Assignment

Central Forces & Poisson's Bracket

2K19/EP/005

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Q1

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} \qquad L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - m\,U(r)$$

$$\left.\begin{aligned} \frac{\partial L}{\partial \dot{r}} &= m\dot{r} \\ \frac{\partial L}{\partial \dot{\theta}} &= mr^2\dot{\theta} \\ \frac{\partial L}{\partial r} &= mr\dot{\theta}^2 - m\frac{\partial U(r)}{\partial r} \\ \frac{\partial L}{\partial \theta} &= 0 \end{aligned}\right\} \qquad \begin{aligned} m\ddot{r} - mr\dot{\theta}^2 + m\frac{\partial U(r)}{\partial r} &= 0 \quad -(1) \\ \frac{d}{dt}\left(mr^2\dot{\theta}\right) = 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} &= 0 \quad -(2) \end{aligned}$$

$$H = \sum p_i\dot{q}_i - L = \left(m\dot{r}^2 + mr^2\dot{\theta}^2\right) - \left(\frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2\right) + m\,U(r)$$

$mr^2\dot{\theta} = mr^2\omega = L$ (angular momentum)

$$dA = \frac{1}{2}r(r\,d\theta), \qquad \frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \boxed{\frac{1}{2}\frac{L}{m}}$$

$= \text{constant}.$

Q2

i) for circular orbit of radius R,

$\dot{r} = 0,\quad \ddot{r} = 0,\quad \left.\frac{\partial U(r)}{\partial r}\right|_R = 0.$

$$\frac{\partial U(r)}{\partial r} = \left. 3Kr^2 m - \frac{M^2}{mr^3}\right|_R = 0$$

$$3KR^2m = \frac{M^2}{mR^3} \qquad M^2 = 3Km^2R^5$$

$$M = \sqrt{3Km^2R^5}.$$

$$K.E = U(R) = mKR^3 + \frac{M^2}{2mR^2} = mKR^3 + \frac{3Km^2R^5}{2mR^2} = \boxed{\frac{5}{2}mKR^3}.$$

ii) Time Period, $T = \frac{2\pi}{\omega}$

$$\omega = \frac{\partial U(r)}{dt} = M/mR^2 = \frac{\sqrt{3Km^2R^5}}{mR^2} = \sqrt{3KR}$$

$$\boxed{T = \frac{2\pi}{\sqrt{3KR}}}$$

Q3

$$e = \frac{r_{max} + r_{min}}{r_{max} - r_{min}} \Rightarrow \frac{r_{max}}{r_{min}} = \frac{1+e}{1-e}$$

as, angular momentum is constant,

$$(r_{max} \cdot V_{min}) = (r_{min} V_{max}).$$

$$\frac{V_{max}}{V_{min}} = \frac{r_{max}}{r_{min}} = \frac{1+e}{1-e}$$

$$= \frac{1 + 0.0167}{1 - 0.0167} = \boxed{1.0339}$$

Q4

$$e = \sqrt{1 + \frac{2EL^2}{m\alpha^2}}$$ for elliptical orbit,

$$r_{max} = a(1+e), \quad r_{min} = a(1-e).$$

as $r_{max} \cdot V_{min} = r_{min} \cdot V_{max}$.

$$\frac{V_{max}}{V_{min}} = \frac{r_{max}}{r_{min}} \qquad \frac{V_{max}}{V_{min}} = \frac{1+e}{1-e}, \quad e = \frac{V_{max} - V_{min}}{V_{max} + V_{min}}$$

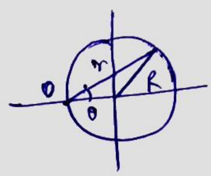
Q5 | eqn of orbit: $\frac{d^2u}{d\theta^2} + u = -\frac{m f(\frac{1}{u})}{L^2u^2}$

$u = \frac{1}{r}$, $L = mr^2\dot{\theta}^2 = \text{const}$.

