Maths Assignment - Calculus of Several Variables.

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A1

$u = f(r)$

$$\underbrace{x = r\cos\theta \quad , \quad y = r\sin\theta}_{\text{Squaring \& Adding}}$$

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta$$

$$x^2 + y^2 = r^2 \quad - (1)$$

Partially differentiate (1) wrt x

$$2x + 0 = 2r\frac{\partial r}{\partial x}$$

$$\frac{\partial r}{\partial x} = x/r. \quad - (2)$$

Partially differentiate (2) wrt y

$$\frac{\partial r}{\partial y} = y/r \quad (3)$$

$u = f(r)$

Partially differentiating wrt x

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[f(r)\right]$$

$$= f'(r) \cdot \frac{\partial r}{\partial x}$$

$$\frac{\partial u}{\partial x} = f'(r) \cdot \frac{x}{r} \quad - (4) \quad [\text{Using } (2)$$

Partially differentiating wrt x

$$\frac{\partial^2 u}{\partial x^2} = \frac{f'(r)}{r} \cdot 1 + x\left[\frac{\frac{\partial}{\partial x}\left[f'(r)\right] r - \frac{\partial r}{\partial x} \cdot f'(r)}{r^2}\right]$$

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$$= \frac{f'(r)}{r} + x\left[\frac{f''(r)\cdot\frac{x}{r}\cdot r - \frac{x}{r}f'(r)}{r^2}\right]$$

$$\frac{\partial^2 u}{\partial x^2} = f'(r)\left[\frac{1}{r} - \frac{x^2}{r^3}\right] + f''(r)\left[\frac{x^2}{r^2}\right] \quad -(5)$$

Similarly $\frac{\partial^2 u}{\partial y^2} = f'(r)\left[1 - \frac{y^2}{r^3}\right] + f''(r)\left[\frac{y^2}{r^2}\right] \quad -(6)$

Adding (5) + (6).

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r)\left[\frac{x^2}{r^2} + \frac{y^2}{r^2}\right] + \frac{f'(r)}{r}\left[2 - \frac{x^2}{r^2} - \frac{y^2}{r^2}\right]$$

$$= f''(r)\left[\frac{x^2+y^2}{r^2}\right] + \frac{f'(r)}{r}\left[2 - \left(\frac{x^2+y^2}{r^2}\right)\right]$$

$$= f''(r) + \frac{f'(r)}{r}. \quad [\text{using } (1)].$$

Hence, proved.

A2

$$w = \sin^{-1}(u) \quad -(1)$$

$$u = \frac{x^2 + y^2 + z^2}{x + y + z}. \quad -(2)$$

$$u = \frac{x^2}{x}\left[\frac{1 + (y/x)^2 + (z/x)^2}{1 + (y/x) + z/x}\right]$$

u is a homogeneous func of order 1.

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Therefore, by euler's thm

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 1 \cdot u. \quad -(3)$$

since $\sin^{-1} u = w$ [- From (1)]

$$\boxed{u = \sin(w)}$$

Putting this value in (3)

$$x\frac{\partial}{\partial x}\left[\sin(w)\right] + y\frac{\partial}{\partial y}\left[\sin(w)\right] + z\frac{\partial}{\partial z}\left[\sin(w)\right] = \sin w.$$

$$x\frac{\partial w}{\partial x} \cdot \cos w + y\frac{\partial w}{\partial y}\cos w + z\frac{\partial w}{\partial z} \cdot \cos w = \sin w$$

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = \frac{\sin w}{\cos w}.$$

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = \tan w$$

Hence, Proved.

A3

$$z = f(x-cy) + g(x+cy). \quad -(1)$$

Partially differentiate wrt x.

$$\frac{\partial z}{\partial x} = f'(x-cy) \cdot 1 + g'(x+cy) \cdot 1.$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x-cy) \cdot 1 + g''(x+cy) \cdot 1 \quad -(2)$$

Partially differentiate (1) wrt y.

$$\frac{\partial z}{\partial y} = f'(x-cy) \cdot (-c) + g'(x+cy) \cdot c.$$

$$\frac{\partial^2 z}{\partial y^2} = f''(x-cy) \cdot (-c)(-c) + g'(x+cy) \cdot c \cdot c.$$

$$\frac{\partial^2 z}{\partial y^2} = c^2\left[f''(x-cy) + g''(x+cy)\right].$$

$\frac{\partial^2 z}{\partial x^2}$

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$$\frac{\partial^2 z}{\partial y^2} = c^2 \left[\frac{\partial^2 z}{\partial x^2} \right] \quad [\text{from } (2)]$$

Hence, proved.

