10. Find the length of the arc of the parabola $(l/r) = 1 + \cos\theta$ cut off by its latus rectum.

6.6 VOLUMES OF SOLIDS OF REVOLUTION

The solids of revolution are solids which can be generated by revolving plane regions about axes, e.g., a sphere. We can find their volume by the application of definite integral.

6.6.1 Cartesian Co-ordinates

(a) *Revolution about x-axis*

Let the area bounded by the curve $y = f(x)$ from $x = a$ to $x = b$ and the x-axis, as shown in Fig. 6.19, be rotated about the x-axis through four right angles, which generates a solid of revolution. Divide the arc AB into n parts by considering subintervals $[x_0, x_1]$, $[x_1, x_2]$ ..., $[x_{n-1}, x_n]$; $a = x_0 < x_1 < x_2 ... < x_n = b$ and let $P'Q' = \Delta x_i = x_{i+1} - x_i$, $i = 0, 1, 2, ... n - 1$.

Then the volume ΔV_i of the solid formed by rotating the strip $P'Q'QP$ about x-axis through four right angles can be approximated as $\Delta V_i = \pi y_i^2 \Delta x_i$, where $P'P = y_i$.

Thus an estimate of the total volume for n strips is given by

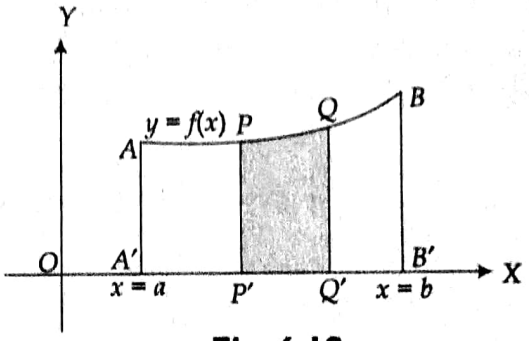

Fig. 6.19

$$\sum_i \pi y_i^2 \Delta x_i \qquad ...(6.45)$$

Let $n \to \infty$ such that max $\Delta x_i \to 0$, the summation (6.45) tends to coincide with the actual volume of the solid of revolution and is given by

$$V = \int_a^b \pi y^2 dx \qquad ...(6.46)$$

(b) *Revolution about the y-axis*

Similarly if the area bounded by curve $x = g(y)$, the y-axis, and the lines $y = c$ and $y = d$, as shown in Fig. 6.20, is revolved about the y-axis, then the volume of the solid of revolution generated is given by

$$V = \int_c^d \pi x^2 \, dy \qquad ...(6.47)$$

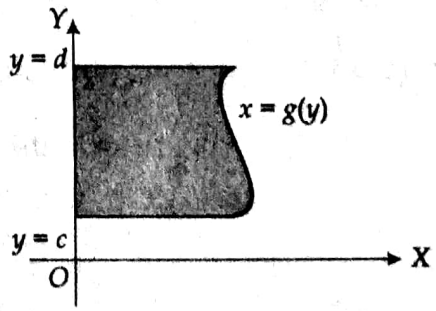

Fig. 6.20

(c) *Revolution about the line y = c*

In case the area bounded by the curve $y = f(x)$, the line $y = c$ and the lines $x = a$, $x = b$, is revolved about the line $y = c$, then the volume of the solid of revolution is given by

$$V = \int_a^b \pi (y - c)^2 dx. \qquad ...(6.48)$$

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(d) Revolution about the line $x = a$

Similarly, if the area bounded by the curve $x = g(y)$, the line $x = a$ and the lines $y = c$, $y = d$ is revolved about the line $x = a$, then the volume of solid of revolution formed is given by

$$V = \int_c^d \pi(x-a)^2 dy. \qquad ...(6.49)$$

(e) Revolution about any line

To obtain the volume of the solid generated by rotation about any axis CD the area bounded by the curve AB, the axis CD and the perpendicular AC and BD on the axis CD, take O, any fix point on the line CD, as the origin and the line OCD as x-axis. A line through O and perpendicular to OD is taken as the y-axis.

If $P(x, y)$ be any point on the curve AB, and $PN \perp OX$, as shown in Fig. 6.21, then the required volume of revolution

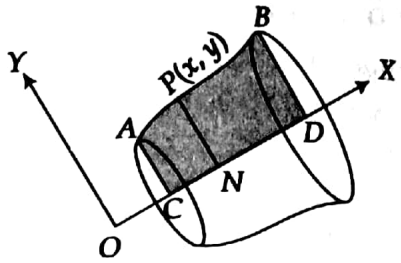

Fig. 6.21

$$V = \pi \int_a^b y^2 dx = \pi \int_{OC}^{OD} (PN)^2 \, d(ON), \qquad ...(6.50)$$

where $OC = a$ and $OD = b$

(f) Revolving area bounded by curves $y = f_1(x)$ and $y = f_2(x)$

The volume of the solid generated by rotating about x-axis the region bounded by the curves $y = f_1(x)$, $y = f_2(x)$ and the ordinates $x = a$ and $x = b$ is

$$V = \pi \int_a^b [f_1(x)^2 - f_2(x)^2] dx, \qquad ...(6.51)$$

when $f_1(x) \geq f_2(x)$, for $x \in [a, b]$.

