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classmate
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ASSIGNMENT - III

Q1) $x^2y'' + 6xy' + (6+x^2)y = 0$ — ①

$x=0$ is a regular singular point.

∴ assumption for the solution:

$$y = \sum_{n=0}^{\infty} a_n (x-x_0)^{n+r} \quad , a_0 \neq 0$$

$$= \sum_{n=0}^{\infty} a_n x^{n+r} = x^r [a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots]$$

$$\therefore y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$\therefore y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n-r-2}$$

on substituting y'', y' and y in ①

$$\Rightarrow x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + 6x \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$+ (x^2+6) \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} + 6\sum_{n=0}^{\infty} (n+r) a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2}$$

$$= -6 \sum_{n=0}^{\infty} a_n x^{n+r}.$$

now comparing coefficients of various powers of x

i) x^r: $a_0 r(r-1) + 6a_0(r) + 6a_0 = 0$

$a_0(r^2+5r+6) = 0 \rightarrow r^2+5r+6=0$

$r = -3, -2$

ii) x^{r+1}: $a_1 r(r+1) + 6a_1(r+1) + 6a_1 = 0$

$a_1(r^2+7r+12) = 0$

$\Rightarrow a_1 = 0 \qquad (\because r^2+7r+12 \neq 0)$

iii) x^{n+r+2} : $a_{n+2}(n+r+2)(n+r+1) + 6a_{n+2}(n+r+2) + 6a_{n+2} + a_n$

$$\Rightarrow a_{n+2} = \frac{-a_n}{(n+r+2)(n+r+7)+6}$$

for $r=-2$

$$a_{n+2} = \frac{-a_n}{n(n+5)+6}$$

$a_2 = -\frac{a_0}{6}$, $a_3 = \frac{-a_1}{12} = 0$

$a_4 = \frac{-a_2}{20} = \frac{a_0}{120}$

$y_1(x) = a_0x^{-2} + a_1x^{-1} + a_2x^0 + a_3x^1 + a_4x^2 + \ldots$

$= \frac{a_0}{x^2} - \frac{a_0}{6} + \frac{a_0}{120}x^2 + \ldots$

for $r=-3$

$$a_{n+2} = \frac{-a_n}{(n-1)(n+4)+6}$$

$a_2 = -a_0/2$; $a_3 = 0$

$a_4 = \frac{-a_2}{30} = \frac{a_0}{60}$

$y_2(x) = x^{-3}(a_0 + a_1x + a_2x^2 + \ldots)$

$= x^{-3}\left(a_0 - \frac{a_0}{2}x^2 + \frac{a_0}{60}x^4 + \ldots\right)$

Q2) Given differential equation : $y'' - y' = 0$
and $y(0) = 2$ and $y'(0) = 0$

let $y(x) = \sum_{n=0}^{\infty} a_n x^n$

$\therefore y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$

$y''(x) = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$

substituting in given differential eq, we get.

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

comparing the coefficients.

$$(n+1)(n+2)\, a_{n+2} = (n+1)\, a_{n+1}$$

$\therefore$ $2a_2 = a_1$, $6a_3 = 2a_2$, $12a_4 = 3a_3$

$a_2 = a_1/2$ $\quad a_3 = \frac{a_1}{6}$; $\quad a_4 = \frac{a_1}{24}$

$\therefore$ solution upto 4^{th} degree is.

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^2 + a_4 x^4$$

$$= a_0 + a_1 x + \frac{a_1}{2} x^2 + \frac{a_1}{3!} x^3 + \frac{a_1}{4!} x^4$$

$y(0) = 2$, $\therefore$ $a_0 = 2$

$y'(0) = 0$; $a_1 = 0$

$\therefore$ $y(x) = 2$

Q3) $x^2 \dfrac{d^2y}{dx^2} + 4x \dfrac{dy}{dx} - 3y = 0$

$$x^2 y'' + 4xy' - 3y = 0$$

let $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$

$$y'(x) = \sum_{n=0}^{\infty} a_n (n+r)\, x^{n+r-1}$$

$$y''(x) = \sum_{n=0}^{\infty} a_n (n+r-1)(n+r)\, x^{n+r-2}$$

substituting in original DE

$$\Rightarrow x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r-2} + 4x \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} - 3 \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r} + 4 \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r} - 3 \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