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = \frac{d}{d\theta}\left(\frac{1}{c\theta}\right) = \frac{-1}{c\theta^2}$$

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = \frac{2}{c\theta^3} = \frac{2c^2}{r^3}$$

$$f(r) = \frac{-L^2}{Mr^2}\left(\frac{2c^2}{r^3} + \frac{1}{r}\right)$$



$r = 2R\cos\theta$

Q6 | $E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r)$

$$\dot{r} = 2R\frac{d}{dt}\cos\theta = -2R\sin\theta\,\dot{\theta}$$

$$\dot{r}^2 = 4R^2\sin^2\theta\,\dot{\theta}^2 = 4R^2(1-\cos^2\theta)\dot{\theta}^2 = (4R^2 - r^2)\dot{\theta}^2$$

$$L = \frac{1}{2}m(4R^2 - r^2 + r^2)\dot{\theta}^2 + V(r)$$

$$= 2mR^2\dot{\theta}^2 + V(r)$$

$l = mr^2\dot{\theta}$, $\dot{\theta}^2 = \frac{l^2}{m^2r^4}$

$$V(r) = L - \frac{2R^2l^2}{mr^4}$$

$$f(r) = -\frac{\partial V(r)}{\partial r} = -\frac{8R^2l^2}{mr^5}$$

$\boxed{\therefore f(r) \propto \frac{1}{r^5}}$

Q7 | for equilibrium, $-\frac{\partial U}{\partial x} = 0$

$$-\frac{d}{dx}\left[\frac{a}{x^{12}} - \frac{b}{x^6}\right] = 0$$

$$-12ax^{-13} + 6bx^{-7} = 0$$

$$\Rightarrow \frac{12a}{x^{13}} = \frac{6b}{x^{7}} \Rightarrow x = \sqrt[6]{\frac{2a}{b}}$$

$$\frac{d^2u}{dx^2} \Rightarrow (12\times13)a\,x^{-14} - (6\times7)\,b\,x^{-8} \Big|_{x=\sqrt[6]{\frac{2a}{b}}}$$

as a & b are +ve, $\frac{d^2u}{dx^2} > 0$, stable equilibrium.

Q8

$$\frac{dV(x)}{dx} = 0, \quad -K\left(e^{-\beta x} + x\left(-\beta e^{-\beta x}\right)\right) = 0$$

$$x\beta \cancel{e^{-\beta x}} = \cancel{e^{-\beta x}} \Rightarrow x = 1/\beta.$$

$$\frac{d^2V(x)}{dx} = -K\left[-\beta e^{-\beta x} + \left[-\beta\left(e^{-\beta x} + x(-\beta e^{-\beta x})\right)\right]\right] > 0$$

for $K, \beta > 0$, stable equilibrium

Q9

$$F = -K/r^n$$

$$U(r) = -\int_{\infty}^{r} F(r)\,dr = -\frac{K}{(n-1)\,r^{n-1}}$$

$$U_{eff}(r) = -\frac{K}{(n-1)\,r^{n-1}} + \frac{L^2}{2mr^2}$$

$$F_{eff} = -\frac{\partial U_{eff}(r)}{\partial r} = -\frac{K}{r^n} + \frac{L^2}{mr^3}.$$

for circular radius r_0,

$$r_0^{\,n-3} = \frac{mK}{L^2}$$

for stable orbit, $\left.\frac{\partial^2 U_{eff}}{\partial r^2}\right|_{r=r_0} > 0$

$$\frac{-nk}{r_0^{n+1}} + \frac{3L^2}{mr_0^4} > 0$$

$$(-n+3)\frac{L^2}{m} > 0 \quad \Rightarrow \underline{\underline{n<3}}.$$ $\therefore$ for $n<3$, orbit is stable.

Q10 The classical poisson bracket of two dynamical variables, u and v, is defined

$$[u,v] = \sum_i \left(\frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i} \right)$$

To prove; $$[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0$$

Consider,

$$= [u,[v,w]] + [v,[w,u]]$$

$$= [u,[v,w]] - [v[u,w]]$$

$$= \sum_i \left([u, \frac{\partial v}{\partial q_i}\frac{\partial w}{\partial p_i} - \frac{\partial v}{\partial p_i}\frac{\partial u}{\partial q_i}] - [v, \frac{du}{\partial q_i}\frac{dw}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial w}{\partial q_i}] \right)$$

$$= \sum_i \left([u, \frac{dv}{\partial q_i}\frac{\partial w}{\partial p_i}] - [u, \frac{\partial v}{\partial p_i}\frac{\partial w}{\partial q_i}] - [v, \frac{\partial u}{\partial q_i}\frac{dw}{\partial p_i}] + [v, \frac{\partial u}{\partial p_i}\frac{\partial w}{\partial q_i}] \right)$$

