7

CHAPTER

Multiple Integrals and Their Applications

Multiple integrals are definite integrals of functions of several variables. Double and triple integrals arise in evaluating quantities such as area, volume, mass, moments, centroid and moments of inertia and are used in many applications in science and engineering. If the number of variables is higher, then one will arrive at hypervolumes which cannot be graphed.

7.1 DOUBLE INTEGRALS

Let $f(x, y)$ be a continuous and single valued function of x and y defined over a simple region R bounded by a closed curve C as shown in Fig. 7.1.

Subdivide the region R by drawing lines parallel to co-ordinate axes. Number the rectangles which are inside R, in some order, say from 1 to n. Choose an arbitrary point (x_k, y_k) in each $\Delta A_k = \Delta x_k \, \Delta y_k$, the area of the kth rectangle and form the sum

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k \qquad ...(7.1)$$

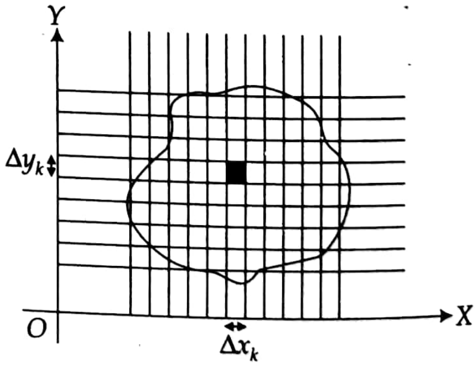


Fig. 7.1

The limit of this sum as $n \to \infty$ and $\Delta A_k \to 0$ is defined as the double integral of $f(x, y)$ over the region R and is denoted by

$$I = \iint_R f(x, y)\,dA, \text{ or } \quad \iint_R f(x, y)\,dx\,dy. \qquad ...(7.2)$$

The continuity of $f(x, y)$ is a sufficient condition for the existence of the double integral, but not a necessary one. The limit under consideration exists for many discontinuous functions as well.

Properties of Double Integrals

If $f(x, y)$ and $g(x, y)$ are defined to be bounded and continuous functions over the region R, then

1. $\displaystyle\iint_R k f(x, y)\,dx\,dy = k\iint_R f(x, y)\,dx\,dy$, for any number k.

2. $\displaystyle\iint_R [f(x, y) \pm g(x, y)]\,dx\,dy = \iint_R f(x, y)\,dx\,dy \pm \iint_R g(x, y)\,dx\,dy.$

3. $\displaystyle\iint_R f(x, y)\,dx\,dy \geq 0$, if $f(x, y) \geq 0$ on R.

4. $\displaystyle\iint_R f(x, y)\,dx\,dy \geq \iint_R g(x, y)\,dx\,dy$, if $f(x, y) \geq g(x, y)$ on R.

5. $\displaystyle\iint_R f(x, y)\,dx\,dy = \iint_{R_1} f(x, y)\,dx\,dy + \iint_{R_2} f(x, y)\,dx\,dy,$

where R is the union of two non-overlapping regions R_1 and R_2.

Evaluation of double Integrals

The double integral in terms of the limit of sums is only applicable to some specific computational problems. In fact, the double integral over a region R is evaluated by two successive single integrations as explained below.

Case I: Let the region R be the rectangular region expressed in the form

$$R = \{(x, y) : a \leq x \leq b,\ c \leq y \leq d\},$$

as shown in Fig. 7.2.

In this case the limits, both for x and y, are constants, so it is immaterial whether we first integrate w.r.t. x or w.r.t. y.

Thus, in this case

$$\iint_R f(x, y)\,dx\,dy = \int_c^d \left(\int_a^b f(x, y)\,dx \right) dy, \text{ or}$$

$$= \int_a^b \left(\int_c^d f(x, y)\,dy \right) dx.$$

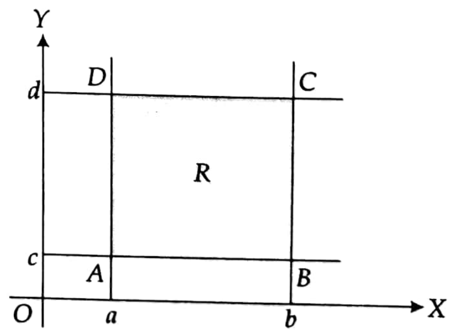


Fig. 7.2

Case II: Let the region R be expressed in the form

$$R = \{(x, y) : a \leq x \leq b,\ g(x) \leq y \leq h(x)\}$$

as shown in Fig. 7.3.

In this case limits for x are constants, but for y are functions of x. We assume $g(x)$ and $h(x)$ to be both integrable functions and $g(x) \leq h(x)$, for $x \in [a, b]$.

limits are f^n of x

Here we first integrate w.r.t. y and then w.r.t. x. Thus

$$\iint_R f(x, y)dxdy = \int_a^b \left(\int_{g(x)}^{h(x)} f(x, y)dy \right) dx.$$

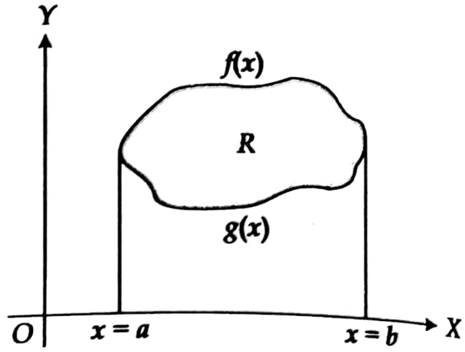


Fig. 7.3

Case III: Let the region R be expressed in the form

$$R = \{(x, y) : g(y) \leq x \leq h(y),\ c \leq y \leq d\}$$

as shown in Fig. 7.4.

In this case limits for x are functions of y, but limits for y are constants, we assume $g(y)$ and $h(y)$ to be both integrable functions and $g(y) \leq h(y)$, for $y \in [c, d]$.

Here we first integrate w.r.t. x and then w.r.t. y. Thus

$$\iint_R f(x, y)dxdy = \int_c^d \left(\int_{g(y)}^{h(y)} f(x, y)dx \right) dy.$$

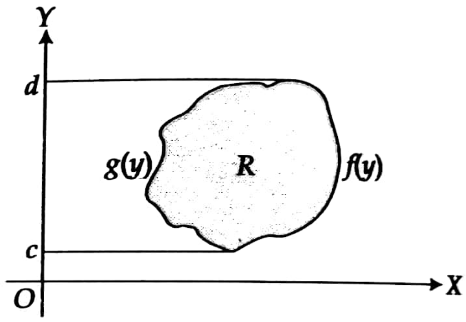


Fig. 7.4

Example 7.1: Evaluate $\iint_R f(x, y)dA$ for $f(x, y) = 1 - 6x^2y$ and $R : \{0 \leq x \leq 2, -1 \leq y \leq 1\}$.

Solution: Let $$I = \iint_R f(x, y)dA = \int_{-1}^{1} \left[\int_0^2 (1 - 6x^2y)dx \right] dy = \int_{-1}^{1} \left[x - 2x^3y \right]_0^2 dy$$

$$= \int_{-1}^{1} (2 - 16y)dy = [2y - 8y^2]_{-1}^{1} = 4.$$

We may verify that the double integral I, evaluated as $I = \int_0^2 \left(\int_{-1}^{1} (1 - 6x^2y)dy \right) dx$ also yields the same value as above, since the limits of integration are constants in this case and, therefore, the order of integration is immaterial.

Example 7.2: Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{1 + x^2 + y^2} dy\, dx.$

Solution: Let

$$I = \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}\,dy\,dx = \int_0^1 \left[\int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}\,dy\right] dx$$

$$= \int_0^1 \left(\frac{1}{\sqrt{1+x^2}}\left[\tan^{-1}\frac{y}{\sqrt{1+x^2}}\right]_0^{\sqrt{1+x^2}}\right) dx$$

$$= \int_0^1 \left[\frac{1}{\sqrt{1+x^2}}(\tan^{-1}1 - \tan^{-1}0)\right] dx$$

$$= \frac{\pi}{4}\int_0^1 \frac{dx}{\sqrt{1+x^2}} = \frac{\pi}{4}\left[\ln\left(x+\sqrt{1+x^2}\right)\right]_0^1$$

$$= \frac{\pi}{4}\left[\ln\left(1+\sqrt{2}\right) - \ln 1\right] = \frac{\pi}{4}\ln\left(1+\sqrt{2}\right)$$

Example 7.3: Calculate $\iint_R \frac{\sin x}{x}\,dA$, where R is the triangle in the xy-plane bounded by the x-axis, the line $y = x$ and the line $x = 1$.

Solution: The region of integration $R = \{(x, y) : 0 \le x \le 1, 0 \le y \le x\}$ is shown in Fig. 7.5. Integrating first w.r.t. y and, then w.r.t. x, we get

$$I = \int_0^1 \frac{\sin x}{x}\left(\int_0^x dy\right) dx = \int_0^1 \frac{\sin x}{x}\,x\,dx$$

$$= \int_0^1 \sin x\,dx = [-\cos x]_0^1 = 1 - \cos 1.$$

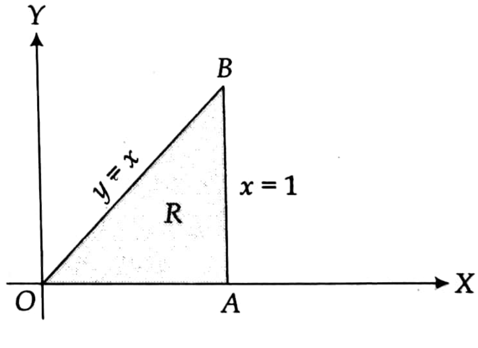


Fig. 7.5

Change of order of Integration

In the preceding example if we represent the region as $R = \{(x, y) : y \le x \le 1, 0 \le y \le 1\}$, then

$$I = \int_0^1 \left(\int_y^1 \frac{\sin x}{x}\,dx\right) dy$$

that is the order of integration is reversed. In this case we observe that $\int \frac{\sin x}{x} dx$ can't be expressed in terms of the elementary function, and thus, it is not easy to calculate the integration in this form.

There is no definite rule to foresee that which order of integration will work well. Sometimes it is convenient to evaluate the integral by changing the order and modify the limits suitably.

In the next example, we illustrate the procedure for changing of order of integration and finding the modified limits.

Example 7.4: Evaluate $\iint_R xy\,dA$, where R is the positive quadrant of the circle $x^2 + y^2 = a^2$ integrating, (a) first w.r.t. to x and then w.r.t. y, (b) first w.r.t. to y and then w.r.t. x.

Solution: The region of integration R is shown in Fig. 7.6. To evaluate the integration over R by integrating first w.r.t. x, imagine an elementary strip PQ through R in the direction of increasing x.

Mark the x values where the strip enters and leaves the region R as shown in Fig. 7.7.

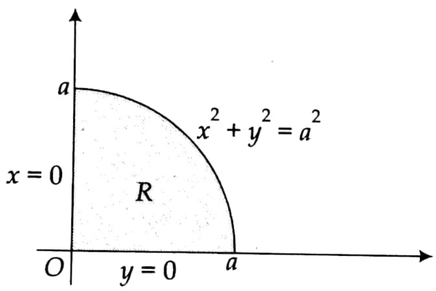


Fig. 7.6

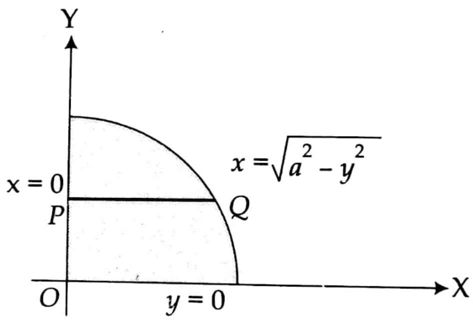


Fig. 7.7

Thus, for x the limits of integration are: $x = 0$ to $x = \sqrt{a^2 - y^2}$.

To cover the entire region R such strips should start with the minimum value of y, that is, $y = 0$ to the maximum value of y, that is, $y = a$. Thus the limits of y are: $y = 0$ to $y = a$, and hence the integral I is therefore expressed as

$$I = \int_0^a \left(\int_0^{\sqrt{a^2 - y^2}} xy\,dx \right) dy = \int_0^a \left(\frac{x^2}{2} y \right)_0^{\sqrt{a^2 - y^2}} dy$$

$$= \frac{1}{2} \int_0^a (a^2 - y^2) y\,dy = \frac{1}{2} \left[\frac{a^2 y^2}{2} - \frac{y^4}{4} \right]_0^a = \frac{a^4}{8}.$$

Next, to evaluate the integration over R by integrating first w.r.t. y, imagine an elementary strip through R in the direction of increasing y. Mark the y values where the strip enters and leaves the region R as shown in Fig. 7.8.

Thus for y, the limits of integration are: $y = 0$ to $y = \sqrt{a^2 - x^2}$.

To cover the entire region R such strip should start with the minimum value of x, that is, $x = 0$, to the maximum value of x, that is, $x = a$. Thus the limits of x are: $x = 0$ to $x = a$, and hence the integral I is therefore expressed as

$$I = \int_0^a \left(\int_0^{\sqrt{a^2-x^2}} xy\,dy \right) dx = \int_0^a \left(\frac{y^2}{2} x \right)_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2} \int_0^a (a^2 - x^2) x\,dx = \frac{1}{2} \left[\frac{a^2x^2}{2} - \frac{x^4}{4} \right]_0^a = \frac{a^4}{8}.$$

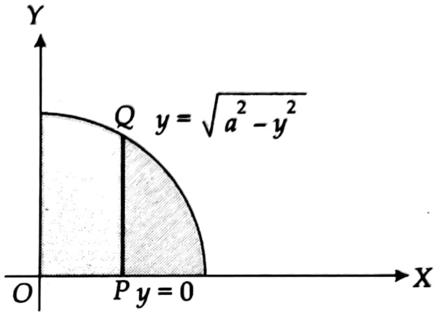


Fig. 7.8

Example 7.5: Change the order of integration and hence evaluate

$$\int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx.$$

Solution: The region of integration is $R = \{(x, y) : 0 \le x \le 1,\ x^2 \le y \le 2 - x\}$, as shown in Fig. 7.9.

In the given integral $I = \int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx$, the integration is first w.r.t. y and then w.r.t. x.

To change the order of integration the elementary strip PQ is to be taken parallel to x-axis. This requires the splitting of the region R into two subregions R_1 and R_2 by the line AB, $y = 1$, refer to Fig. 7.9.

For R_1, the elementary strip PQ goes from $x = 0$ to $x = \sqrt{y}$ and to cover the region min.$y = 0$ and max.$y = 1$. For R_2, the elementary strip $P'Q'$ goes from $x = 0$ to $x = 2 - y$ and to cover the region min.$y = 1$ and max.$y = 2$. Thus the regions R_1 and R_2 are given by

$$R_1 = \{(x, y) : 0 \le x \le \sqrt{y},\ 0 \le y \le 1\}, \text{ and } R_2 = \{(x, y) : 0 \le x \le 2 - y,\ 1 \le y \le 2\}.$$

For R_1, $$I_1 = \int_0^1 \left(\int_0^{\sqrt{y}} xy\,dx \right) dy = \int_0^1 \left(\frac{x^2 y}{2} \right)_0^{\sqrt{y}} dy = \frac{1}{2} \int_0^1 y^2 dy = \left(\frac{y^3}{6} \right)_0^1 = \frac{1}{6};$$

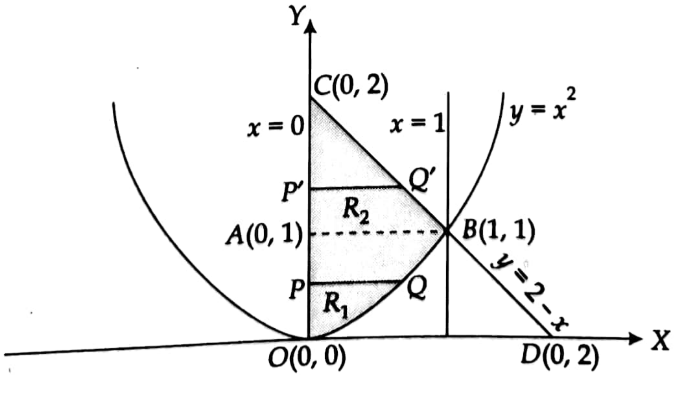


Fig. 7.9

and, for R_2, $$I_2 = \int_1^2 \left(\int_0^{2-y} xy\,dx \right) dy = \int_1^2 \left(\frac{x^2 y}{2} \right)_0^{2-y} dy$$

$$= \frac{1}{2} \int_1^2 y(2-y)^2 dy = \frac{1}{2} \left(2y^2 - \frac{4y^3}{3} + \frac{y^4}{4} \right)_1^2 = \frac{5}{24}.$$

Hence, $$I = \frac{1}{6} + \frac{5}{24} = \frac{3}{8}.$$

Example 7.6: Express $\int_0^{\frac{a}{\sqrt{2}}} \int_0^x x\,dy\,dx + \int_{\frac{a}{\sqrt{2}}}^a \int_0^{\sqrt{a^2-x^2}} x\,dy\,dx$ as a single integral and evaluate it.

Solution: Let $$I_1 = \int_0^{\frac{a}{\sqrt{2}}} \int_0^x x\,dy\,dx \text{ and } I_2 = \int_{\frac{a}{\sqrt{2}}}^a \int_0^{\sqrt{a^2-x^2}} x\,dy\,dx.$$

If R_1 and R_2 are the regions of integration in case of I_1 and I_2 respectively, then

$$R_1 = \left\{(x, y) : 0 \le x \le \frac{a}{\sqrt{2}}, 0 \le y \le x\right\} \text{ and}$$

$$R_2 = \left\{(x, y) : \frac{a}{\sqrt{2}} \le x \le a, 0 \le y \le \sqrt{a^2 - x^2}\right\}.$$

These are shown in Fig. 7.10.

It is obvious that R_1 and R_2 are non-overlapping regions and let $R = R_1 \cup R_2$. Then,

$$I = I_1 + I_2 = \iint_R x\,dx\,dy$$

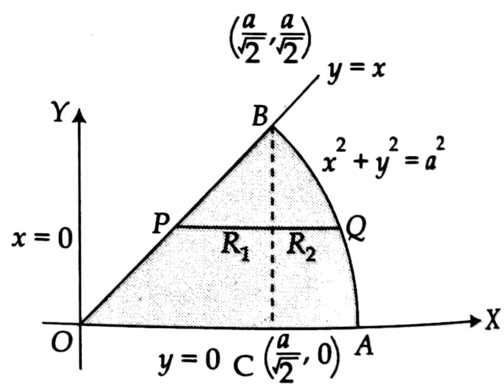


Fig. 7.10

where $R = \left\{(x, y) : y \le x \le \sqrt{a^2 - y^2}, 0 \le y \le \frac{a}{\sqrt{2}}\right\}$.

To evaluate I, we consider an elementary strip PQ parallel to x–axis. The limits for x are: y to $\sqrt{a^2 - y^2}$. To cover the entire region R, the min. y is zero and max.y is $a/\sqrt{2}$ and, therefore,

$$I = \int_0^{\frac{a}{\sqrt{2}}} \left(\int_y^{\sqrt{a^2-y^2}} x\,dx \right) dy = \int_0^{\frac{a}{\sqrt{2}}} \left(\frac{x^2}{2} \right)_y^{\sqrt{a^2-y^2}} dy = \frac{1}{2} \int_0^{\frac{a}{\sqrt{2}}} (a^2 - 2y^2) dy$$

$$= \frac{1}{2}\left(a^2 y - \frac{2y^3}{3}\right)_0^{\frac{a}{\sqrt{2}}} = \frac{1}{2}\left[\frac{a^3}{\sqrt{2}} - \frac{a^3}{3\sqrt{2}}\right] = \frac{a^3}{3\sqrt{2}}.$$

Example 7.7: Evaluate $\int_0^{4a} \int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\,dx$ by changing the order of integration.

Solution: The region of integration $R = \left\{(x, y) : 0 \le x \le 4a, \frac{x^2}{4a} \le y \le 2\sqrt{ax}\right\}$ is shown in Fig. 7.11.

The points of intersection of the two parabolas $y^2 = 4ax$ and $x^2 = 4ay$ are $O(0, 0)$ and $A(4a, 4a)$.

In the given integral, the integration is first w.r.t. y and then w.r.t. x. To change the order of integration consider an elementary strip PQ parallel to x–axis. The limits for x are: $y^2/4a$ to $2\sqrt{ay}$. To cover the entire region R the minimum of y is zero and maximum is $4a$ and, therefore,

$$I = \int_0^{4a}\left(\int_{\frac{y^2}{4a}}^{2\sqrt{ay}} dx\right)dy = \int_0^{4a}\left(2\sqrt{ay} - \frac{y^2}{4a}\right)dy$$

$$= \left(2\sqrt{a}\,\frac{y^{3/2}}{3/2} - \frac{y^3}{12a}\right)_0^{4a} = \frac{32}{3}\sqrt{a}\,a^{3/2} - \frac{64a^3}{12a} = \frac{16}{3}a^2.$$

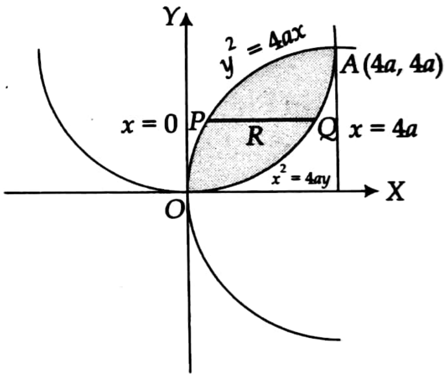


Fig. 7.11

Example 7.8: Evaluate $\int_0^{\infty}\int_0^{x} xe^{-x^2/y}\,dy\,dx$ by change of order of integration.

Solution: The region of integration $R = \{(x, y) : 0 \le x < \infty,\ 0 \le y \le x\}$ is shown in Fig. 7.12.

In the given integral, the integration is first w.r.t. y and then w.r.t. x.

To change the order of integration, take an elementary strip PQ starting from P parallel to x-axis. Thus, the limits for x are: $x = y$ to ∞. To cover the entire region R the minimum y is zero and maximum is infinity, and, therefore

$$I = \int_0^{\infty}\left(\int_y^{\infty} xe^{-x^2/y}\,dx\right)dy.$$

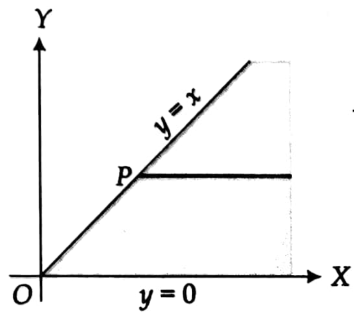


Fig. 7.12

To evaluate I, put $x^2 = t$, this implies $2x\,dx = dt$, we get

$$I = \int_0^{\infty}\left(\frac{1}{2}\int_{y^2}^{\infty} e^{-t/y}\,dt\right)dy = \frac{1}{2}\int_0^{\infty}\left(-ye^{-t/y}\right)_{y^2}^{\infty} dy$$

$$= \frac{1}{2}\int_0^{\infty} ye^{-y}dy = \frac{1}{2}\left[(-ye^{-y})_0^{\infty} - \int_0^{\infty} 1(-e^{-y})dy\right]$$

$$= -\frac{1}{2}\left[e^{-y}\right]_0^{\infty} = \frac{1}{2}.$$

Imp. polar form

Example 7.9: Evaluate $\int_0^{a/\sqrt{2}}\int_0^{y} \ln(x^2+y^2)\,dx\,dy + \int_{a/\sqrt{2}}^{a}\int_0^{\sqrt{a^2-y^2}} \ln(x^2+y^2)\,dx\,dy$ by change of order of integration.

