Chapter 15

Kinetics of Particles and Rigid Bodies

Learning Objectives

After completing this chapter, you will be able to

LO 15.1 List Newton's laws of motion.
LO 15.2 Describe D'Alembert's principle and apply it to solve problems of dynamics.
LO 15.3 Solve problems involving translation of rigid bodies.
LO 15.4 Solve problems involving fixed-axis rotation of rigid bodies.
LO 15.5 Solve problems involving general plane motion of rigid bodies.
LO 15.6 Solve problems involving motion of vehicles on level and banked roads.

15.1 ❑ INTRODUCTION

There are some engineering problems, which require consideration of forces causing motion, in addition to motion characteristics. The branch of dynamics that deals with such problems involving relationship between the motion of particles or a rigid body and its causes is called ***kinetics***. The causes of motion may be forces, moments, or a combination of these. Generally, the Newton's laws of motion are used to solve such problems. However, another principle known as D'Alembert's principle is best suited for certain problems providing simpler solution.

LO 15.1

List Newton's laws of motion.

15.2 ❑ NEWTON'S LAWS OF MOTION

Newton's laws of motion, published by the English scientist Sir Isaac Newton in 1687, form the basis of classical mechanics. They describe the relationship between motion of a body and the forces causing the motion. They relate the velocity and acceleration of a body to change in its momentum.

15.2.1 Newton's First Law

An object tends to remain in its state of rest or of motion in straight line with constant velocity; *unless it experiences a net external force.* This tendency of a body to retain its state is known as the *inertia*. Hence, this is also known as the *law of inertia.* This law implies the following:

(i) Objects in vacuum move with constant velocity, if no external force acts on them.
(ii) External force originates from outside the object. Hence, an external agent is required to change the velocity of the object.

Newton's first law holds true in an ***inertial frame of reference***, also called *Newtonian reference frame* or *Galilean reference frame*. An inertial frame of reference describes time and space homogeneously, isotropically, and in a time-independent manner. An inertial frame is a non-accelerating coordinate system. That means an inertial frame of reference is in a state of constant and rectilinear motion and has no acceleration. The laws of physics are the same in all inertial reference frames. Although the measured velocities and positions might be different in different inertial frames, but the rules governing their relationships are the same. Note that the measured acceleration of an object is the same in all inertial reference frames.

15.2.2 Newton's Second Law

The net force acting on an object is equal to the rate of change of its linear momentum in an inertial frame of reference. Thus, the net force F acting on an object of mass m can be expressed as

$$F = \frac{d(mv)}{dt} \tag{15.1}$$

Here, v is the velocity of object during time interval dt. For mass remaining constant, as is usually the case, Eq. (15.1) becomes,

$$F = m\frac{dv}{dt} = ma \tag{15.2}$$

where, a is the acceleration of the object. Therefore, Newton's second law can also be stated as "*the acceleration of an object is directly proportional to the net external force acting on it and inversely proportional to its mass.*" Direction of acceleration is *same* as that of the net force. In other words, the vector sum of forces acting on an object is equal to mass of the object multiplied by the acceleration.

Myth

Newton's second law is F = ma.

Fact!

No. This is a form of Newton's second law applicable only if the mass of a body remains constant during motion. For a rocket, the mass goes on reducing due to burning of fuel, and hence this relation does not hold true. When stated in its original form, it says that *the net force is equal to the rate of change of momentum*.

The right-hand side of Eq. (15.2) represents the *inertia* or *inertial force.* This inertial force always opposes the motion. It is not a real force but a *fictitious* or *perceivable* force. From Eq. (15.2), the acceleration produced by a force F is given as

$$a = \frac{F}{m} = \frac{F}{(W/g)} \tag{15.3}$$

Thus, the acceleration generated is inversely proportional to a factor (W/g) called ***factor of sluggishness***. It is a measure of *inertia* and is equal to the mass m of the object.

15.2.3 Newton's Second Law for Curvilinear Motion

Myth

Newton's second law of motion always holds true.

Fact!