Comparing coefficients on both sides:

i) x^r : $r(r-1)a_0 + 4ra_0 - 3a_0 = 0$

$\Rightarrow r(r-1) + 4r - 3 = 0$

$r = \frac{-3 \pm \sqrt{21}}{2}$ $\quad r_1 = \frac{-3+\sqrt{21}}{2}$; $r_2 = \frac{-3-\sqrt{21}}{2}$

ii) x^{r+1} : $(r+1)ra_1 + 4(r+1)a_1 - 3a_1 = 0$

$\Rightarrow (r^2+5r+1)a_1 = 0 \Rightarrow a_1 = 0$

iii) x^{n+r} : $(n+r)(n+r-1)a_n + 4(n+r)a_n - 3a_n = 0$

$(n+r)a_n(n+r-1+4) - 3a_n = 0$

$a_n[(n+r)(n+r+3) - 3] = 0$

$\Rightarrow a_n[n^2 + rn + 3n + nr + r^2 + 3r - 3] = 0$

$\Rightarrow a_n(n^2 + 2nr + 3n) = 0$

as, $n^2 + 2rn + 3n \neq 0$

$\therefore a_n = 0$

$\Rightarrow a_1 = a_2 = a_3 = \dots a_n = 0.$

$\therefore \boxed{y = a_0}$ solution.

Q4) $2(1-x)y'' - xy' + y = 0$ —①

as $x=1$ is regular singular point.

$\therefore$ By Forbenius method:

$$y = \sum_{n=0}^{\infty} a_n (x-1)^{n+r} = (x-1)^r \left[a_0 + a_1(x-1) + a_2(x-1)^2 + \dots\right]$$

$$\therefore y' = \sum_{n=0}^{\infty} (n+r) a_n (x-1)^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n (x-1)^{n+r-2}$$

on substituting in eq. ①

$$2(1-x)\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n (x-1)^{n+r-2} - x\sum_{n=0}^{\infty} (n+r) a_n (x-1)^{n+r-1}$$

$$+ \sum_{n=0}^{\infty} a_n (x-1)^{n+r} = 0$$

$$2\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n (x-1)^{n+r-2} - 2\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n (x-1)^{n+r-1}$$

$$- \sum_{n=0}^{\infty} (n+r-1) a_n (x-1)^{n+r} = 0$$

comparing coefficients.

i) $(x-1)^{r-2}$: $2r(r-1)a_0 = 0$ —②

$r = 0, 1$

ii) $(x-1)^{r-1} \Rightarrow 2r(r+1)a_1 - 2r(r-1)a_0 = 0$

from ② $2r(r+1)a_1 = 0 \Rightarrow a_1 = 0$

iii) x^{n+r} : $2(n+r+2)(n+r+1)a_{n+2} - 2(n+r+1)(n+r)a_{n+1} - (n+r-1)a_n = 0$

$\rightarrow n=0$

$\Rightarrow 2(r+2)(r+1)a_2 - 2(r+1)(r)a_1 - (r-1)a_0 = 0$

$$a_2 = \frac{(r-1)}{2(r+1)(r+2)} a_0$$

$n=1 \Rightarrow \quad 2(r+3)(r+2)a_3 - 2(r+2)(r+1)a_2 - ra_1 = 0$

$$a_3 = \frac{(r+1)(r+2)}{(r+2)(r+3)} a_2 = \frac{(r-1)\, a_0}{2(r+2)(r+3)}$$

$$\Rightarrow y = (x-1)^r \left[a_0 + 0 + \frac{(r-1)}{2(r+1)(r+2)} a_0 (x-1)^2 + \frac{(r-1)a_0}{2(r+2)(r+3)} (x-1)^3 + \dots \right]$$

for $r=0$,

$$y_1 = \left[a_0 + \frac{(-1)a_0}{4}(x-1)^2 - \frac{a_0}{2\times 3!}(x-1)^3 + \dots \right]$$

for $r=1$;