Solution: The regions of integration are:

$$R_1 = \left\{(x, y) : 0 \le x \le y,\ 0 \le y \le \frac{a}{\sqrt{2}}\right\},\ R_2 = \left\{(x, y) : 0 \le x \le \sqrt{a^2 - y^2},\ \frac{a}{\sqrt{2}} \le y \le a\right\}$$

These are non-overlapping; the joint region, $R = R_1 \cup R_2$, given by

$$R = \left\{(x, y) : y \le x \le \sqrt{a^2 - y^2}, 0 \le y \le a\right\}$$

is shown in Fig. 7.13. In the given integral the integration is first w.r.t. x and then w.r.t. y. To change the order of integration, take an elementary strip PQ parallel to y-axis. The limits for y are: $y = x$ to $y = \sqrt{a^2 - x^2}$. To cover the entire region R, minimum x is zero and maximum x is $\frac{a}{\sqrt{2}}$, and, therefore

$$I = \int_0^{\frac{a}{\sqrt{2}}}\left(\int_x^{\sqrt{a^2-x^2}} \ln(x^2+y^2)dy\right)dx.$$

To evaluate I, we change to polar co-ordinates. Put $x = r\cos\theta$, $y = r\sin\theta$, we obtain

Imp.

$$I = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{a} (\ln r^2)r\,dr\,d\theta = 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(\int_0^{a} \ln r\,(r\,dr)\right)d\theta$$

$$= \frac{\pi}{2}\left[\ln r.\frac{r^2}{2} - \int \frac{1}{r}.\frac{r^2}{2}dr\right]_0^a = \frac{\pi}{2}\left[\frac{r^2}{2}\ln r - \frac{r^2}{4}\right]_0^a$$

$$= \frac{\pi a^2}{4}\left[\ln a - \frac{1}{2}\right], \quad \text{since } \lim_{r\to 0} r^2 \ln r = 0.$$

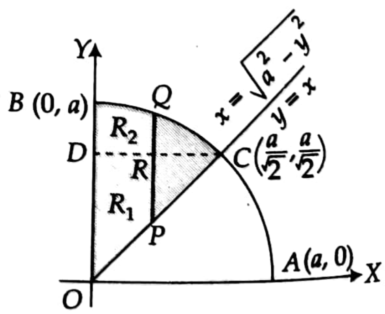


Fig. 7.13

7.2 DOUBLE INTEGRALS IN POLAR CO-ORDINATES

Let $f(r, \theta)$ be defined over a region R bounded by the radii vectors $\theta = \alpha$, $\theta = \beta$ and the continuous curves $r = g_1(\theta)$, $r = g_2(\theta)$ as shown in Fig. 7.14.

Suppose that $a \le g_1(\theta) \le g_2(\theta) \le b$ for every value of $\theta \in [\alpha, \beta]$. Then the region R lies in the region $ABCD$ defined by $\{(r, \theta) : a \le r \le b,\ \alpha \le \theta \le \beta\}$.

Divide the radial interval $[a, b]$ into h parts by concentric circular arc at interval $\Delta r = \dfrac{b-a}{h}$ and divide the angular interval $[\alpha, \beta]$ into k parts by radial lines at interval $\Delta\theta = \dfrac{\beta - \alpha}{k}$.

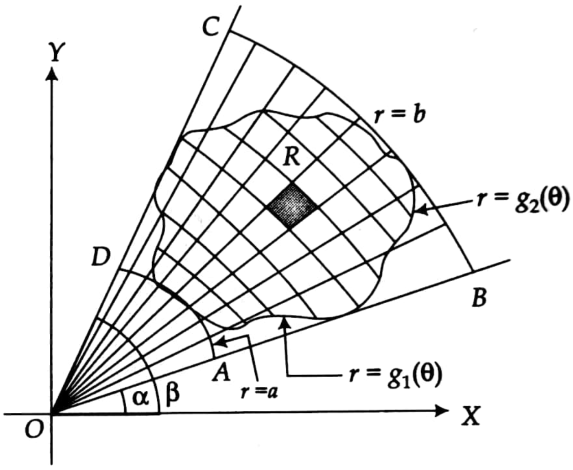


Fig. 7.14

Thus, the whole region $ABCD$ has been divided into polar rectangles. Number these polar rectangles that lie inside R from 1 to n and let their areas be $\Delta A_1, \Delta A_2, \ldots, \Delta A_n$, where

$$\Delta A_i = \frac{1}{2}\left(r_i + \frac{1}{2}\Delta r\right)^2 \Delta\theta - \frac{1}{2}\left(r_i - \frac{1}{2}\Delta r\right)^2 \Delta\theta$$

$$= \frac{1}{2}(2r_i \Delta r \Delta\theta) = r_i \Delta r \Delta\theta;$$

(r_i, θ_i) being the co-ordinates of the centre of the polar rectangle of area ΔA_i. Form the sum

$$S_n = \sum_{i=1}^{n} f(r_i, \theta_i)\Delta A_i = \sum_{i=1}^{n} f(r_i, \theta_i) r_i \Delta r \Delta\theta. \qquad \ldots(7.3)$$

If f is continuous throughout R, then S_n approaches a limit S as $\Delta r \to 0$ and $\Delta\theta \to 0$, that is, as $n \to \infty$. This limit is defined as the double integral of f over R and we write

$$S = \lim_{n\to\infty} \sum_{i=1}^{n} f(r_i, \theta_i) r_i \Delta r \Delta\theta = \iint_R f(r, \theta) r\, dr\, d\theta = \boxed{\int_{\alpha}^{\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r, \theta) r\, dr\, d\theta.} \qquad \ldots(7.4)$$

In general, to evaluate the double integral in polar coordinates, we first integrate w.r.t. r and then w.r.t. θ. However, the order of integration may be changed with suitable changes in the limits.

Example 7.10: Evaluate $\iint r^3 dr\, d\theta$ over the region bounded by the circles $r = 2\cos\theta$ and $r = 4\cos\theta$.

Solution: The region of integration

$$R = \left\{(r, \theta) : 2\cos\theta \le r \le 4\cos\theta, \frac{-\pi}{2} \le \theta \le \frac{\pi}{2}\right\}$$

is shown in Fig. 7.15.

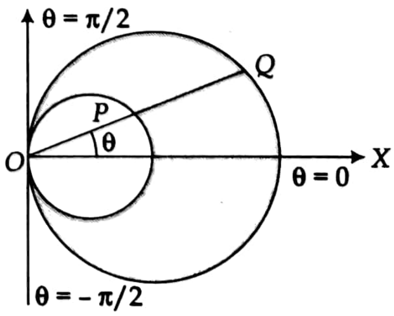


Fig. 7.15

$$\text{Therefore } I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\int_{2\cos\theta}^{4\cos\theta} r^3 dr \right) d\theta = \int_{-\pi/2}^{\pi/2} \left[\frac{r^4}{4} \right]_{2\cos\theta}^{4\cos\theta} d\theta$$

$$= \frac{1}{4} \int_{-\pi/2}^{\pi/2} (256 \cos^4 \theta - 16 \cos^4 \theta) d\theta = 120 \int_0^{\pi/2} \cos^4 \theta \, d\theta = 120 \times \frac{3}{4} \times \frac{1}{2} \frac{\pi}{2} = \frac{45\pi}{2}.$$

Example 7.11: Evaluate $\iint_R \frac{r\,dr\,d\theta}{\sqrt{r^2 + a^2}}$ where R is the area of one loop of the leminiscate $r^2 = a^2 \cos 2\theta$.

Solution: The region of integration

$$R = \left\{ (r, \theta), 0 \le r \le a\sqrt{\cos 2\theta}, \frac{-\pi}{4} \le \theta \le \frac{\pi}{4} \right\}$$

is shown in Fig. 7.16. Therefore,

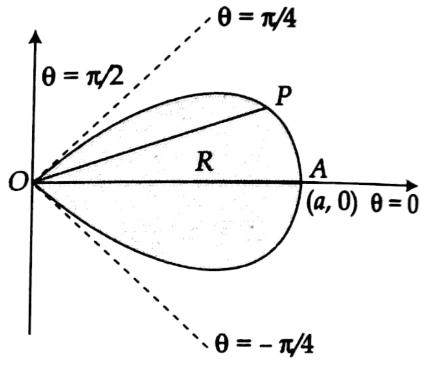


Fig. 7.16

$$I = \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \left(\int_0^{a\sqrt{\cos 2\theta}} \frac{r\,dr}{\sqrt{r^2 + a^2}} \right) d\theta = \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \left[(r^2 + a^2)^{\frac{1}{2}} \right]_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= 2a \int_0^{\frac{\pi}{4}} \left(\sqrt{1 + \cos 2\theta} - 1 \right) d\theta = 2a \int_0^{\frac{\pi}{4}} \left(\sqrt{2} \cos \theta - 1 \right) d\theta = 2a \left[\sqrt{2} \sin \theta - \theta \right]_0^{\pi/4}$$

$$= 2a \left[\sqrt{2} \sin \frac{\pi}{4} - \frac{\pi}{4} \right] = 2a \left(1 - \frac{\pi}{4} \right).$$

Example 7.12: Sketch the region of integration of $\int_a^{ae^{\pi/4}} \int_{2\ln\left(\frac{r}{a}\right)}^{\pi/2} f(r, \theta) . r\,d\theta\,dr$ and change the order of integration.

Solution: The region of integration is

$$R = \left\{ (r, \theta) : a \le r \le ae^{\frac{\pi}{4}}, 2 \ln \frac{r}{a} \le \theta \le \frac{\pi}{2} \right\}$$

Here $r = a$ and $r = ae^{\pi/4}$ are circles with pole as the centre and radii a and $ae^{\pi/4}$ respectively. The curve $\theta = 2 \ln \frac{r}{a}$ is $r = ae^{\theta/2}$, an equiangular spiral. At $\theta = 0$, $r = a$ and at $\theta = \pi/2$, $r = ae^{\pi/4}$.

The region R is shown in Fig. 7.17.

To change the order of integration take a strip PQ passing through the pole O. Thus the limits for r are: a to $r = ae^{\theta/2}$. To cover the entire region R the minimum of θ is zero and maximum is $\pi/2$. Thus the region R can be rewritten as

$$R = \begin{cases} a \le r \le ae^{\theta/2} \\ 0 \le \theta \le \pi/2 \end{cases}$$

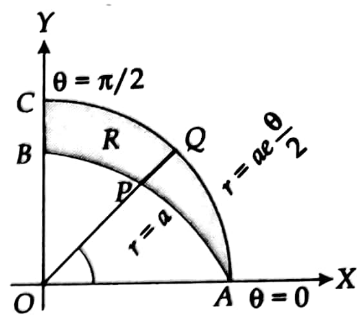


Fig. 7.17

and, therefore, the given integral becomes

$$I = \int_0^{\pi/2} \left(\int_a^{ae^{\theta/2}} f(r, \theta) r\, dr \right) d\theta.$$

EXERCISE 7.1

1. Evaluate the following integrals

(a) $\int_0^1 \int_{x^2}^{x} (x^2 + 3y + 2)\, dy\, dx$ (b) $\int_0^1 \int_x^{x^2} e^{y/x}\, dy\, dx$

(c) $\int_0^{\frac{\pi}{2}} \int_0^{a\cos\theta} r \sin\theta\, dr\, d\theta$ (d) $\int_0^{\pi} \int_0^{a(1+\cos\theta)} r^2 \cos\theta\, dr\, d\theta$

2. (a) Evaluate $\iint \frac{xy\, dx\, dy}{\sqrt{1 - x^2}}$ over the positive quadrant of the disc $x^2 + y^2 \le 1$.

(b) Evaluate $\iint y\, dx\, dy$ over the positive quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

3. Evaluate $\iint_R xy(x + y)\, dx\, dy$, where R is the region bounded by $y = x^2$ and $y = x$.

4. Evaluate $\iint \sin(ax + by)\, dx\, dy$ over the triangular area bounded by $x = 0$, $y = 0$ and $ax + by = 1$.

5. Evaluate, $\iint_R xy\, dx\, dy$, where R is the region bounded by $y = 0$, $x = 4a$ and the curve $x^2 = 4ay$.

6. Evaluate $\iint_R \sqrt{(y^2 - xy)}\, dy\, dx$, where R is the triangular region with vertices (0, 0), (0, 1) and (1, 1).

Evaluate the following integrals (7-12) by changing the order of integration.

7. $\int_0^1 \int_x^{2-x} dy\,dx$

8. $\int_0^\infty \int_x^\infty (e^{-y}/y)\,dy\,dx$

9. $\int_0^1 \int_{4y}^4 e^{x^2}\,dx\,dy$

10. $\int_0^\infty \int_0^x x e^{-x^2/y}\,dy\,dx$

11. $\int_0^a \int_{\sqrt{ax}}^a \frac{y^2\,dy\,dx}{\sqrt{y^4 - a^2x^2}}$

12. $\int_0^a \int_0^x \frac{f'(y)}{\sqrt{(a-x)(x-y)}}\,dy\,dx$

13. Calculate $\iint r^3\,dr\,d\theta$ over the area included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$.

14. Evaluate $\iint r\sin\theta\,dr\,d\theta$ over the cardioid $r = a(1-\cos\theta)$ above the initial line.

15. Show that $\iint r\,dr\,d\theta = a^2$ over the area of the lemniscate $r^2 = a^2\cos 2\theta$.

7.3 TRANSFORMATION OF VARIABLES IN DOUBLE INTEGRAL

Similar to the case of definite integrals in case of a single variable, the evaluation of a double integral sometimes is simplified by the transformation of variables of integration, e.g., from cartesian to polar or to some general system, say (u, v).

Let the variables x, y be defined in a region R_{xy} in the xy-plane be transformed to the new variables u, v as $x = x(u, v)$, $y = y(u, v)$, where $x(u, v)$ and $y(u, v)$ are continuous and have continuous first order derivatives in the region R'_{uv} in the uv– plane corresponding to the region R_{xy} in the xy–plane. Let the inverse transformations be $u = u(x, y)$, $v = v(x, y)$ where $u(x, y)$ and $v(x, y)$ are defined and have continuous first order derivatives in the region R_{xy} in the xy–plane. Then,

$$\iint_{R_{xy}} f(x, y)\,dx\,dy = \iint_{R'_{uv}} f[x(u, v), y(u, v)]\,|J|\,du\,dv = \iint_{R'_{uv}} g(u, v)\,du\,dv.$$

Here $J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$ is the Jacobian of the variable of transformation from (x, y) to (u, v). For example, in case of polar co-ordinates $x = r\cos\theta$, $y = r\sin\theta$ we have $J = r$ and thus

$$\iint_{R_{xy}} f(x, y)\,dx\,dy = \iint_{R'_{r\theta}} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta = \iint_{R'_{r\theta}} g(r, \theta)\,dr\,d\theta$$

where $R'_{r\theta}$ is the region in the $r\theta$–plane corresponding to the region R_{xy} in the xy–plane.

Example 7.13: Evaluate the integral $\iint (a^2 - x^2 - y^2)\,dx\,dy$ over the area of the circle $x^2 + y^2 = a^2$.

Solution: The region of integration R is, $R = \left\{(x, y) : -\sqrt{a^2 - x^2} \le y \le \sqrt{a^2 - x^2}, -a \le x \le a\right\}$

and, thus $I = \displaystyle\int_{-a}^{a}\left(\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (a^2 - x^2 - y^2)\,dy\right)dx.$

To simplify the computation we transform to polar co-ordinates system, by substituting $x = r\cos\theta$, $y = r\sin\theta$, $J = r$, and, therefore, I becomes

$$I = \int_0^{2\pi}\int_0^{a} (a^2 - r^2)\,r\,dr\,d\theta = 2\pi\int_0^{a} (a^2 r - r^3)\,dr = 2\pi\left(a^2\frac{r^2}{2} - \frac{r^4}{4}\right)_0^{a} = \frac{\pi a^4}{2}.$$

Example 7.14: Evaluate $\displaystyle\iint\left[\frac{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}{1 + \frac{x^2}{a^2} + \frac{y^2}{b^2}}\right]^{1/2} dx\,dy$ over the positive quadrant of the ellipse $x^2/a^2 + y^2/b^2 = 1$.

Solution: Applying the transformation $x = au$, $y = bv$, we obtain

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab.$$

The region of integration, the positive quadrant of the ellipse $x^2/a^2 + y^2/b^2 = 1$, transforms to the positive quadrant of the circle $u^2 + v^2 = 1$, and, therefore the integral becomes

$$I = ab\int_0^{1}\int_0^{\sqrt{1-u^2}}\left(\frac{1 - u^2 - v^2}{1 + u^2 + v^2}\right)^{1/2} du\,dv.$$

To simplify the computation further we transform to polar co-ordinates system by substituting $u = r\cos\theta$, $v = r\sin\theta$, which gives $J = \dfrac{\partial(u, v)}{\partial(r, \theta)} = r$ and, therefore, I becomes

$$I = ab\int_0^{\pi/2}\left(\int_0^{1}\left(\frac{1 - r^2}{1 + r^2}\right)^{\frac{1}{2}} r\,dr\right)d\theta = \frac{\pi ab}{2}\int_0^{1}\left(\frac{1 - r^2}{1 + r^2}\right)^{\frac{1}{2}} r\,dr$$

$$= \frac{\pi ab}{4}\int_0^{1}\left(\frac{1 - t}{1 + t}\right)^{\frac{1}{2}} dt = \frac{\pi ab}{4}\int_0^{1}\frac{1 - t}{\sqrt{1 - t^2}}\,dt, \quad \text{using } r^2 = t,$$

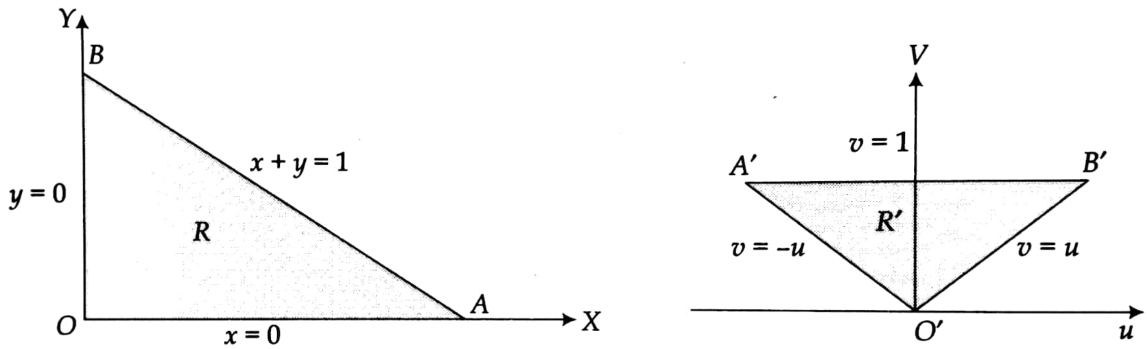
$$= \frac{\pi ab}{4}\left[\int_0^1 \frac{1}{\sqrt{1-t^2}}dt - \int_0^1 \frac{t}{\sqrt{1-t^2}}dt\right] = \frac{\pi ab}{4}\left[\sin^{-1} t + \sqrt{1-t^2}\right]_0^1 = \frac{\pi ab}{4}\left[\frac{\pi}{2}-1\right].$$

Example 7.15: Evaluate $\iint_R \cos\left(\frac{x-y}{x+y}\right)dx\,dy$, where R is the region bounded by $x = 0$, $y = 0$, $x + y = 1$.

Solution: We apply the transformation $u = x - y$, $v = x + y$, which gives

$$x = \frac{u+v}{2}, \text{ and } \quad y = \frac{v-u}{2}.$$

Thus the line $x = 0$ transforms to the line $u + v = 0$ that is, $v = -u$, the line $y = 0$ transforms to the line $v = u$ and the line $x + y = 1$ transforms to $v = 1$. Thus the region R as shown in Fig. 7.18a in the xy plane transforms to the region R' in the uv plane as shown in Fig. 7.18b.



(a) (b)

Fig. 7.18

Also, $J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$. Thus,

$$I = \iint_R \cos\left(\frac{x-y}{x+y}\right)dx\,dy = \frac{1}{2}\iint_{R'} \cos\frac{u}{v}\,du\,dv = \frac{1}{2}\int_0^1\left(\int_{-v}^{v} \cos\frac{u}{v}\,du\right)dv = \frac{1}{2}\int_0^1\left[v\sin\frac{u}{v}\right]_{-v}^{v} dv$$

$$= \frac{1}{2}\int_0^1 v[\sin(1) - \sin(-1)]dv = \sin(1)\int_0^1 v\,dv = \frac{1}{2}\sin(1).$$

EXERCISE 7.2

Evaluate the following integrals by changing the variables to polars coordinates

1. $\int_0^{4a}\int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2}\,dx\,dy$

2. $\int_0^{a}\int_0^{\sqrt{a^2-y^2}} y^2\sqrt{x^2+y^2}\,dx\,dy$

3. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{a\,dx\,dy}{(x^2+y^2+a^2)^{3/2}(x^2+y^2+b^2)^{1/2}}$

4. $\int_0^{2a}\int_0^{\sqrt{(2ax-x^2)}} (x^2+y^2)\,dy\,dx$

5. Change the order of integration in the double integral $\int_0^{2a}\int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x,y)\,dy\,dx.$

6. Evaluate $\iint y^2\,dx\,dy$ over the area outside the circle $x^2+y^2-ax=0$ and inside the circle $x^2+y^2-2ax=0$.

7. Evaluate $\iint \frac{dx\,dy}{(1+x^2+y^2)^{3/2}}$ over the unit square $0\le x\le 1,\ 0\le y\le 1$.

8. Evaluate $\iint \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dx\,dy$ over the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

7.4 APPLICATIONS OF DOUBLE INTEGRALS

Double integrals are used to calculate the areas of bounded regions in the plane and also the mass, moments, centres of mass, etc. of thin plates covering these regions. In addition to this, double integrals are used to calculate the volume of the region below the surface $z = f(x, y)$ and above the xy-plane and volumes of the solids generated by revolution.

In this section we consider these applications of double integrals.

7.4.1 Area, Mass and Centre of Mass of the Bounded Regions in Plane

If we take $f(x, y) = 1$ in the definition of the double integral over a region R, then (7.2) gives *the area A of the bounded plane region R in cartesian co-ordinates, that is,*

$$A = \iint_R dx\,dy. \qquad ...(7.5)$$

In *polar co-ordinates system, the area A of the bounded plane region R* is given by

$$A = \iint_R r\,dr\,d\theta. \qquad ...(7.6)$$

The expression

$$\frac{1}{A}\iint_R f(x, y)\,dx\,dy \qquad \text{...(7.7)}$$

is defined as the *average value of f(x, y) over the region R*, where A is the area of the region R.

For example, if $f(x, y)$ denotes the distance of an arbitrary point $P(x, y)$ in R from a fixed point T then (7.7) gives the average distance of R from T.

Next, if $\rho(x, y)$ is the density function, then

$$M = \iint_R \rho(x, y)\,dx\,dy \qquad \text{...(7.8)}$$

gives *the mass M of the thin plate covering region R* in the xy–plane.

Further, $M_x = \iint_R y\rho(x, y)\,dx\,dy$ and $M_y = \iint_R x\rho(x, y)\,dx\,dy$ ...(7.9)

give *the first moments about x–axis and y–axis*, respectively, and

$$\bar{x} = M_y/M, \quad \bar{y} = M_x/M \qquad \text{...(7.10)}$$

give *the co-ordinates of the centre of mass* of the mass M in R.

7.4.2 Moments of Inertia and Radii of Gyration

If $\rho(x, y)$ is the density function, then

$$I_x = \iint_R y^2\rho(x, y)\,dx\,dy, \quad I_y = \iint_R x^2\rho(x, y)\,dx\,dy \qquad \text{...(7.11)}$$

are the *moments of inertia*, or *second moments of the mass M in R about x–axis and y–axis respectively*. Also

$$I_0 = \iint_R (x^2 + y^2)\rho\,dx\,dy = I_x + I_y \qquad \text{...(7.12)}$$

is the *moment of inertia about the origin, or the polar moment of the mass M in R*, and

$$I = \iint_R d^2(x, y)\rho(x, y)\,dx\,dy \qquad \text{...(7.13)}$$

is the *moment about a line l of the mass M in R*, where $d(x, y)$ is the distance of an arbitrary point $P(x, y)$ from the line l.

The *radii of gyration* R_x, R_y and R_0 respectively about x–axis, y–axis and the origin are defined as

$$I_x = MR_x^2, \quad I_y = MR_y^2 \text{ and } I_0 = MR_0^2. \qquad \text{...(7.14)}$$

Example 7.16: Find the average height of the paraboloid $z = x^2 + y^2$ over the square $0 \le x \le 2$, $0 \le y \le 2$.

Solution: The average height $\bar{h}$ of the paraboloid $z = x^2 + y^2$ over the region $R = \{(x, y),\ 0 \le x, y \le 2\}$, a square of area 4 sq. unit, is given by

$$\bar{h} = \frac{1}{4}\int_0^2\int_0^2 (x^2+y^2)dx\,dy = \frac{1}{4}\int_0^2 \left(x^2y+\frac{y^3}{3}\right)_0^2 dx$$

$$= \frac{1}{4}\int_0^2 \left(2x^2+\frac{8}{3}\right)dx = \frac{1}{4}\left(\frac{2x^3}{3}+\frac{8}{3}x\right)_0^2 = \frac{1}{4}\left(\frac{16}{3}+\frac{16}{3}\right) = 8/3.$$

*

Example 7.17: Find the area bounded by the ellipse $x^2/a^2 + y^2/b^2 = 1$ using the double integration.