No. This law holds true *only* when the motion of a body is formulated with reference to an inertial frame (that is, a frame fixed or moving with constant velocity in a straight line).

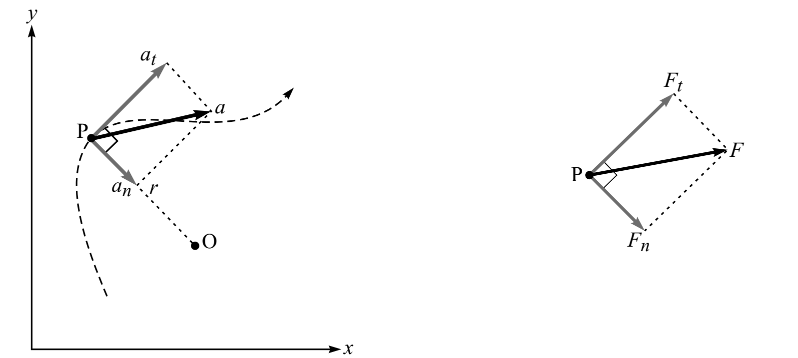
Consider a particle of mass m moving in a curvilinear path with tangential velocity v, as shown in Fig. 15.1*a*. At an instant of time, let the particle be at the point P with radius of curvature r. Let a_t and a_n be the tangential and normal components of acceleration a of the particle, caused by a force F. The tangential component F_t and normal component F_n of this force, as shown in Fig. 15.1*b*, are given as

$$F_t = ma_t \tag{15.4}$$

$$F_n = ma_n \tag{15.5}$$

If ω is the instantaneous angular velocity and α is the instantaneous angular acceleration of the particle then (as seen in Chapter 14) the two components of the acceleration are given as

$$a_t = r\alpha \quad \text{and} \quad a_n = \frac{v^2}{r} = \omega^2 r$$



(*a*) A particle moving in a curvilinear path (*b*) Forces acting on the particle

Fig. 15.1

15.2.4 Newton's Third Law

To every action, there is always an opposite reaction of same magnitude. In other words, this law may be stated as *the mutual actions of two bodies upon each other are always equal and opposite in direction.* This implies that whenever a movement is restrained, reaction is produced.

Example 15.1 A man weighing 800 N dives into a swimming pool from a height of 20 m. It is estimated that he goes down in water by 2 m and then starts rising. Neglecting air resistance, determine the average resistance offered by water to the motion of the man. **[LO 15.1]**

Solution Given: $W_m = 800$ N, $h = 20$ m, $h_w = 2$ m. Assume the downward direction as *positive*.

Let v be the velocity of the man when he reaches the water surface. Using *third* equation of motion and noting that $u = 0$, we have

$$v = \sqrt{2gh} = \sqrt{2 \times 9.81 \times 20} = 19.81 \text{ m/s}$$

When the man goes down through a distance of 2 m inside the water, his velocity changes from 19.81 m/s to zero. This occurs because of retardation due to water resistance. Using the third equation of motion, we get

$$0^2 - 19.81^2 = 2a \times 2 \quad \Rightarrow \quad a = -\frac{19.81^2}{2 \times 2} = -98.11 \text{ m/s}^2 \text{ (retardation)}$$

If F_r is the average resistance of water acting on the man in the upward direction, the net downward force acting on the man, $F_n = W_m - F_r$. Thus,

$$F = ma: \quad (800 - F_r) = \frac{800}{9.81} \times (-98.11) \quad \Rightarrow \quad F_r = \mathbf{8800.8\ N}$$

LO 15.2

Describe D'Alembert's principle and apply it to solve problems of dynamics.

15.3 ❑ D'ALEMBERT'S PRINCIPLE AND DYNAMIC EQUILIBRIUM

D'Alembert's principle is a fundamental principle used in classical mechanics. It is an alternative to Newton's second law and leads to a simpler solution in some special cases.

