CHAPTER 6 Definite Integrals and Their Applications

Integral calculus is related to differential calculus by the fundamental theorem of integral calculus which says roughly that the derivative and integral are inverse operations. Integration is vital to many scientific areas since numerous powerful mathematical tools are based on integration. Applications of definite integrals include computations involving area, arc length, volume and surface of the solid generated.

6.1 ANTIDERIVATIVES: INDEFINITE INTEGRALS

A function $F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$ for all x in the domain of definition of f. We observe that if $F(x)$ is an antiderivative of $f(x)$, then $F(x) + c$, where c is an arbitrary constant, is also antiderivative of $f(x)$. $F(x) + c$ is called the *indefinite integral of $f(x)$ with respect to x*, and denoted by

$$\int f(x)\,dx = F(x) + c.$$

The function f is called the *integrand* and the constant c is called the *constant of integration*.

A number of indefinite integrals are found by reversing derivative formulae. We can evaluate an indefinite integral directly, or by using the method of substitution, or integration by parts.

Next, we list indefinite integrations of some commonly used functions. These results can be derived using the standard methods of integration.

1. $\int x^n dx = \dfrac{x^{n+1}}{n+1} + c,\ (n \neq -1)$
2. $\int \dfrac{1}{x}\,dx = \ln |x| + c$
3. $\int e^{ax} dx = \dfrac{1}{a} e^{ax} + c$
4. $\int \sin x\,dx = -\cos x + c$
5. $\int \cos x\,dx = \sin x + c$
6. $\int \tan x\,dx = -\ln |\cos x| + c$

7. $\int \cot x\,dx = \ln|\sin x| + c$

8. $\int \sec x\,dx = \ln|\sec x + \tan x| + c$

9. $\int \operatorname{cosec} x\,dx = \ln|\operatorname{cosec} x - \cot x| + c$

10. $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

11. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$

12. $\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\frac{x}{a} + c$

13. $\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a} + c$

14. $\int \sin^2 x\,dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + c$

15. $\int \cos^2 x\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + c$

16. $\int \tan^2 x\,dx = \tan x - x + c$

17. $\int \cot^2 x\,dx = -\cot x - x + c$

18. $\int \ln x\,dx = x\ln x - x + c$

19. $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c$

20. $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c$

21. $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

22. $\int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\ln\left(x+\sqrt{a^2+x^2}\right) + c$

23. $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left(x+\sqrt{x^2-a^2}\right) + c$

24. $\int \sinh x\,dx = \cosh x + c$

25. $\int \cosh x\,dx = \sinh x + c$

26. $\int \operatorname{sech}^2 x\,dx = \tanh x + c$

27. $\int \operatorname{cosech}^2 x\,dx = \coth x + c$

28. $\int \tanh x\,dx = \ln|\cosh x| + c$

29. $\int \coth x\,dx = \ln|\sinh x| + c$

6.2 DEFINITE INTEGRALS AND THEIR PROPERTIES

Let $f(x)$ be a function defined and continuous on a closed interval $[a, b]$. Divide the interval $[a, b]$ int[o] [n] subintervals by choosing $n-1$ points, say $x_1, x_2, \ldots, x_{n-1}$ between a and b such that

$$a < x_1 < x_2 < \ldots < x_{n-1} < b.$$

To make the notation consistent denote, $a = x_0$ and $b = x_n$, and let $\Delta x_i = x_i - x_{i-1}$, $\quad i = 1, 2, \ldots n$.

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6.2.2 Properties of Definite Integrals

We state below some important properties of the definite integrals.

1. **Invariance Property**. If $f(x)$ is integrable over $[0, a]$, then

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx. \qquad \text{...(6.3)}$$

2. $$\left.\begin{aligned}\int_{-a}^{a} f(x)dx &= 2\int_0^a f(x)dx, && \text{if } f(x) \text{ is even}\\ &= 0, && \text{if } f(x) \text{ is odd}\end{aligned}\right\} \qquad \text{...(6.4)}$$

3. $$\left.\begin{aligned}\int_{0}^{2a} f(x)dx &= 2\int_0^a f(x)dx, && \text{if } f(2a-x) = f(x)\\ &= 0, && \text{if } f(2a-x) = -f(x)\end{aligned}\right\} \qquad \text{...(6.5)}$$

4. **Shift Property:** If $f(x)$ is integrable and defined for the necessary values of x, then

$$\int_a^b f(x)dx = \int_{a-c}^{b-c} f(x+c)dx. \qquad \text{...(6.6)}$$

5. **Max-Min Inequality:** If M and m are the maximum and minimum values of $f(x)$ on $[a, b]$, then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a). \qquad \text{...(6.7)}$$

6. **Mean-Value Theorem:** If $f(x)$ is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{(b-a)}\int_a^b f(x)dx. \qquad \text{...(6.8)}$$

The expression, $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)dx$ is called the *average, or mean value of* $f(x)$ *on* $[a, b]$.

7. **First Fundamental Theorem of Integral Calculus:** If $f(x)$ is continuous on the closed interval $[a, b]$, then $F(x) = \displaystyle\int_a^x f(t)dt$ is continuous on $[a, b]$, differentiable in (a, b), and $\dfrac{dF}{dx} = f(x), a \le x \le b$.

That is, for every continuous function the process of integration and differentiation are inverse of one another.

8. Second Fundamental Theorem of Integral Calculus: If $f(x)$ is continuous on $[a, b]$ and $F(x)$ is antiderivative of $f(x)$ on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a). \quad \text{...(6)}$$

This theorem gives us a method to evaluate the definite integral of a continuous function from a to b. The existence part is taken care of by the First Fundamental Theorem of Integral Calculus.

or, $$\int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$

(b) Let $$I_n = \int \cos^n x\,dx = \int \cos^{n-1} x \cos x\,dx.$$

Integrating by parts, we obtain

$$I_n = \cos^{n-1} x \sin x - \int (n-1)\cos^{n-2} x(-\sin x)\sin x\,dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x(1-\cos^2 x)dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx$$

$$= \cos^{n-1} x \sin x + (n-1)\,I_{n-2} - (n-1)\,I_n$$

or, $$nI_n = \cos^{n-1} x \sin x + (n-1)\,I_{n-2}$$

or, $$I_n = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n} I_{n-2} \quad ...(6.$$

as the reduction formula for $\int \cos^n x\,dx$.

Specifically, consider the definite integral, $I_n = \int_0^{\pi/2} \cos^n x\,dx$,

Using (6.13), it becomes

$$I_n = \frac{1}{n}[\cos^{n-1} x \sin x]_0^{\pi/2} + \frac{n-1}{n} I_{n-2} = 0 + \frac{n-1}{n} I_{n-2}$$

or, $$\int_0^{\pi/2} \cos^n x\,dx = \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2} x\,dx. \quad ...(6.1$$

From (6.12) and (6.14) we observe that if

$I_n = \int_0^{\pi/2} \sin^n x\,dx$, or $= \int_0^{\pi/2} \cos^n x\,dx$, both can be expressed as

$$I_n = \frac{n-1}{n} I_{n-2} = \frac{n-1}{n}\frac{n-3}{n-2} I_{n-4} = \frac{n-1}{n}\frac{n-3}{n-2}\frac{n-5}{n-4} I_{n-6}, \text{ and so on.}$$

6.4 AREAS OF BOUNDED REGIONS

Definite integrals are applied to find the areas of the bounded regions, the volumes and surface areas of solids of revolution, the length of curves, centroid of area and volume of revolution, etc. In this section, we find the areas of bounded regions.

6.4.1 Area Under a Curve in Cartesian Form

The area A of a region, bounded above by a curve $y = f(x)$, below by the axis of x and on left by ordinate $x = a$ and on right by $x = b$, as shown in Fig. 6.1, is given by

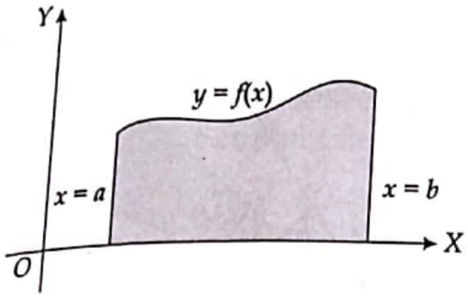


Fig. 6.1

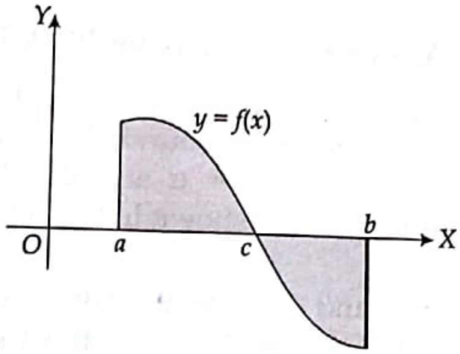


Fig. 6.2

$$A = \int_a^b f(x)dx. \qquad \text{...(6.32)}$$

If the curve $y = f(x)$ is below the x-axis, then the value of the integral $\int_a^b f(x)$ is negative; we take he area as the magnitude of this value.