$$y_2 = [a_0 + 0 + 0 \dots]$$

$$\boxed{y_2 = a_0}$$

Q5) $x(x-2)y'' + 4xy' + 3y = 0$

$x=2$, is a regular singular point assumption of sol: by forbenius method

$$y(x) = \sum_{n=0}^{\infty} a_n (x-2)^{n+r}$$

$$y'(x) = \sum_{n=0}^{\infty} (n+r)(a_n)(x-2)^{n+r-1}$$

$$y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n (x-2)^{n+r-2}$$

$$\Rightarrow x(x-2)\sum_{n=0}^{\infty}(n+r)(n+r-1)\, a_n (x-2)^{n+r-2} + 4x\sum_{n=0}^{\infty}(n+r)\, a_n (x-2)^{n+r-1} + 3\sum_{n=0}^{\infty} a_n (x-2)^{n+r}$$

$\Rightarrow$ comparing coefficients of powers of x:

i) x^{r-1} : $-2r(r-1)a_0 = 0 \Rightarrow r = 0, 1$; $a_0 \neq 0$

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ii) x^r : $r(r-1)a_0 - 2r(r+1)a_1 + 4ra_0 + 3a_0 = 0$

$$\Rightarrow \quad a_1 = \frac{(4r+3)}{2(r+1)r} a_0$$

iii) x^{n+r} :

$$(n+r)(n+r-1)a_n - 2(n+r)(n+r+1)a_{n+1} + 4(n+r)a_n + 3a_n = 0$$

$$a_{n+1} = \frac{(n+r)(n+r-1) + 4(n+r) + 3}{2(n+r)(n+r+1)} a_n$$

$$\Rightarrow \quad a_2 = \frac{r(r+1) + 4(r+1) + 3}{2(r+1)(r+2)} \qquad a_1 = \frac{(4r+3)(r+4)}{4r(r+1)(r+2)} a_0$$

for $r=0$;

$a_2 = \frac{3}{4} a_1$

for $r=1$

$a_1 = \frac{7}{4} a_0$

$a_2 = \frac{35}{24} a_0$

$$y_1 = (x-2)^0 \left[a_0 + a_1(x-2) + \frac{3}{4} a_1 (x-2)^2 + \dots \right]$$

$$\cancel{y_2 = (x-2)\left[a_0 + \frac{7}{4} a_0 (x-2) + \frac{35}{24} a_0 (x-2)^2 + \dots \right]}$$

Q6) $x^2 y'' + xy' + (x^2-1)y = 0$

using forbenius.

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y'(x) = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

substituting these in given DE.

$$x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + x \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} +$$

$$(x^2-1) \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} a_n x^{n+r+2} + \sum_{n=0}^{\infty} (n+r)^2 a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

Comparing coefficients of power of x on both sides:

i) x^r : $r^2 - a_0 1 = 0 \Rightarrow r = \pm 1$

ii) x^{r+1} : $(r+1)^2 a_1 - a_1 = 0 \Rightarrow a_1 = 0$

iii) x^{n+r+2} : $a_n + (n+r+2)^2 a_{n+2} - a_{n+2} = 0$

$$a_{n+2} = \frac{a_n}{1-(n+r+2)^2} \Rightarrow a_2 = \frac{a_0}{1-(r+2)^2}$$

for $r = 1$

$$a_2 = -\frac{a_0}{8} \qquad a_4 = \frac{a_2}{1-(r+4)^2} = \frac{a_2}{-24}$$

$$y_1 = x\left[a_0 - \frac{a_0}{8}x^2 + \frac{a_0 x^4}{4\times 8} + \dots\right]$$

for $r = -1$

$a_n = a_{n+2}$

$\therefore \quad a_0 = a_2 = a_4 = \dots$

$a_1 = a_3 = \dots = 0$

$$\therefore \quad y_2 = a_0 + a_2 x^2 + a_4 x^4 + \dots$$

$$y_2 = a_0 + a_0 x^2 + a_0 x^4 + \dots$$

Q8) If n and m are non-negative integers, then

$$\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = \begin{cases} 0, & m \neq n \\ \frac{2}{2n+1}; & m = n. \end{cases}$$

