{n\pi x}{l}\right)dx = \frac{1}{2l}\int_{-l}^{l} f(x)e^{\frac{-in\pi x}{l}}dx$$

and, $$c_{-n} = \frac{1}{2l}\int_{-l}^{l} f(x)\left(\cos\frac{n\pi x}{l} + i\sin\frac{n\pi x}{l}\right)dx = \frac{1}{2l}\int_{-l}^{l} f(x)e^{\frac{in\pi x}{l}}dx\,.$$

Combining these two, we have

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{\frac{-in\pi x}{l}}dx, \quad n = 0, \pm 1, \pm 2. \qquad ...(17.46)$$

Thus, (17.45) is the complex form of the Fourier series representation of the periodic function $f(x)$ defined over the interval $-l < x < l$, with Fourier coefficients given by (17.46).

Since, the complex form of the Fourier series representation of a function is derived from its real variable definition, the convergence properties of the complex Fourier series are the same as those for the real variable case. Thus, at points of continuity of $f(x)$ the series converges to $f(x)$, while at points of discontinuity it converges to the mid-point.

Example 17.17: Find the complex Fourier series representation of the function

$$f(x) = \begin{cases} 0, & 0 < x \le 1 \\ 1, & 1 < x < 4 \end{cases}$$

when $$f(x) = f(x+4).$$

Solution: The function $f(x)$ is periodic with period 4 defined on the interval (0, 4), with $2l = 4$. Thus the complex Fourier coefficients c_n are given by

$$c_n = \frac{1}{4}\int_0^4 f(x)e^{\frac{-in\pi x}{2}}dx = \frac{1}{4}\int_1^4 e^{\frac{-in\pi x}{2}}dx.$$

For $n = 0$, we get

$$c_0 = \frac{1}{4}\int_1^4 dx = 3/4.$$

For all c_n, except $n = 0$

$$c_n = \frac{1}{4}\left[\frac{-2}{in\pi}e^{\frac{-in\pi x}{2}}\right]_1^4 = \frac{i}{2\pi n}[1 - e^{-in\pi/2}].$$

Hence, the complex Fourier series representation of $f(x)$ is

$$f(x) = \frac{3}{4} + \lim_{k\to\infty}\sum_{n=-k}^{k}\frac{i}{2\pi n}(1 - e^{-in\pi/2})e^{\frac{in\pi x}{2}}, \quad (n \neq 0).$$

Example 17.18: Find the complex Fourier series representation of the function

$$f(x) = e^{-x}, \quad -\pi < x < \pi; \quad f(x) = f(x + 2\pi)$$

Solution: The function $f(x)$ is periodic with period 2π, defined on the interval $(-\pi, \pi)$. Here $l = \pi$, thus the complex Fourier coefficients are

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-x}e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-(1+in)x}\,dx$$

$$= \frac{-1}{2\pi(1+in)}[e^{-(1+in)x}]_{-\pi}^{\pi} = \frac{-1}{2\pi(1+in)}[e^{-(1+in)\pi} - e^{(1+in)\pi}]$$

$$= \frac{-1}{2\pi(1+in)}[e^{-\pi}(\cos n\pi - i\sin n\pi) - e^{\pi}(\cos n\pi + i\sin n\pi)]$$

$$= \frac{(1-in)}{2\pi(1+n^2)}[(e^{\pi} - e^{-\pi})\cos n\pi] = (-1)^n\frac{(1-in)\sinh\pi}{\pi(1+n^2)}.$$

Hence, the complex Fourier series is

$$f(x) = \frac{\sinh\pi}{\pi}\lim_{k\to\infty}\sum_{n=-k}^{k}(-1)^n\left(\frac{1-in}{1+n^2}\right)e^{inx}.$$

17.6.2 Frequency Spectra of a Function $f(x)$

In applications of Fourier series to periodic physical phenomena with fundamental period T, it is sometimes more convenient to work in terms of the angular frequency w, defined as

$$w = 2\pi/T = 2\pi/2l = \pi/l,$$

called the *fundamental angular frequency*. In terms of w, the Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right),$$

can be written as

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nwx + b_n \sin nwx), \qquad ...(17.47)$$

where $\quad a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)dx, \quad a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos nwx\, dx$ and $b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin nwx\, dx; \quad l = \pi/w.$