6.6.2 Polar Co-ordinates

(a) Revolution about the initial line $\theta = 0$

The elementary area for the sector OPQ, as shown in Fig. 6.22, is $r^2 \Delta\theta/2$, and its centroid can be considered to lie on OP at $\left(\frac{2}{3} r\cos\theta, \frac{2}{3} r\sin\theta\right)$. The elementary volume generated when this area is evolved about initial line $\theta = 0$ is $\Delta V = \frac{2\pi}{3} r^3 \sin\theta\Delta\theta$. Hence the volume V of the solid generated by rotating the sector OAB about the initial line, when the area OAB is bounded by the lines $\theta = \alpha$, $\theta = \beta$ and the curve $r = f(\theta)$ is given by

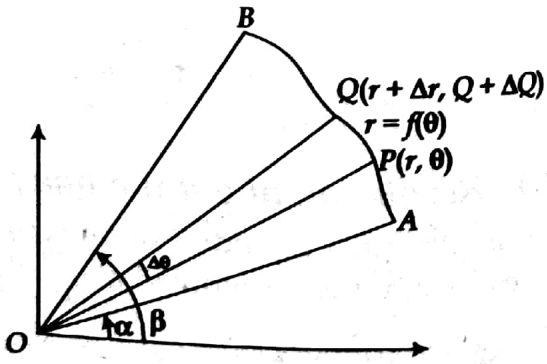

Fig. 6.22

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$$V = \frac{2\pi}{3}\int_{\alpha}^{\beta} r^3 \sin\theta \, d\theta. \qquad ...(6.52)$$

In case the area OAB is rotated about the line $\theta = \pi/2$, then volume V generated is given by

$$V = \frac{2\pi}{3}\int_{\alpha}^{\beta} r^3 \cos\theta \, d\theta. \qquad ...(6.53)$$

6.6.3 Parametric Co-ordinates

When the curve is in parametric form $x = \phi(t)$, $y = \psi(t)$, $t_0 \le t \le t_1$, then the volume of the solid generated by rotating the area bounded by the curve about x-axis is

$$V = \pi\int_{t_0}^{t_1} [\psi(t)]^2 \phi'(t)dt \qquad ...(6.54)$$

and, about y-axis is

$$V = \pi\int_{t_0}^{t_1} [\phi(t)]^2 \psi'(t)dt \qquad ...(6.55)$$

Example 6.23: A segment is cut off a sphere of radius a by a plane at a distance $a/2$ from the centre. Show that the volume of the segment is (5/32) of the volume of the sphere.

Solution: Consider a circle $x^2 + y^2 = a^2$ of radius a. The sphere of radius a is generated by revolving the semi-circular area $AOBDA$ about x-axis, while the segment is generated by revolving the area $ALMA$ about the x-axis, where $OL = \frac{a}{2}$ as shown in Fig. 6.23.

The volume of the sphere is

$$V = \pi\int_{-a}^{a} y^2\,dx = \pi\int_{-a}^{a} (a^2 - x^2)dx = 2\pi\left[a^2x - \frac{x^3}{3}\right]_0^a = \frac{4}{3}\pi a^3.$$

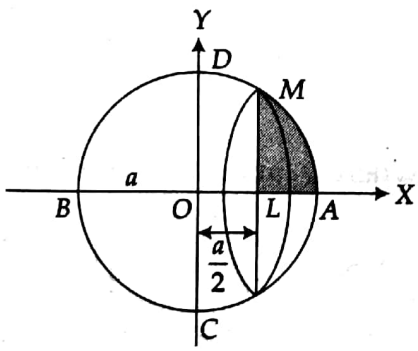

Fig. 6.23

Next, the volume of the segment is

$$V' = \pi\int_{\frac{a}{2}}^{a} y^2dx = \pi\int_{\frac{a}{2}}^{a} (a^2 - x^2)dx = \pi\left[a^2x - \frac{x^3}{3}\right]_{\frac{a}{2}}^{a}$$

$$= \pi\left[a^3 - \frac{a^3}{3} - \frac{a^3}{2} + \frac{a^3}{24}\right] = \frac{5}{24}\pi a^3 = \frac{5}{32}V.$$

This proves the result.