Solution: Using the symmetry we consider the area only in the first quadrant as shown in Fig. 7.19 and thus

$$A = 4\int_0^b \int_0^{a\sqrt{1-\frac{y^2}{b^2}}} dx\,dy = 4a\int_0^b \sqrt{1-\frac{y^2}{b^2}}\,dy$$

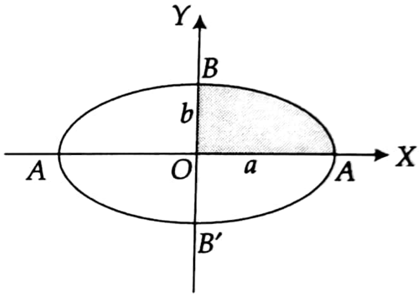


Fig. 7.19

Substituting $y = b\sin\theta$, we get

$$A = 4a\int_0^{\pi/2} \cos\theta\,(b\cos\theta)d\theta$$

$$= 4ab\int_0^{\pi/2} \frac{1+\cos 2\theta}{2}d\theta = 2ab\left[\theta+\frac{\sin 2\theta}{2}\right]_0^{\pi/2} = \pi ab.$$

Example 7.18: Find the area lying inside the circle $x^2 + y^2 - 2ax = 0$ and outside the circle $x^2 + y^2 = a^2$ using double integration.

Solution: The centres of the two given circles are $(a, 0)$ and $(0, 0)$ and both are of radius a. The required area is shown in Fig. 7.20.

Changing to polar co-ordinates, the equations of the circle are $r = a$ and $r = 2a\cos\theta$. Their points of intersection A, A' are given by

$$2a\cos\theta = a, \text{ or } \cos\theta = \frac{1}{2}, \text{ or } \theta = \pm\frac{\pi}{3}.$$

Thus the required area is equal to

$$A = 2\int_0^{\frac{\pi}{3}}\int_a^{2a\cos\theta} r\,dr\,d\theta = 2\int_0^{\pi/3}\left[\frac{r^2}{2}\right]_a^{2a\cos\theta} d\theta$$

$$= \int_0^{\pi/3} (4a^2\cos^2\theta - a^2)d\theta = a^2\int_0^{\pi/3} [2(1+\cos 2\theta) - 1]d\theta$$

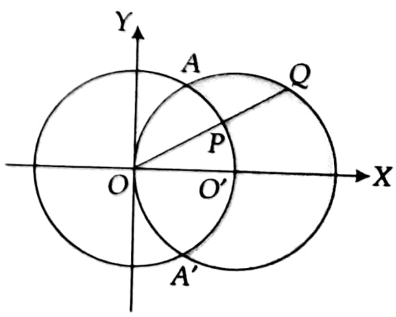


Fig. 7.20

$$= a^2 \int_0^{\pi/2} (1 + 2\cos 2\theta)d\theta = a^2[\theta + \sin 2\theta]_0^{\pi/3} = a^2\left[\frac{\pi}{3} + \sin\frac{2\pi}{3}\right] = a^2\left[\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right].$$

Example 7.19: Find the area included between the curve $r = a(\sec\theta + \cos\theta)$ and its asymptotes.

Solution: The curve $r = a(\sec\theta + \cos\theta)$ is symmetrical about the initial line $\theta = 0$, and has an asymptote $r = a\sec\theta$. The required area is shown in Fig. 7.21. It is given by

$$A = 2\int_0^{\pi/2} \int_{a\sec\theta}^{a(\sec\theta+\cos\theta)} r\,dr\,d\theta$$

$$= 2\int_0^{\pi/2} \left[\frac{r^2}{2}\right]_{a\sec\theta}^{a(\sec\theta+\cos\theta)} d\theta$$

$$= a^2 \int_0^{\pi/2} [(\sec\theta + \cos\theta)^2 - \sec^2\theta]d\theta = a^2 \int_0^{\pi/2} (\cos^2\theta + 2)d\theta$$

$$= \frac{a^2}{2} \int_0^{\pi/2} (\cos 2\theta + 5)d\theta = \frac{a^2}{2}\left[\frac{\sin 2\theta}{2} + 5\theta\right]_0^{\pi/2} = \frac{5\pi a^2}{4}.$$

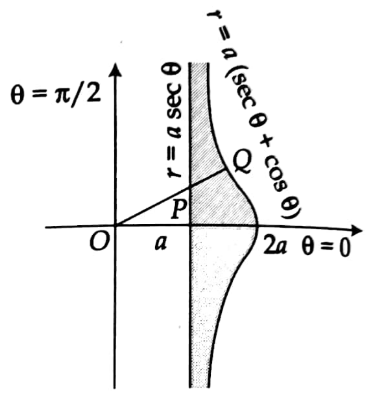


Fig. 7.21

Example 7.20: A thin plate covers the triangular region bounded by the x–axis and the lines $x = 1$ and $y = 2x$ in the first quadrant. The plate's density at the point (x, y) is $\rho(x, y) = 6(x + y + 1)$. Find the plate's mass, first moments, centre of mass, moments of inertia and radii of gyration about the co-ordinate axes.

Solution: The triangular region R as shown in Fig. 7.22, is given by

$R = \{(x, y): 0 \le y \le 2x,\ 0 \le x \le 1\}$.

The mass M of the plate is

$$M = \iint_R \rho(x, y)dy\,dx = \int_0^1 \int_0^{2x} 6(x + y + 1)dy\,dx$$

$$= 6\int_0^1 \left[xy + \frac{y^2}{2} + y\right]_0^{2x} dx = 6\int_0^1 [2x^2 + 2x^2 + 2x]dx$$

$$= 12\int_0^1 (2x^2 + x)dx = 12\left[\frac{2x^3}{3} + \frac{x^2}{2}\right]_0^1 = 14.$$

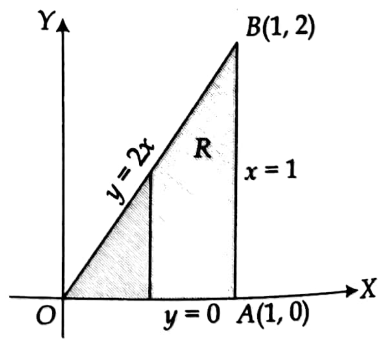


Fig. 7.22

The first moment about x–axis, refer to (7.9), is

$$M_x = \iint_R y\rho(x, y)dy\,dx = 6\int_0^1 \int_0^{2x} y(x + y + 1)dy\,dx = 6\int_0^1 \left(\frac{xy^2}{2} + \frac{y^3}{3} + \frac{y^2}{2}\right)_0^{2x} dx$$

$$= 6\int_0^1 \left(2x^3 + \frac{8}{3}x^3 + 2x^2\right)dx = 6\int_0^1 \left(\frac{14}{3}x^3 + 2x^2\right)dx = 6\left[\frac{7}{6}x^4 + \frac{2x^3}{3}\right]_0^1 = 11.$$

The first moment about the y–axis, refer to (7.9), is

$$M_y = \iint_R x\rho(x, y)dy\,dx = 6\int_0^1\int_0^{2x} x(x + y + 1)dy\,dx = 6\int_0^1 \left(x^2y + \frac{xy^2}{2} + xy\right)_0^{2x} dx$$

$$= 6\int_0^1 (2x^3 + 2x^3 + 2x^2)dx = 12\int_0^1 (2x^3 + x^2)dx = 12\left(\frac{x^4}{2} + \frac{x^3}{3}\right)_o^1 = 10.$$

Thus the co-ordinates of the centre of mass $G(\bar{x}, \bar{y})$, refer to (7.10), are

$$\bar{x} = \frac{M_y}{M} = \frac{10}{14} = \frac{5}{7}, \quad \bar{y} = \frac{M_x}{M} = \frac{11}{14}.$$

The moment of inertia about x–axis, refer to (7.11), is

$$I_x = \iint_R y^2\rho(x, y)dy\,dx = 6\int_0^1\int_0^{2x} y^2(x + y + 1)dy\,dx = 6\int_0^1 \left[\frac{xy^3}{3} + \frac{y^4}{4} + \frac{y^3}{3}\right]_0^{2x} dx$$

$$= 6\int_0^1 \left(\frac{8x^4}{3} + 4x^4 + \frac{8x^3}{3}\right)dx = 6\int_0^1 \left(\frac{20x^4}{3} + \frac{8x^3}{3}\right)dx = 6\left(\frac{4x^5}{3} + \frac{2x^4}{3}\right)_0^1 = 12.$$

The moment of inertia about y–axis is

$$I_y = \iint_R x^2\rho(x, y)dy\,dx = 6\int_0^1\int_0^{2x} x^2(x + y + 1)dy\,dx = 6\int_0^1 \left(x^3y + \frac{x^2y^2}{2} + x^2y\right)_0^{2x} dx$$

$$= 6\int_0^1 (2x^4 + 2x^4 + 2x^3)dx = 12\int_0^1 (2x^4 + x^3)dx = 12\left(\frac{2x^5}{5} + \frac{x^4}{4}\right)_0^1 = \frac{39}{5}.$$

The moment of inertia about the z–axis is

$$I_0 = \iint_R (x^2 + y^2)\rho(x, y)dy\,dx = I_x + I_y = 12 + \frac{39}{5} = \frac{99}{5}.$$

The radii of gyration about the x-axis, y-axis and z-axis are

$$R_x = \sqrt{I_x/M} = \sqrt{12/14} = \sqrt{6/7}, \quad R_y = \sqrt{I_y/M} = \sqrt{\left(\frac{39}{5}\right)\Big/14} = \sqrt{39/70}$$

and, $R_0 = \sqrt{I_0/M} = \sqrt{\left(\frac{99}{5}\right)\Big/14} = \sqrt{99/70}$ respectively.

EXERCISE 7.3

1. Show that the area enclosed between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $16a^2/3$.
2. Show that the area enclosed by the line $y = x$ and the parabola $y = x^2$ in the first quadrant is $9/2$.
3. Find the area enclosed by the curve $r = a(1 + \cos\theta)$.
4. Find the area enclosed by one loop of the curve $r^2 = a^2 \cos 2\theta$.
5. Find by double integration, the area lying inside the circle $r = a \sin\theta$ and outside the cardiod $r = a(1 - \cos\theta)$.
6. Find the area enclosed by the curve $y = 3x/(x^2 + 2)$ and $y^2 = 4(1 - x)$.
7. Find the centroid of the area of the circle $x^2 + y^2 = a^2$ in the first quadrant.
8. Find the average height of the hemisphere $z = \sqrt{a^2 - x^2 - y^2}$ above the disc $x^2 + y^2 \le a^2$ in the xy-plane.
9. Find the centroid of the region of constant density in the first quadrant bounded by the x-axis, the parabola $y^2 = 2x$ and the line $x + y = 4$.
10. Find by double integration the centre of gravity of the area of the cardioid $r = a(1 + \cos\theta)$, the density being constant.
11. Find the centroid of a loop of the lemniscate $r^2 = a^2 \cos 2\theta$.
12. A plane in the form of a quadrant of the ellipse $(x/a)^2 + (y/b)^2 = 1$ is of small but varying thickness, the thickness at any point being proportional to the product of the distances of that point from the axes, show that the co-ordinates of the centroid are $(8a/15, 8b/15)$.
13. Find the centre of mass, moment of inertia and radius of gyration about y-axis of a thin rectangular plate cut from the first quadrant by the lines $x = 6$ and $y = 1$ if the density $\rho(x, y) = x + y + 1$.
14. Find the centre of mass, moments of inertia, radii of gyration about the co-ordinate axes of a thin triangular plate bounded by the lines $y = x$, $y = -x$ and $y = 1$ if density $\rho(x, y) = y + 1$.
15. Find moment of inertia of uniform area bounded by the curve $r^2 = a^2 \cos 2\theta$ about its axis.
16. Find the moment of inertia of a solid right circular cone of uniform density having base radius r and height h about (i) its axis, (ii) an axis through the vertex and perpendicular to its axis. (iii) a diameter of its base.

7.4.3 Area of a Curved Surface: $z = f(x, y)$

Let S' be the projection of the surface $S: z = f(x, y)$, on the xy-plane. Divide the region S' into area elements by drawing lines parallel to the axis of x and y as shown in Fig. 7.23. If the element $PQRS$ of area $\delta x\,\delta y$ is the projection of the surface element δS, then $\delta x\,\delta y = \cos\gamma\,\delta S$, where γ is the angle between $\hat{k}$, the normal to the xy-plane, and $\hat{N}$, the outward normal to δS, the elementary surface.

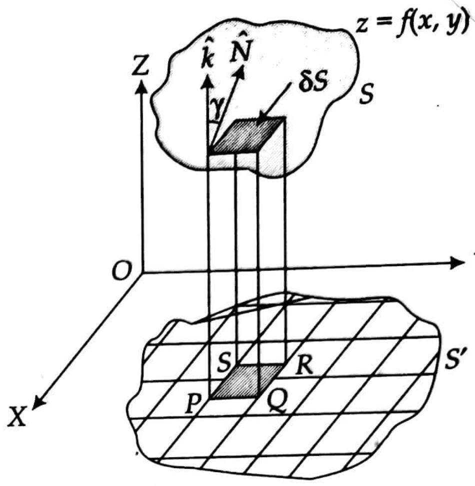


Fig. 7.23

Since the d.c.'s of the normal to S, that is, $f(x, y) - z = 0$ are proportional to $-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1$ and those to the z-axis are 0, 0, 1,

therefore
$$\cos\gamma = \frac{1}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}},$$

and thus
$$\delta S = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\ \delta x\,\delta y.$$

Hence,
$$S = \lim_{\delta S \to 0} \sum \delta S = \iint_{S'} \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\ dx\,dy, \qquad ...(7.15)$$

where integration is over S', the orthogonal projection of S on the xy–plane.

7.4.4 Volume of the Region Below the Surface $z = f(x, y)$ and Above the xy-plane

Consider a surface S, $z = f(x, y)$ and let S' be the orthogonal projection of S on the xy-plane as shown in Fig. 7.23.

Divide S' into elementary rectangular areas $\delta x\,\delta y$ by drawing lines parallel to x-axis and y-axis. With each of these rectangles as base, erect a prism with lengths parallel to z-axis. Then the volume of this typical prism between S' and the surface $z = f(x, y)$ is $\delta V = z\,\delta x\,\delta y = f(x, y)\delta x\,\delta y$.

Therefore, the volume of the solid cylinder bounded above by the surface $z = f(x, y)$ and below by S' orthogonal projection of S in the xy-plane, with generators parallel to the z-axis is given by

$$\boxed{V = \iint_{S'} f(x, y)\,dx\,dy.} \qquad ...(7.16)$$

In terms of polar co-ordinates, it is given by

$$\boxed{V = \iint_{S'} zr\,dr\,d\theta.} \qquad ...(7.17)$$

7.4.5 Volume of Solid of Revolution

Let the region R bounded above by $y = f(x)$, below by x-axis and the ordinates $x = a$ and $x = b$, as shown in Fig. 7.24, is revolved about x-axis. We need to determine the volume of the solid generated as such.

Consider an elementary area $PQRS$, where $P = (x, y)$ and $R = (x + \delta x, y + \delta y)$.

The volume of the solid generated by this elementary rectangle $PQRS$ of area $\delta x\ \delta y$ when revolved about x-axis is $\delta V = \pi[(y + \delta y)^2 - y^2]\delta x \approx 2\pi y\,\delta y\,\delta x$, neglecting the second order differentials.

Therefore, the total volume of the solid generated is given by

$$\boxed{V = 2\pi \iint_R y\,dy\,dx,} \qquad ...(7.18)$$

where $R = \{(x, y) : a \le x \le b,\ 0 \le y \le f(x)\}$.

In case the area is revolved about y–axis, then the volume of the solid generated is

$$\boxed{V = 2\pi \iint_R x\,dx\,dy.} \qquad ...(7.19)$$

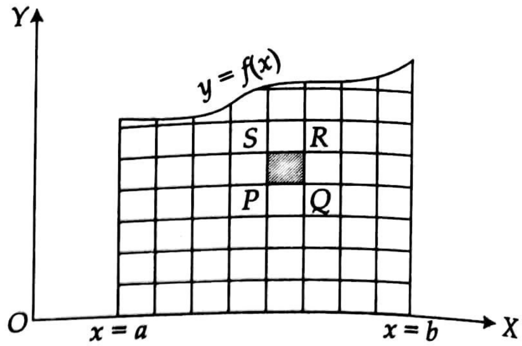


Fig. 7.24

In terms of *polar coordinates,* the corresponding formulae for the volumes of solid generated, when R is revolved about x-axis and y-axis are respectively

$$V = 2\pi \iint_R r\sin\theta . r\,dr\,d\theta = 2\pi \iint_R r^2 \sin\theta\,dr d\theta \qquad ...(7.20)$$

and,

$$V = 2\pi \iint_R r\cos\theta . r\,dr\,d\theta = 2\pi \iint_R r^2 \cos\theta\,dr d\theta. \qquad ...(7.21)$$

In case the area *is revolved about any line L,* then the volume of the solid generated is

$$\boxed{V = 2\pi \iint_R d(x, y)\,dx\,dy,} \qquad ...(7.22)$$

where $d(x, y)$ is the perpendicular distance of an arbitrary point $P(x, y)$ in R from the line L.

Example 7.21: A circular hole of radius b is made centrally through a sphere of radius a, find the volume of the remaining portion of the sphere.

Solution: Let the centre of the sphere be taken as the origin and axis of the hole be taken as the z-axis as shown in Fig. 7.25.

The volume of the upper half of the hole is $\iint_R z\,dx\,dy,$

where $z = \sqrt{a^2 - x^2 - y^2}$ and R is the orthogonal projection of the surface for the hollow portion $z = \sqrt{a^2 - x^2 - y^2}$ in the xy-plane, that is, $R : x^2 + y^2 = b^2$. Hence the volume V_1 of the circular hole is

$$V_1 = 2 \iint_{x^2+y^2=b^2} \sqrt{a^2 - x^2 - y^2}\,dx\,dy.$$

Using the polar co-ordinates, we obtain

$$V_1 = 2\int_0^{2\pi}\int_0^b \sqrt{a^2 - r^2}\,r\,dr\,d\theta = 4\pi\int_0^b \sqrt{a^2 - r^2}\,r\,dr$$

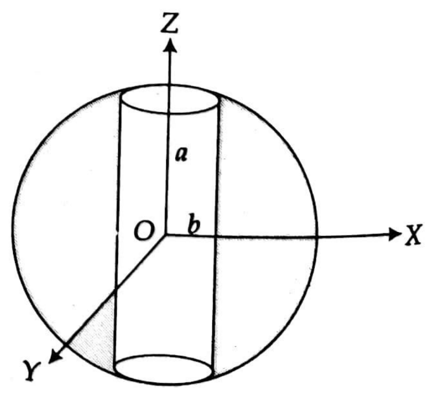


Fig. 7.25

$$= 4\pi\left[-\frac{1}{3}(a^2 - r^2)^{3/2}\right]_0^b = \frac{4\pi}{3}[a^3 - (a^2 - b^2)^{3/2}].$$

Since the volume of the sphere is $V_2 = \frac{4}{3}\pi a^3$, hence the volume V of the remaining portion is

$$V = V_2 - V_1 = \frac{4}{3}\pi a^3 - \frac{4}{3}\pi[a^3 - (a^2 - b^2)^{3/2}] = \frac{4}{3}\pi(a^2 - b^2)^{3/2}.$$

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Example 7.22: Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ using the double integration.

Solution: The required volume V is 8 times the volume in the first octant, thus $V = 8\iint_R z\,dx\,dy$,

where $z = c\sqrt{1 - x^2/a^2 - y^2/b^2}$ and R is the projection of this surface in the xy-plane which is the region in the first quadrant of the ellipse $x^2/a^2 + y^2/b^2 = 1$. Therefore,

$$V = 8\int_0^a\left(\int_0^{b\sqrt{1-x^2/a^2}} c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\,dy\right)dx.$$ Substituting $\boxed{y = b\sqrt{1 - \frac{x^2}{a^2}}\sin\theta}$, we get

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$$V = 8c\int_0^a\left(\int_0^{\pi/2}\sqrt{1 - \frac{x^2}{a^2}}\cos\theta\left(b\sqrt{1 - \frac{x^2}{a^2}}\cos\theta\,d\theta\right)\right)dx = 8bc\int_0^a\left(\int_0^{\pi/2}\left(1 - \frac{x^2}{a^2}\right)\cos^2\theta\,d\theta\right)dx$$

$$= 4bc\int_0^a\left\{\left(1 - \frac{x^2}{a^2}\right)\int_0^{\pi/2}(\cos 2\theta + 1)d\theta\right\}dx = 4bc\int_0^a\left(1 - \frac{x^2}{a^2}\right)\left(\frac{\sin 2\theta}{2} + \theta\right)_0^{\pi/2}dx$$

$$= 2\pi bc\int_0^a\left(1 - \frac{x^2}{a^2}\right)dx = 2\pi\,bc\left[x - \frac{x^3}{3a^2}\right]_0^a = \frac{4\pi bc a}{3}\text{ cubic unit.}$$

Example 7.23: Find by double integration, the volume generated by the revolution of the cardioid $r = a(1 - \cos\theta)$ about its axis $\theta = 0$.

Solution: The volume generated by the revolution of the cardioid about x-axis, that is, about $\theta = 0$ is obtained by considering the area above (or below) the initial line and is given by

$$V = 2\pi\iint_R r^2\sin\theta\,dr\,d\theta,$$

where $R = \{(r,\theta) : 0 \le r \le a(1 - \cos\theta),\ 0 \le \theta \le \pi\}$, as shown in Fig. 7.26.

Thus, $V = 2\pi \int_0^{\pi} \left(\int_0^{a(1-\cos\theta)} r^2 dr \right) \sin\theta \, d\theta$

$$= \frac{2\pi}{3} \int_0^{\pi} \left[r^3 \right]_0^{a(1-\cos\theta)} \sin\theta \, d\theta = \frac{2\pi a^3}{3} \int_0^{\pi} (1 - \cos\theta)^3 \sin\theta \, d\theta$$

$$= \frac{2\pi a^3}{3} \left[\frac{(1 - \cos\pi)^4}{4} \right] = \frac{8\pi a^3}{3}.$$

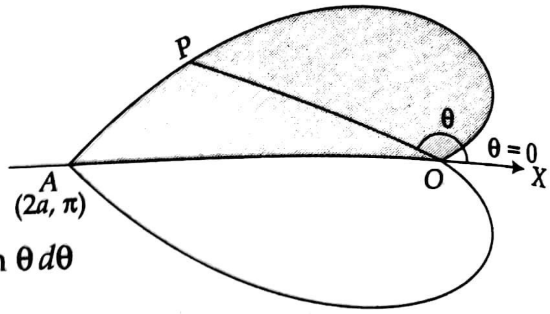

Fig. 7.26

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Example 7.24: Find the volume of the solid generated obtained by revolving the circle $x^2 + y^2 = 4$ about the line $x = 3$.

Solution: The required volume is

$$V = \iint_R 2\pi d(x, y) dx\, dy,$$

where $R = \{(x, y): -\sqrt{4 - x^2} \le y \le \sqrt{4 - x^2}, -2 \le x \le 2\}$ is the region enclosed by the circle $x^2 + y^2 = 4$, as shown in Fig. 7.27, and $d(x, y)$ is the perpendicular distance of an arbitrary point $P(x, y)$ in R from the line $x - 3 = 0$, that is, $d(x, y) = |3 - x| = 3 - x$, for $-2 \le x \le 2$. Thus,

$$V = 2\pi \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (3 - x) dx\, dy = 2\pi \int_{-2}^{2} \left\{ (3 - x) \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy \right\} dx$$

$$= 4\pi \int_{-2}^{2} (3 - x)\sqrt{4 - x^2}\, dx = 4\pi \left[\int_{-2}^{2} 3\sqrt{4 - x^2} - \int_{-2}^{2} \sqrt{4 - x^2}\, x\, dx \right]$$

$$= 4\pi \left[3\left\{ \frac{x}{2}\sqrt{4 - x^2} + 2\sin^{-1}\frac{x}{2} \right\} + \frac{1}{3}(4 - x^2)^{3/2} \right]_{-2}^{2}$$

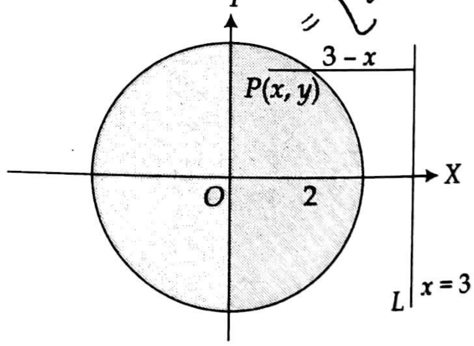

Fig. 7.27

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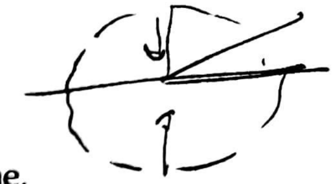
$$= 4\pi[3\pi + 3\pi] = 24\pi^2.$$

Example 7.25: Show that the surface area of the sphere $x^2 + y^2 + z^2 = a^2$ is $4\pi a^2$.

