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ROLL No → 66

BATCH → B-5

MATHS - ASSIGNMENT - 2

1. Find the general solution of the following differential equations.

(a) $y'' - 8y' + 16y = 0$

$\rightarrow$ $(D^2 - 8D + 16)\,y = 0$.

Roots of AE is

$[D^2 - 2\times4\times D + (4)^2] \Rightarrow 0$

$(D-4)(D-4) = 0$

$D = 4, 4$.

CF sol^n is $\Rightarrow$ $\underline{(c_1 + c_2 x)e^{4x}}$.

General sol^n $\longrightarrow$ $(c_1 + c_2 x)\,e^{4x}$.

(b) $y'''' - 4y''' + 8y'' - 8y' + 4y \Rightarrow 0$.

$\Rightarrow$ A.E equation is $\longrightarrow$

$(D^4 - 4D^3 + 8D^2 - 8D + 4)\,y \Rightarrow 0$.

$\Rightarrow (D^4 + 4D^2 + 4 - 4D^3 + 4D^2 - 8D = 0$

$\Rightarrow (D^2)^2 + (-2D)^2 + (2)^2 - 4D^3 + 4D^2 - 8D = 0$

$\Rightarrow (D^2 - 2D + 2)^2 \Rightarrow 0$

$(D^2 - 2D + 2)(D^2 - 2D + 2) = 0$,

$D^2 - 2D + 2 = 0$

$(D-1)^2 = \pm i$

$D = 1 \pm i, 1 \pm i$.

so, sol^n is

$y = e^x[(c_1 x + c_2)\cos x + (c_3 x + c_4)\sin x]$

(c) $4y'''' - 4y''' - 23y'' + 12y' + 36y \Rightarrow 0.$

$\Rightarrow$ It's AE equation is $\Rightarrow$

$\Rightarrow (4D^4 - 4D^3 - 23D^2 + 12D + 36)y \Rightarrow 0$

$\Rightarrow (-2D^2)^2 + (6)^2 + (D)^2 - 24D^2 + 12D - 4D^3 = 0$

$\Rightarrow (-2D^2 + 6 + D)^2 \Rightarrow 0$

- roots are $\Rightarrow$

$2D^2 - D - 6 = 0$

$(2D^2 - 4D + 3D - 6) = 0$

$2D(D-2) + 3(D-2) \Rightarrow 0$

$D = 2, \quad D = \frac{-3}{2}.$

Roots are $2, 2, \frac{-3}{2}, \frac{-3}{2}.$

solution is $\Rightarrow (c_1 + c_2 x)e^{2x} + (c_3 x + c_4)e^{-3/2 x}$

Q2 Find the general solution of the following non-homogeneous differential equations:

(a) $(D^2 + a^2)y \Rightarrow \cot ax.$

$\rightarrow$ Auxiliary equation $\Rightarrow$

$D^2 + a^2 = 0.$

$\boxed{D \Rightarrow \pm ia.}$

so,

complementary function

$u\ y \Rightarrow c_1 \cos ax + c_2 \sin ax$

For, Particular integral $\Rightarrow$

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$$\Rightarrow y \Rightarrow \frac{\cot ax}{(D^2+a^2)}$$

$$y = \frac{\cot ax}{(D+ia)(D-ia)}$$

$$y = \frac{\cot ax}{2ia}\left[\frac{(D+ia)-(D-ia)}{(D+ia)(D-ia)}\right]$$

$$y = \frac{\cot ax}{2ia}\left[\frac{1}{(D-ia)} - \frac{1}{(D+ia)}\right]$$

$$y \Rightarrow \frac{\cot ax}{2ia}\left[\frac{1}{D-ia} - \frac{1}{D+ia}\right]$$

$$y = \frac{e^{iax}}{2ia}\int e^{-iax}\cot ax\,dx - e^{-iax}\int e^{+iax}\cot ax\,dx$$

$$y = \frac{e^{iax}}{2ia}\left[\int\frac{(\cos ax - i\sin ax)\cos ax\,dx}{\sin ax}\right] - \frac{e^{-iax}}{2ia}\left[\int\frac{(\cos ax + i\sin ax)\cos ax}{\sin ax}\,dx\right]$$

$$y = \frac{e^{iax}}{2ia}\left[\int\left(\frac{\cos^2 ax}{\sin ax} - i\cos ax\right).dx\right] - \frac{e^{-iax}}{2ia}\left[\int\left(\frac{\cos^2 ax}{\sin ax} + i\cos ax\right)dx\right]$$

$$y \Rightarrow \frac{e^{iax}}{2ia}\left[\int\left(\frac{1-\sin^2 ax}{\sin ax} - i\cos ax\right)dx\right]$$

$$-\frac{e^{-iax}}{2ia}\left[\int\left(\frac{1-\sin^2 ax}{\sin ax} + i\cos ax\right)dx\right]$$

$$y \Rightarrow \frac{e^{iax}}{2ia}\left[\int (\operatorname{cosec} ax - \sin ax - i\cos ax)\,dx\right.$$

$$\left. - \frac{e^{-iax}}{2ia}\left[\int (\operatorname{cosec} ax - \sin ax + i\cos ax)\,dx\right.\right.$$

$$y \Rightarrow \frac{e^{iax}}{2ia^2}\left[\log|\operatorname{cosec} ax - \cot ax| + \cos ax - i\sin ax\right]$$

$$- \frac{e^{-iax}}{2ia^2}\left[\log|\operatorname{cosec} ax - \cot ax| + \cos ax + i\sin ax\right]$$

$$y = \left(\frac{e^{iax} - e^{-iax}}{2ia^2}\right)\log|\operatorname{cosec} ax - \cot ax|$$

$$+ \frac{\cos ax}{2ia^2}\left(e^{iax} - e^{-iax}\right)$$

$$- \frac{i\sin ax}{2ia^2}\left(e^{iax} + e^{-iax}\right)$$

$$y = \frac{\sin ax}{a^2}\log|\operatorname{cosec} ax - \cot ax| + \cancel{\frac{\cos ax \sin x}{a^2}}$$

$$- \cancel{\frac{\cos ax \sin ax}{a^2}}$$

$$y = \frac{\sin ax}{a^2}\log|\operatorname{cosec} ax - \cot ax|.$$

$\therefore$ Complete sol^n $\Rightarrow$ C.F + I.P $= c_1\cos ax + c_2\sin ax$

$$+ \frac{\sin ax}{a^2}\log|\operatorname{cosec} ax - \cot ax|$$

2nd approach :→

using Method of variation of parameter :→

y_c for $(D^2 + a^2)y = 0$

Roots of AE are $\pm ia$.

