Question based on Hamilton's Equation and Poission Bracket

Key Concepts

Hamiltonian $H = \sum_{k=1}^{n} p_k \dot{q}_k - L$

Generalised momenta $p_k = \frac{\partial L}{\partial \dot{q}_k}$

Poission Bracket $[Q.P]_{q,p} = \left(\frac{\partial Q}{\partial q} \frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p} \frac{\partial P}{\partial q} \right)$

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For a simple Harmonic Oscillator

$$L = \frac{1}{2}\dot{q}^2 - \frac{1}{2}q^2$$

if $A(p,q) = \dfrac{p + iq}{\sqrt{2}}$

and $H(p,q)$ is Hamiltonian of the system

find the poisson bracket $[A,H]_{q,p}$.

Solution:

Given $A = \frac{p + iq}{\sqrt{2}}$

$$L = \frac{1}{2}\dot{q}^2 - \boxed{\frac{1}{2}q^2}$$

Now, Hamiltonian is $H = \sum_k p_k \dot{q}_k - L$

$$H = p\dot{q} - L$$

(here $q_k = q$
$p_k = p$)

Now $p = \frac{\partial L}{\partial \dot{q}} = \frac{\partial}{\partial \dot{q}} \frac{1}{2}\left(\dot{q}^2 - \frac{1}{2}q^2\right)$

$$= \frac{\partial}{\partial \dot{q}} \frac{1}{2}\dot{q}^2 - 0$$

$$p = \frac{1}{2}\frac{\partial}{\partial \dot{q}}\dot{q}^2$$

$$p = \frac{1}{2}\cdot 2\cdot \dot{q}$$

$$p = \dot{q}$$

$\therefore$ Hamiltonian Becomes

$$H = p\dot{q} - L$$

$$= pp - \left(\frac{\dot{q}^2}{2} - \frac{q^2}{2}\right)$$

$$= p^2 - \frac{\dot{p}^2}{2} + \frac{q^2}{2}$$

$$= \frac{2p^2 - p^2}{2} + \frac{q^2}{2} = \frac{p^2}{2} + \frac{q^2}{2}$$

$$[A, H]_{q,p} = \frac{\partial A}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial H}{\partial q}$$

$$= \frac{\partial}{\partial q}\left(\frac{p+iq}{\sqrt{2}}\right)\frac{\partial}{\partial p}\left(\frac{p^2}{2}+\frac{q^2}{2}\right) - \frac{\partial}{\partial p}\left(\frac{p+iq}{\sqrt{2}}\right)\frac{\partial}{\partial q}\left(\frac{p^2+q^2}{2}\right)$$

$$= \left(0+\frac{i}{\sqrt{2}}\frac{\partial q}{\partial q}\right)\left(\frac{1}{2}\frac{\partial p^2}{\partial p}+0\right) - \left(\frac{1}{\sqrt{2}}\frac{\partial p}{\partial p}+0\right)\left(0+\frac{1}{2}\frac{\partial q^2}{\partial q}\right)$$

$$= \left(\frac{i}{\sqrt{2}}\right)\left(\frac{1}{2}\, 2p\right) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\cdot 2q\right)$$

$$= \frac{ip}{\sqrt{2}} - \frac{q}{\sqrt{2}}$$

$$[A, H]_{q,p} = \frac{ip - q}{\sqrt{2}}$$

$$= \frac{ip + (-1)q}{\sqrt{2}}$$

$$= \frac{ip + i^2 q}{\sqrt{2}}$$

$$= \frac{i(p + iq)}{\sqrt{2}}$$

$$= iA \quad \text{(Ans)}.$$