Q4 $u = \tan^{-1}\left(\frac{y^2}{x}\right)$.

$$\tan u = \frac{y^2}{x}$$

$$\tan u = x\left[\frac{y}{x}\right]^2$$

which represents a homogenous func of order 1.

By euler th'm

$$x^2 \frac{\partial^2(\tan u)}{\partial x^2} + 2xy \frac{\partial^2(\tan u)}{\partial x \partial y} + y^2 \frac{\partial^2[\tan u]}{\partial y^2} = 1(1-1)\tan u$$

$$= \text{zero} \quad -(1)$$

$$\frac{\partial^2(\tan u)}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left[\tan u\right]\right].$$

$$= \frac{\partial}{\partial x}\left[\sec^2 u \frac{\partial u}{\partial x}\right].$$

$$= \sec^2 u \frac{\partial^2 u}{\partial x^2} + \left[2\sec u \sec u \tan u \left(\frac{\partial u}{\partial x}\right)^2\right]. \quad -(2)$$

$$\frac{\partial^2(\tan u)}{\partial y^2} = \sec^2 u \frac{\partial^2 u}{\partial x^2} + \left[2\sec u \sec u \tan u \left(\frac{\partial u}{\partial y}\right)^2\right] \quad -(3)$$

$$\frac{\partial^2(\tan u)}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\sec^2 u \frac{\partial u}{\partial y}\right]$$

$$= \sec^2 u \frac{\partial^2 u}{\partial x \partial y} + \left[2\sec u \tan u \sec u \frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right] \quad -(4)$$

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Putting values of $\frac{\partial^2(\tan^2 u)}{\partial x^2}$; $\frac{\partial^2(\tan u)}{\partial y^2}$, $\frac{\partial^2(\tan u)}{\partial x \partial y}$

from (2), (3), (4) in (1), we get.

$$\Rightarrow \sec^2 u\left[x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}\right]$$

$$+ 2\sec^2 u \tan u\left[x^2\left(\frac{\partial u}{\partial x}\right)^2 + 2xy\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + y^2\left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$= \text{zero} = 0$$

$$\Rightarrow \sec^2 u\left[\text{Ans}\right] + 2\sec^2 u \tan u\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right]^2 = 0$$

$$\left[\text{where Ans} = x^2\frac{\partial u^2}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}\right]$$

$$\Rightarrow \left[\text{Ans}\right]\sec^2 u + 2\sec^2 u \tan u\left[1 \cdot \frac{\tan u}{\sec^2 u}\right]^2 = 0$$

$$\sec^2 u\left[\text{Ans}\right] = \frac{-2\sec^2 u \tan^3 u}{\sec^4 u}$$

$$\text{Ans} = -2 \cdot \frac{\tan^3 u}{\sec^3 u} \cdot \frac{1}{\sec u}$$

$$= -2 \cdot \sin^3 u \cdot \cos u$$

$$= -\sin 2u \cdot \sin^2 u$$

$$\left[\because x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{\tan u}{\sec^2 u} \Rightarrow \cancel{\text{or}}\ x\frac{\partial}{\partial x}\left[\tan u\right] + y\left[\frac{\partial}{\partial y}(\tan u)\right] = \tan u \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{\tan u}{\sec u}\right]$$

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Q5 →

$$u = \underbrace{\sin\frac{y}{x}}_{u_1} + \underbrace{x\sin^{-1}\left(\frac{y}{x}\right)}_{u_2} = u_1 + u_2$$

u_1 has order of zero

u_2 has order of one.

By euler's thm

$$x^2\frac{\partial^2 u_1}{\partial x^2} + 2xy\frac{\partial^2 u_1}{\partial x\partial y} + y^2\frac{\partial^2 u_1}{\partial y^2} = 0(0-1)\,u_1$$

$$= 0 \quad (2)$$

$$x^2\frac{\partial^2 u_2}{\partial x^2} + 2xy\frac{\partial^2 u_2}{\partial x\partial y} + y^2\frac{\partial^2 u_2}{\partial y^2} = 1(1-1)\,u_2$$

$$= 0 \quad (3)$$

Adding (2) + (3)

$$x^2\left[\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_2}{\partial x^2}\right] + 2xy\left[\frac{\partial^2 u_1}{\partial x\partial y} + \frac{\partial^2 u_2}{\partial x\partial y}\right] + y^2\left[\frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_2}{\partial y^2}\right] = 0$$

Since $u = u_1 + u_2$

$$x^2\frac{\partial^2}{\partial x^2}[u_1+u_2] + 2xy\frac{\partial^2}{\partial x\partial y}[u_1+u_2] + y^2\frac{\partial^2}{\partial y^2}[u_1+u_2] = 0$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$$

Hence, proved.

