12 CHAPTER

Series Solutions of Differential Equations and Special Functions

A series solution is a standard strategy for solving an initial value problem with variable coefficients when the standard methods do not yield solution in the closed form. The series forms are sometimes more informative about the behaviour of the solution. The methods are broadly classified as power series method and general series method. Equations like that of Legendre and Bessel and their solutions play a basic role in applied mathematics. The study of their solutions constitutes the theory of special functions.

12.1 INTRODUCTION

We have seen in the preceding chapter that a homogeneous linear differential equation with constant coefficients can be solved by algebraic methods and its solutions are elementary functions from calculus. Also some special types of linear differential equations with variable coefficients can be reduced to equations with constant coefficients by applying change of variables and can be solved. However, many differential equations arising in practical applications are linear with variable coefficients but can't be reduced to equations with constant coefficients, and thus are need to be solved by some other methods. For example, Legendre and Bessel equations are two such important equations. Their solutions are also not usual functions from calculus but are *special functions* having applications in engineering and applied mathematics. The study of these solutions is called the *theory of special functions.* The power series method is the standard strategy to solve linear differential equations with variable coefficients. Such a solution being explicit one, giving $y(x)$ as an infinite series in x, may also give important information about the nature of the solution like passing through origin, even or odd, increasing or decreasing on a given interval, etc.

We begin with power series solutions for differential equations admitting such solutions, and then will study an extension of it, called the *Frobenius method* for the general problems which do not admit power series solutions about a particular point.

12.2 POWER SERIES SOLUTIONS

A *power series* in x about a point x_0 is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \ldots$$

where a_0, a_1, a_2, ... are constants, called the *coefficients* of the series and x_0 is a constant, called the *center* of the series. In particular, if x_0 is zero, then the power series is

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$$

If a function $f(x)$ has a power series representation in some open interval about x_0, that is, if $f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$, for $x \in (x_0 - h, x_0 + h)$, $h > 0$, then $f(x)$ is said to be analytic at $x = x_0$.

For example, $f(x) = \dfrac{1}{1-x}$ is analytic at $x = 0$, since $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, $\quad -1 < x < 1$.

Similarly $\sin x$ is analytic at $x = 0$, since $\sin x = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n+1)!} x^{2n+1}$, $\quad$ for all x.

A necessary condition for a function $f(x)$ to be analytic at a point $x = x_0$ is that $f(x)$ is infinitely differentiable at $x = x_0$.

12.2.1 Power Series Solution of First Order Initial Value Problems

Consider the first-order initial value differential equation of the form

$$\boxed{y' + p(x)y = r(x);\ \ y(x_0) = y_0} \qquad \ldots(12.1)$$

where the functions $p(x)$ and $r(x)$ are analytic at x_0.

We state without proof that *an initial value problem whose coefficients are analytic at x_0 has an analytic solution at x_0.*

Thus, we say that the initial value problem (12.1) has the expanded solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n, \quad \text{where } a_n = \frac{1}{n!} y^{(n)}(x_0). \qquad \ldots(12.2)$$

We illustrate this approach by considering the following examples.

Example 12.1: Find the power series solution of

$$y' + (1 + x^2)y = \sin x; \quad y(0) = 1. \qquad \ldots(12.3)$$

Solution: The coefficients of the initial value problem are analytic at the point $x = 0$, hence the solution is of the form

$$y(x) = \sum_{n=0}^{\infty} \frac{1}{n!} y^{(n)}(0)x^n = y(0) + y'(0)x + \frac{1}{2!}y''(0)x^2 + \frac{1}{3!}y'''(0)x^3 + \ldots \qquad \ldots(12.4)$$

Initial condition is $y(0) = 1$. Put $x = 0$ in Eq. (12.3) to obtain

$$y'(0) + y(0) = 0, \text{ or } \quad y'(0) = -y(0) = -1.$$

To determine $y''(0)$, differentiate Eq. (12.3) to obtain

$$y'' + (1 + x^2)y' + 2xy = \cos x. \qquad \ldots(12.5)$$

Put $x = 0$, $y''(0) + y'(0) = 1$, or $\quad y''(0) = 1 - y'(0) = 2$.

To determine $y'''(0)$, differentiate Eq. (12.5) to obtain

$$y''' + (1 + x^2)y'' + 4xy' + 2y = -\sin x.$$

Put $x = 0$, $y'''(0) + y''(0) + 2y(0) = 0$, or $\quad y'''(0) = -(y''(0) + 2(y(0)) = -4$, and so on.

Substituting for $y(0)$, $y'(0)$, $y''(0)$ etc. in (12.4), we obtain

$$y(x) = 1 - x + x^2 - \frac{2}{3}x^3 + \ldots$$

as the required series solution.

Example 12.2: Find the power series solution of

$$y' + (\sin x)y = 1 - x; \quad y(\pi) = -3. \qquad \ldots(12.6)$$

Solution: The coefficients $\sin x$ and $1 - x$ are analytic for all x but, since the initial condition is at $x = \pi$, we obtain the power series solution about $x = \pi$, of the form

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$$y(x) = y(\pi) + y'(\pi)(x - \pi) + \frac{y''(\pi)}{2!}(x - \pi)^2 + \frac{y'''(\pi)}{3!}(x - \pi)^3 + \ldots \qquad \ldots(12.7)$$

Initial condition is $y(\pi) = -3$. Put $x = \pi$ in Eq. (12.6) to obtain $y'(\pi) = 1 - \pi$.

To determine $y''(\pi)$, differentiate Eq. (12.6) to obtain

$$y'' + (\sin x)y' + (\cos x)y = -1. \qquad \ldots(12.8)$$

Put $x = \pi$, $y''(\pi) - y(\pi) = -1$, or $\quad y''(\pi) = y(\pi) - 1 = -4$.

To determine $y'''(\pi)$, differentiate (12.8) to obtain

$$y''' + \sin xy'' + 2\cos xy' - \sin xy = 0.$$

Put $x = \pi$, $y'''(\pi) - 2y'(\pi)$, or $\quad y'''(\pi) = 2y'(\pi) = 2(1 - \pi)$.

Substituting for $y(\pi)$, $y'(\pi)$, $y''(\pi)$ etc. in (12.7), we obtain

$$y(x) = -3 + (1 - \pi)(x - \pi) - 2(x - \pi)^2 + \frac{(1 - \pi)}{3}(x - \pi)^3 + \ldots$$

as the required series solution.

12.2.2 Power Series Solution for Second and Higher Order Equations

The method for obtaining series solution of a first order initial value problem can be extended to second and higher order equations. For a second order differential equation of the form

$$\boxed{y'' + p(x)y' + q(x)y = f(x); \quad y(x_0) = y_0, \quad y'(x_0) = y_1.} \qquad ...(12.9)$$

We have the following result:

If $p(x)$, $q(x)$ and $f(x)$ are analytic at x_0, then the initial value problem (12.9) has a unique solution which is also analytic at x_0.

