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Question Based On

"Frequency of Normal Mode"

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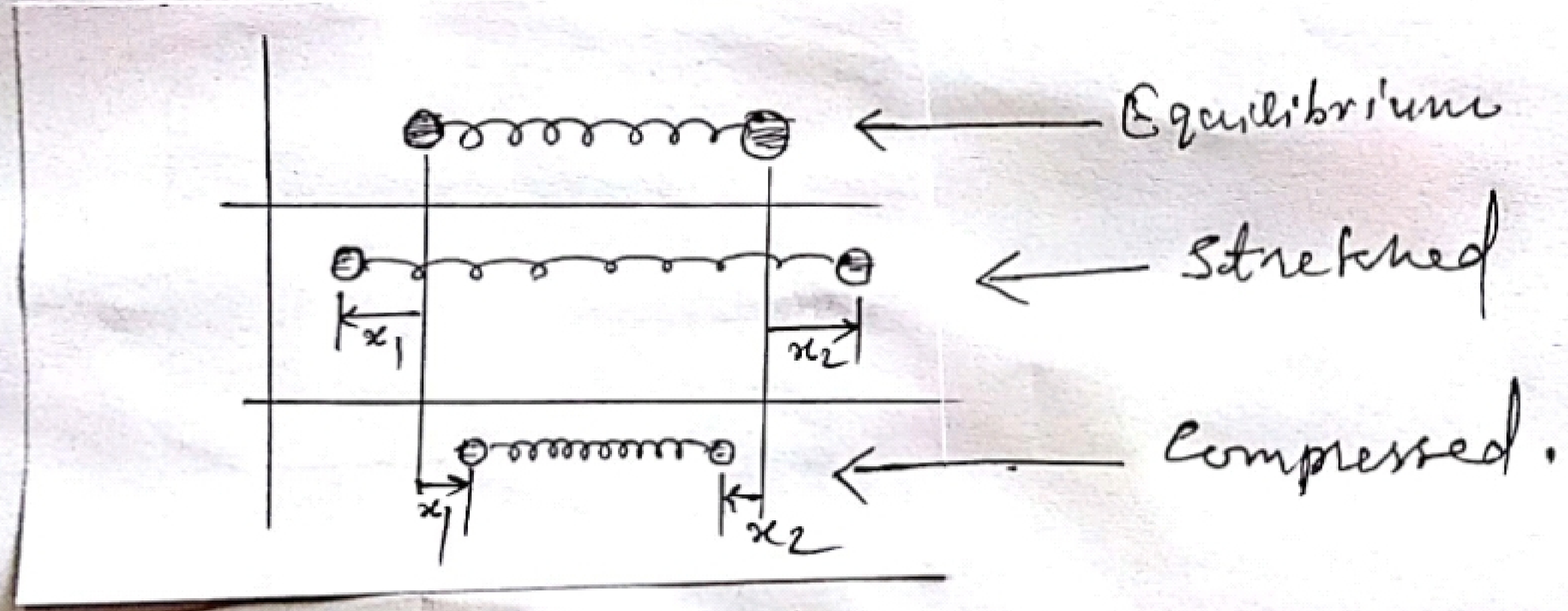
Key Concepts :-

if $V = V(x_1, x_2)$ (Potential Energy of System)

$T = \boxed{T(\dot{x}_1, \dot{x}_2)}$ (Kinetic Energy of System)



Normal Modes : Different ways by which system can vibrate

example 

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Normal frequency: Frequency of Normal Modes is called Normal frequencies. (ω)

Normal frequencies are obtained by solving Secular Equation

$$|\bar{V} - \omega^2 \bar{T}| = 0$$

solving $\omega = \omega_1, \omega_2$

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Here $\bar{V}$ = potential Energy matrix

$$\bar{V} = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}$$

$$V_{11} = \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_1}(V)$$

$$V_{12} = \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}(V)$$

$$V_{21} = \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1}(V)$$

$$V_{22} = \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_2}(V)$$

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$\overline{T}$ = kinetic Energy Matrix

