ENGINEERING MECHANICS AND STRENGTH OF MATERIALS

(In S.I. Units)

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ENGINEERING MECHANICS

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1

Fundamentals of Engineering Mechanics

1.1. INTRODUCTION

Engineering mechanics is that branch of science which deals with the behaviour of a body when the body is at rest or in motion. The engineering mechanics may be divided into Statics and Dynamics. The branch of science, which deals with the study of a body when the body is at rest, is known as *Statics* while the branch of science which deals with the study of a body when the body is in motion, is known as *Dynamics*. Dynamics is further divided into kinematics and kinetics. The study of a body in motion, when the forces which cause the motion are not considered, is called *kinematics* and if the forces are also considered for the body in motion, that branch of science is called *kinetics*. The classification of Engineering Mechanics are shown in Fig. 1.1 below.

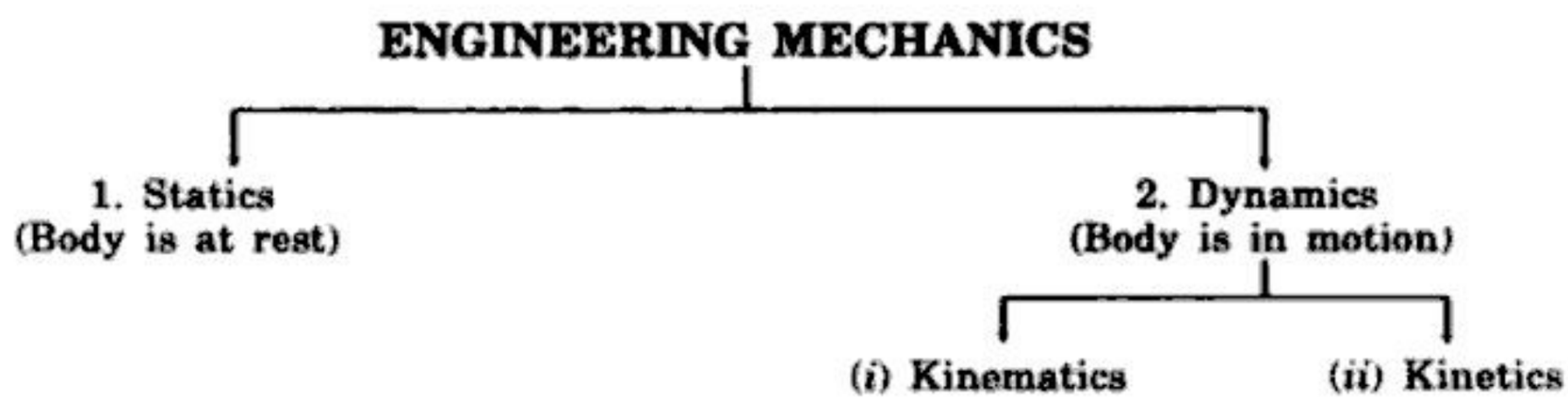

Fig.1.1.

1.2. DEFINITIONS

1.2.1. Vector Quantity. A quantity which is completely specified by magnitude and direction, is known as a vector quantity. Some examples of vector quantities are : velocity, acceleration, force and momentum. A vector quantity is represented by means of a straight line with an arrow as shown in Fig. 1.2. The length of the straight line (*i.e.*, AB) represents the magnitude and arrow represents the direction of the vector. The symbol $\vec{AB}$ also represents this vector, which means it is acting from A to B.

Fig. 1.2. Vector Quantity

1.2.2. Scalar Quantity. A quantity, which is completely specified by magnitude only, is known as a scalar quantity. Some examples of scalar quantity are : mass, length, time and temperature.

1.2.3. A Particle. A particle is a body of infinitely small volume and the mass of the particle is considered to be concentrated at a point. Hence a particle is assumed to a point and the mass of the particle is concentrated at this point.

1.2.4. Law of Parallelogram of Forces. The law of parallelogram of forces is used to determine the resultant of two forces acting at a point in a plane. It states, "If two forces, acting at a point be represented in magnitude and direction by the two adjacent sides of a parallelogram, then their resultant is represented in magnitude and direction by the diagonal of the parallelogram passing through that point".

Let two forces P and Q act at a point O as shown in Fig. 1.3. The force P is represented in magnitude and direction by OA whereas the force Q is presented in magnitude and direction by

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OB. Let the angle between the two forces be 'α'. The resultant of these two forces will be obtained in magnitude and direction by the diagonal (passing through O) of the parallelogram of which OA and OB are two adjacent sides. Hence draw the parallelogram with OA and OB as adjacent sides as shown in Fig. 1.4. The resultant R is represented by OC in magnitude and direction.

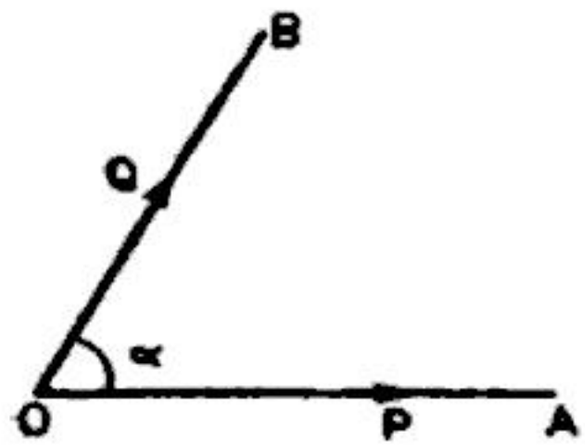

Fig. 1.3.

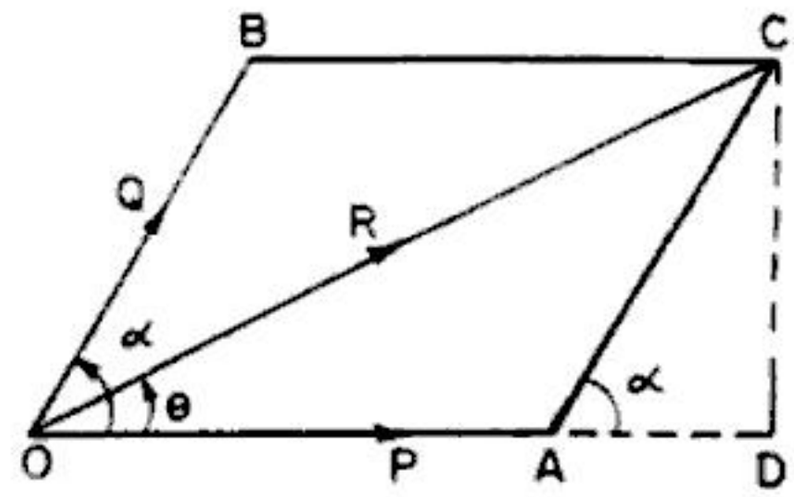

Fig. 1.4.

Magnitude of Resultant (R)

From C draw CD perpendicular to OA produced.

Let α=Angle between two forces P and Q

$=\angle AOB$

Now $\angle DAC=\angle AOB$ (Corresponding angles)

$=\alpha$

In parallelogram $OACB$, AC is parallel and equal to OB.

$\therefore$ $AC=Q$.

In triangle ACD, $AD=AC\cos\alpha=Q\cos\alpha$

and $CD=AC\sin\alpha=Q\sin\alpha$.

In triangle OCD, $OC^2=OD^2+DC^2$.

But $OC=R$, $OD=OA+AD=P+Q\cos\alpha$

and $DC=Q\sin\alpha$.

$$\therefore \quad R^2=(P+Q.\cos\alpha)^2+(Q\sin\alpha)^2$$
$$=P^2+Q^2\cos^2\alpha+2PQ\cos\alpha+Q^2\sin^2\alpha$$
$$=P^2+Q^2(\cos^2\alpha+\sin^2\alpha)+2PQ\cos\alpha$$
$$=P^2+Q^2+2PQ\cos\alpha \qquad (\because \cos^2\alpha+\sin^2\alpha=1)$$

$$\therefore \quad R=\sqrt{P^2+Q^2+2PQ\cos\alpha} \qquad ...(1.1)$$

Equation (1.1) gives the magnitude of resultant force R.

Direction of Resultant

Let θ=Angle made by resultant with OA.

Then from triangle OCD,

$$\tan\theta=\frac{CD}{OD}=\frac{Q\sin\alpha}{P+Q\sin\alpha}$$

$$\therefore \quad \theta=\tan^{-1}\left(\frac{Q\sin\alpha}{P+Q\sin\alpha}\right) \qquad ...(1.2)$$

Equation (1.2) gives the direction of resultant (R).

Two cases are important.

1st Case. If the two forces P and Q act at right angles, then

$\alpha=90°$

From equation (1.1), we get the magnitude of resultant as

$$R=\sqrt{P^2+Q^2+2PQ\cos\alpha}=\sqrt{P^2+Q^2+2PQ\cos 90°}$$

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$$=\sqrt{P^2+Q^2} \qquad (\because \cos 90°=0) \qquad ...(1.2\ A)$$

From equation (1.2), the direction of resultant is obtained as

$$\theta=\tan^{-1}\left(\frac{Q \sin \alpha}{P+Q \cos \alpha}\right)$$

$$=\tan^{-1}\left(\frac{Q \sin 90°}{P+Q \cos 90°}\right)=\tan^{-1}\frac{Q}{P} \qquad (\because \sin 90°=1 \text{ and } \cos 90°=0)$$

2nd Case. The two forces P and Q are equal and are acting at an angle α between them. Then the magnitude and direction of resultant is given as

$$R=\sqrt{P^2+Q^2+2PQ \cos \alpha}$$

$$=\sqrt{P^2+P^2+2P\times P\times \cos \alpha} \qquad (\because P=Q)$$

$$=\sqrt{2P^2+2P^2 \cos \alpha}=\sqrt{2P^2\ (1+\cos \alpha)}$$

$$=\sqrt{2P^2\times 2 \cos^2 \frac{\alpha}{2}} \qquad \left(\because 1+\cos \alpha=2 \cos^2 \frac{\alpha}{2}\right)$$

$$=\sqrt{4P^2 \cos^2 \frac{\alpha}{2}}$$

$$=2P \cos \frac{\alpha}{2} \qquad ...(1.3)$$

and

$$\theta=\tan^{-1}\left(\frac{Q \sin \alpha}{P+Q \cos \alpha}\right)=\tan^{-1}\frac{P \sin \alpha}{P+P \cos \alpha} \qquad (\because P=Q)$$

$$=\tan^{-1}\frac{P \sin \alpha}{P(1+\cos \alpha)}=\tan^{-1}\frac{\sin \alpha}{1+\cos \alpha}$$

$$=\tan^{-1}\frac{2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}}{2 \cos^2 \frac{\alpha}{2}} \qquad \left(\because \sin \alpha=2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}\right)$$

$$=\tan^{-1}\frac{\sin \frac{\alpha}{2}}{\cos \frac{\alpha}{2}}=\tan^{-1}\left(\tan \frac{\alpha}{2}\right)=\frac{\alpha}{2} \qquad ...(1.4)$$

It is not necessary that one of two forces, should be along x-axis. The forces P and Q may be in any direction as shown in Fig. 1.5. If the angle between the two forces is 'α', then their resultant will be given by equation (1.1). The direction of the resultant would be obtained from equation (1.2). But angle θ will be the angle made by resultant with the direction of P.

1.2.5. Law of Triangle of Forces. It states that, "if three forces acting at a point be represented in magnitude and direction by the three sides of a triangle, taken in order, they will be in equilibrium."

1.2.6. Lami's Theorem. It states that, "If three forces acting at a point are in equilibrium, each force will be proportional to the sine of the angle between the other two forces."

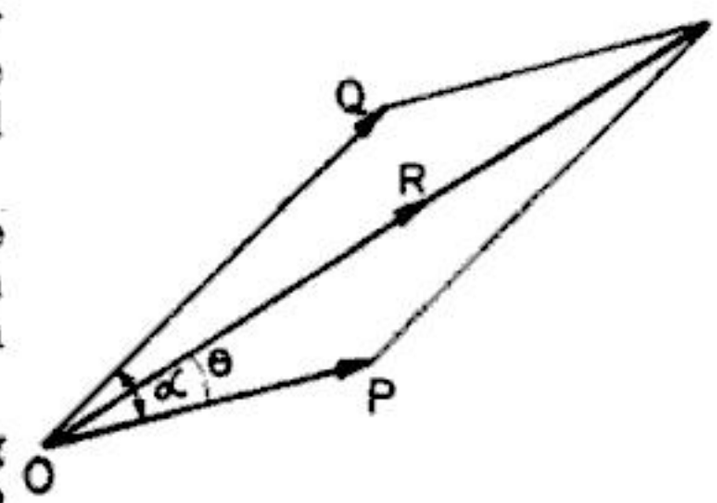

Fig. 1.5.

Suppose the three forces P, Q and R are acting at a point O and they are in equilibrium as shown in Fig. 1.6.

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Let α=Angle between force P and Q.
β=Angle between force Q and R.
γ=Angle between force R and P.

Then according to Lami's theorem,

$P \propto$ sine of angle between Q and $R \propto \sin\beta$.

$$\therefore \quad \frac{P}{\sin\beta}=\text{constant}$$

Similarly $\dfrac{Q}{\sin\gamma}=\text{constant}$ and $\dfrac{R}{\sin\alpha}=\text{constant}$

or
$$\frac{P}{\sin\beta}=\frac{Q}{\sin\gamma}=\frac{R}{\sin\alpha} \qquad ...(1.5)$$

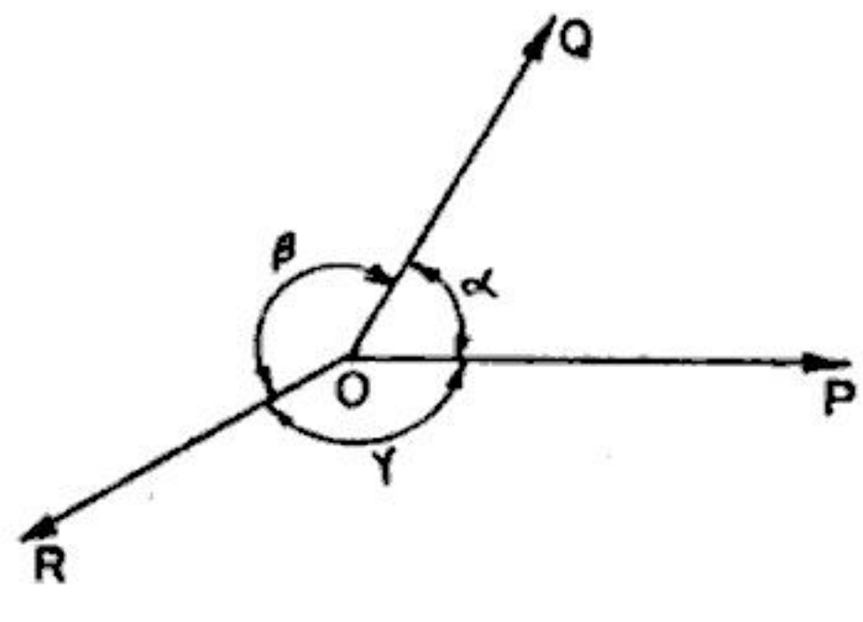

Fig. 1.6.

1.3. SYSTEM OF UNITS

The following system of units are mostly used :

1. C.G.S. (*i.e.*, Centimetre-Gram-Second) system of units.
2. M.K.S. (*i.e.*, Metre-Kilogram-Second) system of units.
3. S.I. (*i.e.*, International) system of units.

1.3.1. C.G.S. System of units. In this system, length is expressed in centimetre, mass in gram and time in second. The unit of force in this system is *dyne*, which is defined as the force acting on a mass of one gram and producing an acceleration of one centimetre per second square.

1.3.2. M.K.S. System of Units. In this system, length is expressed in metre, mass in kilogram and time in second. The unit of force in this system is expressed as kilogram force and is represented as kgf.

1.3.3. S.I. System of Units. S.I. is abbreviation for 'The System International Units'. It is also called the International System of Units. In this system length is expressed in metre mass in kilogram and time in second. The unit of force in this system is Newton and is represented N. Newton is the force acting on a mass of one kilogram and producing an acceleration of one metre per second square. The relation between newton (N) and dyne is obtained as

$$\text{One Newton}=\text{One kilogram mass}\times\frac{\text{One metre}}{s^2}$$

$$=1000 \text{ gm}\times\frac{100 \text{ cm}}{s^2} \qquad (\because \text{ one kg}=1000 \text{ gm})$$

$$=1000\times100\times\frac{\text{gm}\times\text{cm}}{s^2}$$

$$=10^5 \text{ dyne} \qquad \left\{\because \text{ dyne}=\frac{\text{gm}\times\text{cm}}{s^2}\right\}$$

When the magnitude of forces is very large, then the unit of force like kilo-newton and mega-newton is used. Kilo-newton is represented by kN.

One kilo-newton$=10^3$ newton

or $\quad 1 \text{ kN}=10^3 \text{ N}$

and One mega newton$=10^6$ Newton

The large quantities are represented by kilo, mega, giga and terra. They stand for :

Kilo $=10^3$ and represented by.....k

Mega $=10^6$ and represented by.....M

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Giga $=10^9$ and represented by.....G

Tera $=10^{12}$ and represented by.....T

Thus mega newton means 10^6 newton and is represented by MN. Similarly, giga newton means 10^9 N and is represented by GN. The symbol TN stands for 10^{12} N.

The small quantities are represented by milli, micro, nano and pico. They are equal to

Milli $=10^{-3}$ and represented by .....m

Micro $=10^{-6}$ and represented by .....μ

Nano $=10^{-9}$ and represented by .....n

Pico $=10^{-12}$ and represented by .....p.

Thus milli newton means 10^{-3} newton and is represented by mN. Micro newton means 10^{-6} N and is represented by μN.

The relation between kilogram force (kgf) and newton (N) is given by

$$\text{One kgf} = 9.81 \text{ N}$$

Weight of a body is the force with which the body is attracted towards earth. If W=weight of a body, m=mass in kg, then

$$W = m \times g \text{ Newtons}$$

If mass, m of the body is 1 kg, then its weight will be,

$$W = 1 \text{ (kg)} \times 9.81 \frac{\text{m}}{\text{s}^2} = 9.81 \text{ N.} \qquad \left(\because \text{N} = \text{kg} \frac{\text{m}}{\text{s}^2}\right)$$

1.4. TRIGONOMETRIC FORMULAE AND EXPRESSIONS

The following are the trigonometric formulae in a right-angled triangle ABC of Fig. 1.7.

(*i*) $\sin\theta = \dfrac{AC}{BC}$

(*ii*) $\cos\theta = \dfrac{AB}{BC}$

(*iii*) $\tan\theta = \dfrac{AC}{AB}$

(*iv*) $\sin(A+B) = \sin A \cos B + \cos A \sin B$

(*v*) $\sin(A-B) = \sin A \cos B - \sin A \sin B$

(*vi*) $\cos(A+B) = \cos A \cos B - \cos A \sin B$

(*vii*) $\cos(A-B) = \cos A \cos B + \sin A \sin B$

(*viii*) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

(*ix*) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

(*x*) $\sin 2A = 2 \sin A \cos A$

(*xi*) $\sin^2\theta + \cos^2\theta = 1$.

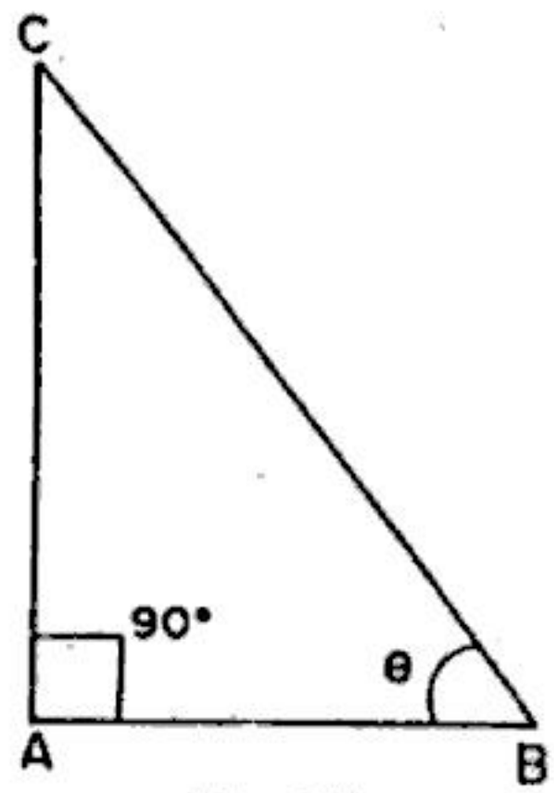

Fig. 1.7.

1.5. DIFFERENTIATION AND INTEGRATION

1.5.1. Differentiations. (*i*) Differentiation of a quantity (say A) with respect to x is written as $\dfrac{d}{dx}(A)$ or $\dfrac{dA}{dx}$

(*ii*) $\dfrac{d}{dx}(x^4) = 4x^3$, $\dfrac{d}{dx}(x^n) = n\, x^{n-1}$ and $\dfrac{d}{dx}(x) = 1$

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(*iii*) $\frac{d}{dx}(8x+5)^4=4\ (8x+5)^3\times 8$

(*iv*) $\frac{d}{dx}(4)=0$ as differentiation of constant is zero.

(*v*) $\frac{d}{dx}\ (u.v)=u\ .\ \frac{dv}{dx}+v\ .\ \frac{du}{dx}$ [when u and v are functions of x]

(*vi*) Differentiation of trigonometrical functions

$$\frac{d}{dx}(\sin\ x)=\cos\ x$$

$$\frac{d}{dx}(\cos\ x)=-\sin\ x$$

$$\frac{d}{dx}(\tan\ x)=\sec^2\ x.$$

1.5.2. Integrations. (*i*) Integration of a quantity (say A) with respect to x is written as $\int A dx$.

(*ii*) $\int x^n dx=\frac{x^{n+1}}{n+1}$

(*iii*) $\int 4dx=4x$

(*iv*) $\int (8x+5)^4\ dx=\frac{(8x+5)^{4+1}}{(4+1)\times 8}$.

Problem 1.1. *Two forces of magnitude 10 N and 8 N are acting at a point. If the angle between the two forces is 60°, determine the magnitude of the resultant force.*

Sol. Given :

Force $P=10$ N

Force $Q=8$ N

Angle between the two forces,

$\alpha=60°$

The magnitude of the resultant force (R) is given by equation (1.1)

$$R=\sqrt{P^2+Q^2+2\ PQ\ \cos\ \alpha}=\sqrt{10^2+8^2+2\times 10\times 8\times\cos\ 60°}$$

$$=\sqrt{100+64+2\times 10\times 8\times\tfrac{1}{2}} \qquad (\because \cos\ 60°=\tfrac{1}{2})$$

$$=\sqrt{100+94+80}=\sqrt{244}=\mathbf{15.62\ N.}\quad \textbf{Ans.}$$

Problem 1.2. *Two equal forces are acting at a point with an angle of 60° between them. If the resultant force is equal to $20\times\sqrt{3}$ N, find magnitude of each force.*

Sol. Given :

Angle between the force, $\alpha=60°$

Resultant, $R=20\times\sqrt{3}$

The forces are equal. Let P is the magnitude of each force.

Using equation (1.3), we have

$$R=2P\ \cos\ \frac{\alpha}{2}$$

or $$20\times\sqrt{3}=2P\times\cos\ \left(\frac{60°}{2}\right)=2P\ \cos\ 30°=2P\times\frac{\sqrt{3}}{2}=P\times\sqrt{3} \qquad \left(\because \cos\ 30°=\frac{\sqrt{3}}{2}\right)$$

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$\therefore \qquad P=\frac{20\times\sqrt{3}}{\sqrt{3}}=20$ N.

$\therefore$ **Magnitude of each force=20 N. Ans.**

Problem 1.3. *The resultant of the two forces, when they act at an angle of 60° is 14 N. If the same forces are acting at right angles, their resultant is $\sqrt{136}$ N. Determine the magnitude of the two forces.*

Sol. Given :

Case I

Resultant, $R_1=14$ N

Angle, $\alpha=60°$

Case II

Resultant, $R_2=\sqrt{136}$ N

Angle, $\alpha=90°$

Let the magnitude of the two forces are P and Q.

Using equation (1.1) for case I.

$$R=\sqrt{P^2+Q^2+2PQ\cos\alpha}$$

or $$14=\sqrt{P^2+Q^2+2PQ\times\cos 60°}=\sqrt{P^2+Q^2+2PQ\times\tfrac{1}{2}}$$

or $$14=\sqrt{P^2+Q^2+PQ}$$

Squaring, $196=P^2+Q^2+PQ$ ...(*i*)

Using equation (1.2 A) for case II,

$$R=\sqrt{P^2+Q^2}$$

or $$\sqrt{136}=\sqrt{P^2+Q^2}$$

or $$136=P^2+Q^2$$ (Squaring both sides) ...(*ii*)

Subtracting equation (*ii*) from equation (*i*), we get

$$196-136=P^2+Q^2+PQ-(P^2+Q^2)$$

or $$60=PQ$$ ...(*iii*)

Multiplying the above equation by two, we get

$$120=2PQ$$ ...(*iv*)

Adding equation (*iv*) to equation (*ii*), we get

$$136+120=P^2+Q^2+2PQ$$

or $$256=P^2+Q^2+2PQ$$

or $$(16)^2=(P+Q)^2$$

or $$16=P+Q$$

$\therefore$ $$P=(16-Q)$$ ...(*v*)

Substituting the value of P in equation (*iii*), we get

$$60=(16-Q)\times Q=16Q-Q^2$$

or $$Q^2-16Q+60=0$$

$\therefore$ This is a quadratic equation.

$$\therefore \qquad Q=\frac{16\pm\sqrt{(-16)^2-4\times 60}}{2}=\frac{16\pm\sqrt{256-240}}{2}=\frac{16\pm 4}{2}$$

$$=\frac{16+4}{2} \text{ and } \frac{16-4}{2}=10 \text{ and } 6.$$

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Substituting the value of Q in equation (v), we get

$$P=(16-10) \text{ or } (16-6)=6 \text{ or } 10.$$

$\therefore$ Hence the two forces are 10 N and 6 N. **Ans.**

Problem 1.4. *Two forces are acting at a point O as shown in Fig. 1.8. Determine the resultant in magnitude and direction.*

Sol. Given :

Force $P=50$ N,

Force $Q=100$ N

Angle between the two forces, $\alpha=30°$.

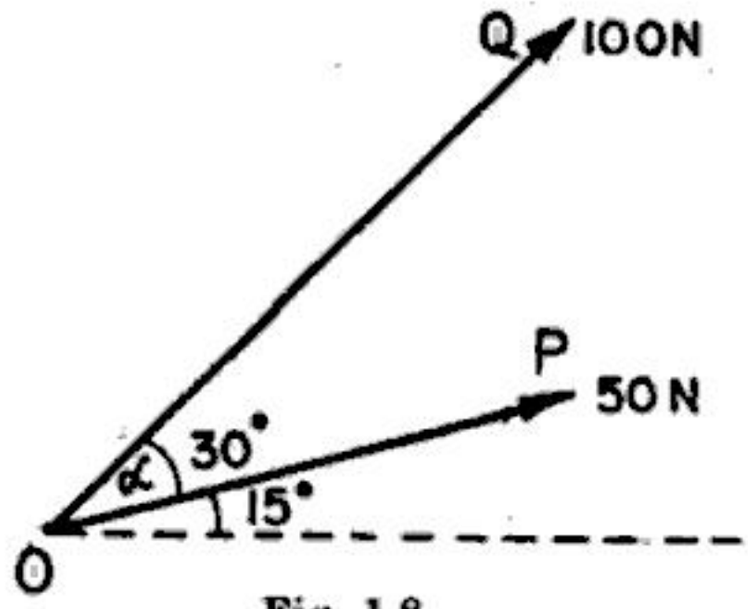

Fig. 1.8.

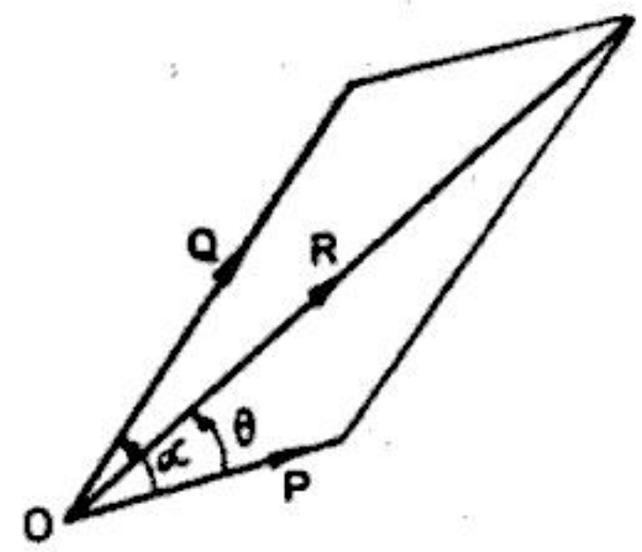

Fig. 1.9.

The magnitude of the resultant R is given by equation (1.1) as

$$R=\sqrt{P^2+Q^2+2PQ \cos \alpha}=\sqrt{50^2+100^2+2\times50\times100\times\cos 30°}$$

$$=\sqrt{2500+10000+8660}=\sqrt{21160}=\mathbf{145.46\ N.}\quad \textbf{Ans.}$$

The resultant R is shown in Fig. 1.9.

The angle made by the resultant with the direction of P is given by equation (1.2) as

$$\tan \theta=\frac{Q \sin \alpha}{P+Q \cos \alpha}$$

or $$\theta=\tan^{-1}\left(\frac{Q \sin \alpha}{P+Q \cos \alpha}\right)=\tan^{-1}\left(\frac{100\times\sin 30°}{50+100 \cos 30°}\right)$$

$$=\tan^{-1} 0.366=20.10°$$

$\therefore$ Angle made by resultant with x-axis$=\theta+15°=20.10+15$

$$=\mathbf{35.10°.}\quad \textbf{Ans.}$$

Problem 1.5. *The resultant of two concurrent forces is 1500 N and the angle between the forces is 90°. The resultant makes an angle of 36° with one of the force. Find the magnitude of each force.*

Sol. Given :

Resultant, $R=1500$ N

Angle between the forces, $\alpha=90°$

Angle made by resultant with one force, $\theta=36°$

Let P and Q are two forces.

Using equation (1.2), $$\tan \theta=\frac{Q \sin \alpha}{P+Q \cos \alpha}$$

or $$\tan 36°=\frac{Q \sin 90°}{P+Q \cos 90°}=\frac{Q\times1}{P+Q\times0}=\frac{Q}{P}$$

or $$0.726=\frac{Q}{P}$$

or $$Q=0.726\ P \qquad ...(i)$$

Using equation (1.1), $$R=\sqrt{P^2+Q^2+2PQ \cos \alpha}$$

or $$R^2=P^2+Q^2+2PQ \cos \alpha$$

or $$1500^2=P^2+(0.726P)^2+2P(0.726P)\times\cos 90° \qquad (\because Q=0.726P)$$

$$1500^2=P^2+0.527P^2+0 \qquad (\because \cos 90°=0)$$

$$=1.527\ P^2$$

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$$\therefore \qquad P=\sqrt{\frac{1500^2}{1.527}}=\frac{1500}{1.2357}=1213.86 \text{ N}$$

Substituting the value of P in equation (*i*), we get

$$Q=0.726\times 1213.86=\mathbf{881.26 \text{ N.}} \quad \textbf{Ans.}$$

Problem 1.6. *The sum of two concurrent forces P and Q is 270 N and their resultant is 180 N. The angle between the force P and resultant R is 90°. Find the magnitude of each force and angle between them.*

Sol. Given :

Sum of two concurrent forces $=270$ N

or $P+Q=270$ N

Resultant, $R=180$ N

Angle between force P and resultant $R=90°$

This means $\theta=90°$

Find : (*i*) Magnitude of P and Q

(*ii*) Angle between P and Q (*i.e.*, angle α)

Using equation (1.2), $$\tan\theta=\frac{Q\sin\alpha}{P+Q\cos\alpha}$$

or $$\tan 90°=\frac{Q\sin\alpha}{P+Q\cos\alpha}$$

But $\tan 90°=\infty$ (*i.e.* infinity). This is only possible when $P+Q\cos\alpha=0$

$$\therefore \qquad P=-Q\cos\alpha \qquad ...(i)$$

Using equation (1.1), $$R=\sqrt{P^2+Q^2+2PQ\cos\alpha}$$

Squaring to both sides, we get $$R^2=P^2+Q^2+2PQ\cos\alpha$$

$$=P^2+Q^2+2P(-P) \quad (\because \text{From equation } (i),\ Q\cos\alpha=-P)$$

$$=P^2+Q^2-2P^2=Q^2-P^2$$

$$=(Q+P)(Q-P)$$

or $$180^2=270(Q-P) \qquad (\because R=180,\ Q+P=270)$$

$$32400=270(Q-P)$$

$$\therefore \qquad Q-P=\frac{32400}{270}=120$$

But $P+Q=270$ (given)

Adding the above two equations, we get

$$2Q=270+120=390$$

$$\therefore \qquad Q=\mathbf{195 \text{ N.}} \quad \textbf{Ans.}$$

and $$P=270-Q=270-195=\mathbf{75 \text{ N.}} \quad \textbf{Ans.}$$

Value of angle α

Substituting the values of P and Q in equation (*i*),

$$P=-Q\cos\alpha$$

or $$75=-195\cos\alpha$$

or $$\cos\alpha=\frac{-75}{195}=-0.3846$$

$$\therefore \qquad \alpha=\cos^{-1}(-0.3846)=\mathbf{112.618°.} \quad \textbf{Ans.}$$

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Problem 1.7. *A weight of 1000 N is supported by two chains as shown in Fig. 1.10. Determine the tension in each chain.*

Sol. Given :

Weight at $C=1000$ N

$\angle CAB=30°$

$\angle CBA=60°$

$\angle ACB=90°$

In right-angled triangle ADC,

$\angle ACD=90°-30°=60°$

In right-angled triangle BDC,

$\angle BCD=90°-60°=30°$

$\therefore \quad \angle ACE=180°-60°=120°$

$\angle BCE=180°-30°=150°$

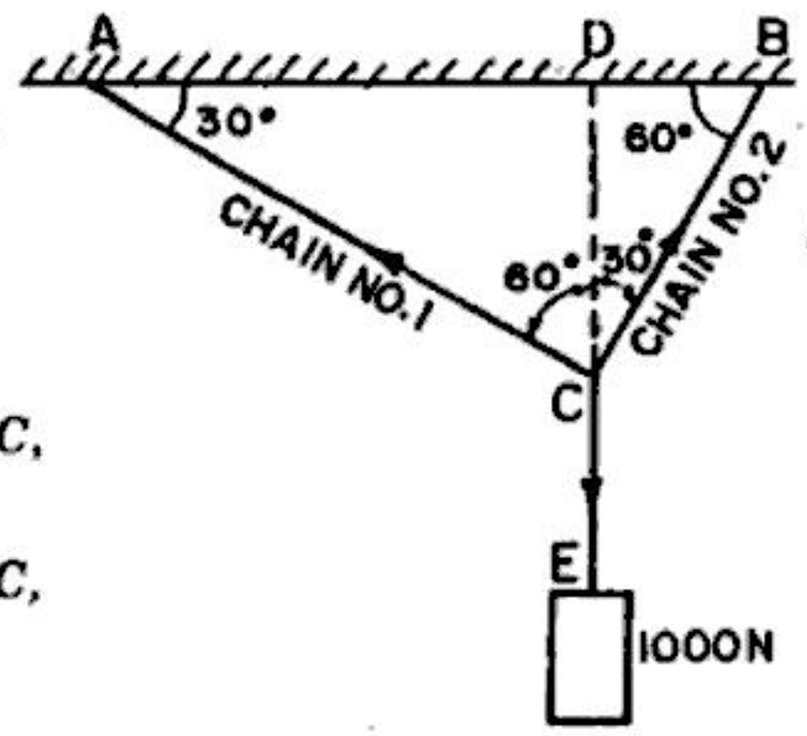

Fig. 1.10.

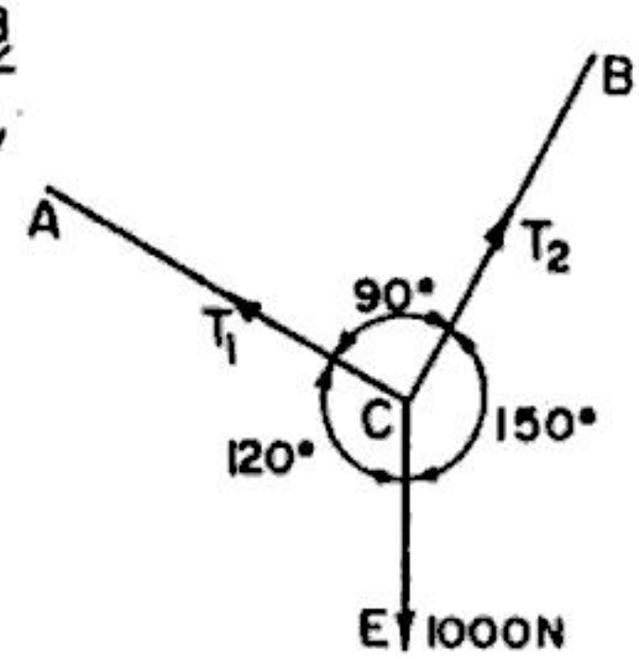

Fig. 1.11.

Let T_1=Tension in chain No. 1

T_2=Tension in chain No. 2.

Applying Lami's theorem at point C (Refer Fig. 1.11).

$$\frac{T_1}{\sin 150°}=\frac{T_2}{\sin 120°}=\frac{1000}{\sin 90°}$$

or $$\frac{T_1}{\sin 150°}=\frac{T_3}{\sin 120°}=1000 \qquad (\because \sin 90°=1)$$

$\therefore \quad T_1=1000 \sin 150°=1000\times.5=$ **500 N. Ans.**

and $\quad T_2=1000 \sin 120°=1000\times.866=$ **866 N. Ans.**

Problem 1.8. *A weight of 900 N is supported by two chains of lengths 4 m and 3 m as shown in Fig. 1.12. Determine the tension in each chain.*

Sol. Given :

Weight at $C=900$ N

Length, $AC=4$ m

Length, $BC=3$ m

Length, $AB=5$ m

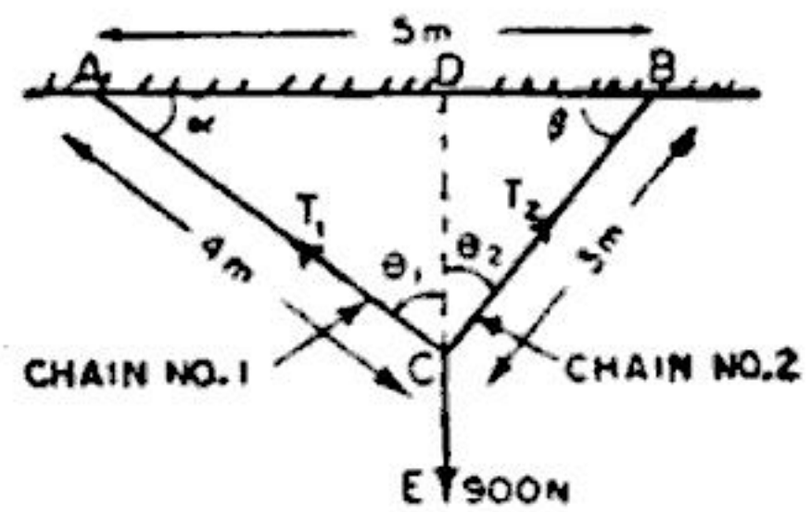

Fig. 1.12.

In triangle ABC,

$$AC^2+BC^2=4^2+3^2=16+9=25$$

$$AB^2=5^2=25$$

$\therefore \quad AB^2=AC^2+BC^2$

$\therefore$ Triangle ABC is a right-angled triangle in which

$\angle ACB=90°$

$$\sin \alpha=\frac{BC}{AB}=\frac{3}{5}=0.6$$

$\therefore \quad \alpha=36° \ 52'$

and $\quad \alpha+\beta=90°$

$\therefore \quad \beta=90°-\alpha=90°-(36° \ 52')=53° \ 8'.$

Let T_1=Tension in chain AC

T_2=Tension in chain BC

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In right-angled triangle ADC,

$$\theta_1 = 90° - \alpha = 90° - 36° \ 52' = 53° \ 8'$$

In right-angled triangle BDC,

$$\theta_1 = 90° - \beta = 90° - 53° \ 8' = 36° \ 52'$$

$$\therefore \quad \angle ACE = 180° - \theta_1 = 180° - 53° \ 8' = 126° \ 52'$$

$$\angle BCE = 180° - \theta_2 = 180° - 36° \ 52' = 143° \ 8'$$

and $\angle ACB = 90°$.

Applying Lami's theorem at C

$$\frac{T_1}{\sin \text{ of } \angle BCE} = \frac{T_2}{\sin \text{ of } \angle ACE} = \frac{900}{\sin 90°}$$

or $$\frac{T_1}{\sin 143° \ 8'} = \frac{T_2}{\sin 126° \ 52'} = \frac{900}{1} \qquad (\because \sin 90° = 1)$$

$$\therefore \quad T_1 = 900 \times \sin 143° \ 8' = 900 \times .597 = \mathbf{537.44 \ N. \ Ans.}$$

$$T_2 = 900 \times \sin 126° \ 52' = \mathbf{720 \ N. \ Ans.}$$

Problem 1.9. *An electric light fixture weighing 15 N hangs from a point C, by two strings AC and BC. AC is inclined at 60° to the horizontal and BC at 45° to the vertical as shown in Fig. 1.13. Using Lami's theorem or otherwise determine the forces in the strings AC and BC.*

(AMIE Summer, 1975)

Sol. Given :

Weight at $C = 15$ N

$\angle OAC = 60°$

$\angle CBD = 45°$

Let T_1 = Force in string BC

T_2 = Force in string AC

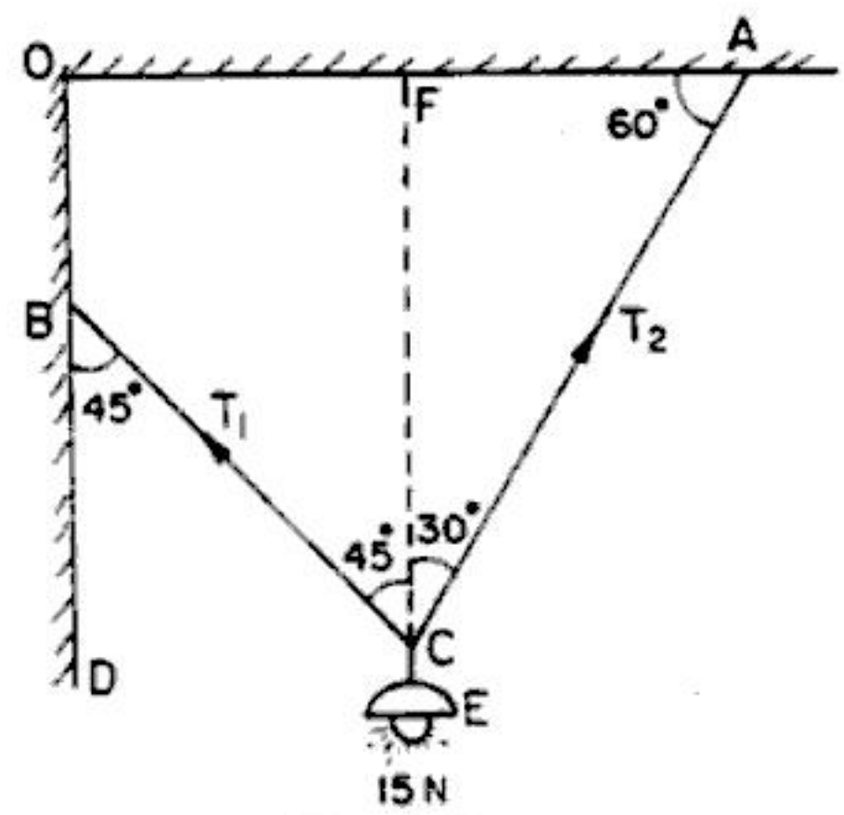

Fig. 1.13.

1st Method

Using Lami's theorem at C

$$\frac{15}{\sin \text{ of } \angle BCA} = \frac{T_1}{\sin \text{ of } \angle ACE} = \frac{T_2}{\sin \text{ of } \angle ACE}$$

But $\angle BCA = 45° + 30° = 75°$

$$\angle ACE = 180° - 30° = 150°$$

$$\angle BCE = 180° - 45° = 135°$$

$$\therefore \quad \frac{15}{\sin 75°} = \frac{T_1}{\sin 150°} = \frac{T_2}{\sin 135°}$$

$$\therefore \quad T_1 = \frac{15 \times \sin 150°}{\sin 75°} = \mathbf{7.76 \ N. \quad Ans.}$$

and $$T_2 = \frac{15 \times \sin 135°}{\sin 75°} = \mathbf{10.98 \ N. \quad Ans.}$$

2nd Method

The point C is in the equilibrium. The forces acting at C are 15 N, T_1 and T_2.

Resolving all forces at C in the horizontal direction

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$$T_1 \sin 45° = T_2 \sin 30°$$

or $$T_1 \times \frac{1}{\sqrt{2}} = T_2 \times \frac{1}{2}$$

$\therefore$ $$T_2 = T_1 \times \frac{2}{\sqrt{2}} = \sqrt{2} \times T_1 \quad ...(i)$$

Resolving all forces at C in the vertical direction,

$$T_1 \cos 45° + T_2 \cos 30° = 15$$

or $$T_1 \times \frac{1}{\sqrt{2}} + T_2 \times \frac{\sqrt{3}}{2} = 15 \quad ...(ii)$$

Substituting the value of T_2 from equation (*i*) into equation (*ii*),

$$T_1 \times \frac{1}{\sqrt{2}} + \sqrt{2} \times T_1 \times \frac{\sqrt{3}}{2} = 15$$

or $$\frac{T_1}{\sqrt{2}} + \frac{\sqrt{3}T_1}{\sqrt{2}} = 15 \quad \left(\because \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{1}{\sqrt{2}}\right)$$

or $$T_1 + \sqrt{3}T_1 = 15 \times \sqrt{2}$$

or $$T_1(1 + \sqrt{3}) = 15 \times \sqrt{2}$$

$\therefore$ $$T_1 = \frac{15 \times \sqrt{2}}{1 \times \sqrt{3}} = \mathbf{7.76\ N.} \quad \textbf{Ans.}$$

Substituting this value of T_1 in equation (*i*), we get

$$T_2 = \sqrt{2} \times T_1 = \sqrt{2} \times 7.76 = \mathbf{10.98\ N.} \quad \textbf{Ans.}$$

1.6. RESOLUTION OF A FORCE

Resolution of a force means "finding the components of a given force in two given directions."

Let a given force be R which makes an angle θ with X-axis as shown in Fig. 1.14. It is required to find the components of the force R along X-axis and Y-axis.

Components of R along X-axis $= R \cos \theta$.

Component of R along Y-axis $= R \sin \theta$.

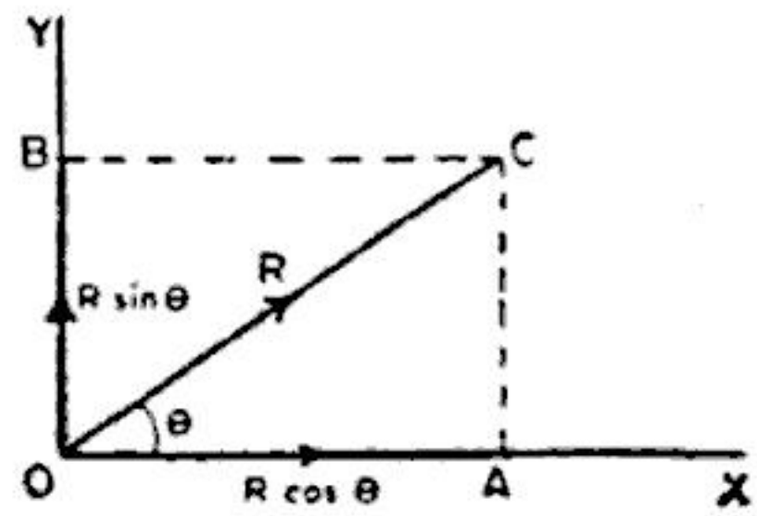

Fig. 1.14.

1.7. RESOLUTION OF A NUMBER OF COPLANAR FORCES

Let a number of coplanar forces (forces acting in one plane are called co-planar forces) R_1, R_2, R_3, ... are acting at a point as shown in Fig. 1.15.

Let θ_1 = Angle made by R_1 with X-axis

θ_2 = Angle made by R_2 with X-axis

θ_3 = Angle made by R_3 with X-axis

H = Resultant component of all forces along X-axis

V = Resultant component of all forces along Y-axis

R = Resultant of all forces

θ = Angle made by resultant with X-axis.

Each force can be resolved into two components, one along X-axis and other along Y-axis.

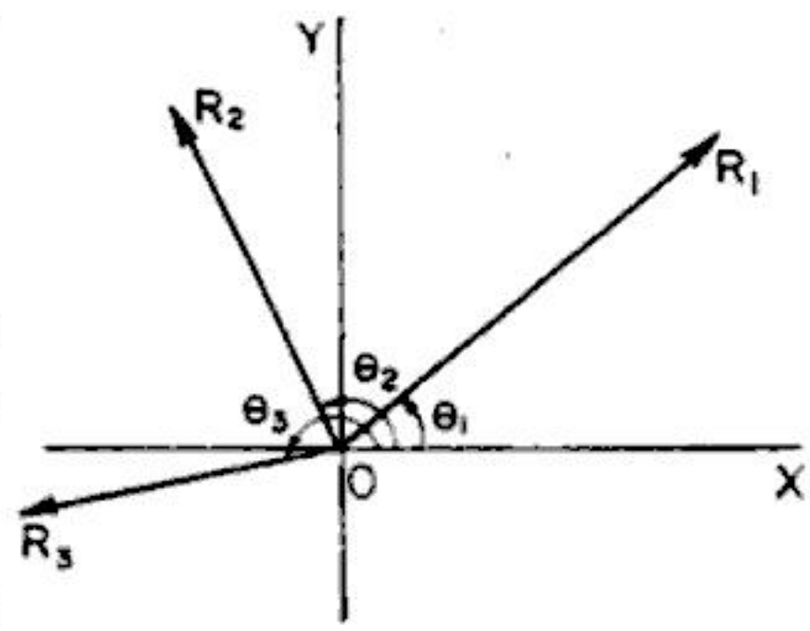

Fig. 1.15.

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Component of R_1 along X-axis$=R_1 \cos \theta_1$

Component of R_1 along Y-axis$=R_1 \sin \theta_1$.

Similarly, the components of R_2 and R_3 along X-axis and Y-axis are $(R_1 \cos \theta_2, R_2 \sin \theta_2)$ and $(R_3 \cos \theta_3, R_3 \sin \theta_3)$ respectively.

Resultant components along X-axis

=Sum of components of all forces along X-axis.

$$\therefore \quad H=R_1 \cos \theta_1+R_2 \cos \theta_2+R_3 \cos \theta_3+.... \quad ...(1.6)$$

Resultant component along Y-axis.

=Sum of components of all forces along Y-axis.

$$\therefore \quad V=R_1 \sin \theta_1+R_2 \sin \theta_2+R_3 \sin \theta_3+.... \quad ...(1.7)$$

Then resultant of all the forces,

$$R=\sqrt{H^2+V^2} \quad ...(1.8)$$

The angle made by R with X-axis is given by,

$$\tan \theta=\frac{V}{H}. \quad ...(1.9)$$

Problem 1.10. *Two forces are acting at a point O as shown in Fig. 1.16. Determine the resultant in magnitude and direction.*

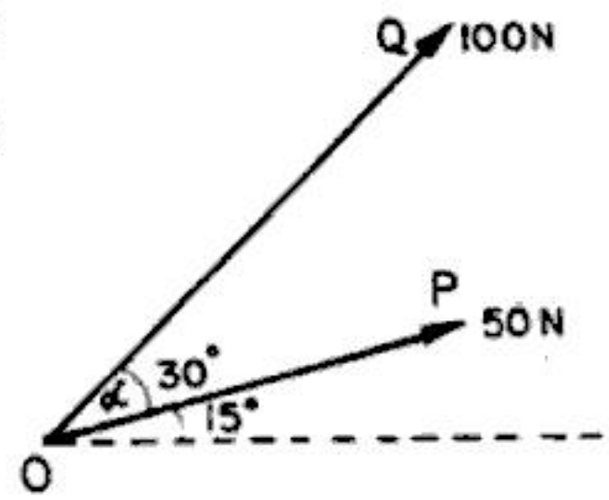

Fig. 1.16.

Sol. The above problem has been solved earlier. Hence it will be solved by resolution of forces.

Force $P=50$ N and force $Q=100$ N.

Let us first find the angles made by each force with X-axis

Angle made by P with X-axis $=15°$

Angle made by Q with X-axis $=15+30=45°$

Let H=Sum of components of all forces along X-axis.

V=Sum of components of all forces along Y-axis.

The sum of components of all forces along X-axis is given by,

$$H=P \cos 15°+Q \cos 45°=50\times\cos 15°+100 \cos 45°=119 \text{ N}$$

The sum of components of all forces along Y-axis is given by,

$$V=P \sin 15°+Q \sin 45°=50 \sin 15°+100 \sin 45°=83.64 \text{ N}$$

The magnitude of the resultant force is given by equation (1.8),

$$R=\sqrt{H^2+V^2}=\sqrt{119^2+83.64^2}=\mathbf{145.46 \text{ N}} \quad \textbf{Ans.}$$

The direction of the resultant force is given by equation (1.9),

$$\tan \theta=\frac{V}{H}=\frac{83.64}{119}$$

$$\therefore \quad \theta=\tan^{-1} \frac{83.64}{119}=\mathbf{35.10°}. \quad \textbf{Ans.}$$

Here θ is the angle made by resultant R with x-axis.

Problem 1.11. *Three forces of magnitude 40 kN, 15 kN and 20 kN are acting at a point O as shown in Fig. 1.17. The angles made by 40 kN, 15 kN and 20 kN forces with X-axis are 60°, 120° and 240° respectively. Determine the magnitude and direction of the resultant force.*

Sol. Given :

$$R_1=40 \text{ kN}, \theta_1=60°$$

$$R_2=15 \text{ kN}, \theta_2=120°$$

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$R_3=20$ kN, $\theta_3=240°$

The sum of components of all forces along X-axis is given by equation (1.6) as

$$H=R_1 \cos \theta_1+R_2 \cos \theta_2+R_3 \cos \theta_2$$
$$=40\times\cos 60°+15\times\cos 120°+20\times\cos 240°$$
$$=40\times\tfrac{1}{2}+15\times(-\tfrac{1}{2})+20\times(-\tfrac{1}{2})$$
$$=20-7.5-10=2.5 \text{ kN.}$$

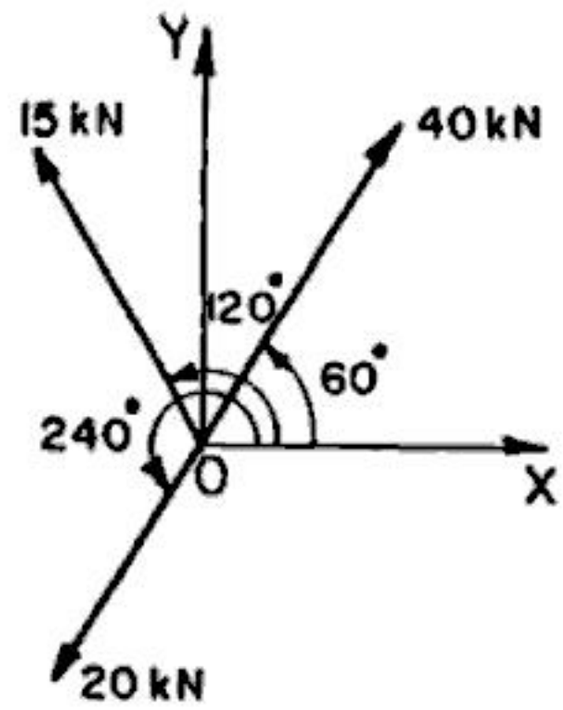

Fig. 1.17.

The resultant component along Y-axis is given by equation (1.7) as

$$V=R_1 \sin \theta_1+R_2 \sin \theta_2+R_3 \sin \theta_3$$
$$=40\times\sin (60°)+15\times\sin (120°)+20\times\sin (240°)$$
$$=40+\frac{\sqrt{3}}{3}+15\times\frac{\sqrt{3}}{2}+20\times\left(\frac{-\sqrt{3}}{2}\right)$$
$$=20\times\sqrt{3}+7.5\times\sqrt{3}-10\times\sqrt{3}=17.5\times\sqrt{3} \text{ kN}$$
$$=30.31 \text{ kN.}$$

The magnitude of the resultant force is given by equation (1.8)

$$R=\sqrt{H^2+V^2}=\sqrt{2.5^2+30.31^2}=\mathbf{30.41 \text{ kN.}} \quad \textbf{Ans.}$$

The direction of the resultant force is given by equation (1.9)

$$\tan \theta=\frac{V}{H}=\frac{30.31}{2.5}=12.124=\tan 85.28°$$

$\therefore$ $\theta=\mathbf{85.28°}$ **or** $\mathbf{85° 16.8'}$. **Ans.**

Problem 1.12. *Four forces of magnitude 10 kN, 15 kN, 20 kN and 40 kN are acting at a point O as shown in Fig. 1.18. The angles made by 10 kN, 15 kN, 20 kN and 40 kN with X-axis are 30°, 60°, 90° and 120° respectively. Find the magnitude and direction of the resultant force.*

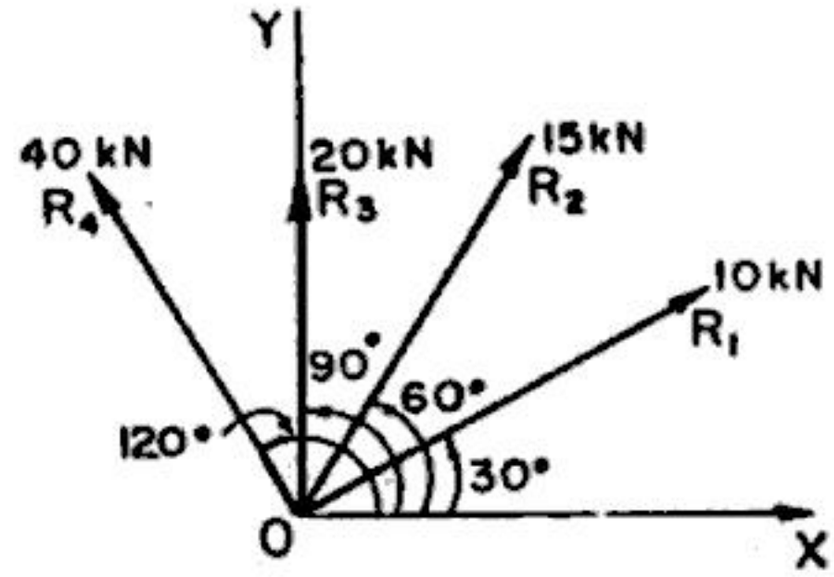

Fig. 1.18.

Sol. Given :

$R_1=10$ kN and $\theta_1=30°$

$R_2=15$ kN and $\theta_2=60°$

$R_3=20$ kN and $\theta_3=90°$

$R_4=40$ kN and $\theta_4=120°$

The resultant components along X-axis is given by (1.6) as

$$H=R_1 \cos \theta_1+R_2 \cos \theta_2+R_3 \cos \theta_3+R_4 \cos \theta_2$$
$$=10 \cos 30°+15 \cos 60°+20 \cos 90°+40 \cos 120°$$
$$=10\times\frac{\sqrt{3}}{2}+15\times\tfrac{1}{2}+20\times 0+40\times(-\tfrac{1}{2})$$
$$(\because \cos 90°=0 \text{ and } \cos 120°=-\tfrac{1}{2})$$
$$=5\times\sqrt{3}+7.5-20$$
$$=8.66+7.5-20=-3.84 \text{ kN.}$$

Negative sign means that H is acting along OX' as shown in Fig. 1.19.

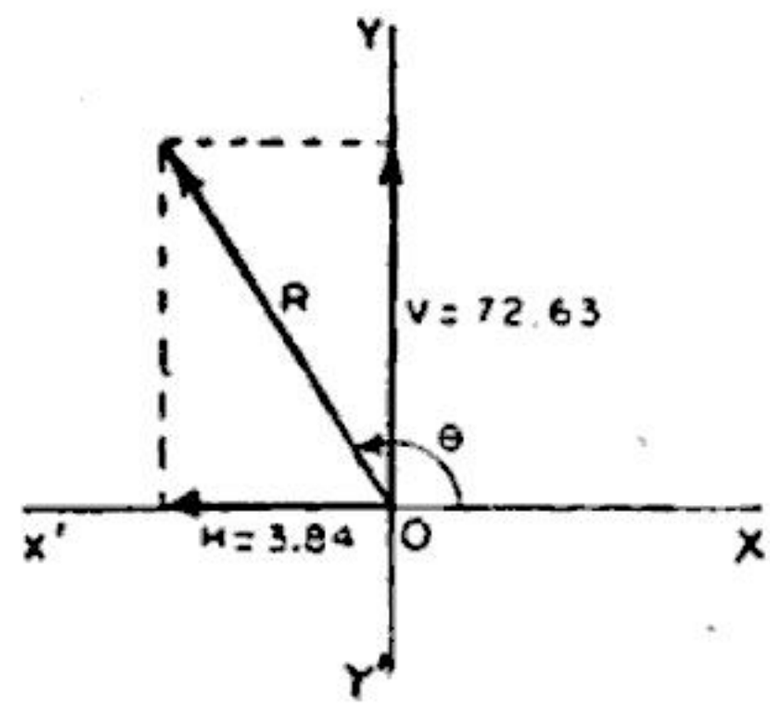

Fig. 1.19.

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The resultant components along Y-axis is given by equation (1.7) as

$$V=R_1 \sin \theta_1+R_2 \sin \theta_2+R_3 \sin \theta_3+R_4 \sin \theta_4$$
$$=10 \sin 30°+15 \sin 60°+20 \sin 90°+40 \sin 120°$$
$$=10\times\frac{1}{2}+15\times\frac{\sqrt{3}}{2}+20\times 1+40\times\frac{\sqrt{3}}{2}$$
$$=5+7.5\times\sqrt{3}+20+20\times\sqrt{3}$$
$$=25+27.5\times\sqrt{3}=72.63 \text{ kN.}$$

Positive sign means that V is acting along OY as shown in Fig. 1.19.

The magnitude of the resultant force is given by equation (1.8) as

$$R=\sqrt{H^2+V^2}=\sqrt{(-3.84)^2+72.63^2}=\sqrt{14.745+5275.117}=\mathbf{72.73 \text{ kN.}} \quad \textbf{Ans.}$$

The direction of the resultant force is given by equation (1.9) as

$$\tan \theta=\frac{V}{H}=\frac{72.63}{-3.84}=-18.91.$$

From Fig. 1.19 it is clear that θ lies between 90° and 180°.

The angle whose tangent is 18.91° is 86.97.

$\therefore$ $\theta=(180°-86.97°)=\mathbf{93.03°}.$ **Ans.**

1.8. MOMENT OF A FORCE

The product of a force and the perpendicular distance of the line of action of the force from a point is known as moment of the force about that point.

P=A force acting on a body as shown in Fig. 1.20.

r=Perpendicular distance between the point O and line of action of the force P.

The moment of the force P about O

$=P\times r$

The tendency of the moment $P\times r$ is to rotate the body in the clockwise direction about O.

Hence this moment is called clockwise moment. If the tendency of rotation is anti-clockwise, the moment is called anti-clockwise moments.

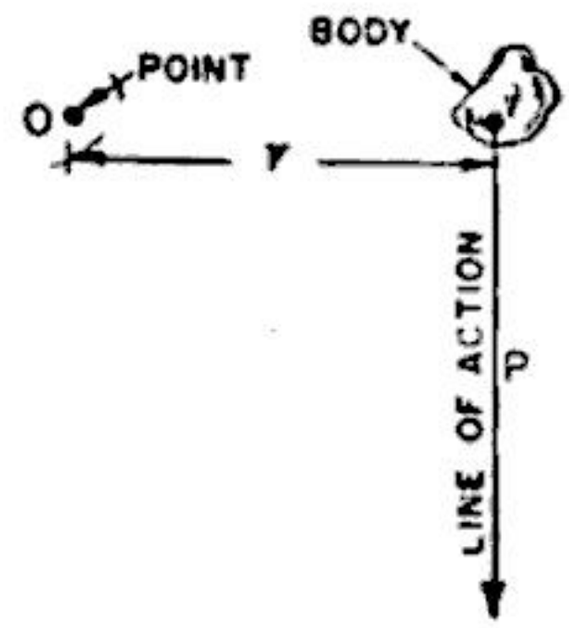

Fig. 1.20.

1.8.1. Units of Moment. In M.K.S. system the moment is expressed as kgf m whereas in S.I. system, moment is expressed as newton metre (N m).

1.8.2. Effect of Force and Moment on a Body. The force acting on a body causes linear displacement while moment causes an angular displacement. Hence a body when acted by a number of coplanar forces will be in equilibrium if :

(*i*) Resultant component of forces along any direction is zero *i.e.* resultant component of forces in the direction of x, in the direction of y and in the direction of z are zero.

(*ii*) Resultant moments of the forces about any point in the plane of the forces is zero or clockwise moment is equal to anti-clockwise moments.

Note. If the resultant component of forces acting on a body along any direction is zero but the net moment of the forces about any point is not zero the body will not be in equilibrium. The body will have the tendency to rotate about the point.

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Problem 1.13. *A beam of span 10 m is carrying a point load of 100 N at a distance 4 m from A. Determine the beam reactions.*

Sol. Given :

Span $AB=10$ m
Load at C, $W=200$ N
Distance, $AC=4$ m
Distance, $BC=10-4=6$ m

Let R_A=Reaction at A and
R_B=Reaction at B

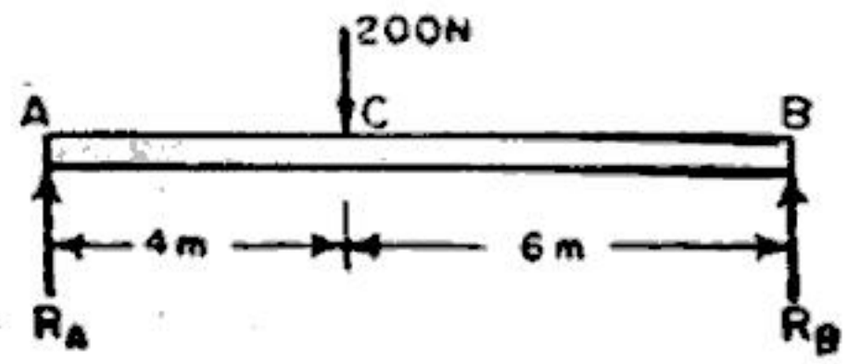

Fig. 1.21.

As the beam is in equilibrium, the clockwise moments of all forces about any point must be equal to anti-clockwise moments about that point. Also the resultant force in any direction must be zero.

Taking moments about A,

Clockwise moment=Anti-clockwise moments

$$200\times4=R_B\times10$$

$$\therefore \quad R_B=\frac{200\times4}{10}=\mathbf{80\ N.}\quad \textbf{Ans.}$$

Also $\quad R_A+R_B=200$ N

$$\therefore \quad R_A=200-R_B=200-80=\mathbf{120\ N.}\ \textbf{Ans.}$$

Problem 1.14. *Four forces of magnitudes 10 N, 20 N, 30 N and 40 N are acting respectively along the four sides of a square ABCD as shown in Fig. 1.22. Determine the magnitude, direction and position of the resultant force.*

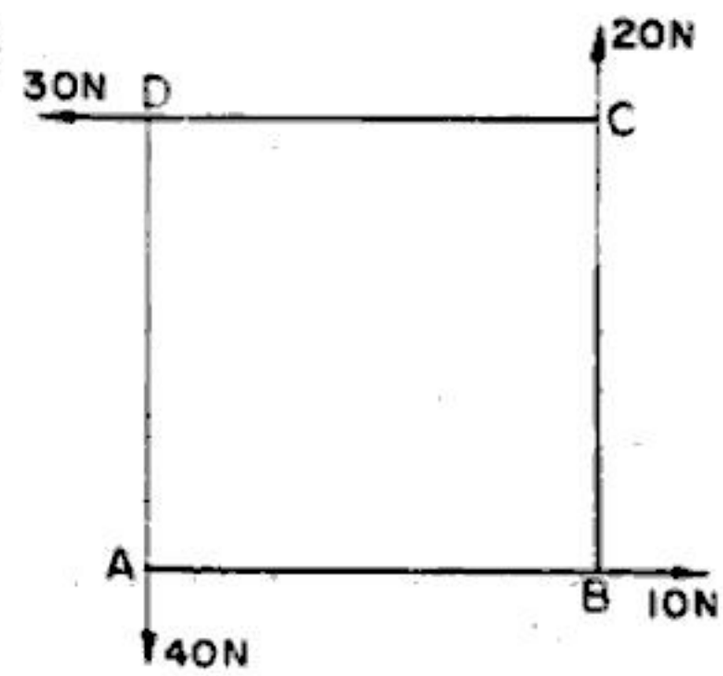

Fig. 1.22.

Sol. Given :

Force along $AB=10$ N
Force along $BC=20$ N
Force along $CD=30$ N
Force along $DA=40$ N

The net force in the horizontal direction is given as,

$$H=10-30=-20\ \text{N}$$

The net force in the vertical direction given as,

$$V=20-40=-20\ \text{N}$$

The resultant force is given by equation (1.8) as

$$R=\sqrt{H^2+V^2}=\sqrt{(-20)^2+(-20)^2}=\sqrt{400+400}=\sqrt{2\times400}$$

$$=20\times\sqrt{2}\ \text{N.}\quad \textbf{Ans.}$$

The direction of the resultant force is given by equation (1.9) as

$$\tan\theta=\frac{V}{H}=\frac{-20}{-20}=1$$

$$\therefore \theta=45^\circ.$$

Since H and V are $-$ve, hence θ lies between 180° and 270°. Hence from Fig. 1.23, it is clear that actual

$$\theta=180+45=225^\circ.\ \textbf{Ans.}$$

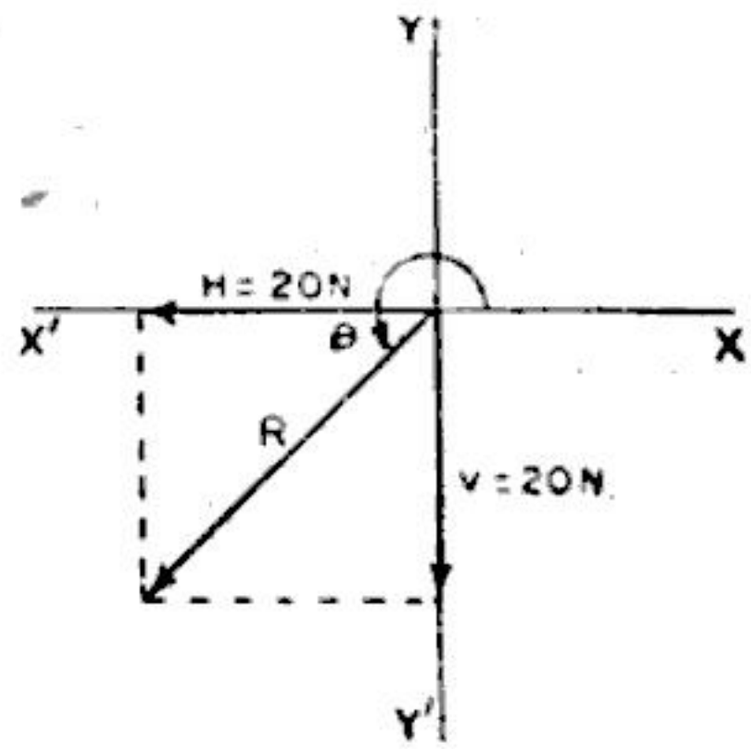

Fig. 1.23.

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Position of the resultant force

The position of the resultant force is obtained by equating the clockwise moments and anti-clockwise moments about A.

Let x=Perpendicular distance between A and line of action of the resultant force and
a=Side of the square $ABCD$.

Taking moments of all forces about A,

$$40\times0+10\times0+20\times a+30\times a=R\times\text{perpendicular distance of } R \text{ from } A$$

$$=20\times\sqrt{2}\times x \qquad (\because R=20\times\sqrt{2})$$

or $$20a+30a=20\times\sqrt{2}\times x$$

or $$x=\frac{50a}{20\times\sqrt{2}}=\frac{5a}{2\times\sqrt{2}}. \quad \textbf{Ans.}$$

1.9. LAW OF MECHANICS

The following basic laws and principles are considered to be the foundation of mechanics :

(*i*) Newton's first and second laws of motion

(*ii*) Newton's third law

(*iii*) The gravitational law of attraction

(*iv*) The parallelogram law

(*v*) The Principle of Transmissibility of forces.

1.9.1. Newton's First and Second Laws of Motion. Newton's first law states, "Every body continues in a state of rest or uniform motion in a straight line unless it is compelled to change that state by some external force acting on it."

Newton's second law states, "The net external force acting on a body in a direction is directly proportional to the rate of change of momentum in that direction."

1.9.2. Newton's Third Law. Newton's third law states, "To every action there is always equal and opposite reaction."

Fig. 1.24 shows two bodies A and B which are placed one above the other on a horizontal surface.

Here F_1=Force exerted by horizontal surface on body A (action)

$-F_1$=Force exerted by body A on horizontal surface (reaction)

F_2=Force exerted by body A on body B (action)

$-F_2$=Force exerted by body B on body A (reaction)

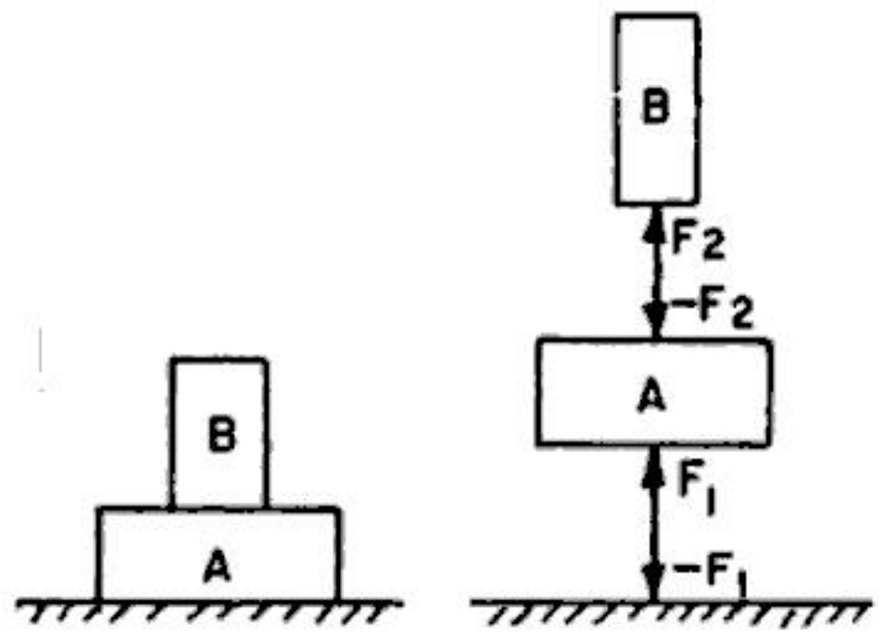

Fig. 1.24.

1.9.3. The Gravitational Law of Attraction. It states that two bodies will be attracted towards each other along their connecting line with a force which is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

Refer to Fig. 1.24(*a*).

Let m_1=Mass of first body

m_2=Mass of second body

r=Distance between the bodies

F=Force of attraction between the bodies.

Then according to the law of gravitational attraction.

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$$F \propto m_1.m_2$$

$$\propto \frac{1}{r^2}$$

or $$F \propto \frac{m_1 m_2}{r^2}$$

or $$F = G\frac{m_1 m_2}{r^2} \quad ...(1.10)$$

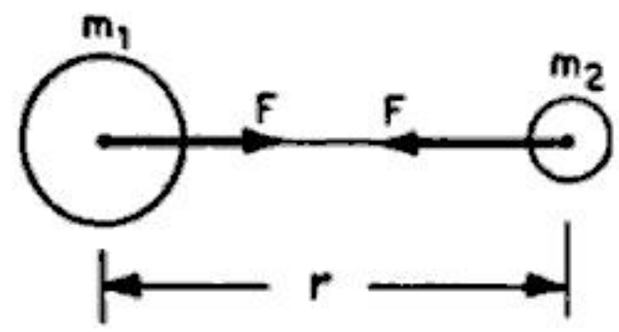

Fig. 1.24 (*a*)

where G=Universal gravitational constant of proportionality.

1.9.4. The Parallelogram Law. This law has been already defined. It states that if two forces acting at a point be represented in magnitude and direction by the two adjacent sides of a parallelogram, then their resultant is represented in magnitude and direction by the diagonal of the parallelogram passing through that point.

1.9.5. The Principle of Transmissibility of Forces. It states that if a force, acting at a point on a rigid* body, is shifted to any other point which is on the line of action of the force, the external effect on the force on the body remains unchanged.

For example, consider a force F acting at point O on a rigid body as shown in Fig. 1.25(*a*). On this rigid body, "there is another point O' in the line of action of the force F. Suppose at this point O', two equal and opposite forces F_1 and F_2 (each equal to F and collinear with F) are applied as shown in Fig. 1.25(*b*). The force F and F_2, being equal and opposite, will cancell each other, leaving a force F_1 at point O' as shown in Fig. 1.25(*c*). But force F_1 is equal to force F.

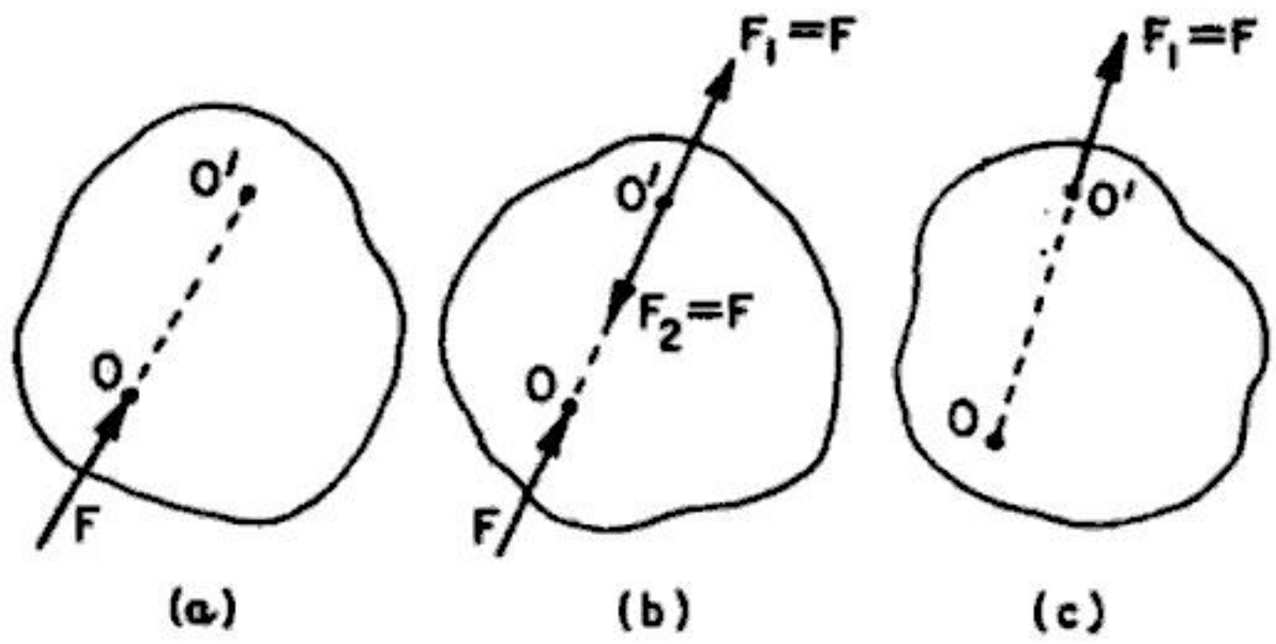

Fig. 1.25.

The original force F acting at point O, has been transferred to point O' which is along the line of action of F without changing the effect of the force on the rigid body. Hence any force acting at a point on a rigid body can be transmitted to act at any other point along its line of action without changing its effect on the rigid body. This proves the principle of transmissibility of a force.

HIGHLIGHTS

1. Engineering mechanics is divided into statics and dynamics. The study of a body at rest is known as statics whereas the study of a body in motion is known as dynamics.
2. A quantity which is completely specified by magnitude and direction is known as vector quantity.
3. A particle is a body of infinitely small volume and is considered to be concentrated at a point.
4. Law of parallelogram of forces states that "If two forces, acting at a point be represented in magnitude and direction by the two adjacent sides of a parallelogram, then their resultant is represented in magnitude and direction by the diagonal of the parallelogram passing through that point.
5. If two forces P and Q act at a point and the angle between the two forces be α, then the resultant is given by

*A body which does not deforms under the action of loads or external forces is known as rigid body.

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$$R=\sqrt{P^2+Q^2+2PQ\cos\alpha}$$

and the angle made by the resultant with the direction of force P is expressed as

$$\tan\theta=\frac{Q\sin\alpha}{P+Q\cos\alpha}$$

6. If the two forces P and Q are equal and are acting at an angle α between them, then the resultant is given by

$$R=2P\cos\frac{\alpha}{2}$$

and angle made by the resultant is expressed as

$$\theta=\frac{\alpha}{2}.$$

7. According to Lami's theorem, "If three forces acting at a point are equilibrium, each force will be proportional to the sine of the angle between the other two forces."
8. The relation between newton and dyne is given by

   One newton $=10^5$ dyne.
9. Moment of a force about a point = Force × perpendicular distance of the line of action of the force from that point.
10. The force causes linear displacement while moment causes angular displacement. A body will be in equilibrium if (*i*) resultant force in any direction is zero and (*ii*) the net moment of the forces about any point is zero.
11. Gravitational law of attraction is given by,

$$F=G\frac{m_1\times m_2}{r^2}$$

where G=Universal gravitational constant

m_1, m_2=Mass of bodies

r=Distance between the bodies

F=Force of attraction between the bodies.

EXERCISE 1

(A) Theoretical Problems

1. What do you mean by scalar and vector quantities ?
2. Define the law of parallelogram of forces. What is the use of this law ?
3. State triangle law of forces and Lami's theorem.
4. Two forces P and Q are acting at a point in a plane. The angle between the forces is 'α'. Prove that the resultant (R) of the two forces is given by

$$R=\sqrt{P^2+Q^2+2PQ\cos\alpha}.$$

5. Define the following terms : dyne, newton, meganewton and moment of a force.
6. Prove that one newton is equal to 10^5 dyne.
7. Explain the terms : clockwise moments and anti-clockwise moments.
8. What is the effect of force and moment on a body ?
9. Indicate whether the following statement is true or false.
   'The resultant components of the forces acting on a body along any direction is zero but the net moment of the forces about any point is not zero, the body will be in equilibrium'. **[Ans. False]**
10. Write the S.I. units of : Force, moment and velocity.
11. What do you mean by resolution of a force ?
12. A number of coplanar forces are acting at a point making different angles with x-axis. Find an expression for the resultant force. Find also the angle made by the resultant force with x-axis.
13. State and explain the principle of transmissibility of forces.
14. State and explain the following laws :
    (*i*) Newton's laws of motion.
    (*ii*) The gravitational law of attraction.

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15. Using gravitation law of attraction, prove that $W = m \times g$.

16. Explain fully the following terms :

(*i*) Resolved part of a given force in a given direction.

(*ii*) Lami's theorem.

(B) Numerical Problems

1. Determine the magnitude of the resultant of the two forces of magnitude 12 N and 9 N acting at a point if the angle between the two forces is 30°. [Ans. 20.3 N]

2. Find the magnitude of two equal forces acting at a point with an angle of 60° between them, if the resultant is equal to $30 \times \sqrt{3}$ N. [Ans. 30 N]

3. The resultant of two forces when they act at right angles is 10 N, whereas when they act at an angle of 60° the resultants is $\sqrt{148}$. Determine the magnitude of the two forces. [Ans. 8 N and 6 N]

4. Three forces of magnitude 30 kN, 10 kN and 15 kN are acting at a point *O*. The angles made by 30 kN force, 10 kN force and 15 kN force with *x*-axis are 60°, 120° and 240° respectively. Determine the magnitude and direction of the resultant force. [Ans. 21.79 kN, 83°24]

5. A weight of 800 N is supported by two chains as shown in Fig. 1.26. Determine the tension in each chain. [Ans. 273.5 N, 751.7 N]

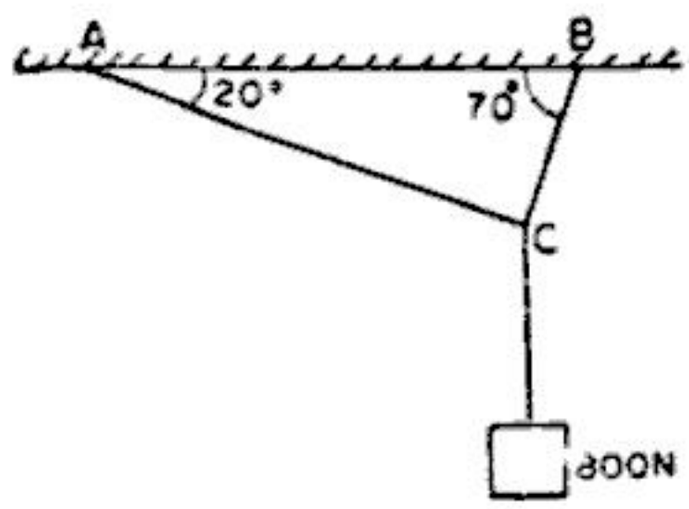

Fig. 1.26.

6. An electric light fixture weighing 20 N hangs from a point *C*, by two strings *AC* and *BC*. *AC* is inclined at 60° to the horizontal and *BC* at 30° to the vertical as shown in Fig. 1.27. Using Lami's theorem or otherwise determine the forces in the strings *AC* and *BC*. [Ans. 8.929 N, 13.05 N]

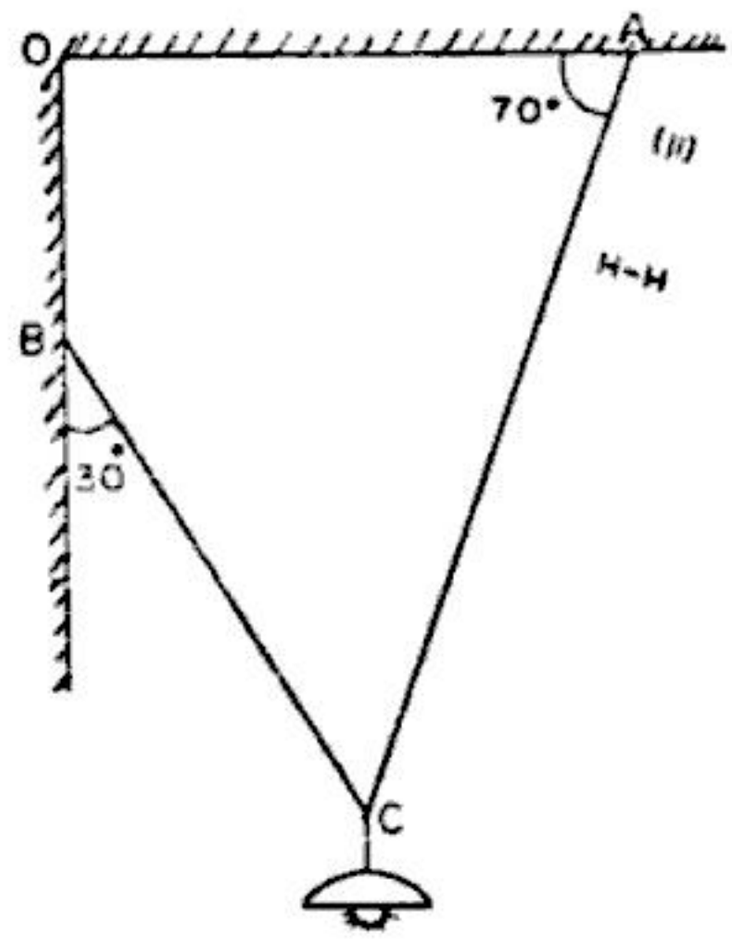

Fig. 1.27.

7. A beam *AB* of span 6 m carries a point load of 100 N at a distance 2 m from *A*. Determine the beam reaction. [Ans. $R_A = 66.67$ N ; $R_B = 33.33$ N]

8. Four forces of magnitudes 20 N, 30 N, 40 N and 50 N are acting respectively along the four sides of a square taken in order. Determine the magnitude, direction and position of the resultant force. $\left[\text{Ans. } 20 \times \sqrt{2} \text{ N, } 225°, \dfrac{7a}{2 \times \sqrt{2}}\right]$

9. Two forces magnitude 15 N and 12 N are acting at a point. If the angle between the two forces is 60°, determine the resultant of the forces in magnitude and direction. [Ans. 23.43 N, 26.3°]

10. Four forces of magnitude *P*, 2*P*, $3 \times \sqrt{3}P$ and 4*P* are acting at a point *O*. The angles made by these forces with *x*-axis are 0°, 60°, 150° and 300° respectively. Find the magnitude and direction of the resultant force. [Ans. *P*, 1200]

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2

Coplanar Collinear and Concurrent Forces

2.1. INTRODUCTION

Coplanar forces means the forces in a plane. The word collinear stands for the forces which are having common lines of action whereas the word concurrent stands for the forces which intersect at a common point. When several forces act on a body, then they are called *a force system* or *a system of forces*. In a system in which all the forces lie in the same plane, it is known as *coplanar force system*. Hence this chapter deals with a system of forces which are acting in the same plane and the forces are either having a common line of action or intersecting at a common point.

2.2. CLASSIFICATION OF A FORCE SYSTEM

A force system may be coplanar or non-coplanar. If in a system all the forces lie in the same plane then the force system is known as coplanar. But if in a system all the forces lie in different planes, then the force system is known as non-coplanar. Hence a force system is classified as shown in Fig. 2.1.

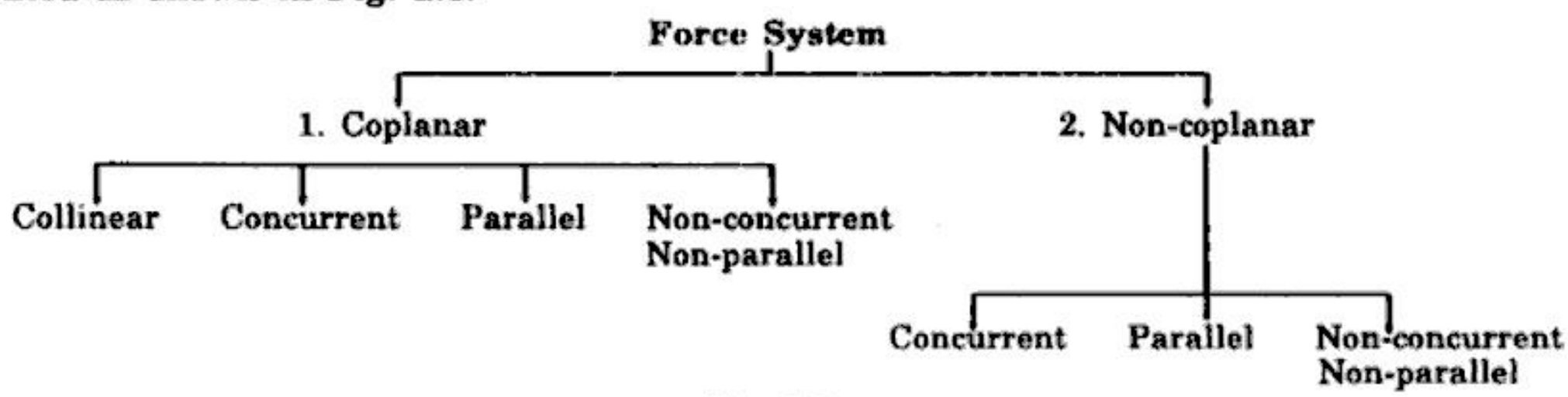

Fig. 2.1

In this chapter, we shall discuss only coplanar force system, in which the forces may be :

(*i*) Collinear (*ii*) Concurrent (*iii*) Parallel

(*iv*) Non-concurrent, non-parallel (or General system of forces).