In terms of the fundamental angular frequency, the complex Fourier series form (17.45) can be expressed as

$$f(x) = \lim_{k\to\infty} \sum_{n=-k}^{k} c_n e^{inwx}, \qquad ...(17.48)$$

where $$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-inwx}dx, \quad l = \pi/w, \; n = 0, \pm 1, \pm 2, ... \qquad ...(17.49)$$

The plot of the points $(nw, |c_n|)$, where w is the fundamental angular frequency and c_n are the Fourier coefficients as defined in (17.49) is called the *frequency spectrum* or *amplitude spectrum* of the function $f(x)$ and the number nw is called the *nth harmonic frequency* of the function $f(x)$.

Example 17.19: Find the frequency spectrum of the periodic pulse defined by

$$f(x) = 3x/4, \quad 0 \le x \le 8 \text{ and } \quad f(x+8) = f(x).$$

Solution: The function $f(x)$ is periodic with period $T = 2l = 8$ defined on [0, 8]. Thus the fundamental angular frequency w is

$$w = \frac{2\pi}{T} = \frac{2\pi}{8} = \frac{\pi}{4}.$$

The complex Fourier coefficients are

$$c_n = \frac{1}{8}\int_0^8 \frac{3}{4} x e^{-\frac{in\pi}{4}x} dx = \frac{3}{32}\left[x\left(\frac{4}{-in\pi}\right)e^{-\frac{in\pi}{4}x} - \left(\frac{4}{-in\pi}\right)^2 e^{-\frac{in\pi x}{4}}\right]_0^8$$

$$= \frac{3i}{n\pi}, \quad n \neq 0, \text{ after simplification.}$$

For $n = 0$, $\quad c_0 = \frac{3}{32}\int_0^8 x\,dx = \frac{3}{32}\left(\frac{x^2}{2}\right)_0^8 = 3.$

The frequency spectrum of $f(x)$ is a plot of points $(nw, |c_n|)$, where

$$nw = \frac{n\pi}{4}, \quad |c_0| = 3 \text{ and } |c_n| = \frac{3}{|n|\pi}, \text{ for } n = \pm 1, \pm 2, \pm 3, \ldots .$$

The plot is shown in Fig. 17.14.

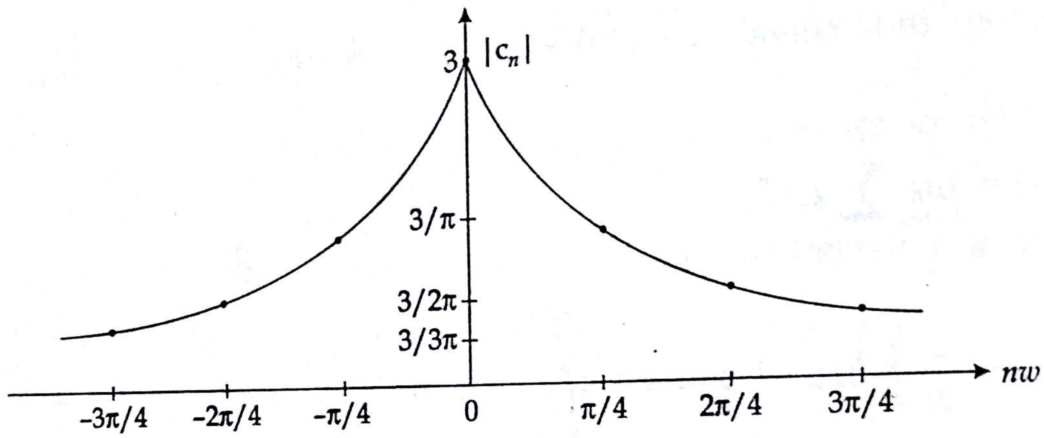

Fig. 17.14

Example 17.20: Find the frequency spectrum of the periodic function

$$f(x) = \begin{cases} 0, & -\pi < x < -\pi/2 \\ 1, & -\pi/2 \le x \le \pi/2 \\ 0, & \pi/2 < x < \pi \end{cases}$$

when $f(x + 2\pi) = f(x)$, for all x.

