$$E \equiv i\hbar \frac{\partial}{\partial t}$$

$$p \equiv -i\hbar \nabla$$

$$\Rightarrow \left.\begin{aligned} p_x &= -i\hbar \frac{\partial}{\partial x} \\ p_y &= -i\hbar \frac{\partial}{\partial y} \\ p_z &= -i\hbar \frac{\partial}{\partial z} \end{aligned}\right\}$$

$$\begin{aligned} [x p_x - p_x x]\psi &= -i\hbar \left[x \frac{\partial \psi}{\partial x} - \frac{\partial}{\partial x}(x\psi)\right] \\ &= -i\hbar \left[\cancel{x \frac{\partial \psi}{\partial x}} - \cancel{x \frac{\partial \psi}{\partial x}} - \psi\right] \\ &= i\hbar \psi. \end{aligned}$$

$$[x p_x - p_x x] \equiv [x, p_x] = i\hbar$$

$\boxed{\text{Commutation Relation}}$

$$\Rightarrow [x, p_x] \equiv [y, p_y] \equiv [z, p_z] \equiv i\hbar$$

$$\text{But } [x, y] = [y, z] = [z, x] = 0$$

$$[x, p_y] = [y, p_z] = [z, p_x] = 0$$

$$[p_x, p_y] = [p_y, p_z] = [p_z, p_x] = 0$$

<u>3.D Schrödinger Equation</u>

$$\nabla^2 \Psi + \frac{2m}{\hbar^2}(E - V)\Psi = 0$$

$$\boxed{i\hbar \frac{\partial \psi}{\partial t} = H\psi}$$

where $H \equiv -\frac{\hbar^2}{2m}\nabla^2 + V$

$$E \rightarrow i\hbar \frac{\partial}{\partial t}$$

$$p \rightarrow -i\hbar \nabla$$

In classical Mechanics Angular Momentum of a particle is defined by

$$\vec{L} = \vec{r} \times \vec{p}$$

$$L = (ix + jy + kz) \times (-i\hbar)\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)$$

$$= -i\hbar\left[k x\frac{\partial}{\partial y} - j x\frac{\partial}{\partial z} - k y\frac{\partial}{\partial x} + i y\frac{\partial}{\partial z} + j z\frac{\partial}{\partial x} - i z\frac{\partial}{\partial y}\right]$$

$$= -i\hbar\left[i\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) + j\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) + \hat{k}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right]$$

$$= -i\hbar\left[L_x + L_y + L_z\right]$$

$$L_x \equiv -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$L_y \equiv -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

$$L_x = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

$$L^2 = L_x^2 + L_y^2 + L_z^2$$

Prove that $[L_x, L_y] \equiv i\hbar L_z$

or

$$[L_x\ L_y]\psi = i\hbar L_z \psi$$

$$L_x \equiv -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$L_y \equiv -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

$$L_z \equiv -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

$$[L_x, L_y]\psi = (L_x L_y - L_y L_x)\psi$$

$$= -\hbar^2\left\{\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\left(z\frac{\partial}{\partial z} - x\frac{\partial}{\partial z}\right) - \left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\right\}\psi$$

$$= -\hbar^2\left\{\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\left(z\frac{\partial\psi}{\partial x} - x\frac{\partial\psi}{\partial z}\right) - \left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\left(y\frac{\partial\psi}{\partial z} - z\frac{\partial\psi}{\partial y}\right)\right\}$$

$$= -\hbar^2\left\{y\frac{\partial}{\partial z}\left(z\frac{\partial\psi}{\partial x}\right) - y\frac{\partial}{\partial z}\left(x\frac{\partial\psi}{\partial z}\right) - z\frac{\partial}{\partial y}\left(z\frac{\partial\psi}{\partial x}\right) + z\frac{\partial}{\partial y}\left(x\frac{\partial\psi}{\partial z}\right)\right.$$

$$- z\frac{\partial}{\partial x}\left(y\frac{\partial\psi}{\partial z}\right) + z\frac{\partial}{\partial x}\left(z\frac{\partial\psi}{\partial y}\right) + x\frac{\partial}{\partial z}\left(y\frac{\partial\psi}{\partial z}\right) - x\frac{\partial}{\partial z}\left(z\frac{\partial\psi}{\partial y}\right)\Big\}$$

$$= -\hbar^2\left\{y\frac{\partial\psi}{\partial x} + yz\frac{\partial^2\psi}{\partial z\,\partial x} - yx\frac{\partial^2\psi}{\partial z^2} - z^2\frac{\partial^2\psi}{\partial y\,\partial x} - zy\frac{\partial^2\psi}{\partial x\,\partial z} + z^2\frac{\partial^2\psi}{\partial x\,\partial y} + xy\frac{\partial^2\psi}{\partial z^2} + zx\frac{\partial^2\psi}{\partial y\,\partial z} - x\frac{\partial\psi}{\partial y} - xz\cdot\frac{\partial^2\psi}{\partial z\,\partial y}\right\}$$

$$= -\hbar^2\left(y\frac{\partial\psi}{\partial x} - x\frac{\partial\psi}{\partial y}\right)$$

$$= -\hbar^2\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)\psi$$

$$= -\hbar^2\cdot\frac{L_z}{i\hbar}\psi = \frac{i^2\hbar^2 L_z\psi}{i\hbar} = i\hbar L_z\psi$$

$$\Rightarrow [L_x, L_y] \equiv i\hbar L_z \Leftarrow$$

$$[L_x, L_y] = i\hbar L_z$$

$$[L_y, L_z] = i\hbar L_x$$

$$[L_z, L_x] = i\hbar L_y.$$

Lowering and Raising Operators

$$L_+ = L_x + iL_y \quad \text{Raising Op.}$$

$$L_- = L_x - iL_y \quad \text{Lowering Op}$$

$$L_x = \frac{L_+ + L_-}{2} \qquad L_y = \frac{L_+ - L_-}{2}$$

$$L^2 = L_z^2 + \frac{1}{2}\left[L_+L_- + L_-L_+\right]$$

$$L_+L_- = L^2 - L_z^2 + \hbar L_z$$

$$L_-L_+ = L^2 - L_z^2 + \hbar L_z$$

$$\left.\begin{aligned} &[L^2, L_\pm] = 0 \\ &[L_z, L_\pm] = \pm\hbar L_\pm \\ &[L_+, L_-] = 2\hbar L_z \end{aligned}\right\} \leftarrow \text{Prove that}$$