CHAPTER 2

Matrices, Determinants and Eigenvalue Problems

There are many problems of interest in science and engineering where the solution oftenly leads to a system of linear algebraic simultaneous equations. Matrices are very elegant and powerful tool to analyze such a system as a single entity. However, the field of applications of matrices and determinants comprises much more than the solution of linear algebraic equations. The study of matrices includes linear transformations and eigenvalue problems also. With the advancement of digital computers, matrices find applications in almost all branches of science and engineering like electrical networks, graph theory, computer graphics, optimization problems, system of differential equations and stochastic processes just to name a few.

2.1 MATRICES: SOME BASIC DEFINITIONS

Matrix. An $m \times n$ matrix is an array of mn entries called elements arranged in m horizontal and n vertical columns in the form

$$\mathbf{A} = [a_{ij}]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad \ldots(2.1)$$

We say that the matrix (2.1) is of order $m \times n$ (m by n) and the element a_{ij}, ($i = 1, 2,\ldots m, j = 1, 2,\ldots n$), common to the ith row and the jth column, is called the *general element*. Normally, the elements are numbers real or complex, although they may occasionally be other objects such as differential operators or even matrices itself. The matrices are generally denoted by the boldface upper case letters like **A**, **B**, **C**, etc.

If all the elements of a matrix are real it is called a *real matrix*, whereas if one or more of its elements are complex it is called a *complex matrix*.

Square Matrix. If $m = n$, then **A** is called a *square matrix* of order n, and in that case diagonal containing the entries $a_{11}, a_{22}, \ldots, a_{nn}$ is called the *principal diagonal* of **A**. The sum of the diagonal elements of a square matrix is called the *trace* of the matrix **A**. A matrix which is not square is called a *rectangular* matrix.

Triangular matrices. A square matrix $A = [a_{ij}]$ is called a *lower triangular matrix* if $a_{ij} = 0$, whenever $i < j$, that is, if all elements above the principal diagonal are zero, and an *upper triangular matrix* if $a_{ij} = 0$, whenever $i > j$, that is, if all the elements below the principal diagonal are zeros. The matrices

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

respectively are the upper triangular and the lower triangular matrices. If a matrix is upper triangular or lower triangular it is said to be *triangular.*

Row and Column Matrices. A matrix having a single row is called a *row matrix* or *row vector* and a matrix having a single column is called a *column matrix* or *column vector*.

Null matrix. A matrix **A** of order $m \times n$ in which all the elements are zero is called a *null matrix* and is normally denoted by **O**.

Diagonal matrix. A square matrix **A** in which all the off-diagonal elements a_{ij} $(i \neq j)$ are zero is called a *diagonal matrix,* normally denoted by **D**. Thus

$$\mathbf{D} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

is a diagonal matrix of order n. Sometimes it is written as $\mathbf{D} = \text{diag}\,[a_{11}, a_{22}, \ldots, a_{nn}]$.

Scalar matrix. A diagonal matrix of order n with all its diagonal elements equal, that is $a_{ii} = d$, $i = 1, 2, \ldots n$, is called a *scalar matrix* of order n. Thus $\mathbf{D} = \text{diag}\,(d, d, \ldots, d)$ is a scalar matrix.

Unit matrix or Identity matrix. If all the diagonal elements are equal to 1, then the matrix $\mathbf{D} = \text{diag}\,(1, 1, \ldots, 1)$ is called a *unit matrix* or an *identity matrix* of order n, and is denoted by $\mathbf{I}_n$.

For example, $\mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a unit matrix of order 3.

Equal matrices. Two matrices $\mathbf{A} = [a_{ij}]$ of order $m \times n$ and $\mathbf{B} = [b_{ij}]$ of order $p \times q$ are *equal,* written as $\mathbf{A} = \mathbf{B}$, if, and only if **A** and **B** are of the same order, so that $m = p$ and $n = q$ and $a_{ij} = b_{ij}$, for each i and j.

2.2 MATRIX ALGEBRA

In case of matrices we define the following basic operations.

I. **Matrix addition and subtraction**

II. **Scalar multiplication**

III. **Matrix multiplication**

I. Matrix addition and subtraction. If $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ are any two matrices of the same order, say $m \times n$, then their sum, denoted by $\mathbf{A} + \mathbf{B}$, is defined as

$$\mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}]$$

and is itself an $m \times n$ matrix.

If **A** and **B** are of the same order, they are said to be *conformable for addition*, otherwise $\mathbf{A} + \mathbf{B}$ is not defined.

Similarly, the difference of two matrices **A** and **B** each of the same order $m \times n$, denoted by $\mathbf{A} - \mathbf{B}$, is defined as

$$\mathbf{A} - \mathbf{B} = [a_{ij} - b_{ij}]$$

and is again an $m \times n$ matrix.

II. Scalar multiplication. If $\mathbf{A} = [a_{ij}]$ is a matrix of order $m \times n$ and λ is any scalar, then the multiplication of the matrix, **A** by a scalar λ, denoted by $\lambda\mathbf{A}$, is defined as

$$\lambda\mathbf{A} = [\lambda a_{ij}]$$

and is itself a matrix of order $m \times n$.

We do not distinguish between $\lambda\mathbf{A}$ and $\mathbf{A}\lambda$ and further, we define $-\mathbf{A} = (-1)\mathbf{A}$ as the *negative of the matrix* **A**.

For example, if $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 2 & 5 \end{bmatrix}$, and $\mathbf{B} = \begin{bmatrix} 5 & 4 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & 3 \end{bmatrix}$, then

$$3\mathbf{A} = \begin{bmatrix} 3 & 6 & 9 \\ 0 & 3 & 6 \\ 3 & 6 & 15 \end{bmatrix}, \mathbf{A} + \mathbf{B} = \begin{bmatrix} 6 & 6 & 4 \\ 2 & 2 & 2 \\ 2 & 4 & 8 \end{bmatrix}, \text{and } \mathbf{A} - \mathbf{B} = \begin{bmatrix} -4 & -2 & 2 \\ -2 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

Properties of matrix addition and scalar multiplication. If **A**, **B** and **C** are $m \times n$ matrices, **O** is an $m \times n$ null matrix and α, β are scalars, real or complex, then

(i) $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ *(commutativity)*

(ii) $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$ *(associativity)*

(iii) $\mathbf{A} + \mathbf{O} = \mathbf{O} + \mathbf{A} = \mathbf{A}$ *(additive identity)*

(iv) $\mathbf{A} + (-\mathbf{A}) = \mathbf{O}$ *(additive inverse)*

(v) $\alpha(\beta\mathbf{A}) = (\alpha\beta)\mathbf{A}$

(vi) $(\alpha + \beta)\mathbf{A} = \alpha\mathbf{A} + \beta\mathbf{A}$

(vii) $\alpha(\mathbf{A} + \mathbf{B}) = \alpha\mathbf{A} + \alpha\mathbf{B}$

The proofs follow directly from the definitions.

III. Matrix Multiplication. The product of the two matrices **A** and **B**, denoted by **AB**, is defined only when the number of columns in **A** is equal to the number of rows in **B**. Two such matrices are said to be *conformable for multiplication*.

If $\mathbf{A} = [a_{ij}]$ is a matrix of order $m \times n$ and $\mathbf{B} = [b_{kj}]$ is a matrix of order $n \times p$, then their product **AB** is

defined as $\mathbf{AB} = \left[\sum_{k=1}^{n} a_{ik}\, b_{kj}\right]$

which itself is a matrix of order $m \times p$ and, if we denote $\mathbf{AB} = \mathbf{C} = [c_{ij}]$, then $c_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj}$.

For example, if $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2\times 3}$ and $\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}_{3\times 2}$, then

$$\mathbf{AB} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{bmatrix}_{2\times 2}$$

In the product **AB**, **B** is said to be *pre-multiplied* by **A**, or **A** is said to be *post-multiplied* by **B**.

If **A** is square matrix, then the product **AA** is defined as $\mathbf{A}^2$. Similarly we can define the higher powers of **A**. If $\mathbf{A}^2 = \mathbf{A}$, then the matrix **A** is called *idempotent*.

Properties of matrix multiplication. If **A**, **B**, **C**, **I** and **O** are matrices of suitable orders and α, β are scalars, then

(a) $\mathbf{IA} = \mathbf{AI} = \mathbf{A}$

(b) $\mathbf{OA} = \mathbf{AO} = \mathbf{O}$

(c) $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$

(d) $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$

(e) $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$

(f) $\alpha(\mathbf{AB}) = (\alpha\mathbf{A})\,\mathbf{B} = \mathbf{A}(\alpha\mathbf{B})$

(g) $\mathbf{A}(\alpha\mathbf{B} + \beta\mathbf{C}) = \alpha\mathbf{AB} + \beta\mathbf{AC}$

The proofs follow directly from the definitions. In addition to this, we note that

(h) $\mathbf{AB} \neq \mathbf{BA}$, that is, in general, *matrix multiplication is not commutative*.

(i) $\mathbf{AB} = \mathbf{AC}$, does not necessarily imply that $\mathbf{B} = \mathbf{C}$

(j) $\mathbf{AB} = \mathbf{O}$, does not necessarily imply that $\mathbf{A} = \mathbf{O}$ or $\mathbf{B} = \mathbf{O}$.

For example, if $\mathbf{A} = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$, then **AB** and **BA** both are defined and we can

see easily that $\mathbf{AB} = \begin{bmatrix} 0 & -4 \\ 10 & 3 \end{bmatrix}$ and $\mathbf{BA} = \begin{bmatrix} -10 & 2 & 21 \\ -16 & 2 & 37 \\ -2 & -2 & 11 \end{bmatrix}$; and thus $\mathbf{AB} \neq \mathbf{BA}$.

Also, if $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$, then $\mathbf{AB} = \mathbf{O}$ but neither $\mathbf{A} = \mathbf{O}$, nor $\mathbf{B} = \mathbf{O}$.

We note that if for two matrices **A** and **B**, both the product matrices **AB** and **BA** are defined and if matrix **A** is of order $p \times q$, then matrix **B** must be of the order $q \times p$.

2.2.1 Partitioning of Matrices

The matrices encountered in modern applications may be of quite large order and such large matrices create special computational problems. It is often advantageous to work instead with a number of smaller matrices through the use of partitioning.

Any matrix **A** may be partitioned into a number of submatrices called blocks, by vertical lines that extend from bottom to top and horizontals lines that extend from left to right. There is more than one way in which a matrix can be partitioned. For example, consider a matrix **A** of order 3×3 given as

$$\mathbf{A} = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix}.$$

One way in which this matrix can be partitioned is as follows:

$$\mathbf{A} = \left[\begin{array}{cc|c} 3 & -1 & 2 \\ \hline 1 & 2 & 0 \\ 2 & 1 & 0 \end{array}\right]$$

This can now be written in block matrix form as $\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$, where the submatrices are

$$\mathbf{A}_{11} = [3, -1],\ \mathbf{A}_{12} = [2],\ \mathbf{A}_{21} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \text{ and } \mathbf{A}_{22} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The addition and scalar multiplication of block matrices follow the same rules as those for ordinary matrices but care must be exercised when multiplying block matrices. Consider the partition of the matrix **A** defined above and the matrix **B** of order 3×4 given as

$$\mathbf{B} = \left[\begin{array}{c|ccc} 1 & 2 & 2 & 1 \\ 3 & 1 & 1 & 0 \\ \hline 2 & 3 & 0 & 2 \end{array}\right]$$

which are conformable for the product **AB**, which itself is a 3×4 matrix. If **B** is partitioned as indicated above, then $\mathbf{B} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix}$ where the submatrices are

$$\mathbf{B}_{11} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \mathbf{B}_{12} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \mathbf{B}_{21} = [2], \text{ and } \mathbf{B}_{22} = [3, 0\ 2].$$

Using the definition of the matrix multiplication, we may write the matrix product in the condensed form as

$$\mathbf{AB} = \begin{bmatrix} \mathbf{A}_{11}\mathbf{B}_{11} + \mathbf{A}_{12}\mathbf{B}_{21} & \mathbf{A}_{11}\mathbf{B}_{12} + \mathbf{A}_{12}\mathbf{B}_{22} \\ \mathbf{A}_{21}\mathbf{B}_{11} + \mathbf{A}_{22}\mathbf{B}_{21} & \mathbf{A}_{21}\mathbf{B}_{12} + \mathbf{A}_{22}\mathbf{B}_{22} \end{bmatrix}$$

where the partitioned matrices have been multiplied as though their elements were ordinary numbers. This result holds because of appropriate partitioning, such that each product of submatrices is conformable for multiplication and the matrices sums are conformable for additions. We can check that

$$\mathbf{A}_{11}\,\mathbf{B}_{11} + \mathbf{A}_{12}\,\mathbf{B}_{21} = [4],\ \mathbf{A}_{11}\,\mathbf{B}_{12} + \mathbf{A}_{12}\,\mathbf{B}_{22} = [11 \quad 5 \quad 7]$$

$$\mathbf{A}_{21}\,\mathbf{B}_{11} + \mathbf{A}_{22}\,\mathbf{B}_{21} = \begin{bmatrix} 7 \\ 5 \end{bmatrix},\ \mathbf{A}_{21}\,\mathbf{B}_{12} + \mathbf{A}_{22}\,\mathbf{B}_{22} = \begin{bmatrix} 4 & 4 & 1 \\ 5 & 5 & 2 \end{bmatrix},$$

and so $\quad \mathbf{AB} = \begin{bmatrix} [4] & [11 \quad 5 \quad 7] \\ \begin{bmatrix} 7 \\ 5 \end{bmatrix} & \begin{bmatrix} 4 & 4 & 1 \\ 5 & 5 & 2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 4 & 11 & 5 & 7 \\ 7 & 4 & 4 & 1 \\ 5 & 5 & 5 & 2 \end{bmatrix}$

The result can be confirmed by direct matrix multiplication.

Matrix partitioning is particularly useful in applying the multiplication of matrices if one of these can be partitioned in such a way that some of its submatrices are null matrices. Then the computational time is drastically reduced.

2.3 SPECIAL MATRICES

In this section we shall discuss some special matrices like *symmetric, skewsymmetric, orthogonal* in case of matrices over real, and *Hermitian, skew-Hermitian* and *unitary* in case of matrices over complex number system. Also we shall introduce the *transpose* and *conjugate* of a matrix.

2.3.1 Transpose of a Matrix

If **A** is a matrix of order $m \times n$, then the *transpose* of **A**, denoted by **A**′ or $\mathbf{A}^T$, is obtained by interchanging the rows and columns of the matrix **A**. That is, if $\mathbf{A} = [a_{ij}]_{m \times n}$ is a matrix of order $m \times n$, then the transpose of **A** is the $n \times m$ matrix, $\mathbf{A}^T = [a_{ji}]_{n \times m}$.

For example, if $\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 3 & 3 \\ 4 & 6 & 7 \end{bmatrix}$, then $\mathbf{A}^T = \begin{bmatrix} 2 & -1 & 4 \\ 0 & 3 & 6 \\ 1 & 3 & 7 \end{bmatrix}$

Properties of the Transpose. The basic properties of the transpose are

(a) $(\mathbf{A}^T)^T = \mathbf{A}$
(b) $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$.
(c) $(\alpha\mathbf{A})^T = \alpha\mathbf{A}^T$
(d) $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$,

The proofs follow directly from the definitions.

In general, if $\mathbf{A}_1 \mathbf{A}_2 \ldots \mathbf{A}_k$ is defined then $(\mathbf{A}_1 \mathbf{A}_2 \ldots \mathbf{A}_k)^T = \mathbf{A}_k^T \mathbf{A}_{k-1}^T \ldots \mathbf{A}_1^T$.

2.3.2 Symmetric and Skew-symmetric Matrices

A matrix $\mathbf{A} = [a_{ij}]$ is

(a) *Symmetric*, if $a_{ij} = a_{ji}$ for all i and j, that is $\mathbf{A} = \mathbf{A}^T$.
(b) *Skew-symmetric* or *antisymmetric*, if $a_{ij} = -a_{ji}$ for all i and j, that is, $\mathbf{A} = -\mathbf{A}^T$.

We note that for either of these properties to apply $\mathbf{A}$ must be square, otherwise $\mathbf{A}$ and $\mathbf{A}^T$ will be of different orders. Further, for $\mathbf{A}$ to be skew-symmetric all of its leading diagonal elements must be zero, since $a_{ij} = -a_{ji}$ for $i = j$, gives $2a_{ii} = 0$, that is, $a_{ii} = 0$ for all i. For example, if

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}, \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0 & b & c \\ -b & 0 & f \\ -c & -f & 0 \end{bmatrix}$$

Then, $\mathbf{A}$ is symmetric and $\mathbf{B}$ is skew-symmetric.

Further, *every square matrix A can be expressed as the sum of a symmetric and a skew-symmetric matrix*, for, the square matrix $\mathbf{A}$ can be written as $\mathbf{A} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T) + \frac{1}{2}(\mathbf{A} - \mathbf{A}^T)$, and we can verify that the matrix $\mathbf{B} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$ is symmetric and the matrix $\mathbf{C} = \frac{1}{2}(\mathbf{A} - \mathbf{A}^T)$ is skew-symmetric.

2.3.3 Orthogonal Matrix

A square matrix $\mathbf{A} = [a_{ij}]$ of order n is *orthogonal* matrix, if $\mathbf{AA}^T = I_n$.

For example, the matrix $\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is orthogonal, since

$$\mathbf{AA}^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}_2.$$

Also we note that *if* **A** *and* **B** *are orthogonal matrices, then the product matrices* **AB** *and* **BA** *are also orthogonal matrices.*

2.3.4 Conjugate of a Matrix

If **A** is a complex matrix of order $m \times n$, then the *conjugate of* **A**, denoted by $\overline{\mathbf{A}}$, is obtained by replacing the elements with their corresponding complex conjugates. That is, if $\mathbf{A} = [a_{ij}]$ is a matrix of order $m \times n$, then the conjugate of **A** is the matrix $\overline{\mathbf{A}} = [\bar{a}_{ij}]$, where $\bar{a}_{ij}$ is the complex conjugate of a_{ij}.

We note that *in case of real matrices,* **A** *and its conjugate* $\overline{\mathbf{A}}$ *are the same.*

2.3.5 Hermitian and Skew-Hermitian Matrices

A complex matrix $\mathbf{A} = [a_{ij}]$ is

(a) *Hermitian*, if $a_{ij} = \bar{a}_{ji}$ for all i and j, that is, if $\mathbf{A} = (\overline{\mathbf{A}})^{\mathrm{T}}$

(b) *Skew-Hermitian*, if $a_{ij} = -\bar{a}_{ji}$ for all i and j, that is, if $\mathbf{A} = -(\overline{\mathbf{A}})^{\mathrm{T}}$.

Sometimes Hermitian matrix is denoted by $\mathbf{A}^{\mathrm{H}}$ or $\mathbf{A}^{\theta}$.

We note that for either of these properties to apply **A** must be square, otherwise **A** and $\mathbf{A}^{\mathrm{H}}$ will be of different orders. Further, for **A** to be Hermitian all of its leading diagonal elements must be real and for **A** to be skew-Hermitian all of its leading diagonal elements must be either zero or purely imaginary.

Also, every complex square matrix **A** *can be expressed as the sum of a Hermitian and a skew-Hermitian matrix*, for, the matrix **A** can be written as $\mathbf{A} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^{\mathrm{H}}) + \frac{1}{2}(\mathbf{A} - \mathbf{A}^{\mathrm{H}})$ and we can verify that the matrix $\mathbf{B} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^{\mathrm{H}})$ is Hermitian and the matrix $\mathbf{C} = \frac{1}{2}(\mathbf{A} - \mathbf{A}^{\mathrm{H}})$ is skew-Hermitian.

2.3.6 Unitary Matrix

A complex square matrix $\mathbf{A} = [a_{ij}]$ of order n is *unitary matrix*, if $\mathbf{A}\mathbf{A}^{\mathrm{H}} = \mathbf{I}_n$

For example the matrix $\mathbf{A} = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$ is unitary, since

$$\mathbf{A}\mathbf{A}^{\mathrm{H}} = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix} \begin{bmatrix} \frac{1-i}{2} & \frac{1-i}{2} \\ \frac{-1-i}{2} & \frac{1+i}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}_2.$$

Also we note that *if* **A** *and* **B** *are unitary matrices, then the product matrices* **AB** *and* **BA** *are also unitary matrices.*

Example 2.1: If $\mathbf{A}=\begin{bmatrix}1 & 3 & 0\\ -1 & 2 & 1\\ 0 & 0 & 2\end{bmatrix}$, and $\mathbf{B}=\begin{bmatrix}2 & 3 & 4\\ 1 & 2 & 3\\ -1 & 1 & 2\end{bmatrix}$, compute **AB** and **BA** and show that $\mathbf{AB}\neq\mathbf{BA}$.

Solution: Considering the rows of **A** and columns of **B**, we have

$$\mathbf{AB}=\begin{bmatrix}1.2+3.1+0.-1 & 1.3+3.2+0.1 & 1.4+3.3+0.2\\ -1.2+2.1+1.-1 & -1.3+2.2+1.1 & -1.4+2.3+1.2\\ 0.2+0.1+2.-1 & 0.3+0.3+2.1 & 0.4+0.3+2.2\end{bmatrix}=\begin{bmatrix}5 & 9 & 13\\ -1 & 2 & 4\\ -2 & 2 & 4\end{bmatrix}$$

Again considering the rows of **B** and columns of **A**, we have

$$\mathbf{BA}=\begin{bmatrix}2.1+3.-1+4.0 & 2.3+3.2+4.0 & 2.0+3.1+4.2\\ 1.1+2.-1+3.0 & 1.3+2.2+3.0 & 1.0+2.1+3.2\\ -1.1+1.-1+2.0 & -1.3+1.2+2.0 & -1.0+1.1+2.2\end{bmatrix}=\begin{bmatrix}-1 & 12 & 11\\ -1 & 7 & 8\\ -2 & -1 & 5\end{bmatrix},$$

which is not equal to **AB**.

Example 2.2: If $\mathbf{A}=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$, show that $\mathbf{A}^2-5\mathbf{A}=2\mathbf{I}$, where **I** is the unit matrix of order 2. Hence, determine $\mathbf{A}^4$.

Solution: We have,

$$\mathbf{A}^2=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}=\begin{bmatrix}7 & 10\\ 15 & 22\end{bmatrix}, \text{ and } \quad 5\mathbf{A}=5\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}=\begin{bmatrix}5 & 10\\ 15 & 20\end{bmatrix}$$

Hence, $$\mathbf{A}^2-5\mathbf{A}=\begin{bmatrix}7 & 10\\ 15 & 22\end{bmatrix}-\begin{bmatrix}5 & 10\\ 15 & 20\end{bmatrix}=\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}=2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=2\mathbf{I}.$$

From the above result $\mathbf{A}^2=5\mathbf{A}+2\mathbf{I}$, hence

$$\mathbf{A}^4=\mathbf{A}^2\mathbf{A}^2=(5\mathbf{A}+2\mathbf{I})(5\mathbf{A}+2\mathbf{I})=25\mathbf{A}^2+10\mathbf{AI}+10\mathbf{IA}+4\mathbf{I}^2$$

$$=25\mathbf{A}^2+20\mathbf{A}+4\mathbf{I}, \text{ since } \mathbf{AI}=\mathbf{IA}=\mathbf{A} \text{ and } \mathbf{I}^2=\mathbf{I}$$

$$=25\begin{bmatrix}7 & 10\\ 15 & 22\end{bmatrix}+20\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}+4\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}175 & 250\\ 375 & 550\end{bmatrix}+\begin{bmatrix}20 & 40\\ 60 & 80\end{bmatrix}+\begin{bmatrix}4 & 0\\ 0 & 4\end{bmatrix}=\begin{bmatrix}199 & 290\\ 405 & 634\end{bmatrix}$$

Example 2.3: If **A, B, C** are three matrices such that

$$A = [x\ y\ z],\ B = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \text{ and } C = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \text{ then find } ABC.$$

Solution: Since associative law holds for matrices multiplication, therefore **ABC** can be written as **A**(**BC**), or (**AB**)**C**.

$$\text{Now, } BC = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax + hy + gz \\ hx + by + fz \\ gx + fy + cz \end{bmatrix}, \text{ therefore,}$$

$$A(BC) = [x\ y\ z]\begin{bmatrix} ax + hy + gz \\ hx + by + fz \\ gx + fy + cz \end{bmatrix}$$

$$= [x(ax + hy + gz) + y(hx + by + fz) + z(gx + fy + cz)]$$

$$= [ax^2 + by^2 + cz^2 + 2hxy + 2gzx + 2fyz].$$

Example 2.4: Express the matrix $\begin{bmatrix} 3 & 1 & -2 \\ 2 & 1 & 7 \\ -4 & 5 & 3 \end{bmatrix}$ as the sum of two matrices, one symmetric and one skew-symmetric.

Solution:

$$\text{Let } A = \begin{bmatrix} 3 & 1 & -2 \\ 2 & 1 & 7 \\ -4 & 5 & 3 \end{bmatrix}, \text{ then } A^T = \begin{bmatrix} 3 & 2 & -4 \\ 1 & 1 & 5 \\ -2 & 7 & 3 \end{bmatrix},$$

Consider

$$\frac{1}{2}(A + A^T) = \begin{bmatrix} 3 & 3/2 & -3 \\ 3/2 & 1 & 6 \\ -3 & 6 & 3 \end{bmatrix} = B, \text{ and } \frac{1}{2}(A - A^T) = \begin{bmatrix} 0 & -1/2 & 1 \\ 1/2 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix} = C, \text{ say.}$$

$$\text{Then, } \quad B + C = \begin{bmatrix} 3 & 3/2 & -3 \\ 3/2 & 1 & 6 \\ -3 & 6 & 3 \end{bmatrix} + \begin{bmatrix} 0 & -1/2 & 1 \\ 1/2 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -2 \\ 2 & 1 & 7 \\ -4 & 5 & 3 \end{bmatrix} = A,$$

here we can check that the matrix **B** is symmetric, that is, $B^T = B$ and the matrix **C** is skew-symmetric, that is, $C^T = -C$.

Example 2.5: If the matrix $\begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$ is orthogonal, then find the values of a, b and c.

Solution:

Let $$\mathbf{A} = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}, \text{ then } \mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix}, \text{ and}$$

$$\mathbf{AA}^{\mathrm{T}} = \begin{bmatrix} 0 & 2b & c \\ 0 & b & -c \\ a & -b & c \end{bmatrix}\begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix} = \begin{bmatrix} 4b^2+c^2 & 2b^2-c^2 & -2b^2+c^2 \\ 2b^2-c^2 & a^2+b^2+c^2 & a^2-b^2-c^2 \\ -2b^2+c^2 & a^2-b^2-c^2 & a^2+b^2+c^2 \end{bmatrix}$$

If **A** is orthogonal, then $\mathbf{AA}^{\mathrm{T}} = \mathbf{I}$; thus

$$\begin{bmatrix} 4b^2+c^2 & 2b^2-c^2 & -2b^2+c^2 \\ 2b^2-c^2 & a^2+b^2+c^2 & a^2-b^2-c^2 \\ -2b^2+c^2 & a^2-b^2-c^2 & a^2+b^2+c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equating the corresponding elements, we obtain

$$4b^2 + c^2 = 1, \quad 2b^2 - c^2 = 0, \quad a^2 + b^2 + c^2 = 1.$$

Solving for a, b and c we get, $a = \pm\dfrac{1}{\sqrt{2}}, b = \pm\dfrac{1}{\sqrt{6}}$, and $c = \pm\dfrac{1}{\sqrt{3}}$.

Imp.

Example 2.6: Show that the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is nilpotent matrix of order 3.

Solution: A square matrix **A** such that $\mathbf{A}^p = 0$, but $\mathbf{A}^{p-1} \neq 0$, p being positive integer, is called *nilpotent matrix of order p*. We have

$$\mathbf{A}^2 = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 1+5-6 & 1+2-3 & 3+6-9 \\ 5+10-12 & 5+4-6 & 15+12-18 \\ -2-5+6 & -2-2+3 & -6-6+9 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix} \neq \mathbf{0}, \text{ and}$$

$$\mathbf{A}^3 = \mathbf{A}.\mathbf{A}^2 = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0+3-3 & 0+3-3 & 0+9-9 \\ 0+6-6 & 0+6-6 & 0+18-18 \\ 0-3+3 & 0-3+3 & 0-9+9 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{0}.$$

Thus **A** is nilpotent of order 3.