Solution: The orthogonal projection of the sphere $x^2 + y^2 + z^2 = a^2$ in the xy-plane is the circle $x^2 + y^2 = a^2$.

For the surface $z^2 = a^2 - x^2 - y^2$, we have $\frac{\partial z}{\partial x} = -\frac{x}{z}$, $\frac{\partial z}{\partial y} = -\frac{y}{z}$.

Thus, $\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 + 1 = \dfrac{x^2}{z^2} + \dfrac{y^2}{z^2} + 1 = \dfrac{a^2}{a^2 - x^2 - y^2}$. Therefore,



$$S = 2\iint_R \frac{a}{\sqrt{a^2 - x^2 - y^2}}\,dx\,dy,$$ where R is the circle $x^2 + y^2 = a^2$ in the xy-plane.

Changing to polar co-ordinates $x = r\cos\theta,\ y = r\sin\theta$

$$S = 2\int_0^{2\pi}\int_0^{a} \frac{a}{\sqrt{a^2 - r^2}}\,r\,dr\,d\theta = -2\pi a\int_0^{a}(a^2 - r^2)^{-\frac{1}{2}}(-2r)dr = -4\pi a\left[(a^2 - r^2)^{1/2}\right]_0^a = 4\pi a^2.$$

Example 7.26: Find the area of the portion of the surface of the sphere $x^2 + y^2 + z^2 = 9$ lying inside the cylinder $x^2 + y^2 = 3y$.

Solution: The projection of the one-fourth of the desired surface area in the xy-plane is the semicircle $x^2 + y^2 = 3y$ in the first quadrant as shown in the Fig. 7.28. For the surface of the sphere $x^2 + y^2 + z^2 = 9$, we have

$$\frac{\partial z}{\partial x} = -\frac{x}{z},\quad \frac{\partial z}{\partial y} = -\frac{y}{z}.$$

Thus $$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1 = \frac{x^2 + y^2 + z^2}{z^2} = \frac{9}{9 - x^2 - y^2}.$$

Thus the required surface area S is

$$S = 4\iint_R \frac{3}{\sqrt{9 - x^2 - y^2}}\,dx\,dy,$$

where R is the semicircle $x^2 + y^2 = 3y$ in the first quadrant.

Changing to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, the region is

$$R = \left\{(r,\theta): 0 \le r \le 3\sin\theta,\ 0 \le \theta \le \frac{\pi}{2}\right\},$$ and hence

$$S = 4\int_0^{\frac{\pi}{2}}\int_0^{3\sin\theta} \frac{3}{\sqrt{9 - r^2}}\,r\,dr\,d\theta = -12\int_0^{\pi/2}\left[(9 - r^2)^{\frac{1}{2}}\right]_0^{3\sin\theta} d\theta$$

$$= 36\int_0^{\pi/2}(1 - \cos\theta)\,d\theta = 18(\pi - 2).$$

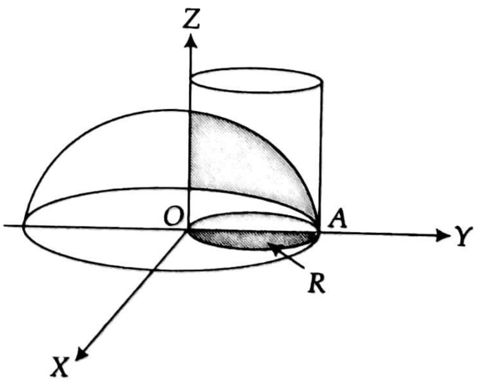


Fig. 7.28

EXERCISE 7.4

1. Find the volume of the sphere $x^2 + y^2 + z^2 = a^2$, using double integration.
2. Find the volume of the region bounded by the surfaces $y = x^2$ and $x = y^2$ and the planes $z = 0$, $z = 3$.
3. Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.
4. Using double integration find the volume of the tetrahedron bounded by the coordinate planes and the plane $x/a + y/b + z/c = 1$ in the first octant.
5. Using double integration show that the volume generated by the revolution of the cardioid $r = a(1 + \cos\theta)$ about its axis is $8\pi a^3/3$.
6. The area bounded by the parabola $y^2 = 4x$ and the straight lines $x = 1$ and $y = 0$ in the first quadrant is revolved about the line $y = 2$. Find by double integral the volume of the solid generated.
7. Find the volume generated by the revolution of the curve $y^2(2a - x) = x^3$ about its asymptote through four right angles.
8. Find the area of the portion of the cylinder $x^2 + z^2 = 4$ lying inside the cylinder $x^2 + y^2 = 4$.
9. Compute the area of that part of the plane $x + y + z = 2a$ which lies in the first octant and is bounded by the cylinder $x^2 + y^2 = a^2$.
10. Find area of the surface of the cylinder $x^2 + z^2 = a^2$ that lies inside the cylinder $x^2 + y^2 = a^2$.

7.5 TRIPLE INTEGRALS

Let $f(x, y, z)$ be a continuous and single valued function of x, y and z defined over a closed and bounded region D in space. Subdivide the region D into a number of parallelopipeds by drawing planes parallel to the coordinate planes. Number the parallelopipeds which are inside D in some order say from 1 to n. Choose an arbitrary point (x_k, y_k, z_k) in each ΔV_k, where $\Delta V_k = \Delta x_k \Delta y_k \Delta z_k$ is the volume of the kth parallelopiped, and form the sum

$$S_n = \sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta x_k \Delta y_k \Delta z_k. \qquad \text{...(7.23)}$$

The limit of this sum as $n \to \infty$ and $\Delta V_k \to 0$ is defined as the triple integral of $f(x, y, z)$ over the region D and is denoted by

$$I = \iiint_D f(x, y, z)\,dV, \text{ or } \iiint_D f(x, y, z)\,dx\,dy\,dz. \qquad \text{...(7.24)}$$

Similar to double integrals, here also the continuity of $f(x, y, z)$ is a sufficient condition for the existence of the triple integrals, but not a necessary one. Also triple integrals satisfy the properties similar to that of double integrals.

Further, as in case of double integrals, the triple integrals are also hardly evaluated as the limits of the sums. These are evaluated by three successive integrations. If the region D is given by

$$D = \{(x, y, z) : x_1 \le x \le x_2,\; y_1(x) \le y \le y_2(x),\; z_1(x, y) \le z \le z_2(x, y)\},$$

then the triple integral is evaluated as

$$\iiint\limits_D f(x, y, z)\,dx\,dy\,dz = \int_{x_1}^{x_2}\left[\int_{y_1(x)}^{y_2(x)}\left[\int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)\,dz\right]dy\right]dx.$$

However, the order of integration depends on the form of the problem given.

Example 7.27: Evaluate $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a} (x^2 + y^2 + z^2)\,dx\,dy\,dz.$

Solution: Let

$$I = \int_{-c}^{c}\left(\int_{-b}^{b}\left(\int_{-a}^{a}(x^2 + y^2 + z^2)dx\right)dy\right)dz = \int_{-c}^{c}\left(\int_{-b}^{b}\left[\frac{x^3}{3} + y^2x + z^2x\right]_{-a}^{a} dy\right)dz$$

$$= 2\int_{-c}^{c}\left(\int_{-b}^{b}\left[\frac{a^3}{3} + a(y^2 + z^2)\right]dy\right)dz = 2\int_{-c}^{c}\left[\frac{a^3}{3}y + a\left(\frac{y^3}{3} + z^2y\right)\right]_{-b}^{b} dz$$

$$= 4\int_{-c}^{c}\left(\frac{ba^3}{3} + \frac{ab^3}{3} + z^2ba\right)dz = 4\left[\frac{ba^3}{3}z + \frac{ab^3z}{3} + \frac{z^3ba}{3}\right]_{-c}^{c}$$

$$= \frac{8}{3}\left[a^3bc + ab^3c + abc^3\right] = \frac{8abc}{3}(a^2 + b^2 + c^2).$$

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Example 7.28: Evaluate the triple integral $\iiint xyz\,dx\,dy\,dz$ over the volume enclosed by three co-ordinate planes and the plane $x + y + z = 1$.

Solution: Let D be the volume enclosed by three co-ordinate planes and the plane $x + y + z = 1$. The plane $x + y + z = 1$ meets the co-ordinate axis in $A(1, 0, 0)$, $B(0, 1, 0)$ and $C(0, 0, 1)$. The projection of the region D on the xy-plane is the region bounded by the lines $x = 0$, $y = 0$ and $x + y = 1$, as shown in Fig. 7.29.

Hence, $D = \{(x, y, z): 0 \le x \le 1, 0 \le y \le 1 - x, 0 \le z \le 1 - x - y\}$ and, therefore, the triple integral is

$$I = \iiint\limits_D xyz\,dx\,dy\,dz = \int_0^1\left(\int_0^{1-x}\left(\int_0^{1-x-y} xyz\,dz\right)dy\right)dx$$

$$= \int_0^1\left(\int_0^{1-x}\left[\frac{xyz^2}{2}\right]_0^{1-x-y} dy\right)dx = \frac{1}{2}\int_0^1\left(\int_0^{1-x} xy(1 - x - y)^2 dy\right)dx$$

$$= \frac{1}{2}\int_0^1\left(\int_0^{1-x}(xy + x^3y + xy^3 - 2x^2y - 2xy^2 + 2x^2y^2)dy\right)dx$$

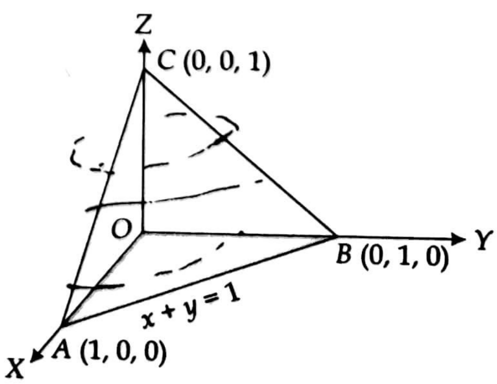

Fig. 7.29

$$= \frac{1}{2}\int_0^1 \left[\frac{xy^2}{2} + \frac{x^3y^2}{2} + \frac{xy^4}{4} - x^2y^2 - \frac{2xy^3}{3} + \frac{2x^2y^3}{3}\right]_0^{1-x} dx$$

$$= \frac{1}{2}\int_0^1 \left[\frac{x(1-x)^2}{2} + \frac{x^3(1-x)^2}{2} + \frac{x(1-x)^4}{4} - x^2(1-x)^2 - \frac{2x(1-x)^3}{3} + \frac{2x^2(1-x)^3}{3}\right] dx$$

$$= \frac{1}{24}\int_0^1 (x + 8x^2 - 30x^3 + 32x^4 - 11x^5)dx$$

$$= \frac{1}{24}\left[\frac{x^2}{2} + \frac{8x^3}{3} - \frac{30x^4}{4} + \frac{32x^5}{5} - \frac{11x^6}{6}\right]_0^1 = \frac{1}{24}\left[\frac{1}{2} + \frac{8}{3} - \frac{15}{2} + \frac{32}{5} - \frac{11}{6}\right] = \frac{7}{720}.$$

Example 7.29: Evaluate the triple integral $\iiint_D y\,dx\,dy\,dz$, where D is the region bounded by the surfaces $x = y^2$, $x = y + 2$, $4z = x^2 + y^2$ and $z = y + 3$.

Solution: The variable z varies from $(x^2 + y^2)/4$ to $y + 3$. The projection of D on the xy-plane is the region bounded by the curves $x = y^2$ and $x = y + 2$, which intersect when $y^2 = y + 2$, that is, when $y = -1, 2$. For $-1 \le y \le 2$, we have, $y^2 \le x \le y + 2$, hence, the required region is

$$D = \left\{(x, y, z) :- 1 \le y \le 2, y^2 \le x \le y + 2, \frac{x^2 + y^2}{4} \le z \le y + 3\right\}. \text{ Thus,}$$

$$I = \iiint_D y\,dx\,dy\,dz = \int_{-1}^{2}\left(\int_{y^2}^{y+2}\left(\int_{\frac{x^2+y^2}{4}}^{y+3} y\,dz\right)dx\right)dy$$

$$= \int_{-1}^{2}\left(\int_{y^2}^{y+2} y\left(y + 3 - \frac{x^2 + y^2}{4}\right)dx\right)dy = \int_{-1}^{2}\left[\left(y^2 + 3y - \frac{y^3}{4}\right)x - \frac{x^3y}{12}\right]_{y^2}^{y+2} dy$$

$$= \int_{-1}^{2}\left[\left(y^2 + 3y - \frac{y^3}{4}\right)(y + 2 - y^2) - \frac{y}{12}\{(y+2)^3 - y^6\}\right] dy$$

$$= \int_{-1}^{2}\left[\frac{16y}{3} + 4y^2 - 3y^3 - \frac{4y^4}{3} + \frac{y^5}{4} + \frac{y^7}{12}\right] dy$$

$$= \left[\frac{8y^2}{3} + \frac{4y^3}{3} - \frac{3y^4}{4} - \frac{4y^5}{15} + \frac{y^6}{24} + \frac{y^8}{96}\right]_{-1}^{2} = \frac{92}{15} - \frac{433}{480} = \frac{837}{160}.$$

7.6 TRANSFORMATION OF VARIABLES IN TRIPLE INTEGRALS

The method is analogous to double integrals except that here we work in three dimensions instead of two. We define x, y, z as functions of the three new variables u, v, w as $x = x(u, v, w)$, $y = y(u, v, w)$, $z = z(u, v, w)$ having continuous first order partial derivatives.

Suppose that the region D in the xyz-space is tranformed to the region G in the uvw-space, and the function $f(x, y, z)$ becomes $g(u, v, w)$, then

$$\iiint_D f(x, y, z)\,dx\,dy\,dz = \iiint_G g(u, v, w)\,|J|\,du\,dv\,dw, \qquad ...(7.25)$$

where
$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \qquad ...(7.26)$$

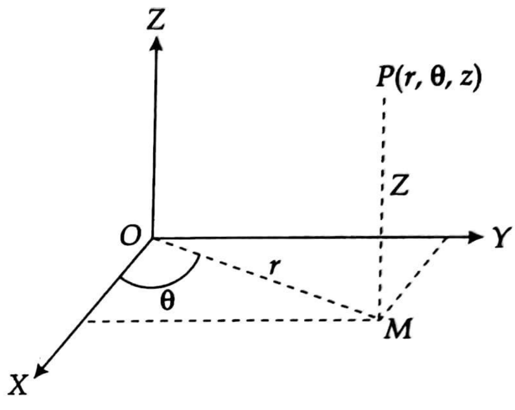

Fig. 7.30

is the Jacobian of the variable of transformation.

For example, in case of change to cylindrical co-ordinates r, θ and z from the cartesian co-ordinates x, y and z, refer to Fig. 7.30, we have

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z, \qquad ...(7.27)$$

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r. \qquad ...(7.28)$$

Thus, $$\iiint_D f(x, y, z)\,dx\,dy\,dz = \iiint_G g(r, \theta, z)\,r\,dr\,d\theta\,dz. \qquad ...(7.29)$$

In case of change to *spherical co-ordinates r, θ and φ* from the cartesian co-ordinates x, y and z, refer to Fig. 7.31, we have

$$\left.\begin{aligned} x &= r\sin\phi\cos\theta, \\ y &= r\sin\phi\sin\theta, \\ z &= r\cos\phi, \end{aligned}\right\} \quad 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi. \qquad ...(7.30)$$

$$J = \frac{\partial(x, y, z)}{\partial(r, \phi, \theta)} = \begin{vmatrix} \sin\phi\cos\theta & r\cos\phi\cos\theta & -r\sin\phi\sin\theta \\ \sin\phi\sin\theta & r\cos\phi\sin\theta & r\sin\phi\cos\theta \\ \cos\phi & -r\sin\phi & 0 \end{vmatrix}$$

$$= r^2\sin\phi, \qquad ...(7.31)$$

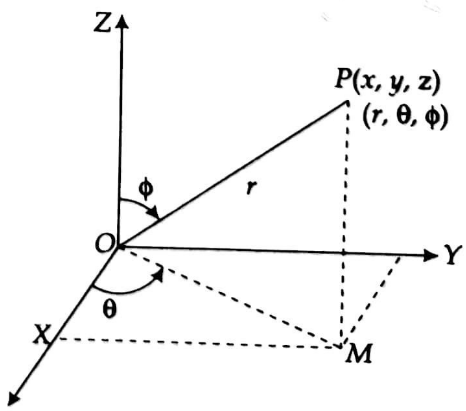

Fig. 7.31

refer to Example 5.23 (ii).

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Example 7.30: Evaluate $\int_0^3 \int_0^4 \int_{x=y/2}^{y/2+1} \left(\frac{2x-y}{2}+\frac{z}{3}\right) dx\, dy\, dz$ by applying the transformation $x = u + v$, $y = 2v$, $z = 3w$.

Solution: The region of integration D in the xyz-space is given by

$$D = \{(x, y, z) : y/2 \le x \le y/2 + 1,\ 0 \le y \le 4,\ 0 \le z \le 3\}.$$

Under the transformation $x = u + v$, $y = 2v$, $z = 3w$ the region D transforms to the region G in the uvw-space given by $G = \{(u, v, w): 0 \le u \le 1, 0 \le v \le 2, 0 \le w \le 1\}$.

Also,
$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6,$$

and
$$\frac{2x-y}{2} + \frac{z}{3} = \frac{2(u+v)-2v}{2} + \frac{3w}{3} = u + w.$$

Thus the given integral becomes

$$I = \int_0^1 \left(\int_0^2 \left(\int_0^1 (u+w)\,|J|\, du \right) dv \right) dw = 6\int_0^1 \left(\int_0^2 \left[\frac{u^2}{2} + uw \right]_0^1 dv \right) dw$$

$$= 6\int_0^1 \left(\int_0^2 \left(w + \frac{1}{2} \right) dv \right) dw = 6\int_0^1 \left[wv + \frac{1}{2}v \right]_0^2 dw = 6\int_0^1 (2w+1)\,dw = 6\left[w^2 + w \right]_0^1 = 12.$$

Example 7.31: Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz\, dy\, dx$ using

(i) cartesian co-ordinates,
(ii) cylindrical co-ordinates,
(iii) spherical polar co-ordinates.

Solution: (i) The region of integration is clearly the volume of the sphere $x^2 + y^2 + z^2 = a^2$ in the positive octant. We have

$$I = \int_0^a \left(\int_0^{\sqrt{a^2-x^2}} \left(\int_0^{\sqrt{a^2-x^2-y^2}} dz \right) dy \right) dx = \int_0^a \left(\int_0^{\sqrt{a^2-x^2}} \left(\sqrt{a^2-x^2-y^2} \right) dy \right) dx$$

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$$= \int_0^a \left(\int_0^t \left(\sqrt{t^2 - y^2} \right) dy \right) dx, \text{ where } \quad t = \sqrt{a^2 - x^2}$$

$$= \int_0^a \left[\frac{y\sqrt{t^2 - y^2}}{2} + \frac{t^2}{2} \sin^{-1} \frac{y}{t} \right]_0^t dx = \int_0^a \frac{t^2}{2} \sin^{-1}(1) dx$$

$$= \frac{\pi}{4} \int_0^a (a^2 - x^2) dx = \frac{\pi}{4} \left[a^2 x - \frac{x^3}{3} \right]_0^a = \frac{\pi a^3}{6}.$$

(ii) Changing to cylindrical co-ordinates r, θ and z by substituting $x = r\cos\theta, y = r\sin\theta, z = z$, the equation of the sphere $x^2 + y^2 + z^2 = a^2$ becomes $r^2 + z^2 = a^2$. The region of integration, the volume of the sphere $x^2 + y^2 + z^2 = a^2$ in the positive octant transforms to

$$\{(r, \theta, z) : 0 \le r \le a,\ 0 \le \theta \le \pi/2,\ 0 \le z \le \sqrt{a^2 - r^2}\}.$$

The volume element $dx\,dy\,dz$ becomes $|J|\,dr\,d\theta\,dz = r\,dr\,d\theta\,dz$. Thus,

$$I = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx = \int_0^a \int_0^{\pi/2} \int_0^{\sqrt{a^2-r^2}} r\,dr\,d\theta\,dz = \int_0^a \int_0^{\pi/2} \sqrt{a^2 - r^2}\, r\,dr\,d\theta$$

$$= -\frac{\pi}{4} \int_0^a \sqrt{a^2 - r^2}\,(-2r)dr = -\frac{\pi}{6} \left[(a^2 - r^2)^{3/2} \right]_0^a = -\frac{\pi}{6}[-a^3] = \frac{\pi a^3}{6}.$$

(iii) Changing to spherical polar co-ordinates r, θ and ϕ by substituting

$$x = r\sin\phi\cos\theta, \quad y = r\sin\phi\sin\theta, \quad z = r\cos\phi.$$

the equation of the sphere $x^2 + y^2 + z^2 = a^2$ becomes $r = a$ and region of integration transforms to

$$\{(r, \theta, \phi) : 0 \le r \le a,\ 0 \le \theta \le \frac{\pi}{2},\ 0 \le \phi \le \frac{\pi}{2}\}.$$ In +ve quad

The volume element $dxdydz$ becomes $|J|\,dr\,d\theta\,d\phi = r^2 \sin\phi\,dr\,d\theta\,d\phi$. Thus,

$$I = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dx\,dy\,dz = \int_0^a \int_0^{\pi/2} \int_0^{\pi/2} r^2 \sin\phi\,dr\,d\theta\,d\phi$$

$$= \left(\int_0^a r^2\,dr \right) \left(\int_0^{\pi/2} d\theta \right) \left(\int_0^{\pi/2} \sin\phi\,d\phi \right) = \left[\frac{r^3}{3} \right]_0^a [\theta]_0^{\pi/2} [-\cos\phi]_0^{\pi/2} = \frac{\pi a^3}{6}.$$

~~XXXXX~~

Example 7.32: Evaluate $\displaystyle\int_0^1 \int_0^{\sqrt{(1-x^2)}} \int_{\sqrt{x^2+y^2}}^1 \frac{dz\,dy\,dx}{\sqrt{x^2 + y^2 + z^2}}.$

V. Imp

Solution: Changing to spherical polar co-ordinates (r, θ, ϕ), we have $x = r\sin\phi\cos\theta$, $y = r\sin\phi\sin\theta$, $z = r\cos\phi$,

we have $\sqrt{x^2+y^2+z^2} = r$, and $dx\,dy\,dz = r^2\sin\phi\,dr\,d\theta\,d\phi$.

The region of integration is common to the cone $z^2 = x^2 + y^2$ and the cylinder $x^2 + y^2 = 1$ bounded by the plane $z = 1$ in the positive octant. Thus θ varies from 0 to $\pi/2$, ϕ from 0 to $\pi/4$ and r from 0 to $\sec\phi$, as shown in Fig. 7.32.

Hence the given integral becomes

$$I = \int_0^{\pi/4}\int_0^{\pi/2}\int_0^{\sec\phi} r\sin\phi\,dr\,d\theta\,d\phi = \left(\int_0^{\pi/2} d\theta\right)\int_0^{\pi/4}\left(\frac{r^2}{2}\right)_0^{\sec\phi}\sin\phi\,d\phi$$

$$= \frac{\pi}{4}\int_0^{\pi/4}\sec\phi\tan\phi\,d\phi = \frac{\pi}{4}\left[\sec\phi\right]_0^{\pi/4} = \frac{(\sqrt{2}-1)\pi}{4}.$$

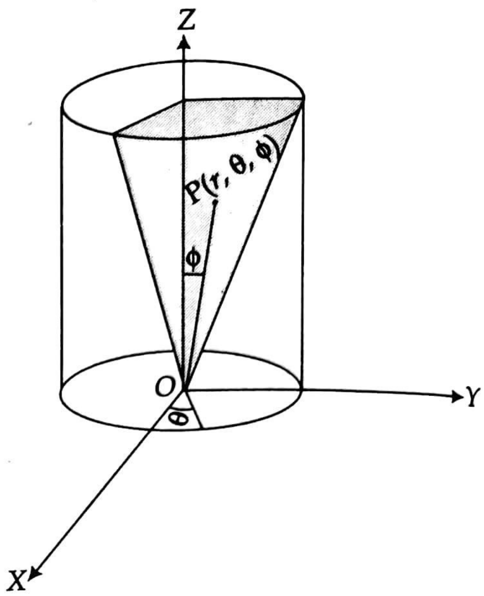

Fig. 7.32

EXERCISE 7.5

1. Evaluate the following triple integrals

(a) $\int_0^{\ln 2}\int_0^{x}\int_0^{x+\ln y} e^{x+y+z}\,dz\,dy\,dx$ (b) $\int_0^{1}\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz\,dz\,dy\,dx.$

2. Evaluate $\iiint_V \sqrt{x^2+y^2}\,dx\,dy\,dz$, where V is the volume bounded by surfaces $z = 0$, $z = 1$, $x^2 + y^2 = 1$.