$\therefore y_c(x) = c_1 \cos ax + c_2 \sin ax$.

$$W = \begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix} \Rightarrow a\begin{bmatrix} \cos ax & \sin ax \\ -\sin ax & \cos ax \end{bmatrix} = a.$$

$$y_p(x) = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$$

$$\Rightarrow \frac{-\cos ax}{a} \int (\cot ax) \times \sin ax + \frac{\sin ax x}{a} \int (\cot ax) \cos ax$$

$$\Rightarrow \frac{-\cos ax}{a} \times \int \cos ax \, dx + \frac{\sin ax}{a} \int \left(\frac{1 - \sin^2 ax}{\sin ax} \right) dx.$$

$$\Rightarrow \frac{-\cos ax \sin ax}{a^2} + \frac{\sin ax}{a} \times \int (\operatorname{cosec} ax - \sin ax) dx$$

$$\Rightarrow \cancel{\frac{-\cos ax \sin ax}{a^2}} + \frac{\sin ax}{a^2} \log|\operatorname{cosec} ax - \cot ax| + \cancel{\frac{\sin ax}{a^2}} \cos ax$$

$$\Rightarrow \frac{\sin ax}{a^2} \log|\operatorname{cosec} ax - \cot ax|$$

$$y(x) = c_1 \cos ax + c_2 \sin ax + \frac{\sin ax}{a^2} \log|\operatorname{cosec} ax - \cot ax|$$

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(b) $(D^3 - D^2 - 6D)y \Rightarrow x^2 + 1 + 3^x$

$\Rightarrow$ It's AE equation roots are $\div\rightarrow$

$(D^3 - D^2 - 6D) = 0$

$D(D^2 - D - 6) = 0$

$D = 0 \qquad D^2 - 3D + 2D - 6 = 0$

$D(D-3) + 2(D-3) = 0$

$(D+2)(D-3) = 0$

$D = -2, 0, 3.$

so, its C.F is

$y = c_1 e^{-2x} + c_2 + c_3 e^{3x}.$

P.I $\div\rightarrow$

$$y = \frac{x^2}{(D^3 - D^2 - 6D)} + \frac{1}{(D^3 - D^2 - 6D)} + \frac{3^x}{(D^3 - D^2 - 6D)}$$

$$y = \frac{x^2}{(D^3 - D^2 - 6D)} + \frac{e^{0x}}{D^3 - D^2 - 6D} + \frac{3^x}{D^3 - D^2 - 6D}$$

(3)

For $\dfrac{x^2}{D^3 - D^2 - 6D} \Rightarrow \dfrac{-1}{6D}\left[\dfrac{1}{1 + \frac{D}{6} - \frac{D^2}{6}}\right] x^2$

$$\Rightarrow \frac{-1}{6D}\left[1 + \left(\frac{D}{6} - \frac{D^2}{6}\right)\right]^{-1} x^2$$

$$\Rightarrow \frac{-1}{6D}\left[1 - \left(\frac{D}{6} - \frac{D^2}{6}\right)x^2 + \left(\frac{D}{6} - \frac{D^2}{6}\right)^2 x^2\right]$$

$$\Rightarrow \frac{-1}{6D}\left[\left(1-\frac{D}{6}+\frac{D^2}{6}+\frac{D^2}{36}\right)x^2\right]$$ ($\because$ Highter terms make derivative $x^2 = 0$)

$$\Rightarrow \frac{-1}{6D}\left[1-\frac{D}{6}+\frac{7D^2}{36}\right]x^2$$

$$\Rightarrow \frac{-1}{6D}\left[x^2-\frac{2x}{6}+\frac{7}{36}\times 2\right]$$

$$\Rightarrow \frac{-1}{6D}\left[\frac{7}{18}-\frac{x}{3}+x^2\right]$$

$$\Rightarrow \frac{-1}{6}\int \frac{7}{18}-\frac{x}{3}+x^2$$

$$= \frac{-1}{6}\left[\frac{7x}{18}-\frac{x^2}{6}+\frac{x^3}{3}\right]$$

$$\Rightarrow \frac{-17x}{108}+\frac{x^2}{36}-\frac{x^3}{18} \quad \text{......} (i)$$

For,

$$\frac{1}{D^3-D^2-6D}$$

$$\Rightarrow \frac{e^{0x}}{D^3-D^2-6D}$$ ($\because f(0) = 0$ So PI $\Rightarrow \frac{x}{f'(0)}$)

$$\Rightarrow \frac{x\times 1}{-6}$$

$$\Rightarrow \frac{-x}{6} \quad \text{......} (ii)$$

For $\dfrac{3^x}{D^3 - D^2 - 6D}$

$\Rightarrow \dfrac{e^{x\log 3}}{D^3 - D^2 - 6D}$

$\Rightarrow \dfrac{e^{x\log 3}}{(\log 3)^3 - (\log 3)^2 - 6\log 3}$ ..... (iii)

So, complete solution is

$$c_1 e^{-2x} + c_2 + c_3 e^{3x} \frac{-17x}{108} - \frac{x}{6} + \frac{x^2}{36} - \frac{x^3}{18}$$
$$+ \frac{e^{x\log 3}}{(\log 3)^3 - (\log 3)^2 - 6\log 3}.$$

(c) $(D^4 + D^2 + 1)y \Rightarrow e^{-x/2}\cos\left(\frac{\sqrt{3}x}{2}\right)$.

