$$L^2 Y(\theta,\phi) = \lambda\hbar^2 Y(\theta,\phi)$$

$\lambda\hbar^2$ – Eigen values of L^2

$Y(\theta,\phi)$ – Eigen function

We will prove that $\lambda = l(l+1)$ takes the value as $l(l+1)$ where $l = 0, 1, 2 \ldots$ and the corresponding Eigen functions are spherical Harmonics. For each value of l there will be $(2l+1)$ fold degeneracy i.e. there will be $(2l+1)$ eigen functions belonging to the same eigen value $l(l+1)\hbar^2$

$$L^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

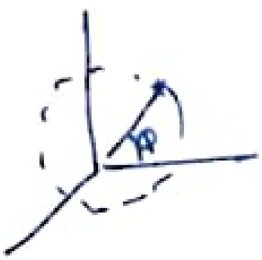
$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\cdot\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi} + \lambda Y(\theta\phi) = 0$$

Method of separation of variables

$$Y(\theta,\phi) = \Theta(\theta)\,\Phi(\phi)$$

$$\frac{\sin^2\theta}{\Theta}\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\frac{d\Theta}{d\theta} + \lambda\,\Theta(\theta)\right]\right. = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = m^2$$

$$\frac{d^2\Phi}{d\phi^2} + m^2\Phi(\phi) = 0$$



$$\Phi(\phi) \approx e^{im\phi}$$

For single value $e^{im(\phi+2\pi)} = e^{i(m\phi)}$

when $e^{im2\pi} = 1$

That is possible when $m = 0, \pm 1, \pm 2 \ldots$

$$\Phi(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}$$ Normalization condition

$$\int_0^{2\pi} \Phi_{m'}^{*}(\phi)\,\Phi_m(\phi)\,d\phi = \delta_{mm'}$$

$$L^2 Y(\theta,\phi) = \lambda\hbar^2 Y(\theta,\phi)$$

$\uparrow$ Eigen function of L^2 $\quad$ $\uparrow$ Eigen value of L^2

* We will prove that $\lambda = l(l+1)$ where $l = 0, 1, 2$ and the corresponding Eigen functions are spherical Harmonics.

* For each value of 'l', there will be $(2l+1)$ fold degeneracy i.e. there will be $(2l+1)$ eigenfunctions belonging to the same eigen value $l(l+1)\hbar^2$

$$L^2 = L_x^2 + L_y^2 + L_z^2$$

$$L^2 = -\hbar^2\left[\frac{1}{\sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

$$\Rightarrow \frac{1}{\sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\cdot\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\cdot\frac{\partial^2 Y}{\partial\phi^2} + \lambda Y(\theta,\phi) = 0$$

Method of Separation of variables $\left[\times \frac{\sin^2\theta}{Y(\theta,\phi)}\right.$ Substituting Y

$$Y(\theta,\phi) = F(\theta)\,\Phi(\phi)$$

$$\Rightarrow \frac{\sin^2\theta}{F(\theta)}\left[\frac{1}{\sin\theta}\cdot\frac{d}{d\theta}\left(\sin\theta\frac{dF(\theta)}{d\theta}\right) + \lambda F(\theta)\right] = -\frac{1}{\Phi(\phi)}\frac{d^2\Phi}{d\phi^2} = m^2$$

Variables are separated and are equal $\Rightarrow$ Both are equal to m^2 (say).

$$\frac{d^2\Phi}{d\phi^2} = m^2\,\Phi(\phi)$$

Wave function to be single valued

$$\Phi(\phi) \sim e^{im\phi}$$

$$\Phi(\phi+2\pi) = \Phi(\phi)$$

$$e^{im(\phi+2\pi)} = e^{im\phi} \Rightarrow e^{im\times 2\pi} = 1$$

$$\Rightarrow$$

This means $m = 0, \pm 1, \pm 2$

Normalization

$$\int_0^{2\pi} |\Phi_m(\phi)|^2 d\theta = 1 \Rightarrow \Phi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}.$$

Orthonormality $\longrightarrow$ $\int^{2\pi} \Phi_{m'}^* \Phi_m \, d\theta = \delta_{m'm} = 1$ when $m = m'$
$= 0$ when $m \neq$

$$\frac{\sin^2\theta}{F(\theta)}\left[\frac{1}{\sin\theta}\cdot\frac{d}{d\theta}\left(\sin\theta\cdot\frac{dF(\theta)}{d\theta}\right)+\lambda F(\theta)\right]-m^2=0$$

$\times \frac{F(\theta)}{\sin^2\theta}$

$$\frac{1}{\sin\theta}\cdot\frac{d}{d\theta}\left(\sin\theta\cdot\frac{dF}{d\theta}\right)+\left(\lambda-\frac{m^2}{\sin^2\theta}\right)F(\theta)=0$$

Associated Legendre's Equation

Solution is difficult but is obtained

<u>$m=0$</u>

$$\frac{1}{\sin\theta}\cdot\frac{d}{d\theta}\left(\sin\theta\frac{dF}{d\theta}\right)+\lambda F(\theta)=0$$