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$$f(x,y) = e^x \cos y.$$

$x_0 = 1$, $y_0 = \pi/4$ $\qquad f(x_0,y_0) = \frac{e}{\sqrt{2}}$

$x = x_0 + h$

$h = x - x_0$

$h = x - 1$ - (1)

$y = y_0 + k$

$y = \pi/4 + k$

$k = y - \pi/4$. - (2)

By Taylor thm

$$f(x,y) = f(x_0,y_0) + \frac{1}{1!}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right] f(x_0,y_0)$$

$$+ \frac{1}{2!}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^2 f(x_0,y_0) \quad \ldots\ldots$$

Till second degree

$$f(x,y) = f(x_0,y_0) + \left[h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right]$$

$$+ \left[h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right] \quad (3)$$

$\frac{\partial f}{\partial x} = f_x = e^x \cos y \Rightarrow f_x(1, \pi/4) = \frac{e}{\sqrt{2}}$

$f_y = -e^x \sin y \Rightarrow f_y(1, \pi/4) = -\frac{e}{\sqrt{2}}$

$f_{xx} = e^x \cos y \Rightarrow f_{xx}(1, \pi/4) = e/\sqrt{2}$

$f_{xy} = -e^x \sin y \Rightarrow f_{xy}(1, \pi/4) = -e/\sqrt{2}$.

$f_{yy} = -e^x \cos y \Rightarrow f_{yy}(1, \pi/4) = -e/\sqrt{2}$.

Putting value back in (3), we get

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$$= \frac{e}{\sqrt{2}} + \left[h\frac{e}{\sqrt{2}} - k\frac{e}{\sqrt{2}} \right] + \frac{1}{2!}\left[h^2\frac{e}{\sqrt{2}} - 2hk\frac{e}{\sqrt{2}} - k^2\frac{e}{\sqrt{2}} \right]$$

Putting value of h, k from ①, ②.

$$= \frac{e}{\sqrt{2}} + \left[(x-1)\frac{e}{\sqrt{2}} - \left(y-\frac{\pi}{4}\right)\left(\frac{e}{\sqrt{2}}\right) \right] + \frac{1}{2!}\left[(x-1)^2\frac{e}{\sqrt{2}} - 2(x-1)\left(y-\frac{\pi}{4}\right)\left(\frac{e}{\sqrt{2}}\right) - \left(y-\frac{\pi}{4}\right)^2\left(\frac{e}{\sqrt{2}}\right) \right]$$

This expression is approximation of $e^x \cos y$ about pt $\left(1, \frac{\pi}{4}\right)$.

Q7

$$x^2 + xy + y^2 - 16 = 0 \quad \text{curve eqn}$$

Let 'd' be dist of pt (x,y) on curve.

$$d = \sqrt{x^2+y^2}$$

$$d^2 = x^2 + y^2$$

which has constraint that $x^2 + xy + y^2 - 16 = 0$

$$F(x,y,\lambda) = x^2+y^2 + \lambda(x^2 + xy + y^2 - 16)$$

For maxima or minima

$$\frac{\partial F}{\partial x} = 2x[1+\lambda] + \lambda y = 0 \quad - ①$$

$$\frac{\partial F}{\partial y} = 2y[1+\lambda] + \lambda x = 0 \quad - ②$$

$$\Downarrow$$

$$y = \frac{-\lambda x}{2[1+\lambda]}$$

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Putting value of y in (1)