For a rigid body, D'Alembert's principle simply states that *the net external force on a body plus the negative of inertia force is equal to zero*. When a force F is applied on a body of mass m, an acceleration a is produced (Fig. 15.2). If a *fictitious force* (equal to inertia force ma) is applied to the body in a direction opposite to that of acceleration a, the body comes to equilibrium. We can then write

$$\boxed{F - ma = 0} \tag{15.6}$$

This is the most common form of D'Alembert's principle. This implies that the body is in ***dynamic equilibrium*** under the action of the real force F and the fictitious force, ma. This fictitious force is called ***inertial force***. It acts through the centre of mass of the rigid body. By using D'Alembert's principle, a dynamics problem effectively changes into a problem of *statics*.

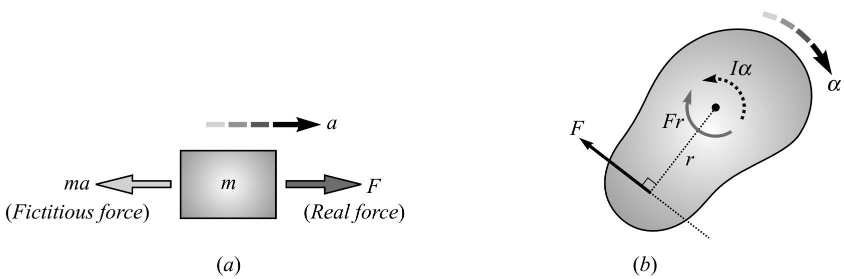


Fig. 15.2 *Body in dynamic equilibrium.*

Similarly, in case of a rotating body, net moment (Fr) about mass centre can be balanced by *inertial torque* ($I\alpha$), where I is the centroidal mass moment of inertia, α is the angular acceleration. Therefore, according to D'Alembert's principle,

$$\boxed{Fr - I\alpha = 0} \tag{15.7}$$

Thus, by using this principle, a system can be analysed exactly as a static system subjected to the inertial force and/or moment and the external forces. This is also called *inertial method of analysis*. The advantage of this method is that in the equivalent static system, one can take moments about any point (not just the centre of mass). This often leads to simpler calculations because any force can be eliminated from the moment equations by choosing the appropriate point about which moments are taken.

Example 15.2 Determine the acceleration produced in a body of 40-kg mass when it is acted upon by a force of 140 N, as shown in Fig. 15.3*a*. Use D'Alembert's principle.

[LO 15.2]

Solution Given: $m = 40$ kg, $F = 140$ N. Using D'Alembert's principle, we apply an inertial force $F_i = ma$ in a direction opposite to that of acceleration (Fig. 15.3*b*). The body is then in dynamic equilibrium. Hence,

$$\Sigma F_x = 0: \quad F - F_i = 0 \quad \Rightarrow \quad 140 - 40a = 0 \quad \Rightarrow \quad a = \mathbf{3.5\ m/s^2}$$

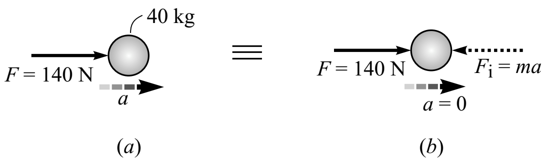


Fig. 15.3

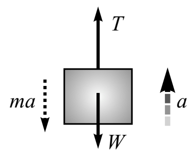


Fig. 15.4

Example 15.3 An elevator cage of a mineshaft, weighing 5 kN, is lifted or lowered by means of a wire rope. Starting from rest, it moves upwards with constant acceleration and acquires a velocity of 3 m/s within a distance of 3 m. Calculate the tensile force in the cable during the accelerated motion.