Example 12.3: Find the power series solution of the initial value problem

$$y'' - e^x y' + 2y = 1; \quad y(0) = -3, y'(0) = 1. \qquad ...(12.10)$$

Solution: The coefficients e^x, 2 and 1 are analytic at $x = 0$; let the solution be

$$y(x) = y(0) + y'(0)x + \frac{1}{2!}y''(0)\,x^2 + \frac{1}{3!}y'''(0)x^3 + ... \qquad ...(12.11)$$

Initial conditions are $y(0) = -3, y'(0) = 1$.

Put $x = 0$ in (12.10) to obtain $y''(0) - y'(0) + 2y(0) = 1$, or $y''(0) = 1 + y'(0) - 2y(0) = 8$.

To obtain $y'''(0)$, differentiate (12.10), we have

$$y''' - e^x y'' - (e^x - 2)\,y' = 0 \qquad ...(12.12)$$

Put $x = 0$, $y'''(0) - y''(0) + y'(0) = 0$, or $y'''(0) = y''(0) - y'(0) = 7$.

To obtain $y^{(iv)}(0)$, differentiate (12.12) we have

$$y^{(iv)} - e^x y''' - 2(e^x - 1)y'' - e^x y' = 0. \qquad ...(12.13)$$

Put $x = 0$, $y^{(iv)}(0) - y'''(0) - y'(0) = 0$, or $y^{(iv)}(0) = y'(0) + y'''(0) = 8$.

Substituting for $y(0), y'(0), y''(0), y'''(0)$ etc. in (12.11), we have

$$y(x) = -3 + x + 4x^2 + \frac{7}{6}x^3 + \frac{1}{3}x^4 + ...$$

as the required series solution.

Example 12.4: Find the power series solution of $y'' + \cos x\, y' + 4y = 2x - 1$ about $x = 0$.

Solution: We rewrite the given differential equation as the initial value problem, say

$$y'' + \cos x\, y' + 4y = 2x - 1; \quad y(0) = a, \quad y'(0) = b, \qquad ...(12.14)$$ Imp. step

with a and b as two arbitrary constants.

The coefficients $\cos x$, 4 and $(2x - 1)$ are analytic for all real values of x. Let the solution be

$$y(x) = y(0) + y'(0)x + \frac{1}{2!}y''(0)x^2 + \frac{1}{3!}y'''(0)\,x^3 + ... \qquad ...(12.15)$$

Initial conditions are $y(0) = a$ and $y'(0) = b$.

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To find $y''(0)$, put $x = 0$ in Eq. (12.14), we obtain

$y''(0) + y'(0) + 4y(0) = -1$, or $y''(0) = -1 - 4y(0) - y'(0) = -1 - 4a - b.$

To find $y'''(0)$, differentiate (12.14), we obtain

$$y''' + \cos x y'' + (-\sin x + 4)y' = 2. \qquad \text{...(12.16)}$$

Put $x = 0$ to obtain $y'''(0) = 2 - 4y'(0) - y''(0) = 2 - 4b + 1 + 4a + b = 4a - 3b + 3.$

Thus, from (12.15), we obtain

$$y(x) = a + bx - \frac{(1 + 4a + b)}{2!}x^2 + \frac{(3 - 3b + 4a)}{3!}x^3 + \ldots$$

as the desired general solution.

Example 12.5: Find the power series solution of

$$(1 - x^2)y'' - 2xy' + 2y = 0 \qquad \text{...(12.17)}$$

about $x = 0$.

Solution: Let the initial conditions be $y(0) = a$ and $y'(0) = b$, where a and b are two arbitrary constants. Substituting $x = 0$ in (12.17), we have $y''(0) + 2y(0) = 0$, which gives, $y'' = -2y(0) = -2a$.

To find $y'''(0)$ differentiate (12.17) w.r.t. x to obtain

$$(1 - x^2)y''' - 4xy'' = 0. \qquad \text{...(12.18)}$$

Put $x = 0$ in (12.18) to obtain $y'''(0) = 0$. To find $y^{iv}(0)$ differentiate (12.18) to obtain

$$(1 - x^2)y^{iv} - 6xy''' - 4y'' = 0. \qquad \text{...(12.19)}$$

Put $x = 0$ it gives, $y^{iv}(0) = 4y''(0) = -8a$.

Hence, the power series solution for the given differential equation is

$$y = a + bx - ax^2 - \frac{1}{3}ax^4 + \ldots$$

where a and b are two arbitrary constants.

In the above example, we observe that coefficients $-2x/(1 - x^2)$ and $2/(1 - x^2)$ of the differential equation (12.17) are analytic at $x = 0$, and therefore the power series solution exists. However, these coefficients are not analytic at $x = \pm 1$. Thus, the power series solution cannot exist in any interval I which contains 1, or -1. Therefore, $(-1, 1)$ is the *interval of convergence* and 1 is the *radius of convergence* for the series solution obtained. ?

12.2.3 Power Series Solution by Developing the Recurrence Relation

So far we have obtained the coefficients of the power series by differentiating the given differential equation step by step. Practically this procedure suits when we need to find only a few coefficients. Otherwise the coefficients can be obtained in a more systematic way by developing a recurrence relation between them, which allows us to generate coefficients once some preceding coefficients are known. We illustrate the procedure through a few examples given next.

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Example 12.6: Find the power series solution of

$$y'' + 2xy' + (1 + x^2)y = 0; \quad y(0) = 3; \quad y'(0) = -1 \qquad ...(12.20)$$

about $x = 0$ by developing the recurrence relation between the coefficients.

Solution: Let

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \qquad ...(12.21)$$

be the solution of Eq. (12.20). We have $y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1}$, and $y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$.

Substituting expressions for y, y' and y'' in (12.20), we obtain

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 2x\sum_{n=1}^{\infty} na_n x^{n-1} + (1 + x^2)\sum_{n=0}^{\infty} a_n x^n = 0,$$

or,
$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} + 2\sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+2} = 0. \qquad ...(12.22)$$

We shift the indices in the first and last summation to make the index of x in each series in (12.22) same. For the first summation on left side of Eq. (12.22), replace $(n - 2)$ by n

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \text{ becomes } \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n.$$

Similarly, for the last summation on left side of Eq. (12.22), replace $n + 2$ by n

$$\sum_{n=0}^{\infty} a_n x^{n+2} \text{ becomes } \sum_{n=2}^{\infty} a_{n-2}\, x^n.$$

Thus (12.22) reduces to

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} a_n x^n + \sum_{n=2}^{\infty} a_{n-2}x^n = 0. \qquad ...(12.23)$$

Rewriting it as

$$(2a_2 + a_0) + (6a_3 + 3a_1)x + \sum_{n=2}^{\infty} [(n+2)(n+1)a_{n+2} + (2n+1)a_n + a_{n-2}]x^n = 0 \qquad ...(12.24)$$

The condition for Eq. (12.24) to hold for all x in some open interval about zero is that the coefficient of each power of x on the right side of it must be zero. Therefore,

$2a_2 + a_0 = 0$, $6a_3 + 3a_1 = 0$, and $(n+2)(n+1)a_{n+2} + (2n+1)a_n + a_{n-2} = 0$, for $n \geq 2$.