EXERCISE 2.1

1. If $\mathbf{A} = \begin{bmatrix} 1 & -2 & 1 & 7 & -9 \\ 8 & 2 & -5 & 0 & 0 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} -5 & 1 & 8 & 21 & 7 \\ 12 & -6 & -2 & -1 & 9 \end{bmatrix}$, find $4\mathbf{A} + 8\mathbf{B}$.

2. If $\mathbf{A} = \begin{bmatrix} -4 & 6 & 2 \\ -2 & -2 & 3 \\ 1 & 1 & 8 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} -2 & 4 & 6 & 12 & 5 \\ -3 & -3 & 1 & 1 & 4 \\ 0 & 0 & 1 & 6 & -9 \end{bmatrix}$, determine which of $\mathbf{AB}$ and $\mathbf{BA}$ is defined.

   Also carry out that product.

3. If $\mathbf{A} = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$, prove that $\mathbf{A}^3 - 4\mathbf{A}^2 - 3\mathbf{A} + 11\mathbf{I} = 0$

4. Express the matrix $\begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 1 \\ 3 & 5 & 7 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix.

5. If $\mathbf{A} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, show that $\mathbf{A}^n = \begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}$

6. Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ symmetric matrices.
   (a) Give an example to show that $\mathbf{AB}$ need not be symmetric.
   (b) Prove that $\mathbf{AB}$ is symmetric if, and only if $\mathbf{AB} = \mathbf{BA}$.

7. If $\mathbf{A}$ and $\mathbf{B}$ are square matrices of the same order and $\mathbf{A}$ is symmetric, then show that $\mathbf{B}'\mathbf{AB}$ is also symmetric.

8. If $\mathbf{A}$ and $\mathbf{B}$ are symmetric matrices, then prove that $\mathbf{AB} - \mathbf{BA}$ is skew- symmetric.

9. If $\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ 4 & 5 & 0 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$, verify that $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$

10. Show that the following matrices are orthogonal

    (a) $\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$ (b) $\begin{bmatrix} -2/3 & 1/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix}$

11. If $\mathbf{A}$ is Hermitian, is $i\mathbf{A}$ also Hermitian? Explain. What about $\mathbf{A}^2$? Is it Hermitian?

12. Check which of the following matrices are Hermitian

(a) $\begin{bmatrix} 3 & 1+4i \\ 1-4i & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 2+i & 0 & 3-5i \\ 7 & 1 & 4i \\ 2 & i & 3 \end{bmatrix}$

13. Show that the matrix $\begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix}$ is nilpotent and find its order.

14. Show that the matrix $\begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} & 0 \\ \frac{1+i}{2} & \frac{1-i}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is unitary matrix.

15. Write the matrix $\begin{bmatrix} 1+i & 3+i & 3+2i \\ -1+3i & 2 & 4+i \\ -3-2i & 2+3i & 4+2i \end{bmatrix}$ as the sum of a Hermitian and a skew-Hermitian matrix.

16. Classify the following matrices as orthogonal, Hermitian, skew-Hermitian or unitary.

(a) $\begin{bmatrix} 1 & 2i & -3 \\ -2i & 2 & 1+4i \\ -3 & 1-4i & 3 \end{bmatrix}$ (b) $\frac{1}{\sqrt{2}}\begin{bmatrix} e^{i\theta} & e^{i\left(\frac{\pi}{2}+\theta\right)} \\ e^{i\left(\frac{\pi}{2}-\theta\right)} & -e^{i(\pi-\theta)} \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1+i & 2+i \\ -1+i & 0 & -2+3i \\ -2+i & 2+3i & 0 \end{bmatrix}$

17. If $\mathbf{U}$ is a unitary matrix, then show that $\overline{\mathbf{U}}$, $\mathbf{U}^T$ and $\mathbf{U}^n$ (n, a positive integer) are also **unitary** matrices.

2.4 DETERMINANTS

In this section we introduce a scalar quantity associated with every square matrix, called the *determinant* of the matrix. In addition to their numerous applications, determinants play a key role in the theory of system of linear algebraic equations.

2.4.1 Some Basic Definitions

Determinant. If $\mathbf{A}$ is a square matrix of order n, then *determinant* of $\mathbf{A}$, denoted by **det A**, or $|\mathbf{A}|$, is defined as

$$|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \quad \text{...(2.2)}$$

The determinant has a value which is real, if the matrix is real, and may be real or complex, if the matrix is complex. The vertical bars are used to distinguish **det A**, which is a number, from the matrix **A**, which is an $n \times n$ array of numbers.

Consider a $|\mathbf{A}|$ of order 2×2 given by $|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$.

Its value is given by $|\mathbf{A}| = a_{11}a_{22} - a_{12}a_{21}$. For example,

$$\begin{vmatrix} 3 & -1 \\ 2 & 6 \end{vmatrix} = (3)(6) - 2(-1) = 18 + 2 = 20, \quad \text{and} \quad \begin{vmatrix} 1+i & i \\ -3i & 2 \end{vmatrix} = (1+i)2 - (-3i)\,i = -1 + 2i.$$

Minors and cofactors. The *minor* $\mathbf{M}_{ij}$, associated with the element a_{ij} in the ith row and jth column of the nth order **det A**, is the determinant of order $n-1$ obtained from det **A** by deleting the elements in the ith row and jth column.

The *cofactor* $\mathbf{A}_{ij}$ associated with the element a_{ij} in det **A** is defined in terms of the minor $\mathbf{M}_{ij}$, as $\mathbf{A}_{ij} = (-1)^{i+j}\mathbf{M}_{ij}$, for $i, j = 1, 2, \ldots, n$. For example, if

$$|\mathbf{A}| = \begin{vmatrix} 4 & 7 & -2 \\ 0 & 3 & 2 \\ 1 & 5 & 6 \end{vmatrix}, \text{then } \mathbf{M}_{11} = \begin{vmatrix} 3 & 2 \\ 5 & 6 \end{vmatrix}, \mathbf{M}_{22} = \begin{vmatrix} 4 & -2 \\ 1 & 6 \end{vmatrix}, \mathbf{M}_{32} = \begin{vmatrix} 4 & -2 \\ 0 & 2 \end{vmatrix}$$

and, $$\mathbf{A}_{11} = (-1)^{1+1}\begin{vmatrix} 3 & 2 \\ 5 & 6 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 5 & 6 \end{vmatrix}, \mathbf{A}_{22} = (-1)^{2+2}\begin{vmatrix} 4 & -2 \\ 1 & 6 \end{vmatrix} = \begin{vmatrix} 4 & -2 \\ 1 & 6 \end{vmatrix}, \mathbf{A}_{32} = (-1)^{3+2}\begin{vmatrix} 4 & -2 \\ 0 & 2 \end{vmatrix} = -\begin{vmatrix} 4 & -2 \\ 0 & 2 \end{vmatrix}$$

We note that a *determinant of order n has n^2 minors and corresponding number of cofactors.*

2.4.2 Expansion of a Determinant

A determinant of order n can be expanded through the elements of any row or any column and *the value of the determinant is the sum of the products of the element of the ith row (or, the jth column) and their corresponding co-factors,* thus

$$|\mathbf{A}| = \sum_{j=1}^{n} a_{ij}\mathbf{A}_{ij} \quad \left(\text{or, } \sum_{i=1}^{n} a_{ij}\mathbf{A}_{ij}\right)$$

In terms of minors it is $$|\mathbf{A}| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij}\mathbf{M}_{ij} \quad \left(\text{or, } \sum_{i=1}^{n} (-1)^{i+j} a_{ij}\mathbf{M}_{ij}\right)$$

Let $\det A = \begin{vmatrix} 0 & 2 & -1 \\ 4 & 3 & 5 \\ 2 & 0 & -4 \end{vmatrix}$, then by definition, $\det A = \sum_{j=1}^{3} a_{ij}\,\mathbf{A}_{ij}$

For $i = 1$, $\mathbf{det\ A} = a_{11}\,\mathbf{A}_{11} + a_{12}\,\mathbf{A}_{12} + a_{13}\,\mathbf{A}_{13} = a_{11}\mathbf{M}_{11} - a_{12}\mathbf{M}_{12} + a_{13}\mathbf{M}_{13}$

$$= (0)\begin{vmatrix} 3 & 5 \\ 0 & -4 \end{vmatrix} - (2)\begin{vmatrix} 4 & 5 \\ 2 & -4 \end{vmatrix} + (-1)\begin{vmatrix} 4 & 3 \\ 2 & 0 \end{vmatrix} = 0 - 2(-16 - 10) - (0 - 6) = 58.$$

We can expand the determinant through any row or column, the value remains the same. But, in general, it is convenient to choose that row or column which contains most zeros in it. In fact we note that if all the elements of any row or column are zero, then $\mathbf{det\ A} = 0$. If determinant of a square matrix **A** is zero, matrix is said to be *singular*, otherwise *non-singular*.

Also we note that, *the sum of the products of the elements of any row (or, column) with the corresponding cofactors of any other row (or, column) is zero.* Thus, we have the following results:

$$\sum_{k=1}^{n} a_{ik}\,\mathbf{A}_{jk} = \begin{cases} |\mathbf{A}|, & i = j \\ 0, & i \neq j \end{cases} \qquad \ldots(2.3)$$

$$\sum_{k=1}^{n} a_{ki}\,\mathbf{A}_{kj} = \begin{cases} |\mathbf{A}|, & i = j \\ 0, & i \neq j \end{cases} \qquad \ldots(2.4)$$

We observe that for large n the cofactor expansion process is very laborious and time-consuming even with the fast computers. However the scientific calculations, in general, involve determinant of higher orders. Next, we study the various properties of determinants which can be used to simplify the evaluation of determinants.

2.4.3 Properties of Determinants

Following are some important properties satisfied by the determinants.

1. *A determinant remains unaltered by changing its rows into columns and columns into rows, that is,* $|A| = |A^T|$.

2. *If any two rows (columns) of a determinant are interchanged, then the numerical value of the determinant remains unchanged but changes in sign.*

In general, if any row (column) is shifted over p rows (column), then the value of the resulting determinant is $(-1)^p$ times the original determinant.

3. *If the corresponding elements of any two rows (columns) of a determinant are the same, then determinant vanishes.*

4. *If each element of a row (column) is multiplied by the same scalar, then the value of the determinant is multiplied by the same scalar.*

As a consequence to this, if α is a common factor of each element of a row (column) then the factor α can be taken out of the determinant. But we must note that when we multiply a matrix by a scalar α, then every element of the matrix is multiplied by α, and therefore $|\alpha\,\mathbf{A}| = \alpha^n\,|\mathbf{A}|$, where **A** is a square matrix of order n.

5. *If each element of a row (column) can be expressed as sum of two or more terms, then the determinant can be expressed as the sum of two or more determinants.*

6. *If a non-zero constant multiple of the elements of any row (column) are added to the corresponding elements of some other row (column), then the value of the determinant remains unchanged.*

In case the elements of the jth row are multiplied by a constant $k \neq 0$ and are added to the corresponding element of the ith row, then this operation is symbolized as $R_i \to R_i + kR_j$ and is called the *elementary row operation* Similarly, corresponding column operation is symbolized as $C_i \to C_i + kC_j$, and is called the *elementary column operation.*

7. *If the elements of a determinant are functions of x and two rows (columns) become identical when x = a, then x – a is a factor of the value of the determinant.*

For example, if $|\mathbf{A}| = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$, then **det A** vanishes for $x = y$, thus, $(x - y)$ is a factor of the value of the **det A**. Similarly, $(y - z)$, $(z - x)$ are factors of **det A** and, in fact, we can verify that $|\mathbf{A}| = (x - y)(y - z)(z - x)$.

This result is known as *factor theorem on determinants.*

8. *The determinants of diagonal, lower triangular and upper triangular matrices are product of their principal diagonal elements.*

Addition and multiplication of two determinants The two determinants can be added or multiplied only when they are of the same order and these operations are performed on the lines of matrices addition and multiplication. Since the value of a determinant does not change by interchanging the rows and columns, so multiplication can be carried out row by row multiplication rule, or column by column multiplication rule. Further, in general,

$$\det(\mathbf{A} + \mathbf{B}) \neq \det \mathbf{A} + \det \mathbf{B}, \qquad \ldots(2.5)$$

but,

$$\det(\mathbf{AB}) = (\det \mathbf{A})(\det \mathbf{B}). \qquad \ldots(2.6)$$

Example 2.7: Evaluate (a) $\begin{vmatrix} 1 & 4 & 9 & 16 \\ 4 & 9 & 16 & 25 \\ 9 & 16 & 25 & 36 \\ 16 & 25 & 36 & 49 \end{vmatrix}$ (b) $\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{vmatrix}$

Solution: (a) Let $\Delta = \begin{vmatrix} 1 & 4 & 9 & 16 \\ 4 & 9 & 16 & 25 \\ 9 & 16 & 25 & 36 \\ 16 & 25 & 36 & 49 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 9 & 16 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 11 \\ 7 & 9 & 11 & 13 \end{vmatrix}$, $[R_2 \to R_2 + (-1)R_1,$
$R_3 \to R_3 + (-1)R_2,$
$R_4 \to R_4 + (-1)R_3]$

$$= \begin{vmatrix} 1 & 4 & 9 & 16 \\ 3 & 5 & 7 & 9 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{vmatrix} \quad [R_3 \to R_3 + (-1)R_2, \; R_4 \to R_4 + (-1)R_3]$$

$= 0$, since R_3 and R_4 are identical.

(b) Since there are two zeros in the second row, therefore, expanding by this row, we get

$$\Delta = -\begin{vmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix} + 0 - 3\begin{vmatrix} 0 & 1 & 3 \\ 2 & 3 & 1 \\ 3 & 0 & 2 \end{vmatrix} + 0$$

$$= -\left[1\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} + 0\right] - 3\left[0 - 2\begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix} + 3\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix}\right]$$

(Expanding by C_1) (Expanding by C_1)

$= -[(0-1) - 3(4-3)] - 3[-2(2-0) + 3(1-9)] = 4 + 84 = 88.$

Example 2.8: Evaluate (a) $\begin{vmatrix} b+c & a+b & a \\ c+a & b+c & b \\ a+b & c+a & c \end{vmatrix}$ (b) $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix}$

Solution: (a) Let $\Delta = \begin{vmatrix} b+c & a+b & a \\ c+a & b+c & b \\ a+b & c+a & c \end{vmatrix}$

$$= \begin{vmatrix} a+b+c & a+b & a \\ a+b+c & b+c & b \\ a+b+c & c+a & c \end{vmatrix}, \quad [C_1 \to C_1 + C_3]$$

$$= (a+b+c)\begin{vmatrix} 1 & a+b & a \\ 1 & b+c & b \\ 1 & c+a & c \end{vmatrix} \quad \left[C_1 \to \frac{1}{a+b+c}C_1\right]$$

$$= (a+b+c)\begin{vmatrix} 1 & a+b & a \\ 0 & c-a & b-a \\ 0 & c-b & c-a \end{vmatrix}, \quad [R_2 \to R_2 - R_1; R_3 \to R_3 - R_1]$$

$= (a+b+c)\,[(c-a)^2 - (c-b)(b-a)]$ [expanding by C_1]
$= (a+b+c)\,[(c^2 + a^2 - 2ca) - (cb - ca - b^2 + ba)]$
$= (a+b+c)\,(a^2 + b^2 + c^2 - bc - ca - ab) = a^3 + b^3 + c^3 - 3\,abc.$

(b) Let
$$\Delta = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} = \frac{1}{xyz}\begin{vmatrix} x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \\ xyz & xyz & xyz \end{vmatrix}, \quad [C_1 \to xC_1, C_2 \to yC_2, C_3 \to zC_3]$$

$$= \frac{xyz}{xyz}\begin{vmatrix} x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \\ 1 & 1 & 1 \end{vmatrix}, \quad [R_3 \to \frac{1}{xyz}R_3]$$

$$= (-1)^2\begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}, \quad [R_2 \leftrightarrow R_3 \text{ and then } R_1 \leftrightarrow R_2]$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ x^2 & y^2-x^2 & z^2-x^2 \\ x^3 & y^3-x^3 & z^3-x^3 \end{vmatrix}, \quad [C_2 \to C_2 - C_1; C_3 \to C_3 \to C_1]$$

$$= \begin{vmatrix} y^2-x^2 & z^2-x^2 \\ y^3-x^3 & z^3-x^3 \end{vmatrix}, \quad [\text{Expanding by } R_1]$$

$$= (y-x)(z-x)\begin{vmatrix} y+x & z+x \\ y^2+xy+x^2 & z^2+zx+x^2 \end{vmatrix}, \quad [C_1 \to \frac{1}{y-x}C_1, C_2 \to \frac{1}{z-x}C_2]$$

$$= (y-x)(z-x)\begin{vmatrix} y+x & z-y \\ y^2+xy+x^2 & (z^2-y^2)+(z-y)x \end{vmatrix}, \quad [C_2 \to C_2 - C_1]$$

$$= (y-x)(z-x)(z-y)\begin{vmatrix} y+x & 1 \\ y^2+xy+x^2 & x+y+z \end{vmatrix}, \quad [C_2 \to \frac{1}{z-y}C_2]$$

$$= (y-x)(z-x)(z-y)[(y+x)(x+y+z) - (y^2+xy+x^2)]$$

$$= (x-y)(y-z)(z-x)(xy+yz+zx).$$

Example 2.9: Prove that (a) $\begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix} = (x-a)^3(x+3a)$

(b) $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3.$

Solution:

(a) Let $$\Delta = \begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix} = \begin{vmatrix} x+3a & a & a & a \\ x+3a & x & a & a \\ x+3a & a & x & a \\ x+3a & a & a & x \end{vmatrix}, \quad [C_1 \to C_1 + C_2 + C_3 + C_3]$$

$$= (x+3a)\begin{vmatrix} 1 & a & a & a \\ 1 & x & a & a \\ 1 & a & x & a \\ 1 & a & a & x \end{vmatrix} \quad [C_1 \to \frac{1}{x+3a} C_1]$$

$$= (x+3a)\begin{vmatrix} 1 & a & a & a \\ 0 & x-a & 0 & 0 \\ 0 & 0 & x-a & 0 \\ 0 & 0 & 0 & x-a \end{vmatrix}, \quad [R_2 \to R_2 - R_1, R_3 \to R_3 - R_1, R_4 \to R_4 - R_1]$$

$$= (x + 3a)\,(x - a)^3,$$

since the determinant of an upper triangular matrix is the product of its principal diagonal elements.

(b) Let $$\Delta = \begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = \begin{vmatrix} 2(a+b+c) & a & b \\ 2(a+b+c) & b+c+2a & b \\ 2(a+b+c) & a & c+a+2b \end{vmatrix}, \quad [C_1 \to C_1 + C_2 + C_3]$$

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}, \quad [C_1 \to \frac{1}{2(a+b+c)} C_1]$$

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 0 & b+c+a & 0 \\ 0 & 0 & c+a+b \end{vmatrix}, \quad [R_2 \to R_2 - R_1; R_3 \to R_3 - R_1]$$

$$= 2(a+b+c)^3.$$

Example 2.10: Without actual expansion, show that $$\begin{vmatrix} 1 & a & a^2 & a^3+bcd \\ 1 & b & b^2 & b^3+cda \\ 1 & c & c^2 & c^3+dab \\ 1 & d & d^2 & d^3+abc \end{vmatrix} = 0.$$

Solution:

Let $\Delta = \begin{vmatrix} 1 & a & a^2 & a^3+bcd \\ 1 & b & b^2 & b^3+cda \\ 1 & c & c^2 & c^3+dab \\ 1 & d & d^2 & d^3+abc \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} + \begin{vmatrix} 1 & a & a^2 & bcd \\ 1 & b & b^2 & cda \\ 1 & c & c^2 & dab \\ 1 & d & d^2 & abc \end{vmatrix} = \Delta_1 + \Delta_2$, say.

Consider, $\Delta_2 = \begin{vmatrix} 1 & a & a^2 & bcd \\ 1 & b & b^2 & cda \\ 1 & c & c^2 & dab \\ 1 & d & d^2 & abc \end{vmatrix} = \dfrac{1}{abcd}\begin{vmatrix} a & a^2 & a^3 & abcd \\ b & b^2 & b^3 & abcd \\ c & c^2 & c^3 & abcd \\ d & d^2 & d^3 & abcd \end{vmatrix}$

$$= \begin{vmatrix} a & a^2 & a^3 & 1 \\ b & b^2 & b^3 & 1 \\ c & c^2 & c^3 & 1 \\ d & d^2 & d^3 & 1 \end{vmatrix} = (-1)^3 \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} = -\Delta_1$$

Therefore, $\Delta = 0$.

Example 2.11: Without actual expansion, show that $\begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix} = 0.$

Solution:

Let $\Delta = \begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix}$

$$= \begin{vmatrix} \cos A\cos P+\sin A\sin P & \cos A\cos Q+\sin A\sin Q & \cos A\cos R+\sin A\sin R \\ \cos B\cos P+\sin B\sin P & \cos B\cos Q+\sin B\sin Q & \cos B\cos R+\sin B\sin R \\ \cos C\cos P+\sin C\sin P & \cos C\cos Q+\sin C\sin Q & \cos C\cos R+\sin C\sin R \end{vmatrix}$$

$$= \begin{vmatrix} \cos A & \sin A & 0 \\ \cos B & \sin B & 0 \\ \cos C & \sin C & 0 \end{vmatrix}\begin{vmatrix} \cos P & \sin P & 0 \\ \cos Q & \sin Q & 0 \\ \cos R & \sin R & 0 \end{vmatrix} = 0 \times 0 = 0.$$

Example 2.12: Solve the equation $\begin{vmatrix} a+x & b+x & c+x \\ b+x & c+x & a+x \\ c+x & a+x & b+x \end{vmatrix} = 0$, when $a \neq b \neq c$.

Solution: The equation is $\begin{vmatrix} a+x & b+x & c+x \\ b+x & c+x & a+x \\ c+x & a+x & b+x \end{vmatrix} = 0$

or, $\begin{vmatrix} a+b+c+3x & b+x & c+x \\ a+b+c+3x & c+x & a+x \\ a+b+c+3x & a+x & b+x \end{vmatrix} = 0, \quad [C_1 \to C_1 + C_2 + C_3]$

or, $(a+b+c+3x)\begin{vmatrix} 1 & b+x & c+x \\ 1 & c+x & a+x \\ 1 & a+x & b+x \end{vmatrix} = 0, \quad [C_1 \to \dfrac{1}{a+b+c+3x} C_1]$

or, $(a+b+c+3x)\begin{vmatrix} 1 & b+x & c+x \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix} = 0, \quad [R_2 \to R_2 - R_1, R_3 \to R_3 - R_1]$

or, $(a+b+c+3x)\,[(c-b)(b-c) - (a-b)(a-c)] = 0$

or, $(a+b+c+3x)\,[(b-c)^2 + (a-b)(a-c)] = 0$

or, $(a+b+c+3x)\,[a^2 + b^2 + c^2 - ab - bc - ca] = 0$

or, $\frac{1}{2}(a+b+c+3x)\,[(a-b)^2 + (b-c)^2 + (c-a)^2] = 0$

or, $x = -\frac{1}{3}(a+b+c)$, since $(a-b)^2 + (b-c)^2 + (c-a)^2$ is non-negative.

Example 2.13: If a, b, c are all different and $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$, show that $abc = -1$.

Solution:

$$\text{Let } \Delta = \begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$$

$$= (-1)^2\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (1 + abc)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= (1 + abc)\begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} \quad [R_2 \to R_2 - R_1, R_3 \to R_3 - R_1]$$

$$= (1 + abc)\begin{vmatrix} b-a & (b-a)(b+a) \\ c-a & (c-a)(c+a) \end{vmatrix} = (1 + abc)(b - a)(c - a)\begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix}$$

$$= (1 + abc)(b - a)(c - a)(c - b) = (1 + abc)(a - b)(b - c)(c - a).$$

Since $\Delta = 0$ and $a \neq b \neq c$, therefore, $1 + abc = 0$, that is, $abc = -1$.

Example 2.14: Show that $\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = 4(a + b)(b + c)(c + a)$

Solution: Let $\Delta = \begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix}$. Put $a = -b$, it becomes

$$\begin{vmatrix} 2b & 0 & c-b \\ 0 & -2b & b+c \\ c-b & c+b & -2c \end{vmatrix} = \begin{vmatrix} 2b & 0 & c+b \\ 0 & -2b & c+b \\ c-b & c+b & -(c+b) \end{vmatrix} \quad [C_3 \to C_3 + C_1]$$

$$= (c+b)\begin{vmatrix} 2b & 0 & 1 \\ 0 & -2b & 1 \\ c-b & c+b & -1 \end{vmatrix} = (c+b)\begin{vmatrix} c+b & c+b & 0 \\ c-b & c-b & 0 \\ c-b & c+b & -1 \end{vmatrix} \quad [R_1 \to R_1 + R_3, R_2 \to R_2 + R_3]$$

$$= (c + b)^2 (c - b)\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ c-b & c+b & -1 \end{vmatrix} = 0$$

Therefore $(a + b)$ is a factor of Δ. Similarly $(b + c)$, $(c + a)$ are factors of Δ. Since Δ is of degree 3 in a, b, c, therefore any other factor of Δ must be independent of a, b and c. Thus,

$$\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = k(a + b)(b + c)(c + a),$$

where k is a constant. To evaluate k, we put arbitrary values for a, b and c, say $a = b = 1$ and $c = 0$, we get

$$\begin{vmatrix} -2 & 2 & 1 \\ 2 & -2 & 1 \\ 1 & 1 & 0 \end{vmatrix} = k(2)\,(1)\,(1).$$

Simplifying, we get $2k = 8$, that is, $k = 4$.

Example 2.15: Show that $\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2$, where the capital letters denote the cofactor of the corresponding small letters.

Solution: Consider $\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$= \begin{vmatrix} a_1A_1 + b_1B_1 + c_1C_1 & a_2A_1 + b_2B_1 + c_2C_1 & a_3A_1 + b_3B_1 + c_3C_1 \\ a_1A_2 + b_1B_2 + c_1C_2 & a_2A_2 + b_2B_2 + c_2C_2 & a_3A_2 + b_3B_2 + c_3C_2 \\ a_1A_3 + b_1B_3 + c_1C_3 & a_2A_3 + b_2B_3 + c_2C_3 & a_3A_3 + b_3B_3 + c_3C_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1A_1 + b_1B_1 + c_1C_1 & 0 & 0 \\ 0 & a_2A_2 + b_2B_2 + c_2C_3 & 0 \\ 0 & 0 & a_3A_3 + b_3B_3 + c_3C_3 \end{vmatrix}$$

$$= \begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix} = \Delta^3, \text{ where } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Thus, $$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2$$

Example 2.16: Show that $\begin{vmatrix} a^2+\lambda & ab & ac & ad \\ ab & b^2+\lambda & bc & bd \\ ac & bc & c^2+\lambda & cd \\ ad & bd & cd & d^2+\lambda \end{vmatrix} = \lambda^3\,(a^2 + b^2 + c^2 + d^2 + \lambda)$.

Solution:

Let $$\Delta = \begin{vmatrix} a^2+\lambda & ab & ac & ad \\ ab & b^2+\lambda & bc & bd \\ ac & bc & c^2+\lambda & cd \\ ad & bd & cd & d^2+\lambda \end{vmatrix}$$

$$= abcd \begin{vmatrix} a+\frac{\lambda}{a} & b & c & d \\ a & b+\frac{\lambda}{b} & c & d \\ a & b & c+\frac{\lambda}{c} & d \\ a & b & c & d+\frac{\lambda}{d} \end{vmatrix} \left[R_1 \to \frac{1}{a}R_1, R_2 \to \frac{1}{b}R_2, R_3 \to \frac{1}{c}R_3, R_4 \to \frac{1}{d}R_4\right]$$

$$= \frac{abcd}{abcd} \begin{vmatrix} a^2+\lambda & b^2 & c^2 & d^2 \\ a^2 & b^2+\lambda & c^2 & d^2 \\ a^2 & b^2 & c^2+\lambda & d^2 \\ a^2 & b^2 & c^2 & d^2+\lambda \end{vmatrix} [C_1 \to aC_1, C_2 \to bC_2, C_3 \to cC_3, C_4 \to dC_4]$$

$$= \begin{vmatrix} a^2+b^2+c^2+d^2+\lambda & b^2 & c^2 & d^2 \\ a^2+b^2+c^2+d^2+\lambda & b^2+\lambda & c^2 & d^2 \\ a^2+b^2+c^2+d^2+\lambda & b^2 & c^2+\lambda & d^2 \\ a^2+b^2+c^2+d^2+\lambda & b^2 & c^2 & d^2+\lambda \end{vmatrix} [C_1 \to C_1 + C_2 + C_3 + C_4]$$

$$= (a^2+b^2+c^2+d^2+\lambda) \begin{vmatrix} 1 & b^2 & c^2 & d^2 \\ 1 & b^2+\lambda & c^2 & d^2 \\ 1 & b^2 & c^2+\lambda & d^2 \\ 1 & b^2 & c^2 & d^2+\lambda \end{vmatrix} \left[C_1 \to \frac{1}{a^2+b^2+c^2+\lambda}C_1\right]$$

$$= (a^2+b^2+c^2+d^2+\lambda) \begin{vmatrix} 1 & b^2 & c^2 & d^2 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{vmatrix} [R_2 \to R_2 - R_1, R_3 - R_1, R_4 \to R_4 - R_1]$$

$= \lambda^3(a^2+b^2+c^2+d^2+\lambda)$, since the value of the triangular determinant is the product of its principal diagonal elements.