In case the curve $y = f(x)$ crosses the x-axis at a point c, $a < c < b$, as shown Fig. 6.2, then area is ven by

$$A = \int_a^c f(x)dx + \left|\int_c^b f(x)dx\right| \qquad \text{...(6.33)}$$

The area bounded by the curve $x = f(y)$, the axis of y and the two abscissas $y = c$ and $y = d$ as vn in Fig. 6.3, is given by

$$A = \int_c^d f(y)dy. \qquad \text{...(6.34)}$$

he area of the region bounded above by the curve $y = f(x)$, below by the curve $y = g(x)$ and the $= a$, $x = b$, as shown in Fig. 6.4, is given by

$$A = \int_a^b [f(x) - g(x)]dx. \qquad \text{...(6.35)}$$

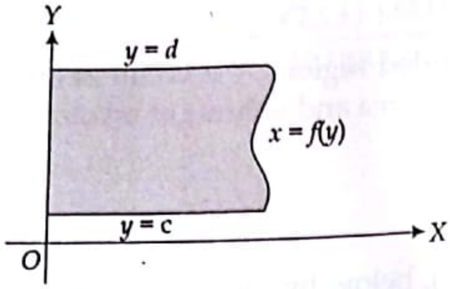


Fig. 6.3

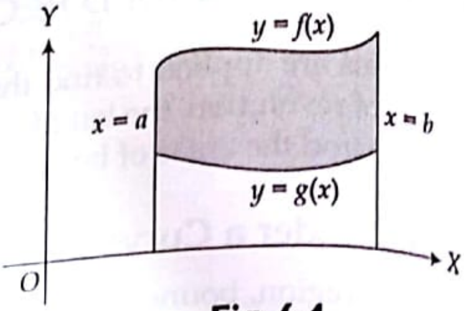


Fig. 6.4

6.4.2 Area Under a Curve in Polar Form

If $r = f(\theta)$ is the equation of a curve in polar form, then the area $OABO$ of the sector bounded by the curve $r = f(\theta)$ and the radial lines $\theta = \alpha$ and $\theta = \beta$, when $f(\theta)$ is continuous on $[\alpha, \beta]$ as shown in Fig. 6.5 can be found as follows.

Let $P(r, \theta)$ and $Q(r + \Delta r, \theta + \Delta\theta)$ be two points on the curve $r = f(\theta)$, Q being close to P. If ΔA is the area of the elementary strip OPQ, then for small value of $\Delta\theta$, PQ can be approximated with a circular arc of radius r, and thus

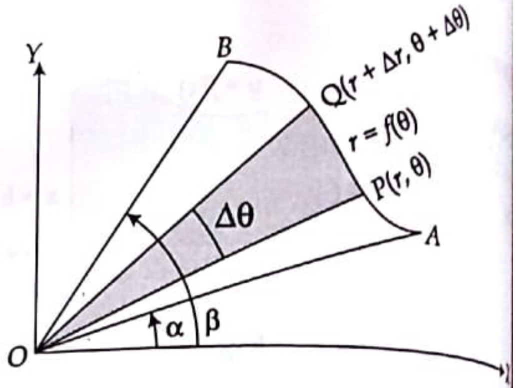


Fig. 6.5

$$\Delta A = \frac{1}{2} r^2 \Delta\theta.$$

Hence, the area A of the sector $OABO$, when θ goes from α to β, is given by

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta. \quad \text{...(6.36)}$$

6.4.3 Area of a Curve in Parametric Form

If a curve is in the parametric form as $x = \phi(t)$, $y = \psi(t)$, $a \le t \le b$, where $\phi(t)$, $\psi(t)$ are continuous on $[a, b]$, then the area bounded above by the curve $y = f(x)$, below by x-axis and the ordinates $x = \phi(a)$ and $x = \phi(b)$, is given by

$$A = \int_{\phi(a)}^{\phi(b)} y dx = \int_{a}^{b} \psi(t)\phi'(t) dt. \quad \text{...(6.37)}$$

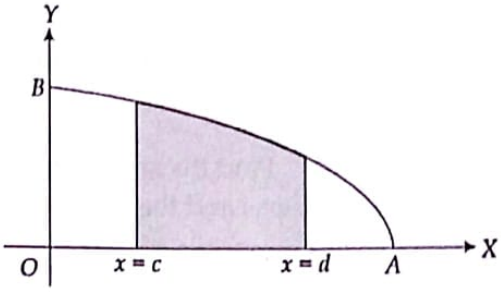
Example 6.11: Find the area bounded by the ellipse $x^2/a^2 + y^2/b^2 = 1$, the ordinates $x = c$, $x = d$ and the x-axis, in the first quadrant.

Solution: Equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. In the first quadrant, it is $y = \frac{b}{a}\sqrt{1 - \frac{x^2}{a^2}}$.

The required area A, as shown in Fig. 6.6, is

$$A = \int_c^d y\,dx = \frac{b}{a}\int_c^d (a^2 - x^2)^{1/2}dx$$

$$= \frac{b}{a}\left[\frac{1}{2}x(a^2 - x^2)^{1/2} + \frac{1}{2}a^2 \sin^{-1}\frac{x}{a}\right]_c^d$$



$$= \frac{b}{2a}\left[d(a^2 - d^2)^{1/2} - c(a^2 - c^2)^{1/2} + a^2\left(\sin^{-1}\frac{d}{a} - \sin^{-1}\frac{c}{a}\right)\right] \text{ sq. unit.}$$

Fig. 6.6

Example 6.12: Find the area bounded by the curve $x^2 = 4y$ and the straight line $x = 4y - 2$.

Solution: The points of intersection P and Q of the parabola $x^2 = 4y$ and the straight line $x = 4y - 2$ are given by

$$x = x^2 - 2, \text{ or } \quad x^2 - x - 2 = 0$$

or, $(x+1)(x-2) = 0$, or $\quad x = -1, 2$.

The required area A, as shown in Fig. 6.7, is

$$A = \int_{-1}^{2}\left(\frac{x+2}{4} - \frac{x^2}{4}\right)dx = \frac{1}{4}\int_{-1}^{2}(x + 2 - x^2)dx$$

$$= \frac{1}{4}\left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2} = \frac{1}{4}\left(\frac{10}{3} + \frac{7}{6}\right) = \frac{9}{8} \text{ sq. unit.}$$

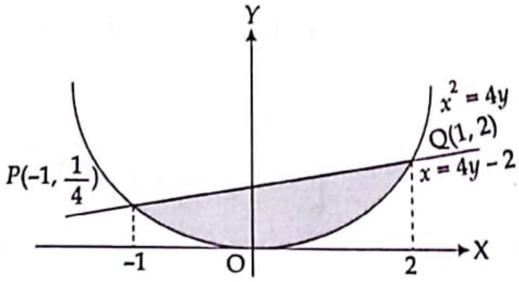


Fig. 6.7

Example 6.13: Find the area between the curve $y^2(2a - x) = x^3$ and its asymptotes.

Solution: The curve $y^2(2a - x) = x^3$ is symmetric about x-axis. It passes through the origin (0, 0) and for real and finite y, $x \in [0, 2a)$.

In fact, $y = 0$ is the cuspidal tangent at (0, 0) and $x = 2a$ is asymptote to this curve parallel to the y-axis. The required area as shown in Fig. 6.8, is

$$A = 2\int_0^{2a} \frac{x^{3/2}}{\sqrt{2a - x}}\,dx.$$

Put $x = 2a\sin^2\theta$, it gives $dx = 4a\sin\theta\cos\theta\,d\theta$.

Thus, $$A = 2\int_0^{\pi/2} \frac{(2a)^{3/2}\sin^3\theta . 4a\sin\theta\cos\theta}{(2a)^{1/2}\cos\theta}\,d\theta$$

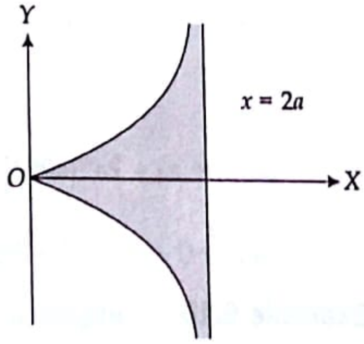


Fig. 6.8

$$= 16a^2 \int_0^{\pi/2} \sin^4\theta\, d\theta = 16a^2\, \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = 3\pi a^2 \text{ sq. unit.}$$

Example 6.14: Find the area of the region in the first quadrant that is bounded above by $y =$ and below by x-axis and the line $y = x - 2$.

Solution: The parabola $y = \sqrt{x}$ and the line $y = x - 2$, meet where

$\sqrt{x} = x - 2$ or, $x^2 - 5x + 4 = 0$ or $x = 1, 4$.

Only the value $x = 4$, satisfies the equation $\sqrt{x} = x - 2$. The value $x = 1$ is an extraneous roots. Corresponding to $x = 4$, the point is $Q(4, 2)$.

The required area A, as shown in Fig. 6.9, is given by

$$A = \int_0^2 \{(2 + y) - y^2\} dy, \quad \text{refer to (6.34)}$$

$$= \left[2y + \frac{y^2}{2} - \frac{y^3}{3}\right]_0^2 = 4 + \frac{4}{2} - \frac{8}{3} = \frac{10}{3} \text{ sq. unit.}$$

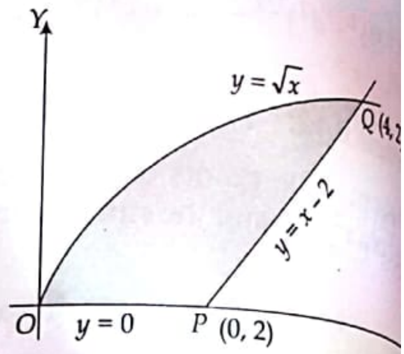


Fig. 6.9

Example 6.15: For the cycloid $x = a(t + \sin t)$, $y = a(1 - \cos t)$ obtain the area between its base a the portion of the curve from cusp to cusp.