Solution: The function $f(x)$ is periodic with period $T = 2\pi$ defined over the interval $(-\pi, \pi)$. The fundamental angular frequency w is

$$w = 2\pi/T = 2\pi/2\pi = 1.$$

The complex Fourier coefficients are

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{-inx}dx.$$

For $n = 0$, $c_0 = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} dx = \frac{1}{2}$; and for all other $n \neq 0$

$$c_n = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{-inx}dx = \frac{1}{n\pi}\left[\frac{e^{in\pi/2} - e^{-in\pi/2}}{2i}\right] = \frac{1}{n\pi}\sin\frac{n\pi}{2}, \quad n = \pm 1, \pm 2, \ldots.$$

The frequency spectrum of $f(x)$ is a plot of points $(nw, |c_n|)$. Here

$$nw = n,\ |c_0| = \frac{1}{2} \text{ and } |c_n| = \frac{1}{n\pi}\left|\sin\frac{n\pi}{2}\right|, \text{ for } n = \pm 1, \pm 2, \ldots .$$

The plot is as shown in Fig. 17.15.

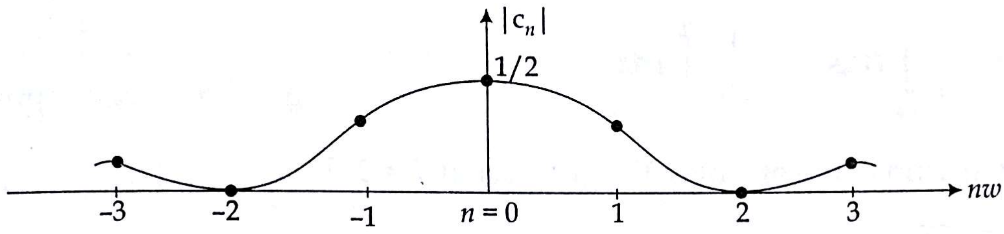

Fig. 17.15

EXERCISE 17.4

Find the complex Fourier series representation of $f(x)$ on the given interval

1. $f(x) = \begin{cases} -1, & -1 < x < 0 \\ 1, & 0 \le x < 1 \end{cases}, \quad f(x+2) = f(x)$

2. $f(x) = e^x, \quad 0 < x < 1, \quad f(x+1) = f(x).$

3. $f(x) = |E \sin \lambda x|, \quad 0 < x < \pi/\lambda, \quad f(x + \pi/\lambda) = f(x)$

4. $f(x) = e^{-|x|}, \quad -2 < x < 2, \quad f(x+4) = f(x)$

Find the frequency spectrum of $f(x)$ for the following problems

5. $f(x) = \begin{cases} 0, & -\pi/2 < x < 0 \\ \sin x, & 0 \le x < \pi/2 \end{cases}, \quad f(x+\pi) = f(x)$

6. $f(x) = \begin{cases} -1, & -1 < x < 0 \\ 1, & 0 \le x < 1 \end{cases}, \quad f(x+2) = f(x)$

7. $f(x) = |E \sin \lambda x|, \quad 0 < x < \pi/\lambda, \quad f(x + \pi/\lambda) = f(x)$

8. Plot some points of the frequency spectrum of the function defined by

$$f(x) = 4 + \sum_{n=-\infty, n\neq 0}^{\infty} \frac{26}{n(12-5i)} e^{2nix}.$$

17.7 NUMERICAL HARMONIC ANALYSIS

So far we have derived the Fourier series expansion of a function $f(x)$ when it was known analytically. However, in many practical problems the analytic nature of the periodic function $f(x)$

is not known but one may be in a position to observe only a set of values of x and y; y being dependent on x, say $y = f(x)$. In such a case, to evaluate the Fourier coefficients, the Euler's formulas studied previously need some modifications given as below.

Let (x_i, y_i), $i = 1, 2, \ldots k$ be the given set of values, where the x_i's are equally spaced. The Fourier coefficients are

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} y dx$$

$$= [\text{Mean value of } y \text{ over the one period } T = 2\pi]$$

$$= \frac{1}{k}\sum_{i=1}^{k} y_i, \quad \ldots(17.50)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = 2.\frac{1}{2\pi}\int_{-\pi}^{\pi} y\cos nx\, dx$$

$$= 2[\text{Mean value of } y\cos nx \text{ over the one period } T = 2\pi]$$

$$= \frac{2}{k}\sum_{i=1}^{k} y_i \cos nx_i, \quad \ldots(17.51)$$

and,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = 2.\frac{1}{2\pi}\int_{-\pi}^{\pi} y\sin nx\, dx$$

$$= 2[\text{Mean value of } y \sin nx \text{ over the one period } T = 2\pi]$$

$$= \frac{2}{k}\sum_{i=1}^{k} y_i \sin nx_i. \quad \ldots(17.52)$$

Then the Fourier series for $y = f(x)$ is

$$y = a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx), \quad \ldots(17.53)$$

where the Fourier coefficients are given by (17.50), (17.51) and (17.52).