Example 2.17: Prove that

$$\begin{vmatrix} (a-x)^2 & (b-x)^2 & (c-x)^2 \\ (a-y)^2 & (b-y)^2 & (c-y)^2 \\ (a-z)^2 & (b-z)^2 & (c-z)^2 \end{vmatrix} = 2(a-b)(b-c)(c-a)(x-y)(y-z)(z-x)$$

Solution:

$$\text{Let} \quad \Delta = \begin{vmatrix} a^2-2ax+x^2 & b^2-2bx+x^2 & c^2-2cx+x^2 \\ a^2-2ay+y^2 & b^2-2by+y^2 & c^2-2cy+y^2 \\ a^2-2az+z^2 & b^2-2bz+z^2 & c^2-2cz+z^2 \end{vmatrix} = \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} \begin{vmatrix} 1 & -2x & x^2 \\ 1 & -2y & y^2 \\ 1 & -2z & z^2 \end{vmatrix}$$

$$= (-1)^2 \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} (-2) \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 2 \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$= 2(a-b)(b-c)(c-a)(x-y)(y-z)(z-x).$$

Alternatively, this result can be proved using *factor theorem* also as follows:

Substituting $a = b$ in Δ, the first and second columns become equal and hence $\Delta = 0$, thus $(a-b)$ is a factor of Δ. Similarly, $(b-c)$, $(c-a)$, $(x-y)$, $(y-z)$ and $(z-x)$ are factors of Δ. Also since Δ is a determinant of degree 6, thus any other factor must be independent of a, b, c, x, y, and z and therefore,

$$\Delta = \begin{vmatrix} (a-x)^2 & (b-x)^2 & (c-x)^2 \\ (a-y)^2 & (b-y)^2 & (c-y)^2 \\ (a-z)^2 & (b-z)^2 & (c-z)^2 \end{vmatrix} = k(a-b)(b-c)(c-a)(x-y)(y-z)(z-x)$$

where k is a constant. To evaluate k, we put arbitrary values for x, y, z, a, b, c, say $x = 0, y = 1, z = -1$, $a = 0, b = 1, c = -1$, we get

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 4 \\ 1 & 4 & 0 \end{vmatrix} = k(-1)(2)(-1)(-1)(2)(-1)$$

Simplifying it gives $8 = 4k$, or $k = 2$. Thus,

$$\begin{vmatrix} (a-x)^2 & (b-x)^2 & (c-x)^2 \\ (a-y)^2 & (b-y)^2 & (c-y)^2 \\ (a-z)^2 & (b-z)^2 & (c-z)^2 \end{vmatrix} = 2(a-b)(b-c)(c-a)(x-y)(y-z)(z-x).$$

EXERCISE 2.2

1. For $\mathbf{A} = \begin{bmatrix} -3 & 0 & 4 \\ 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$, verify $|\mathbf{AB}| = |\mathbf{A}|\,|\mathbf{B}|$.

2. Without actual expansion prove that the following determinants vanish.

(a) $\begin{vmatrix} 21 & 17 & 7 & 10 \\ 24 & 22 & 6 & 10 \\ 6 & 8 & 2 & 3 \\ 5 & 7 & 1 & 2 \end{vmatrix}$ (b) $\begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix}$ (c) $\begin{vmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{vmatrix}$

3. Evaluate

(a) $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 3 & 4 \\ 1 & 2 & 4 & 4 \\ 1 & 2 & 3 & 5 \end{vmatrix}$ (b) $\begin{vmatrix} 3 & -2 & 1 & 2 \\ 2 & 3 & -2 & 4 \\ 3 & 2 & 3 & 4 \\ -2 & 4 & 0 & 5 \end{vmatrix}$

Prove the results in Problems **(4-10)** without a direct expansion of the determinant by using the properties of the determinants.

4. $\begin{vmatrix} 1+a & a & a \\ b & 1+b & b \\ c & c & 1+c \end{vmatrix} = (1 + a + b + c)$

5. $\begin{vmatrix} x^2+a^2 & ab & ac \\ ab & x^2+b^2 & bc \\ ac & cb & x^2+c^2 \end{vmatrix} = x^4(x^2 + a^2 + b^2 + c^2)$

6. $\begin{vmatrix} k & 1 & 1 & 1 \\ 1 & k & 1 & 1 \\ 1 & 1 & k & 1 \\ 1 & 1 & 1 & k \end{vmatrix} = (k + 3)\,(k - 1)^3.$

7. $\begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 \\ \alpha^3 & \beta^3 & \gamma^3 & \delta^3 \end{vmatrix} = (\alpha - \beta)\,(\alpha - \gamma)\,(\alpha - \delta)\,(\beta - \gamma)\,(\beta - \delta)\,(\gamma - \delta).$

8. $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (a+c)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc\,(a+b+c)^3.$

9. $\begin{vmatrix} 4 & 5 & 6 & x \\ 5 & 6 & 7 & y \\ 6 & 7 & 8 & z \\ x & y & z & 0 \end{vmatrix} = (x-2y+z)^2.$ 10. $\begin{vmatrix} a & b & a & a \\ a & b & b & b \\ b & b & b & a \\ a & a & b & a \end{vmatrix} = -(a-b)^4.$

11. Solve the following equations:

(a) $\begin{vmatrix} x+1 & 2x+1 & 3x+1 \\ 2x & 4x+3 & 6x+3 \\ 4x+1 & 6x+4 & 8x+4 \end{vmatrix} = 0$ (b) $\begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 2x+3 & 3x+4 & 4x+5 \\ 3x+5 & 5x+8 & 10x+17 \end{vmatrix} = 0$

12. Prove that if a, b, c are all different and $\begin{vmatrix} a & a^3 & a^4-1 \\ b & b^3 & b^4-1 \\ c & c^3 & c^4-1 \end{vmatrix} = 0$, then $(abc)(bc+ca+ab) = a+b+c$.

13. Show that

(a) $\begin{vmatrix} \sin^2 A & \sin A \cos A & \cos^2 A \\ \sin^2 B & \sin B \cos B & \cos^2 B \\ \sin^2 C & \sin C \cos C & \cos^2 C \end{vmatrix} = -\sin(A-B)\sin(B-C)\sin(C-A)$

(b) $\begin{vmatrix} \sin(a+\alpha) & \sin(b+\alpha) & \sin(c+\alpha) \\ \sin(a+\beta) & \sin(b+\beta) & \sin(c+\beta) \\ \sin(a+\gamma) & \sin(b+\gamma) & \sin(c+\gamma) \end{vmatrix} = 0$

14. Show that

$$\begin{vmatrix} yz-x^2 & zx-y^2 & xy-z^2 \\ zx-y^2 & xy-z^2 & yz-x^2 \\ xy-z^2 & yz-x^2 & zx-y^2 \end{vmatrix} = \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}^2$$

15. If $p = ax + by + cz$, $q = ay + bz + cx$, and $r = az + bx + cy$, then show that

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix} = p^3 + q^3 + r^3 - 3pqr.$$

2.5 THE INVERSE OF A MATRIX

In this section, we shall exclusively consider the square matrices.

Let **A** *be any matrix. A matrix* **B**, *if it exists, is called the inverse of the matrix* **A** *if the products* **AB** *and* **BA** *are defined and* **AB** = **BA** = *I*. Since in case of the existence of the inverse of **A**, the product **AB** and **BA** both are defined and equal thus, **A** and **B** both are square matrices of the same order. Further, since $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}| = |\mathbf{I}| = 1$, thus, both $|\mathbf{A}|$ and $|\mathbf{B}|$ must be non-zero *that is both the matrix and its inverse must be non-singular.*

Interchanging the order of **A** and **B** we observe that *if* **B** *is the inverse of* **A**, *then* **A** *must be the inverse of* **B**.

The inverse of a square matrix **A** of order n, if exists, is denoted by $\mathbf{A}^{-1}$. Thus

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_n,$$

where $\mathbf{I}_n$ is a unit matrix of order n.

It is easy to find non-zero square matrices that have no inverse. For example, consider $\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}$. If **B** is the inverse of **A** and say, $\mathbf{B} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$\mathbf{AB} = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ a & b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which gives $2a = 1$, $a = 0$, $2b = 0$ and $b = 0$, which are impossible conditions, thus **A** does not has an inverse. We can check that **A** is a *singular* matrix, since $|\mathbf{A}| = 0$.

On the other hand, some matrices do have inverses, for example, we have

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & -2 \\ 1 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus the matrices $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 & -2 \\ 1 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ are inverse of each other

In fact, we can check in this case that both matrices are non-singular.

2.5.1 Method to Find the Inverse of a Square Matrix *A*

The inverse of a non-singular square matrix **A** *is given by*

$$\mathbf{A}^{-1} = \frac{adj(\mathbf{A})}{|\mathbf{A}|}, \ |A| \neq 0, \qquad \text{...(2.7)}$$

where $adj(\mathbf{A})$ = *adjoint matrix of* **A** = *transpose of the matrix of cofactors of* **A**.

To establish it, consider $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, then

$$\mathbf{A}\, adj(\mathbf{A}) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}, \quad \ldots(2.8)$$

where A_{ij} is the cofactor of a_{ij} in A.

Performing the multiplication on the right side of (2.8) and using (2.3), we obtain

$$\mathbf{A}\, adj(\mathbf{A}) = \begin{bmatrix} |\mathbf{A}| & 0 & \cdots & 0 \\ 0 & |\mathbf{A}| & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & |\mathbf{A}| \end{bmatrix} = |\mathbf{A}|\, \mathbf{I}_n.$$

Since $|\mathbf{A}| \neq 0$, dividing both sides by $|\mathbf{A}|$, we get $\mathbf{A}\dfrac{adj(\mathbf{A})}{|\mathbf{A}|} = \mathbf{I}_n$, which gives (2.7).

Also, *if* **A** *has an inverse, then it is unique.*

If possible let **B** and **C** both be two different inverses of the matrix **A**. Then

$$\mathbf{B} = \mathbf{BI} = \mathbf{B}(\mathbf{AC}) = (\mathbf{BA})\mathbf{C} = \mathbf{IC} = \mathbf{C}.$$

Hence the inverse of **A** is unique

2.5.2 Properties of Inverse Matrices

1. *The unit matrix* **I** *is its own inverse, that is,* $\mathbf{I} = \mathbf{I}^{-1}$
2. *The inverse of the inverse matrix is the matrix itself, that is,* $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.
3. *Inverse of the transpose is the transpose of inverse, that is,* $(\mathbf{A}^{T})^{-1} = (\mathbf{A}^{-1})^{T}$.
4. *If* **A** *and* **B** *are non-singular matrices, then* **AB** *is also non-singular and* $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.
5. *If* **A** *is non-singular, then* $(\mathbf{A}^{-1})^{m} = (\mathbf{A}^{m})^{-1}$ *for any positive integral, value of* m.
6. *If* **A** *is invertible, then* $det(\mathbf{A}^{-1}) = \dfrac{1}{\det \mathbf{A}}$.
7. *The inverse of a non-singular upper or lower triangular matrix is respectively an upper or lower triangular matrix.*

8. *If* $\mathbf{D} = diag(d_{11}, d_{22}, \ldots, d_{nn})$, $d_{ii} \neq 0$, *then* $\mathbf{D}^{-1} = diag(1/d_{11}, 1/d_{22}, \ldots, 1/d_{nn})$.
9. *If* $\mathbf{AB} = \mathbf{O}$ *and* $\mathbf{A}$ *is a non-singular matrix, then* $\mathbf{B}$ *must be a null matrix. Similarly, if* $\mathbf{B}$ *is non-singular, then* $\mathbf{A}$ *must be a null matrix.*
10. *If* $\mathbf{AB} = \mathbf{AC}$ *and* $\mathbf{A}$ *is non-singular, then* $\mathbf{B} = \mathbf{C}$.
11. *The inverse of the sum of two matrices is, in general, not equal to the sum of their individual inverses, that is,* $(\mathbf{A} + \mathbf{B})^{-1} \neq \mathbf{A}^{-1} + \mathbf{B}^{-1}$

Example 2.18: Find the inverse of $\mathbf{A} = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 1 & 4 \\ 1 & 5 & -2 \end{bmatrix}$, if it exists.

Solution: We have

$$|\mathbf{A}| = \begin{vmatrix} 3 & 2 & -1 \\ 0 & 1 & 4 \\ 1 & 5 & -2 \end{vmatrix} = 3\begin{vmatrix} 1 & 4 \\ 5 & -2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix} = 3(-2-20) + (8+1) = -57 \neq 0. \text{ Thus } \mathbf{A}^{-1} \text{ exists.}$$

Since *adj* **A** is the transpose of the cofactor matrix, therefore,

$$adj\ \mathbf{A} = \begin{bmatrix} -22 & 4 & -1 \\ -1 & -5 & -13 \\ 9 & -12 & 3 \end{bmatrix}^T = \begin{bmatrix} -22 & -1 & 9 \\ 4 & -5 & -12 \\ -1 & -13 & 3 \end{bmatrix}$$

Thus, $$\mathbf{A}^{-1} = \frac{adj.\mathbf{A}}{|\mathbf{A}|} = -\frac{1}{57}\begin{bmatrix} -22 & -1 & 9 \\ 4 & -5 & -12 \\ -1 & -13 & 3 \end{bmatrix}$$

We may verify that $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_3$.

Example 2.19: Show that

$$\begin{bmatrix} 1 & -\tan(\theta/2) \\ \tan(\theta/2) & 1 \end{bmatrix}\begin{bmatrix} 1 & \tan(\theta/2) \\ -\tan(\theta/2) & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Solution: Let $\mathbf{A} = \begin{bmatrix} 1 & -\tan(\theta/2) \\ \tan(\theta/2) & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & \tan(\theta/2) \\ -\tan(\theta/2) & 1 \end{bmatrix}$

First we calculate $\mathbf{B}^{-1}$. We have

$$|B| = \begin{bmatrix} 1 & \tan(\theta/2) \\ -\tan(\theta/2) & 1 \end{bmatrix} = 1 + \tan^2(\theta/2) = \sec^2(\theta/2) \neq 0 \text{ for any value of } \theta.$$

Also, $adj\,(\mathbf{B}) = \begin{bmatrix} 1 & \tan(\theta/2) \\ -\tan(\theta/2) & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & -\tan(\theta/2) \\ \tan(\theta/2) & 1 \end{bmatrix}$

Thus $\quad \mathbf{B}^{-1} = \dfrac{adj(\mathbf{B})}{|\mathbf{B}|} = \dfrac{1}{\sec^2(\theta/2)} \begin{bmatrix} 1 & -\tan(\theta/2) \\ \tan(\theta/2) & 1 \end{bmatrix}$

Hence $\mathbf{AB}^{-1} = \dfrac{1}{\sec^2(\theta/2)} \begin{bmatrix} 1 & -\tan(\theta/2) \\ \tan(\theta/2) & 1 \end{bmatrix} \begin{bmatrix} 1 & -\tan(\theta/2) \\ \tan(\theta/2) & 1 \end{bmatrix}$

$$= \frac{1}{\sec^2(\theta/2)} \begin{bmatrix} 1-\tan^2(\theta/2) & -2\tan(\theta/2) \\ 2\tan(\theta/2) & 1-\tan^2(\theta/2) \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2(\theta/2) - \sin^2(\theta/2) & -2\sin(\theta/2)\cos(\theta/2) \\ 2\sin(\theta/2)\cos(\theta/2) & \cos^2(\theta/2) - \sin^2(\theta/2) \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

2.6 SOLUTION OF LINEAR SYSTEM OF EQUATIONS ($n \times n$ FORM)

Consider the system of n equations in n unknowns

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \end{array}\right\} \qquad \ldots(2.9)$$

In matrix form the system of Eqs. (2.9) is

$$\mathbf{Ax} = \mathbf{b}, \qquad \ldots(2.10)$$

where $\mathbf{A} = [a_{ij}]_{n\times n}$, $\mathbf{x} = [x_1, \ x_2, \ \cdots, \ x_n]^T$ and $\mathbf{b} = [b_1, \ b_2, \ \cdots, \ b_n]^T$

The matrix $\mathbf{A}$ is called the *coefficient matrix* and $\mathbf{x}$ the *solution vector*. If $\mathbf{b} \neq \mathbf{0}$, then the system of Eqs (2.9) is called *non-homogeneous;* and in case $\mathbf{b} = \mathbf{0}$, the system is said to be *homogeneous*. Further, the system of equations is *consistent* if it has at least one solution and *inconsistent* if it has no solution at all.

Next we discuss the solution of non-homogeneous system of equations.

2.6.1 Method of Determinants: Cramer's Rule

We explain this method by considering the system of three linear equations in three unknowns:

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{array}$$

The determinant of the coefficient matrix **A** is, $|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$.

We have, $x_1|\mathbf{A}| = \begin{vmatrix} x_1a_{11} & a_{12} & a_{13} \\ x_1a_{21} & a_{22} & a_{23} \\ x_1a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 & a_{12} & a_{13} \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 & a_{22} & a_{23} \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix} = |\mathbf{A}_1|$, say.

Thus, $x_1 = \dfrac{|\mathbf{A}_1|}{|\mathbf{A}|}$, provided $|\mathbf{A}| \neq 0$. Similarly, $x_2 = \dfrac{|\mathbf{A}_2|}{|\mathbf{A}|}$, and $x_3 = \dfrac{|\mathbf{A}_3|}{|\mathbf{A}|}$, where $|\mathbf{A}_i|$ is the determinant of the matrix $\mathbf{A}_i$ obtained by replacing the ith column of **A** by the right-hand side column vector $\mathbf{b} = [b_1, b_2, b_3]^T$.

This result can be generalized to system of n linear equations in n variables. The following three cases arise:

(a) When $|\mathbf{A}| \neq 0$, the system of equation is *consistent and has the unique solution* given by

$$x_i = \frac{|\mathbf{A}_i|}{|\mathbf{A}|},\ i = 1, 2, \ldots n.$$

(b) When $|\mathbf{A}| = 0$ and at least one of the $|\mathbf{A}_i| \neq 0$, then the system of equations has no solution, and thus the system is *inconsistent*.

(c) When $|\mathbf{A}| = 0$ and all the $|\mathbf{A}_i| = 0$, $i = 1, 2, \ldots, n$, then the system of equation is *consistent and has infinite number of solutions*.

2.6.2 The Matrix Method

The system of Eqs. (2.9) in the matrix form is

$$\mathbf{Ax} = \mathbf{b} \qquad \ldots(2.10a)$$

Let **A** be non-singular, pre multiplying (2.10a) by $\mathbf{A}^{-1}$ we obtain

$$\mathbf{A}^{-1}(\mathbf{Ax}) = \mathbf{A}^{-1}\mathbf{b}, \text{ or } \mathbf{Ix} = \mathbf{A}^{-1}\mathbf{b}, \quad \text{or} \quad \mathbf{x} = \mathbf{A}^{-1}\mathbf{b} \qquad \ldots(2.11)$$

The unique solution is obtained by equating the values of x_i, $i = 1, 2, \ldots n$ to the corresponding elements in the resultant product matrix on the right side of (2.11).

In case $\mathbf{b} = \mathbf{0}$, then $\mathbf{x} = \mathbf{0}$; that is, the trivial solution is the only solution.

When the *system of equation is homogeneous* of the form

$$\mathbf{Ax} = \mathbf{0}, \qquad \ldots(2.12)$$

then trivial solution $\mathbf{x} = \mathbf{0}$ is always a solution of this system, thus *a homogeneous system is always consistent.*

If **A** is non-singular, then $\mathbf{x} = \mathbf{A}^{-1}(\mathbf{0}) = \mathbf{0}$ is the only solution of the homogeneous system (2.12).

A non-trivial solution of the homogeneous system (2.12) *exists if, and only if* **A** *is singular and in this case the homogeneous system has infinite number of solutions.*

These solutions are at least *one-parameter family* of solutions. $|A| = 0$

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Example 2.20: Show that the system of equations $3x + y + 2z = 3, 2x - 3y - z = -3, x + 2y + z = 4$, has a unique solution. Find the solution by (a) matrix method (b) Cramer's rule.

Solution: Here $|\mathbf{A}| = \begin{vmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 3(-3+2) - 1(2+1) + 2(4+3) = 8 \neq 0.$

Since $|\mathbf{A}| \neq 0$, thus the system of equations has a unique solution.

(a) *The matrix method*. We have

$$adj(\mathbf{A}) = \begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix}. \quad \text{Thus } A^{-1} = \frac{1}{8}\begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix}$$

Therefore, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix}\begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 8 \\ 16 \\ -8 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, which gives $x = 1$, $y = 2$, and $z = -1$.

(b) *Cramer's rule*. We have

$$|\mathbf{A}_1| = \begin{vmatrix} 3 & 1 & 2 \\ -3 & -3 & -1 \\ 4 & 2 & 1 \end{vmatrix} = 3(-3+2) - 1(-3+4) + 2(-6+12) = 8$$

$$|\mathbf{A}_2| = \begin{vmatrix} 3 & 3 & 2 \\ 2 & -3 & -1 \\ 1 & 4 & 1 \end{vmatrix} = 3(-3+4) - 3(2+1) + 2(8+3) = 16$$

$$|\mathbf{A}_3| = \begin{vmatrix} 3 & 1 & 3 \\ 2 & -3 & -3 \\ 1 & 2 & 4 \end{vmatrix} = 3(-12+6) - 1(8+3) + 3(4+3) = -8.$$

Therefore, $\quad x = \dfrac{|\mathbf{A}_1|}{|\mathbf{A}|} = 1, \quad y = \dfrac{|\mathbf{A}_2|}{|\mathbf{A}|} = 2$, and $\quad z = \dfrac{|\mathbf{A}_3|}{|\mathbf{A}|} = -1.$

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Example 2.21: Solve the homogeneous system of equations $x + 3y - 2z = 0, 2x - y + 4z = 0, x - 11y + 14z = 0$.

Solution: Here, $|\mathbf{A}| = \begin{vmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{vmatrix} = 1(-14+44) - 3(28-4) - 2(-22+1) = 30 - 72 + 42 = 0.$

Since $|\mathbf{A}| = 0$, hence the given system has infinite number of solutions. Rewriting the first two equations, in terms of z as $x + 3y = 2z$, $2x - y = -4z$.

Solving these for x and y, we obtain $x = -10z/7, y = 8z/7$. Hence the solution is

$$x = -10\alpha/7, \quad y = 8\alpha/7, \quad z = \alpha$$

where α is arbitrary. This is one parameter family of solution; α being the parameter. If satisfies the third equation also.

Remark. The case when the system of linear equations has infinite number of solutions will be dealt in detail in Section (2.10.2).

EXERCISE 2.3

1. Given the matrix $\mathbf{A} = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}$, compute $adj(\mathbf{A})$ and prove that

$$\mathbf{A}(adj(\mathbf{A})) = (adj\mathbf{A})\,\mathbf{A} = |\mathbf{A}|\,\mathbf{I}_3.$$

2. Find the inverse of the following matrices. Verify that in each case $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$.

(a) $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 3 & 4 \\ 3 & -1 & 6 \\ -1 & 5 & 1 \end{bmatrix}$

3. If $\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 9 & 3 \\ 1 & 4 & 2 \end{bmatrix}$, and $\mathbf{B} = \begin{bmatrix} 1 & 2 & 6 \\ 2 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$, then verify that

(a) $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ (b) $\det(\mathbf{A}^{-1}) = \dfrac{1}{\det \mathbf{A}}$. (c) $(\mathbf{A} + \mathbf{B})^{-1} \neq \mathbf{A}^{-1} + \mathbf{B}^{-1}$

(d) $(\mathbf{A}^{-1})^{\mathrm{T}} = (\mathbf{A}^{\mathrm{T}})^{-1}$ (e) $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$ (f) $[adj(\mathbf{A})]^{-1} = adj(\mathbf{A}^{-1})$.

4. If the matrix $\mathbf{A}$ is nilpotent with $\mathbf{A}^p = 0$, show that

$$(\mathbf{I} - \mathbf{A})^{-1} = \mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \ldots + \mathbf{A}^p.$$

Solve the following system of equations by Cramer's rule.

5. $x - y + z = 4$, $2x + y - 3z = 0$, $x + y + x = 2$

6. $x + y + z = 3$, $x + 2y + 3z = 4$, $x + 4y + 9z = 6$

7. $x + y + z = 6.6$, $x - y + z = 2.2$, $x + 2y + 3z = 15.2$

8. $2x + y + z = 0$, $3x + 2y + 3z = 18$, $x + 4y + 9z = 16$

Solve the following system of equations by matrix method.

9. $2x + 5y + 3z = 1$, $-x + 2y + z = 2$, $x + y + z = 0$

10. $3x - y + z = 6$, $4x - y + 2z = 7$, $2x - y + z = 4$

11. $x - y + z = 4,\quad 2x + y - 3z = 0,\quad x + y + z = 2$
12. $2x - z = 1\quad 5x + y = 7,\quad y + 3z = 5$
13. Solve the system of equations $2yz - zx + xy = 3xyz,\quad 3yz + 2zx + 4xy = 19xyz,\quad 6yz + 7zx - xy = 17xyz.$
14. Solve the system of equations $x^2z^3/y = e^8,\quad y^2z/x = e^4,\quad x^3y/z^4 = 1.$
15. Determine the values of k for which the system of equations $x - ky + z = 0,\quad kx + 3y - kz = 0,\quad 3x + y - z = 0$ has (i) only trivial solution, (ii) non-trivial soloution
16. If the system of equations $x + ay + az = 0,\quad bx + y + bz = 0,\quad cx + cy + z = 0, a, b, c \neq 0, 1$

    has a non-trivial solution, then show that $\dfrac{a}{1-a} + \dfrac{b}{1-b} + \dfrac{c}{1-c} = -1.$

2.7 ELEMENTARY TRANSFORMATIONS. RANK, NORMAL AND ECHELONS FORM OF A MATRIX. INVERSE BY GAUSS-JORDEN METHOD

First we discuss elementary transformations on matrices.

2.7.1 Elementary Row and Column Transformations

The three *elementary row transformations* that are performed on a matrix are:

(i) Interchange of two rows, denoted by $R_i \leftrightarrow R_j$

(ii) Multiplication of a row by a non-zero constant, denoted by $R_i \to \alpha R_i$

(iii) Addition of a constant multiplication of one row to another row, denoted by $R_i \to R_i + \alpha R_j$

The corresponding three transformations when performed on columns of a matrix are called *elementary column transformations* and are denoted by $C_i \leftrightarrow C_j$, $C_i \to \alpha C_i$ and $C_i \Rightarrow C_i + \alpha C_j$, respectively.

For example, consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 6 & 4 & -3 & 2 \\ 2 & 0 & 1 & 7 & 4 \\ 5 & 2 & 8 & 2 & 3 \end{bmatrix}$

Applying $R_1 \leftrightarrow R_3$, **A** gives, $\mathbf{A}_1 = \begin{bmatrix} 5 & 2 & 8 & 2 & 3 \\ 2 & 0 & 1 & 7 & 4 \\ 1 & 6 & 4 & -3 & 2 \end{bmatrix}$

Applying $R_1 \to 2R_1$, $\mathbf{A}_1$ gives, $\mathbf{A}_2 = \begin{bmatrix} 10 & 4 & 16 & 4 & 6 \\ 2 & 0 & 1 & 7 & 4 \\ 1 & 6 & 4 & -3 & 2 \end{bmatrix}$

Applying $R_2 \to R_2 + 2R_3$, $\mathbf{A}_2$ gives, $\mathbf{A}_3 = \begin{bmatrix} 10 & 4 & 16 & 4 & 6 \\ 4 & 12 & 9 & 1 & 8 \\ 1 & 6 & 4 & -3 & 2 \end{bmatrix}$

Similarly, we can apply elementary column transformations.