From the first two conditions, we obtain $a_2 = -\frac{1}{2}a_0$ and $a_3 = -\frac{1}{2}a_1$.

The third condition is the recurrence relation and holds for $n = 2, 3, 4, \ldots$. Rewriting this as

$$a_{n+2} = -\frac{[(2n+1)a_n + a_{n-2}]}{(n+1)(n+2)} \qquad \ldots(12.25)$$

For $n = 2$, $\quad a_4 = -\frac{(5a_2 + a_0)}{12} = \frac{a_0}{8}$, using $a_2 = -\frac{a_0}{2}$

For $n = 3$, $\quad a_5 = -\frac{(7a_3 + a_1)}{20} = \frac{a_1}{8}$, using $a_3 = -\frac{1}{2}a_1$.

Similarly, $\quad a_6 = -\frac{a_0}{48}$, $a_7 = -\frac{a_1}{48} \ldots$

Substituting for these coefficients in (12.21) and grouping the terms with coefficient a_0 and a_1, we obtain

$$y(x) = a_0\left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 + \ldots\right) + a_1\left(x - \frac{1}{2}x^3 + \frac{1}{8}x^5 - \frac{1}{48}x^7 + \ldots\right) \qquad \ldots(12.26)$$

Here a_0 and a_1 are arbitrary constants and the functions

$$y_1(x) = 1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 + \ldots$$

$$y_2(x) = x - \frac{1}{2}x^3 + \frac{1}{8}x^5 - \frac{1}{48}x^7 + \ldots$$

are two linearly independent solutions of the given differential equation.

To find a_0 and a_1, we use the initial conditions $y(0) = 3$ and $y'(0) = -1$ in (12.26), which gives $a_0 = 3$ and $a_1 = -1$. Hence, the required power series solution is

$$\boxed{y(x) = 3 - x - \frac{3}{2}x^2 + \frac{1}{2}x^3 + \frac{3}{8}x^4 - \frac{1}{8}x^5 \ldots - \ldots}$$

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Example 12.7: Find the power series solution for $y'' + xy' - y = e^{3x}$ about $x = 0$.

Solution: Substituting $y(x) = \sum_{n=0}^{\infty} a_n x^n$, and $e^{3x} = \sum_{n=0}^{\infty} \frac{3^n}{n!} x^n$ in the given equation, we obtain

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{3^n}{n!} x^n. \qquad \ldots(12.27)$$

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Shifting the indices in the first summation on left side of Eq. (12.27) by replacing $n - 2$ with n, (12.27) becomes

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} na_nx^n - \sum_{n=0}^{\infty} a_nx^n = \sum_{n=0}^{\infty} \frac{3^n}{n!}x^n.$$ Rewriting it as

$$(2a_2 - a_0) + \sum_{n=1}^{\infty} [(n+2)(n+1)a_{n+2} + (n-1)a_n]x^n = 1 + \sum_{n=1}^{\infty} \frac{3^n}{n!}x^n. \qquad ...(12.28)$$

Equating coefficients of like powers of x on both sides of Eq. (12.28), we obtain

$$2a_2 - a_0 = 1, \text{ or } \quad a_2 = \frac{(1+a_0)}{2}, \text{ and, the recurrence relation}$$

$$(n+2)(n+1)a_{n+2} + (n-1)a_n = \frac{3^n}{n!}, \text{ for } \quad n = 1, 2, ...$$

which can be rewritten as

$$a_{n+2} = \frac{(3^n/n!) - (n-1)a_n}{(n+2)(n+1)}, \text{ for } n = 1, 2, ...$$

It gives, $a_3 = \frac{1}{2}$, $a_4 = (9 - 2a_2)/24 = (8 - a_0)/24$ etc.

Hence the desired series solution is

$$y(x) = a_0 + a_1x + \frac{(1+a_0)}{2}x^2 + \frac{1}{2}x^3 + \left(\frac{1}{3} - \frac{a_0}{24}\right)x^4 + ...$$

where a_0 and a_1 are two arbitrary constants.

EXERCISE 12.1

Find the power series solution of the following initial value problems about the point where the initial conditions are given

1. $y' + e^x y = x^2; \quad y(0) = 4$
2. $y' + (1 + x^2)y = \sin x; \quad y(0) = a$
3. $y' + 4xy = 3e^{x-1}; \quad y(1) = 1$
4. $y' + x(1 - 2x)y = 1; \quad y(0) = 1$
5. $(x+1)y' - (x+2)y = 0; \quad y(-2) = 1$
6. $y'' - xy' + e^x y = 4; \quad y(0) = 1, \quad y'(0) = 4$
7. $y'' - xy = 2x; \quad y(1) = 3, \quad y'(1) = 0$
8. $y'' + \frac{1}{x+2}y' - xy = 0; \quad y(0) = y'(0) = 1$
9. $y'' + \frac{1}{x-1}y' + \frac{1}{x+2}y = 2; \quad y(0) = y'(0) = 3.$
10. $(2 + x^2)y'' - 2xy' + 3y = 0; \quad y(1) = 1, \quad y'(1) = -1.$

Find the Maclaurin's series expansion solution of the following differential equations:

11. $y'' - 2xy' + x^2y = 0;\quad y(0) = a,\quad y'(0) = b.$

12. $y'' + xy' + (1 - x^2)y = x;\quad y(0) = a,\quad y'(0) = b.$

13. $(4 + x^2)y'' - 6xy' + 8y = 0;\quad y(0) = a,\quad y'(0) = b.$

14. $(1 + x^2)y'' + 3xy' + (1 - x^2)y = 0;\quad y(0) = a,\quad y'(0) = b.$

15. $2y'' + 3x^2y' + (1 - x^2)y = 2x;\quad y(0) = a,\quad y'(0) = b.$

Using the power series method, find the solution for the following differential equations about $x = 0$. Find also the recurrence relation between the coefficients.

16. $y'' + w^2y = 0,\quad w > 0$ is constant.