Solution: The cycloid $x = a(t + \sin t)$, $y = a(1 - \cos t)$ is symmetrical about y-axis. The required a A, as shown in Fig. 6.10, is

$$A = 2\int_0^{2a} x\, dy = 2\int_0^{\pi} a(t + \sin t)a \sin t\, dt$$

$$= 2a^2 \int_0^{\pi} [t \sin t + \sin^2 t] dt$$

$$= 2a^2\left[(-t\cos t)_0^{\pi} - \int_0^{\pi} 1.(-\cos t)\, dt + \frac{1}{2}\int_0^{\pi} (1 - \cos 2t)\, dt\right]$$

$$= 2a^2(0 + \pi) + 2a^2 [\sin t]_0^{\pi} + a^2\left[t - \frac{1}{2}\sin 2t\right]_0^{\pi}$$

$$= 2\pi a^2 + 0 + \pi a^2 - 0 = 3\pi a^2 \text{ sq. unit.}$$

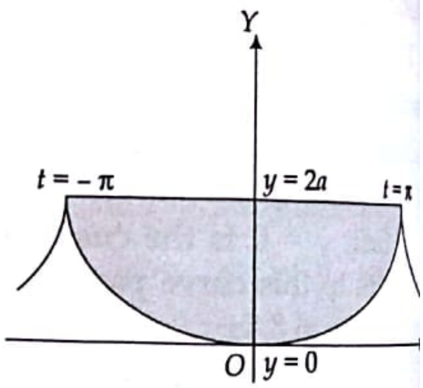


Fig. 6.10

ample 6.16: Find the area of the loop of the curve $x^3 + y^3 = 3axy$.

ution: The required area is as shown in Fig. 6.11.

To calculate the area we transform the curve $x^3 + y^3 = 3xy$ into polar form.

Put $x = r\cos\theta$, $y = r\sin\theta$, the equation becomes

$$r^3(\cos^3\theta + \sin^3\theta) = 3ar^2\cos\theta\sin\theta$$

$$r = \frac{3a\sin\theta\cos\theta}{\cos^3\theta + \sin^3\theta}.$$

The loop is completed as θ goes from 0 to $\frac{\pi}{2}$, thus the required area is given as

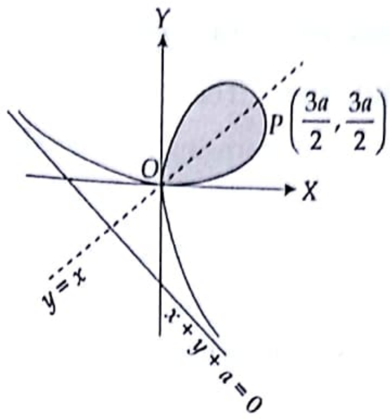


Fig. 6.11

$$A = \int_0^{\pi/2} \frac{1}{2} r^2 d\theta = \frac{9a^2}{2} \int_0^{\pi/2} \frac{\sin^2\theta\cos^2\theta}{(\cos^3\theta + \sin^3\theta)^2} d\theta.$$

Dividing the numerator and denominator of the integrand by $\cos^6\theta$, we obtain

$$A = \frac{9a^2}{2} \int_0^{\pi/2} \frac{\tan^2\theta\sec^2\theta d\theta}{(1 + \tan^3\theta)^2}.$$

Substituting $1 + \tan^3\theta = t$, which gives $3\tan^2\theta\sec^2\theta d\theta = dt$, we obtain

$$A = \frac{3a^2}{2} \int_1^{\infty} \frac{dt}{t^2} = \frac{3a^2}{2}\left[-\frac{1}{t}\right]_1^{\infty} = \frac{3a^2}{2} \text{ sq. unit.}$$

Example 6.17: Find the area of the region that lies inside the circle $r = a\cos\theta$ and outside the cardiod $r = a(1 - \cos\theta)$. AE imp

Solution: The required area is as shown in Fig. 6.12.

The points of intersection of the circle $r = a\cos\theta$ and the cardiod $r = a(1 - \cos\theta)$ are given by

$a\cos\theta = a(1 - \cos\theta)$, which gives $\cos\theta = \frac{1}{2}$ or $\theta = \pm\frac{\pi}{3}$.

Hence the area $A = \frac{1}{2} \cdot 2\int_0^{\pi/3} (r_1^2 - r_2^2) d\theta$

$$= a^2 \int_0^{\pi/3} [\cos^2\theta - (1 - \cos\theta)^2] d\theta$$

$$= a^2 \int_0^{\pi/3} (2\cos\theta - 1) d\theta = a^2[2\sin\theta - \theta]_0^{\pi/3}$$

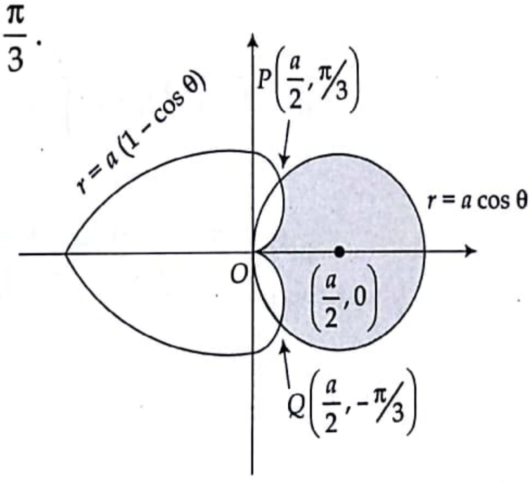


Fig. 6.12

$$= a^2\left[2\frac{\sqrt{3}}{2} - \frac{\pi}{3}\right] = \frac{a^2}{3}(3\sqrt{3} - \pi) \text{ sq. unit.}$$

Example 6.18: Find the area between the curve $r = a(\sec\theta + \cos\theta)$ and its asymptote.

Solution: The equation of the curve is $r = a(\sec\theta + \cos\theta)$.

Transforming it into cartesian form by substituting $x = r\cos\theta$ and $y = r\sin\theta$, we obtain

$$r = a\left(\frac{r}{x} + \frac{x}{r}\right), \text{ or } xr^2 = a(x^2 + r^2).$$

Substituting $r^2 = x^2 + y^2$ and simplifying, we get $y^2 = \dfrac{x^2(2a - x)}{x - a}$.

The curve is symmetrical about x-axis, $x = a$ is asymptote to the curve and $x = 2a$ is the t[angent] to the curve at the point $P(2a, 0)$. For real and finite y, $x \in (a, 2a]$. The required area, as shown [in Fig.] 6.13, is

$$A = 2\int_a^{2a} y\,dx = 2\int_a^{2a} x\left(\frac{2a - x}{x - a}\right)^{1/2} dx = 2\int_a^{2a} x\left(\frac{a - (x - a)}{x - a}\right)^{1/2} dx.$$

Substituting $(x - a) = a\sin^2 t$, which gives $dx = 2a\sin t\cos t\,dt$, and thus

$$A = 2\int_0^{\pi/2} a(1 + \sin^2 t)\left\{\frac{a - a\sin^2 t}{a\sin^2 t}\right\}^{1/2} 2a\sin t\cos t\,dt$$

$$= 4a^2\int_0^{\pi/2} (1 + \sin^2 t)\cos^2 t\,dt$$

$$= 4a^2\left[\int_0^{\pi/2} \cos^2 t\,dt + \int_0^{\pi/2} \sin^2 t\cos^2 t\,dt\right]$$

$$= 4a^2\left[\frac{1}{2}\cdot\frac{\pi}{2} + \frac{2-1}{2+2}\cdot\frac{2-1}{2}\cdot\frac{\pi}{2}\right] = 4a^2\left[\frac{\pi}{4} + \frac{\pi}{16}\right] = \frac{5}{4}\pi a^2 \text{ sq. units.}$$

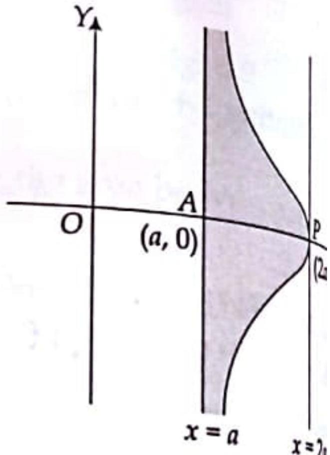


Fig. 6.13

EXERCISE 6.3

1. Find the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.
2. Find the area bounded by the parabola $y^2 = 4ax$ and its latus rectum.

3. Find the area bounded by the parabola $y^2 = x$ and the straight lines $x = 0, x = 1$ and $y - x = 0$.
4. Find the area enclosed by the curve $a^2x^2 = y^3(a - y)$.
5. Find the area enclosed by the curves $x^2 = 4ay$ and $y = \dfrac{8a^3}{x^2 + 4a^2}$.
6. Show that the area cut off a parabola by any double ordinate is two-third of the corresponding rectangle formed by that double ordinate and its distance from the vertex.
7. Find the area enclosed by the curve $x^2(x^2 + y^2) = a^2(x^2 - y^2)$.
8. Find the area bounded by the curve $xy^2 = 4a^2(2a - x)$ and its asymptote.
9. Find the area between the curve $x^2y^2 = a^2(y^2 - x^2)$ and its asymptotes.
10. Obtain the area common to two circles $r = a\sqrt{2}$ and $r = 2a\cos\theta$.
11. Find the area common to the two cardiodes $r = a(1 + \cos\theta)$ and $r = a(1 - \cos\theta)$.
12. Find the area common to the two ellipses $a^2x^2 + b^2y^2 = 1$ and $b^2x^2 + a^2y^2 = 1$, $(0 < a < b)$.
13. Show that the area enclosed by the curve $|x| + |y| = 2a$, $(a > 0)$ is $8a^2$.
14. Find the area of one loop of the curve $r = a\sin n\theta$, $n \in N$ and is odd. What is the total area of all the loops?
15. Show that the larger of the two areas into which the circle $x^2 + y^2 = 64a^2$ is divided by the parabola $y^2 = 12ax$ is $\dfrac{16}{3}a^2(8\pi - \sqrt{3})$.
16. Show that the area bounded by cissoid $x = a\sin^2 t$, $y = a\sin^3 t/\cos t$ and its asymptote is $3\pi a^2/4$.
17. Find the area enclosed by the curve $x = a\cos^3 t$, $y = b\sin^3 t$.
18. Find the area included between the portion of the cycloid from cusp to cusp and its base $x = a(t - \sin t)$, $y = a(1 - \cos t)$.
19. Find the area of the region in the first quadrant bounded on the left by the y-axis, below by the line $y = x/4$, above left by the curve $y = 1 + \sqrt{x}$, and above right by the curve $y = 2/\sqrt{x}$.
20. The Fig. 6.14 here shows the triangle AOC inscribed in the region cut from the parabola $y = x^2$ by the line $y = a^2$. Find the limit of the ratio of the area of the triangle to the area of the parabolic region as a approaches zero.