2.2.1. Coplanar Collinear. Fig. 2.2 shows three forces F_1, F_2 and F_3 acting in a plane. These three forces are in the same line *i.e.*, these three forces are having a common line of action. This system of forces is known as coplanar collinear force system. Hence in coplanar collinear system of forces, all the forces act in the same plane and have a common line of action.

2.2.2. Coplanar Concurrent. Fig. 2.3 shows three forces F_1, F_2 and F_3 acting in a plane and these forces intersect or meet at a common point O. This system of forces is known as coplanar concurrent force system. Hence in coplanar concurrent system of forces, all the forces act in the same plane and they intersect at a common point.

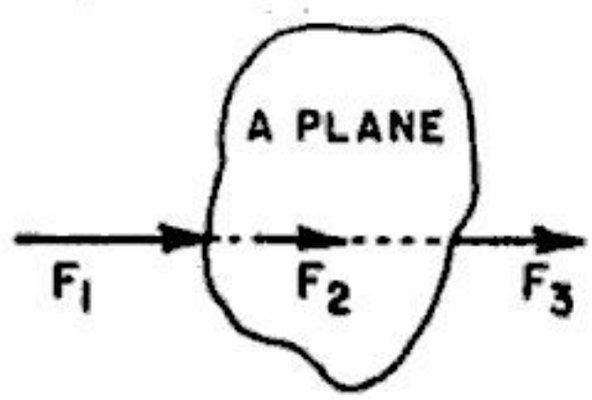

Fig. 2.2. Coplanar Collinear Forces.

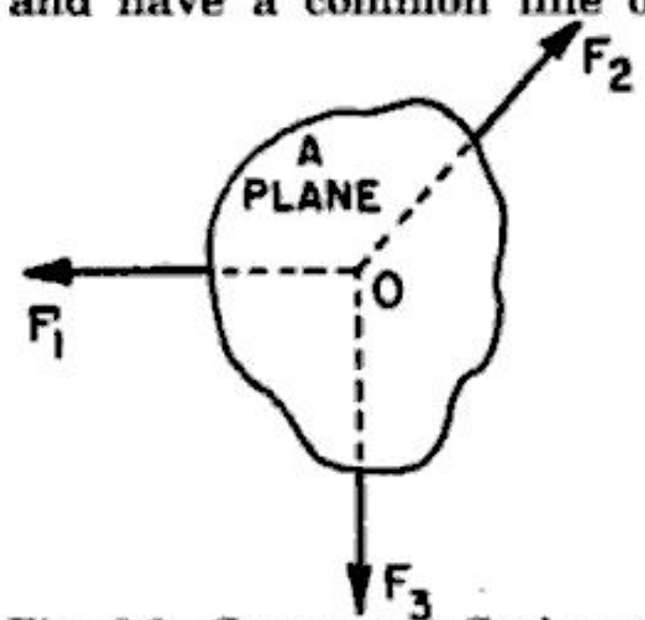

Fig. 2.3. Concurrent Coplanar Forces.

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2.2.3. Coplanar Parallel. Fig. 2.4 shows three forces F_1, F_2 and F_3 acting in a plane and these forces are parallel. This system of forces is known as coplanar parallel force system. Hence in coplanar parallel system of forces, all the forces act in the same plane and are parallel.

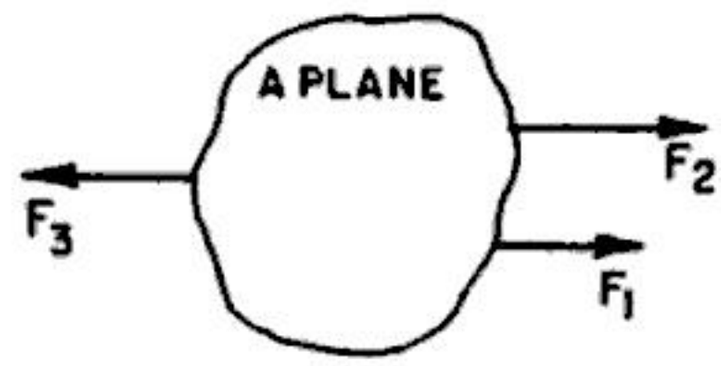

Fig. 2.4. Coplanar Parallel Forces.

2.2.4 Coplanar Non-concurrent Non-parallel. Fig. 2.5 shows four forces F_1, F_2, F_3 and F_4 acting in a plane. The lines of action of these forces lie in the same plane but they are neither parallel nor meet or intersect at a common point. This system of forces is known as coplanar non-concurrent non-parallel force system. Hence in coplanar non-concurrent non-parallel system of forces, all the forces act in the same plane but the forces are neither parallel nor meet at a common point. This force system is also known as *general system of forces.*

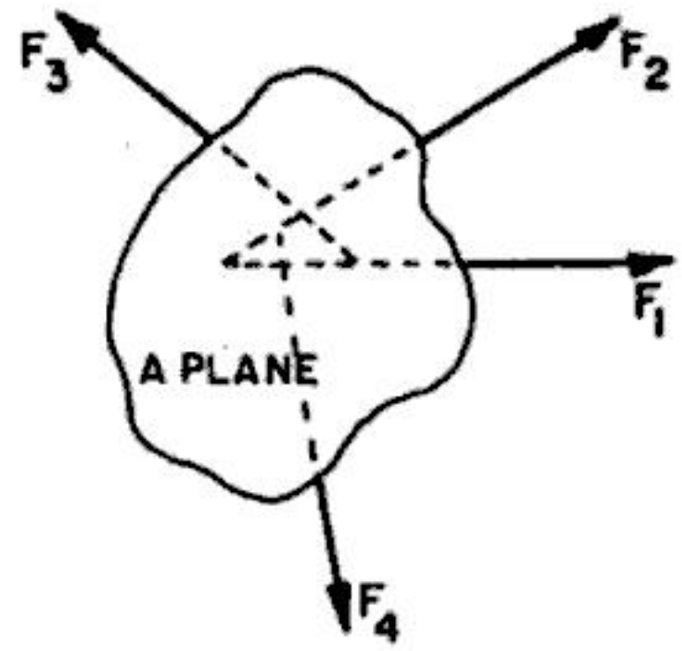

Fig. 2.5. Non-concurrent Non-parallel.

2.3. RESULTANT OF SEVERAL FORCES

When a number of coplanar forces are acting on a rigid* body, then these forces can be replaced by a single force which has the same effect on the rigid body as that of all the forces acting together, then this single force is known as the *resultant* of several forces. Hence a single force which can replace a number of forces acting on a rigid body, without causing any change in the external effects on the body, is known as the *resultant force.*

2.4. RESULTANT OF COPLANAR FORCES

The resultant of coplanar forces may be determined by following two methods :

1. Graphical method
2. Analytical method.

The resultant of the following coplanar forces will be determined by the above two methods :

(*i*) Resultant of collinear coplanar forces

(*ii*) Resultant of concurrent coplanar forces.

2.5. RESULTANT OF COLLINEAR COPLANAR FORCES

As defined in Art. 2.2.1, collinear coplanar forces are those forces which act in the same plane and have a common line of action. The resultant of these forces are obtained by analytical method or graphical method.

2.5.1. Analytical method. The resultant is obtained by adding all the forces if they are acting in the same direction. If any one of the forces is acting in the opposite direction, then resultant is obtained by subtracting that force.

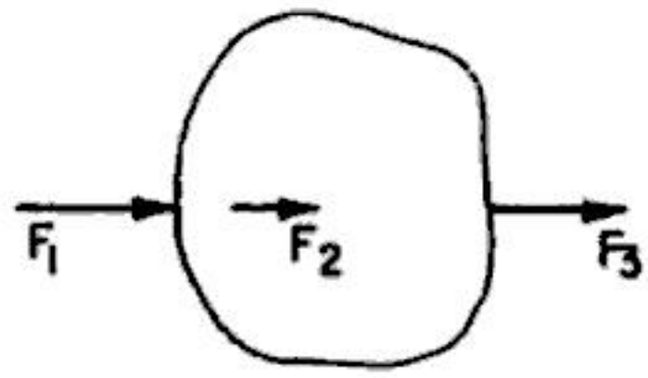

Fig. 2.6.

Fig. 2.6 shows three collinear coplanar forces F_1, F_2 and F_3 acting on a rigid body in the same direction. Their resultant R will be sum of these forces.

*Rigid body is a body which does not deform under the action of loads or external forces. In case of rigid body, the distance between any two points of the body remains constant, when this body is subjected to loads. Though all the bodies do deform to same extent under the action of loads, but in many situation, this deformation is negligible small.

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$\therefore \qquad R = F_1 + F_2 + F_3$ ...(2.1)

If any one of these forces (say force F_2) is acting in the opposite direction, as shown in Fig. 2.7, then their resultant will be given by

$$R = F_1 - F_2 + F_3 \quad ...(2.2)$$

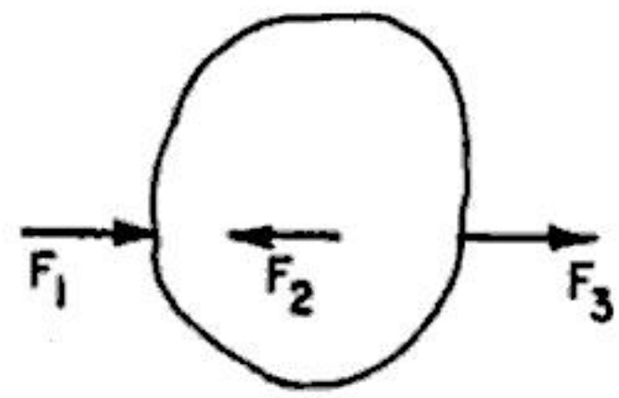

Fig. 2.7.

2.5.2. Graphical Method. Some suitable scale is chosen and vectors are drawn to the chosen scale. These vectors are added/or subtracted to find the resultant. The resultant of the three collinear forces F_1, F_2 and F_3 acting in the same direction will be obtained by adding all the vectors. In Fig. 2.8, the force $F_1 = ab$ to some scale, force $F_2 = bc$ and force $F_3 = cd$. Then the length ad represents the magnitude of the resultant on the scale chosen.

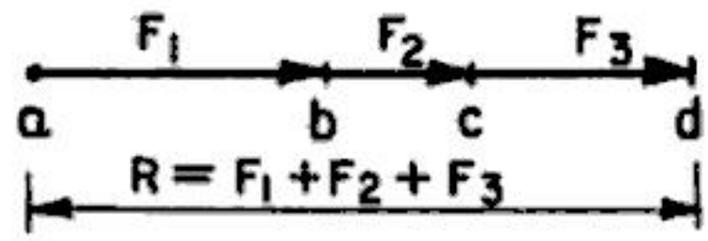

Fig. 2.8.

The resultant of the forces F_1, F_2 and F_3 acting on a body shown in Fig. 2.7 will be obtained by subtracting the vector F_2. This resultant is shown in Fig. 2.9, in which the force $F_1 = ab$ to some suitable scale. This force is acting from a to b. The force F_2 is taken equal to bc on the same scale in opposite direction. This force is acting from b to c. The force F_3 is taken equal to cd. This force is acting from c to d. The resultant force is represented in magnitude by ad on the chosen scale.

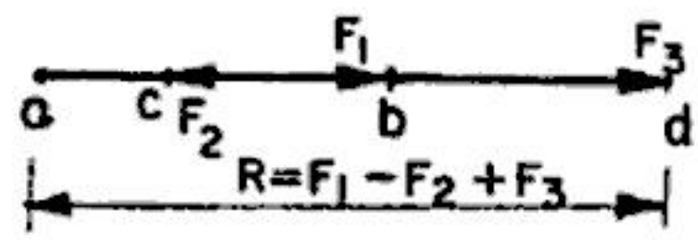

Fig. 2.9.

Problem 2.1. *Three collinear horizontal forces of magnitude 200 N, 100 N and 300 N are acting on a rigid body. Determine the resultant of the forces analytically and graphically when*

(i) all the forces are acting in the same direction,

(ii) the force 100 N acts in the opposite direction.

Sol. Given :

$$F_1 = 200 \text{ N}, \; F_2 = 100 \text{ N and } F_3 = 300 \text{ N}$$

(a) Analytical Method

(*i*) When all the forces are acting in the same direction, then resultant is given by equation (2.1) as

$$R = F_1 + F_2 + F_3 = 200 + 100 + 300 = \mathbf{600 \text{ N.}} \quad \textbf{Ans.}$$

(*ii*) When the force 100 N acts in the opposite direction, then resultant is given by equation (2.2) as

$$R = F_1 + F_2 + F_3 = 200 - 100 + 300 = \mathbf{400 \text{ N.}} \quad \textbf{Ans.}$$

(b) Graphical Method

Select a suitable scale. Suppose 100 N=1 cm. Then to this scale, we have

$$F_1 = \frac{200}{100} = 2 \text{ cm},$$

$$F_2 = \frac{100}{100} = 1 \text{ cm},$$

and $$F_3 = \frac{300}{100} = 3 \text{ cm}.$$

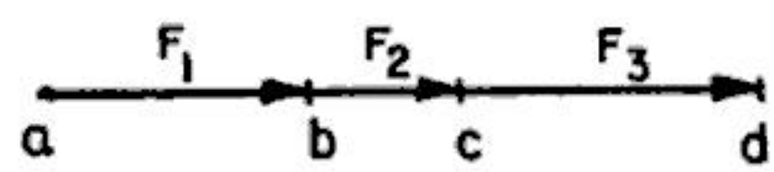

Fig. 2.10.

(i) When all the forces act in the same direction.

Draw vectors $ab = 2$ cm to represent F_1,

vector $bc = 1$ cm to represent F_2 and vector

$cd = 3$ cm to represent F_3 as shown in Fig. 2.10.

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Measure vector ad which represents the resultant.

By measurement length $ad=6$ cm

$\therefore$ Resultant=Length $ad\times$chosen scale ($\because$ Chosen scale is 1 cm=100 N)

$=6\times100=600$ N. **Ans.**

(*ii*) When force 100 N=F_2, acts in the opposite direction

Draw length $ab=2$ cm to represent force F_1.

From b, draw $bc=1$ cm in the opposite direction to represent F_2. From c draw $cd=3$ cm to represent F_3 as shown in Fig. 2.10(*a*).

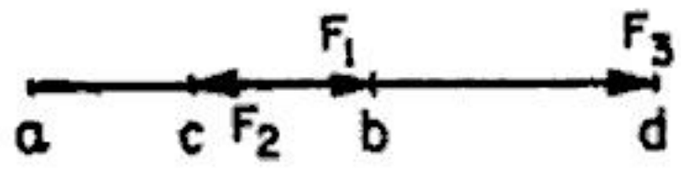

Fig. 2.10. (*a*)

Measure length ad. This gives the resultant.

By measurement, length $ad=4$ cm

$\therefore$ Resultant=Length $ad\times$chosen scale

$=4\times100=400$ N. **Ans.**

2.6. RESULTANT OF CONCURRENT COPLANAR FORCES

As defined in Art. 2.2.2, concurrent coplanar forces are those forces which act in the same plane and they intersect or meet at a common point. We will consider the following two cases :

(*i*) When two forces act at a point

(*ii*) When more than two forces act at a point.

2.6.1. When two forces act at a point

(a) Analytical Method

In Art. 1.2.4, we have mentioned that when two forces act at a point, their resultant is found by the law of parallelogram of forces. The magnitude of resultant is obtained from equation (1.1) and the direction of resultant with one of the forces is obtained from equation (1.2).

Suppose two forces P and Q act at point O as shown in Fig. 2.11 and α is the angle between them. Let θ is the angle made by the resultant R with the direction of force P.

Forces P and Q form two sides of a parallelogram and according to the law, the diagonal through the point O gives the resultant R as shown.

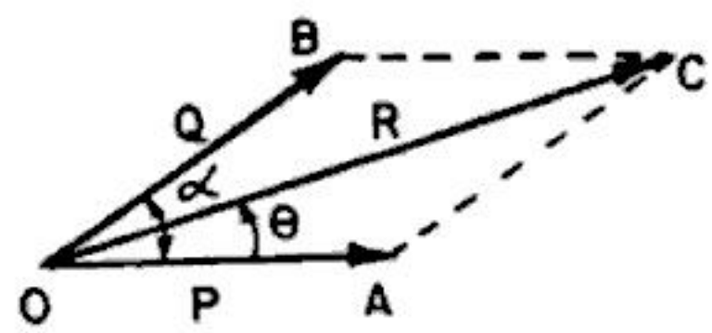

Fig. 2.11.

The magnitude* of resultant is given by

$$R=\sqrt{P^2+Q^2+2PQ\cos\alpha}$$

The above method of determining the resultant is also known as the *cosine law method.*

The direction* of the resultant with the force P is given by

$$\theta=\tan^{-1}\left(\frac{Q\sin\alpha}{P+Q\cos\alpha}\right)$$

(b) Graphical Method

(*i*) Choose a convenient scale to represent the forces P and Q.

(*ii*) From point O, draw a vector $Oa=P$.

(*iii*) Now from point O, draw another vector $Ob=Q$ and at an angle of α as shown in Fig. 2.12.

(*iv*) Complete the parallelogram by drawing lines $ac \parallel$ to Ob and $bc \parallel$ to Oa.

*Refer Art. 1.2.4 for the derivation of magnitude and direction of resultant on page 1.

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(*v*) Measure the length *OC*.

Then resultant *R* will be equal to length *OC*×chosen scale.

(*vi*) Also measure the angle θ, which will give the direction of resultant.

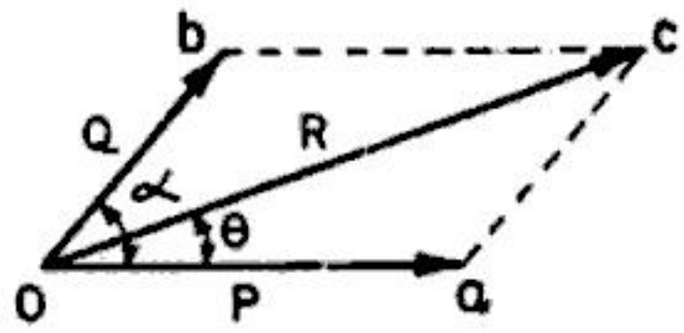

Fig. 2.12.

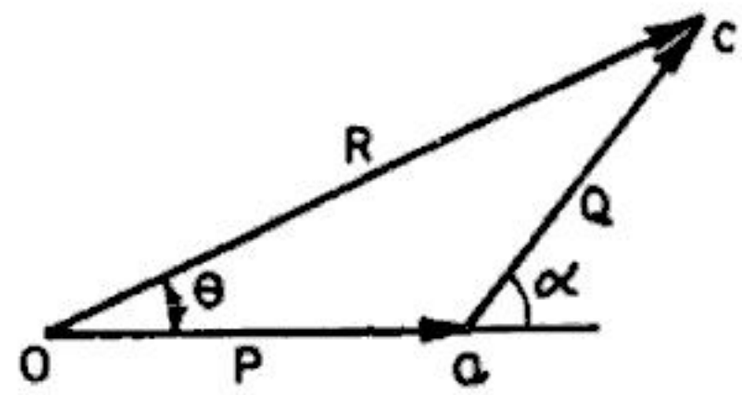

Fig. 2.13.

The resultant can also be determine graphically by drawing a triangle *oac* as explained below and shown in Fig. 2.13.

(*i*) Draw a line *oa* parallel to *P* and equal to *P*.

(*ii*) From *a*, draw a vector *ac* at an angle α with the horizontal and cut *ac* equal to *Q*.

(*iii*) Join *oc*. Then *oc* represents the magnitude and direction of resultant *R*.

Magnitude of resultant *R*=Length *OC*×chosen scale. The direction of resultant is given by angle θ. Hence measure the angle θ.

2.6.2. When more than two forces act at a point

(a) Analytical Method

The resultant of three or more forces acting at a point is found analytically by a method which is known as rectangular components method (Refer to Art. 1.7). According to this method all the forces acting at a point are resolved into horizontal and vertical components and then *algebraic summation** of horizontal and vertical components is done separately. The summation of horizontal component is written as ΣH and that of vertical as ΣV. Then resultant *R* is given by

$$R=\sqrt{(\Sigma H)^2+(\Sigma V)^2}.$$

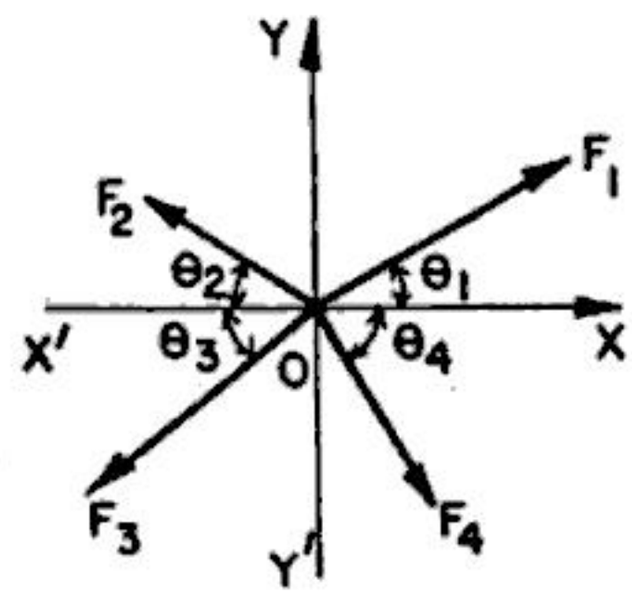

Fig. 2.14.

The angle made by the resultant with horizontal is given by

$$\tan\theta=\frac{(\Sigma V)}{(\Sigma H)}$$

∴ Let four forces F_1, F_2, F_3 and F_4 act at a point *O* as shown in Fig. 2.14.

The inclination of the forces is indicated with respect to horizontal direction. Let

θ_1=Inclination of force F_1 with *OX*, θ_2=Inclination of force F_2 with *OX'*

θ_3=Inclination of force F_3 with *OX'* θ_4=Inclination of force F_4 with *OX*.

The force F_1 is resolved into horizontal and vertical components and these components are shown in Fig. 2.14 (*a*). Similarly, Fig. 2.14 (*b*), (*c*) and (*d*) shows the horizontal and vertical components of forces F_2, F_3 and F_4 respectively. The various horizontal components are :

$F_1 \cos\theta_1 \rightarrow (+)$ $F_2 \cos\theta_2 \leftarrow (-)$

$F_3 \cos\theta_3 \leftarrow (-)$ $F_4 \cos\theta_4 \rightarrow (+)$

*Summation means addition. Algebraic summation of horizontal components means that if all the horizontal components are in the same direction then they are added. But if one horizontal component is in opposite direction then it is subtracted.

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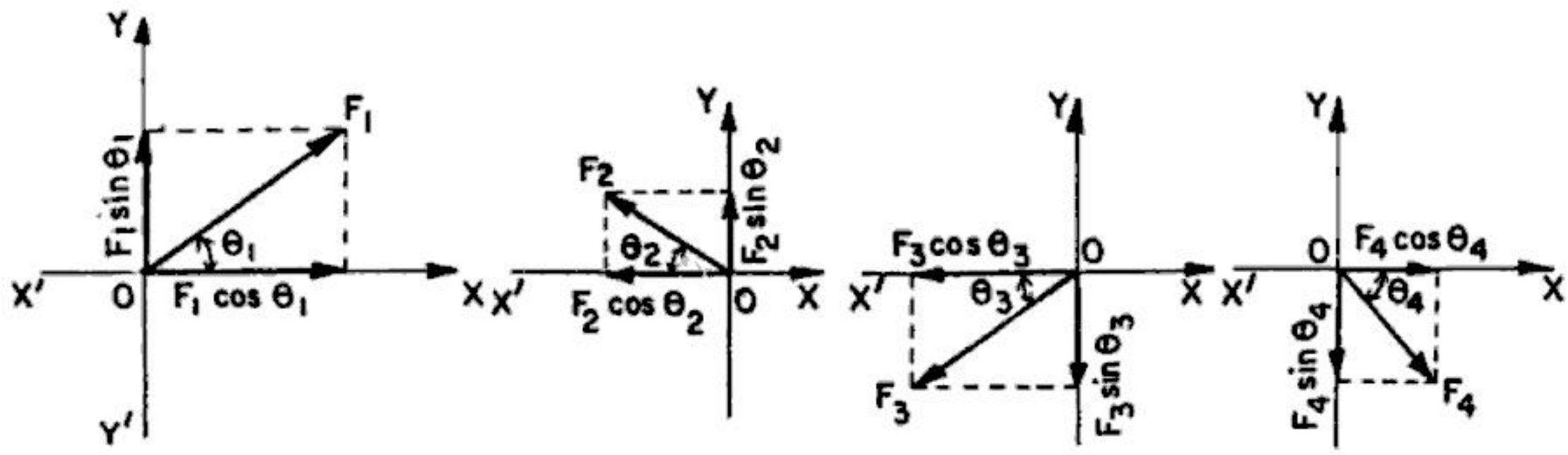

Fig. 2.14. (*a*) Fig. 2.14. (*b*) Fig. 2.14. (*c*) Fig. 2.14 (*d*)

∴ Summation or algebraic sum of horizontal components :

$$\Sigma H = F_1 \cos\theta_1 - F_2 \cos\theta_2 - F_3 \cos\theta_3 + F_4 \cos\theta_4$$

Similarly, various vertical components of all forces are :

$F_1 \sin\theta_1 \uparrow (+)$

$F_2 \sin\theta_2 \uparrow (+)$

$F_3 \sin\theta_3 \downarrow (-)$

$F_4 \sin\theta_4 \downarrow (-)$

∴ Summation or algebraic sum of vertical components :

$$\Sigma V = F_1 \sin\theta_1 + F_2 \sin\theta_2 - F_3 \sin\theta_3 - F_4 \sin\theta_4$$

Then the resultant will be given by

$$R = \sqrt{(\Sigma H)^2 + (\Sigma V)^2} \quad ...(2.1)$$

And the angle (θ) made by resultant with x-axis is given by

$$\tan\theta = \frac{(\Sigma V)}{(\Sigma H)} \quad ...(2.2)$$

(b) Graphical Method

The resultant of several forces acting at a point is found graphically with the help of the *polygon law of forces*, which may be stated as

"If a number of coplanar forces are acting at a point such that they can be represented in magnitude and direction by the sides of a polygon taken in the same order, then their resultant is represented in magnitude and direction by the closing side of the polygon taken in the opposite order.

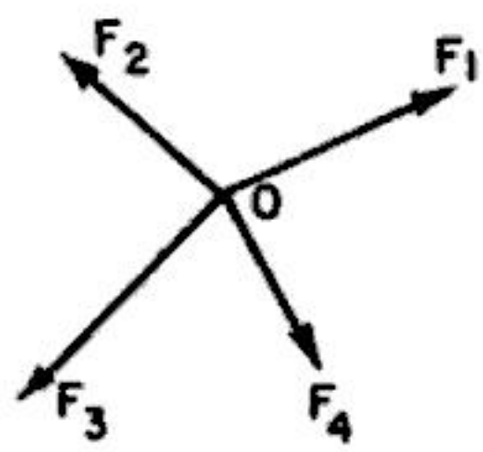

Fig. 2.15.

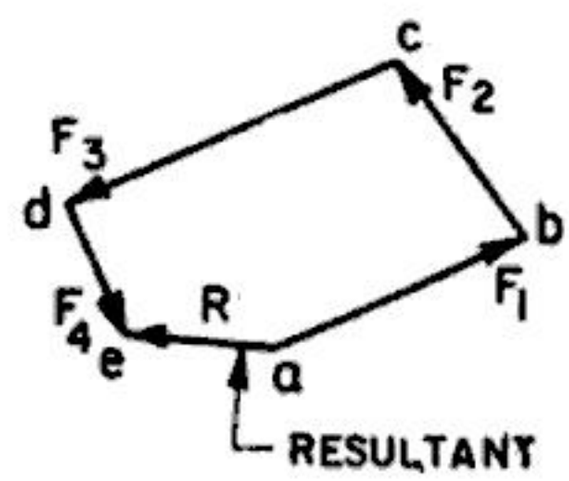

Fig. 2.15. (*a*)

Let the four forces F_1, F_2, F_3 and F_4 act at a point O as shown in Fig. 2.15. The resultant is obtained graphically by drawing polygon of forces as explained below and shown in Fig. 2.15 (*a*).

(*i*) Choose a suitable scale to represent the given forces.

(*ii*) Take any point a. From a, draw vector ab parallel to OF_1. Cut ab=force F_1 to the scale.

(*iii*) From point b, draw bc parallel to OF_2. Cut bc=force F_2.

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(*iv*) From point C, draw cd parallel to OF_3. Cut cd=force F_3.

(*v*) From point d, draw de parallel to OF_4. Cut de=force F_4.

(*vi*) Join point a to e. This is the closing side of the polygon. Hence ae represents the resultant in magnitude and direction. Magnitude of resultant R=Length $ae\times$scale.

The resultant is acting from a to e.

Problem 2.2. *Two forces of magnitude 240 N and 200 N are acting at a point O as shown in Fig. 2.16. If the angle between the forces is 60°, determine the magnitude of the resultant force. Also determine the angle β and γ as shown in the figure.*

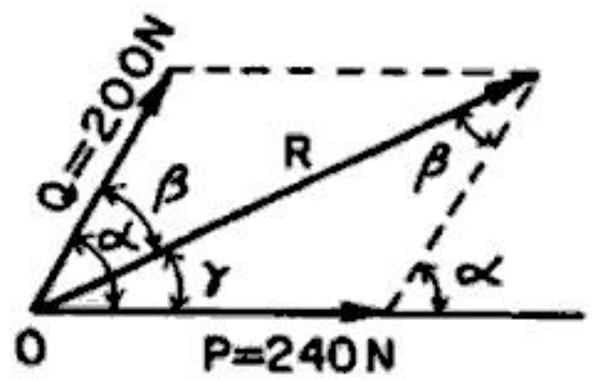

Fig. 2.16.

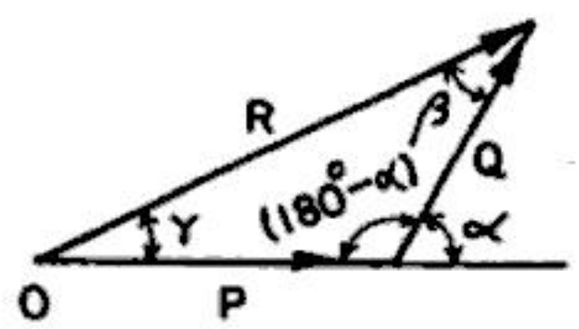

Fig. 2.16. (*a*)

Sol. Given :

Force P=240 N, Q=200 N

Angle between the forces, $\alpha=60°$

The magnitude of resultant R is given by,

$$R=\sqrt{P^2+Q^2+2PQ\cos\alpha}=\sqrt{240^2+200^2+2\times240\times200\times\cos 60°}$$

$$=\sqrt{57600+40000+48000}=\mathbf{381.57\ N.}\quad \textbf{Ans.}$$

Now refer to Fig. 2.16 (*a*). Using sine formula, we get

$$\frac{P}{\sin\beta}=\frac{Q}{\sin\gamma}=\frac{R}{\sin(180°-\alpha)} \qquad ...(i)$$

or

$$\frac{P}{\sin\beta}=\frac{R}{\sin(180°-\alpha)}$$

$$\therefore \quad \sin\beta=\frac{P\sin(180°-\alpha)}{R}=\frac{240\sin(180-60)}{381.57} \qquad (\because P=240\text{ N}, \alpha=60°, R=381.57\text{ N})$$

$$=\frac{240\times\sin 120°}{381.57}=0.5447$$

$$\therefore \quad \beta=\sin^{-1}0.5447=\mathbf{33°}\quad \textbf{Ans.}$$

From equation (*i*), also we have

$$\frac{Q}{\sin\gamma}=\frac{R}{\sin(180°-\alpha)}$$

$$\therefore \quad \sin\gamma=\frac{Q\sin(180°-\alpha)}{R}=\frac{200\sin(180°-60°)}{381.57}=\frac{200\sin 120°}{381.57}=0.4539$$

$$\therefore \quad \gamma=\sin^{-1}0.4539=\mathbf{26.966°}.\quad \textbf{Ans.}$$

Problem 2.3. *Two forces P and Q are acting at a point O as shown in Fig. 2.17. The resultant force is 400 N and angles β and γ are 35° and 25° respectively. Find the two forces P and Q.*

Sol. Given :

Resultant, R=400 N

Angles, $\beta=35°$, $\gamma=25°$

∴ Angle between the two forces,

$$\alpha=\beta+\gamma=35°+25°=60°$$

Refer to Fig. 2.17 (*a*). Using sine formula for ΔOAC, we get

$$\frac{P}{\sin\beta}=\frac{Q}{\sin\gamma}=\frac{R}{\sin(180°-\alpha)} \qquad ...(i)$$

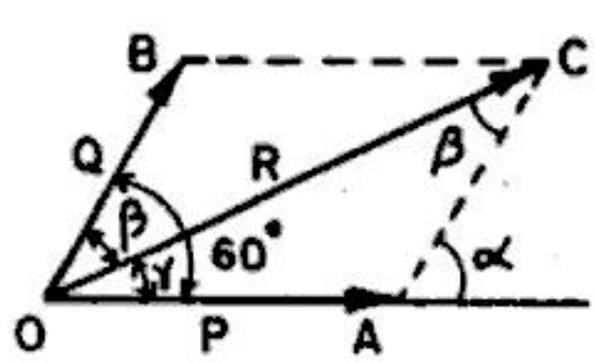

Fig. 2.17.

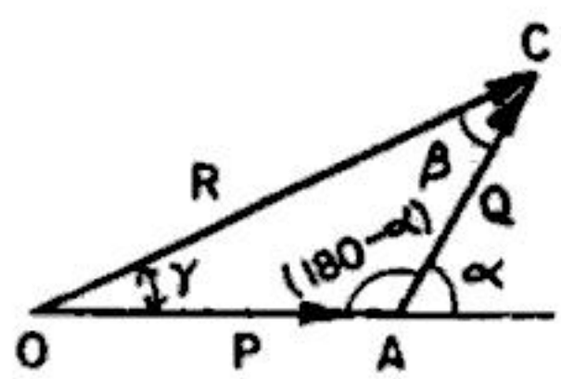

Fig. 2.17. (*a*)

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$$\frac{P}{\sin\beta}=\frac{R}{\sin(180°-\alpha)}$$

$$\therefore \quad P=\frac{R\sin\beta}{\sin(180°-\alpha)}$$

$$=\frac{400\times\sin 35°}{\sin(180°-60°)} \qquad (\because R=400,\ \beta=35,\ \alpha=60°)$$

$$=\frac{400\times 0.5736}{0.866}=\mathbf{264.93\ N.} \quad \textbf{Ans.}$$

Also from equation (*i*), we have

$$\frac{Q}{\sin\gamma}=\frac{R}{\sin(180°-\alpha)}$$

$$\therefore \quad Q=\frac{R\sin\gamma}{\sin(180°-\alpha)}=\frac{400\times\sin 25°}{\sin(180°-60°)}=\frac{400\times 0.4226}{0.866}$$

$$=\mathbf{195.19\ N.} \quad \textbf{Ans.}$$

Problem 2.4. *Two forces P and Q are acting at a point O as shown in Fig. 2.18. The force P=240 N and force Q=200 N. If the resultant of the forces is equal to 400 N, then find the values of angles β, γ and α.*

Sol. Given :

Forces, $P=240$ N, $Q=200$ N

Resultant, $R=400$ N

Let β=Angle between R and Q,

γ=Angle between R and P.

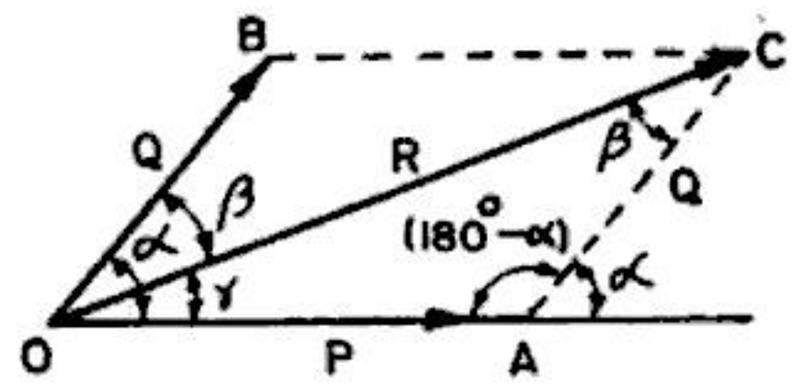

Fig. 2.18.

From Fig. 2.18, it is clear that, $\alpha=\beta+\gamma$.

Let us first calculate the angle α (*i.e.*, angle between the two forces)

Using the relation,

$$R=\sqrt{P^2+Q^2+2PQ\cos\alpha}$$

or $$R^2=P^2+Q^2+2PQ\cos\alpha$$

or $$400^2=240^2+200^2+2\times 240\times 200\times\cos\alpha$$

or $$16000=57600+40000+96000\times\cos\alpha$$

$$\therefore \quad \cos\alpha=\frac{16000-57600-40000}{96000}=0.65$$

$$\therefore \quad \alpha=\cos^{-1}0.65=49.458°=49°\ (0.458\times 60')=49°\ 27.5'$$

Now using sine formula for ΔOAC of Fig. 2.18, we get

$$\frac{P}{\sin\beta}=\frac{Q}{\sin\gamma}=\frac{R}{\sin(180°-\alpha)} \qquad ...(i)$$

or $$\frac{P}{\sin\beta}=\frac{R}{\sin(180°-\alpha)}$$

$$\therefore \quad \sin\beta=\frac{P\sin(180°-\alpha)}{R}$$

$$=\frac{240\sin(180°-49.458)}{400} \qquad (\because P=240,\ \alpha=49.458°)$$

$$=\frac{240\sin(130.542°)}{400}=0.4559$$

$$\therefore \quad \beta=\sin^{-1}0.4559=\mathbf{27.12°.} \quad \textbf{Ans.}$$

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Also from equation (*i*), we have

$$\frac{Q}{\sin\gamma}=\frac{R}{\sin(180^\circ-\alpha)}$$

$$\therefore \quad \sin\gamma=\frac{Q\sin(180^\circ-\alpha)}{R}=\frac{200\times\sin(180^\circ-49.458)}{400}$$

$$=\frac{200\times\sin(130.542^\circ)}{400}=0.3799$$

$$\therefore \quad \gamma=\sin^{-1}0.3799=\mathbf{22.33^\circ}. \quad \textbf{Ans.}$$

Problem 2.5. *A force of 100 N is acting at a point making an angle of 30° with the horizontal. Determine the components of this force along X and Y directions.*

Sol. Given :

Force, $F=100$ N

Angle made by F with horizontal,

$\theta=30^\circ$

Let F_x=Component along x-axis

F_y=Component along y-axis

Then $F_x=F\cos\theta$

$=100\cos 30^\circ=100\times0.866$

$=$**86.6 N. Ans.**

and $F_y=F\sin\theta=100\sin 30^\circ$

$=100\times0.5=$**50 N. Ans.**

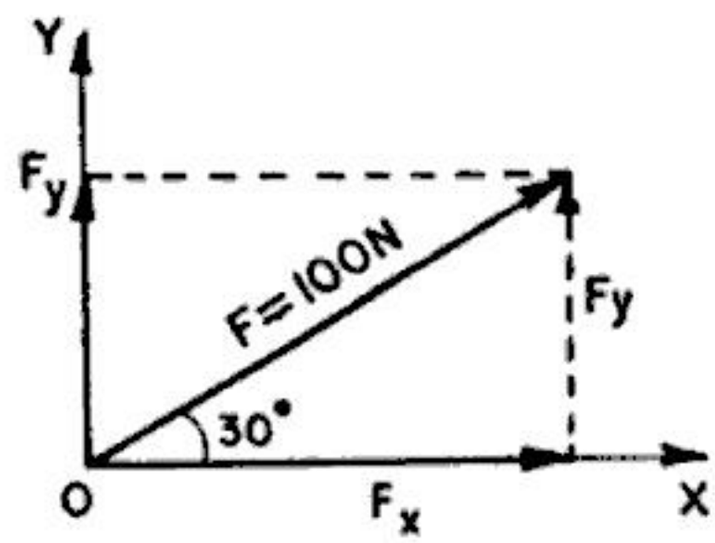

Fig. 2.19.

Problem 2.6. *A small block of weight 100 N is placed on an inclined plane which makes an angle θ=30° with the horizontal. What is the component of this weight ; (ii) parallel to the inclined plane and (iii) perpendicular to the inclined plane ?*

Sol. Given :

Weight of block, $W=100$ N

Inclination of plane, $\theta=30^\circ$

The weight of block $W=100$ N is acting vertically downwards through the C.G. of the block. Resolve this weight into two components *i.e.*, one perpendicular to the inclined plane and other parallel to the inclined plane as shown in Fig. 2.20. The perpendicular (normal) component makes an angle of 30° with the direction of W.

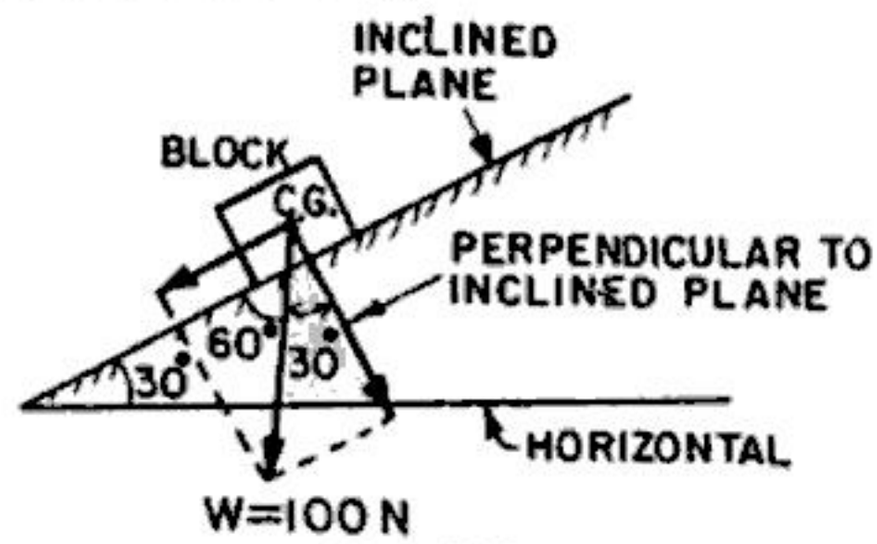

Fig. 2.20.

Hence component of the weight perpendicular to the inclined plane

$=W\cos 30^\circ=100\times0.866=$**8.66 N. Ans.**

Component of the weight (W) parallel to the inclined plane

$=W\sin 30^\circ=100\times0.5=$**50 N. Ans.**

Problem 2.7. *Fig. 2.21 shows a particular position of the connecting rod BA and crank AO. At this position, the connecting rod of the engine exerts a force 2500 N on the crank pin at A. Resolve this force into horizontal and vertical components at A. Also resolve the given force at A along AO and along a direction perpendicular to AO.*

Sol. Given :

Length $BA=50$ cm

Length $AO=25$ cm.

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Force exerted by connecting rod BA at $A=2500$ N. This force is acting along BA at point A.

Let α=Angle made by BA with horizontal.

This angle can be calculated by drawing a perpendicular AC from point A on the horizontal axis. Now the side AC is common in triangles ABC and AOC.

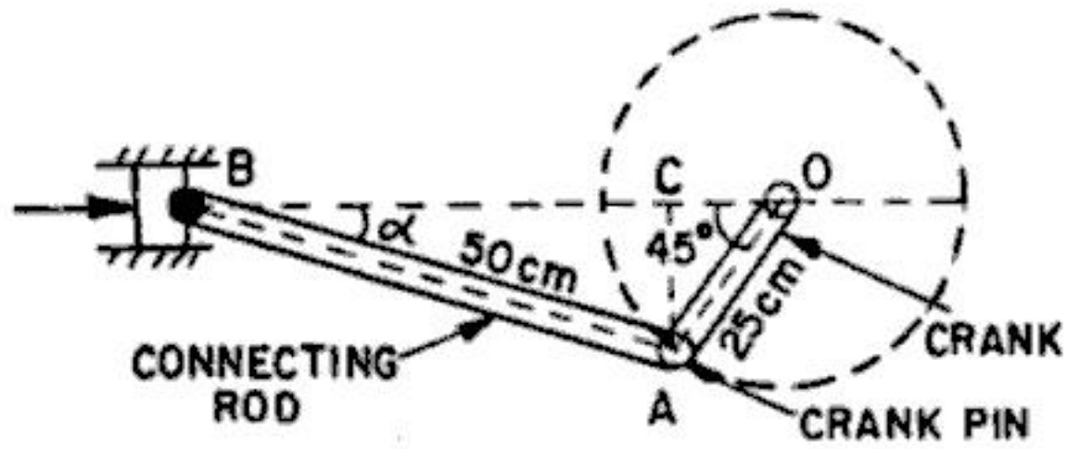

Fig. 2.21.

In triangle ABC, $\sin \alpha=\dfrac{AC}{AB}$

or $AC=AB \sin \alpha=50 \sin \alpha$ $(\because AO=50)$

In triangle AOC, $\sin 45°=\dfrac{AC}{AO}$

or $AC=AO \sin 45°=25 \sin 45°$ $(\because AO=25)$

Equating the two values of AC, we get

$$50 \sin \alpha=25 \sin 45°$$

$$\therefore \quad \sin \alpha=\frac{25 \sin 45°}{50}=0.3535$$

or $$\alpha=\sin^{-1} 0.3535=20.7°$$

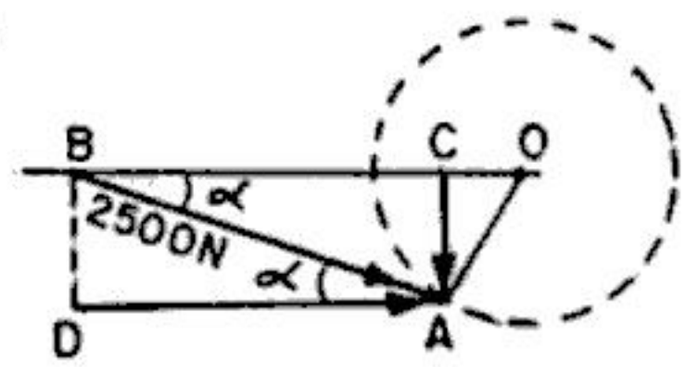

Fig. 2.21 (a)

Now the force 2500 N is acting along BA at point A as shown in Fig. 2.21 (a).

$\therefore$ Horizontal component of this force at $A=2500 \cos \alpha=2500 \cos 20.7°$

$=$**2338.61 N. Ans.**

Vertical component $=2500 \sin \alpha=2500\times\sin 20.7$

$=$**883.75 N. Ans.**

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The force of 2500 N is acting along BA at point A as shown in Fig. 2.21 (b). This force at point A is represented by AD. Hence BAD is a straight line. Resolve the force AD along AO and perpendicular to AO *i.e.*, in the direction AE where AE is perpendicular to AO at point A.

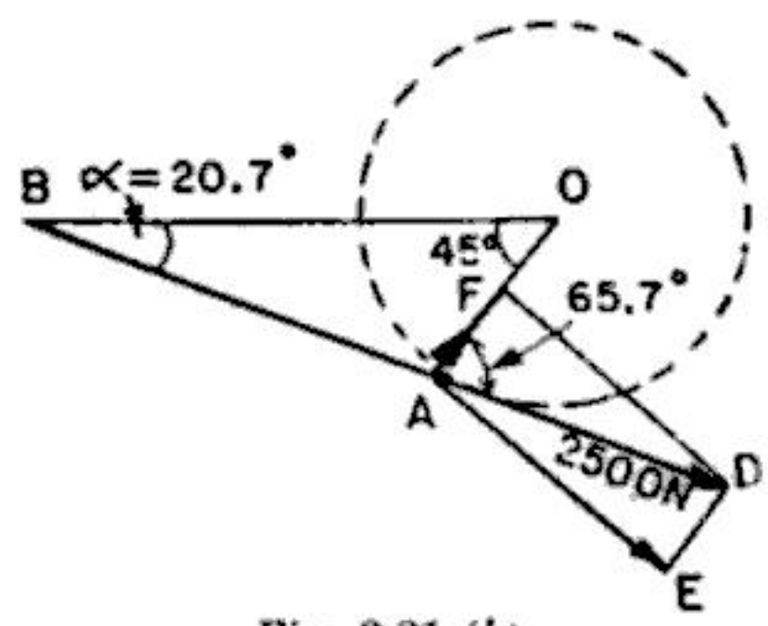

Fig. 2.21 (b)

Now angle $OAD=45+20.7=65.7°$

Component of force AD along

$$AO=AD \cos 65.7°=2500 \cos 65.7°$$

$=$**1028.78 N. Ans.**

Component of force AD along

$$AE=AD \sin 65.7°=2500\times\sin 65.7$$

$=$**2278.5 N. Ans.**

Problem 2.8. *The four coplanar forces are acting at a point as shown in Fig. 2.22. Determine the resultant in magnitude and direction analytically and graphically.*

Sol. Given :

Forces, $F_1=104$ N, $F_2=156$ N,

$F_3=252$ N and $F_4=228$ N.

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(*a*) *Analytical Method.* Resolve each force along horizontal and vertical axes. The horizontal components along OX will be considered as +ve whereas along OX' as −ve. Similarly, vertical components in upward direction will be +ve whereas in downward direction as −ve.

(*i*) *Consider force* $F_1=104$ *N*. Horizontal and vertical components are shown in Fig. 2.22 (*a*).

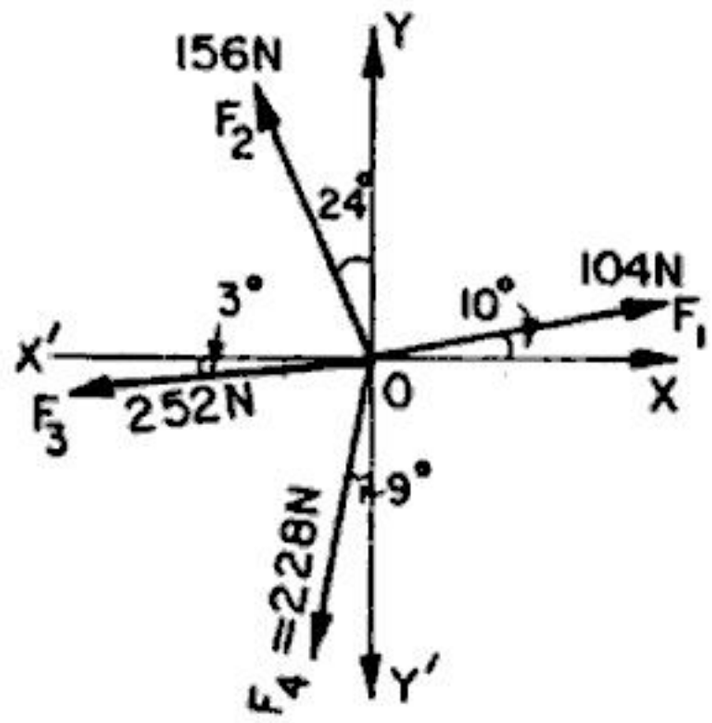

Fig. 2.22.

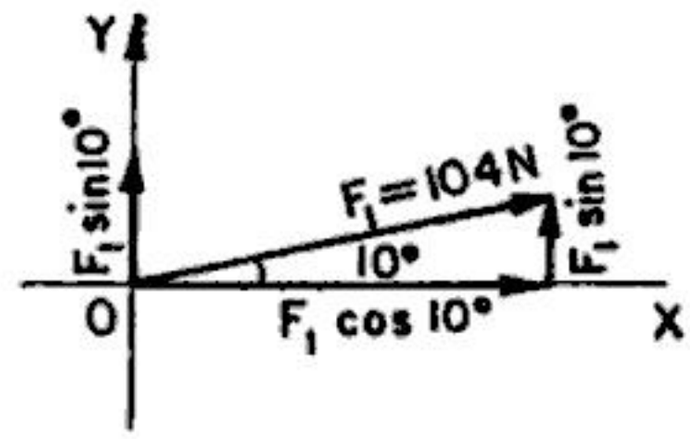

Fig. 2.22. (*a*)

Horizontal component,

$$Fx_1=F_1 \cos 10°=104\times0.9848=102.42 \text{ N}$$

Vertical component,

$$Fy_1=F_1 \sin 10°=104\times0.1736=18.06 \text{ N.}$$

(*ii*) *Consider force* $F_2=156$ *N*. Horizontal and vertical components are shown in Fig. 2.22 (*b*).

Angle made by F_2 with horizontal axis $OX'=90-24=66°$

∴ Horizontal components,

$$Fx_2=F_2 \cos 66°=156\times0.4067$$
$$=63.44 \text{ N.}$$

It is negative as it is acting along OX'.

Vertical component,

$$Fy_2=F_2 \sin 66°=156\times0.9135$$
$$=142.50 \text{ N.} \quad (+\text{ve})$$

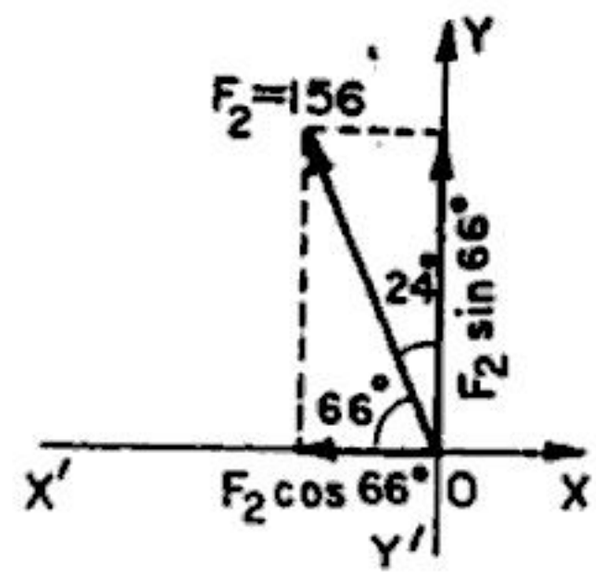

Fig. 2.22 (*b*)

(*iii*) *Consider force* $F_3=252$ *N*. Horizontal and vertical components are shown in Fig. 2.22 (*c*).

Horizontal component,

$$Fx_3=F_3 \cos 3°=252\times0.9986$$
$$=251.64 \text{ N.} \quad (-\text{ve})$$

Vertical component,

$$Fy_3=F_3 \sin 3°=252\times0.0523$$
$$=13.18 \text{ N.} \quad (-\text{ve})$$

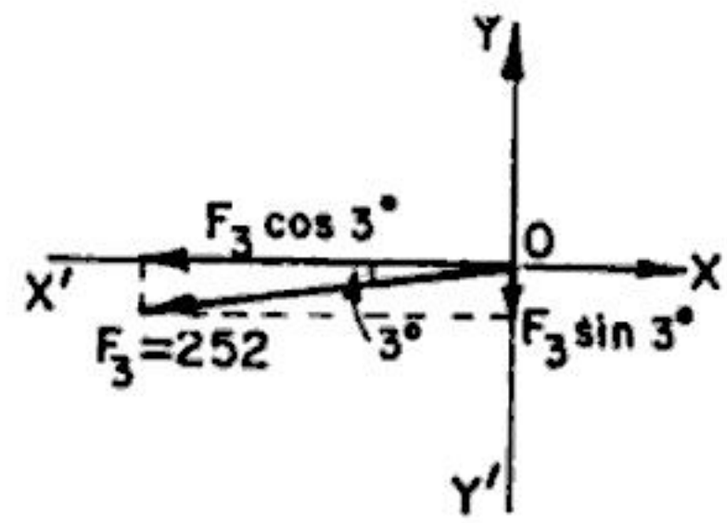

Fig. 2.22 (*c*)

(*iv*) *Consider force* $F_4=228$ *N*. Horizontal and vertical components are shown in Fig. 2.22 (*d*).

Angle made by F_4 with horizontal axis $OX'=90-9=81°$.

∴ Horizontal component,

$$Fx_4=F_4 \cos 81°=228\times0.1564$$
$$=35.66 \text{ N} \quad (-\text{ve})$$

Vertical component,

$$Fy_4=F_4 \sin 81°=228\times0.9877$$
$$=225.2 \text{ N.} \quad (-\text{ve})$$

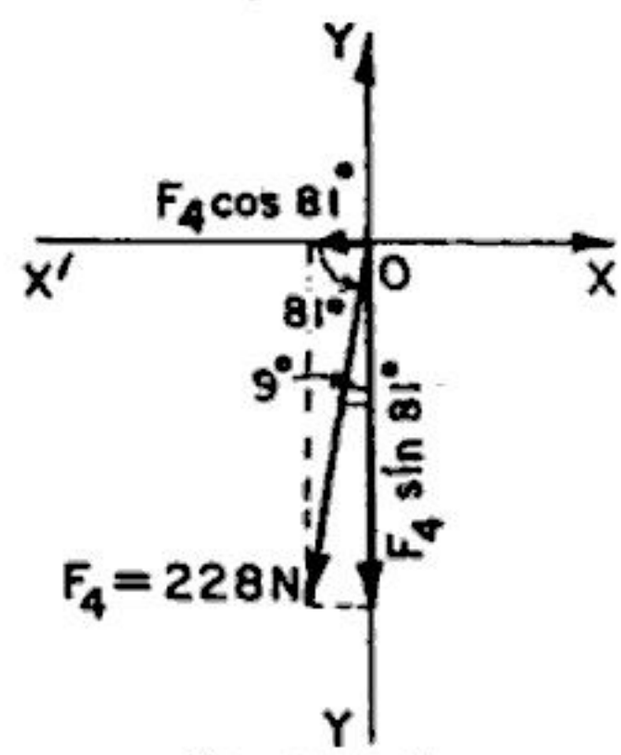

Fig. 2.22 (*d*)

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Now algebraic sum of horizontal components is given by,

$$\Sigma H = Fx_1 - Fx_2 - Fx_3 - Fx_4$$
$$= 102.4 - 63.44 - 251.64 - 35.66$$
$$= -248.32 \text{ N.}$$

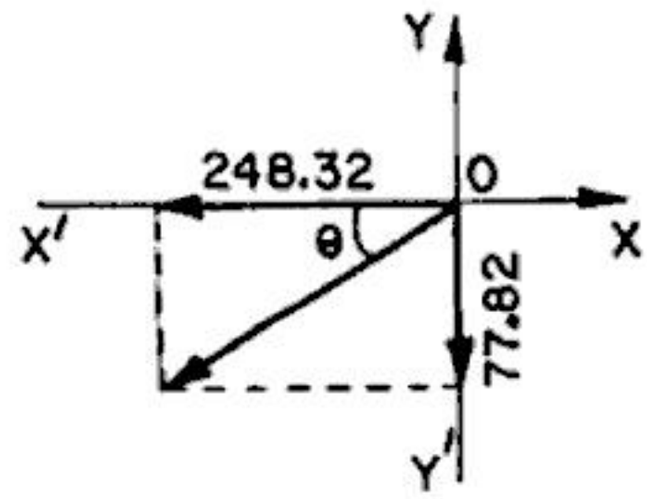

Fig. 2.22 (e)

−ve sign means that ΣH is acting along OX' as shown in Fig. 2.22 (e).

Similarly, the algebraic sum of vertical components is given by,

$$\Sigma V = 18.06 = 142.50 + 13.18 - 225.2$$
$$= -77.82 \text{ N.}$$

−ve sign means that ΣV is acting along OY' as shown in Fig. 2.22 (e).

The magnitude of resultant (*i.e.* R) is obtained by using equation (2.1).

$$\therefore \quad R = \sqrt{(\Sigma H)^2 + (\Sigma V)^2}$$
$$= \sqrt{(248.32)^2 + (77.82)^2} = \mathbf{260.2\ N} \quad \textbf{Ans.}$$

The direction of resultant is given by equation (2.2).

$$\therefore \quad \tan\theta = \frac{\Sigma V}{\Sigma H} = \frac{77.82}{248.32} = 0.3134$$

$$\therefore \quad \theta = \tan^{-1} 0.3134 = 17.4°. \quad \textbf{Ans.}$$

(b) Graphical Method

Fig. 2.23 (*a*), shows the point at which four forces 104 N, 156 N, 252 N and 228 N are acting. The resultant force is obtained graphically by drawing polygon of forces as explained below and shown in Fig. 2.23 (*b*) :

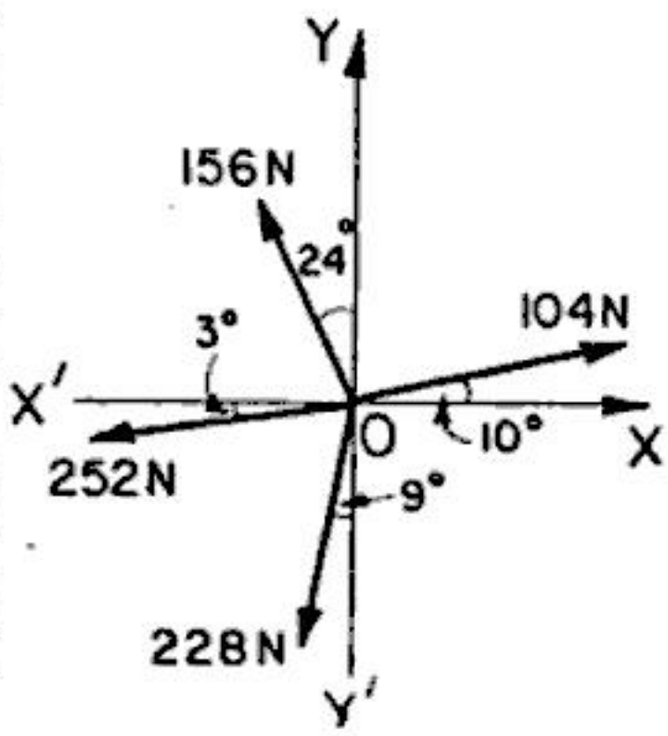

Fig. 2.23 (*a*)

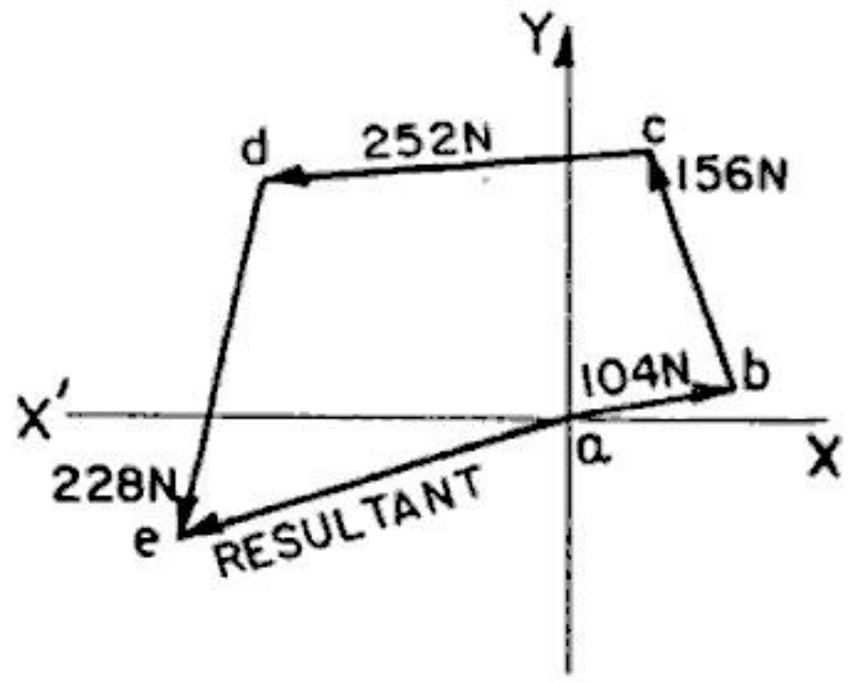

Fig. 2.23 (*b*)

(*i*) Choose a suitable scale to represent the given forces. Let the scale is 25 N = 1 cm. Hence the force 104 N will be represented by $\frac{104}{25} = 4.16$ cm, force 156 N will be represented by $\frac{156}{25} = 6.24$ cm force 252 N will be represented by $\frac{252}{25} = 10.08$ cm and the force 228 N will be represented by $\frac{228}{25} = 9.12$ cm.

(*ii*) Take any point *a*. From point *a*, draw vector *ab* parallel to line of action of force 104 N. Cut *ab* = 4.16 cm. Then *ab* represents the force 104 N in magnitude and direction.

(*iii*) From point *b*, draw vector *bc* parallel to force 156 N and cut *bc* = 6.24 cm. Then vector *cd* represents the force 156 N in magnitude and direction.

(*iv*) From point *c*, draw a vector *cd* parallel 252 N force and cut *cd* = 10.08 cm. Then vector *cd* represents the force 252 N in magnitude and direction.

(*v*) Now from point *d*, draw the vector *de* parallel to 228 N force and cut *de* = 9.12 cm. Then vector *de* represents the force 228 N in magnitude and direction.

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(*vi*) Join point *a* to *e*. The line *ae* is the closing side of the polygon. Hence the side *ae* represents the resultant in magnitude and direction. Measure the length of *ae*.

By measurement, length $ae=10.4$ cm

$\therefore$ Resultant, $R=$Length $ae\times$Scale

$=10.4\times25$ ($\because$ 1 cm=25 N)

$=$**260 N. Ans.**

Now measure angle made by *ae* with horizontal. This angle is 17.4° with axis OX'. **Ans.**

Problem 2.9. *The resultant of four forces which are acting at a point O as shown in Fig. 2.24, is along Y-axis. The magnitude of forces F_1, F_3 and F_4 are 10 kN, 20 kN and 40 kN respectively. The angles made by 10 kN, 20 kN and 40 kN with X-axis are 30°, 90° and 120° respectively. Find the magnitude and direction of force F_2 if resultant is 72 kN.*

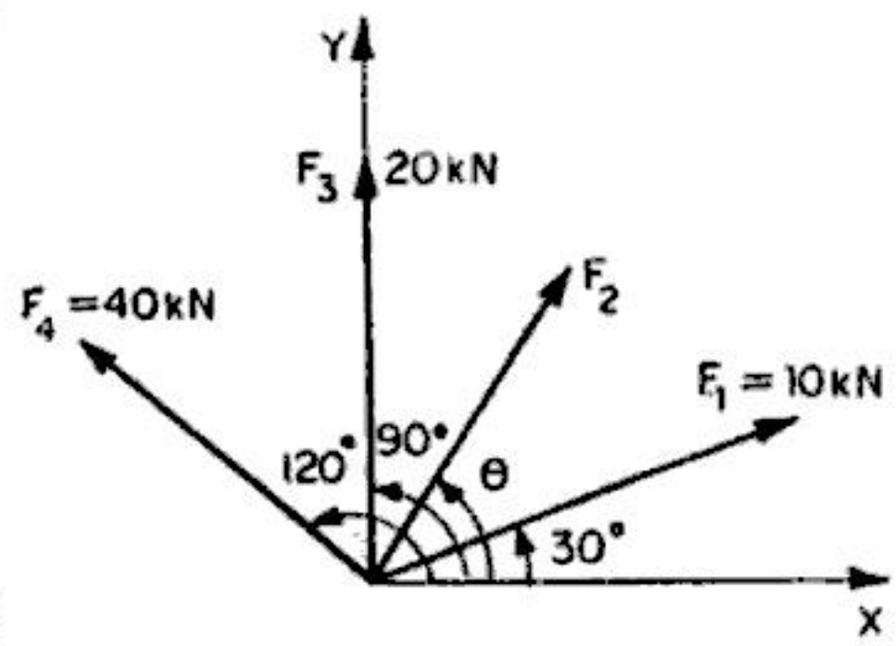

Fig. 2.24.

Sol. Given :

$F_1=10$ kN, $\theta_1=30°$,

$F_2=?$, $\theta_2=\theta$

$F_3=20$ kN, $\theta_3=90°$

$F_4=40$ kN, $\theta_4=120°$

Resultant, $R=72$ kN

Resultant is along Y-axis.

Hence the algebraic sum of horizontal component should be zero and algebraic sum of vertical components should be equal to the resultant.

$\therefore$ $\Sigma H=0$ and $\Sigma V=R=72$ kN

But $\Sigma H=F_1\cos 30°+F_2\cos\theta+F_3\cos 90°+F_4\cos 120°$

$=10\times0.866+F_2\cos\theta+20\times0+40\times(-\frac{1}{2})=8.66+F_2\cos\theta+0-20$

$=F_2\cos\theta-11.34$

$\therefore$ $\Sigma H=0$

or $F_2\cos\theta-11.34=0$

or $F_2\cos\theta=11.34$ ...(*i*)

Now $\Sigma V=F_1\sin 30°+F_2\sin\theta+F_3\sin 90°+F_4\sin 120°$

$=10\times\frac{1}{2}+F_2\sin\theta+20\times1+40\times0.866=5+F_2\sin\theta+20+34.64$

$=F_2\sin\theta+59.64$

But $\Sigma V=R$

$\therefore$ $F_2\sin\theta+59.64=72$

$\therefore$ $F_2\sin\theta=72-59.64=12.36$ ...(*ii*)

Dividing equation (*ii*) and (*i*),

$$\frac{F_2\sin\theta}{F_2\cos\theta}=\frac{12.36}{11.34}$$

or $\tan\theta=1.0899$

$\therefore$ $\theta=\tan^{-1}1.0899=$**47.46° Ans.**

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Substituting the value of θ in equation (*ii*), we get

$$F_2 \sin (47.46°) = 12.36$$

or

$$F_2 = \frac{12.36}{\sin (47.46°)} = \frac{12.36}{0.7368} = 16.77 \text{ kN.} \quad \textbf{Ans.}$$

HIGHLIGHTS

1. Coplanar forces means the forces are acting in one plane.
2. Concurrent forces means the forces are intersecting at a common point.
3. Collinear forces means the forces are having same line of action.
4. The resultant of coplanar forces are determined by analytical and graphical methods.
5. The resultant (R) of three collinear forces F_1, F_2 and F_3 acting in the same direction, is given by

$$R = F_1 + F_2 + F_3.$$

If the force F_2 is acting in opposite direction then their resultant will be,

$$R = F_1 - F_2 + F_3.$$

6. The resultant of the two forces P and Q having an angle α between them and acting at a point, is given by consine law method as

$$R = \sqrt{P^2 + Q^2 + 2PQ \cos \alpha}.$$

And the direction of the resultant with the force P is given by,

$$\tan \theta = \frac{\theta \sin \alpha}{P + \theta \cos \alpha}.$$

7. The resultant of three or more forces acting at a point is given by,

$$R = \sqrt{(\Sigma H)^2 + (\Sigma V)^2}$$

where ΣH = Algebraic sum of horizontal components of all forces,

ΣV = Algebraic sum of vertical components of all forces

The angle made by the resultant with horizontal is given by,

$$\tan \theta = \frac{(\Sigma V)}{(\Sigma H)}.$$

8. The resultant of several forces acting at a point is found graphically by using polygon law of forces.
9. Polygon law of forces states that if a number of coplanar forces are acting at a point such that they can be represented in magnitude and direction by the sides of a polygon taken in the same order, then their resultant is represented in magnitude and direction by the closing side of the polygon taken in the opposite order.

EXERCISE 2

(A) Theoretical Questions

1. Define and explain the following terms :
   (*i*) Coplanar and non-coplanar forces
   (*ii*) Collinear and concurrent forces
   (*iii*) Parallel and non-parallel forces.
2. What is the difference between collinear and concurrent forces ?
3. State and explain the following laws of forces :
   (*i*) Law of parallelogram of forces
   (*ii*) Law of triangle of forces
   (*iii*) Law of polygon of forces.
4. Derive an expression for the resultant in magnitude and direction of two coplanar concurrent forces using cosine law method.
5. Explain in detail the method of finding resultant in magnitude and direction of three or more forces acting at a point by analytical and graphical method.
6. Explain the procedure of resolving a given force into two components at right angles to each other.

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7. Three collinear forces F_1, F_2 and F_3 are acting on a body. What will be the resultant of these forces, if

(*a*) all are acting in the same direction

(*b*) force F_3 is acting in opposite direction.

8. State the law of parallelogram of forces and show that the resultant $R = \sqrt{P^2 + Q^2}$ when the two forces P and Q are acting at right angles to each other. Find the value of R if the angle between the forces is zero.

(B) Numerical Problems

1. Three collinear horizontal forces of magnitude 300 N, 100 N and 250 N are acting on rigid body. Determine the resultant of the forces analytically and graphically when : (*i*) all the forces are acting in the same direction ; (*ii*) the force 100 N acts in the opposite direction. [**Ans.** (*i*) 650 N, (*ii*) 450 N]

2. Two forces of magnitude 15 N and 12 N are acting at a point. The angle between the forces is 60°. Find the resultant in magnitude. [**Ans.** 20.43 N]

3. A force of 1000 N is acting at a point, making an angle of 60° with the horizontal. Determine the components of this force along horizontal and vertical directions. [**Ans.** 500 N, 866 N]

4. A small block of weight 100 N is placed on an inclined plane which makes an angle of 60° with the horizontal. Find the components of this weight (*i*) perpendicular to the inclined plane and (*ii*) parallel to the inclined plane. [**Ans.** 50 N, 86.6 N]

5. Two forces P and Q are acting at a point O as shown in Fig. 2.25. The force P=264.9 N and force Q=195.2 N. If the resultant of the forces is equal to 400 N then find the values of angles β, γ, α.

[**Ans.** β=35°, γ=25°, α=60°]

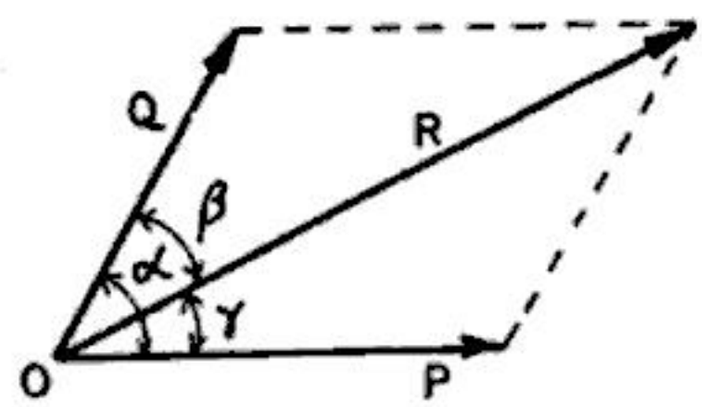

Fig. 2.25.

6. A small block of unknown weight is placed on an inclined plane which makes an angle of 30° with horizontal plane. The component of this weight parallel to the inclined plane is 100 N. Find the weight of the block. [**Ans.** 200 N]

7. In question 6, find the component of the weight perpendicular to the inclined plane. [**Ans.** 173.2 N]

8. The four coplanar forces are acting at a point as shown in Fig. 2.26. Determine the resultant in magnitude and direction analytically and graphically.

[**Ans.** 1000 N, θ=60° with OX]

9. The four coplanar forces are acting at a point as shown in Fig. 2.27. One of the forces is unknown and its magnitude is shown by P. The resultant is having a magnitude 500 N and is acting along x-axis. Determine the unknown force P and its inclination with x-axis.

[**Ans.** P=286.5 N and θ=53° 15']

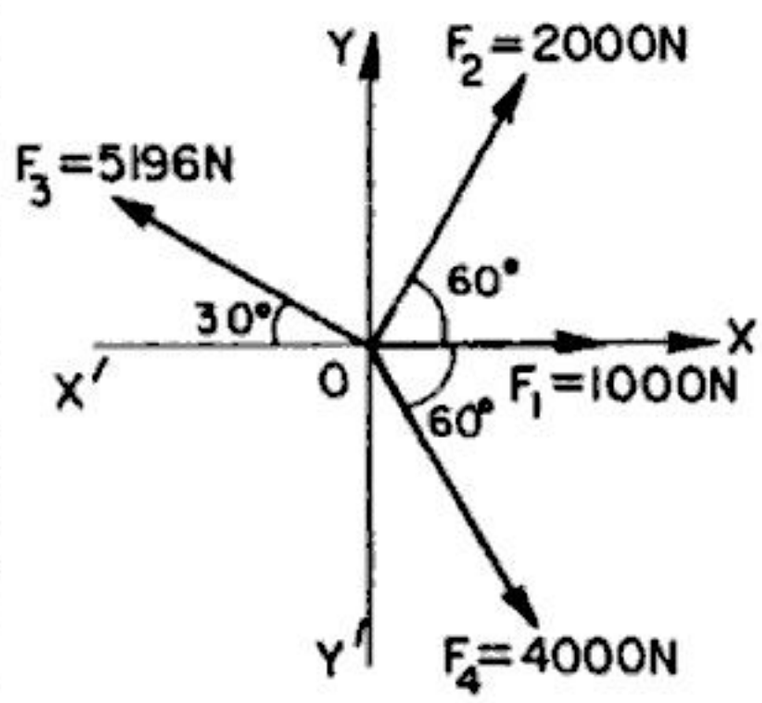

Fig. 2.26.

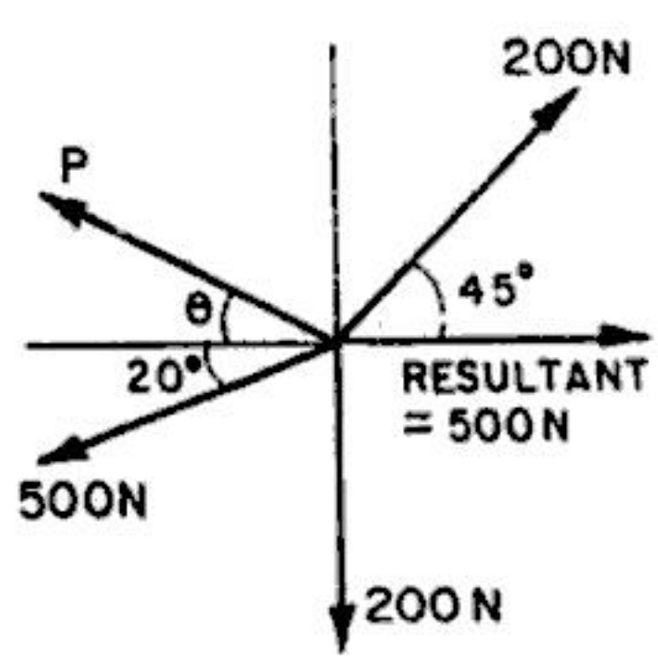

Fig. 2.27.

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3

Coplanar Parallel Forces

3.1. INTRODUCTION

The forces, which are having their line of actions parallel to each other, are known parallel forces. The two parallel forces will not intersect at a point. The resultant of two coplanar concurrent forces (*i.e.*, forces intersecting at the same point) can be directly determined by the method of parallelogram of forces. This method along with other methods for finding resultant of collinear and concurrent coplanar forces, were discussed in earlier chapters.

The parallel forces are having their lines of action parallel to each other. Hence, for finding the resultant of two parallel forces, (two parallel forces do not intersect at a point) the parallelogram cannot be drawn. The resultant of such forces can be determined by applying the *principle of moments*. Hence in this chapter first the concepts of moment and principle of moments will be dealt with. Thereafter the methods of finding resultant of parallel and even non-parallel forces will be explained.

3.2. MOMENT OF A FORCE

The product of a force and the perpendicular distance of the line of action of the force from a point is known as moment of the force about that point.

Let F=A force acting on a body as shown in Fig. 3.1.

r=Perpendicular distance from the point O on the line of action of force F.

Then moment (M) of the force F about O is given by,

$$M=F\times r$$

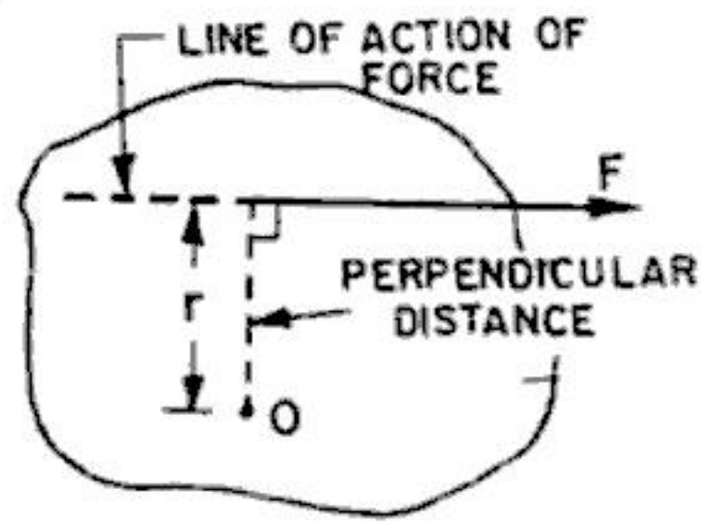

Fig. 3.1.

The tendency of this moment is to rotate the body in the clockwise direction about O. Hence this moment is called *clockwise moment*. If the tendency of a moment is to rotate the body in anti-clockwise direction, then that moment is known as *anti-clockwise moment*. If clockwise moment is taken −ve then anti-clockwise moment will be +ve.

In S.I. system, moment is expressed in N m (Newton metre).

Fig. 3.2 shows a body on which three forces F_1, F_2 and F_3 are acting. Suppose it is required to find the resultant moments of these forces about point O.

Let r_1=Perpendicular distance from O on the line of action of force F_1.

r_2 and r_3=Perpendicular distances from O on the lines of action of force F_2 and F_3 respectively.

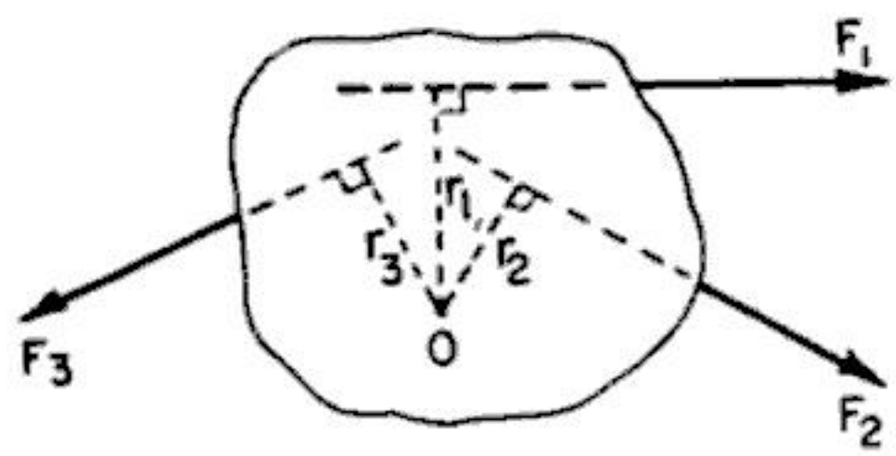

Fig. 3.2.

Moment of F_1 about $O=F_1\times r_1$ (clockwise) (−)

Moment of F_2 about $O=F_2\times r_2$ (clockwise) (−)

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Moment of F_3 about $O = F_3 \times r_3$ (anti-clockwise) (+)

The resultant moment will be *algebraic sum* of all the moments.

$\therefore$ The resultant moment of F_1, F_2 and F_3 about O

$$= -F_1 \times r_1 - F_2 \times r_2 + F_3 \times r_3.$$

Problem 3.1. *Four forces of magnitude 10 N, 20 N, 30 N and 40 N are acting respectively along the four sides of a square ABCD as shown in Fig. 3.4. Determine the resultant moment about the point A. Each side of the square is given 2 m.*

Sol. Given :

Length $AB = BC = CD = DA = 2$ m

Force at $B = 10$ N,

Force at $C = 20$ N,

Force at $D = 30$ N,

Force at $A = 40$ N.

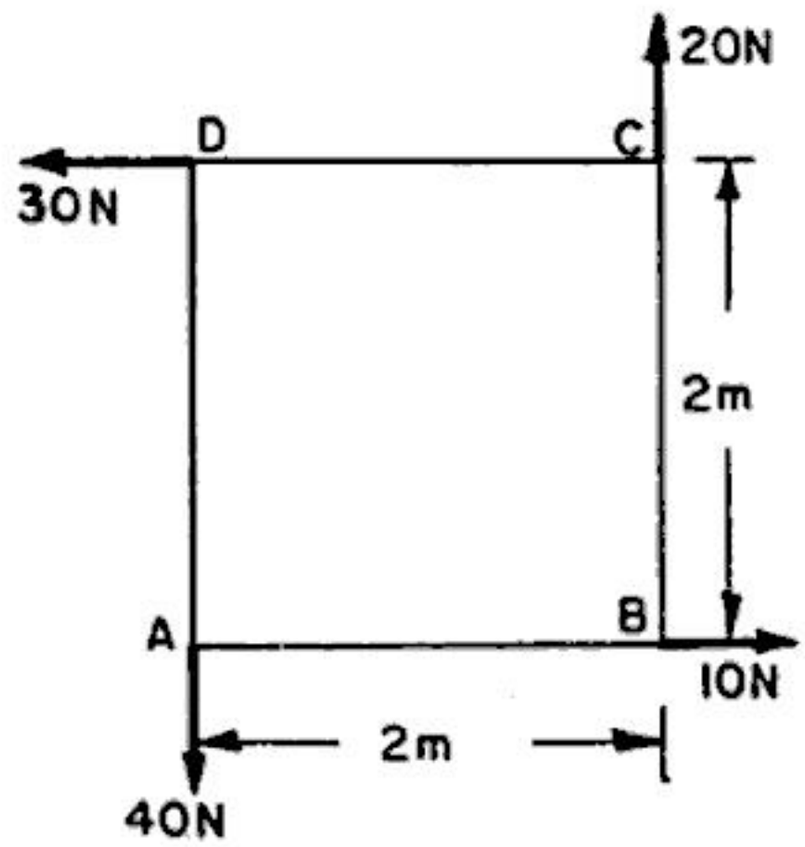

Fig. 3.3.

The resultant moment about point A is to be determined.

The forces at A and B passes through point A. Hence perpendicular distance from A on the lines of action of these fores will be zero.

Hence their moments about A will be zero. The moment of the force at C about point A.

$=$Force at $C \times \perp$ distance from A on the line of action of force at C.

$=(20 \text{ N}) \times (\text{Length } AB)$.

$=20 \times 2$ N m $=40$ N m (anti-clockwise).

The moment of force at D about point A.

$=$Force at $D \times \perp$ distance from A on the line of action of force at D.

$=(30 \text{ N}) \times (\text{Length } AD)$.

$=30 \times 2$ Nm $=60$ Nm (anti-clockwise).

$\therefore$ Resultant moment of all forces about A.

$=40+60=100$ Nm (anti-clockwise). **Ans.**

3.3. PRINCIPLE OF MOMENTS (OR VARIGNON'S PRINCIPLE)

Principle of moments states that the moment of the resultant of a number of forces about any point is equal to the *algebraic sum* of the moments of all the forces of the system about the same point.

And according to Varignon's principle, the moment of a force about any point is equal to the *algebraic sum* of the moments of its components about *that point.*

Proof of Varignon's Principle

Fig. 3.4 (*a*) shows two forces F_1 and F_2 acting at point O. These forces are represented in magnitude and direction by OA and OB. Their resultant R is represented in magnitude and direction by OC which is the diagonal of parallelogram

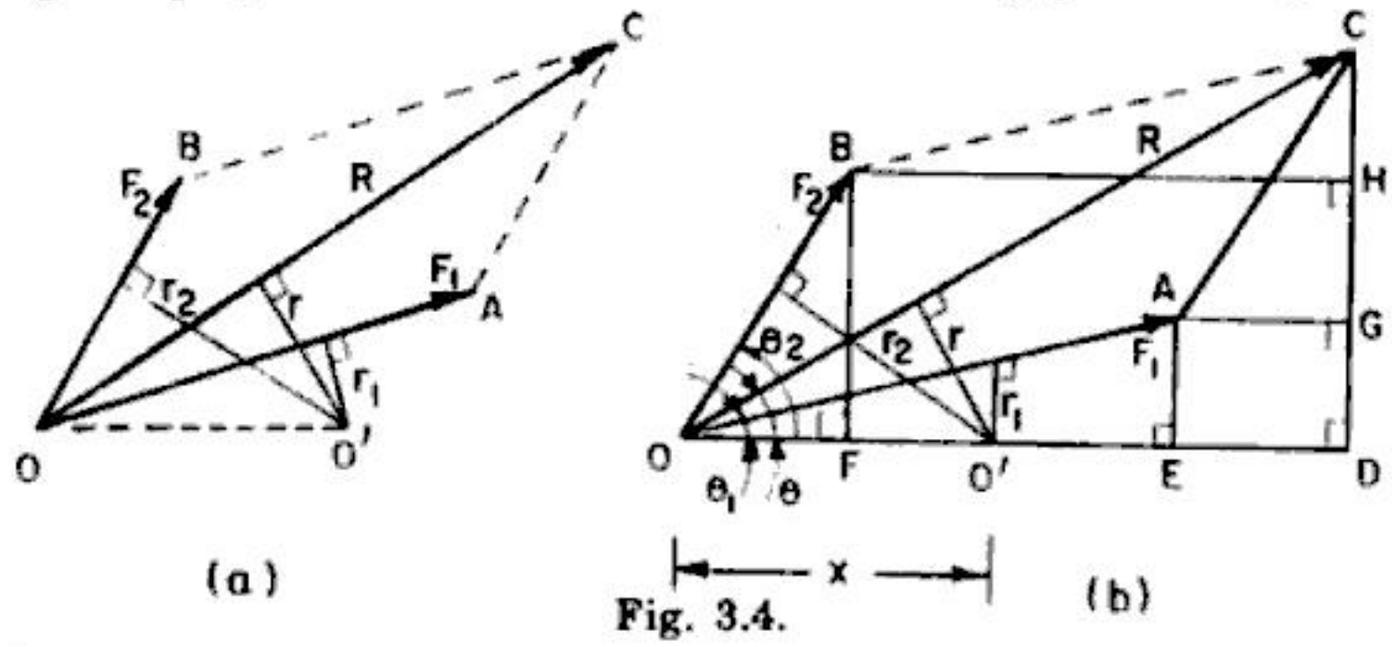

Fig. 3.4.

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$OACB$. Let O' is the point in the plane about which moments of F_1, F_2 and R are to be determined. From point O', draw perpendiculars on OA, OC and OB.

Let r_1=Perpendicular distance between F_1 and O'.

r=Perpendicular distance between R and O'.

r_2=Perpendicular distance between F_2 and O'.

Then according to Varignon's principle ;

Moment of R about O' must be equal to algebraic sum of moments of F_1 and F_2 about O'.

or
$$R \times r = F_1 \times r_1 + F_2 \times r_2$$

Now refer to Fig. 3.4 (*b*). Join OO' and produce it to D. From points C, A and B draw perpendiculars on OD meeting at D, E and F respectively. From A and B also draw perpendiculars on CD meeting the line CD at G and H respectively.