17. $y'' + x^2y = 0$

18. $y'' + x^2y' + 4y = 1 - x^2$

19. $y'' + xy' + (1 + x)y = 0;\quad y(0) = -1,\quad y'(0) = 0$

20. $(1 + 2x)y'' - y' + y = 0;\quad y(0) = 0,\quad y'(0) = 1.$

12.3 LEGENDRE EQUATION. LEGENDRE POLYNOMIALS. FOURIER-LEGENDRE SERIES

In this section, we apply the power series method to solve an important differential equation of applied mathematics, the *Legendre equation* given by

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0, \qquad \text{...(12.29)}$$

where $\alpha \geq 0$ is a real parameter. This equation arises in a variety of applications, particular in connection with physical problems with spherical symmetry.

The coefficients of the Legendre equation are analytic at the origin and the leading coefficient $(1 - x^2)$ only vanishes at $x = \pm 1$, so a power series solution of the Legendre equation exists about the point $x = 0$ in the interval $-1 < x < 1$. Solutions of Eq. (12.29) are called *Legendre functions*, and are examples of *special functions*. In case $\alpha = n$, a non-negative integer, then one of the solution of Legendre Eq. (12.29) is a polynomial of degree n. These polynomials, multiplied by some suitable constants, are called *Legendre polynomials* which are of great practical importance because of their important properties, particularly orthogonality.

12.3.1 Solution of Legendre Equation

Let the solution be

$$y = \sum_{n=0}^{\infty} a_n x^n. \qquad \text{...(12.30)}$$

Substitute y and its derivatives in Eq. (12.29), we obtain

$$(1 - x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=1}^{\infty} na_n x^{n-1} + \alpha(\alpha+1)\sum_{n=0}^{\infty} a_n x^n = 0.$$

This can be rewritten as

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n - 2\sum_{n=1}^{\infty} na_n x^n + \alpha(\alpha+1)\sum_{n=0}^{\infty} a_n x^n = 0$$

or,
$$[2a_2 + \alpha(\alpha+1)a_0] + [6a_3 - 2a_1 + \alpha(\alpha+1)a_1]x$$
$$+ \sum_{n=2}^{\infty} [(n+2)(n+1)\, a_{n+2} - n(n+1)\, a_n + \alpha(\alpha+1)\, a_n]\, x^n = 0. \qquad \text{...(12.31)}$$

Equating the coefficients of constant terms on the left side of (12.31) to zero, we obtain

$$2a_2 + \alpha(\alpha+1)a_0 = 0. \qquad \text{...(12.32)}$$

Similarly, equating the coefficients of terms containing x to zero, we obtain

$$6a_3 - 2a_1 + \alpha(\alpha+1)a_1 = 0. \qquad \text{...(12.33)}$$

Equating the coefficients of the general term x^n, $n \geq 2$ to zero, we have

$$(n+2)(n+1)a_{n+2} + \{\alpha(\alpha+1) - n(n+1)\}a_n = 0. \qquad \text{...(12.34)}$$

From (12.32), $a_2 = \dfrac{-\alpha(\alpha+1)}{2}\, a_0 = \dfrac{-\alpha(\alpha+1)}{2!}\, a_0$

From (12.33), $a_3 = \dfrac{[2-\alpha(\alpha+1)]}{6}\, a_1 = \dfrac{-(\alpha-1)(\alpha+2)}{3!}\, a_1$

From (12.34), we obtain the recurrence relation

$$a_{n+2} = -\frac{(\alpha-n)(\alpha+n+1)}{(n+2)(n+1)}\, a_n, \quad n \geq 2. \qquad \text{...(12.35)}$$

It gives, $a_4 = -\dfrac{(\alpha-2)(\alpha+3)}{4.3}\, a_2 = \dfrac{(\alpha-2)\alpha(\alpha+1)(\alpha+3)}{4!}\, a_0$

$$a_5 = -\frac{(\alpha-3)(\alpha+4)}{5.4}\, a_3 = \frac{(\alpha-3)(\alpha-1)(\alpha+2)(\alpha+4)}{5!}\, a_1$$

$$a_6 = -\frac{(\alpha-4)(\alpha-2)\alpha(\alpha+1)(\alpha+3)(\alpha+5)}{6!}\, a_0, \text{ and so on.}$$

Substituting for these coefficients in Eq. (12.30), we obtain

$$y(x) = a_0\left[1 - \alpha(\alpha+1)\frac{x^2}{2!} + (\alpha-2)\alpha(\alpha+1)(\alpha+3)\frac{x^4}{4!} - \ldots\right]$$

$$+ a_1\left[x - (\alpha-1)(\alpha+2)\frac{x^3}{3!} + (\alpha-3)(\alpha-1)(\alpha+2)(\alpha+4)\frac{x^5}{5!} - \ldots\right] = a_0 y_1 + a_1 y_2(x),$$

$(\alpha-5)(\alpha-3)(\alpha-1)(\alpha+2)\ (\alpha+4)\ (\alpha+6)$

where $$y_1(x) = 1 - \alpha(\alpha+1)\frac{x^2}{2!} + (\alpha - 2)\alpha(\alpha+1)(\alpha+3)\frac{x^4}{4!} - \ldots \qquad \ldots(12.36)$$

and, $$y_2(x) = x - (\alpha-1)(\alpha+2)\frac{x^3}{3!} + (\alpha-3)(\alpha-1)(\alpha+2)(\alpha+4)\frac{x^5}{5!} - \ldots \qquad \ldots(12.37)$$

The series for $y_1(x)$ and $y_2(x)$ converge for $-1 < x < 1$, and further since $y_1(x)$ contains even powers of x only and $y_2(x)$ contains odd powers of x only, hence these are the two linearly independent solutions of the Legendre Eq. (12.29).

12.3.2 Legendre Polynomials

From the form of the solutions (12.36) and (12.37), we observe that if $\alpha = n$ is even, the series $y_1(x)$ will reduce to a polynomial of degree n in even powers of x, whereas if $\alpha = n$ is odd, the series $y_2(x)$ will reduce to a polynomial of degree n in odd powers of x. For example, for $n = 0$,

$$y_1(x) = 1; \quad n = 2, \quad y_1(x) = 1 - 3x^2; \quad n = 4, \quad y_1(x) = 1 - 10x^2 + \frac{35}{3}x^4,$$

and, for $n = 1$,

$$y_2(x) = x; \quad n = 3, \quad y_2(x) = x - \frac{5}{3}x^3; \quad n = 5, \quad y_2(x) = x - \frac{14}{3}x^3 + \frac{21}{5}x^5.$$

These polynomials multiplied with suitable constants are called Legendre polynomials and are denoted by $P_n(x)$, where n denotes the degree of the polynomial. The constants are selected such that $P_n(1) = 1$, for $n = 0, 1, 2, \ldots$. Thus, for $\alpha = n$, one of the linearly independent solutions of the Legendre Eq. (12.29) is a Legendre polynomial of degree n and the second independent solution is an infinite series denoted by $Q_n(x)$.