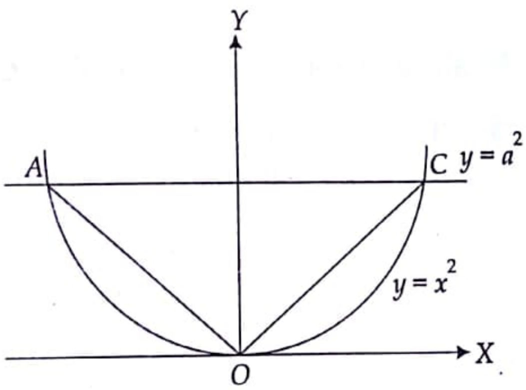


Fig. 6.14

5 ARC LENGTHS OF PLANE CURVES

this section we apply definite integral to find arc lengths of curves in a plane. We consider it for rves in cartesian, parametric and polar forms.

Cartesian Form: Consider a portion AB of the curve $y = f(x)$, $a \le x \le b$ and let dy/dx be contin... on the interval $[a, b]$. Then the arc length between A and B as shown in Fig. 6.15, is given by

$$s = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx, \quad \text{refer to (4.22)} \quad ...(6.38)$$

If the curve is defined as $x = g(y)$, $c \le y \le d$, then

$$s = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy, \quad \text{refer to (4.23)} \quad ...(6.39)$$

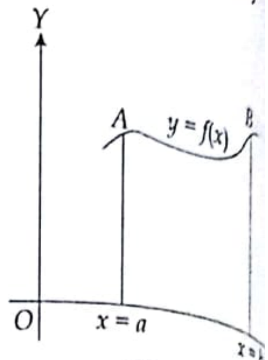


Fig. 6.15

Parametric Form: If x and y are given as functions of parameter t, say $x = \phi(t)$ and $y = \psi(t)$, $t_0 \le t \le t_1$, where $\phi(t)$, $\psi(t)$ have continuous first order derivatives on $[t_0, t_1]$, then the arc length is given by

$$s = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\, dt, \quad \text{refer to (4.24)} \quad ...(6$$

Polar Form: In case the curve is given in polar form $r = f(\theta)$, $\alpha \le \theta \le \beta$, where $f(\theta)$ has contin... first order derivative on $[\alpha, \beta]$, then the arc length is given by

$$s = \int_{\alpha}^{\beta} \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta, \quad \text{refer to (4.29)} \quad ...(6$$

$$s = \int_{\alpha}^{\beta} \sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}\, dr, \quad \text{refer to (4.30)} \quad ...(6$$

Example 6.19: Find the total length of the curve given by $x = a\cos^3\theta$, $y = a\sin^3\theta$.

Solution: The curve as shown in Fig. 6.16 is symmetrical about both the axes. Thus the t... length s of the curve, refer to (6.40), is

$$s = 4\int_0^{\pi/2} \sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= 4\int_0^{\pi/2} \sqrt{(-3a\cos^2\theta\sin\theta)^2+(3a\sin^2\theta\cos\theta)^2}\, d\theta$$

$$= 4\int_0^{\pi/2} 3a\cos\theta\sin\theta\, d\theta = 6a\int_0^{\pi/2} \sin 2\theta\, d\theta$$

$$= 6a\left(\frac{-\cos 2\theta}{2}\right)_0^{\pi/2} = 6a\left(\frac{1+1}{2}\right) = 6a.$$

Example 6.20: Find the length of the curve $y = \left(\frac{x}{2}\right)^{2/3}$ from $x = 0$ to $x = 2$.

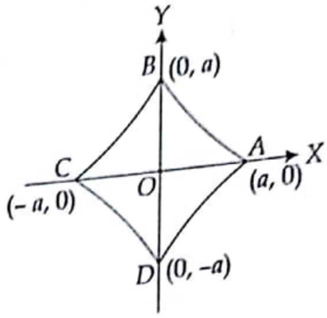


Fig. 6.16

Solution: The curve is $y = \left(\frac{x}{2}\right)^{2/3}, \quad 0 \le x \le 2$ ....(6.43)

The derivative $\frac{dy}{dx} = \frac{2}{3}\left(\frac{x}{2}\right)^{-\frac{1}{3}}\left(\frac{1}{2}\right) = \frac{1}{3}\left(\frac{x}{2}\right)^{-\frac{1}{3}}$ is discontinuous at $x = 0$, so we can't find the arc length using (6.38). Rewriting (6.43) as

$$x = 2y^{3/2}, \quad 0 \le y \le 1.$$

The derivative $\frac{dx}{dy} = 3y^{1/2}$ is continuous in the interval $[0, 1]$. Thus the arc length, refer to (6.39) is

$$s = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^1 \sqrt{1 + 9y}\, dy = \left[\frac{2}{3}\frac{(1+9y)^{3/2}}{9}\right]_0^1 = \frac{2}{27}(10\sqrt{10} - 1).$$

Example 6.21: Find the arc length of the loop of the curve

$$9ay^2 = (x - 2a)\,(x - 5a)^2.$$

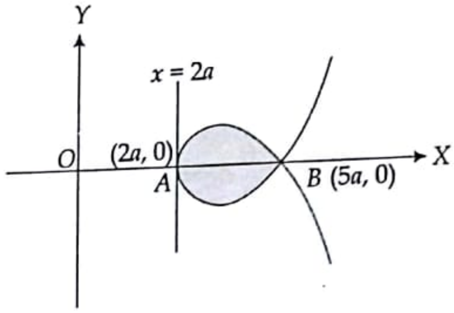


Fig. 6.17

Solution: The curve

$$9ay^2 = (x - 2a)\,(x - 5a)^2 \qquad ...(6.44)$$

is symmetrical about x-axis, and the loop of the curve lies between $x = 2a$ and $x = 5a$ as shown in Fig. 6.17.

Differentiating w.r.t x, we get

$$18ay\frac{dy}{dx} = (x - 5a)^2 + 2(x - 2a)(x - 5a) \text{ or, } \frac{dy}{dx} = \frac{(x - 5a)(x - 3a)}{6ay}.$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{(x - 5a)^2(x - 3a)^2}{36a^2y^2} = \frac{(x - a)^2}{4a(x - 2a)}, \text{ using (6.44) and simplifying.}$$

Thus the arc length of the loop of the curve, refer to (6.38), is

$$s = 2\int_{2a}^{5a} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = 2\int_{2a}^{5a} \frac{(x - a)}{2\sqrt{a}\,\sqrt{x - 2a}}\, dx.$$

Put $x - 2a = t$, which gives $dx = dt$, thus

$$s = \frac{1}{\sqrt{a}}\int_0^{3a} \frac{a + t}{\sqrt{t}}\, dt = \frac{1}{\sqrt{a}}\int_0^{3a}\left(at^{-\frac{1}{2}} + t^{\frac{1}{2}}\right) dt = \frac{1}{\sqrt{a}}\left[2a\,t^{\frac{1}{2}} + \frac{2}{3}t^{3/2}\right]_0^{3a} = \frac{1}{\sqrt{a}}\left[2a\sqrt{3a} + \frac{2}{3}(3a)^{3/2}\right] = 4\sqrt{3}a.$$

Example 6.22: Find the perimeter of the cardiod $r = a(1 + \cos\theta)$ and show that the upper half of the curve is bisected at $\theta = \pi/3$.

Solution: The curve $r = a(1 + \cos\theta)$ is symmetrical about the line $\theta = 0$, as shown in Fig. 6.18.

The perimeter, refer to (6.14), is

$$s = 2\int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= 2\int_0^{\pi} [a^2(1+\cos\theta)^2 + (-a\sin\theta)^2]^{\frac{1}{2}}\, d\theta$$

$$= 2\sqrt{2}a\int_0^{\pi} (1+\cos\theta)^{\frac{1}{2}}\, d\theta$$

$$= 4a\int_0^{\pi} \cos\frac{\theta}{2}\, d\theta = 8a\left[\sin\frac{\theta}{2}\right]_0^{\pi} = 8a.$$

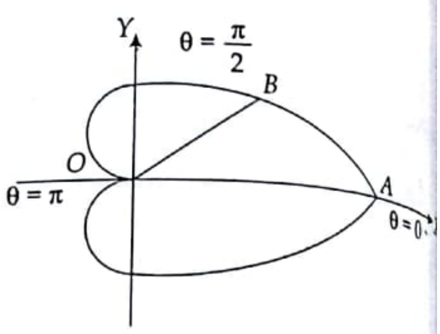


Fig. 6.18

The length of the upper half is $4a$, and let the upper half is bisected at $\theta = \theta_1$, then

$$\int_0^{\theta_1} \left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{\frac{1}{2}} d\theta = 2a, \text{ which gives}$$

$$2a\int_0^{\theta_1} \cos\frac{\theta}{2} d\theta = 2a, \text{ or } \int_0^{\theta_1} \cos\frac{\theta}{2} d\theta = 1, \text{ or, } \sin\frac{\theta_1}{2} = \frac{1}{2}, \quad \text{or} \quad \theta_1 = \pi/3.$$

EXERCISE 6.4

1. Find the length of the portion of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ from $x = 0$ to $x = a$ in the first quadrant.
2. Find the parameter of the curve $x^2 + y^2 = a^2$.
3. Prove that the length of the arc of the parabola $y^2 = 4ax$ cut off by its latus rectum is $a\left[\sqrt{2} + \ln\left(\sqrt{2}+1\right)\right]$.
4. Find the perimeter of the one loop of the curve $x^2(a^2 - x^2) = 8a^2y^2$.
5. Find the entire length of the curve given by $r = a\sin^3(\theta/3)$.
6. Prove that the whole length of an arc of the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$ is $8a$.
7. Prove that the cardiod $r = a(1 + \cos\theta)$ is divided by the line $4r\cos\theta = 3a$ into two parts such that the length of the arcs on either side of this line are equal.
8. Show that on the curve $x = a(2\cos t - \cos 2t)$, $y = a(2\sin t - \sin 2t)$, $s = 16a\sin^2(\psi/6)$ where s is the length of the arc of the curve from the point $t = 0$ to the point where the tangent makes an angle ψ with the tangent at the point $t = 0$.
9. Find the length of the arc of equiangular spiral $r = ae^{\theta\cot\alpha}$ between the points for which radii vectors are r_1 and r_2.