3. Evaluate $\iiint z(x^2+y^2+z^2)\,dx\,dy\,dz$ over the volume of the cylinder $x^2+y^2 = a^2$, $z = 0$, $z = h$ by changing to cylindrical co-ordinates.

4. Evaluate $\int_0^{a}\int_0^{\sqrt{a^2-z^2}}\int_0^{\sqrt{a^2-y^2-z^2}} (x^2+y^2+z^2)\,dx\,dy\,dz.$

5. Evaluate $\int_0^{a}\int_0^{x}\int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx$ stating precisely the region of integration.

6. Evaluate $\int_0^{1}\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}}$ by changing to spherical polar co-ordinates.

7. Evaluate $\iiint x^2 dx\,dy\,dz$ over the volume bounded by the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

8. Evaluate $\iiint (x + y + z)$ over the tetrahedron bounded by the planes $x = 0, y = 0, z = 0$ and $x + y + z = 1$.

9. Evaluate $\iiint x^2 y^2 z^2 dx\,dy\,dz$ over the volume bounded by $xy = 4$, $xy = 9$, $yz = 1$, $yz = 4$, $zx = 25$, $zx = 49$.

10. Evaluate $\iiint \frac{dx\,dy\,dz}{(x^2 + y^2 + z^2)^{3/2}}$ over the region bounded by $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + z^2 = b^2, a > b$.

7.7 APPLICATIONS OF TRIPLE INTEGRALS

Triple integrals are used to calculate the volume, mass, moment, centroid, moment of inertia, etc. in case of objects in three dimensions.

7.7.1 Volume, Mass, the Centre of Mass of the Bounded Regions in Space

In case we take $f(x, y, z) = 1$ in (7.24), then I gives the volume V of the bounded region D in the cartesian co-ordinates. Thus,

$$\boxed{V = \iiint_D dx\,dy\,dz} \qquad \text{...(7.32)}$$

in cylindrical co-ordinates (7.32) becomes

$$\boxed{V = \iiint_D r\,dr\,d\theta\,dz} \qquad \text{...(7.33)}$$

and, in spherical co-ordinates we have

$$\boxed{V = \iiint_D r^2 \sin\phi\,dr\,d\theta\,d\phi.} \qquad \text{...(7.34)}$$

If $f(x, y, z)$ is an integrable function defined over a region D of measurable volume V, then the expression,

$$\frac{1}{V}\iiint_D f(x, y, z)dx\,dy\,dz \qquad \text{...(7.35)}$$

is defined as *the average value of* $f(x, y, z)$ over D.

For example, if $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$, then (7.35) gives the average distance of D from the orgin (0, 0, 0).

If $\rho(x, y, z)$ is the density function, then

$$M = \iiint_D \rho(x, y, z)dx\,dy\,dz \qquad ...(7.36)$$

gives the mass M of the solid bounded by the region D in space.

The expressions

$$M_{yz} = \iiint_D \rho f(x, y, z)dx\,dy\,dz, \quad M_{zx} = \iiint_D y\rho(x, y, z)dx\,dy\,dz, \quad M_{xy} = \iiint_D z\rho(x, y, z)dx\,dy\,dz \qquad ...(7.37)$$

are called the *first moments about the co-ordinate planes;* and

$$\bar{x} = M_{yz}/M, \quad \bar{y} = M_{zx}/M, \quad \bar{z} = M_{xy}/M \qquad ...(7.38)$$

give the co-ordinates of the *centre of mass* or *the centroid* of the mass M in D.

7.7.2 Moments of Inertia of the Solid Covering Region D in Space

If $\rho(x, y, z)$ is the density function, then

$$I_x = \iiint_D (y^2 + z^2)\rho dx\,dy\,dz, \; I_y = \iiint_D (x^2 + z^2)\rho dx\,dy\,dz, \quad I_z = \iiint_D (x^2 + y^2)\rho dx\,dy\,dz. \qquad ...(7.39)$$

are the *moments of inertia,* or *the second moments of the mass M in D about x-axis, y-axis and z-axis* respectively.

In general, if $r = (x, y, z)$ is the distance of an arbitrary point (x, y, z) in D from a line L, then

$$I_L = \iiint_D r^2\rho(x, y, z)dx\,dy\,dz \qquad ...(7.40)$$

is the *moment of the mass M in D about the line L.*

Example 7.33: Find the volume bounded above by the surface $z = 1 - (x^2 + y^2)$, on the sides by the planes $x = 0, y = 0, x + y = 1$ and below by the plane $z = 0$.

Solution: The region of integration is $D = \{(x, y, z): 0 \le z \le 1 - (x^2 + y^2),\; 0 \le y \le 1 - x,\; 0 \le x \le 1\}$.

Hence, the volume bounded by the region D is

$$V = \int_0^1\int_0^{1-x}\int_0^{1-x^2-y^2} dz\,dy\,dx = \int_0^1\int_0^{1-x} (1 - x^2 - y^2)dy\,dx = \int_0^1\left[y - x^2y - \frac{y^3}{3}\right]_0^{1-x} dx$$

$$= \int_0^1\left[(1-x) - (1-x)\,x^2 - \frac{(1-x)^3}{3}\right]dx = \left[x - \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} + \frac{(1-x)^4}{12}\right]_0^1 = \frac{1}{3}.$$

* V. Imp.

Example 7.34: Find the volume of the portion of the sphere $x^2 + y^2 + z^2 = a^2$ lying inside the cylinder $x^2 + y^2 = ay$.

Solution: The volume inside the cylinder bounded by the sphere is twice the volume of the shaded region as shown in Fig. 7.33. Its projection on the xy-plane is the circle $x^2 + y^2 = ay$.

Changing to cylindrical co-ordinates (r, θ, z), we have

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

The equation of the sphere $x^2 + y^2 + z^2 = a^2$ becomes $r^2 + z^2 = a^2$; and that of the circle $x^2 + y^2 = ay$ in the xy-plane becomes $r = a\sin\theta$. The volume element $dx\,dy\,dz = r\,dr\,d\theta\,dz$ and the region of integration is $D = \{(r, \theta, z) : -\sqrt{a^2 - r^2} \le z \le \sqrt{a^2 - r^2}, 0 \le r \le a\sin\theta, 0 \le \theta \le \pi\}$.

Thus volume

$$V = \int_0^{\pi}\int_0^{a\sin\theta}\int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta = 2\int_0^{\pi}\int_0^{a\sin\theta} r\sqrt{a^2-r^2}\,dr\,d\theta$$

$$= -\int_0^{\pi}\int_0^{a\sin\theta}\sqrt{a^2-r^2}\,(-2r)\,dr\,d\theta = -\frac{2}{3}\int_0^{\pi}\left[(a^2-r^2)^{3/2}\right]_0^{a\sin\theta} d\theta$$

$$= \frac{2a^3}{3}\int_0^{\pi}(1-\cos^3\theta)\,d\theta = \frac{2a^3}{3}\int_0^{\pi}\left[1 - \frac{3}{4}\cos\theta - \frac{1}{4}\cos 3\theta\right]d\theta$$

$$= \frac{2a^3}{3}\left[\theta - \frac{3}{4}\sin\theta - \frac{1}{12}\sin 3\theta\right]_0^{\pi} = \frac{2\pi a^3}{3} \text{ cubic unit.}$$

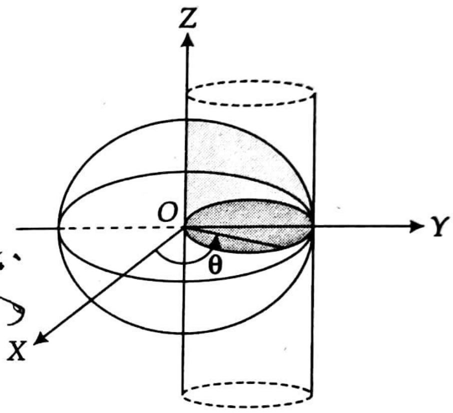

Fig. 7.33

Example 7.35: Find the volume of the upper region D cut from the solid sphere $x^2 + y^2 + z^2 \le 1$ by the cone, $\phi = \pi/3$.

Solution: Using the spherical polar coordinates $x = r\sin\phi\cos\theta$, $y = r\sin\phi\sin\theta$, $z = r\cos\phi$, the volume

$$V = \iiint_D r^2\sin\phi\,dr\,d\theta\,d\phi,$$

where D is the region shown in Fig. 7.34 and is given by

$$D = \{(r, \theta, \phi) : 0 \le r \le 1,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi/3\}.$$

Thus,

$$V = \int_0^{\pi/3}\int_0^{2\pi}\int_0^{1} r^2\sin\phi\,dr\,d\theta\,d\phi = \left(\int_0^1 r^2 dr\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi/3}\sin\phi\,d\phi\right)$$

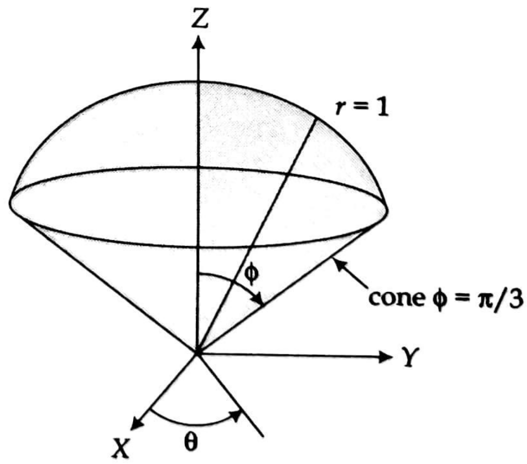

Fig. 7.34

$$= \left(\frac{r^3}{3}\right)_0^1 [\theta]_0^{2\pi} [-\cos\phi]_0^{\pi/3} = \frac{1}{3} \times 2\pi \times \frac{1}{2} = \frac{\pi}{3} \text{ cubic unit.}$$

Example 7.36: A solid fills the region between two concentric spheres of radii a and b, $0 < a < b$. The density at each point is inversely proportional to its square of distance from the origin. Find the total mass.

Solution: The region of integration D is shown as the shaded area in Fig. 7.35. The density $\rho(x, y, z)$ at a point (x, y, z) is given by

$$\rho(x, y, z) = \frac{k}{x^2 + y^2 + z^2}, \text{ where } k \text{ is a constant. Thus,}$$

$$M = \iiint_D \frac{k}{x^2 + y^2 + z^2} \, dx\, dy\, dz,$$

where D is given by $a^2 \le x^2 + y^2 + z^2 \le b^2$.

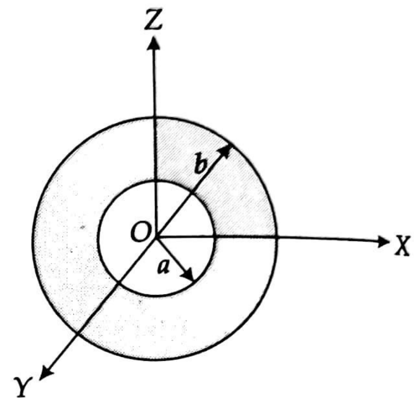

Fig. 7.35

Changing to spherical polar co-ordinates

$x = r\sin\phi\cos\theta$, $y = r\sin\phi\sin\theta$, $z = r\cos\phi$, $a < r < b$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$.

The volume element $dxdydz = r^2 \sin\phi \, dr d\theta d\phi$, and the region of integration is $\{(r, \theta, \phi) : a \le r \le b, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\}$. Thus,

$$M = k \int_0^{2\pi} \int_0^{\pi} \int_a^b \frac{r^2 \sin\phi}{r^2} \, dr d\phi d\theta = k(b-a) \int_0^{2\pi} \int_0^{\pi} \sin\phi \, d\phi \, d\theta = 2k\pi(b-a)\left[-\cos\phi\right]_0^{\pi} = 4k\pi(b-a) \text{ units.}$$

Example 7.37: If the density at any point of the positive octant of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ varies as xyz, find the co-ordinates of the centroid of the solid.

Solution: The density $\rho(x, y, z)$ at a point (x, y, z) is given by $\rho(x, y, z) = kxyz$, where k is a constant. Thus if M is the mass of the solid octant, then

$$M = \iiint_D kxyz \, dx\, dy\, dz,$$

where integration is to be taken over the region D, the positive octant of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

Applying the transformation $x = aX$, $y = bY$, $z = cZ$, the given ellipsoid transforms to the sphere $X^2 + Y^2 + Z^2 = 1$ and the volume element $dx\, dy\, dz = abc\, dx\, dy\, dz$, therefore, M becomes

$$M = ka^2b^2c^2 \iiint_{D'} XYZ \, dX\, dY\, dZ,$$

where region D' is the positive octant of the sphere $X^2 + Y^2 + Z^2 = 1$.

On the same lines M_{yz}, the moment about the yz-plane is given by

$$M_{yz} = ka^3b^2c^2 \iiint_{D'} X^2YZ\,dX\,dY\,dZ,$$

and if, $G(\bar{x}, \bar{y}, \bar{z})$ is the centroid of the solid octant, then

$$\bar{x} = \frac{M_{yz}}{M} = \frac{a\iiint_D X^2YZ\,dX\,dY\,dZ}{\iiint_D XYZ\,dX\,dY\,dZ}.$$

Changing to spherical polar co-ordinates $X = r\sin\phi\cos\theta$, $Y = r\sin\phi\sin\theta$, $Z = r\cos\phi$, we have volume element $dX\,dY\,dZ = r^2\sin\phi\,dr\,d\theta\,d\phi$, and thus

$$\bar{x} = \frac{a\int_0^1\int_0^{\pi/2}\int_0^{\pi/2} r^6 \sin\theta\cos^2\theta\sin^4\phi\cos\phi\,dr\,d\theta\,d\phi}{\int_0^1\int_0^{\pi/2}\int_0^{\pi/2} r^5 \sin\theta\cos\theta\sin^3\phi\cos\phi\,dr\,d\theta\,d\phi}$$

$$= \frac{a\left(\int_0^1 r^6\,dr\right)\left(\int_0^{\pi/2}\sin\theta\cos^2\theta\,d\theta\right)\left(\int_0^{\pi/2}\sin^4\phi\cos\phi\,d\phi\right)}{\left(\int_0^1 r^5\,dr\right)\left(\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta\right)\left(\int_0^{\pi/2}\sin^3\phi\cos\phi\,d\phi\right)} = \frac{6a}{7}\,\frac{\left(\frac{1}{3.1}\right)\left(\frac{3.1}{5.3.1}\right)}{\left(\frac{1}{2}\right)\frac{2}{4.2}} = \frac{16a}{35}.$$

Similarly, $\bar{y} = \dfrac{16b}{35}$, $\bar{z} = \dfrac{16c}{35}$, and thus the centroid is $\left(\dfrac{16a}{35}, \dfrac{16b}{35}, \dfrac{16c}{35}\right)$.

Example 7.38: Find the moment of inertia of the mass of the tetrahedron bounded by the co-ordinate planes and the plane $x + y + z = 1$ about the z-axis, assuming the density to be uniform throughout.

Solution: Let ρ be the uniform density then the moment of inertia of the mass of the tetrahedron about the z-axis is $I_z = \rho\iiint_D (x^2 + y^2)\,dx\,dy\,dz$, where integration is over the region D given as

$$D = \{(x, y, z): 0 < x < 1,\ 0 < y < 1 - x,\ 0 < z < 1 - x - y\}.$$

Thus, $$I_z = \rho\int_0^1\int_0^{1-x}\int_0^{1-x-y} (x^2 + y^2)\,dz\,dy\,dx = \rho\int_0^1\int_0^{1-x} (x^2 + y^2)(1 - x - y)\,dy\,dx$$

$$= \rho\left[\int_0^1 \int_0^{1-x} (1-x-y)x^2 dy\, dx + \int_0^1 \int_0^{1-x} (1-x-y)y^2 dy\, dx\right]$$

$$= \rho\left[\int_0^1 \left\{(1-x)y - \frac{y^2}{2}\right\}_0^{1-x} x^2 dx + \int_0^1 \left\{\frac{(1-x)y^3}{3} - \frac{y^4}{4}\right\}_0^{1-x} dx\right]$$

$$= \rho\left[\frac{1}{2}\int_0^1 x^2(1-x)^2 dx + \frac{1}{12}\int_0^1 (1-x)^4 dx\right]$$

$$= \rho\left[\frac{1}{2}\left[\frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5}\right]_0^1 + \frac{1}{12}\left[-\frac{(1-x)^5}{5}\right]_0^1\right] = \rho\left(\frac{1}{60} + \frac{1}{60}\right) = \frac{\rho}{30}.$$

Example 7.39: Find the centre of mass of a solid of constant density ρ bounded below by the disk $x^2 + y^2 \le 4$ in the plane $z = 0$ and above by the paraboloid $z = 4 - x^2 - y^2$.

Solution: If $G(\bar{x}, \bar{y}, \bar{z})$ is the centre of mass of the solid region shown in Fig. 7.36, then by symmetry

$$\bar{x} = \bar{y} = 0, \text{ and } \bar{z} = \frac{M_{xy}}{M},$$

where $M_{xy} = \iiint_D \rho z\, dx\, dy\, dz$ is the moment about the xy-plane and $M = \iiint_D \rho\, dx\, dy\, dz$ is the mass of the solid region. The region of integration D is

$$\{(x, y, z): 0 \le z \le 4 - x^2 - y^2,\ -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}\ \ -2 \le x \le 2\}.$$

Thus, $M_{xy} = \rho \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{4-x^2-y^2} z\, dz\, dy\, dx = \frac{\rho}{2}\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - x^2 - y^2)^2 dy\, dx.$

Changing to polar co-ordinates $x = r\cos\theta$, $y = r\sin\theta$, so that, $dx\, dy = r\, dr\, d\theta$, we have

$$M_{xy} = \frac{\rho}{2}\int_0^{2\pi}\left(\int_0^2 (4-r^2)^2 r\, dr\right) d\theta = -\frac{\rho}{4}\left\{\int_0^2 (4-r^2)^2(-2r)dr\right\}\left\{\int_0^{2\pi} d\theta\right\}$$

$$= -\frac{\pi\rho}{6}\left[(4-r^2)^3\right]_0^2 = \frac{32\pi\rho}{3}.$$

Also, $$M = \rho \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{4-x^2-y^2} dz\,dy\,dx$$

$$= \rho \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - x^2 - y^2)dy\,dx = \rho \int_{0}^{2\pi}\int_{0}^{2} (4 - r^2)\,r\,dr\,d\theta$$

$$= -\frac{\rho}{2}\int_{0}^{2\pi}\int_{0}^{2} (4 - r^2)(-2r)dr\,d\theta = -\frac{\pi\rho}{2}\left[(4 - r^2)^2\right]_0^2 = 8\pi\rho.$$

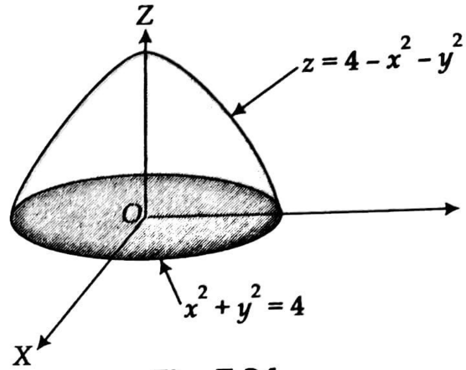

Fig. 7.36

Therefore, $\bar{z} = (M_{xy}/M) = \frac{4}{3}$; thus the centre of mass is (0, 0, 4/3).

Example 7.40: If $f(x, y, z) = xyz$ is the density of a solid cube bounded by the co-ordinate planes $x = 2$, $y = 2$, and $z = 2$ in the first octant, then find the average density of the solid.

Solution: The average density $\bar{\rho}$ of the solid cube is given by

$$\bar{\rho} = \frac{1}{V}\int_0^2\int_0^2\int_0^2 xyz\,dx\,dy\,dz,$$

where V is the volume of the cube; which is $2 \times 2 \times 2 = 8$ cubic units. Thus,

$$\bar{\rho} = \frac{1}{8}\int_0^2\int_0^2\int_0^2 xyz\,dx\,dy\,dz = \frac{1}{8}\left(\int_0^2 x\,dx\right)\left(\int_0^2 y\,dy\right)\left(\int_0^2 z\,dz\right) = \frac{1}{8}\left(\frac{x^2}{2}\right)_0^2\left(\frac{y^2}{2}\right)_0^2\left(\frac{z^2}{2}\right)_0^2 = 1 \text{ mass/volume.}$$

EXERCISE 7.6

1. Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.
2. Find the volume of the region enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.
3. Find the volume bounded above by the sphere $x^2 + y^2 + z^2 = 2a^2$ and below by the paraboloid $az = x^2 + y^2$.
4. Find the volume cut off from the cylinder $x^2 + y^2 = ax$ by the planes $z = mx$ and $z = nx$, $n > m$.
5. Show that the volume of the solid surrounded by the surface $(x/a)^{2/3} + (y/b)^{2/3} + (z/c)^{2/3} = 1$ is $4\pi abc/35$.
6. Find the volume of the region bounded by the paraboloid $az = x^2 + y^2$ and the cylinder $x^2 + y^2 = b^2$.
7. Find the region between the planes $x + y + 2z = 2$ and $2x + 2y + z = 4$ in the first octant.
8. Find the average distance from a point $P(x, y, z)$ in the cube in the first octant bounded by the co-ordinate planes and the planes $x = 1$, $y = 1$ and $z = 1$.

9. Find the x co-ordinate of the centre of gravity of the solid lying inside the cylinder $x^2 + y^2 = 2ax$ between the plane $z = 0$ and the paraboloid $x^2 + y^2 = az$.
10. Find the centre of mass of the solid hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geq 0$, if the density at any point is proportional to the distance from the origin.
11. Find the mass of the solid bounded by the planes $x + z = 1$, $x - z = -1$, $y = 0$ and the surface $y = \sqrt{z}$. The density of the solid is $\rho(x, y, z) = 2y + 5$.
12. Find the moment of inertia of a solid right circular cylinder about its axis and about a diameter of the base.
13. Find the centre of gravity of the volume common to the cylinder $x^2 + y^2 = ax$ and the sphere $x^2 + y^2 + z^2 = a^2$ above the plane $z = 0$.
14. Obtain the moment of inertia of the sphere of radius about a diameter in terms of mass M of the sphere.
15. A hemisphere of radius r has a cylindrical hole of radius a drilled through it, the axis of the hole being along the radius normal to the plane face of the hemisphere. Find its radius of gyration about a diameter of this face.

7.8 IMPROPER INTEGRALS AND THEIR CONVERGENCE

In the definite integral $\int_a^b f(x)dx$, in general, we assume two conditions. First, the interval of integration from a to b is finite and second, the integrand $f(x)$ is bounded for all x in $[a, b]$. In practice, we frequently come across problems that fail to meet one or both of these conditions. For example, we might be interested to find the area under the curve $y = (\ln x)/x^2$ from $x = 1$ to $x = \infty$, which is an example of infinite domain, such integrals are called *improper integrals*.

7.8.1 Kinds of Improper Integrals

If in the definite integral $\int_a^b f(x)dx$, a or b, or both a and b are infinite, then the integral is called *improper integral of the first kind*, or *improper integral with infinite limits*. But if $f(x)$ becomes infinite at $x = a$ or $x = b$ or at one or more points within the interval (a, b), then the integral is called *improper integral of the second kind*, or *improper integral with unbounded integrand*. For example, the integral $\int_1^\infty \frac{dx}{x^p}$, is an improper integral of the first kind, while the integral $\int_0^3 \frac{dx}{(x-1)^{2/3}}$ is an improper integral of the second kind, since the integrand $f(x) = 1/(x - 1)^{2/3}$ is unbounded at $x = 1$.