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

$$T_{11} = \frac{\partial}{\partial \dot{x}_1} \frac{\partial}{\partial \dot{x}_1} (T)$$

$$T_{12} = \frac{\partial}{\partial \dot{x}_1} \frac{\partial}{\partial \dot{x}_2} (T)$$

$$T_{21} = \frac{\partial}{\partial \dot{x}_2} \frac{\partial}{\partial \dot{x}_1} (T)$$

$$T_{22} = \frac{\partial}{\partial \dot{x}_2} \frac{\partial}{\partial \dot{x}_2} (T)$$

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You can also find $T_{11}, T_{12}, T_{21}, T_{22}$ by comparing K.E. of system with

$$T = \frac{1}{2}\left[T_{11}\dot{x}_1^2 + T_{12}\dot{x}_1\dot{x}_2 + T_{21}\dot{x}_2\dot{x}_1 + T_{22}\dot{x}_2^2 \right]$$

[for two masses]

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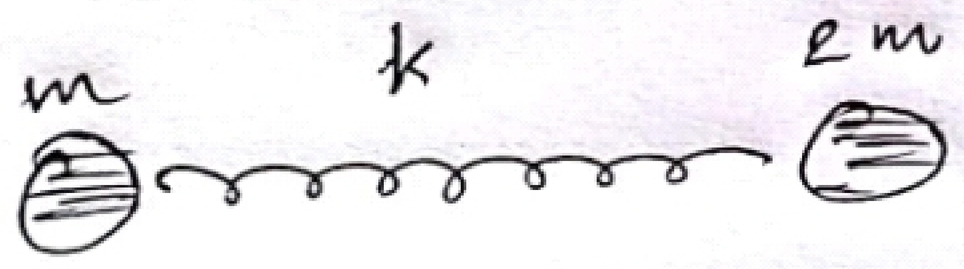
Question: Gate 2011

Two Bodies of masses m and 2m are connected by a spring constant, the frequency of normal mode is