Equivalents Matrices. Two $m \times n$ matrices **A** and **B** are said to be *equivalent* if one can be obtained from the other by applying a sequence of elementary transformations.

The equivalence between the matrices **A** and **B** is denoted by **A** ~ **B**.

2.7.2 Elementary Matrices

An $n \times n$ *elementary matrix* is a matrix that is obtained from an $n \times n$ unit matrix by performing a single elementary row (column) transformation.

We will denote by $\mathbf{R}_{ij}$, the elementary matrix obtained from the unit matrix **I** by interchanging its ith and jth rows; by $\mathbf{R}_{i(\alpha)}$ the elementary matrix obtained from the unit matrix **I** by multiplying its ith row with the scalar $\alpha \neq 0$; and by $\mathbf{R}_{i+j(\alpha)}$, the elementary matrix obtained from the unit matrix **I** by adding α times the jth row to its ith row.

The notations for elementary matrices obtained from corresponding column operations will be $\mathbf{C}_{ij}$, $\mathbf{C}_{i(\alpha)}$, and $\mathbf{C}_{i+j(\alpha)}$ respectively.

For example, if $\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then $\mathbf{R}_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\mathbf{C}_{1(5)} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

$$\mathbf{R}_{2+3(2)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and } \mathbf{C}_{1+2(-1)} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We note from the definition of the elementary matrices and properties of the determinants, that

$|\mathbf{R}_{ij}| = |\mathbf{C}_{ij}| = -1$, $|\mathbf{R}_{i(\alpha)}| = |\mathbf{C}_{i(\alpha)}| = \alpha$, and $|\mathbf{R}_{i+j(\alpha)}| = |\mathbf{C}_{i+j(\alpha)}| = 1$.

Next we state an important result:

Let **R** *be an $m \times m$ elementary matrix obtained by performing an elementary row operation on the unit matrix* $\mathbf{I}_m$ *and let* **A** *be an $m \times n$ matrix, then the matrix* **RA** *is the matrix that is obtained from the matrix* **A** *by performing the same row operation.*

Similarly, if **C** *is an $n \times n$ elementary matrix obtained by performing an elementary column operation on the unit matrix* $\mathbf{I}_n$, *then the matrix* **AC** *is the matrix that is obtained from the matrix* **A** *by performing the same column operation.*

These two results can be stated jointly as follows.

Elementary row (column) transformation of a matrix **A** *can be performed by pre-multiplying (post-multiplying)* **A** *by the corresponding elementary matrix.*

For example, consider $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$. Then

$$\mathbf{R}_{13}\,\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} a_{31} & a_{32} & a_{33} & a_{34} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{11} & a_{12} & a_{13} & a_{14} \end{bmatrix}$$

It is the same as obtained from **A** by interchanging Ist and 3rd rows.

Also, $$\mathbf{AC}_{1(5)} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5a_{11} & a_{12} & a_{13} & a_{14} \\ 5a_{21} & a_{22} & a_{23} & a_{24} \\ 5a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

It is the same as obtained from **A** by multiplying the first column by 5.

2.7.3 Rank of a Matrix

A matrix **A**, *not necessarily square, is of rank r if it contains at least one square submatrix of order* $r \times r$ *with non-zero determinant but no square submatrix of order larger than* $r \times r$ *with non-zero determinant.*

Thus, *rank is the order of the largest non-zero minor of* **A**. If a matrix **A** has a non-zero minor of order r, then its rank is greater than or equal to r. If all the minors of order $r + 1$ of a matrix A are zeros, then its rank is less than or equal to r.

The rank of a matrix **A** is denoted by $\rho(\mathbf{A})$.

We have the following results concerning the rank of a matrix;

1. *For a rectangular matrix* **A** *of order* $m \times n$, *the rank* $(\mathbf{A}) \le \min\{m, n\}$.
2. *For a square matrix* **A** *of order n, the rank* $(\mathbf{A}) = n$, *if* $|\mathbf{A}| \neq 0$, *otherwise rank* $(\mathbf{A}) < n$. *If rank* $(\mathbf{A}) = n$, *then the square matrix is said to be non-singular, otherwise singular.*
3. *Every non-zero matrix* **A** *is of rank greater than or equal to one.*
4. *The rank of a matrix is zero if, and only if it is a null matrix.*
5. *For any matrix* **A**, *the rank of* **A** *is equal to rank of* $\mathbf{A}^{\mathrm{T}}$.
6. *The rank of the product of two matrices cannot exceed the rank of either matrix.*
7. *The rank of a matrix remains unaltered even if it is subjected to elementary row or column transformations, that is, equivalent matrices have the same rank.*

The last result is quite useful in finding the rank of a matrix.

2.7.4 Normal form of a Matrix

Every non-zero matrix **A** of order $m \times n$ with rank r can be reduced by a sequence of elementary transformations to the form

$$\begin{bmatrix} \mathbf{I}_r & \mathbf{O}_{r \times n-r} \\ \mathbf{O}_{m-r \times r} & \mathbf{O}_{m-r \times n-r} \end{bmatrix}_{m \times n}$$

called the *normal form* of **A**, where $\mathbf{I}_r$ is a unit matrix of order $r \times r$; and $\mathbf{O}_{r \times n-r}$, $\mathbf{O}_{m-r \times r}$ and $\mathbf{O}_{m-r \times n-r}$ are null matrices of the orders indicated.

Next, since each elementary row (column) transformation over a matrix **A** can be affected by pre (post) multiplying with a suitable elementary matrix and each elementary matrix is non-singular, therefore, we have the following result:

Corresponding to every matrix **A** *of order* $m \times n$ *with rank* r, *there exist non-singular matrices* **P** *of order* $m \times m$ *and* **Q** *of order* $n \times n$ *such that*

$$\mathbf{PAQ} = \begin{bmatrix} \mathbf{I}_r & \mathbf{O}_{r \times n-r} \\ \mathbf{O}_{m-r \times r} & \mathbf{O}_{m-r \times n-r} \end{bmatrix}$$

We should note that for a given matrix **A**, the matrices **P** and **Q** are not necessarily unique.

Example 2.22: Determine the rank of the following matrices using elementary row transformations:

(a) $\begin{bmatrix} 2 & 1 & -3 & 4 \\ 2 & 4 & -2 & 5 \\ 0 & 3 & 1 & 3 \\ 2 & 1 & -3 & -2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 5 \\ 1 & 5 & 5 & 7 \\ 8 & 1 & 14 & 17 \end{bmatrix}$

Solution: (a) Let

$$A = \begin{bmatrix} 2 & 1 & -3 & 4 \\ 2 & 4 & -2 & 5 \\ 0 & 3 & 1 & 3 \\ 2 & 1 & -3 & -2 \end{bmatrix} \begin{matrix} \text{Operate} \\ R_2 \to R_2 - R_1, \\ R_4 \to R_4 - R_1 \end{matrix} \sim \begin{bmatrix} 2 & 1 & -3 & 4 \\ 0 & 3 & 1 & 1 \\ 0 & 3 & 1 & 3 \\ 0 & 0 & 0 & -6 \end{bmatrix}$$

$$\begin{matrix} \text{Operate} \\ R_3 \to R_3 - R_2 \end{matrix} \sim \begin{bmatrix} 2 & 1 & -3 & 4 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -6 \end{bmatrix} \begin{matrix} \text{Operate} \\ R_4 \to R_4 + 3R_3 \end{matrix} \sim \begin{bmatrix} 2 & 1 & -3 & 4 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank of the matrix obtained is 3 because a minor of order 3×3, that is $\begin{vmatrix} 2 & 1 & 4 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{vmatrix} = 12 \neq 0$, and the only minor of order 4×4 is zero. Thus, the rank of given matrix is also 3.

(b) Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 5 \\ 1 & 5 & 5 & 7 \\ 8 & 1 & 14 & 17 \end{bmatrix} \begin{matrix} \text{Operate} \\ R_2 \to R_2 - 2R_1, \\ R_3 \to R_3 - R_1, \\ R_4 \to R_4 - 8R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 3 & 2 & 3 \\ 0 & -15 & -10 & -15 \end{bmatrix} \begin{matrix} \text{Operate} \\ R_3 \to R_3 + R_2, \\ R_4 \to R_4 - 5R_2 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank of the matrix obtained is two because a minor of order 2×2, e.g., $\begin{vmatrix} 1 & 2 \\ 0 & -3 \end{vmatrix} = -3 \neq 0$ and every minor of order 3×3 is zero. Hence the rank of given matrix is also 2.

Example 2.23: Determine the rank of the following matrix using elementary column transformations only $\begin{bmatrix} 1 & 1 & -1 & 1 \\ -1 & 1 & -3 & -3 \\ 1 & 0 & 1 & 2 \\ 1 & -1 & 3 & 3 \end{bmatrix}$

Solution: Let

$$A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ -1 & 1 & -3 & -3 \\ 1 & 0 & 1 & 2 \\ 1 & -1 & 3 & 3 \end{bmatrix} \begin{matrix} \text{Operate} \\ C_2 \to C_2 - C_1, \\ C_3 \to C_3 + C_1, \\ C_4 \to C_4 - C_1 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & -4 & -2 \\ 1 & -1 & 2 & 1 \\ 1 & -2 & 4 & 2 \end{bmatrix} \begin{matrix} \text{Operate} \\ C_3 \to C_3 + 2C_2, \\ C_4 \to C_4 + C_2 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix}$$

The rank of this matrix is two, since there are non-zero minors of order 2 and every minor of order three is zero. Hence, the rank of given matrix is also 2.

Imp.

Example 2.24: Find the rank of the matrix $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ using elementary transformations.

Solution: Let

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix} \begin{matrix} \text{Operate} \\ R_1 \leftrightarrow R_2 \end{matrix} \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix} \begin{matrix} \text{Operate} \\ C_2 \to C_2 + C_1, \\ C_3 \to C_3 + 2C_1, \\ C_4 \to C_4 + 4C_1 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 5 & 3 & 7 \\ 3 & 4 & 9 & 10 \\ 6 & 9 & 12 & 17 \end{bmatrix}$$

$$\begin{matrix} \text{Operate} \\ R_2 \to R_2 - 2R_1, \\ R_3 \to R_3 - 3R_1, \\ R_4 \to R_4 - 6R_1 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix} \begin{matrix} \text{Operate} \\ R_2 \to R_2 - R_3, \\ R_4 \to R_4 - 2R_3 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 1 & -6 & -3 \end{bmatrix}$$

$$\begin{matrix}\text{Operate}\\ C_3 \to C_3 + 6C_2,\\ C_4 \to C_4 + 3C_2\end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 33 & 22 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{matrix}\text{Operate}\\ R_3 \to R_3 - 4R_2,\\ R_4 \to R_4 - R_2\end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank of this matrix is 3, since minor of order 4 is zero and there is a non-zero minor $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 33 \end{vmatrix}$ of order three. Hence the rank of the given matrix is also 3.

Example 2.25: Reduce the following matrix into its normal form and hence find its rank.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$$

Solution: The matrix is

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} \begin{matrix}\text{Operate}\\ C_2 \to C_2 - C_1,\\ C_3 \to C_3 - 2C_1\end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \begin{matrix}\text{Operate}\\ R_2 \to R_2 - R_1\end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

$$\begin{matrix}\text{Operate}\\ C_3 \to C_3 - C_2\end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{matrix}\text{Operate}\\ R_3 \to R_3 + R_2\end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which is the normal form of A. Thus rank of A is 2.

Example 2.26: For the matrix $\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$ find non-singular matrices **P** and **Q** such that **PAQ** is in the normal form. Hence, find the rank of **A**.

Solution: Write **A** = **IAI**, that is,

$$\begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{A} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We shall perform every elementary row(column) transformation of A by subjecting the pre factor (post factor) of A to the same operation.

Operate $R_2 \to R_2 + (-4)R_1$, we obtain

$$\begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 5 & -8 & 14 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operate $C_2 \to C_2 + C_1$, $C_3 \to C_3 + (-2)C_1$, $C_4 \to C_4 + 3C_1$, we obtain

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & -8 & 14 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operate $R_2 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 5 & -8 & 14 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -4 & 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operate $C_4 \to C_4 + (-2)C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & -2 \\ 0 & 5 & -8 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -4 & 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operate $R_3 \to R_3 + (-3) R_2$, $R_4 \to R_4 + (-5)R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -8 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -3 \\ -4 & 1 & 0 & -5 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operate $C_4 \to C_4 + 2C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -8 & -12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -3 \\ -4 & 1 & 0 & -5 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -2 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operate $R_4 \to R_4 + 8R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -3 \\ -4 & 1 & 8 & -29 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -2 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operate $R_4 \to -\frac{1}{12} R_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -3 \\ 1/3 & -1/12 & -2/3 & 29/12 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -2 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The L.H.S is in its normal form. Hence,

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -3 \\ 1/3 & -1/12 & -2/3 & 29/12 \end{bmatrix}, \mathbf{Q} = \begin{bmatrix} 1 & 1 & -2 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and the rank of **A** = 4.

2.7.5 The Echelon Form of a Matrix

A matrix of order $m \times n$ is said to be in row (column) echelon form, if

(1) *the entries in a row (column) appear to the right (below) of the first non-zero entry*

(2) *the number of zeros preceding the first non-zero element in the ith row (column) is less than that the number of such zeros in the (i + 1)th row (column); and*

(3) *all rows (columns) that consist entirely of zeros lie at the bottom (right) of the matrix.*

For example, the matrices

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 & 0 & 5 & 7 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

are in their row echelon forms and the matrices

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix}$$

are in their column echelon form.

We note that in case of a square matrix the row echelon form is an upper triangular matrix and the column echelon form is a lower triangular matrix.

A matrix **A** is reduced to its row-echelon form by performing a sequence of appropriate row transformations over it, and is reduced to its column echelon form by performing a sequence of appropriate column transformations over it.

The rank of a matrix can also be found from its echelon form. In fact *rank is equal to the number of non-zero rows (columns) in its row (column) echelon form*. In addition to this echelon approach is applied to test whether a given set of vectors are linearly independent or not, which we shall discuss in Section 2.8.2.

2.7.6 Gauss-Jorden Method of Finding the Inverse

The method is stated as follows.

The elementary row operations which reduce a given square matrix **A** *of order $n \times n$ to a unit matrix* $\mathbf{I}_n$ *when applied to the unit matrix* $\mathbf{I}_n$ *give the inverse of the matrix* **A**.

Proof. Let the successive row operations which reduce the given square matrix **A** of order $n \times n$ to $\mathbf{I}_n$ result from the pre-multiplication of **A** by the elementary matrices $\mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_k$, so that

$$\mathbf{R}_k \mathbf{R}_{k-1} \ldots \mathbf{R}_2 \mathbf{R}_1 \mathbf{A} = \mathbf{I}$$

Post-multiplying this by $\mathbf{A}^{-1}$, we obtain

$$(\mathbf{R}_k \mathbf{R}_{k-1} \ldots \mathbf{R}_2 \mathbf{R}_1 \mathbf{A})\mathbf{A}^{-1} = \mathbf{I}\mathbf{A}^{-1}, \text{ or } \mathbf{R}_k \mathbf{R}_{k-1} \ldots \mathbf{R}_2 \mathbf{R}_1 (\mathbf{A}\mathbf{A}^{-1}) = \mathbf{A}^{-1}$$

or, $$\mathbf{R}_k \mathbf{R}_{k-1} \ldots \mathbf{R}_2 \mathbf{R}_1 \mathbf{I} = \mathbf{A}^{-1}, \text{ or } \mathbf{A}^{-1} = \mathbf{R}_k \mathbf{R}_{k-1} \ldots \mathbf{R}_2 \mathbf{R}_1 \mathbf{I}.$$

To find $\mathbf{A}^{-1}$ we write the augmented matrix [**A**; **I**], where **I** is the unit matrix of the same order as that of **A**. Then perform the same row operations on both **A** and **I**. As and when **A** reduces to **I**, the other matrix represents $\mathbf{A}^{-1}$.

This method also fails to work when det **A** = 0. In such a case the elementary row operations applied produce one or more rows of zeros at the bottom so that **A** cannot be reduced to I_n. In fact, in this case **A** reduces to its normal form $\begin{bmatrix} \mathbf{I}_r & 0 \\ 0 & 0 \end{bmatrix}$ since the rank of **A** in this case is less than n.

Remark: The method can also be applied by subjecting [**A**; **I**] to a sequence of elementary column operations only.

Example 2.27: Find the inverse of $\mathbf{A} = \begin{bmatrix} 1 & 3 & 0 \\ -2 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ by applying Gauss-Jorden method.

Solution: Consider the augmented matrix

$$[\mathbf{A}\,|\,\mathbf{I}] = \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ -2 & 3 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right], \quad \begin{array}{l}\text{Operate} \\ R_2 \to R_2 + 2R_1\end{array} \sim \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 9 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l}\text{Operate} \\ R_2 \leftrightarrow R_3\end{array} \sim \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 9 & 1 & 2 & 1 & 0 \end{array}\right] \begin{array}{l}\text{Operate} \\ R_1 \to R_1 - 3R_2, \\ R_3 \to R_3 - 9R_2\end{array} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & 0 & -3 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -8 & 2 & 1 & -9 \end{array}\right]$$

$$\begin{array}{l}\text{Operate} \\ R_3 \to (-1/8)\, R_3\end{array} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & 0 & -3 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1/4 & -1/8 & 9/8 \end{array}\right]$$

$$\begin{array}{l}\text{Operate} \\ R_2 \to R_2 - R_3, \\ R_1 \to R_1 + 3R_3\end{array} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/4 & -3/8 & 3/8 \\ 0 & 1 & 0 & 1/4 & 1/8 & -1/8 \\ 0 & 0 & 1 & -1/4 & -1/8 & 9/8 \end{array}\right]$$

$$\text{Hence, } A^{-1} = \frac{1}{8}\begin{bmatrix} 2 & -3 & 3 \\ 2 & 1 & -1 \\ -2 & -1 & 9 \end{bmatrix}$$

EXERCISE 2.4

1. If $\mathbf{P} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{Q} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \mathbf{R} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then

'for the following' matrices verify by direct calculation that, (a) pre-multiplication by **P** multiplies row 1 by 3; (b) pre-multiplication by **Q** interchanges rows 1 and 3; and (c) post-multiplication by **R** adds twice column 2 to column 1.

(a) $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \\ 3 & 1 & 2 & 4 \\ 4 & 1 & 2 & 3 \end{bmatrix}$

2. Using the elementary row transformations, find the rank of the following matrices:

(a) $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 3 & 1 & 0 & 4 \\ 3 & 1 & 2 & -1 & 1 \\ 4 & -1 & 3 & -2 & -2 \\ 5 & 4 & 3 & -1 & 5 \end{bmatrix}$

3. Using the elementary column transformations, find the rank of the following matrices:

(a) $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 0 & -1 & 0 \\ 4 & 1 & 0 & 5 \\ 0 & 1 & 3 & 6 \\ 6 & 1 & -2 & 6 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 3 & 1 & 0 & 4 \\ 3 & 1 & 2 & -1 & 1 \\ 4 & -1 & 3 & -2 & -2 \\ 5 & 4 & 3 & -1 & 5 \end{bmatrix}$

4. Using elementary row and column operations, find the rank of the following matrices:

(a) $\begin{bmatrix} 2 & 1 & -2 \\ -1 & -1 & 1 \\ 3 & 1 & -2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 3 & 1 & 0 & 4 \\ 3 & 1 & 2 & -1 & 1 \\ 4 & -1 & 3 & -2 & -2 \\ 5 & 4 & 3 & -1 & 5 \end{bmatrix}$

5. Using elementary row and column operations reduce the following matrices to their normal forms and hence find their ranks.

(a) $\begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 7 & 11 & 15 & 19 \\ 7 & 15 & 21 & 27 \end{bmatrix}$

6. For the following matrices find non-singular matrices **P** and **Q** such that **PAQ** is in normal form. Also find their ranks.

(a) $\begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 0 & -1 & 0 \\ 4 & 1 & 0 & 5 \\ 0 & 1 & 3 & 6 \\ 6 & 1 & -2 & 6 \end{bmatrix}$

7. For the two matrices given in the preceding problem verify that the rank of the product does not exceed the rank of the either.

8. Illustrate with an example that rank(**A**) = rank(**B**) does not necessarily imply that rank($\mathbf{A}^2$) = rank($\mathbf{B}^2$).

9. Reduce the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 5 \\ 1 & 5 & 5 & 7 \\ 8 & 1 & 14 & 17 \end{bmatrix}$ to its row echelon form and find its rank.

10. Reduce the matrix $\begin{bmatrix} 1 & 1 & -1 & 1 \\ -1 & 1 & -3 & -3 \\ 1 & 0 & 1 & 2 \\ 1 & -1 & 3 & 3 \end{bmatrix}$ to its column echelon form and find its rank.

11. Use Gauss-Jorden method to find the inverse of the following matrices:

(a) $\begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 4 & 4 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

2.8 VECTOR SPACES

In this section we discuss vector space and other related concepts.

2.8.1 Vector Space

A *non-empty* set V of elements which may be vectors, matrices or functions, etc. denoted by $\boldsymbol{a}, \mathbf{b}, \ldots$ is called a *vector space* and these elements are called *vectors*, if in V are defined two algebraic operations: *Vector addition* and *scalar multiplication* as follows:

I. *Vector addition* associates with every pair of vectors $\boldsymbol{a}$ and $\mathbf{b}$ of V a unique vector of V, called the sum of $\boldsymbol{a}$ and $\mathbf{b}$ denoted by $\boldsymbol{a} + \mathbf{b}$, such that the following axioms are satisfied.

(i) $\boldsymbol{a} + \mathbf{b} = \mathbf{b} + \boldsymbol{a}$ (commutativity)

(ii) $(\boldsymbol{a} + \mathbf{b}) + \mathbf{c} = \boldsymbol{a} + (\mathbf{b} + \mathbf{c})$ (associativity)

(iii) $\boldsymbol{a} + \mathbf{0} = \boldsymbol{a}$ (existence of a unique zero vector in V)

(iv) $\boldsymbol{a} + (-\boldsymbol{a}) = \mathbf{0}$ (existence of additive inverse in V)

II. *Scalar multiplication* associates with every vector $\boldsymbol{a}$ of V and any scalar α, a unique vector of V, called the scalar-product of α and $\boldsymbol{a}$, denoted by $\alpha\, \boldsymbol{a}$, such that the following axioms are satisfied.

(v) $\alpha\, (\boldsymbol{a} + \mathbf{b}) = \alpha\, \boldsymbol{a} + \alpha\, \mathbf{b}$ (distributivity)

(vi) $(\alpha + \beta)\boldsymbol{a} = \alpha\, \boldsymbol{a} + \beta\, \boldsymbol{a}$ (distributivity)

(vii) $\alpha\, (\beta\, \boldsymbol{a}) = (\alpha\, \beta)\boldsymbol{a}$ (associativity)

(viii) $1\boldsymbol{a} = \boldsymbol{a}$ (existence of multiplicative identity)

The vector addition and scalar multiplication defined above are not necessarily the usual addition and multiplication operators. Thus the vector space depends not only on the set V of vectors, but also on the definitions of the algebraic operations on V.

If both the elements of V and the scalars α, β are real, then V is called a *real vector space*, otherwise V is called a *complex vector space*.

Examples of a few vector spaces under the usual operations of vector addition and scalars are:

1. All ordered n-tuples of real numbers as vectors and real numbers as scalars forms $\mathbb{R}^n$, n-dimensional real vector space. In particular, for $n = 3$ we have $\mathbb{R}^3$ consisting of ordered triplets vectors in space.
2. The set of all $m \times n$ matrices under usual operations of matrices addition and scalar multiplication forms a vector space.
3. The set of all constant, linear and quadratic polynomials in x together forms a vector space under the usual addition of two polynomials and multiplication of a polynomial by a real number.

But we must note that set of all quadratic polynomials in x does not form a vector space under the usual addition and scalar multiplication by a real, since sum of the two quadratic polynomial may not be a quadratic polynomial.

2.8.2 Linear Dependence and Independence of Vectors

Let V be a vector space. A finite set of m vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_m$ each with the same number of components of V is said to be linearly dependent, if there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_m$, not all zero, such that

$$\alpha_1\, \boldsymbol{a}_1 + \alpha_2\, \boldsymbol{a}_2 + \ldots + \alpha_m\, \boldsymbol{a}_m = 0 \qquad \ldots(2.13)$$

because then we can express at least one of the vectors as a linear combination of the others. For example, if in (2.13) $\alpha_1 \neq 0$, then we can rewrite (2.13) as

$$\boldsymbol{a}_1 = \left(\frac{-\alpha_2}{\alpha_1}\right)\boldsymbol{a}_2 + \left(\frac{-\alpha_3}{\alpha_1}\right)\boldsymbol{a}_3 + \ldots + \left(\frac{-\alpha_m}{\alpha_1}\right)\boldsymbol{a}_m$$

or, $$a_1 = \beta_2 a_2 + \beta_3 a_3 + \ldots + \beta_m a_m, \text{ where } \beta_i = \left(\frac{-\alpha_i}{\alpha_1}\right), i = 1, 2, \ldots, m$$

If (2.13) is satisfied only for $\alpha_1 = \alpha_2 = \ldots = \alpha_m = 0$, then the set of vectors is said to be *linearly independent*.

For example, the three vectors $a_1 = [1, 1, 1, 3]$, $a_2 = [1, 2, 3, 4]$, and $a_3 = [3, 4, 5, 10]$ are linearly dependent since it is easy to check that these vectors satisfy the relation

$$2a_1 + a_2 - a_3 = 0. \qquad \ldots(2.14)$$

Method to check linear independence (dependence) of vectors. For a given set of vectors, it is easy to verify an equation of the form (2.14) but it is not so obvious to construct it to find out whether the given set of vectors are linearly independent or dependent. A test to check this for m vectors, with m components each, is as follows.

In this case, (2.13) gives a homogeneous system of m algebraic equations in m unknowns $\alpha_1, \alpha_2, \ldots, \alpha_m$. Non-trivial solution exists if the determinant of the coefficient matrix is zero, that is, *vectors are linearly dependent, if the determinant of the coefficient matrix is zero. In case of non-zero determinant the vectors are linearly independent.*

In general, to check the linear dependence or independence of m vectors $a_1, a_2, \ldots, a_m$ with n components each, when n may or may not be equal to m, find the rank of the matrix with rows (or columns) as vectors $a_1, a_2, \ldots, a_m$. *If the matrix with row (or, column) vectors $a_1, a_2, \ldots, a_m$ has rank m, the vectors are linearly independent. However, if the rank is less than m the vectors are linearly dependent.* The rank of the matrix formed so gives the number of linearly independent vectors in the set of m vectors.

In fact, we reduce the matrix of m row (column) vectors to its row (or, column) echelon form, the number of non-zero rows (or columns) gives the number of linearly independent vectors, in the set of m vectors.

Also we have the following result which follows immediately from above. *The m vectors, with n components each $n < m$ are always linearly dependent.* For example, three or more vectors in the plane are always linearly dependent.

Another result of interest is *a set of vectors containing* **0** *as one of its elements is always linearly dependent,* since it can always be expressed as

$$\alpha_1 \mathbf{0} + \alpha_2 a_2 + \alpha_3 a_3 + \ldots + \alpha_m a_m = \hat{0}$$

for $\alpha_1 \neq 0$, and $\alpha_2 = \alpha_3 = \ldots = \alpha_m = 0$.

2.8.3 Dimension, Basis and Span of a Vector Space

Let V be a vector space. The maximum number of linearly independent vectors in V is called the *dimension* of V and is denoted by *dim* (V).

The set S consisting of the maximum possible number of linearly independent vector of V is called a *basis* of V. Thus the number of vectors in a basis of V equals the *dimension of* V.

The set of all linear combinations of the linearly independent vectors $a_1, a_2, \ldots, a_p$ with the same number of components is called the *span* of these vectors. We must note that *a basis set S spans the vector space V*.

For example, consider the vector space $\mathbb{R}^3$. Following sets of vectors of $\mathbb{R}^3$ are linearly independent and thus each set forms a basis of $\mathbb{R}^3$.