AE :-

$D^4 + D^2 + 1 \Rightarrow 0.$

$(D^2+1)^2 - D^2 = 0.$

$(D^2 + D + 1)(D^2 - D + 1) = 0.$

$D \Rightarrow \dfrac{-1 \pm \sqrt{3}i}{2}, \dfrac{1 \pm \sqrt{3}i}{2}$

CF :- $y \Rightarrow e^{-x/2}\left[c_1 \cos\frac{\sqrt{3}x}{2} + c_2 \sin\frac{\sqrt{3}x}{2}\right]$

$+ e^{x/2}\left[c_3 \cos\frac{\sqrt{3}}{2} + c_4 \sin\frac{\sqrt{3}x}{2}\right].$

$$P.I \longrightarrow \frac{e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right)}{(D^4 + D^2 + 1)}$$

$$\longrightarrow \frac{e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right)}{\left[\left(D-\frac{1}{2}\right)^4 + \left(D-\frac{1}{2}\right)^2 + 1\right]}$$

$$\Rightarrow \frac{e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right)}{\left[\left(D-\frac{1}{2}\right)^2 + \left(D-\frac{1}{2}\right)+1\right]\left[\left(D-\frac{1}{2}\right)^2 - \left(D-\frac{1}{2}\right)+1\right]}$$

$$\Rightarrow \frac{e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right)}{\left[D^2 + \frac{3}{4}\right]\left[D^2 - 2D + \frac{7}{4}\right]}$$

$D^2 = \frac{-3}{4}$

Since $f(D) = 0$,

So, $\frac{\partial}{\partial D}\left[D^2 + \frac{3}{4}\right]\left[D^2 - 2D + \frac{7}{4}\right]$.

$$\Rightarrow \frac{e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right)x}{2D\left(D^2 - 2D + \frac{7}{4}\right) + \left(D^2 + \frac{3}{4}\right)(2D - 2)}$$

put $D^2 = \frac{-3}{4}$.

$$\Rightarrow \frac{e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right)x}{2D(1 - 2D)}$$

$$\Rightarrow e^{-x/2} \frac{\cos\left(\frac{\sqrt{3}x}{2}\right) x}{2D - 4D^2}$$

$$\Rightarrow e^{-x/2} \left[\frac{\cos\left(\frac{\sqrt{3}}{2}x\right) x}{2D - 4\times\left(\frac{-3}{4}\right)}\right]$$

$$\Rightarrow e^{-x/2} \left[\frac{\cos\left(\frac{\sqrt{3}}{2}x\right) x}{2D + 3}\right]$$

$$\Rightarrow e^{-x/2} \frac{(2D-3)}{(2D+3)(2D-3)} \left[x\cos\left(\frac{\sqrt{3}}{2}x\right)\right]$$

$$\Rightarrow \frac{e^{-x/2}}{4D^2 - 9} (2D-3) \left[x\cos\left(\frac{\sqrt{3}x}{2}\right)\right]$$

$$\Rightarrow \frac{e^{-x/2}}{-12} [2D-3] \left[x\cos\frac{\sqrt{3}x}{2}\right]$$

$$D^2 = \frac{-3}{4}$$

$$\Rightarrow \frac{-e^{-x/2}}{12} \left[2\times D\left(x\cos\frac{\sqrt{3}x}{2}\right) - 3x\cos\frac{\sqrt{3}x}{2}\right]$$

$$\Rightarrow \frac{-e^{x/2}}{12} \left[2\left(\cos\frac{\sqrt{3}x}{2} - \frac{\sqrt{3}x}{2}\sin\frac{\sqrt{3}x}{2}\right) - 3x\cos\frac{\sqrt{3}x}{2}\right)$$

$$\Rightarrow \frac{-e^{x/2}}{6}\cos\frac{\sqrt{3}x}{2} + \frac{e^{x/2}\sqrt{3}x}{12}\sin\frac{\sqrt{3}x}{2} + \frac{e^{x/2}x}{4}\cos\frac{\sqrt{3}x}{2}$$

$\Rightarrow$ complete solution $\Rightarrow$

$$e^{-x/2}\left[c_1\cos\frac{\sqrt{3}x}{2} + c_2\sin\frac{\sqrt{3}x}{2}\right] + e^{x/2}\left[c_3\cos\frac{\sqrt{3}x}{2} + c_4\sin\frac{\sqrt{3}x}{2}\right]$$

$$-\frac{e^{x/2}}{6}\cos\frac{\sqrt{3}x}{2} + \frac{\sqrt{3}xe^{x/2}}{72}\sin\frac{\sqrt{3}x}{2} + \frac{xe^{x/2}}{24}\cos\frac{\sqrt{3}x}{2}$$

(d) $(D^4+2D^2+1)y \Rightarrow x^2\cos x$

$\rightarrow$ AE $\Rightarrow$ $(D^4+2D^2+1)=0$

$(D^2+1)^2=0$.

$(D^2+1)(D^2+1)=0$.

$D \Rightarrow \pm i, \pm i$

CF $\Rightarrow$ $(c_1x+c_2)\cos x + (c_3x+c_4)\sin x$

PI $\Rightarrow$ $y \Rightarrow \dfrac{x^2\cos x}{D^4+2D^2+1}$

$$\Rightarrow \text{Real part of}\left(\frac{x^2e^{ix}}{D^4+2D^2+1}\right)$$

$$y \Rightarrow \text{Real part of } e^{ix}\left[\frac{x^2}{(D+i)^4+2(D+i)^2+1}\right]$$

$$y \Rightarrow \text{Real part of } e^{ix}\times\frac{x^2}{[D^2+2iD]^2}$$

$$y \Rightarrow \text{Real part of } e^{ix}\times\frac{x^2}{-4D^2\left[1+\frac{D}{2i}\right]^2}$$

$$\Rightarrow \text{Real part of } e^{ix} \times \left[\frac{-1}{4D^2}\left(1 - \frac{Di}{2}\right)^{-2} x^2\right]$$