$$0 = 2x[1+\lambda] + \frac{\lambda^2(-x)}{2[1+\lambda]}$$

$$0 = x[4(1+\lambda)^2 - \lambda^2]$$

$$\boxed{\lambda = -1}$$

$$2+3\lambda = 0 \quad , \quad 2+\lambda = 0$$

$$\boxed{\lambda = -2/3} \quad , \quad \boxed{\lambda = -2}$$

For $\lambda = -2$

$\therefore$ (1) becomes

$$-x - y = 0$$

$$x + y = 0$$

$$\boxed{x = -y}$$

Also

$$x^2 + xy + y^2 = 16$$

$$x^2 - x^2 + x^2 = 16$$

$$\boxed{x = \pm 4}$$

Pts

$(4,-4), (-4,4)$

$$D^2 = 16 + 16$$

$$D^2 = \underline{32}$$

For $\lambda = -2/3$

(2) becomes

$$x - y = 0$$

$$\boxed{x = y}$$

Also

$$x^2 + y^2 + xy = 16$$

$$3x^2 = 16$$

$$x = \pm 4/\sqrt{3}$$

Pts

$\left(\frac{4}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right) \quad \left(\frac{-4}{\sqrt{3}}, \frac{-4}{\sqrt{3}}\right)$

$$D^2 = \frac{16}{3} + \frac{16}{3}$$

$$D^2 = 32/3$$

$\therefore$ $D^2_{max} = 32$, farthest pts $= (4,-4), (-4,4)$

$D^2_{min} = 32/3$, nearest pts $\rightarrow \left(\frac{4}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right), \left(\frac{-4}{\sqrt{3}}, \frac{-4}{\sqrt{3}}\right)$

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Q8

Surface Area $= 2(xy + xz + yz)$

if x, y, z are lengths of side of parallelopiped.

Also $\boxed{x+y+z = a}$

$\therefore F(x,y,z,\lambda) = 2(xy + xz + yz) + \lambda(x+y+z-a)$.

For maxima or minima

$$\frac{\partial F}{\partial x} = 0 = 2(y+z) + \lambda \quad (1)$$

$$\frac{\partial F}{\partial y} = 0 = 2(x+z) + \lambda \quad (2)$$

$$\frac{\partial F}{\partial z} = 0 = 2(x+y) + \lambda \quad (3)$$

Adding these three eqn

$$2 \times 2[x+y+z] + 3\lambda = 0$$

$$4a + 3\lambda = 0$$

$$\lambda = -4a/3$$

$\therefore$ Using $\lambda = -4a/3$ in eqn (1)

$$2(y+z) + (-4a/3) = 0$$

$$2(a-x) - 4a/3 = 0$$

$$2a/3 - 2x = 0$$

$$\boxed{x = a/3}$$

Similarly $\boxed{y = a/3}$, $\boxed{z = a/3}$

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Surface Area Max m $= 2\left(\frac{a^2}{9} + \frac{a^2}{9} + \frac{a^2}{9}\right)$

$= \frac{2a^2}{3}$ sq units

Dimensions are $\left(\frac{a}{3}, \frac{a}{3}, \frac{a}{3}\right)$

A cube of side length $a/3$.

Q9 Let the eqn of circle be $x^2+y^2+z^2=1$

Let a pt be (x,y,z) be in first octant of sphere and therefore volume of that box having corner pt (x,y,z) will be

$V = (2x \times 2y \times 2z)$ cubic units.

$V = 8xyz,$

constrained that $x^2+y^2+z^2-1=0$

$F(x,y,z,\lambda) = 8(xyz) + \lambda(x^2+y^2+z^2-1)$

For maxima or minima

$\frac{\partial F}{\partial x} = 0 = 8yz + 2\lambda x$ — (1)

$\frac{\partial F}{\partial y} = 0 = 8xz + 2\lambda y$ — (2)

$\frac{\partial F}{\partial z} = 0 = 8xy + 2\lambda z$ — (3)

$[x,y,z \neq 0]$

(1).x + (2).y + 3(4) becomes

$8xyz + 2\lambda x^2 + 8xyz + 2\lambda y^2 + 8xyz + 2\lambda z^2 = 0$

$$24xyz + 2\lambda[x^2+y^2+z^2] = 0$$

$$\lambda = (-xyz)(12)$$

$\therefore$ Putting back value in (1)

$$8yz - 24x^2yz = 0$$

$$yz[1-3x^2] = 0$$

$$3x^2 = 1$$

$$\boxed{x = 1/\sqrt{3}}$$

[$\because$ x, y, z cant be zero, and in first octant they are +ve].