Let θ_1=Angle made by F_1 with OD,

θ=Angle made by R with OD, and

θ_2=Angle made by F_2 with OD.

In Fig. 3.4 (*b*), $OA=BC$ and also OA parallel to BC, hence the projection of OA and BC on the same vertical line CD will be equal *i.e.*, $GD=CH$ as GD is the projection of OA on CD and CH is the projection of BC on CD.

Then from Fig. 3.4 (*b*), we have

$P_1 \sin \theta_1 = AE = GD = CH$

$F_1 \cos \theta_1 = OE$

$F_2 \sin \theta_1 = BF = HD$

$F_2 \cos \theta_2 = OF = ED$ $\because$ ($OB=AC$ and also $OB \parallel AC$. Hence projections of OB and AC on the same horizontal line OD will be equal *i.e.*, $OF=ED$)

$R \sin \theta = CD$

$R \cos \theta = OD$

Let the length $OO'=x$.

Then $x \sin \theta_1 = r_1$, $x \sin \theta = r$ and $x \sin \theta_2 = r_2$

Now moment of R about O'

$= R \times (\perp$ distance between O' and $R) = R \times r$

$= R \times x \sin \theta$ $\quad (\because r = x \sin \theta)$

$= (R \sin \theta) \times x$

$= CD \times x$ $\quad (\because R \sin \theta = CD)$

$= (CH + HD) \times x = (F_1 \sin \theta_1 + F_2 \sin \theta_2) \times x$

$(\because CH = F_1 \sin \theta_1$ and $HD = F_2 \sin \theta_2)$

$= F_1 \times x \sin \theta_1 + F_2 \times x \sin \theta_2$

$= F_1 \times r_1 + F_2 \times r_2$ $\quad (\because x \sin \theta_1 = r_1$ and $x \sin \theta_2 = r_2)$

=Moment of F_1 about O'+Moment of F_2 about O'.

Hence moment of R about any point is the algebraic sum of moments of its components (*i.e.*, F_1 and F_2) about the same point. Hence Varignon's principle is proved.

The principle of moments (or Varignon's principle) is not restricted to only two concurrent forces but is also applicable to any coplanar force system, *i.e.*, concurrent or non-concurrent or parallel force system.

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Problem 3.2. *A force of 100 N is acting at a point A as shown in Fig. 3.5. Determine the moments of this force about O.*

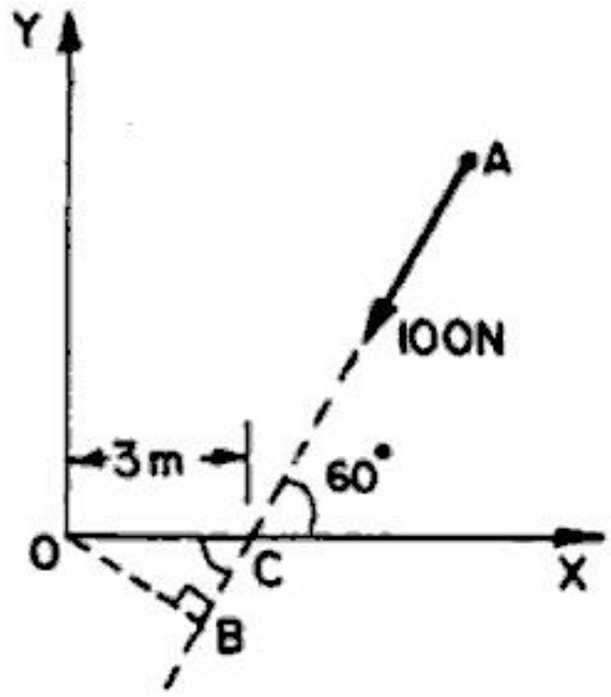

Fig. 3.5.

Sol. Given :

Force at $A = 100$ N

Draw a perpendicular from O on the line of action of force 100 N. Hence OB is the perpendicular on the line of action of 100 N as shown in Fig. 3.5.

1st Method

Triangle OBC is a right-angled triangle. And angle $OCB = 60°$.

$$\therefore \sin 60° = \frac{OB}{OC}$$

$$\therefore \quad OB = OC \sin 60°$$

$$= 3 \times 0.866 = 2.598 \text{ m}$$

Moment of the force 100 N about O

$$= 100 \times OB = 100 \times 2.598$$

$$= 259.8 \text{ Nm (clockwise).} \quad \textbf{Ans.}$$

2nd Method

The moment of force 100 N about O, can also be determined by using Varignon's principle. The force 100 N is replaced by its two rectangular components at any convenient point. Here the convenient point is chosen as C. The horizontal and vertical components of force 100 N acting at C are shown in Fig. 3.6.

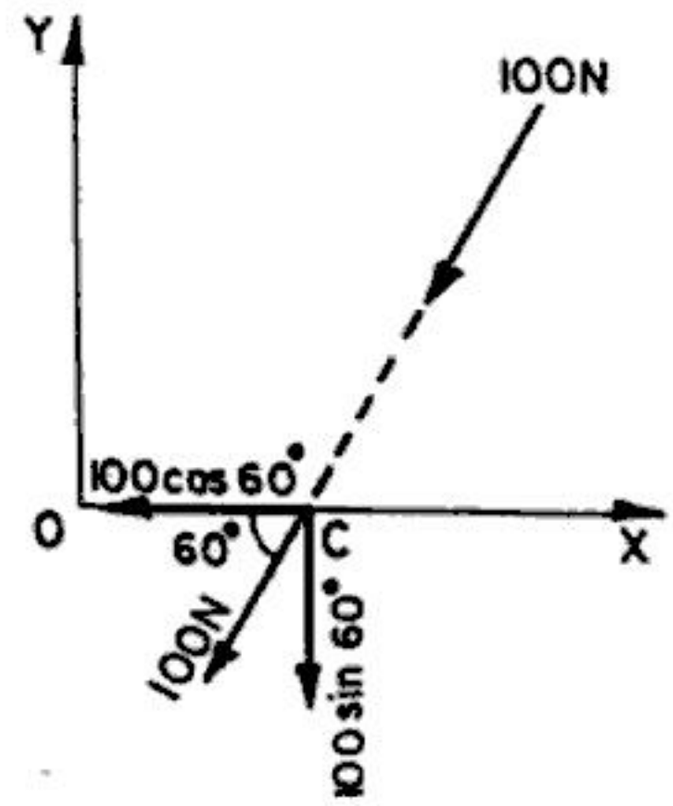

Fig. 3.6.

(*i*) The horizontal component

$$= 100 \times \cos 60° = 50 \text{ N}$$

But this force is passing through O and hence has no moment about O.

The vertical component

$$= 100 \times \sin 60° = 100 \times 0.866$$

$$= 86.6 \text{ N}$$

This force is acting vertical downwards at C. Moment of this force about O.

$$= 86.6 \times OC$$

$$= 86.6 \times 3 \qquad (\because OC = 3 \text{ m})$$

$$= 259.8 \text{ N (clockwise).} \quad \textbf{Ans.}$$

3.4. TYPES OF PARALLEL FORCES

The following are the important types of parallel forces :

1. Like parallel forces,
2. Unlike parallel forces.

3.4.1. Like parallel forces. The parallel forces which are acting in the same direction, are known as like parallel forces. In Fig. 3.7, two parallel forces F_1 and F_2 are shown. They are acting in the same direction. Hence they are called as like parallel forces. These forces may be equal or unequal in magnitude.

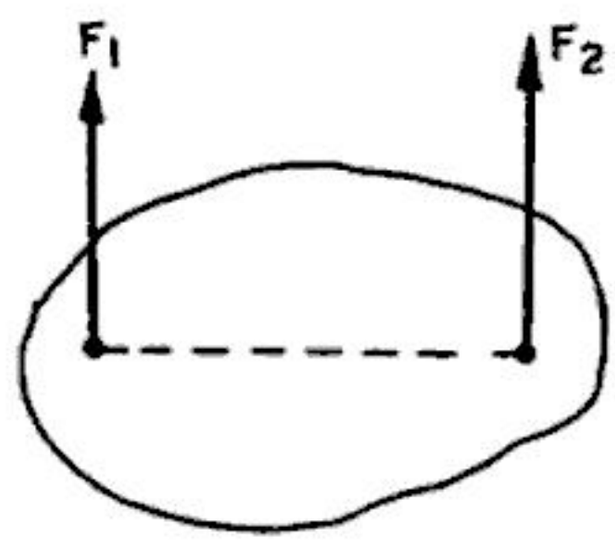

Fig. 3.7.

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3.4.2. Unlike Parallel Forces. The parallel forces which are acting in the opposite direction, are known as unlike parallel forces. In Fig. 3.8, two parallel forces F_1, F_2 are acting in opposite direction. Hence they are called as unlike parallel forces. These forces may be equal or unequal in magnitude.

The unlike parallel forces may be divided into : (*i*) unlike equal parallel forces, and (*ii*) unlike unequal parallel forces.

Unlike equal parallel forces are those which are acting in opposite direction and are equal in magnitude.

Unlike unequal parallel forces are those which are acting in opposite direction and are unequal in magnitude.

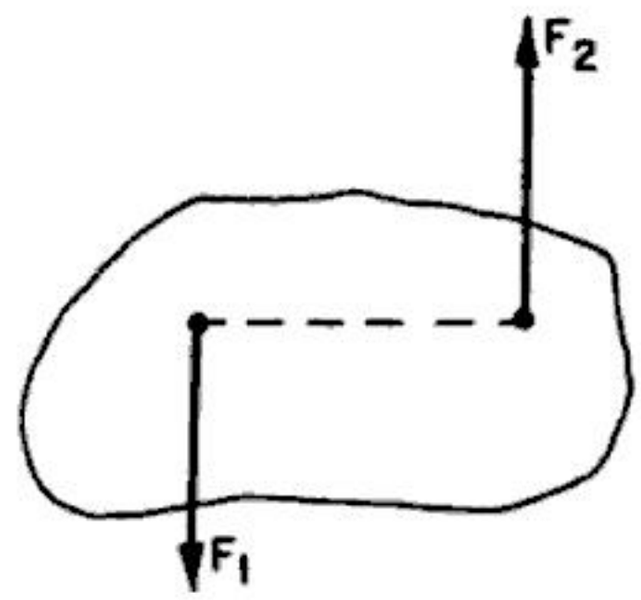

Fig. 3.8.

3.5. RESULTANT OF TWO PARALLEL FORCES

The resultant of following two parallel forces will be considered :

1. Two parallel forces are like.
2. Two parallel forces are unlike and are unequal in magnitude.
3. Two parallel forces are unlike but equal in magnitude.

3.5.1. Resultant of two like parallel forces. Fig. 3.9 shows a body on which two like parallel forces F_1 and F_2 are acting. It is required to determine the resultant (R) and also the point at which the resultant R is acting. For the two parallel forces which are acting in the same direction, obviously the resultant R is given by,

$$R=F_1+F_2$$

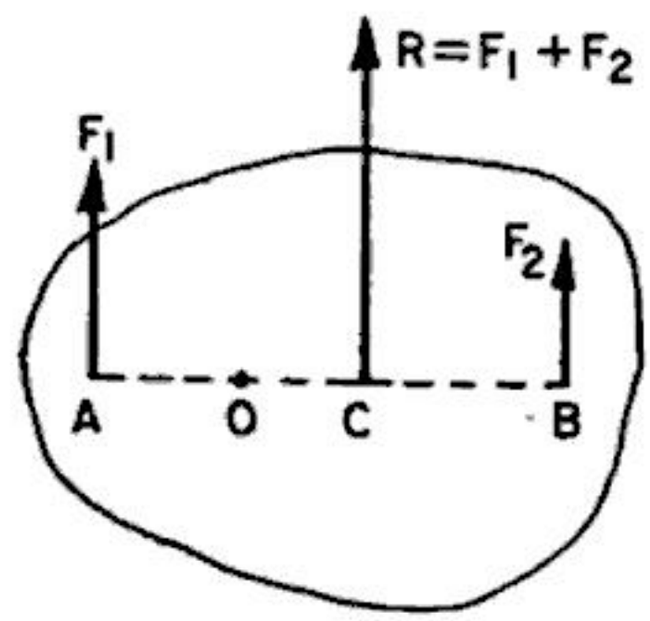

Fig. 3.9.

In order to find the point at which the resultant is acting, Varignon's principle (or method of moments) is used. According to this, the algebraic sum of moments of F_1 and F_2 about any point should be equal to the moment of the resultant (R) about that point. Now arbitrarily choose any point O along line AB and take moments of all forces about this point.

Moment of F_1 about $O=F_1\times AO$ (clockwise)(−)

Moment of F_2 about $O=F_2\times BO$ (anti-clockwise)(+)

∴ Algebraic sum of moments of F_1 and F_2 about O

$$=-F_1\times AO+F_2\times BO$$

Moment of resultant about $O=R\times CO$ (anti-clockwise) (+)

But according to principle of moments the algebraic sum of moments of F_1 and F_2 about O should be equal to the moment of resultant about the same point O.

$$\therefore \quad -F_1\times AO+F_2\times BO=+R\times CO$$

$$=(F_1+F_2)\times CO \qquad (\because R=F_1+F_2)$$

or $$F_1(AO+CO)=F_2(BO-CO)$$

or $$F_1\times AC=F_2\times BC \qquad (\because AO+CO=AC \text{ and } BO-CO=BC)$$

or $$\frac{F_1}{F_2}=\frac{BC}{AC}$$

The above relation shows that the resultant R acts at the point C, parallel to the lines of action of the given forces F_1 and F_2 in such a way that the resultant divides the distance AB

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in the ratio inversely proportional to the magnitudes of F_1 and F_2. Also the point C lies in line AB *i.e.*, point C is not outside AB.

The location of the point C, at which the resultant R is acting, can also be determined by taking moments about points A of Fig. 3.9. As the force F_1 is passing through A, the moment of F_1 about A will be zero.

The moment of F_2 about $A = F_2 \times AB$ (anti-clockwise)(+)

Algebraic sum of moments of F_1 and F_2 about O

$$= O + F_2 \times AB = F_2 \times AB \text{ (anti-clockwise)(+)} \quad ...(i)$$

The moment of resultant R about A

$$= R \times AC \text{ (anti-clockwise)(+)} \quad ...(ii)$$

But according to the principle of moments, the algebraic sum of moments of F_1 and F_2 about A should be equal to the moment of resultant about the same point A. Hence equating equations (*i*) and (*ii*),

$$F_2 \times AB = R \times AC$$

But $R = (F_1 + F_2)$ hence the distance AC should be less than AB. Or in other words, the point C will lie inside AB.

3.5.2. Resultant of Two Unlike Parallel Forces (Unequal in magnitude). Fig. 3.10 shows a body on which two unlike parallel forces F_1 and F_2 are acting which are unequal in magnitude. Let us assume that force F_1 is more than F_2. It is required to determine the resultant R and also the point at which the resultant R is acting. For the two parallel forces, which are acting in opposite direction, obviously the resultant is given by,

$$R = F_1 - F_2$$

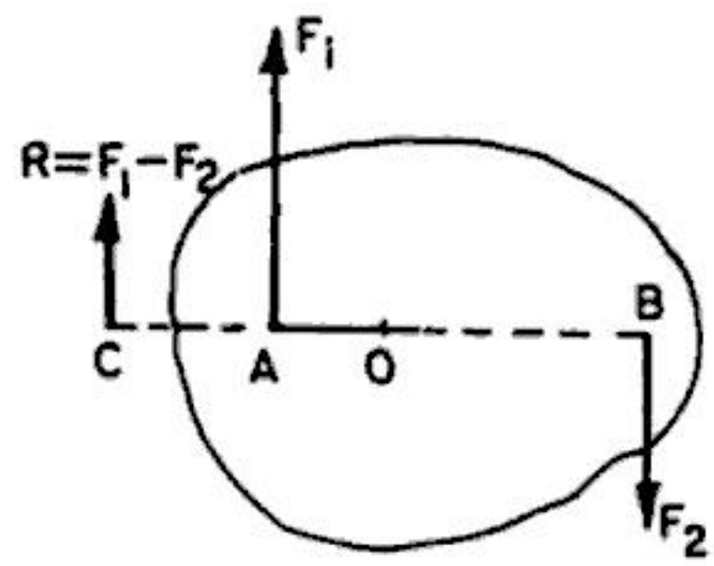

Fig. 3.10.

Let the resultant R is acting at C as shown in Fig. 3.10.

In order to find the point C, at which the resultant is acting, principle of moments is used.

Choose arbitrarily any point O in line AB. Take the moments of all forces (*i.e.*, F_1, F_2 and R) about this point.

Moment of F_1 about $O = F_1 \times AO$ (clockwise)

Moment of F_2 about $O = F_2 \times BO$ (clockwise)

Algebraic sum of moments of F_1 and F_2 about O

$$= F_1 \times AO + F_2 \times BO \quad ...(i)$$

Moment of resultant force R about O

$$= R \times CO \text{ (clockwise)}$$

$$= (F_1 - F_2) \times CO \quad (\because R = F_1 - F_2)$$

$$= F_1 \times CO - F_2 \times CO \quad ...(ii)$$

But according to the principle of moments, the algebraic sum of moments of all forces about any point should be equal to the moment of resultant about that point. Hence equating equations (*i*) and (*ii*), we get

$$F_1 \times AO + F_2 \times BO = F_1 \times CO - F_2 \times CO$$

or $$F_2(BO + CO) = F_1(CO - AO)$$

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$$F_2 \times BC = F_1 \times AC \qquad (\because BO+CO=BC \text{ and } CO-AO=AC)$$

or
$$\frac{BC}{AC}=\frac{F_1}{F_2}$$

or
$$\frac{F_1}{F_2}=\frac{BC}{AC}$$

But $F_1 > F_2$, hence BC will be more than AC. Hence point C lies outside of AB and on the same side as the larger force F_1. *Thus in case of two unlike parallel forces the resultant lies outside the line joining the points of action of the two forces and on the same side as the larger force.*

The location of the point C, at which the resultant R is acting, can also be determined by taking moments about point A, of Fig. 3.10. As the force F_1 is passing through A, the moment of F_1 about A will be zero.

The moment of F_2 about $A = F_2 \times AB$ (clockwise)$(-)$

Algebraic sum of moments of F_1 and F_2 about A

$$= O + F_2 \times AB = F_2 \times AB \text{ (clockwise)}(-) \qquad ...(i)$$

The moment of resultant R about A should be equal to the algebraic sum of moments of F_1 and F_2 (*i.e.*, $=F_2 \times AB$) according to the principle of moments. Also the moment of resultant R about A should be clockwise. As R is acting upwards [$\because F_1 > F_2$ and $R=(F_1-F_2)$ so R is acting in the direction of F_1], the moment of resultant R about A would be clockwise only if the point C is towards the left of point A. Hence the point C will be outside the line AB and on the side of F_1 (*i.e.*, larger force).

Now the moment of resultant R about A

$$= R \times AC \text{ (clockwise)}(-) \qquad ...(ii)$$

Equating equations (*i*) and (*ii*),

$$F_2 \times AB = R \times AC$$
$$= (F_1 - F_2) \times AC \qquad (\because R = F_1 - F_2)$$

As F_1, F_2 and AB are known, hence AC can be calculated. Or in other words, the location of point C is known.

3.5.3. Resultant of two unlike parallel forces which are equal in magnitude. When two equal and opposite parallel forces act on a body, at some distance apart, the two forces from a *couple* which has a tendency to rotate the body. The perpendicular distance between the parallel forces is known as *arm of the couple*.

Fig. 3.11 shows a body on which two parallel forces, which are acting in opposite direction but equal in magnitude, are acting. These two forces will form a couple which will have a tendency to rotate the body in clockwise direction. The moment of the couple is the product of either one of the forces and perpendicular distance between the forces.

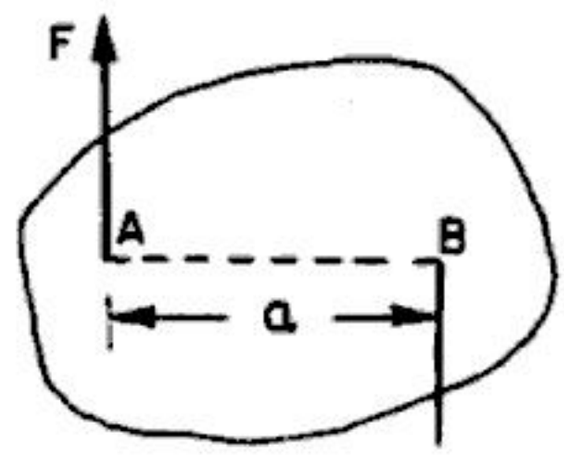

Fig. 3.11.

Let F = Force at A or at B

a = Perpendicular distance (or arm of the couple)

The moment (M) of the couple is given by,

$$M = F \times a.$$

The units of moment will be Nm.

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Problem 3.3. *Three like parallel forces 100 N, 200 N and 300 N are acting at points A, B and C respectively on a straight line ABC as shown in Fig. 3.12. The distances are AB=30 cm and BC=40 cm. Find the resultant and also the distance of the resultant from point A on line ABC.*

Sol. Given :

Force at $A=100$ N

Force at $B=200$ N

Force at $C=300$ N

Distance $AB=30$ cm, $BC=40$ cm. As all the forces are parallel and acting in the same direction, their resultant R is given by

$R=100+200+300=600$ N

Let the resultant is acting at a distance of x cm from the point A as shown in Fig. 3.12.

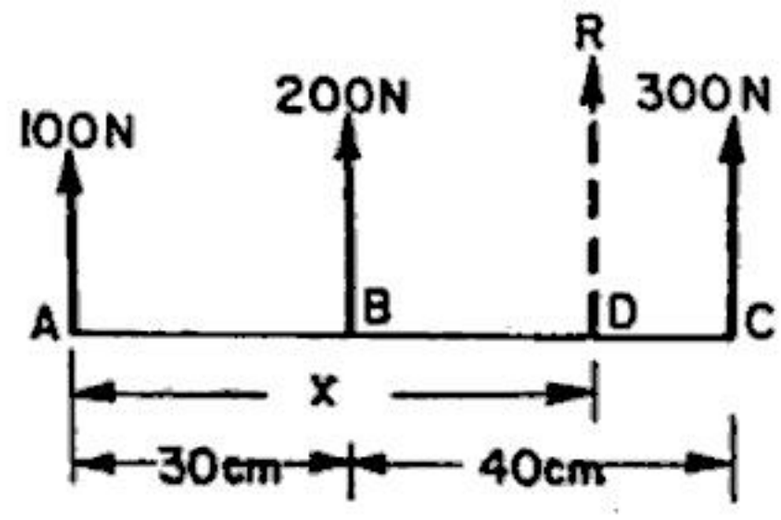

Fig. 3.12.

Now take the moments of all forces about point A. The force 100 N is passing A, hence its moment about A will be zero.

$\therefore$ Moment of 100 N force about $A=0$

Moment of 200 N force about $A=200\times30=6000$ N cm (anti-clockwise)

Moment of 300 N force about $A=300\times AC$

$=300\times70=21000$ N cm (anti-clockwise)

Algebraic sum of moments of all forces about A

$=0+6000+21000=27000$ N cm (anti-clockwise)

Moment of resultant R about $A=R\times x$

$=600\times x$ N cm ($\because R=600$)

But algebraic sum of moments of all forces about A

=Moment of resultant about A

or $27000=600\times x$

or $x=\dfrac{27000}{600}=$ **45 cm. Ans.**

Problem 3.4. *The three like parallel forces of magnitude 50 N, F and 100 N are shown in Fig. 3.13. If the resultant R=250 N and is acting at a distance of 4 m from A, then find*

(i) Magnitude of force F.

(ii) Distance of F from A.

Sol. Given :

Forces at $A=50$ N, at $B=F$ and $D=100$ N

$R=250$ N, Distance $AC=4$ m, $CD=3$ m.

(i) Magnitude of force F

The resultant R of three like forces is given by,

$R=50+F+100$

or $250=50+F+100$ ($\because R=250$)

$\therefore$ $F=250-50-100=$ **100 N. Ans.**

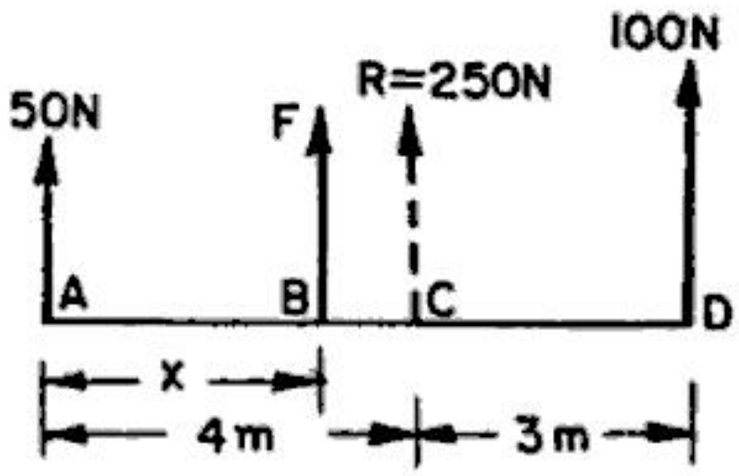

Fig. 3.13.

(ii) Distance of F from A

Take the moments of all forces about point A.

Moment of force 50 N about $A=0$ ($\because$ force 50 N is passing through)

Moment of force F about $A=F\times x$ (anti-clockwise)

Moment of force 100 N about $A=100\times AD=100\times7=700$ N m (anti-clockwise)

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∴ Algebraic sum of moments of all forces about A

$=0+F\times x+700$ N m

$=F\times x+700$ N m (anti-clockwise)

Moment of resultant R about A

$=R\times 4=250\times 4=1000$ N m (anti-clockwise)

But algebraic sum of moments of all forces about A must be equal to the moment of resultant R about A.

∴ $F\times x+700=1000$

or $F\times x=1000-700=300$

or $x=\frac{300}{F}=\frac{300}{100}$ (∵ $F=100$ N)

$=3$ m. **Ans.**

Problem 3.5. *Four parallel forces of magnitudes 100 N, 150 N, 25 N and 200 N are shown in Fig. 3.14. Determine the magnitude of the resultant and also the distance of the resultant from point A.*

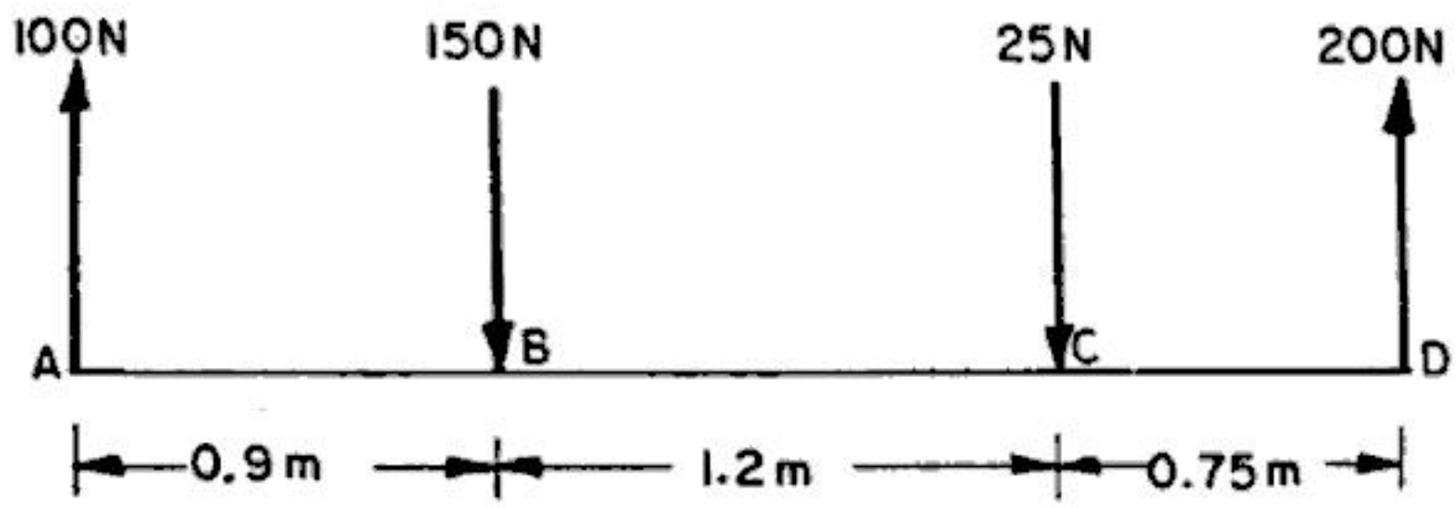

Fig. 3.14.

Sol. Given :

Forces are 100 N, 150 N, 25 N and 200 N.

Distances $AB=0.9$ m, $BC=1.2$ m, $CD=0.75$ m.

As all the forces are acting vertically, hence their resultant R is given by

$R=100-150-25+200$ (Taking upward force +ve and downward as −ve)

$=300-175=125$ N

+ve sign shows that R is acting vertically upwards. To find the distance of R from point A, take the moments of all forces about point A.

Let x=Distance of R from A in metre.

As the force 100 N is passing through A, its moment about A will be zero.

Moment of 150 N force about $A=150\times AB=150\times 0.9$ (clockwise)(−)

$=-135$ Nm

Moment of 25 N force about $A=25\times AC=25\times(0.9+1.2)=25\times 2.1$ (clockwise)(−)

$=-52.5$ Nm.

Moment of 200 N force about $A=200\times AD=200\times(0.9+1.2+0.75)$

$=200\times 2.85$ (anti-clockwise) (+)

$=570$ Nm

Algebraic sum of moments of all forces about A

$=-135-52.5+570=382.5$ Nm ...(*i*)

+ve sign shows that this moment is anti-clockwise. Hence the moment of resultant R about A must be 382.5 Nm, *i.e.*, moment of R should be anti-clockwise about A. The moment of R about A will be anti-clockwise if R is acting upwards and towards the right of A.

Now moment of R about $A=R\times x$ But $R=125$

$=125\times x$ (anti-clockwise) (+)

$=+125\times x$ ...(*ii*)

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Equating (*i*) and (*ii*),

$$382.5 = 125 \times x$$

or

$$x = \frac{382.5}{125} = 3.06 \text{ m.} \quad \textbf{Ans.}$$

∴ Resultant ($R = 125$ N) will be 125 N upwards and is acting at a distance of 3.06 m to the right of point A.

3.6. RESOLUTION OF A FORCE INTO A FORCE AND A COUPLE

A given force F applied to a body at any point A can always be replaced by an equal force applied at another point B together with a couple which will be equivalent to the original force. This is proved as given below :

Let the given force F is acting at point A as shown in Fig. 3.15 (*a*).

This force is to be replaced at the point B. Introduce two equal and opposite forces at B, each of magnitude F and acting parallel to the force at A as shown in Fig. 3.15 (*b*). The force system of Fig. 3.15 (*b*) is equivalent to the single force acting at A of Fig. 3.15 (*a*). In Fig. 3.15 (*b*) three equal forces are acting. The two forces *i.e.*, force F at A and the oppositely directed force F at B (*i.e.*, vertically downward force at B) from a couple. The moment of this couple is $F \times x$ clockwise where x is the perpendicular distance between the lines of action of forces at A and B. The third force is acting at B in the same direction in which the force at A is acting. In Fig. 3.15 (*c*), the couple is shown by curved arrow with symbol M. The force system of Fig. 3.15 (*c*) is equivalent to Fig. 3.15 (*b*). Or in other words the Fig. 3.15 (*c*) is equivalent to Fig. 3.15 (*a*). Hence the given force F acting at A has been replaced by an equal and parallel force applied at point B in the same direction together with a couple of moment $F \times x$.

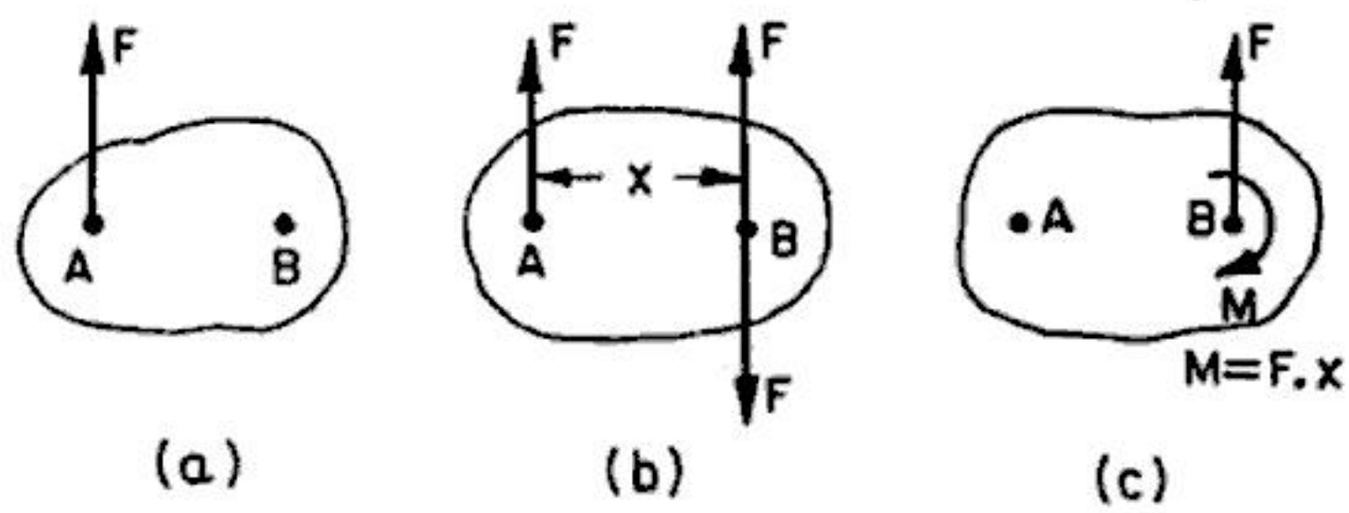

Fig. 3.15.

Thus a force acting at a point in a rigid body can be replaced by an equal and parallel force at any other point in the body, and a couple.

Problem 3.6. *A system of parallel forces are acting on a rigid bar as shown in Fig. 3.16. Reduce this system to :*

(i) a single force

(ii) a single force and a couple at A

(iii) a single force and a couple at B.

Sol. Given :

Forces at A, C, D and B are 32.5 N, 150 N, 67.5 N and 10 N respectively.

Distances $AC = 1$ m,

$CD = 1$ m

and $BD = 1.5$ m.

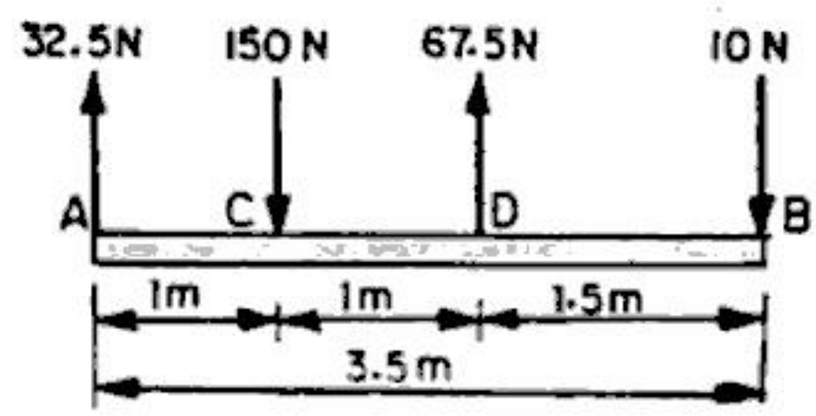

Fig. 3.16.

(i) Single force system. The single force system will consist only resultant force in magnitude and location. All the forces are acting in the vertical direction and hence their resultant (R) in magnitude is given by

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$R=32.5-150+67.5-10=$ **−60 N. Ans.**

Negative sign shows that resultant is acting vertically downwards.

Let x=Distance of resultant from A towards right. To find the location of the resultant take the moments of all forces about A, we get

Moment of resultant about A=*Algebraic sum of moments of all forces about A*

or $$R\times x=-50\times AC+67.5\times AD-10\times AB$$

(Taking clockwise moment −ve and anti-clockwise moment +ve)

or $$(-60)\times x=50\times 1+67.5\times 2-10\times 3.5 \qquad (\because R=-60)$$

or $$-60x=-150+135-35=-50$$

$\therefore$ $$x=\frac{-50}{-60}=\mathbf{0.833\ m.}\quad \textbf{Ans.}$$

Hence the given system of parallel forces is equivalent to a single force 60 N acting vertically downwards at point E at a distance of 0.833 m from A shown in Fig. 3.16 (*a*).

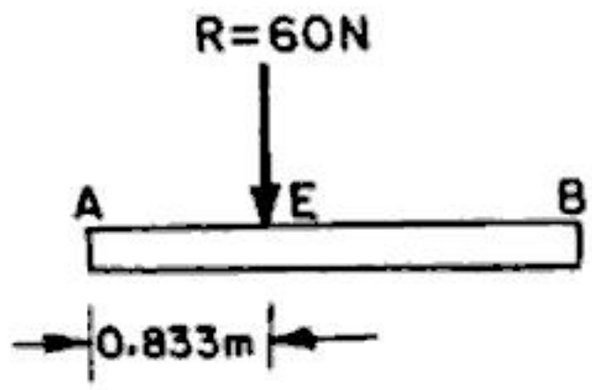

Fig. 3.16 (*a*)

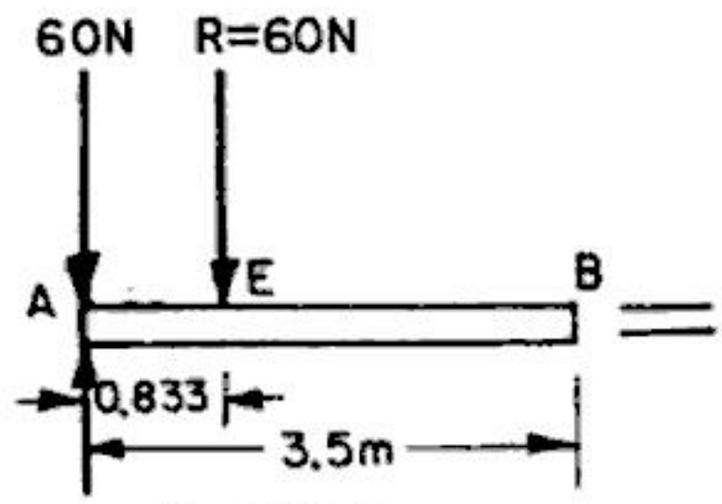

Fig. 3.16 (*b*)

(*ii*) *A single force and a couple at A*. The resultant force R acting at point E as shown in Fig. 3.16 (*a*) can be replaced by an equal force applied at point A in the same direction together with a couple. This is shown in Fig. 3.16 (*c*).

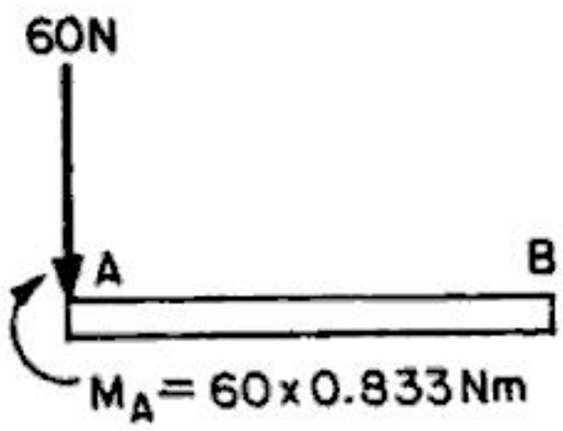

Fig. 3.16. (*c*)

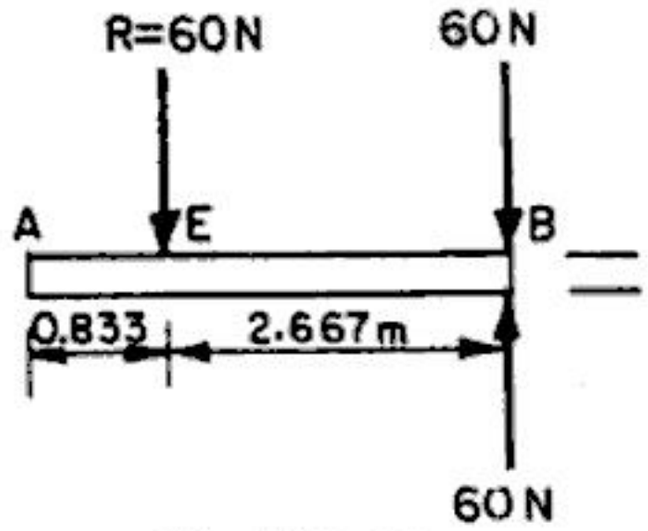

Fig. 3.16. (*d*)

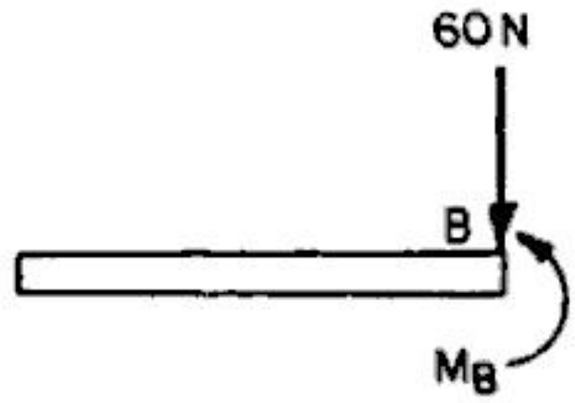

Fig. 3.16. (*e*)

The moment of the couple=60×0.833 Nm (clockwise)

=**−49.98 Nm. Ans.** (−ve sign is due to clockwise)

(*iii*) *A single force and a couple at B*. First find distance BE. But from Fig. 3.16 (*b*), the distance $BE=AB-AE=3.5-0.833=2.667$ m.

Now if the force $R=60$ N is moved to the point B, it will be accompanied by a couple of moment $60\times BE$ or 60×2.667 Nm. This is shown in Fig. 3.16 (*e*).

The moment of the couple=60×2.667 Nm (anti-clockwise)

=**160 Nm. Ans.**

3.7. GENERAL CASE OF PARALLEL FORCES IN A PLANE

Fig. 3.17 shows a number of parallel forces acting on a body in one plane. The forces F_1, F_2 and F_4 are acting in one direction, whereas the forces F_3 and F_5 are acting in the opposite direction. Let R_1=Resultant of forces F_1, F_2 and F_5 and R_2=Resultant of forces F_3 and F_5. The

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resultants R_1 and R_2 are acting in opposite direction and are parallel to each other. Now three important cases are possible.

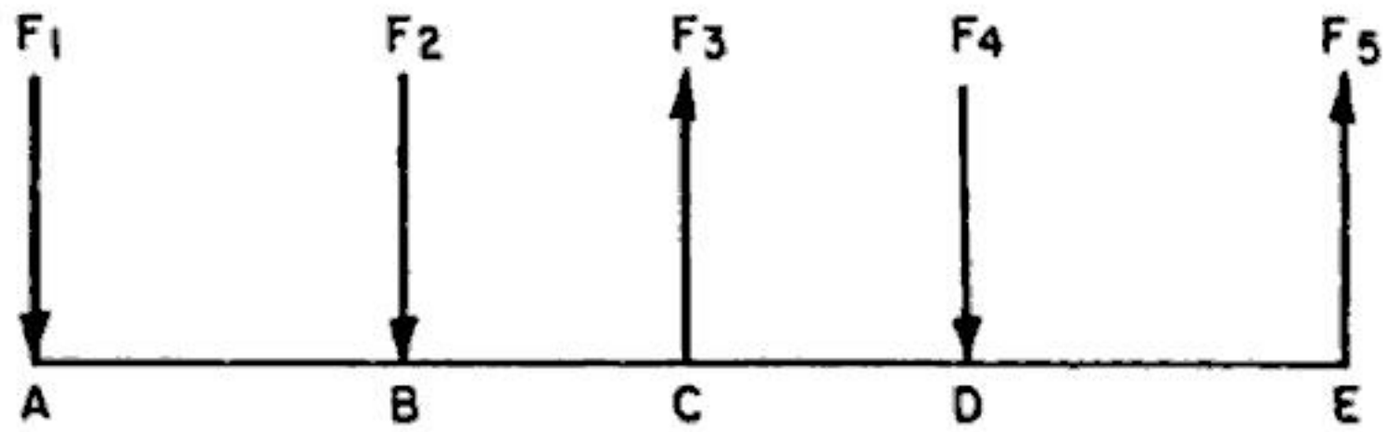

Fig. 3.17.

1. R_1 may not be equal to R_2. Then we shall have two unequal parallel forces (R_1 and R_2) acting in the opposite direction. The resultant R of these two forces (R_1 and R_2) can be easily obtained. The point of application of resultant R can be obtained by equating the moment of R about any point to the algebraic sum of the moments of individual forces about the same point.

2. R_1 is equal to R_2. Then we shall have two equal parallel forces (R_1 and R_2) acting in the opposite direction. The resultant R of these two forces will be zero. Now the system may reduce to a couple or the system is in equilibrium. To distinguish between these two cases, the algebraic sum of moments of all forces (F_1, F_2, ...F_5) about any point is taken. If the sum of moments is not zero, the system reduces a resultant couple. The calculated moment gives the moment of this couple.

3. R_1 is equal to R_2 and sum of moments of all forces (F_1, F_2, F_3, F_4, F_5, ...) about any point is zero, then the system will not be subjected to any resultant couple but the system will be in equilibrium.

Problem 3.7. *Determine the resultant of the parallel force system shown in Fig. 3.18.*

Sol. Given :

Forces at A, B, C, D and E are 4 N, 8 N, 8 N, 16 N and 12 N respectively.

Distances $AB=0.6$ m, $BC=0.9$ m, $CD=1.2$ m, and $DE=0.6$ m

Since all the forces are vertical and parallel, hence their resultant is given by

$$R=-4+8-8+16-12=0$$

As the resultant force on the system is zero, there will be two possibilities. The system has a resultant couple or the system is in equilibrium. To distinguish between these two possibilities, take the sum of moments of all forces about any point. Let us take the moments about point A.

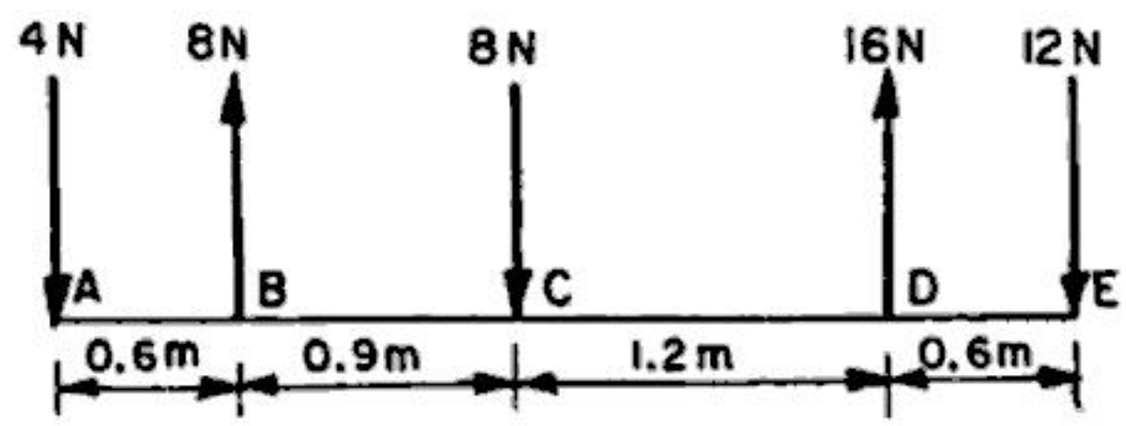

Fig. 3.18.

$\therefore$ Algebraic sum of moments of all forces about A

$$=4\times0+8\times AB-8\times AC+16\times AD-12\times AE$$

$$=0+8\times0.6-8\times(0.6+0.9)+16\times(0.6+0.9+1.2)-12\times(0.6+0.9+1.2+0.6)$$

$$=0+4.8-12+16\times2.7-12\times3.3 \text{ Nm}=4.8-12+43.2-39.6$$

$$=48-51.6=-3.6 \text{ Nm}$$

As the algebraic sum of moments of all forces about any point is not zero, the system will have a resultant couple of magnitude -3.6 Nm *i.e.*, a clockwise couple. **Ans.**

Problem 3.8. *Determine the resultant of the parallel forces acting on a body as shown in Fig. 3.19.*

Sol. Since all the forces are vertical and parallel, hence their resultant is given by

$$R=-20+20+40-30-10=0$$

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Taking moments of all forces about the point A, we get

Resultant moment

$$=20\times0+20\times2+40\times2.5-30\times3-10\times5$$

$$=0+40+100-90-50=140-140=0$$

As the resultant moment is zero and also the resultant force on the body is zero, the body will be in equilibrium. **Ans.**

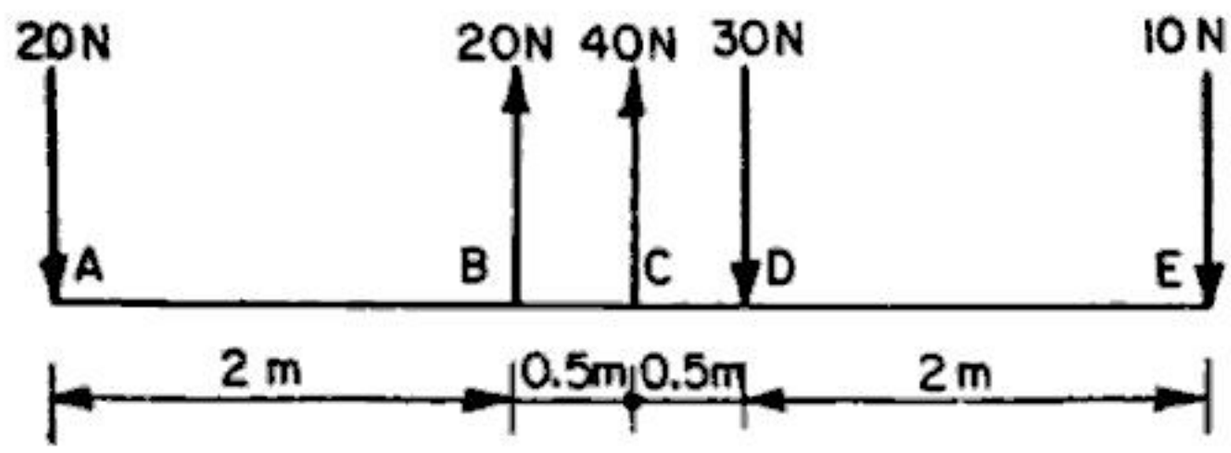

Fig. 3.19.

HIGHLIGHTS

1. Parallel forces are having their lines of action parallel to each other.
2. The moment of a force about any point is the product of force and perpendicular distance between the point and line of action of force.
3. Anticlockwise moment is taken +ve whereas clockwise moment is taken −ve.
4. Varignon's principle states that the moment of a force about any point is equal to the algebraic sum of moments of its components about that point.
5. Like parallel forces are parallel to each other and are acting in the same direction, whereas the unlike parallel forces are acting in opposite direction.
6. The resultant of two like parallel forces is the sum of the two forces and acts at a point between the line in such a way that the resultant divides the distance in the ratio inversely proportional to the magnitudes of the forces.
7. When two equal and opposite parallel forces act on a body at some distance apart, the two forces form a couple which has a tendency to rotate the body. The moment of this couple is the product of either one of the forces and perpendicular distance between the forces.
8. A given force F applied to a body at any point A can always be replaced by an equal force applied at another point B in the same direction together with a couple.
9. If the resultant of a number of parallel forces is not zero, the system can be reduced to a single force, whose magnitude is equal to the algebraic sum of all forces. The point of application of this single force is obtained by equating the moment of this single force about any point to the algebraic sum of moments of all forces acting on the system about the same point.
10. If the resultant of a number of parallel forces is zero, then the system may have a resultant couple or may be in equilibrium. If the algebraic sum of moments of all forces about any point is not zero, then system will have a resultant couple. But if the algebraic sum of moments of all forces about any point is zero, the system will be in equilibrium.

EXERCISE 3

A. Theoretical Questions

1. Define the terms : Coplanar parallel forces, like parallel forces and unlike parallel forces.

2. Define and explain the moment of a force. Differentiate between clockwise moment and anticlockwise moment.

3. (*a*) State the Varignon's principle. Also give the proof of Varignon's principle.

(*b*) Differentiate between :

(*i*) Concurrent and non-concurrent forces,

(*ii*) Coplanar and non-coplanar forces

(*iii*) Moment of a force and couple.

4. Define moment of a force about a point and show that the algebraic sum of the moments of two coplanar forces about a point is equal to the moment of their resultant about that point.

5. What are the different types of parallel forces ? Distinguish between like and unlike parallel forces.

6. Prove that the resultant of two like parallel forces F_1 and F_2 is $F_1 + F_2$. Also prove that the resultant divides the line of joining the points of action of F_1 and F_2 internally in the inverse ratio of the forces.

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7. Prove that in case of two unlike parallel forces the resultant lies outside the line joining the points of action of the two forces and on the same side as the larger force.

8. Describe the method of finding the line of action of the resultant of a system of parallel forces.

9. The resultant of a system of parallel forces is zero, what does it signify ?

10. Describe the method of finding the resultant of two unlike parallel forces which are equal in magnitude.

11. Prove that a given force F applied to a body at any point A can always be replaced by an equal force applied at another point B together with a couple.

12. State the principle of moment.

13. Indicate whether the following statements are True or False.

(*i*) Force is an agency which tends to cause motion.

(*ii*) The tension member of a frame work is called a street.

(*iii*) The value of g reduces slightly as we move from poles towards the equator.

(*iv*) Coplanar forces are those which have the same magnitude and direction.

(*v*) A couple consists of two unequal and parallel forces acting on a body, having the same line of action.

(*vi*) A vector diagram of a force represents its magnitude, direction, sense and point of application.

(*vii*) The force of gravitation on a body is called its weight.

(*viii*) The centre of gravity of a body is the point, through which the resultant of parallel forces passes in whatever position may the body be placed.

[**Ans.** (*i*) True (*ii*) False (*iii*) True (*iv*) False (*v*) False (*vi*) False (*vii*) True (*viii*) True.]

B. Numerical Problems

1. Four forces of magnitudes 20 N, 40 N, 60 N and 80 N are acting respectively along the four sides of a square $ABCD$ as shown in Fig. 3.20. Determine the resultant moment about point A.
Each side of square is 2 m. [**Ans.** 200 Nm anti-clockwise]

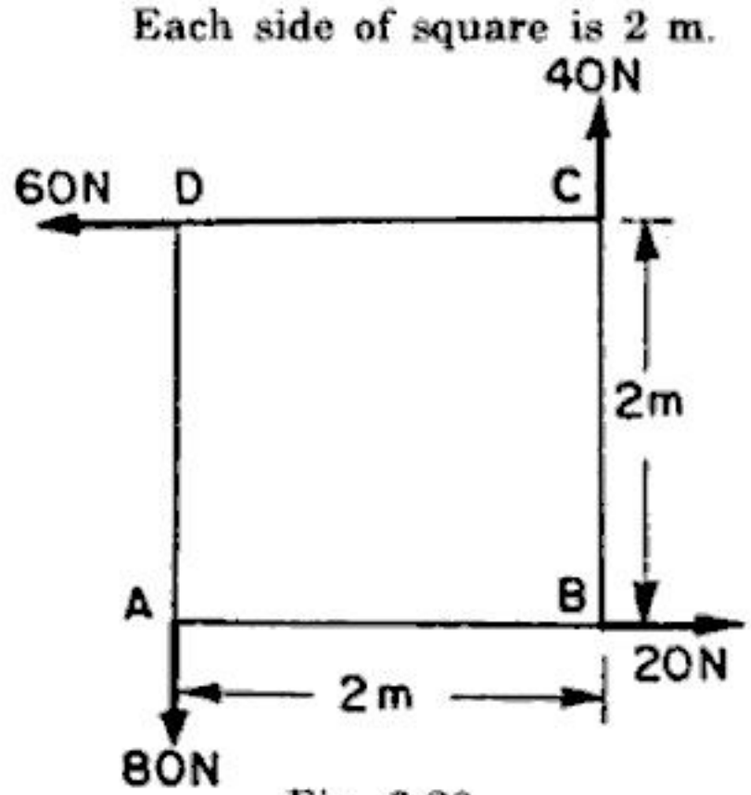

Fig. 3.20.

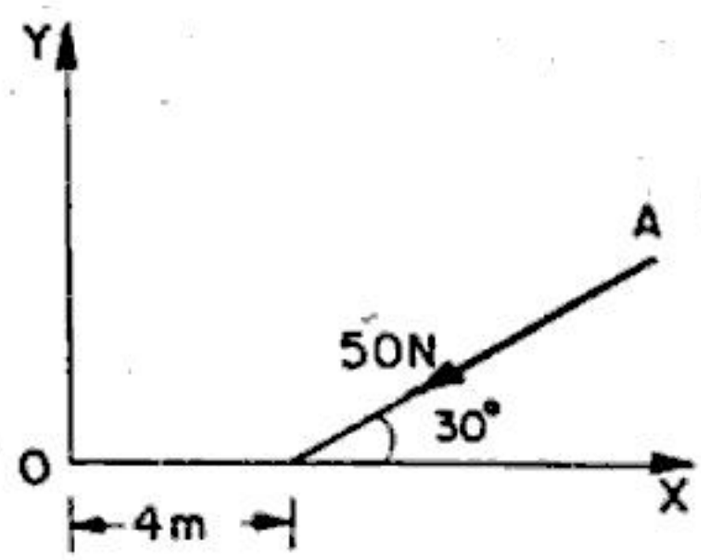

Fig. 3.21

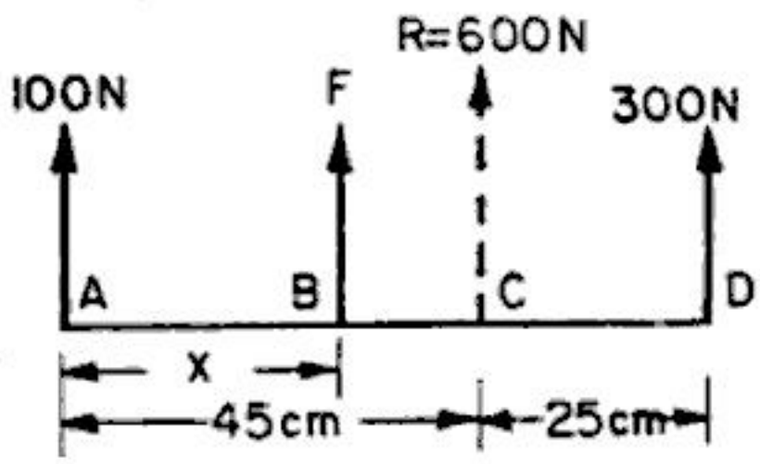

Fig. 3.22

2. A force of 50 N is acting at a point A as shown in Fig. 3.21. Determine the moment of this force about O. [**Ans.** 100 Nm clockwise]

3. Three like parallel forces 20 N, 40 N and 60 N are acting at points A, B, and C respectively on a straight line ABC. The distances are $AB=3$ m and $BC=4$ m. Find the resultant and also the distance of the resultant from point A on line ABC. [**Ans.** 120 N, 4.5 m]

4. The three like parallel forces 101 N, F and 300 N are acting as shown in Fig. 3.22. If the resultant $R=600$ N and is acting at a distance of 45 cm from A, then find the magnitude of force F and distance of F from A. [**Ans.** 200 N, 30 cm]

5. Four parallel forces of magnitudes 100 N, 200 N, 50 N and 400 N are shown in Fig. 3.23. Determine the magnitude of the resultant and also the distance of the resultant from point A.
[**Ans.** $R=350$ N, 3.07 m]

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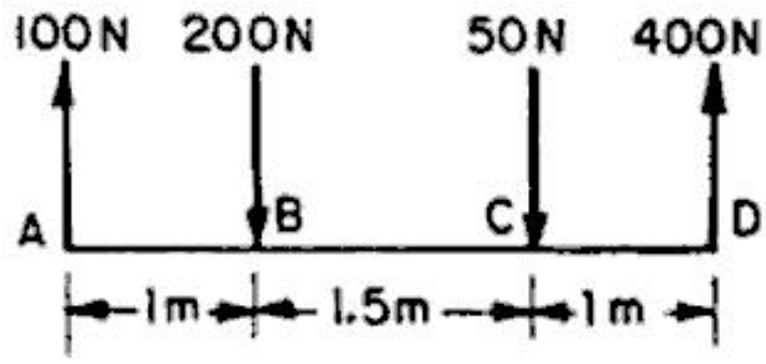

Fig. 3.23.

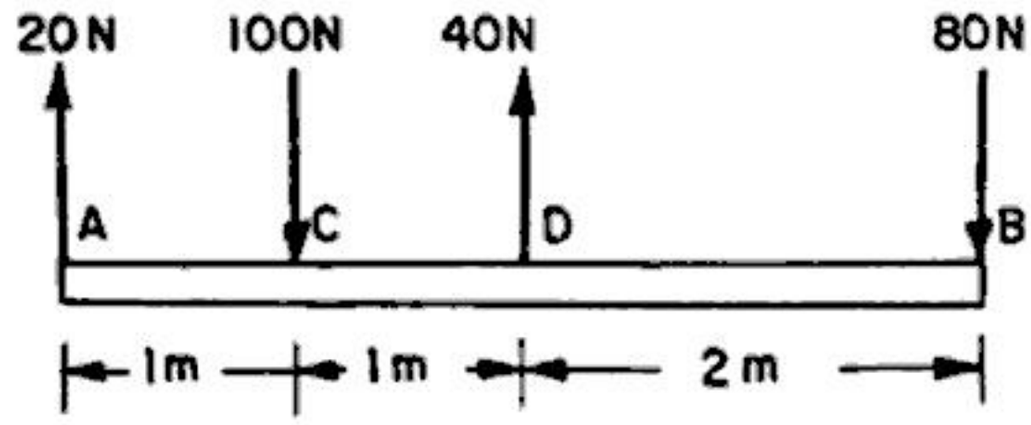

Fig. 3.24.

6. A system of parallel forces are acting on a rigid bar as shown in Fig. 3.24. Reduce this system to :

(*i*) a single force,
(*ii*) a single force and a couple at *A*,
(*iii*) a single force and a couple at *B*. [**Ans.** (*i*) R=120 N at 2.83 m from *A*
(*ii*) R=120 N and M_A = −340 Nm
(*iii*) R=120 N and M_B = 120 Nm]

7. Five forces are acting on a body as shown in Fig. 3.25. Determine the resultant.
[**Ans.** R=0, Resultant couple=10 Nm]

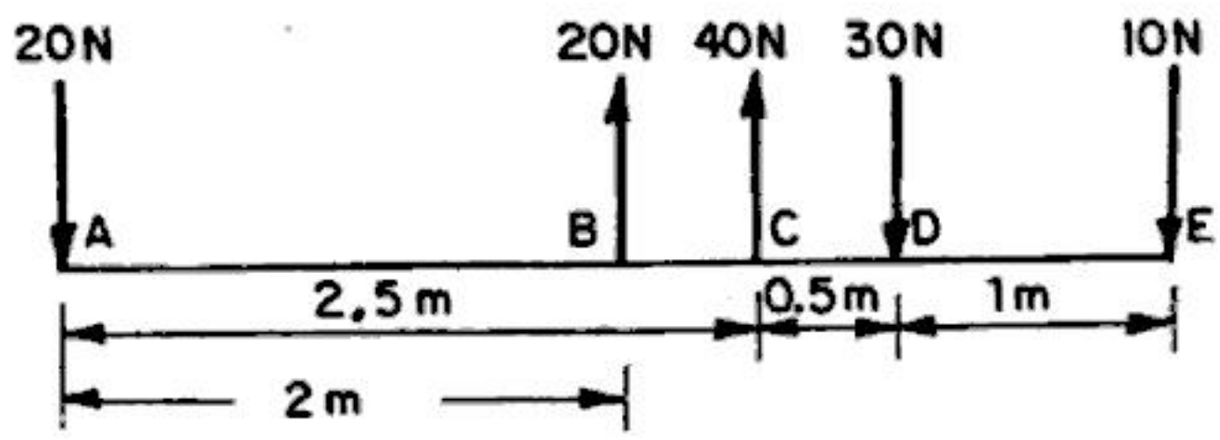

Fig. 3.25.

8. Determine the resultant of the parallel forces shown in Fig. 3.26.
[**Ans.** Body is in equilibrium]

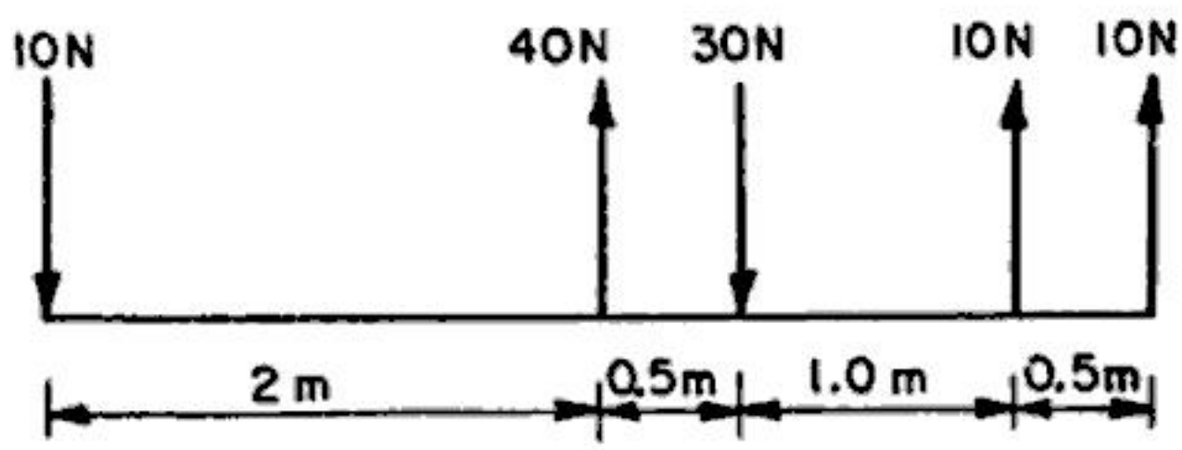

Fig. 3.26.

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4

Conditions of Equilibrium

4.1. INTRODUCTION

When some external forces (which may be concurrent or parallel) are acting on a stationary body, the body may start moving or may start rotating about any point. But if the body does not start moving *and also* does not start rotating about any point, then the body is said to be in equilibrium. In this chapter, the conditions of equilibrium for concurrent forces (*i.e.*, forces meeting at a point) and for parallel forces will be described. Also the concept of free body diagram, different types of support reactions and determination of reactions will be explained.

4.2. PRINCIPLE OF EQUILIBRIUM

The principle of equilibrium states that, a stationary body which is subjected to coplanar forces (concurrent or parallel) will be in equilibrium if the algebraic sum of all the external forces is zero *and also* the algebraic sum of moments of all the external forces about any point in their plane is zero. Mathematically, it is expressed by the equations :

$$\Sigma F=0 \qquad ...(4.1)$$

$$\Sigma M=0 \qquad ...(4.2)$$

The sign Σ is known as sigma which is a Greek letter. This sign represents the *algebraic sum* of forces or moments.

The equation (4.1) is also known as force law of equilibrium whereas the equation (4.2) is known as moment law of equilibrium.

The forces are generally resolved into horizontal and vertical components. Hence equation (4.1) is written as

$$\Sigma F_x=0 \qquad ...(4.3)$$

and

$$\Sigma F_y=0 \qquad ..(4.4)$$

where ΣF_x=Algebraic sum of all horizontal components

and ΣF_y=Algebraic sum of all vertical components.

Hence the conditions of equilibrium are

$$\Sigma F_x=0, \quad \Sigma F_y=0$$

and

$$\Sigma M=0.$$

For the concurrent forces, the lines of action of all forces meet at a point, and hence the moment of those force about that very point will be zero or $\Sigma M=0$ automatically.

Thus for concurrent force system, the condition $\Sigma M=0$ becomes redundant and only two conditions, *i.e.*, $\Sigma F_x=0$ and $\Sigma F_y=0$ are required.

4.3. FORCE LAW OF EQUILIBRIUM

Force law of equilibrium is given by equation (4.1) or by equations (4.3) and (4.4). Let us apply this law to the following important force system :

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(*i*) Two force system

(*ii*) Three force system

(*iii*) Four force system.

4.3.1. Two force system. When a body is subjected to two forces, then the body will be in equilibrium if the two forces are *collinear, equal and opposite* as shown in Fig. 4.1.

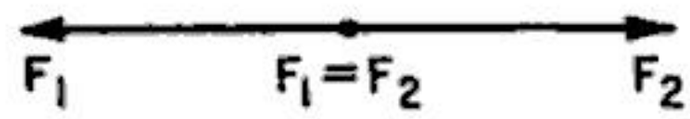

Fig. 4.1.

If the two forces acting on a body are equal and opposite but are parallel, as shown in Fig. 4.1 (*a*), then the body will not be in equilibrium. This is due to the fact that the three conditions of equilibrium will not be satisfied. This is proved as given below :

(*i*) Here $\Sigma F_x=0$ as there is no horizontal force acting on the body. Hence 1st condition of equilibrium is satisfied.

(*ii*) Also here $\Sigma F_y=0$ as $F_1=F_2$.

Hence second condition of equilibrium is also satisfied.

(*iii*) ΣM about any point should be zero. The resultant moment about point A is given by

$$M_A=-F_2\times AB \text{ (−ve sign is due to clockwise moment)}$$

But M_A is not equal to zero. Hence the third condition is not satisfied.

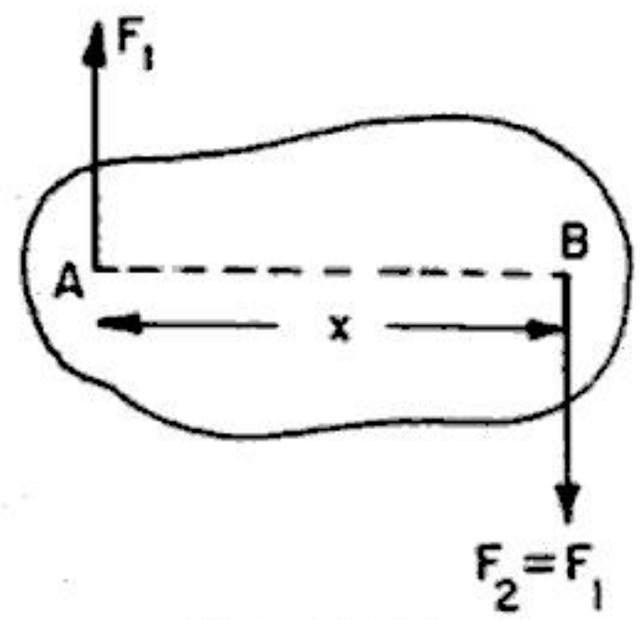

Fig. 4.1. (*a*)

Hence a body will not be in equilibrium under the action of two equal and opposite parallel forces.

Two equal and opposite parallel forces produce a couple and moment of the couple is $-F_1\times AB$ [see Fig. 4.1 (*a*)].

4.3.2. Three force system. The three forces acting on a body which is in equilibrium may be either concurrent or parallel. Let us first consider that the body is in equilibrium when three forces, acting on the body, are concurrent. This is shown in Fig. 4.2.

(*a*) *When three forces are concurrent.* The three concurrent forces F_1, F_2 and F_3 are acting on a body at point O and the body is in equilibrium. The resultant of F_1 and F_2 is given by R. If the force F_3 is collinear, equal and opposite to the resultant R, then the body will be in equilibrium. The force F_3 which is equal and opposite to the resultant R is known as *equilibrant*. Hence for three concurrent forces acting on a body when the body is in equilibrium, the resultant of the two forces should be equal and opposite to the third force.

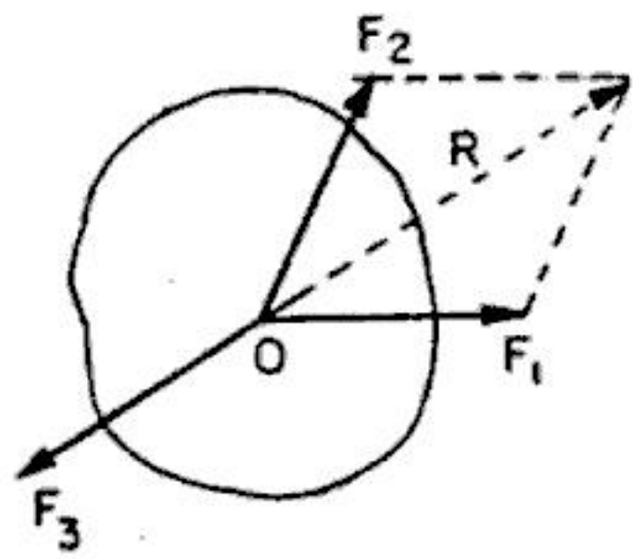

Fig. 4.2.

(*b*) *When three forces are parallel.* Fig. 4.3 shows a body on which three parallel forces F_1, F_2 and F_3 are acting and the body is in equilibrium. If three forces F_1, F_2 and F_3 are acting in the same direction, then there will be a resultant $R=F_1+F_2+F_3$ and body will not be in equilibrium. The three forces are acting in opposite direction and their magnitude is so adjusted that there is no resultant force and body is in equilibrium. Let us suppose that F_2 is acting in opposite direction as shown in Fig. 4.3.

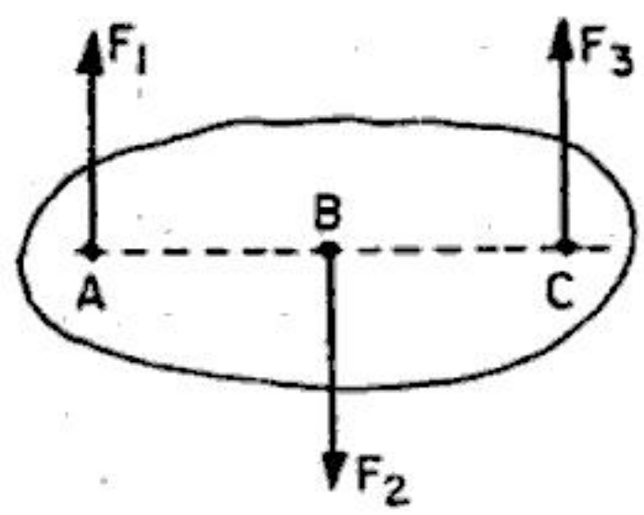

Fig. 4.3.

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Now let us apply the three conditions of equilibrium :

(*i*) $\Sigma F_x=0$ as there is no horizontal force acting on the body

(*ii*) $\Sigma F_y=0$ *i.e.*, $F_1+F_3=F_2$

(*iii*) $\Sigma M=0$ about any point.

Taking the moments of F_1, F_2 and F_3 about point A,

$$\Sigma M_A=-F_2\times AB+F_3\times AC$$

(Moment of F_3 is anti-clockwise whereas moment of F_2 is clockwise)

For equilibrium, ΣM_A should be zero

i.e., $$-F_2\times AB+F_3\times AC=0$$

If the distances AB and AC are such that the above equation is satisfied, then the body will be in equilibrium under the action of three parallel forces.

4.3.3. Four force system. The body will be in equilibrium if the resultant force in horizontal direction is zero (*i.e.* $\Sigma F_x=0$), resultant force in vertical direction is zero (*i.e.*, $\Sigma F_y=0$) and moment of all forces about any point in the plane of forces is zero (*i.e.*, $\Sigma M=0$).

Problem 4.1. *Two forces F_1 and F_2 are acting on a body and the body is in equilibrium. If the magnitude of the force F_1 is 100 N and is acting at O along x-axis as shown in Fig. 4.4, then determine the magnitude and direction of force F_2.*

Sol. Given :

Force, $F_1=100$ N

The body is in equilibrium under the action of two forces F_1 and F_2.

When two forces are acting on a body and the body is in equilibrium, then the two forces should be collinear, equal and opposite.

$\therefore$ $F_2=F_1=100$ N

The force F_2 should pass through O, and would be acting in the opposite direction of F_1.

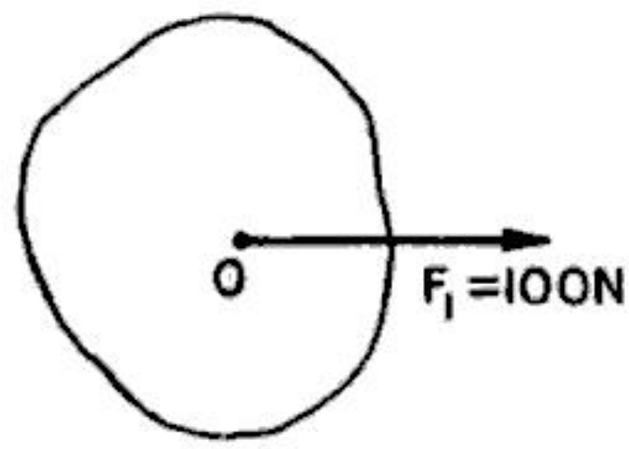

Fig. 4.4

Problem 4.2. *Three forces F_1, F_2 and F_3 are acting on a body as shown in Fig. 4.5 and the body is in equilibrium. If the magnitude of force F_3 is 400 N, find the magnitudes of force F_1 and F_2.*

Sol. Given :

Force, $F_3=400$ N.

As the body is in equilibrium, the resultant force is x-direction should be zero and also the resultant force in y-direction should be zero.

(*i*) For $\Sigma F_x=0$, we get

$$F_1 \cos 30^\circ-F_2 \cos 30^\circ=0$$

or $$F_1-F_2=0$$

or $$F_1=F_2 \quad ...(i)$$

(*ii*) For $\Sigma F_y=0$, we get

$$F_1 \sin 30^\circ+F_2 \sin 30^\circ-400=0$$

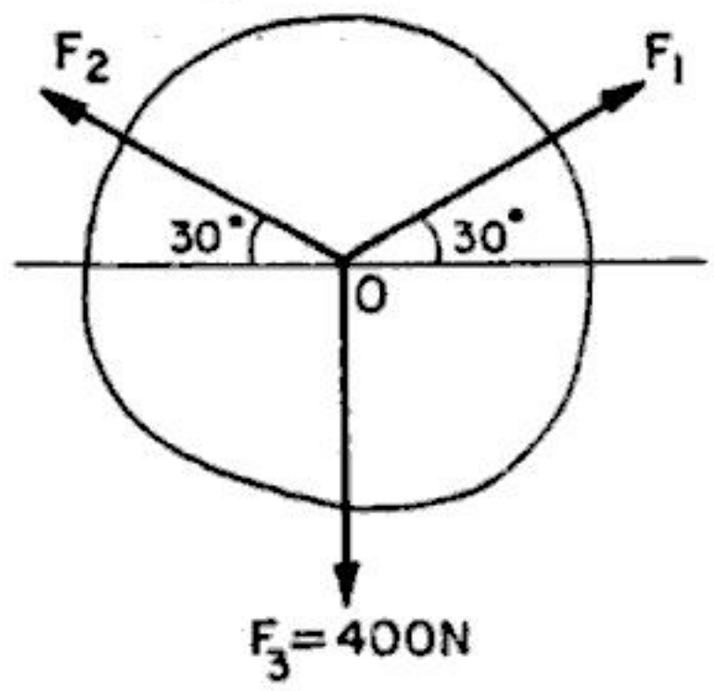

Fig. 4.5.

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or $F_1 \times 0.5 + F_2 \times 0.5 = 400$

or $F_1 \times 0.5 + F_2 \times 0.5 = 400$ $(\because F_2 = F_1)$

or $F_1 = 400$ **N. Ans.**

Also $F_2 = F_1 = 400$ **N. Ans.**

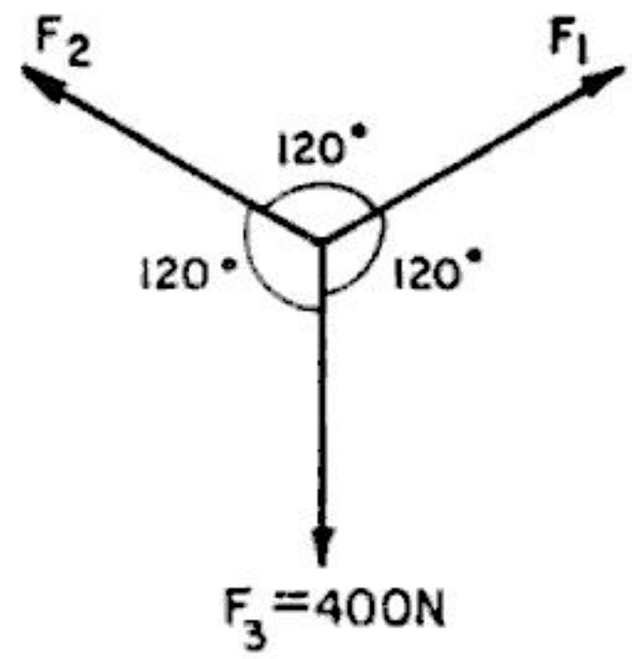

Fig. 4.5. (*a*)

2nd Method

If three forces are acting on a body at a point and the body is in equilibrium, Lami's Theorem can be applied.

Using Lami's theorem,

$$\frac{F_1}{\sin 120^\circ} = \frac{F_2}{\sin 120^\circ} = \frac{400}{\sin 120^\circ}$$

or $F_1 = F_2 = 400$ **N. Ans.**

Problem 4.3. *Three parallel forces F_1, F_2 and F_3 are acting on a body as shown in Fig. 4.6 and the body is in equilibrium. If force $F_1 = 250$ N and $F_3 = 1000$ N and the distance between F_1 and $F_2 = 1.0$ m, then determine the magnitude of force F_2 and the distance of F_2 from force F_3.*

Sol. Given :

Force, $F_1 = 250$ N

Force, $F_3 = 1000$ N

Distance $AB = 1.0$ m

The body is in equilibrium.

Find F_2 and distance BC.

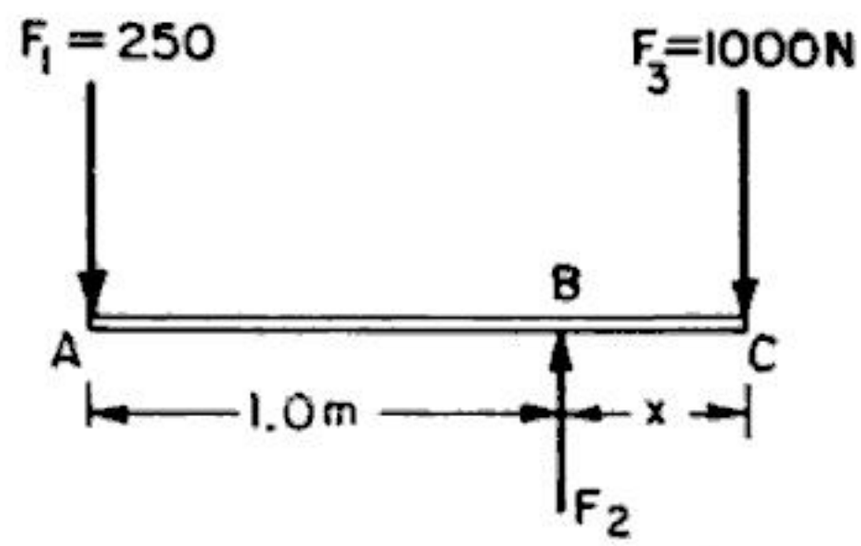

Fig. 4.6.

For the equilibrium of the body, the resultant force in the vertical direction should be zero (here there is no force in horizontal direction).

$\therefore$ For $\Sigma F_y = 0$, we get

$$F_1 + F_3 - F_2 = 0$$

or $$250 + 1000 - F_2 = 0$$

or $$F_2 = 250 + 1000 = \mathbf{1250\ N.\quad Ans.}$$

For the equilibrium of the body, the moment of all forces about any point must be zero.

Taking moments of all forces about point A and considering distance $BC = x$, we get

$$F_2 \times AB - AC \times F_3 = 0$$

or $$1250 \times 1.0 - (1 + x) \times 1000 = 0 \qquad (\because AC = AB + BC = 1 + x)$$

or $$1250 - 1000 - 1000x = 0$$

or $$250 = 1000x$$

or $$x = \frac{250}{1000} = \mathbf{0.25\ m.\quad Ans.}$$

Problem 4.4. *The five forces F_1, F_2, F_3, F_4 and F_5 are acting at a point on a body as shown in Fig. 4.7 and the body is in equilibrium. If $F_1 = 18$ N, $F_2 = 22.5$ N, $F_3 = 15$ N and $F_4 = 30$ N, find the force F_5 in magnitude and direction.*

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Sol. Given :

Forces, $F_1=18$ N, $F_2=2.25$ N,

$F_3=15$ N and $F_4=30$ N.

The body is in equilibrium. Find force F_5 in magnitude and direction. This problem can be solved analytically and graphically.

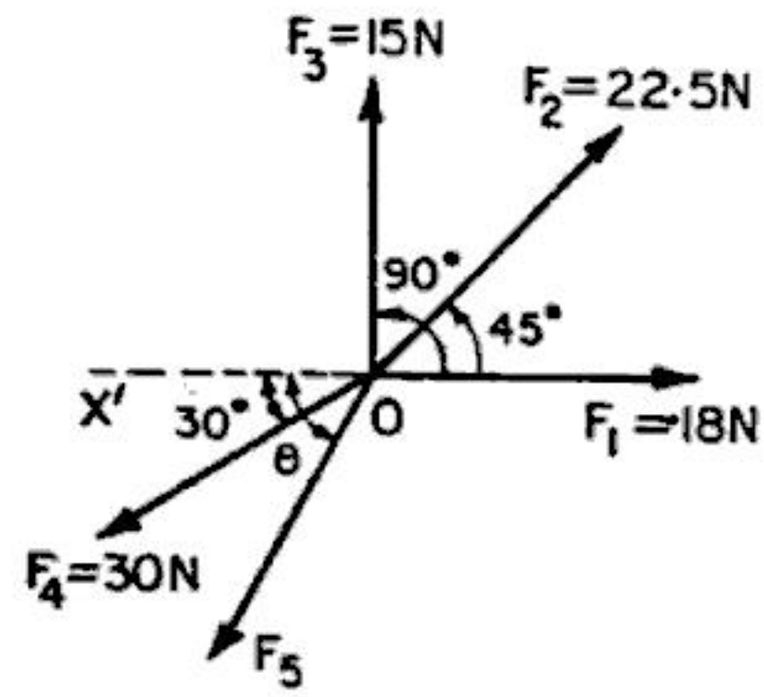

Fig. 4.7.

1. Analytical Method

Let θ=Angle made by force F_5 with horizontal axis $O-X'$.

As the body is in equilibrium, the resultant force in x-direction and y-direction should be zero.

(*i*) For $\Sigma F_x=0$, we get

$$F_1+F_2 \cos 45°-F_4 \cos 30°-F_5 \cos \theta=0$$

or $\quad 18+22.5\times0.707-30\times0.866-F_5 \cos \theta=0$

or $\quad 18+15.9-25.98-F_5 \cos \theta=0$

or $\quad F_5 \cos \theta=18+15.9-25.98$

or $\quad F_5 \cos \theta=7.92 \qquad ...(i)$

(*ii*) For $\Sigma F_y=0$, we get

$$F_2 \sin 45°+F_3-F_4 \sin 30°-F_5 \sin \theta=0$$

or $\quad 22.5\times0.707+15-30\times0.5-F_5 \sin \theta=0$

or $\quad 15.9+15-15-F_5 \sin \theta=0$

or $\quad F_5 \sin \theta=15.9 \qquad ...(ii)$

Dividing equation (*ii*) by equation (*i*), we get

$$\frac{F_5 \sin \theta}{F_5 \cos \theta}=\frac{15.9}{7.92}$$

or $\quad \tan \theta=2.0075$

$\therefore \quad \theta=\tan^{-1} 2.0075=$ **63.52°. Ans.**

Substituting the value of θ in equation (*i*), we get

$F_5 \cos 63.52°=7.92$

$\therefore \quad F_5=\dfrac{7.92}{\cos 63.52}=$ **17.76 N. Ans.**

2. Graphical Method

(*i*) First draw a space diagram with given four forces F_1, F_2, F_3 and F_4 at correct angles as shown in Fig. 4.8 (*a*).

(*ii*) Now choose a suitable scale, say 1 cm=5 N for drawing a force diagram. Take any point O in the force diagram as shown in Fig. 4.8 (*b*).

(*iii*) Draw line Oa parallel to force F_1 and cut $Oa=F_1=18$ N to the same scale.

(*iv*) From a, draw the line ab parallel to F_2 and cut $ab=F_2=22.5$ N

(*v*) From b, draw the line bc parallel to F_3 and cut $bc=F_3=15$ N

(*vi*) From c, draw the line cd parallel to F_4 and cut $cd=F_4=30$ N

(*vii*) Now join d to O. Then the closing side dO represents the force F_5 in magnitude and direction. Now measure the length dO.

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By measurement, length $dO = 3.55$ cm.

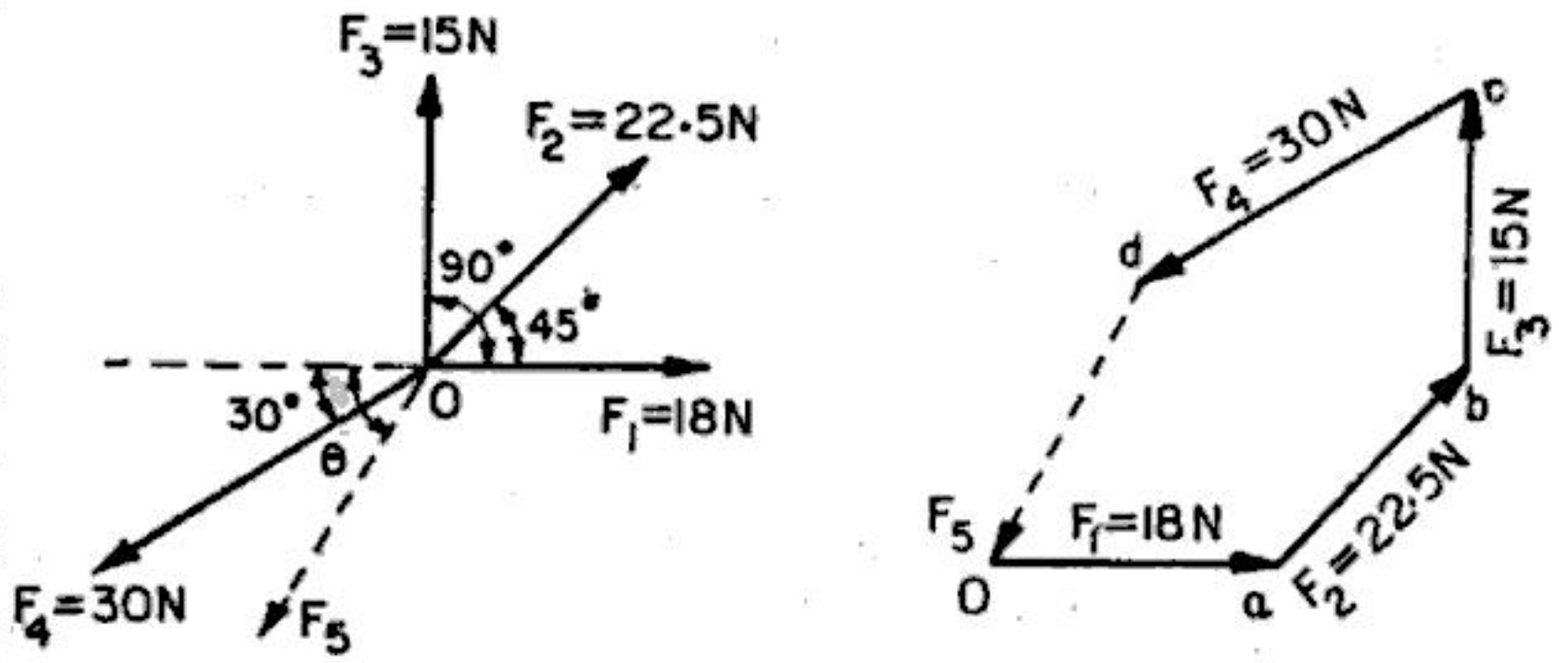
$\therefore$ Force

F_5 = Length $dO \times$ Scale

$= 3.55 \times 5 = 17.75$ N. **Ans.**

The direction is obtained in the space diagram by drawing the force F_5 parallel to line dO.