$$\Rightarrow \text{R.P of } e^{ix}\left[\frac{-1}{4D^2}\left(1 + Di + 3\left(\frac{iD}{2}\right)^2 + \ldots\right)x^2\right]$$

$$\Rightarrow \text{R.P of } e^{ix}\left[\frac{-1}{4D^2}\left[x^2 + 2ix - \frac{3}{4} \times 2\right)\right.$$

$$\Rightarrow \text{R.P of } e^{ix}\left[-\frac{1}{4D^2}\left[x^2 + 2ix - \frac{3}{2}\right]\right)$$

$$\Rightarrow \text{R.P of}\left(\frac{-e^{ix}}{4}\right)\left[\frac{1}{D^2}\left(x^2 + 2ix - \frac{3}{2}\right)\right]$$

$$\Rightarrow \text{R.P. of}\left(-\frac{e^{ix}}{4}\right)\left[\frac{1}{D}\left(\frac{x^3}{3} + ix^2 - \frac{3}{2}x\right)\right]$$

$$\Rightarrow \text{R.P. of}\left(-\frac{e^{ix}}{4}\right)\left[\frac{x^4}{12} + \frac{ix^3}{3} - \frac{3}{4}x^2\right]$$

$$\Rightarrow \text{R.P of}\left[\frac{-x^4 e^{ix}}{48} - \frac{ix^3 e^{ix}}{12} + \frac{3}{16}x^2 e^{ix}\right]$$

$$\Rightarrow -\frac{x^4 \cos x}{48} + \frac{\sin x \, x^3}{12} + \frac{3}{16}x^2 \cos x$$

$$\Rightarrow \frac{1}{48}\left[4x^3 \sin x - (x^4 - 9x^2)\cos x\right]$$

C.S $= CF + PI$

$\Rightarrow (c_1x+c_2)\cos x + (c_3x+c_4)\sin x + \frac{1}{48}\left[4x^3\sin x - (x^4-9x^2)\cos x\right]$.

3. consider the differential equations

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0.$$

(a) show that e^x and xe^x are two linearly independent solutions for all $x \in R$.

$\Rightarrow$ $y = c_1e^x + c_2xe^x$.

$y_1 = e^x$ and $y_2 = xe^x$.

$y_1' = e^x$ and $y_2' = e^x + xe^x$

$$w(x) = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix}.$$

$\Rightarrow e^x(e^x + xe^x) - e^x(xe^x)$

$\Rightarrow e^{2x}. \neq 0$ (exponential never equal to 0).

so, By Wronskian,

e^x and xe^x are linearly independent.

Fundamental theorem $\Rightarrow$

~~Sin~~ e^x and xe^x form set of solutions of (1).

(b) General solution $\Rightarrow$

$y = c_1e^x + c_2xe^x \Rightarrow (c_1+c_2x)e^x$

(c) Find a particular solution which satifies $y(0) = 1$, $y'(0) = 4$.

$\rightarrow$ $y(0) = 1$

$[c_1 + c_2(0)]e^0 = 1$

$\boxed{c_1 = 1}$

$y'(x) \Rightarrow (c_1 + c_2 x)e^x + c_2 e^x$

$y'(0) \Rightarrow (c_1 + c_2)$

$4 - 1 \Rightarrow c_2$

$3 = c_2$.

$\therefore$ Particular solution is

$y = \underline{(1+3x)\, e^x}$.

(4) consider $(D-m_1)(D-m_2)y = 0$ such that $m_1 = m_2$. and $m_1, m_2 \in R$. Then show that general solution of the equation reads. $(c_1 x + c_2)e^{m_1 x}$.

$\Rightarrow$ Let $(D-m_1)y = z$.

$(D-m_1)(D-m_1)y = 0$

$(D-m_1)z = 0$

$\frac{dz}{dx} - m_1 z = 0$

$z \times e^{-m_1 x} \Rightarrow c$

$\boxed{z = c_1 e^{m_1 x}}$ ...... (i).

From eq(i)

$$\frac{dy}{dx} - m_1 y = c_1 e^{m_1 x}$$

$$y \times e^{-m_1 x} = \int c_1 e^{m_1 x - m_1 x}$$

$$y e^{-m_1 x} = c_1 x + c_2$$

$$\boxed{y = (c_1 x + c_2) e^{m_1 x}}$$

6 (5) Find the solution of the following differential equations using method of variation of parameters.

(a) $y'' - 3y' + 2y \Rightarrow \frac{e^x}{1+e^x}$.

$\rightarrow$ AE $\longrightarrow$

$$(D^2 - 3D + 2) = 0$$

$$(D-2)(D-1) = 0$$

$$D = 2, 1.$$

$$CF = c_1 e^{x} + c_2 e^{2x}.$$

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$$\text{wronskian}(w) = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} \Rightarrow e^{3x}.$$

$$P.S \Rightarrow -y_1 \int \frac{y_2 X}{\omega} dx + y_2 \int \frac{y_1 X}{\omega} dx$$

$$\Rightarrow -e^x \int \frac{e^{2x} \times e^x}{(1+e^x) e^{3x}} dx + e^{2x} \int \frac{e^x \times e^x}{(1+e^x) e^{3x}} dx$$

$$\Rightarrow -e^{x}\int \frac{dx}{(1+e^{x})} + e^{2x}\int \frac{dx}{(1+e^{x})e^{x}}$$

$$\Rightarrow -e^{x}\int \left(\frac{e^{-x}}{e^{-x}+1}\right)dx + e^{2x}\int \frac{(e^{x}+1)-e^{x}}{(e^{x}+1)e^{x}}$$

$$\Rightarrow +e^{x}\ln|e^{-x}+1| + e^{2x}\int\left(\frac{1}{e^{x}} - \frac{1}{e^{x}+1}\right)dx$$

$$\Rightarrow e^{x}\ln|e^{-x}+1| + e^{2x}\int (e^{-x})dx - \left(\frac{e^{-x}}{e^{-x}+1}\right)dx$$

$$\Rightarrow e^{x}\ln|e^{-x}+1| - e^{2x}\times e^{-x} + e^{2x}\ln|e^{-x}+1|$$

$$\Rightarrow e^{x}\ln|e^{-x}+1| - e^{x} + e^{2x}\ln|e^{-x}+1|.$$

C.S $\Rightarrow$ C.F + P.I

$$\Rightarrow c_1e^{x} + c_2e^{2x} + e^{x}\ln|e^{-x}+1| - e^{x} + e^{2x}\ln|e^{-x}+1|$$

(b) $y'' - y \Rightarrow e^{-x}\sin(e^{-x}) + \cos(e^{-x})$.