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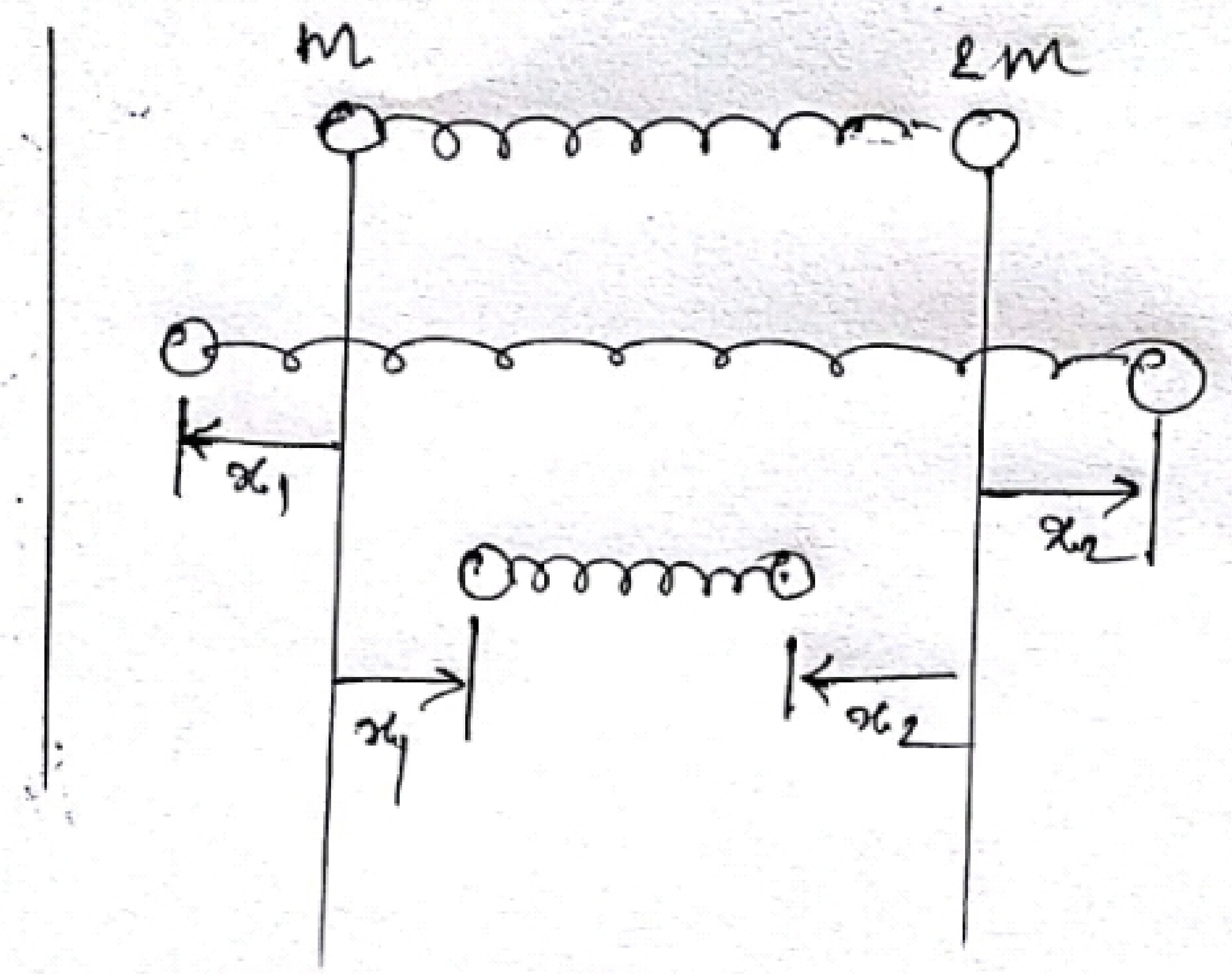
Solution



$$T = \frac{1}{2} m v_1^2 + \frac{1}{2}(2m) v_2^2$$

$$= \frac{1}{2} m \left(\frac{dx_1}{dt}\right)^2 + \frac{1}{2}(2m)\left(\frac{dx_2}{dt}\right)^2$$

$$= \frac{1}{2} m (\dot{x}_1)^2 + \frac{1}{2}(2m)(\dot{x}_2)^2$$

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$$V = \frac{1}{2} k \left(x_2 - x_1\right)^2$$ (Potential Energy)

$x_2 - x_1$ = Increase in length of spring or decrease in length of spring.

Solving Secular eq.

$$\left| \bar{V} - \omega^2 \bar{T} \right| = 0.$$ to find 'ω' (Normal freq)

where $\bar{V} = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}$

$$V_{11} = \frac{\partial^2}{\partial x_1^2} V$$

$$V_{11} = \frac{\partial^2}{\partial x_1^2} \left\{ \frac{1}{2} k \left(x_2 - x_1\right)^2 \right\}$$

$$V_{11} = \frac{1}{2} k \frac{\partial^2}{\partial x_1^2} \left(x_2 - x_1\right)^2$$

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$$V_{11} = \frac{1}{2}k\frac{\partial}{\partial x_1}\left[\frac{\partial}{\partial x_1}(x_2-x_1)^2\right]$$

$$= \frac{1}{2}k\frac{\partial}{\partial x_1}\left[2(x_2-x_1)\cdot\frac{\partial}{\partial x_1}(x_2-x_1)\right]$$

$$= \frac{1}{2}k\frac{\partial}{\partial x_1}\left[2(x_2-x_1)\cdot\left(0-\frac{\partial x_1}{\partial x_1}\right)\right]$$