The process of finding the Fourier series for a function given by numerical values is known as *numerical harmonic analysis*. The term $(a_1\cos x + b_1 \sin x)$ is called *the fundamental* or *first harmonic*, the term $(a_2\cos 2x + b_2\sin 2x)$ is called *the second harmonic* and so on.

Example 17.21: Given that x is a function of θ over the interval $0 \le \theta \le 2\pi$. Find the Fourier series expansion of x upto the second harmonic on the basis of the following data

θ : 0, $\pi/6$ $\pi/3$ $\pi/2$ $2\pi/3$ $5\pi/6$ π $7\pi/6$ $4\pi/3$ $3\pi/2$ $5\pi/3$ $11\pi/6$

x : 298 356 373 337 254 155 80 51 60 93 147 221

Solution: The Fourier series for $x = f(\theta)$ upto the second harmonic is

$$x \approx a_0 + \sum_{n=1}^{2} (a_n \cos n\theta + b_n \sin n\theta),$$

where the Fourier coefficients are given by

$$a_0 = \frac{1}{12}\sum_{i=1}^{12} x_i, \quad a_n = \frac{1}{6}\sum_{i=1}^{12} x_i \cos n\theta_i, \text{ and } \quad b_n = \frac{1}{6}\sum_{i=1}^{12} x_i \sin n\theta_i.$$

To evaluate the coefficients we form the following table:

θ	$\sin\theta$	$\cos\theta$	$\sin 2\theta$	$\cos 2\theta$	x	$x\sin\theta$	$x\cos\theta$	$x\sin 2\theta$	$x\cos 2\theta$
0	0.00	1.00	0.00	1.00	298	0.00	298.00	0.00	298.00
$\pi/6$	0.50	0.87	0.87	0.50	356	178.00	309.72	309.72	178.00
$\pi/3$	0.87	0.50	0.87	−0.50	373	324.51	186.50	324.51	−186.50
$\pi/2$	1.00	0.00	0.00	−1.00	337	337.00	0.00	0.00	−337.00
$2\pi/3$	0.87	−0.50	−0.87	−0.50	254	220.98	−127.00	−220.98	−127.00
$5\pi/6$	0.50	−0.87	−0.87	−0.50	155	77.50	−134.85	−134.85	−77.50
π	0.00	−1.00	0.00	1.00	80	0.00	−80.00	0.00	80.00
$7\pi/6$	−0.50	−0.87	0.87	0.50	51	−25.50	−44.37	44.37	25.50
$4\pi/3$	−0.87	−0.50	0.87	−0.50	60	−52.20	−30.00	52.20	−30.00
$3\pi/2$	−1.00	0.00	0.00	−1.00	93	−93.00	0.00	0.00	−93.00
$5\pi/3$	−0.87	0.50	−0.87	−0.50	147	−102.90	73.50	−102.90	−73.50
$11\pi/6$	−0.50	0.87	−0.87	0.50	221	−110.50	192.27	−192.27	110.50
				Total:	2425	753.89	643.77	54.18	−77.50

Find the first four terms in a series of sines to represent T and calculate T for $\theta = 75°$.