Measure the angle θ, which is equal to 63.5°. Or the force F_5 is making an angle of $180 + 63.5 = 243.5°$ with the force F_1.



(a) Space diagram (b) Force diagram

Fig. 4.8.

Problem 4.5. *Fig. 4.8 (c) shows the coplanar system of forces acting on a flat plate. Determine : (i) the resultant and (ii) x and y intercepts of the resultant.*

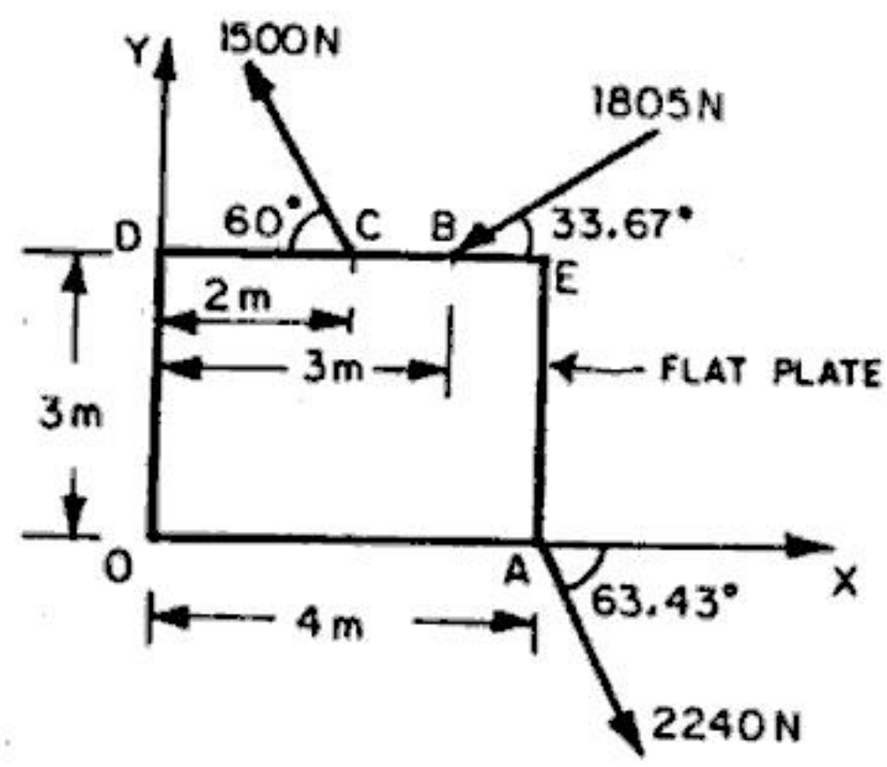


Fig. 4.8 (c)

Sol. Given :

Force at $A = 2240$ N. Angle with x-axis $= 63.43°$

Force at $B = 1805$ N. Angle with x-axis $= 33.67°$

Force at $C = 1500$ N. Angle with x-axis $= 60°$

Lengths $OA = 4$ m, $DB = 3$ m, $DC = 2$ m and $OD = 3$ m

Each force is resolved into X and Y components as shown in Fig. 4.8 (*d*).

(*i*) Force at $A = 2240$ N.

Its X-component $= 2240 \times \cos 63.43° = 1001.9$ N,

Its Y-component $= 2240 \times \sin 63.43° = 2003.4$ N

(*ii*) Force at $B = 1805$ N.

X-component $= 1805 \times \cos 33.67°$

$= 1502.2$ N

Y-component $= 1805 \times \sin 33.67°$

$= 1000.7$ N

(*iii*) Force at $C = 1500$ N.

X-component $= 1500 \times \cos 60° = 750$ N

Y-component $= 1500 \times \sin 60° = 1299$ N

The net force along X-axis,

$$R_x = \Sigma F_x$$
$$= 1001.9 - 1502.2 - 750$$
$$= -1250.3 \text{ N}$$

The net force along Y-axis,

$$R_y = \Sigma F_y$$
$$= -2003.4 - 1000.7 + 1299$$
$$= -1705.1 \text{ N}$$

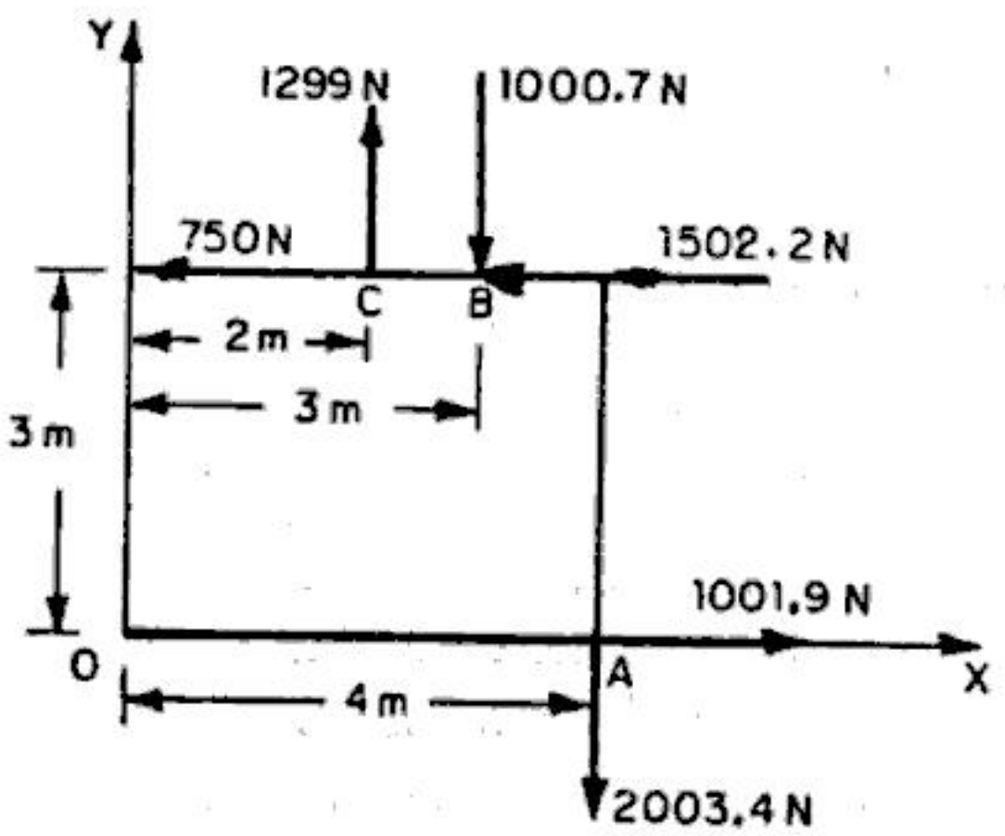


Fig. 4.8 (*d*)

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(i) The resultant force is given by,

$$R=\sqrt{R_x^2+R_y^2}=\sqrt{(-1250.3)^2+(-1705.1)^2}$$
$$=\sqrt{1563250+2907366}=\mathbf{2114.4\ N.}\quad \textbf{Ans.}$$

The angle made by the resultant with x-axis is given by

$$\tan\theta=\frac{R_y}{R_x}=\frac{-1705.1}{-1250.3}=1.363$$

$$\therefore\quad \theta=\tan^{-1}1.363=53.70$$

The net moment* about point O,

$$M_0=2003.4\times4+1000.7\times3-1299\times2-1502.2\times3-750\times3$$
$$=8012.16+3002.1-2598-4506.6-2250=11014.26-9354.6$$
$$=1659.66\ \text{Nm}$$

As the net moment about O is clockwise, hence the resultant must act towards right of origin O, making an angle=53.7° with x-axis as shown in Fig. 4.8 (*e*). The components R_x and R_y are also negative. Hence this condition is also satisfied.

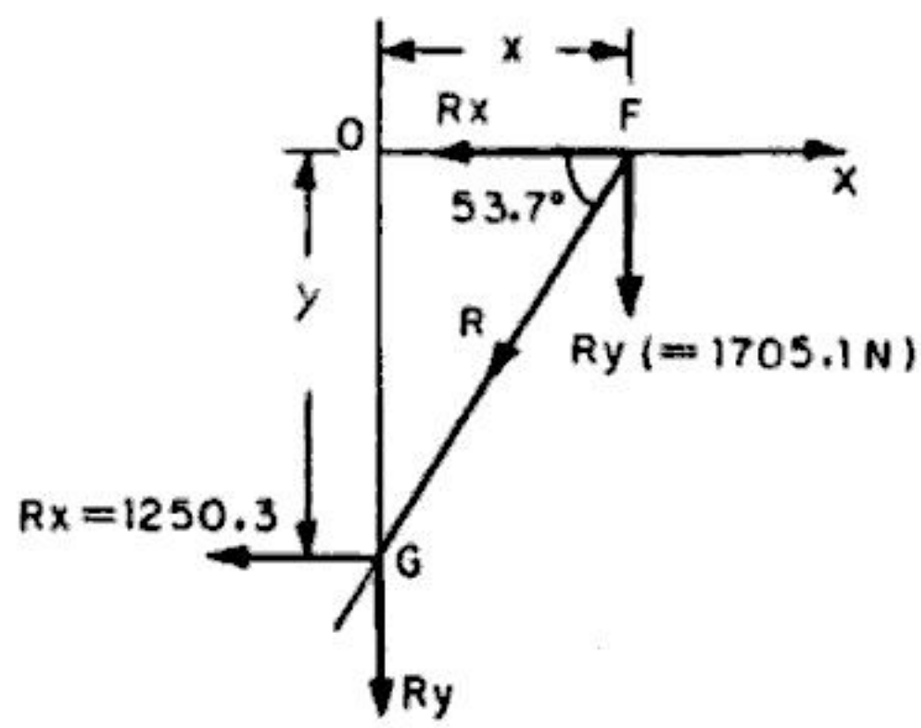

Fig. 4.8 (*e*)

(ii) Intercepts of resultant on x-axis and y-axis [Refer to Fig. 4.8 (*e*)].

Let x=Intercept of resultant along x-axis.

y=Intercept of resultant along y-axis.

The moment of a force about a point is equal to the sum of the moments of the components of the force about the same point. Resolving the resultant (R) into its component R_x and R_y at F.

Moment of R about O=Sum of moments of R_x and R_y at O

But moment of R about O=1659.66 $\qquad (M_0=1659.66)$

$\therefore\quad 1659.66=R_x\times O+R_y\times x$

(as R_x at F passes through O hence it has no moment)

$\therefore\quad 1659.66=1705.1\times x \qquad (\because R_y=1705.1)$

$\therefore\quad x=\dfrac{1659.66}{1705.10}=0.97$ m right of O. **Ans**

To find y-intercept, resolve the resultant R at G into its component R_x and R_y.

$\therefore$ Moment of R about O=Sum of moments of R_x and R_y at O

or $\qquad 1659.66=R_x\times y+R_y\times O.$

(At G, R_y passes through O and hence has no moment)

$\therefore\quad 1659.66=1250.3\times y$

$\therefore\quad y=\dfrac{1659.66}{1250.30}=1.32$ m below O. **Ans.**

---

*Considering clockwise moment positive and anti-clockwise moment as negative. At A, the X component of 1001.9 N passes through O and hence has no moment.

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Problem 4.6. *A lamp weighing 5 N is suspended from the ceiling by a chain. It is pulled aside by a horizontal cord until the chain makes an angle of 60° with the ceiling as shown in Fig. 4.9. Find the tensions in the chain and the cord by applying Lami's theorem and also by graphical method.*

Sol. Given :

Weight of lamp=5 N

Angle made by chain with ceiling=60°

Cord is horizontal as shown in Fig. 4.9.

(i) By Lami's Theorem

Let T_1=Tension (or pull) in the cord

T_2=Tension (or pull) in the chain.

Now from the geometry, it is obvious that angles between T_1 and lamp will be 90°, between lamp and T_2 150° and between T_2 and T_1 120° [Refer to Fig. 4.9 (*b*)].

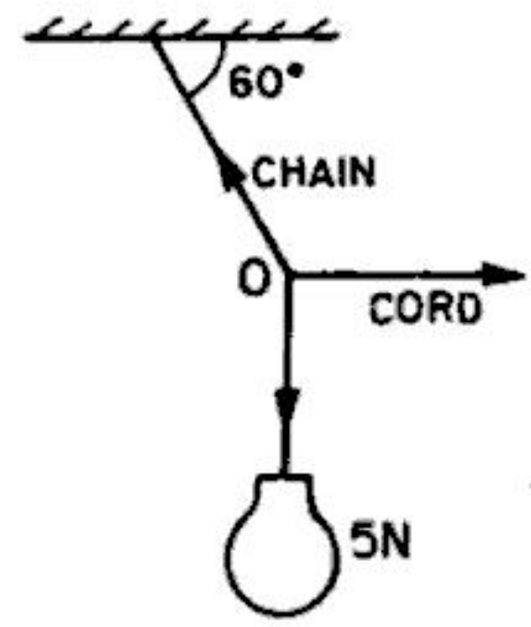

Fig. 4.9 (*a*)

Applying Lami's theorem, we get

$$\frac{T_1}{\sin 150^\circ}=\frac{T_2}{\sin 90^\circ}=\frac{5}{\sin 120^\circ}$$

$$\therefore \quad T_1=5\times\frac{\sin 150^\circ}{\sin 120^\circ}=\mathbf{2.887\ N.} \quad \textbf{Ans.}$$

and $$T_2=5\times\frac{\sin 90^\circ}{\sin 120^\circ}=\mathbf{5.774\ N.} \quad \textbf{Ans.}$$

(ii) By Graphical Method

(1) First draw the space diagram at correct angles as shown in Fig. 4.9 (*b*). Now choose a suitable scale say 1 cm=1 N for drawing a force diagram as shown in Fig. 4.9 (*c*). Take any point O in the force diagram.

(2) From O, draw the line Oa vertically downward to represent the weight of the lamp. Cut Oa=5 N.

(3) From a, draw the line ab parallel to T_2. The magnitude of T_2 is unknown. Now from O, draw the line Ob horizontally (*i.e.*, parallel to T_1) cutting the line ab at point b.

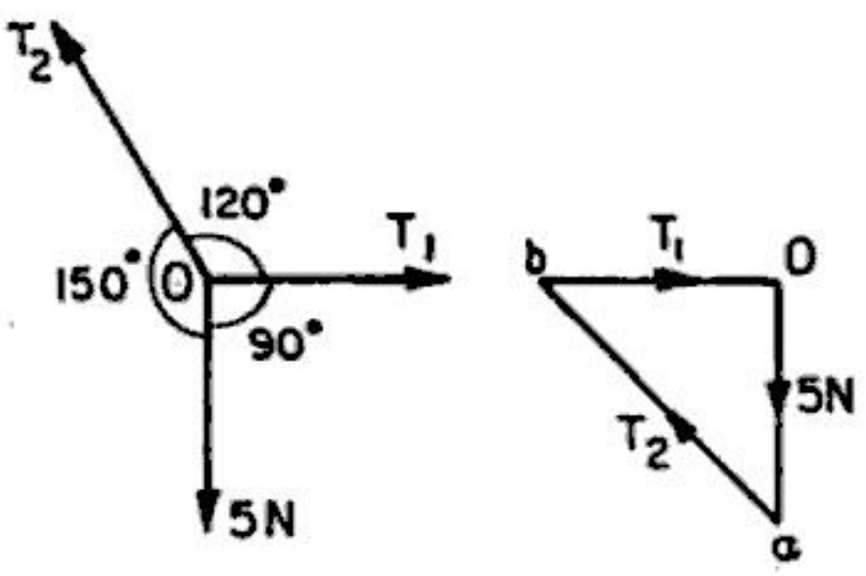

Fig. 4.9.

(4) Now measure the lengths ab and bO.

Then ab represents T_2 and bO represents T_1. By measurements, ab=5.77 cm and bO=2.9 cm.

$\therefore$ Pull in the cord=bO=2.9 cm×scale=2.9×1=**2.9 N. Ans.**

Pull in the chain=ab=5.77 cm×scale=5.77×1=**5.77 N. Ans.**

Problem 4.7. *On a horizontal line PQRS 12 cm long, where PQ=QR=RS=4 cm, forces of 1000, 1500, 1000 and 500 N are acting at P, Q, R and S respectively, all downwards, their lines of action making angles of 90, 60, 45 and 30 degrees respectively with PS. Obtain the resultant of the system completely in magnitude, direction and position graphically and check the answer analytically.*

Sol. Given :

$PQ=QR=RS=4$ cm

Force at P=1000 N. Angle with PS=90°

Force at Q=1500 N. Angle with QS=60°

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Force at $R=1000$ N. Angle with $RS=45°$

Force at $S=500$ N. Angle with $PS=30°$

Graphical Method

Draw the space diagram of the forces as shown in Fig. 4.9 (*d*). The procedure is as follows :

(*i*) Draw a horizontal line $PQRS=12$ cm in which take $PQ=QR=RS=4$ cm.

(*ii*) Draw the line of action of forces P, Q, R, S of magnitude 1000 N, 1500 N, 1000 N and 500 N respectively at an angle of 90°, 60°, 45° and 30° respectively with line PS as shown in Fig. 4.9.

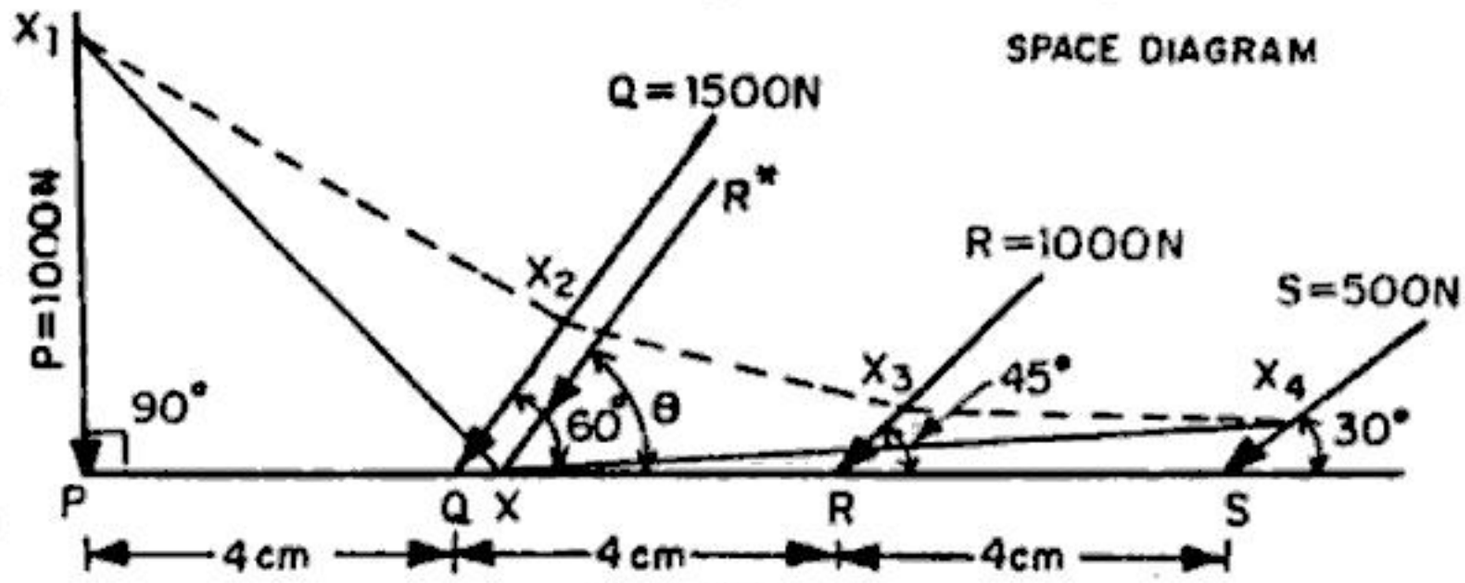

Fig. 4.9. (*d*)

Magnitude and direction of Resultant force (R^*)

To find the magnitude and direction of the resultant force, the force diagram is drawn as shown in Fig. 4.9 (*e*) as given below :

(*i*) Draw the vector *ab* to represent the force 1000 N to a scale of 1 cm =500 N. The vector *ab* is parallel to the line of action of force P.

(*ii*) From point *b*, draw vector $bc=1500$ N and parallel to the line of action of force Q. Similarly the vectors, $cd=1000$ N and parallel to line of action of force R and $de=500$ N and parallel to the line of action of force S, are drawn.

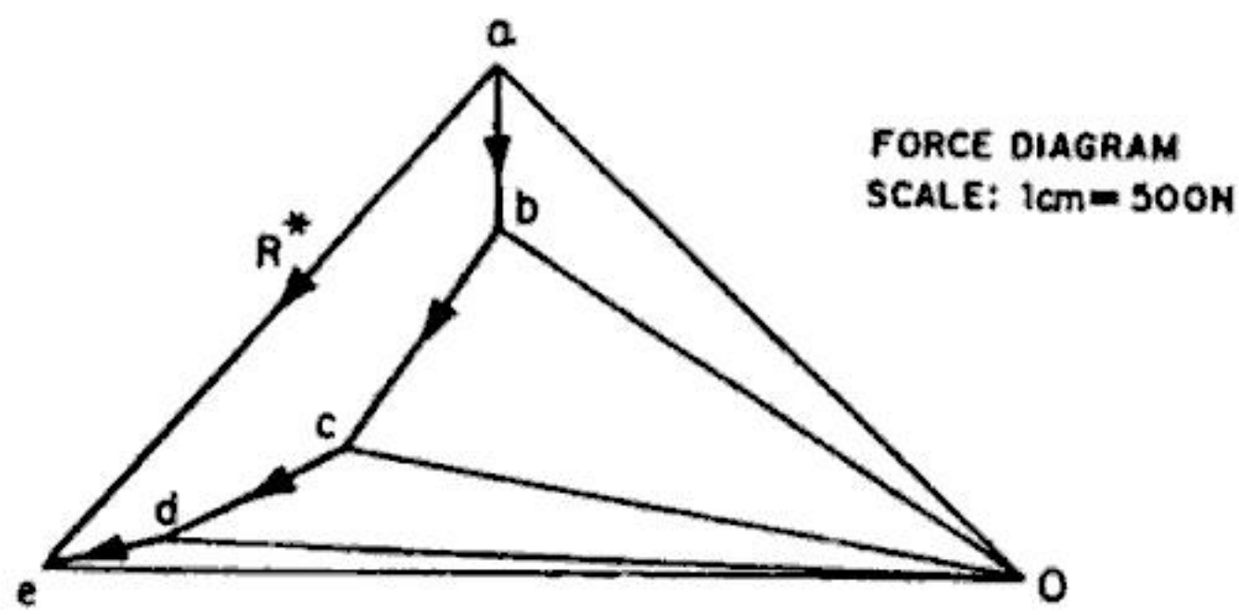

Fig. 4.9. (*e*)

(*iii*) Join *ae* which gives the magnitude of the resultant. Measuring *ae*, the resultant force is equal to 3770 N.

(*iv*) To get the line of action of the resultant, choose any point O on force diagram (called the pole) and join *Oa*, *Ob*, *Oc*, *Od* and *Oe*.

(*v*) Now choose any point X_1 on the line of action of force P and draw a line parallel to *Oa*.

(*vi*) Also from the point X_1, draw another line parallel to *Ob*, which cuts the line of action of force Q at X_2. Similarly from point X_2, draw a line parallel to *Oc* to cut the line of action of force R at X_3. From point X_3, draw a line parallel to *Od* to cut the line of action of force S at X_4.

(*vii*) From point X_4, draw a line parallel *Oe*.

(*viii*) Produce the first line (*i.e.* the line from X_1 and parallel to *Oa*) and the last line (*i.e.* the line from X_4 and parallel to *Oe*) to intersect at X. Then the resultant must pass through this point.

(*ix*) From point X, draw a line parallel to *ae* which determines the line of action of resultant force. Measure PX. By measurements :

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Resultant force, $R^*=3770$ N

Point of action, $PX=4.20$ cm

Direction, $\theta=60°\ 30'$ with PS.

Analytical Method

In analytical method, all the forces acting can be resolved horizontally and vertically. Resultant of all vertical and horizontal forces can be calculated separately and then the final resultant can be obtained.

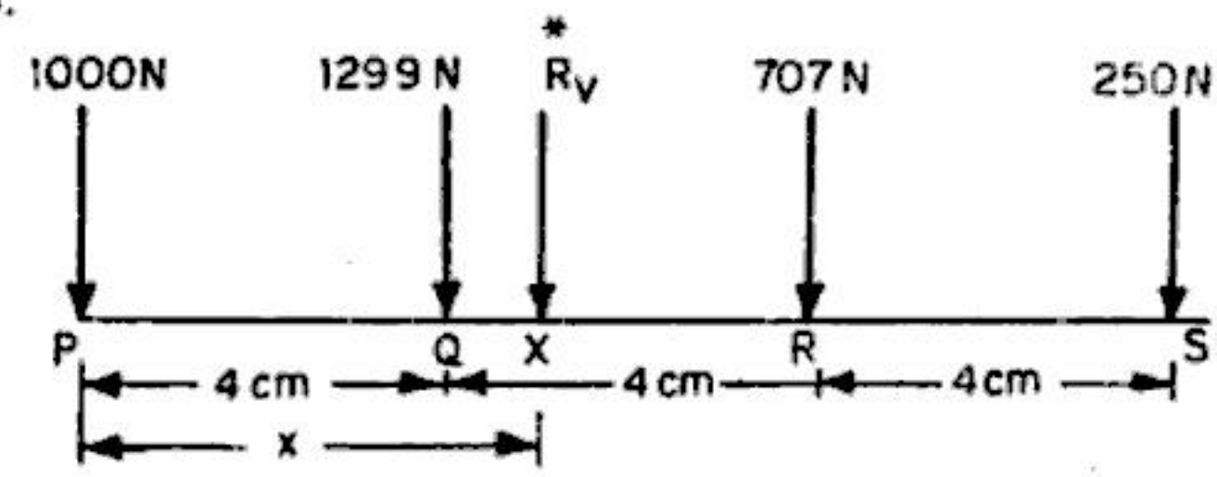

Fig. 4.9 (f)

Resolving all forces and considering the system for vertical forces only.

Vertical force at $P=1000$ N

Vertical force at $Q=1500 \sin 60°=1299$ N

Vertical force at $R=1000 \sin 45°=707$ N

Vertical force at $S=500 \sin 30°=250$ N

Let R_V^*=the resultant of all vertical forces and acting at a distance x cm from P.

$$=1000+1299+707+250=3256 \text{ N}$$

Taking moments of all vertical forces about point P,

$$R_V^* \times x=1299\times4+707\times8+250\times12=13852$$

$$\therefore \quad x=\frac{13852}{R_V^*}$$

$$=\frac{13852}{3256}=4.25 \text{ cm}$$

Now consider the system for horizontal forces only Horizontal force at $P=0$

Horizontal force at $Q=1500\times\cos 60°=750$ N

Horizontal force at $R=1000\times\cos 45°=707$ N

Horizontal force at $S=500\times\cos 30°=433$ N

Resultant of all horizontal forces will be,

$$R_H^*=0+750+707+433=1890 \text{ N}$$

The resultant R^* of R_V^* and R_H^* will also pass through point X which is at a distance of 4.25 cm from P.

$$\therefore \quad R^*=\sqrt{R_V^{*2}+R_H^{*2}}$$

$$=\sqrt{3256^2+1890^2}=3764 \text{ N. Ans.}$$

The resultant will make an angle θ with PS and is given by

$$\tan\theta=\frac{R_V^*}{R_H^*}=\frac{3256}{1890}=1.723$$

$$\therefore \quad \theta=\tan^{-1} 1.723=59.9°$$

Thus the resultant of 3764 N makes an angle 59.9° with PS and passing through point X which is at a distance of 4.25 cm from point P.

This result confirms closely with the values obtained by graphical method.

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4.4. ACTION AND REACTION

From the Newton's third law of motion, we know that to every action there is equal and opposite reaction. Hence reaction is always equal and opposite to the action.

Fig. 4.10 (*a*) shows a ball placed on a horizontal surface (or horizontal plane) such that it is free to move along the plane but cannot move vertically downward. Hence the ball will exert a force vertically downwards at the support as shown in Fig. 4.10 (*b*). This force is known as *action*. The support will exert an equal force vertically upwards on the ball at the point of contact as shown in Fig. 4.10 (*c*).

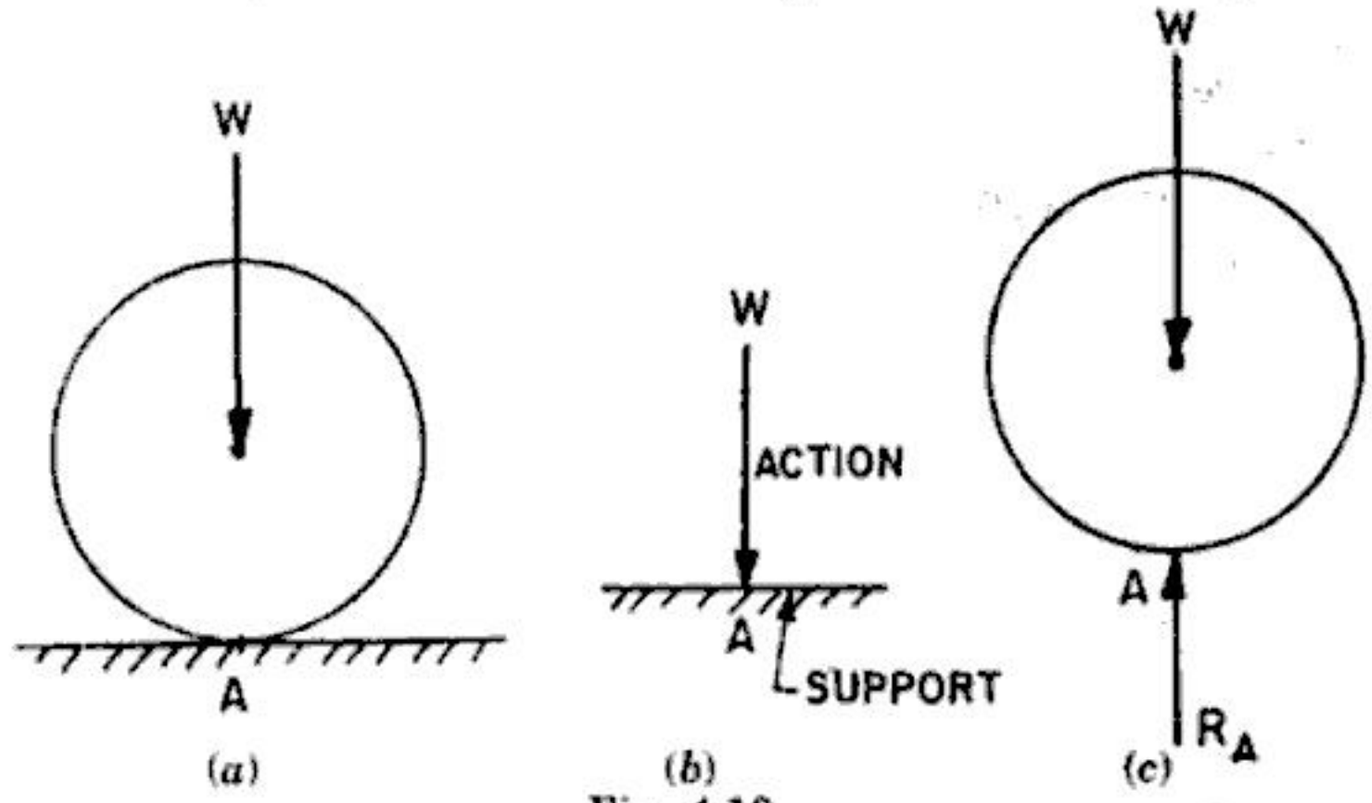

Fig. 4.10.

The force, exerted by the support on the ball, is known as *reaction*. Hence *'any force on a support causes an equal and opposite force from the support so that action and reaction are two equal and opposite forces.'*

4.5. FREE BODY DIAGRAM

The equilibrium of the bodies which are placed on the supports can be considered if we remove the supports and replace them by the reactions which they exert on the body. In Fig. 4.10 (*a*), if we remove the supporting surface and replace it by the reaction R_A that the surface exerts on the balls as shown in Fig. 4.10 (*c*), we shall get *free-body diagram*.

The point of application of the reaction R_A will be the point of contact A, and from the law of equilibrium of two forces, we conclude that the reaction R_A must be vertical and equal to the weight W.

Hence Fig. 4.10 (*c*), in which the ball is *completely isolated* from its support and in which all forces acting on the ball are shown by vectors, is known a free-body diagram. Hence to draw the free-body diagram of a body we remove all the supports (like wall, floor, hinge or any other body) and replace them by the reactions which these support exert on the body. Also the body should be completely isolated.

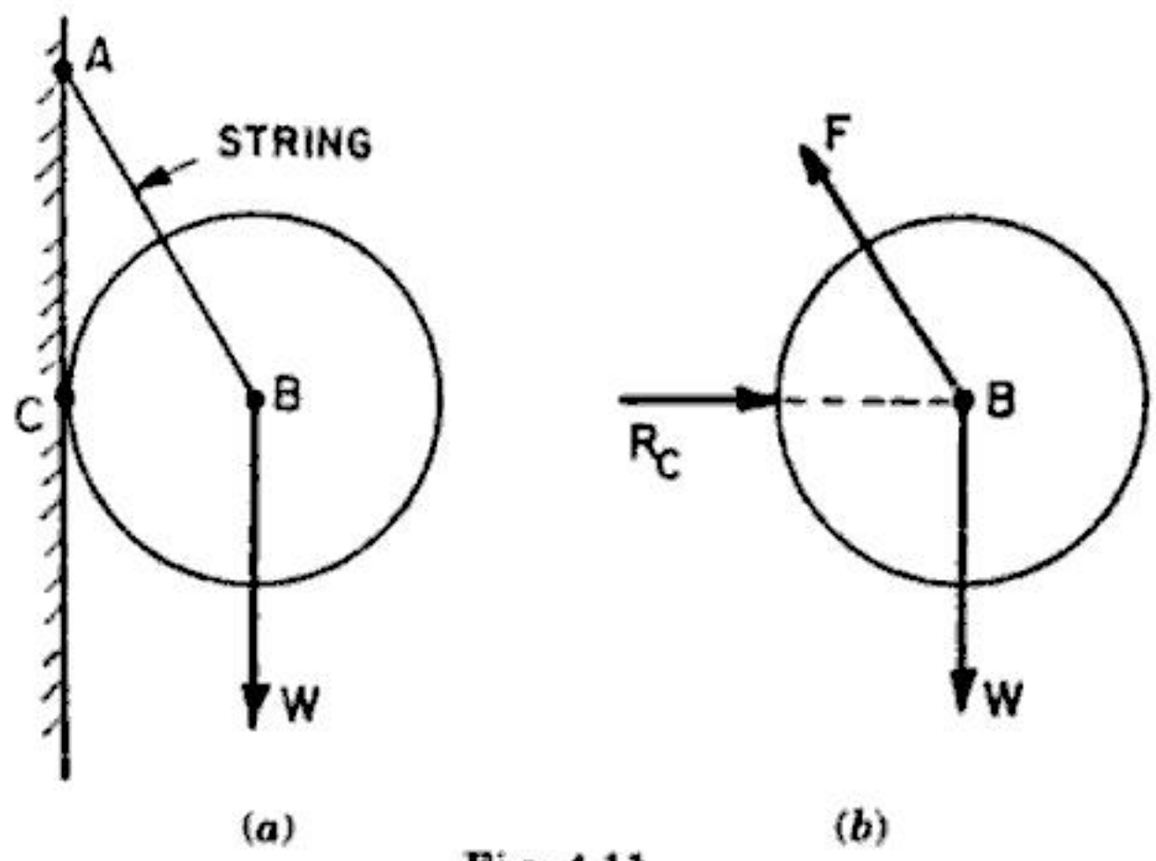

Fig. 4.11.

Problem 4.8. *Draw the free body diagram of ball of weight W supported by a string AB and resting against a smooth vertical wall at C as shown in Fig. 4.11 (a).*

Sol. Given :

Weight of ball $= W$

The ball is supported by a string AB and is resting against a vertical wall at C.

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To draw the free-body diagram of the ball, isolate the ball completely (*i.e.*, isolate the ball from the support and string). Then besides the weight W acting at B, we have two reactive forces to apply one replacing the string AB and another replacing the vertical wall AC. Since the string is attached to the ball at B and since a string can pull only along its length, we have the reactive force F applied at B and parallel to BA. The magnitude of F is unknown.

The reaction R_C will be acting at the point of contact of the ball with vertical wall *i.e.* at point C. As the surface of the wall is *perfectly smooth**, the reaction R_C will be normal to the vertical wall (*i.e.* reaction R_C will be horizontal in this case) and will pass through the point B. The magnitude of R_C is also unknown. The complete free-body diagram is shown in Fig. 4.11 (*b*).

Problem 4.9. *A circular roller of weight 100 N and radius 10 cm hangs by a tie rod AB=20 cm and rests against a smooth vertical wall at C as shown in Fig. 4.12 (a). Determine : (i) the force F in the tie rod, and (ii) the reaction R_C at point C.*

Sol. Given :

Weight of roller, $W=100$ N

Radius of roller, $BC=10$ cm

Length of tie rod, $AB=20$ cm

From ΔABC, we get

$$\sin\theta=\frac{BC}{AB}=\frac{10}{20}=0.5$$

$$\therefore \quad \theta=\sin^{-1} 0.5=30°$$

The free-body diagram of the roller is shown in Fig. 4.12 (*b*) in which

R_C=Reaction at C

F=Force in the tie rod AB

Free-body diagram shows the equilibrium of the roller. Hence the resultant force in x-direction and y-direction should be zero.

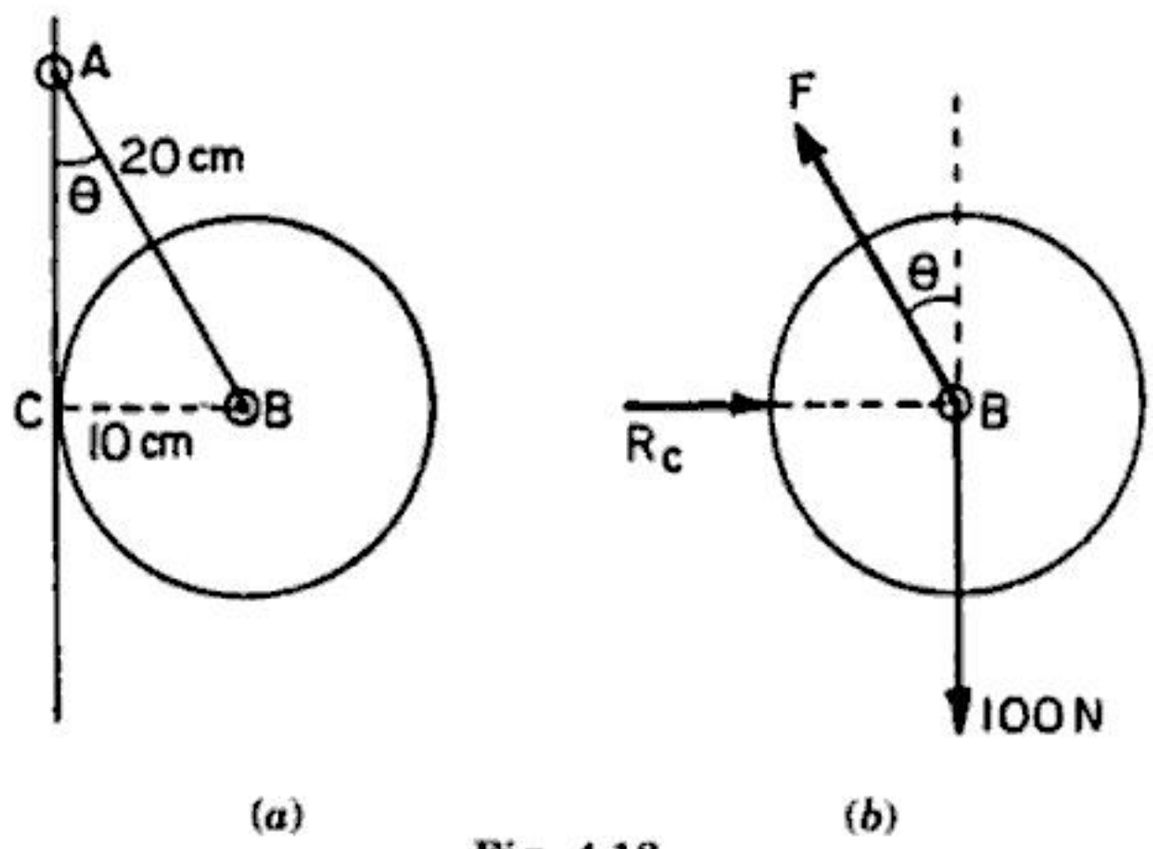

Fig. 4.12.

For $\Sigma F_x=0$, we get

$$R_C-F\sin\theta=0$$

or $$R_C=F\sin\theta \qquad ...(i)$$

For $\Sigma F_y=0$, we get

$$100-F\cos\theta=0$$

or $$100=F\cos\theta$$

or $$F=\frac{100}{\cos\theta}=\frac{100}{\cos 30°} \qquad (\because \theta=30°)$$

$$=115.47 \text{ N.} \quad \textbf{Ans.}$$

Substituting the value of F in equation (*i*),

$$R_C=115.47\times\sin 30°=57.73 \text{ N.} \quad \textbf{Ans.}$$

Problem 4.10. *Draw the free-body diagram of a ball of weight W, supported by a string AB and resting against a smooth vertical wall at C and also resting against a smooth horizontal floor at D as shown in Fig. 4.13 (a).*

---

*The reaction at a perfectly smooth surface is always normal to the surface.

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Sol. Given :

To draw the free-body diagram of the ball, the ball should be isolated completely from the vertical support, horizontal support and string AB. Then the forces acting on the isolated ball as shown in Fig. 4.13 (b), will be :

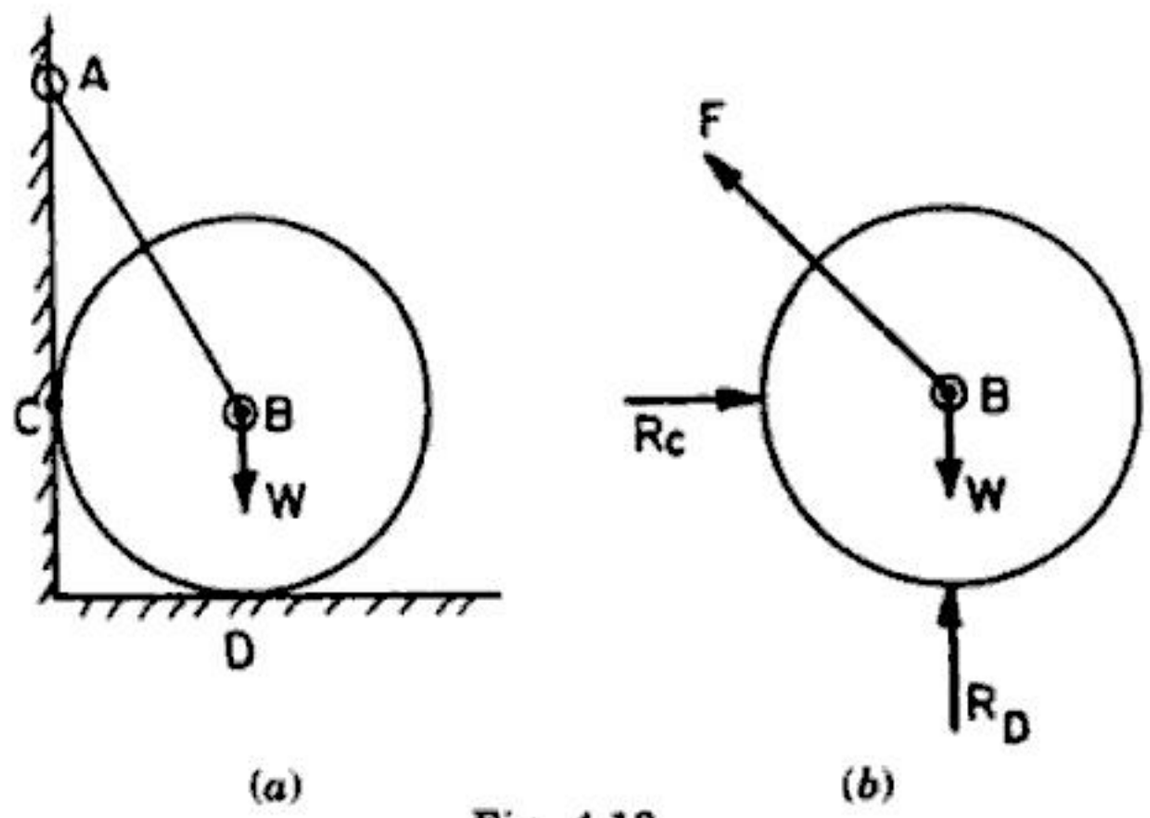
(i) Reaction R_C at point C, normal to AC.

(ii) Force F in the direction of string.

(iii) Weight W of the ball.

(iv) Reaction R_D at point D, normal to horizontal surface.

The reactions R_C and R_D will pass through the centre of the ball *i.e.*, through point B.



(a) (b)

Fig. 4.13.

Problem 4.11. *A ball of weight 120 N rests in a right-angled groove, as shown in Fig. 4.14 (a). The sides of the groove are inclined to an angle of 30° and 60° to the horizontal. If all the surfaces are smooth, then determine the reactions R_A and R_C at the points of contact.*

Sol. Given :

Weight of ball, $W=120$ N

Angle of groove $= 90°$

Angle made by side FD with horizontal$=30°$

Angle made by side ED with horizontal$=60°$

$\therefore$ Angle $FDH=30°$ and angle $EDG=60°$

Consider the equilibrium of the ball. For this draw the free body diagram of the ball as shown in Fig. 4.14 (b).

The forces acting on the isolated ball will be :

(i) Weight of the ball$=120$ N and acting vertically downwards.

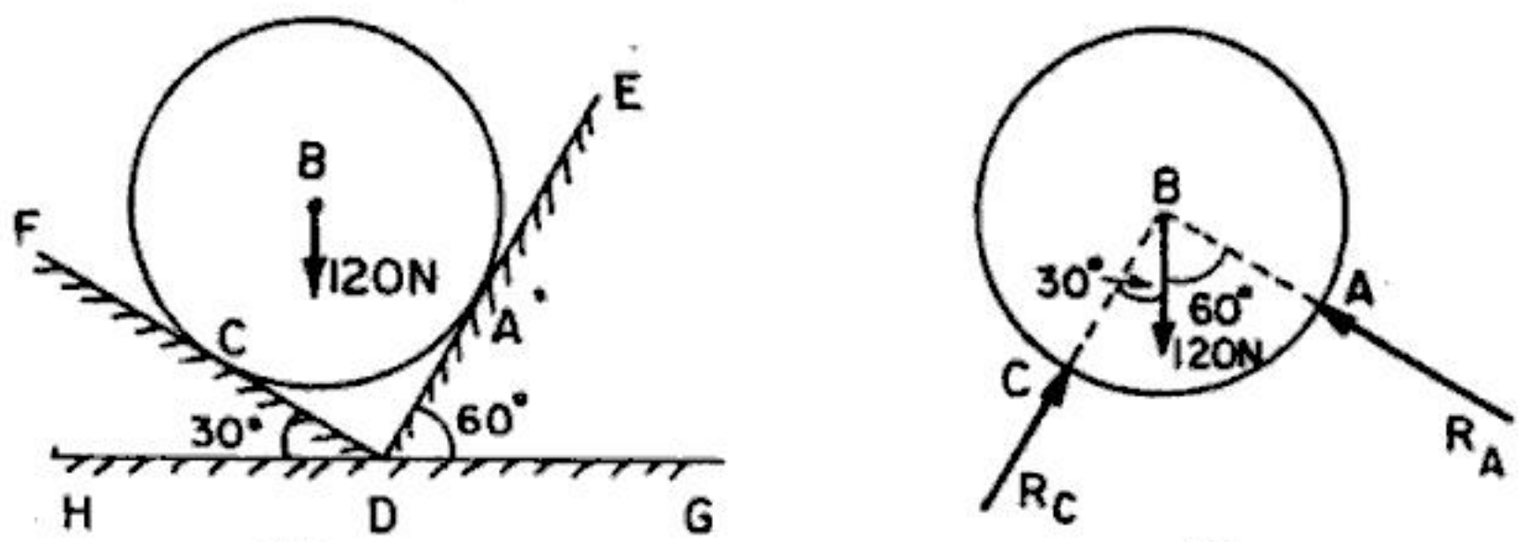
(ii) Reaction R_C acting at C and normal to FD.

(iii) Reaction R_A acting at A and normal to DE.

The reactions R_A and R_C will pass through B, *i.e.*, centre of the ball. The angles made by R_A and R_C at point B will be obtained as shown in Fig. 4.14 (c).

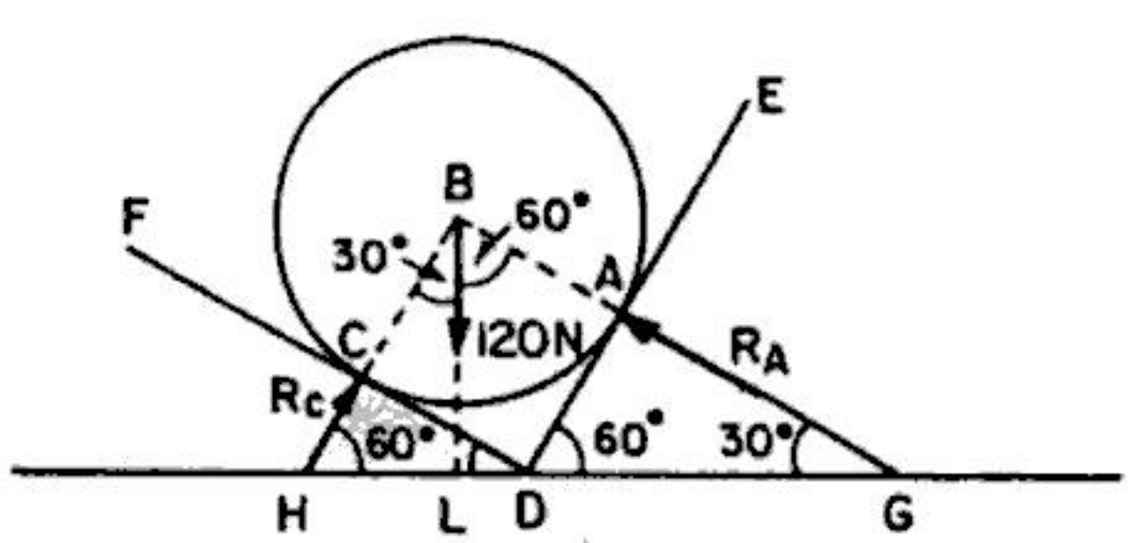
In ΔHDC, $\angle CDH=30°$ and $\angle DCH=90°$. Hence $\angle DHC$ will be 60°. Now in ΔHBL, $\angle BLH=90°$ and angle $LHB=60°$. Hence $\angle HBL$ will be 30°.

Similarly, $\angle GBL$ may be calculated. This will be equal to 60°.



(a) (b)

Fig. 4.14.



(c)

Fig. 4.14.

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equilibrium ? What is the magnitude of the pin reaction at B ? Assume the bars to be weightless and pins at A and B to be smooth.

Sol. Given :

Length $AB=1.6$ m, Length $BD=1.2$ m, $BC=0.8$ m and $CD=0.4$ m

Horizontal force at $C=200$ N

Let M=Couple applied A to bar AB to keep the system in equilibrium.

R_B=Reaction at B

R_{BX} and R_{BY} are the horizontal and vertical components of reaction R_B.

The free-body diagram of the sloping bar BCD is shown in Fig. 4.19 (*b*). The reaction R_D at point D is vertical. Also show R_{BX} and R_{BY} at B.

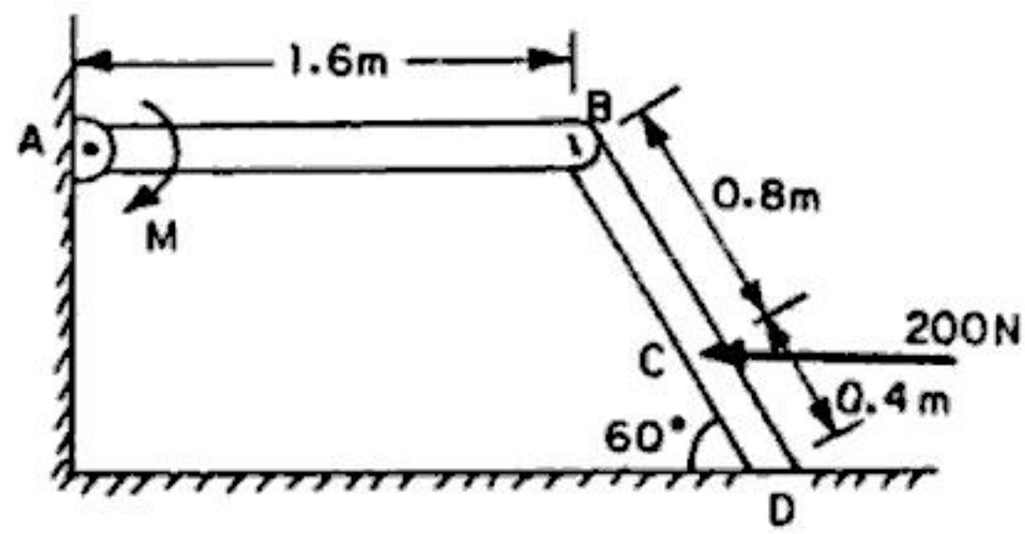

Fig. 4.18.

Since the bar is in equilibrium, first apply $\Sigma F_x=0$, which gives

$$R_{BX}=200 \text{ N}$$

Now apply $\Sigma F_y=0$, then

$$R_{BY}=R_D \quad ...(i)$$

The magnitude of R_{BY} and R_D is unknown. To find their values apply ΣM at the pin B.

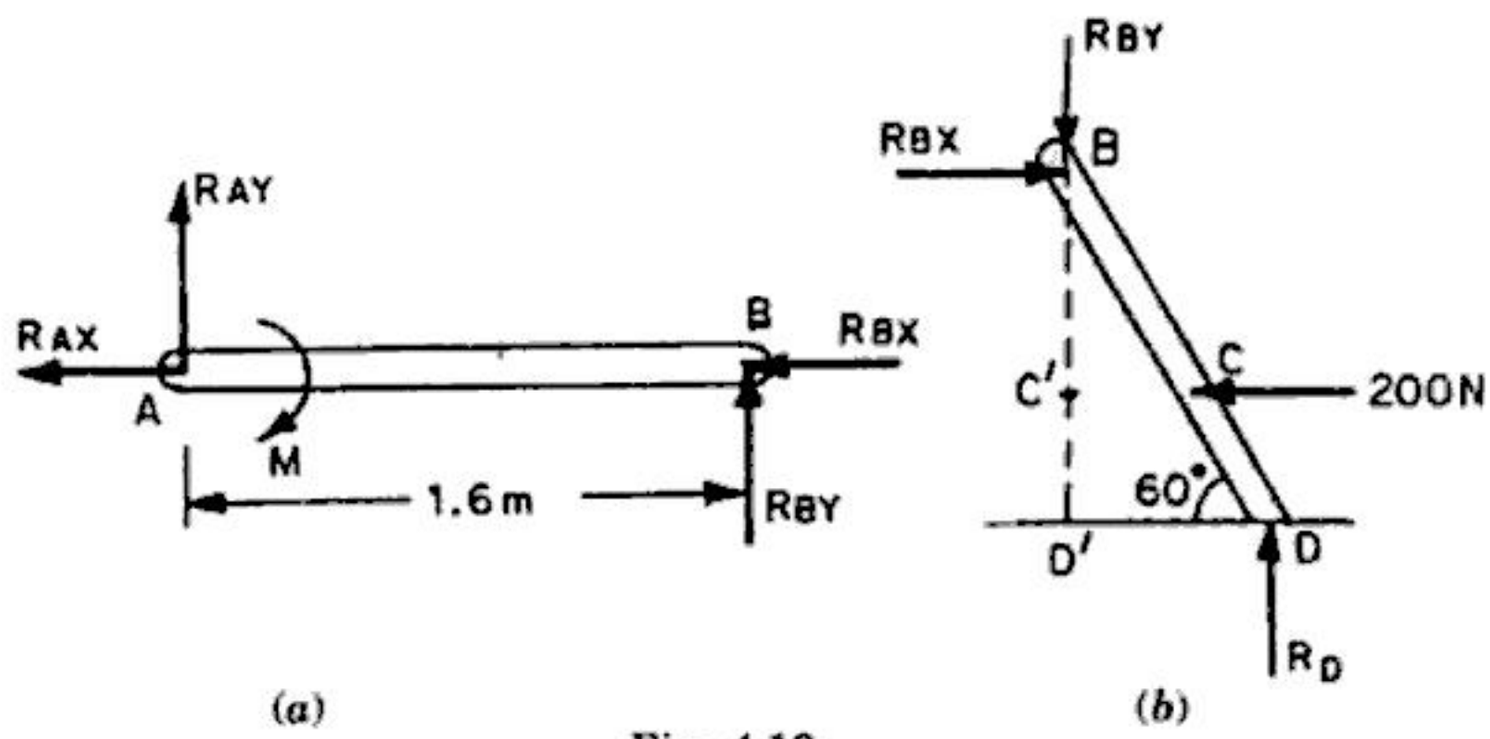

Fig. 4.19.

Taking moments of all forces at point B, we get

$$R_D \times DD' = 200 \times BC'$$

or $$R_D \times BD \cos 60° = 200 \times BC \sin 60° \quad (\because DD' = BD \cos 60° \text{ and } BC' = BC \sin 60°)$$

or $$R_D \times 1.2 \cos 60° = 200 \times 0.8 \sin 60°$$

or $$R_D = \frac{200 \times 0.8 \times 0.866}{1.2 \times \cos 60°} = 230.93 \text{ N}$$

$\therefore$ From equation (*i*),

$$R_{BY} = R_D = 230.93 \text{ N}$$

Reaction at B,

$$R_B = \sqrt{R_{BX}^2 + R_{BY}^2} = \sqrt{200^2 + 230.93^2} = \mathbf{305.4 \text{ N.}} \quad \textbf{Ans.}$$

Now draw the free-body diagram of the horizontal bar AB as shown in Fig. 4.19 (*a*). Show the reactions R_{BX} and R_{BY} at B. Also show the reactions R_{A_y} and R_{A_x} at A. The applied couple M is also shown in the figure.

Apply $\Sigma M_A = 0$

$$M = R_{BY} \times 1.6 = 230.93 \times 1.6$$

$$= \mathbf{369.44 \text{ Nm.}} \quad \textbf{Ans.}$$

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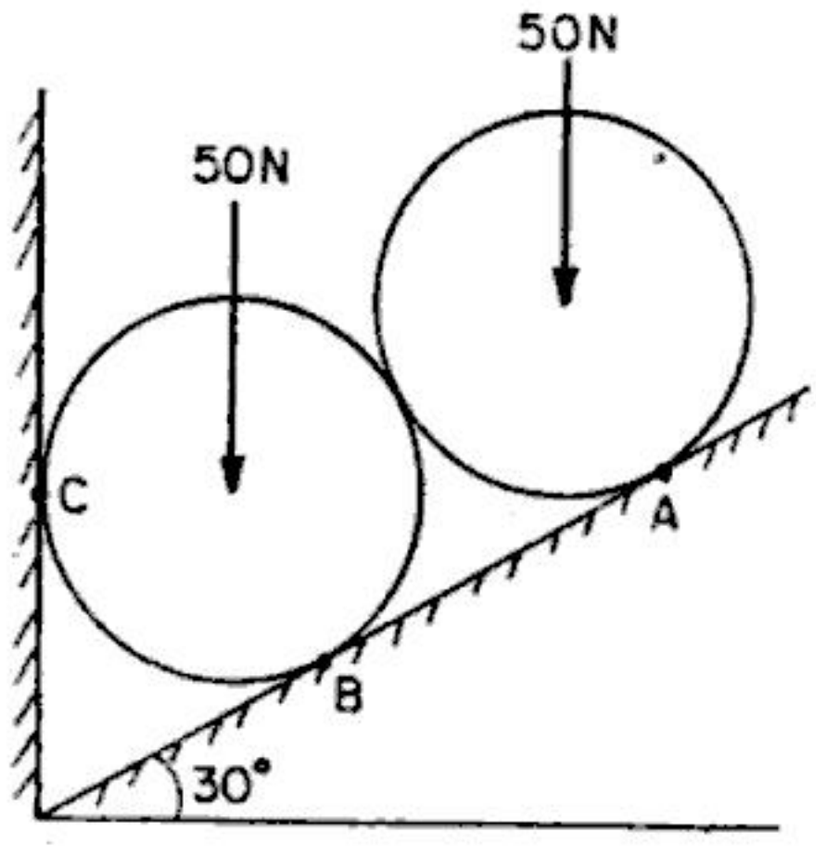

Fig. 4.27.

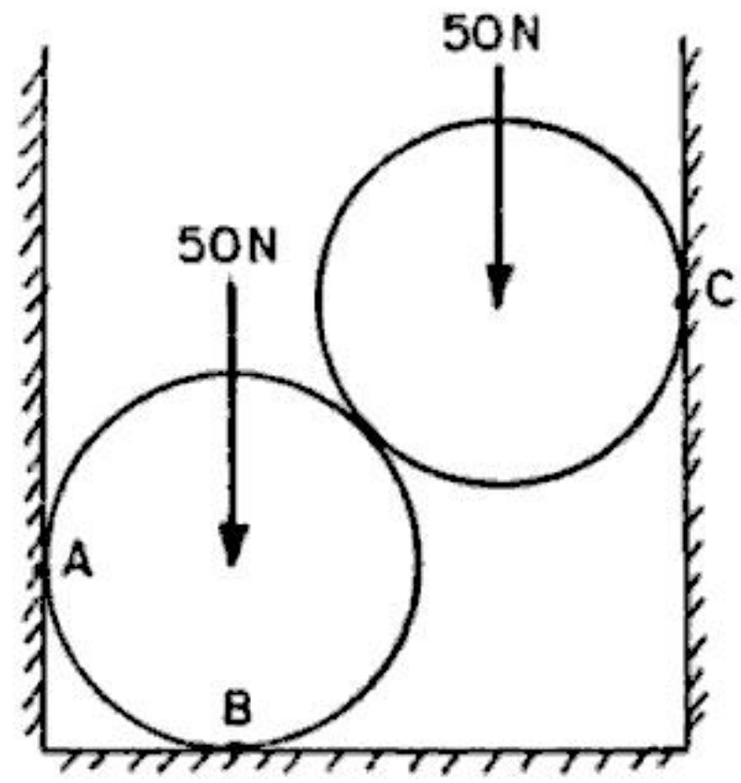

Fig. 4.28.

12. Two spheres, each of weight 50 N and of radius 10 cm rest in a horizontal channel of width 36 cm as shown in Fig. 4.28. Find the reactions on the points of contact A, B and C.

[Ans. $R_A = R_C = 66.67$ N, $R_B = 100$ N]

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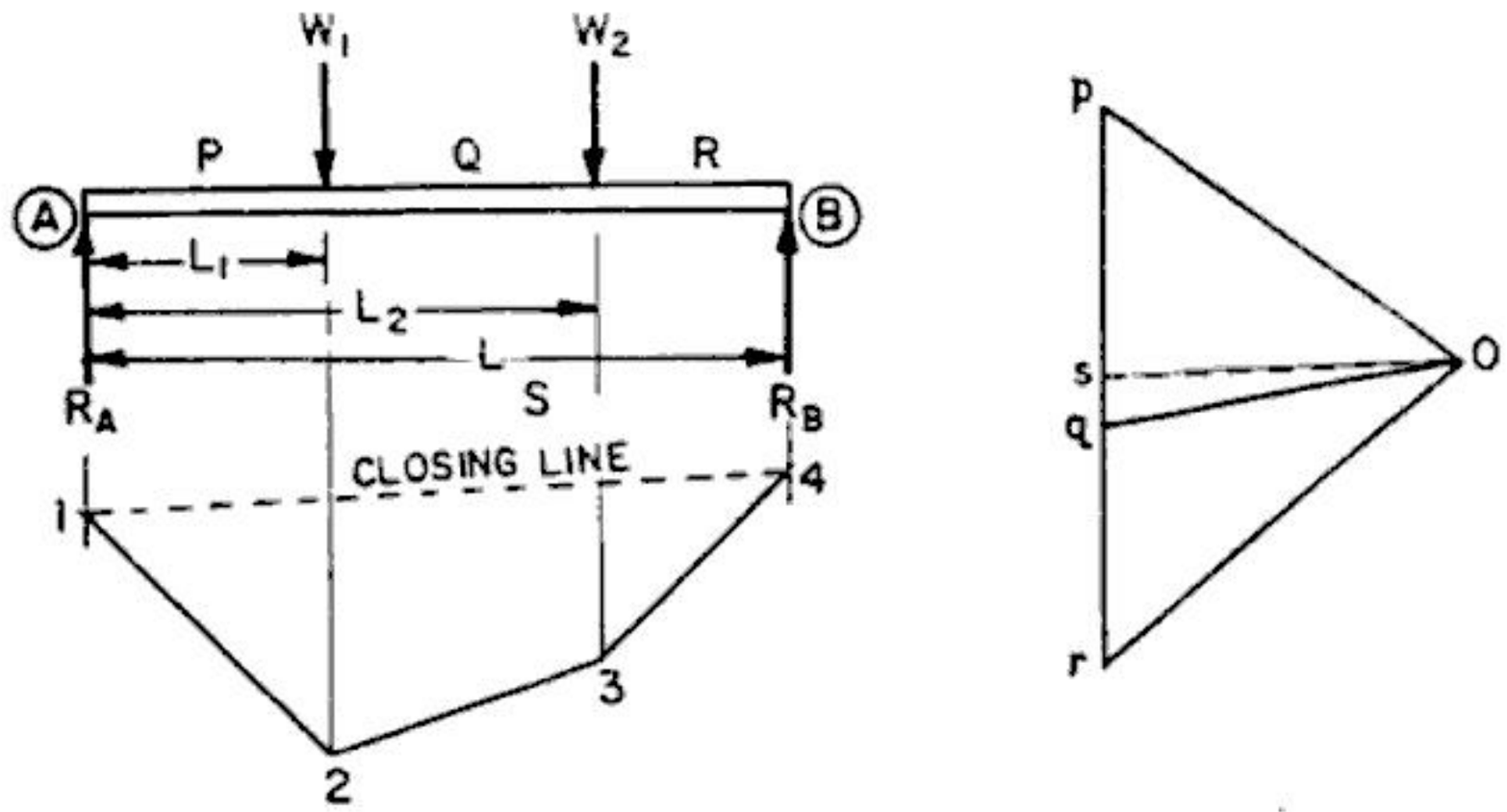
(*vi*) Now join the point 1 to point 4. The line 1-4 is known as closing line. Now from point *O* (*i.e.*, from vector diagram) draw line *Os* parallel to line 1-4.

(*vii*) Now in the vector diagram the length *sp* represents the magnitude of reaction R_A to the same scale. Similarly, the length *rs* represents the magnitude of reaction R_B to the same scale.



(*a*) Space diagram (*b*) Vector diagram

Fig. 5.11.

5.5. PROBLEMS ON SIMPLE SUPPORTED BEAMS

Problem 5.1. *A simply supported beam AB of span 6 m carries point loads of 3 kN and 6 kN at a distance of 2 m and 4 m from the left end A as shown in Fig. 5.12. Find the reactions at A and B analytically and graphically.*

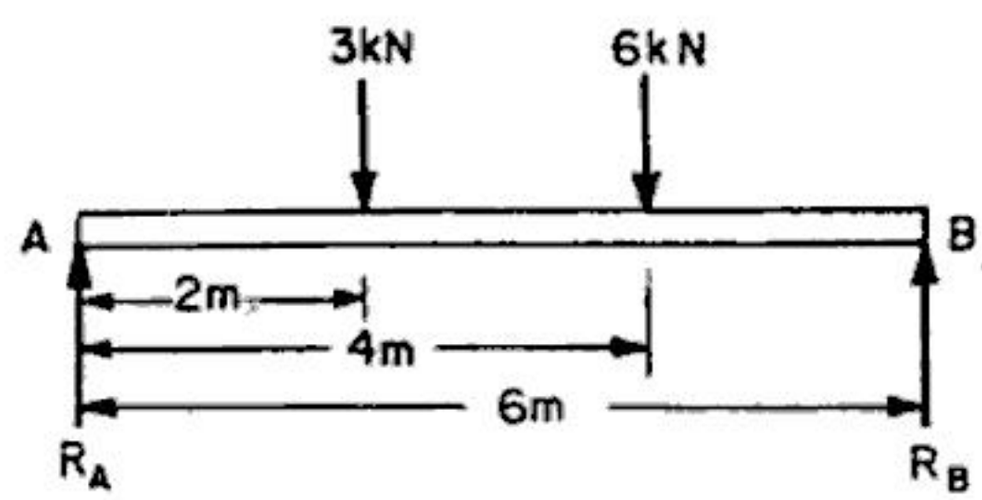


Fig. 5.12.

Sol. Given :

Span of beam = 6 m

Let R_A = Reaction at *A*

R_B = Reaction at *B*

(*a*) **Analytical Method**. As the beam is in equilibrium, the moments of all the forces about any point should be zero.

Now taking the moment of all forces about *A*, and equating the resultant moment to zero, we get

$$R_B \times 6 - 3 \times 2 - 6 \times 4 = 0$$

or $$6R_B = 6 + 24 = 30$$

or $$R_B = \frac{30}{6} = 5 \text{ kN.} \quad \textbf{Ans.}$$

Also for equilibrium, $\Sigma F_y = 0$

$\therefore$ $$R_A + R_B = 3 + 6 = 9$$

$\therefore$ $$R_A = 9 - R_B = 9 - 5 = 4 \text{ kN.} \quad \textbf{Ans.}$$

(*b*) **Graphical Method**. First of all draw the space diagram of the beam to a suitable scale. Let 1 cm length in space diagram represents 1 m length of beam. Hence take *AB* = 6 cm, distance of load 3 kN from *A* = 2 cm and distance of 6 kN from *A* = 4 cm as shown in Fig. 5.13 (*a*).

Now name all the loads and reactions according to Bow's notation *i.e.* load 3 kN is named by *PQ*, load 6 kN by *QR*, reaction R_B by *SR* and reaction R_A by *SP*.

Now the vector diagram is drawn according to the following steps : [Refer to Fig. 5.13 (*b*)].

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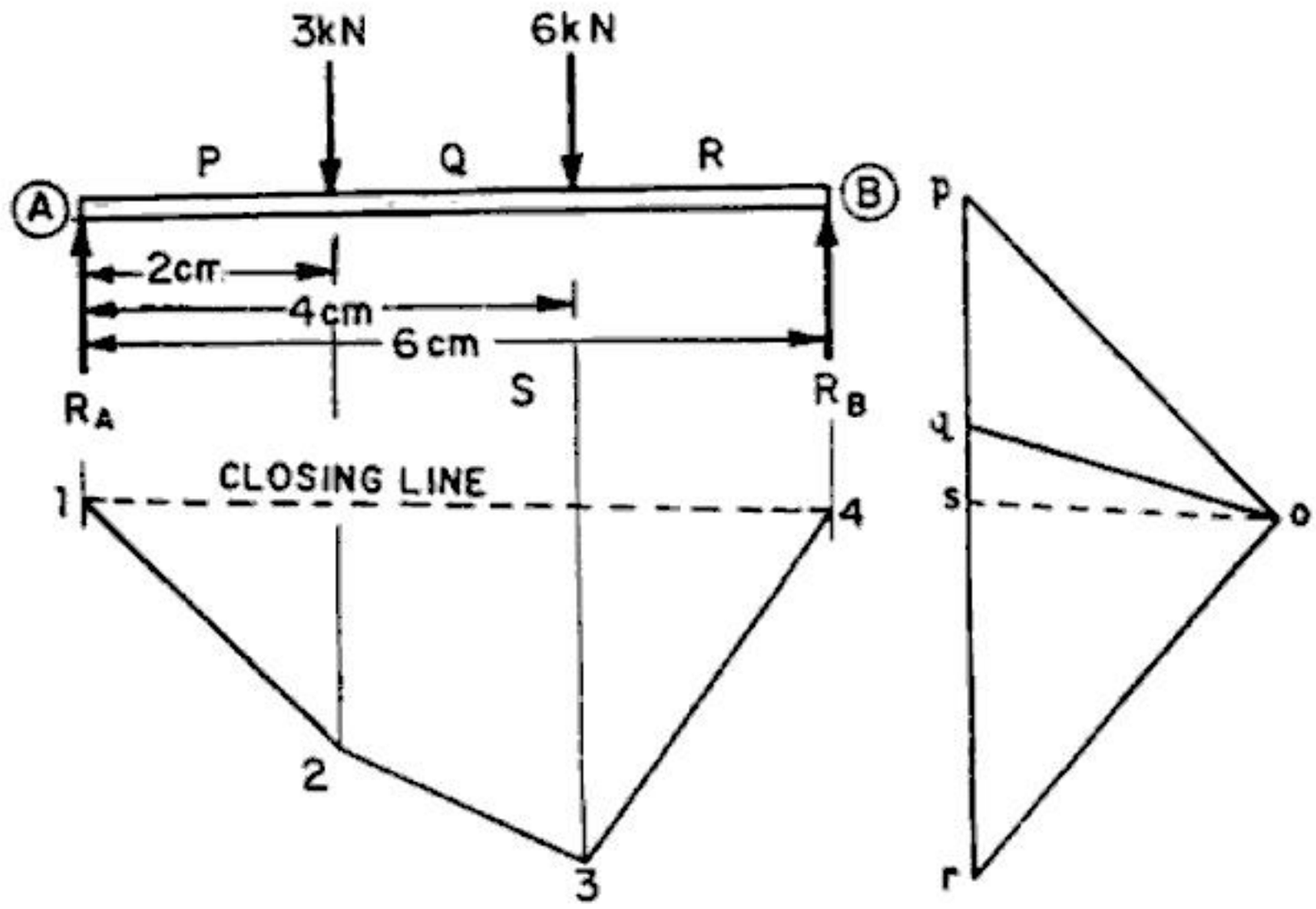
1. Choose a suitable scale to represent various loads. Let 1 cm represents 1 kN load. Hence load *PQ* (*i.e.*, 3 kN) will be equal to 3 cm and load *QR* (*i.e.*, 6 kN) 6 cm.

2. Now take any point *p* and draw line *pq* parallel to load *PQ* (*i.e.*, 3 kN). Take *pq*=3 cm to represent the load of 3 kN.

3. Through *q*, draw line *qr* parallel to load *QR* (*i.e.*, 6 kN). Cut *qr* equal to 6 cm to represent the load of 6 kN.

4. Now take any point *O*. Join the point *O* to the points *p*, *q* and *r* as shown in Fig. 5.13 (*b*).



(*a*) Space diagram (*b*) Vector diagram

Fig. 5.13.

5. Now in Fig. 5.13 (*a*), extend the lines of action of the loads (3 kN and 6 kN), and the two reactions. Take any point 1, on the line of action of the reaction R_A. Through 1, draw the line 1-2 parallel to *pO*, intersecting the line of action of load 3 kN at point 2.

6. From point 2, draw line 2-3 parallel to *qO*, intersecting the line of action of load 6 kN at 3. Similarly, from point 3, draw a line 3-4 parallel to *rO*, intersecting the line of action of reaction R_B at point 4.

7. Join 1 to 4. The line 1-4 is known as closing line. From the vector diagram, from point *O*, draw line *Os* parallel to line 1-4.

8. Measure the length *sp* and *rs*. The length *sp* represents the reaction R_A and length *rs* represents the reaction R_B.

By measurement, *sp*=4 cm

and *rs*=5 cm

$\therefore$ R_A=Length *sp*×scale=4×1 kN=**4 kN. Ans.**

R_B=Length *rs*×scale=5×1 kN=**5 kN. Ans.**

Problem 5.2. *A simply supported beam AB of length 9 m, carries a uniformly distributed load of 10 kN/m for a distance of 6 m from the left end. Calculate the reactions at A and B.*

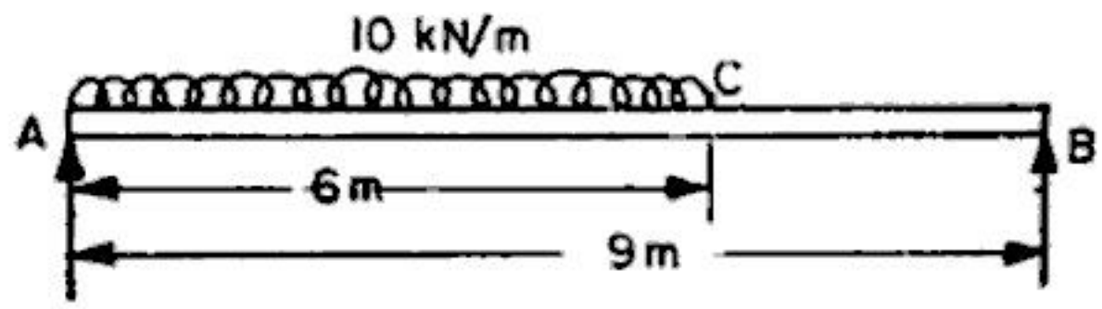


Fig. 5.14.

Sol. Given :

Length of beam=9 m

Rate of U.D.L.=10 kN/m

Length of U.D.L.=6 m

Total load due to U.D.L.

=(Length of U.D.L.)×Rate of U.D.L.

=6×10=60 kN

This load of 60 kN will be acting at the middle point of *AC* *i.e.*, at a distance of $\frac{6}{2}$=3 m from *A*.

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Let R_A=Reaction at A and

R_B=Reaction at B

Taking the moments of all forces about point A, and equating the resultant moment to zero, we get

$$R_B \times 9 - (6 \times 10) \times 3 = 0$$

or $$9R_B - 180 = 0$$

$$\therefore \quad R_B = \frac{180}{9} = \mathbf{20\ kN.} \quad \textbf{Ans.}$$

Also for equilibrium, $\Sigma F_y = 0$

or $$R_A + R_B = 6 \times 10 = 60$$

$$\therefore \quad R_A = 60 - R_B = 62 - 20 = \mathbf{40\ kN.} \quad \textbf{Ans.}$$

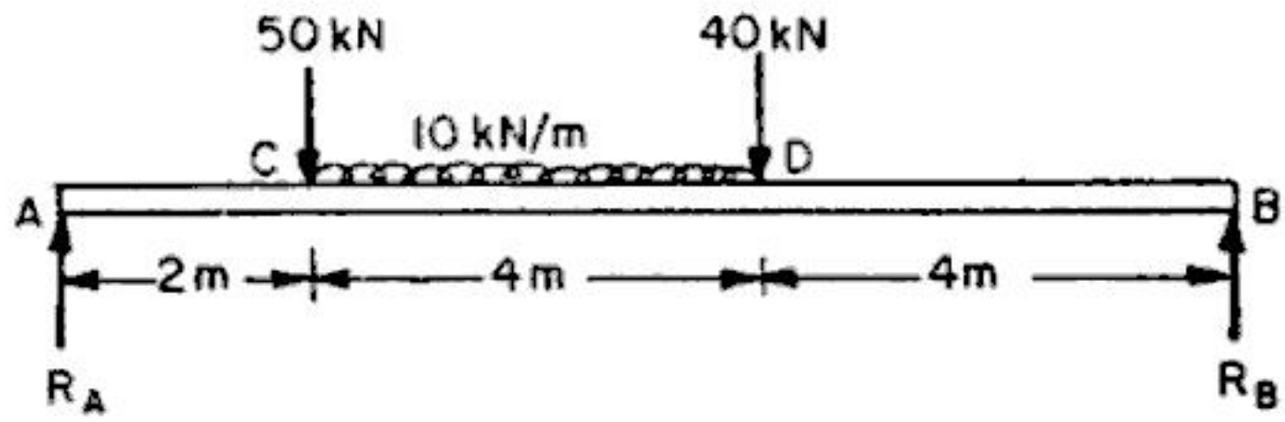

Fig. 5.15.

Problem 5.3. *A simply supported beam of length 10 m, carries the uniformly distributed load and two point loads as shown in Fig. 5.15. Calculate the reactions R_A and R_B.*

Sol. Given :

Length of beam =10 m

Length of U.D.L. =4 m

Rate of U.D.L. =10 kN/m

$\therefore$ Total load due to U.D.L.=4×10=40 kN

This load of 40 kN due to U.D.L. will be acting at the middle point of CD, *i.e.*, at a distance of $\frac{4}{2}$=2 m from C (or at a distance of 2+2=4 m from point A).

Let R_A=Reaction at A

and R_B=Reaction at B

Taking the moments of all forces about point A and equating the resultant moment to zero, we get

$$R_B \times 10 - 50 \times 2 - 40 \times (2+4) - (10 \times 4)\left(2 + \frac{4}{2}\right) = 0$$

or $$10R_B - 100 - 240 - 160 = 0$$

or $$10R_B = 100 + 240 + 160 = 500$$

$$\therefore \quad R_B = \frac{500}{10} = \mathbf{50\ kN.} \quad \textbf{Ans.}$$

Also for equilibrium of the beam, $\Sigma F_y = 0$

$$\therefore \quad R_A + R_B = \text{Total load on the beam} = 50 + 10 \times 4 + 40 = 130$$

$$\therefore \quad R_A = 130 - R_B = 130 - 50 = \mathbf{80\ kN.} \quad \textbf{Ans.}$$

Problem 5.4. *A simply supported beam of span 9 m carries a uniformly varying load from zero at end A to 900 N/m at end B. Calculate the reactions at the two ends of the support.*

Sol. Given :

Span of beam =9 m

Load at end A=0

Load at end B=900 N/m

Total load on the beam =Area of ABC

$$= \frac{AB \times BC}{2} = \frac{9 \times 900}{2} = 4050 \text{ N}$$

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or $5R_B - 1000 - 6666.66 = 0$

or $5R_B = 1000 + 6666.66 = 16666.66$

or $R_B = \frac{16666.66}{5} = 3333.33$ N. **Ans.**

Also for the equilibrium of the beam, $\Sigma F_y = 0$

$\therefore$ $R_A + R_B$ = Total load on the beam

$= 6000$ ($\because$ Total load on beam = 6000 N)

$\therefore$ $R_A = 6000 - R_B = 6000 - 3333.33 = 2666.67$ N. **Ans.**

5.6. PROBLEMS ON OVERHANGING BEAMS

If the end portion of a beam is extended beyond the support, then the beam is known as overhanging beam. Over hanging portion may be at one end of the beam or at both ends of the beam as shown in Fig. 5.18.

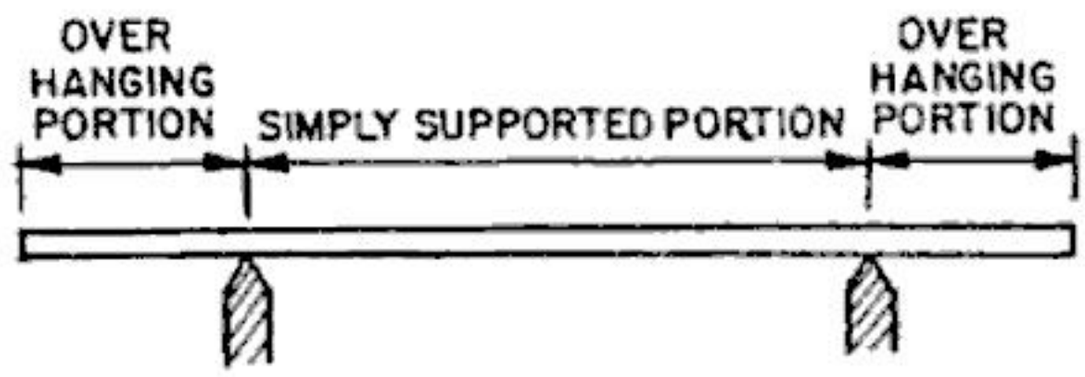

Fig. 5.18.

Problem 5.6. *A beam AB of span 8 m, overhanging on both sides, is loaded as shown in Fig. 5.19. Calculate the reactions at both ends.*

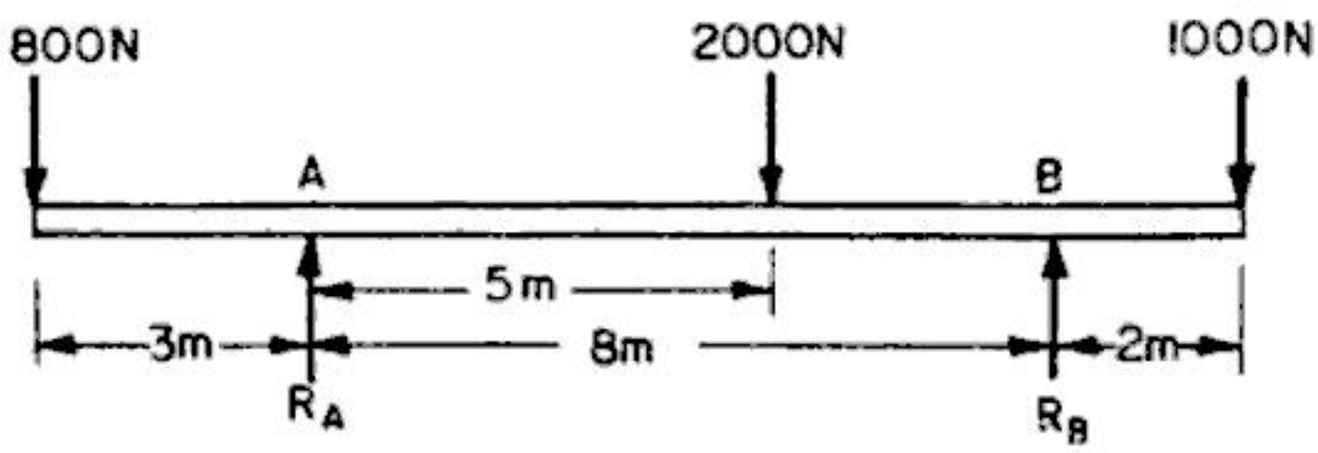

Fig. 5.19.

Sol. Given :

Span of beam = 8 m

Let R_A = Reaction at A

and R_B = Reaction at B.

Taking the moments of all forces about point A and equating the resultant moment to zero, we get

$$R_B \times 8 + 800 \times 3 - 2000 \times 5 - 1000 \times (8+2) = 0$$

or $8R_B + 2400 - 10000 - 10000 = 0$

or $8R_B = 20000 - 2400 = 17600$

$\therefore$ $R_B = \frac{17600}{8} = 2200$ N. **Ans.**

Also for the equilibrium of the beam, we have

$$R_A + R_B = 800 + 2000 + 1000 = 3800$$

$\therefore$ $R_A = 3800 - R_B = 3800 - 2200 = 1600$ N. **Ans.**

Problem 5.7. *A beam AB of span 4 m, overhanging on one side upto a length of 2 m, carries a uniformly distributed load of 2 kN/m over the entire length of 6 m and a point load of 2 kN/m as shown in Fig. 5.20. Calculate the reactions at A and B.*

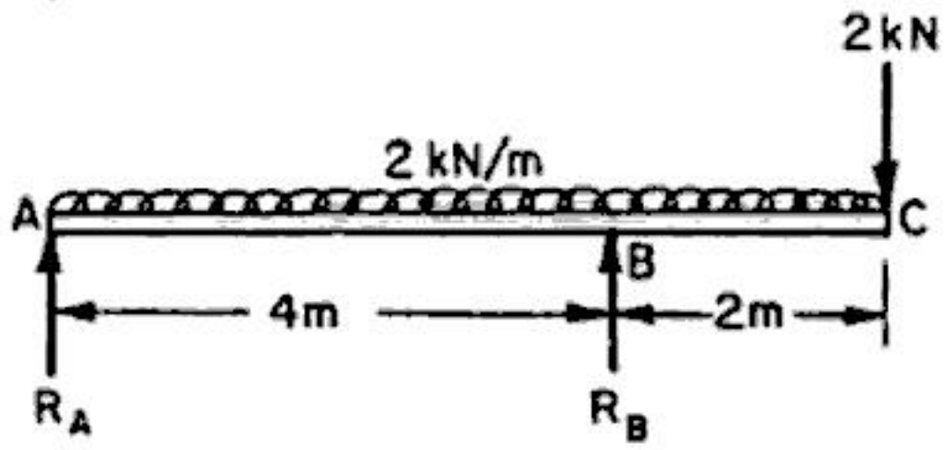

Fig. 5.20.

Sol. Given :

Span of beam = 4 m

Total length = 6 m

Rate of U.D.L. = 2 kN/m

Total load due to U.D.L. = 2×6 = 12 kN

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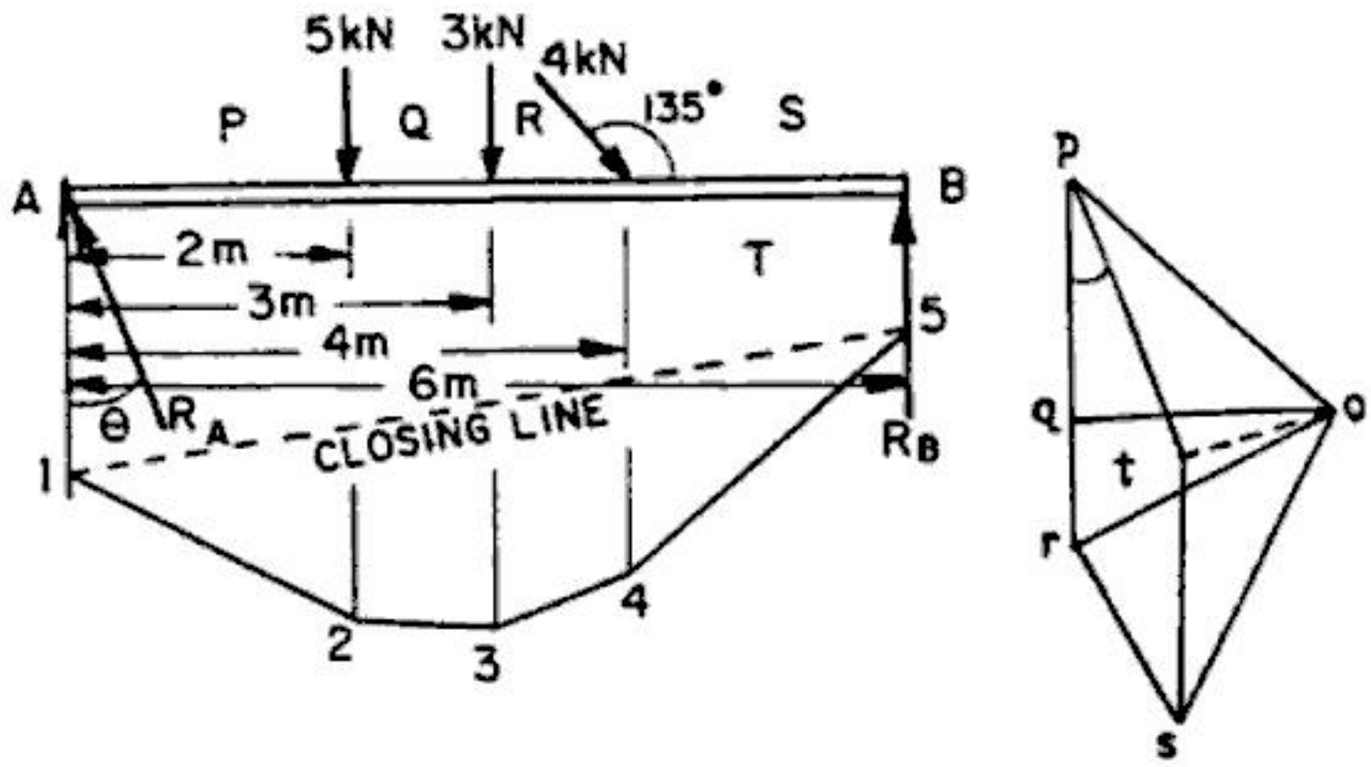
(*iv*) Now take any point O, and join Op, Oq, Or and Os.