Let us take $x=\cos\theta \Rightarrow dx=-\sin\theta\, d\theta$

$x=\cos\theta$
$-1 \le x \le +1$

$$\sin\theta\cdot\frac{dF(\theta)}{d\theta}=\sin\theta\cdot\frac{dF}{dx}\cdot\frac{dx}{d\theta}$$

$$=-\sin^2\theta\cdot\frac{dF}{dx}=-\frac{dF}{dx}(1-x^2)$$

is function of x

$$-\frac{1}{\sin\theta}\cdot\frac{d}{d\theta}\left(\frac{dF}{dx}\right)\frac{dF}{dx}(1-x^2)+\lambda F(\theta)$$

$$\frac{d}{dx}\left\{\frac{dF}{dx}(1-x^2)\right\}+\lambda F(x)=0$$

$$\Rightarrow (1-x^2)\frac{d^2F}{dx^2}-2x\frac{dF}{dx}+\lambda F(x)=0$$

$\Rightarrow$ Legendre's Equation

<u>Power Series</u> Frobenius Method

$$F(x)=\sum_{r=0,1}^{\infty} a_r x^{r+s}$$

$$(1-x^2)\sum_r a_r(r+s)(r+s-1)x^{r+s-2}-2x\sum a_r(r+s)x^{r+s-1}+\lambda\sum a_r x^{s+r}=0$$

$$\Rightarrow \sum\left[(r+s)(r+s-1)a_r x^{r+s-2}-\left\{(r+s)(r+s-1)+2(r+s)-\lambda\right\}a_r x^{r+s}\right]=0$$

Since this is valid for all values of x, we equate the coeff of x to 0

$s(s-1)a_0=0$, $s(s+1) \Rightarrow s=0$ or $s=1$

$s(s+1)a_1=0$

$$(1-x^2)\frac{d^2F}{dx^2} - 2x\frac{dF}{dx} + \lambda F(x) = 0$$

Legendre's Equation

Power Series

$$F(x) = \sum_{r=0}^{\infty} a_r x^{r+s}$$

$$(1-x^2)\sum (r+s)(r+s-1)\, a_r\, x^{r+s-2} - \sum 2(r+s)\, a_r x^{r+s} + \lambda \sum a_r x^{r+s}$$

$$\sum\Big[(r+s)(r+s-1)\, a_r x^{r+s-2} - (r+s)(r+s-1)\, a_r x^{r+s} - 2(r+s)\, a_r x^{r+s} + \lambda\, a_r x^{r+s}\Big] = 0$$

$$\sum (r+s)(r+s-1)\, a_r x^{r+s-2} - \sum\{\underbrace{(r+s)(r+s-1) + 2(r+s)} - \lambda\}\, a_r x^{r+s} = 0$$

$$\sum (r+s)(r+s-1)\, a_r x^{r+s-2} - \sum\{(r+s)(r+s+1) - \lambda\}\, a_r x^{r+s}$$

$r=0$ $\quad \times\ x^{2-s}$

$\cancel{\sum s(s-1)\, a_0}$

$$\sum\{(r+s)(r+s+1) - \lambda\}\, a_r x^{r+2} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} (r+s)(r+s-1)\, a_r x^r - \sum\{(r+s)(r+s+1) - \lambda\}\, x^{r+2}$$

$$s(s-1)\, a_0 + s(s+1)\, a_1 x + \sum_{r=0}^{\infty} a_{r+2}\, (r+2+s)\,(r+s+1)\ \cancel{s(s+2)\, x}\ \cancel{(r+s)(r+s+1)}$$

$$= \sum\{(r+s)(r+s+1) - \lambda\}\, a_r x^{r+2}$$

$$= \{s.(s+1) - \lambda\}\, a_0 x^2 +$$

$$s(s-1)\, a_0 = 0$$

$$s(s+1)\, a_1 = 0$$

$$\frac{a_{r+2}}{a_r} = \frac{(r+s)(r+s+1) - \lambda}{(r+s+2)(r+s+1)}$$

Recurrence Relation

$s \Rightarrow 0$

$$\frac{a_{r+2}}{a_r} = \frac{r(r+1) - \lambda}{(r+2)(r+1)} \Rightarrow$$ a_0 and a_2, a_2 & a_4 are related.

$$F(x) = [a_0 + a_2 x^2 + a_4 x^4 + a_6 x^6 + \cdots] + [a_1 x + a_3 x^3 + \cdots]$$

$\frac{a_{r+2}}{a_r} \to 1$ for large r $\quad \boxed{\lambda = l(l+1)}$

$$\psi \longrightarrow r, \theta, \phi$$

$$L_z \psi(r, \theta\, \phi) = -i\hbar \left[x \frac{\partial \psi}{\partial y} - y \frac{\partial \psi}{\partial x} \right)$$

$$L_z \psi(r, \theta, \phi) = -i\hbar \frac{\partial \psi}{\partial \phi}$$

$$\Rightarrow \boxed{L_z = -i\hbar \frac{\partial}{\partial \phi}}$$

$$L_x = i\hbar \left(\sin\phi \frac{\partial}{\partial \theta} + \cot\theta \cdot \cos\phi \frac{\partial}{\partial \phi} \right)$$

$$L_y = i\hbar \left(-\cos\phi \frac{\partial}{\partial \theta} + \cot\theta \cdot \sin\phi \frac{\partial}{\partial \phi} \right)$$

$$L^2 = L_x^2 + L_y^2 + L_z^2$$

$$L_x^2 = L_x \cdot L_x$$

$$L^2 = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \cdot \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2\theta} \cdot \frac{\partial^2}{\partial \phi^2} \right]$$

$$L_z^2 = L_z \cdot L_z$$

$$L_z^2 \psi(r, \theta, \phi) = L_z \cdot L_z \cdot \psi(r, \theta \cdot \phi)$$

$$= (-i\hbar)^2 \cdot \frac{\partial^2 \psi}{\partial \phi^2} = -\hbar^2 \frac{\partial^2 \psi}{\partial \phi^2}$$

$$\boxed{\begin{aligned} L_z Y_{lm}(\theta, \phi) &= m\hbar\, Y_{lm}(\theta, \phi) \\ L^2 Y_{lm}(\theta, \phi) &= l(l+1)\hbar^2\, Y_{lm}(\theta, \phi) \end{aligned}}$$

$Y_{lm}(\theta, \phi) = ?$ Eigenfunction