Example 6.24: Find the volume of the solid generated by revolving the region bounded by the curves $y = 3 - x^2$ and $y = -1$ about the line $y = -1$.

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Solution: The region $BCAB$ bounded by the parabola $y = 3 - x^2$ and the line $y = -1$, as shown in Fig. 6.24, is revolved about the line $y = -1$.

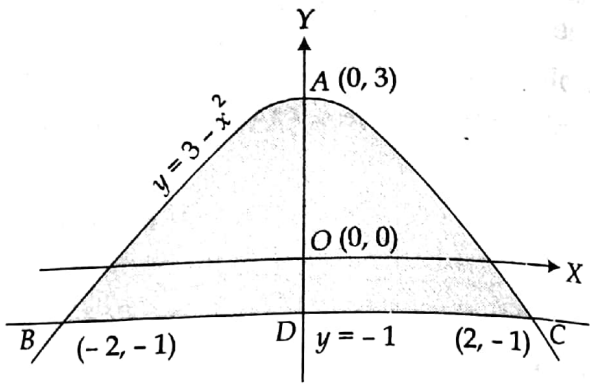

Fig. 6.24

The volume generated is

$$V = \pi \int_{-2}^{2} (1 + y)^2 dx = \pi \int_{-2}^{2} (1 + 3 - x^2)^2 dx$$

$$= 2\pi \int_{0}^{2} (4 - x^2)^2 dx$$

$$= 2\pi \int_{0}^{2} (16 - 8x^2 + x^4) dx = 2\pi \left[16x - \frac{8x^3}{3} + \frac{x^5}{5} \right]_0^2$$

$$= 2\pi \left[32 - \frac{64}{3} + \frac{32}{5} \right] = \frac{512}{15}\pi$$

Example 6.25: Find the volume of the solid of revolution obtained by revolving the region bounded between the curves $y^2 = x^3$ and $x^2 = y^3$ about the x-axis.

Solution: The two curves are $y^2 = x^3$ and $x^2 = y^3$.

Their points of intersection are given by

$$x^2 = y^3 = (x^{3/2})^3 = x^{9/2}$$

or, $$x^2(1 - x^{5/2}) = 0, \text{ or } \quad x = 0, 1.$$

Thus points of intersection are $O(0, 0)$ and $A(1, 1)$.

The region $OPAR$ bounded by the curves is shown in Fig. 6.25.

The required volume of revolution is given by

$$V = \pi \int_0^1 (y_1^2 - y_2^2) dy$$

where y_1 and y_2 are the upper and lower curves of the bounded region.

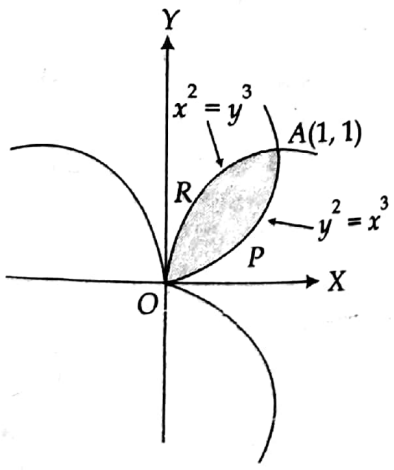

Fig. 6.25

Thus, $$V = \pi \int_0^1 \left[(x^{2/3})^2 - (x^{3/2})^2 \right] dx$$

$$= \pi \int_0^1 (x^{4/3} - x^3) dx$$

$$= \pi \left[\frac{3}{7} x^{7/3} - \frac{x^4}{4} \right]_0^1 = \pi \left[\frac{3}{7} - \frac{1}{4} \right]$$

$$= \frac{5\pi}{28} \text{ cubic unit.}$$

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Example 6.26: The area cut off the parabola $y^2 = 4ax$ by the chord joining the vertex to one end of the latus rectum is revolved about the chord. Find the volume of the solid generated.

Solution: The equation of the chord passing through the points $O(0, 0)$ and $L(a, 2a)$ is $y = 2x$. The region bounded by the parabola $y^2 = 4ax$ and the chord $y = 2x$ is as shown in Fig. 6.26.