$\Rightarrow$ A.E $\Rightarrow$

$(D^2-1)y = 0.$

$\boxed{D \Rightarrow \pm 1.}$

$$\boxed{CF = c_1e^{x} + c_2e^{-x}}$$

$$w(x) = \begin{vmatrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \end{vmatrix} \Rightarrow -2.$$

$$\text{P.I.} \Rightarrow -e^{x}\int \frac{e^{-x}\left[e^{-x}\sin(e^{-x}) + \cos(e^{-x})\right]}{-2}\,dx \quad (1)$$

$$+ e^{-x}\int \frac{e^{x}\left[e^{-x}\sin(e^{-x}) + \cos(e^{-x})\right]}{-2}\,dx \quad (2)$$

solving integral (1)

$$\frac{+e^{x}}{2}\int e^{-x}\left(e^{-x}\sin e^{-x} + \cos e^{-x}\right)dx$$

let $e^{-x} \Rightarrow t$, $-e^{-x}dx = dt$

$$\Rightarrow -\frac{e^{x}}{2}\int \cancel{t}\,(t\sin t + \cos t)\,dt.$$

$$\Rightarrow -\frac{e^{x}}{2}\int \underset{\text{I}}{t\sin t\,dt} + \cos t\,dt.$$

Applying by part.

$$\Rightarrow -\frac{e^{x}}{2}\left[(-t\cos t) + \int \cos t\,dt + \int \cos t\,dt\right.$$

$$\Rightarrow -\frac{e^{x}}{2}\left[-t\cos t + 2\sin t\right] \Rightarrow -\frac{e^{x}}{2} \times \left[-e^{-x}\cos(e^{-x}) + 2\sin e^{-x}\right]$$

$$\Rightarrow \frac{\cos(e^{-x}) - 2e^{x}\sin(e^{-x})}{2}$$

solving integral 2 $\longrightarrow$

$$\Rightarrow \frac{-e^{-x}}{2}\int e^{x}\left[e^{-x}\sin(e^{-x}) + \cos(e^{-x})\right]dx$$

$$\Rightarrow -\frac{e^{-x}}{2}\int e^{x}\left[\cos(e^{-x}) + e^{-x}\sin(e^{-x})\right]dx.$$

$$\Rightarrow \frac{-e^{-x}}{2} \times e^{x} \times \cos(e^{-x}) \qquad \left(\because \int e^{x}(f(x) + f'(x))dx \Rightarrow e^{x}f(x)\right).$$

$$\Rightarrow \frac{-e^{-x+x}}{2}\cos(e^{-x})$$

$$\Rightarrow \frac{-\cos(e^{-x})}{2}.$$

so,

$$PI \Rightarrow \frac{\cos(e^{-x}) - 2e^{x}\sin(e^{-x}) - \cos(e^{-x})}{2}.$$

$$\Rightarrow \underline{-e^{x}\sin(e^{-x})}.$$

$$C.S \Rightarrow CF + P.I.$$

$$= c_1e^{x} + c_2e^{-x} - e^{x}\sin(e^{-x}).$$

(c) $y'' + y = \dfrac{1}{1+\sin x}.$

→ ~~CF~~ AE :→

$(D^2+1) = 0$, $D \Rightarrow \pm i$

so,

CF solution is

$y = c_1 \sin x + c_2 \cos x$.

~~wronskian test~~

$$\text{wronskian}(W) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} \Rightarrow -1.$$

$$P.I = +\sin x \int \frac{\cos x}{(1+\sin x)} - \cos x \int \frac{\sin x}{1+\sin x} dx$$

$$P.I = \sin x \ln|1+\sin x| - \cos x \int \frac{\sin x(1-\sin x)}{(1+\sin x)(1-\sin x)} dx$$

$$P.I \Rightarrow \sin x \ln|1+\sin x| - \cos x \int \frac{(\sin x - \sin^2 x)}{\cos^2 x} dx$$

$$P.I = \sin x \ln|1+\sin x| - \cos x \int (\sec x \tan x - \tan^2 x) dx$$

$$P.I = \sin x \ln|\sin x+1| - \cos x \int (\sec x \tan x - \sec^2 x + 1) dx$$

$$P.I = \sin x \ln|\sin x+1| - \cos x(\sec x) + \cos x(\tan x) - (x)\cos x$$

$$P.I. = \sin x \ln|\sin x+1| - 1 + \sin x - x\cos x.$$

complete solⁿ :→

$c_1 \sin x + c_2 \cos x + \sin x \ln|\sin x + 1| - 1 + \sin x - x\cos x$

(6) Find the complete solution of the following Euler's - cauchy equation.

(a) $x^3 \frac{d^3y}{dx^3} - 4x^2\frac{d^2y}{dx^2} + 8x\frac{dy}{dx} - 8y \Rightarrow 4\ln x$

→ Let $\ln x \Rightarrow t$.

$\frac{dy}{dx} \Rightarrow \frac{dy}{dt} \times \frac{dt}{dx}$

$\frac{dy}{dx} \Rightarrow \frac{dy}{dt}\frac{1}{x}$

$\boxed{\frac{x\,dy}{dx} \Rightarrow \frac{dy}{dt}} \Rightarrow Dy$

similarly

$x^2\frac{d^2y}{dx^2} \Rightarrow \frac{d^2y}{dt^2} - \frac{dy}{dt} \Rightarrow D(D-1)y$

and $x^3\frac{d^3y}{dx^3} \Rightarrow D(D-1)(D-2)y$.