The first five Legendre polynomials obtained by setting the condition $P_n(1) = 1$ are

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), P_3(x) = \frac{1}{2}(5x^3 - 3x), \text{ and } P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3).$$

It is a little bit cumbersome to obtain $P_n(x)$ like this. These polynomials are explicitly given by the formula known as *Rodrigue's formula*, proved next.

V. Imp

12.3.3 Rodrigue's Formula PROOF imp

C(1-3) = 1

a = -1

The Rodrigue's formula is

$$\boxed{P_n(x) = \frac{1}{n!2^n}\frac{d^n}{dx^n}[(x^2-1)^n].} \qquad \ldots(12.38)$$

To prove it, let $u = (x^2 - 1)^n$. Differentiating with respect to x, we obtain

$$u_1 = 2nx(x^2-1)^{n-1} = \frac{2nxu}{(x^2-1)}, \text{ or } (1 - x^2)u_1 + 2nxu = 0, \text{ where } u_1 = \frac{du}{dx}.$$

Leibnitz theorem: if $y = uv$ where $u = u(x)$; $v = v(x)$

$\frac{d^n}{dx^n}(uv) = uv_n + {}^nC_1 u_1 v_{n-1} + {}^nC_2 u_2 v_{n-2} + \cdots {}^nC_n u_n v_{n-n} + \cdots + u_n v$

$u_n = \frac{d^n(u)}{dx^n}$

$v_n = \frac{d^n v}{dx^n}$

Differentiating it $(n + 1)$ times using the Leibnitz theorem, we obtain

$$[(1 - x^2)u_{n+2} + (n + 1)(-2x)u_{n+1} + \frac{1}{2!}(n + 1)(n)(-2)u_n] + 2n[xu_{n+1} + (n + 1)u_n] = 0$$

or,
$$(1 - x^2)u''_n - 2xu'_n + n(n + 1)u_n = 0, \qquad \text{...(12.39)}$$

where $u'_n = \frac{du_n}{dx}$ and $u''_n = \frac{d^2u_n}{dx^2}$.

Eq. (12.39) is Legendre differential equation in $y = cu_n$, where c is an arbitrary constant. Since $P_n(x)$ is the finite series solution of the Legendre equation, therefore,

$$P_n(x) = cu_n = c\frac{d^n}{dx^n}[(x^2 - 1)^n]. \qquad \text{...(12.40)}$$

The constant c is determined by setting $P_n(1) = 1$. From (12.40), we have

$$P_n(1) = c\frac{d^n}{dx^n}[(x^2 - 1)^n]_{x=1} = c\frac{d^n}{dx^n}[(x - 1)^n(x + 1)^n]_{x=1}$$

$$= c[n!(x + 1)^n + \text{terms containing } (x - 1) \text{ and its higher powers}]_{x=1}$$

$= cn!2^n$. Thus, $\boxed{c = \frac{1}{n!2^n}}$.

Substituting this value of c in (12.40) we obtain (12.38), the Rodrigue's formula.

From (12.38), we have

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \quad P_3(x) = \frac{1}{2}(5x^3 - 3x^2),$$

$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$ and, in general,

$$P_n(x) = \sum_{r=0}^{N} \frac{(-1)^r(2n - 2r)!}{2^n r!(n - r)!(n - 2r)!} x^{n-2r}, \qquad \text{...(12.41)}$$

which is proved as follows. By Binomial theorem

$$(x^2 - 1)^n = \sum_{r=0}^{n} c^n_r (x^2)^{n-r}(-1)^r = \sum_{r=0}^{n} (-1)^r \frac{n!}{r!(n - r)!} x^{2n-2r},$$

and thus, from (12.38)

$$P_n(x) = \frac{1}{n!2^n}\sum_{r=0}^{n}(-1)^r \frac{n!}{r!(n - r)!}\frac{d^n}{dx^n}(x^{2n-2r}) = \sum_{r=0}^{N}\frac{(-1)^r(2n - 2r)!}{2^n r!(n - r)!(n - 2r)!}x^{n-2r},$$

which is (12.41). Here $N = \frac{n}{2}$, or $\frac{(n-1)}{2}$, whichever is integer, that is, $N = \left\lfloor \frac{n}{2} \right\rfloor$, so that the power of x for the last term in the series (12.41) is either 0 or 1.

The graphs of some of the even and the odd polynomials are shown in Figs. 12.1a & b.

$P_n(x)$ is of degree n, and contains only even powers of x if n is even, and only odd powers of x if n is odd. These polynomials are defined for all real x, but the relevant interval for the Legendre differential equation is $-1 < x < 1$. The Legendre polynomials belong to an important class of polynomials called the *orthogonal polynomials*, to be discussed in Section 12.3.6.

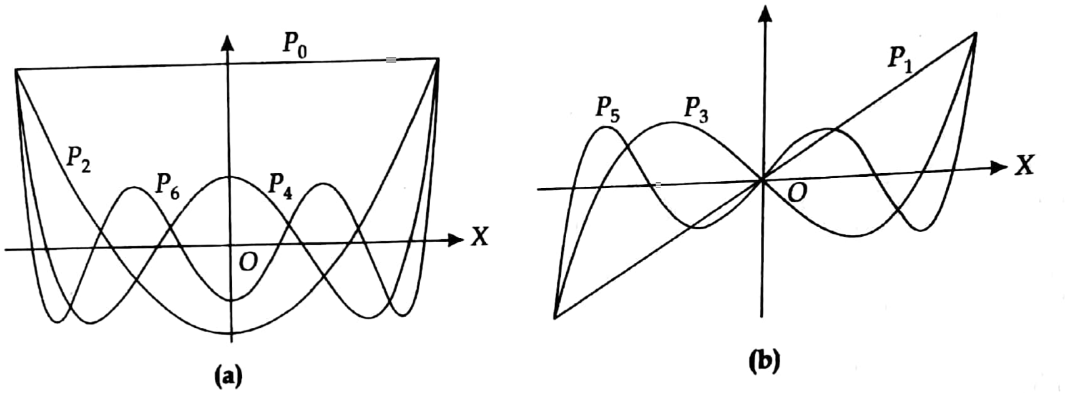

Fig. 12.1

12.3.4 Generating Function for Legendre Polynomials

Many properties of Legendre polynomials can be derived by using the concept of generating function. We claim that the generating function of $P_n(x)$ can be given by

$$L(x, t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n, \qquad \text{...(12.42)}$$

that is, if $L(x, t)$ is expanded as a power series in t, then the coefficient of t^n is $P_n(x)$, the Legendre polynomial of degree n.