Similarly $y = \frac{1}{\sqrt{3}}$, $z = 1/\sqrt{3}$.

$$V_{max} = 8(xyz)$$

$$= 8\left(\frac{1}{\sqrt{3}}\right)^3$$

$$= \frac{8}{3\sqrt{3}} \text{ cubic units}$$

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12 Let (x, y, z) be the point on plane

$$x + y + z = 2a \quad -(1)$$

And (x_1, y_1, z_1) be the point on the sphere

$$x_1^2 + y_1^2 + z_1^2 = a^2 \quad -(2)$$

$$d = \sqrt{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2}$$

$$d^2 = (x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2 \quad -(3)$$

where d denotes distance b/w two points (x, y, z) & (x_1, y_1, z_1)

and furthermore x, y, z are constrained to be

$x + y + z - 2a = 0$ and x_1, y_1, z_1 be $x_1^2 + y_1^2 + z_1^2 - a^2 = 0$

$$F(x, y, z, x_1, y_1, z_1, \lambda_1, \lambda_2)$$

$$= (x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2 + \lambda_1(x+y+z-2a) + \lambda_2(x_1^2 + y_1^2 + z_1^2 - a^2)$$

For maxima or minima

$$\frac{\partial F}{\partial x} = 0 = 2(x-x_1) + \lambda_1 \quad -(4)$$

$$\frac{\partial F}{\partial y} = 0 = 2(y-y_1) + \lambda_1 \quad -(5)$$

$$\frac{\partial F}{\partial z} = 0 = 2(z-z_1) + \lambda_1 \quad -(6)$$

$$\frac{\partial F}{\partial x_1} = 0 = -2(x-x_1) + 2\lambda_2 x_1 \quad -(7)$$

$$\frac{\partial F}{\partial y_1} = 0 = -2(y-y_1) + 2\lambda_2 y_1 \quad -(8)$$

$$\frac{\partial F}{\partial z_1} = 0 = -2(z-z_1) + 2\lambda_2 z_1 \quad -(9)$$

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From (4) & (7)

$$\lambda_1 + 2\lambda_2 x_1 = 0 \Rightarrow x_1 = \frac{-\lambda_1}{2\lambda_2}$$

From (5) & (8)

$$\lambda_1 + 2\lambda_2 y_1 = 0 \Rightarrow y_1 = \frac{-\lambda_1}{2\lambda_2}$$

From (6) & (9)

$$\lambda_1 + 2\lambda_2 z_1 = 0 \Rightarrow z_1 = \frac{-\lambda_1}{2\lambda_2}$$

Hence, since λ_1 & λ_2 are constants, therefore

$$\boxed{x_1 = y_1 = z_1}$$

From (2)

$$3x_1^2 = a^2$$

$$\boxed{x_1 = \pm a/\sqrt{3}} = y_1 = z_1 \quad -10$$

From (4), (5), (6)

$$x = -\frac{\lambda_1}{2} + x_1$$

$$y = -\frac{\lambda_1}{2} + y_1$$

$$z = -\frac{\lambda_1}{2} + z_1$$

Since $x_1 = y_1 = z_1 = \pm a/\sqrt{3}$, therefore $\boxed{x = y = z}$

From (1)

$$3x - 2a = 0$$

$$\boxed{x = 2a/3}$$

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distance, d becomes

when $x = y = z = \frac{2a}{3}$, $x_1 = y_1 = z_1 = a/\sqrt{3}$.

$$d^2 = 3\left(\frac{2a}{3} - a/\sqrt{3}\right)^2$$

$$d = \sqrt{3}\sqrt{\left(\frac{2a}{3} - a/\sqrt{3}\right)^2} \quad \text{or} \quad d = \sqrt{3}\sqrt{\frac{4a^2}{9} + \frac{a^2}{3} - \frac{4a^2}{3\sqrt{3}}}$$

$$d = \sqrt{3}\left(\frac{2a}{3} - \frac{a}{\sqrt{3}}\right)$$

$$d = \frac{a}{\sqrt{3}}\left[2 - \sqrt{3}\right] \quad \text{or} \quad d = \frac{a}{\sqrt{3}}\sqrt{7 - 4\sqrt{3}}.$$

when $x = y = z = -\frac{2a}{3}$, $x_1 = y_1 = z_1 = -a/\sqrt{3}$

$$d = \frac{a}{\sqrt{3}}\left[2 + \sqrt{3}\right) \quad \text{or} \quad d = a/\sqrt{3}\sqrt{7 + 4\sqrt{3}}$$

$$\boxed{\begin{aligned} d_{min} &= \frac{a}{\sqrt{3}}\sqrt{7 - 4\sqrt{3}} \\ d_{max} &= \frac{a}{\sqrt{3}}\sqrt{7 + 4\sqrt{3}} \end{aligned}}$$

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Q11 $f(x,y) = \tan^{-1}(xy)$.