10. Find the length of the arc of the parabola $(l/r) = 1 + \cos\theta$ cut off by its latus rectum.

6.6 VOLUMES OF SOLIDS OF REVOLUTION

The solids of revolution are solids which can be generated by revolving plane regions about axes, e.g., a sphere. We can find their volume by the application of definite integral.

6.6.1 Cartesian Co-ordinates

(a) Revolution about x-axis

Let the area bounded by the curve $y = f(x)$ from $x = a$ to $x = b$ and the x-axis, as shown in Fig. 6.19, be rotated about the x-axis through four right angles, which generates a solid of revolution. Divide the arc AB into n parts by considering subintervals $[x_0, x_1]$, $[x_1, x_2]$ ..., $[x_{n-1}, x_n]$; $a = x_0 < x_1 < x_2 ... < x_n = b$ and let $P'Q' = \Delta x_i = x_{i+1} - x_i$, $i = 0, 1, 2, ... n - 1$.

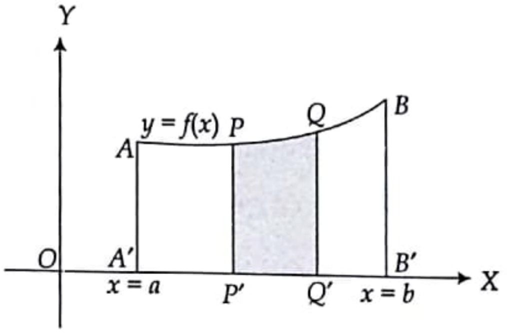


Fig. 6.19

Then the volume ΔV_i of the solid formed by rotating the strip $P'Q'QP$ about x-axis through four right angles can be approximated as $\Delta V_i = \pi y_i^2 \Delta x_i$, where $P'P = y_i$.

Thus an estimate of the total volume for n strips is given by

$$\sum_i \pi y_i^2 \Delta x_i \quad ...(6.45)$$

Let $n \to \infty$ such that $\max \Delta x_i \to 0$, the summation (6.45) tends to coincide with the actual volume of the solid of revolution and is given by

$$V = \int_a^b \pi y^2 dx \quad ...(6.46)$$

(b) Revolution about the y-axis

Similarly if the area bounded by curve $x = g(y)$, the y-axis, and the lines $y = c$ and $y = d$, as shown in Fig. 6.20, is revolved about the y-axis, then the volume of the solid of revolution generated is given by

$$V = \int_c^d \pi x^2 \, dy \quad ...(6.47)$$

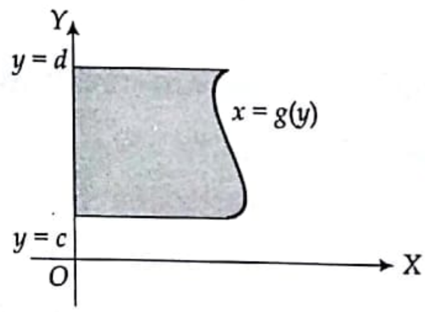


Fig. 6.20

(c) Revolution about the line y = c

In case the area bounded by the curve $y = f(x)$, the line $y = c$ and the lines $x = a$, $x = b$, is revolved about the line $y = c$, then the volume of the solid of revolution is given by

$$V = \int_a^b \pi (y - c)^2 dx. \quad ...(6.48)$$

(d) Revolution about the line $x = a$

Similarly, if the area bounded by the curve $x = g(y)$, the line $x = a$ and the lines $y = c$, $y = d$ is revolved about the line $x = a$, then the volume of solid of revolution formed is given by

$$V = \int_c^d \pi(x-a)^2 dy. \quad \text{...(6.4}$$

(e) Revolution about any line

To obtain the volume of the solid generated by rotation about any axis CD the area bounded by the curve AB, the axis CD and the perpendicular AC and BD on the axis CD, take O, any fix point on the line CD, as the origin and the line OCD as x-axis. A line through O and perpendicular to OD is taken as the y-axis.

If $P(x, y)$ be any point on the curve AB, and $PN \perp OX$, as shown in Fig. 6.21, then the required volume of revolution

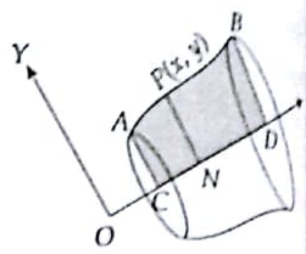


Fig. 6.21

$$V = \pi\int_a^b y^2 dx = \pi\int_{OC}^{OD} (PN)^2\, d(ON), \quad \text{...(6.50)}$$

where $OC = a$ and $OD = b$

(f) Revolving area bounded by curves $y = f_1(x)$ and $y = f_2(x)$

The volume of the solid generated by rotating about x-axis the region bounded by the curve $y = f_1(x)$, $y = f_2(x)$ and the ordinates $x = a$ and $x = b$ is

$$V = \pi\int_a^b [f_1(x)^2 - f_2(x)^2]dx, \quad \text{...(6.51}$$

when $f_1(x) \geq f_2(x)$, for $x \in [a, b]$.

6.6.2 Polar Co-ordinates

(a) Revolution about the initial line $\theta = 0$

The elementary area for the sector OPQ, as shown in Fig. 6.22, is $r^2\,\Delta\theta/2$, and its centroid can be considered to lie on OP at $\left(\frac{2}{3}r\cos\theta, \frac{2}{3}r\sin\theta\right)$. The elementary volume generated when this area is evolved about initial line $\theta = 0$ is $\Delta V = \frac{2\pi}{3}r^3 \sin\theta\Delta\theta$. Hence the volume V of the solid generated by rotating the sector OAB about the initial line, when the area OAB is bounded by the lines $\theta = \alpha$, $\theta = \beta$ and the curve $r = f(\theta)$ is given by

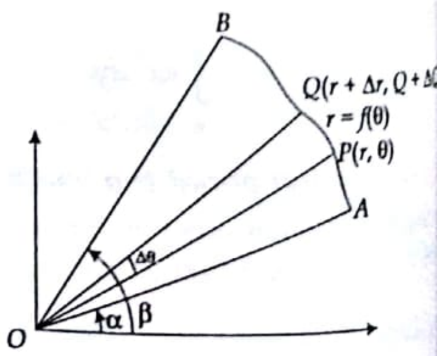


Fig. 6.22

$$V = \frac{2\pi}{3}\int_{\alpha}^{\beta} r^3 \sin\theta\, d\theta. \qquad ...(6.52)$$

In case the area OAB is rotated about the line $\theta = \pi/2$, then volume V generated is given by

$$V = \frac{2\pi}{3}\int_{\alpha}^{\beta} r^3 \cos\theta\, d\theta. \qquad ...(6.53)$$

6.6.3 Parametric Co-ordinates

When the curve is in parametric form $x = \phi(t)$, $y = \psi(t)$, $t_0 \le t \le t_1$, then the volume of the solid generated by rotating the area bounded by the curve about x-axis is

$$V = \pi\int_{t_0}^{t_1} [\psi(t)]^2 \phi'(t)\, dt \qquad ...(6.54)$$

and, about y-axis is

$$V = \pi\int_{t_0}^{t_1} [\phi(t)]^2 \psi'(t)\, dt \qquad ...(6.55)$$

all

Example 6.23: A segment is cut off a sphere of radius a by a plane at a distance $a/2$ from the centre. Show that the volume of the segment is (5/32) of the volume of the sphere.

Solution: Consider a circle $x^2 + y^2 = a^2$ of radius a. The sphere of radius a is generated by revolving the semi-circular area $AOBDA$ about x-axis, while the segment is generated by revolving the area $ALMA$ about the x-axis, where $OL = \frac{a}{2}$ as shown in Fig. 6.23.

The volume of the sphere is

$$V = \pi\int_{-a}^{a} y^2\, dx = \pi\int_{-a}^{a} (a^2 - x^2)\, dx = 2\pi\left[a^2 x - \frac{x^3}{3}\right]_0^a = \frac{4}{3}\pi a^3.$$

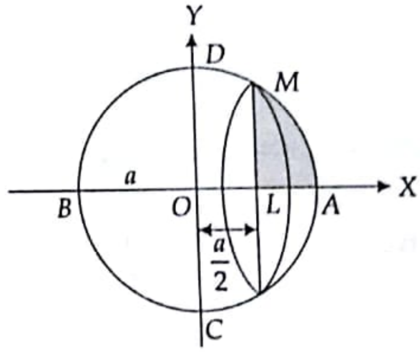


Fig. 6.23

Next, the volume of the segment is

$$V' = \pi\int_{\frac{a}{2}}^{a} y^2 dx = \pi\int_{\frac{a}{2}}^{a} (a^2 - x^2)\, dx = \pi\left[a^2 x - \frac{x^3}{3}\right]_{\frac{a}{2}}^{a}$$

$$= \pi\left[a^3 - \frac{a^3}{3} - \frac{a^3}{2} + \frac{a^3}{24}\right] = \frac{5}{24}\pi a^3 = \frac{5}{32}V.$$

This proves the result.

Example 6.24: Find the volume of the solid generated by revolving the region bounded by the curves $y = 3 - x^2$ and $y = -1$ about the line $y = -1$.