Let $P(x, y)$ be any point on the parabola and PN be the perpendicular from P on OL, then

$$PN = \frac{|y-2x|}{\sqrt{1^2+(-2)^2}} = \frac{1}{\sqrt{5}}|y-2x|.$$

Next, $$ON = \sqrt{OP^2 - PN^2}$$

$$= \sqrt{x^2+y^2-\frac{1}{5}(y-2x)^2}$$

$$= \sqrt{\frac{5x^2+5y^2-y^2-4x^2+4xy}{5}}$$

$$= \frac{x+2y}{\sqrt{5}}.$$

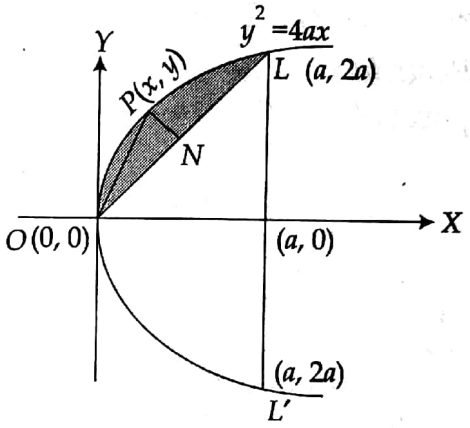

Fig. 6.26

Therefore, $$d(ON) = \frac{1}{\sqrt{5}}(dx + 2dy).$$

From the equation $y^2 = 4ax$, we have, $2y\,dy = 4a\,dx$ or $dy = \frac{2a}{y}dx$. Thus,

$$d(ON) = \frac{1}{\sqrt{5}}\left(dx + \frac{4a}{y}dx\right) = \frac{1}{\sqrt{5}}\left(1+\frac{4a}{y}\right)dx.$$

The required volume generated, refer to (6.50), is

$$V = \int_0^a \pi(PN)^2\,d(ON) = \pi\int_0^a \frac{(y-2x)^2}{5}\,\frac{1}{\sqrt{5}}\left(1+\frac{4a}{y}\right)dx$$

$$= \frac{\pi}{5\sqrt{5}}\int_0^a\left[(y-2x)^2 + \frac{4a}{y}(y-2x)^2\right]dx$$

$$= \frac{\pi}{5\sqrt{5}}\int_0^a\left[y^2+4x^2-4xy+4ay+\frac{16ax^2}{y}-16ax\right]dx$$

$$= \frac{\pi}{5\sqrt{5}}\int_0^a\left[4ax+4x^2-8x\sqrt{ax}+8a\sqrt{ax}+8x\sqrt{ax}-16ax\right]dx$$

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$$= \frac{\pi}{5\sqrt{5}} \int_0^a \left[4ax + 4x^2 + 8a\sqrt{ax} - 16ax\right] dx$$

$$= \frac{\pi}{5\sqrt{5}} \left[2ax^2 + \frac{4}{3}x^3 + \frac{16}{3}a^{3/2}x^{3/2} - 8ax^2\right]_0^a$$

$$= \frac{\pi a^3}{5\sqrt{5}} \left[2 + \frac{4}{3} + \frac{16}{3} - 8\right] = \frac{2\pi a^3}{15\sqrt{5}} \text{ cubic unit.}$$

Example 6.27: The area lying inside the cardioid $r = 2a(1 + \cos\theta)$ and outside the parabola $r(1 + \cos\theta) = 2a$ is revolved about the initial line $\theta = 0$. Find the volume of the solid generated.

Solution: The points of intersection of the two curves, the cardiod $r = 2a(1 + \cos\theta)$ and the parabola $r = 2a/(1+\cos\theta)$, are given by

$$2a(1 + \cos\theta) = \frac{2a}{(1 + \cos\theta)}$$

or, $$(1 + \cos\theta)^2 = 1, \text{ or } \cos\theta = 0, \text{ or } \theta = \frac{\pi}{2}, \frac{-\pi}{2}.$$

The bounded region is shown in Fig. 6.27.

The required volume is generated by rotating just the upper half (or, the lower half) portion for which θ varies from 0 to $\frac{\pi}{2}$. Therefore, $V = \frac{2\pi}{3}\int_0^{\pi/2} (r_1^3 - r_2^3)\sin\theta\, d\theta$, where r_1 is the value of r for the cardiod and r_2 is the value of r for the parabola, thus

$$V = \frac{2\pi}{3}\int_0^{\pi/2} \left[8a^3(1+\cos\theta)^3 - \frac{8a^3}{(1+\cos\theta)^3}\right] \sin\theta\, d\theta$$

$$= \frac{16\pi a^3}{3}\int_0^{\pi/2} \left[(1+\cos\theta)^3 - (1+\cos\theta)^{-3}\right] \sin\theta\, d\theta$$

$$= \frac{-16\pi a^3}{3}\left[\frac{(1+\cos\theta)^4}{4} - \frac{(1+\cos\theta)^{-2}}{-2}\right]_0^{\pi/2}$$

$$= \frac{-16\pi a^3}{3}\left[\frac{1}{4} + \frac{1}{2} - 4 - \frac{1}{8}\right] = 18\pi a^3.$$