→ $D(D-1)(D-2)y - 4D(D-1)y + 8Dy - 8 \Rightarrow 4t$

→ $D(D^2-3D+2)y - 4(D^2-D)y + 8Dy - 8 \Rightarrow 4t$

→ $(D^3 - 3D^2 + 2D - 4D^2 + 4D + 8D - 8)y = 4t$

→ $(D^3 - 7D^2 + 14D - 8)y \Rightarrow 4t$

→

$AE \Rightarrow D^3 - 7D^2 + 14D - 8$

$\Rightarrow (D-1)(D-2)(D-4)$

$D = 1, 2, 4.$

So $CF = c_1 e^{t} + c_2 e^{2t} + c_3 e^{4t}$

$$PI \Rightarrow \frac{4t}{(D-1)(D-2)(D-4)}$$

$$PI \Rightarrow \left[\frac{1}{3}\left(\frac{1}{D-1}\right) - \frac{1}{2}\left(\frac{1}{D-2}\right) + \frac{1}{6}\left(\frac{1}{D-4}\right)\right] 4t$$

$$PI \Rightarrow \left[\frac{4}{3}\left(\frac{t}{D-1}\right) - \frac{4}{2}\left(\frac{t}{D-2}\right) + \frac{4}{6}\left(\frac{t}{D-4}\right)\right]$$

$$\Rightarrow \frac{4}{3} e^{t} \int t e^{-t} dt - 2e^{2t} \int t e^{-2t} dt + \frac{2}{3} e^{4t} \int t e^{-4t} dt$$

$$\Rightarrow \frac{4e^{t}}{3}\left[-te^{-t} - e^{-t}\right] - 2e^{2t}\left[-\frac{te^{-2t}}{2} - \frac{e^{-2t}}{4}\right]$$

$$+ \frac{2}{3} e^{4t}\left[-\frac{te^{-4t}}{4} - \frac{e^{-4t}}{16}\right]$$

$$\Rightarrow \frac{-4t}{3} - \frac{4}{3} + \frac{2t}{2} + \frac{1}{2} + \frac{-t}{6} - \frac{1}{24}$$

$$\Rightarrow -\frac{t}{2} - \frac{7}{8}.$$

$$C.S \Rightarrow c_1 e^{\ln x} + c_2 e^{2\ln x} + c_3 e^{4\ln x} - \frac{\ln x}{2} - \frac{7}{8}$$

$$\Rightarrow c_1 x + c_2 x^2 + c_3 x^4 - \frac{\ln x}{2} - \frac{7}{8}.$$

(b) $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \ln x \sin(\ln(x))$.

$\rightarrow$ let $x = e^t$. $\quad D = \frac{d}{dt}$.

$\ln x = t$.

$[D(D-1) + D + 1]y = t\sin t$.

$[D^2 - \cancel{D + D} + 1]y = t\sin t$

$(D^2+1)y = t\sin t$.

~~CF~~. A.E. $\Rightarrow D^2+1$

C.F $= D^2+1 \Rightarrow 0$

$\boxed{D = \pm i}$

$\boxed{y = c_1\cos t + c_2\sin t.}$

$$P.I = \frac{t\sin t}{D^2+1}$$

$$= \text{Im. P of}\left(\frac{te^{it}}{D^2+1}\right)$$

$$\Rightarrow \text{I.P of } e^{it}\left(\frac{t}{(D+i)^2+1}\right)$$

$$\Rightarrow \text{I.P of } e^{it}\left(\frac{t}{D^2+2iD.}\right)$$

$$\Rightarrow \text{I.P of } e^{it} \times \frac{1}{2iD}\left[\frac{t}{1+\frac{D}{2i}}\right]$$

$\Rightarrow$ I.P of $\frac{e^{it}}{2iD}\left[1-\frac{D}{2i}\right]t$

$\Rightarrow$ I.P of $\frac{e^{it}}{2iD}\left[t-\frac{Dt}{2i}\right]$

$\Rightarrow$ I.P of $\frac{e^{it}}{2iD}\times\left[t-\frac{1}{2i}\right]$

$\Rightarrow$ I.P of $\frac{e^{it}}{2i}\times\left[\frac{t^2}{2}-\frac{t}{2i}\right]$

I.P of $\frac{-(\cos t+i\sin t)\, i}{2}\left[\frac{t^2}{2}+\frac{t\,i}{2}\right]$

$\Rightarrow \frac{(-\cos t\, i \cancel{+} \sin t)}{2}\left[\frac{t^2}{2}+\frac{t\,i}{2}\right]$

$= \frac{-(\cos t)t^2}{4}+\frac{t\sin t}{4}$

complete solⁿ :-

$\Rightarrow c_1\cos t+c_2\sin t-\frac{(\cos t)t^2}{4}+\frac{t\sin t}{4}$

$\Rightarrow c_1\cos(\ln x)+c_2\sin(\ln x)-\frac{\cos(\ln x)(\ln x)^2}{4}$
$+\frac{(\ln x)\sin(\ln x)}{4}$

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7. Solve the following system of simultaneous differential equations.

(a) $\frac{dx}{dt} + 4x + 3y = t$ and $\frac{dy}{dt} + 2x + 5y = e^t$.

$\Rightarrow$ let $\frac{d}{dt} = D$

$(D+4)x + 3y = t$ ... (i)

$2x + (D+5)y = e^t$ ... (ii)

multiplying eq (ii) by 2 and operating $(D+4)$ in (ii) and subtracting (ii) from (i)

$[6 - (D+5)(D+4)]y \Rightarrow 2t - (D+4)e^t$

$[6 - (D^2+9D+20)]y = 2t - e^t - 4e^t$

$(D^2+9D+20-6)y \Rightarrow 5e^t - 2t$

$(D^2+9D+14)y \Rightarrow 5e^t - 2t$

A.E $\Rightarrow$ $D^2 + 9D + 14$.