Solution: The region $BCAB$ bounded by the parabola $y = 3 - x^2$ and the line $y = -1$, as shown in Fig. 6.24, is revolved about the line $y = -1$.

The volume generated is

$$V = \pi \int_{-2}^{2} (1 + y)^2 dx = \pi \int_{-2}^{2} (1 + 3 - x^2)^2 dx$$

$$= 2\pi \int_{0}^{2} (4 - x^2)^2 dx$$

$$= 2\pi \int_{0}^{2} (16 - 8x^2 + x^4) dx = 2\pi \left[16x - \frac{8x^3}{3} + \frac{x^5}{5} \right]_0^2$$

$$= 2\pi \left[32 - \frac{64}{3} + \frac{32}{5} \right] = \frac{512}{15}\pi$$

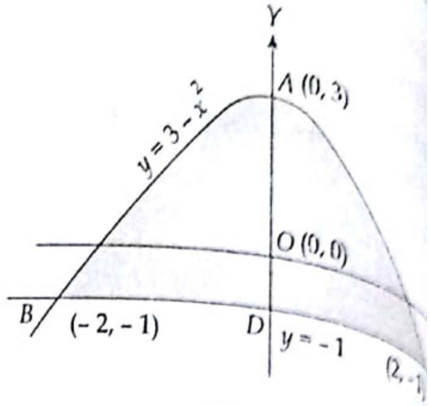


Fig. 6.24

Example 6.25: Find the volume of the solid of revolution obtained by revolving the reg bounded between the curves $y^2 = x^3$ and $x^2 = y^3$ about the x-axis.

Solution: The two curves are $y^2 = x^3$ and $x^2 = y^3$.

Their points of intersection are given by

$$x^2 = y^3 = (x^{3/2})^3 = x^{9/2}$$

or,

$$x^2(1 - x^{5/2}) = 0, \text{ or } \quad x = 0, 1.$$

Thus points of intersection are $O(0, 0)$ and $A(1, 1)$.

The region $OPAR$ bounded by the curves is shown in Fig. 6.25.

The required volume of revolution is given by

$$V = \pi \int_{0}^{1} (y_1^2 - y_2^2) dy,$$

where y_1 and y_2 are the upper and lower curves of the bounded region.

Thus,

$$V = \pi \int_{0}^{1} \left[(x^{2/3})^2 - (x^{3/2})^2 \right] dx$$

$$= \pi \int_{0}^{1} (x^{4/3} - x^3) dx$$

$$= \pi \left[\frac{3}{7} x^{7/3} - \frac{x^4}{4} \right]_0^1 = \pi \left[\frac{3}{7} - \frac{1}{4} \right]$$

$$= \frac{5\pi}{28} \text{ cubic unit.}$$

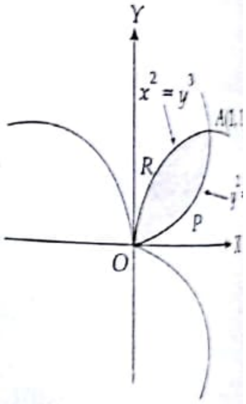


Fig. 6.25

Example 6.26: The area cut off the parabola $y^2 = 4ax$ by the chord joining the vertex to one end of the latus rectum is revolved about the chord. Find the volume of the solid generated.

Solution: The equation of the chord passing through the points $O(0, 0)$ and $L(a, 2a)$ is $y = 2x$. The region bounded by the parabola $y^2 = 4ax$ and the chord $y = 2x$ is as shown in Fig. 6.26.

Let $P(x, y)$ be any point on the parabola and PN be the perpendicular from P on OL, then

$$PN = \frac{|y - 2x|}{\sqrt{1^2 + (-2)^2}} = \frac{1}{\sqrt{5}}\,|y - 2x|.$$

Next, $ON = \sqrt{OP^2 - PN^2}$

$$= \sqrt{x^2 + y^2 - \frac{1}{5}(y - 2x)^2}$$

$$= \sqrt{\frac{5x^2 + 5y^2 - y^2 - 4x^2 + 4xy}{5}}$$

$$= \frac{x + 2y}{\sqrt{5}}.$$

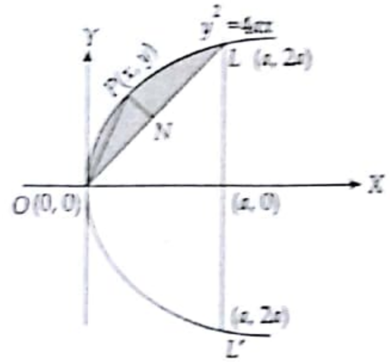


Fig. 6.26

Therefore, $d(ON) = \frac{1}{\sqrt{5}}(dx + 2dy).$

From the equation $y^2 = 4ax$, we have, $2y\,dy = 4a\,dx$ or $dy = \frac{2a}{y}dx$. Thus,

$$d(ON) = \frac{1}{\sqrt{5}}\left(dx + \frac{4a}{y}dx\right) = \frac{1}{\sqrt{5}}\left(1 + \frac{4a}{y}\right)dx.$$

The required volume generated, refer to (6.50), is

$$V = \int_0^a \pi(PN)^2\, d(ON) = \pi\int_0^a \frac{(y - 2x)^2}{5}\,\frac{1}{\sqrt{5}}\left(1 + \frac{4a}{y}\right)dx$$

$$= \frac{\pi}{5\sqrt{5}}\int_0^a \left[(y - 2x)^2 + \frac{4a}{y}(y - 2x)^2\right]dx$$

$$= \frac{\pi}{5\sqrt{5}}\int_0^a \left[y^2 + 4x^2 - 4xy + 4ay + \frac{16ax^2}{y} - 16ax\right]dx$$

$$= \frac{\pi}{5\sqrt{5}}\int_0^a \left[4ax + 4x^2 - 8x\sqrt{ax} + 8a\sqrt{ax} + 8x\sqrt{ax} - 16ax\right]dx$$

$$= \frac{\pi}{5\sqrt{5}} \int_0^a \left[4ax + 4x^2 + 8a\sqrt{ax} - 16ax\right] dx$$

$$= \frac{\pi}{5\sqrt{5}} \left[2ax^2 + \frac{4}{3}x^3 + \frac{16}{3}a^{3/2}x^{3/2} - 8ax^2\right]_0^a$$

$$= \frac{\pi a^3}{5\sqrt{5}} \left[2 + \frac{4}{3} + \frac{16}{3} - 8\right] = \frac{2\pi a^3}{15\sqrt{5}} \text{ cubic unit.}$$

Example 6.27: The area lying inside the cardioid $r = 2a(1 + \cos\theta)$ and outside the parabola $r(1 + \cos\theta) = 2a$ is revolved about the initial line $\theta = 0$. Find the volume of the solid generated.

Solution: The points of intersection of the two curves, the cardiod $r = 2a(1 + \cos\theta)$ and the parabola $r = 2a/(1 + \cos\theta)$, are given by

$$2a(1 + \cos\theta) = \frac{2a}{(1 + \cos\theta)}$$

or, $(1 + \cos\theta)^2 = 1$, or $\cos\theta = 0$, or $\theta = \frac{\pi}{2}, \frac{-\pi}{2}$.

The bounded region is shown in Fig. 6.27.

The required volume is generated by rotating just the upper half (or, the lower half) portion for which θ varies from 0 to $\frac{\pi}{2}$. Therefore, $V = \frac{2\pi}{3}\int_0^{\pi/2} (r_1^3 - r_2^3)\sin\theta\, d\theta$, where r_1 is the value of r for the cardiod and r_2 is the value of r for the parabola, thus

$$V = \frac{2\pi}{3}\int_0^{\pi/2}\left[8a^3(1 + \cos\theta)^3 - \frac{8a^3}{(1 + \cos\theta)^3}\right]\sin\theta\, d\theta$$

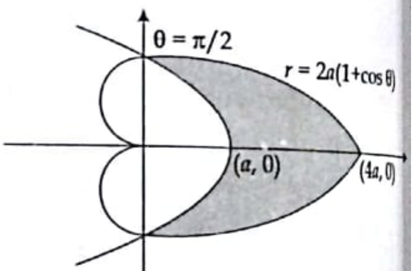


Fig. 6.27

$$= \frac{16\pi a^3}{3}\int_0^{\pi/2}\left[(1 + \cos\theta)^3 - (1 + \cos\theta)^{-3}\right]\sin\theta\, d\theta$$

$$= \frac{-16\pi a^3}{3}\left[\frac{(1 + \cos\theta)^4}{4} - \frac{(1 + \cos\theta)^{-2}}{-2}\right]_0^{\pi/2}$$

$$= \frac{-16\pi a^3}{3}\left[\frac{1}{4} + \frac{1}{2} - 4 - \frac{1}{8}\right] = 18\pi a^3.$$

Example 6.28: The area bounded by an arc of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, $0 \le \theta \le 2\pi$ and the x-axis is revolved around x-axis. Find the volume of the solid generated.

Solution: The area bounded by an arc of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ is shown in Fig. 6.28.