This integral relationship is called orthogonality of Legendre's polynomial on [-1,1]

proof: let $P_m(x)$ and $P_n(x)$ be solution of following differential equations respectively:

$$(1-x^2)u'' - 2xu' + m(m+1)u = 0 \quad \text{—①}$$
$$(1-x^2)v'' - 2xv' + n(n+1)v = 0 \quad \text{—②}$$

multiply ① by v and ② by u and then subtract ① from ② we get.

$$\Rightarrow (1-x^2)[u''v - uv''] - 2x(u'v - uv') + [m(m+1) - n(n+1)]uv = 0$$

$$\frac{d}{dx}[(1-x^2)(u'v - uv')] = [n(n+1) - m(m+1)]uv$$

as $u = P_m(x)$ and $v = P_n(x)$

$$\Rightarrow \int_{-1}^{1} P_n(x)\, P_m(x)\, dx = \frac{1}{n(n+1) - m(m+1)} \left[(1-x^2)(P_m'(x) P_n(x) - P_m(x) P_n'(x)\right]$$

If $m \neq n$ then $\int_{-1}^{1} P_m(x) P_n(x)\, dx = 0$

for $m = n$,

$$\int_{-1}^{1} P_n^2(x) = \frac{1}{(2^n n!)^2} \int_{-1}^{1} D^n(x^2-1)^n \cdot D^n(x^2-1)^n\, dx$$

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(from Rodrigue formulae)

$$= \frac{1}{(2^n n!)^2} \left[\left. D^{n-1}(x^2-1)^n \, D^n(x^2-1)^n \right|_{-1}^{1} - \int_{-1}^{1} D^{n-1}(x^2-1)^n D^{n+1}(x^2-1)^n dx \right]$$

$$= \frac{1}{(2^n n!)^2} (-1)^n \int_{-1}^{1} D^n(x^2-1)^n \, D^{2n}(x^2-1)^n dx$$

$$= \frac{1}{(2^n n!)^2} (-1)^n (2n)! \int_{-1}^{1} (x^2-1)^n dx$$

$$= \frac{2}{(2n+1)}$$

Q9) $$\int_0^1 x J_n(\alpha x) J_n(\beta x) dx = \begin{cases} 0 & ; \text{ if } \alpha \neq \beta \\ \frac{1}{2}[J_{n+1}(\alpha)]^2 & ; \text{ if } \alpha = \beta \end{cases}$$

let $J_n(\alpha x)$ and $J_n(\beta x)$ be solutions of below diff. equation respectively:

$J_n(\alpha x)$: $x^2 u'' + xu' + (\alpha^2 x^2 - n^2) u = 0$ — ①

$J_n(\beta x)$: $x^2 v'' + xv' + (\beta^2 x^2 - n^2) v = 0$ — ②

① × v − ② × u

$$\Rightarrow x^2(u''v - v''u) + x(u'v - v'u) + (\alpha^2 - \beta^2) x^2 uv = 0$$

$$\Rightarrow \frac{d}{dx}[x(u'v - uv')] = (\beta^2 - \alpha^2) x \, uv \quad \text{— ③}$$

$u \to J_n(\alpha x)$ $\qquad$ $v \to J_n(\beta x)$

$u' \to \alpha J_n'(\alpha x)$ $\qquad$ $v' \to \beta J_n'(\beta x)$

∴ eq ③ becomes.

$$x J_n(\alpha x) J_n(\beta x) = \frac{1}{\beta^2 - \alpha^2} \frac{d}{dx}\left[x(\alpha J_n'(\alpha x) J_n(\beta x) - J_n(\alpha x) \beta J_n'(\beta x))\right]$$

integrating both sides:

$$\int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx = \frac{1}{(\beta^2-\alpha^2)}\left[x(\alpha J_n'(\alpha x) J_n(\beta x) - J_n(\alpha x)\,\beta J_n'(\beta x))\right]_0^1$$

$$= \frac{1}{(\beta^2-\alpha^2)}\left[\alpha J_n'(\alpha) J_n(\beta) - J_n(\alpha)\,\beta J_n'(\beta)\right]$$

if $\alpha \neq \beta$

$$\int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx = \frac{1}{(\beta^2-\alpha^2)}\left[\alpha J_n'(\alpha) J_n(\beta) - J_n(\alpha)\,\beta J_n'(\beta)\right] = 0$$