$(D^2 + 7D + 2D + 14.) \Rightarrow 0$

$D(D+7) + 2(D+7) \Rightarrow 0$

$(D+2)(D+7) = 0$

$D = -2, -7$.

so, $CF = c_1 e^{-2t} + c_2 e^{-7t}$.

$$P.I \Rightarrow \frac{5e^t}{(D+2)(D+7)} - \frac{2t}{(D+2)(D+7)}$$

$$P.I \Rightarrow \frac{5e^t}{3 \times 8} - \frac{2t}{(D^2+9D+14)}$$

$$P.I = \frac{5e^t}{24} - \frac{2t}{14\left[1+\frac{9D}{14}+\frac{D^2}{14}\right]}$$

$$P.I \Rightarrow \frac{5e^t}{24} - \frac{1}{7}\left[1+\frac{9D}{14}\right]^{-1} t$$

$$\Rightarrow \frac{5e^t}{24} - \frac{1}{7}\left[1-\frac{9D}{14}\right] t$$

$$\Rightarrow \frac{5e^t}{24} - \frac{1}{7}\left[t - \frac{9}{14}\right]$$

$$\Rightarrow \frac{5e^t}{24} - \frac{t}{7} + \frac{9}{98}.$$

C.S.

$$y \Rightarrow c_1 e^{-2t} + c_2 e^{-7t} + \frac{5e^t}{24} - \frac{t}{7} + \frac{9}{98}. \quad (iii)$$

$$\frac{dy}{dt} \Rightarrow -2c_1 e^{-2t} - 7c_2 e^{-7t} + \frac{5e^t}{24} - \frac{1}{7} \ldots (iv)$$

putting eq (iii) and (iv) in eq (i)

$$-\left(-2c_1e^{-2t}-7c_2e^{-7t}+\frac{5e^t}{24}-\frac{1}{7}\right)+2x$$
$$+5\left(c_1e^{-2t}+c_2e^{-7t}+\frac{5e^t}{24}-\frac{t}{7}+\frac{9}{98}\right)\Rightarrow e^t$$

$$\Rightarrow \cancel{3x}\left(-2c_1e^{-2t}-7c_2e^{-7t}+\frac{5e^t}{24}-\frac{1}{7}\right)+2x$$
$$+5c_1e^{-2t}+5c_2e^{-7t}+\frac{25e^t}{24}-\frac{5t}{7}+\frac{45}{98}\Rightarrow e^t$$

$$\Rightarrow 2x+3c_1e^{-2t}-2c_2e^{-7t}+\frac{5}{4}e^t-\frac{5t}{7}+\frac{31}{98}\Rightarrow e^t$$
$$\Rightarrow 2x\Rightarrow -3c_1e^{-2t}+2c_2e^{-7t}-\frac{e^t}{4}+\frac{5t}{7}-\frac{31}{98}$$
$$\Rightarrow x=\frac{-3c_1}{2}e^{-2t}+c_2e^{-7t}-\frac{e^t}{8}+\frac{5t}{14}-\frac{31}{196}$$

So,

$$y\Rightarrow c_1e^{-2t}+c_2e^{-7t}+\frac{5e^t}{24}-\frac{t}{7}+\frac{9}{98}$$
$$x=\frac{-3c_1}{2}e^{-2t}+c_2e^{-7t}-\frac{e^t}{8}+\frac{5t}{14}-\frac{31}{196}$$

(b) $\frac{d^2x}{dt^2}+2x-y\Rightarrow 0$ and $\frac{d^2y}{dt^2}-x+2y\Rightarrow 0$

$\Longrightarrow$ let $\frac{d}{dt}\Rightarrow D$

$(D^2+2)x-y\Rightarrow 0 \quad \ldots (i) \qquad (D^2+2)y-x\Rightarrow 0 \quad \ldots (ii)$

operating (ii) equation with (D^2+2) and adding eq (i)

$$[(D^2+2)^2-1]y \Rightarrow 0$$

$$\rightarrow (D^4+4+4D^2-1)y \Rightarrow 0$$

$$\rightarrow (D^4+4D^2+3)y \Rightarrow 0.$$

$$A.E \Rightarrow D^4+4D^2+3.$$

$$D^4+4D^2+3=0$$

$$D^4+4D^2+4-1=0$$

$$(D^2+2)^2 \Rightarrow 1$$

$$D^2+2 \Rightarrow 1, \quad D^2+2 \Rightarrow -1.$$

$$\boxed{D^2 \Rightarrow -1} \quad D^2 \Rightarrow -3.$$

$$D \Rightarrow i, -i \quad D \Rightarrow -\sqrt{3}i, \sqrt{3}i.$$

C.F $\Rightarrow$

$$y \Rightarrow c_1\cos t + c_2\sin t + c_3\sin\sqrt{3}t + c_4\cos\sqrt{3}t \quad \ldots (iii)$$

$$\frac{dy}{dt} \Rightarrow -c_1\sin t + c_2\cos t + \sqrt{3}c_3\cos\sqrt{3}t - \sqrt{3}c_4\sin\sqrt{3}t$$

$$\frac{d^2y}{dt^2} \Rightarrow -c_1\cos t - c_2\sin t - 3c_3\sin\sqrt{3}t - 3c_4\cos\sqrt{3}t \quad (iv)$$

putting eq (iii) and (iv) in eq (i)

$$-c_1\cos t - c_2\sin t - 3c_3\sin\sqrt{3}t - 3c_4\cos\sqrt{3}t$$
$$+2c_1\cos t + 2c_2\sin t + 2c_3\sin\sqrt{3}t + 2c_4\cos\sqrt{3}t$$
$$\Rightarrow x.$$

$c_1 \cos t + c_2 \sin t - c_3 \sin\sqrt{3}t - c_4 \cos\sqrt{3}t$
$\Rightarrow x$

So,

$\rightarrow x = c_1 \cos t + c_2 \sin t - c_3 \sin\sqrt{3}t - c_4 \cos\sqrt{3}t$

$y \Rightarrow c_1 \cos t + c_2 \sin t + c_3 \sin\sqrt{3}t + c_4 \cos\sqrt{3}t$

(8) Solve the initial value problems.