(i) $[1, 0, 0], [0, 1, 0], [0, 0, 1]$ (ii) $[1, -1, 0], [0, 1, -1], [0, 0, 1]$

We observe that *there can be more than one basis for the same vector space.*

As another example, the three different basis for $\mathbb{R}^2$ are:

(i) $[1, 0], [0, 1]$ (ii) $[1, 1], [1, -1]$, and (iii) $[1, 0], [0, -1]$

Obviously, $\dim(\mathbb{R}^2) = 2$, and $\dim(\mathbb{R}^3) = 3$.

If elements of a vector space V are the real 2×2 matrices, then the set

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

forms a basis of V and spans V, since any element $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of V can be expressed as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The dimension of the vector space V is four, the number of elements in S. Similarly, the real $m \times n$ matrices, with fixed m and n form an mn-dimensional vector space.

2.8.4 Subspaces

Let V be a vector space defined with two algebraic operations of vector addition and scalar multiplication. A non-empty subset W of V that itself forms a vector space under the same two algebraic operations as defined for the vectors of V, is called a *subspace* of the vector space V.

The vector space V is also taken as a subspace of itself.

To show that W is a subspace of V, it is sufficient to show only that (i) W is closed under vector addition and scalar multiplication, and (ii) the existence of the '*zero element*' and the '*additive inverse*' in W.

For example, if V is the set of all $n \times n$ real square matrices with usual algebraic operations of matrix addition and scalar multiplication, then the subset W consisting of all symmetrical matrices of order $n \times n$ forms a subspace of V, but the subset W' consisting of all $n \times n$ matrices having real positve elements does not form a subspace of V since W' is not closed under scalar multiplication and also the null matrix of order $n \times n$ in V does not belong to W'.

Example 2.28: Test whether the following set of vectors in $\mathbb{R}^4$ is linearly dependent or independent: $\mathbf{x}_1 = (2, 1, 1, 0)$, $\mathbf{x}_2 = (0, 2, 0, 1)$, $\mathbf{x}_3 = (1, 1, 0, 2)$, $\mathbf{x}_4 = (0, 2, 1, 1)$.

Solution: Consider the vector equation, $\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \alpha_3\mathbf{x}_3 + \alpha_4\mathbf{x}_4 = \mathbf{0}$.

Substituting for $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$, we obtain

$$\alpha_1 (2, 1, 1, 0) + \alpha_2(0, 2, 0, 1) + \alpha_3(1, 1, 0, 2) + \alpha_4 (0, 2, 1, 1) = \mathbf{0}.$$

Comparing, we get

$$\left.\begin{aligned} 2\alpha_1 + 0.\alpha_2 + \alpha_3 + 0.\alpha_4 &= 0 \\ \alpha_1 + 2\alpha_2 + \alpha_3 + 2\alpha_4 &= 0 \\ \alpha_1 + 0.\alpha_2 + 0.\alpha_3 + \alpha_4 &= 0 \\ 0.\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 &= 0 \end{aligned}\right\} \quad ...(2.15)$$

Determinant of the coefficient matrix of the system of Eqs. (2.15) is

$$|\mathbf{A}| = \begin{vmatrix} 2 & 0 & 1 & 0 \\ 1 & 2 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 \end{vmatrix} = 2\begin{vmatrix} 2 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 2\{-(4-1)\} + \{1(-1) - 1(2-2)\} = -7 \neq 0$$

Therefore, the set of equations (2.15) has only a trivial solution $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ and hence the vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ are linearly independent.

Example 2.29: Verify that the following set of vectors in $\mathbb{R}^3$ is linearly dependent:

$$\mathbf{x}_1 = (1, 0, 1), \mathbf{x}_2 = (1, 1, 1), \mathbf{x}_3 = (1, 1, 2), \mathbf{x}_4 = (1, 2, 1)$$

Also find the number of linearly independent vectors.

Solution: Consider the matrix **A** with column vectors as $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$, we have

$$|\mathbf{A}| = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 2 & 1 \end{bmatrix} \begin{matrix} \text{Operate} \\ R_3 \to R_3 - R_1 \end{matrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{matrix} \text{Operate} \\ C_2 \Rightarrow C_2 - C_1, \\ C_3 \to C_3 - C_1, \\ C_4 \to C_4 - C_1 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{matrix} \text{Operate} \\ C_3 \to C_3 - C_2, \\ C_4 \to C_4 - 2C_2 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Thus the rank of **A** is 3, hence the given set of vectors is linearly dependent and the number of linearly independent vectors is three.

Alternatively, it is a set of m (= 4) vectors with n (= 3) components each and $m > n$, hence is linearly dependent.

Example 2.30: Test whether the following sets of vectors are linearly dependent or independent. Find the dimension and the basis of the given set of vectors.

(a) $(1, 3, 5), (2, -1, 4), (-2, 8, 2)$

(b) $(2, 3, 6, -3, 4), (4, 2, 12, -3, 6), (4, 10, 12, -9, 10)$

(c) $(3, 0, 2, 2), (-1, 7, 4, 9), (7, -7, 0, -5)$

Solution: (a) Consider the matrix **A** with row vectors $(1,3,5)$, $(2,-1,4)$ and $(-2,8,2)$, we have

$$\mathbf{A}=\begin{bmatrix}1&3&5\\2&-1&4\\-2&8&2\end{bmatrix}\begin{matrix}\text{Operate}\\R_2\to R_2-2R_1,\\R_3\to R_3+2R_1\end{matrix}\sim\begin{bmatrix}1&3&5\\0&-7&-6\\0&14&12\end{bmatrix}\begin{matrix}\text{Operate}\\R_3\to R_3+2R_2\end{matrix}\sim\begin{bmatrix}1&3&5\\0&-7&-6\\0&0&0\end{bmatrix}$$

This is the row-echelon form of **A** and since all the rows in the row-echelon form of **A** are not non-zero, the given set of vectors is linearly dependent.

The dimension is two equal to the number of non-zero rows and a basis can be taken as the set $\{(1,3,5),(0,-7,-6)\}$.

(b) Consider the matrix **A** with row vectors as the given vectors, we have

$$\mathbf{A}=\begin{bmatrix}2&3&6&-3&4\\4&2&12&-3&6\\4&10&12&-9&10\end{bmatrix}$$

$$\begin{matrix}\text{Operate}\\R_2\to R_2-2R_1,\\R_3\to R_3-2R_1\end{matrix}\sim\begin{bmatrix}2&3&6&-3&4\\0&-4&0&3&-2\\0&4&0&-3&2\end{bmatrix}\begin{matrix}\text{Operate}\\R_3\to R_2+R_3\end{matrix}\sim\begin{bmatrix}2&3&6&-3&4\\0&-4&0&3&-2\\0&0&0&0&0\end{bmatrix}$$

Since all the rows in the row echelon form of **A** are not non-zero, the given set of vectors is linearly dependent with dimension 2, the number of non-zero rows. A basis can be taken as the set $\{(2,3,6,-3,4),(0,-4,0,3,-2)\}$.

(c) Consider the matrix **A** with row vectors $(3,0,2,2)$, $(-1,7,4,9)$ and $(7,-7,0,-5)$, we have

$$\mathbf{A}=\begin{bmatrix}3&0&2&2\\-1&7&4&9\\7&-7&0&-5\end{bmatrix}\begin{matrix}\text{Operate}\\R_2\to 3R_2,\\R_3\to 3R_3\end{matrix}\sim\begin{bmatrix}3&0&2&2\\-3&21&12&27\\21&-21&0&-15\end{bmatrix}$$

$$\begin{matrix}\text{Operate}\\R_2\to R_2+R_1,\\R_3\to R_3-7R_1\end{matrix}\sim\begin{bmatrix}3&0&2&2\\0&21&14&29\\0&-21&-14&-29\end{bmatrix}\begin{matrix}\text{Operate}\\R_3\to R_3+R_2,\end{matrix}\sim\begin{bmatrix}3&0&2&2\\0&21&14&29\\0&0&0&0\end{bmatrix}$$

Since all the rows in the row echelon form of **A** are not non-zero, the given set of vectors are linearly dependent with dimension 2, the number of non-zero rows. A basis can be taken as the set $\{(3,0,2,2),(0,21,14,29)\}$.

Example 2.31: In the following determine a basis for the subspace S of $\mathbb{R}^n$ and determine the dimension of the subspace.

(a) S consists of all vectors on or parallel to the plane $x+y+z=0$ in $\mathbb{R}^3$.

(b) S consists of all vectors $(0,x,y,0,y)$ in $\mathbb{R}^5$

Solution: (a) A vector (x,y,z) in $\mathbb{R}^3$ is in S when $z=-x-y$, thus vectors in S can be expressed as

$$(x,y,z)=(x,y,-x-y)=x(1,0,-1)+y(0,1,-1)$$

Thus the two vectors $(1, 0, -1)$ and $(0, 1, -1)$ span S and also the vectors are linearly independent, since one can't be expressed as a scalar multiple of the second, and hence form a basis for S and $\dim(S) = 2$.

(b) The set S consists of all vectors of the form $(0, x, y, 0, y)$ in $\mathbb{R}^5$ can be expressed as

$$(0, x, y, 0, y) = x(0, 1, 0, 0, 0) + y(0, 0, 1, 0, 1)$$

Thus, the two vectors $(0, 1, 0, 0, 0)$ and $(0, 0, 1, 0, 1)$ span S and also the two are linearly independent vectors. Hence, form a basis for S and $\dim(S) = 2$.

Example 2.32: Let V be the set of all ordered pairs (x, y), where x, y are real numbers. The two algebraic operations of addition and scalar multiplication are defined as

$$(x_1, y_1) + (x_2, y_2) = (2x_1 - 3x_2, y_1 - y_2)$$

$$\alpha(x_1, y_1) = (\alpha x_1/3, \alpha y_1/3)$$

Show that V is not a vector-space.

Solution: Consider the commutative law for addition

$$(x_2, y_2) + (x_1, y_1) = (2x_2 - 3x_1, y_2 - y_1) \neq (x_1, y_1) + (x_2, y_2).$$

Thus the law does not hold.

Next, consider the associative law for addition

$$\{(x_1, y_1) + (x_2, y_2)\} + (x_3, y_3) = (2x_1 - 3x_2, y_1 - y_2) + (x_3, y_3) = (4x_1 - 6x_2 - 3x_3, y_1 - y_2 - y_3)$$

and, $(x_1, y_1) + \{(x_2, y_2) + (x_3, y_3)\} = (x_1, y_1) + (2x_2 - 3x_3, y_2 - y_3) = (2x_1 - 6x_2 - 9x_3, y_1 - y_2 + y_3)$

Hence, $\{(x_1, y_1) + (x_2, y_2)\} + (x_3, y_3) \neq (x_1, y_1) + \{(x_2, y_2) + (x_3, y_3)\}$

Thus, associative law for addition is also not satisfied.

Consider $1.(x_1, y_1) = (x_1/3, y_1/3) \neq (x_1, y_1)$

Thus, the existence of multiplicative identity is not satisfied.

Hence V is not a vector-space.

Example 2.33: Find the span of (a) $\mathbf{x}_1 = (5, 1)$, $\mathbf{x}_2 = (1, 3)$ in $\mathbb{R}^2$ (b) $\mathbf{x}_1 = (1, 2, 2)$, $\mathbf{x}_2 = (-1, 0, 2)$ in $\mathbb{R}^3$

Solution: (a) Let $\mathbf{y} = (y_1, y_2)$ be any given vector in $\mathbb{R}^2$. We try to express $\mathbf{y} = \alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2$.

That is, $(y_1, y_2) = \alpha_1 (5, 1) + \alpha_2(1, 3) = (5\alpha_1 + \alpha_2, \alpha_1 + 3\alpha_2)$

Thus $5\alpha_1 + \alpha_2 = y_1, \alpha_1 + 3\alpha_2 = y_2$

It is obvious, that for given y_1, y_2 the above system of equation is solvable for α_1, α_2 and hence $\{x_1, x_2\}$ spans $\mathbb{R}^2$.

(b) Let $\mathbf{y} = (y_1, y_2, y_3)$ be any given vector in $\mathbb{R}^3$. We try to express $\mathbf{y} = \alpha_1\mathbf{x}_1 + \alpha_1\mathbf{x}_2$.

That is, $(y_1, y_2, y_3) = \alpha_1 (1, 2, 2) + \alpha_2 (-1, 0, 2) = (\alpha_1 - \alpha_2, 2\alpha_1, 2\alpha_1, + 2\alpha_2)$.

Thus $\alpha_1 - \alpha_2 = y_1, 2\alpha_1 = y_2, 2\alpha_1 + 2\alpha_2 = y_3$. ...(2.16)

Solving first two equations in (2.16) for α_1, α_2 and substituting in the third equation, we get

$2y_1 - 2y_2 + y_3 = 0$

Thus, the span of $\{\mathbf{x}_1, \mathbf{x}_2\}$ is the set of all possible vectors $\mathbf{y} = (y_1, y_2, y_3)$ such that $2y_1 - 2y_2 + y_3 = 0$. That is, the span of $\{\mathbf{x}_1, \mathbf{x}_2\}$ is the subspace of $\mathbb{R}^3$ consisting of all vectors in the plane $2y_1 - 2y_2 + y_3 = 0$.

EXERCISE 2.5

1. Is the given set, under usual addition and scalar multiplication a vector space or not? Give reason. In case the answer is yes, find a basis and determine the dimension:
   (a) All 2×2 matrices such that $a_{11} + a_{22} = 0$
   (b) All polynomials in x of degree less than or equal to 4.
   (c) All $m \times n$ matrices with positive enteries
   (d) All vectors (x, y, z) in $\mathbb{R}^3$ such that $2x + 3z = 0$
2. Show that the following sets are linearly dependent by expressing one of the vectors as a linear combination of the others
   (a) $\{(1, 1), (1, 2), (3, 4)\}$
   (b) $\{(1, 2, 3), (3, 2, 1), (5, 5, 5)\}$
   (c) $\{(1, 0, 0,), (0, 1,0), (3, 3, 0), (2, -7, 9)\}$
3. Determine whether the following set is linearly independent or linearly dependent. If it is linearly dependent, then find a linear combination of the vectors.
   (a) $\{(2, 3, 0), (1, -2, 4)\ (1, 1, 0), (1, 1, 1)\}$
   (b) $\{(4, -1, 1, 2), (3, 0, 2, 5), (0, 0, 0, 0)\}$
   (c) $\{(1, 3, 4, 2), (3, -5, 2, 2), (2, -1, 3, 2)\}$
   (d) $\{(1, -1, 0), (0, 1, -1), (0, 0, 1)\}$
   (e) $\{(1, 9, 9, 8), (2, 0, 0, 3), (2, 0, 0, 8)\}$.
4. Show whether the given sets are identical. Explain
   (a) span $\{(2, -1, -1), (3, 1, 0)\}$; and span $\{(2, -1, -1), (5, 5, 2)\}$
   (b) span $\{(1, -1, 0), (0, 0, 1), (1, 2, 3)\}$; and span $\{(1, -1, 0), (0, 1, -1), (0, 0, 1)\}$
   (c) span $\{(1, 2, -1), (-3, 0, 0)\}$; and span $\{(1, 0, 0), (1, 3, 0)\}$
5. Let V be the set of all ordered triplets (x, y, z) in $\mathbb{R}^3$ with vector addition defined as $(x, y, z) + (u, v, w) = (3x + 4u, y - 2v, z + w)$ and scalar multiplication defined as $\alpha(x, y, z) = (\alpha x, \alpha y, \alpha z/3)$. Discuss whether V is a vector space or not.
6. Let V be the set of all positive real numbers with addition defined as $x + y = xy$ and scalar multiplication defined as $\alpha x = x^{\alpha}$. Discuss whether V is a vector space or not.
7. Prove that the set S of all vectors lying in any plane in $\mathbb{R}^3$ that passes through the origin forms a subspace of $\mathbb{R}^3$. Does the set of all vectors lying in any plane in $\mathbb{R}^3$ that don't pass through origin form a subspace of $\mathbb{R}^3$?
8. Let V be the set of all 2×2 complex matrices with algebraic operations of usual matrix addition and scalar multiplication and S consisting of all matrices of the form: $\begin{bmatrix} z & x+iy \\ x-iy & t \end{bmatrix}$, where x, y, z, t are real numbers. Check whether S forms a subspace of V when (a) scalars are real numbers, (b) scalars are complex numbers.

9. Find a basis of the row space and of the column space of the matrix

$$\begin{bmatrix} 3 & 1 & 4 \\ 0 & 5 & 8 \\ -3 & 4 & 4 \\ 1 & 2 & 4 \end{bmatrix}$$

10. Show that a 'zero vector space', that is, a vector space consisting of the zero vector alone, has no basis.

2.9 MATRICES AS LINEAR TRANSFORMATIONS

The matrices of order $n \times m$ can be viewed as linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$. We discuss this aspect in this section.

2.9.1 Linear Transformation

Let X and Y be two vector spaces both real or complex, over the same field of scalars. If to each vector $\mathbf{x}$ in X is assigned a unique vector $\mathbf{y}$ in Y, then we say that a *mapping* or *transformation* of X into Y is given. Let this transformation be denoted by T. The vector $\mathbf{y}$ in Y assigned to vector $\mathbf{x}$ in X is called the *image* of $\mathbf{x}$ under T and is denoted by $T(\mathbf{x})$ and $\mathbf{x}$ is called the *pre-image* of $\mathbf{y}$.

The transformation T is called a *linear transformation*, if for all vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ in X and scalars α,

$$T(\mathbf{x}_1 + \mathbf{x}_2) = T(\mathbf{x}_1) + T(\mathbf{x}_2), \text{ and } T(\alpha\mathbf{x}_1) = \alpha T(\mathbf{x}_1).$$

These conditions are equivalent to

$$T(\alpha\mathbf{x}_1 + \beta\mathbf{x}_2) = \alpha T(\mathbf{x}_1) + \beta T(\mathbf{x}_2)$$

for $\mathbf{x}_1$ and $\mathbf{x}_2$ in X and any scalars α and β.

Sum and product of linear transformations. Let T_1 and T_2 be two linear transformations from X into Y. The *sum* $T_1 + T_2$ is defined as

$$(T_1 + T_2)(\mathbf{x}) = T_1(\mathbf{x}) + T_2(\mathbf{x}), \mathbf{x} \in X.$$

It can be verified that $T_1 + T_2$ is also a linear transformation, and $T_1 + T_2 = T_2 + T_1$.

If X, Y and Z are three vector spaces, all real or complex, on the same scalar field and T_1 is defined from X into Y and T_2 is defined from Y into Z, then the *product* T_2T_1 is defined from X into Z as

$$T_2T_1(\mathbf{x}) = T_2(T_1(\mathbf{x})).$$

The transformation T_2T_1 is also called *composite transformation*.

It can be verified that T_2T_1 is also a linear transformation and, in general, $T_2T_1 \neq T_1T_2$.

2.9.2 Matrix Representation of a Linear Transformation

Let $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$. Then any real $m \times n$ matrix $\mathbf{A} = [a_{ij}]_{m\times n}$ gives a transformation of $\mathbb{R}^n$ into $\mathbb{R}^m$,

$$\mathbf{y} = \mathbf{A}\mathbf{x}, \quad \text{...(2.17)}$$

where $\mathbf{x}$ is in $\mathbb{R}^n$ and $\mathbf{y}$ is in $\mathbb{R}^m$.

The transformation defined by (2.17) is linear transformation, since

$$\mathbf{A}(\alpha\mathbf{x}_1 + \beta\mathbf{x}_2) = \alpha\mathbf{A}\mathbf{x}_1 + \beta\mathbf{A}\mathbf{x}_2.$$

It can also be shown that every linear transformation T of $\mathbb{R}^n$ into $\mathbb{R}^m$ can be given in terms of an $m \times n$ matrix $\mathbf{A}$, after selecting a basis for $\mathbb{R}^n$ and a basis for $\mathbb{R}^m$.

In two-dimensional space $\mathbb{R}^2$ the standard basis is $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$

In three-dimensional space $\mathbb{R}^3$ the standard basis is $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$

For example, the matrix representing the linear transformation that maps (x_1, x_2) to $(2x_1 - 5x_2, 3x_1 + 4x_2)$ is $\mathbf{A} = \begin{bmatrix}2 & -5\\3 & 4\end{bmatrix}$ since this transformation can be represented as

$$y_1 = 2x_1 - 5x_2$$
$$y_2 = 3x_1 + 4x_2$$

Similarly, the matrix representing the linear transformation (x_1, x_2, x_3) to $(x_2 + x_3, x_2 - x_3)$ is

$$\mathbf{A} = \begin{bmatrix}0 & 1 & 1\\0 & 1 & -1\end{bmatrix}_{2\times 3}$$

If in (2.17), the matrix of transformation $\mathbf{A}$ is square matrix of order $n \times n$, then (2.17) maps $\mathbb{R}^n$ into $\mathbb{R}^n$. If the matrix $\mathbf{A}$ is singular the transformation (2.17) is called *singular transformation*. If $\mathbf{A}$ is *non-singular*, then (2.17) is called *non-singular* or *regular transformation*. Further if $\mathbf{A}$ is non-singular, then $\mathbf{A}^{-1}$ exists; pre-multiplying (2.17) by $\mathbf{A}^{-1}$, gives the *inverse transformation*

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{y} \qquad \ldots(2.18)$$

The inverse transformation is also a linear transformation.

Orthogonal transformation The linear transformation $\mathbf{y} = \mathbf{A}\mathbf{x}$ is said to be orthogonal if it transforms

$$y_1^2 + y_2^2 + \ldots + y_n^2 \text{ into } x_1^2 + x_2^2 + \ldots + x_n^2. \qquad \ldots(2.19)$$

The matrix of orthogonal transformation is an *orthogonal matrix*, (refer to Section 2.3.3), for (2.19) gives

$$\mathbf{x}^T\mathbf{x} = \mathbf{y}^T\mathbf{y} = (\mathbf{A}\mathbf{x})^T(\mathbf{A}\mathbf{x}) = \mathbf{x}^T\mathbf{A}^T\mathbf{A}\mathbf{x}.$$

It is possible only if, $\mathbf{A}^T\mathbf{A} = \mathbf{I}$. Thus $\mathbf{A}$ is an orthogonal matrix.

For example, the transformation $y_1 = x_1$, $y_2 = \cos\theta\, x_2 - \sin\theta\, x_3$, $y_3 = \sin\theta\, x_2 + \cos\theta\, x_3$

is orthogonal. We can verify that it transforms $x_1^2 + x_2^2 + x_3^2$ to $y_1^2 + y_2^2 + y_3^2$, and the coefficient matrix

$$A = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{vmatrix}$$ is orthogonal.

Example 2.34: Test whether the transformation (x_1, x_2, x_3) to $(2x_1 + x_2 + x_3, x_1 + x_2 + 2x_3, x_1 - 2x_3)$ is non-singular. If so write the inverse transformation.

Solution: The given transformation is

$$y_1 = 2x_1 + x_2 + x_3$$
$$y_2 = x_1 + x_2 + 2x_3$$
$$y_3 = x_1 + 0x_2 - 2x_3$$

The matrix of transformation is $\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix}$

Here, $|\mathbf{A}| = -1 \neq 0$. Hence the transformation is non-singular. The inverse transformation is given by $\mathbf{x} = \mathbf{A}^{-1}\mathbf{y}$. We can see easily that $\mathbf{A}^{-1} = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{bmatrix}$. Thus the inverse transformation is

$$x_1 = 2y_1 - 2y_2 - y_3$$
$$x_2 = -4y_1 + 5y_2 + 3y_3$$
$$x_3 = y_1 - y_2 - y_3$$

Example 2.35: Find the composite transformation which expresses x_1, x_2 in terms of z_1, z_2 when $x_1 = 3y_1 + 2y_2$, $x_2 = -y_1 + 4y_2$, and $y_1 = z_1 + 2z_2$, $y_2 = 3z_1$.

Solution: The matrix of transformation from $\mathbf{y}$ to $\mathbf{x}$ is $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}$

The matrix of transformation from $\mathbf{z}$ to $\mathbf{y}$ is $\mathbf{B} = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$

Hence, the matrix of transformation from $\mathbf{z}$ to $\mathbf{x}$ is

$$\mathbf{C} = \mathbf{AB}$$

$$= \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ 11 & -2 \end{bmatrix}$$

Therefore, $x_1 = 9z_1 + 6z_2$, $x_2 = 11z_1 - 2z_2$.

EXERCISE 2.6

1. Find the inverse transformation of the following:

(a) $y_1 = 3x_1 + 2x_2$
$y_2 = 4x_1 + x_2$

(b) $y_1 = (-2x_1 + x_2 + 2x_3)/3$
$y_2 = (2x_1 + 2x_2 + x_3)/3$
$y_3 = (x_1 - 2x_2 + 2x_3)/3$

(c) $y_1 = x_1$
$y_2 = x_2 \cos\theta + x_3 \sin\theta$
$y_3 = -x_2 \sin\theta + x_3 \cos\theta$

2. Obtain the transformation from (x_1, x_2, x_3) to (z_1, z_2, z_3) when

$$\begin{aligned} y_1 &= 2x_1 + x_2 \\ y_2 &= x_2 - 2x_3 \\ y_3 &= -x_1 + 2x_2 + x_3 \end{aligned} \quad \text{and} \quad \begin{aligned} z_1 &= y_1 + y_2 + y_3 \\ z_2 &= y_1 + 2y_2 + 3y_3 \\ z_3 &= y_1 + 3y_2 + 5y_3 \end{aligned}$$

3. Find a, b, and c such that the following transformation is orthogonal

$$y_1 = ax_1 + bx_2 + cx_3, \qquad y_2 = -x_4,$$
$$y_3 = cx_1 + ax_2 - bx_3, \qquad y_4 = -bx_1 + cx_2 - ax_3.$$

4. Prove that the composite transformation of two orthogonal transformations is also orthogonal.

2.10 SOLUTION OF LINEAR SYSTEM OF EQUATIONS: GENERAL FORM

In this section we shall consider the non-homogeneous as well as homogeneous system of equations of the $m \times n$ form.

2.10.1 System of Linear Non-homogeneous Equations

Consider the system of m linear non-homogeneous equations in n unknowns $x_1, x_2, \ldots x_n$ given by

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{array}\right\} \qquad \ldots(2.20)$$

The system of Eqs. (2.20) in the matrix form is

$$\mathbf{Ax} = \mathbf{b}, \qquad \ldots(2.21)$$

where $\mathbf{A} = [a_{ij}]_{m\times n}$ is the coefficient matrix,

$\mathbf{x} = (x_1, x_2, \ldots x_n)^T$ is the solution vector; and

$\mathbf{b} = (b_1, b_2, \ldots b_n)^T$.

Using the concept of the rank, we discuss first the *consistency of the system of* Eqs. (2.20). Consider the ranks of the *coefficient matrix* **A** and the *augmented matrix* $\tilde{\mathbf{A}}$, where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \quad \text{and} \quad \tilde{\mathbf{A}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}_{m\times(n+1)}$$

We have the following result:

Theorem 2.1 (Fundamental Theorem for Linear System) *The system of Eqs. (2.20) is consistent, if and only if the coefficient matrix* **A** *and the augmented matrix* $\tilde{\mathbf{A}}$ *have the same rank.*

Proof. Let $\mathbf{c}_{(1)}, \mathbf{c}_{(2)}, \ldots \mathbf{c}_{(n)}$ be the column vectors of the coefficient matrix **A**, then (2.21) can be written as

$$\mathbf{c}_{(1)}\, x_1 + \mathbf{c}_{(2)}\, x_2 + \ldots + \mathbf{c}_{(n)} x_n = \mathbf{b} \qquad \ldots(2.22)$$

Let the rank of **A** be r. Since r is the rank of **A**, so the matrix **A** has r linearly independent columns and without loss of generality, we can take these to be $\mathbf{c}_{(1)}, \mathbf{c}_{(2)}, \ldots, \mathbf{c}_{(r)}$. Thus, each of the column vectors $\mathbf{c}_{(r+1)}, \mathbf{c}_{(r+2)}, \ldots, \mathbf{c}_{(n)}$ will be a linear combination of $\mathbf{c}_{(1)}, \mathbf{c}_{(2)}, \ldots \mathbf{c}_{(r)}$.

Now if the system (2.22) is consistent, then there must exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$\alpha_1\, \mathbf{c}_{(1)} + \alpha_2 \mathbf{c}_{(2)} + \ldots + \alpha_n\, \mathbf{c}_{(n)} = \mathbf{b}$$

which shows that **b** is a linear combination of $\mathbf{c}_{(1)}, \mathbf{c}_{(2)}, \ldots \mathbf{c}_{(n)}$ and hence that of $\mathbf{c}_{(1)}, \mathbf{c}_{(2)}, \ldots, \mathbf{c}_{(r)}$. Thus, the rank of the matrix $\mathbf{A} = [\mathbf{c}_{(1)}, \mathbf{c}_{(2)}, \ldots, \mathbf{c}_{(n)}]$ is equal to the rank of the augmented matrix $\tilde{\mathbf{A}} = [\mathbf{c}_{(1)}, \mathbf{c}_{(2)}, \ldots, \mathbf{c}_{(n)}, \mathbf{b}]$.