[LO 15.2]

Solution Given: $W = 5$ kN, $v = 3$ m/s, $s = 3$ m. Thus,

$$v^2 - u^2 = 2as: \quad 3^2 - 0 = 2a \times 3 \quad \Rightarrow \quad a = 1.5\ \text{m/s}^2$$

In the FBD of the cage (Fig. 15.4), we have applied inertial force ma to bring the cage in dynamic equilibrium. Thus,

$$\Sigma F_y = 0: \quad T - W - ma = 0 \quad \Rightarrow \quad T = W + ma = W\left(1 + \frac{a}{g}\right)$$

$$\Rightarrow \quad T = 5 \times 1000\left(1 + \frac{1.5}{9.81}\right) = \mathbf{5764.53\ N}$$

LO 15.3

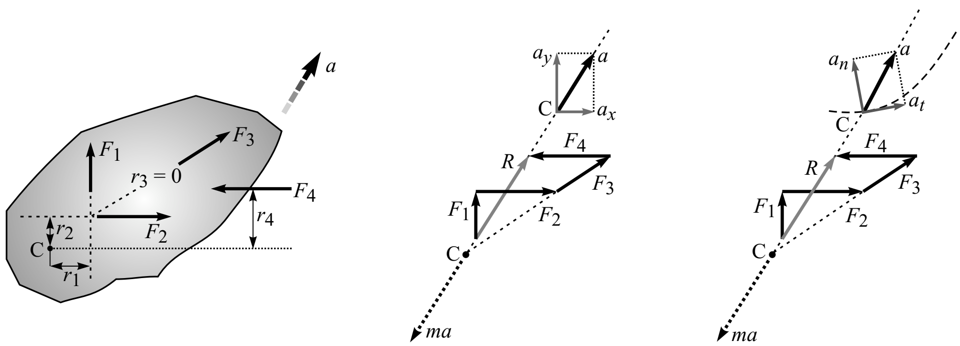
Solve problems involving translation of rigid bodies.

15.4 ❑ RIGID BODY IN PURE TRANSLATION

Consider a rigid body of mass m in pure translation under the action of a system of forces, as shown in Fig. 15.5*a*. The body has only curvilinear or rectilinear acceleration a, but no rotation. Here, all elements of the rigid body will experience the same amount of acceleration. So, according to Newton's second law, the generalized force equilibrium equation of motion is

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \ldots = \Sigma\mathbf{F} = m\mathbf{a} \quad \Rightarrow \quad \mathbf{R} = m\mathbf{a} \tag{15.8}$$

where, $\Sigma\mathbf{F} = \mathbf{R}$, the resultant force of the force system acting on the rigid body (Fig. 15.5*b*). In terms of magnitudes, above equation becomes, $R = ma$.



(a) A rigid body in translation (b) Rectilinear motion (c) Curvilinear motion

Fig. 15.5

Furthermore, as there is no rotation, net moment of all the forces about the mass centre C of the body should be zero. That is,

$$\Sigma M|_C = 0 \quad \text{or} \quad \sum_{i=1}^{n} F_i r_i = 0$$

where r_i is the perpendicular distance of ith force F_i from the mass centre C (Fig. 15.5a).

For a planar *rectilinear* translation (Fig. 15.5b), the equations of motion are as follows.

$$\Sigma F_x = ma_x, \quad \Sigma F_y = ma_y \quad \text{and} \quad \Sigma M|_C = 0 \tag{15.9}$$

Similarly, for a *curvilinear* translation (Fig. 15.5c), the equations of motion are as follows.

$$\Sigma F_t = ma_t, \quad \Sigma F_n = ma_n \quad \text{and} \quad \Sigma M|_C = 0 \tag{15.10}$$

LO 15.4

Solve problems involving fixed-axis rotation of rigid bodies.

15.5 ❑ RIGID BODY UNDER FIXED-AXIS ROTATION

Consider a rigid body rotating counterclockwise about the z-axis passing through the point O, as shown in Fig. 15.6. Let the angular acceleration of a small element of mass dm at radial distance r from the point O be α. The tangential acceleration is then $a_t = r\alpha$. The inertial force acting on the elemental mass is $(dm)(r\alpha)$. Moment of this inertial force about O is given as