if $\alpha = \beta$

$$\int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx = \frac{1}{(\beta^2-\alpha^2)}\left(\alpha J_n'(\alpha) J_n(\beta) - J_n(\alpha)\,\beta J_n'(\beta)\right)$$

$$= 0$$

if $\alpha = \beta$

$$\int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx = \frac{1}{(\beta^2-\alpha^2)}\left(\alpha J_n'(\alpha) J_n(\beta) - J_n(\alpha)\,\beta J_n'(\beta)\right)$$

$$= \frac{1}{2}\left[J_{n+1}(\alpha)\right]^2$$

$$\left(\because \frac{d}{dx} J_n(\alpha) = J_{n+1}(\alpha)\right)$$

Q10) —

Q11) let $v = (x^2-1)^n$

$\Rightarrow v_1 = n(x^2-1)^{n-1}\, 2x$

$= 2nx\,(x^2-1)^{n-1}$

$\Rightarrow (x^2-1)\, v_1 = 2nx\,(x^2-1)^n = 2nxv$

$(1-x^2)\, v_1 + 2nxv = 0$ — ①

Differentiate ① $(n+1)$ times w.r.t. x.

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$$(1-x^2)V_{n+2} - 2xV_{n+1} + n(n+1)V_n = 0$$

let $V_n = y$

$$\therefore (1-x^2)y'' - 2xy' + n(n+1)y = 0$$

$$\therefore y = cP_n(x)$$

$$\Rightarrow P_n(x) = \frac{1}{c}y = \frac{1}{c}D^n(x^2-1)^n$$

$$P_n(x) = \frac{1}{c}\left[D^n(x-1)^n(x+1)^n\right]$$

also. $P_n(1) = 1$

$$1 = \frac{1}{c}D^n(x-1)^n(x+1)^n$$

$$c = \left[D^n(x-1)^n(x+1)^n\right]_{x=1}$$

$$= \left[(D^n(x-1)^n)(x+1)^n\right]_{x=1}$$

$$= n!\,2^n$$

$$\Rightarrow P_n(x) = \frac{1}{n!\,2^n}D^n(x^2-1)^n$$

$$P_{10}(x) = \frac{1}{10!\,2^{10}}D^{10}(x^2-1)^{10}$$

$$= \frac{1}{10!\,2^{10}}\left[{}^{10}C_0(x^2)^{10}(-1)^0 + {}^{10}C_1(x^2)^9(-1) \cdots\cdots\right.$$

$$\left.\cdots\ {}^{10}C_9(x^2)^1(-1)^9 + {}^{10}C_{10}(-1)^{10}\right]$$

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Classroom Question:

~~Solve~~ Prove: $$\frac{(-1)^n (2n)!}{2^{2n} (n!)^2} \int_{-1}^{1} (x^2-1)^n \, dx = \frac{2}{(2n+1)}$$

put $x = \sin\theta$

$dx = \cos\theta \, d\theta$

$$\Rightarrow \frac{\cancel{(-1)^n} (2n)!}{2^{2n} (n!)^2} \cancel{(-1)^n} \int_{-\pi/2}^{\pi/2} \cos^{(2n+1)} \theta \, d\theta$$

$$\Rightarrow \frac{(2n)!}{2^{2n} (n!)^2} \, 2 \int_{0}^{\pi/2} \cos^{2n+1} \theta \, d\theta$$

$$\Rightarrow \frac{(2n)!}{2^{2n} (n!)^2} \, 2 \left[\frac{(2n)(2n-2)(2n-4) \ldots\ldots 2.}{(2n+1)(2n-1)(2n-3) \ldots\ldots 1} \right]$$

using reduction formulae.

$$\Rightarrow \frac{(2n-1)\cancel{(2n-3)} \ldots\ldots 1}{2^n \; n!} \times 2 \left[\frac{(2n)(2n-2)(2n-4) \ldots\ldots 2}{(2n+1)(2n-1)\cancel{(2n-3)} \ldots 1} \right]$$

$$\Rightarrow \frac{2 \times 2^n \; n!}{(2n+1) \; 2^n \; n!} = \frac{2}{2n+1}$$

Hence proved.