Initially, we shall assume that the integrand $f(x)$ is of the same sign within the range of integration, generally it is assumed that $f(x) \geq 0$; also we will consider $f(x)$ to be continuous over each finite subinterval contained in the range of integration.

7.8.2 Convergence of Improper Integrals of the First Kind

When the limits involved exist, we evaluate such integrals by the following procedure.

(a) If f is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx. \qquad ...(7.41)$$

(b) If f is continuous on $(-\infty, b)$, then

$$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx. \qquad ...(7.42)$$

(c) If f is continuous on $[a, b]$ and c is any finite constant including zero in (a, b), then

$$\int_{-\infty}^\infty f(x)dx = \lim_{a\to-\infty} \int_a^c f(x)dx + \lim_{b\to\infty} \int_c^b f(x)dx. \qquad ...(7.43)$$

In each case if the limits on the right exist and are finite, we say that the corresponding improper integral *converges* and the limit is the *value* of the improper integral. In case the limit fails to exist we say that the improper integral *diverges*.

Example 7.41: Evaluate the following improper integrals, if they exist.

(a) $\displaystyle\int_{-\infty}^0 x \sin x\, dx$ (b) $\displaystyle\int_0^\infty \frac{dx}{a^2+x^2}$, $(a > 0)$

(c) $\displaystyle\int_{-\infty}^\infty \frac{1}{1+x^2} dx$ (d) $\displaystyle\int_a^\infty \frac{dx}{x^p}$, $(a > 0)$, $p \neq 1$.

Solution: (a) $$\int_{-\infty}^0 x \sin x dx = \lim_{a\to-\infty} \int_a^0 x \sin x\, dx = \lim_{a\to-\infty} [-x \cos x + \sin x]_a^0$$

$$= \lim_{a\to-\infty} (a \cos a - \sin a).$$

Since $\cos a$ and $\sin a$ oscillate between ± 1, thus improper integral diverges to $-\infty$.

(b) $$\int_0^\infty \frac{dx}{a^2+x^2} = \lim_{b\to\infty} \int_0^b \frac{dx}{a^2+x^2} = \lim_{b\to\infty} \left[\frac{1}{a} \tan^{-1} \frac{x}{a}\right]_0^b = \frac{1}{a} \lim_{b\to\infty} \tan^{-1} \frac{b}{a} = \frac{\pi}{2a}.$$

Thus, the improper integral converges to $\dfrac{\pi}{2a}$.

(c) $$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2} = \lim_{a\to-\infty} \int_{a}^{0} \frac{dx}{1+x^2} + \lim_{b\to\infty} \int_{0}^{b} \frac{dx}{1+x^2}$$

$$= \lim_{a\to-\infty} [\tan^{-1} x]_a^0 + \lim_{b\to\infty} [\tan^{-1} x]_0^b = -\lim_{a\to-\infty} \tan^{-1} a + \lim_{b\to\infty} \tan^{-1} b = -(-\pi/2) + \pi/2 = \pi.$$

Thus, the improper integral converges to π.

(d) $$\int_{a}^{\infty} \frac{dx}{x^p} = \lim_{b\to\infty} \int_{a}^{b} \frac{dx}{x^p} = \lim_{b\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_a^b = -\lim_{b\to\infty} \left[\frac{b^{-(p-1)}}{p-1}\right] + \frac{a^{1-p}}{p-1}.$$

But, $-\lim_{b\to\infty} \left[\frac{b^{-(p-1)}}{p-1}\right] \to 0$, if $p > 1$ and diverges to $+\infty$, if $p < 1$. Hence,

$$\int_{a}^{\infty} \frac{dx}{x^p} = \frac{a^{1-p}}{p-1}, \text{ if } \quad p > 1.$$

For $p = 1$, $\int_{a}^{\infty} \frac{dx}{x} = \lim_{b\to\infty} \int_{a}^{b} \frac{dx}{x} = \lim_{b\to\infty} [\ln x]_a^b = \lim_{b\to\infty} \ln b - \ln a$, which diverges to ∞ as $b \to \infty$.

Thus, the improper integral converges to $\frac{a^{1-p}}{p-1}$, if $p > 1$.

Example 7.42: Evaluate the following improper integrals with infinite limits.

(a) $\int_{e^2}^{\infty} \frac{dx}{x(\ln x)^3}$ (b) $\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+5}$ (c) $\int_{0}^{\infty} e^{-x} \sin x\, dx$.

Solution: (a) We have,

$$\int_{e^2}^{\infty} \frac{dx}{x(\ln x)^3} = \lim_{b\to\infty} \int_{e^2}^{b} \frac{dx}{x(\ln x)^3} = \lim_{b\to\infty} \left[\frac{1}{-2(\ln x)^2}\right]_{e^2}^{b} = -\left(\frac{1}{2}\right) \lim_{b\to\infty} \left[\frac{1}{(\ln b)^2} - \frac{1}{4}\right] = \frac{1}{8}.$$

(b) We have,

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+5} = \lim_{a\to-\infty} \int_{a}^{c} \frac{dx}{x^2+2x+5} + \lim_{b\to\infty} \int_{c}^{b} \frac{dx}{x^2+2x+5}, \quad a < c < b$$

$$= \lim_{a\to-\infty} \left[\frac{1}{2} \tan^{-1} \frac{x+1}{2}\right]_a^c + \lim_{b\to\infty} \left[\frac{1}{2} \tan^{-1} \frac{x+1}{2}\right]_c^b$$

$$= \frac{1}{2}\tan\frac{c+1}{2} + \frac{\pi}{4} + \frac{\pi}{4} - \frac{1}{2}\tan^{-1}\frac{c+1}{2} = \pi/2.$$

(c) We have,

$$\int_0^\infty e^{-x}\sin x\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\sin x\,dx = \lim_{b\to\infty}\left[-\frac{e^{-x}}{2}(\sin x + \cos x)\right]_0^b$$

$$= -\frac{1}{2}\lim_{b\to\infty}[e^{-b}(\sin b + \cos b) - 1] = \frac{1}{2}.$$

Example 7.43: Evaluate the integral $I = \int_0^\infty \frac{x^2}{1+x^4}dx$.

Solution: Apply $x = 1/t$, then I becomes

$$I = \int_0^\infty \frac{x^2}{1+x^4}dx = \int_\infty^0 \frac{1/t^2}{1+1/t^4}(-1/t^2dt) = \int_0^\infty \frac{1}{1+t^4}dt.$$

Adding an other integral I to the both sides, we obtain

$$2I = \int_0^\infty \frac{1}{1+t^4}dt + \int_0^\infty \frac{t^2}{1+t^4}dt = \int_0^\infty \frac{1+t^2}{1+t^4}dt = \int_0^\infty \frac{1/t^2+1}{t^2+1/t^2}dt.$$

Apply $z = t - \frac{1}{t}$, we obtain

$$I = \frac{1}{2}\int_{-\infty}^\infty \frac{dz}{z^2+2} = \frac{1}{2}\left[\lim_{a\to-\infty}\int_a^0 \frac{dz}{z^2+2} + \lim_{b\to\infty}\int_0^b \frac{dz}{z^2+2}\right]$$

$$= -\frac{1}{2\sqrt{2}}\lim_{a\to-\infty}\tan^{-1}\frac{a}{\sqrt{2}} + \frac{1}{2\sqrt{2}}\lim_{b\to\infty}\tan^{-1}\frac{b}{\sqrt{2}} = \frac{\pi}{2\sqrt{2}}.$$

7.8.3 Comparison Tests

It may not always be possible to discuss the convergence or divergence of certain improper integrals directly. For example, the integral $\int_0^\infty e^{-x^2}dx$ can't be integrated directly. We introduce some comparison tests which are used to discuss the convergence or divergence of such improper integrals. Though by the applications of test we can't find the value to which the improper integral converges, yet we may be able to find a bound to the integral by applying the comparison tests. In such cases we approximate the integral numerically.

Direct comparison test: *If $f(x)$ and $g(x)$ are continuous on $[a, \infty)$ and $0 \le f(x) \le g(x)$ for all $x \ge a$, then*

1. $\int_a^\infty f(x)dx$ *converges, if* $\int_a^\infty g(x)dx$ *converges,*

2. $\int_a^\infty g(x)dx$ *diverges, if* $\int_a^\infty f(x)dx$ *diverges.*

For example, $\int_1^\infty e^{-x^2}\, dx$ converges, since $0 \le e^{-x^2} \le e^{-x}$ for all $x \ge 1$, and

$$\int_1^\infty e^{-x} dx = \lim_{b\to\infty} \int_1^b e^{-x}dx = \lim_{b\to\infty} [-e^{-x}]_1^b = -\lim_{b\to\infty} [e^{-b} - e^{-1}] = \frac{1}{e}.$$

Thus, we can say that the value of $\int_1^\infty e^{-x^2} dx$ is less than $\frac{1}{e}$.

Similarly, $\int_1^\infty \frac{1}{\sqrt{x^2 - 0.1}}$ diverges because $\frac{1}{\sqrt{x^2 - 0.1}} \ge \frac{1}{x}$ for all $x \ge 1$, and

$$\int_1^\infty \frac{1}{x} dx = \lim_{b\to\infty} \int_1^b \frac{1}{x} dx = \lim_{b\to\infty} [\ln x]_1^b = \lim_{b\to\infty} [\ln b - 0] \to \infty.$$

Example 7.44: Test the convergence of the integral $\int_0^\infty x^{100} e^{-0.01x}\, dx$.

Solution: Here $f(x) = x^{100} e^{-0.01x} = \dfrac{x^{100}}{e^{0.01x}}$

$$= \frac{x^{100}}{1 + (0.01)x + \frac{(0.01x)^2}{2!} + \ldots} < \frac{x^{100}}{\frac{(0.01x)^{102}}{102!}} = \frac{(102)!(10)^{102}}{x^2}.$$

The improper integral $\int_0^\infty \frac{(102)!(10)^{102}}{x^2} dx$ converges (since, $p = 2 > 1$), and hence by the direct comparison test the integral $\int_0^\infty x^{100} e^{-0.01x} dx$ also converges.

Limit comparison test: *If $f(x)$ and $g(x)$ are two positive functions continuous on $[a, \infty)$ and $\lim_{x\to\infty} \frac{f(x)}{g(x)} = l,\ 0 < l < \infty$, then $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ both converge or diverge simultaneously. However, it should be clear that the two integrals do not have the same value in case of convergence. Further, in case $l =$ 0, we can conclude only that convergence of $\int_a^\infty g(x)dx$ implies the convergence of $\int_a^\infty f(x)dx$.*

The integral $\int_1^\infty \frac{3}{e^x+5}dx$ converges by the limit comparison test, since $\int_1^\infty \frac{dx}{e^x}$ converges and

$$\lim_{x\to\infty} \frac{1/e^x}{3/(e^x+5)} = \lim_{x\to\infty} \frac{e^x+5}{3e^x} = \lim_{x\to\infty} \frac{1+5e^{-x}}{3} = \frac{1}{3}$$ is a positive finite limit.

Example 7.45: Check for the convergence of the integrals

(a) $\int_1^\infty \frac{\sin^2 x}{x^2}dx$ (b) $\int_1^\infty e^{-x}x^p\,dx$, p is real.

Solution: (a) We have, $0 \le \frac{\sin^2 x}{x^2} \le \frac{1}{x^2}$ on $[1, \infty)$. But $\int_1^\infty \frac{1}{x^2}dx$ is convergent, therefore, by the direct comparison test $\int_1^\infty \frac{\sin^2 x}{x^2}dx$ is also convergent.

(b) Consider $g(x) = \frac{1}{x^2}$. Then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{e^{-x}x^p}{x^{-2}} = \lim_{x\to\infty} \frac{x^{(p+2)}}{e^x} = 0.$

But $\int_1^\infty \frac{1}{x^2}dx$ is convergent, therefore, by the limit comparison test $\int_1^\infty e^{-x}x^p dx$ is also convergent.

7.8.4 Convergence Tests for Improper Integrals of the Second Kind

Here we introduce tests for the convergence of the improper integrals of the form $\int_a^b f(x)\,dx$, where a and b are finite constants but $f(x)$ has infinite discontinuity at $x = a$, or $x = b$, or $x = a$ and $x = b$ both, or $f(x)$ has infinite discontinuities at one or more finite number of points $c_1, c_2 ..., c_k$ in (a, b). When the limit(s) involved exist, we evaluate such integrals as follows.

(a) If $f(x)$ has infinite discontinuity at $x = a$, then

$$\int_a^b f(x)\,dx = \lim_{h\to 0} \int_{a+h}^b f(x)\,dx. \qquad ...(7.44)$$

(b) If $f(x)$ has infinite discontinuity at $x = b$, then

$$\int_a^b f(x)dx = \lim_{k\to 0}\int_a^{b-k} f(x)dx. \qquad ...(7.45)$$

(c) If $f(x)$ has infinite discontinuity at $x = a$ and $x = b$, both, then for $a < c < b$,

$$\int_a^b f(x)dx = \lim_{h\to 0}\int_{a+h}^{c} f(x)dx + \lim_{k\to 0}\int_c^{b-k} f(x). \qquad ...(7.46)$$

(d) If $f(x)$ has infinite discontinuity at c, $a < c < b$, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx = \lim_{k\to 0}\int_a^{c-k} f(x)dx + \lim_{h\to 0}\int_{c+h}^{b} f(x)dx \qquad ...(7.47)$$

(e) If $f(x)$ has infinite discontinuities at c_1, c_2, $a < c_1 < c_2 < b$, then

$$\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \int_{c_2}^{b} f(x)dx. \qquad ...(7.48)$$

The improper integrals on the right of (7.48) are evaluated separately on the lines as discussed above. In case the each integral converges separately, we say that the given improper integral on the left of (7.48) converges and its limit is the sum of the limits of the improper integrals on the left. If any of the integral on the left of (7.48) fails to converge, then the given improper integral does not converge.

Example 7.46: Discuss the convergence of the following integrals

(a) $\displaystyle\int_a^b \frac{dx}{(a-x)^2}$ (b) $\displaystyle\int_0^3 \frac{dx}{(x-1)^{2/3}}$ (c) $\displaystyle\int_0^2 \frac{dx}{\sqrt{4-x^2}}$ (d) $\displaystyle\int_0^3 \frac{dx}{3x-x^2}$

Solution: (a) The integrand $f(x) = \dfrac{1}{(a-x)^2}$ has infinite discontinuity at $x = a$. We have

$$\int_a^b \frac{dx}{(a-x)^2} = \lim_{h\to 0}\int_{a+h}^{b} \frac{dx}{(a-x)^2} = \lim_{h\to 0}\left[\frac{1}{(a-x)}\right]_{a+h}^{b} = \frac{1}{a-b} - \lim_{h\to 0}\frac{1}{(-h)} \to \infty.$$

Hence the improper integral diverges.

(b) The integrand $f(x) = \dfrac{1}{(x-1)^{2/3}}$ has infinite discontinuity at $x = 1$. We have

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}} = \lim_{k\to 0}\int_0^{1-k} \frac{dx}{(x-1)^{2/3}} + \lim_{h\to 0}\int_{1+h}^{3} \frac{dx}{(x-1)^{2/3}}$$

$$= 3\lim_{k\to 0}\left[(x-1)^{1/3}\right]_0^{1-k} + 3\lim_{h\to 0}\left[(x-1)^{1/3}\right]_{1+h}^{3}$$

$$= 3\left[\lim_{k\to 0}\left\{(-k)^{\frac{1}{3}} - (-1)^{\frac{1}{3}}\right\}\right] + 3\left[\lim_{h\to 0}\left\{(2)^{\frac{1}{3}} - (h)^{\frac{1}{3}}\right\}\right] = 3 + 3\,(2)^{1/3}.$$

Thus the improper integral converges to $3\left(1+\sqrt[3]{2}\right)$.

(c) The integrand $f(x) = \dfrac{1}{\sqrt{4-x^2}}$ has infinite discontinuity at $x = 2$. We have

$$\int_0^2 \frac{dx}{\sqrt{4-x^2}} = \lim_{k\to 0}\int_0^{2-k} \frac{dx}{\sqrt{4-x^2}} = \lim_{k\to 0}\left[\sin^{-1}\frac{x}{2}\right]_0^{2-k} = \lim_{k\to 0}\sin^{-1}\left(1-\frac{k}{2}\right) = \sin^{-1}1 = \pi/2.$$

Thus the improper integral converges to $\pi/2$.

(d) The integrand $f(x) = \dfrac{1}{3x-x^2}$ has infinite discontinuities both at $x = 0$ and $x = 3$. Take any point, say $x = 2$, inside $(0, 3)$ at which $f(x)$ is defined. We have

$$\int_0^3 \frac{dx}{3x-x^2} = \int_0^2 \frac{dx}{3x-x^2} + \int_2^3 \frac{dx}{3x-x^2} = \lim_{h\to 0}\int_h^2 \frac{dx}{3x-x^2} + \lim_{k\to 0}\int_2^{3-k} \frac{dx}{3x-x^2}$$

$$= \frac{1}{3}\lim_{h\to 0}\left[\ln\left(\frac{x}{3-x}\right)\right]_h^2 + \frac{1}{3}\lim_{k\to 0}\left[\ln\left(\frac{x}{3-x}\right)\right]_2^{3-k}$$

$$= \frac{1}{3}\lim_{h\to 0}\left[\ln 2 - \ln\left(\frac{h}{3-h}\right)\right] + \frac{1}{3}\lim_{k\to 0}\left[\ln\frac{3-k}{k} - \ln 2\right].$$

Since the limits on the right side do not exist therefore, the given improper integral diverges.

Remark. Direct comparison tests and limit comparison tests discussed in case of improper integrals of first kind can also be applied to ascertain the convergence or divergence of the improper integrals of the second kind also.

Example 7.47: Discuss the convergence of the following improper integrals

(a) $\displaystyle\int_1^2 \frac{\sqrt{x}}{\ln x}\,dx$ (b) $\displaystyle\int_0^{\pi/2} \frac{\cos^m x}{x^n}\,dx.$

Solution: The integrand $f(x) = \dfrac{\sqrt{x}}{\ln x}$ has $x = 1$ as its point of infinite discontinuity in the interval

[1, 2]. Also $f(x) = \dfrac{\sqrt{x}}{\ln x} > 0, \quad 1 < x \le 2$. Consider $g(x) = \dfrac{1}{x \ln x}$, then we have

$$\lim_{x\to 1+} \frac{f(x)}{g(x)} = \lim_{h\to 0} \frac{f(1+h)}{g(1+h)} = \lim_{h\to 0} (1+h)^{3/2} = 1.$$

Thus improper integrals $\int_1^2 f(x)dx$ and $\int_1^2 g(x)dx$ converge or diverge together. We have

$$\int_1^2 g(x)dx = \int_1^2 \frac{dx}{x\ln x} = \lim_{h\to 0}\int_{1+h}^2 \frac{dx}{x\ln x} = \lim_{h\to 0}[\ln(\ln x)]_{1+h}^2 = \lim_{h\to 0}[\ln(\ln 2) - \ln(\ln(1+h))] = \infty.$$

Therefore, $\int_1^2 \dfrac{dx}{x\ln x}$ is divergent and hence by the limit comparison test the improper integral $\int_1^2 \dfrac{\sqrt{x}}{\ln x}dx$ is also divergent.

(b) The integrand $f(x) = \dfrac{\cos^m x}{x^n}$ has $x = 0$ as its point of infinite discontinuity in the domain $\left[0, \dfrac{\pi}{2}\right]$.

Also, $f(x) = \dfrac{\cos^m x}{x^n} < \dfrac{1}{x^n}$ for $x \in (0, \pi/2]$. Take $g(x) = \dfrac{1}{x^n}$. Consider

$$\int_0^{\pi/2} g(x)dx = \int_0^{\pi/2} \frac{1}{x^n}dx = \lim_{h\to 0}\int_h^{\pi/2}\frac{1}{x^n}dx = \lim_{h\to 0}\left[\frac{x^{-n+1}}{-n+1}\right]_h^{\pi/2}$$

$$= \lim_{h\to 0}\frac{1}{1-n}\left[(\pi/2)^{-n+1} - \frac{1}{h^{n-1}}\right] = \frac{1}{1-n}\left(\frac{\pi}{2}\right)^{1-n}, \text{ for } n<1.$$

Therefore $\int_0^{\pi/2} g(x)dx$ is convergent for $n < 1$, and hence, by the direct comparison test, $\int_0^{\pi/2}\dfrac{\cos^m x}{x^n}dx$ is also convergent for $n < 1$.

7.8.5 Cauchy Principal Value

We have seen that if in the case of the definite integral $\int_a^b f(x)dx$, the integrand $f(x)$ has infinite discontinuity at an interior point c such that $a < c < b$, then we write

$$\int_a^b f(x)dx = \lim_{k\to 0}\int_a^{c-k} f(x)dx + \lim_{h\to 0}\int_{c+h}^{b} f(x)dx \qquad ...(7.49)$$

where k and h tend to zero independently. It may sometimes happen that the two limits on the right of (7.49) do not converge separately but if we set $h = k$, then we may find a finite answer since the unbounded parts of the two limits cancel out. The value so obtained is called the *Cauchy Principal value* of the integral, written as $pr.v.\int_a^b f(x)dx$.

Example 7.48: Evaluate the principal value of the integral $\int_1^4 \frac{dx}{(x-2)^3}$.

Solution: The integrand $f(x) = \frac{1}{(x-2)^3}$ has a point of infinite discontinuity at $x = 2$ in the domain [1, 4]. Write,

$$\int_1^4 \frac{dx}{(x-2)^3} = \int_1^2 \frac{dx}{(x-2)^3} + \int_2^4 \frac{dx}{(x-2)^3} = \lim_{k\to 0}\int_1^{2-k} \frac{dx}{(x-2)^3} + \lim_{h\to 0}\int_{2+h}^{4} \frac{dx}{(x-2)^3}$$

$$= \lim_{k\to 0}\frac{1}{2}\left(-\frac{1}{(x-2)^2}\right)_1^{2-k} + \lim_{h\to 0}\frac{1}{2}\left(-\frac{1}{(x-2)^2}\right)_{2+h}^{4} = \lim_{k\to 0}\left(\frac{1}{2}-\frac{1}{2k^2}\right) + \lim_{h\to 0}\left(-\frac{1}{8}+\frac{1}{2h^2}\right).$$

Since, both the limits on the right diverge to infinity and hence the given improper integral diverges. But in case we set $k = h$, then we get

$$\int_1^4 \frac{dx}{(x-2)^3} = \lim_{h\to 0}\left\{\frac{1}{2}-\frac{1}{2k^2}-\frac{1}{8}+\frac{1}{2h^2}\right\} = \frac{3}{8}. \text{ Thus, } \quad pr.v\int_1^4 \frac{dx}{(x-2)^3} = \frac{3}{8}.$$

7.8.6 Absolute Convergence of Improper Integrals

So far we have assumed that $f(x)$ is of the same sign throughout the range of integration. In case $f(x)$ changes sign within the interval of integration, we consider the absolute convergence of the improper integral.

The improper integral $\int_a^b f(x)dx$ *is said to be absolutely convergent,* if $\int_a^b |f(x)|\,dx$ *is convergent.* Also, *an absolutely convergent improper integral is convergent but the converse may not be true.* Further, since $|f|$ is always non-negative within the range of integration thus all the comparison tests discussed earlier to check the convergence of improper integrals may be applied to check the absolute convergence also.

For example, consider the improper integral $\int_2^{\infty} \frac{\sin x}{3x^2+1} dx$. Here the integrand $f(x) = \frac{\sin x}{3x^2+1}$ is not everywhere positive in the interval of integration $[2, \infty]$. But $\left|\frac{\sin x}{3x^2+1}\right| \leq \frac{1}{3x^2+1} < \frac{1}{3x^2}$ and both $\frac{1}{3x^2+1}$, and $\frac{1}{3x^2} > 0$ for $x \in [2, \infty]$.

Since, the integral $\int_2^{\infty} \frac{dx}{3x^2}$ is convergent, ($p = 2 > 1$); thus by direct comparison test $\int_2^{\infty} \frac{dx}{3x^2+1}$, and hence, $\int_2^{\infty} \left|\frac{\sin x}{3x^2+1}\right| dx$ is also convergent. Thus the integral $\int_2^{\infty} \frac{\sin x}{3x^2+1}$ converges absolutely.

Example 7.49: Prove that $I = \int_0^{\infty} \frac{\sin x}{x} dx$ converges conditionally.