Conversely, if rank of **A** is equal to rank of $\tilde{\mathbf{A}}$, then **b** must be a linear combination of the column vectors of **A**, say

$$\mathbf{b} = \alpha_1\, \mathbf{c}_{(1)} + \alpha_2\, \mathbf{c}_{(2)} + \ldots + \alpha_n\, \mathbf{c}_{(n)} \qquad \ldots(2.23)$$

since otherwise rank of $\tilde{\mathbf{A}}$ = rank **A** + 1. But (2.23) implies that the given system is consistent and has a solution $x_1 = \alpha_1, x_2 = \alpha_2, \ldots, x_n = \alpha_n$ by comparing (2.22) with (2.23).

This completes the proof.

Remarks. **1.** The system (2.20) has a unique solution if the common rank r of **A** and $\tilde{\mathbf{A}}$ equals n, the number of unknowns.

2. If r is less than n, then the system has infinitely many solutions and all these solutions are obtained by determining r suitable unknowns in terms of the remaining $(n - r)$ unknowns to which arbitrary values can be assigned. Solutions form the $(n - r)$ parameters family of solutions.

3. The solution(s), if exists, can be obtained by reducing the augmented matrix $\tilde{A}$ in the row-echelon form.

Example 2.36: Test for consistency and solve the system of equations

$$x + y + z = 6,\; x - y + 2z = 5,\; 3x + y + z = 8,\; 2x - 2y + 3z = 7.$$

Solution: Consider the augmented matrix

$$\tilde{\mathbf{A}} = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -1 & 2 & 5 \\ 3 & 1 & 1 & 8 \\ 2 & -2 & 3 & 7 \end{bmatrix} \quad \begin{matrix} \text{Operate} \\ R_2 - R_1, \\ R_3 - 3R_1, \\ R_4 - 2R_1, \end{matrix} \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & -2 & -2 & -10 \\ 0 & -4 & 1 & -5 \end{bmatrix}$$

Operate $R_3 - R_2$, $R_4 - 2R_2$ $\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & -3 & -9 \\ 0 & -0 & -1 & -3 \end{bmatrix}$ Operate $R_4 - \frac{1}{3}R_3 \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & -3 & -9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

3

It is clear that rank of **A** = rank of $\tilde{\mathbf{A}}$ = 3. Thus the system of equations is consistent and has a unique solution.

To find the solution, we have the equations

$$x + y + z = 6$$
$$-2y + z = -1$$
$$-3z = -9$$

Using the backward substitution, we obtain $x = 1, y = 2$, and $z = 3$.

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Example 2.37: Find the values of λ and μ so that the equations

$$2x + 3y + 5z = 9, \quad 7x + 3y - 2z = 8, \quad 2x + 3y + \lambda z = \mu$$

have (i) no solution, (ii) a unique solution, and (iii) an infinite number of solutions.

Solution: The coefficient and the augmented matrices are respectively

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix}, \text{and } \tilde{\mathbf{A}} = \begin{bmatrix} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & \lambda & \mu \end{bmatrix}$$

The system has the unique solution if, and only if the rank of A is three, and for this

$$|\mathbf{A}| = \begin{vmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{vmatrix} = 15(5 - \lambda) \neq 0.$$

That is, $\lambda \neq 5$, thus for a unique solution $\lambda \neq 5$ and μ can have any value.

If $\lambda = 5$, then system will have no solution for those values of μ for which the matrices

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & 5 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & 5 & \mu \end{bmatrix}$$

have different ranks, but first matrix is of rank 2 and second is not of rank 2 unless $\mu = 9$. Thus, the system will have no solution if $\lambda = 5$ and $\mu \neq 9$. Further, if $\lambda = 5$ and $\mu = 9$, the system will have infinite number of solutions.

Example 2.38: Test for the consistency of the following system of equations

$$x + y + 2z + w = 5,\ 2x + 3y - z - 2w = 2,\ 4x + 5y + 3z = 7.$$

Solution: The augmented matrix is

$$\tilde{\mathbf{A}} = \begin{bmatrix} 1 & 1 & 2 & 1 & 5 \\ 2 & 3 & -1 & -2 & 2 \\ 4 & 5 & 3 & 0 & 7 \end{bmatrix} \begin{matrix} \text{Operate} \\ R_3 - 2R_2 \\ R_2 - 2R_1 \end{matrix} \sim \begin{bmatrix} 1 & 1 & 2 & 1 & 5 \\ 0 & 1 & -5 & -4 & -8 \\ 0 & -1 & 5 & 4 & 3 \end{bmatrix}$$

$$R_3 \to R_3 + R_2 \sim \begin{bmatrix} 1 & 1 & 2 & 1 & 5 \\ 0 & 1 & -5 & -4 & -8 \\ 0 & 0 & 0 & 0 & -5 \end{bmatrix}$$

Now the rank of $\mathbf{A} = 2$, and rank of $\tilde{\mathbf{A}} = 3$, since $\begin{vmatrix} 2 & 1 & 5 \\ -5 & -4 & -8 \\ 0 & 0 & -5 \end{vmatrix} = (-5)(-8+5) = 15 \neq 0.$

Imp. Hence, the given system of equations is inconsistent.

Example 2.39: Test for consistency of the following system of equations

$$2x - 3y + 5z = 1,\ 3x + y - z = 2,\ x + 4y - 6z = 1,$$

and, if consistent, solve the system.

Solution: The augmented matrix is

$$\tilde{\mathbf{A}} = \begin{bmatrix} 2 & -3 & 5 & 1 \\ 3 & 1 & -1 & 2 \\ 1 & 4 & -6 & 1 \end{bmatrix} \text{Operate } R_1 \leftrightarrow R_3 \sim \begin{bmatrix} 1 & 4 & -6 & 1 \\ 3 & 1 & -1 & 2 \\ 2 & -3 & 5 & 1 \end{bmatrix}$$

$$\begin{matrix} \text{Operate} \\ R_2 - 3R_1 \\ R_3 - 2R_1 \end{matrix} \sim \begin{bmatrix} 1 & 4 & -6 & 1 \\ 0 & -11 & 17 & -1 \\ 0 & -11 & 17 & -1 \end{bmatrix} \text{Operate } R_3 - R_2 \sim \begin{bmatrix} 1 & 4 & -6 & 1 \\ 0 & -11 & 17 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here, the rank of $A = \text{rank } \tilde{A} = 2$. Hence, the system is consistent but since the common value of the rank is less than the number of unknowns, the system has infinite number of solutions.

The two independent equations are: $x + 4y - 6z = 1, -11y + 17z = -1$.

Solving these in terms of z, we get $x = \dfrac{7 - 2z}{11}, y = \dfrac{1 + 17z}{11}$.

Thus, the infinite solutions are given by $x = \dfrac{7 - 2t}{11}, y = \dfrac{1 + 17t}{11}, z = t$, where t is a parameter.

2.10.2 System of Linear Homogeneous Equations

Consider the system of m linear homogeneous equations in n unknowns $x_1, x_2, \ldots, x_n$ given by

$$\left.\begin{array}{llll} a_{11}x_1 + a_{12}x_2 & + \ldots + & a_{1n}x_n & = 0 \\ a_{21}x_1 + a_{22}x_2 & + \ldots + & a_{2n}x_n & = 0 \\ \quad \cdots & + \ldots + & \quad \cdots & \\ a_{m1}x_1 + a_{m2}x_2 & + \ldots + & a_{mn}x_n & = 0 \end{array}\right\} \qquad \ldots(2.24)$$

In the matrix form the system of equations (2.24) is

$$\mathbf{Ax} = \mathbf{0} \qquad \ldots(2.25)$$

Since the coefficient matrix $\mathbf{A}$ and the augmented matrix $\tilde{\mathbf{A}}$ in this case have the same rank, say r, hence the system (2.24) is always consistent.

If $r = n$, the number of unknowns, the system (2.24) has only trivial solution

$$x_1 = 0, x_2 = 0, \ldots, x_n = 0.$$

If $r < n$, the system has infinitely many non-zero solutions obtained by determining r suitable unknowns in terms of the remaining $(n - r)$ variables to which arbitrary values can be assigned.

We note that in case the number of equations m is less than the number of variables n, then the homogeneous system (2.24) always has solutions other than the trivial solution $x_1 = x_2 = \ldots = x_n = 0$.

In case the number of equations m is equal to the number of variables then the necessary and sufficient condition for the system to have non-trivial solution is that $\boxed{|\mathbf{A}| = 0}$, as discussed earlier in Section 2.6.

Example 2.40: Solve the equations

$$4x + 2y + z + 3w = 0, \quad 6x + 3y + 4z + 7w = 0, \quad 2x + y + w = 0$$

Solution: The coefficient matrix is

$$A = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix} \begin{array}{c} \text{Operate} \\ R_2 - \frac{3}{2}R_1 \\ R_3 - \frac{1}{2}R_1 \end{array} \sim \begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 5/2 & 5/2 \\ 0 & 0 & -1/2 & -1/2 \end{bmatrix} \text{Operate } R_3 + \frac{1}{5}R_2 \sim \begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 5/2 & 5/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, the rank of $\mathbf{A}$ is 2 less than the number of variables, thus there are infinite number of solutions.

The two independent equations are $4x + 2y + z + 3w = 0$, $z + w = 0$.

Solving these for, say z and y in terms of x and w, we have $z = -w$, $y = -2x - w$.

Hence the solutions are: $x = \alpha$, $y = -2\alpha - \beta$, $z = -\beta$, $w = \beta$, where α and β are two parameters.

Example 2.41: Find the values for λ for which the following equations $3x + y - \lambda z = 0$, $4x - 2y - 3z = 0$, $2\lambda x + 4y + \lambda z = 0$ possess a non-trivial solution, and also find the corresponding solutions.

Solution: The system will have non-trivial solution when the determinant of the coefficient matrix is zero, that is $\begin{vmatrix} 3 & 1 & -\lambda \\ 4 & -2 & -3 \\ 2\lambda & 4 & \lambda \end{vmatrix} = 0$ which gives $\lambda^2 + 8\lambda - 9 = 0$, or $\lambda = 1, -9$.

For $\lambda = 1$, the system of equations is

$$3x + y - z = 0,\ 4x - 2y - 3z = 0,\ 2x + 4y + z = 0.$$

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Taking any two equations, say the first two, we have

$$\frac{x}{-5} = \frac{y}{5} = \frac{z}{-10}, \text{ which gives, } x = \frac{z}{2}, y = \frac{-z}{2}.$$

Hence, the solution is $x = \frac{\alpha}{2}, y = \frac{-\alpha}{2}, z = \alpha$, where α is a parameter.

Similarly for $\lambda = -9$, we obtain the solution as $x = \frac{-3\beta}{2}$, $y = \frac{-9\beta}{2}$, $z = \beta$, where β is another parameter.

EXERCISE 2.7

Test for consistency the following systems of linear equations and solve, if consistent

1. $5x + 3y + 7z = 4$, $3x + 26y + 2z = 9$, $7x + 2y + 10z = 5$
2. $x - 4y + 5z = 8$, $3x + 7y - z = 3$, $x + 15y - 11z = 14$.
3. $x + 2y - 3z - 4u = 6$, $x + 3y + z - 2u = 4$, $2x + 5y - 2z - 5u = 10$
4. $x + 2y + z = 3$, $2x + 3y + 2z = 5$, $3x - 5y + 5z = 2$, $3x + 9y - z = 4$
5. $2x + 6y + 11 = 0$, $6x + 20y - 6z + 3 = 0$, $6y - 18z + 1 = 0$

   Solve the following homogeneous systems of linear equations

6. $x - 2y + z - w = 0$, $x + y - 2z + 3w = 0$, $4x + y - 5z + 8w = 0$, $5x - 7y + 2z - w = 0$.
7. $3x + 2y + z = 0$, $2x + 3z = 0$, $y + 5z = 0$, $x + 2y + 3z = 0$
8. $x + 3y + 2z = 0$, $2x - y + 3z = 0$, $3x - 5y + 4z = 0$, $x + 17y + 4z = 0$.
9. $x + y - z + w = 0$, $2x + 3y + z + 4w = 0$, $3x + 2y - 6z + w = 0$
10. $x + y + z + w = 0$, $-x + y + z - w = 0$, $-x - y + z + w = 0$, $x + y - z + w = 0$.
11. Discuss the consistency of the system of equations $2x - 3y + 6z - 5w = 3$, $y - 4z + w = 1$, $4x - 5y + 8z - 9w = \lambda$ for various values of λ, if consistent, find the solution
12. Find the value of λ for which the following equations have non-trivial solution. Find the corresponding families of solutions $(\lambda - 1)x + (3\lambda + 1)y + 2\lambda z = 0$, $(\lambda - 1)x + (4\lambda - 2)y + (\lambda + 3)z = 0$, $2x + (3\lambda + 1)y + 3(\lambda - 1)z = 0$.

13. Show that the equations $3x + 4y + 5z = a, 4x + 5y + 6z = b, 5x + 6y + 7z = c$ are inconsistent unless $a + c = 2b$.

14. Find the value of λ for which the following system of equations possess a non-trivial solution and also find the solution in each case

$$2x - 2y + z = \lambda x, \quad 2x - 3y + 2z = \lambda y, \quad -x + 2y = \lambda z.$$

2.11 MATRIX EIGENVALUE PROBLEMS. CAYLEY-HAMILTON THEOREM

Let $\mathbf{A} = [a_{ij}]$ be a square matrix of order $n \times n$. The matrix eigenvalue problem concerns with set of equations of the form

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x}, \qquad \ldots(2.26)$$

where $\mathbf{x}$ is an unknown vector and λ is an unknown scalar and our aim is to determine both. Obviously, $\mathbf{x} = \mathbf{0}$ is a solution of (2.26) but trivial solution is of no practical interest. We are interested in non-trivial solutions. These are called *eigenvectors* of $\mathbf{A}$ and the values of λ for which such non-trivial solutions exist are called *eigenvalues* or *characteristic values* of $\mathbf{A}$. Geometrically, solving (2.26) implies looking for those vectors $\mathbf{x}$ which are transformed by the mapping $\mathbf{Ax}$ into vector $\lambda\mathbf{x}$ with components proportional to $\mathbf{x}$ itself, λ being the constant of proportionality.

The problem of determining the eigenvalues and eigenvectors of a matrix is called an eigenvalue problem.

Matrix eigenvalue problems find numerous applications in the field of engineering and physical sciences e.g. in the study of Markov processes, stretching of elastic membrane, vibrations of beams, population models, etc.

2.11.1 Eigenvalues and Eigenvectors

The set of equations (2.26) can be written as

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}. \qquad \ldots(2.27)$$

This homogeneous system of equations has a non-trivial solution if, and only if the determinant of the coefficient matrix $(\mathbf{A} - \lambda\mathbf{I})$ is zero, that is,

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} a_{11-\lambda} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22-\lambda} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn-\lambda} \end{vmatrix} = 0. \qquad \ldots(2.28)$$

The $|\mathbf{A} - \lambda\mathbf{I}|$ is called the *characteristic determinant* and the equation $|\mathbf{A} - \lambda\mathbf{I}| = 0$ is called the *characteristic equation of the matrix* $\mathbf{A}$. Expanding (2.28) we obtain characteristic equation of $\mathbf{A}$ of the form

$$|\mathbf{A} - \lambda\mathbf{I}| = (-1)^n \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \ldots + c_{n-1}\lambda + c_n = 0 \qquad \ldots(2.29)$$

a polynomial of degree n in λ, here c's are expressible in terms of the elements a_{ij}. Thus, the eigenvalues of a square matrix $\mathbf{A}$ are the roots of its characteristic equation (2.28).

The characteristic equation (2.28) has n roots which may be real or complex, simple or repeated, denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$ called the *eigenvalues*. The set of the eigenvalues is called the *spectrum of* **A** and the largest of absolute values of the eigenvalues of **A** is called the *spectral radius* of **A**.

After determining the eigenvalues λ_i's we solve the homogeneous equations $(\mathbf{A} - \lambda_i \mathbf{I})\,\mathbf{x} = 0$ for each λ_i to get the corresponding *eigenvector* $\mathbf{x}_i$. Corresponding to n distinct eigenvalues we get n independent eigenvectors. But corresponding to repeated eigenvalues, it may or may not be possible to get linearly independent eigenvectors.

Also, *if* **x** *is an eigenvector of a matrix* **A** *corresponding to an eigenvalues* λ, *then so is* $k\mathbf{x}$ *for any* $k \neq 0$, *since* $\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \Rightarrow \mathbf{A}\,(k\mathbf{x}) = \lambda(k\mathbf{x})$.

Thus the eigenvector corresponding to an eigenvalue is not unique in this sense.

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Example 2.42: Find the eigenvalues and eigenvectors of the following matrices:

(a) $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$ V.V.Imp * 85 (b) $\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$

Solution: (a) The characteristic equation of the matrix **A** is $|\mathbf{A} - \lambda\mathbf{I}| = 0$, that is, $\begin{vmatrix} 1-\lambda & -2 \\ -5 & 4-\lambda \end{vmatrix} = 0$

or, $\lambda^2 - 5\lambda - 6 = 0$, or $\lambda = 6, -1$. Thus the eigenvalues of **A** are 6 and -1.

Corresponding to $\lambda = 6$, the eigenvector **x** is given by, $(\mathbf{A} - 6\mathbf{I})\mathbf{x} = \mathbf{O}$,

or, $$\begin{bmatrix} -5 & -2 \\ -5 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This gives only one independent equation $5x_1 + 2x_2 = 0$. From this, we obtain $\dfrac{x_1}{2} = \dfrac{x_2}{-5}$ and thus the eigenvector is $[2, -5]^T$.

Corresponding to $\lambda = -1$, the eigenvector **x** is given by, $(\mathbf{A} + \mathbf{I})\mathbf{x} = \mathbf{O}$

or, $$\begin{bmatrix} 2 & -2 \\ -5 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This gives only one independent solution $x_1 - x_2 = 0$, or $x_1 = x_2$, and thus the eigenvector is $[1, 1]^T$.

(b) The characteristic equation for the matrix **A** is $|\mathbf{A} - \lambda\mathbf{I}| = 0$, which gives

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 2 & 3-\lambda & 2 \\ 3 & 3 & 4-\lambda \end{vmatrix} = 0.$$

On expansion, we obtain $\lambda^3 - 9\lambda^2 + 15\lambda - 7 = 0$, or $(\lambda - 1)(\lambda - 1)(\lambda - 7) = 0$.

Therefore, the eigenvalues of A are $\lambda = 1, 1, 7$.

For $\lambda = 1$, if $\mathbf{x}$ is the eigenvector, then we have $(\mathbf{A} - \mathbf{I})\mathbf{x} = \mathbf{O}$, which gives

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This gives only one independent equation $x_1 + x_2 + x_3 = 0$.
Taking $x_1 = 0, x_2 = 1$, and $x_3 = -1$; $x_1 = 1, x_2 = 0, x_3 = -1$.
We can have two linearly independent eigenvectors as $[0, 1, -1]^T$ and $[1, 0, -1]^T$.
For $\lambda = 7$, if $\mathbf{x}$ is the eigenvector, then we have $(\mathbf{A} - 7\mathbf{I})\mathbf{x} = \mathbf{O}$, which gives

$$\begin{bmatrix} -5 & 1 & 1 \\ 2 & -4 & 2 \\ 3 & 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} -5x_1 + x_2 + x_3 &= 0 \\ 2x_1 - 4x_2 + 2x_3 &= 0 \\ 3x_1 + 3x_2 - 3x_3 &= 0 \end{aligned}$$

From the first two equations we have

$$\frac{x_1}{6} = \frac{x_2}{2+10} = \frac{x_3}{20-2}, \text{ or } \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{3}.$$ Therefore the eigenvector is $[1, 2, 3]^T$.

Example 2.43: Find the eigenvalues and the eigenvectors of the following matrices:

(a) $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (c) $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Solution: (a) The characteristic equation for the matrix $\mathbf{A}$ is $|\mathbf{A} - \lambda\mathbf{I}| = 0$. It gives

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0, \text{ or } (1-\lambda)^3 = 0.$$

Therefore the eigenvalues for $\mathbf{A}$ are $\lambda = 1, 1, 1$.

Since eigenvalues are repeated, we will be interested to know whether the matrix $\mathbf{A}$, which is of order 3×3, has three independent eigenvectors or it has less than three independent eigenvectors.

For $\lambda = 1$, the eigenvector $\mathbf{x}$ is given by $(\mathbf{A} - \mathbf{I})\mathbf{x} = \mathbf{0}$, that is, $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

which gives $x_2 = 0, x_3 = 0$ and x_1 arbitrary. Choosing $x_1 = 1$, hence the only eigenvector is $[1, 0, 0]^T$.
(b) The eigenvalues are $\lambda = 1, 1, 1$. The eigenvector $\mathbf{x}$ corresponding to $\lambda = 1$ in this case is given by $(\mathbf{A} - \mathbf{I})\mathbf{x} = \mathbf{0}$, that is,

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which gives $x_2 = 0$, and x_1 and x_3 arbitrary. Taking $x_1 = 0, x_3 = 1$ and $x_1 = 1, x_3 = 0$, the two independent eigenvectors are $[0\ 0\ 1]^T$ and $[1\ 0\ 0]$.

(c) The eigenvalues are $\lambda = 1, 1, 1$. The eigenvector **x** corresponding to $\lambda = 1$ in this case is given by $(\mathbf{A} - \mathbf{I})\mathbf{x} = \mathbf{0}$, that is,

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

which gives x_1, x_2, x_3 arbitrary. Taking $x_1 = 1, x_2 = x_3 = 0$; $x_2 = 1, x_1 = x_3 = 0$; $x_1 = x_2 = 0, x_3 = 1$. The three independent eigenvectors are $[1\ 0\ 0]^T$, $[0\ 1\ 0]^T$, $[0\ 0\ 1]^T$.

Remark. The order M_λ of an eigenvalue λ as a root of the characteristic equation is called the *algebraic multiplicity* of λ. The number m_λ of linear independent eigenvectors corresponding to λ is called the *geometric multiplicity* of λ. The difference $M_\lambda - m_\lambda$ is called the *defect* of λ. In the above example the algebraic multiplicity of $\lambda = 1$ is 3, but, defect of λ is 2 in (a), 1 in (b) and 0 in (c).

2.11.2 Properties of Eigenvalues and Eigenvectors

1. *The sum of the eigenvalues of a matrix* **A** *is the sum of the elements of the principal diagonal, that is, equal to trace* (**A**).

We prove this for a square matrix **A** of order 3×3. We have

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix} = -\lambda^3 + (a_{11} + a_{22} + a_{33})\lambda^2 - \ldots \qquad \ldots(2.30)$$

If λ_1, λ_2 and λ_3 are the eigenvalues of the matrix **A**, then

$$|\mathbf{A} - \lambda\mathbf{I}| = (-1)^3 (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = -\lambda^3 + (\lambda_1 + \lambda_2 + \lambda_3)\lambda^2 - \ldots \qquad \ldots(2.31)$$

Comparing the right-hand sides of (2.30) and (2.31), we obtain

$$\lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33} = \text{trace } (\mathbf{A}).$$

2. *The product of the eigenvalue of a matrix* **A** *is equal to its determinant.*

Set $\lambda = 0$ in (2.31), we obtain $|\mathbf{A}| = (-1)^6 \lambda_1 \lambda_2 \lambda_3$, that is, $\lambda_1 \lambda_2 \lambda_3 = |\mathbf{A}|$.

From this result, we observe that *if* **A** *is singular then at least one of its eigenvalue is zero and conversely.*

3. *If λ is an eigenvalue of a matrix* **A**, *then $1/\lambda$ is the eigenvalue of $\mathbf{A}^{-1}$, provided the inverse exists.*

Let **x** be the eigenvector of **A** corresponding to the eigenvalue λ, then $\mathbf{Ax} = \lambda\mathbf{x}$. Pre-multiplying both sides of this by $\mathbf{A}^{-1}$, we get $\mathbf{A}^{-1}(\mathbf{Ax}) = \mathbf{A}^{-1}(\lambda\mathbf{x})$, or $\mathbf{Ix} = \lambda(\mathbf{A}^{-1}\mathbf{x})$, or $\mathbf{A}^{-1}\mathbf{x} = \dfrac{1}{\lambda}\mathbf{x}$.

Thus $1/\lambda$ is an eigenvalue of $\mathbf{A}^{-1}$.

4. *The matrices* $\mathbf{A}$ *and* $\mathbf{A}^T$ *has the same eigenvalues.*
This result follows from the fact that determinant $|\mathbf{A} - \lambda\mathbf{I}| = |\mathbf{A}^T - \lambda\mathbf{I}|$.

5. *If λ is an eigenvalue of an orthogonal matrix, then $1/\lambda$ is also its eigenvalue.*
Since, a matrix is orthogonal if $\mathbf{A}\mathbf{A}^T = \mathbf{I}_n$, that is, if $\mathbf{A}^T = \mathbf{A}^{-1}$. Thus the result follows from properties 3 and 4 above.

6. *If λ is an eigenvalue of a matrix* $\mathbf{A}$, *then λ^m is eigenvalue for* $\mathbf{A}^m$, *m being a positive integer.*
For, if $\mathbf{x}$ is the corresponding eigenvector of $\mathbf{A}$, then $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$.

Pre-multiplying by $\mathbf{A}$ on both sides, we obtain $\mathbf{A}^2\mathbf{x} = \mathbf{A}(\lambda\mathbf{x}) = \lambda(\mathbf{A}\mathbf{x}) = \lambda(\lambda\mathbf{x}) = \lambda^2\mathbf{x}$, and so on. In general, we have $\mathbf{A}^m\mathbf{x} = \lambda^m\mathbf{x}$

Thus λ^m is an eigenvalue of $\mathbf{A}^m$.

7. *If λ is an eigenvalue for* $\mathbf{A}$, *then $\lambda - k$ is an eigenvalue for* $\mathbf{A} - k\mathbf{I}$ *for any scalar k.*
For, $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ implies $\mathbf{A}\mathbf{x} - k\mathbf{I}\mathbf{x} = \lambda\mathbf{x} - k\mathbf{x}$, or $(\mathbf{A} - k\mathbf{I})\mathbf{x} = (\lambda - k)\mathbf{x}$
Thus $(\lambda - k)$ is an eigenvalue of $\mathbf{A} - k\mathbf{I}$. This property is called *spectral shift.*

8. *If λ is an eigenvalue of* $\mathbf{A}$, *then $k\lambda$ is an eigenvalue of $k\mathbf{A}$ for any non-zero scalar k.*
For, $\quad \mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ implies $k\mathbf{A}\mathbf{x} = k\lambda\mathbf{x}$, or $(k\mathbf{A})\mathbf{x} = (k\lambda)\mathbf{x}$

Thus $k\lambda$ is an eigenvalue of $k\mathbf{A}$.