The volume generated when this area is revolved around x-axis is given by

$$V = \pi \int_0^{2\pi} y^2\, dx$$

$$= \pi \int_0^{2\pi} a^2(1 - \cos\theta)^2\, a(1 - \cos\theta)d\theta = \pi a^3 \int_0^{2\pi} (1 - \cos\theta)^3 d\theta$$

$$= \pi a^3 \int_0^{2\pi} (1 - 3\cos\theta + 3\cos^2\theta - \cos^3\theta)d\theta.$$

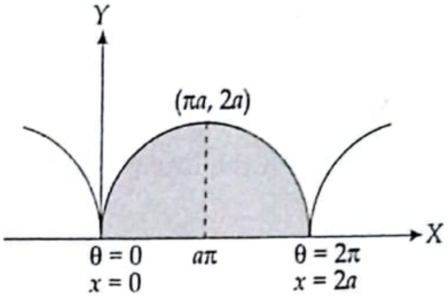


Fig. 6.28

$$= \pi a^3 \int_0^{2\pi} \left[1 - 3\cos\theta + \frac{3}{2}(1 + \cos 2\theta) - \frac{1}{4}(\cos 3\theta + 3\cos\theta)\right] d\theta$$

$$= \pi a^3 \int_0^{2\pi} \left[\frac{5}{2} - \frac{15}{4}\cos\theta + \frac{3}{2}\cos 2\theta - \frac{1}{4}\cos 3\theta\right] d\theta$$

$$= \pi a^3 \left[\frac{5}{2}\theta - \frac{15}{4}\sin\theta + \frac{3}{4}\sin 2\theta - \frac{1}{12}\sin 3\theta\right]_0^{2\pi} = 5\pi^2 a^3 \text{ cubic units.}$$

EXERCISE 6.5

1. Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$, $x = 4$ about the line $y = 1$.
2. Find the volume of the solid generated by revolving about x-axis the area bounded by the parabola $y = (x^2/4) + 2$ and the straight line $8y = 5x + 14$.
3. The area bounded by the parabola $y^2 = 4x$ and the straight line $4x - 3y + 2 = 0$ is rotated about the y-axis. Find the volume of the solid generated.
4. The ellipse $(x^2/a^2) + (y^2/b^2) = 1$ is revolved about y-axis. Find the volume of the solid generated.
5. A cone is formed by revolving about y-axis the line joining the origin to the point (a, b). Find the volume of the cone generated.
6. Find the volume of the solid generated by the revolution of the circle $x^2 + (y - b)^2 = a^2$, $b > a$ about the axis of x.
7. Find the volume of the solid generated by revolving the area included between the curve $(y + 8)/x = x - 2$ and the x-axis about the line $x + 5 = 0$.
8. The loop of the curve $r = a\cos\theta$ lying between $\theta = -\pi/6$ and $\theta = \pi/6$ revolves about the initial line. Find the volume of the solid generated.
9. The area of the cardioid $r = a(1 + \cos\theta)$ included between $\theta = -\pi/2$ and $\theta = \pi/2$ is rotated about the line $\theta = \pi/2$. Show that the volume generated is $2[2 + (5\pi/8)]\pi a^3$.

10. Find the volume of the solid obtained by revolving the cissoid $y^2(2a - x) = x^3$ about its asymptotes.
11. Find the volume of the solid generated by revolving the astroid $x = \cos^3\theta$, $y = a\sin^3\theta$ about x-axis.
12. Show that the volume of the solid generated by revolving the leminiscate $r^2 = a^2\cos 2\theta$ about the line $\theta = \pi/2$, is $\pi a^3/4\sqrt{2}$.

6.7 SURFACE AREAS OF SOLIDS OF REVOLUTION

In this section, we find the surface areas of the solids of revolution by the application of definite integral. We discuss the various cases as given below.

6.7.1 Cartesian Co-ordinates

(a) Revolution about x-axis

An estimate of the curved surface area of the slice generated may be found by assuming that it is formed by rotating the chord PQ instead of the arc PQ, refer to Fig. 6.19, about x-axis. Under this assumption the solid formed is a slice of a cone with surface area approximately given by

$$\Delta S_i = 2\pi y_i \Delta s_i$$

where $\Delta s_i = \widehat{PQ}$. Hence the curved surface area S of the solid of revolution is given by

$$S = \int_{x=a}^{x=b} 2\pi y\, ds = \int_a^b 2\pi y \frac{ds}{dx}\, dx = 2\pi \int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx, \quad \text{refer to (4.22)} \qquad \text{...(6.56)}$$

(b) Revolution about the y-axis

Similarly if the area bounded by the curve $x = g(y)$, the y-axis and the lines $y = c$ and $y = d$ is revolved about the y-axis, refer to Fig. (6.20), then surface area of the solid of revolution generated is given by

$$S = \int_c^d 2\pi x\, ds = 2\pi \int_c^d x \frac{ds}{dy}\, dy = 2\pi \int_c^d x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy, \quad \text{refer to (4.23)} \qquad \text{...(6.57)}$$

6.7.2 Polar Co-ordinates

(a) Revolution about the initial line θ = 0

The formula for the surface area of revolution in terms of polar co-ordinates, when the area is revolved around the initial line $\theta = 0$, is given by

$$S = \int 2\pi y\, ds = \int_{\theta=\alpha}^{\beta} 2\pi y \frac{ds}{d\theta}\, d\theta = \int_\alpha^\beta 2\pi r \sin\theta \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}\, d\theta, \quad \text{refer to (4.29).} \qquad \text{...(6.58)}$$

(b) Revolution about the line $\theta = \pi/2$

Similarly when the area is revolved around the line $\theta = \frac{\pi}{2}$, the surface area S is given by

$$S = \int 2\pi x\, ds = \int_{\theta=\alpha}^{\beta} 2\pi x \frac{ds}{d\theta}\, d\theta = \int_{\alpha}^{\beta} 2\pi r \cos\theta \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}\, d\theta, \qquad ...(6.59)$$

refer to (4.29).

6.7.3 Parametric Co-ordinates

In terms of parametric co-ordinates $x = \phi(t)$, $y = \psi(t)$, $t_0 \le t \le t_1$, the surface of the solid of revolution obtained by rotating the area about the x-axis is,

$$S = \int 2\pi y\, ds = \int 2\pi y \frac{ds}{dt}\, dt = \int_{t_0}^{t_1} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad ...(6.60)$$

and about y-axis is

$$S = \int_{t_0}^{t_1} 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt. \qquad ...(6.61)$$

refer to (4.24).

Example 6.29: Find the surface area of the solid generated by revolving the loop of the curve $3ay^2 = x(x-a)^2$ about x-axis.

Solution: The loop of the given curve extends from $x = 0$ to $x = a$ as shown in Fig. 6.29.

Therefore, the required surface area S obtained by revolving the loop about x-axis is

$$S = \int 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \qquad ...(6.62)$$

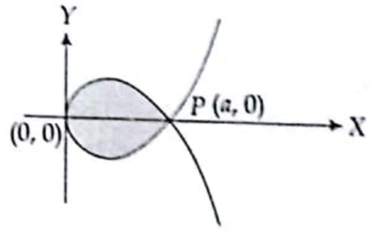


Fig. 6.29

For the given equation $3ay^2 = x(x-a)^2$, we have

$$6ay\frac{dy}{dx} = (x-a)^2 + 2x(x-a) \text{ or, } \frac{dy}{dx} = \frac{(x-a)(3x-a)}{6ay}.$$

Thus,
$$y\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = y\sqrt{1 + \frac{(x-a)^2(3x-a)^2}{36a^2y^2}}$$

$$= \sqrt{y^2 + \frac{(x-a)^2(3x-a)^2}{36a^2}} = \sqrt{\frac{x(x-a)^2}{3a} + \frac{(x-a)^2(3x-a)^2}{36a^2}}$$

$$= \frac{(x-a)}{6a}\sqrt{12ax + (3x-a)^2} = \frac{(x-a)(3x+a)}{6a} = \frac{3x^2 - 2ax - a^2}{6a}.$$

Hence, (6.62) becomes

$$S = \frac{\pi}{3a}\int_0^a (3x^2 - 2ax - a^2)dx = \frac{\pi}{3a}[x^3 - ax^2 - a^2x]_0^a = \frac{-\pi a^2}{3}.$$

Thus $S = \frac{\pi a^2}{3}$, taking the numerical value.

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Example 6.30: Find the surface area of the solid generated by revolving an arc of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, $0 \le \theta \le 2\pi$ about the x-axis.

Solution: The area bounded by the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, $0 \le \theta \le 2\pi$ and the x-axis is shown in Fig. 6.28. The surface area of the solid generated by revolving the area about x-axis is given by

$$S = 2\pi\int_0^{2\pi} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

We have, $$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1-\cos\theta)} = \frac{2a\sin\theta/2\cos\theta/2}{2a\sin^2\theta/2} = \cot\theta/2.$$ Thus

$$\therefore \quad S = 2\pi\int_0^{2\pi} a(1-\cos\theta)\sqrt{1+\cot^2\frac{\theta}{2}}\,a(1-\cos\theta)d\theta.$$

$$= 8\pi a^2\int_0^{2\pi}\sin^4\frac{\theta}{2}\operatorname{cosec}\frac{\theta}{2}d\theta = 8\pi a^2\int_0^{2\pi}\sin^3\frac{\theta}{2}d\theta = 2\pi a^2\int_0^{2\pi}\left(3\sin\frac{\theta}{2} - \sin\frac{3\theta}{2}\right)d\theta$$

$$= 2\pi a^2\left[-6\cos\frac{\theta}{2} + \frac{2}{3}\cos\frac{3\theta}{2}\right]_0^{2\pi} = 2\pi a^2\left[6 - \frac{2}{3} + 6 - \frac{2}{3}\right] = \frac{64}{3}\pi a^2.$$

Example 6.31: The lemniscate $r^2 = a^2\cos 2\theta$ revolves about a tangent at the pole. Show that the surface generated is $4\pi a^2$.

Solution: The curve $r^2 = a^2\cos 2\theta$ has two loops so the required surface area is twice the surface area generated by one loop when it revolves around the tangent, say MOM' as shown in Fig. 6.