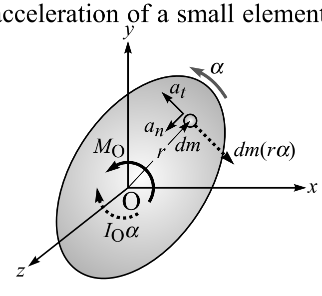


Fig. 15.6

$$dM = (dm)(r\alpha)r = \alpha r^2\, dm$$

The above moment is integrated over the entire body to get net moment about the point O. Hence,

$$M_O = \int_m dM = \int_m \alpha r^2 dm = \alpha \int_m r^2 dm = \alpha I_O$$

where, $I_O = \int_m r^2 dm$ is the *mass moment of inertia* of the rigid body about the z-axis; and $I_O\alpha$ is the *angular momentum* during fixed-axis rotation. This net moment is equal but opposite to the external moment M_O about the z-axis. Hence, the moment *dynamic equilibrium* equation is

$$\boxed{M_O - I_O\alpha = 0} \tag{15.11}$$

Equation (15.11) is called ***Euler's equation of motion***.

LO 15.5

Solve problems involving general plane motion of rigid bodies.

15.6 ❑ GENERAL PLANE MOTION OF A RIGID BODY

General plane motion of a rigid body is a combination of rectilinear or curvilinear translation and fixed-axis rotation. Therefore, both the force equilibrium and the moment equilibrium exist for this type of motion. For rectilinear translation, forces are generally resolved along cartesian *x*- and *y*-axes. In case of curvilinear motion, resolution is done along tangential and normal directions, as explained in Section 15.4. Hence, for rectilinear motion, we have

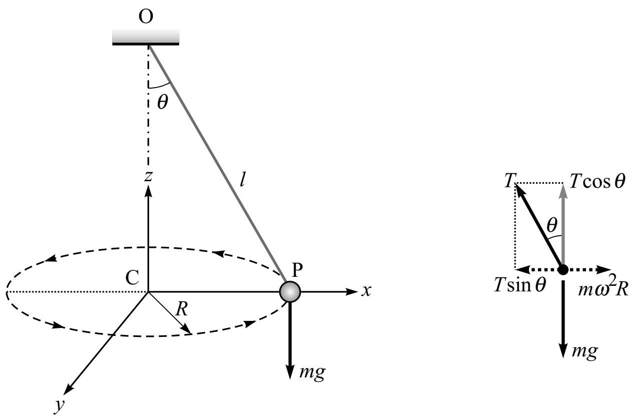
$$\sum_{i=1}^{n} F_{xi} = ma_x, \quad \sum_{i=1}^{n} F_{yi} = ma_y \quad \text{and} \quad \Sigma M_O = I_O\alpha \tag{15.12}$$

For curvilinear motion, we can write

$$\sum_{i=1}^{n} F_{ti} = ma_t, \quad \sum_{i=1}^{n} F_{ni} = ma_n \quad \text{and} \quad \Sigma M_O = I_O\alpha \tag{15.13}$$

Example 15.4 Figure 15.7*a* shows a bob of a conical pendulum of mass *m*, which describes a horizontal circle of radius *R*. The circle is defined by equations, $x = R\sin\omega t$ and $y = R\cos\omega t$. The length of the inextensible string is *l*. Determine the tension in the string and prove that it is constant during such motion.

[LO 15.2]



(*a*) Pendulum describing motion (*b*) FBD of bob

Fig. 15.7

Solution Let *T* be the tension in the string. The bob experiences an acceleration of $\omega^2 R$ towards the centre. According to D'Alembert's principle, let us apply an inertia force $m\omega^2 R$ opposite to this acceleration as shown in the FBD of the pendulum bob (Fig. 15.7*b*). Considering its horizontal equilibrium, we have

$$\Sigma F_x = 0: \quad m\omega^2 R - T\sin\theta = 0 \quad \Rightarrow \quad T\sin\theta = m\omega^2 R \tag{i}$$

From the geometry of Fig. 15.7*a*, $\sin\theta = \dfrac{R}{l}$

Substituting this into Eq. (*i*), $T\dfrac{R}{l} = m\omega^2 R \quad \Rightarrow \quad T = \boldsymbol{ml\omega^2}$

Since *l*, *m* and ω do not vary with time, the tension *T* in the string is constant.