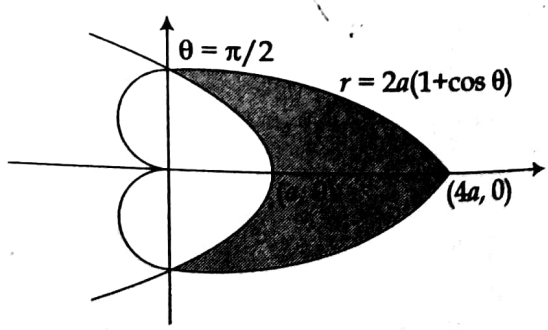

Fig. 6.27

Example 6.28: The area bounded by an arc of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, $0 \le \theta \le 2\pi$ and the x-axis is revolved around x-axis. Find the volume of the solid generated.

Solution: The area bounded by an arc of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ is shown in Fig. 6.28.

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The volume generated when this area is revolved around x-axis is given by

$$V = \pi \int_0^{2\pi} y^2\, dx$$

$$= \pi \int_0^{2\pi} a^2(1 - \cos\theta)^2\, a(1 - \cos\theta)d\theta = \pi a^3 \int_0^{2\pi} (1 - \cos\theta)^3 d\theta$$

$$= \pi a^3 \int_0^{2\pi} (1 - 3\cos\theta + 3\cos^2\theta - \cos^3\theta)d\theta.$$

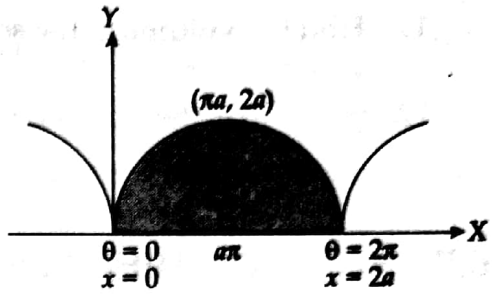

Fig. 6.28

$$= \pi a^3 \int_0^{2\pi} \left[1 - 3\cos\theta + \frac{3}{2}(1 + \cos 2\theta) - \frac{1}{4}(\cos 3\theta + 3\cos\theta)\right] d\theta$$

$$= \pi a^3 \int_0^{2\pi} \left[\frac{5}{2} - \frac{15}{4}\cos\theta + \frac{3}{2}\cos 2\theta - \frac{1}{4}\cos 3\theta\right] d\theta$$

$$= \pi a^3 \left[\frac{5}{2}\theta - \frac{15}{4}\sin\theta + \frac{3}{4}\sin 2\theta - \frac{1}{12}\sin 3\theta\right]_0^{2\pi} = 5\pi^2 a^3 \text{ cubic units.}$$

EXERCISE 6.5

1. Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$, $x = 4$ about the line $y = 1$.
2. Find the volume of the solid generated by revolving about x-axis the area bounded by the parabola $y = (x^2/4) + 2$ and the straight line $8y = 5x + 14$.
3. The area bounded by the parabola $y^2 = 4x$ and the straight line $4x - 3y + 2 = 0$ is rotated about the y-axis. Find the volume of the solid generated.
4. The ellipse $(x^2/a^2) + (y^2/b^2) = 1$ is revolved about y-axis. Find the volume of the solid generated.
5. A cone is formed by revolving about y-axis the line joining the origin to the point (a, b). Find the volume of the cone generated.
6. Find the volume of the solid generated by the revolution of the circle $x^2 + (y - b)^2 = a^2$, $b > a$ about the axis of x.
7. Find the volume of the solid generated by revolving the area included between the curve $(y + 8)/x = x - 2$ and the x-axis about the line $x + 5 = 0$.
8. The loop of the curve $r = a\cos\theta$ lying between $\theta = -\pi/6$ and $\theta = \pi/6$ revolves about the initial line. Find the volume of the solid generated.
9. The area of the cardioid $r = a(1 + \cos\theta)$ included between $\theta = -\pi/2$ and $\theta = \pi/2$ is rotated about the line $\theta = \pi/2$. Show that the volume generated is $2[2 + (5\pi/8)]\pi a^3$.

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10. Find the volume of the solid obtained by revolving the cissoid $y^2(2a - x) = x^3$ about its asymptotes.
11. Find the volume of the solid generated by revolving the astroid $x = \cos^3\theta$, $y = a\sin^3\theta$ about x-axis.
12. Show that the volume of the solid generated by revolving the leminiscate $r^2 = a^2\cos 2\theta$ about the line $\theta = \pi/2$, is $\pi a^3/4\sqrt{2}$.