(*v*) Now in space diagram [*i.e.* Fig. 5.23 (*a*)], extend the lines of actions of loads PQ, QR, RS and reaction R_B.

(*vi*) Take any point 1, vertically below the point A as shown in Fig. 5.23 (*a*). From point 1, draw line 1-2 parallel to line pO, intersecting the line of action of 5 kN at point 2.

(*vii*) Similarly, draw lines 2-3, 3-4 and 4-5 parallel to qO, rO and sO respectively. Join point 1 to 5. Line 1-5 is the closing line in space diagram.



(*a*) Space diagram (*b*) Vector diagram

Fig. 5.23.

(*viii*) From O in vector diagram, draw a line parallel to closing line 1-5. Now through s, draw a line st vertical (as the reaction R_B is vertical), intersecting the line through O at t. Join t to p.

(*ix*) The length st represents the reaction R_B in magnitude and direction whereas the length tp gives the magnitude and direction of reaction R_A. At point A, draw a line parallel to tp as shown in Fig. 5.23 (*a*). By measurement, we get

$$R_A = \text{length } tp = 6.43 \text{ kN}$$

$$R_B = \text{length } st = 5.052 \text{ kN}$$

and $$\theta = 26.1°.$$

Problem 5.10. *A beam AB 10 m long is hinged at A and supported on rollers over a smooth surface inclined at 30° to the horizontal at B. The beam is loaded as shown in Fig. 5.24. Determine reactions at A and B.*

Sol. Given :

Length of beam = 10 m

Let R_A = Reaction at A

and R_B = Reaction at B

The reaction R_B will be normal to the support as the beam at B is supported on the rollers. But the support at B is making an angle 30° with the horizontal or 60° with the vertical as shown in Fig. 5.24.

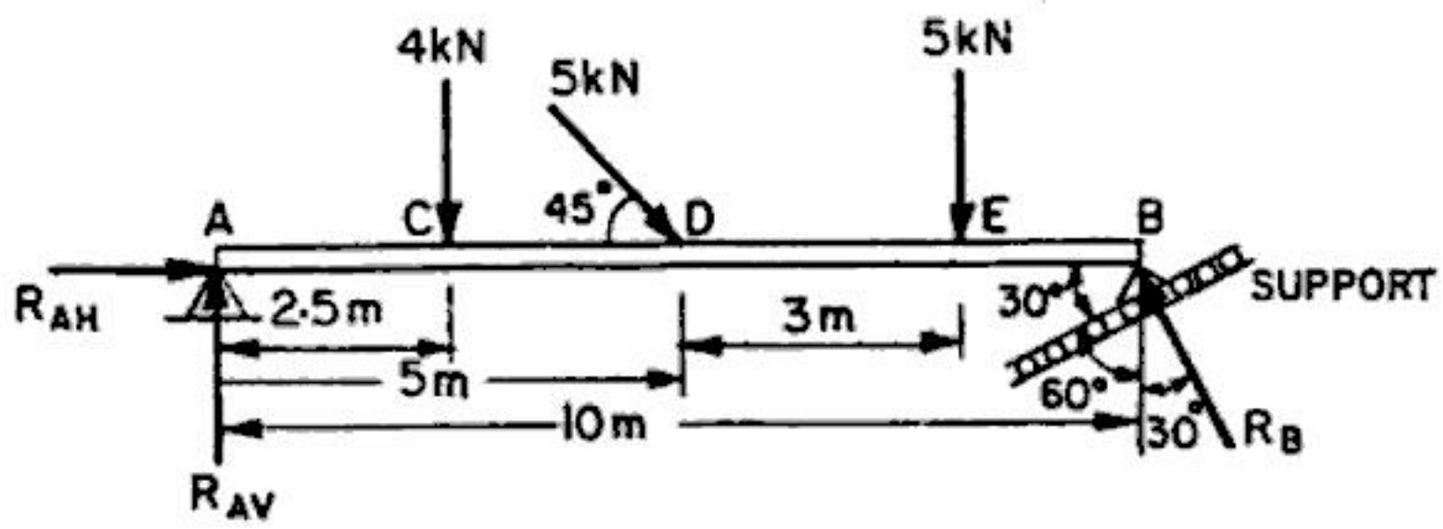


Fig. 5.24.

Hence the reaction R_B is making an angle of 30° with the vertical

The vertical component of R_B

$$= R_B \cos 30°$$

and horizontal component of R_B

$$= R_B \sin 30°$$

These components are shown in Fig. 5.24 (*a*).

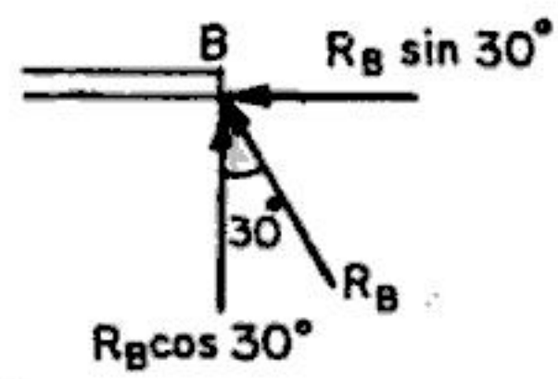


Fig. 5.24. (*a*)

Resolving the load of 5 kN acting at D into horizontal and vertical components, we get

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or $6928-1979.6+13500-169.14\ R_B=0$

or $18448.4=169.14\ R_B$

$$\therefore \quad R_B=\frac{18448.4}{169.14}=109.07$$

Let R_A=Reaction at the point A

The reaction at A can be resolved in two components *i.e.* R_{Ax} and R_{Ay}.

For equilibrium, $\Sigma F_x=0$

or $R_{Ax}+100\cos 30°-70\sin 45° -R_B\sin 20°=0$

or $R_{Ax}=R_B\sin 20°+70\sin 45° -100\cos 30°$

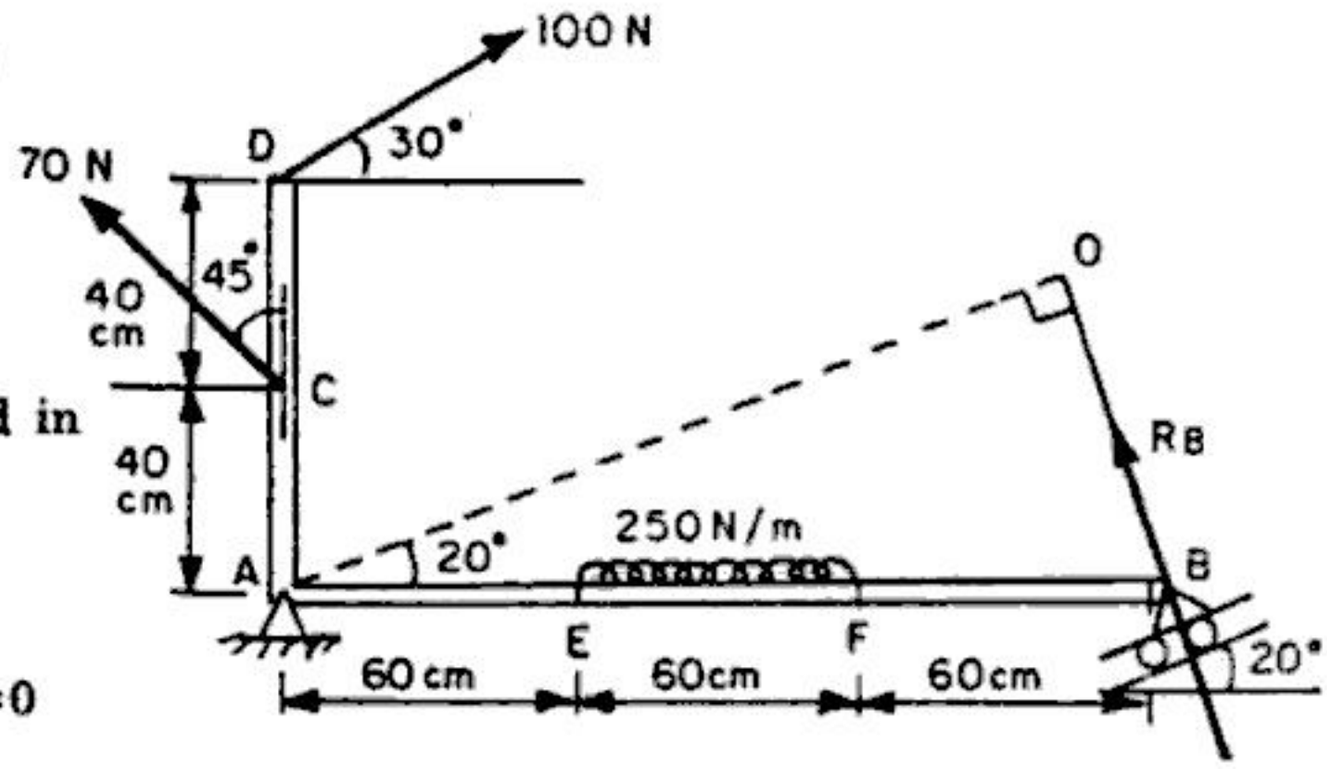

Fig. 5.27. (*a*)

$=109.07\times0.342+70\times0.707-100\times0.866$

$=37.3+49.49-86.6=0.19$ N

For equilibrium, $\Sigma F_y=0$

or $R_{Ay}+100\sin 30°+70\cos 45°+R_B\cos 20°=150$

or $R_{Ay}=150-100\sin 30°-70\cos 45°-R_B\cos 20°$

$=150-50-49.49-109.07\times0.9396=-51.98$ N

(−ve sign means, R_{Ay} will be acting vertically downward)

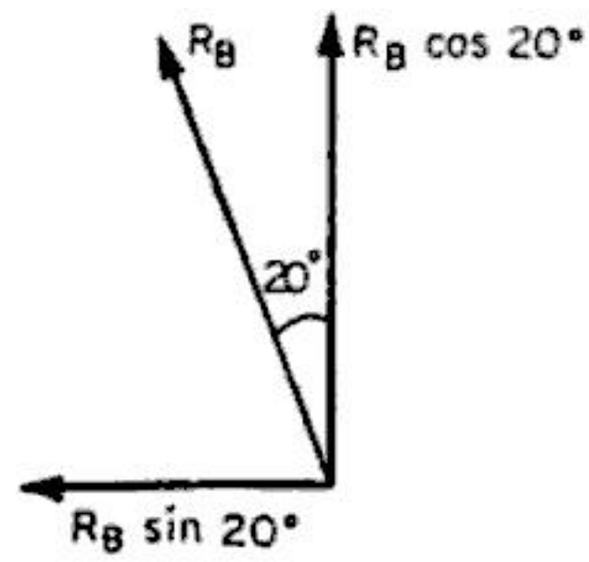

Fig. 5.27. (*b*)

Refer to Fig. 5.27 (*c*)

$$\therefore \quad R_A=\sqrt{R_{Ax}^2+R_{Ay}^2}$$

or $$R_A=\sqrt{0.19^2+(-51.98)^2}=\sqrt{0.0361+2701.92}$$

$$=\mathbf{51.9803\ kN.}\quad \textbf{Ans.}$$

The angle made by R_A with x-axis is given by

$$\tan\theta=\frac{R_{Ay}}{R_{Ax}}=\frac{51.98}{0.19}=273.57$$

$$\therefore \quad \theta=\tan^{-1}273.57=\mathbf{89.79°.}\quad \textbf{Ans.}$$

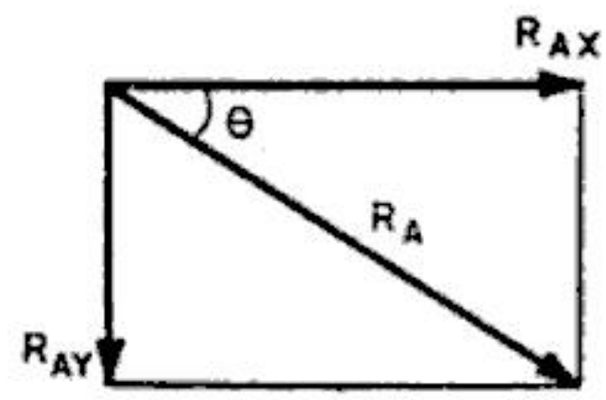

Fig. 5.27. (*c*)

5.8. PROBLEMS WHEN BEAMS ARE SUBJECTED TO COUPLES

In this section, the reactions of the beam will be calculated when beams are subjected to clockwise or anticlockwise couple along with the other loads. While taking the moments about any point, the magnitude and sense of the couple is taken into consideration. But when the total load on the beam is calculated the magnitude and sense of the couple is not considered.

Problem 5.13. *A simply supported beam AB of 7 m span is subjected to : (i) 4 kN m clockwise couple at 2 m from A, (ii) 8 kN m anti-clockwise couple at 5 m from A and (iii) a triangular load with zero intensity at 2 m from A increasing to 4 kN per m at a point 5 m from A. Determine reactions at A and B.*

Sol. Given :

Span of beam=7 m

Couple at C (*i.e.* at 2 m from A)=4 kN m (clockwise)

Couple at D (*i.e.* at 5 m from A)=8 kN m (anticlockwise)

Triangular load from C to D with :

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6

Analysis of Perfect Frames

6.1. INTRODUCTION

A structure made up of several bars (or members) riveted or welded together is known as frame. If the frame is composed of such members which are just sufficient to keep the frame in equilibrium, when the frame is supporting an external load, then the frame is knwon as *perfect frame*. Though in actual practice the members are welded or riveted together at their joints, yet for calculation purposes the joints are assumed to be hinged or pin-joined. In this chapter, we shall discuss how to determine the forces in the members of a perfect frame, when it is subject to some external load.

6.2. TYPES OF FRAMES

The different types of frames are :

(*i*) Perfect frame and (*ii*) Imperfect frame.

Imperfect frame may be a deficient frame or a redundant frame.

6.2.1. Perfect Frame. The frame which is composed of such members, which are just sufficient to keep the frame in equilibrium, when the frame is supporting an external load, is known as perfect frame. The simplest perfect frame is a triangle as shown in Fig. 6.1 which consists three members and three joints. The three members are : *AB*, *BC* and *AC* whereas the three joints are *A*, *B* and *C*. This frame can be easily analysed by the condition of equilibrium.

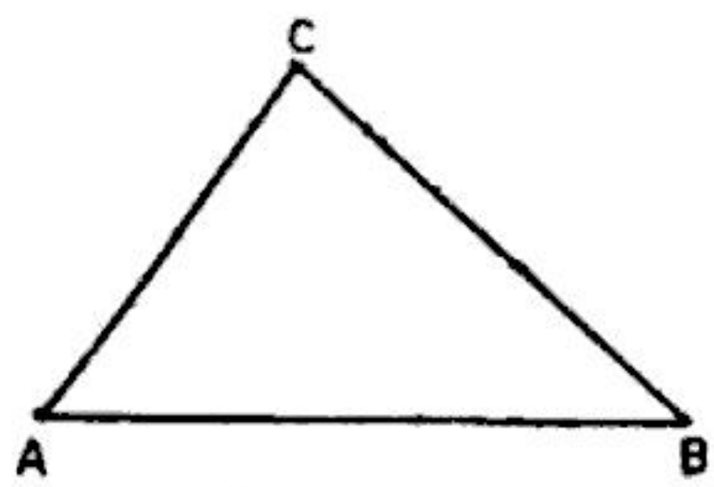

Fig. 6.1.

Let the two members *CD* and *BD* and a joint *D* are added to the triangular frame *ABC*. Now, we get a frame *ABCD* as shown in Fig. 6.2 (*a*). This frame can also be analysed by the conditions of equilibrium. This frame is also known as perfect frame.

Suppose we add a set of two members and a joint again, we get a perfect frame as shown in Fig. 6.2 (*b*). Hence for a perfect frame, the number of joints and number of members are given by,

$$n=2j-3$$

where n=Number of members, and

j=Number of joints.

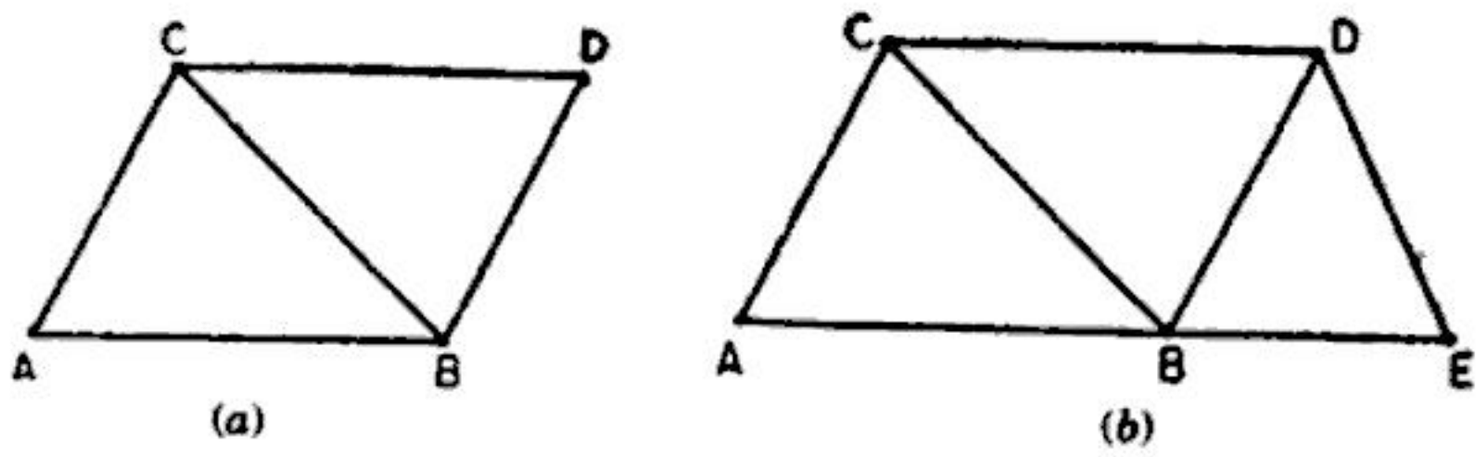

Fig. 6.2.

6.2.2. Imperfect Frame. A frame in which number of members and number of joints are not given by

$$n=2j-3$$

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Resolving the forces vertically, we get

$F_4 \sin 30° = R_B = 0.667$

or $$F_4 = \frac{0.667}{\sin 30°} = 1.334 \text{ kN} \quad \text{(Compressive)}$$

Resolving the forces horizontally, we get

$$F_5 = F_4 \cos 30° = 1.334 \times 0.866$$
$$= 1.155 \text{ kN (Tensile)}$$

Joint D

Let F_3=Force in member CD. The forces F_2 and F_5 have been already calculated in magnitude and direction. The forces F_2 and F_5 are tensile and hence they will be pulling the joint D as shown in Fig. 6.11. Let the direction* of F_3 is assumed as shown in Fig. 6.11.

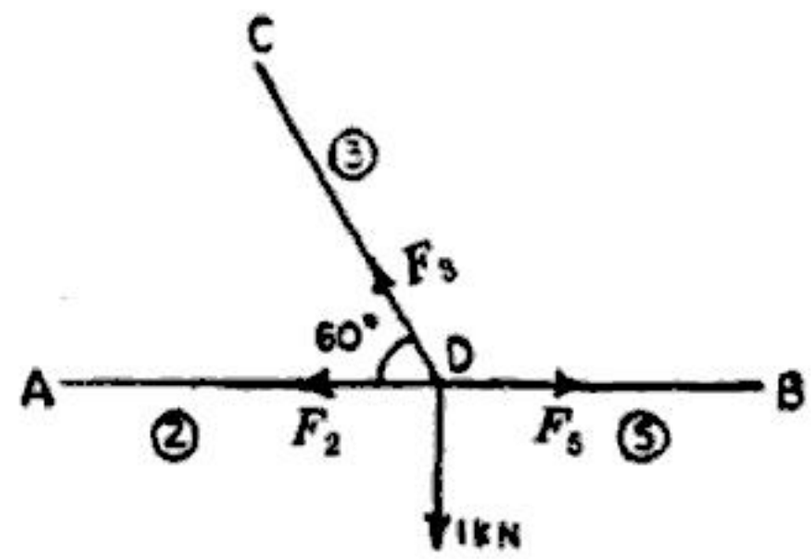

Fig. 6.11.

Resolving the forces vertically, we get

$$F_3 \sin 60° = 1$$

$$\therefore \quad F_3 = \frac{1}{\sin 60°} = \frac{1}{0.866} = 1.1547 \text{ kN (Tensile)}$$

Hence the forces in the members are :

F_1=0.666 kN (Compressive)

F_2=0.5767 kN (Tensile) F_3=1.1547 kN (Tensile)

F_4=1.334 kN (Compressive) F_5=1.155 kN (Tensile). **Ans.**

Problem 6.3. *A truss of span 5 m is loaded as shown in Fig. 6.12. Find the reactions and forces in the members of the truss.*

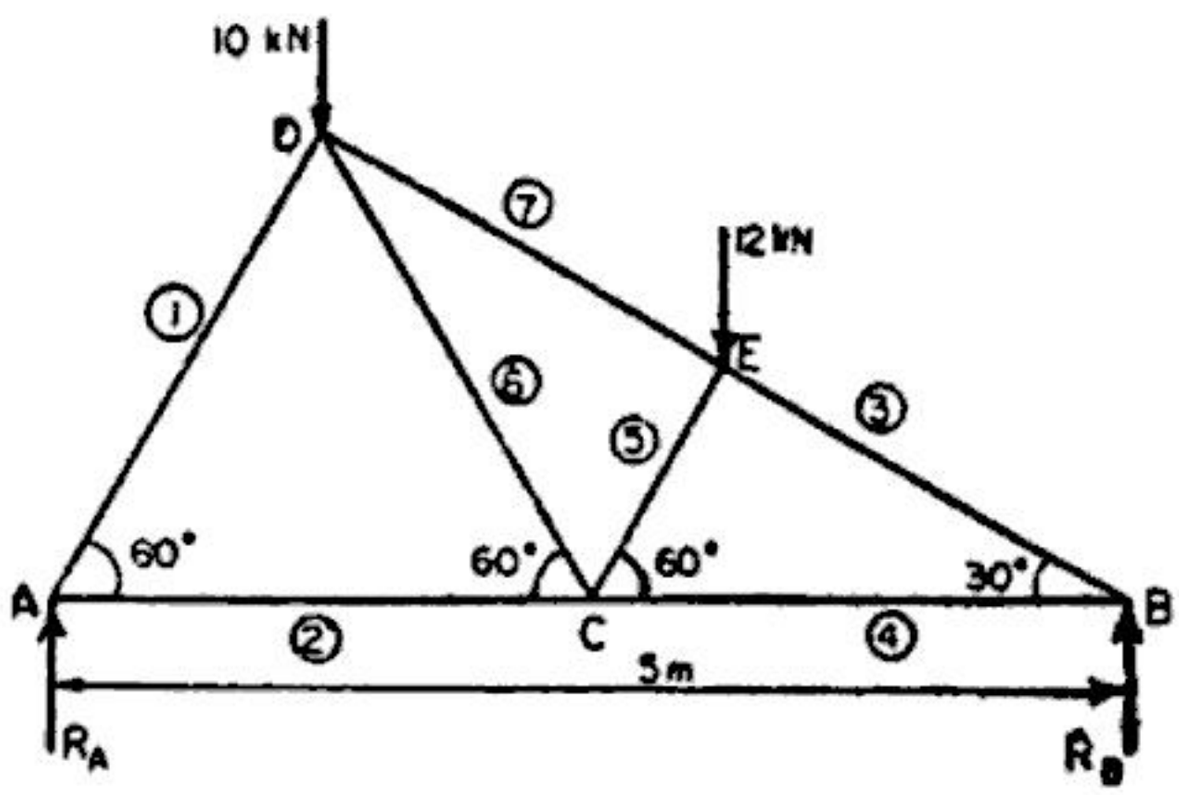

Fig. 6.12.

Sol. Let us first determine the reactions R_A and R_B.

*The horizontal force F_5 is more than F_2. Hence the horizontal component of F_3 must be in the direction of F_2. This is only possible if F_3 is acting away from D.

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Resolving forces horizontally, we get

$$F_{CD}=F_{CG}\ \sin\ \theta=12.5\times\frac{3}{5}=7.5\ \text{kN (Compressive)}$$

Now consider the equilibrium of joint G.

Joint G

The force in member CG is 12.5 kN (Tensile).

Hence at the joint G, this force will be pulling the joint G as shown in Fig. 6.17 (*c*).

Resolving the forces vertically, we get

$$12.5\ \cos\ \theta+F_{GD}=9$$

$$\therefore \qquad F_{GD}=9-12.5\ \cos\ \theta=9-12.5\times\frac{4}{5} \qquad \left(\because\ \cos\theta=\frac{4}{5}\right)$$

$$=9-10=-1\ \text{kN.}$$

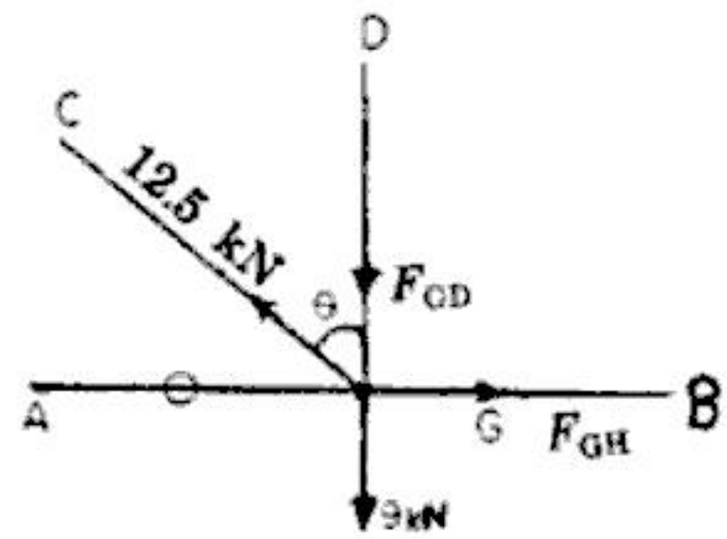

Fig. 6.17. (*c*)

As the magnitude of F_{GD} is negative, hence its assumed direction is wrong. The correct direction will be as shown in Fig. 6.17 (*d*).

Then, $F_{GD}=1$ kN (Compressive)

Resolving the forces horizontally, we get

$$12.5\ \sin\ \theta=F_{GH}$$

or $$F_{GH}=12.5\times\frac{3}{5} \qquad \left(\because\ \sin\ \theta=\frac{3}{5}\right)$$

$$=7.5\ \text{kN (Tensile)}$$

Now consider the equilibrium of joint D.

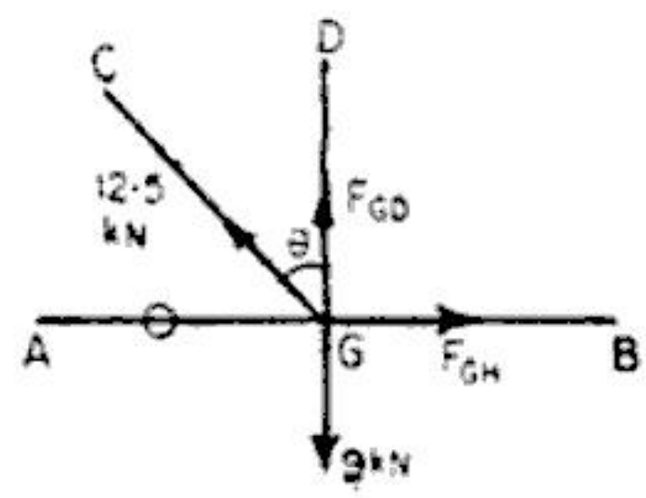

Fig. 6.17. (*d*)

Joint D

The forces in the members CD and GD have been already calculated. They are 7.5 kN and 1 kN respectively. Both are compressive.

Let F_{DH}=Force in member DH, and

F_{DE}=Force in member DE

Resolving the forces vertically, we get

$$F_{DH}\ \cos\ \theta=1\ \text{kN}$$

$$\therefore \qquad F_{DH}=\frac{1}{\cos\ \theta}=\frac{1}{(4/5)}=\frac{5}{4}=1.25\ \text{kN (Tensile)}$$

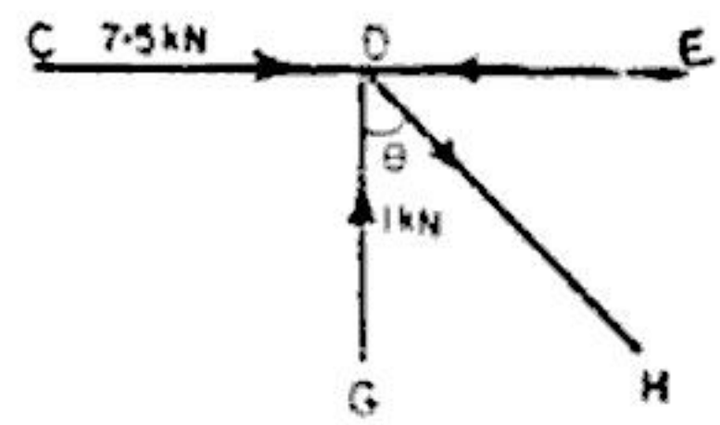

Fig. 6.17. (*e*)

Resolving the forces horizontally, we get

$$7.5+F_{DH}\ \sin\ \theta=F_{DE}$$

or $$F_{DE}=7.5+1.25\times\frac{3}{5} \qquad \left(\because\ \sin\ \theta=\frac{3}{5}\right)$$

$$=7.5+0.75=8.25\ \text{kN (Compressive)}$$

Now consider the equilibrium of joint E.

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Problem 6.6. *Determine the forces in all the members of a cantilever truss shown in Fig. 6.20.*

Sol. Here the calculations can be started from end C. Hence consider the equilibrium of the joint C.

Joint C

Let F_{CD}=Force in member CD, and

F_{CA}=Force in member CA.

Their assumed directions are shown in Fig. 6.20.

Resolving the force vertically, we get

$$F_{CD} \times \sin 60° = 1000$$

$$\therefore \quad F_{CD} = \frac{1000}{\sin 60°} = \frac{1000}{0.866} = 1154.7 \text{ N (Tensile)}$$

Resolving the forces horizontally, we get

$$F_{CA} = F_{CD} \times \cos 60°$$

$$= 1154.7 \times 0.5 = 577.35 \text{ N (Compressive)}$$

Now consider the equilibrium of the joint D.

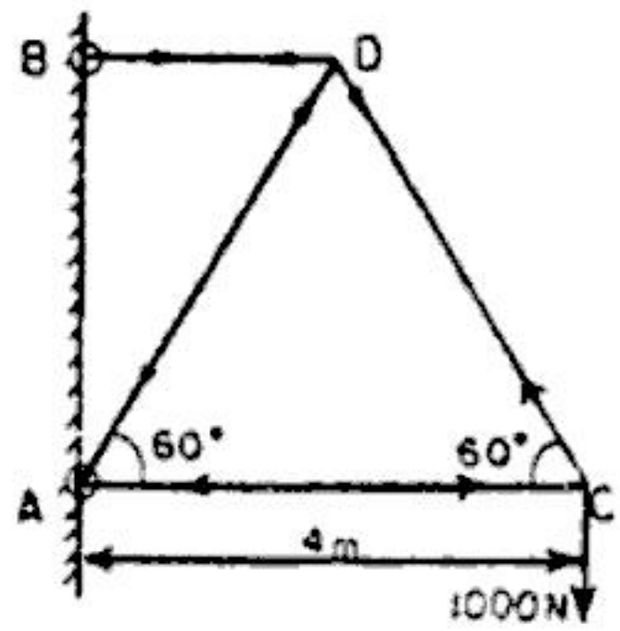

Fig. 6.20.

Joint D

[See Fig. 6.20 (*a*)]

The force F_{CD}=1154.7 N (tensile) is already calculated.

Let F_{AD}=Force in member AD, and

F_{BD}=Force in member BD

Their assumed directions are shown in Fig. 6.20 (*a*).

Resolving the forces vertically, we get

$$F_{AD} \cos 30° = 1154.7 \cos 30°$$

$$\therefore \quad F_{AD} = \frac{1154.7 \cos 30°}{\cos 30°} = 1154.7 \text{ N (Compressive)}$$

Resolving the forces horizontally, we get

$$F_{BD} = F_{AD} \sin 30° + F_{DC} \sin 30°$$

$$= 1154.7 \times 0.5 + 1154.7 \times 0.5 = 1154.7 \text{ N (Tensile)}$$

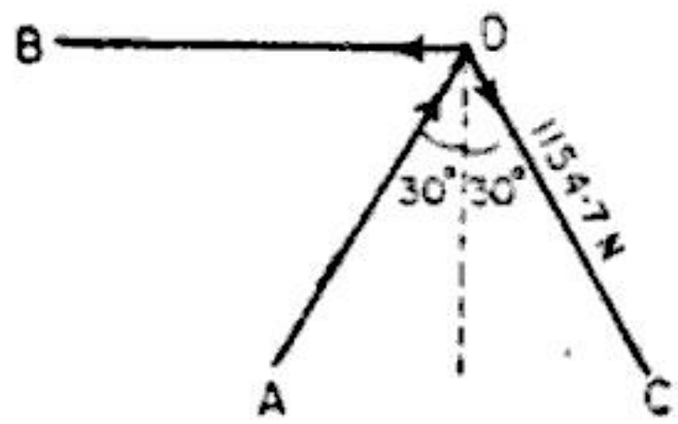

Fig. 6.20. (*a*)

Now the forces are shown in a tabular form below :

Member	*Force in the member*	*Nature of force*
AC	577.35 N	Compressive
CD	1154.7 N	Tensile
AD	1154.7 N	Compressive
BD	1154.7 N	Tensile

Problem 6.7. *Determine the forces in all the members of a cantilever truss shown in Fig. 6.21.*

Sol. Start the calculations from joint C.

From triangle ACE, we have

$$\tan \theta = \frac{AS}{AC} = \frac{3}{4}$$

Also $\quad EC = \sqrt{3^2 + 4^2} = 5$

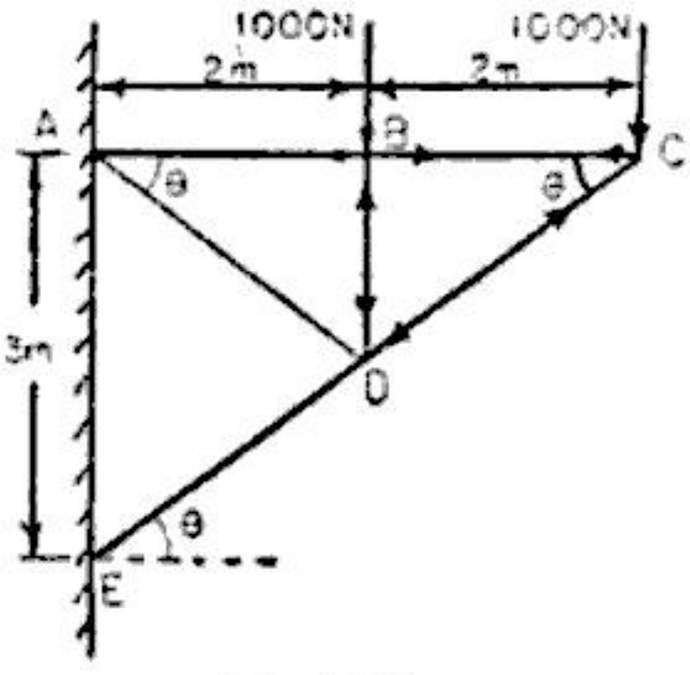

Fig. 6.21.

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consists of a horizontal reaction H_A and a vertical reaction R_A.

Taking moment about A, we get

$R_B \times 12 = 8 \times 1.5 + 3 \times 4 + 6 \times 8 = 72$

$\therefore \quad R_B = \frac{72}{12} = 6 \text{ kN}(\uparrow)$

and $\quad R_A = \text{Total vertical loads} - R_B$

$= (3+6) - 6 = 3 \text{ kN } (\uparrow)$

and $\quad H_A = \text{Sum. of all horizontal loads}$

$= 8 \text{ kN } (\leftarrow)$

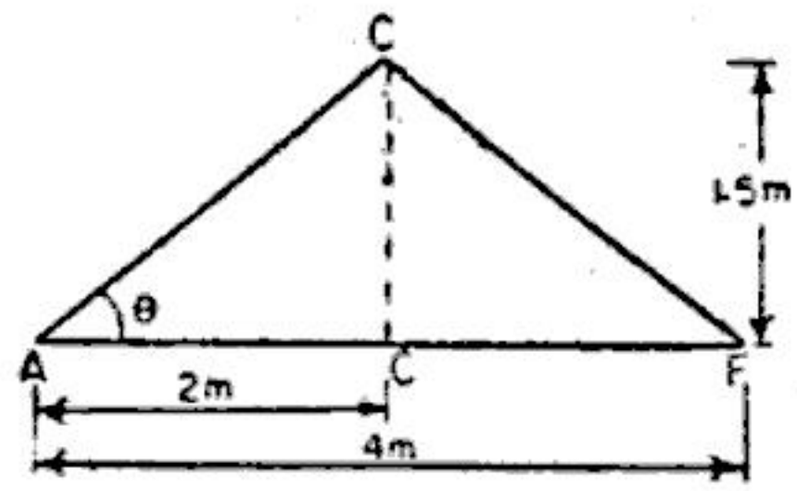

Fig. 6.23. (a)

In the triangle ACC^*,

$$AC = \sqrt{AC^{*2} + CC^{*2}} = \sqrt{2^2 + 1.5^2} = 2.5$$

$$\therefore \quad \cos\theta = \frac{AC^*}{AC} = \frac{2}{2.5} = 0.8$$

and

$$\sin\theta = \frac{CC^*}{AC} = \frac{1.5}{2.5} = 0.6$$

Now the forces in the members can be calculated. Consider the joint A.

Joint A

The reactions R_A and H_A are known in magnitude and direction. Let the directions of the forces F_{CA} and F_{FA} are as shown in Fig. 6.23 (b).

Resolving the forces vertically, we get

$F_{CA} \times \sin\theta = 3 \text{ kN}$

$\therefore \quad F_{CA} = \frac{3}{\sin\theta} = \frac{3}{0.6} = 5 \text{ kN}$ (Compressive)

Resolving the forces horizontally, we get

$F_{FA} = F_{CA} \cos\theta + H_A = 5 \times 0.8 + 8 = 12 \text{ kN}$ (Tensile)

Now consider joint C.

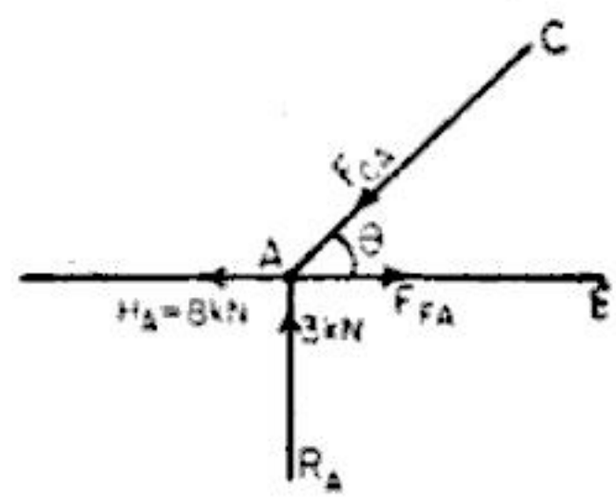

Fig. 6.23. (b)

Joint C

The force F_{CA} is known in magnitude and direction. The assumed directions of the forces F_{CD} and F_{CF} are shown in Fig. 6.23 (c).

Resolving forces vertically, we get

$F_{CA} \sin\theta = F_{CF} \sin\theta$

$\therefore \quad F_{CA} = F_{CF} = 5 \text{ kN} \qquad (\because F_{CA} = 5 \text{ kN})$

$\therefore \quad F_{CF} = 5 \text{ kN}$ (Tensile)

Resolving forces horizontally, we get

$F_{CD} = F_{CA} \cos\theta + F_{CF} \cos\theta = 5 \times 0.8 + 5 \times 0.8 = 8 \text{ kN}$

(Compressive)

Now consider the joint F.

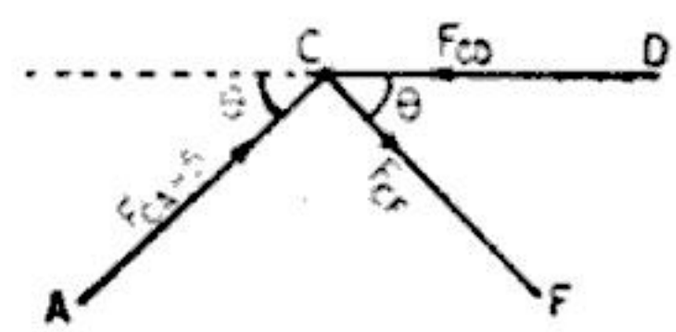

Fig. 6.23. (c)

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Joint F

The forces F_{FA} and F_{FC} are known in magnitude and directions. The assumed directions of the forces F_{DF} and F_{GF} are shown in Fig. 6.23 (*d*).

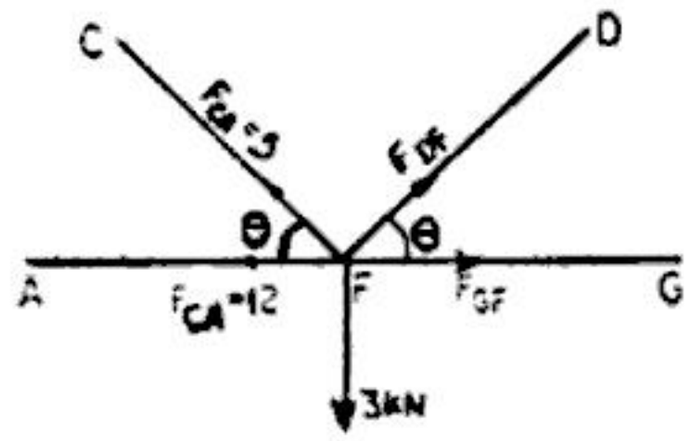

Fig. 6.23. (*d*)

Resolving the forces vertically, we get

$5 \times \sin\theta + F_{DF} \sin\theta = 3$

or $$F_{DF} = -\frac{5 \sin\theta + 3}{\sin\theta}$$

$$= -5 + \frac{3}{\sin\theta} = -5 + \frac{3}{0.6} = -5 + 5 = 0$$

Resolving the forces horizontally, we get

$12 + 5 \cos\theta = F_{GF} + F_{DF} \cos\theta$

or $12 + 5 \times 0.8 = F_{GF} + 0$

or $12 + 4 = F_{GF}$

$\therefore$ $F_{GF} = 12 + 4 = 16$ kN (Tensile)

Now consider the joint *D*.

Joint D

The forces F_{DC} and F_{FD} are known in magnitude and direction. The assumed directions of F_{DG} and F_{DE} are shown in Fig. 6.23 (*e*).

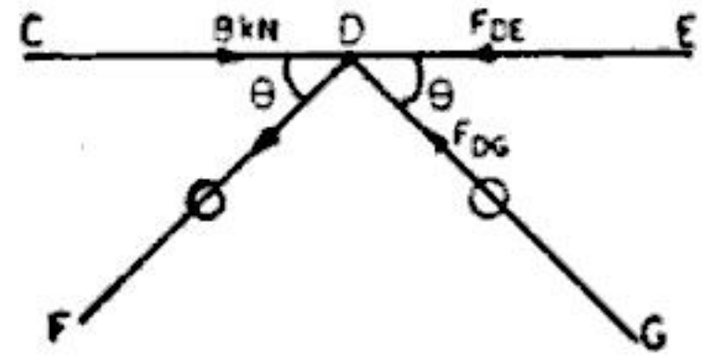

Fig. 6.23. (*e*)

Resolving vertically, we get

$F_{DG} \sin\theta = F_{DF} \times \sin\theta = 0$

$\therefore$ $F_{DG} = 0$

Resolving forces horizontally, we get

$F_{DE} = F_{CD} = 8$ kN

$\therefore$ $F_{DE} = 8$ kN (Compressive)

Now consider the joint *G*.

Joint G

The forces F_{DG} and F_{FG} are known in magnitude and direction. The assumed directions of F_{GE} and F_{GB} are shown in Fig. 6.23 (*f*).

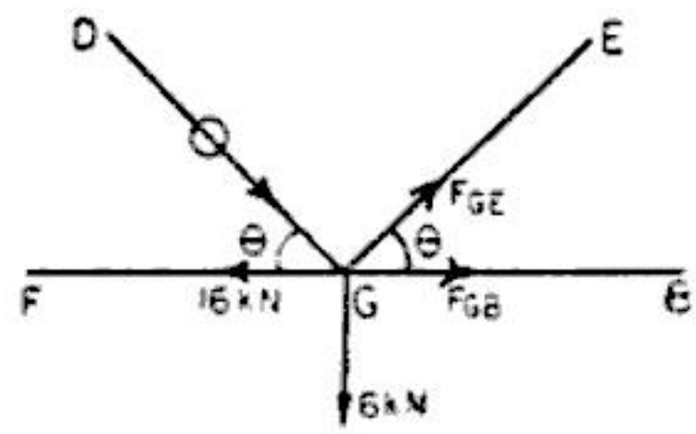

Fig. 6.23. (*f*)

Resolving the forces vertically, we get

$F_{GE} \sin\theta = F_{DG} \sin\theta + 6 = 6$

or $$F_{GE} = \frac{6}{\sin\theta} = \frac{6}{0.6}$$

$= 10$ kN (Tensile)

Resolving forces horizontally,

$F_{GB} = 16 - F_{GE} \cos\theta$

$= 16 - 10 \times 0.8 = 8$ kN (Tensile)

Now consider the joint *E*.

Joint E

The forces F_{GE} and F_{DE} are known in magnitude and directions. Let F_{BE} is acting in a direction as shown in Fig. 6.23 (*g*).

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Resolving forces vertically, we get

$$F_{GE} \sin \theta = F_{BE} \sin \theta$$

$$\therefore \quad F_{BE} = F_{GE} = 10$$

$$\therefore \quad F_{BE} = 10 \text{ kN}$$

(Compressive)

If we have calculated the forces in member BE and BG, considering joint B, we would have got the same results.

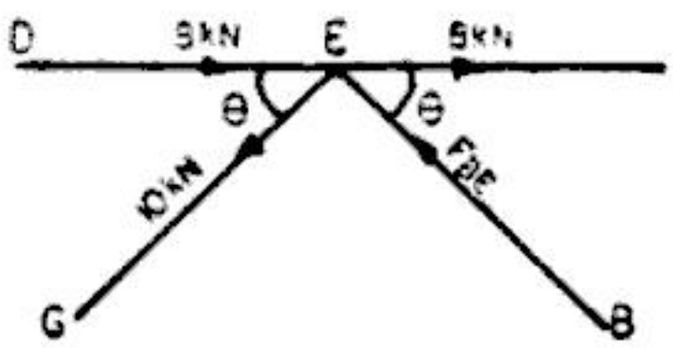

Fig. 6.23. (g)

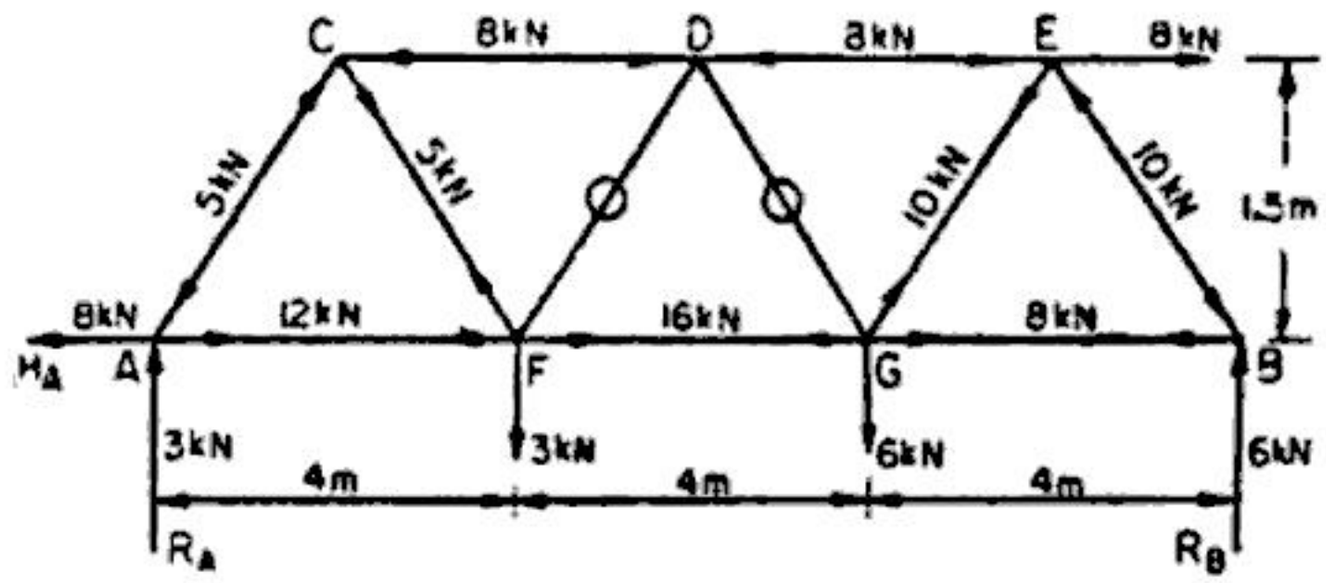

Fig. 6.23. (h)

Now the forces in each member are shown in Fig. 6.23 (h).

6.5.4. Method of Joints applied to Trusses carrying inclined loads. If a truss carries inclined loads, hinged at one end and supported on rollers at the other end, then the support reaction at the roller supported end will be normal, whereas the support reactions at the hinged end will consists of :

(i) Horizontal reaction and (ii) Vertical reaction.

The inclined loads are resolved into horizontal and vertical components.

The horizontal reaction will be obtained by adding algebraically all the horizontal components of the inclined loads ; whereas the vertical reaction will be obtained by subtracting the roller support reaction from the total vertical components of the inclined loads. Now the forces in the members of truss can be determined.

Problem 6.10. *Determine the forces in the truss shown in Fig. 6.24 which is subjected to inclined loads.*

Sol. The truss is supported on roller at B and hence R_B will be vertical.

The truss is hinged at A and hence the support reactions at A will consists of a horizontal reaction H_A and a vertical reaction R_A.

Now length $AC = 4 \times \cos 30°$

$$= 4 \times 0.866$$

$$= 3.464 \text{ m}$$

and length $AD = 2 \times AC$

$$= 2 \times 3.464$$

$$= 6.928 \text{ m}$$

Now taking moments about A, we get

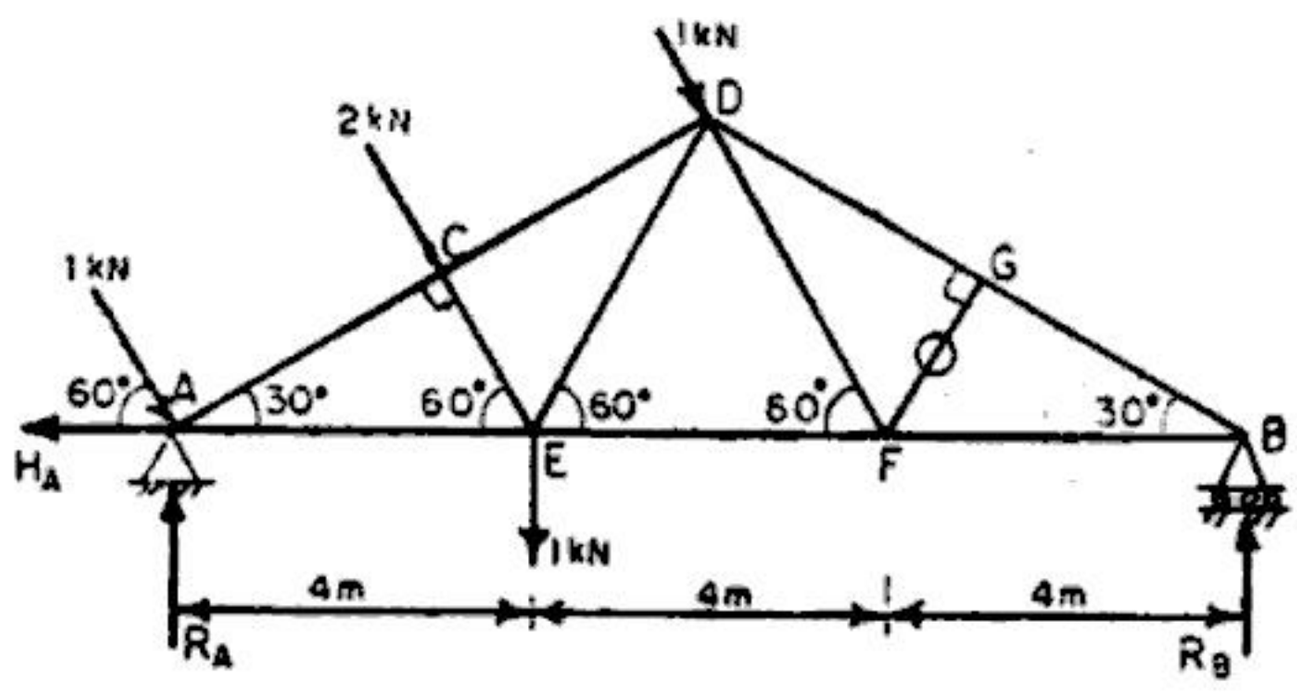

Fig. 6.24.

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At the joint G, two forces, *i.e.*, F_{BG} and F_{DG} are in the same straight line and hence the third force, *i.e.*, F_{GF} should be zero.

$\therefore \quad F_{GF}=0$

Now consider the joint F.

Joint F [See Fig. 6.24 (*d*)]

Resolving forces vertically, we get

$$F_{DF}\times\sin 60^\circ=0$$

$$\therefore \quad F_{DF}=0$$

Resolving horizontally, we get

$$F_{FB}=F_{EF}=2.58 \text{ kN}$$

$$\therefore \quad F_{FB}=2.58 \text{ kN} \quad \text{(Compressive)}$$

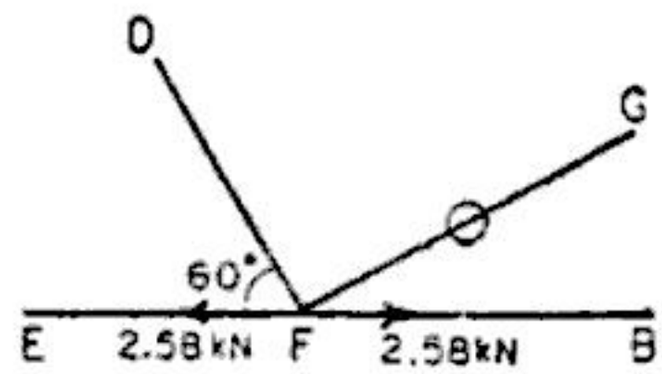

Fig. 6.24. (*d*)

Now consider the joint B.

Joint B

Resolving vertically, we get

$F_{BG}\times\sin 30^\circ=1.49$

$$\therefore \quad F_{BG}=\frac{1.49}{0.5}=2.98 \text{ kN} \quad \text{(Compressive)}$$

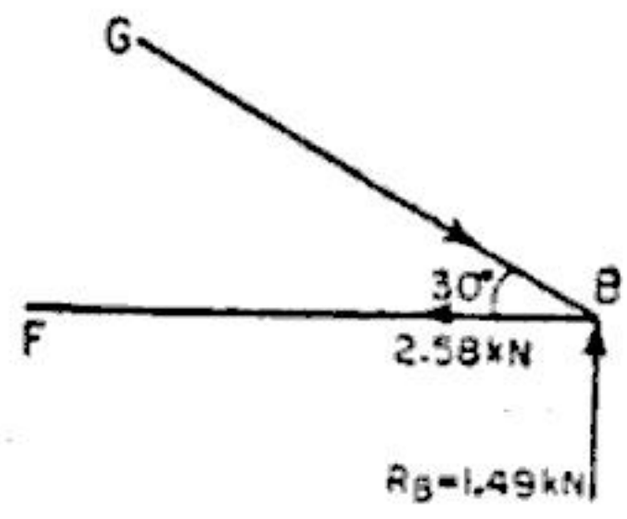

Fig. 6.24. (*e*)

Joint G

$F_{GD}=F_{BG}\doteq 2.98$ kN (Compressive)

The forces are shown in a tabular form as

Member	*Force in the member*	*Nature of force*
AC	4.216 kN	Compressive
AE	5.15 kN	Tensile
CE	2 kN	Compressive
CD	4.216 kN	Compressive
ED	3.155 kN	Tensile
EF	2.58 kN	Tensile
DF	0	Nil
DG	2.98 kN	Compressive
GB	2.98 kN	Compressive
FB	2.58 kN	Compressive
FG	0	Nil

6.6. METHOD OF SECTIONS

When the forces in a few members of a truss are to be determined, then the method of section is mostly used. This method is very quick as it does not involve the solution of other joints of the truss.

In this method, a section line is passed through the members, in which forces are to be determined as shown in Fig. 6.25. The section line should be drawn in such a way that it does not cut more than three members in which the forces are unknown. The part of the truss, on any one side of the section line, is treated as a free body in equilibrium under the action of

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the right part). This part is shown in Fig. 6.30 (*a*). Let F_1, F_2 and F_3 are the forces members 1, 2 and 3 respectively. Let their directions are assumed as shown in Fig. 6.30 (*a*).

Taking moments of all the forces acting on the left part about point D, we get

$$10\times3=F_3\times4$$

$$\therefore \quad F_3=\frac{10\times3}{4}=7.5 \text{ kN (Tensile).} \quad \textbf{Ans.}$$

Now taking the moments of all the forces acting on the left part about point G, we get

$$10\times3+F_1\times4=0$$

$$\therefore \quad F_1=\frac{-30}{4}=-7.5 \text{ kN}$$

Negative sign shows that force F_1 is compressive

$$\therefore \quad F_1=7.5 \text{ kN (Compressive).} \quad \textbf{Ans.}$$

Now taking the moments about the point C, we get

$$F_2\times3-9\times3+F_3\times4=0$$

or $$F_2\times3-27+7.5\times4=0 \qquad (\because F_3=7.5)$$

or $$F_2=\frac{27-7.5\times4}{3}=\frac{-3}{3}=-1.0 \text{ kN}$$

Negative sign shows that force F_2 is compressive.

$$\therefore \quad F_2=1.0 \text{ kN (Compressive).} \quad \textbf{Ans.}$$

Problem 6.14. *For the pin-joined truss shown in Fig. 6.31, find the forces in the members marked 1, 2 and 3 with the single load of 80 kN as shown.*

Sol. First calculate reactions R_A and R_B.

Taking moments about A,

$$R_B\times16=80\times12$$

$$\therefore \quad R_B=\frac{80\times12}{16}=60 \text{ kN}$$

$$\therefore \quad R_A=\text{Total vertical load}-R_B$$

$$=80-60=20 \text{ kN}$$

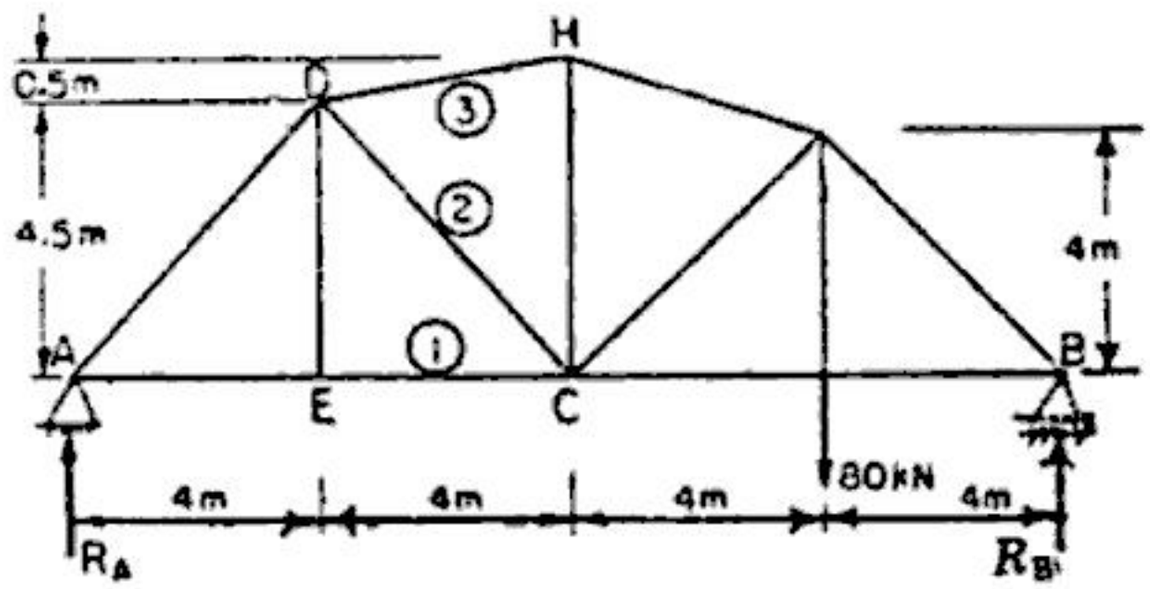

Fig. 6.31.

Let us now find the forces in the members 1, 2 and 3 by the method of section. Take a section Y-Y passing through the members 1, 2 and 3. Now consider the equilibrium of left portion shown in Fig. 6.31 (*a*).

Let F_1, F_2 and F_3 are the forces in the members 1, 2 and 3 respectively. Their assumed directions are also shown in Fig. 6.31 (*a*).

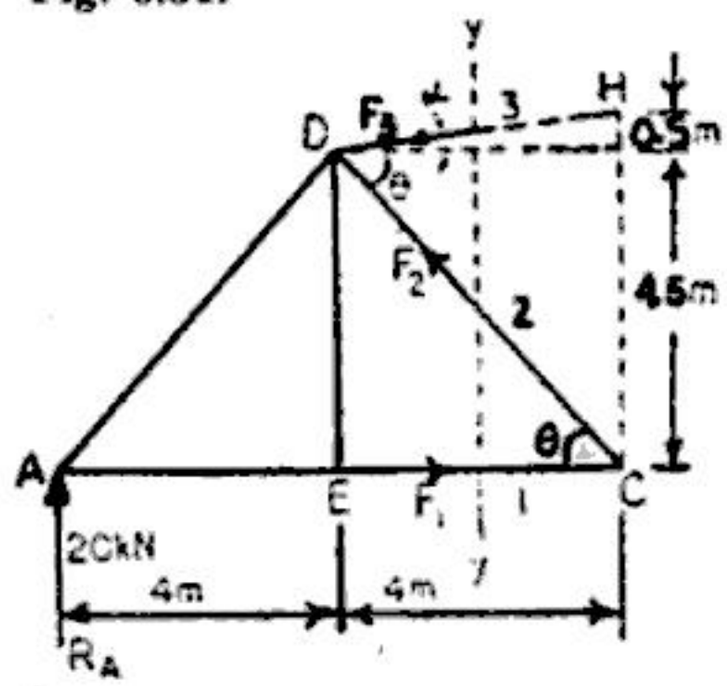

Fig. 6.31. (*a*)

Taking moments of all forces (here R_A, F_1, F_2 and F_3) about point D, we get

$$R_A\times4=F_1\times4.5$$

$$\therefore \quad F_1=\frac{R_A\times4}{4.5}=\frac{20\times4}{4.5}=17.78 \text{ kN (Tensile)} \quad \textbf{Ans.}$$

Now taking the moments about C, we get

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is shown in Fig. 6.33 (*a*). Let F_{DG}, F_{FD} and F_{EF} are the forces in members *DG*, *FD* and *EF* respectively. Let their directions are assumed as shown in Fig. 6.33 (*a*). Taking moments of all forces acting on right part about point *F*, we get

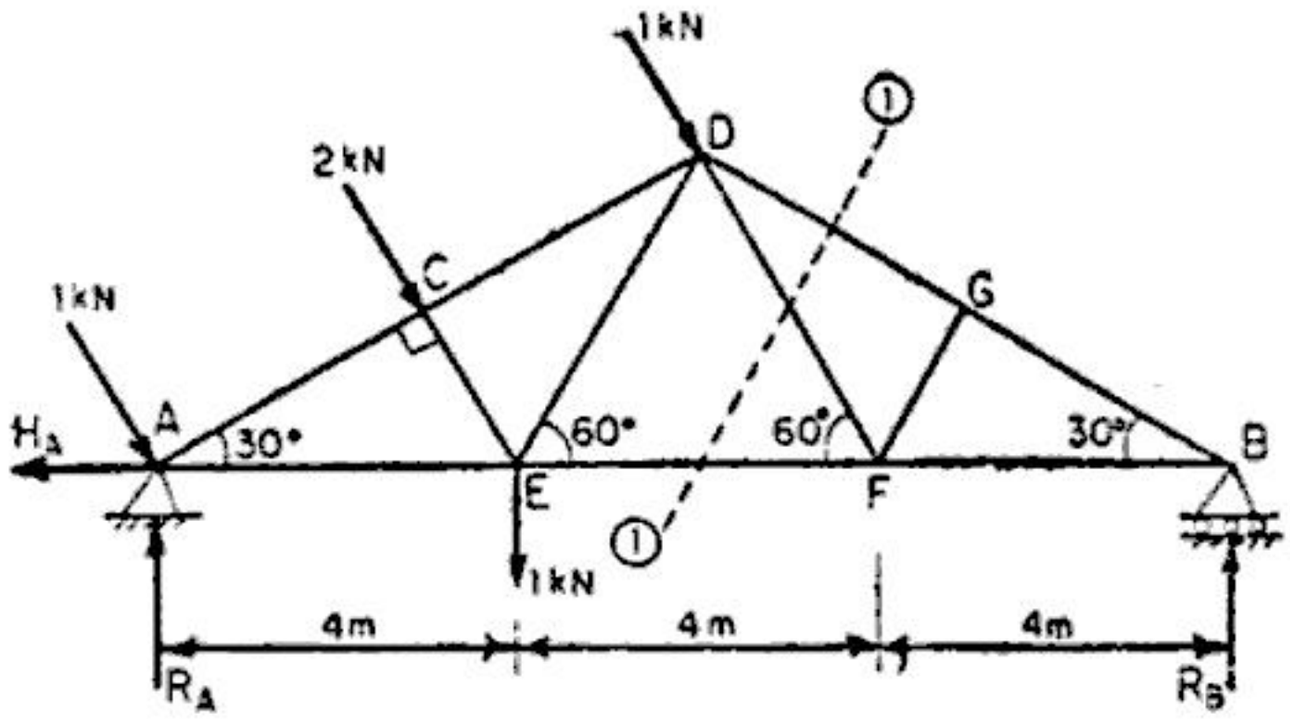

Fig. 6.33.

$$R_B \times 4 + F_{DG} \times FG = 0$$

or $$1.49 \times 4 + F_{DG} \times (4 \times \sin 30°) = 0 \quad (\because FG = 4 \times \sin 30°)$$

or $$F_{DG} = \frac{-1.49 \times 4}{4 \times \sin 30°} = -2.98 \text{ kN}$$

−ve sign shows that the force F_{DG} is compressive.

$\therefore$ $F_{DG} = 2.98$ kN (Compressive). **Ans.**

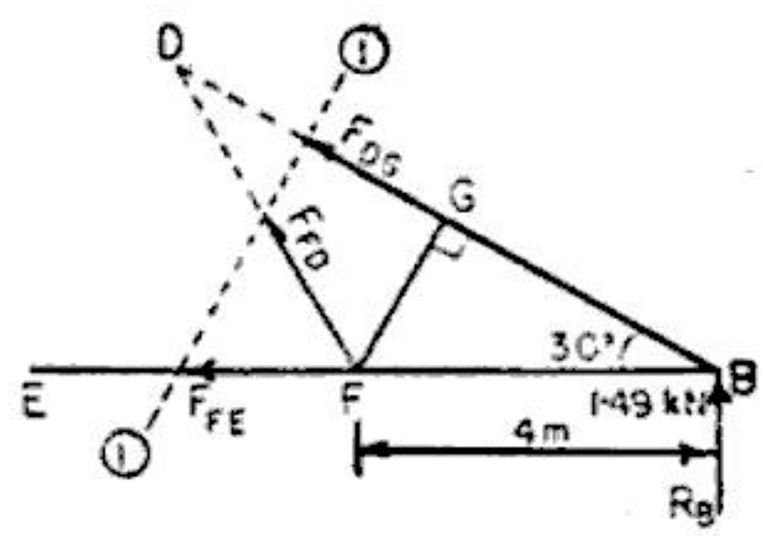

Fig. 6.33. (*a*)

Now taking the moments about point *D*, we get

$$R_B \times BD \cos 30° = F_{FE} \times BD \times \sin 30°$$

or $$R_B \times \cos 30° = F_{FE} \times \sin 30°$$

$\therefore$ $$F_{FE} = \frac{1.49 \times \cos 30°}{\sin 30°} = \frac{1.49 \times 0.866}{0.5}$$

$$= 2.58 \text{ kN (Tensile)}. \quad \textbf{Ans.}$$

Now taking the moments of all forces acting on the right part about *B*, we get

$F_{FD} \times \perp$ distance between F_{FD} and $B = 0$

$\therefore$ $F_{FD} = 0$. **Ans.** ($\because$ $\perp$ distance between F_{FD} and *B* is not zero)

6.7. GRAPHICAL METHOD

The force in a perfect frame can also be determined by a graphical method. The analytical methods (such as method of joints and method of sections) give absolutely correct results, but sometimes it is not possible to get the results from analytical methods. Then a graphical method can be used conveniently to get the results. The graphical method also provides reasonable accurate results.

The naming of the various members of a frame are done according to *Bow's notations*. According to this notation a force is designated by two capital letters which are written on either side of the line of action of the force. A force with letters *A* and *B* on either side of the line of action is shown in Fig. 6.34. This force will be called *AB*.

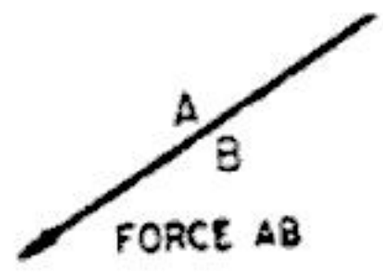

Fig. 6.34.

The following steps are necessary for obtaining a graphical solution of a frame.

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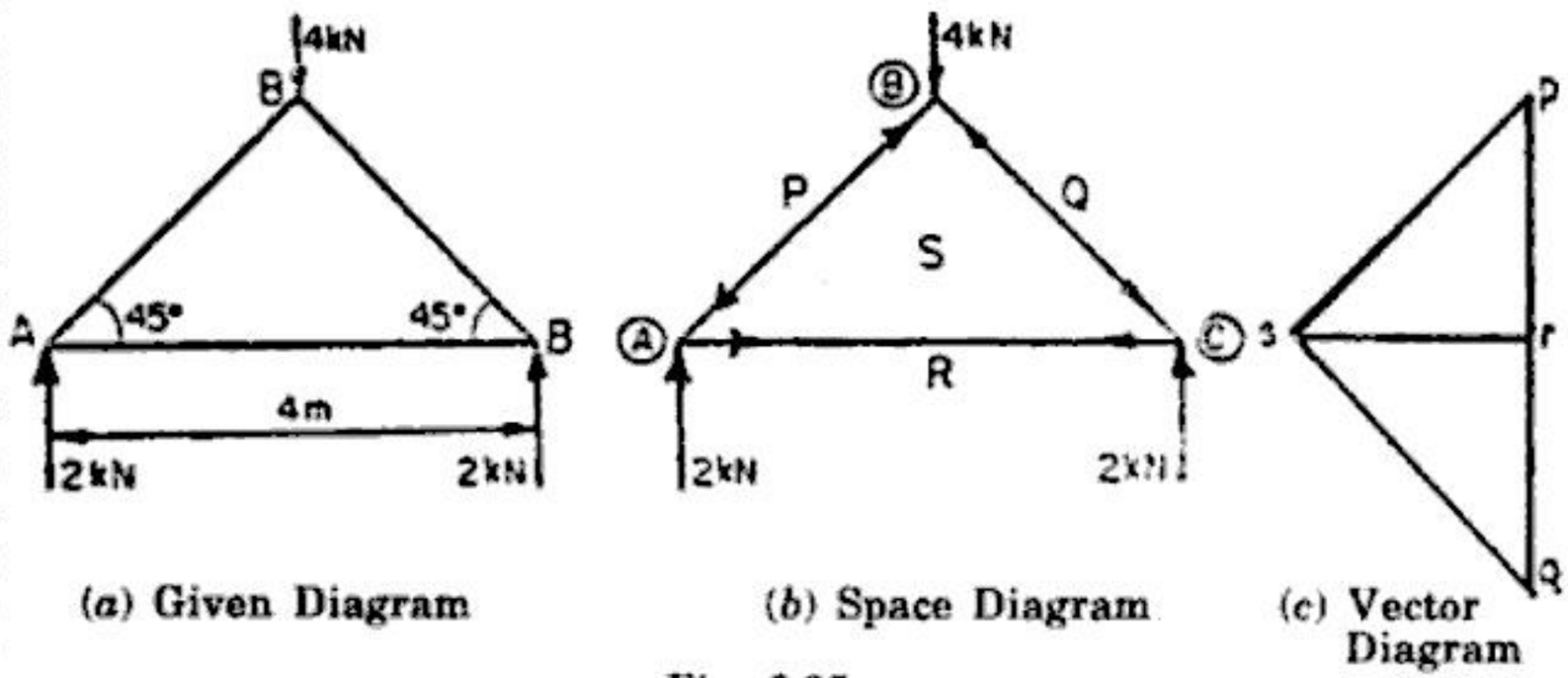
(*i*) Making a space diagram

(*ii*) Constructing a vector diagram

(*iii*) Preparing a force table.

1. Making a space diagram. The given truss or frame is drawn accurately according to some linear scale. The loads and support reactions in magnitude and directions are also shown on the frame. Then the various members of the frame are named according to Bow's notation. Fig. 6.35 (*a*) shows a given truss and the forces in the members *AB*, *BC* and *AC* are to be determined. Fig. 6.35 (*b*) shows the space diagram to same linear scale. The member *AB* is named as *PS* and so on.



(*a*) Given Diagram (*b*) Space Diagram (*c*) Vector Diagram

Fig. 6.35.

2. Constructing a vector diagram. Fig. 6.35 (*c*) shows a vector diagram, which is drawn as given below :

(*i*) Take any point *p* and draw *pq* parallel to *PQ* vertically downwards. Cut *pq*=4 kN to same scale.

(*ii*) Now from *q* draw *qr* parallel to *QR* vertically upwards and cut *qr*=2 kN to the same scale.

(*iii*) From *r* draw *rp* parallel to *RP* vertically upwards and cut *rp*=2 kN to the same scale.

(*iv*) Now from *p*, draw a line *ps* parallel to *PS* and from *r*, draw a line *rs* parallel to *RS*, meeting the first line at *s*. This is vector diagram for joint (*A*). Similarly the vector diagrams for joint (*B*) and (*C*) can be drawn.

3. Preparing a force table. The magnitude of a force in a member is known by the length of the vector diagram for the corresponding member, *i.e.*, the length *ps* of the vector diagram will give the magnitude of force in the member *PS* of the frame.

Nature of the force (*i.e.*, tensile or compressive) is determined according to the following procedure :

(*i*) In the space diagram, consider any joint. Move round that joint in a *clockwise direction*. Note the order of two capital letters by which the members are named. For example, the members at the joint (*A*) in space diagram Fig. 6.35 (*b*) are named as *PS*, *SR* and *RP*.

(*ii*) Now consider the vector diagram. Move on the vector diagram in the order of the letters (*i.e.*, *ps*, *sr* and *rp*).

(*iii*) Now mark the arrows on the members of the space diagram of that joint (here joint *A*).

(*iv*) Similarly, all the joints can be considered and arrows can be marked.

(*v*) If the arrow is pointing towards the joint, then the force in the member will be compressive whereas if the arrow is away from the joint, then the force in the member will be tensile.

Problem 6.17. *Find the forces in the members AB, AC and BC of the truss shown in Fig. 6.36.*

Sol. First determine the reactions R_B and R_C.

From Fig. 6.36 (*a*), $AB = BC \times \cos 60° = 5 \times \frac{1}{2} = 2.5$ m

Distance of line of action of 20 kN from point *B*

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$$=AB \cos 60° = 2.5 \times \frac{1}{2} = 1.25 \text{ m}$$

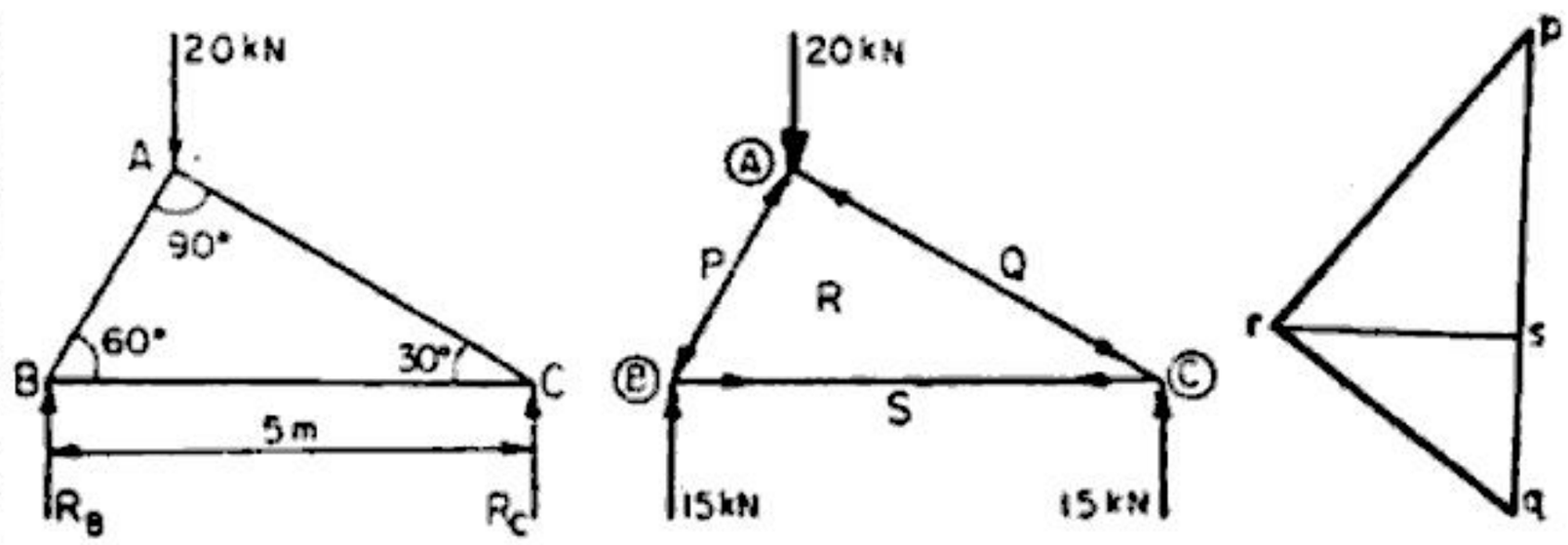
Now taking moments about B, we get

$$R_C \times 5 = 20 \times 1.25 = 25$$

$$\therefore \quad R_C = \frac{25}{5} = 5 \text{ kN}$$

and $$R_B = 20 - 5 = 15 \text{ kN}$$

Now draw the space diagram for the truss alongwith load of 20 kN and the reactions R_B and R_C equal to 15 kN and 5 kN respectively as shown in Fig. 6.36 (*b*). Name the members *AB*, *AC* and *BC* according to Bow's notations as *PR*, *QR* and *RS* respectively. Now construct the vector diagram as shown in Fig. 6.36 (*c*) and as explained below :



(*a*) Given Diagram (*b*) Space Diagram (*c*) Vector Diagram

Fig. 6.36.

(*i*) Take any point *p* and draw a vertical line *pq* downward equal to 20 kN to some suitable scale. From *q* draw a vertical line *qs* upward equal to 5 kN to the same scale to represent the reaction at *C*. Then *sp* will represent the reaction R_B to the scale.

(*ii*) Now draw the vector diagram for the joint (*B*). From *p*, draw a line *pr* parallel to *PR* and from *s* draw a line *sr* parallel to *SR*, meeting the first line at *r*. Now *prs* is the vector diagram for the joint (*B*). Now mark the arrows on the joint *B*. The arrow in member *PR* will be towards to joint *B*, whereas the arrow in the member *RS* will be away from the joint *B* as shown in Fig. 6.36 (*b*).

(*iii*) Similarly draw the vector diagrams for joint *A* and *C*. Mark the arrows on these joints in space diagram.

Now measure the various sides of the vector diagram. The forces are obtained by multiplying the scale factor. The forces in the members are given in a tabular form as :

Member		*Force in member*	*Nature of force*
According to given truss	*According to Bow's notation*		
AB	*PR*	17.3 kN	Compressive
AC	*QR*	10.0 kN	Compressive
BC	*RS*	8.7 kN	Tensile

Problem 6.18. *A truss of span 7.5 m carries a point load of 1000 N at joint D as shown in Fig. 6.37. Find the reactions and forces in the member of the truss.*

Sol. First determine the reactions R_A and R_B

Taking moments about *A*, we get

$$R_B \times 7.5 = 5 \times 1000$$

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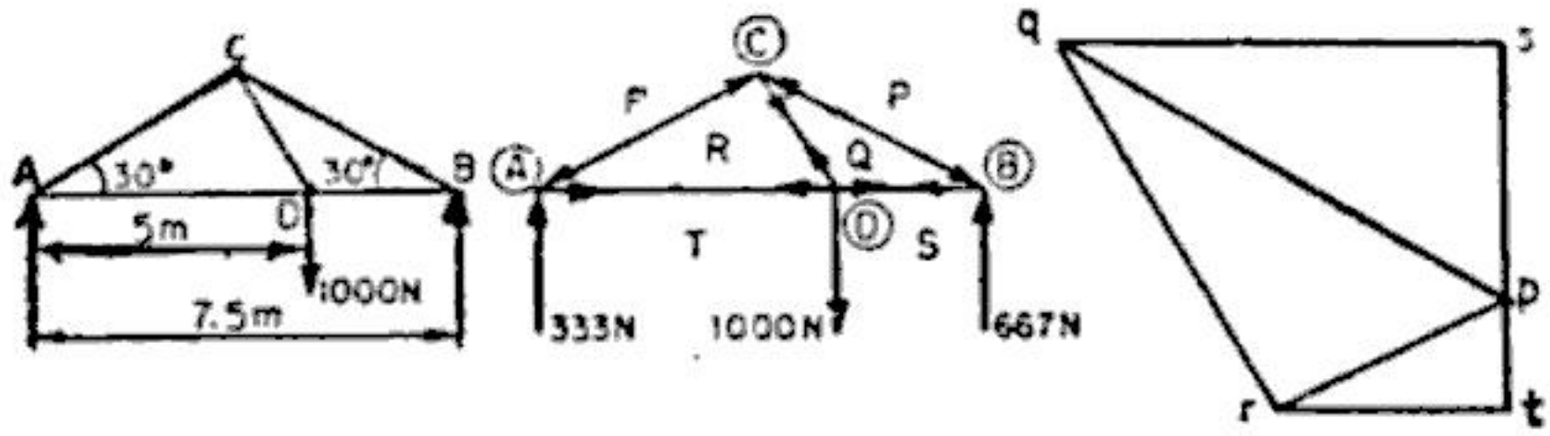
$$\therefore \qquad R_A = \frac{5000}{7.5} = 667 \text{ N}$$

and $$R_A = 1000 - 667 = 333 \text{ N}.$$

Now draw the space diagram for the truss alongwith load of 1000 N and reactions R_A and R_B equal to 333 N and 667 N respectively as shown in Fig. 6.37 (*b*). Name the members *AC*, *CB*, *AD*, *CD* and *DB* according to Bow's notations as *PR*, *PQ*, *RT*, *QR* and *QS* respectively. Now construct the vector diagram as shown in Fig. 6.37 (*c*) and as explained below :



(*a*) Given Diagram (*b*) Space Diagram (*c*) Vector Diagram

Fig. 6.37.

(*i*) Take any point *s* and draw a vertical line *st* downward equal to load 1000 N to some suitable scale. From *t* draw a vertical line *tp* upward equal to 333 N to the same scale to represent the reaction at *A*. Then *ps* will represent the reaction R_B to the scale.

(*ii*) Now draw the vector diagram for the joint *A*. From *p*, draw a line *pr* parallel to *PR* and from *t* draw a line *tr* parallel to *RT*, meeting the first line at *r*. Now *prt* is the vector diagram for the joint *A*. Now mark the arrows on the joint *A*. The arrow in the member *PR* will be towards the joint *A*, whereas the arrow in the member *RT* will be away from the joint *A* as shown in Fig. 6.37 (*b*).

(*iii*) Similarly draw the vector diagrams for the joint *C*, *B* and *D*. Mark the arrows on these joints as shown in Fig. 6.37 (*b*).

Now measure the various sides of the vector diagrams. The forces in the members are obtained by multiplying the scale factor to the corresponding sides of the vector diagram. The forces in members are given in a tabular form as :

Member		*Force in member*	*Nature of force*
According to given truss	*According to Bow's notation*		
AC	*PR*	666 N	Compressive
AD	*RT*	576.7 N	Tensile
CB	*PQ*	1333 N	Compressive
CD	*QR*	1155 N	Tensile
DB	*QS*	1555 N	Tensile

Problem 6.19. *Determine the forces in all the members of a cantilever truss shown in Fig. 6.38.*

Sol. In this case the vector diagram can be drawn without knowing the reactions. First of all draw the space diagram for the truss along with loads of 1000 N of joints *B* and *C*. Name the members *AB*, *BC*, *CD*, *DE*, *AD* and *BD* according to Bow's notation as *PT*, *QS*, *SR*, *RV*, *VT* and *ST* respectively. Now construct the vector diagram as shown in Fig. 6.38 (*c*) and as explained below :

(*i*) The vector diagram will be started from joint *C* where forces in two members are unknown. Take any point *q* and draw a vertical line *qr* downward equal to load 1000 N to some suitable scale. From *r*, draw a line *rs* parallel to *RS* and from *q* draw a line *qs* parallel to *QS*, meeting the first line at *s*. Now *qrS* is the vector diagram for the joint *C*. Now mark

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the arrows on the joint C. The arrow in the member RS will be towards the joint C, whereas the arrow in the member SQ will be away from the joint C as shown in Fig. 6.38 (*b*).

(*ii*) Now draw the vector diagram for the joints B and D similarly.

Mark the arrows on these joints as shown in Fig. 6.38 (*b*).

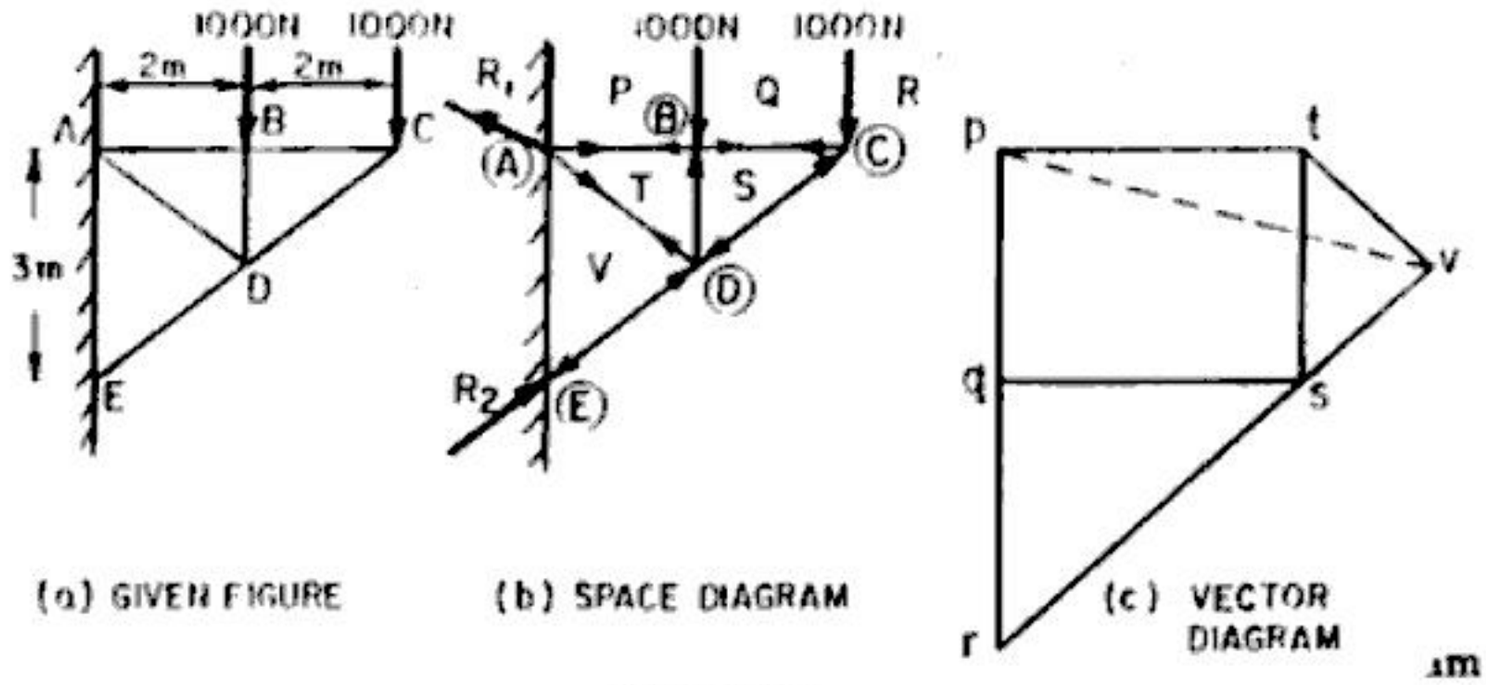

Fig. 6.38.

Now measure the various sides of the vector diagram. The forces in the members are given in a tabular form as :

Member		*Force in member*	*Nature of force*
According to given truss	*According to Bow's notation*		
AB	*PT*	1333 N	Tensile
BC	*QS*	1333 N	Tensile
CD	*SR*	1666 N	Compressive
DE	*RV*	2500 N	Compressive
AD	*VT*	833 N	Tensile
BD	*ST*	1000 N	Compressive

From the vector diagram, the reactions R_1 and R_2 at A and E can be determined in magnitude and directions.