Solution: Rewrite I as the sum of the two integrals as

$$I = \int_0^{\infty} \frac{\sin x}{x} dx = \int_0^{\pi/2} \frac{\sin x}{x} dx + \int_{\pi/2}^{\infty} \frac{\sin x}{x} dx. \qquad \text{...(7.50)}$$

The first integral on the right of (7.50) is a proper integral, since $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

Next, consider the second integral

$$\int_{\pi/2}^{\infty} \frac{\sin x}{x} dx = \lim_{b \to \infty} \int_{\pi/2}^{b} \frac{\sin x}{x} dx = \lim_{b \to \infty} \left[\left[-\frac{\cos x}{x} \right]_{\pi/2}^{b} - \int_{\pi/2}^{b} \frac{\cos x}{x^2} dx \right] = -\int_{\pi/2}^{\infty} \frac{\cos x}{x^2} dx,$$

$$\text{since } \lim_{b \to \infty} \left[-\frac{\cos x}{x} \right]_{\pi/2}^{b} = 0.$$

The improper integral $\int_{\pi/2}^{\infty} \frac{\cos x}{x^2} dx$ converges absolutely, since $\left|\frac{\cos x}{x^2}\right| \leq \frac{1}{x^2}$ and the integral $\int_{\pi/2}^{\infty} \frac{dx}{x^2}$ converges. Therefore the integral $\int_{\pi/2}^{\infty} \frac{\sin x}{x} dx$ converges and thus from (7.50), the integral

$\int_0^{\infty} \frac{\sin x}{x} dx$ converges.

Next, $\left|\frac{\sin x}{x}\right| \geq \left|\frac{\sin^2 x}{x}\right| = \frac{1-\cos 2x}{2x}$, but the integral

$$\int_{\pi/2}^{\infty} \frac{1-\cos 2x}{2x} dx = \lim_{b\to\infty}\left[\frac{1}{2}\int_{\pi/2}^{b} \frac{dx}{x} - \frac{1}{2}\int_{\pi/2}^{\infty} \frac{\cos 2x}{x} dx\right] = \frac{1}{2}\left[\lim_{b\to\infty} \ln b - \frac{1}{2}\ln \pi/2 - \frac{1}{2}\int_{\pi/2}^{\infty} \frac{\cos 2x}{x} dx\right]$$

tends to ∞, since the integral $\int_{\pi/2}^{\infty} \frac{\cos 2x}{x} dx$ converges. Thus, the integral $\int_{\pi/2}^{\infty} \left|\frac{1-\cos 2x}{2x}\right| dx$, and, therefore, the integral $\int_{\pi/2}^{\infty} \left|\frac{\sin x}{x}\right| dx$ diverges, and hence, the integral $\int_0^{\infty} \frac{\sin x}{x} dx$ converges conditionally only.

Example 7.50: Prove that the integral $I = \int_0^{\infty} \sin (x^2) dx$ converges.

Solution: Put $x = \sqrt{t}$, we have $I = \int_0^{\infty} \sin (x^2) dx = \frac{1}{2}\int_0^{\infty} \frac{\sin t}{\sqrt{t}} dt$.

Writing the integral on the right as the sum of two integrals as

$$\int_0^{\infty} \frac{\sin t}{\sqrt{t}} dt = \int_0^{\pi/2} \frac{\sin t}{\sqrt{t}} dt + \int_{\pi/2}^{\infty} \frac{\sin t}{\sqrt{t}} dt. \qquad \text{...(7.51)}$$

The first integral on the right of (7.51) is convergent, since $\lim_{t\to 0^+} \frac{\sin t}{\sqrt{t}} = 0$. Consider the second integral

$$\int_{\pi/2}^{\infty} \frac{\sin t}{\sqrt{t}} dt = \left[-\frac{\cos t}{\sqrt{t}}\right]_{\pi/2}^{\infty} - \frac{1}{2}\int_{\pi/2}^{\infty} \frac{\cos t\, dt}{t^{3/2}} = -\frac{1}{2}\int_{\pi/2}^{\infty} \frac{\cos t}{t^{3/2}} dt. \qquad \text{...(7.52)}$$

The last integral in (7.52) converges absolutely, since $\frac{|\cos t|}{t^{3/2}} \leq \frac{1}{t^{3/2}}$ and the integral $\int_{\pi/2}^{\infty} \frac{1}{t^{3/2}} dt$ converges, hence the integral $\int_{\pi/2}^{\infty} \frac{\sin t}{\sqrt{t}} dt$, and thus I converges.

EXERCISE 7.7

1. Show whether the following integrals converge or diverge.

(a) $\int_0^\infty \frac{dx}{x^4+2}$ (b) $\int_0^\infty \frac{x^{3.2}\,dx}{x^4+100}$ (c) $\int_4^\infty \frac{\sin^2 x\,dx}{\sqrt{x}(x-1)}$ (d) $\int_0^1 \frac{dx}{x^2\cos x}$

2. Evaluate the following improper integrals, if they exist

(a) $\int_0^\infty x\sin x\,dx$ (b) $\int_0^\infty e^{-ax}\cos px\,dx,\ a>0$

(c) $\int_1^\infty \frac{dx}{x\sqrt{x^2-1}}$ (d) $\int_0^1 \frac{x^p - x^{-p}}{x-1}\,dx.$

3. Enter the change $x = 1/\xi$ in the improper integral $\int_2^\infty \frac{dx}{x^4-2}$; check whether the resultant integral is improper or not.

4. For what range of α does the given integrals converge ?

(a) $\int_0^\infty \frac{dx}{x^\alpha+3}$ (b) $\int_0^\infty \frac{x^\alpha dx}{x+1}$ (c) $\int_1^2 (x^2-1)^\alpha dx$

(d) $\int_1^\infty \frac{dx}{x^\alpha}$ (e) $\int_0^1 \frac{dx}{x^\alpha}.$

5. Discuss the convergence of the integrals

(a) $\int_{-\infty}^\infty xe^{-x^2}\,dx$ (b) $\int_{-\pi/2}^{\pi/2} \tan x\,dx$ (c) $\int_{-1}^{+1} \sqrt{\frac{1+x}{1-x}}\,dx$

6. Check for the absolute convergence of the following improper integrals

(a) $\int_0^1 \frac{\sin(1/x)}{x^p}\,dx$ (b) $\int_2^\infty \frac{\sin x}{x(\ln x)^2}\,dx$ (c) $\int_{-\infty}^\infty \frac{\sin 3x}{1+x^4}\,dx$

7. Find the Cauchy principal value of $\int_{-\infty}^\infty \frac{x}{8-x^3}\,dx.$

8. Prove that the following integrals converge

(a) $\int_0^\infty \cos(x^2)dx$ (b) $\int_0^\infty 2x\cos(x^4)dx$.

9. Prove the convergence of the integral $I = \int_0^{\pi/2} \ln(\sin x)dx$ and evaluate it.

10. Prove that the integral $\int_0^\pi \frac{dx}{(\sin x)^k}$ converges if $k < 1$, and diverges if $k \geq 1$.

7.9 THE GAMMA FUNCTION

Euler's gamma function with parameter α, denoted by $\Gamma(\alpha)$, is defined as the integral

$$\boxed{\Gamma(\alpha) = \int_0^\infty e^{-x}x^{\alpha-1}dx, \quad \alpha > 0} \qquad ...(7.53)$$

This integral arises frequently in many science and engineering applications and has been studied extensively.

Convergence of the gamma integral We observe that gamma integral is improper for two reasons, first, the upper limit is ∞ and second, the integrand has infinite discontinuity at $x = 0$, for $0 < \alpha < 1$. To check its convergence or divergence, we rewrite the gamma integral as

$$\int_0^\infty e^{-x}x^{\alpha-1}dx = \int_0^\tau e^{-x}x^{\alpha-1}dx + \int_\tau^\infty e^{-x}x^{\alpha-1}dx, \qquad 0 < \tau < \infty$$

$$= I_1 + I_2, \text{ say.}$$

First consider the convergence of the integral I_1, at $x = 0$ and $0 < \alpha < 1$.

Take $f(x) = e^{-x}x^{\alpha-1}$ and $g(x) = x^{\alpha-1}$; we have

$$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{h\to 0} \frac{f(h)}{g(h)} = \lim_{h\to 0} e^{-h} = 1.$$

Since, $\int_0^\tau g(x)dx = \int_0^\tau \frac{dx}{x^{1-\alpha}}$ converges for $1 - \alpha < 1$, or $\alpha > 0$. Thus I_1 is also convergent for all $\alpha > 0$.

Next, consider the convergence of the integral I_2 at ∞.

Take $f(x) = e^{-x}x^{\alpha-1}$ and $g(x) = 1/x^2$, we have

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^{\alpha+1}}{e^x} = 0, \text{ for all } \quad x \geq \tau.$$

Since $\int_\tau^\infty \frac{dx}{x^2}$ is convergent, therefore, by the limit comparison test the integral $\int_\tau^\infty e^{-x} x^{\alpha-1} dx$ also converges for all α.

Hence the gamma integral $\Gamma(\alpha) = \int_0^\infty e^{-x}x^{\alpha-1}dx$ *is convergent for* $\alpha > 0$.

Replacing x by x^2 in (7.53), we obtain

$$\boxed{\Gamma(\alpha) = \int_0^\infty e^{-x^2} x^{2\alpha-2}(2x)dx = 2\int_0^\infty e^{-x^2} x^{2\alpha-1} dx,} \quad ...(7.54)$$

another form of the gamma function with parameter α.

Integrating (7.53) by parts, we obtain

$$\Gamma(\alpha+1) = \int_0^\infty e^{-x}x^\alpha dx = -\left[x^\alpha e^{-x}\right]_0^\infty + \alpha\int_0^\infty e^{-x}x^{\alpha-1}dx.$$

Thus, $$\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha). \quad ...(7.55)$$

Also $\Gamma(1) = \int_0^\infty e^{-x}dx = 1$. Hence, if α is a positive integer m, then by the repeated applications of (7.55), we obtain

$$\Gamma(m+1) = m!, \quad m = 0, 1, 2, ... \quad ...(7.56)$$

Thus the gamma function can be regarded as a *generalized factorial function*.

Also from (7.55), we obtain

$$\Gamma(\alpha) = \frac{\Gamma(\alpha+1)}{\alpha} = \frac{\Gamma(\alpha+2)}{\alpha(\alpha+1)} = ... = \frac{\Gamma(\alpha+k+1)}{\alpha(\alpha+1)\ldots(\alpha+k)}.$$

Thus, $$\Gamma(\alpha) = \frac{\Gamma(\alpha+k+1)}{\alpha(\alpha+1)\ldots(\alpha+k)}, \ (\alpha \neq 0, -1, -2, ..., -k) \quad ...(7.57)$$

We can use (7.57) to define the gamma function for negative α, $\alpha \neq -1, -2, ...,$ choosing k to be the smallest integer such that $\alpha + k + 1 > 0$. In fact, expressions (7.55) and (7.57) may be considered together to give a definition of $\Gamma(\alpha)$ for all α not equal to zero or a negative integer. The graph of $y = \Gamma(\alpha)$ is shown in Fig. 7.37. Clearly it is a continuous function of α for $\alpha > 0$.

Consider, $\Gamma(1/2) = \int_0^\infty e^{-x}x^{-\frac{1}{2}}dx = 2\int_0^\infty e^{-t^2}dt$, where $x = t^2$. Write,

$$[\Gamma(1/2)]^2 = \left[2\int_0^\infty e^{-t^2}dt\right]\left[2\int_0^\infty e^{-s^2}ds\right] = 4\int_0^\infty\int_0^\infty e^{-(s^2+t^2)}\,ds\,dt$$

$$= 4\int_{\theta=0}^{\pi/2}\int_{0}^{\infty} re^{-r^2}\,dr\,d\theta, \text{ where } s = r\cos\theta,\ t = r\sin\theta$$

$$= 2\pi\int_0^{\infty} re^{-r^2}dr = -\pi\int_0^{\infty} e^{-r^2}(-2r)dr = -\pi\left[e^{-r^2}\right]_0^{\infty} = \pi.$$

It gives $\Gamma(1/2) = \sqrt{\pi}$, a value of practical importance.

Also from (7.57) for $k = 0$, $\Gamma(\alpha) = \dfrac{\Gamma(\alpha+1)}{\alpha}$, $\alpha < 0$.

Substituting $\alpha = -\dfrac{1}{2}$, we obtain $\Gamma(-1/2) = -2\sqrt{\pi}$, another value used frequently.

An approximation of the gamma function for large positive α is given by the *stirling formula*

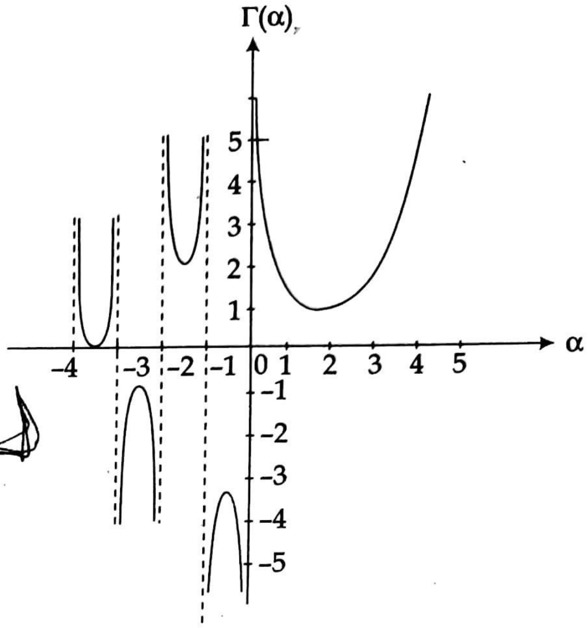

Fig. 7.37

$$\Gamma(\alpha+1) \approx \sqrt{2\pi\alpha}\left(\frac{\alpha}{e}\right)^{\alpha},$$

where e is the base of natural logarithm.

We will find that many integrals which occur in practical applications are not themselves gamma function integrals but can be evaluated by reducing to the gamma functions by making suitable change of variables.

7.10 THE BETA FUNCTION

The beta function with parameters l and m, denoted by $B(l, m)$ is defined as the integral

$$B(l, m) = \int_0^1 x^{l-1}(1-x)^{m-1}dx,\ l > 0, m > 0. \quad ...(7.58)$$

Convergence of the beta function The integral $B(l, m)$ is an improper integral for $0 < l < 1$, $0 < m < 1$. It has points of infinite discontinuity at $x = 0$, when $l < 1$, and at $x = 1$ when $m < 1$.

When $l < 1$ and $m < 1$, take a number $c \in (0, 1)$ and write the improper integral as

$$I = \int_0^1 x^{l-1}(1-x)^{m-1}dx = \int_0^c x^{l-1}(1-x)^{m-1}dx + \int_c^1 x^{l-1}(1-x)^{m-1}dx = I_1 + I_2, \text{ say.}$$

Integral $I_1 = \int_0^c x^{l-1}(1-x)^{m-1}dx$ is improper because $x = 0$ is a point of infinite discontinuity, and

integral $I_2 = \int_c^1 x^{l-1}(1-x)^{m-1}dx$ is improper because $x = 1$ is a point of infinite discontinuity.

First, consider the convergence of I_1 at $x = 0$, $0 < l < 1$. The integrand is $f(x) = x^{l-1}(1-x)^{m-1}$. Consider another function $g(x) = x^{l-1}$, we have

$$\lim_{x \to 0^+} \frac{f(x)}{g(x)} = \lim_{x \to 0^+} (1-x)^{m-1} = 1.$$

The integral $\int_0^c g(x)dx = \int_0^c \frac{dx}{x^{1-l}}$ is convergent when $1 - l < 1$, or $l > 0$.

Therefore by limit comparison test $\int_0^c f(x)dx$, that is, I_1 is also convergent for $l > 0$.

Next consider the convergence of I_2 at $x = 1$, for $0 < m < 1$. The integrand is $f(x) = x^{l-1}(1-x)^{m-1}$.

Consider another function $g(x) = (1-x)^{m-1}$. We have, $\lim_{x \to 1^-} \frac{f(x)}{g(x)} = \lim_{x \to 1^-} x^{l-1} = 1$.

The integral $\int_c^1 g(x)dx = \int_c^1 \frac{dx}{(1-x)^{l-m}} = \int_0^{1-c} \frac{dx}{x^{1-m}}$ is convergent, when $1 - m < 1$, or $m > 0$.

Therefore, by limit comparison test, $I_2 = \int_c^1 f(x)dx$ is also convergent for $m > 0$.

Hence the beta integral $B(l, m)$ converges for $l > 0$, $m > 0$.

By substituting $x = (1 - t)$ in (7.58), it is very easy to see that the *beta function is symmetric with respect to its parameters l, m,* that is,

$$B(l, m) = B(m, l). \qquad \text{...(7.59)}$$

Next, substituting $x = \sin^2 \theta$ so that, $dx = 2 \sin \theta \cos \theta \, d\theta$, in (7.58), we obtain

$$\boxed{B(l, m) = 2\int_0^{\pi/2} \sin^{2l-1}\theta \cos^{2m-1}\theta \, d\theta,} \qquad \text{...(7.60)}$$

another form of the beta function.

Relation between the beta and the gamma function The relation is

$$\boxed{B(l, m) = \frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m)}, \quad l > 0, \ m > 0.} \qquad \text{...(7.61)}$$

To prove (7.61), consider

$\Gamma(l) = \int_0^\infty e^{-x} x^{l-1} dx = 2\int_0^\infty e^{-t^2} t^{2l-1} dt$, where $x = t^2$. Similarly, $\Gamma(m) = 2\int_0^\infty e^{-s^2} s^{2m-1} ds$. Therefore,

$$\Gamma(l)\Gamma(m) = \left(2\int_0^\infty e^{-t^2} t^{2l-1} dt\right)\left(2\int_0^\infty e^{-s^2} s^{2m-1} ds\right) = 4\int_0^\infty \int_0^\infty e^{-(s^2+t^2)} s^{2m-1} t^{2l-1} ds\, dt. \qquad \text{...(7.62)}$$

Changing to polar co-ordinates $s = r\cos\theta$, $t = r\sin\theta$ so that $ds\,dt = r\,dr\,d\theta$, (7.62) becomes

$$\Gamma(l)\Gamma(m) = 4\int_0^{\pi/2}\int_0^{\infty} e^{-r^2}\,.\,r^{2(l+m)-1}\cos^{2m-1}\theta\,\sin^{2l-1}\theta\,d\theta\,dr$$

$$= \left[2\int_0^{\pi/2}\sin^{2l-1}\theta\cos^{2m-1}\theta d\theta\right]\left[2\int_0^{\infty} e^{-r^2} r^{2(l+m)-1}dr\right]$$

$= B(l, m)\,\Gamma(l + m)$, using (7.60) and (7.54). This gives (7.61).

Next, substituting $l = m = \dfrac{1}{2}$ in (7.61), we obtain

$$B\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma(1/2)\,\Gamma(1/2)}{\Gamma(1)} = \pi, \qquad \text{...(7.63)}$$

and, for $l = m = 1$, we have

$$B(1, 1) = \frac{\Gamma(1)\,\Gamma(1)}{\Gamma(2)} = 1. \qquad \text{...(7.64)}$$

The results (7.63) and (7.64) can be obtained from (7.60) also.

Another form of the beta function $B(l, m)$ is

$$B(l, m) = \int_0^{\infty}\frac{x^{l-1}}{(1+x)^{l+m}}\,dx, \qquad \text{...(7.65)}$$

obtained from (7.58) by substituting $x = t/(1 + t)$.

The beta function has its applications in statistics and also in science and engineering because of its close relation to the gamma function. Also a few definite integrals of some trigonometric function, which arise in some practical problems, can be evaluated in terms of beta function. For example, set $p = 2l - 1$ and $q = 2m - 1$ in (7.60), we obtain

$$\int_0^{\pi/2}\sin^p x\cos^q x\,dx = \frac{1}{2}B\left(\frac{p+1}{2},\frac{q+1}{2}\right), \qquad \text{...(7.66)}$$

an important result to remember.

In particular, set $p = n$ and $q = 0$, we obtain

$$\int_0^{\pi/2}\sin^n x\,dx = \frac{1}{2}B\left(\frac{n+1}{2},\frac{1}{2}\right). \qquad \text{...(7.67)}$$

Similarly,

$$\int_0^{\pi/2}\cos^n x\,dx = \frac{1}{2}B\left(\frac{1}{2},\frac{n+1}{2}\right). \qquad \text{...(7.68)}$$

Example 7.51: Express the following integrals in terms of gamma functions

(a) $\int_0^\infty x^{2/3} e^{-\sqrt{x}}\, dx$ (b) $\int_0^1 \frac{dx}{\sqrt{1-x^4}}$ (c) $\int_0^\infty a^{-bx^2}\, dx$ (d) $\int_0^1 x^{3/2}(1-\sqrt{x})^{1/2} dx.$

Solution: (a) Consider $I = \int_0^\infty x^{2/3} e^{-\sqrt{x}}\, dx$. Set $\sqrt{x} = t$, I becomes

$$I = \int_0^\infty (t^2)^{2/3}\, e^{-t} \cdot 2t\, dt = 2\int_0^\infty e^{-t}\, t^{7/3} dt = 2\Gamma(10/3) = 2\,\frac{7}{3}\,\frac{4}{3}\,\frac{1}{3}\,\Gamma(1/3) = \frac{56}{27}\Gamma(1/3).$$

(b) Consider $I = \int_0^1 \frac{dx}{\sqrt{1-x^4}}$. Set $x^2 = \sin\theta$, it becomes

$$I = \int_0^{\pi/2} \frac{\frac{1}{2}(\sin\theta)^{-\frac{1}{2}} \cos\theta\, d\theta}{\sqrt{1-\sin^2\theta}} = \frac{1}{2}\int_0^{\pi/2} (\sin\theta)^{-\frac{1}{2}}\, d\theta = \frac{1}{2} B\left(\frac{1}{4}, \frac{1}{2}\right), \text{ using (7.67)}$$

$$= \frac{1}{2}\,\frac{\Gamma(1/4)\,\Gamma(1/2)}{\Gamma(3/4)} = \frac{\sqrt{\pi}}{2}\,\frac{\Gamma(1/4)}{\Gamma(3/4)}.$$

(c) Consider $I = \int_0^\infty a^{-bx^2} dx = \int_0^\infty e^{-(b\ln a)x^2}\, dx.$

Set $(b \ln a)\, x^2 = t$, so that, $2(b \ln a)\, x dx = dt$, thus I becomes

$$I = \frac{1}{2\sqrt{b \ln a}} \int_0^\infty e^{-t}\, t^{-1/2} dt = \frac{\Gamma(1/2)}{2\sqrt{b\ln a}} = \frac{\sqrt{\pi}}{2\sqrt{b \ln a}}.$$

(d) Consider $I = \int_0^1 x^{3/2}(1 - \sqrt{x})^{1/2} dx$. Set $\sqrt{x} = t$, it becomes

$$I = 2\int_0^1 t^3(1 - t)^{1/2}\, t\, dt = 2\int_0^1 t^4(1 - t)^{1/2} dt = 2B(5, 3/2)$$

$$= 2\frac{\Gamma(5)\,\Gamma(3/2)}{\Gamma(13/2)} = 2\frac{4!\,(1/2)\sqrt{\pi}}{\frac{11}{2}\,\frac{9}{2}\,\frac{7}{2}\,\frac{5}{2}\,\frac{3}{2}\,\frac{1}{2}\sqrt{\pi}} = \frac{512}{3465}.$$

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Example 7.52: Given $\int_0^\infty \frac{x^{n-1}}{1+x}dx = \frac{\pi}{\sin n\pi}, 0 < n < 1,$ show that $\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$.

Solution: Consider
$$\Gamma(n)\,\Gamma(1-n) = B(n, 1-n) = B(1-n, n)$$
$$= \int_0^1 x^{-n}(1-x)^{n-1}dx. \qquad \text{...(7.69)}$$

Set $x = \frac{1}{1+y}$. It gives $dx = -\frac{dy}{(1+y)^2}$. Thus (7.69) becomes

$$\Gamma(n)\,\Gamma(1-n) = \int_\infty^0 \left(\frac{1}{1+y}\right)^{-n}\left(\frac{y}{1+y}\right)^{n-1}\left(-\frac{dy}{(1+y)^2}\right) = \int_0^\infty \frac{y^{n-1}}{1+y}dy = \frac{\pi}{\sin n\pi}.$$

Example 7.53: Show that

(a) $\int_0^1 x^m(\ln x)^n dx = \frac{(-1)^n\, n!}{(m+1)^{n+1}}$, where $n > 0$ is an integer and $m > -1$.

(b) $\int_0^1 y^{n-1}\left(\ln\frac{1}{y}\right)^{m-1} dy = \frac{\Gamma(m)}{n^m}$, where $m, n > 0$.