To prove this, write the Maclaurin series for $(1-u)^{-1/2}$, $-1 < u < 1$, we have

$$(1-u)^{\frac{-1}{2}} = 1 + \frac{1}{2}u + \frac{(1/2)(3/2)}{2!}u^2 + \frac{(1/2)(3/2)(5/2)}{3!}u^3 + \ldots$$

$$= 1 + \frac{2!}{(1!)^2 2^2}u + \frac{4!}{(2!)^2 2^4}u^2 + \ldots + \frac{(2n)!}{(n!)^2 2^{2n}}u^n + \ldots \qquad \text{...(12.43)}$$

Setting $u = (2x-t)t$, (12.43) becomes

$$(1-2xt+t^2)^{-1/2} = 1 + \frac{2!}{(1!)^2 2^2}(2x-t)t + \frac{4!}{(2!)^2 2^4}(2x-t)^2 t^2 + \ldots$$

$$+\frac{(2n-2r)!}{[(n-r)!]^2 2^{2n-2r}}(2x-t)^{n-r}t^{n-r}+\ldots+\frac{(2n)!}{[n!]^2 2^{2n}}(2x-t)^n t^n+\ldots$$

Now, the term in t^n from the term containing $(2x-t)^{n-r}t^{n-r}$

$$=\frac{(2n-2r)!}{[(n-r)!]^2 2^{2n-2r}}\cdot c_r^{n-r}.(2x)^{n-2r}(-t)^r\ t^{n-r}$$

$$=\frac{(2n-2r)!}{[(n-r)!]^2 2^{2n-2r}}\cdot\frac{(n-r)!}{r!(n-2r)!}(-1)^r 2^{n-2r}x^{n-2r}t^n=\frac{(-1)^r(2n-2r)!}{2^n r!(n-r)!(n-2r)!}x^{n-2r}t^n$$

Collecting all such terms in t^n which will occur, till the term containing $(2x-t)^n t^n$, we get the term in t^n

$$=\sum_{r=0}^{N}\frac{(-1)^r(2n-2r)!}{2^n r!(n-r)!(n-2r)!}x^{n-2r}t^n=P_n(x)t^n, \qquad \ldots(12.44)$$

where $N=\left\lfloor\frac{n}{2}\right\rfloor$, the integral value less than or equal to $n/2$. Thus $(1-2xt+t^2)^{-1/2}$ is the generating function for the Legendre polynomials $P_n(x)$.

Example 12.8: SS * Using the generating function of the Legendre polynomials prove that for each integral $n\geq 0$,

(a) $P_n(1)=1$, (b) $P_n(-1)=(-1)^n$, (c) $P_{2n}(0)=(-1)^n\dfrac{(2n+1)!}{2^{2n}(n!)^2}$, and $P_{2n+1}(0)=0$.

Solution:

(a) Set $x=1$ in (12.42), we have $L(1,t)=\dfrac{1}{\sqrt{1-2t+t^2}}=\dfrac{1}{1-t}=\sum\limits_{n=0}^{\infty}P_n(1)t^n$.

Also for $-1<t<1$, $\dfrac{1}{1-t}=\sum\limits_{n=0}^{\infty}t^n$, and thus, $P_n(1)=1$, for each integral $n\geq 0$.

(b) Similarly set $x=-1$ in (12.42), we have $L(-1,t)=\dfrac{1}{\sqrt{1+2t+t^2}}=\dfrac{1}{1+t}=\sum\limits_{n=0}^{\infty}P_n(-1)t^n$.

Also for $-1<t<1$, $\dfrac{1}{1+t}=\sum\limits_{n=0}^{\infty}(-1)^n t^n$, and thus $P_n(-1)=(-1)^n$, for each integral $n\geq 0$.

(c) Further, set $x = 0$ in (12.42), we obtain $L(0, t) = \dfrac{1}{\sqrt{1+t^2}} = (1 + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(0)t^n.$

Also for $t^2 < 1$, $\quad (1 + t^2)^{-1/2} = \sum_{n=0}^{\infty} (-1)^n \dfrac{(2n+1)!}{2^{2n}(n!)^2} t^{2n}$

and thus $\quad P_{2n}(0) = (-1)^n \dfrac{(2n+1)!}{2^{2n}(n!)^2}$ and $P_{2n+1}(0) = 0,\quad$ for $n = 0, 1, 2, \ldots$

12.3.5 Recurrence Relations for Legendre Polynomials

The Legendre polynomials $P_n(x)$ satisfy the following recurrence relations

1. $P_n'(x) = xP_{n-1}'(x) + nP_{n-1}(x)$ ...(12.45)
2. $P_{n+1}'(x) - P_{n-1}'(x) = (2n+1)P_n(x)$ ...(12.46)
3. $xP_n'(x) = nP_n(x) + P_{n-1}'(x)$ ...(12.47)
4. $(1 - x^2)P_{n-1}'(x) = n[xP_{n-1}(x) - P_n(x)]$ ...(12.48)
5. $(1 - x^2)P_n'(x) = n[P_{n-1}(x) - xP_n(x)]$ ...(12.49)
6. $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$ ...(12.50)

Proof. We use Rodrigue's formula (12.38) to prove these relations.

1. We have, $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}[(x^2-1)^n]$. Differentiating w.r.t. x, to obtain

$$P_n'(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}[n(x^2-1)^{n-1}2x] = \frac{1}{2^{n-1}(n-1)!}\frac{d^n}{dx^n}[x(x^2-1)^{n-1}]$$

$$= \frac{1}{2^{n-1}(n-1)!}\left[x\frac{d^n}{dx^n}(x^2-1)^{n-1} + n\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^{n-1}\right] \qquad \text{(using Leibnitz's rule)}$$

$$= x\frac{d}{dx}\left[\frac{1}{2^{n-1}(n-1)!}\frac{d^{n-1}}{dx^{n-1}}[(x^2-1)]^{n-1}\right] + n\frac{1}{2^{n-1}(n-1)!}\frac{d^{n-1}}{dx^{n-1}}[(x^2-1)]^{n-1}$$

$$= xP_{n-1}'(x) + nP_{n-1}(x).$$

2. We have, $\quad P_{n+1}'(x) = \dfrac{d}{dx}\left[\dfrac{1}{2^{n+1}(n+1)!}\dfrac{d^{n+1}}{dx^{n+1}}[(x^2-1)^{n+1}]\right]$

$$= \frac{1}{2^{n+1}(n+1)!}\frac{d^{n+1}}{dx^{n+1}}[(n+1)(x^2-1)^n.2x] = \frac{1}{2^n n!}\frac{d^{n+1}}{dx^{n+1}}[x(x^2-1)^n]$$

$$= \frac{1}{2^n n!}\frac{d^n}{dx^n}[xn(x^2-1)^{n-1}2x + (x^2-1)^n]$$

$$= \frac{1}{2^{n-1}(n-1)!}\frac{d^n}{dx^n}[x^2(x^2-1)^{n-1}] + P_n(x)$$

$$= \frac{1}{2^{n-1}(n-1)!}\frac{d^n}{dx^n}[(x^2-1)^n + (x^2-1)^{n-1}] + P_n(x)$$

$$= \frac{2n}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n] + \frac{d}{dx}\left[\frac{1}{2^{n-1}(n-1)!}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^{n-1}\right] + P_n(x)$$

$$= 2nP_n(x) + P'_{n-1}(x) + P_n(x)$$

Thus, $P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x)$.