6.7 SURFACE AREAS OF SOLIDS OF REVOLUTION

In this section, we find the surface areas of the solids of revolution by the application of definite integral. We discuss the various cases as given below.

6.7.1 Cartesian Co-ordinates

(a) Revolution about x-axis

An estimate of the curved surface area of the slice generated may be found by assuming that it is formed by rotating the chord PQ instead of the arc PQ, refer to Fig. 6.19, about x-axis. Under this assumption the solid formed is a slice of a cone with surface area approximately given by

$$\Delta S_i = 2\pi y_i \Delta s_i$$

where $\Delta s_i = \widehat{PQ}$. Hence the curved surface area S of the solid of revolution is given by

$$S = \int_{x=a}^{x=b} 2\pi y\, ds = \int_a^b 2\pi y \frac{ds}{dx} dx = 2\pi \int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx, \quad \text{refer to (4.22)} \qquad ...(6.56)$$

(b) Revolution about the y-axis

Similarly if the area bounded by the curve $x = g(y)$, the y-axis and the lines $y = c$ and $y = d$ is revolved about the y-axis, refer to Fig. (6.20), then surface area of the solid of revolution generated is given by

$$S = \int_c^d 2\pi x\, ds = 2\pi \int_c^d x \frac{ds}{dy} dy = 2\pi \int_c^d x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy, \quad \text{refer to (4.23)} \qquad ...(6.57)$$

6.7.2 Polar Co-ordinates

(a) Revolution about the initial line $\theta = 0$

The formula for the surface area of revolution in terms of polar co-ordinates, when the area is revolved around the initial line $\theta = 0$, is given by

$$S = \int 2\pi y\, ds = \int_{\theta=\alpha}^{\beta} 2\pi y \frac{ds}{d\theta} d\theta = \int_{\alpha}^{\beta} 2\pi r \sin\theta \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}\, d\theta, \quad \text{refer to (4.29).} \qquad ...(6.58)$$

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(b) Revolution about the line $\theta = \pi/2$

Similarly when the area is revolved around the line $\theta = \frac{\pi}{2}$, the surface area S is given by

$$S = \int 2\pi x\, ds = \int_{\theta=\alpha}^{\beta} 2\pi x \frac{ds}{d\theta}\, d\theta = \int_{\alpha}^{\beta} 2\pi r \cos\theta \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}\, d\theta, \qquad ...(6.59)$$

refer to (4.29).

6.7.3 Parametric Co-ordinates

In terms of parametric co-ordinates $x = \phi(t)$, $y = \psi(t)$, $t_0 \le t \le t_1$, the surface of the solid of revolution obtained by rotating the area about the x-axis is,

$$S = \int 2\pi y\, ds = \int 2\pi y \frac{ds}{dt}\, dt = \int_{t_0}^{t_1} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad ...(6.60)$$

and about y-axis is

$$S = \int_{t_0}^{t_1} 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt. \qquad ...(6.61)$$

refer to (4.24).

Example 6.29: Find the surface area of the solid generated by revolving the loop of the curve $3ay^2 = x(x-a)^2$ about x-axis.

Solution: The loop of the given curve extends from $x = 0$ to $x = a$ as shown in Fig. 6.29.

Therefore, the required surface area S obtained by revolving the loop about x-axis is

$$S = \int 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \qquad ...(6.62)$$

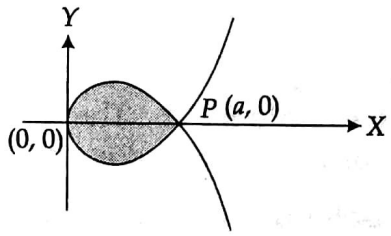

Fig. 6.29

For the given equation $3ay^2 = x(x-a)^2$, we have

$$6ay\frac{dy}{dx} = (x-a)^2 + 2x(x-a) \text{ or, } \frac{dy}{dx} = \frac{(x-a)(3x-a)}{6ay}.$$

Thus,
$$y\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = y\sqrt{1 + \frac{(x-a)^2(3x-a)^2}{36a^2y^2}}$$

$$= \sqrt{y^2 + \frac{(x-a)^2(3x-a)^2}{36a^2}} = \sqrt{\frac{x(x-a)^2}{3a} + \frac{(x-a)^2(3x-a)^2}{36a^2}}$$

$$= \frac{(x-a)}{6a}\sqrt{12ax + (3x-a)^2} = \frac{(x-a)(3x+a)}{6a} = \frac{3x^2 - 2ax - a^2}{6a}.$$

Hence, (6.62) becomes

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$$S = \frac{\pi}{3a}\int_0^a (3x^2 - 2ax - a^2)dx = \frac{\pi}{3a}[x^3 - ax^2 - a^2x]_0^a = \frac{-\pi a^2}{3}.$$

Thus $S = \dfrac{\pi a^2}{3}$, taking the numerical value.