Solution: The half-range sine series to represent T is

$$T \approx b_1 \sin\theta + b_2 \sin 2\theta + b_3 \sin 3\theta + b_4 \sin 4\theta,$$

where the coefficients b_i's are given by

$$b_n = \frac{2}{6}\Sigma T \sin n\theta = \frac{1}{3}\Sigma T \sin n\theta.$$

To calculate b_i's we form the following table:

θ	T	sinθ	sin 2θ	sin 3θ	sin 4θ	T sin θ	T sin 2θ	T sin 3θ	T sin 4θ
0	0	0.00	0.00	0.00	0.00	0.00	0.00	0.0	0.00
30	5224	0.50	0.87	1.00	0.87	2612.00	4544.88	5224.00	4544.88
60	8097	0.87	0.87	0.00	−0.87	7044.39	7044.39	0	−7044.39
90	7850	1.00	0.00	−1.00	0.00	7850.00	0.00	−7850.00	0.00
120	5499	0.87	−0.87	0.00	0.87	4784.13	−4784.13	0.00	4784.13
150	2626	0.50	−0.87	1.00	−0.87	1313.00	−2284.62	2626.00	−2284.62
					Total:	23603.52	4520.52	0.00	0.00

Thus, $b_1 = \frac{23603.52}{3} = 7867.67 \approx 7868,$

$b_2 = \frac{4520.52}{3} = 1506.84 \approx 1507,$

$b_3 = 0, \quad b_4 = 0.$

Hence, the Fourier series is given by

$$T \approx 7868 \sin\theta + 1507 \sin 2\theta.$$

When $\theta = 75°$, then

$$T = 7868 \sin 75° + 1507 \sin 150°$$
$$= 7868 \times 0.9659 + 1507 \times 0.50$$
$$\approx 8353.20.$$

Example 17.23: The following table gives the variations of a periodic current over a fundamental period of T second

t(sec)	:	0	$T/6$	$T/3$	$T/2$	$2T/3$	$5T/6$	T
A(amp)	:	1.98	1.30	1.05	1.30	−0.88	−0.25	1.98

Show that there is direct current part of 0.75 amp. in the variable current and obtain the amplitude of the first harmonic.

Solution: The series is periodic over the interval $(0, T)$ hence the period $2l = T$, that is, $l = T/2$. Thus the current A is given as

$$A = a_0 + a_1 \cos \frac{2\pi t}{T} + b_1 \sin \frac{2\pi t}{T} + a_2 \cos \frac{4\pi t}{T} + b_2 \sin \frac{4\pi t}{T}.$$

Here a_0 represents the direct current part and $\sqrt{a_1^2 + b_1^2}$ gives the amplitude of the first harmonic.

To calculate the coefficients, we form the following table:

t	$2\pi t/T$	$\cos(2\pi t/T)$	$\sin(2\pi t/T)$	A	$A\cos(2\pi t/T)$	$A\sin(2\pi t/T)$
0	0	1.00	0.000	1.98	1.98	0.00
T/6	$\pi/3$	0.50	0.87	1.30	0.65	1.13
T/3	$2\pi/3$	−0.50	0.87	1.05	−0.53	0.91
T/2	π	−1.0	0.00	1.30	−1.30	0.00
2T/3	$4\pi/3$	−0.5	0.87	−0.88	0.44	0.76
5T/6	$5\pi/3$	0.5	0.87	−0.25	−.13	0.22
			Total:	4.5	1.11	3.02

Here, $\Sigma A = 4.5$, $\Sigma A \cos(2\pi t/T) = 1.11$, $\Sigma A \sin(2\pi t/T) = 3.02$. Hence

$$a_0 = \frac{4.5}{6} = 0.75, \quad a_1 = \frac{1.11}{3} = 0.37. \quad b_1 = \frac{3.02}{3} = 1.01.$$

Thus, the direct current part is 0.75 amp. and amplitude of the first harmonic is $\sqrt{(0.37)^2 + (1.01)^2} = 1.07$ amp.

EXERCISE 17.5

1. The following values of y give the displacement of a certain machine part for the rotation x of the flywheel

x :	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
y :	1.98	2.15	2.77	−0.22	−0.31	1.43	1.98

Express y in Fourier series upto the third harmonic.

2. The following values of y give the displacement in inches of a certain machine part for the rotation x of the flywheel. Expand y in the form of a Fourier series upto fourth hormonic

x :	0	30°	60°	90°	120°	150°	180°
y :	0	9.2	14.4	17.8	17.3	11.7	0

3. Obtain the first three coefficients in the Fourier cosine series for y, where y is given in the following table:

x :	0	1	2	3	4	5
y :	4	8	15	7	6	2

4. The turning moment T on the crank-shaft of a steam engine for the crank angle θ in degr is recorded as follow. Express T in a series of sines upto the fourth harmonic

θ :	0°	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165° 1
T :	0	2.7	5.2	7	8.1	8.3	7.9	6.8	5.5	4.1	2.6	1.2

5. A part of a machine has an oscillatory motion. The displacement y at a time t is given belo

t :	0	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16	0.18	0.20
y :	0	0.64	1.13	1.34	0.95	0.00	−0.92	−1.33	−1.17	−0.66	0.0

Find constants in the equation $y = A\sin(10\pi t + \alpha_1) + B\sin(20\pi t + \alpha_2)$.