By Taylor thm

$$f(x_0+h, y_0+k) = f(x_0,y_0) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x_0,y_0)$$

$$+ \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0,y_0) \quad \ldots$$

Let $x_0 = 1$, $y_0 = 1$, $f(1,1) = \pi/4$

$\therefore$ $h = x - x_0$ $\boxed{h = x-1}$ $k = y - y_0$ $\boxed{k = y-1}$

$$f(x,y) = f(x_0,y_0) + h f_x + k f_y + \frac{h^2 f_{xx}}{2} + \frac{k^2 f_{yy}}{2} + \frac{2hk}{2} f_{xy} \ldots$$

$f_x = \frac{1}{1+(xy)^2} \cdot y$ $f_x(1,1) = \frac{1}{2}$

$f_y = \frac{1}{1+(xy)^2} \cdot x$ $f_y(1,1) = \frac{1}{2}$.

$f_{xx} = y\left[\frac{1}{[1+(xy)^2]^2} \cdot -2xy^2\right]$. $f_{xx}(1,1) = \frac{-1}{2}$

$f_{yy} = x\left[\frac{-1}{[1+(xy)^2]^2} \cdot 2yx^2\right]$ $f_{yy}(1,1) = -\frac{1}{2}$.

$$f_{xy} = \frac{[1+(xy)^2][1] - [x][2xy^2]}{[1+(xy)^2]^2}$$

$$f_{xy}(1,1) = \frac{2-2}{4} = 0.$$

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Therefore, upto ~~two terms~~ second degree

$$f(x,y) = \frac{\pi}{4} + \frac{h}{2} + \frac{k}{2} - \frac{h^2}{4} - \frac{k^2}{4} + 0.$$

Since $\boxed{h = x-1}$, $\boxed{k = y-1}$

$$f(x,y) = \frac{\pi}{4} + \frac{(x-1)}{2} + \frac{(y-1)}{2} - \frac{(x-1)^2}{4} - \frac{(y-1)^2}{4}$$

$$= 0.7854 + \frac{(x-1)}{2} + \frac{(y-1)}{2} - \frac{(x-1)^2}{4} - \frac{(y-1)^2}{4}$$

$$f(1.1, 0.8) = 0.7854 + \frac{0.1}{2} + \frac{(-0.2)}{2} - \frac{(0.1)^2}{4} - \frac{(-0.2)^2}{4}$$

$$= 0.7854 + 0.05 - 0.1 - 0.0025 - 0.01.$$

$$= \underline{\underline{0.7229}}$$

Q12 $f(x,y) = y^2 + x^2y + x^4.$

For minima or maxima

$f_x = 0, \quad f_y = 0$

$f_x = 2xy + 4x^3 = 0 \qquad f_y = 2y + x^2 = 0$

$x[2y + 4x^2] = 0$

$x[2y + x^2 + 3x^2] = 0$

$x[0 + 3x^2] = 0$

$x^3 = 0 \qquad \boxed{x = 0} \qquad$ then $y = -\frac{x^2}{2} = 0.$

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Critical pt $(0,0)$.

$f_{xx} = 2y + 12x^2$, At $(0,0)$

$f_{xx} = 0$.

$f_{yy} = 2$

$f_{xy} = 2x$, At $(0,0)$, $f_{xy} = 0$

$f_{xx} f_{yy} - (f_{xy})^2 = 0$

Test is inconclusive

$$f(x,y) = y^2 + 2\left(\frac{x^2}{2}\right)(y) + \frac{x^4}{4} + \frac{3x^4}{4}$$

$$f(x,y) = \left(y + \frac{x^2}{2}\right)^2 + \frac{3x^4}{4}$$

$\Delta f = f(x+h, y+k) - f(x,y)$

At $(0,0)$

$\Delta f = f(h,k) - f(0,0)$

$f(0,0) = 0$.

$\Delta f = f(h,k)$

$$= \left(k + \frac{h^2}{2}\right)^2 + \frac{3h^4}{4}$$

For any value of h, k

$\Delta f > 0$,

Hence, minimum will occur at $(0,0)$.