Solution: (a) Consider $I = \int_0^1 x^m(\ln x)^n dx$. Set $\ln x = t$, so that, $x = e^t$ and $dx = e^t dt$, thus I becomes

$$I = \int_{-\infty}^0 e^{mt}t^n e^t dt = \int_{-\infty}^0 e^{(m+1)t}t^n dt.$$

Further setting $(m+1)\,t = -\tau$, we obtain

$$I = \frac{(-1)^n}{(m+1)^{n+1}}\int_0^\infty e^{-\tau}\tau^n d\tau = \frac{(-1)^n\,\Gamma(n+1)}{(m+1)^{n+1}} = \frac{(-1)^n\, n!}{(m+1)^{n+1}}.$$

(b) Consider $I = \int_0^1 y^{n-1}\left(\ln\frac{1}{y}\right)^{m-1} dy$. Set $\ln\frac{1}{y} = t$, so that, $y = e^{-t}$, thus I becomes

$$I = -\int_\infty^0 e^{-(n-1)t}t^{m-1}e^{-t}dt = \int_0^\infty e^{-nt}t^{m-1}dt.$$

Further setting $nt = \tau$, we obtain

$$I=\int_0^\infty \frac{e^{-\tau}\tau^{m-1}}{n^{m-1}}\frac{d\tau}{n}=\frac{1}{n^m}\int_0^\infty e^{-\tau}\tau^{m-1}d\tau=\frac{\Gamma(m)}{n^m}.$$

Example 7.54: Show that $\displaystyle\int_0^1 \frac{x^2\,dx}{\sqrt{(1-x^4)}}\times\int_0^1\frac{dx}{\sqrt{(1+x^4)}}=\frac{\pi}{4\sqrt{2}}.$

Solution: Consider, $I_1=\displaystyle\int_0^1\frac{x^2\,dx}{\sqrt{(1-x^4)}}dx.$ Set $x^2=\sin\theta$, I_1 becomes

$$I_1=\frac{1}{2}\int_0^{\pi/2}\frac{\sin\theta\cos\theta}{\cos\theta\sqrt{\sin\theta}}d\theta=\frac{1}{2}\int_0^{\pi/2}\sqrt{\sin\theta}\,d\theta=\frac{1}{4}B\left(\frac{3}{4},\frac{1}{2}\right).$$

Next consider, $I_2=\displaystyle\int_0^1\frac{dx}{\sqrt{(1+x^4)}}$. Set $x^2=\tan\theta$, I_2 becomes

$$I_2=\frac{1}{2}\int_0^{\pi/4}\frac{\sec^2\theta\,d\theta}{\sqrt{\tan\theta}\sec\theta}=\frac{1}{2}\int_0^{\pi/4}\frac{d\theta}{\sqrt{\sin\theta\cos\theta}}$$

$$=\frac{1}{\sqrt{2}}\int_0^{\pi/4}\frac{d\theta}{\sqrt{\sin 2\theta}}=\frac{1}{2\sqrt{2}}\int_0^{\pi/2}\frac{d\theta}{\sqrt{\sin\theta}}=\frac{1}{4\sqrt{2}}B\left(\frac{1}{4},\frac{1}{2}\right).$$

Thus $$\int_0^1\frac{x^2\,dx}{\sqrt{(1-x^4)}}\times\int_0^1\frac{dx}{\sqrt{(1+x^4)}}=\frac{1}{16\sqrt{2}}B\left(\frac{3}{4},\frac{1}{2}\right)B\left(\frac{1}{4},\frac{1}{2}\right)$$

$$=\frac{1}{16\sqrt{2}}\frac{\Gamma(3/4)\,\Gamma(1/2)}{\Gamma(5/4)}\frac{\Gamma(1/4)\,\Gamma(1/2)}{\Gamma(3/4)}=\frac{\pi}{4\sqrt{2}}.$$

Example 7.55: Show that

(a) $\displaystyle\int_0^\infty xe^{-ax}\cos bx\,dx=\frac{a^2-b^2}{(a^2+b^2)^2}$ (b) $\displaystyle\int_0^\infty xe^{-ax}\sin bx\,dx=\frac{2ab}{(a^2+b^2)^2}$

Solution: Let $$I=\int_0^\infty xe^{-ax}\cos bx\,dx+i\int_0^\infty xe^{-ax}\sin bx\,dx=\int_0^\infty xe^{-ax}(\cos bx+i\sin bx)dx$$

$$=\int_0^\infty xe^{-ax}e^{ibx}dx=\int_0^\infty xe^{-(a-ib)x}dx=\frac{\Gamma(2)}{(a-ib)^2},\ \text{since}\int_0^\infty xe^{-mx}dx=\frac{\Gamma(2)}{m^2}$$

$$= \frac{1}{(a-ib)^2} = \frac{(a+ib)^2}{(a-ib)^2(a+ib)^2} = \frac{a^2-b^2+2iab}{(a^2+b^2)^2} = \frac{a^2-b^2}{(a^2+b^2)^2} + i\frac{2ab}{(a^2+b^2)^2}$$

Equating the real and imaginary parts on both sides, we obtain

$$\int_0^\infty xe^{-ax}\cos bx\,dx = \frac{a^2+b^2}{(a^2+b^2)^2} \text{ and } \int_0^\infty xe^{-ax}\sin bx\,dx = \frac{2ab}{(a^2+b^2)^2}.$$

Example 7.56: If l, m are positive real numbers then

$$\iint_D x^{l-1}y^{m-1}dx\,dy = \frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m+1)}, \qquad ...(7.70)$$

where D is the domain $x \geq 0$, $y \geq 0$ and $x + y \leq 1$.

Solution: The region of integration is $D = \{(x, y): 0 \leq y \leq 1-x,\ 0 \leq x \leq 1\}$.

$$\text{Let } I = \iint_D x^{l-1}y^{m-1}dx\,dy = \int_0^1\left(\int_0^{1-x} x^{l-1}\,y^{m-1}\,dy\right)dx = \frac{1}{m}\int_0^1 x^{l-1}(1-x)^m dx = \frac{1}{m}B(l, m+1)$$

$$= \frac{1}{m}\frac{\Gamma(l)\,\Gamma(m+1)}{\Gamma(l+m+1)} = \frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m+1)}.$$

Example 7.57: If l, m, n are positive real numbers, then

$$\iiint_D x^{l-1}y^{m-1}z^{n-1}dx\,dy\,dz = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)}, \qquad ...(7.71)$$

where D is the region $x \geq 0$, $y \geq 0$, $z \geq 0$, $x + y + z \leq 1$.

Solution: The region of integration is $D = \{(x, y, z): 0 \leq z \leq 1-x-y,\ 0 \leq y \leq 1-x,\ 0 \leq x \leq 1\}$.

$$\text{Let, } I = \iiint_D x^{l-1}y^{m-1}z^{n-1}dx\,dy\,dz = \int_0^1\int_0^{1-x}\int_0^{1-x-y} x^{l-1}y^{m-1}z^{n-1}dz\,dy\,dx.$$

Consider $I' = \displaystyle\int_0^{1-x}\int_0^{1-x-y} y^{m-1}z^{n-1}dz\,dy$. Set $1 - x = h$, I' becomes

$$I' = \int_0^h\int_0^{h-y} y^{m-1}z^{n-1}dz\,dy = \frac{1}{n}\int_0^h y^{m-1}(h-y)^n dy.$$

Next, set $y = hY$, I' becomes

$$I' = \frac{1}{n}\int_0^1 (hY)^{m-1}h^n(1-Y)^n h\,dY = \frac{h^{m+n}}{n}\int_0^1 Y^{m-1}(1-Y)^n dY$$

$$= \frac{h^{m+n}}{n} B(m, n+1) = \frac{(1-x)^{m+n}}{n} B(m, n+1).$$

Thus $$I = \frac{B(m,n+1)}{n} \int_0^1 x^{l-1}(1-x)^{m+n} dx$$

$$= \frac{B(m,n+1)}{n} B(l, m+n+1) = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)}.$$

Remark: The integral (7.70) is known as *Dirichlet's integral for two variables,* and the integral (7.71) is called *Dirichlet's integral for three variables.* Dirichlet's integrals are used to evaluate some specific area and volume integrals, as illustrated in the example considered next.

Example 7.58: Evaluate $\iiint_D xyz\,dx\,dy\,dz$, where D is the region enclosed by the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

Solution: Since the region of integration $x^2/a^2 + y^2/b^2 + z^2/c^2 \le 1$ is symmetrical in all the eight octants, therefore the given integral is

$$I = 8 \iiint_{D'} xyz\,dx\,dy\,dz,$$

where D' is given as $x \ge 0$, $y \ge 0$, $z \ge 0$ and $x^2/a^2 + y^2/b^2 + z^2/c^2 \le 1$.

Set $x^2/a^2 = u$, $y^2/b^2 = v$, and $z^2/c^2 = w$, so that, $2xdx = a^2du$, $2ydy = b^2dv$, and $2zdz = c^2\,dw$. The integral I becomes

$$I = a^2\,b^2\,c^2 \iiint_{D''} du\,dv\,dw,$$

where D'' is given as $u \ge 0$, $v \ge 0$, $w \ge 0$ and $u + v + w \le 1$. Rewriting I as,

$$I = a^2b^2c^2 \iiint_{D''} u^{1-1}\,v^{1-1}\,w^{1-1} du\,dv\,dw = a^2\,b^2\,c^2\,\frac{\Gamma(1)\,\Gamma(1)\,\Gamma(1)}{\Gamma(4)}, \text{ using (7.71)}$$

$$= \frac{1}{6}\,a^2\,b^2\,c^2.$$

EXERCISE 7.8

1. Evaluate the following improper integrals in terms of gamma function.

(a) $\int_0^\infty \sqrt{x}\,e^{-x^2} dx$ (b) $\int_0^\infty e^{-x^3}\,dx$ (c) $\int_{-\infty}^\infty e^{-x^2} dx$ (d) $\int_0^\infty \frac{x^a}{a^x} dx.$

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3.

2. Evaluate the following integrals using the gamma and beta functions.

(a) $\int_0^1 x^m(1-x^n)^p dx$ (b) $\int_0^{\pi/2} \sin^7\theta \cos^5\theta \, d\theta$ (c) $\int_0^{\pi/2} \sin^{10}\theta \, d\theta$

(d) $\int_0^a \frac{x^{3/2}}{\sqrt{a^2-x^2}} dx$ (e) $\int_0^1 \frac{dx}{\sqrt{-\ln x}}$ (f) $\int_0^1 \frac{dx}{\sqrt{1-x^3}}$

(g) $\int_0^\infty e^{-ax} x^{m-1} \sin bx \, dx.$

3. Prove that

(a) $\int_0^{\pi/2} \sqrt{\sin\theta} \, d\theta \times \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} = \pi$

(b) $\int_{-1}^1 (1-x^2)^n dx = \frac{2^{2n+1}+(n!)^2}{(2n+1)!}$, n is a positive integer

(c) $\int_0^\infty x^m e^{-\alpha x^n} dx = \frac{1}{n\alpha^{(m-1)/n}} \Gamma\left(\frac{m+1}{n}\right)$ $m, n, \alpha > 0$

(d) $\int_0^m x^n \left(1-\frac{x}{m}\right)^{m-1} dx = m^{n+1} B(m, n+1)$, $m, n > 0$.

4. Prove that

(a) $B(m, n) = B(m+1, n) + B(m, n+1)$

(b) $\frac{B(m+1, n)}{m} = \frac{B(m, n+1)}{m} = \frac{B(m, n)}{m+n}$

(c) $B(l, m) = \int_0^1 \frac{x^{l-1}+x^{m-1}}{(1+x)^{l+m}} dx.$

5. Show that

(a) $\Gamma(2n) = \frac{2^{2n-1}}{\sqrt{\pi}} \Gamma\left(n+\frac{1}{2}\right) \Gamma(n)$, and hence, $\Gamma(1/4)\,\Gamma(3/4) = \pi\sqrt{2}$

(b) $B(n, n) = \dfrac{\Gamma(n)\sqrt{\pi}}{2^{2n-1}\,\Gamma\left(n+\frac{1}{2}\right)}$.

6. Show that $\iint x^{m-1}y^{n-1}dx\,dy$ over the positive quadrant of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is

$\dfrac{a^m b^n}{2n} B\left(\dfrac{m}{2}, \dfrac{n}{2}+1\right)$.

7. Evaluate the integral $\iiint_D x^{p-1}y^{q-1}z^{r-1}dx\,dy\,dz,\quad p, q, r > 0$, where D is the region of the tetrahedron bounded by $x \geq 0, y \geq 0, z \geq 0$ and $x + y + z \leq 1$.

8. The plane $x/a + y/b + z/c = 1$ meets the axes in A, B and C. Apply Dirichlet's integral to find the volume of the tetrahedron $OABC$. Also find its mass if the density at any point is $kxyz$, where k is a constant.

9. Evaluate $\iiint_D x^{l-1}y^{m-1}z^{n-1}dx\,dy\,dz$, using Dirichlet's integrals where the region of integration D is given by $x, y, z \geq 0$ and $(x/a)^p + (y/b)^q + (z/c)^r \leq 1$.

7.11 THE ERROR FUNCTION

The *error function of x*, denoted by $erf(x)$, is defined as

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt. \qquad ...(7.72)$$

The graph of $erf(x)$ is shown in Fig. 7.38.
It shows that it is odd in x.
In the series form, it can be expressed as

$$erf(x) = \frac{2}{\sqrt{\pi}}\left[x - \frac{x^3}{1!3} + \frac{x^5}{2!5} - \frac{x^7}{3!7} + \dots\right]. \qquad ...(7.73)$$

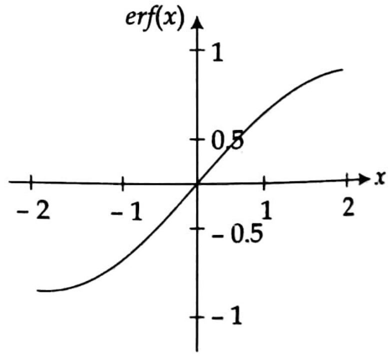

Fig. 7.38

The *complementary error function,* denoted by $erf_c(x)$, is defined as

$$erf_c(x) = \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-t^2}\,dt. \qquad ...(7.74)$$

Set $t^2 = u$ in (7.72), we obtain

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^{x^2} e^{-u} \frac{1}{2} u^{-1/2} du = \frac{1}{\sqrt{\pi}} \int_0^{x^2} u^{-1/2} e^{-u} du. \quad ...(7.75)$$

This is *another form of the error function.*

The error functions arise in the theory of probability and solution of some partial differential equations and find applications in physics and various engineering disciplines. Next, we study some properties of the error function.

Properties of Error Function

1. **$erf(0) = 0$ and $erf(\infty) = 1$.** The result $erf(0) = 0$ is obvious. For $erf(\infty) = 1$, from (7.75)

$$erf(\infty) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} u^{-1/2} e^{-u} du = \frac{1}{\sqrt{\pi}} \Gamma(1/2) = \frac{1}{\sqrt{\pi}} \sqrt{\pi} = 1.$$

2. **$erf(x) + erf_c(x) = 1$.** Consider

$$erf(x) + erf_c(x) = \frac{2}{\sqrt{\pi}} \int_0^{x} e^{-t^2} dt + \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-t^2} dt = 1, \quad \text{refer to (7.54)}$$

3. **$erf(-x) = -erf(x)$.** By definition, $erf(-x) = \frac{2}{\sqrt{\pi}} \int_0^{-x} e^{-t^2} dt$. Set $t = -y$, we have

$$erf(-x) = \frac{2}{\sqrt{\pi}} \int_0^{x} e^{-y^2} (-dy) = -\frac{2}{\sqrt{\pi}} \int_0^{x} e^{-y^2} dy = -erf(x).$$

4. **Derivative of error function: $\frac{d}{dx}[erf(ax)] = \frac{2a}{\sqrt{\pi}} e^{-a^2x^2}$.** By definition

$erf(ax) = \frac{2}{\sqrt{\pi}} \int_0^{ax} e^{-t^2} dt$. Differentiating under the integral sign, using Leibnitz's rule,

$$\frac{d}{dx} erf(ax) = \frac{2}{\sqrt{\pi}} \left[\int_0^{ax} \frac{\partial}{\partial x} (e^{-t^2}) dt + \frac{d}{dx}(ax)\, e^{-a^2x^2} - \frac{d}{dx}(0)\, 1 \right] = \frac{2}{\sqrt{\pi}} [a e^{-a^2x^2}] = \frac{2a}{\sqrt{\pi}} e^{-a^2x^2}.$$

5. **Integral of error function: $\int_0^t erf(ax) dx = t\, erf(at) + \frac{1}{a\sqrt{\pi}} \left[e^{-a^2t^2} - 1 \right]$.**

$$\int_0^t erf(ax) dx = [x erf(ax)]_0^t - \int_0^t x \left(\frac{d}{dx} erf(ax) \right) dx = t erf(at) - \int_0^t x \frac{2a}{\sqrt{\pi}} e^{-a^2x^2} dx$$

$$= t\, erf(at) - \frac{a}{\sqrt{\pi}} \int_0^{t^2} e^{-a^2 x}\, dx, \quad \text{(replacing } x^2 \text{ by } x)$$

$$= t\, erf(at) - \frac{a}{\sqrt{\pi}} \left[\frac{e^{-a^2 x}}{-a^2} \right]_0^{t^2} = t\, erf(at) + \frac{1}{a\sqrt{\pi}} \left(e^{-a^2 t^2} - 1 \right).$$

EXERCISE 7.9

1. Show that

(a) $erf_c(-x) + erf_c(x) = 2$ (b) $\displaystyle\int_a^b e^{-x^2} dx = \frac{\sqrt{\pi}}{2} [erf(b) - erf(a)]$

(c) $\displaystyle\int_0^\infty e^{-x^2 - 2ax} dx = \frac{\sqrt{\pi}}{2} e^{a^2} [1 - erf(a)].$

2. Show that

(a) $\displaystyle\frac{d}{dx} [erf_c(ax)] = -\frac{2a}{\sqrt{\pi}} e^{-a^2 x^2}$ (b) $\displaystyle\int_0^t erf_c(ax) dx = t\,erf_c(at)\ \frac{1}{a\sqrt{\pi}} \left[e^{-a^2 t^2} - 1 \right].$

3. Prove that $\displaystyle erf_c(x/2) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma\left(1 - \frac{k}{2}\right)} x^k.$

4. Expanding $erf(x)$ in series, show that $\displaystyle\int_0^\infty e^{-pt} erf(\sqrt{t}) dt = \frac{1}{p\sqrt{p+1}}.$

ANSWERS

Exercise 7.1 (p. 423)

1. (a) $7/12$ (b) $1/2$ (c) $a^2/6$ (d) $\frac{5}{8}\pi a^3$

2. (a) $1/6$ (b) $\frac{ab^2}{3}$ 3. $\frac{3}{56}$

4. $\dfrac{(\sin 1 - \cos 1)}{ab}$ 5. $\dfrac{64a^4}{3}$ 6. 6 7. 1

8. 1 9. $(1/8)(e^{16} - 1)$ 10. $1/2$ 11. $\pi a^2/6$

12. $-\pi[f(a) - f(0)]$ 13. 22.5π 14. $4a^2/3$.

Exercise 7.2 (p. 427)

1. $8\left(\dfrac{\pi}{2} - \dfrac{5}{3}\right)$ 2. $\dfrac{\pi a^5}{20}$ 3. $\dfrac{2\pi}{a+b}$ 4. $\dfrac{3\pi a^4}{4}$

6. $\dfrac{15\pi a^4}{64}$ 7. $\dfrac{\pi}{6}$ 8. $2\pi ab/3$.

Exercise 7.3 (p. 432)

3. $(3/2)a^2\pi$ 4. $a^2/2$ 5. $a^2\left(1 - \dfrac{\pi}{4}\right)$ 6. $\dfrac{3}{2}\ln 3 - 2/3$

7. $(4a/3\pi, 4a/3\pi)$ 8. $2a/3$ 9. $\bar{x} = 64/35$, $\bar{y} = 5/7$

10. $(5a/6, 0)$ 11. $\left(\dfrac{\pi a\sqrt{2}}{8}, 0\right)$ 13. $\bar{x} = 11/3$, $\bar{y} = 14/27$, $I_y = 432$, $R_y = 4$

14. $\bar{x} = 0$, $\bar{y} = 7/10$, $I_x = 9/10$, $I_y = 3/10$, $I_0 = 6/5$

15. $\dfrac{\rho a^4}{48}(3\pi - 8)$ 16. $\dfrac{\pi\rho r^4 h}{10}$, $\dfrac{\pi\rho r^2 h}{20}(r^2 + 4h^2)$, $\dfrac{\pi\rho r^2 h}{60}(3r^2 + 2h^2)$.

Exercise 7.4 (p. 438)

1. $\dfrac{4}{3}\pi a^3$ 2. 1 3. 16π 4. $\dfrac{1}{6}abc$

6. $\dfrac{10\pi}{3}$ 7. $2\pi^2 a^3$ 8. 32 9. $\dfrac{3\pi a^2}{4}$

10. $8a^2$.

Exercise 7.5 (p. 444)

1. (a) $\dfrac{8}{3}\ln 2 - \dfrac{19}{9}$ (b) $1/48$ 2. $\pi/6$ 3. $\dfrac{\pi}{4}a^2h^2(a^2 + h^2)$

4. $\dfrac{\pi a^5}{10}$ 5. $\dfrac{1}{8}(e^{4a} - 6e^{2a} + 8e^{a} - 3)$ 6. $\dfrac{\pi^2}{8}$ 7. $\dfrac{4}{15}\pi a^3 bc$

8. $1/4$ 9. 2325.04 10. $4\pi \ln(a/b)$.

Exercise 7.6 (p. 451)

1. $\dfrac{4\pi abc}{3}$ 2. $8\pi\sqrt{2}$ 3. $\left(\dfrac{4}{3}\sqrt{2}-2\right)\pi a^3$ 4. $(n-m)\pi a^3/8$

6. $\dfrac{\pi b^4}{2a}$ 7. 2 8. 1 9. $4a/3$

10. $(0, 0, 2a/5)$ 11. 3 12. $\dfrac{1}{2}\rho\pi r^6 h$, $\dfrac{1}{12}\rho\pi r^2 h\,(3r^2+4h^2)$

13. $\left(0, 0, \dfrac{45a\pi}{64(3\pi-4)}\right)$ 14. $\dfrac{2}{5}Ma^2$ 15. $[(4r^2+a^2)/10]^{1/2}$

Exercise 7.7 (p. 464)

1. (a) converges (b) diverges (c) converges (d) diverges.
2. (a) diverges (b) $a/(a^2+p^2)$ (c) $\pi/2$ (d) $\dfrac{1}{p}-\pi\cot p\pi$, $(-1<p<1)$.
3. The resultant integral is not improper.
4. (a) $1<\alpha<\infty$ (b) $-1<\alpha<0$ (c) $\alpha>-1$ (d) $\alpha>1$ (e) $\alpha<1$.
5. (a) converges to zero (b) divergent (c) converges to π.
6. (a) converges absolutely for $p<1$ (b) converges absolutely
   (c) converges absolutely.
7. $-\sqrt{3}\,\pi/6$ 9. $-(\pi/2)\ln 2$

Exercise 7.8 (p. 474)

1. (a) $(1/2)\,\Gamma(3/4)$ (b) $1/3\,\Gamma(1/3)$ (c) $\sqrt{\pi}$ (d) $\dfrac{\Gamma(a+1)}{(\ln a)^{a+1}}$
2. (a) $\dfrac{1}{n}B\left(\dfrac{m+1}{n}, p+1\right)$ (b) $1/120$ (c) $\dfrac{63\pi}{512}$
   (d) $\dfrac{1}{2}a^{3/2}B\left(\dfrac{5}{4}, \dfrac{1}{2}\right)$ (e) $\sqrt{\pi}$ (f) $\dfrac{\sqrt{\pi}\,\Gamma(1/3)}{\Gamma(5/4)}$
   (g) $\dfrac{\Gamma(m)}{r^m}\sin m\theta$; $r=\sqrt{a^2+b^2}$, $\theta=\tan^{-1}\dfrac{b}{a}$.
7. $\dfrac{\Gamma(p)\,\Gamma(q)\,\Gamma(r)}{\Gamma(p+q+r+1)}$ 8. $abc/6$; $ka^2b^2c^2/720$ 9. $\dfrac{a^l\, b^m\, c^n}{pqr}\cdot\dfrac{\Gamma(l/p)\,\Gamma(m/q)\,\Gamma(n/r)}{\Gamma(l/p+m/q+n/r+1)}$.