9. *Corresponding to two distinct eigenvalues, eigenvectors are also distinct (linearly independent).*
Let $\mathbf{x}$ be the common eigenvector corresponding to the two distinct eigenvalues λ_1 and λ_2, then

$$\mathbf{A}\mathbf{x} = \lambda_1\mathbf{x} \text{ and } \mathbf{A}\mathbf{x} = \lambda_2\mathbf{x}, \text{ which gives } \lambda_1\mathbf{x} = \lambda_2\mathbf{x}, \text{ or } (\lambda_1 - \lambda_2)\mathbf{x} = \mathbf{0}$$

Since $\lambda_1 \neq \lambda_2$, thus $\mathbf{x} = \mathbf{0}$, a contradiction since eigenvectors are non-zero.
But the result is not true otherwise, refer to Example 2.43.
In fact we have the following result which we state without proof. *If λ is an eigenvalue of multiplicity of a square matrix* $\mathbf{A}$ *of order n, then the number of linearly independent eigenvectors associated with λ is $k = n - r$, where $r = rank\,(\mathbf{A} - \lambda\mathbf{I})$, $1 \leq k \leq m$.*

10. *Eigenvalues of diagonal and triangular matrices, upper and lower both, are same as the diagonal elements.*

11. *For a real matrix* $\mathbf{A}$, *if $\alpha + i\beta$ is an eigenvalue, then its conjugate $\alpha - i\beta$ is also an eigenvalue of* $\mathbf{A}$. *This result does not hold in case* $\mathbf{A}$ *is a complex matrix.*

Theorem 2.2 (Cayley-Hamilton Theorem) *Every square matrix satisfies its own characteristic equation.*

Proof. Let $\mathbf{A}$ be a square matrix of order $n \times n$, then its characteristic equation is

$$|\mathbf{A} - \lambda\mathbf{I}| = 0 \text{ or, } (-1)^n\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \ldots + c_{n-1}\lambda + c_n = 0.$$

Let $\mathbf{P}$ be the adjoint of the matrix $\mathbf{A} - \lambda\mathbf{I}$, then elements of $\mathbf{P}$ will be the polynomials of degree $(n-1)$ in λ. Thus the matrix $\mathbf{P}$ can be expressed in the form of a matrix polynomial

$$\mathbf{P} = \mathbf{P}_1\lambda^{n-1} + \mathbf{P}_2\lambda^{n-2} + \ldots + \mathbf{P}_{n-1}\lambda + \mathbf{P}_n,$$

where $\mathbf{P}_i$'s are all the square matrices of order n, whose elements are functions of the elements of $\mathbf{A}$.
Also we know that, $[\mathbf{A} - \lambda\mathbf{I}]\ adj.\ (\mathbf{A} - \lambda\mathbf{I}) = |\mathbf{A} - \lambda\mathbf{I}|\ \mathbf{I}$; here $\mathbf{I}$ is a unit matrix of order $n \times n$.

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Therefore, $[A - \lambda I][P_1\lambda^{n-1} + P_2\lambda^{n-2} + \ldots + P_{n-1}\lambda + P_n] = [(-1)^n\lambda^n + c_1\lambda^{n-1} + \ldots + c_{n-1}\lambda + c_n]I.$

Equating the coefficients of various powers of λ, we get

$$AP_1 = (-1)^n I$$
$$AP_1 - P_2 = c_1 I$$
$$AP_2 - P_3 = c_2 I$$
$$\ldots$$
$$AP_{n-1} - P_n = c_{n-1} I$$
$$AP_n = c_n I.$$

Pre-multiply these equations from top to bottom, respectively by $A^n, A^{n-1}, \ldots, A, I$ and add the terms on the left get cancelled in pair and we obtain

$$O = (-1)^n A^n + c_1 A^{n-1} + \ldots + c_{n-1} A + c_n I, \quad \ldots(2.32)$$

the desired result.

The result (2.32) gives an alternate method to find the inverse of a matrix A, pre-multiplying (2.32) by A^{-1} and rearranging the terms, we obtain

$$A^{-1} = -\frac{1}{c_n}[(-1)^n A^{n-1} + c_1 A^{n-2} + \ldots + c_{n-1} I] \quad \ldots(2.33)$$

Also (2.32) can be used to obtain A^n in terms of lower powers of n as

$$A^n = (-1)^{n+1}[c_1 A^{n-1} + c_2 A^{n-2} + \ldots + c_{n-1} A + c_n I]. \quad \ldots(2.34)$$

Example 2.44: Verify Cayley Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ and hence find A^{-1}.

Solution: The characteristic equation of the matrix A is $|A - \lambda I| = 0$, which gives

$$\begin{vmatrix} 1-\lambda & 2 & -2 \\ -1 & 3-\lambda & 0 \\ 0 & -2 & 1-\lambda \end{vmatrix} = 0, \text{ or } \lambda^3 - 5\lambda^2 + 9\lambda - 1 = 0.$$

To verify Cayley-Hamilton theorem we are to verify that

$$A^3 - 5A^2 + 9A - I = 0 \quad \ldots(2.35)$$

We have $A^2 = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix},$

$$\mathbf{A}^3=\mathbf{A}^2\mathbf{A}=\begin{bmatrix}-1 & 12 & -4\\ -4 & 7 & 2\\ 2 & -8 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & -2\\ -1 & 3 & 0\\ 0 & -2 & 1\end{bmatrix}=\begin{bmatrix}-13 & 42 & -2\\ -11 & 9 & 10\\ 10 & -22 & -3\end{bmatrix}$$

Therefore, $\mathbf{A}^3-5\mathbf{A}^2+9\mathbf{A}-\mathbf{I}$

$$=\begin{bmatrix}-13 & 42 & -2\\ -11 & 9 & 10\\ 10 & -22 & -3\end{bmatrix}-5\begin{bmatrix}-1 & 12 & -4\\ -4 & 7 & 2\\ 2 & -8 & 1\end{bmatrix}+9\begin{bmatrix}1 & 2 & -2\\ -1 & 3 & 0\\ 0 & -2 & 1\end{bmatrix}-\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}-13+5+9-1 & 42-60+18-0 & -2+20-18-0\\ -11+20-9-0 & 9-35+27-1 & 10-10+0-0\\ 10-10+0-0 & -22+40-18-0 & -3-5+9-1\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

This verifies Cayley Hamiltan theorem.

Pre-multiplying (2.35) by $\mathbf{A}^{-1}$ and rearranging the terms, we obtain

$$\mathbf{A}^{-1}=\mathbf{A}^2-5\mathbf{A}+9\mathbf{I}=\begin{bmatrix}-1 & 12 & -4\\ -4 & 7 & 2\\ 2 & -8 & 1\end{bmatrix}-5\begin{bmatrix}1 & 2 & -2\\ -1 & 3 & 0\\ 0 & -2 & 1\end{bmatrix}+9\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}3 & 2 & 6\\ 1 & 1 & 2\\ 2 & 2 & 5\end{bmatrix}$$

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Example 2.45: For $\mathbf{A}=\begin{bmatrix}1 & 0 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}$, show $\mathbf{A}^n=\mathbf{A}^{n-2}+\mathbf{A}^2-\mathbf{I}$, for $n\geq 3$, and hence find $\mathbf{A}^n$.

Solution: The characteristic equation of the matrix $\mathbf{A}$ is $|\mathbf{A}-\lambda\mathbf{I}|=0$, which gives

$$\begin{vmatrix}1-\lambda & 0 & 0\\ 1 & -\lambda & 1\\ 0 & 1 & -\lambda\end{vmatrix}=0, \text{ or } \lambda^3-\lambda^2-\lambda+1=0.$$

Using Cayley-Hamilton theorem, we have $\mathbf{A}^3-\mathbf{A}^2-\mathbf{A}+\mathbf{I}=\mathbf{0}$

$$\mathbf{A}^3-\mathbf{A}^2=\mathbf{A}-\mathbf{I} \qquad \ldots(2.36)$$

Pre-multiplying both sides successively by $\mathbf{A}$, we obtain

$$\mathbf{A}^4-\mathbf{A}^3=\mathbf{A}^2-\mathbf{A}$$

$$\mathbf{A}^5-\mathbf{A}^4=\mathbf{A}^3-\mathbf{A}^2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\cancel{\mathbf{A}^{n-1}}-\mathbf{A}^{n-2}=\cancel{\mathbf{A}^{n-3}}-\mathbf{A}^{n-4}$$

$$\mathbf{A}^n-\cancel{\mathbf{A}^{n-1}}=\mathbf{A}^{n-2}-\cancel{\mathbf{A}^{n-3}}$$

$A^n = A^{n+2}$

Adding these equations along with (2.36), we obtain

$$A^n - A^2 = A^{n-2} - I$$

$$\text{or,} \quad A^n = A^{n-2} + A^2 - I, n \geq 3. \qquad \ldots(2.37)$$

Using the Eq. (2.37) recursively, we obtain

$$\begin{aligned} A^n &= (A^{n-4} + A^2 - I) + A^2 - I \\ &= A^{n-4} + 2(A^2 - I) \\ &= A^{n-6} + 3(A^2 - I) \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &= A^{n-(n-2)} + \frac{n-2}{2}(A^2 - I) \\ &= A^2 + \frac{n-2}{2}(A^2 - I) = \frac{n}{2}A^2 - \frac{1}{2}(n-2)I. \end{aligned} \qquad \ldots(2.38)$$

We have,
$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Thus from (2.38)

$$A^n = \frac{n}{2}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - \frac{1}{2}(n-2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ n/2 & 1 & 0 \\ n/2 & 0 & 1 \end{bmatrix}$$

EXERCISE 2.8

1. For the matrix, $A = \begin{bmatrix} -2 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 6 \end{bmatrix}$ verify that if λ_i's are the eigenvalues of A, then

   (a) $\lambda_i + 2$ are the eigenvalues of $A + 2I$,
   (b) $3\lambda_i$ are the eigenvalues of $3A$
   (c) $\dfrac{1}{\lambda_i}$ are the eigenvalues of A^{-1},
   (d) λ^2 are the eigenvalues of A^2.

2. Find the sum and product of the eigenvalues of the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 5 & 6 \\ 7 & 4 & 3 & 2 \\ 4 & 3 & 0 & 5 \end{bmatrix}$

Find the eigenvalues and the eigenvectors of the matrices (3–5)

3. $\begin{bmatrix} 4 & -6 & -6 \\ 0 & -2 & 0 \\ 1 & -1 & -1 \end{bmatrix}$ 4. $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ 5. $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Verify Cayley-Hamilton theorem for the matrices (6–8). Find inverse, if it exists.

6. $\begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ 7. $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ 8. $\begin{bmatrix} 2 & -1 & +1 \\ -1 & 2 & -1 \\ 1 & -1 & -2 \end{bmatrix}$

9. If $\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$, then show $\mathbf{A}^8 - 5\mathbf{A}^7 + 7\mathbf{A}^6 - 3\mathbf{A}^5 + \mathbf{A}^4 - 5\mathbf{A}^3 + 8\mathbf{A}^2 - 2\mathbf{A} + \mathbf{I} = \mathbf{0}$.

10. Find $\mathbf{A}^8$, if $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.

2.12 SIMILAR MATRICES. DIAGONALIZATION

In this section we explore the properties of eigenvectors further. If an $n \times n$ square matrix **A** has n linearly independent eigenvectors they can be used for transforming **A** into diagonal form, an aspect important from application point of view.

2.12.1 Similar Matrices

A square matrix $\hat{\mathbf{A}}$ of order $n \times n$ is called *similar* to a square matrix **A** of order $n \times n$ if

$$\hat{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} \qquad \ldots(2.39)$$

for some non-singular matrix **P**.

The transformation (2.39) which gives $\hat{\mathbf{A}}$ from **A**, is called a *similarity transformation* and **P** *similarity matrix*.

Pre-multiplying both sides of (2.39) by **P** and post-multiplying by $\mathbf{P}^{-1}$, we get

$$\mathbf{P}\hat{\mathbf{A}}\mathbf{P}^{-1} = \mathbf{A} \qquad \ldots(2.40)$$

Therefore, **A** is similar to $\hat{\mathbf{A}}$, if, and only if $\hat{\mathbf{A}}$ is similar to **A**.

The similarity transformations are important since they preserve eigenvalues.

We have the following result:

Theorem 2.3. *If $\hat{\mathbf{A}}$ is similar to* **A**, *then $\hat{\mathbf{A}}$ has the same eigenvalue as* **A**, *and further if* **x** *is an eigenvector of* **A** *corresponding to the eigenvalue λ, then $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$ is an eigenvector of $\hat{\mathbf{A}}$ corresponding to the same eigenvalue λ, where* **P** *is the similarity matrix.*

Proof. Since λ is an eigenvalue and $\mathbf{x}$ is the corresponding eigenvector of $\mathbf{A}$, thus $\mathbf{Ax} = \lambda\mathbf{x}$ Pre-multiplying both sides by $\mathbf{P}^{-1}$, we get ...(2.41)

$$\mathbf{P}^{-1}\mathbf{Ax} = \mathbf{P}^{-1}\lambda\mathbf{x} = \lambda\mathbf{P}^{-1}\mathbf{x} = \lambda\mathbf{y}$$

Consider

$$\begin{aligned}\mathbf{P}^{-1}\mathbf{Ax} &= \mathbf{P}^{-1}\mathbf{AIx}\\ &= \mathbf{P}^{-1}\mathbf{APP}^{-1}\mathbf{x}, \quad \text{since } \mathbf{I} = \mathbf{PP}^{-1}\\ &= \hat{\mathbf{A}}\,(\mathbf{P}^{-1}\mathbf{x}) = \hat{\mathbf{A}}\mathbf{y}.\end{aligned}$$

Hence from (2.41),

$$\hat{\mathbf{A}}\mathbf{y} = \lambda\mathbf{y}$$

Thus λ is an eigenvalue of $\hat{\mathbf{A}}$ and the corresponding eigenvector is $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$. Further, $\mathbf{P}^{-1}\mathbf{x} \neq \mathbf{0}$, since $\mathbf{P}^{-1}\mathbf{x} = \mathbf{0}$ would give $\mathbf{PP}^{-1}\mathbf{x} = \mathbf{P0} = \mathbf{0}$, or $\mathbf{Ix} = \mathbf{0}$, or $\mathbf{x} = \mathbf{0}$, which is a contradiction.

We must note that the converse of the above result is not true. *Two matrices which have the same eigenvalues may not be similar.*

Also, if $\mathbf{A}$ is similar to $\hat{\mathbf{A}}$ and $\hat{\mathbf{A}}$ is similar to $\hat{\mathbf{A}}_1$, then $\mathbf{A}$ is also similar to $\hat{\mathbf{A}}_1$, since

$$\hat{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{AP},\ \hat{\mathbf{A}}_1 = \mathbf{Q}^{-1}\hat{\mathbf{A}}\mathbf{Q}, \text{ for some non-singular matrices } \mathbf{P} \text{ and } \mathbf{Q}, \text{ gives}$$

$$\hat{\mathbf{A}}_1 = \mathbf{Q}^{-1}(\mathbf{P}^{-1}\mathbf{AP})\,\mathbf{Q} = (\mathbf{PQ})^{-1}\mathbf{A}(\mathbf{PQ}) = \mathbf{R}^{-1}\mathbf{AR},$$

where $\mathbf{R} = \mathbf{PQ}$ is non-singular.

2.12.2 Diagonalization of a Matrix

Generally, it is difficult to find a non-singular matrix $\mathbf{P}$ which satisfies the equation $\hat{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{AP}$ for any two matrices $\hat{\mathbf{A}}$ and $\mathbf{A}$.

However, it is possible to find $\mathbf{P}$ when $\hat{\mathbf{A}}$ or $\mathbf{A}$ is a diagonal matrix. Thus, for a given matrix $\mathbf{A}$ our interest is to find a non-singular matrix $\mathbf{P}$ such that

$$\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}, \qquad \text{...(2.42)}$$

where $\mathbf{D}$ is a diagonal matrix. If such a matrix exists we say that $\mathbf{A}$ is *diagonalizable matrix* and the transformation (2.42) is called the *diagonalization transformation.*

Further, since similar matrices have the same eigenvalues thus the diagonal elements of $\mathbf{D}$ are the eigenvalues of $\mathbf{A}$.

We have the following result:

Theorem 2.4. *If a square matrix $\mathbf{A}$ of order $n \times n$ has n linearly independent eigenvectors, then*

$$\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP} \qquad \text{...(2.43)}$$

is diagonal matrix, with the eigenvalues of $\mathbf{A}$ as the elements on the principal diagonal, and the similarity matrix $\mathbf{P}$ is the matrix with eigenvectors of $\mathbf{A}$ as column vectors.

Proof. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be n linearly independent eigenvectors corresponding respectively to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the matrix $\mathbf{A}$. Further let

$$\mathbf{P} = [\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_n] \text{ and } \mathbf{D} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n).$$

We have, $\mathbf{AP} = \mathbf{A}[\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_n] = [\mathbf{Ax}_1, \mathbf{Ax}_2, \ldots, \mathbf{Ax}_n]$

$$= [\lambda_1\mathbf{x}_1, \lambda_2\mathbf{x}_2, \ldots, \lambda_n\mathbf{x}_n] = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n]\,\mathbf{D} = \mathbf{PD}. \qquad \ldots(2.44)$$

Since, the columns of **P** are linearly independent vectors, the matrix **P** is of rank n and, therefore, is invertible. Pre-multiplying both sides of (2.44) by $\mathbf{P}^{-1}$, we obtain $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$, which is (2.43).

The matrix **P** here is called the *modal matrix* of **A** and the diagonal matrix **D** is called the *spectral matrix* of **A**.

We must note that in the above result the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ not necessarily be distinct since even if eigenvalues are repeated the corresponding eigenvectors may be independent.

2.12.3 Calculation of Power of a Matrix

Diagonalization of a matrix is quite useful for obtaining powers of a matrix.

Let **A** be a square matrix of order n with n linearly independent eigenvectors, then $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$. It gives $\mathbf{A} = \mathbf{PDP}^{-1}$. Thus,

$$\mathbf{A}^2 = \mathbf{AA} = (\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1}) = \mathbf{PD}^2\mathbf{P}^{-1}$$

$$\mathbf{A}^3 = \mathbf{A}^2\mathbf{A} = (\mathbf{PD}^2\mathbf{P}^{-1})(\mathbf{PDP}^{-1}) = \mathbf{PD}^3\mathbf{P}^{-1}$$

In general, we have $\mathbf{A}^n = \mathbf{PD}^n\mathbf{P}^{-1}$ for positive integers n.

Example 2.46: Examine whether the following matrices are diagonizable or not.

(a) $\begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$

Solution: (a) The matrix $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$ has eigenvalues $\lambda = 6, -1$, and the corresponding linearly independent eigenvectors are $[2,\ -5]^T$ and $[1,\ 1]^T$, refer to Example 2.42a. Hence, the given matrix is diagonizable.

(b) The matrix $\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$ has eigenvalues $\lambda = 1, 1, 7$, refer to Example 2.42b.

Corresponding to $\lambda = 1$, there are two linearly independent eigenvectors $[0,\ 1,\ -1]^T$ and $[1,\ 0,\ -1]^T$. Corresponding to $\lambda = 7$, the eigenvector is $[1,\ 2,\ 3]^T$. Since, the matrix has three linearly independent eigenvectors, hence it is diagonizable.

Example 2.47: Show that the matrix $\mathbf{A} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is digonalizable. Find **P** such that $\mathbf{P}^{-1}\mathbf{AP}$ is a diagonal matrix and hence find $\mathbf{A}^2$.

Solution: The characteristic equation of the matrix **A** is $\begin{vmatrix} 1-\lambda & 6 & 1 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0.$

It gives $(3-\lambda)\,[(1-\lambda)(2-\lambda)-6] = 0$, or $(\lambda+1)(\lambda-3)(\lambda-4) = 0$. Thus, the eigenvalues of A are $\lambda = -1$, 3, 4.

For $\lambda = -1$, the eigenvector $\mathbf{x}_1$ is given by

$$\begin{bmatrix} 2 & 6 & 1 \\ 1 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} 2x_1 + 6x_2 + x_3 &= 0 \\ x_1 + 3x_2 &= 0 \\ 4x_3 &= 0 \end{aligned}$$

This gives $x_3 = 0$, $x_1 = -3x_2$. Take $x_2 = 1$, the eigenvector is $\mathbf{x}_1 = [-3,\ 1,\ 0]^T$

For $\lambda = 3$, the eigenvector $\mathbf{x}_2$ is given by

$$\begin{bmatrix} -2 & 6 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} -2x_1 + 6x_2 + x_3 &= 0 \\ x_1 - x_2 &= 0 \end{aligned}$$

This gives $x_1 = x_2$, $x_3 = -4x_2$. Take $x_2 = 1$, the eigenvector is $\mathbf{x}_2 = [1,\ 1,\ -4]^T$.

For $\lambda = 4$, the eigenvector $\mathbf{x}_3$ is given by

$$\begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} -3x_1 + 6x_2 + x_3 &= 0 \\ x_1 - 2x_2 &= 0 \\ -x_3 &= 0 \end{aligned}$$

This gives $x_3 = 0$, $x_1 = 2x_2$. Take $x_2 = 1$, the eigenvector $\mathbf{x}_3 = [2,\ 1,\ 0]^T$

Since **A** has three linearly independent eigenvectors, thus it is diagonizable. The modal matrix **P** of **A** is $\mathbf{P} = \begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}$

To find $\mathbf{P}^{-1}$, we have $|\mathbf{P}| = -20$, and $adj\,\mathbf{P} = \begin{bmatrix} 4 & -8 & 1 \\ 0 & 0 & 5 \\ -4 & -12 & -4 \end{bmatrix}$

Thus, $\mathbf{P}^{-1} = \dfrac{adj\,\mathbf{P}}{|\mathbf{P}|} = \dfrac{1}{20}\begin{bmatrix} -4 & 8 & -1 \\ 0 & 0 & -5 \\ 4 & 12 & 4 \end{bmatrix}$, and hence the spectral matrix **D** is

$$D = P^{-1}AP = \frac{1}{20}\begin{bmatrix} -4 & 8 & -1 \\ 0 & 0 & -5 \\ 4 & 12 & 4 \end{bmatrix}\begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

To find A^2, we have $A^2 = PD^2P^{-1}$?

$$= \frac{1}{20}\begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}\begin{bmatrix} (-1)^2 & 0 & 0 \\ 0 & (3)^2 & 0 \\ 0 & 0 & (4)^2 \end{bmatrix}\begin{bmatrix} -4 & 8 & 1 \\ 0 & 0 & -5 \\ 4 & 12 & 4 \end{bmatrix}$$

$$= \frac{1}{20}\begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 16 \end{bmatrix}\begin{bmatrix} -4 & 8 & 1 \\ 0 & 0 & -5 \\ 4 & 12 & 4 \end{bmatrix} = \frac{1}{20}\begin{bmatrix} 140 & 360 & 80 \\ 60 & 200 & 20 \\ 0 & 0 & 180 \end{bmatrix} = \begin{bmatrix} 7 & 18 & 4 \\ 3 & 10 & 1 \\ 0 & 0 & 9 \end{bmatrix}$$

Example 2.48: The eigenvectors of a 3×3 matrix **A** corresponding to the eigenvalues 1, 2, 3 are $[-1, -1, 1]^T$, $[0, 1, 0]^T$ and $[0, -1, 1]^T$ respectively. Find the matrix **A**.

Solution: The modal and spectral matrices of **A** are, respectively

$$P = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

We have $P^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ and, therefore,

$$A = PDP^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & -1 \\ 2 & 0 & 3 \end{bmatrix}$$

EXERCISE 2.9

1. For the matrices $A = \begin{bmatrix} 10 & -3 & 5 \\ 0 & 1 & 0 \\ -15 & 9 & -10 \end{bmatrix}$ and $P = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 5 \end{bmatrix}$, find $\hat{A} = P^{-1}AP$. Also find the eigenvectors **y** of $\hat{A}$ and show that $x = Py$ are the eigenvectors of **A**.

Test the following matrices (2–4) for diagonalization. If diagonalizable, find **P** such that $P^{-1}AP$ is a diagonal matrix.

2. $\begin{bmatrix} 5 & 10 & -10 \\ 10 & 5 & -20 \\ 5 & -5 & -10 \end{bmatrix}$ 3. $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$ 4. $\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$

Find the matrix **A** whose eigenvalues and the corresponding eigenvectors are given as below.

5. Eigenvalues 1, –1, 2; Eigenvectors $[1,\ 1,\ 0]^T$, $[1,\ 0,\ 1]^T$, $[3,\ 1,\ 1]^T$
6. Eigenvalues 0, 0, 3; Eigenvectors $[1,\ 2,\ -1]^T$, $[-2,\ 1,\ 0]^T$, $[3,\ 0,\ 1]^T$
7. Eigenvalues 3, –4, 0; Eigenvectors $[-1,\ 3,\ -1]^T$, $[1,\ -1,\ 3]^T$, $[2,\ 1,\ 4]$
8. Find a matrix **P** which transforms the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ to diagonal form. Hence, calculate $\mathbf{A}^4$.
9. If $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$, then show that $\mathbf{A}^8 = \begin{bmatrix} -12099 & 12355 & 6305 \\ -12100 & 12356 & 6305 \\ -13120 & 13120 & 6561 \end{bmatrix}$
10. Show that the matrix $\mathbf{A} = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$, $a \neq b$ is transformed to a diagonal form $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$, when **P** $= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, where $\theta = \frac{1}{2}\tan^{-1}\left(\frac{2h}{a-b}\right)$.

2.13 SPECIAL MATRICES EIGENVALUES

In case of *real square matrices* we have already considered three special types of matrices namely, *symmetric, skew-symmetric* and *orthogonal*, refer to Section 2.3. These matrices occur quite frequently in applications.

In case of *complex square matrices*, the three special types of matrices are the generalization of the matrices considered in case of reals. These are *Hermitian, skew-Hermitian* and *unitary matrices*, refer to Section 2.3.

It is quite interesting to note that the spectra, that is the set of eigenvalues of these special matrices can be located as follows, refer to Fig. 2.1.

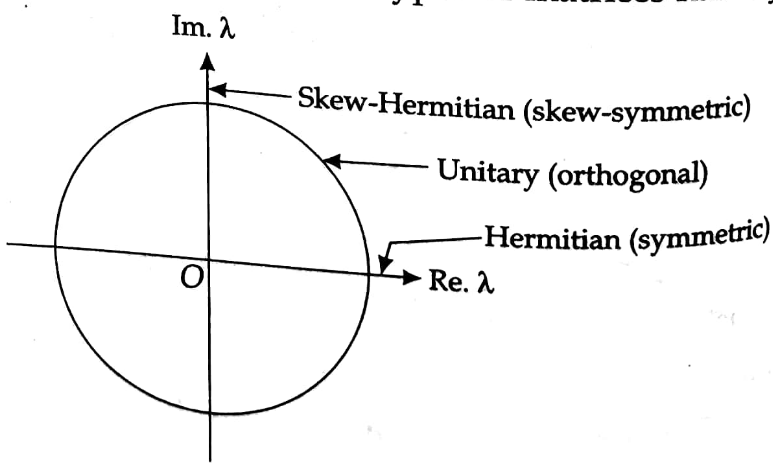

Fig. 2.1

We have the following result concerning the eigenvalues of special matrices:

Theorem (2.5) (Eigenvalues of Special Matrices)

(a) *The eigenvalues of a Hermitian matrix (and thus, of a symmetric matrix) are real.*

(b) *The eigenvalues of a skew-Hermitian matrix (and thus, of a skew-symmetric matrix) are pure imaginary or zero.*

(c) *The eigenvalues of a unitary matrix (and thus, of an orthogonal matrix) have absolute value 1.*

Proof. Let λ be an eigenvalue of a matrix $\mathbf{A}$ and $\mathbf{x}$ be the corresponding eigenvector. Then $\mathbf{Ax} = \lambda\mathbf{x}$. Pre-multiplying both sides of this *by* $\bar{\mathbf{x}}^T$, we get

$$\bar{\mathbf{x}}^T \mathbf{Ax} = \lambda \bar{\mathbf{x}}^T \mathbf{x}, \quad \text{or} \quad \lambda = \frac{\bar{x}^T Ax}{\bar{x}^T \mathbf{x}}. \qquad \ldots(2.45)$$

We note that $\bar{\mathbf{x}}^T \mathbf{Ax}$ and $\bar{\mathbf{x}}^T \mathbf{x}$ are scalars and the denominator $\bar{\mathbf{x}}^T \mathbf{x}$ is always real and positive. Therefore, the nature of λ depends upon the nature of the numerator $\bar{\mathbf{x}}^T \mathbf{Ax}$.

(a) *The matrix* $\mathbf{A}$ *is Hermitian, that is,* $\mathbf{A}^T = \bar{\mathbf{A}}$.

Since $\bar{\mathbf{x}}^T \mathbf{Ax}$ is scalar, we have

$$\bar{\mathbf{x}}^T \mathbf{Ax} = (\bar{\mathbf{x}}^T \mathbf{Ax})^T = \mathbf{x}^T \mathbf{A}^T \bar{\mathbf{x}} = \mathbf{x}^T \bar{\mathbf{A}} \bar{\mathbf{x}} = \overline{(\bar{\mathbf{x}}^T \mathbf{Ax})} \qquad \ldots(2.46)$$

Hence $\bar{\mathbf{x}}^T \mathbf{Ax}$ is real, and thus from (2.45) λ is real.

(b) *The matrix* $\mathbf{A}$ *is skew-Hermitian, that is* $\mathbf{A}^T = -\bar{\mathbf{A}}$

Proceeding as in (2.46), in this case, we obtain $\bar{\mathbf{x}}^T \mathbf{Ax} = -\overline{(\bar{\mathbf{x}}^T \mathbf{Ax})}$ so $\bar{\mathbf{x}}^T \mathbf{Ax}$ is purely imaginary or zero, therefore, from (2.45), λ is purely imaginary or $\lambda = 0$.

