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D'Alembert's Principle : Lagrange's Equation of Motion

$$\sum_i \left(F_i^{(a)} - \dot{\vec{p}}_i\right) \cdot \delta \vec{r}_i = 0$$ for dynamic equilibrium

2nd term :- $\sum_i \dot{\vec{p}}_i \cdot \delta\vec{r}_i = \sum_i m_i \ddot{\vec{r}}_i \cdot \delta\vec{r}_i = \sum_{i=1}^{N} m_i \cdot \ddot{\vec{r}}_i \cdot \sum_{j=1}^{3N} \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j$

$$= \sum_{i,j} m_i \boxed{\ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}} \delta q_j = \sum_{i,j} \left\{ \frac{d}{dt}\left(m_i \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right) - m_i \dot{\vec{r}}_i \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right) \right\} \delta q_j$$

$$\ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \Rightarrow \frac{d}{dt}\left(\dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right) - \dot{\vec{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)$$

Ⓐ $\rightarrow \frac{1}{2}\frac{\partial}{\partial \dot{q}_j}(\vec{v}_i \cdot \vec{v}_i) = \frac{1}{2}\frac{\partial}{\partial \dot{q}_j}\vec{v}_i^{\,2}$

Ⓑ $\Rightarrow \frac{1}{2}\frac{\partial}{\partial q_j}(\vec{v}_i)^2$

Write $\frac{\partial \vec{r}_i}{\partial q_j} = \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j}$

$$\Rightarrow \sum_{i,j}\left[\frac{d}{dt}\left(m_i \dot{\vec{r}}_i \left(\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j}\right)\right) - m_i \dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial q_j}\right]\delta q_j = \sum_{i,j}\left[\frac{d}{dt}\left(m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j}\right) - m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial q_j}\right]\delta q_j$$

$$\Rightarrow \sum_{i,j}\left[\frac{d}{dt}\left(m_i \underbrace{\vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j}}_{Ⓐ}\right) - m_i \underbrace{\vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial q_j}}_{Ⓑ}\right]\delta q_j$$

$$\sum_{i,j} m_i \, \dot{r}_i \cdot \frac{\partial r_i}{\partial q_j} \delta q_j = \cancel{\sum_i \left[\frac{d}{dt} \left\{ \frac{\partial}{\partial \dot{q}_j} \left(\sum_j \right.\right.\right.}$$

$$\Downarrow$$

$$\sum_{i,j} \left\{ \frac{d}{dt} \left[\frac{\partial}{\partial \dot{q}_j} \underbrace{\left(\frac{1}{2} m_i v_i^2 \right)}_{T} \right] - \frac{\partial}{\partial q_j} \left(\frac{1}{2} m_i v_i^2 \right) \right\} \delta q_j$$

$$T = \sum_i \frac{1}{2} m_i \vec{v}_i^{\,2}$$

$$\Rightarrow \left[\sum_j \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right] \delta q_j$$

$$\cancel{\sum_i \vec{F}_i^{(a)} \cdot \delta q_j} \qquad \sum_i F_i^{(a)} \cdot \delta \vec{r}_i = \sum_{i,j} \vec{F}_i \cdot \underbrace{\frac{\partial \vec{r}_i}{\partial q_j}}_{Q_j} \delta q_j = \sum_j Q_j \, \delta q_j$$

→ forces of acting on the system of particles

$$\sum_i \left(\vec{F}_i - \dot{\vec{p}}_i \right) \cdot \delta r_i = 0 = \sum_j Q_j \cdot \delta q_j - \sum_j \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) \delta q_j + \frac{\partial T}{\partial q_j} \delta q_j$$

$$\Rightarrow \sum_j \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) \delta q_j - \frac{\partial T}{\partial q_j} \delta q_j = \sum_j Q_j \cdot \delta q_j$$

$$\Rightarrow \sum_j \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - Q_j \right] \delta q_j = 0$$

N no. of particles f Degrees of freedom

$f = 3N - K$

"f" no. of eq^n

'f' eq^{n}s

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j$$

1. Q_j is conservative in nature

$\vec{F}_i \rightarrow$ conservative force, $= -\vec{\nabla}_i V$

$$Q_j = -\frac{\partial V}{\partial q_j}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial}{\partial q_j}(T-V) = 0.$$

2. Dissipative forces $\rightarrow Q_j$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j$$

$\hookrightarrow$ dissipative forces will be taken in Q_j

$L = T - V$.

$\frac{\partial V}{\partial \dot{q}_j} = 0$ as potential is ~~ve~~ position dependent

$$\frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_j}\right) - \frac{\partial}{\partial q_j}(T-V) = 0 \quad \text{or} \quad \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0}$$

EOM

$m\ddot{\vec{r}} = \vec{F}$ Newtonian

$\frac{d^2 r}{dt^2}$

L = Lagrangian of the system = $T - V$

Langrange's EOM

1. Invariant w.r.t coord^n transformation
2. emphasizes on the Work & energy (scalar qts) rather than forces (vector qts.)
3. Independent generalised coord^{n}s $\rightarrow$ actual particles