$$= \frac{1}{2}k\frac{\partial}{\partial x_1}\left[2(x_2-x_1)(-1)\right]$$

$$= \frac{1}{2}k(-2)\frac{\partial}{\partial x_1}(x_2-x_1)$$

$$= \frac{1}{2}k(-2)(-1)$$

$$V_{11} = k$$

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Similary $V_{12} = \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} V$

$$= \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\left\{\frac{1}{2} k (x_2 - x_1)^2\right\}$$

$$= \boxed{-k} \quad \text{(on solving)}$$

$$V_{21} = \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1}(V)$$

$$= \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1}\left\{\frac{1}{2} k (x_2 - x_1)^2\right\}$$

$$= -k$$

$$V_{22} = \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_2}\left\{\frac{1}{2} k (x_2 - x_1)^2\right\}$$

$$= k$$

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$$\therefore \; \bar{V} = \begin{bmatrix} V_{11} & V_{12} \\ V_{22} & V_{22} \end{bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$$

Now

$$\bar{T} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

$$T_{11} = \frac{\partial^2}{\partial \dot{x}_1^2}\left(\frac{1}{2} m \dot{x}_1^2 + \frac{1}{2}(2m)\dot{x}_2^2 \right)$$

$$= \frac{\partial^2}{\partial \dot{x}_1^2}\left(\frac{1}{2} m \dot{x}_1^2 \right) + 0$$

$$= \frac{1}{2} m \frac{\partial}{\partial \dot{x}_1}\left\{ \frac{\partial}{\partial \dot{x}_1}\left(\dot{x}_1^2\right) \right\}$$

$$= \frac{1}{2} m \frac{\partial}{\partial \dot{x}_1} 2\dot{x}_1 = m \frac{\partial}{\partial \dot{x}_1} \dot{x}_1 = m.$$

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$$T_{12} = \frac{\partial}{\partial \dot{x}_1}\frac{\partial}{\partial \dot{x}_2}\left(\frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}(2m)\dot{x}_2^2\right) = 0$$

$$T_{21} = \frac{\partial}{\partial \dot{x}_2}\frac{\partial}{\partial \dot{x}_1}\left(\frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}(2m)\dot{x}_2^2\right) = 0$$

$$T_{22} = \frac{\partial}{\partial \dot{x}_2}\frac{\partial}{\partial \dot{x}_2}\left(\frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}(2m)\dot{x}_2^2\right) = 2m$$

$$\therefore \boxed{\bar{T} = \begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix}}$$

putting the values in $\left|\bar{V} - \omega^2 \bar{T}\right| = 0.$

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$$\left| \begin{pmatrix} k & -k \\ -k & k \end{pmatrix} - \omega^2 \begin{pmatrix} m & 0 \\ 0 & 2m \end{pmatrix} \right| = 0$$

$$\left| \begin{pmatrix} k & -k \\ -k & k \end{pmatrix} - \begin{pmatrix} \omega^2 m & 0 \\ 0 & 2\omega^2 m \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} k-\omega^2 m & -k \\ -k & k-2\omega^2 m \end{vmatrix} = 0.$$

$$\left(k-\omega^2 m\right)\left(k-2\omega^2 m\right) - (-k)(-k) = 0$$

$$\Rightarrow k\left(k-2\omega^2 m\right) - \omega^2 m\left(k-2\omega^2 m\right) - k^2 = 0$$

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$$k^2 - 2w^2mk - w^2mk + 2w^4m^2 - k^2 = 0$$

$$k^2 - 3w^2mk + 2(w^2m)^2 - k^2 = 0$$

$$-3w^2mk + 2w^4m^2 = 0$$

$$\Rightarrow \quad 2w^4m^2 = +3w^2mk$$

$$w^4 = \frac{3}{2}\frac{w^2mk}{m^2}$$

$$w^2 = \frac{3}{2}\frac{k}{m}$$

$$w = \pm\sqrt{\frac{3}{2}\frac{k}{m}}$$

$$\therefore \quad w = +\sqrt{\frac{3}{2}\frac{k}{m}}$$

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