(c) *The matrix A is unitary, that is,* $\bar{\mathbf{A}}^T = \mathbf{A}^{-1}$.

Consider
$$\mathbf{Ax} = \lambda\mathbf{x} \qquad \ldots(2.47)$$
and its conjugate transpose
$$(\bar{\mathbf{A}}\bar{\mathbf{x}})^T = (\bar{\lambda}\bar{\mathbf{x}})^T \qquad \ldots(2.48)$$

From (2.47) and (2.48), we have

$$(\bar{\mathbf{A}}\bar{\mathbf{x}})^T (\mathbf{Ax}) = (\bar{\lambda}\bar{\mathbf{x}})^T (\lambda\mathbf{x}) = |\lambda|^2 \bar{\mathbf{x}}^T \mathbf{x}.$$

or,
$$\bar{\mathbf{x}}^T \bar{\mathbf{A}}^T \mathbf{Ax} = |\lambda|^2 \bar{\mathbf{x}}^T \mathbf{x}, \text{ or } \bar{\mathbf{x}}^T \mathbf{x} = |\lambda|^2 \bar{\mathbf{x}}^T \mathbf{x}, \text{ since } \bar{\mathbf{A}}^T = \bar{\mathbf{A}}^{-1}.$$

Since $\bar{\mathbf{x}}^T \mathbf{x} \neq 0$, we have $|\lambda|^2 = 1$, and hence, $|\lambda| = 1$.

This completes the proof.

Remark. The numerator $\bar{\mathbf{x}}^T \mathbf{Ax}$ in (2.45) is called a *form* in the components $x_1, x_2, \ldots, x_n$; and $\mathbf{A}$ is called its *coefficient matrix*.

Example 2.49: For the Hermitian, skew-Hermitian and unitary matrices given respectively as

$$\mathbf{A} = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}, \text{ and } \mathbf{C} = \begin{bmatrix} \frac{1}{2}i & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2}i \end{bmatrix}.$$

find the eigenvalues.

Solution: For the *Hermitian matrix* **A**, the characteristic equation is $(4-\lambda)(7-\lambda) - 10 = 0$, or $\lambda^2 - 11\lambda + 18 = 0$, which gives eigenvalues as $\lambda = 9, 2$.

For the *skew-Hermitian matrix* **B**, the characteristic equation is $(3i - \lambda)(-i - \lambda) - 5 = 0$, or $\lambda^2 - 2i\lambda + 8 = 0$, which gives eigenvalues as $\lambda = 4i, -2i$.

Similarly, for the *unitary matrix* **C**, the characteristic equation is $\lambda^2 - i\lambda - 1 = 0$, which gives the eigenvalues as $\lambda = \frac{1}{2}i \pm \frac{1}{2}\sqrt{3}$. We observe that $|\lambda| = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$.

2.14 QUADRATIC FORMS. REDUCTION TO CANONICAL FORM

In this section we discuss quadratic forms and their reduction to canonical form. The quadratic forms find applications in physics and geometry e.g., in conic sections and quadratic surfaces.

2.14.1 Quadratic Form

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ be an arbitrary vector in $\mathbb{R}^n$, then quadratic form is an expression of the form

$$\mathbf{Q} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j.$$

For example, in case of $n = 3$, the quadratic form is

$$\mathbf{Q} = a_{11}x_1^2 + (a_{12} + a_{21})x_1x_2 + a_{22}x_2^2 + (a_{23} + a_{32})x_2x_3 + a_{33}x_3^2 + (a_{31} + a_{13})x_3^2$$

$$= [x_1, x_2, x_3]\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \mathbf{x}^T\mathbf{A}\mathbf{x} \qquad \text{...(2.49)}$$

Using the definition of matrices multiplication, set $c_{ij} = \frac{a_{ij} + a_{ji}}{2}$, then the matrix $\mathbf{C} = [c_{ij}]$ is symmetric, since $c_{ij} = c_{ji}$, and also, $c_{ij} + c_{ji} = a_{ij} + a_{ji}$. The form (2.49) can be expressed as

$$\mathbf{Q} = \mathbf{x}^T\mathbf{C}\mathbf{x}, \qquad \text{...(2.50)}$$

where the matrix $\mathbf{C} = [c_{ij}]$ is symmetric with $c_{ij} = \frac{a_{ij} + a_{ji}}{2}$. For example, for $n = 3$.

$c_{11} = a_{11}, c_{22} = a_{22}, c_{33} = a_{33}, c_{12}\, c_{21} = \dfrac{a_{12} + a_{21}}{2}, \quad c_{13} = c_{31} = \dfrac{a_{13} + a_{31}}{2}$, and $c_{23} = c_{32} = \dfrac{a_{23} + a_{32}}{2}$.

Example 2.50: Obtain the corresponding symmetric matrix for the quadratic form
(a) $\mathbf{Q} = 3x_1^2 + 10x_1x_2 + 2x_2^2$ (b) $\mathbf{Q} = x_1^2 + 3x_2^2 + 2x_3^2 + 2x_1x_2 + 6x_2\, x_3$.

Solution: (a) Here $a_{11} = 3, a_{12} + a_{21} = 10, a_{22} = 2$. Therefore, $c_{11} = 3, c_{12} = c_{21} = \dfrac{10}{2} = 5, c_{22} = 2$.

The symmetric matrix is $\mathbf{C} = \begin{bmatrix} 3 & 5 \\ 5 & 2 \end{bmatrix}$

(b) Here $a_{11} = 1, a_{22} = 3, a_{33} = 2, a_{12} + a_{21} = 2, a_{23} + a_{22} = 6, a_{13} + a_{31} = 0$

Therefore, $c_{11} = 1, c_{22} = 3, c_{33} = 2, c_{12} = c_{21} = 1, c_{23} = c_{32} = 3, c_{13} = c_{31} = 0$

The symmetric matrix is $\mathbf{C} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 3 \\ 0 & 3 & 2 \end{bmatrix}$

2.14.2 Definiteness

A real quadratic form $\mathbf{Q} = \mathbf{x}^T\mathbf{A}\mathbf{x}$ and its matrix $\mathbf{A} = [a_{ij}]$ are said to be

(a) *positive definite,* if $\mathbf{Q} > 0$ for all $\mathbf{x} \neq \mathbf{0}$
(b) *negative definite,* if $\mathbf{Q} < 0$ for all $\mathbf{x} \neq \mathbf{0}$
(c) *semi-positive definite,* if $\mathbf{Q} \geq 0$ for all $\mathbf{x} \neq \mathbf{0}$
(d) *semi-negative definite,* if $\mathbf{Q} \leq 0$ for all $\mathbf{x} \neq \mathbf{0}$
(e) *indefinite,* if $\mathbf{Q}$ is not of definite sign for all $\mathbf{x} \neq \mathbf{0}$

We state following results (without proofs) in connection with the definiteness of a matrix **A**:

1. *A necessary and sufficient condition for positive definiteness is that all the leading minors, that is,*

$$A_1 = a_{11} \quad A_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, A_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \ldots, A_n = \det \mathbf{A}$$

are positive.

2. *The eigenvalues of a positive definite matrix are real and positive, and the eigenvalues of a negative definite matrix are real and negative. In case of semi-positive definite (semi-negative definite) matrix at least one eigenvalue is zero, and rest are positive (negative). In case of indefiniteness some eigenvalues are positive and some are negative.*

Example 2.51: Examine the nature of the following matrices

(a) $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$ (b) $\mathbf{A} = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Solution: (a) The quadratic form of the matrix **A** is given by

$$Q = \mathbf{x}^T\mathbf{A}\mathbf{x} = 3x_1^2 + 3x_1x_2 + 4x_2^2 = 3\left[x_1^2 + x_1x_2 + \frac{1}{4}x_2^2\right] + \frac{13}{4}x_2^2 = 3\left[x_1 + \frac{1}{2}x_2\right]^2 + \frac{13}{4}x_2^2 > 0$$

unless **x** = 0. Hence **A** is *positive definite*.

Remark. We can verify that eigenvalues of **A** are 2 and 5 both positive; and the leading minors of **A** are 3 and 10, again both positive. We must note it is sufficient to verify only one of these three aspects to verify the nature of a matrix.

(b) The characteristic equation of the matrix **A** is $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$, which gives eigenvalues as $\lambda = 5, -3, -3$. Since, the eigenvalues are positive as well as negative. The matrix **A** is *indefinite.*

Example 2.52: Find the nature of the quadratic form:

(a) $Q = x^2 + 5y^2 + z^2 + 2xy + 2yz + 6zx$ (b) $Q = 2x^2 + 2y^2 + 3z^2 + 2xy - 4yz - 4zx$

Solution: (a) The matrix of the quadratic form is $\mathbf{A} = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.

Its characteristic equation gives

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0, \text{ or } (\lambda + 2)(\lambda - 3)(\lambda - 6) = 0.$$

Hence its eigenvalues are $\lambda = -2, 3, 6$. Since the two of these being positive and one is negative, thus the given quadratic form is indefinite.

(b) The matrix of the quadratic form is $\mathbf{A} = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & -2 \\ -2 & -2 & 3 \end{bmatrix}$. Its characteristic equation gives

$$\begin{vmatrix} 2-\lambda & 1 & -2 \\ 1 & 2-\lambda & -2 \\ -2 & -2 & 3-\lambda \end{vmatrix} = 0, \text{ or } \lambda^3 - 7\lambda^2 + 7\lambda - 1 = 0$$

Solving we obtain, $\lambda = 1, \lambda = 3 \pm \sqrt{8}$ as its eigenvalues which are all positive. So the given quadratic form is positive definite.

2.14.3 Reduction of Quadratic Form to Canonical Form

If by any real non-singular linear transformation, a real quadratic form can be expressed as a sum and difference of the squares of the new variables, then this latter expression is called the canonical form of the given quadratic form.

For example, consider the quadratic form for the three components x_1, x_2, x_3. It is

$$\mathbf{Q} = \mathbf{x}^T\mathbf{A}\mathbf{x} = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + (a_{12}+a_{21})x_1x_2 + (a_{23}+a_{32})x_2x_3 + (a_{31}+a_{13})x_3x_1,$$

where the coefficient matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is symmetric.

Let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of the matrix $\mathbf{A}$ (not necessarily all different) and $\mathbf{x}_1 = [x_{11}, x_{21}, x_{31}]^T$, $\mathbf{x}_2 = [x_{12}, x_{22}, x_{32}]^T$, $\mathbf{x}_3 = [x_{13}, x_{23}, x_{33}]^T$ be the corresponding *linearly independent eigenvectors* in the normalized form, that is, each element in an eigenvectors is divided by square root of sum of the squares of all three elements in that eigenvector.

The modal matrix $\mathbf{P}$ given by $\mathbf{P} = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$ is orthogonal matrix and the transformation defined by

$$\mathbf{x} = \mathbf{P}\mathbf{y} \qquad \ldots(2.51)$$

is the *orthogonal transformation.*

This transforms the quadratic form $\mathbf{Q}$ to the canonical form as follows:

We have, $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$, or $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$, where $\mathbf{D} = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$ is the spectral matrix.

Consider
$$\begin{aligned}
\mathbf{Q} &= \mathbf{x}^T\mathbf{A}\mathbf{x} = \mathbf{x}^T\mathbf{P}\mathbf{D}\mathbf{P}^{-1}\mathbf{x} \\
&= \mathbf{x}^T(\mathbf{P}^T)^T\mathbf{D}\mathbf{P}^{-1}\mathbf{x}, \\
&= \mathbf{x}^T(\mathbf{P}^{-1})^T\mathbf{D}\mathbf{P}^{-1}\mathbf{x}, \text{ since } \mathbf{P} \text{ is orthogonal, that is, } \mathbf{P}^T = \mathbf{P}^{-1} \\
&= (\mathbf{P}^{-1}\mathbf{x})^T\mathbf{D}\mathbf{P}^{-1}\mathbf{x} \\
&= \mathbf{y}^T\mathbf{D}\mathbf{y}, \text{ since } \mathbf{P}^{-1}\mathbf{x} = \mathbf{y}, \text{ refer to (2.51)} \\
&= [y_1, y_2, y_3]\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2,
\end{aligned} \qquad \ldots(2.52)$$

which is the desired canonical form.

The form (2.52) is also known as *sum of the squares form* or *the principal axes form.*

The number of positive square terms in the canonical form is called the *index* of the quadratic form and the difference between the number of positive and negative square terms is called the *signature* of the form.

Example 2.53: Transform the quadratic form $\mathbf{Q} = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$ to the canonical form. What type of conic section does it represent? Also find the matrix of transformation and the transformation.

Solution: Here the coefficient matrix is $\mathbf{A} = \begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. The characteristic equation of $\mathbf{A}$ is $(17-\lambda)^2 - 225 = 0$, which gives $\lambda = 2$ and $\lambda = 32$ as its two eigenvalues.

Hence the given quadratic form becomes $\mathbf{Q} = 2y_1^2 + 32y_2^2$. Thus $\mathbf{Q} = 128$ represents the ellipse

$$2y_1^2 + 32y_2^2 = 128, \quad \text{or} \quad \frac{y_1^2}{64} + \frac{y_2^2}{4} = 1.$$

To find **P**, the matrix of transformation, we find eigenvectors of **A**.

For $\lambda = 2$, $(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{x} = \mathbf{0}$, gives $\begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 15x_1 - 15x_2 = 0$, or $x_1 = x_2$

Taking $x_2 = 1$, the normalized eigenvector is $[1/\sqrt{2}, 1/\sqrt{2}]^T$.

Similarly for $\lambda_2 = 32$, the normalized eigenvector is $[-1/\sqrt{2}, 1/\sqrt{2}]^T$.

Hence, the matrix of transformation is $\mathbf{P} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ and the transformation is

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, which gives $x_1 = \frac{1}{\sqrt{2}}y_1 - \frac{1}{\sqrt{2}}y_2$, and $x_2 = \frac{1}{\sqrt{2}}y_1 + \frac{1}{\sqrt{2}}y_2$.

It represents 45° rotation.

Example 2.54: Reduce the quadratic form $\mathbf{Q} = 3x_1^2 + 5x_2^2 + 3x_3^2 - 2x_2x_3 + 2x_3x_1 - 2x_1x_2 = 144$ to the canonical form. What does it represent? Find the matrix of transformation.

Solution: For the given quadratic form **Q** the coefficient matrix is

$$\mathbf{A} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}, \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The characteristic equation, $|\mathbf{A} - \lambda\mathbf{I}| = 0$, is $\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0.$

It gives eigenvalues as $\lambda_1 = 2$, $\lambda_2 = 3$ and $\lambda_3 = 6$. Hence the quadratic form **Q** reduces to the canonical form $2y_1^2 + 3y_2^2 + 6y_3^2$ and thus $\mathbf{Q} = 144$ gives

$$2y_1^2 + 3y_2^2 + 6y_3^2 = 144, \quad \text{or} \quad \frac{y_1^2}{72} + \frac{y_2^2}{48} + \frac{y_3^2}{24} = 1$$

which is an ellipsoid.

To find the matrix of transformation P, we find eigenvectors of the matrix A, which, we can verify are $[1,\ 0,\ -1]^T$, $[1,\ 1,\ 1]^T$, and $[1,\ -2,\ 1]^T$. In the normalized form these are

$$[1/\sqrt{2},\ 0,\ -1/\sqrt{2}]^T, [1/\sqrt{3},\ 1/\sqrt{3},\ 1/\sqrt{3}]^T, [1/\sqrt{6},\ -2/\sqrt{6},\ 1/\sqrt{6}]^T.$$

Hence the matrix of transformation **P** is

$$P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$$

EXERCISE 2.10

1. Are the following matrices Hermitian, skew-Hermitian or unitary? Find their eigenvalues and eigenvectors.

   (a) $\begin{bmatrix} 2 & 3+4i \\ 3-4i & 2 \end{bmatrix}$ (b) $\begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$

2. Find the coefficient matrix for the following quadratic forms and hence find the nature of the form

   (a) $3x_1^2 + 3x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_1x_3 - 2x_2x_3$ (b) $2x_1x_2 + 2x_2x_3 + 2x_3x_1$

   (c) $|x_1|^2 + |x_2|^2 + 3|x_3|^2 + i\bar{x}_1x_3 - ix_1\bar{x}_3$ (d) $8x_1^2 + 7x_2^2 + 3x_3^2 - 12x_1x_2 - 8x_2x_3 + 4x_3x_1$

3. Examine the nature of the following matrices

   (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$

4. Find $\bar{\mathbf{x}}^T\mathbf{A}\mathbf{x}$ when

   (a) $\mathbf{A} = \begin{bmatrix} i & -2+3i \\ 2+3i & 0 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} i \\ 4 \end{bmatrix}$ (b) $\mathbf{A} = \begin{bmatrix} -i & 1 & 2+i \\ -1 & 0 & 3i \\ -2+i & 3i & i \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

5. Reduce the following quadratic forms to canonical forms by an orthogonal transformation and give the matrix of transformation

   (a) $x_1^2 + 3x_2^2 + 3x_3^2 - 2x_2x_3$ (b) $4x_1^2 + 3x_2^2 + x_2^2 - 8x_1x_2 - 6x_2x_3 + 4x_3x_1$

6. Find out what type of conic section is represented by following quadratic forms by transforming to canonical forms. Also find the transformation in each case.

   (a) $7x_1^2 + 6x_1x_2 + 7x_2^2 = 200$ (b) $32x_1^2 - 60x_1x_2 + 7x_2^2 = -52$

   (c) $9x_1^2 - 6x_1x_2 + x_2^2 = 40$

Exercise 2.1 (p. 74)

1. $\begin{pmatrix} -36 & 0 & 68 & 196 & 20 \\ 128 & -40 & -36 & -8 & 72 \end{pmatrix}$

2. $AB = \begin{bmatrix} -10 & -34 & -16 & -30 & -14 \\ 10 & -2 & -11 & -8 & -45 \\ -5 & 1 & 15 & 61 & -63 \end{bmatrix}$; BA is not defined.

4. $\begin{bmatrix} 1 & 4 & 7/2 \\ 4 & 8 & 3 \\ 7/2 & 3 & 7 \end{bmatrix} + \begin{bmatrix} 0 & -2 & 1/2 \\ 2 & 0 & -2 \\ -1/2 & 2 & 0 \end{bmatrix}$

12. (a) Hermitian (b) not Hermitian. 13. $n = 3$

15. $\begin{bmatrix} 1 & 1-i & 2i \\ 1+i & 2 & 3-i \\ -2i & 3+i & 4 \end{bmatrix} + \begin{bmatrix} i & 2+2i & 3 \\ -2+2i & 0 & 1+2i \\ -3 & -1+2i & 2i \end{bmatrix}$

16. (a) Hermitian (b) Unitary (c) Skew-Hermitian.

Exercise 2.2 (p. 88)

3. (a) 1 (b) 286 11. (a) $0, 0, -1/2$ (b) $-1, -1, 2$

Exercise 2.3 (p. 96)

2. (a) $\begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$ (b) $\begin{bmatrix} 31/2 & -17/2 & -11 \\ 9/2 & -5/2 & -3 \\ -7 & 4 & 5 \end{bmatrix}$

5. $x = 2, y = -1, z = 1$

6. $x = 2, y = 1, z = 0$ 7. $x = 1.2, y = 2.2, z = 3.2$ 8. $x = 7, y = -9, z = 5$

9. $x = -1, y = 0, z = 1$ 10. $x = 2, y = z = 1$ 11. $x = 2, y = -1, z = 1$

12. $x = 1, y = 2, z = 1$ 13. $x = 1, y = 1/2, z = 1/3$ 14. $x = y = z = e^2$

15. (i) $k \neq 2, -3$ (ii) $k = 2$, or -3

Exercise 2.4 (p. 106)

2. (a) 2 (b) 3 (c) 3 (d) 2. 3. (a) 2 (b) 2 (c) 4 (d) 2.

4. (a) 3 (b) 3 (c) 3 (d) 2. 5. (a) 2 (b) 2

6. (a) 3 (b) 4.

9. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$; Rank is 2. 10. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix}$; Rank is 2.

11. (a) $\frac{1}{10}\begin{bmatrix} -7 & 2 & 3 \\ -13 & -2 & 7 \\ 8 & 2 & -2 \end{bmatrix}$ (b) $\frac{1}{21}\begin{bmatrix} 1 & 10 & -7 \\ 1 & -11 & 14 \\ -3 & 12 & 0 \end{bmatrix}$

(c) $\frac{1}{2}\begin{bmatrix} 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & -1/3 & 1/3 & 1 \\ 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 1 & 1/3 & -1/3 & 0 \end{bmatrix}$

Exercise 2.5 (p. 115)

1. (a) yes!; $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$; 3 1. (b) yes!; $[1, x, x^2, x^3, x^4]$; 5.

1. (c) No; set is not closed under matrices addition, scalar multiplication also additive identity, that is, null matrix is not in the set.

1. (d) Yes!; $(3, 0, -2), (0, 1, 0)$; 2.

3. (a) linearly dependent; $(2, 3, 0) = -\frac{1}{2}(1, -2, 4) + (1, 1, 0) + \frac{4}{3}(1, 1, 1)$

(b) linearly dependent; $(0,0,0,0) = 0(3, 0, 2, 5) + 0(4, -1, 1, 2)$

(c) linearly dependent $(2, -1, 3, 2) = \frac{1}{2}(1, 3, 4, 2) + \frac{1}{2}(3, -5, 2, 2)$

(d) linearly independent (e) linearly independent

4. (a) Yes! (b) Yes (c) No. 5. No; 6. Yes! 7. No

8. (a) Yes (b) No. 9. e.g., $(3, 1, 4), (0, 5, 8)$; and $\left(1, 0, -1, \frac{1}{3}\right)^T$ $(1, 2, 1, 1,)^T$.

Exercise 2.6 (p. 119)

1. (a) $x_1 = -0.2y_1 + 0.4y_2$
$x_2 = 0.8y_1 - 0.6y_2$

(b) $x_1 = (-2y_1 + 2y_2 + y_3)/3$
$x_2 = (y_1 + 2y_2 - 2y_3)/3$
$x_3 = (2y_1 + y_2 + 2y_3)/3$

(c) $x_1 = y_1$
$x_2 = \cos\theta y_2 - \sin\theta y_3$
$x_3 = \sin\theta y_2 + \cos\theta y_3$.

2. $z_1 = x_1 + 4x_2 - x_3$
$z_2 = -x_1 + 9x_2 - x_3$
$z_3 = -3x_1 + 14x_2 - x_3$

3. $a = 2/7, b = 3/7, c = 6/7$

Exercise 2.7 (p. 124)

1. $x = 7/11, y = 3/11, z = 0$
2. Inconsistent.
3. $x = 11\alpha + 10, y = -4\alpha - 2, z = \alpha, u = 0$; α arbitrary.
4. $x = -1, y = 1, z = 2$
5. Inconsistent
6. $x = \alpha - \frac{5}{3}\beta, y = \alpha - \frac{4}{3}\beta, z = \alpha, w = \beta$; α, β are arbitrary
7. $x = y = z = 0$; the only solution
8. $x = 11\alpha, y = \alpha, z = -7\alpha$; α arbitrary.
9. $x = 4\alpha + \beta, y = -3\alpha - 2\beta, z = \alpha, w = \beta$
10. $x = y = z = w = 0$; the only solution
11. Consistent for $\lambda = 7, x = 3\alpha + \beta + 3, y = 4\alpha - \beta + 1, z = \alpha, w = \beta$.
12. Consistent for $\lambda = 0, 3$. For $\lambda = 0$; $x = y = z = \alpha$; α being parameter.
For $\lambda = 3$; $x = -5\alpha - 3\beta, y = \alpha, z = \beta$; α, β being parameters.
14. For $\lambda = 1$; $x = 2\alpha - \beta, y = \alpha, z = \beta$. For $\lambda = -3, x = -\alpha, y = -2\alpha, z = \alpha$, α arbitrary.

Exercise 2.8 (p. 132)

1. Eigenvalues of A: $1, -3, 7$.
2. 10, 15
3. $2, [3\ 0\ 1]^T; -2, [1\ 1\ 0]^T; 1, [2\ 0\ 1]^T$
4. $1, 1, 1; [0\ 3\ -2]^T$
5. $0, [1\ 2\ 2]^T; 3, [2\ 1\ -2]^T; 15, [2\ -2\ 1]^T$
6. $\frac{1}{3}\begin{bmatrix} -3 & -2 & 4 \\ 3 & 1 & -2 \\ -3 & 0 & 3 \end{bmatrix}$
7. $\frac{1}{4}\begin{bmatrix} 12 & 4 & 6 \\ -5 & -1 & -3 \\ -1 & -1 & -1 \end{bmatrix}$
8. $\frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$
9. $\begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$
10. $625\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Exercise 2.9 (p. 137)

1. $\hat{\mathbf{A}} = \begin{bmatrix} 355 & -42 & 560 \\ 0 & 1 & 0 \\ -225 & 27 & -355 \end{bmatrix}$ Eigenvalues are: $-5, 1, 5$

$\mathbf{y} = [-14\ 0\ 9]^T, [-14\ 2\ 9]^T, [-8\ 0\ 5]^T$ $\quad \mathbf{x} = [-1\ 0\ 3]^T, [-1\ 2\ 3]^T, [-1\ 0\ 1]^T$

2. $\mathbf{P}=\begin{bmatrix}1&2&0\\1&0&1\\0&1&1\end{bmatrix}$ $\mathbf{D}=\begin{bmatrix}15&0&0\\0&0&0\\0&0&-15\end{bmatrix}$ 3. Not diagonalizable.

4. $\mathbf{P}=\begin{bmatrix}3&2&2\\-1&0&1\\3&1&0\end{bmatrix}$ $\mathbf{D}=\begin{bmatrix}1&0&0\\0&2&0\\0&0&2\end{bmatrix}$ 5. $\begin{bmatrix}6&-5&-7\\1&0&-1\\3&-3&-4\end{bmatrix}$

6. $\frac{3}{8}\begin{bmatrix}3&6&15\\0&0&0\\1&2&3\end{bmatrix}$ 7. $\frac{1}{10}\begin{bmatrix}73&2&-37\\-115&10&55\\177&18&-93\end{bmatrix}$

8. $\mathbf{P}=\frac{1}{6}\begin{bmatrix}-3&0&3\\2&-2&2\\1&2&1\end{bmatrix}$ $\mathbf{A}^4=\begin{bmatrix}251&485&235\\485&1051&485\\235&485&251\end{bmatrix}$

Exercise 2.10 (p. 145)

1. (a) Hermitian, $\lambda=-3,7;\mathbf{x}=[-3-4i \quad 5]^T,[3+4i \quad 5]^T$

(b) Skew-Hermitian, $\lambda=-i,i$ $\mathbf{x}=[0 \quad -1 \quad 1]^T,[0 \quad 1 \quad 1]^T,[1 \quad 0 \quad 0]^T$

2. (a) $\begin{bmatrix}3&1&1\\1&3&-1\\1&-1&3\end{bmatrix}$; positive definite (b) $\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}$; indefinite

(c) $\begin{bmatrix}1&0&i\\0&1&0\\-i&0&3\end{bmatrix}$; positive definite. (d) $\begin{bmatrix}8&-6&2\\-6&7&-4\\2&-4&3\end{bmatrix}$; positive semi-definite

3. (a) positive definite (b) positive definite (c) positive definite 4. (a) $17i$ (b) $16i$

5. (a) $y_1^2+2y_2^2+4y_3^2$, $\mathbf{P}=\frac{1}{\sqrt{2}}\begin{bmatrix}\sqrt{2}&0&0\\0&1&1\\0&1&-1\end{bmatrix}$ (b) $4y_1^2-y_2^2+y_3^2$, $\mathbf{P}=\begin{bmatrix}1&1&-3\\0&1&-1\\0&0&1\end{bmatrix}$

6. (a) Ellipse $\frac{y_1^2}{50}+\frac{y_2^2}{20}=1;\ x_1=(y_1+y_2)/\sqrt{2},\ x_2=(-y_1+y_2)/\sqrt{2}$

(b) Hyperbola $\frac{y_1^2}{4}-\frac{y_2^2}{1}=1;\ x_1=(2y_1+3y_2)/\sqrt{13},\ x_2=(3y_1-2y_2)/\sqrt{13}$

(c) Straight lines $y_2=\pm 2;\ x_1=(y_1+3y_2)/\sqrt{10},\ x_2=(3y_1-y_2)/\sqrt{10}$