Example 6.30: Find the surface area of the solid generated by revolving an arc of the cycloid, $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, $0 \le \theta \le 2\pi$ about the x-axis.

Solution: The area bounded by the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, $0 \le \theta \le 2\pi$ and the x-axis is shown in Fig. 6.28. The surface area of the solid generated by revolving the area about x-axis is given by

$$S = 2\pi\int_0^{2\pi} y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx.$$

We have, $$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1-\cos\theta)} = \frac{2a\sin\theta/2\cos\theta/2}{2a\sin^2\theta/2} = \cot\theta/2. \text{ Thus}$$

$$\therefore \quad S = 2\pi\int_0^{2\pi} a(1-\cos\theta)\sqrt{1+\cot^2\frac{\theta}{2}}\,a(1-\cos\theta)d\theta.$$

$$= 8\pi a^2\int_0^{2\pi}\sin^4\frac{\theta}{2}\operatorname{cosec}\frac{\theta}{2}d\theta = 8\pi a^2\int_0^{2\pi}\sin^3\frac{\theta}{2}d\theta = 2\pi a^2\int_0^{2\pi}\left(3\sin\frac{\theta}{2} - \sin\frac{3\theta}{2}\right)d\theta$$

$$= 2\pi a^2\left[-6\cos\frac{\theta}{2} + \frac{2}{3}\cos\frac{3\theta}{2}\right]_0^{2\pi} = 2\pi a^2\left[6 - \frac{2}{3} + 6 - \frac{2}{3}\right] = \frac{64}{3}\pi a^2.$$

Example 6.31: The lemniscate $r^2 = a^2\cos 2\theta$ revolves about a tangent at the pole. Show that the surface generated is $4\pi a^2$.

Solution: The curve $r^2 = a^2\cos 2\theta$ has two loops so the required surface area is twice the surface area generated by one loop when it revolves around the tangent, say MOM' as shown in Fig. 6.30.

Let $P(r, \theta)$ be any point on the loop of the lemniscate and $PN \perp MOM'$, then $PN = r\sin\left(\theta + \dfrac{\pi}{4}\right)$.

Also from $r^2 = a^2\cos 2\theta$, we obtain

$$2r\frac{dr}{d\theta} = -2a^2\sin 2\theta, \text{ or } \frac{dr}{d\theta} = \frac{-a^2}{r}\sin 2\theta.$$

Then surface area S generated is given by

$$S = 2\int_{-\pi/4}^{\pi/4} 2\pi\, PN\, ds = 4\pi\int_{-\pi/4}^{\pi/4} r\sin\left(\theta + \frac{\pi}{4}\right)\frac{ds}{d\theta}\,d\theta$$

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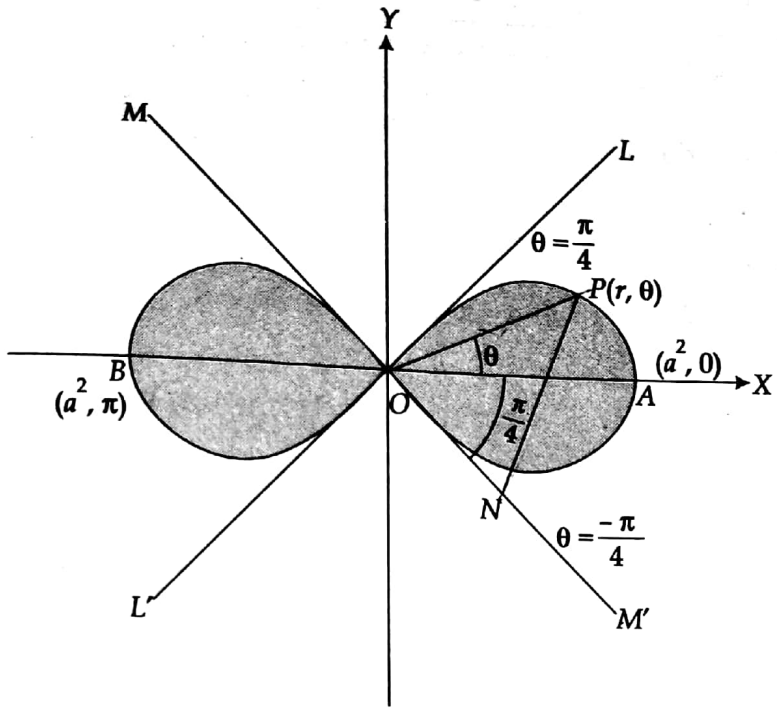