ANSWERS

Exercise 17.1 (p. 163)

1. $\frac{\pi^2}{12} + \frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \dots$

2. $-\frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right)$

3. $-1 + \pi\sin x - \frac{1}{2}\cos x + \frac{2}{2^2-1}\cos 2x + \frac{2}{3^2-1}\cos 3x + \dots$

4. $-\frac{\pi^2}{3} + 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \dots\right] + 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \dots\right]$

5. $\frac{\pi^2}{3} - 4\left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \dots\right)$

7. $\frac{\sinh 8}{8} + \sinh 8\sum_{n=1}^{\infty}\left[\frac{16(-1)^n}{64+\pi^2 n^2}\cos\left(\frac{n\pi x}{2}\right) + \frac{2n\pi(-1)^n}{64+\pi^2 n^2}\sin\left(\frac{n\pi x}{2}\right)\right]$

8. $\frac{k}{2} + \frac{2k}{\pi}\left(\cos\frac{\pi x}{2} - \frac{1}{3}\cos\frac{\pi x}{\cancel{3}\,2} + \frac{1}{5}\cos\frac{\pi x}{\cancel{5}\,2} - \dots\right)$

9. $4\left(\frac{1}{2} - \frac{1}{1.3}\cos 2\pi x - \frac{1}{3.5}\cos 4\pi x - \frac{1}{5.7}\cos 6\pi x \dots\right)$

Exercise 17.2 (p. 171)

1. $\frac{1}{5} - \frac{8}{\pi^4}\left[\frac{\pi^2-6}{1^4}\cos\pi x - \frac{2^2\pi^2-6}{2^4}\cos 2\pi x + \frac{3^2\pi^2-6}{3^4}\cos 3\pi x + \dots\right]$

2. $1 - \frac{1}{2}\cos x - \frac{2}{1.3}\cos 2x + \frac{2}{2.4}\cos 3x - \frac{2}{3.5}\cos 4x + \ldots$

3. $\frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right)$

4. $\frac{2}{\pi} + \frac{4}{\pi}\left(\frac{1}{3}\cos 2x - \frac{1}{15}\cos 4x + \ldots\right)$

5. $\frac{2}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\right)$

6. $\frac{\pi}{2} + 1 - \frac{4}{\pi}\left(\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right)$

7. $\frac{8}{3} + \frac{16}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2} - \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} - \ldots\right]$

8. $1, \frac{4}{\pi}\left[\sin\frac{\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} + \frac{1}{5}\sin\frac{5\pi x}{2} + \ldots\right]$

9. $\frac{8}{3} + \frac{32}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[\cos\frac{n\pi}{2} - \frac{2}{n\pi}\sin\frac{n\pi}{2}\right]\cos\frac{n\pi x}{4}$

10. $\frac{4}{\pi}\left[\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \ldots\right]$

11. $\frac{2}{\pi} - \frac{4}{\pi}\left[\frac{1}{1.3}\cos\frac{2\pi x}{l} + \frac{1}{3.5}\cos\frac{4\pi x}{l} + \frac{1}{5.7}\cos\frac{6\pi x}{l} + \ldots\right]$

12. $\frac{1}{\pi} + \frac{1}{\pi}\left[\cos x - \frac{2}{3}\cos 2x - \cos 3x - \frac{2}{15}\cos 4x + \frac{1}{3}\cos 5x - \ldots\right]$

Exercise 17.3 (p. 177)

1. $\frac{-\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}$

2. $\pi^2 - x^2 = \frac{2\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos nx}{n^2}, \quad x = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}$