$\frac{d^2y}{dx^2} + y \Rightarrow \sin(x+a), \quad y(0) = y'(0) \Rightarrow 0$

$\Rightarrow$ let $\frac{d}{dx} \Rightarrow D$

$(D^2+1)y \Rightarrow \sin(x+a).$

A.E $\rightarrow$

$D^2+1 = 0$

$\boxed{D \Rightarrow i, -i}$

C.F. $\Rightarrow c_1 \cos x + c_2 \sin x.$ (i)

$PI = \frac{\sin(x+a)}{(D^2+1)}$

$\Rightarrow \frac{x\sin(x+a)}{2D}$ $\left[\because f(D) = 0\right.$ $\therefore \frac{x}{f'(D)}$

$\Rightarrow \frac{1}{2}\left[-x\cos(x+a) + \sin(x+a)\right]$

C.S ⇒ C.F + P.I

$$y \Rightarrow c_1 \cos x + c_2 \sin x - \frac{x\cos(x+a)}{2} + \frac{\sin(x+a)}{2}$$

For $x=0,\ y=0$

$$0 = c_1 + \frac{\sin a}{2}$$

$$c_1 = -\frac{\sin a}{2}$$

$$y' \Rightarrow -c_1 \sin x + c_2 \cos x - \cancel{\frac{\cos(x+a)}{2}} + \frac{x\sin(x+a)}{2} + \cancel{\frac{\cos(x+a)}{2}}$$

$$y' \Rightarrow -c_1 \sin x + c_2 \cos x + \frac{x\sin(x+a)}{2}$$

at $x=0,\ y' \Rightarrow 0$

$\therefore c_2 \Rightarrow 0$

So,

$$y \Rightarrow -\frac{\sin a \cos x}{2} - \frac{x\cos(x+a)}{2} + \frac{\sin(x+a)}{2}$$

$$y \Rightarrow \frac{\sin x \cos a + \cancel{\sin a \cos x} - \cancel{\cos x \sin a} - x\cos(x+a)}{2}$$

$$y = \frac{\sin x \cos a - x\cos(x+a)}{2}$$

(9) The position of a particle executing simple harmonic motion at the end of 1st, 2nd and 3rd second of its motion are x_1, x_2 and x_3 respectively. Show that time period is $\dfrac{2\pi}{\cos^{-1}\left(\dfrac{x_1+x_3}{2x_2}\right)}$

$\Rightarrow$ General form of SHM :→

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \quad \text{and } T = \frac{2\pi}{\omega}$$

Solving :→

$x = c_1 \sin\omega t + c_2 \cos\omega t$.

at 1st second, 2nd second, 3rd second,

$x_1 = c_1 \sin\omega + c_2 \cos\omega$.

$x_2 = c_1 \sin 2\omega + c_2 \cos 2\omega$.

$x_3 \Rightarrow c_1 \sin 3\omega + c_2 \cos 3\omega$.

$$\Rightarrow \frac{x_1+x_3}{2x_2} \Rightarrow \frac{c_1[\sin\omega + \sin 3\omega] + c_2[\cos\omega + \cos 3\omega]}{2[c_1 \sin 2\omega + c_2 \cos 2\omega]}$$

$$\Rightarrow \frac{x_1+x_3}{2x_2} \Rightarrow \frac{2c_1 \sin 2\omega \cos\omega + 2c_2 \cos 2\omega \cos\omega}{2[c_1 \sin 2\omega + c_2 \cos 2\omega]}$$

$$\Rightarrow \frac{x_1+x_3}{2x_2} \Rightarrow \frac{\cancel{2}\cancel{[c_1 \sin 2\omega + c_2 \cos 2\omega]} \cos\omega}{\cancel{2}\cancel{[c_1 \sin 2\omega + c_2 \cos 2\omega]}}$$

$$\cos^{-1}\left(\frac{x_1+x_3}{2x_2}\right) \Rightarrow \omega .$$

$$T \Rightarrow \frac{2\pi}{\omega} = \frac{2\pi}{\cos^{-1}\left(\frac{x_1+x_3}{2x_2}\right)} .$$

(10) consider the equation :→

$$y'' + y = 0$$

(a) verify that the boundary value problem for eq^n with boundary conditions $y(0)=1$, $y(\pi/2)=1$ has a unique solution.

⇒ let $\frac{d}{dx} = D$

$$(D^2+1)y = 0$$

$\therefore$ $y = c_1 \sin x + c_2 \cos x$

at $x=0$,

$c_2 = 1$.

at $x = \pi/2$

$c_1 = 1$.

so, $\boxed{y \Rightarrow \sin x + \cos x}$

so, $y = \sin x + \cos x$ is a unique solution.

(b) verify that the boundary value problem for equation (1) with boundary conditions $y(0)=1$, $y(\pi)=1$ has no solution.

$\Rightarrow \quad y = c_1 \sin x + c_2 \cos x$

at $x = 0$, $y = 1$.

$c_2 = 1$.

$\therefore \boxed{y = c_1 \sin x + \cos x}$

at $x = \pi$.

$y = c_1 \sin \pi + \cos \pi$.

$\boxed{y = -1}$ so, not possible $y(\pi) = 1$.

Hence, no solution.

(4) verify that the boundary value problem for equation ① with boundary conditions $y(0) = 1$, $y(2\pi) = 1$ has infinite many solutions.

$y = c_1 \sin x + c_2 \cos x$.

$\boxed{1 = c_2}$ $(x = 0)$.

$\boxed{1 = c_2}$ (at $x = 2\pi$)

so c_1 can have any arbitrary value.
so, infinite no. of equations, solutions are possible.