From this recurrence relation, there follows another important result concerning the integral of Legendre polynomials, as

$$\int P_n(x)dx = \frac{1}{2n+1}[P_{n+1}(x) - P_{n-1}(x)]. \qquad ...(12.51)$$

3. Replacing n by $(n+1)$ in (12.45) to obtain

$$P'_{n+1}(x) = xP'_n(x) + (n+1)P_n(x),$$

or, $$xP'_n(x) = P'_{n+1}(x) - (n+1)P_n(x) \qquad ...(12.52)$$

Also from (12.46), $$P'_{n+1}(x) = P'_{n-1}(x) + (2n+1)P_n(x) \qquad ...(12.53)$$

Substituting for $P'_{n+1}(x)$ from (12.53) in (12.52) we get (12.47).

4. Subtracting x times multiple of (12.45) from (12.47) to obtain

$$0 = [nP_n(x) + P'_{n-1}(x)] - x[xP'_{n-1}(x) + nP_{n-1}(x)]$$

or, $(1-x^2)P'_{n-1}(x) = n[xP_{n-1}(x) - P_n(x)]$, which is (12.48)

5. Multiplying (12.47) by x and subtracting it from (12.45) to obtain

$$(1-x^2)P'_n(x) = [xP'_{n-1}(x) + nP_{n-1}(x)] - x[nP_n(x) + P'_{n-1}(x)]$$

or, $(1-x^2)P'_n(x) = n[P_{n-1}(x) - xP_n(x)]$, which is (12.49)

6. Replacing n by $(n+1)$ in (12.48) to obtain

$$(1-x^2)P'_n(x) = (n+1)[xP_n(x) - P_{n+1}(x)]$$

or, $$(n+1)P_{n+1}(x) = (n+1)xP_n(x) - (1-x^2)P'_n(x) \qquad ...(12.54)$$

Substituting for $(1-x^2)P'_n(x)$ from (12.49) in (12.54), we obtain (12.50) as

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x),$$

These recurrence relations can also be proved using generating function for $P_n(x)$. For example, to prove (12.49), we can begin as follows.

Differentiating the generating function (12.42) with respect to t, we obtain

$$\frac{\partial L(x,t)}{\partial t} = -\frac{1}{2}(1-2xt+t^2)^{-3/2}(-2x+2t) = \frac{(x-t)}{(1-2xt+t^2)^{3/2}}.$$ This gives

$$(1-2xt+t^2)\frac{\partial L(x,t)}{\partial t} - (x-t)\,L(x,t) = 0. \qquad ...(12.5$$

Substituting $L(x,t) = \sum_{n=0}^{\infty} P_n(x)t^n$ in (12.55) to obtain

$$(1-2xt+t^2)\sum_{n=1}^{\infty} nP_n(x)t^{n-1} - (x-t)\sum_{n=0}^{\infty} P_n(x)t^n = 0$$

or,
$$\sum_{n=1}^{\infty} nP_n(x)t^{n-1} - \sum_{n=1}^{\infty} 2nxP_n(x)t^n + \sum_{n=1}^{\infty} nP_n(x)t^{n+1} - \sum_{n=0}^{\infty} xP_n(x)t^n + \sum_{n=0}^{\infty} P_n(x)t^{n+1} = 0.$$

Making the index of t same in every summation, we obtain

$$\sum_{n=0}^{\infty} (n+1)P_{n+1}(x)t^n - \sum_{n=1}^{\infty} 2nxP_n(x)t^n + \sum_{n=2}^{\infty} (n-1)P_{n-1}(x)t^n - \sum_{n=0}^{\infty} xP_n(x)t^n + \sum_{n=1}^{\infty} P_{n-1}(x)t^n = 0. \qquad ...(12.56)$$

Rewriting it as

$$[P_1(x) - xP_0(x)] + [2P_2(x) - 3xP_1(x) + P_0(x)]t$$

$$+ \sum_{n=2}^{\infty} [(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x)]t^n = 0. \qquad ...(12.57)$$

For (12.57) to be true the coefficient of t^n must be zero for $n = 0, 1, 2, \ldots$

Thus, $P_1(x) - xP_0(x) = 0$, $2P_2(x) - 3xP_1(x) + P_0(x) = 0$

and, for $n \geq 2$, $(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$.

This is valid for $n = 0$ and 1 also; this establishes (12.49) for all integers $n \geq 0$.

Example 12.9: Using the recurrence relation $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$,

(a) Generate the Legendre polynomials P_2, P_3, P_4 given that $P_0(x) = 1$ and $P_1(x) = x$.

(b) Show that the coefficient of x^n in $P_n(x)$ is $a_n = \dfrac{(2^n)!}{2^n(n!)^2}$.

Solution: (a) The recurrence relation is $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$.

For $n = 1$, it gives, $\quad 2P_2(x) = 3xP_1(x) - P_0(x) = 3x^2 - 1$, (using $P_0(x) = 1$ and $P_1(x) = x$)

or, $\quad P_2(x) = (3x^2 - 1)/2.$

For $n = 2$, we have $\quad 3P_3(x) = 5xP_2(x) - 2P_1(x) = 5x(3x^2 - 1)/2 - 2x = 3(5x^3 - 3x)/2$

or, $\quad P_3(x) = (5x^3 - 3x)/2.$

For $n = 3$, we have

$$4P_4(x) = 7xP_3(x) - 3P_2(x) = 7x(5x^3 - 3x)/2 - 3(3x^2 - 2)/2 = (35x^4 - 30x^2 + 3)/2$$

or, $\quad P_4(x) = (35x^4 - 30x^2 + 3)/8.$

(b) We have, $P_1(x) = x$, thus $a_1 = 1$;

$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}, \quad \text{thus } a_2 = \frac{3}{2} = \frac{4!}{2^2(2!)^2}; \; P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x, \quad \text{thus } a_3 = \frac{5}{2} = \frac{6!}{2^3(3!)^2}$$

To prove it in general, equate the coefficient of the highest power of x, that is, x^{n+1} on both sides of the recurrence relation, we obtain $(n+1)a_{n+1} = (2n+1)a_n$, $n \geq 0$.