$x(\pi^2 - x^2) = 12\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n^3}$

3. $f(x) = \frac{1}{4}\pi + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{\pi n^2}\cos nx + \frac{(-1)^{n+1}}{n}\sin nx\right]$

$$g(x) = \frac{1}{4}x\pi + \frac{1}{4}\pi^2 \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{\pi n^3}\sin nx + \frac{(-1)^{n+1}}{n^2}(-\cos nx + (-1)^n)\right]$$

4. $x\cos x + \sin x = \frac{1}{2}\pi\sin x + 2\pi\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}n\sin(nx)$

5. $f'(x) = \frac{1}{\pi}\left(\frac{-2}{3}\sin x + \frac{6}{5}\sin 3x + \ldots\right) + \frac{1}{\pi}\left(\frac{-2}{3}\cos x + \frac{3\pi}{2}\cos 2x - \ldots\right)$

8. $\frac{\pi^4}{96}$

Exercise 17.4 (p. 183)

1. $f(x) = 1 + \lim_{k\to\infty}\sum_{n=-k}^{k}\frac{i}{n\pi}(1-(-1)^n)e^{-in\pi x}$

2. $f(x) = e - \lim_{k\to\infty}\sum_{n=-k}^{k}\frac{e^{-in\pi x}}{1-2n\pi i}, n = 0, \pm 1, \pm 2, \ldots$

3. $f(x) = -2\frac{E}{\pi}\lim_{k\to\infty}\sum_{n=-k}^{k}\frac{1}{(4n^2-1)}e^{2n\lambda ix}.$

4. $f(x) = \sum_{n=-\infty}^{\infty}\frac{2}{4+n^2\pi^2}[\{1-(-1)^n e^{-2}]e^{\frac{-in\pi x}{2}}$

5. $\left(2n, \frac{4|n|}{\pi(4n^2-1)}\right)$

6. $\left(n\pi, \frac{[1-(-1)^n]}{|n|\pi}\right)$

7. $\left(2n\lambda, \left|\frac{2E}{(4n^2-1)\pi}\right|\right)$

8. The frequency spectrum consists of the point (0, 4) and the points $(2n, 2/|n|)$.

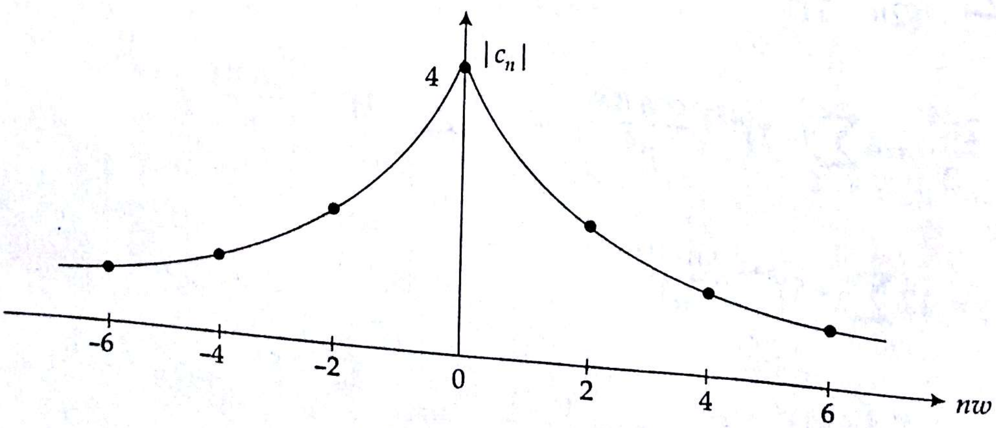

Fig. 17.16

Exercise 17.5 (p. 187)

1. $y \approx 1.3 + 0.92 \cos x + 1.097 \sin x - 0.42 \cos 2x - 0.681 \sin 2x + 0.36 \cos 3x$
2. $y \approx 11.73 - (7.73 \cos 2x + 1.57 \sin 2x) + (-2.83 \cos 4x + .116 \sin 4x)$
3. $y \approx 7 - 2.8 \cos x - 1.5 \cos 2x + 2.7 \cos 3x$
4. $T \approx 7.8 \sin \theta + 1.5 \sin 2\theta - 9.2 \sin 3\theta + 11.6 \sin 4\theta$
5. $A = 1.317$, $B = -0.1524$, $\alpha_1 = -0.0083$, $\alpha_2 = -0.315$.