Reaction $R_2 = rv = 2500$ N. This will be towards point E.

Reaction $R_1 = vp = 2000$ N. This will be away from the point A as shown in Fig. 6.38 (*b*). The reaction R_1 is parallel to vp.

HIGHLIGHTS

1. The relation between number of joints (j) and number of members (n) in a perfect frame is given by

$$n = 2j - 3.$$

2. Different frame is a frame in which number of numbers are less than $(2j - 3)$ whereas a redundant frame is a frame in which number of members are more than $(2j - 3)$.
3. The reaction on a roller support is at right angles to the roller base :
4. The forces in the members of a frame are determined by :
   (*i*) Method of joints (*ii*) Method of sections and (*iii*) Graphical method.
5. The force in a member will be compressive if the member pushes the joint to which it is connected whereas the force in the member will be tensile if the member pulls the joint to which it is connected.
6. While determining forces in a member by method of joints, the joint should be selected in such a way that at any time there are only two members, in which the forces are unknown.
7. If three forces act at a joint and two of them are along the same straight line then third force would be zero.
8. If a truss (or frame) carries horizontal loads, then the support reaction at the hinged end will consists of (*i*) horizontal reaction and (*ii*) vertical reaction.
9. If a truss carries inclined loads, then the support reaction at the hinged end will consists of : (*i*) horizontal reaction and (*ii*) vertical reaction. They will be given as :

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7

Centre of Gravity and Moment of Inertia

7.1. CENTRE OF GRAVITY

Centre of gravity of a body is the point through which the whole weight of the body acts. A body is having only one centre of gravity for all positions of the body. It is represented by C.G. or simply G.

7.2. CENTROID

The point at which the total area of a plane figure (like rectangle, square, triangle, quadrilateral, circle etc.) is assumed to be concentrated, is known as the centroid of that area. The centroid is also represented by C.G. or simply G. The centroid and centre of gravity are at the same point.

7.3. CENTROID OR CENTRE OF GRAVITY OF SIMPLE PLANE FIGURES

(*i*) The centre of gravity (C.G.) of a uniform rod lies at its middle point.

(*ii*) The centre of gravity of a triangle lies at the point where the three medians* of the triangle meet.

(*iii*) The centre of gravity of a rectangle or of a parallelogram is at the point, where its diagonal meet each other. It is also the point of intersection of the lines joining the middle points of the opposite sides.

(*iv*) The centre of gravity of a circle is at its centre.

7.4. CENTRE OF GRAVITY OF PLANE FIGURES BY THE METHOD OF MOMENTS

Fig. 7.1 shows a plane figure of total area A whose centre of gravity is to be determined. Let this area A is composed of a number of small areas a_1, a_2, a_3, a_4, ... etc.

$\therefore \quad A = a_1 + a_2 + a_3 + a_4 + ...$

Let x_1 = The distance of the C.G. of the area a_1 from axis OY

x_2 = The distance of the C.G. of the area a_2 from axis OY

x_3 = The distance of the C.G. of the area a_3 from axis OY

x_4 = The distance of the C.G. of the area a_4 from axis OY and so on.

The moments of all small areas about the axis OY

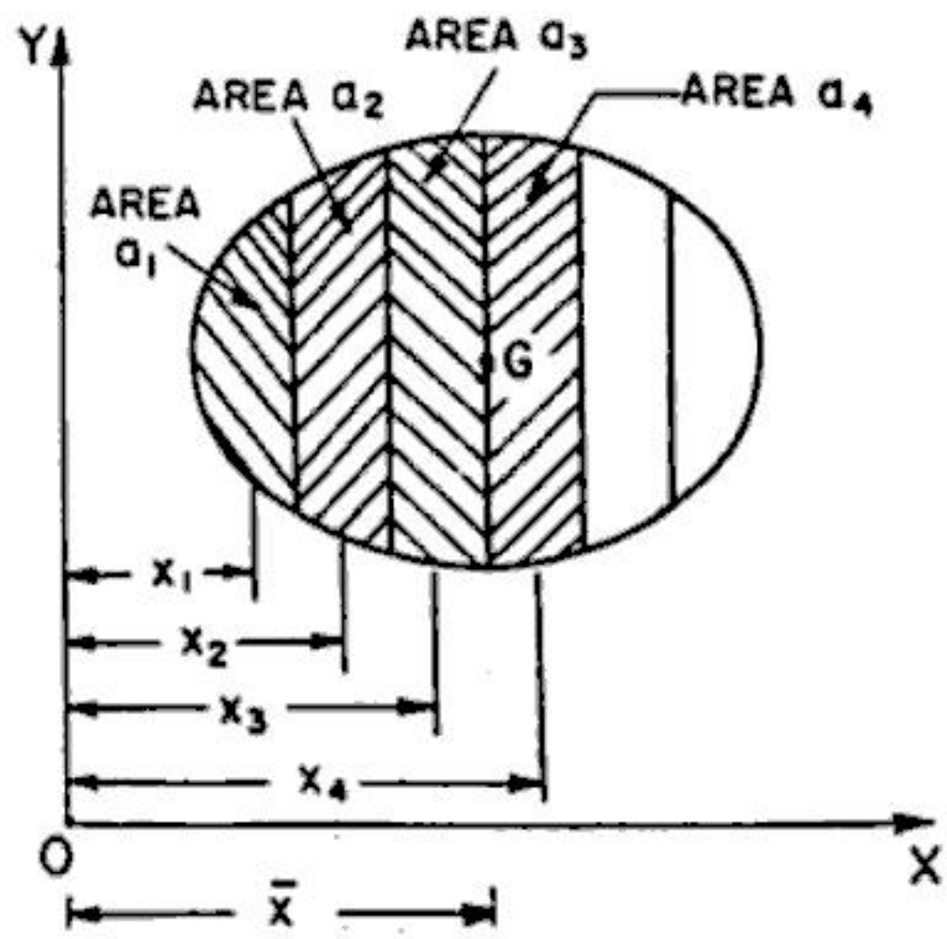

Fig. 7.1.

*The line connecting the vertex and the middle point of the opposite side of a triangle is known as median of the triangle.

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y_2=Distance of C.G. of rectangle *EFGH* from bottom line *ML*

$=2+\frac{15}{2}=2+7.5=9.5$ cm

a_3=Area of rectangle *JKLM*$=20\times2=40$ cm^2

y_3=Distance of C.G. of rectangle *JKLM* from bottom line *ML*

$=\frac{2}{2}=1.0$ cm

Now using equation (7.2), we have

$$\bar{y}=\frac{a_1y_1+a_2y_2+a_3y_3}{A}$$

$$=\frac{a_1y_1+a_2y_2+a_3y_3}{a_1+a_2+a_3} \qquad (\because A=a_1+a_2+a_3)$$

$$=\frac{20\times18+30\times9.5+40\times1}{20+30+40}=\frac{360+285+40}{90}=\frac{685}{90}$$

=7.611 cm. Ans.

Problem 7.3. *Find the centre of gravity of the L-section shown in Fig. 7.4.*

Sol. The given *L*-section is not symmetrical about any section. Hence in this case, there will be two axis of references. The lowest line of the figure (*i.e.*, line *GF*) will be taken as axis of reference for calculating $\bar{y}$. And the left line of the *L*-section (*i.e.*, line *AG*) will be taken as axis of reference for calculating $\bar{x}$.

The given *L*-section is split up into two rectangles *ABCD* and *DEFG*, as shown in Fig. 7.4.

To Find $\bar{y}$

Let $\bar{y}$=Distance of the C.G. of the *L*-section from bottom line *GF*

a_1=Area of rectangle *ABCD* $=10\times2=20$ cm^2

y_1=Distance of C.G. of rectangle *ABCD* from bottom line *GF*

$=2+\frac{10}{2}=2+5=7$ cm

a_2=Area of rectangle *DEFG*$=8\times2=16$ cm^2

y_2=Distance of C.G. of rectangle *DEFG* from bottom line *GF*

$=\frac{2}{2}=1.0$ cm.

Using equation (7.2), we have

$$\bar{y}=\frac{a_1y_1+a_2y_2}{A} \qquad \text{where } A=a_1+a_2$$

$$=\frac{a_1y_1+a_2y_2}{a_1+a_2}$$

$$=\frac{20\times7+16\times1}{20+16}=\frac{140+16}{36}$$

$$=\frac{156}{36}=\frac{13}{3}=4.33 \text{ cm.}$$

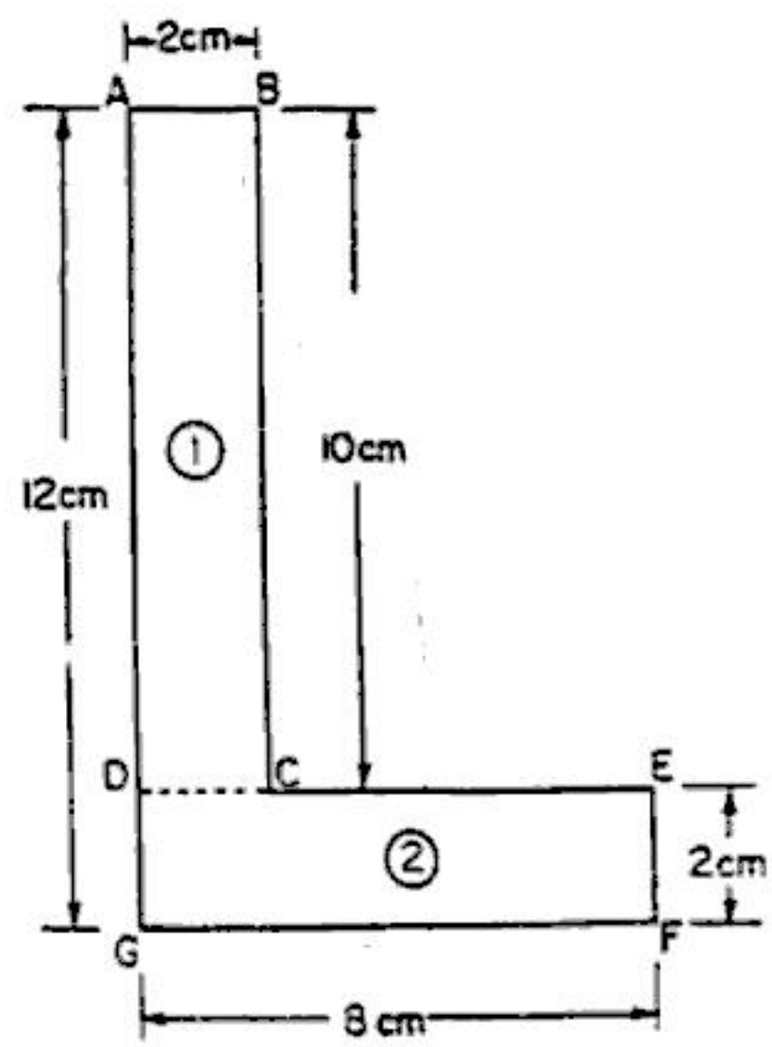

Fig. 7.4.

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$\therefore$ Distance of C.G. of area dA from y-axis$=x$

and distance of C.G. of area dA from x-axis$=\frac{y}{2}$

$$\therefore \qquad x^*=x \quad \text{and} \quad y^*=\frac{y}{2}$$

Let $\bar{x}$=Distance of C.G. of area OAB from axis OY

$\bar{y}$=Distance of C.G. of area OAB from axis OX

Using equation (7.2 A), we get

$$\bar{x}=\frac{\int x^* dA}{\int dA}=\frac{\int_0^6 x\times y dx}{\int_0^6 y dx} \qquad \left(\because \begin{array}{c} dA=ydx \\ x^*=x \end{array}\right)$$

But $y=\frac{x^2}{9}$ from equation (*ii*).

$$\therefore \qquad \bar{x}=\frac{\int_0^6 x\times\frac{x^2}{9}\times dx}{\int_0^6 \frac{x^2}{9}dx}=\frac{\frac{1}{9}\int_0^6 x^3dx}{\frac{1}{9}\int_0^6 x^2dx}=\frac{\int_0^6 x^3dx}{\int_0^6 x^2dx}=\frac{\left[\frac{x^4}{4}\right]_0^6}{\left[\frac{x^3}{3}\right]_0^6}=\frac{\frac{1}{4}\times 6^4}{\frac{1}{3}\times 6^3}$$

$$=\frac{1}{4}\times\frac{3}{1}\times 6=4.5. \qquad \textbf{Ans.}$$

Using equation (7.2 B), we get

$$\bar{y}=\frac{\int y^* dA}{\int dA}$$

where y^*=distance of C.G. of area dA from x-axis

$$=\frac{y}{2} \text{ (here)}$$

$$dA=ydx$$

$$\therefore \qquad \int y^* dA=\int \frac{y}{2}\times dA=\int_0^6 \frac{y}{2}\times ydx=\int_0^6 \frac{y^2}{2}\, dx$$

$$=\frac{1}{2}\int_0^6 y^2dx=\frac{1}{2}\int_0^6 \left(\frac{x^2}{9}\right)^2 dx \qquad \left(\because y=\frac{x^2}{9}\right)$$

$$=\frac{1}{2}\int_0^6 \frac{x^4}{81}\, dx=\frac{1}{2}\times\frac{1}{81}\int_0^6 x^4\, dx=\frac{1}{2}\times\frac{1}{81}\left[\frac{x^5}{5}\right]_0^6$$

$$=\frac{1}{2}\times\frac{1}{81}\times\frac{6^5}{5}=\frac{6^5}{810}$$

Also $$\int dA=\int_0^6 ydx=\int_0^6 \frac{x^2}{9}\, dx=\frac{1}{9}\left[\frac{x^3}{3}\right]_0^6=\frac{1}{9}\times\frac{6^3}{3}=\frac{6^3}{27}$$

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$$=\frac{\int R\ dx}{\int dL}=\frac{R\int_0^R dx}{\int dL}=\frac{R\left[x\right]_0^R}{\left(\frac{2\pi R}{4}\right)}$$

($\because \int dL$ is total length of arc of one quadrant of a circle)

$$=\frac{R\times R}{\frac{2\pi R}{4}}=\frac{2R}{\pi}. \quad \textbf{Ans.}$$

Similarly, the value of $\bar{x}$ can be calculated. Due to symmetry this value will also be equal to $\frac{2R}{\pi}$.

$$\therefore \qquad \bar{x}=\bar{y}=\frac{2R}{\pi}. \quad \textbf{Ans.}$$

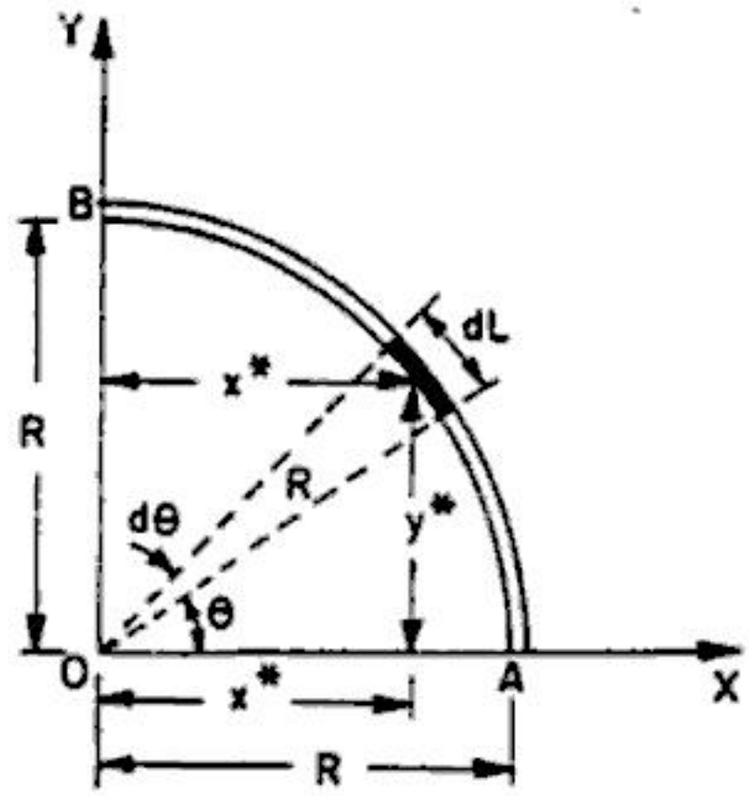

Fig. 7.9. (b)

2nd Method

Here $\quad dL=R\ d\theta$

$y^*=R\ \sin\ \theta$

$x^*=R\ \cos\ \theta$

$$\text{Now} \quad \bar{y}=\frac{\int y^*\ dL}{\int dL}=\frac{\int_0^{\pi/2}(R\ \sin\ \theta)\times(R\ d\theta)}{\int_0^{\pi/2} R\ d\theta}$$

$$=\frac{\int_0^{\pi/2} R^2\ \sin\ \theta\ d\theta}{\int_0^{\pi/2} R\ d\theta}=\frac{R^2\int_0^{\pi/2}\sin\ \theta\ d\theta}{R\int_0^{\pi/2} d\theta}=\frac{R^2\left[-\cos\theta\right]_0^{\pi/2}}{R\left[\theta\right]_0^{\pi/2}}=\frac{-R\left[\cos\left(\frac{\pi}{2}\right)-\cos 0\right]}{\left[\frac{\pi}{2}-0\right]}$$

$$=\frac{-R[0-1]}{\frac{\pi}{2}}=\frac{2R}{\pi}. \quad \textbf{Ans.}$$

Similarly,

$$\bar{x}=\frac{\int x^* dL}{\int dL}=\frac{\int_0^{\pi/2}(R\ \cos\ \theta)\times(R\ d\theta)}{\int_0^{\pi/2} R\ d\theta}=\frac{R^2\int_0^{\pi/2}\cos\ \theta\ d\theta}{R\int_0^{\pi/2} d\theta}$$

$$=\frac{R\left[\sin\ \theta\right]_0^{\pi/2}}{R\left[\theta\right]_0^{\pi/2}}=\frac{R[\sin\ 90^\circ-\sin\ 0^\circ]}{\left(\frac{\pi}{2}-0\right)}=\frac{R}{\left(\frac{\pi}{2}\right)}=\frac{2R}{\pi}. \quad \textbf{Ans.}$$

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Problem 7.9. *Determine the centre of gravity of the area of the circular sector OAB of radius R and central angle α as shown in Fig. 7.10.*

Sol. Divide the area OAB into a large number of triangular elements each of altitude R and base $Rd\theta$ as shown in Fig. 7.10. Such triangular element is shown by OCD in which altitude $OC=R$ and base $CD=Rd\theta$. The area dA of this triangular element is given by,

$$dA=\frac{OC\times CD}{2}=\frac{R\times Rd\theta}{2}=\frac{R^2d\theta}{2}$$

The C.G. of this triangular element is at G

where $OG=\frac{2}{3}\times OC=\frac{2}{3}\times R$

The distance of C.G. of area dA from y-axis is given by,

$$x^*=OG\times\cos\theta=\frac{2}{3}R\times\cos\theta$$

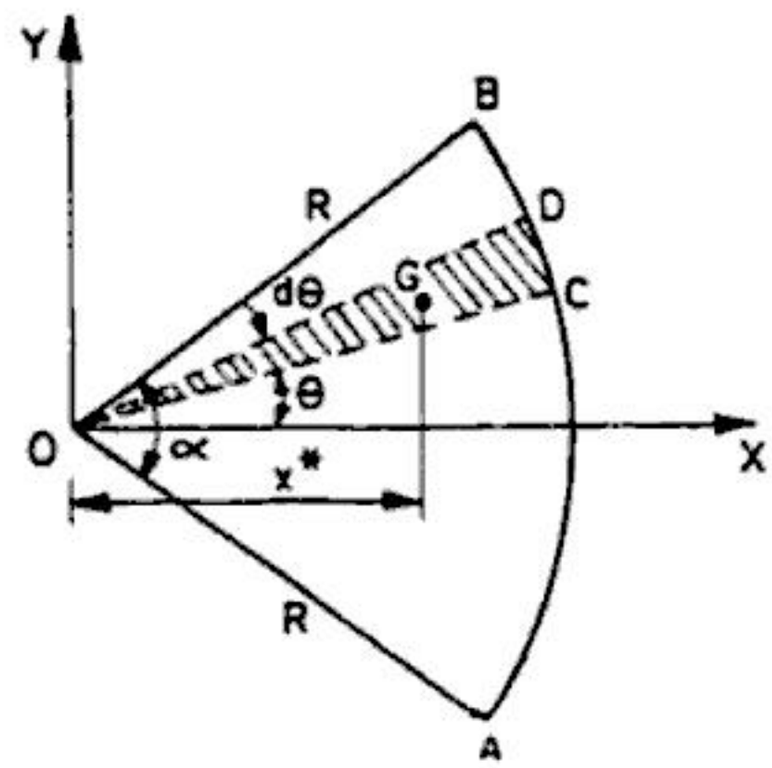

Fig. 7.10.

Now using equation (7.2 A),

$$\bar{x}=\frac{\int x^*\,dA}{\int dA}=\frac{2\int_0^{\alpha/2}\left(\frac{2}{3}R\cos\theta\right)\left(\frac{R^2\,d\theta}{2}\right)}{2\int_0^{\alpha/2}\frac{R^2}{2}\,d\theta}$$

$$=\frac{\frac{R^3}{3}\int_0^{\alpha/2}\cos\theta\,d\theta}{\frac{R^2}{2}\int_0^{\alpha/2}d\theta}=\frac{2R}{3}\,\frac{\left[\sin\theta\right]_0^{\alpha/2}}{\left[\theta\right]_0^{\alpha/2}}$$

$$=\frac{2R}{3}\,\frac{\sin\left(\frac{\alpha}{2}\right)}{\left(\frac{\alpha}{2}\right)}=\frac{4R}{3\alpha}\sin\left(\frac{\alpha}{2}\right).\quad\textbf{Ans.}$$

The area OAB is symmetrical about the x-axis, hence

$\bar{y}=0.$ **Ans.**

For a semi-circle, $\alpha=\pi=180°$, hence

$$\bar{x}=\frac{4R}{3\alpha}\sin\left(\frac{\pi}{2}\right)=\frac{4R}{3\times\pi}\sin\left(\frac{180}{2}\right)=\frac{4R}{3\pi}.\quad\textbf{Ans.}$$

7.6. MOMENT OF INERTIA

Consider a think lamina of area A as shown in Fig. 7.11.

Let x=Distance of the C.G. of area A from the axis OY.

y=Distance of the C.G. of area A from the axis OX.

Then moment of area about the axis OY.

=Area × perpendicular distance of C.G. of area from axis OY.

$=Ax$ ...(7.3)

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Equation (7.3) is known as *first moment of area* about the axis OY.

If the moment of area given by equation (7.3) is again multiplied by the perpendicular distance between the C.G. of the area and axis OY (*i.e.*, distance x), then the quantity $(Ax) \cdot x = Ax^2$ is known as moment or the area or *second moment of area* or moment of inertia about the axis OY.

Similarly, the moment of area (or first moment of area) about the axis $OX = Ay$.

And second moment of area (or moment of inertia) about the axis $OX = (Ay) \cdot y = Ay^2$.

If, instead of area, the mass (m) of the body is taken into consideration then the second moment is known as second moment of mass. This second moment of mass is also known as moment of inertia.

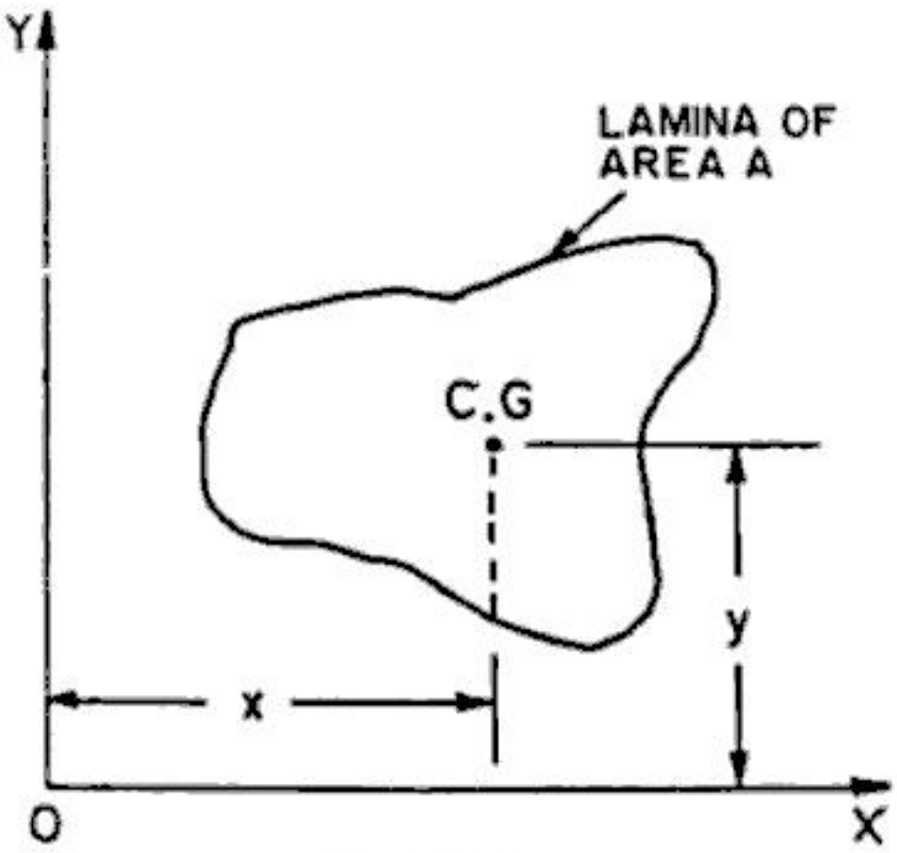

Fig. 7.11.

Hence moment of inertia when mass is taken into consideration about the axis $OY = mx^2$ and about the axis $OX = my^2$.

Hence the product of the area (or mass) and the square of the distance of the centre of gravity of the area (or mass) from as axis is known as moment of inertia of the area (or mass) about that axis. Moment of inertia is represented by I. Hence moment of inertia about the axis OX is represented by I_{xx} whereas about the axis OY by I_{yy}.

Consider a plane area which is split up into small areas a_1, a_2, a_3 .... etc. Let the C.G. of the small areas from a given axis be at a distance of r_1, r_2, r_3.... etc. as shown in Fig. 7.12.

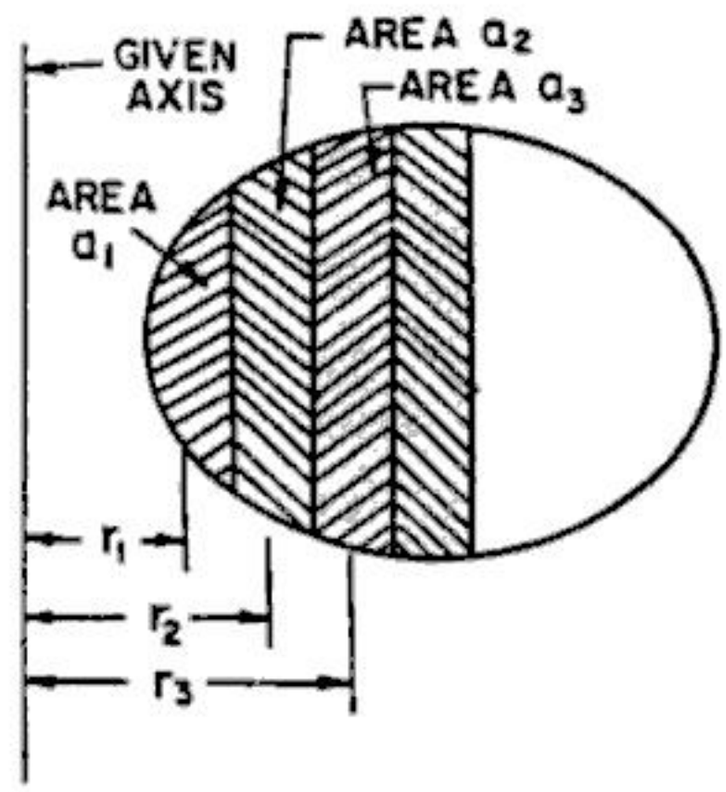

Fig. 7.12.

Then the moment of inertia of the plane area about the given axis is given by

$$I = a_1 r_1^2 + a_2 r_2^2 + a_3 r_3^2 + ... \qquad ...(7.4)$$

or

$$I = \Sigma a r^2. \qquad ...(7.5)$$

7.7. RADIUS OF GYRATION

Radius of gyration of a body (or a given lamina) is defined as the distance from an axis of reference where the whole mass (or area) of a body is assumed to be concentrated so as not to alter the moment of inertia about the given axis.

For the Fig. 7.12, the moment of inertia about the given axis is given by equation (7.4) as

$$I = a_1 r_1^2 + a_2 r_2^2 + a_3 r_3^2 + ... \qquad ...(i)$$

Let the whole mass (or area) of the body is concentrated at a distance k from the axis of reference, then the moment of inertia of the whole area about the given axis will be equal to Ak^2.

If $Ak^2 = I$, then k is known as radius of gyration about the given axis.

$$\therefore \qquad k = \sqrt{\frac{I}{A}}. \qquad ...(7.6)$$

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Proof. A lamina of plane area A is shown in Fig. 7.14.

Let X-X=The axis in the plane of area A and passing through the C.G. of the area.

AB=The axis in the plane of area A and parallel to axis X-X.

h=Distance between AB and X-X.

Consider a strip parallel to X-X axis at a distance y from the X-X axis.

Let the area of the strip=dA

Moment of inertia of area dA about X-X axis=dAy^2.

$\therefore$ Moment of inertia of the total area about X-X axis,

$$I_{XX} \quad \text{or} \quad I_G=\Sigma dAy^2 \qquad ...(i)$$

Moment of inertia of the area dA about AB

$$=dA(h+y)^2=dA[h^2+y^2+2hy].$$

$\therefore$ Moment of inertia of the total area A about AB,

$$I_{AB}=\Sigma dA[h^2+y^2+2hy]=\Sigma dAh^2+\Sigma dAy^2+\Sigma dA\ 2hy.$$

As h or h^2 is constant and hence they can taken outside the summation sign. Hence the above equation becomes

$$I_{AB}=h^2\Sigma dA+\Sigma dAy^2+2h\Sigma dAy.$$

But $\Sigma dA=A$. Also from equation (*i*), $\Sigma dAy^2=I_G$. Substituting these values in the above equation, we get

$$I_{AB}=h^2 \,.\, A+I_G+2h\,\Sigma dAy. \qquad ...(ii)$$

But $dA\ .\ y$ represents the moment of area of strip about X-X axis. And ΣdAy represents the moments of the total area about X-X axis. But the moments of the total area about X-X axis is equal to the product of total area (A) and the distance of the C.G. of the total area from X-X axis. As the distance of the C.G. of the total area from X-X axis is zero, hence ΣdAy will be equal to zero.

Substituting this value in equation (*ii*), we get

$$I_{AB}=h^2 \,.\, A+I_G+0$$

or

$$I_{AB}=I_G+Ah^2. \qquad ...(7.8)$$

7.10. DETERMINATION OF MOMENT OF INERTIA

The moment of inertia of the following sections will be determined by the method of integration :

1. Moment of inertia of a rectangular section,
2. Moment of inertia of a circular section,
3. Moment of inertia of a triangular section,
4. Moment of inertia of a uniform thin rod.

7.10.1. Moment of Inertia of a Rectangular Section

1st Case. Moment of inertia of the Rectangular section about the X-X axis passing through the C.G. of the section.

Fig. 7.15 shows a rectangular section $ABCD$ having width=b and depth=d. Let X-X is the horizontal axis passing through the C.G. of the rectangular section. We want to determined the moment of inertia of the rectangular section about X-X axis. The moment of inertia of the given section about X-X axis is represent by I_{XX}.

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Distance of the strip from the base $=(h-y)$

$\therefore$ Moment of inertia of the strip about the base

$=$ Area of strip $\times$ (Distance of strip from base)^2

$$=\frac{by}{h} \cdot dy \cdot (h-y)^2=\frac{by}{h}(h-y)^2 \cdot dy.$$

The moment of inertia of the whole triangular section about the base (I_{BC}) is obtained by integrating the above equation between the limits O to h.

$$\therefore \quad I_{BC}=\int_0^h \frac{by}{h}[h-y]^2\, dy=\frac{b}{h}\int_0^h y(h-y)^2\, dy$$

($\because$ b and h are constants and can be taken outside the integral sign)

$$=\frac{b}{h}\int_0^h y(h^2+y^2-2hy)\, dy=\frac{b}{h}\int_0^h (yh^2+y^3-2hy^2)\, dy$$

$$=\frac{b}{h}\left[\frac{y^2h^2}{2}+\frac{y^4}{4}-\frac{2hy^3}{2}\right]_0^h=\frac{b}{h}\left[\frac{h^2 \cdot h^2}{2}+\frac{h^4}{4}-\frac{2h \cdot h^3}{3}\right]=\frac{b}{h}\left[\frac{h^4}{2}+\frac{h^4}{4}-\frac{2h^4}{3}\right]$$

$$=\frac{b}{h} \cdot h^4\left[\frac{6+3-8}{12}\right]=bh^3 \cdot \frac{1}{12}$$

$$=\frac{bh^3}{12}. \qquad ...(7.14)$$

2nd Case. Moment of inertia of the triangular section about an axis passing through the C.G. and parallel to the base.

Consider a triangular section of base $=b$ and height $=h$ as shown in Fig. 7.21. Let X-X is the axis passing through the C.G. of the triangular section and parallel to the base.

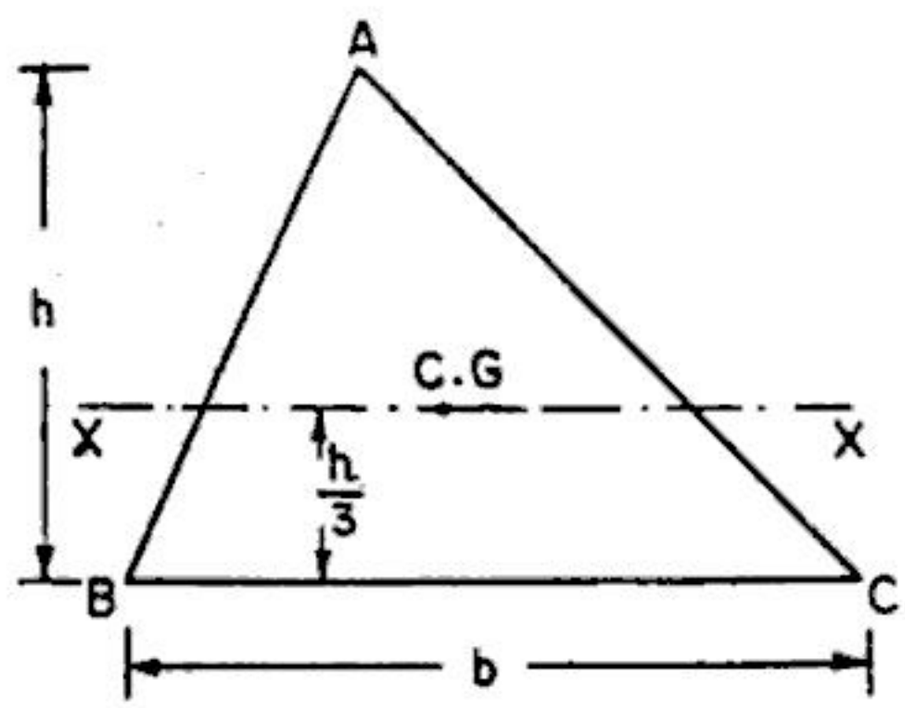

Fig. 7.21.

The distance between the C.G. of the triangular section and base $AB=\frac{h}{3}$.

Now from the theorem of parallel axis, given by equation (7.8), we have

Moment of inertia about BC = Moment of inertia about C.G. + Area $\times$ (Distance between X-X and BC)^2

or $$I_{BC}=I_G+A\times\left(\frac{h}{3}\right)^2$$

$$\therefore \quad I_G=I_{BC}-A\times\left(\frac{h}{3}\right)^2$$

$$=\frac{bh^3}{12}-\left(\frac{b\times h}{2}\right)\cdot\left(\frac{h}{3}\right)^2 \qquad \left(\because\ I_{BC}=\frac{bh^3}{12} \text{ and Area}=\frac{b\times h}{2}\right)$$

$$=\frac{bh^3}{12}-\frac{bh^3}{18}=\frac{bh^3\,(3-2)}{36}=\frac{bh^3}{36}. \qquad ...(7.15)$$

7.10.4. Moment of Inertia of a uniform thin rod. Consider a uniform thin rod AB of length L as shown in Fig. 7.22.

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$$=I_{G_2}+a_2h_2^2=166.667+20\times1.5^2$$

$$=166.667+45=211.667 \text{ cm}^4.$$

And moment of inertia of the rectangle (3) about the horizontal axis, passing through the C.G. of the given section

$$=I_{G_3}+a_3h_3^2=13.333+40\times4.5^2$$

$$=13.333+810=823.333 \text{ cm}^4$$

Now moment of inertia of the given section about the horizontal axis, passing through the C.G. of the given section

=Sum of the moment of inertia of the rectangles (1), (2) and (3) about the horizontal axis, passing through the C.G. of the given section

$$=1131.667+211.667+823.333$$

$$=\mathbf{2166.667 \text{ cm}^4}. \quad \textbf{Ans.}$$

Problem 7.12. *Find the moment of inertia of the area shown shaded in Fig. 7.25, about edge AB.*

Sol. Given :

Radius of semi-circle, $R=10$ cm

Width of rectangle, $b=20$ cm

Depth of rectangle, $d=25$ cm

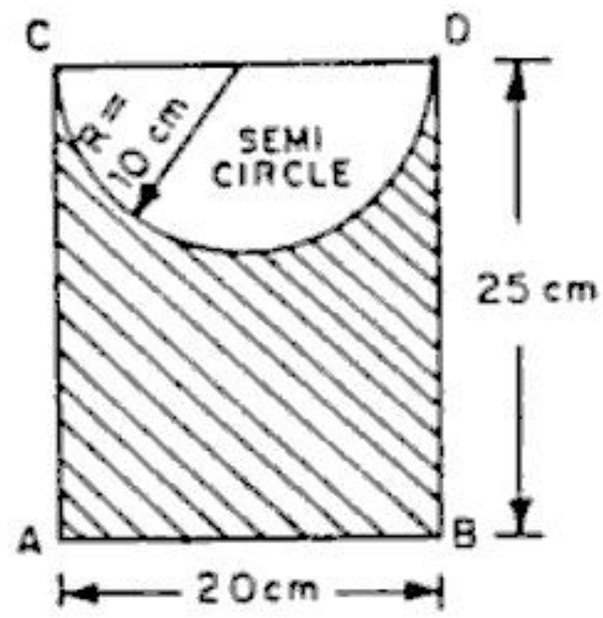

Fig. 7.25.

Moment of inertia of the shaded portion about AB

=M.O.I. of rectangle $ABCD$ about AB − M.O.I. of semi-circle on DC about AB

M.O.I. of rectangle $ABCD$ about AB

$$=\frac{bd^3}{3} \qquad \text{[see equation (7.11)]}$$

$$=\frac{20\times25^3}{12}=104{,}167 \text{ cm}^4$$

M.O.I. of semi-circle about DC

$$=\frac{1}{2}\times\text{[M.O.I. of a circle of radius 10 cm about a diameter]}$$

$$=\frac{1}{2}\times\left[\frac{\pi}{64}d^4\right]=\frac{1}{2}\times\frac{\pi}{64}\times20^4=3{,}925 \text{ cm}^4$$

Distance of C.G. of semi-circle from DC

$$=\frac{4r}{3\pi}=\frac{4\times10}{3\pi}=4.24 \text{ cm}$$

Area of semi-circle, $A=\frac{\pi r^2}{2}=\frac{\pi\times10^2}{2}=157.1 \text{ cm}^2$

M.O.I. of semi-circle about a line through its C.G. parallel to CD

=M.O.I. of semi-circle about CD − Area×[Distance of C.G. of semi-circle from DC]^2

$$=3925-157.1\times4.24^2$$

$$=3925-2824.28=1100.72 \text{ cm}^4$$

Distance of C.G. of semi-circle from AB

$$=25-4.24=20.76 \text{ cm}$$

M.O.I. of semi-circle about $AB=1100.72+157.1\times20.76^2$

$$=1100.72+67706.58=68807.30 \text{ cm}^4$$

∴ M.O.I. of shaded portion about AB

$$=104{,}167-68807.30=35359.7 \text{ cm}^4. \quad \textbf{Ans.}$$

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[**Hint.** $a_1 = 10 \times 30 = 300 \text{ mm}^2$, $x_1 = 5$ mm, $y_1 = 15$.

$a_2 = 40 \times 10 = 400 \text{ mm}^2$, $x_2 = 10 + 20 = 30$ mm, $y_2 = 5$ mm

$a_3 = 10 \times 20 = 200 \text{ mm}^2$, $x_3 = 5$ mm, $y_3 = -10$ mm

$a_4 = 10 \times 10 = 100 \text{ mm}^2$, $x_4 = 45$ mm, $y_4 = 10 + 5 = 15$ mm

$$\bar{x} = \frac{a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4}{(a_1 + a_2 + a_3 + a_4)} = \frac{1500 + 12000 + 1000 + 4500}{1000}$$

$= 1.5 + 12 + 1 + 4.5 = 19$ mm. **Ans.**

$$\bar{y} = \frac{a_1y_1 + a_2y_2 + a_3y_3 + a_4y_4}{a_1 + a_2 + a_3 + a_4} = \frac{4500 + 2000 - 2000 + 1500}{1000}$$

$= 4.5 + 2 - 2 + 1.5 = 6$ mm. **Ans.**]

8. A thin homogeneous wire is bent into a triangular shape ABC such that $AB = 240$ mm, $BC = 260$ mm and $AC = 100$ mm. Locate the C.G. of the wire with respect to co-ordinate axes. Angle at A is right angle.

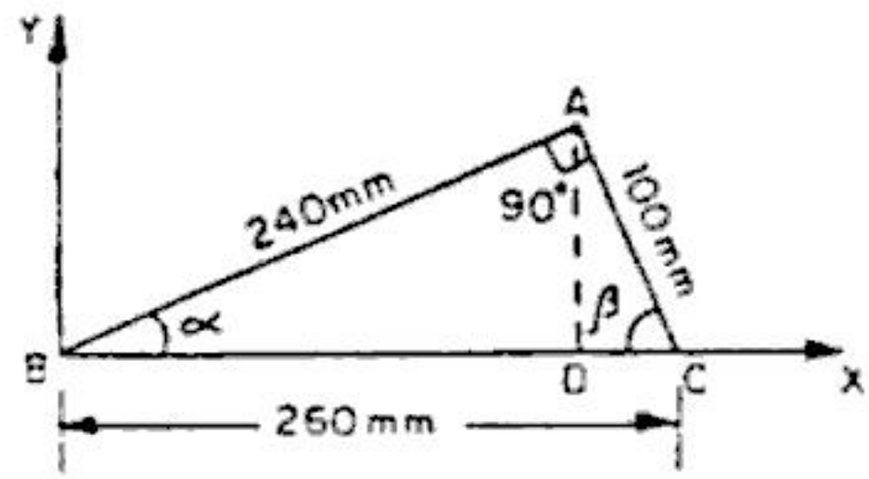

Fig. 7.30.

[**Hint.** First determine angles α and β. Use sine rule

$$\frac{BC}{\sin 90^\circ} = \frac{AC}{\sin \alpha} = \frac{AB}{\sin \beta}. \quad \therefore \quad \sin \alpha = \frac{AC \times \sin 90^\circ}{BC} = \frac{100}{260}$$

$\therefore \quad \alpha = 22.62^\circ$. Also $\sin \beta = \dfrac{AB}{BC} \times \sin 90^\circ = \dfrac{240}{260}$

$\therefore \quad \beta = 67.38^\circ$

Using equation,

$$\bar{x} = \frac{L_1x_1 + L_2x_2 + L_3x_3}{(L_1 + L_2 + L_3)}, \quad \text{where} \quad L_1 = AB = 240,$$

x_1=distance of C.G. of AB from y-axis

$$= \frac{240}{2} \times \cos \alpha = 120 \times \cos 22.62^\circ = 110.77 \text{ mm}$$

$L_2 = BC = 260$ mm, x_2=Distance of C.G. of BC from y-axis=130

$L_3 = AC = 100$ mm, x_3=Distance of C.G. of AC from y-axis

$$= BD + \frac{100}{2} \cos \beta = 240 \cos \alpha + 50 \cos \beta$$

$$= 240 \times \cos 22.62^\circ + 50 \cos 67.38^\circ = 240.77$$

$$\therefore \quad \bar{x} = \frac{240 \times 110.77 + 260 \times 130 + 100 \times 240.77}{240 + 260 + 100} = 140.77 \text{ mm.} \quad \textbf{Ans.}$$

$$\bar{y} = \frac{L_1y_1 + L_2y_2 + L_3y_3}{L_1 + L_2 + L_3}$$

where $y_1 = \dfrac{240}{2} \sin \alpha = 120 \times \sin 22.62^\circ = 46.154$

$y_2 = 0$, $y_3 = \dfrac{100}{2} \sin \beta = 50 \sin 67.38^\circ$

$= 46.154$

$$\therefore \quad \bar{y} = \frac{240 \times 46.154 + 260 \times 0 + 100 \times 46.154}{600}$$

$= 26.154$ mm. **Ans.**]

9. Determine the C.G. of the uniform plane lamina shown in Fig. 7.31. All dimensions are in cm. [**Hint.** The Fig is symmetrical about y-y axis.

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$$\therefore \quad \bar{y} = \frac{a_1y_1 + a_2y_2 + a_3y_3 + a_4y_4}{a_1 + a_2 + a_3 + a_4}$$

where $a_1 = 40 \times 30 = 1200 \text{ cm}^2, \quad y_1 = \frac{30}{2} = 15 \text{ cm},$

$$a_2 = 30 \times 20 = 600 \text{ cm}^2, \quad y_2 = 30 + \frac{30}{2} = 45 \text{ cm}$$

$$a_3 = -\frac{\pi \times 10^2}{2} = -50\pi, \quad y_3 = \frac{4r}{3\pi} = \frac{4 \times 10}{3\pi} = \frac{40}{3\pi}$$

$$a_4 = -\frac{20 \times 10}{2} = -100, \quad y_4 = 60 - \frac{10}{3} = \frac{170}{3}$$

$$\therefore \quad \bar{y} = \frac{1200 \times 15 + 600 \times 45 - 50\pi \times \frac{40}{3\pi} - 100 \times \frac{170}{3}}{1200 + 600 - 50\pi - 100}$$

$$= \frac{18000 + 27000 - 666.7 - 5666.7}{1700 - 50\pi}$$

$$= \frac{38666.6}{1542.92} = 25.06 \text{ cm}$$

from Origin 0. **Ans.**]

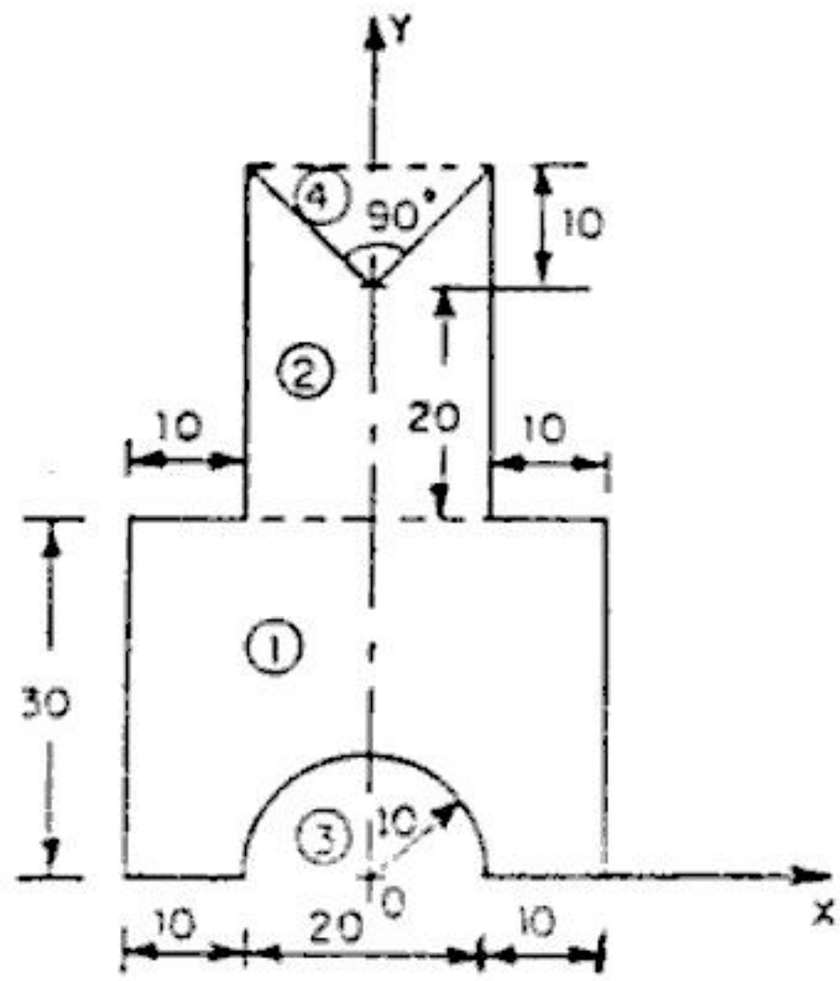

Fig. 7.31.

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8

Friction

8.1. INTRODUCTION

When a solid body slides over a stationary solid body, a force is exerted at the surface of contact by the stationary body on the moving body. This force is called the *force of friction* and is always acting in the direction opposite to the direction of motion. The property of the bodies by virtue of which a force is exerted by a stationary body on the moving body to resist the motion of the moving body is called *friction*. Friction acts parallel to the surface of contact and depends upon the nature of surface of contact.

8.2. DEFINITIONS

For defining the terms like co-efficient of friction (μ) and angle of friction (ϕ), consider a solid body placed on a horizontal plane surface as shown in Fig. 8.1.

Let W=Weight of body acting through C.G. downward,

R=Normal reaction of body acting through C.G. upward,

P=Force acting on the body through C.G. and parallel to the horizontal surface.

If P is small, the body will not move as the force of friction acting on the body in the direction opposite to P will be more than P. But if the magnitude of P goes on increasing, a stage comes, when the solid body is on the point of motion. At this stage, the force of friction acting on the body is called *limiting force of friction*. The limiting force of friction is denoted by F.

Resolving the forces on the body vertically and horizontally, we get

$$R=W$$
$$F=P.$$

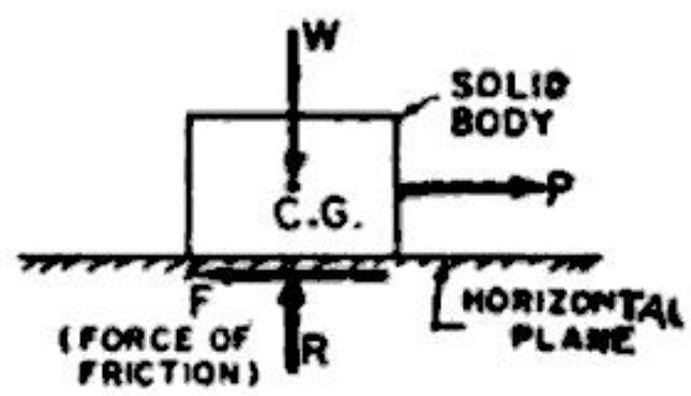

Fig. 8.1. Solid body on horizontal surface.

If the magnitude of P is further increased the body will start moving. The force of friction, acting on the body when the body is moving, is called *kinetic friction*.

8.2.1. Co-efficient of Friction (μ). It is defined as the ratio of the limiting force of friction (F) to the normal reaction (R) between two bodies. It is denoted by the symbol μ. Thus

$$\mu=\frac{\text{Limiting force of friction}}{\text{Normal reaction}}=\frac{F}{R}.$$

$$\therefore \qquad F=\mu R \qquad \text{...(8.1)}$$

8.2.2. Angle of Friction (ϕ). It is defined as the angle made by the resultant of the normal reaction (R) and the limiting force of friction (F) with the normal reaction (R). It is denoted by ϕ. Fig. 8.2 shows a solid body resting on a rough horizontal plane.

Let S=Resultant of the normal reaction (R) and limiting force of friction (F)

Then angle of friction=ϕ

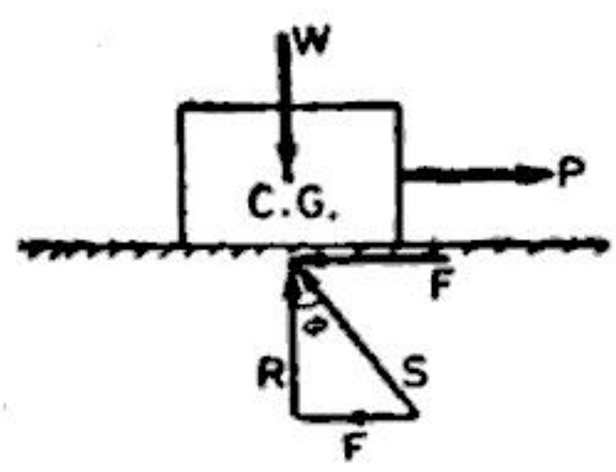

Fig. 8.2. Angle of friction.

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=Angle between S and R

From Fig. 8.2, we have

$$\tan \phi = \frac{F}{R} = \frac{\mu R}{R} \qquad [\because \ F = \mu R \text{ from (8.1)}]$$

$$= \mu = \text{Co-efficient of friction} \qquad ...(8.2)$$

Thus the tangent of the angle of friction is equal to the co-efficient of friction.

A block of weight W is placed on a rough horizontal plane surface as shown in Fig. 8.3 and a force P is applied at an angle θ with the horizontal such that the block just tends to move.

Let R = Normal reaction

μ = Co-efficient of friction

F = Force of friction

$= \mu R$

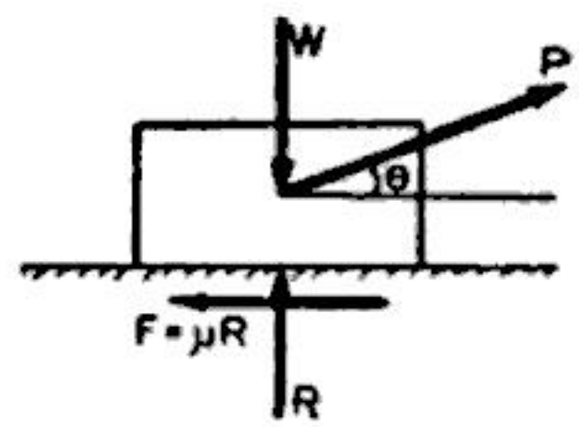

Fig. 8.3.

In this case the normal reaction R will not be equal to weight of the body. The normal reaction is obtained by resolving the forces on the block horizontally and vertically. The force P is resolved in two components *i.e.*, $P \cos \theta$ in the horizontal direction and $P \sin \theta$ in the vertical direction.

Resolving forces on the block horizontally, we get

$$F = P \cos \theta$$

or

$$\mu R = P \cos \theta \qquad ...(i) \ (\because F = \mu R)$$

Resolving forces on the block vertically, we get

$$R + P \sin \theta = W$$

$$\therefore \quad R = W - P \sin \theta \qquad ...(ii)$$

From equation (*ii*), it is clear that normal reaction is not equal to the weight of the block.

If in equation (*ii*), the values of W, P and θ are known, then value of normal reaction (R) can be obtained. This value of R can be substituted in equation (*i*) to determine the value of co-efficient of friction μ.

Note. (*i*) The force of friction is always equal to μR (*i.e.*, $F = \mu R$).

(*ii*) The normal reaction (R) is not equal to the weight of the body always.

8.2.3. Cone of Friction. It is defined as the right circular cone with vertex at the point of contact of the two bodies (or surfaces), axis in the direction of normal reaction (R) and semi-vertical angle equal to angle of friction (ϕ). Fig. 8.4 shows the cone of friction in which,

O = Point of contact between two bodies

R = Normal reaction and also axis of the cone of friction

ϕ = Angle of friction.

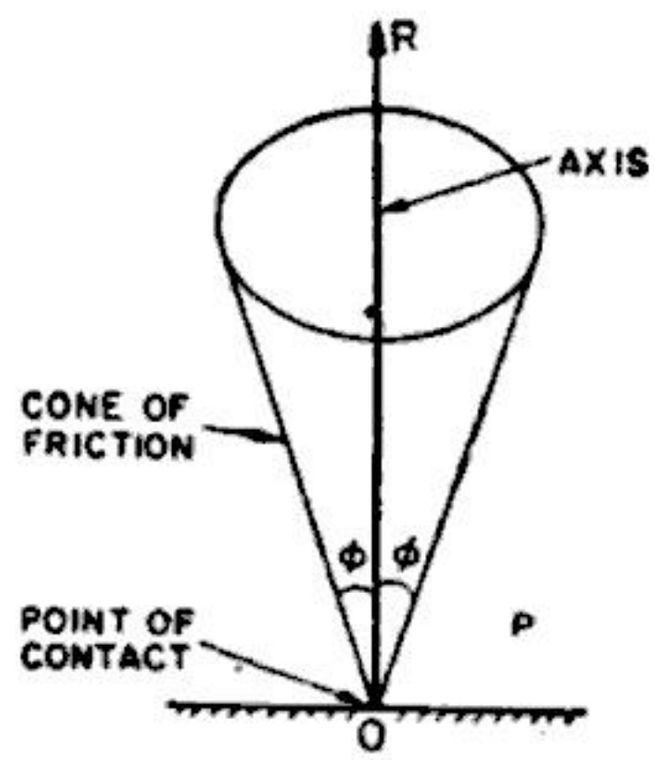

Fig. 8.4. Cone of friction.

8.3. TYPES OF FRICTION

The friction is divided into following two types depending upon the nature of the two surfaces in contact :

1. Static friction, and
2. Dynamic friction.

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If the two surfaces, which are in contact, are at rest, the force experienced by one surface is called static friction. But if one surface starts moving and the other is at rest, the force experienced by the moving surface is called dynamic friction. If between the two surfaces, no lubrication (oil or grease) is used, the friction, that exists between two surface is called '*Solid Friction*' or '*Dry Friction*'.

8.4. LAWS OF SOLID FRICTION

The friction, that exists between two surfaces which are not lubricated, is known as solid friction. The two surfaces may be at rest or one of the surface is moving and other surface is at rest. The following are the laws of solid friction :

1. The force of friction acts in the opposite direction in which surface is having tendency to move.
2. The force of friction is equal to the force applied to the surface, so long as the surface is at rest.
3. When the surface is on the point of motion, the force of friction is maximum and this maximum frictional force is called the limiting friction force.
4. The limiting frictional force bears a constant ratio to the normal reaction between two surfaces.
5. The limiting frictional force does not depend upon the shape and areas of the surfaces in contact.
6. The ratio between limiting friction and normal reaction is slightly less when the two surfaces are in motion.
7. The force of friction is independent of the velocity of sliding.

The above laws of solid friction are also called laws of static and dynamic friction.

Problem 8.1. *A body of weight 100 Newtons is placed on a rough horizontal plane. Determine the co-efficient of friction if a horizontal force of 60 Newtons just causes the body to slide over the horizontal plane.*

Sol. Given :

Weight of body, $W=100$ N

Horizontal force applied,

$$P=60 \text{ N}$$

∴ Limiting force of friction,

$$F=P=60 \text{ N}$$

Let μ=co-efficient of friction.

The normal reaction of the body is given as

$$R=W=100 \text{ N}$$

Using equation (8.1),

$$F=\mu R$$

or $$\mu=\frac{F}{R}=\frac{60}{100}=0.6. \quad \textbf{Ans.}$$

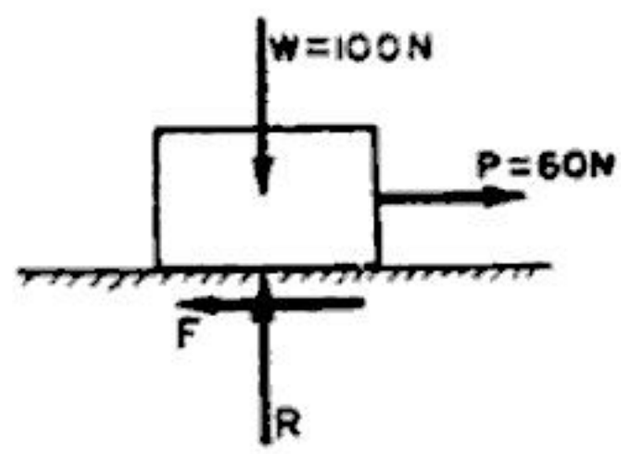

Fig. 8.5.

Problem 8.2. *A body of weight 200 N is placed on a rough horizontal plane. If the co-efficient of friction between the body and the horizontal plane is 0.3, determine the horizontal force required to just slide the body on the plane.*

Sol. Given :

Weight of body, $W=200$ N

Co-efficient of friction, $\mu=0.3$

Normal reaction, $R=W=200$ N

Let F=horizontal force which causes the body to just slide over the plane.

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or $P \cos(\theta-\phi)=W \sin \phi$ $[\because \cos\theta \cos\phi+\sin\theta \sin\phi=\cos(\theta-\phi)]$

$$\therefore \quad P=\frac{W \sin \phi}{\cos(\theta-\phi)}$$

The force P will be least, if the denominator *i.e.*, $\cos(\theta-\phi)$ is maximum. But $\cos(\theta-\phi)$ will be maximum, if

$$\cos(\theta-\phi)=1$$

or $\theta-\phi=0$

or $\theta=\phi$

$\therefore$ $P_{least}=W \sin \phi$ or $W \sin \theta$.

Hence the force P will be least if the angle of inclination of P with the horizontal is equal to the angle of friction ϕ.

Problem 8.7. *A man wishing to slide a stone block of weight 1000 N over a horizontal concrete floor, ties a rope to the block and pulls it in a direction inclined upward at an angle of 20° to the horizontal. Calculate the minimum pull necessary to slide the block if the co-efficient of friction $\mu=0.6$. Calculate also the pull required if the inclination of the rope with the horizontal is equal to the angle of friction and prove that this is the least force required to slide the block.*

(A.M.I.E., Winter 1990)

Sol. Given :

Weight, $W=1000$ N

Angle with horizontal, $\theta=20°$

Co-efficient of friction, $\mu=0.6$

Let P=Force applied

R=Normal reaction

F=Force of friction$=\mu R$

The forces acting on the block are shown in Fig. 8.10 (*a*).

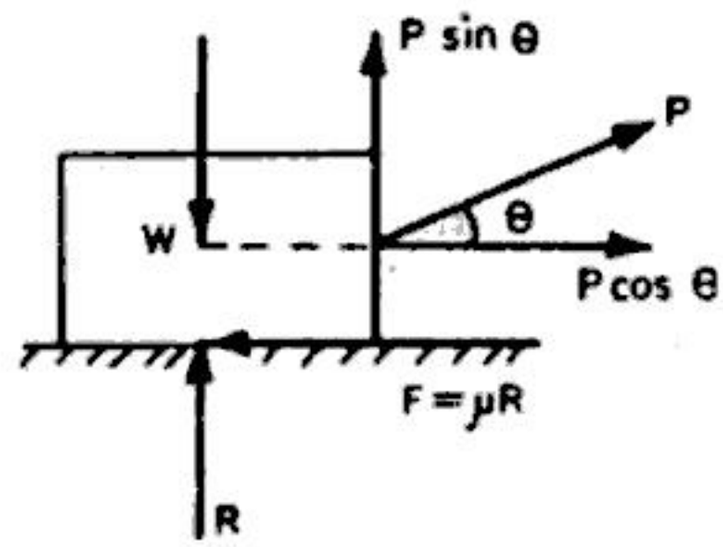

Fig. 8.10. (*a*)

Resolving forces horizontally,

$$P \cos\theta=\mu R$$

or $P \cos 20°=0.6\times R$ ...(*i*)

Resolving forces vertically,

$$R+P \sin\theta=W$$

or $R+P \sin 20°=1000$

or $R=1000-P \sin 20°$ ...(*ii*)

Substituting the value of R in equation (*i*), we get

$$P \cos 20°=0.6\,(1000-P \sin 20°)=600-0.6\,P \sin 20°$$

or $P \cos 20°+0.6\,P \sin 20°=600$

or $P\,(\cos 20°+0.6\times\sin 20°)=600$ ...(*iii*)

$$\therefore \quad P=\frac{600}{(\cos 20°+0.6 \sin 20°)}=\frac{600}{(0.9397+0.6\times 0.342)}$$

$$=\frac{600}{1.1449}=\mathbf{524\ N.} \quad \textbf{Ans.}$$

Pull required if the inclination of the rope with the horizontal is equal to angle of friction.

Let ϕ=Angle of friction

=the angle made by rope with horizontal (given)

$=20°$

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Thus the force P will be minimum if the angle of inclination of the force with the inclined plane is equal to the angle of friction.

(*b*) *Least force when the body is on the point of motion down the plane.*

When the body is on the point of motion down the plane, the force of friction $F=\mu R$ is acting up the plane as shown in Fig. 8.14.

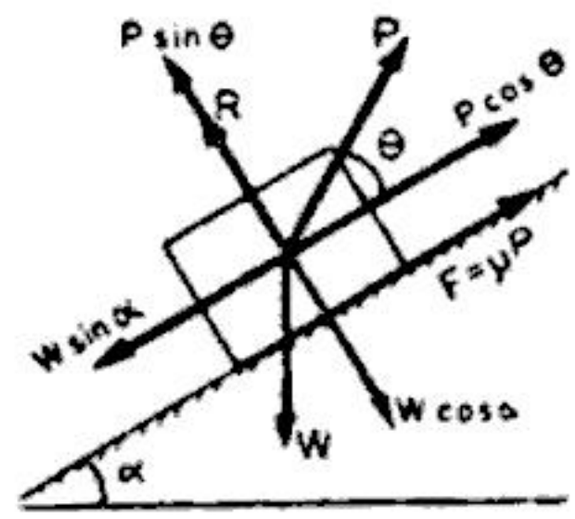

Fig. 8.14. Body moving down.

Resolving the forces along the plane

$$W \sin \alpha = P \cos \theta + \mu R \qquad ...(iii)$$

Resolving forces perpendicular to the inclined plane

$$R + P \sin \theta = W \cos \alpha$$

or $$R = W \cos \alpha - P \sin \theta$$

Substituting the value of R in equation (*iii*), we get

$$W \sin \alpha = P \cos \theta + \mu (W \cos \alpha - P \sin \theta)$$
$$= P \cos \theta + \mu W \cos \alpha - \mu P \sin \theta$$

or $$W \sin \alpha - \mu W \cos \alpha = P \cos \theta - \mu P \sin \theta$$

or $$W [\sin \alpha - \mu \cos \alpha] = P [\cos \theta - \mu \sin \theta]$$

or $$P = \frac{W (\sin \alpha - \mu \cos \alpha)}{(\cos \theta - \mu \sin \theta)}$$

$$= \frac{W (\sin \alpha - \tan \phi \cos \alpha)}{(\cos \theta - \tan \phi \sin \theta)} \qquad (\because \mu = \tan \phi)$$

$$= \frac{W \left(\sin \alpha - \frac{\sin \phi}{\cos \phi} \cos \alpha\right)}{\left(\cos \theta - \frac{\sin \phi}{\cos \phi} \sin \theta\right)}$$

$$= \frac{W (\sin \alpha \cos \phi - \sin \phi \cos \alpha)}{(\cos \theta \cos \phi - \sin \phi \sin \theta)}$$

$$= \frac{W \sin (\alpha - \phi)}{\cos (\theta + \phi)}. \qquad ...(8.5)$$

Problem 8.10. *A body of weight 500 N is pulled up an inclined plane, by a force of 350 N. The inclination of the plane is 30° to the horizontal and the force is applied parallel to the plane. Determine the co-efficient of friction.*

Sol. Given :

Weight of body, $W = 500$ N

Force applied, $P = 350$ N

Inclination, $\alpha = 30°$

Let μ = Co-efficient of Q friction

R = Normal reaction

F = Force of friction $= \mu R$.

The body is in equilibrium under the action of the forces shown in Fig. 8.15.

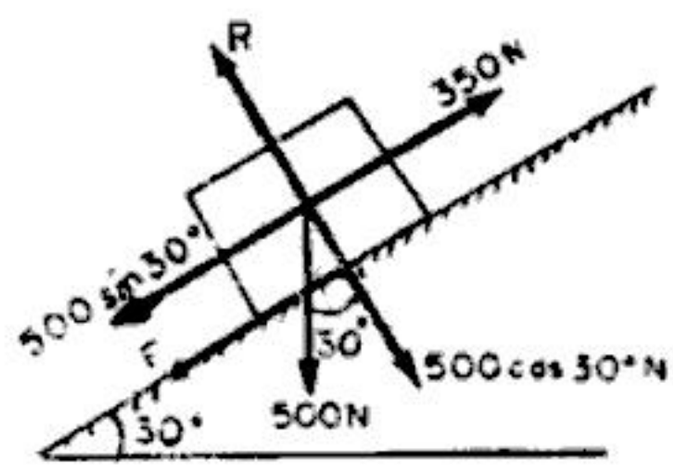

Fig. 8.15. Body moving up.

Resolving the forces along the plane,

$$500 \sin 30° + F = 350$$

or $$500 \sin 30° + \mu R = 350 \qquad (\because F = \mu R) \qquad ...(i)$$

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Resolving the forces along the plane,

$$P = W \sin\theta + \mu \times R \quad \text{...(iii)}$$

Resolving the forces normal to the plane,

$$R = W \cos\theta \quad \text{...(iv)}$$

Substituting the value of R in equation (*iii*),

$$P = W \sin\theta + \mu \times W \cos\theta = W(\sin\theta + \mu \cos\theta)$$
$$= 100(\sin 11.53° + \mu \cos 11.53°) = 100(0.2 + \mu \times 0.9798)$$
$$= \mathbf{100(0.2 + 0.9798 \times \mu)\ N.} \quad \textbf{Ans.}$$

Problem 8.14. *(a) Define co-efficient of friction and limiting friction (b) Block A weighing 15 N is a rectangular prism resting on a rough inclined plane as shown in Fig. 8.20. The block is tied up by a horizontal string which has a tension of 5 N. Find :*

(i) The frictional force on the block,

(ii) The normal reaction of the inclined plane, and

(iii) The co-efficient of friction between the surfaces of contact.

(AMIE Nov., 1967)

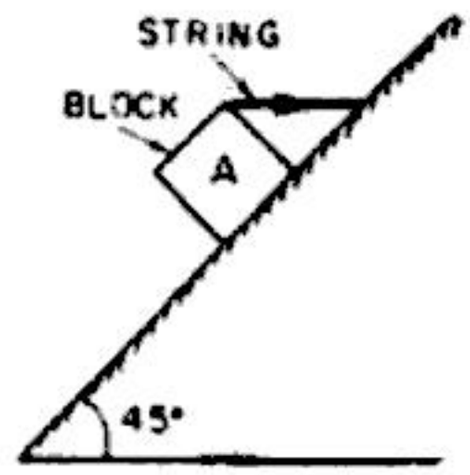

Fig. 8.20.

Sol. (*a*) For definition of co-efficient of friction and limiting friction, please refer to Art. 8.2 and Art. 8.2.1.

(*b*) Given :

Weight of block, $W = 15$ N

Tension in string, $T = 5$ N

Inclination of plane, $\alpha = 45°$

Let F = Frictional force,

R = Normal reaction, and

μ = Co-efficient of friction.

Since there is tension in string which means if string is removed, the block will slide down the plane. Hence force of friction will be acting in the upward direction.

The block A is in equilibrium under the action of the forces shown in Fig. 8.21. The forces are :

1. The weight of block, $W = 15$ N
2. Horizontal tension in the string, $T = 5$ N
3. Normal reaction, R
4. Force of friction, $F = \mu R$, acting upward.

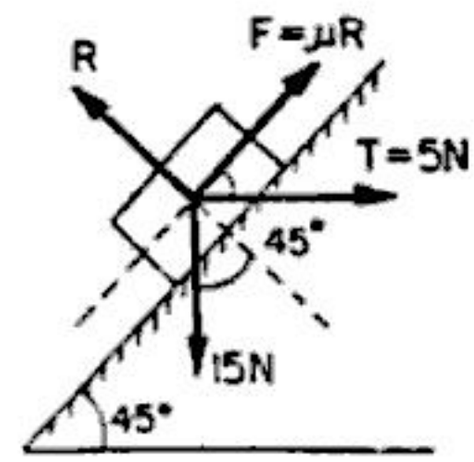

Fig. 8.21.
Forces acting on the Block.

Resolving forces along the inclined plane,

$$15 \sin 45° = F + 5 \cos 45°$$

$$\therefore \quad F = 15 \sin 45° - 5 \cos 45° = 15 \times .707 - 5 \times .707$$
$$= 10 \times .707 = 7.07 \text{ N}$$

Resolving forces normal to inclined plane,

$$R = 15 \cos 45° + T \cos 45° = 15 \cos 45° + 5 \cos 45°$$
$$= 15 \times .707 + 5 \times .707 = 20 \times .707 = 14.14 \text{ N}$$

Using equation (8.1), we get

$$F = \mu R$$

$$\therefore \quad \mu = \frac{F}{R} = \frac{7.07}{14.14} = 0.5.$$

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As the motion of the two bodies is about to take place down the inclined plane, the force of frictions F_1 and F_2 will be acting upward. The two bodies are in equilibrium under the action of forces shown in Fig. 8.26.

Forces on the first body

Resolving forces along the plane,

$$400 \sin \alpha = T + F_1 = T + \mu_1 R_1 = T + 0.15\ R_1 \quad ...(i)$$

Resolving forces normal to the plane,

$$400 \cos \alpha = R_1$$

Substituting the value of R_1 in equation (*i*),

$$400 \sin \alpha = T + 0.15 \times 400 \cos \alpha$$
$$= T + 60 \cos \alpha$$
$$\therefore \quad T = 400 \sin \alpha - 60 \cos \alpha \quad ...(ii)$$

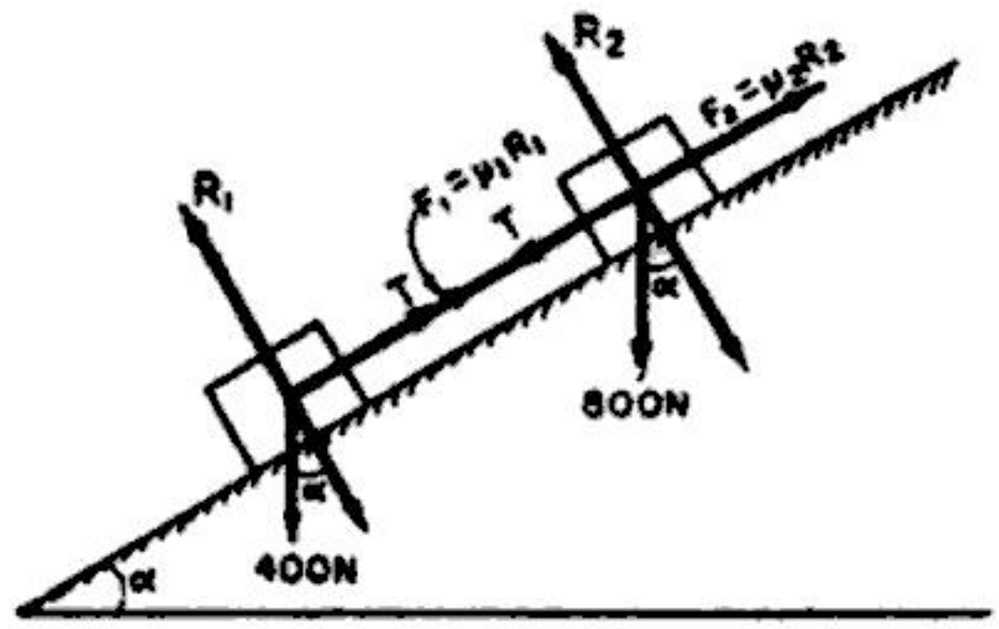

Fig. 8.26.

Forces on the Second body

Resolving forces along the plane,

$$800 \sin \alpha + T = F_2 = \mu_2 R_2 = 0.40\ R_2 \quad ...(iii)$$

Resolving forces normal to the plane,

$$R_2 = 800 \cos \alpha$$

Substituting the values of R_2 in equation (*iii*),

$$800 \sin \alpha + T = 0.40 \times 800 \cos \alpha = 320 \cos \alpha$$

or $$T = 320 \cos \alpha - 800 \sin \alpha \quad ...(iv)$$

Equating the values of T, given by equations (*ii*) and (*iv*),

$$400 \sin \alpha - 60 \cos \alpha = 320 \cos \alpha - 800 \sin \alpha$$

or $$400 \sin \alpha + 800 \sin \alpha = 320 \cos \alpha + 60 \cos \alpha$$

or $$1200 \sin \alpha = 380 \cos \alpha$$

or $$\frac{\sin \alpha}{\cos \alpha} = \frac{380}{1200} = 0.3166$$

or $$\tan \alpha = 0.3166$$

$$\therefore \quad \alpha = \tan^{-1} 0.3166 = \mathbf{17.56^\circ \text{ or } 17^\circ\ 34'}. \quad \textbf{Ans.}$$

Substituting the value of α in equation (*ii*),

$$T = 400 \sin 17.56^\circ - 60 \cos 17.56^\circ$$
$$= 400 \times 0.3017 - 60 \times 0.9534$$
$$= 120.68 - 57.20 = \mathbf{63.48\ N}. \quad \textbf{Ans.}$$

Problem 8.18. *Two blocks A and B are connected by a horizontal rod and are supported on two rough planes as shown in Fig. 8.27. If the weight of block B is 1500 N and coefficient of friction of block A and B are 0.25 and 0.35 respectively, find the smallest weight of block A for which equilibrium can exist.*

Sol. Given : Weight of block B, $W_B = 1500$ N

Co-efficient of friction for block A, $\mu_A = 0.25$

Co-efficient of friction for block B, $\mu_B = 0.35$

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T=Tension in the string

R_1=Normal reaction on the lower surface of weight 30 N

F_1=Force of friction=μR_1

Resolving the forces along the plane,

$$T=30 \sin \theta+\mu R_1$$

$$=30 \sin \theta+\frac{1}{3} R_1 \quad ...(i)$$

Resolving the forces normal to the plane,

$$R_1=30 \cos \theta \quad ...(ii)$$

Substituting the value of R_1 in equation (*i*),

$$T=30 \sin \theta+\frac{1}{3}\times 30 \cos \theta$$

$$=30 \sin \theta+10 \cos \theta \quad ...(iii)$$

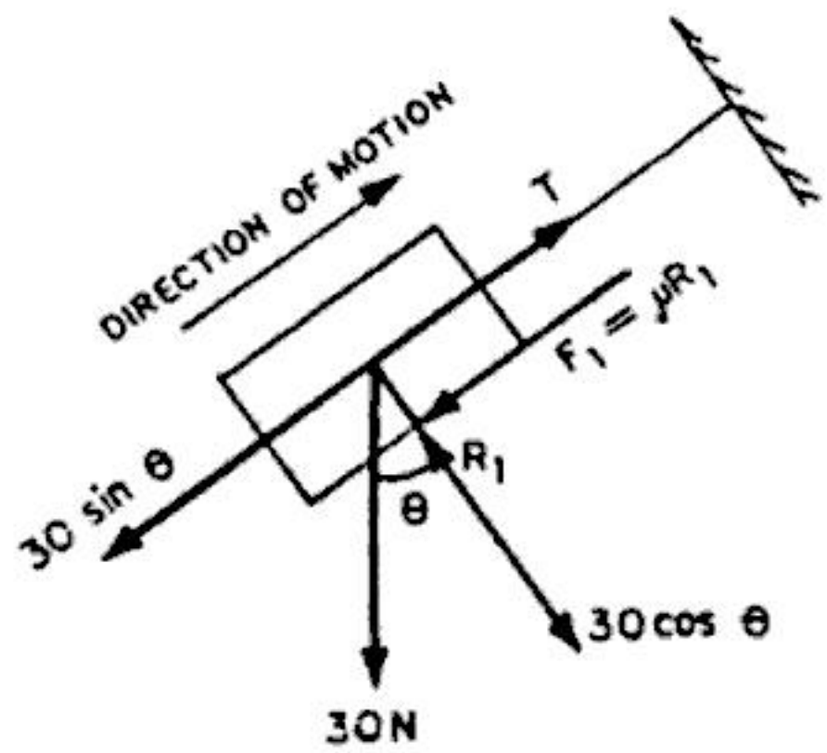

Fig. 8.29. (*e*)

Now consider the equilibrium of weight 90 N

The weight 90 N will be in equilibrium under the action of forces shown in Fig. 8.29 (*f*).

Resolving the forces along the plane,

$$90 \sin \theta=\mu R_1+\mu R_2$$

$$=\frac{1}{3}R_1+\frac{1}{3}R_2$$

$$=\frac{1}{3}\times 30 \cos \theta+\frac{1}{3}R_2 \quad (\because R_1=30 \cos \theta)$$

$$=10 \cos \theta+\frac{1}{3} R_2 \quad ...(iv)$$

Resolving the forces normal to the plane,

$$R_2=R_1+90 \cos \theta$$

$$=30 \cos \theta+90 \cos \theta \quad (\because R_1=30 \cos \theta)$$

$$=120 \cos \theta \quad ...(v)$$

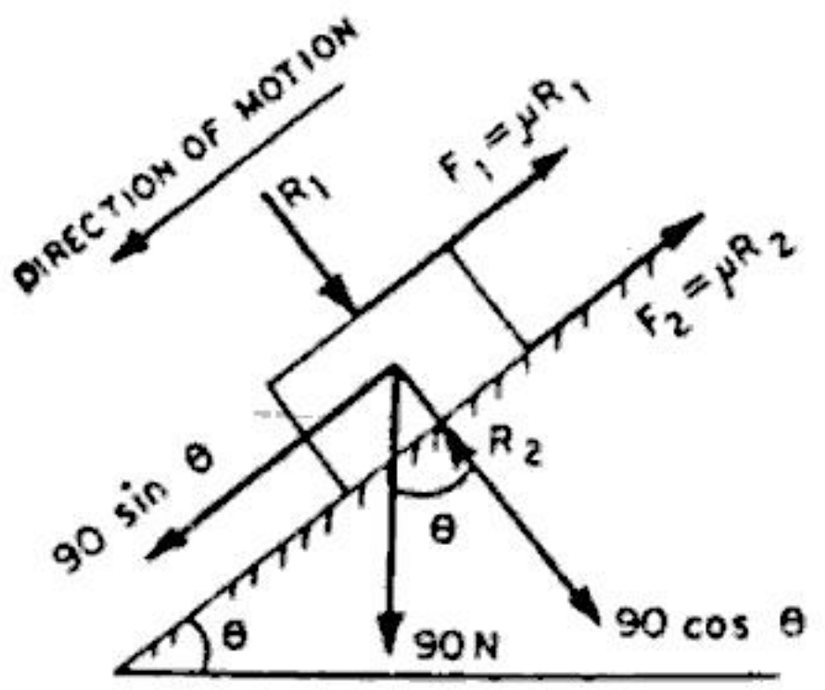

Fig. 8.29. (*f*)

Substituting the value of R_2 in equation (*iv*), we get

$$90 \sin \theta=10 \cos \theta+\frac{1}{3}\times 120 \cos \theta$$

$$=10 \cos \theta+40 \cos \theta=50 \cos \theta$$

$$\therefore \quad \frac{\sin \theta}{\cos \theta}=\frac{50}{90}=0.5555$$

or $\tan \theta=0.5555$

$$\therefore \quad \theta=\tan^{-1} 0.5555=\mathbf{29.05°}. \quad \textbf{Ans.}$$

8.7. ANALYSIS OF LADDER FRICTION

Fig. 8.30 shows a ladder AC resting on the ground and leaning against a wall.

Let R_A=Reaction at A

R_C=Reaction at C

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F_A=Force of friction at A

$=\mu R_A$

F_C=Force of friction at C

$=\mu R_C$.

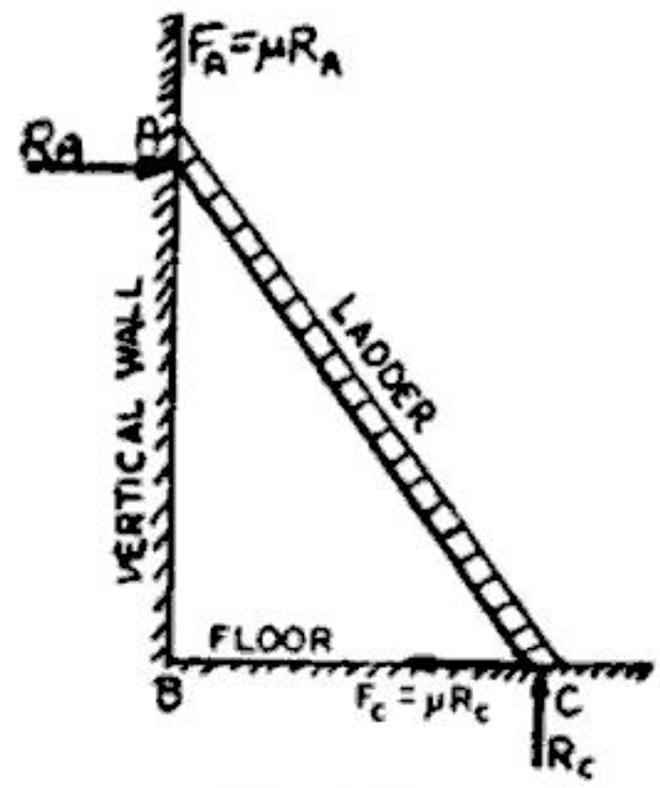

Fig. 8.30.

Due to the self weight of the ladder or when some man stands on the ladder, the upper end A of the ladder tends to slip downwards, and hence the force of friction between the ladder and the vertical wall $F_A=\mu R_A$ will be acting upwards as shown in Fig. 8.30. Similarly, the lower end C of the ladder will tend to move towards right and hence a force of friction between ladder and floor $F_C=\mu R_C$ will be acting towards left.

For the equilibrium of the system, the algebraic sum of the horizontal and vertical components of the forces must be zero. Also the moments of all the forces about any point must be zero.

Note. If the vertical wall is smooth, there will be no force of friction between the ladder and the vertical wall.

Problem 8.21. *A uniform ladder of length 10 m and weighing 20 N is placed against a smooth vertical wall with its lower end 8 m from the wall. In this position the ladder is just to slip. Determine :*

(i) the co-efficient of friction between the ladder and the floor, and

(ii) frictional force acting on the ladder at the point of contact between ladder and floor.

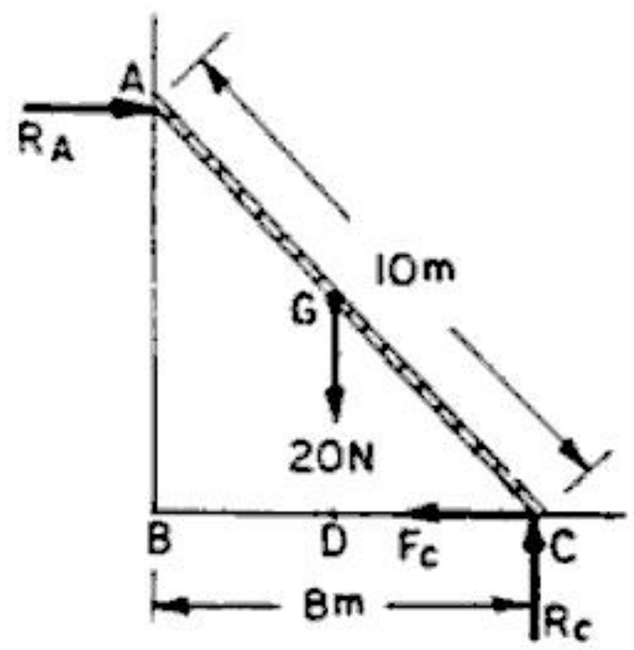

Fig. 8.31.

Sol. Given :

Weight of ladder, $W=20$ N

Length of ladder, $AC=10$ m

Distance of lower end of ladder from wall, *i.e.*,

$BC=8$ m.

In right-angled triangle ABC,

$$AB=\sqrt{AC^2-BC^2}$$
$$=\sqrt{10^2-8^2}=\sqrt{100-64}$$
$$=\sqrt{36}=6 \text{ m.}$$

As vertical wall is smooth, hence there will be no force of friction between ladder and wall.

In the position of the ladder shown in Fig. 8.31, the ladder is just to slip. Hence the lower end C of the ladder will tend to move towards right and hence a force of friction between ladder and floor (*i.e.*, $F_A=\mu R_C$) will be acting towards left.

Let R_A=Reaction at A

R_C=Reaction at C

μ=Co-efficient of friction between ladder and floor at C

F_C=Force of friction at $C=\mu R_C$

Ladder is uniform and the weight of ladder ($W=20$ N) is acting at the middle point of AC at G. The line of action of W will pass through the middle point of BC. Hence distance $CD=\frac{1}{2}\times BC=\frac{8}{2}=4.0$ m.

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Resolving the force vertically, we get

$$R_C = 20 \text{ N}$$

Resolving the forces horizontally, we get

$$R_A = F_C = \mu \times R_C = \mu \times 20 = 20\ \mu \qquad ...(i)$$

Taking the moments of all forces about point C

Clockwise moments = Anticlockwise moments

$$R_A \times AB = 20 \times CD \qquad \text{(The moments of } R_C \text{ and } F_C \text{ about point } C \text{ is zero)}$$

or $$20\ \mu \times 6 = 20 \times 4 \qquad [\because\ R_A = 20\ \mu \text{ from equation } (i)]$$

$$\therefore \quad \mu = \frac{20 \times 4}{20 \times 6} = 0.67. \quad \textbf{Ans.}$$

(*ii*) Frictional force acting at C is given as

$$F_C = \mu \times R_C = 0.67 \times 20 = \mathbf{13.40\ N.} \quad \textbf{Ans.}$$

Problem 8.22. *The weight of 14 m long bar as shown in Fig. 8.32 is 600 N and if may be considered to be concentrated at a point 6 m from the bottom. It rests against a smooth vertical wall at A and on a rough horizontal floor at B. The co-efficient of static friction between the bar and the floor is 1/3. Establish by calculations if the bar would stand in the 60° position as shown.*

(AMIE Summer, 1989)

Sol. Given :

Length, $AB = 14$ m

Weight, $W = 600$ N

The weight is acting at D, which is 6 m from point B.

$\therefore$ Length $BD = 6$ m and length $AD = 8$ m

Vertical wall is smooth and hence $F_A = 0$

Co-efficient of friction, $\mu = 1/3$

Angle $CBA = 60°$

Let us find :

(*i*) actual force of friction available at point B, and

(*ii*) Force of friction required for equilibrium at point B.