It gives,
$$a_{n+1} = \frac{2n+1}{(n+1)}a_n = \frac{2n+1}{n+1}\cdot\frac{2n-1}{n}a_{n-1} = \frac{(2n+1)(2n-1)(2n-3)\ldots 3.1.}{(n+1)n(n-1)\ldots 2.1.}a_1$$

$$= \frac{(2n+1)(2n-1)(2n-3)\ldots 3.1.}{(n+1)n(n-1)\ldots 2.1.}, \quad \text{since } a_1 = 1.$$

Hence,
$$a_{n+1} = \frac{(2n+2)!}{[(n+1)!]^2 2^{n+1}}$$

Replacing $n+1$ by n, we get $\quad a_n = \dfrac{(2n)!}{2^n(n!)^2}$, the desired result.

12.3.6 Orthogonality of the Legendre Polynomials on [−1, 1]

We have the following theorem.

Theorem 12.1 (Orthogonality of Legendre Polynomials): *If n and m are non-negative integers, then*

$$\int_{-1}^{1} P_m(x)\,P_n(x)dx = \begin{cases} 0, & m \neq n \\ \dfrac{2}{2n+1}, & m = n \end{cases} \qquad \ldots(12.58)$$

This integral relationship is called *orthogonality* of the Legendre polynomials on [−1, 1].

Proof. We prove (12.58) using the fact that $P_n(x)$ is a solution of the Legendre equation

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0,$$

for $\alpha = n$. Thus for some non-negative integers m and n, we have

$$(1 - x^2)P''_n(x) - 2xP'_n(x) + n(n+1)P_n(x) = 0 \quad ...(12.59)$$

and, $$(1 - x^2)P''_m(x) - 2xP'_m(x) + m(m+1)P_m(x) = 0. \quad ...(12.60)$$

Multiplying (12.59) by $P_m(x)$ and (12.60) by $P_n(x)$ and subtracting the resultant equations, we obtain

$$(1 - x^2)(P''_n P_m - P''_m P_n) - 2x(P'_n P_m - P'_m P_n) + [n(n+1) - m(m+1)]P_n P_m = 0$$

or, $$\frac{d}{dx}[(1 - x^2)(P'_n P_m - P'_m P_n)] + [n(n+1) - m(m+1)]\, P_n P_m = 0.$$

Integrating it over the interval $[-1, 1]$, we obtain

$$[(1 - x^2)(P'_n P_m - P'_m P_n)]_{-1}^{+1} + [n(n+1) - m(m+1)] \int_{-1}^{1} P_n P_m \, dx = 0.$$

The first term vanishes at $x = \pm 1$, and thus for $m \neq n$, we get

$$\int_{-1}^{1} P_m(x) P_n(x) dx = 0, \quad m \neq n.$$

The case $m = n$ can be proved using the generating function (12.42) given by

$$(1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) t^n.$$

Squaring both sides of it and integrating w.r.t. x over $[-1, 1]$, we obtain

$$\int_{-1}^{1} \frac{dx}{(1 - 2xt + t^2)} = \int_{-1}^{1} \left[\sum_{n=0}^{\infty} P_n(x) t^n\right]^2 dx. \quad ...(12.61)$$

From left side of Eq. (12.61), we obtain

$$\int_{-1}^{1} \frac{dx}{(1 - 2xt + t^2)} = \left[\frac{\ln(1 - 2xt + t^2)}{-2t}\right]_{-1}^{1} = -\frac{1}{2t}[\ln(1 - 2t + t^2) - \ln(1 + 2t + t^2)]$$

$$= \frac{1}{t}[\ln(1 + t) - \ln(1 - t)] = 2\left[1 + \frac{t^2}{3} + \frac{t^4}{5} + \dots \frac{t^{2n}}{2n+1} + \dots\right] \quad ...(12.62)$$

Also right side of Eq. (12.61), using the orthogonal property of the Legendre polynomials for $m \neq n$ gives

$$\int_{-1}^{1} \left[\sum_{n=0}^{\infty} P_n(x) t^n\right]^2 dx = \sum_{n=0}^{\infty} \left(\int_{-1}^{1} P_n^2(x)\, dx\right) t^{2n}. \quad ...(12.63)$$

From (12.62) and (12.63), equating the coefficient of t^{2n} gives

$$\int_{-1}^{1} P_n^2(x)dx = \frac{2}{2n+1}, \quad \text{for } n \geq 0.$$

This proves (12.58).

12.3.7 Fourier-Legendre Series

Suppose $f(x)$ is defined and have continuous derivatives over the interval $[-1, 1]$. We want to explore the possibility of expanding $f(x)$ in a series of Legendre polynomials, that is,

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x), \qquad \text{...(12.64)}$$

where the coefficients c_i's are constants.

To determine these coefficients, multiply the proposed expansion (12.64) by $P_m(x)$ and integrate the resulting equation over $[-1, 1]$, interchanging the summation and the integral, we get

$$\int_{-1}^{1} f(x)P_m(x)dx = \sum_{n=0}^{\infty} c_n\left(\int_{-1}^{1} P_n(x)P_m(x)dx\right). \qquad \text{...(12.65)}$$

Because of the orthogonal property, all terms in the summation on the right side of Eq. (12.65) are zeros, except when $n = m$, and thus

$$\int_{-1}^{1} f(x)P_m(x)dx = c_m \frac{2}{2m+1},$$

or,
$$c_m = \frac{2m+1}{2}\int_{-1}^{1} f(x)P_m(x)dx, \quad m = 0, 1, 2, \ldots \qquad \text{...(12.66)}$$

The coefficients c_m's are called the *Fourier-Legendre coefficients* of f, and the resultant series is called the *Fourier-Legendre series*.

As a special case of Fourier-Legendre expansion, any polynomial $f(x)$ is a linear combination of Legendre polynomials. Thus in case of a polynomial, this series can be obtained by just solving for x^n in terms of $P_n(x)$ and writing each power of x in $f(x)$ in terms of Legendre polynomials.

For example, let $f(x) = 4x^3 - 2x^2 - 3x + 8$. Write

$$f(x) = c_0P_0(x) + c_1P_1(x) + c_2P_2(x) + c_3P_3(x)$$

or,
$$4x^3 - 2x^2 - 3x + 8 = c_0(1) + c_1(x) + c_2\left(\frac{3x^2-1}{2}\right) + c_3\left(\frac{5x^3-3x}{2}\right).$$