Fig. 6.30

$$= 4\pi \int_{-\pi/4}^{\pi/4} r \sin\left(\theta + \frac{\pi}{4}\right)\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{\frac{1}{2}} d\theta, \quad \text{refer to (4.29)}$$

$$= 4\pi \int_{-\pi/4}^{\pi/4} r \sin\left(\theta + \frac{\pi}{4}\right)\left[r^2 + \frac{a^4}{r^2}\sin^2 2\theta\right]^{\frac{1}{2}} d\theta$$

$$= 4\pi \int_{-\pi/4}^{\pi/4} \sin\left(\theta + \frac{\pi}{4}\right)\sqrt{r^4 + a^4 \sin^2 2\theta}\, d\theta$$

$$= 4\pi \int_{-\pi/4}^{\pi/4} \sin\left(\theta + \frac{\pi}{4}\right)\sqrt{a^4 \cos^2 2\theta + a^4 \sin^2 2\theta}\, d\theta = 4\pi a^2 \int_{-\pi/4}^{\pi/4} \sin\left(\theta + \frac{\pi}{4}\right) d\theta$$

$$= 4\pi a^2 \left[-\cos\left(\theta + \frac{\pi}{4}\right)\right]_{-\pi/4}^{\pi/4} = -4\pi a^2 \left[\cos\frac{\pi}{2} - \cos 0\right] = 4\pi a^2.$$

Example 6.32: Find the surface of the solid generated by the revolution of the asteroid $x = a\cos^3 t, y = a\sin^3 t$ about the y-axis.

Solution: The asteroid is symmetrical about the axes, as shown in Fig. 6.31.

For its portion in the first quadrant, $0 \le t \le \pi/2$, we have

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$$\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2} = \sqrt{9a^2\cos^4 t\sin^2 t + 9a^2\sin^4 t\cos^2 t}$$

$$= 3a\sin t\cos t.$$

Hence, the required surface area is

$$S = 2\times 2\pi\int_0^{\pi/2} x\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt = 4\pi\int_0^{\pi/2} a\cos^3 t.3a$$

$$\sin t\cos t\,dt$$

$$= 12\pi a^2\int_0^{\pi/2}\sin t\cos^4 t\,dt = 12\pi a^2\frac{3.1}{5.3.1} = \frac{12\pi a^2}{5}$$

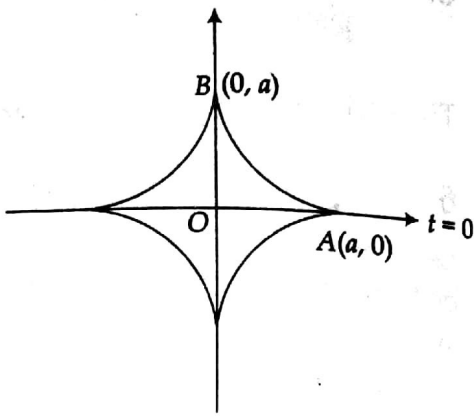

Fig. 6.31

EXERCISE 6.6

1. The portion between the consecutive cusps of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$ is revolved about the x-axis. Prove that the ratio of the area of the surface generated to the area of the cycloid is 64:9.
2. A quadrant of a circle of radius a revolves around its chord. Show that the surface of the spindle generated is $2\pi a^2\sqrt{2}\,[1 - \pi/4]$.
3. Find the area of the surface generated by revolving the loop of the curve $3ay^2 = x(x-a)^2$ about x-axis.
4. Find the volume and surface area of the solid generated by revolving the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ about x-axis.
5. A circular arc revolves about the chord. Show that the area of the surface generated is $4\pi a^2(\sin\alpha - \alpha\cos\alpha)$ where 2α is the angle subtended by the arc at the centre.
6. Prove that surface area of the solid generated by revolving tractrix $x = a\cos t + \frac{a}{2}\ln\tan^2(t/2)$, $y = a\sin t$ about x-axis is $4\pi a^2$.
7. Prove that the surface and volume of the solid generated by revolving the loop of the curve $x = t^2$, $y = t - t^3/3$ about x-axis are respectively 3π and $3\pi/4$.
8. Find the volume and surface area of the right circular cone obtained by the revolution of a right angled triangle about a side which contains the right angle.
9. Find the area of the surface generated by revolving about the x-axis a closed contour formed by the curves $y = x^2$ and $x = y^2$.
10. An arc of the catenary $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$ whose end points have abscissas 0 and x respectively, revolves about the x-axis. If S and V are respectively the surface and the volume generated show that $S = 2V/a$.

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