The force of friction required for equilibrium will be obtained from moment equation *i.e.*, $\Sigma M = 0$. The actual force of friction available will be obtained from summation of forces in horizontal and vertical directions *i.e.*, $\Sigma F_x = 0$ and $\Sigma F_y = 0$

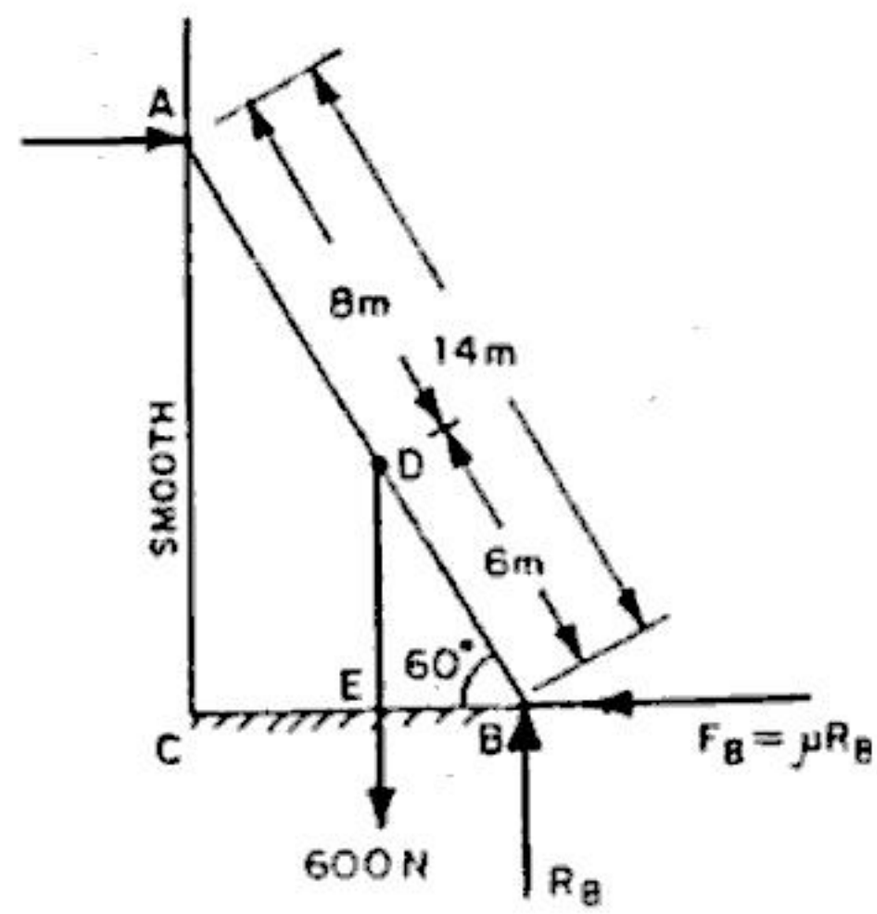

Fig. 8.32.

Let R_A = Normal reaction at A

R_B = Normal reaction at B

F_B = Force of friction at $B = \mu R_B$

The forces acting on the ladder are shown in Fig. 8.32.

Resolving forces horizontally,

$$R_A = F_B \qquad ...(i)$$

Resolving forces vertically,

$$R_B = 600 \text{ N} \qquad ...(ii)$$

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Now R_A=Normal reaction at A

R_B=Normal reaction at B.

The forces acting on the ladder are shown in Fig. 8.33.

Resolving forces vertically,

$$R_A=850+750=1600 \text{ N} \quad ...(i)$$

Resolving forces horizontally,

$$R_B=F_A=\mu R_A=\mu\times1600=1600\ \mu\text{N} \quad ...(ii)$$

In right-angled ΔABC, $BC=AB \sin 65°=6 \sin 65°$

$$=6\times.9063=5.437 \text{ m}$$

$$AC=AB \cos 65°=6 \cos 65°=6\times.4226=2.5357 \text{ m}$$

As G is the middle point of AB and GD is normal to AC.

$\therefore$ D is the middle point of AC

$$\therefore \quad AD=\frac{AC}{2}=\frac{2.5357}{2}=1.267 \text{ m}.$$

Now $AH=AE \cos 65°=(AB-BE) \cos 65°=(6-4) \cos 65°$

$$=2\times\cos 65°=2\times.4226=.8452 \text{ m}.$$

Taking the moments of all forces about A, we get

$$R_B\times BC=850\times AD+750\times AH.$$

Substituting the values of BC, AD, AH and R_B from equation (*ii*),

$$1600\mu\times5.437=850\times1.267+750\times.8452$$

or $$8699.2\ \mu=1076.95+633.9+1710.85$$

$$\therefore \quad \mu=\frac{1710.85}{8699.24}=\mathbf{0.199.} \quad \textbf{Ans.}$$

Problem 8.24. (S.I. Units). *A uniform ladder of weight 200 N of length 4.5 m rests on a horizontal ground and leans against a rough vertical wall. The co-efficient of friction between the ladder and floor is 0.4 and between ladder and vertical wall is 0.2. When a weight of 990 N is placed on the ladder at a distance of 1.2 m from the top of the ladder, the ladder is at the point of sliding. Find :*

(i) The angle made by the ladder with horizontal,

(ii) Reaction at the foot of the ladder, and

(iii) Reaction at the top of the ladder.

Sol. Given :

Weight of ladder, $W=200$ N

Length of ladder,

$$L=AB=4.5 \text{ m}$$

Co-efficient of friction between ladder and floor,

$$\mu_A=0.4.$$

Co-efficient of friction between ladder and wall,

$$\mu_B=0.2.$$

Weight on ladder,

$$W_1=990 \text{ N}$$

Distance $BE=1.2$ m.

Let α=Angle made by ladder with horizontal

R_A=Normal reaction at A

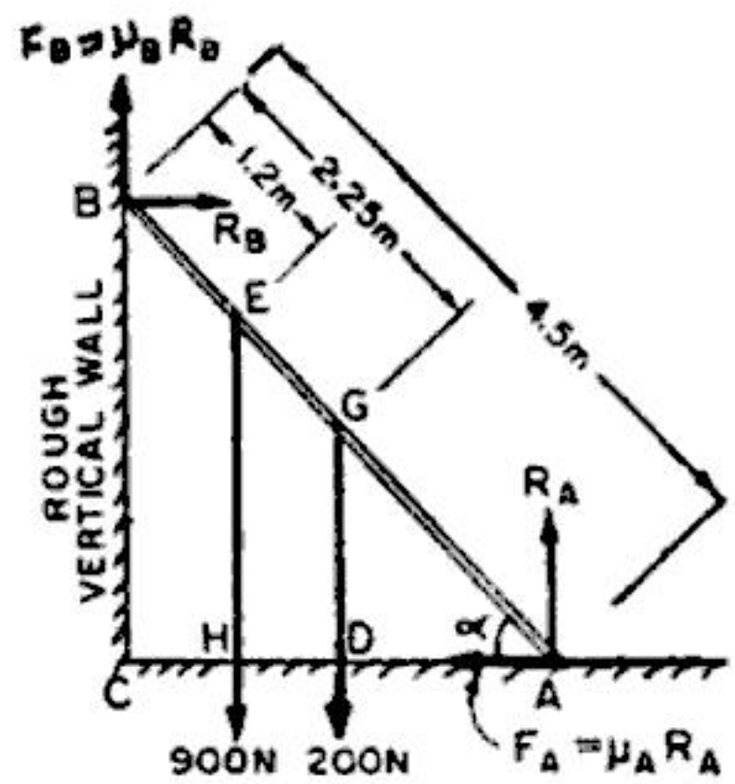

Fig. 8.34.

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(*iii*) Reaction R_2 on the face AB (The reaction R_2 is the resultant of normal reaction on the rubbing face AB and force of friction on surface AB). The reaction R_2 will be inclined at an angle ϕ_2 with the normal.

When the force P is applied on the wedge, the surface CA will be moving towards left and hence force of friction on this surface will be acting towards right. Similarly, the force of friction on face AB will be acting from A to B. These forces are shown in Fig. 8.37.

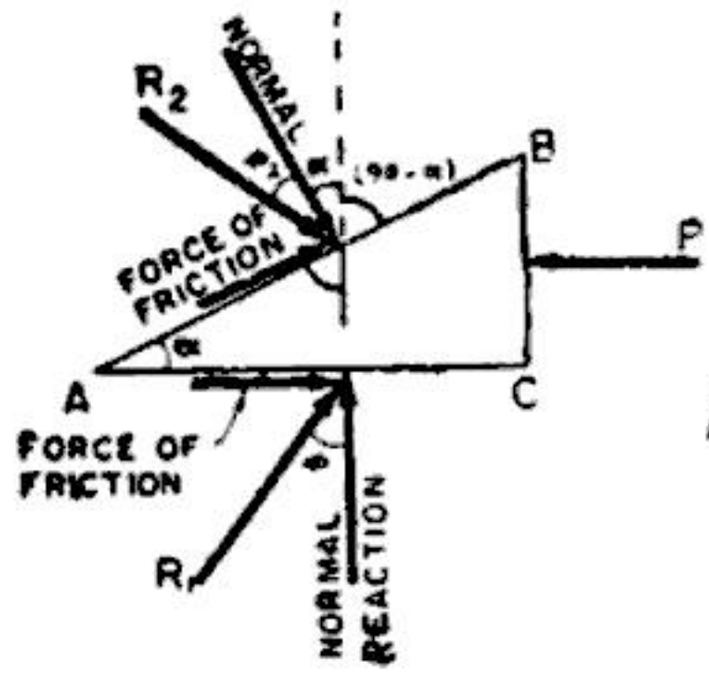

Fig. 8.37. Equilibrium of wedge.

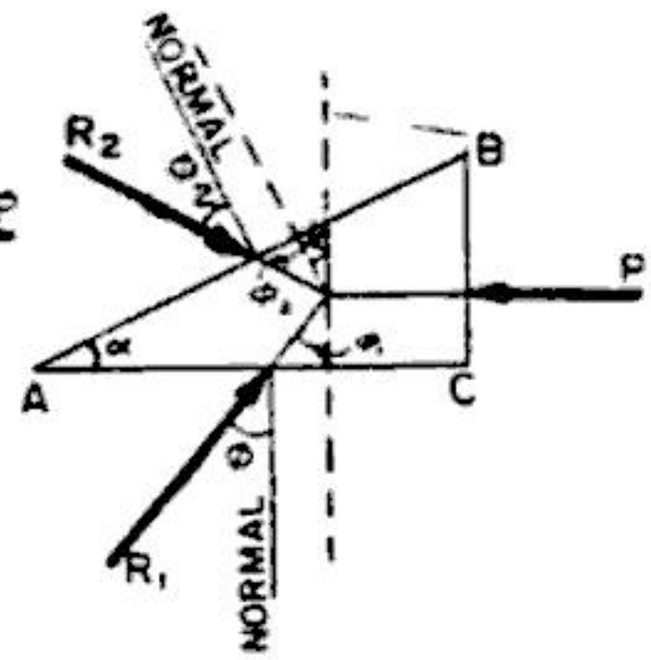

Fig. 8.38.

Resolving the forces horizontally, we get

$$R_1 \sin \phi_1 + R_2 \sin (\phi_2 + \alpha) = P.$$

Resolving the forces vertically, we get

$$R_1 \cos \phi_1 = R_2 \cos (\phi_2 + \alpha).$$

By Lami's Theorem

The wedge is in equilibrium under the action of three forces namely R_1, R_2 and P. These forces, when produced, will meet at a point as shown in Fig. 8.38.

Applying Lami's theorem, we get

$$\frac{P}{\sin (180 - \phi_1 - \phi_2 - \alpha)} = \frac{R_1}{\sin (90 + \alpha + \phi_2)} = \frac{R_2}{\sin (90 + \phi_1)}$$

Equilibrium of Body placed on the Wedge

The forces acting on the body are shown in Fig. 8.39. They are :

(*i*) The weight W of the body.

(*ii*) Reaction R_3 on the face GD. (The reaction R_3 is the resultant of the normal reaction on the rubbing face GD and force of friction on surface GD).

(*iii*) Reaction R_2 on the face GF (The reaction R_2 is the resultant of the normal reaction on the rubbing face GF and force of friction on surface GF).

These forces are shown in Fig. 8.39.

Resolving the forces R_2, R_3 and W horizontally, we get

$$R_3 \cos \phi_3 = R_2 \sin (\alpha + \phi_2)$$

Resolving forces vertically, $W + R_3 \sin \phi_3 = R_2 \cos (\alpha + \phi_2)$

By Lami's Theorem

The forces R_3, R_2 and W are produced to meet at a point as shown in Fig. 8.39.

The body is in equilibrium under the action of these three forces.

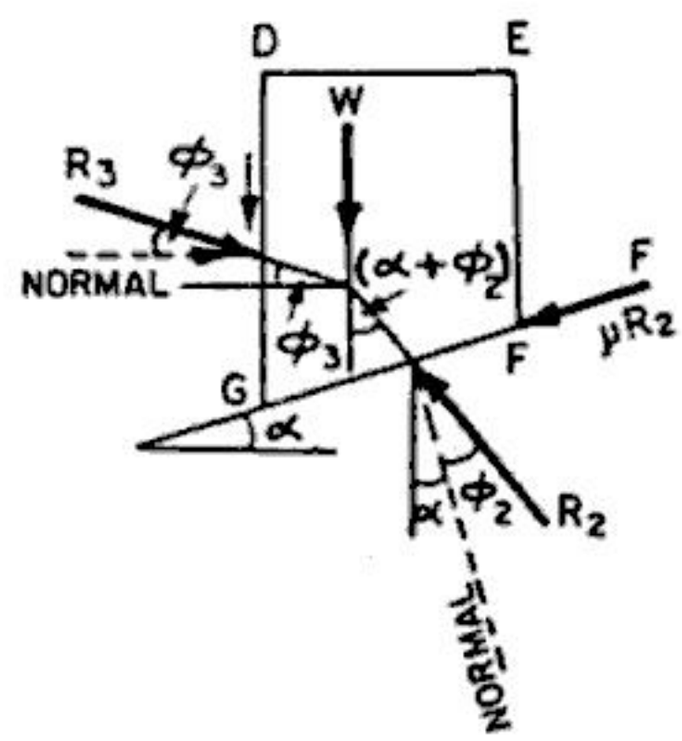

Fig. 8.39.

Hence applying Lami's theorem, we get

$$\frac{W}{\sin (90 + \phi_3 + \alpha + \phi_2)} = \frac{R_2}{\sin (90 - \phi_3)} = \frac{R_3}{\sin [180 - (\alpha + \phi_1)]}.$$

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15. If the force for the S. No. 14 is applied to the body horizontally, then the force required to drag the body is given by

$P = W \tan(\alpha+\phi)$.... when the body is on the point of moving up the plane.

$= W \tan(\alpha-\phi)$.... when the body is on the point of moving down the plane.

16. Wedge is a piece of metal or wood which is usually of a triangular or trapezoidal in cross-section. It is used either lifting loads or used for slight adjustments in the position of a body *i.e.*, for tightening fits or keys for shafts.

17. The problems on wedges are solved by : (*i*) resolving the forces on the wedge and block horizontally and vertically or (*ii*) by applying Lami's Theorem.

EXERCISE 8

(A) Theoretical Questions

1. Define the following terms : Friction, limiting force of friction, co-efficient of friction and angle of friction.

2. Explain the difference between co-efficient of friction and angle of friction. [*AMIE Winter, 1981*]

3. (*a*) State the laws of static and dynamic friction. [*AMIE (S) 1982, (S) 1986*]

(*b*) State the laws of solid friction. [*AMIE Summer, 1983, (S) 1989*]

4. Prove that the angle of friction is equal to the angle of the inclined plane, when a solid body of weight W placed on the inclined plane, is about to slide down.

5. What do you mean by 'angle of repose' ? Prove that angle of repose is equal to the angle of friction.

6. A body of weight W is placed on an inclined rough plane. The inclination of the plane with the horizontal is less than the angle of friction. The body will

(*a*) be in equilibrium (*b*) move downwards

(*c*) move upwards (*d*) none of the above. [**Ans.** (*a*)]

7. If in the above question, the inclination of the plane with the horizontal is more than the angle of friction, the body will

(*a*) be in equilibrium (*b*) move downwards

(*c*) move upwards (*d*) none of the above. [**Ans.** (*b*)]

8. A body of weight W is placed on a rough inclined plane having inclination α to the horizontal. The force P is applied horizontally to drag the body. If the body is on the point of motion up the plane, prove that P is given by

$$P = W \tan(\alpha + \phi)$$

where ϕ = Angle of friction.

9. In the above question, if the body is on the point of motion down the plane, prove that the force P is given by

$$P = W \tan(\alpha - \phi).$$

10. A body of weight W is placed on a rough inclined plane having inclination α to the horizontal. A force P is applied to the body in such a way that it makes an angle θ to the inclined plane. Prove that the force required to drag the body when the body is on the point of motion up the plane is given by

$$P = \frac{W \sin(\alpha + \phi)}{\cos(\theta - \phi)}$$

where α=Inclination of the inclined plane with horizontal

θ=Inclination of the force with the inclined plane

ϕ=Angle of friction.

11. Prove that in question 10, (*i*) the force P will be minimum if the angle of inclination of the force with the inclined plane is equal to the angle of friction and (*ii*) least force is given by

$$P_{min} = W \sin(\alpha + \phi).$$

12. Define statics and kinetic friction and state the laws of solid friction. [*AMIE Summer, 1983*]

13. Derive an expression for the least force required to drag a body on a rough horizontal plane.

14. What is a wedge ? Explain how a wedge is used to raise heavy loads.

(*AMIE Winter, 1982 ; AMIE Summer, 1986*)

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$$\text{M.A.} = \frac{\text{Weight lifted}}{\text{Effort applied}} = \frac{W}{P} \qquad ...(9.5)$$

6. **Ideal Machine.** If the friction in a machine is negligible, the machine is known as ideal machine. In ideal machine the efficiency is 100% and output is equal to input. Equating the input and output given by equations (9.1) and (9.2), we have for ideal machine

$$P \times y = W \times x$$

or

$$\frac{y}{x} = \frac{W}{P}$$

But from equation (9.4), $\frac{y}{x}$=V.R. and from equation (9.5) $\frac{W}{P}$=M.A. Hence for ideal machine

$$\text{V.R.} = \text{M.A.} \qquad ...(9.6)$$

In actual practice, no machine is ideal. A part of the work done on the machine is always lost in overcoming friction and hence the work done *on* the machine is always greater than the work done *by* the machine.

7. **Efficiency of a machine (η) in terms of mechanical advantage (M.A.) and velocity ratio (V.R.)**

Efficiency of machine is given by equation (9.3) as

$$\eta = \frac{\text{Output}}{\text{Input}} = \frac{W \times x}{P \times y} = \frac{W/P}{y/x}$$

$$= \frac{\text{M.A.}}{\text{V.R.}} \qquad \left(\because \frac{W}{P} = \text{M.A. and } \frac{y}{x} = \text{V.R.}\right) \qquad ...(9.7)$$

8. **Efficiency of a machine (η) in terms of ideal effort and actual effort.**

Let P = Actual effort applied to an actual machine

P' = Ideal effort applied to an ideal machine.

For an ideal machine,

$$\text{Input} = \text{Output}$$

$$P' \times y = W \times x$$

$$\therefore \quad P' = \frac{W \times x}{y} \qquad ...(9.8)$$

For an actual machine, efficiency is given by equation (9.3),

$$\eta = \frac{Wx}{P \times y} = \frac{1}{P}\left(\frac{Wx}{y}\right)$$

$$= \frac{P'}{P} \qquad \left[\because \frac{Wx}{y} = P' \text{ from Equation (9.8)}\right]$$

$$= \frac{\text{Ideal effort}}{\text{Actual effort}} \qquad ...(9.9)$$

9.3. REVERSIBLE AND IRREVERSIBLE MACHINE

Let P is the actual effort applied to a machine to lift a load W. If the actual effort P is removed, the load W may or may not fall down to the initial position. If the load W falls down, the machine is said to be reversible. In this case work is done by the machine in reverse direction. And if the load W does not fall down, the machine is said to be *irreversible* or *self locking*. In this case work is not done by the machine in reverse direction.

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in Fig. 9.1. For an ideal machine the straight line will pass through the origin. But for an actual machine, the straight line will intercept the y-axis as shown in Fig. 9.1. The intercept OA = Effort applied to the machine to overcome friction. If the effort applied is less than OA, the load will not be lifted. The law of a machine is given mathematically as :

$$y = mx + c$$

where y = Effort = P

x = Load = W and c = constant

$\therefore$ The law of machine becomes

$$P = mW + c \qquad ...(9.10)$$

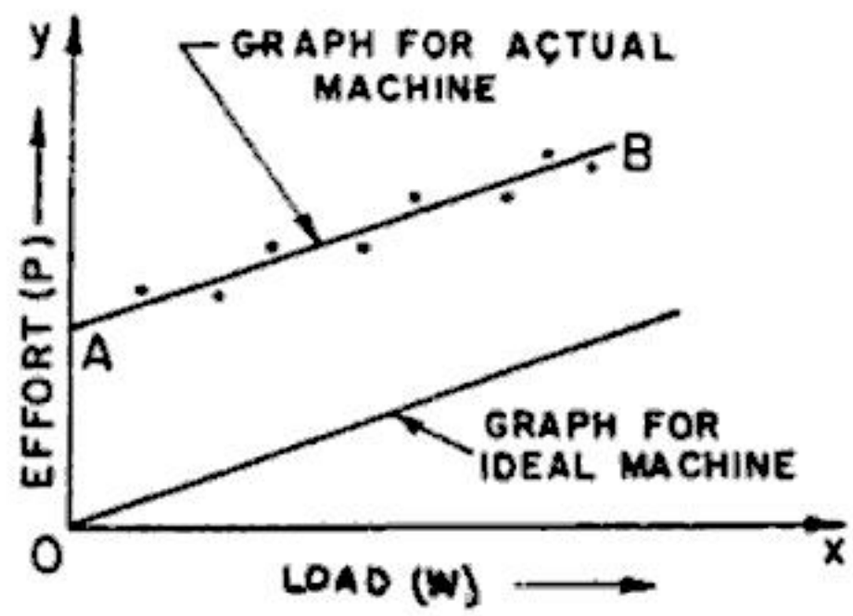

Fig. 9.1. Law of a machine.

In equation (9.10),

m = Slope of the line AB and is equal to a constant which is known co-efficient of friction.

c = Intercept of the line on Y-axis and is equal to the effort required to overcome friction.

9.5.1. Expression for maximum mechanical advantage. Equation (9.10) gives the law of a machine as

$$P = mW + C$$

where P = Effort applied

W = Load lifted

From the above equation, the value of W is obtained as

$$mW = P - C$$

or $$W = \frac{P-C}{m} = \frac{P}{m}\left[1 - \frac{C}{P}\right]$$

or $$\frac{W}{P} = \frac{1}{m}\left[1 - \frac{C}{P}\right] \qquad ...(i)$$

But $$\frac{W}{P} = \text{Mechanical Advantage}$$

From Fig. 9.1, it is clear that with the increase of W (load), the effort (P) increases. With the increase of P, the value of $\frac{C}{P}$ decreases as C is constant. Hence from equation (i) it is obvious that with the decrease of $\frac{C}{P}$, the mechanical advantage increases. If the value of P is so large, that the value $\frac{C}{P}$ becomes negligible, then mechanical advantage will be maximum and its value will be obtained from equation (i), as

$$\frac{W}{P} = \frac{1}{m} \qquad \left\{\because \frac{C}{P} \text{ is negligible}\right\}$$

$$\boxed{\therefore \quad \text{Max. M.A.} = \frac{1}{m}} \qquad ...(9.10\text{ A})$$

9.5.2. Expression for Maximum Efficiency

Let V.R. = Velocity ratio of a machine

Then $$\eta = \frac{\text{M.A.}}{\text{V.R.}} = \frac{\frac{W}{P}}{\text{V.R.}} = \frac{W}{P \times (\text{V.R.})}$$

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9.7. FRICTION OF A MACHINE EXPRESSED IN TERMS OF LOAD

Let P = Effort required to lift a load W,

W = Actual load lifted by P,

W' = Ideal load lifted by P, and

F_W = Frictional force of the machine in terms of load

The ideal load (W') lifted by the effort P will be more than the actual load (W) lifted by the same effort. Hence the difference of ideal load and actual load lifted by the same effort (P) is equal to the frictional force of the machine in terms of load.

∴ Frictional force of the machine in terms of load

= Ideal load lifted − Actual load lifted

or $$F_W = W' - W \quad \text{...(iii)}$$

But from equation (9.7),

$$\eta = \frac{\text{M.A.}}{\text{V.R.}} = \frac{\left(\frac{\text{Load}}{\text{Effort}}\right)}{\text{V.R.}} \quad \text{...(iv)}$$

For ideal machine, $\eta = 100\%$ or 1.0

$$\text{Load} = W'$$

and $$\text{Effort} = P$$

Substituting these values in equation (iv), we get

$$1.0 = \frac{W'}{P \times \text{V.R.}}$$

∴ $$W' = P \times \text{V.R.}$$

Substituting the value of W' in equation (iii), we get

$$\boxed{F_W = (P \times \text{V.R.}) - W} \quad \text{...(9.12)}$$

Problem 9.7. *The efficiency of a machine is 80% when an effort of 15 N is required to lift a load of 130 N. Calculate the velocity ratio and the frictional force of the machine in terms of effort and load.*

Sol. Given :

Efficiency, $\eta = 80\% = \frac{80}{100} = 0.8$

Effort, $P = 15$ N

Load, $W = 130$ N

Using equation (9.7), we get

$$\eta = \frac{\text{M.A}}{\text{V.R.}} = \frac{W}{P \cdot \text{V.R.}} \quad \left(\because \text{M.A.} = \frac{W}{P}\right)$$

or $$0.80 = \frac{130}{\frac{15}{\text{V.R.}}} = \frac{130}{15 \times \text{V.R.}}$$

∴ $$\text{V.R.} = \frac{130}{15 \times 0.8} = \mathbf{10.833.} \quad \textbf{Ans.}$$

Frictional force of the machine in terms of effort is given by equation (9.11) as

$$F_P = P - \frac{W}{\text{V.R.}} = 15 - \frac{130}{10.833} = 15 - 12 = \mathbf{3\ N.} \quad \textbf{Ans.}$$

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Problem 9.12. *In a lifting machine a load of 200 N was lifted by an effort of 15 N. The velocity ratio of the machine is 40. If the effort is removed, will there be a reversal of the machine ? Determine the frictional load of the machine.*

Sol. Given :

Load lifted, $W=200$ N

Effort applied, $P=15$ N

Velocity ratio, V.P.=40.

(*i*) The machine will be non-reversible if its efficiency is less than 50%.

Now efficiency, $$\eta=\frac{\text{M.A.}}{\text{V.R.}}=\frac{W}{P}\Big/\text{V.R.}=\frac{15}{40}=\frac{200}{15\times40}=0.3333 \quad \text{or} \quad 33.33\%.$$

As the efficiency is less than 50%, the machine is non-reversible.

(*ii*) Friction load of the machine is given by

$$F_w=P\times(\text{V.R.})-W=15\times40-200$$
$$=600-200=400 \text{ N. Ans.}$$

9.8. IMPORTANT LIFTING MACHINES

The following are the important lifting machines :

1. Simple wheel and axle.
2. Differential wheel and axle.
3. Worm and worm wheel.
4. Single purchase crab winch.
5. Double purchase crab winch.
6. Pulleys :
   (*a*) First system of pulleys
   (*b*) Second system of pulleys
   (*c*) Third system of pulleys.
7. Weston's differential pulley block.
8. Screw jack :
   (*a*) Simple Screw Jack
   (*b*) Differential Screw Jack.

9.8.1. Simple wheel and axle. Fig. 9.2 shows a simple wheel and axle which consists of two cylinders *A* and *B* of different diameters keyed to the same shaft. The bigger cylinder *A* is called the wheel, and the smaller cylinder *B* the axle.

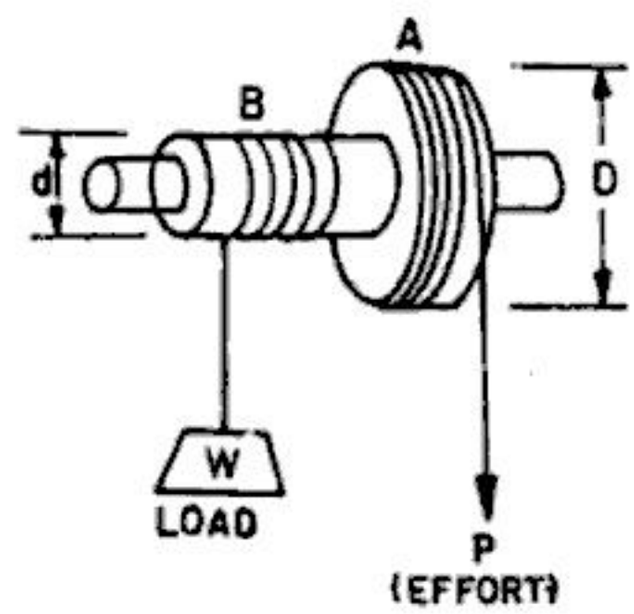

Fig. 9.2.

A string is wound round the axle *B*, which carries the load *W* to be lifted. A second string is wound round the wheel *A*, to which effort is applied. The two strings are wound in the *opposite directions*. Hence a downward motion of *P* will raise the load *W*.

Let W=Load lifted,

P=Effort applied,

D=Diameter of wheel, and

d=Diameter of axle.

As the wheel and axle are keyed to the same shaft, hence when wheel makes one revolution, the axle will also make one revolution.

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$$\therefore \quad \text{V.R.} = \frac{2\pi L}{2 \times \frac{2\pi r}{T}} = \frac{L \times T}{2r} \quad \text{...(9.16)}$$

If worm is n-threaded, then

$$\text{V.R.} = \frac{L \times T}{n \times r} \quad \text{...(9.17)}$$

Problem 9.15. *The number of teeth on the worm-wheel of a single-threaded worm and worm wheel is 60. Calculate the velocity ratio if the diameter of effort wheel is 25 cm and that of load drum is 12.5 cm. The effort required to lift a load of 600 N by this machine is 20 N. Find the efficiency of the machine.*

Sol. Given :

No. of teeth on worm-wheel, $T = 60$

Dia. of effort wheel $= 25$ cm

$\therefore$ Radius of effort wheel, $L = \frac{25}{2} = 12.5$ cm.

Dia. of load drum $= 12.5$ cm.

$\therefore$ Radius of load drum, $r = \frac{12.5}{2} = 6.25$ cm

Effort, $P = 20$ N

Load, $W = 600$ N

(*i*) V.R. for a single-threaded worm and worm wheel is given by equation (9.15).

$$\therefore \quad \text{V.R.} = \frac{LT}{r} = \frac{12.5 \times 60}{6.25}$$

$$= 120. \quad \textbf{Ans.}$$

(*ii*) Efficiency is given by,

$$\eta = \frac{\text{M.A.}}{\text{V.R.}}$$

$$= \frac{\left(\frac{W}{P}\right)}{\text{V.R.}} = \frac{\left(\frac{600}{20}\right)}{120}$$

$$= \frac{30}{120} = 0.25 = 25\%. \quad \textbf{Ans.}$$

9.8.4. Single Purchase Crab Winch. Fig. 9.5 shows a single purchase crab winch, which consists of an 'effort axle' and a load axle. On effort axle, a small toothed wheel known as pinion is mounted. On load axle, a large toothed wheel known as spur wheel is mounted in such a way that the spur wheel meshes with the pinion. Both the axles are suitably mounted on a rigid frame as shown.

A rope is fixed to the load axle and is wound a few turns round it. The free end of the rope carries the load W.

Effort is applied at the end of the lever which is fixed to the effort axle. By rotating the lever the pinion is rotated which in turn rotates the spur wheel and the rope is wound on the load axle, thus lifting the load attached to the free end of the rope.

Let W = Load lifted,

P = Effort applied,

T_1 = Number of teeth on the pinion,

T_2 = Number of teeth on the spur wheel,

L = Length of lever arm, and

D = Diameter of the load axle.

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But $\quad$ M.A. $=\dfrac{W}{P^*}$ where P^* is ideal effort

$\therefore \quad \dfrac{W}{p^*}=16$

or $\quad p^*=\dfrac{W}{16}=\dfrac{1440}{16}=90$ N

But actual effort, $P=100$ N

$\therefore$ Effort wasted in friction

$=$Actual effort$-$Ideal effort$=100-90=20$ N. **Ans.**

(*iii*) *Load wasted in friction*

Let $\quad W^*=$Ideal load

For ideal machine,

M.A. $=$ V.R. $=16$

or $\quad \dfrac{W^*}{P}=16$

or $\quad W^*=16\times P \qquad$ where $P=$Actual effort$=100$ N

$\therefore \quad =16\times 100=1600$ N

$\therefore$ Load wasted in friction

$=$Ideal load$-$Actual load$=1600-1440=160$ N. **Ans.**

(*b*) **Second system of pulleys.** Fig. 9.13 (*b*) and (*c*) shows the second system of pulleys which consists of two blocks each containing a number of pulleys. The upper block is fixed to a fixed support whereas the lower block is movable. Both the blocks carry either equal number of pulleys or the upper block may have one pulley more than the lower one.

The same string is passed round all the pulleys. One end of the string is fixed to the lower block [if the upper block has one pulley more than the lower one as shown in Fig. 9.13 (*b*)], and the other end of the string is free and effort is applied to this free end.

If both the blocks have the same number of pulleys, then one end of the string is fixed to the upper block as shown in Fig. 9.13 (*c*) and effort is applied to the other free end of the string. In both cases, the weight is attached to the lower block.

The pulleys being smooth, the tension (T) in the string throughout will be equal to the applied effort P.

Let n=number of segments supporting the lower block with weight W.

The force supporting the lower block$=n\times T$

$=n\times P \qquad (\because \quad T=P)$

When lower block is in equilibrium,

$W=n\times P$

or $\quad \dfrac{W}{P}=n$

$\therefore \quad$ M.A. $=n$

Since the system is ideal (*i.e.*, without friction), hence

M.A. $=$ V.R.

$\therefore \quad$ V.R. $=n \qquad$ ...(9.21)

where n=number of segments supporting the movable block or load.

=always total number of pulleys in two blocks.

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Expression for P in terms of co-efficient of friction and pitch of the screw.

From equation (9.30 *c*)

$$P=\frac{Wd}{2L}\tan(\phi-\alpha)=\frac{Wd}{2L}\frac{\tan\phi-\tan\alpha}{1+\tan\phi\tan\alpha}$$

$$=\frac{Wd}{2L}\frac{\mu-\frac{p}{\pi d}}{1+\mu\frac{p}{\pi d}} \qquad \left(\because \tan\phi=\mu,\ \tan\alpha=\frac{d}{\pi d}\right)$$

$$=\frac{Wd}{2L}\frac{\mu\pi d-p}{\pi d+\mu p}. \qquad ...(9.31)$$

Problem 9.22. *(a) Find the effort required to apply at the end of a handle, fitted to the screw head of a screw-jack to lift a load of 1500 N. The length of the handle is 70 cm. The mean diameter and the pitch of the screw-jack are 6 cm and 0.9 cm respectively. The co-efficient of friction is given as 0.095.*

(b) If instead of raising the load of 1500 N, the same load is lowered, determine the effort required to apply at the end of the handle.

Sol. Given :

Load, $W=1500$ N

Length of handle, $L=70$ cm$=0.70$ m

Mean dia. of screw-jack, $d=6$ cm$=0.6$ m

Pitch of the screw-jack, $p=0.9$ cm$=0.009$ m

Co-efficient of friction, $\mu=0.095$.

(*a*) **Effort (P) required at the end of the handle for raising the load (W).**

Let P=Effort required

Using equation (9.29), we have

$$P=\frac{Wd}{2L}\left(\frac{p+\mu\pi d}{\pi d-p\mu}\right)=\frac{1500\times.06}{2\times0.70}\left(\frac{.009+.095\times\pi\times0.6}{\pi\times.06-.009\times.095}\right)\ \text{N}$$

$$=64.2857\left(\frac{.0269}{.1885-.000855}\right)=\mathbf{9.21\ N.}\quad \textbf{Ans.}$$

(*b*) The effort (P) required at the end of the handle for lowering the load W is given by equation (9.31) as

$$P=\frac{Wd}{2L}\frac{(\mu\pi d-p)}{(\pi d+\mu p)}=\frac{1500\times.06}{2\times0.70}\left(\frac{0.095\times\pi\times.06-.009}{\pi\times0.6+.095\times.009}\right)$$

$$=64.2857\left(\frac{.0089}{.1885+.000855}\right)=\mathbf{3.02\ N.}\quad \textbf{Ans.}$$

Problem 9.23. *A screw jack is used to lift a load of 3 kN. The screw of the screw-jack is square threaded with two threads to 1.2 cm. If the co-efficient of friction between the nut and screw is 0.09 and the outer diameter of the screw is 6 cm, find the force required at the end of the handle of length 60 cm to lift the load.*

Sol. Given :

Load lifted, $W=3$ kN$=3\times1000=3000$ N

No. of square thread in 1.2 cm=2

$\therefore$ Pitch of the screw, $p=\frac{1.2}{\text{No. of threads}}=\frac{1.2}{2}$

$=0.6$ cm$=.006$ m.

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Problem 9.26. *A screw-jack has a square thread, 7.5 cm mean diameter and 1.5 cm pitch. The load on the jack revolves with the screw. The co-efficient of friction at the screw threads is 0.05.*

(i) Find the tangential force to be applied to the jack at 36 cm radius so as to lift a load of 600 N.

(ii) State whether the jack is self-locking. If it is, find the torque necessary to lower the load. If not, find the torque, which must be applied to keep the load from descending.

(AMIE Winter, 1981)

Sol. Given :

Mean diameter, $d=7.5$ cm$=0.075$ m

Pitch of threads, $p=1.5$ cm$=0.015$ m

Co-efficient of friction, $\mu=0.05$.

Part First

Let P=Tangential force required at 36 radius to lift the load

W=Load lifted=600 N

L=Length of handle=36 cm=0.36 m

Using equation (9.29), we get

$$P=\frac{Wd}{2L}\left(\frac{p+\mu\pi d}{\mu d-p\mu}\right)=\frac{600\times.075}{2\times0.36}\left(\frac{.015+0.05\times\pi\times.075}{\pi\times.075-0.015\times.05}\right)$$

$$=62.5\times\left(\frac{0.02678}{0.2356-0.00075}\right)$$

$$=7.126 \text{ N.} \quad \textbf{Ans.}$$

Part Second

The jack will be self-locking if its efficiency is less than 50%. Hence first find the efficiency of the jack. The efficiency of the screw-jack is given by the equation (9.32) as

$$\eta=\frac{\tan\alpha}{\tan(\alpha+\phi)} \qquad ...(i)$$

where from equation (9.13), $\tan\alpha=\frac{p}{\pi d}=\frac{0.015}{\pi\times0.75}=0.0636$

and $\tan\phi=\mu=0.05$.

From trigonometry, $\tan(\alpha+\phi)=\frac{\tan\alpha+\tan\phi}{1-\tan\alpha\tan\phi}=\frac{.0636+.05}{1-.0636\times.05}$

$$=\frac{0.1136}{0.9968}=.0114.$$

Substituting the values of $\tan\alpha$ and $\tan(\alpha+\phi)$ in equation (*i*), we get

$$\eta=\frac{0.0636}{.0114}=0.5579=55.79\%.$$

As the efficiency is more than 50%, the jack is not self-locking. **Ans.**

Part Third

As the screw-jack is not self-locking, the weight W(=600 N) will start coming down. The effort (P) applied at the mean radius of the screw jack to balance the load, which is descending, is given by equation [9.30 (*b*)] as

$$P'=W\tan(\alpha-\phi) \qquad \text{[Here } \alpha>\phi \text{ as } \tan\alpha=.0636 \text{ and } \tan\phi=.05\text{]}$$

$$=W\frac{\tan\alpha-\tan\phi}{1+\tan\alpha\tan\phi} \qquad \left(\because \tan(\alpha-\phi)=\frac{\tan\alpha-\tan\phi}{1+\tan\alpha\tan\phi}\right)$$

$$=600\left(\frac{.0636-.05}{1+.0636\times.05}\right)=\frac{600\times.01366}{1.003180}$$

$$=8.17 \text{ kg wt}$$

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10.4. PRINCIPLE OF VIRTUAL WORK

It states, "If a system of forces acting on a body or a system of bodies be in equilibrium and if the system is imagined to undergo a small displacement consistent with the geometrical conditions, then the algebraic sum of the virtual work done by the forces of the system is zero".

10.5. UNITS OF WORK

Work is the product of force and distance. In S.I. system, unit of force is Newton (N) whereas the unit of distance is in metre (m). Hence the unit of work in S.I. system is Newton-metre and written as N m.

The unit of force in M.K.S. system is kilogram force (written as kgf) and unit of distance is metre (m). Hence unit of work in M.K.S. system is written as kgf-m.

One kgf-m=9.81 N m. ($\because$ 1 kgf=9.81 N)

10.6. FORCES TO BE OMITTED WHILE APPLYING THE PRINCIPLE OF VIRTUAL WORK

The following forces may be omitted while applying the principle of virtual work :

(*i*) Tension produced in an inextensible string. As the string is inextensible, its length does not undergo any change. Hence $\delta l=0$ and $T\times\delta l=0$.

(*ii*) Reactions at smooth pins and hinges.

(*iii*) Reactions between the body and any surface on which the body rolls without sliding.

(*iv*) Forces normal to the direction of motion.

(*v*) The mutual action and reaction between the two bodies whose equilibrium is being considered together. As the action and reaction are equal and opposite, so the work done by them cancel each other.

For most of the problems solved by virtual work method, it is assumed that there is no friction in hinges, bearing or along sliding surfaces. This means the system is assumed to be frictionless.

10.7. IMPORTANT POINTS TO BE REMEMBERED WHILE APPLYING THE PRINCIPLE OR VIRTUAL WORK

1. If the tension T in an inextensible string is to be calculated, replace the string by two forces each equal to T acting at the ends of the string. Now apply the principle of virutal work and calculate the value of T.

2. If the reaction R at a rigid support is to be calculated, as the rigid support is not supposed to move, the virtual work done by rigid support will be zero. Hence the reaction R will not appear in the equation of virtual work. To overcome this difficulty, remove the support and replace it by a force R. This force can be given a virtual displacement and the value of R will be obtained by applying principle of virtual work.

3. If all the forces act in the same direction, give the system a virtual displacement in the direction of the forces.

4. Virtual displacements are denoted by δx or $\delta\theta$ whereas actual displacements are denoted by dx or $d\theta$.

10.8. USES OF THE PRINCIPLE OF VIRTUAL WORK

The principle of virtual work is used in solving problems of statics in the following cases:

1. Beams 2. Framed structure 3. Lifting machine.

10.8.1. The Principle of Virtual Work for Problems on Beams. The principle of virtual work is applied for the problems on beams as given below:

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Virtual work done by point load at D

$= -$Load at $D \times$Virtual displacement at D.

(Work done is $-$ve as load at D is acting downward but virtual displacement is upward)

$= -20 \times \delta_D$

$= -20 \times 0.6\ \delta_B$ [$\because$ From equation (*ii*), $\delta_D = 0.6\ \delta_B$]

$= -12\ \delta_B$ kN m.

Virtual work done by point load at C

$= -$Load at $C \times$Virtual displacement at C

(Load at C is acting downward but virtual displacement is upward and hence work is $-$ve)

$= -15 \times \delta_C$

$= -15 \times 0.4\ \delta_B$ [$\because$ From equation (*i*), $\delta_C = 0.4\ \delta_B$]

$= -6\ \delta_B$ kN m

Virtual work done by reaction

$R_A = R_A \times$Virtual displacement at A

$= R_A \times 0$

($\because$ End A is intact and hence virtual displacement is zero)

Total Virtual work done=Virtual work done by R_B+Virtual work done by point load at D+Virtual work done by point load at C+Virtual work done by R_A

$= R_B \times \delta_B + (-12\ \delta_B) + (-6\ \delta_B) + 0$

$= R_B \times \delta_B - 12\ \delta_B - 6\ \delta_B$

$= R_B \times \delta_B - 18\ \delta_B$.

But from the principle of virtual work, the total virtual work done should be zero.

$\therefore \quad R_B \times \delta_B - 18\ \delta_B = 0$

or $$R_B = \frac{18\ \delta_B}{\delta_B} = \mathbf{18\ kN.}\quad \textbf{Ans.}$$

But $\quad R_A + R_B = 15 + 20 = 35$

$\therefore \quad R_A = 35 - R_B = 35 - 18 = \mathbf{17\ kN.}$ **Ans.**

Problem 10.3. *By the principle of virtual work, determine the reactions for the beam shown in Fig. 10.6.*

Sol. Given :

Point load at $C = 4$ kN

Point load at $D = 12$ kN

Point load at $E = 8$ kN.

Distances $AD = 6$ m, $AB = 10$ m, $AE = 10 + 2 = 12$ m, $AC = 2$ m.

Let R_A=Reaction at A

R_B=Reaction at B

To determine the reaction R_B, let the beam is given a virtual displacement at B in the upward direction, keeping the end A intact. All the points lying towards right from point A will be have upward virtual displacements whereas the points lying towards left from point A will have downward virtual displacements.

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$\therefore \quad -540\ \delta_B + R_B \times \delta_B = 0$

$\therefore \quad R_B = \dfrac{540\ \delta_B}{\delta_B} = \mathbf{540\ N.}$ **Ans.**

Reaction at hinge C

Consider the beam AC alone. It is in equilibrium as shown in Fig. 10.8. The reaction R_C at hinge must act vertically downward to keep the beam in equilibrium.

Let a virtual displacement at C is given in upward direction.

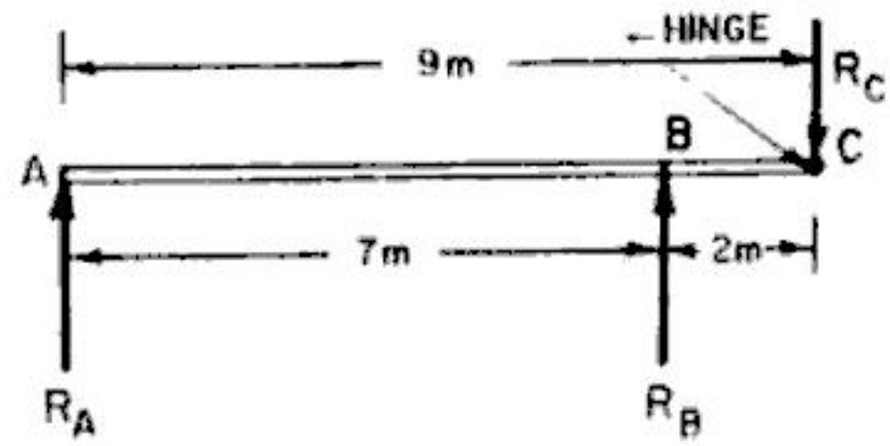

Fig. 10.8.

Now the virtual work done by R_C

$= -R_C \times$ Virtual displacement at C

$= -R_C \times \delta_C$ (−ve sign as R_C acting downward but virtual displacement is upward)

Virtual work done by $R_B = R_B \times \delta_B$

$= 540 \times \delta_B$ ($\because R_B = 540$)

Virtual work done by $R_A = R_A \times 0 = 0$

$\therefore$ Total virtual work done $= -R_C \times \delta_C + 540 \times \delta_B + 0$

$= -R_C \times \delta_C + 540 \times \delta_B$

$= -R_C \times \dfrac{9}{7}\ \delta_B + 540 \times \delta_B$ $\left[\because \text{From equation } (i),\ \delta_C = \dfrac{9}{7}\ \delta_B\right]$

But from the principle of virtual work, total virtual work done should be zero.

$\therefore \quad -R_C \times \dfrac{9}{7}\ \delta_B + 540\ \delta_B = 0$

$\therefore \quad R_C \times \dfrac{9}{7}\ \delta_B = 540\ \delta_B$

or $\quad R_C = \dfrac{7}{9} \times 540 = \mathbf{420\ N.}$ **Ans.**

Problem 10.5. *Two beams AC and CD of lengths 4 m and 6 m respectively are hinged at C. These are supported on rollers at the left and right ends (A and D). A hinged support is provided at B, 4 m from D. Two loads of 4 kN and 3 kN at a distance of 3 m from A and 2 m from D respectively are acting on the two beams. Find the reaction at B, using the principle of virtual work.*

Sol. Given :

Length	$AC = 4$ m
Length	$CD = 6$ m
Length	$BD = 4$ m
Length	$AE = 3$ m
Length	$FD = 2$ m
Point load at	$E = 4$ kN
Point load at	$F = 3$ kN
Let	R_A = Reaction at A
	R_B = Reaction at support B
	R_D = Reaction at D.

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x=Horizontal distance of A from O

y=Vertical distance of C from x-axis

Hence total virtual work done

$$=P\times\delta x+W\times\delta y$$

But according to principal of virtual work, the total virtual work done should be zero.

$$\therefore \quad P\times\delta x+W\times\delta y=0 \qquad ...(i)$$

The values of δx and δy are obtained as given below :

From ΔOAD of Fig. 10.16,

$$\tan\theta=\frac{OD}{OA}=\frac{9}{x}$$

$$\therefore \quad x=\frac{9}{\tan\theta}=9\cot\theta$$

Hence differential dx (or δx) is obtained as

$$\delta x=9\,(-\text{cosec}^2\,\theta)\times\delta\theta$$

Also $\quad y=4\sin\theta$

$$\therefore \quad \delta y=4\cos\theta\,.\,\delta\theta$$

Substituting the values of δx and δy in equation (*i*),

$$P\times 9\,(-\text{cosec}^2\,\theta)\times\delta\theta+W\times 4\cos\theta\,\delta\theta=0$$

or $$-9P\,\text{cosec}^2\,\theta+4W\cos\theta=0$$

or $$4W\cos\theta=9P\,\text{cosec}^2\,\theta$$

$$\therefore \quad P=\frac{4\,W\cos\theta}{9\,\text{cosec}^2\,\theta}$$

$$=\frac{4}{9}\times W\times\cos\theta\times\sin^2\theta$$

$$=\frac{4}{9}\times 600\times\cos 60^\circ\times\sin^2 60 \qquad (\because\ W=600\text{ N and }\theta=60^\circ)$$

$$=\frac{4}{9}\times 600\times\frac{1}{2}\times(0.866)^2=\mathbf{100\ N.}\quad \textbf{Ans.}$$

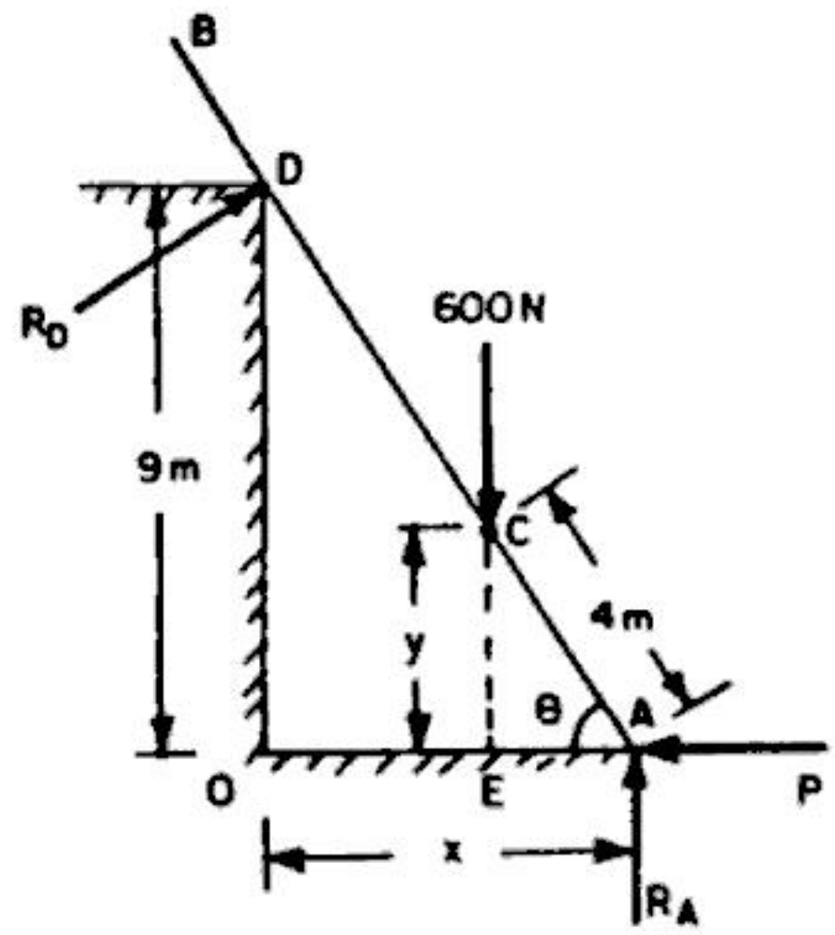

Fig. 10.16.

Problem 10.11. *A uniform bar AB of weight W is supported as shown in Fig. 10.17 (a) and is subjected to a load P. Applying the principle of virtual work derive an expression (in terms of P, θ and W), for the effort F required at B as shown to keep the bar in equilibrium. Hence calculate the value of F if P=800 N, W=600 N and θ=60°. Neglect friction.* (AMIE, Winter 1990)

Sol. Given :

1st Part

Friction is to be neglected. Hence force of friction at B and D will be zero.

Fig. 10.17 (*a*) shows the given system, whereas the Fig. 10.17 (*b*) shows its free body diagram, with all the forces acting on the bar. The virtual work done by normal reactions R_B and R_D will be zero. The virtual work will be done by W, P and effort F applied at B. The weight W and force P are acting vertically downwards, whereas the effort F is acting horizontally at B. Hence for virtual work, we require virtual vertical displacements at E and C and virtual horizontal displacement at B.

Let δx_B=Virtual horizontal displacement at B

δy_E=Virtual vertical displacement at E

δy_C=Virtual vertical displacement at C

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Suppose the beam AB is given a virtual displacement in the upward direction at B, keeping end A intact as shown in Fig. 10.20 (b).

Let δ_C=Virtual displacement at C in the upward direction.

δ_D=Virtual displacement at D in the upward direction.

The triangles ADD' and ACC' of Fig. 10.16 are similar.

$$\therefore \quad \frac{DD'}{AD}=\frac{CC'}{AC}$$

$$\text{or} \quad \frac{\delta_D}{3}=\frac{\delta_C}{4.5} \qquad (\because\ AD=3,\ AC=4.5,\ DD'=\delta_D \text{ and } CC'=\delta_C)$$

$$\therefore \quad \delta_D=\frac{3\delta_C}{4.5} \qquad ...(i)$$

Now virtual work done by reaction R_C

$=R_C\times$Virtual displacement at C

$=R_C\times\delta_C$. (as reaction R_C and virtual displacement are in the upward direction, virtual work done will be positive)

Virtual work done by point load 1200 N at D

$=1200\times$Virtual displacement at D

$=1200\times(-\delta_D)$

$=-1200\times\delta_D$

$=-1200\times\dfrac{3\delta_C}{4.5}$ $\left[\because\ \delta_D=\dfrac{3\delta_C}{4.5} \text{ from equation } (i)\right]$

$=-\dfrac{3600}{4.5}\ \delta_C$ (Work done will be −ve, as load is acting downward but displacement is upward)

$\therefore$ Total virtual work done$=R_C\times\delta_C-\dfrac{3600}{4.5}\ \delta_C$.

But according to the principle of virtual work, the total virtual work done should be zero.

$$\therefore \quad R_C\times\delta_C-\frac{3600}{4.5}\ \delta_C=0$$

$$\text{or} \quad R_C=\frac{3600}{4.5}=800 \text{ N. Ans.}$$

Problem 10.14. *State the principle of virtual work and apply the same to obtain the reaction at the support C due to the vertical load P in Fig. 10.21 given below. Beams AB and BD are horizontal and supported as given below.*

Sol. Given :

Statement of principle of virtual work is given in Art. 10.4.

Fig. 10.21 (a) shows the system described in the problem. Fig. 10.21 (b) shows the free-body diagram of the two given horizontal beams AB and BD. The reactions R_A, R_B, R_C, R_D and given load P are indicated in this figure.

Let the system is given virtual displacement at B in the upward direction, keeping ends A and D fixed. Then the point B is displaced to B', C is displaced to C' and E to E' as shown in Fig. 10.21 (c).

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or $$M - FR\sin\theta\left[1+\frac{R\cos\theta}{L\sqrt{1-\left(\frac{R}{L}\right)^2\sin^2\theta}}\right]=0$$

or $$M = FR\sin\theta\left[1+\frac{R\cos\theta}{L\sqrt{1-\left(\frac{R}{L}\right)^2\sin^2\theta}}\right] \quad \textbf{Ans.}$$

HIGHLIGHTS

1. Virtual displacement is an imaginary displacement. Hence virtual work is the work done by a force on a body due to small virtual displacement (*i.e.*, imaginary displacement).
2. Work is the product of force and distance moved by a body in the direction of the force. Work done is given by

   Work done $= P \times S$....if force and distance are in the same direction

   $= P\cos\theta \times S$....if force is making angle θ with the direction of motion of body

   where P = Force and

   S = Distance moved by body.
3. Work done will be zero if the force is acting at an angle of 90° with the direction of motion of the body.
4. If the point of application of the force moves in the direction of the force, the work done is positive whereas if the point of application of the force moves in a direction opposite to the force, the work done is negative.
5. The unit of work in M.K.S. system is kgf-m whereas in S.I. system the unit of work is Nm (Newton metre).
6. Principle of virtual work states, "If a system of forces acting on a body or a system of bodies be in equilibrium and if the system is imagined to undergo a small displacement consistent with the geometrical conditions, then the algebraic sum of the virtual work done by the forces of the system is zero."
7. Virtual displacements are denoted by δx or $\delta\theta$ whereas actual displacements are denoted by dx or $d\theta$.
8. For most of the problems solved by virtual work method, it is assumed that there is no friction in hinges, bearing or along sliding surfaces.
9. For determining the force in a member of a framed structure the member, in which force is to be determined, is replaced by two forces each equal to P acting at the ends of the member. Then virtual work principle is applied and force is determined.
10. If all the forces act in the same direction, the system is given virtual displacement in the direction of forces.

EXERCISE 10

(A) Theoretical Questions

1. (*a*) Explain the terms : Virtual displacement, work and virtual work. *(S 1989)*

   (*b*) Define virtual displacement and virtual work. *(AMIE, W 1988)*

2. State and clearly explain the principle of virtual work. Also explain how it can be used in solving problems in statics. *(AMIE Summer, 1990)*

3. Write the unit of work in S.I. system and in M.K.S. system. *(AMIE Summer, 1976)*

4. What are the forces which should be omitted while applying the principle of virtual work ?

5. What is the use of the principle of virtual work ?

6. (*a*) How will you determine the force in a member of a framed structure with the help of principle of virtual work ?

   (*b*) While solving a problem by the principle of virtual work, the virtual displacements must be infinitesimally small and must be consistent with the constraints of the system. Explain this statement.

7. If the point of application of the force moves in a direction opposite to the force, the work done is

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Using the equation,

$$S_1 = ut + \tfrac{1}{2} gt^2$$

or $$60 = 0 \times t + \tfrac{1}{2} \times 9.81 \times t^2$$

$$\therefore \quad t = \sqrt{\frac{60 \times 2}{9.81}} = 3.497 \text{ sec.}$$

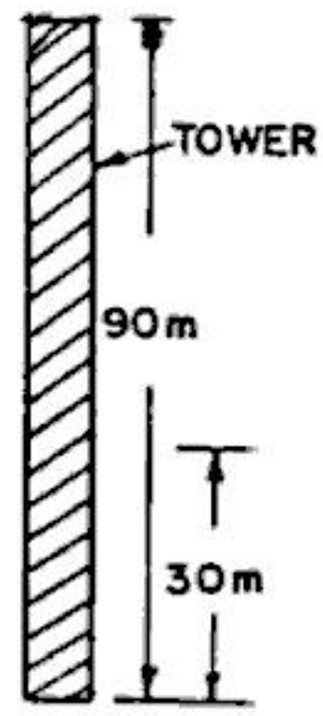

Fig. 11.2.

For the second particle, we have (particle is moving up)

Time, $t = 3.497$ sec (the time is same)

Distance, $S_2 = 30$ m

Acceleration, $g = -9.81$ m/sec.

Let u = Initial velocity with which the second particle should be projected up.

Using the equation,

$$S_2 = ut + \tfrac{1}{2} gt^2$$

or $$30 = u \times 3.497 + \tfrac{1}{2}(-9.81) \times 3.497^2$$

$$= 3.497\, u - 59.98$$

$$\therefore \quad u = \frac{30 + 59.98}{3.497} = \mathbf{25.73 \text{ m/sec.}} \quad \textbf{Ans.}$$

Problem 11.9. *An aeroplane is flying horizontally at a height of 800 m. A bomb is released from the aeroplane when the speed of aeroplane is 600 kmph (kilometre per hour). Determine the time required for the bomb to reach the ground and the horizontal distance travelled by the bomb during flight.*

Sol. Given :

Height of aeroplane, $h = 800$ m

Speed of aeroplane, $U = 600$ kmph $= 600 \times 1000$ m per hour

$$= \frac{600 \times 1000}{60 \times 60} = 166.67 \text{ m/sec.}$$

As the aeroplane is moving in the horizontal direction, when the bomb is released. This means the velocity of bomb, initially will be in the horizontal direction only. This velocity is equal to the velocity of aeroplane, *i.e.*, 166.67 m/sec. But at the same time, the bomb is attracted downwards due to gravitational effect. The initial velocity of the bomb in the downward direction is zero.

1st Case

Consider the motion of bomb in the downward direction.

Initial velocity in the downward direction, $u = 0$

Height of bomb, when it is released, $h = 800$ m

Acceleration due to gravity, $g = 9.81$ m/sec^2

Let t = time taken by bomb to reach the ground.

Using the equation,

$$h = ut + \tfrac{1}{2} gt^2$$

$$\therefore \quad 800 = 0 \times t + \tfrac{1}{2} \times 9.81 \times t^2 = 4.905\, t^2$$

$$\therefore \quad t = \sqrt{\frac{800}{4.905}} = 12.77 \text{ sec.}$$

$\therefore$ **Time required to reach the ground = 12.77 sec.** **Ans.**

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$$=\frac{20}{100}\times 49.05=9.81 \text{ m/sec.}$$

∴ The velocity of the stone after breaking the glass pane

$$=49.05-9.81=39.24 \text{ m/sec.}$$

Now the distance travelled by the stone, after breaking the glass pane, is one second is given by

$$s=ut+\tfrac{1}{2}gt^2$$

$$=39.24\times 1+\tfrac{1}{2}\times 9.81\times 1^2 \quad (\because t=1 \text{ sec and initial velocity of stone after breaking}=39.24 \text{ m/sec})$$

$$=29.24+4.905$$

$$=44.145 \text{ m.} \quad \textbf{Ans.}$$

Problem 11.13. *An object falls from rest from an unknown height. In the last second of its motion, the object travels a distance of 53.90 m. If $g=9.80$ m/sec², determine :*

(i) the height from which the object falls and

(ii) total time taken by the object in falling.

Sol. Given :

Initial velocity of the object, $u=0$

Distance travelled in the last second=53.90

Acceleration due to gravity, $g=9.80$ m/sec²

Let h=Height from which the object falls and
t=Total time of fall.

Then h is also equal to the distance travelled by the object time 't'

$$\therefore \quad h=ut+\tfrac{1}{2}gt^2$$

$$=0\times t+\tfrac{1}{2}\times 9.80\times t^2 \quad [\because u=0]$$

$$=4.9\ t^2 \quad ...(i)$$

If the distance travelled in $(t-1)$ second is h', then we have

$$h'=u(t-1)+\tfrac{1}{2}g(t-1)^2 \quad \{\text{In equation } s=ut+\tfrac{1}{2}gt^2, \text{ substitute } t=(t-1)\}$$

$$=0\times(t-1)+\tfrac{1}{2}\times 9.80(t-1)^2 \quad (\because u=0)$$

$$=4.9\ (t-1)^2 \quad ...(ii)$$

∴ Distance travelled by the object in the last second of its motion

$$=h-h'$$

$$=4.9\,t^2-4.9\,(t-1)^2$$

$$=4.9\,[t^2-(t-1)^2]$$

$$=4.9\,[t^2-(t^2+1-2t)]$$

$$=4.9\,[t^2-t^2-1+2t]$$

$$=4.9\,[2t-1]$$

But according to the given condition, the distance travelled by the object in the last second of its motion=53.90.

$$\therefore \quad 4.9\,[2t-1]=53.90$$

$$\therefore \quad 2t-1=\frac{53.90}{4.90}=11.0$$

$$\therefore \quad t=\frac{11+1}{2}=\frac{12}{2}=6 \text{ sec.} \quad \textbf{Ans.}$$

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11. If V_A and V_B are the velocities at points A and B of a link AB and AO and BO are the distances of the points A and B from the instantaneous centre, then we have

$$\frac{V_A}{V_B}=\frac{AO}{BO}.$$

(A) Theoretical Questions

1. Define the terms : angular velocity and angular acceleration.
2. What is the relation between angular velocity and linear velocity ?
3. Prove that linear acceleration is equal to r times the angular acceleration, where r is the radius of rotation.
4. What do you mean by circular motion ? Give some examples of circular motion.
5. Derive the following equations of motion of a body moving along a circular path with uniform angular acceleration :

   (*i*) $\omega=\omega_0+\alpha t$ (*ii*) $\theta=\omega_0 t+\frac{1}{2}\alpha t^2$ (*iii*) $\omega^2-\omega_0^2=2\alpha\theta$.
6. What is the relations between angular velocity and r.p.m. of a body ?
7. Define the terms : motion of translation, motion of rotation, combined motion of rotation and translation.
8. What do you mean by instantaneous centre of rotation ? How can it be located for a body moving with combined motion of rotation and translation ? [*A.M.I.E., (W) 1982*]
9. A link AB is having linear velocities at A and B as V_A and V_B. If the link is having a combined motion of rotation and translation, how will you determine the angular velocity of the link AB ?
10. How will you find the velocity and acceleration of a piston of a reciprocating engine mechanism by : (*a*) graphical method, and (*b*) analytical method ?
11. Distinguish between the linear velocity of a point on a body rotating about a fixed axis and its angular velocity. Derive the relation between them. [*A.M.I.E., Summer 1982*]

(B) Numerical Problems

1. A body is rotating with an angular velocity of 8 radian/sec. After 5 seconds, the angular velocity of the body becomes 28 rad/sec. Determine the angular acceleration of the body. [**Ans.** 4 rad/sec^2]
2. A wheel, rotating about a fixed axis at 30 r.p.m. is uniformly accelerated for 50 seconds, during which time it makes 40 revolution. Find : (*i*) angular velocity at the end of this interval, and (*ii*) time required for the speed to reach 80 revolution per minute. [**Ans.** (*i*) 6.91 rad/sec, (*ii*) 69.44 sec]
3. A wheel is rotating about its axis with a constant acceleration of 2 rad/sec^2. If the initial and final angular velocities are 5 rad/sec and 15 rad/sec, determine the total angle turned through during the time interval this change of angular velocity took place. [**Ans.** 50 rad.]
4. (*a*) A flywheel starts rotating from rest and is given an acceleration of 1.5 rad/sec^2. Find the angular velocity and speed in r.p.m. after 60 seconds. [**Ans.** 90 rad/sec, 859.4 r.p.m.]

   (*b*) If the flywheel is brought to rest with a uniform angular retardation of 0.5 rad/sec^2, determine the time taken by the flywheel in seconds to come to rest. [**Ans.** 3 Minute]
5. A flysheel is rotating at 150 r.p.m. and after 8 seconds it is rotating at 120 r.p.m. If the retardation is uniform, determine number of revolution made by the flywheel and the time taken by the flywheel before it comes to rest from the speed of 150 r.p.m. [**Ans.** 50 rev., 40 sec]
6. The angle of rotation of a body is given by the equation

$$\theta=t^3-4t^2+10t+5$$

   where θ is expressed in radians and t in seconds. Determine :

   (*i*) Angular velocity and (*ii*) angular acceleration of the body when $t=0$ and $t=5$ seconds.

   [**Ans.** (*i*) 10 rad/sec, 45 rad/sec, (*ii*) 6 rad/sec^2, 22 rad/sec^2]
7. A link AB is moving in a vertical plane. At a certain instant, when the link is inclined at 60° to the horizontal, the point A is moving horizontally at 5 m/sec, while B is moving vertically upwards. Find the velocity of B. [**Ans.** 2.88 m/sec]

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From equation (11.24) it is clear that T'=half the time of flight.

Problem 11.31. *A particle is projected at an angle of 60° with the horizontal. The horizontal range of the particle is 5 kilometres. Find : (i) the velocity of projection, and (ii) the maximum height attained by the projectile.*

Sol. (*i*) Given :

Angle of projection, $\alpha=60°$

Horizontal range, $R=5$ kilometres$=5\times1000=5000$ m

Let u=Velocity of projection

Using equation (11.25) for horizontal range,

$$R=\frac{u^2}{g}\times\sin 2\alpha$$

or
$$500=\frac{u^2}{9.81}\times\sin(2\times60)$$

or
$$u^2=\frac{5000\times9.81}{\sin 120°}=\frac{5000\times9.81}{0.866}=56639.7$$

$\therefore$
$$u=\sqrt{56639.7}=\mathbf{237.99\ m/sec.}\quad \textbf{Ans.}$$

(*ii*) **Maximum height attained by the projectile**

Let h_{max}=Maximum height attained by projectile. Using equation (11.23) for maximum height,

$$h_{max}=\frac{u^2\sin^2\alpha}{2g}=\frac{237.99^2\times\sin^2 60°}{2\times9.81}$$

$$=\frac{237.99^2\times(0.866)^2}{2\times9.81}=\mathbf{2164.98\ m.}\quad \textbf{Ans.}$$

Problem 11.32. *A particle is projected in air with a velocity 100 m/sec and at an angle of 30° with the horizontal. Find :*

(*i*) *The horizontal range,*

(*ii*) *The maximum height by the particle, and*

(*iii*) *The time of flight.*

Sol. Given :

Velocity of projection, $u=100$ m/sec

Angle of projection, $\alpha=30°$

Let R=Horizontal range

h_{max}=Maximum height attained by the particle, and

T=Time of flight.

(*i*) Using equation (11.25) for horizontal range,

$$R=\frac{u^2\times\sin 2\alpha}{g}=\frac{100^2\times\sin(2\times30°)}{9.81}$$

$$=\frac{100.00\times\sin 60}{9.81}=\mathbf{882.77\ m.}\quad \textbf{Ans.}$$

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where R = Horizontal range between the point of projection and the mark, when angle of projection $=\alpha$

R_1 = Horizontal range when $\alpha=15°$

and R_2 = Horizontal range when $\alpha=45°$.

Velocity of projection in all cases $=u$.

Using equation (11.25) for horizontal range, we have

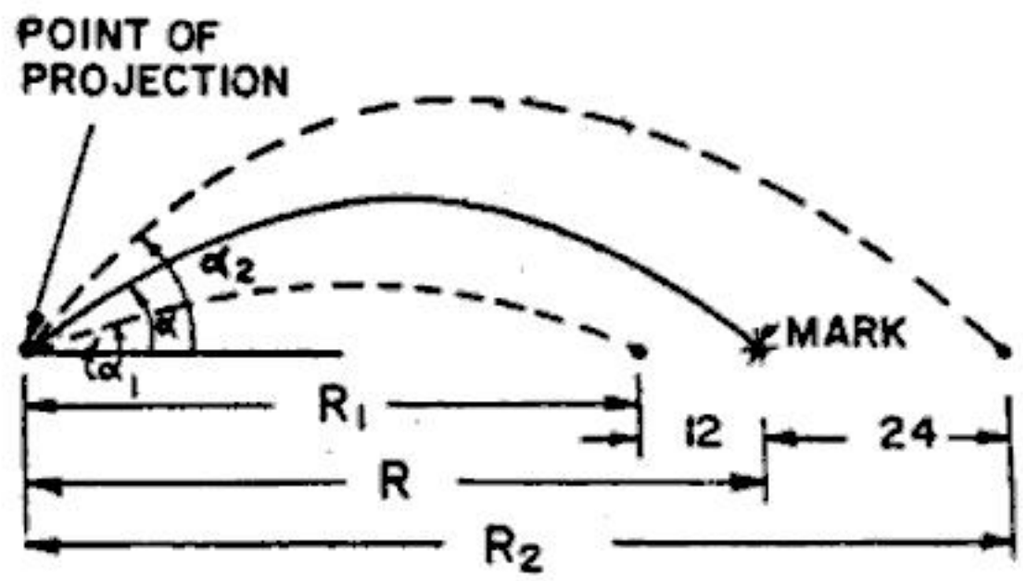

Fig. 11.19.

$$R=\frac{u^2 \sin 2\alpha}{g} \qquad ...(i)$$

$$\therefore \quad R_1=\frac{u^2 \sin 2\alpha_1}{g}=\frac{u^2 \sin (2\times 15°)}{g}$$

or $$(R-12)=\frac{u^2 \sin 30}{9.81} \qquad (\because R_1=R-12)$$

or $$R-12=\frac{u^2\times 0.5}{9.81} \qquad (\because \sin 30°=0.5) \qquad ...(ii)$$

Similarly, $$R_2=\frac{u^2\times \sin 2\alpha_2}{g}=\frac{u^2 \sin (2\times 45)}{g}$$

or $$R+24=\frac{u^2 \sin 90°}{9.81} \qquad (\because R_2=R+24)$$

or $$R+24=\frac{u^2\times 1}{9.81} \qquad (\because \sin 90°=1) \qquad ...(iii)$$

Dividing equations (*ii*) and (*iii*), we have

$$\frac{R-12}{R+24}=\frac{\frac{u^2\times 0.5}{9.81}}{\frac{u^2}{9.81}}=\frac{0.5}{1}=\frac{1}{2}$$

or $$(R+24)=2\,(R-12)=2R-24$$

or $$2R-R=24+24$$

or $$R=48 \text{ m} \qquad ...(iv)$$

Substituting the value of R in equation (*iii*), we get

$$48+24=\frac{u^2\times 1}{9.81}$$

or $$72=\frac{u^2\times 1}{9.81}$$

or $$u^2=9.81\times 72 \qquad ...(v)$$

Substituting the values of u^2 and R in equation (*i*), we get

$$48=\frac{(9.81\times 72)\times \sin 2\alpha}{g}$$

$$=\frac{9.81\times 72\times \sin 2\alpha}{9.81}=72\times \sin 2\alpha$$

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(*i*) Horizontal distance OB is given by equation (11.32)

Range of $OB = u \times t$

$= 30 \times 14.278 =$ **428.340 m. Ans.**

(*ii*) *Velocity of bomb hitting the target B*

The vertical downward component of velocity at B is given by equation (11.33) as

$$v = g \times t = 9.81 \times 14.278 \text{ m/sec}$$
$$= 140.06 \text{ m/sec}$$

The horizontal component of velocity is constant throughout the motion and is equal to 30 m/sec. Hence at target, the horizontal velocity,

$\therefore$ $u = 30$ m/sec

$\therefore$ Resultant velocity $= \sqrt{u^2 + v^2}$

$$= \sqrt{30^2 + 140.06^2} = \sqrt{900 + 19618.8}$$
$$= 143.24 \text{ m/sec.}$$

$\therefore$ Velocity of bomb hitting the target = **143.24 m/sec. Ans.**

(*iii*) *The direction with which the bomb hits the target*

Let θ = Inclination of resultant velocity with the horizontal at target

Then using equation (11.30), we have

$$\tan \theta = \frac{v}{u} = \frac{140.06}{30} = 4.668$$

$\therefore$ $\theta = \tan^{-1} 4.668 =$ **77° 54.5′. Ans.**

Problem 11.44. *A bomb is released from an aeroplane flying at a speed of 1000 km/hr on a straight level course 2000 m above the ground. Find the time required for the bomb to reach the ground and the horizontal distance travelled by the bomb after its release. Assume $g = 9.8$ m/s² and ground to be horizontal. Neglect air resistance.* (AMIE, Summer 1988)

Sol. Given :

Height of aeroplane, $H = 2000$ m

Velocity of aeroplane, $u = 1000$ km/hr

$$= \frac{1000 \times 1000}{60 \times 60}$$
$$= 277.78 \text{ m/s.}$$

Value of $g = 9.8$ m/s²

Let t = Time required by the bomb to reach the ground

R = Horizontal distance travelled by bomb after its release.

(*i*) The time required by the bomb to reach the ground is obtained from equation (11.31). Hence using equation (11.31),

$$H = \frac{1}{2} \times g \times t^2$$

or $$\frac{2H}{g} = t^2$$

or $$t = \sqrt{\frac{2H}{g}} = \sqrt{\frac{2 \times 2000}{9.8}}$$

$$=$$ **20.2 seconds. Ans.**

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12

Laws of Motion and Simple Harmonic Motion

12.1. LAWS OF MOTION

When a body is at rest or moving in a straight line or rotating about an axis, the body obeys certain laws of motion. These laws are called Newton's laws of motion. There are three laws of motion. These laws for linear motions and angular motions are given in the next articles.

12.2. NEWTON'S LAWS FOR LINEAR MOTION

First Law. It states that a body continues in its state of rest or of uniform motion in a straight line unless it is compelled by an external force to change that state.

Second Law. It states that the rate of change of momentum of a body is proportional to the external force applied on the body and takes place in the direction of the force.

Third Law. It states that to every action, there is always an equal and opposite reaction.

12.3. DEFINITIONS

Before discussing Newton's laws of motion, let us define certain terms like mass, weight and momentum.

12.3.1. Mass. The quantity of matter contained in a body is known as mass of the body. Mass is a scalar quantity. In C.G.S. units, the mass is expressed in gramme (gm) whereas in S.I. units the mass is expressed in kilogramme (kg).

12.3.2. Weight. Weight of a body is defined as the force, by which the body is attracted towards the centre of the earth. Mathematically weight of a body is given by

$$\text{Weight} = \text{mass} \times \text{acceleration due to gravity}$$

$$= \text{mass} \times g \qquad \text{...(12.1)}$$

If mass is taken in kilogram (kg) and acceleration due to gravity in metre per second square (m/sec^2), then weight is expressed in newton (N). But if mass is taken in gramme (gm) and acceleration due to gravity in centimetre per second square (cm/sec^2), then weight is expressed in dyne. The relation between newton (N) and dyne is given as

$$\text{One Newton} = 10^5 \text{ dyne.}$$

12.3.3. Momentum. The product of the mass of a body and its velocity is known as momentum of the body. Momentum is a vector quantity. Mathematically, momentum is given by

$$\text{Momentum} = \text{Mass} \times \text{Velocity.}$$

12.4. NEWTON'S FIRST LAW OF MOTION

It consists of two parts. First part states that a body continues in its state of rest unless it is compelled by an external force to change that state. A book lying on a table remains at rest, unless it is lifted by some external force.

Second part states that a body continues in its state of uniform motion in a straight line unless it is compelled by an external force to change that state. In actual practice, we see that when a body is moving with a uniform velocity in a straight line, the body does not continue

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Using equation (12.5), we get

$$T=W\left(1+\frac{f}{g}\right)$$

or $$600=500\left(1+\frac{f}{9.80}\right)$$

or $$\frac{600}{500}=1+\frac{f}{9.80}$$

or $$1.2=1+\frac{f}{9.80}$$

or $$f=(1.2-1.0)\times 9.80$$

$$=0.2\times 9.80=\mathbf{1.96\ m/s^2}.\quad \textbf{Ans.}$$

Problem 12.18. *An elevator weighs 2500 N and is moving vertically downwards with a constant acceleration. Write the equation for the elevator cable tension. Starting from rest it travels a distance of 35 metres during an interval of 10 seconds. Find the cable tension during this time. Neglect all other resistances to motion. What are the limits of cable tension ?*

(AMIE, Summer 1979)

Sol. Given :

Weight of elevator, $W=2500$ N

Initial velocity, $u=0$

Distance travelled, $s=35$ m

Time, $t=10$ sec.

Let T=Tension in the cable supporting an elevator in N.

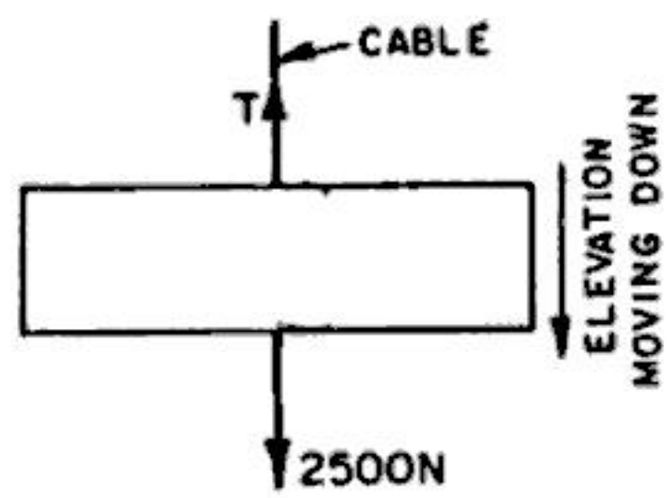

Fig. 12.14.

1st Part. The equation for the elevator cable tension is obtained as given below : (see Fig. 12.14). The elevator is moving down.

∴ Net acceleration force in the downward direction

$$=(W-T)$$

$$=(2500-T)\ \text{N}.$$

The net accelerating force produces an acceleration 'f' in the downward direction.

Hence using the relation,

Net force=mass×acceleration

$$(2500-T)=\frac{2500}{9.81}\times f \qquad \left(\because \text{mass}=\frac{W}{g}=\frac{2500}{9.81}\right)$$

or $$T=2500-\frac{2500}{9.81}\times f$$

∴ $$T=2500\left(1-\frac{f}{9.81}\right)\ \text{N} \qquad ...(i)$$

Hence the above equation (*i*) represents the equation for the elevator cable tension when the elevator is moving downwards.

2nd Part. Limits of cable tension are obtained from equation (*i*) as given below :

(*i*) When $f=0$, and this value is substituted in equation (*i*), the value of T is obtained as

$$T=2500\left(1-\frac{0}{9.81}\right)=2500\ \text{N}.$$

(*ii*) When $f=9.81$ m/s², and this value is substituted in equation (*i*), the value of T is obtained as

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(*ii*) Using equation (12.14) for tension,

$$T=\frac{W_1W_2(1+\sin\theta)}{(W_1+W_2)}$$

$$=\frac{15\times40\ (1+.2588)}{(15+40)}=\frac{15\times40\times1.2588}{55}$$

$$=\mathbf{13.732\ N.}\quad \textbf{Ans.}$$

Problem 12.31. *If in the problem 12.30, the inclined surface is a rough one, having a co-efficient of friction between the weight 40 N and the inclined surface equal to 0.2 and the weight hanging free is 25 N, determine :*

(*i*) *the acceleration of the system,*

(*ii*) *the tension in the string, and*

(*iii*) *the distance moved by the weight 25 N in 3 seconds starting from rest. Take* $g=9.80\ m/s^2$.

Sol. Given : (From problem 12.30)

$W_1=25$ N, $W_2=40$ N, $g=9.80$ m/s²

$\theta=15°$.

Co-efficient of friction, $\mu=0.2$.

Let f=Acceleration of the system,

T=Tension in the string, and

s=Distance travelled by weight 15 N in 3 seconds starting from rest.

As the inclined surface is a rough one, hence acceleration and tension are obtained by using equations (12.15) and (12.16).

(*i*) Using equation (12.15) for acceleration,

$$f=\frac{g\ (W_1-W_2\sin\theta-\mu\ W_2\cos\theta)}{(W_1+W_2)}$$

$$=\frac{9.80(25-40\times\sin 15°-0.2\times40\times\cos 15°)}{(25+40)}\qquad (\because\ W_1=25\ \text{N})$$

$$=\frac{9.80(25-40\times.2588-0.2\times40\times.9659)}{65}$$

$$=\frac{9.80(25-10.352-7.727)}{65}=\frac{9.80\times6.921}{65}$$

$$=\mathbf{1.043\ m/s^2.}\quad \textbf{Ans.}$$

(*ii*) Using equation (12.16) for tension,

$$T=\frac{W_1W_2}{(W_1+W_2)}(1+\sin\theta+\mu\cos\theta)$$

$$=\frac{25\times40}{(25+40)}(1+\sin 15°+0.2\cos 15°)$$

$$=\frac{1000}{65}(1+0.2588+0.2\times0.9659)$$

$$=\frac{1000}{65}\times1.45198=\mathbf{22.338\ N.}\quad \textbf{Ans.}$$

(*iii*) *Distance moved by the weight 25 N in 3 seconds, starting from rest.*

Initial velocity, $u=0$

Time, $t=3$ seconds

Acceleration, $f=1.043$ m/s²

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The moment of inertia of the sphere about any axis is same as sphere is symmetrical.

12.15. RADIUS OF GYRATION (K)

It is defined as the distance of a body from the axis of reference where whole mass of the body is assumed to be concentrated so as not to alter the moment of inertia about the given axis.

The moment of inertia of a body in terms of radius of gyration is given by

$$I = Mk^2$$

$\therefore$ Radius of gyration, $k = \sqrt{\frac{I}{M}}$. ...(12.22)

(A) Radius of Gyration for Circular Lamina

From equation (12.19), moment of inertia of a circular section about the central axis is given as,

$$I_{zz} = \frac{MR^2}{2}$$ where R=Radius and M=Mass

But moment of inertia in terms of radius of gyration is given by

$$I_{zz} = Mk^2.$$

Equating the two values of I_{zz},

$$\therefore \quad \frac{MR^2}{2} = Mk^2$$

$$\therefore \quad k^2 = \frac{R^2}{2}$$

or $$k = \frac{R}{\sqrt{2}}.$$ ...(12.23)

(B) Radius of Gyration of a Solid Sphere

From equation (12.21), the moment of inertia of a solid sphere about any axis is given as,

$$I = \frac{2\,MR^2}{5}$$

But moment of inertia in terms of radius of gyration is given by

$$I = Mk^2$$

$\therefore$ Equating the two values of I.

$$\therefore \quad Mk^2 = \frac{2\,M}{5}\,R^2$$

$$\therefore \quad k = \sqrt{\frac{2R^2}{5}} = R \times \sqrt{\frac{2}{5}} = 0.6324\,R.$$ ...(12.24)

Problem 12.32. *Calculate the moment of inertia and radius of gyration with respect to the axis of rotation of : (i) a circular lamina of diameter 60 cm and of mass .001 kg/cm², (ii) a circular cylinder of diameter 80 cm of height 15 cm and of mass 0.002 kg/cm³ and (iii) a solid sphere of diameter 40 cm and of mass .0015 kg/cm³.*

Sol. Given :

(i) For a circular lamina :

Diameter, $D = 60$ cm

$\therefore$ Radius, $R = \frac{D}{2} = \frac{60}{2} = 30$ cm

Mass per unit area, $m = 0.001$ kg/cm²

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5. A body of mass 20 kg falls on the muddy ground from a height of 39.2 m. The body penetrates into the ground. Find the distance through which the body will penetrate into the ground, if the resistance by the ground to penetration is constant and equal to 980 N. Take $g=9.8$ m/sec^2.

[Ans. 9.8 m]

6. A man weighing 588 N dives into a swimming pool from a tower of height 9.8 m. He was found to go down in water by 1.5 m and then started rising. Find the average resistance of water.

[Ans. 452 kgf]

7. A bullet weighing 80 gmf and moving with a velocity of 350 m/sec is fired into a log of wood and it penetrates to a depth of 15 cm. If the bullet moving with the same velocity, were fired into a similar piece of wood 10 cm thick, with what velocity would it emerge ? Find also the force of resistance, assuming it to be uniform. [Ans. 202 m/sec, 32666 N]

8. A car, moving on a straight level road, skidded for a total distance of 40 metres after the brakes were applied. Determine the speed of the car, just before the brakes were applied, if the co-efficient of friction between the car tyres and the road is 0.4. Take $g=9.80$ m/sec^2. [Ans. 63.75 km/hr]

9. The tractive force, exerted by a railway car weighing 40 kN, is 1600 N. If the frictional resistance is 4 N per kN of the railway car's weight, determine the acceleration when the railway car is moving on a level track. [Ans. 0.353 m/sec^2]

10. A train of weight 2000 kN starts from rest and attains a speed of 100 km/hr in 4 minutes. If the frictional resistance of the track is 8 N per kN of the train's weight, find the average pull required. Take $g=9.8$ m/sec^2. [Ans. 39.588 kN]

11. A body of weight 300 N is initially stationary on a 45° inclined plane. What distance along the inclined plane must the body slide, before it reaches a speed of 3 m/sec. The co-efficient of friction between the body and the plane is equal to 0.2. [Ans. 81.1 cm]

12. A truck weighing 5 tonnes just moves freely (engine is not running) at a speed of 40 kilometre per hour down a slope of 1 in 45, the road resistance at this speed just being sufficient to prevent any acceleration. Find the road resistance per tonne weight of truck.

What power will the engine have to exert to run up the same slope at double the speed when the road resistance remains the same. [Ans. 22.22 kgf, 65.83]

13. A train weighing 150 tonnes moves down a slope of 1 in 200 at 20 km/hr and engine develops 40 h.p. What horse power will be required to pull the train up the slope at the same speed ?

[Ans. 150.99]

14. A lift carries a weight of 110 N and is moving with a uniform acceleration of 3 m/sec^2. Determine the tension in the cables supporting the lift, when

(*i*) lift is moving upwards, and

(*ii*) lift is moving downwards. Take $g=9.80$ m/sec^2. [Ans. 143.67 N, 76.34 N]

15. A lift has an upward acceleration of 1.5 m/sec^2. What pressure will a man weighing 500 N exert on the floor of the lift ? What pressure would he exert if the lift had an acceleration of 1.5 m/sec^2 downward ? Take $g=9.8$ m/sec^2. [Ans. 576.5 N, 423.5 N]

16. An elevator weighs 2000 N and is moving vertically downwards with a uniform acceleration. Write the equation for the elevator cable tension. Starting from rest it travels a distance of 30 m during an interval of 12 seconds. Find the cable tension during this time. Neglect all other resistances to motion. What are the limits of cable tension ? [Ans. 1915.2 N, at $f=0$, $T=2000$ and at $f=9.81$, $T=0$]

17. An elevator weighing 6000 N is ascending with an acceleration of 2 m/sec^2. During this ascent its operator whose weight is 600 N is standing on the scales placed on the floor. What is the scale reading ? What will be the total tension in the cables of the elevator during this motion ?

[Ans. 722.3 N, 7945.5 N]

18. Two bodies of weight 60 N and 40 N are connected to the two ends of a light in extensible string. The string is passing over a smooth pulley. Determine :

(*i*) the acceleration of the system, and

(*ii*) the tension in the string. Take $g=9.80$ m/sec^2. [Ans. 1.96 m/sec^2, 48 N]

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$$=\frac{\pi}{32}\times 100^4 \text{ mm}^4$$
$$=9817500 \text{ mm}^4$$

$\therefore$ $$q=\frac{C\times J}{L}$$
$$=\frac{(0.8\times 10^5)\times 9817500}{1000} \text{ N mm}$$
$$=7.854\times 10^8 \text{ N mm}$$

and $$I=Mk^2$$
$$=\left(\frac{5000}{9.81}\right)\times 450^2 \text{ kg mm}^2$$

Substituting the values of q and I in equation (*i*), we get

$$f=\frac{1}{2\pi}\sqrt{\frac{7.854\times 10^8 \text{ Nmm}}{\left(\frac{5000}{9.81}\right)\times 450^2 \text{ kg mm}^2}}$$

But $1 \text{ N}^*=1 \text{ kg}\times 1 \text{ m/s}^2$
$=1 \text{ kg}\times 1000 \text{ mm/s}^2$

Substituting the value of N in the above equation, we get

$$f=\frac{1}{2\pi}\sqrt{\frac{(7.854\times 10^8)\times(1 \text{ kg}\times 1000 \text{ mm/s}^2) \text{ mm}}{\left(\frac{5000}{9.81}\right)\times 450^2\times \text{kg mm}^2}}$$

$$=\frac{1}{2\pi}\sqrt{\frac{7.854\times 10^8\times 1000}{\left(\frac{5000}{9.81}\right)\times 450^2 \text{ s}^2}}$$

$$=13.88 \text{ cycles/sec.}$$ **Ans.**

(*ii*) **Frequency of transverse vibration**

It is given by equation (12.53) as

$$f=\frac{1}{2\pi}\sqrt{\frac{g}{\delta}} \qquad ...(ii)$$

where $$\delta=\frac{WL^3}{3EI} \qquad ...(iii)$$

In the above equation, I is the moment of inertia.