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$\rightarrow \quad P(q_1, q_2, q_3) = 2q_1 + 8q_2 + 24q_3.$

Constrained that

$$q_1^2 + 2q_2^2 + 4q_3^2 = 4500000000$$
$$= 4.5 \times 10^9$$

$$\therefore F(q_1, q_2, q_3, \lambda) = (2q_1 + 8q_2 + 24q_3) + \lambda(q_1^2 + 2q_2^2 + 4q_3^2 - 4.5 \times 10^9)$$

For maxima,

$$\frac{\partial F}{\partial q_1} = 0 = 2 + 2q_1\lambda \quad -(1)$$

$$\frac{\partial F}{\partial q_2} = 0 = 8 + 4q_2\lambda \quad -(2)$$

$$\frac{\partial F}{\partial q_3} = 0 = 24 + 8\lambda q_3 \quad -(3)$$

$(2) - 4 \cdot (1)$

$$4q_2\lambda - 8q_1\lambda = 0$$

$$4q_2 - 8q_1 = 0 \qquad [\because \lambda \neq 0]$$

$$\boxed{q_2 = 2q_1} \quad -(4)$$

$(3) - 3 \cdot (2)$

$$8\lambda q_3 - 12\lambda q_2$$

$$\boxed{q_3 = \tfrac{3}{2} q_2} \quad -(5).$$

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Putting values of q_1 and q_3 in

$$q_1^2 + 2q_2^2 + 4q_3^2 = 450000000$$

$$q_2^2\left[\frac{1}{4} + 2 + \cancel{4}\times\frac{9}{\cancel{4}}\right] = 450\times10^8$$

$$q_2^2\left[\frac{1+8+36}{4}\right] = 45\times10^8$$

$$q_2^2 = 4\times10^8$$

$$\boxed{q_2 = 2\times10^4}$$

$$\boxed{q_1 = \frac{q_2}{2} = 10^4}$$

$$\boxed{q_3 = \frac{3q_2}{2} = 3\times10^4}$$

For max^m profit,

manufacture must produce

$q_1 = 10^4$ units

$q_2 = 2\times10^4$ units

$q_3 = 3\times10^4$ units.

Q14

$z = f(x, y)$.

$x = e^{2u} + e^{-2v}$

$y = e^{-2u} + e^{2v}$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\left[2e^{2u}\right] + \frac{\partial z}{\partial y}\left[-2e^{-2u}\right] \quad (1)$$

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$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$= \frac{\partial z}{\partial x}\left[-2e^{-2v}\right] + \frac{\partial z}{\partial y}\left[2e^{2v}\right] \quad (2)$$

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\left[2e^{2u}\right] + \frac{\partial z}{\partial y}\left[-2e^{-2u}\right]$$

$$+ \frac{\partial z}{\partial x}\left[+2e^{-2v}\right] - \frac{\partial z}{\partial y}\left[2e^{2v}\right]$$

$$= \frac{\partial z}{\partial x}\left[2e^{2u} + 2e^{-2v}\right]$$

$$- \frac{\partial z}{\partial y}\left[2e^{-2u} + 2e^{2v}\right]$$

$$= \frac{\partial z}{\partial x}\left[2x\right] - \frac{\partial z}{\partial y}\left[2y\right]$$

$$= 2\left[x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}\right]$$

Hence, proved.

Q15

$$z^2 = xy + 1$$

$$\text{dist} = \sqrt{x^2 + y^2 + z^2}$$

$$d^2 = x^2 + y^2 + z^2$$

$$d^2 = x^2 + y^2 + xy + 1.$$

$$\partial x^2 \qquad x \qquad L \qquad x^2 \qquad J$$

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$d^2 = f(x,y) = x^2 + y^2 + xy + 1$,

$f_x = 2x + y$, $f_y = 2y + x$

For critical pts, $f_x = 0$ & $f_y = 0$

$2x + y = 0$, $2y + x = 0$.

Therefore $x = 0$, $y = 0$

Pt $(0, 0)$

$f_{xx} = 2$, $f_{xy} = 1$, $f_{yy} = 2$

$f_{xx} f_{yy} - f_{xy} = 4 - 1 = 3 > 0$

$f_{xx} > 0$

Minima will occur.

$\therefore z^2 = xy + 1$

$z^2 = 0 + 1$

$z^2 = 1$

$\boxed{z = \pm 1}$

Pts are $(0,0,1)$, $(0,0,-1)$.

$\boxed{d_{min} = 1}$

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