Let $P(r, \theta)$ be any point on the loop of the lemniscate and $PN \perp MOM'$, then $PN = r\sin\left(\theta + \frac{\pi}{4}\right)$

Also from $r^2 = a^2\cos 2\theta$, we obtain

$$2r\frac{dr}{d\theta} = -2a^2\sin 2\theta, \text{ or } \frac{dr}{d\theta} = \frac{-a^2}{r}\sin 2\theta.$$

Then surface area S generated is given by

$$S = 2\int_{-\pi/4}^{\pi/4} 2\pi\, PN\, ds = 4\pi\int_{-\pi/4}^{\pi/4} r\sin\left(\theta + \frac{\pi}{4}\right)\frac{ds}{d\theta}\,d\theta$$

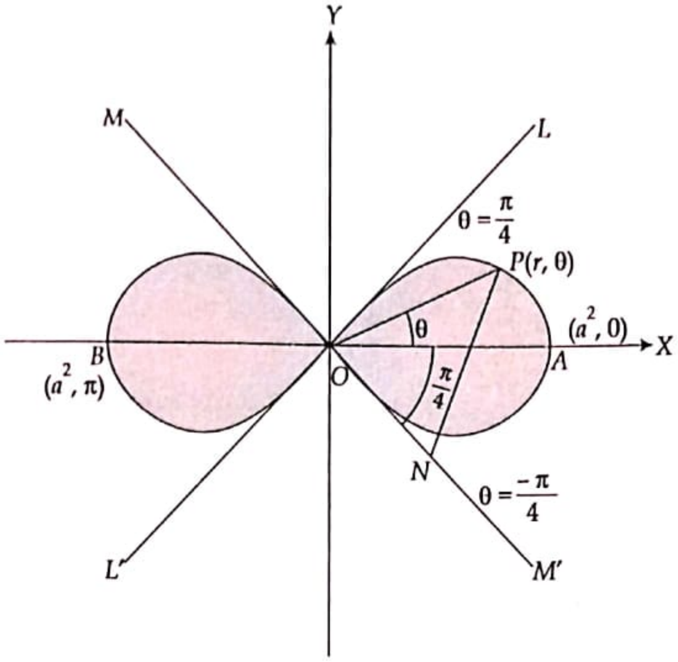

Fig. 6.30

$$= 4\pi \int_{-\pi/4}^{\pi/4} r \sin\left(\theta + \frac{\pi}{4}\right)\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{\frac{1}{2}} d\theta, \quad \text{refer to (4.29)}$$

$$= 4\pi \int_{-\pi/4}^{\pi/4} r \sin\left(\theta + \frac{\pi}{4}\right)\left[r^2 + \frac{a^4}{r^2}\sin^2 2\theta\right]^{\frac{1}{2}} d\theta$$

$$= 4\pi \int_{-\pi/4}^{\pi/4} \sin\left(\theta + \frac{\pi}{4}\right)\sqrt{r^4 + a^4 \sin^2 2\theta}\, d\theta$$

$$= 4\pi \int_{-\pi/4}^{\pi/4} \sin\left(\theta + \frac{\pi}{4}\right)\sqrt{a^4 \cos^2 2\theta + a^4 \sin^2 2\theta}\, d\theta = 4\pi a^2 \int_{-\pi/4}^{\pi/4} \sin\left(\theta + \frac{\pi}{4}\right) d\theta$$

$$= 4\pi a^2\left[-\cos\left(\theta + \frac{\pi}{4}\right)\right]_{-\pi/4}^{\pi/4} = -4\pi a^2\left[\cos\frac{\pi}{2} - \cos 0\right] = 4\pi a^2.$$

Example 6.32: Find the surface of the solid generated by the revolution of the asteroid $x = a\cos^3 t, y = a\sin^3 t$ about the y-axis.

Solution: The asteroid is symmetrical about the axes, as shown in Fig. 6.31.

For its portion in the first quadrant, $0 \le t \le \pi/2$, we have

$$\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2} = \sqrt{9a^2\cos^4 t\sin^2 t+9a^2\sin^4 t\cos^2 t}$$

$$= 3a\sin t\cos t.$$

Hence, the required surface area is

$$S = 2\times 2\pi\int_0^{\pi/2} x\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt = 4\pi\int_0^{\pi/2} a\cos^3 t.3a$$

$\sin t\cos t\,dt$

$$= 12\pi a^2\int_0^{\pi/2}\sin t\cos^4 t\,dt = 12\pi a^2\frac{3.1}{5.3.1} = \frac{12\pi a^2}{5}$$

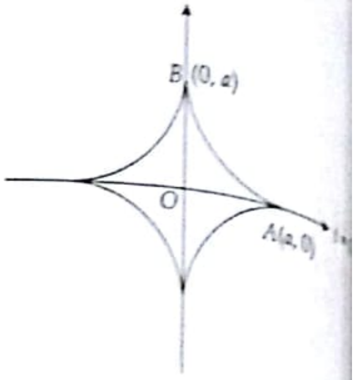

Fig. 6.31

EXERCISE 6.6

1. The portion between the consecutive cusps of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$ is revolved about the x-axis. Prove that the ratio of the area of the surface generated to the area of the cycloid is 64:9.
2. A quadrant of a circle of radius a revolves around its chord. Show that the surface of the spindle generated is $2\pi a^2\sqrt{2}\,[1 - \pi/4]$.
3. Find the area of the surface generated by revolving the loop of the curve $3ay^2 = x(x - a)^2$ about x-axis.
4. Find the volume and surface area of the solid generated by revolving the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ about x-axis.
5. A circular arc revolves about the chord. Show that the area of the surface generated is $4\pi a^2(\sin\alpha - \alpha\cos\alpha)$ where 2α is the angle subtended by the arc at the centre.
6. Prove that surface area of the solid generated by revolving tractrix $x = a\cos t + \frac{a}{2}\ln\tan^2(t/2)$, $y = a\sin t$ about x-axis is $4\pi a^2$.
7. Prove that the surface and volume of the solid generated by revolving the loop of the curve $x = t^2$, $y = t - t^3/3$ about x-axis are respectively 3π and $3\pi/4$.
8. Find the volume and surface area of the right circular cone obtained by the revolution of a right angled triangle about a side which contains the right angle.
9. Find the area of the surface generated by revolving about the x-axis a closed contour formed by the curves $y = x^2$ and $x = y^2$.
10. An arc of the catenary $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$ whose end points have abscissas 0 and x respectively, revolves about the x-axis. If S and V are respectively the surface and the volume generated show that $S = 2V/a$.

ANSWERS

Exercise 6.1 (p. 366)

4. 32 atm. 5. $2v/3$ 8. (a) $\pi^2/4$ (b) $(\pi/2)\ln 2$.

Exercise 6.2 (p. 378)

1. (a) $\frac{3\sqrt{3}}{32}+\frac{5\pi}{48}$ (b) $\frac{2}{315}$ (c) $\frac{35\pi}{256}$ (d) $\frac{16\pi}{1155}$

2. (a) $\frac{16}{35}a^7$ (b) $\frac{1}{a^{2n-1}}\frac{(2n-3)(2n-5)\dots 3.1}{(2n-2)(2n-4)\dots 4.2}\cdot\frac{\pi}{2}$ (c) $\frac{1}{15}$

4. $\frac{64\sqrt{2}}{15}$ 8. $(3\pi-8)/12$ 9. $24/85$

Exercise 6.3 (p. 384)

1. πab 2. $\frac{8}{3}a^2$ 3. $\frac{1}{6}$ 4. $\frac{\pi a^2}{8}$

5. $2a^2\left(\pi-\frac{2}{3}\right)$ 7. $a^2(\pi-2)$ 8. $4\pi a^2$ 9. $4a^2$

10. $(\pi-1)a^2$ 11. $\frac{a^2}{2}(3\pi-8)$ 12. $\frac{4}{ab}\tan^{-1}\left(\frac{a}{b}\right)$ 14. $\frac{\pi a^2}{4n}, \frac{\pi a^2}{4}$

17. $\frac{3}{8}\pi ab$ 18. $3\pi a^2$ 19. $\frac{11}{3}$

Exercise 6.4 (p. 388)

1. $\frac{3a}{2}$ 2. $2\pi a$ 4. $\frac{\pi a}{\sqrt{2}}$ 5. $\frac{3a\pi}{2}$

9. $(r_2-r_1)\sec\alpha$ 10. $l\left[\sqrt{2}+\ln\left(\sqrt{2}+1\right)\right]$.

Exercise 6.5 (p. 395)

1. $\frac{7\pi}{6}$ 2. $\frac{891}{1280}\pi$ 3. $\pi/20$ 4. $\frac{4}{3}\pi ab^2$

5. $\frac{1}{3}\pi a^2 b$ 6. $2\pi^2a^2b$ 7. 432π 8. $\frac{7\pi a^3}{96}$

10. $2\pi^2 a^3$

11. $\dfrac{32\pi a^3}{105}$.

Exercise 6.6 (p. 400)

3. $\dfrac{\pi a^2}{3}$

4. $\dfrac{12\pi a^2}{5}$

8. $\frac{1}{2}\pi r^2 h;\ \pi r\sqrt{r^2+h^2}$, where r is the base and h is the height and it is rotated about h.

9. $\dfrac{67\sqrt{5}\pi}{48} - \dfrac{\pi}{32}\ln\left(2+\sqrt{5}\right) - \dfrac{\pi}{6}$.

Exercise 6.7 (p. 409)

2. $a\left(\pi - \dfrac{4}{3}\right), \dfrac{2a}{3}$

3. $(3/2, 12/5)$

4. $(9/5, 11/10)$

5. $\left(a, \dfrac{5a}{8}\right)$

6. $\left(\dfrac{\pi}{6(4-\pi)}, \dfrac{12-\pi^2}{12-3\pi}\right)$

7. $\left(2a, \dfrac{10a}{7}\right)$

8. $\dfrac{8\sqrt{2}\pi a^3}{15}$

10. $\dfrac{1}{3}\pi a^2 b$

11. $\pi\left(\dfrac{a^2 b\sqrt{3}}{2} \mp \dfrac{a^3}{3}\right)$

13. $\dfrac{4}{3}\pi a^3 \sin\alpha,\ 4\pi a^2 \sin\alpha$.

14. $\bar{x} = \bar{y} = 4(a^2 + ab + b^2)/3\pi(a+b),\ (2a/\pi, 2a/\pi)$.