$\therefore$ $$I=\frac{\pi}{64}\times d^4$$
$$=\frac{\pi}{64}\times 100^4=4908700 \text{ mm}^4$$

Substituting the values of W, L, E and I in equation (*iii*), we get

$$\delta=\frac{WL^3}{3EI}$$

---

* One newton is a force acting on a mass of one kg and produces an acceleration of one metre per second square.

$\therefore$ $1 \text{ N}=1 \text{ kg}\times 1 \text{ m/s}^2=1 \text{ kg}\times 1000 \text{ mm/s}^2$

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EXERCISE 12

(A) Theoretical Questions

1. Explain the terms. Simple harmonic motion, amplitude, frequency, oscillation and period of simple harmonic motion.

2. Under what conditions can a simple pendulum be assumed to have simple harmonic motion ? Derive an expression for the time period of a simple pendulum of length 'l' performing S.H.M. Find the length of a second's pendulum.

3. For the simple harmonic motion, explain the terms :

(*i*) Amplitude, (*ii*) Periodic time, and (*iii*) Frequency. *(AMIE, Winter 1982)*

4. Explain S.H.M. as the projection of a point moving in a circle. Prove that the velocity of the projection of a point moving in a circle is given by

$$v=-\omega\sqrt{r^2-x^2}$$

where ω=Angular velocity

r=Radius of the circle

x=Distance from the centre of the circle and time is measured from the extreme end.

5. For the question 4, prove that acceleration of the projection of the point moving in a circle is given by

$$f=-\omega^2 r\cos\theta$$

and hence show that acceleration is maximum at the centre of the circle.

6. Prove that the time period (T) of oscillation of a verticle elastic string with upper end fixed is given by

$$T=2\pi\sqrt{\frac{\text{Static extension}}{\text{Acceleration due to gravity}}}.$$

7. Define the stiffness of a spring. Two springs of stiffness C_1 and C_2 are connected in series. The upper end of the spring is fixed whereas the lower end carries a weight W. Prove that the period of oscillation is given by

$$T=2\pi\sqrt{\frac{W(C_1+C_2)}{C_1C_2\times g}}.$$

8. Define the terms : simple pendulum, beat of a pendulum and second's pendulum.

9. Prove that the gain or loss of oscillations due to change in acceleration due to gravity and length of pendulum is given by

$$\frac{dn}{n}=\frac{1}{2}\left[\frac{dg}{g}-\frac{dl}{l}\right].$$

10. A mass m is suspended from the ceiling by a spring whose constant is k. Derive an expression for the displacement of mass at time t, if it is released from the unextended spring position at t=0. Also derive the expression for the velocity at time t. *[A.M.I.E., (W) 1984]*

11. (*a*) What do you understand by free vibrations, forced vibrations and damped vibrations.

(*b*) Distinguish between free-torsional vibration and free-transverse vibration. *[S 1989]*

(B) Numerical Problems

1. Find the velocity and acceleration after 0.4 seconds from the extreme position of a body, moving with sample harmonic motion with an amplitude of 1 m and period of complete oscillation of 1.8 seconds.

[Ans. −3.436 m/sec, −2.118 m/sec^2]

2. A body moving with simple harmonic motion, has an amplitude of 0.8 m and period of complete oscillation is 2.0 seconds. Find the velocity and acceleration of the body $\frac{1}{5}$th of a second after passing the mid-position. **[Ans.** −2.033 m/sec, −4.64 m/sec^2]

3. The piston of an engine moves with simple harmonic motion. The crank rotates at 110 r.p.m. and the stroke is 200 cm. Find the velocity and acceleration of the piston, when it is at a distance of 60 cm from the centre. **[Ans.** −9.215 m/sec, −79.61 m/sec^2]

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$$=\frac{(750+1000)}{2}\times 25=\mathbf{21875\ Nm.}\quad \textbf{Ans.}$$

Problem 13.7. *Determine the work done by electric motor in winding up a uniform cable which hangs from a hoisting drum if its free length is 10 m and weighs 500 N. The drum is rotated by the motor.* (A.M.I.E., Summer 1984)

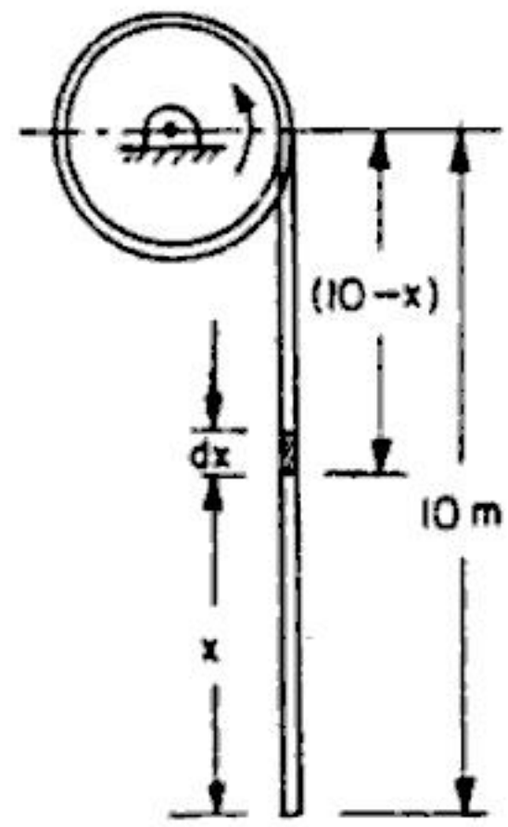

Fig. 13.8.

Sol. Given :

Free length of cable, $L=10$ m

Weight of 10 m length cable $=500$ N

Weight of cable per m length $=\frac{500}{10}=50$ N

Refer to Fig. 13.8.

Consider an element of length dx at a distance of x m from free end. This element is to be raised to a height of $(10-x)$ m. The work done on this element is equal to the product of its weight and the height through which it is raised.

Weight of element=
Weight of cable per m length
×length of element
$=50\times dx=50dx$

Work done on the element,

$$dW=\text{Weight of element}\times\text{Height through which it is raised}$$
$$=(50dx)\times(10-x)$$
$$=50\times(10-x)dx$$

Total work done will be obtained by integrating the above equation from 0 to 10 m.

$$\therefore\quad W=\int_0^{10}50(10-x)dx$$
$$=50\left[10x-\frac{x^2}{2}\right]_0^{10}$$
$$=50\left[10\times 10-\frac{10^2}{2}\right]=50[100-50]$$
$$=50\times 50=\mathbf{2500\ N\ m.}\quad \textbf{Ans.}$$

13.4. POWER

The rate of doing work is known as power. Hence, power can be obtained by dividing the total work done by time. Or in other words, the power is the work done per second. The unit of power in S.I. system is Nm/s or Watt. It is denoted by W. The power in kilowatts is obtained by dividing watt by 1000.

$$\therefore\quad \text{Power}=\text{Work done per second}$$
$$=\frac{\text{Force}\times\text{Distance}}{\text{Time}}=\text{Force}\times\frac{\text{Distance}}{\text{Time}}$$
$$=\text{Force}\times\text{Velocity}\qquad\text{...(13.3)}$$

The force and velocity should be in the same direction.

Problem 13.8. *A train of weight 2000 kN is pulled by an engine on a level track at a constant speed of 36 kilometre per hour. The resistance due to friction is 10 N per kN of the trains weight. Find the power of the engine.*

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Problem 14.10. *The bar shown in Fig. 14.8 is subjected to a tensile load of 160 kN. If the stress in the middle portion is limited to 150 N/mm², determine the diameter of the middle portion. Find also the length of the middle portion if the total elongation of the bar is to be 0.2 mm. Young's modulus is given as equal to 2.1×10^5 N/mm².*

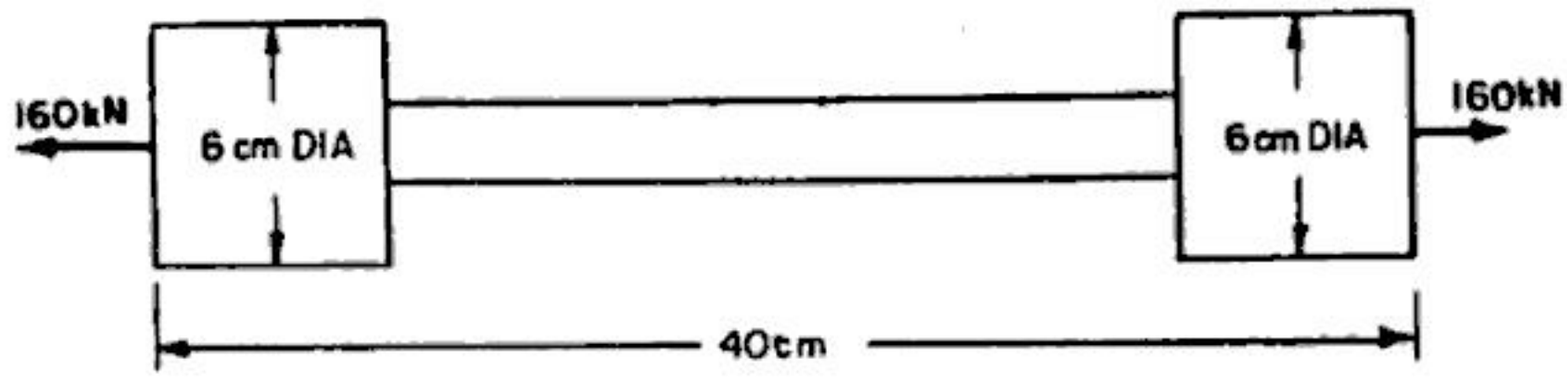

Fig. 14.8

Sol. Given : Tensile load, $P = 160 \text{ kN} = 160 \times 10^3 \text{ N}$

Stress in middle portion, $p_2 = 150 \text{ N/mm}^2$

Total elongation, $dL = 0.2 \text{ mm}$

Total length of the bar, $L = 40 \text{ cm} = 400 \text{ mm}$

Young's modulus, $E = 2.1 \times 10^5 \text{ N/mm}^2$

Diameter of both end portions, $D_1 = 6 \text{ cm} = 60 \text{ mm}$

∴ Area of cross-section of both end portions,

$$A_1 = \frac{\pi}{4} \times 60^2 = 900\,\pi \text{ mm}^2.$$

Let D_2 = Diameter of the middle portion

L_2 = Length of middle portion in mm.

∴ Length of both end portions of the bar,

$$L = (400 - L_2) \text{ mm}$$

Using equation (14.1), we have

$$\text{Stress} = \frac{\text{Load}}{\text{Area}}.$$

For the middle portion, we have

$$p_2 = \frac{P}{A_2} \quad \text{where } A_2 = \frac{\pi}{4} D_2^2$$

or
$$150 = \frac{160000}{\frac{\pi}{4} D_2^2}$$

∴
$$D_2^2 = \frac{4 \times 160000}{\pi \times 150} = 1358 \text{ mm}^2$$

or
$$D_2 = \sqrt{1358} = 36.85 \text{ mm} = \mathbf{3.685 \text{ cm}}. \quad \textbf{Ans.}$$

∴ Area of cross-section of middle portion,

$$A_3 = \frac{\pi}{4} \times 36.85 = 1066 \text{ mm}^2$$

Now using equation (14.8), we get

Total extension,
$$dL = \frac{P}{E}\left[\frac{L_1}{A_1} + \frac{L_2}{A_2}\right]$$

or
$$0.2 = \frac{160000}{2.1 \times 10^5}\left[\frac{(400 - L_2)}{900\pi} + \frac{L_2}{1066}\right] \quad [\because \; L_1 = (400 - L_2) \text{ and } A_2 = 1066]$$

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section FG will coincide with section EF. But the section EF is normal to the line of action of the loading. This means the plane normal to the axis of loading will carry the maximum normal stress.

$\therefore$ Maximum normal stress, $= p \cos^2 \theta = p \cos^2 0° = p$ ...(16.4)

From equation (16.3), it is observed that the tangential stress (*i.e.* shear stress) across the section FG will be maximum when $\sin 2\theta$ is maximum. And $\sin 2\theta$ will be maximum when

$$\sin 2\theta = 1 \quad \text{or} \quad 2\theta = 90° \quad \text{or} \quad 270° \quad \text{or} \quad \theta = 45° \quad \text{or} \quad 135°.$$

This means the shear stress will be maximum on two planes inclined at 45° and 135° to the normal section EF.

$\therefore$ Max. value of shear stress, $= \frac{p}{2} \sin 2\theta = \frac{p}{2} \sin 90° = \frac{p}{2}$. ...(16.5)

From equations (16.4) and (16.5) it is seen that maximum normal stress is equal to p whereas the maximum shear stress is equal to $p/2$ or equal to half the value of greatest normal stress.

Second Method

Fig. 16.2 shows a rectangular bar $ABCD$ of uniform cross-sectional area A and of unit thickness. The bar is subjected to a principal tensile stress p_1 on the faces AD and BC as shown in Fig. 16.2.

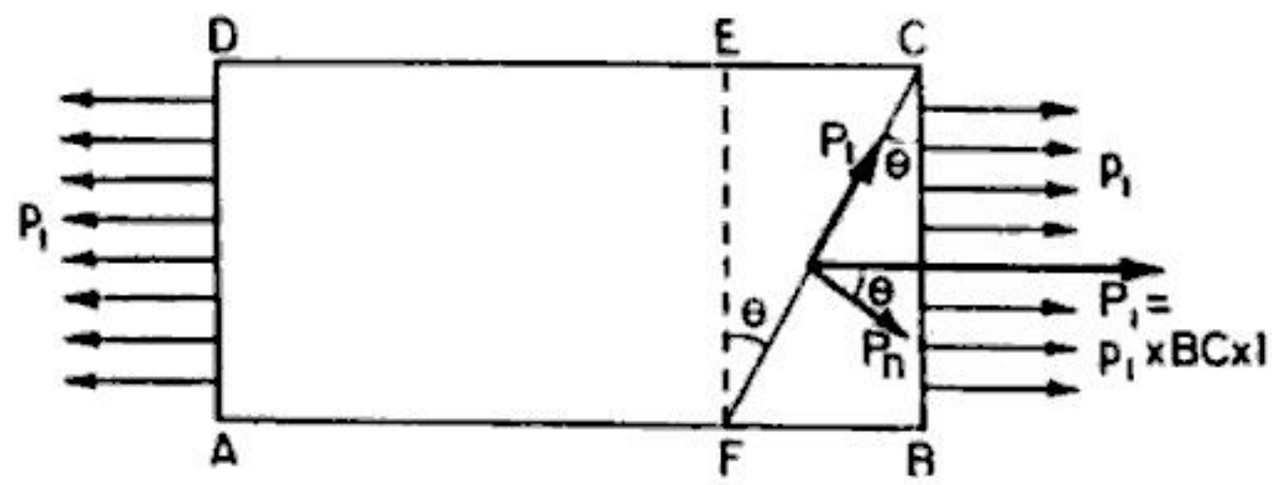

Fig. 16.2

Let the stresses on the oblique plane FC are to be calculated. The plane FC is inclined at an angle θ with the normal cross-section EF (or BC).

The tensile force on BC, P_1 = Stress (p_1) × Area of cross-section

$= p_1 \times BC \times 1.$ ($\because$ Area = $BC \times 1$)

The above tensile force P_1 is also acting on the inclined section FC, in the axial direction as shown in Fig. 16.2. This force F_1 is resolved into two component, *i.e.* one normal to the plane FC and other along the plane FC.

Let P_n = Component of the force P_1, normal to the section FC and

$= P_1 \cos \theta$

$= p_1 \times BC \times 1 \times \cos \theta$ ($\because$ $P_1 = p_1 \times BC \times 1$)

P_t = Component of the force P_1, along the section $FC = P_1 \sin \theta$

$= p_1 \times BC \times 1 \times \sin \theta$

p_n = Normal stress across the section FC

p_t = Shear stress (or tangential stress) across the section FC.

Then normal stress, $p_n = \frac{\text{Force normal to section } FC}{\text{Area of section } FC}$

$= \frac{P_n}{FB \times 1}$ ($\because$ bar is of unit thickness)

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$$= \frac{P_t}{FC \times 1} = \frac{p_1 \times BC \times \sin\theta - p_2 \times BF \times \cos\theta}{FC}$$

$$= p_1 \times \frac{BC}{FC} \times \sin\theta - p_2 \times \frac{BF}{FC} \times \cos\theta$$

$$= p_1 \times \cos\theta \times \sin\theta - p_2 \times \sin\theta \times \cos\theta$$

$$\left(\because \text{ In triangle } FBC, \frac{BC}{FC} = \cos\theta, \frac{BF}{FC} = \sin\theta\right)$$

$$= (p_1 - p_2)\cos\theta \sin\theta$$

$$= \frac{(p_1 - p_2)}{2} \times 2\cos\theta \sin\theta \qquad \text{(Multiplying and dividing by 2)}$$

$$= \frac{(p_1 - p_2)}{2} \sin 2\theta \qquad ...(16.7)$$

The resultant stress on the section FC will be given as

$$p_R = \sqrt{p_n^2 + p_t^2} \qquad ...(16.8)$$

Obliquity [Refer to Fig. 16.4 (*b*).] The angle made by the resultant stress with the normal of the oblique plane, is known as obliquity. It is denoted by ϕ. Mathematically,

$$\tan\phi = \frac{p_t}{p_n} \qquad ...[16.8 (A)]$$

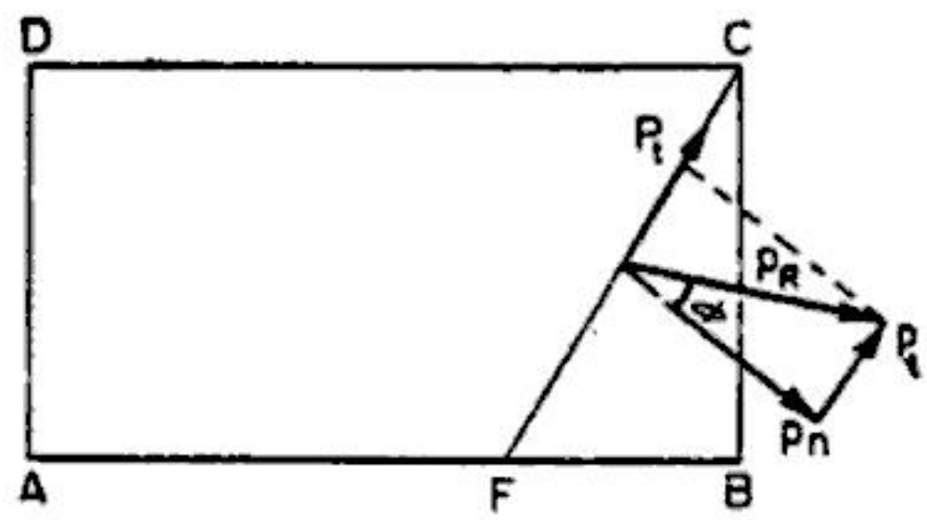

Fig. 16.4 (*b*)

Maximum shear stress. The shear stress is given by equation (16.7). The shear stress (p_t) will be maximum when

$$\sin 2\theta = 1$$

or $$2\theta = 90° \text{ or } 270°$$

$$(\because \sin 90° = 1 \text{ and also } \sin 270° = 1)$$

or $$\theta = 45° \text{ or } 135°$$

And maximum shear stress, $(p_t)_{max} = \dfrac{p_1 - p_2}{2}$ ...(16.9)

Hence the planes, which are at an angle of 45° or 135° with the normal cross-section BC [see Fig. 16.4 (*b*)], carry the maximum shear stresses.

Principal planes. Principal planes are the planes on which shear stress is zero. To located the position of principal planes, the shear stress given by equation (16.7) should be equated to zero.

$\therefore$ For principal planes, $\dfrac{p_1 - p_2}{2} \sin 2\theta = 0$

or $$\sin 2\theta = 0 \qquad [\because (p_1 - p_2) \text{ cannot be equal to zero}]$$

or $$2\theta = 0 \text{ or } 180°$$

$\therefore$ $$\theta = 0 \text{ or } 90°$$

when $\theta = 0$,

$$p_n = \frac{p_1 + p_2}{2} + \frac{p_1 - p_2}{2}\cos 2\theta = \frac{p_1 + p_2}{2} + \frac{p_1 - p_2}{2}\cos 0°$$

$$= \frac{p_1 + p_2}{2} + \frac{p_1 - p_2}{2} \times 1 \qquad (\because \cos 0° = 1)$$

$$= p_1$$

when $\theta = 90°$,

$$p_n = \frac{p_1 + p_2}{2} + \frac{p_1 - p_2}{2}\cos 2 \times 90° = \frac{p_1 + p_2}{2} + \frac{p_1 - p_2}{2}\cos 180°$$

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(*a*) Let q = Shear stress on the two planes.

Using equation (16.15) for greatest principal stress, we get

Greatest principal stress $= \frac{p_1 + p_2}{2} + \sqrt{\left(\frac{p_1 - p_2}{2}\right)^2 + q^2}$

or $$150 = \frac{120 + (-90)}{2} + \sqrt{\left(\frac{120 - (-90)}{2}\right)^2 + q^2}$$

$$= \frac{120 - 90}{2} + \sqrt{\left(\frac{120 + 90}{2}\right)^2 + q^2} = 15 + \sqrt{105^2 + q^2}$$

or $150 - 15 = \sqrt{105^2 + q^2}$ or $135 = \sqrt{105^2 + q^2}$

Squaring both sides, we get $135^2 = 105^2 + q^2$

or $q^2 = 135^2 - 105^2 = 18225 - 11025 = 7200$

$\therefore$ $q = \sqrt{7200}$ = **84.853 N/mm²**. **Ans.**

(*b*) *Maximum shear stress at the point*

Using equation (16.18) for maximum shear stress,

$$(p_t)_{max} = \frac{1}{2}\sqrt{(p_1 - p_2)^2 + 4q^2}$$

$$= \frac{1}{2}\sqrt{[120 - (-90)]^2 + 4 \times 7200} \qquad (\because \ q^2 = 7200)$$

$$= \frac{1}{2}\sqrt{210^2 + 28800} = \frac{1}{2}\sqrt{44100 + 28800} = \frac{1}{2} \times 270 = \mathbf{135\ N/mm^2}. \quad \textbf{Ans.}$$

Problem 16.14. *At a certain point in a strained material, the stresses on two planes, at right angles to each other are 20 N/mm² and 10 N/mm² both tensile. They are accompanied by a shear stress of a magnitude of 10 N/mm². Find graphically or otherwise, the location of principal planes and evaluate the principal stresses.*

(AMIE, Summer 1984)

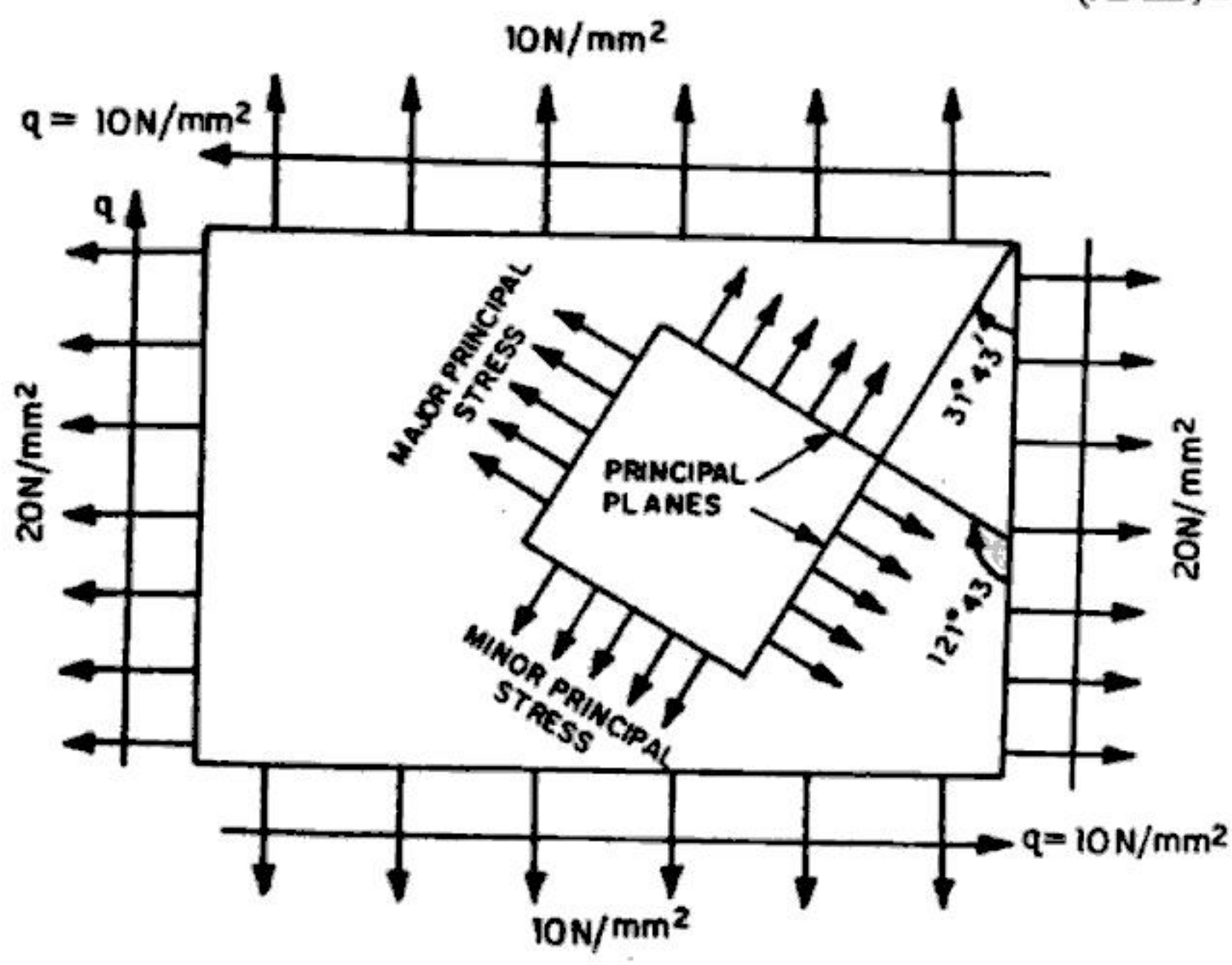

Fig. 16.14 (*a*)

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The directions of principal stresses are given by equation (16.14), as

$$\tan 2\theta = \frac{2q}{(p_1 - p_2)} = \frac{2 \times 400}{(692.82 - 600)} = \frac{800}{92.82} = 8.618$$

$\therefore$ $$2\theta = \tan^{-1} 8.618 = 83.38° \text{ or } 263.38°$$

$\therefore$ $$\theta = \mathbf{41.69° \text{ or } 131.99°}. \quad \textbf{Ans.}$$

(*iii*) *The maximum shear stress and their planes.*

The maximum shear stress is given by equation (16.18).

$\therefore$ $$(p_t)_{max} = \frac{1}{2}\sqrt{(p_1 - p_2)^2 + 4q^2} = \sqrt{\left(\frac{p_1 - p_2}{2}\right)^2 + q^2}$$

$$= \sqrt{\left(\frac{692.82 - 600}{2}\right)^2 + 400^2} = \mathbf{402.68\ kgf/cm^2}. \quad \textbf{Ans.}$$

The direction of maximum shear stress is given by equation (16.17), as

$$\tan 2\theta = \frac{p_2 - p_1}{2q} = \frac{600 - 692.82}{2 \times 400} = \frac{-92.82}{800} = -0.116$$

$\therefore$ $$2\theta = \tan^{-1} -0.116 = -6.61°.$$

–ve sign shows that 2θ is lying in 2nd and 4th quadrant.

$\therefore$ $$2\theta = (180 - 6.61)° \text{ or } (360 - 6.61)° = 173.39° \text{ or } 353.39°$$

$\therefore$ $$\theta = \mathbf{86.695° \text{ or } 176.695°}. \quad \textbf{Ans.}$$

Problem 16.18. *At a certain point in a material under stress the intensity of the resultant stress on a vertical plane is 1000 kgf/cm² inclined at 30° to the normal to that plane and the stress on a horizontal plane has a normal tensile component of intensity 600 kgf/cm² as shown in Fig. 16.16 (c). Find the magnitude and direction of the resultant stress on the horizontal plane and the principal stresses.* (AMIE, Winter 1990)

Sol. Given :

Resultant stress on vertical plane AB = 1000 kgf/cm²

Inclination of the above stress = 30°

Normal stress on horizontal plane BC = 600 kgf/cm²

The resultant stress on plane AB is resolved into normal and tangential component.

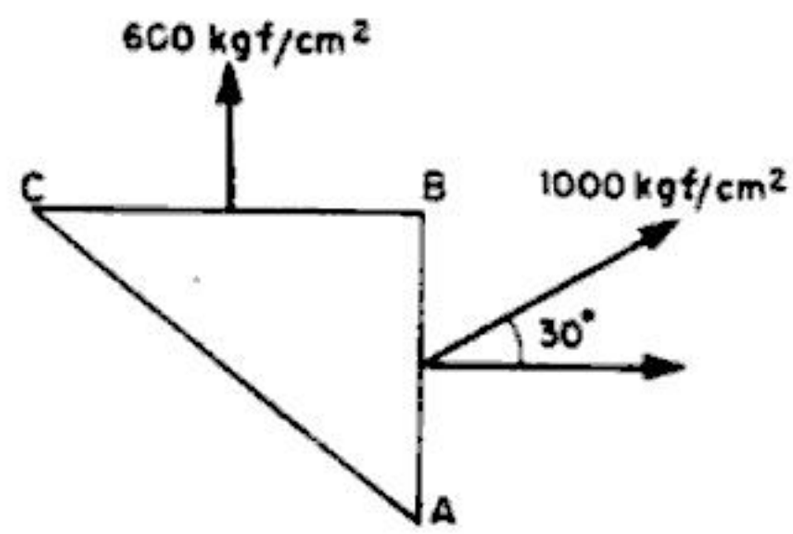

Fig. 16.16 (*c*)

The normal component

$$= 1000 \times \cos 30° = 866 \text{ kgf/cm}^2$$

Tangential component

$$= 1000 \times \sin 30° = 500 \text{ kgf/cm}^2.$$

Hence a shear stress of magnitude 500 kgf/cm² is acting on plane AB. To maintain the wedge in equilibrium, another shear stress of the same magnitude but opposite in direction must act on the plane BC. The free-body diagram of the element $ABCD$ is shown in Fig. 16.16 (*d*), showing normal and shear stresses acting on different faces in which :

$$p_1 = 866 \text{ kgf/cm}^2, p_2 = 600 \text{ kgf/cm}^2 \quad \text{and} \quad q = 500 \text{ kgf/cm}^2$$

(*i*) *Magnitude and direction of resultant stress on horizontal plane BC.*

Normal stress on plane BC, $p_2 = 600$ kgf/cm²

Tangential stress on plane BC, $q = 500$ kgf/cm²

$\therefore$ Resultant stress $$= \sqrt{p_2^2 + q^2}$$

$$= \sqrt{600^2 + 500^2} = \mathbf{781.02\ kgf/cm^2}. \quad \textbf{Ans.}$$

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Proof. (See Fig. 16.19).

$$CD = OC - OD$$

$$= p_1 - p_2 \qquad (\because \quad OC = OA = p_1 \text{ and } OD = OB = p_2)$$

In right angled triangle OEC, $\angle EOC = \theta$.

$\therefore$ $\angle OCE = 90° - \theta$.

In right angled triangle DCF, $\angle DCF = 90° - \theta$.

$\therefore$ $\angle CDF = \theta$

$\therefore$ $DF = CD \cos\theta = (p_1 - p_2)\cos\theta \qquad (\because \quad CD = p_1 - p_2)$

$CF = CD \sin\theta = (p_1 - p_2)\sin\theta$.

In right angled triangle CGF, $\angle GCF = 90 - \theta$

$\therefore$ $CG = CF \cos(90 - \theta) = CF \sin\theta$

$$= (p_1 - p_2)\sin\theta . \sin\theta \qquad [\because \quad CF = (p_1 - p_2)\sin\theta]$$

$$= (p_1 - p_2)\sin^2\theta$$

$\therefore$ $OG = OC - CG = p_1 - (p_1 - p_2)\sin^2\theta$

$$[\because \quad OC = p_1, CG = (p_1 - p_2)\sin^2\theta]$$

$$= p_1 - p_1\sin^2\theta + p_2\sin^2\theta = p_1[1 - \sin^2\theta] + p_2\sin^2\theta$$

$$= p_1\cos^2\theta + p_2\sin^2\theta \qquad (\because \quad 1 - \sin^2\theta = \cos^2\theta) \qquad ...(i)$$

But from equation (16.5 (A)], normal stress across the oblique section is given by

$$p_n = p_1\cos^2\theta + p_2\sin^2\theta \qquad ...(ii)$$

Equating equation (*i*) and (*ii*), we get

$$OG = p_n = \text{Normal stress}$$

Tangential stress, $p_t = GF = CF \sin(90 - \theta) = CF\cos\theta$

$$= (p_1 - p_2)\sin\theta\cos\theta = \frac{p_1 - p_2}{2} 2\sin\theta\cos\theta = \frac{p_1 - p_2}{2}\sin 2\theta.$$

$\therefore$ GF represents the tangential stress.

16.5.2. Important Points. The normal stress tangential stress and resultant stress on the oblique plane by the above method (if any one or both of p_1 and p_2, are compressive) are obtained in the same manner. Only the position of point F will change. The position of point F will be as follows :

(*i*) The point F will be in first quadrant if p_1 and p_2 are tensile stresses (*i.e.*, p_1 and p_2 are +ve).

(*ii*) The point F will be in second quadrant if p_1 is compressive and p_2 is tensile (*i.e.*, p_1 is – ve and p_2 is +ve).

(*iii*) The point F will be in third quadrant if p_1 and p_2 are compressive (*i.e.*, p_1 and p_2 are –ve).

(*iv*) The point F will be in fourth quadrant if p_1 is tensile and p_2 is compressive (*i.e.*, p_1 is +ve and p_2 is –ve).

Problem 16.21. *Solve the problem 16.5 by graphical method.*

Sol. The data given in problem 16.5, is

$$p_1 = 120 \text{ N/mm}^2, p_2 = 60 \text{ N/mm}^2, \theta = 30°.$$

Scale

Take $1 \text{ cm} = 20 \text{ N/mm}^2$

Then $p_1 = \frac{120}{20} = 6 \text{ cm and } p_2 = \frac{60}{20} = 3 \text{ cm}.$

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Length AE = Resultant stress on oblique plane,
Angle ϕ = Obliquity, and

$$\text{Radius of Mohr's circle} = CO \text{ or } OB = \frac{p_1 + p_2}{2}.$$

Proof. (See Fig. 16.24).

$$CO = OB = OE = \text{Radius of Mohr's circle} = \frac{p_1 + p_2}{2}$$

$$AO = OC - AC = \frac{p_1 + p_2}{2} - p_2 = \frac{p_1 + p_2 - 2p_2}{2} = \frac{p_1 - p_2}{2}$$

$\therefore$ $$AD = AO + OD = AO + OB \cos 2\theta \qquad (\because \ OD = OE \cos 2\theta)$$

$$= \frac{p_1 - p_2}{2} + \frac{p_1 + p_2}{2} \cos 2\theta \qquad \left(\because \ OE = \text{Radius} = \frac{p_1 + p_2}{2}\right)$$

$$= p_n \text{ or Normal stress}$$

And $$ED = OE \sin 2\theta$$

$$= \frac{p_1 + p_2}{2} \sin 2\theta \qquad \left(\because \ OE = \frac{p_1 + p_2}{2}\right)$$

$$= p_t \text{ or Tangential (or shear) stress.}$$

Problem 16.24. *Solve problem 16.6, by using Mohr's circle method.*

Sol. Given :

The data given in problem 16.6, is $p_1 = 200$ N/mm^2
$p_2 = -100$ N/mm^2 (compressive)
$\theta = 30°$.

It is required to determine the resultant stress and the maximum shear stress by Mohr's circle method. First choose a suitable scale.

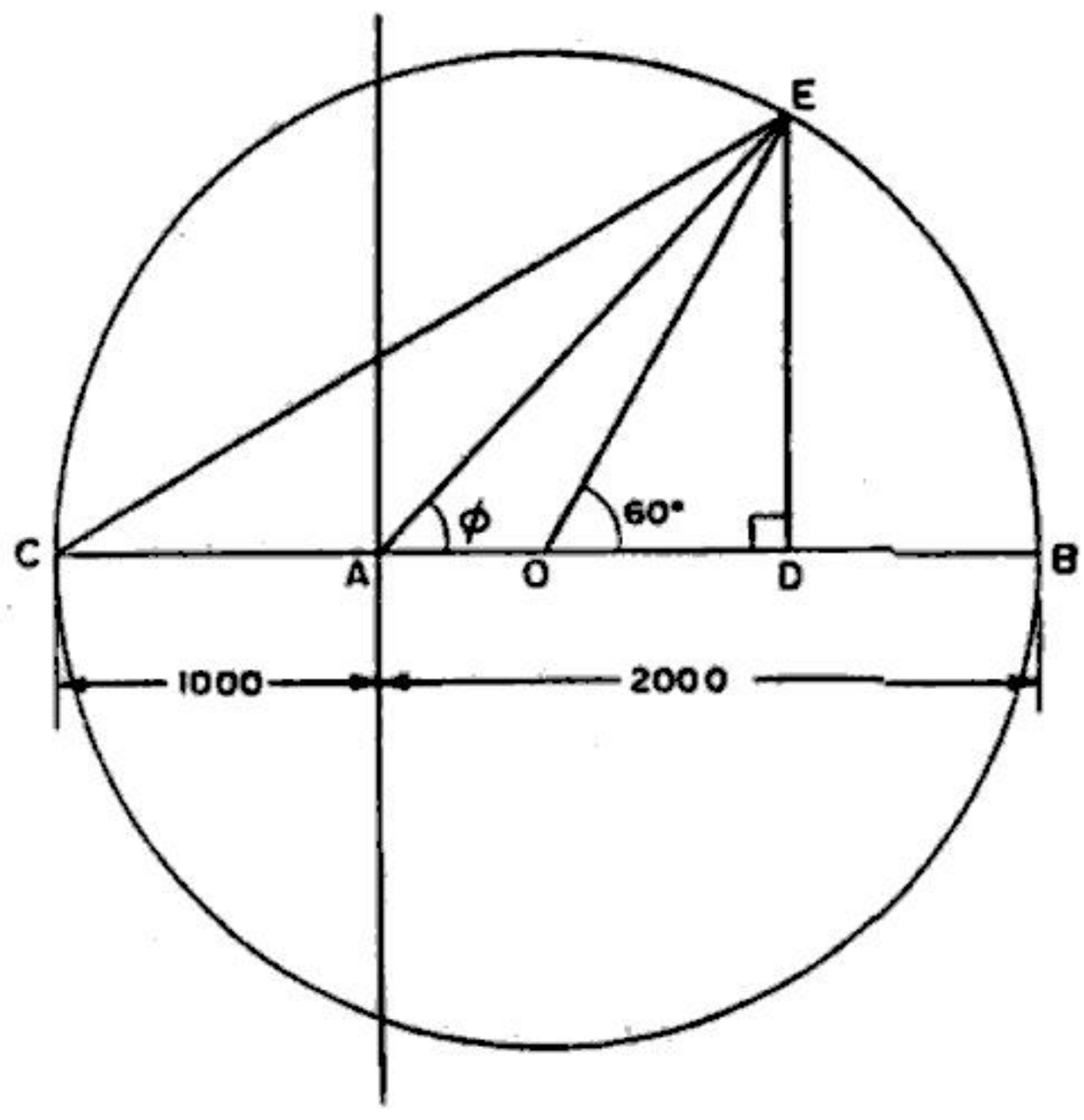

Fig. 16.25

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Mohr's circle of stresses is drawn as given in Fig. 16.31.

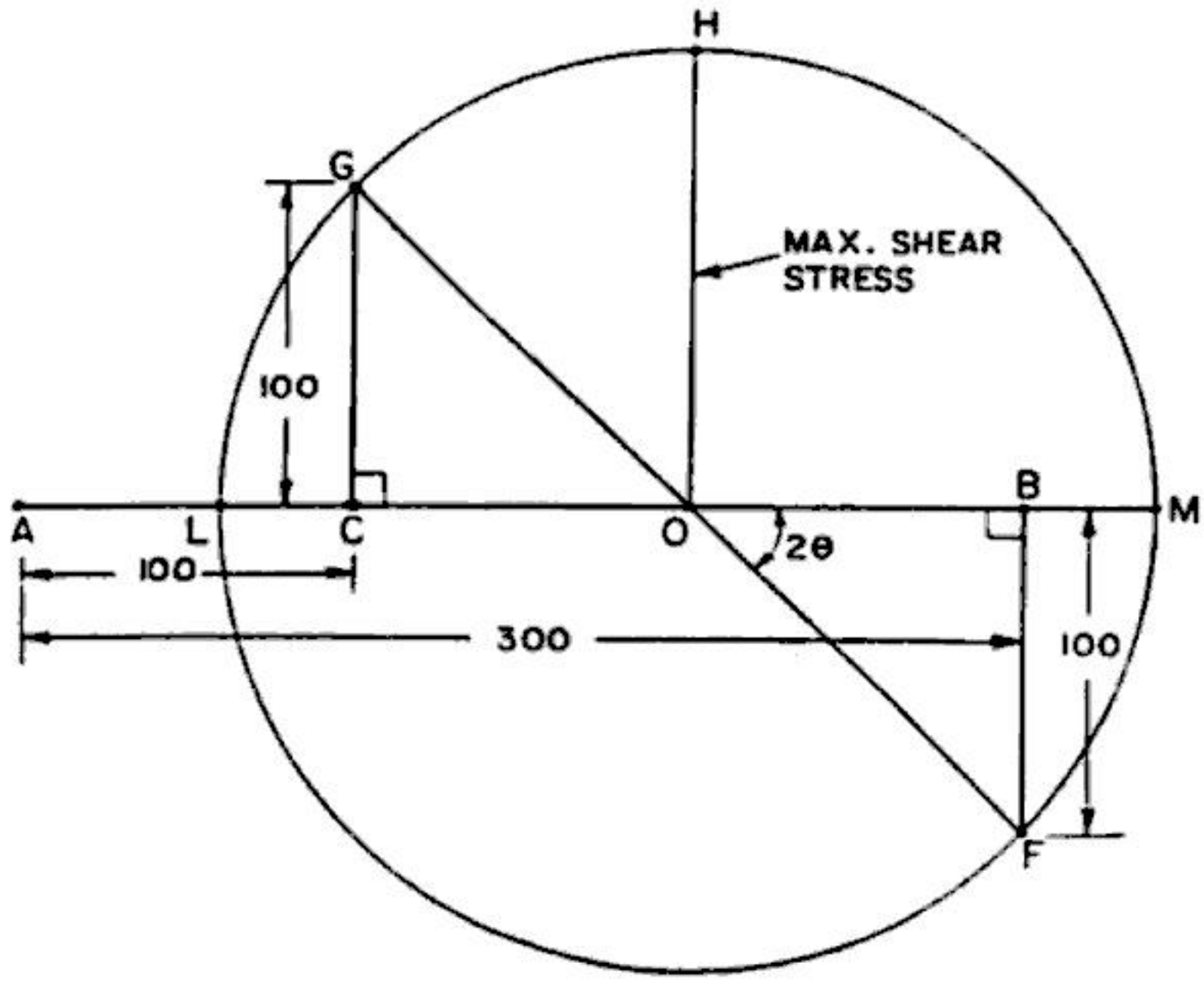

Fig. 16.31

Take any point A and draw a horizontal line through A.

Take $AB = p_1 = 15$ cm and $AC = p_2 = 5$ cm towards right side of A. Draw perpendiculars at B and C and cut off $BF = CG = q = 5$ cm. Bisect BC at O. Now with O as centre and radius equal to OG (or OF), draw a circle cutting the horizontal line through A at L and M as shown in Fig. 16.31. Then AM and AL represents the major and minor principal stresses respectively. And OH represents the maximum shear stress.

By measurements, we have

Length AM = 17.1 cm

Length AL = 2.93 cm

Length OH = Radius of Mohr's circle = 7.05 cm

$\angle FOB$ (or 2θ) = 45°.

∴ Major principal stress = Length AM × Scale = 17.1 × 2 (∵ 1 cm = 2 N/mm²)

= **34.2 N/mm²**. **Ans.**

Minor principal stress = Length AL × Scale = 2.93 × 2 (∵ 1 cm = 2 N/mm²)

= **5.86 N/mm²**

$\angle FOB$ or $2\theta = 45°$

∴ $\theta = \frac{45}{2} = \mathbf{22.5°}$. **Ans.**

The second principal plane is given by $\theta + 90°$.

∴ Second principal plane = 22.5 + 90 = **112.5°**. **Ans.**

The greatest shear stress = Length OH × Scale

= 7.05 × 20 = **14.1 N/mm²**. **Ans.**

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17

Strain Energy and Impact Loading

17.1. INTRODUCTION

Whenever a body is strained, the energy is absorbed in the body. The energy, which is absorbed in the body due to straining effect is known as *strain energy*. The straining effect may be due to gradually applied load or suddenly applied load or load with impact. Hence the strain energy will be stored in the body when the load is applied gradually or suddenly or with an impact. The strain energy stored in the body is equal to the work done by the applied load in stretching the body.

17.2. SOME DEFINITIONS

Before deriving the expressions for the strain energy stored in a body due to gradually applied load or suddenly applied load or load with an impact, the following terms will be defined :

1. Resilience, 2. Proof resilience, and 3. Modulus of resilience.

17.2.1. Resilience. The total strain energy stored in a body is commonly known as resilience. Whenever the straining force is removed from the strained body, the body is capable of doing work. Hence the resilience is also defined as the capacity of a strained body for doing work on the removal of the straining force.

17.2.2. Proof resilience. The maximum strain energy, stored in a body, is known as proof resilience. The strain energy stored in the body will be maximum when the body is stressed upto elastic limit. Hence the proof resilience is the quantity of strain energy stored in a body when strained upto elastic limit.

17.2.3. Modulus of resilience. It is defined as the proof resilience of a material per unit volume. It is an important property of a material. Mathematically,

$$\text{Modulus of resilience} = \frac{\text{Proof resilience}}{\text{Volume of the body}}.$$

17.3. EXPRESSION FOR STRAIN ENERGY STORED IN A BODY WHEN THE LOAD IS APPLIED GRADUALLY

In Art. 17.1, we have mentioned that the strain energy stored in a body is equal to the work done by the applied load in stretching the body.

Fig. 17.1 shows load extension diagram of a body under tensile test upto elastic limit. The tensile load P increases gradually from zero to the value of P and the extension of the body increases from zero to the value of x.

The load P performs work in stretching the body. This work will be stored in the body as *strain energy* which is recoverable after the load P is removed.

Let P = Gradually applied load,

x = Extension of the body

A = Cross-sectional area

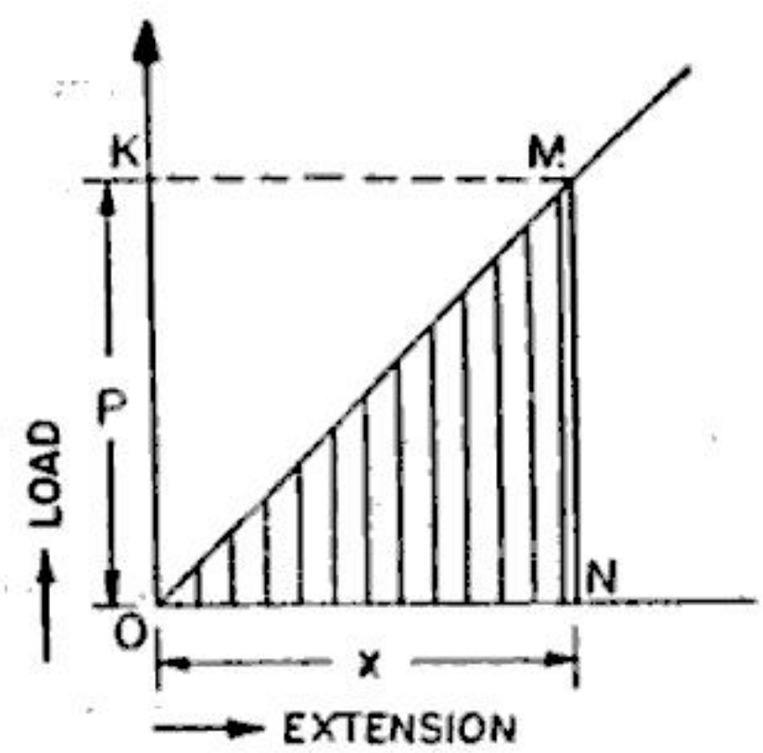

Fig. 17.1

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Suddenly applied load

The instantaneous stress produced by a sudden load is given by equation (17.5) as

$$p = 2 \times \frac{P}{A}$$

or

$$100 = 2 \times \frac{P}{1000}$$

$$\therefore \quad P = \frac{1000 \times 100}{2} = \mathbf{50000\ N = 50\ kN.} \quad \textbf{Ans.}$$

Problem 17.4. *A steel rod is 2 m long and 50 mm in diameter. An axial pull of 100 kN is suddenly applied to the rod. Calculate the instantaneous stress induced and also the instantaneous elongation produced in the rod. Take E = 200 GN/m².*

Sol. Given :

Length, $L = 2\ \text{m} = 2 \times 1000 = 2000\ \text{mm}$

Diameter, $d = 50\ \text{mm}$

$\therefore$ Area, $A = \frac{\pi}{4} \times 50^2 = 625\,\pi\ \text{mm}^2$

Suddenly applied load, $P = 100\ \text{kN} = 100 \times 1000\ \text{N}$

Value of $E = 200\ \text{GN/m}^2$

$= 200 \times 10^9\ \text{N/m}^2$ $\quad (\because\ G = \text{Giga} = 10^9)$

$= \frac{200 \times 19^9}{10^6}\ \text{N/mm}^2$ $\quad (\because\ 1\ \text{m} = 1000\ \text{mm}\ \therefore\ \text{m}^2 = 10^6\ \text{mm}^2)$

$= 200 \times 10^3\ \text{N/mm}^2$

Using equation (17.5) for suddenly applied load,

$$p = 2 \times \frac{P}{A} = 2 \times \frac{100 \times 1000}{625\,\pi}\ \text{N/mm}^2 = \mathbf{101.86\ N/mm^2.} \quad \textbf{Ans.}$$

Let dL = Elongation

Then $$dL = \frac{P}{E} \times L = \frac{101.86}{200 \times 10^3} \times 2000 = \mathbf{1.0186\ mm.} \quad \textbf{Ans.}$$

Problem 17.5. *A uniform metal bar has a cross-sectional area of 700 mm² and a length of 1.5 m. If the stress at the elastic limit is 160 N/mm², what will be its proof resilience ? Determine also the maximum value of an applied load, which may be suddenly applied without exceeding the elastic limit. Calculate the value of the gradually applied load which will produce the same extension as that produced by the suddenly applied load above.*

Take E = 2 × 10⁵ N/mm².

Sol. Given :

Area, $A = 700\ \text{mm}^2$

Length, $L = 1.5\ \text{m} = 1500\ \text{mm}$

$\therefore$ Volume of bar, $V = A \times L = 700 \times 1500 = 1050000\ \text{mm}^2$

Stress at elastic limit, $p^* = 160\ \text{N/mm}^2$

Young's modulus, $E = 2 \times 10^5\ \text{N/mm}^2$

(*i*) Proof resilience is given by equation (17.3), as

Proof resilience $$= \frac{p^{*2}}{2E} \times \text{Volume} = \frac{160^2}{2 \times 2 \times 10^5} \times 1050000$$

$$= 67200\ \text{N-mm} = \mathbf{67.2\ N\text{-}m.} \quad \textbf{Ans.}$$

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Let p = Maximum instantaneous stress induced in the vertical bar,
δL = Maximum elongation, and
U = Strain energy stored.

(*i*) Using equation (17.7),
$$p = \frac{P}{A}\left(1 + \sqrt{1 + \frac{2AEh}{P.L}}\right) + \frac{100}{150}\left(1 + \sqrt{1 + \frac{2 \times 150 \times 2 \times 10^5 \times 20}{100 \times 1500}}\right.$$
$$= \frac{100}{150}(1 + \sqrt{1 + 8000}) = \mathbf{60.23\ N/mm^2}. \quad \textbf{Ans.}$$

(*ii*) Using equation (17.6),
$$\delta L = \frac{p}{E} \times L = \frac{60.23 \times 1500}{2 \times 10^5} = \mathbf{0.452\ mm}. \quad \textbf{Ans.}$$

(*iii*) Strain energy is given by,
$$U = \frac{p^2}{2E} \times V = \frac{60.23^2}{2 \times 2 \times 10^5} \times 225000 = 2045 \text{ N-mm}$$
$$= \mathbf{2.045\ N\text{-}m}. \quad \textbf{Ans.}$$

Problem 17.11. *The maximum instantaneous extension, produced by an unknown falling weight through a height of 4 cm in a vertical bar of length 3 m and of cross-sectional area 5 cm², is 2.1 mm.*

Determine :

(*i*) *the instantaneous stress induced in the vertical bar, and*

(*ii*) *the value of unknown weight. Take $E = 2 \times 10^5$ N/mm².*

Sol. Given :

Instantaneous extension, $\delta L = 2.1$ mm

Length of bar, $L = 3$ m $= 3000$ mm

Area of bar, $A = 5$ cm² $= 500$ mm²

∴ Volume of bar, $V = 500 \times 3000 = 1500000$ mm³

Height through which weight falls, $h = 4$ cm $= 40$ mm

Modulus of elasticity, $E = 2 \times 10^5$ N/mm²

Let p = Instantaneous stress produced, and
P = Unknown weight.

We know $E = \dfrac{\text{stress}}{\text{strain}}$ or Stress $= E \times$ strain

∴ Instantaneous stress $= E \times$ Instantaneous strain $= E \times \dfrac{\delta L}{L}$
$$= 2 \times 10^5 \times \frac{2.1}{3000} \text{ N/mm}^2 = \mathbf{140\ N/mm^2}. \quad \textbf{Ans.}$$

Equating the work done by the falling weight to the strain energy stored, we get
$$P(h + \delta L) = \frac{p^2}{2E} \times V$$
or
$$P(40 + 2.1) = \frac{140^2}{2 \times 2 \times 10^5} \times 1500000 = 73500$$
or
$$P = \frac{73500}{42.1} = \mathbf{1745.8\ N}. \quad \textbf{Ans.}$$

Note. The value of P can also be obtained by using equation (17.7).

Problem 17.12. *An unknown weight falls through a height of 10 mm on a collar rigidly attached to the lower end of a vertical bar 500 cm long and 600 mm² in section. If the maximum extension of the rod is to be 2 mm, what is the corresponding stress and magnitude of the unknown weight ? Take $E = 2.0 \times 10^5$ N/mm².*

(AMIE, Winter 1984)

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3. Calculate instantaneous stress produced in a bar 10 cm^2 in area and 4 m long by the sudden application of a tensile load of unknown magnitude, if the extension of the bar due to suddenly applied load is 1.35 mm. Also determine the suddenly applied load. Take $E = 2 \times 10^5$ N/mm^2. [**Ans.** 67.5 N/mm^2, 33.75 kN]
4. A uniform metal bar has a cross-sectional area of 6 cm^2 and a length of 1.4 m. If the stress at the elastic limit is 1.5 tonne/cm^2, find the proof resilience of the bar. Determine also the maximum value of an applied load, which may be suddenly applied without exceeding the elastic limit. Calculate the value of the gradually applied load which will produced the same extension as that produced by the suddenly applied load above. Take $E = 2000$ tonnes/cm^2.
5. A tension bar 6 m long is made up of two parts, 4 metre of its length has a cross-sectional area 12.5 cm^2 while the remaining 2 m length has a cross-sectional area of 25 cm^2. An axial load of 5 tonnes is gradually applied. Find the total strain energy produced in the bar and compare this value with that obtained in a uniform bar of the same length and having the same volume when under the same load. Take $E = 2 \times 10^6$ kgf/cm^2.
[**Ans.** 242 kgf/cm, 1.054]
6. A load of 20 kgf falls through a height of 2.5 cm on to a collar rigidly attached to the lower end of a vertical bar 2 m long and of 3 cm^2 cross-sectional area. The upper end of the vertical bar is fixed. Determine : (*i*) maximum instantaneous stress induced in the vertical bar, (*ii*) maximum instantaneous elongation, and (*ii*) strain energy stored in the vertical rod. Take $E = 2 \times 10^6$ kgf/cm^2. [**Ans.** (*i*) 584 kgf/cm^2 (*ii*) 0.0584 cm (*iii*) 51.15 kgf-cm]
7. The maximum instantaneous, produced by an unknown falling weight through a height of 4 cm in a vertical bar of length 5 m and of cross-sectional area 5 cm^2, is 1.80 mm. Determine : (*i*) the instantaneous stress induced in the vertical bar and (*ii*) the values of unknown weight.
Take $E = 2 \times 10^6$ kgf/cm^2. [**Ans.** (*i*) 720 kgf/cm^2 and (*ii*) 77.51 kgf]
8. An unknown weight falls through a height of 20 mm on a collar rigidly attached to the lower end of a vertical bar 5 m long and 800 mm^2 in section. If the maximum extension of the rod is to be 2.5 mm, what is the corresponding stress and magnitude of the unknown weight ? Take $E = 2.0 \times 10^6$ kgf/cm^2. [**Ans.** 1000 kgf/cm^2, 444.44 kgf]
9. A bar 1.5 cm diameter gets stretched by 2.5 mm under a steady load of 100 kgf. What stress would be produced in the same bar by a weight of 120 kgf, which falls vertically through a distance 5 cm on to a rigid collar attached at its end ? The bar is initially unstressed. Take $E = 2.0 \times 10^6$ kgf/cm^2. [**Ans.** 1309.44 kgf/cm^2]
10. A vertical round steel rod 2 m long is securely held at its upper end. A weight can slide freely on the rod and its fall is arrested by a stop provided at the lower end of the rod. When the weight falls from a height of 2.5 cm above the stop, the maximum stress reached in the rod is estimated to be 1450 kgf/cm^2. Determine the stress in the rod if the load had been applied gradually and also the maximum stress if the load had fallen from a height of 4.5. Take $E = 2.0 \times 10^6$ kgf/cm^2. [**Ans.** 39.743 kgf/cm^2, 193.42 kgf/cm^2]
11. A vertical compound tie member fixed rigidly at its upper end, consists of a steel rod 3 m long and 20 mm diameter, placed within an equally long brass tube 20 mm internal diameter and 20 mm external diameter. The rod and the tube are fixed together at the ends. The compound member is then suddenly loaded in tension by a weight of 1200 kgf falling through a height of 5 mm on to a flange fixed to its lower end. Calculate the maximum stresses in steel and brass. Assume $E_s = 2 \times 10^6$ kgf/cm^2 and $E_b = 1.0 \times 10^6$ kgf/cm^2. [**Ans.** 1173.5 kgf/cm^2, 586.76 kgf/cm^2]
12. A circular rod 5 cm in diameter and 3 metre long hangs vertically and has a collar securely attached to the lower end. Find the maximum stress induced : (*i*) when a weight of 250 kgf falls through 15 cm on the collar, (*ii*) when a weight of 2500 kgf falls 1.5 cm on the collar. Take $E = 2.1 \times 10^6$ kgf/cm^2.
[**Ans.** (*i*) 1635 kgf/cm^2, (*ii*) 1767 kgf/cm^2]
13. The shear stress in a material at a point is given as 45 N/mm^2. Determine the local strain energy per unit volume stored in the material due to shear stress. Take $C = 8 \times 10^4$ N/mm^2. [**Ans.** 0.01265 N/mm^2]

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The bending moment at a section is the algebraic sum of the moments of forces and reactions acting on one side of the section. Hence bending moment at the section X-X is 500 Nm.

The bending moment will be considered positive when the moment of the forces and reaction on the left portion is clockwise, and on the right portion anti-clockwise. In Fig. 18.12, the bending moment at the section X-X is positive.

Similarly the bending moment will be considered negative when the moment of the forces and reactions on the left portion is anti-clockwise, and on the right portion clockwise as shown in Fig. 18.13. In Fig. 18.13, the bending moment at the section X-X is negative.

18.6. IMPORTANT POINTS FOR DRAWING SHEAR FORCE AND BENDING MOMENT DIAGRAMS

In Art. 18.2, it is mentioned that the shear force diagram is one which shows the variation of the shear force along the length of the beam. And a bending moment diagram is one which show the variation of the bending moment along the length of beam. In these diagrams, the shear force or bending moment are represented by ordinates whereas the length of the beam represents abscissa.

The following are the important points for drawing shear force and bending moment diagrams :

1. Consider the left or the right portion of the section.

2. Add the forces (including reaction) normal to the beam on one of the portion. If right portion of the section is chosen, a force on the right portion acting downwards is positive while a force on the right portion acting upwards is negative.

If the left portion of the section is chosen, a force on the left portion acting upwards is positive while a force on the left portion acting downwards is negative.

3. The positive values of shear force and bending moments are plotted above the base line, and negative values below the base line.

4. The shear force diagram will increase or decrease suddenly *i.e.* by a vertical straight line at a section where there is a vertical point load.

5. The shear force between any two vertical loads will be constant and hence the shear force diagram between two vertical loads will be horizontal.

6. The bending moment at the two supports of a simply supported beam and at the free end of a cantilever will be zero.

18.7. SHEAR FORCE AND BENDING MOMENT DIAGRAMS FOR A CANTILEVER WITH A POINT LOAD AT THE FREE END

Fig. 18.14 shows a cantilever AB of length L fixed at A and free at B and carrying a point load W at the free end B.

Let F_x = Shear force at X, and

M_x = Bending moment at X.

Take a section X at a distance x from the free end. Consider the right portion of the section.

The shear force at this section is equal to the resultant force acting on the right portion at the given section. But the resultant force acting on the right portion at the section X is W and acting in the downward direction. But a force on the right portion acting downwards is considered positive. Hence shear force at X is positive.

$$\therefore \qquad F_x = + W$$

The shear force will be constant at all sections of the cantilever between A and B as there is no other load between A and B. The shear force diagram is shown in Fig. 18.14 (*b*).

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Refer to Fig. 18.18.

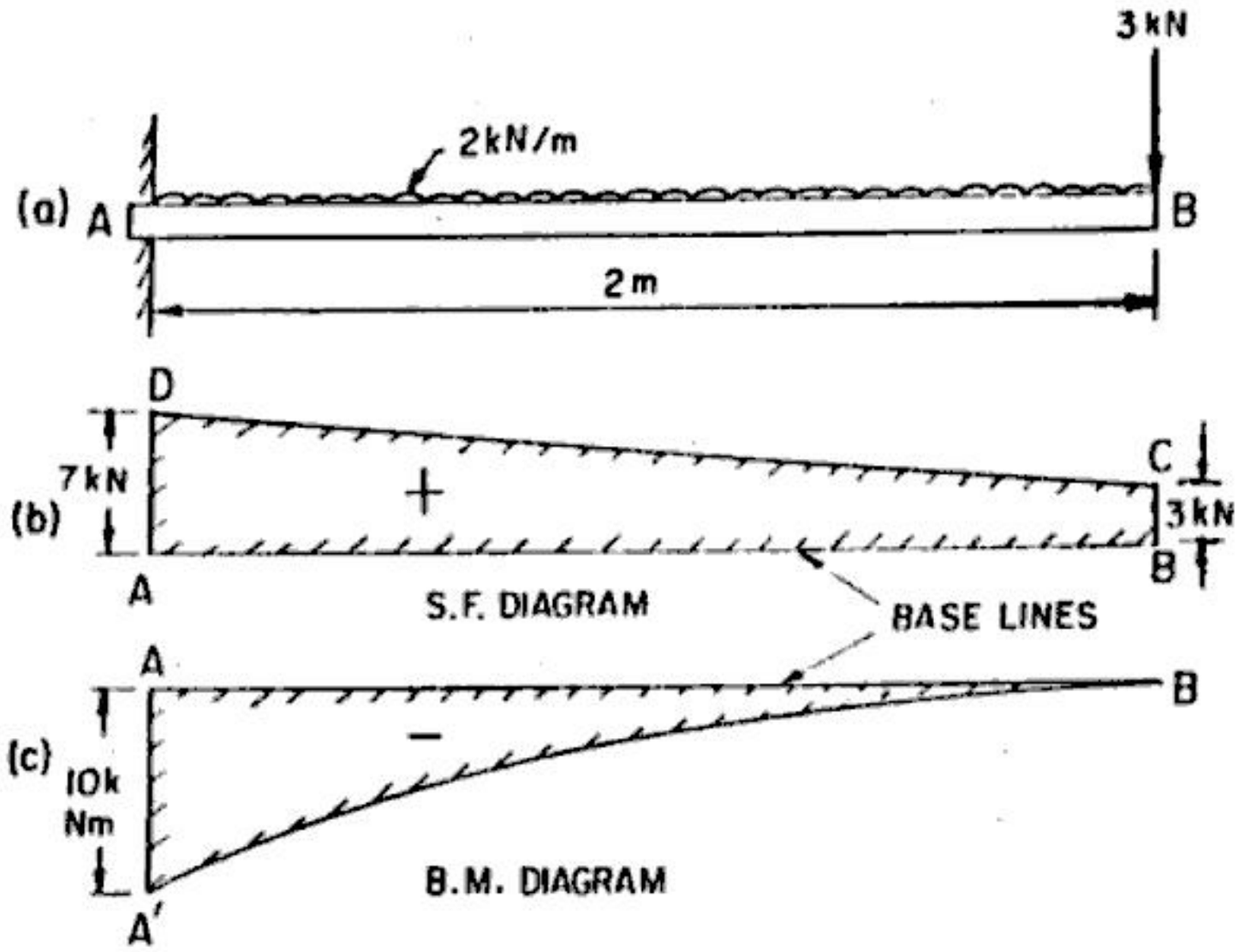

Fig. 18.18

Shear Force Diagram

The shear force at B = 3 kN

Consider any section at a distance x from the free end B. The shear force at the section is given by,

$$F_x = 3.0 + w.x$$

(+ve sign is due to downward force on right portion of the section)

$$= 3.0 + 2 \times x \qquad (\because \quad w = 2 \text{ kN/m})$$

The above equation shows that shear force follows a straight line law.

At B, $x = 0$ hence $\qquad F_B = 3.0$ kN

At A, $x = 2$ m hence $\qquad F_A = 3 + 2 \times 2 = 7$ kN.

The shear force diagram is shown in Fig. 18.18 (b) in which $F_B = BC = 3$ kN and $F_A = AD = 7$ kN. The points C and D are joined by a straight line.

Bending Moment Diagram

The bending moment at any section at a distance x from the free end B is given by,

$$M_x = -\left(3x + wx.\frac{x}{2}\right)$$

$$= -\left(3x + \frac{2x^2}{2}\right) \qquad (\because \quad w = 2 \text{ kN/m})$$

$$= -(3x + x^2) \qquad ...(i)$$

(The bending moment will be negative as for the right portion of the section, the moment of loads at x is clockwise).

The equation (i) shows that the B.M. varies according to the parabolic law. From equation (i), we have

At B, $x = 0$ hence $\qquad M_B = -(3 \times 0 + 0^2) = 0$

At A, $x = 2$ m hence $\qquad M_A = -(3 \times 2 + 2^2) = -10$ kN/m

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From equation (*i*), it is clear that the shear force varies according to straight line law. The values of shear force at different points are :

At A, $x = 0$ hence $F_A = + \frac{w.L}{2} - \frac{w.0}{2} = + \frac{w.L}{2}$

At B, $x = L$ hence $F_B = + \frac{w.L}{2} - w.L = - \frac{w.L}{2}$

At C, $x = \frac{L}{2}$ hence $F_C = + \frac{w.L}{2} - w.\frac{L}{2} = 0$

The shear force diagram is drawn as shown in Fig. 18.27 (*b*).

The bending moment at the section X at a distance x from left end A is given by,

$$M_x = + R_A.x - w.x.\frac{x}{2}$$

$$= \frac{w.L}{2}.x - \frac{w.x^2}{2} \qquad \left(\because R_A = \frac{w.L}{2}\right) \quad ...(ii)$$

From equation (*ii*), it is clear that B.M. varies according to parabolic law.

The values of B.M. at different points are :

At A, $x = 0$ hence $M_A = \frac{w.L}{2}.0 - \frac{w.0}{2} = 0$

At B, $x = L$ hence $M_B = \frac{w.L}{2}.L - \frac{w}{2}.L^2 = 0$

At C, $x = \frac{L}{2}$ hence $M_C = \frac{w.L}{2}.\frac{L}{2} - \frac{w}{2}.\left(\frac{L}{2}\right)^2 = \frac{w.L^2}{4} - \frac{w.L^2}{8} = + \frac{w.L^2}{8}$

Thus the B.M. increases according to parabolic law from zero at A to $+ \frac{w.L^2}{8}$ at the middle point of the beam and from this value the B.M. decreases to zero at B according to the parabolic law.

Now the B.M. diagram is drawn as shown in Fig. 18.27 (*c*).

Problem 18.9. *Draw the shear force end bending moment diagram for a simply supported beam of length 9 m and carrying a uniformly distributed load of 10 kN/m for a distance of 6 m from the left end. Also calculate the maximum B.M. on the section.*

Sol. First calculate reactions R_A and R_B.

Taking moments of the forces about A, we get

$$R_B \times 9 = 10 \times 6 \times \frac{6}{2} = 180$$

$$\therefore \quad R_B = \frac{180}{9} = 20 \text{ kN}$$

$$\therefore \quad R_A = \text{Total load on beam} - R_B = 10 \times 6 - 20 = 40 \text{ kN}.$$

Shear Force Diagram

Consider any section at a distance x from A between A and C. The shear force at the section is given by,

$$F_x = + R_A - 10\,x = + 40 - 10\,x \qquad ...(i)$$

Equation (*i*) shows that shear force varies by a straight line law between A and C.

At A, $x = 0$ hence $F_A = + 40 - 0 = 40$ kN

At C, $x = 6$ m hence $F_C = + 40 - 10 \times 6 = - 20$ kN

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The shear force varies between A and C by a straight line law

The shear force between C and D is constant and equal to -5 kN

The shear force at B is -15 kN

The shear force between D and B varies by a straight line law.

The shear force diagram is drawn as shown in Fig. 18.30 (*b*).

The shear force is zero at point E between A and C. Let us find the location of E from A. Let the point E be at a distance x from A.

The shear force at $E = R_A - 10 \times x = 25 - 10x$

But shear force at $E = 0$

$\therefore \quad 25 - 10x = 0 \quad$ or $\quad 10x = 25$

or $\quad x = \dfrac{25}{10} = 2.5$ m

B.M. Diagram

B.M. at A is zero

B.M. at B is zero

B.M. at C, $\quad M_C = R_A \times 3 - 10 \times 3 \times \dfrac{3}{2} = 25 \times 3 - 45 = 75 - 45 = 30$ kNm

At E, $x = 2.5$ and hence

B.M. at E, $\quad M_E = R_A \times 2.5 - 10 \times 2.5 \times \dfrac{2.5}{2} = 25 \times 2.5 - 5 \times 6.25$

$= 62.5 - 31.25 = 31.25$ kNm

B.M. at D, $\quad M_D = 25(3 + 2) - 10 \times 3 \times \left(\dfrac{3}{2} + 2\right) = 125 - 105 = 20$ kNm

The B.M. between AC and between BD varies according to parabolic law. But B.M. between C and D varies according to straight line law. Now the bending moment diagram is drawn as shown in Fig. 18.30 (*c*).

Problem 18.12. *A simply supported beam of length 10 m, carries the uniformly distributed load and two point loads as shown in Fig. 18.31. Draw the S.F. and B.M. diagram for the beam. Also calculate the maximum bending moment.*

Sol. First calculate the reactions R_A and R_B.

Taking moments of all forces about A, we get

$$R_B \times 10 = 50 \times 2 + 10 \times 4 \times \left(2 + \frac{4}{2}\right) + 40(2 + 4)$$

$$= 100 + 160 + 240 = 500$$

$$\therefore \quad R_B = \frac{500}{10} = 50 \text{ kN}$$

and $\quad R_A =$ Total load on beam $- R_B$

$= (50 + 10 \times 4 + 40) - 50 = 130 - 50 = 80$ kN

S.F. Diagram

The S.F. at A, $\quad F_A = R_A = +80$ kN

The S.F. will remain constant between A and C and equal to $+80$ kN

The S.F. just on R.H.S. of C $\quad = R_A - 50 = 80 - 50 = 30$ kN

The S.F. just on L.H.S. of D $\quad = R_A - 50 - 10 \times 4 = 80 - 50 - 40 = -10$ kN

The S.F. between C and D varies according to straight line law.

The S.F. just on R.H.S. of D $\quad = R_A - 50 - 10 \times 4 - 40 = 80 - 50 - 40 - 40 = -50$ kN

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$$\sin\psi = \frac{dy}{ds}$$

and
$$\cos\psi = \frac{dx}{ds}$$

Now equation (*i*) can be written as

$$R = \frac{ds}{d\psi} = \frac{\left(\frac{ds}{dx}\right)}{\left(\frac{d\psi}{dx}\right)} = \frac{\left(\frac{1}{\cos\psi}\right)}{\left(\frac{d\psi}{dx}\right)}$$

or
$$R = \frac{\sec\psi}{\left(\frac{d\psi}{dx}\right)} \qquad ...(iii)$$

Differentiating equation (*ii*) w.r.t. x, we get

$$\sec^2\psi . \frac{d\psi}{dx} = \frac{d^2y}{dx^2} \quad \text{or} \quad \frac{d\psi}{dx} = \frac{\left(\frac{d^2y}{dx^2}\right)}{\sec^2\psi}$$

Substituting this value of $\frac{d\psi}{dx}$ in equation (*iii*), we get

$$R = \frac{\sec\psi}{\left(\frac{\frac{d^2y}{dx^2}}{\sec^2\psi}\right)} = \frac{\sec\psi . \sec^2\psi}{\frac{d^2y}{dx^2}} = \frac{\sec^3\psi}{\left(\frac{d^2y}{dx^2}\right)}$$

Taking the reciprocal to both sides, we get

$$= \frac{1}{R} = \frac{\frac{d^2y}{dx^2}}{\sec^3\psi} = \frac{\frac{d^2y}{dx^2}}{(\sec^2\psi)^{3/2}} = \frac{\frac{d^2y}{dx^2}}{(1+\tan^2\psi)^{3/2}}$$

For a practical beam, the slope $\tan\psi$ at any point is a small quantity. Hence $\tan^2\psi$ can be neglected.

$$\therefore \quad \frac{1}{R} = \frac{d^2y}{dx^2} \qquad ...(iv)$$

From the bending equation, we have

$$\frac{M}{I} = \frac{E}{R} \quad \text{or} \quad \frac{1}{R} = \frac{M}{EI} \qquad ...(v)$$

Equating equations (*iv*) and (*v*), we get

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$\therefore \quad M = EI\frac{d^2y}{dx^2} \qquad ...(19.3)$$

Differentiating the above equation w.r.t. x, we get

$$\frac{dM}{dx} = EI\frac{d^3y}{dx^3}$$

But
$$\frac{dM}{dx} = F \text{ shear force (See page 131)}$$

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The deflection (*i.e.* y) at any point in CB is obtained by substituting the values of C_3 and C_4 in equation (*vii*). Hence, we get from equation (*vii*),

$$EI.y = \frac{W.b}{6L}.x^3 - \frac{W}{6}(x-a)^3 + \left(EI.i_C - \frac{W.b.a^2}{2L}\right)x + \frac{W.ab}{3}(a-b) - EI.L.i_C \quad ...(xi)$$

The deflection at the point C is obtained by substituting $x = a$ in the above equation. Let y_C = the deflection at C. Hence, we get

$$EI.y_C = \frac{W.b.a^3}{6L} - \frac{W}{6}(a-a)^3 + \left(EI.i_C - \frac{W.b.a^2}{2L}\right)a + \frac{W.a.b}{3}(a-b) - EI.L.i_C$$

or
$$y_C = \frac{1}{EI}\left[\frac{W.b.a^3}{6L} - 0 + EI.a.i_C - \frac{W.b.a^3}{2L} + \frac{W.a.b}{3}(a-b) - EI.L.i_C\right]$$

$$= \frac{1}{EI}\left[\frac{W.b.a^3}{6L} - \frac{W.b.a^3}{2L} + \frac{W.a.b}{3}(a-b) + EI.a.i_C - EI.L.i_C\right] \quad ...(A)$$

The deflection at the point C can also be obtained by substituting $x = a$ in equation (v). Hence, we get

$$EI.y_C = \frac{W.b.a^3}{6L} + \left(EI.i_C - \frac{W.b.a^2}{2L}\right).a$$

or
$$y_C = \frac{1}{EI}\left[\frac{W.b.a^3}{6L} + EI.i_C.a - \frac{W.b.a^3}{2L}\right] \quad ...(B)$$

Equating the two values of y_C given by equations (A) and (B), we get

$$\frac{1}{EI}\left[\frac{W.b.a^3}{6L} + EI.i_C.a - \frac{W.b.a^3}{2L}\right] = \frac{1}{EI}\left[\frac{W.b.a^3}{6L} - \frac{W.b.a^3}{2L} + \frac{W.a.b}{3}(a-b) + EI.a.i_C - EI.L.i_C\right]$$

or
$$0 = \frac{W.a.b}{3}(a-b) - EI.L.i_C$$

or
$$EI.L.i_C = \frac{W.a.b}{3}(a-b)$$

or
$$i_C = \frac{W.a.b}{3EI.L}(a-b) \quad ...(19.8)$$

The above equation gives the value of i_C (*i.e.* slope at point C). Substituting this value of i_C in equation (iv), we get the slope at any point in AC. Hence, we get from equation (iv),

$$EI\frac{dy}{dx} = \frac{W.b}{2L}.x^2 + EI \times \frac{W.a.b}{3EI.L}(a-b) - \frac{W.b.a^2}{2L}$$

$$= \frac{W.b}{2L}.x^2 + \frac{W.a.b}{3L}(a-b) - \frac{W.b.a^2}{2L} = \frac{W.b}{6L}[3x^2 + 2a(a-b) - 3a^2]$$

$$= \frac{W.b}{6L}[3x^2 - 2ab - a^2] \quad ...(C)$$

As the length AC is more than length CB, hence maximum slope will be at the support A, where $x = 0$. Let the slope at A is represented by i_A. Hence $\frac{dy}{dx}$ at A will be equal to i_A. Substituting $x = 0$ in equation (C), we get

$$EI\left(\frac{dy}{dx}\right)_{\text{at } A} = \frac{W.b}{6L}[3 \times 0 - 2ab - a^2]$$

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$\therefore$ Turning force on the ring $= \text{Stress} \times \text{Area} = q \times dA$

$$= \frac{f_s}{R_0} r \times 2\pi r dr \qquad \left(\because \; q = \frac{f_s}{R_0} r\right)$$

$$= 2\pi \frac{f_s}{R_0} r^2 \, dr$$

Turning moment (dT) on the ring,

$$dT = \text{Turning force} \times \text{Distance of the ring from centre}$$

$$= 2\pi \frac{f_s}{R_0} r^2 dr \times r = 2\pi \frac{f_s}{R_0} r^3 \, dr$$

The total turning moment (or total torque T) is obtained by integrating the above equation between the limits R_i and R_0.

$$\therefore \qquad T = \int_{R_i}^{R_0} dT = \int_{R_i}^{R_0} 2\pi \frac{f_s}{R_0} r^3 \, dr = 2\pi \frac{f_s}{R_0} \int_{R_i}^{R_0} r^3 \, dr$$

($\because$ f_s and R_0 are constant and can be taken outside the integral)

$$= 2\pi \frac{f_s}{R_0} \left[\frac{r^4}{4} \right]_{R_i}^{R_0} = 2\pi \frac{f_s}{R_0} \left[\frac{R_0^4 - R_i^4}{4} \right]$$

$$= \frac{\pi}{2} f_s \left[\frac{R_0^4 + R_i^4}{R_0} \right] \qquad \text{...(21.5)}$$

Let D_0 = Outer diameter of the shaft

D_i = Inner diameter of the shaft.

Then $R_0 = \dfrac{D_0}{2}$ and $R_i = \dfrac{D_i}{2}$.

Substituting the values of R_0 and R_i in equation (21.5),

$$T = \frac{\pi}{2} f_s \left[\frac{\left(\frac{D_0}{2}\right)^4 - \left(\frac{D_i}{2}\right)^4}{\left(\frac{D_0}{2}\right)} \right] = \frac{\pi}{2} f_s \left[\frac{\frac{D_0^4}{16} - \frac{D_i^4}{16}}{\frac{D_0}{2}} \right]$$

$$= \frac{\pi}{2} f_s \left[\frac{D_0^4 - D_i^4}{16} \times \frac{2}{D_0} \right] = \frac{\pi}{16} f_s \left[\frac{D_0^4 - D_i^4}{D_0} \right] \qquad \text{...(21.6)}$$

21.5. POWER TRANSMITTED BY SHAFTS

Once the expression for torque (T) for a solid or a hollow shaft is obtained, power transmitted by the shafts can be determined.

Let N = r.p.m. of the shaft

T = Mean torque transmitted in N-m

ω = Angular speed of shaft.

Then $$\text{Power} = \frac{2\pi N T^*}{60} \text{ watts} \qquad \text{...(21.7)}$$

*The torque T obtained by this formula is the average (or mean) torque.

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Sol. Given :

Let D_0 = Outer diameter of the hollow shaft

D_i = Inside diameter of the hollow shaft = 60% of D_0

$$= \frac{60}{100} \times D_0 = 0.6\, D_0$$

D = Diameter of the solid shaft

P = Power transmitted by hollow shaft *or* by solid shaft

N = Speed of each shaft

f_s = Maximum shear stress induced in each shaft. Since material of both shafts is same and hence shear stress will be same.

Power by solid shaft or hollow shaft is given by

$$P = \frac{2\pi NT}{60}$$

$\therefore$ $$T = \frac{60 \times P}{2\pi N} = \text{constant}$$ ($\because$ P and N are same for solid and hollow shafts)

$\therefore$ Torque transmitted by solid shaft is the same as the torque transmitted by hollow shaft.

Torque transmitted by solid shaft is given by equation (21.4) as

$$T = \frac{\pi}{16} f_s D^3 \qquad ...(i)$$

Torque transmitted by hollow shaft is given by equation (21.6) as

$$T = \frac{\pi}{16} f_s \left[\frac{D_0^4 - D_i^4}{D_0}\right] = \frac{\pi}{16} f_s \left[\frac{D_0^4 - (.6D_0)^4}{D_0}\right]$$

$$= \frac{\pi}{16} f_s \left[\frac{D_0^4 - 0.1296 D_0^4}{D_0}\right]$$

$$= \frac{\pi}{16} f_s \times 0.8704\, D_0^3 . \qquad ...(ii)$$

Since torque transmitted is the same and hence equating equation (*i*) and (*ii*)

$$\frac{\pi}{16} f_s D^3 = \frac{\pi}{16} f_s \times 0.8704\, D_0^3.$$

$\therefore$ $$D = (0.8704)^{1/3} D_0 = 0.9548 D_0.$$

$\therefore$ Area of solid shaft $$= \frac{\pi}{4} D^2 = \frac{\pi}{4} (0.9548 D_0)^2 = 0.716\, D_0^2$$

Area of hollow shaft $$= \frac{\pi}{4} [D_0^2 - D_i]^2 = \frac{\pi}{4} [D_0^2 - (0.6 D_0^2)]$$

$$= \frac{\pi}{4} [D_0^2 - 0.36 D_0^2] = \frac{\pi}{4} \times 0.64 D_0^2 = 0.502 D_0^2.$$

For the shafts of the same material, the weight of the shafts is proportional to the areas.

$\therefore$ Saving in material = Saving in area

$$= \frac{\text{Area of solid shaft} - \text{Area of hollow shaft}}{\text{Area of solid shaft}}$$

$$= \frac{0.716 D_0^2 - 0.502 D_0^2}{0.716 D_0^2} = 0.2988.$$

$\therefore$ Percentage saving in material = 0.2988 × 100 = **29.88.** **Ans.**

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But we know, $T^* = \frac{\pi}{16} \times q \times D^3$ or $7159000 = \frac{\pi}{16} \times q \times 100^3$

$\therefore$ $q = \frac{7159000 \times 16}{\pi \times 10^6} = 36.5 \text{ N/mm}^2$

Using equation (21.20) for strain energy,

$$U = \frac{q^2}{4C} \times \text{Volume of shaft} = \frac{36.5^2}{4 \times 8 \times 10^4} \times \text{Volume of shaft}$$

$$= \frac{36.5^2}{4 \times 8 \times 10^4} \times \frac{\pi}{4} \times 100^2 \times 4000 \qquad \left(\because \text{ Volume} = \frac{\pi}{4} \times D^2 \times l\right)$$

$$= \mathbf{130793 \text{ Nmm.}} \quad \textbf{Ans.}$$

Problem 21.30. *A hollow shaft of internal diameter 10 cm, is subjected to pure torque and attains a maximum shear stress q on the outer surface of the shaft. If the strain energy stored in the hollow shaft is given by* $\frac{q^2}{3C} \times V$, *determine the external diameter of the shaft.*

Sol. Given :

Internal dia., $d = 10$ cm

Maximum shear stress $= q$

Strain energy stored, $U = \frac{q^2}{3C} \times V$ where V = Volume

Let D = External diameter of the hollow shaft.

Using equation (21.21) for the strain energy in hollow shaft

$$U = \frac{q^2}{4CD^2}(D^2 + d^2) \times V$$

Equating the two values of strain energy, we get

$$\frac{q^2}{4CD^2}(D^2 + d^2) \times V = \frac{q^2}{3C} \times V$$

or $\frac{D^2 + d^2}{4D^2} = \frac{1}{3}$ $\left(\because \text{ Cancelling } \frac{q^2}{C} \times V \text{ to both sides}\right)$

or $3D^2 + 3d^2 = 4D^2$ or $3d^2 = 4D^2 - 3D^2 = D^2$

or $\frac{D^2}{d^2} = 3$ or $\frac{D}{d} = \sqrt{3} = 1.732$

$\therefore$ $D = 1.732 \times d = 1.732 \times 10 = \mathbf{17.32 \text{ cm.}}$ **Ans.**

21.13. SPRINGS

Springs are the elastic bodies which absorb energy due to resilience. The absorbed energy may be released as and when required. A spring which is capable of absorbing the greatest amount of energy for the given stress, without getting permanently distorted, is known as the best spring. The two important types of springs are :

1. Laminated or leaf springs and
2. Helical springs.

21.13.1. Laminated or leaf spring. The laminated springs are used to absorb shocks in railway wagons, coaches and road vehicles (such as cars, lorries etc.).

Fig. 21.11 shows a laminated spring which consists of a number of parallel strips of a metal having different lengths and same width, placed one over the other. Initially all the plates are bent to the same radius

*Please see equation 21.4 on page 241. Here $q = f_s$.

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Now using the relation, $\bar{y} = \dfrac{a_1 \times y_1 + a_2 \times y_2}{a_1 + a_2} = \dfrac{10 \times 7.5 + 7 \times 3.5}{(10 + 7)}$

$$= \frac{75 + 24.5}{17} = \frac{99.5}{17} = 5.85 \text{ cm.}$$

Now calculate the moment of inertia about x-x axis and y-y axis.

$$I_{XX} = \left[\frac{10 \times 1^3}{12} + a_1 \times (2.15 - 0.5)^2\right] + \left[\frac{1 \times 7^3}{12} + a_2 \times (5.85 - 3.5)^2\right]$$

$$= \left[\frac{10}{12} + 10 \times 1.65^2\right] + \left[\frac{343}{12} + 7 \times 2.35^2\right] = 95.298 \text{ cm}^4$$

and $$I_{YY} = \frac{1 \times 10^3}{12} + \frac{7 \times 1^3}{12} = 83.33 + 0.583 = 83.916 \text{ cm}^4.$$

The least value of moment of inertia is about y-axis.

$\therefore$ $$I = I_{YY} = 83.916 \text{ cm}^4 = 839160 \text{ mm}^4.$$

As the strut is fixed at both ends, hence its effective length (L_e) will be half of its actual length.

$\therefore$ $$L_e = \frac{L}{2} = \frac{3000}{2} = 1500 \text{ mm}$$

Let P = Euler's critical load.

Using equation (23.5), $$P = \frac{\pi^2 \times EI}{L_e^2} = \frac{\pi^2 \times 2 \times 10^5 \times 839160}{1500^2}$$

$$= 736190 \text{ N} = \mathbf{736.19 \text{ kN}}. \quad \textbf{Ans.}$$

Problem 23.11. *A strut length l, moment of inertia of cross-section = I uniform throughout and modulus of material = E, is fixed at its lower end, and its upper end is elastically supported laterally by a spring of stiffness k. Show from the first principles that the crippling load P is given by*

$$\frac{\tan \alpha l}{\alpha l} = 1 - \frac{P}{kl}, \quad \text{where } \alpha^2 = \frac{P}{EI}.$$ (AMIE Summer, 1984)

Sol. Given :

Length of strut $= l$

Moment of inertia $= I$

Young's modulus $= E$

Stiffness of spring $= k$

Crippling load $= P$.

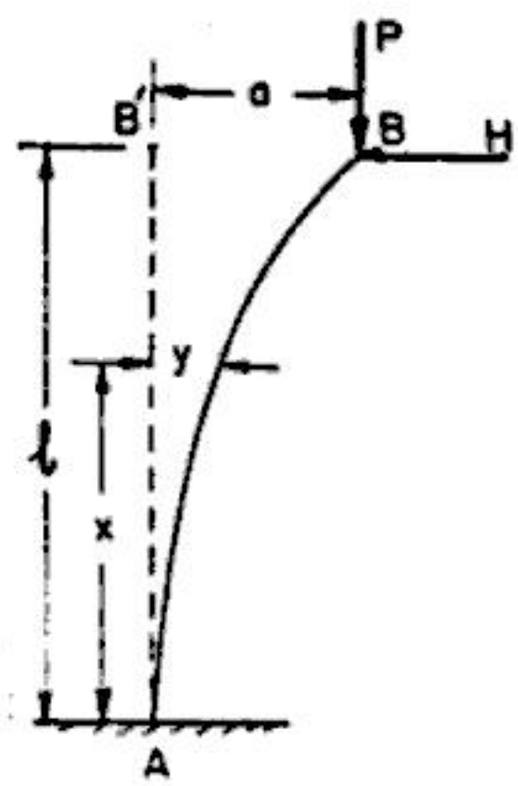

Fig. 23.11

The strut is shown in Fig. 23.11 in which

H = Lateral force due to spring of stiffness k

a = Deflection of the end B

Then H = Stiffness of spring × Deflection of end B

$= k \times a$. ...(i)

Consider any section at a distance x from the fixed end A.

Let y = Deflection at the section.

Then moment at the section = Moment due to crippling load + Moment due to lateral force H

$$= P(a - y) - H(l - x) = P \cdot a - P \cdot y - H(l - x)$$

But moment is also $$= EI \frac{d^2y}{dx^2}$$

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