$$= \sum_i^n \left([u, \frac{dv}{\partial q_i}]\frac{\partial w}{\partial p_i} + [v, \frac{\partial u}{\partial q_i}]\frac{\partial w}{\partial p_i} - [u, \frac{\partial v}{\partial p_i}]\frac{\partial w}{\partial q_i} - [v, \frac{\partial u}{\partial q_i}]\frac{\partial w}{\partial p_i} \right)$$

$$+ \sum_i \left([u, \frac{\partial w}{\partial p_i}]\frac{\partial v}{\partial q_i} - [u, \frac{\partial w}{\partial q_i}]\frac{\partial v}{\partial p_i} + \frac{\partial u}{\partial p_i}[v, \frac{\partial w}{\partial q_i}] - \frac{\partial u}{\partial q_i}[v, \frac{\partial w}{\partial p_i}] \right)$$

$$= \sum_i \left(\frac{\partial}{\partial q_i}[u,v] \frac{\partial w}{\partial p_i} - \frac{\partial w}{\partial q_i} \frac{\partial}{\partial p_i}[u,v] \right)$$

$$+ \sum_i \left(\frac{\partial v}{\partial q_i}\left[u, \frac{\partial w}{\partial p_i}\right] - \left[u, \frac{\partial w}{\partial q_i}\right] \frac{\partial v}{\partial p_i} + \frac{\partial u}{\partial p_i}\left[v, \frac{\partial w}{\partial q_i}\right] - \frac{\partial u}{\partial q_i}\left[v, \frac{\partial w}{\partial p_i}\right] \right)$$

$$= [[u,v], w]$$

$$+ \sum_i \sum_j \left(\frac{\partial^2 w}{\partial p_j p_i} \left(\frac{du}{\partial q_j} \frac{dv}{\partial q_i} - \frac{\partial v}{\partial q_j} \frac{\partial u}{\partial q_i} \right) + \frac{\partial^2 w}{\partial q_i q_j} \left(\frac{\partial u}{\partial p_j} \frac{\partial v}{\partial p_i} - \frac{\partial v}{\partial p_j} \frac{\partial u}{\partial p_i} \right) + \cdots \right.$$

taking $j = i$.

$$\therefore [u,[v,w]] + [v,[w,u]]$$

$$= [[u,v],w] = -[w,[u,v]]$$

$$\therefore \quad [u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0.$$

Q12 |

$$[X, YZ] = [X,Y]Z + Y[X,Z]$$

$$[X, YZ] = \sum_i \left(\frac{dx}{\partial q_i} \frac{\partial (YZ)}{\partial p_i} - \frac{\partial u}{\partial p_i} \frac{\partial (YZ)}{\partial q_i} \right)$$

$$= \sum_i \left[\frac{\partial x}{\partial q_i} \left(Y \frac{\partial z}{\partial p_i} + Z \frac{\partial Y}{\partial p_i} \right) - \frac{dx}{\partial p_i} \left(Y \frac{\partial Z}{\partial q_i} + Z \frac{\partial Y}{\partial q_i} \right) \right]$$

$$= \sum_i \left[Z \left(\frac{dx}{\partial q_i} \frac{dY}{\partial p_i} - \frac{dx}{\partial p_i} \frac{dY}{\partial q_i} \right) + Y \left(\frac{\partial X}{\partial q_i} \frac{\partial Z}{\partial p_i} - \frac{\partial X}{\partial p_i} \frac{\partial Z}{\partial q_i} \right) \right]$$

$$= \cancel{\sum} \; Z[X,Y] + Y[X,Z].$$

Q13

$$\frac{d}{dt} = \frac{\partial}{\partial x_i}\frac{\partial x_i}{\partial t} = \frac{\partial}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial}{\partial t}\frac{\partial t}{\partial t}$$

for Hamiltonian,

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial t} = \frac{\partial f}{\partial q_i}\frac{\partial q_i}{\partial t} + \frac{\partial f}{\partial p_i}\frac{\partial p_i}{\partial t} + \frac{\partial f}{\partial t}$$

as $$[f, H] = \left(\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i}\right)$$

$$[f, H] = \left(\frac{\partial f}{\partial q_i}\frac{\partial q_i}{\partial t} + \frac{\partial f}{\partial p_i}\frac{\partial p_i}{\partial t}\right)$$

$\therefore$ $$\frac{df}{dt} = [f, H] + \frac{\partial f}{\partial t}.$$

END