Learning Objectives:

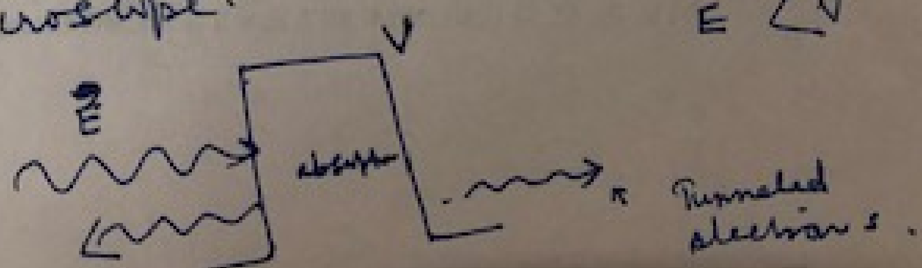
* Tunneling of quantum particle across a potential barrier
* Important physical parameters that affect tunneling
* Physical phenomena where quantum tunneling is observed.
* Explain how quantum tunneling is utilized in modern technology.

Tunneling is a phenomenon in which particles penetrate a potential energy barrier with a height greater than the total energy of the particles.

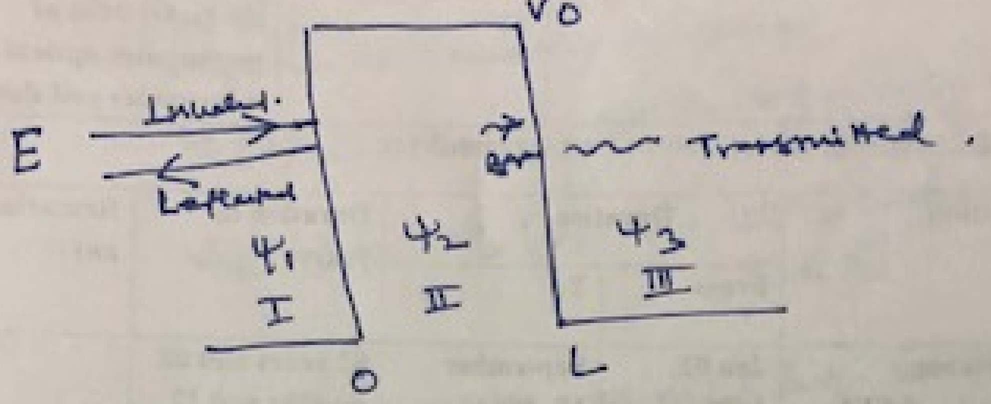
⟹ Violets the principle of classical Mech. & hence interesting

Wide range of Applications: α-decay, quantum tunnel Diode, scanning tunneling microscope.

$E < V$



$E < V_0$



$$V(x) = 0 \text{ when } x < 0$$
$$= V_0 \text{ when } 0 \leq x \leq L$$
$$= 0 \text{ when } x > L$$

General Schrödinger Equation

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E - V_0)\psi = 0 \qquad (1)$$

ψ_1, ψ_2 and ψ_3 are the wave functions in region I, II and III

$$\frac{d^2\psi_1}{dx^2} + \frac{2m}{\hbar^2} E \psi_1 = 0 \text{ for } -\infty < x < 0 \qquad (2)$$

$$\frac{d^2\psi_2}{dx^2} + \frac{2m}{\hbar^2}(E - V_0)\psi_2 = 0 \text{ for } 0 \leq x \leq L \qquad (3)$$

$$\frac{d^2\psi_3}{dx^2} + \frac{2m}{\hbar^2} E \psi_3 = 0 \text{ for } L < x < \infty \qquad (4)$$

Continuity Conditions

* Region at boundary requires

$\psi_1(0) = \psi_2(0)$ at the boundary of region I & II (5)

$\psi_2(0) = \psi_3(0)$ at the boundary of region II & III (6)

* First derivative should be continuous (7)

$$\left.\frac{d\psi_1(x)}{dx}\right|_{x=0} = \left.\frac{d\psi_2(x)}{dx}\right|_{x=0}$$

$$\left.\frac{d\psi_2(x)}{dx}\right|_{x=L} = \left.\frac{d\psi_3(x)}{dx}\right|_{x=L} . \quad (8)$$

$$\psi_1 = Ae^{ikx} + Be^{-ikx}$$

$$\psi_3 = Fe^{ikx} + Ge^{-ikx}$$

where $k = \frac{\sqrt{2mE}}{\hbar}$ or $k^2 = \frac{2mE}{\hbar^2}$

wave number

and $e^{\pm ikx} = \cos kx \pm i\sin kx$

In region I, incident & reflected wave so A & B both exist

In region III, There is No reflection so constant $G = 0$

$\psi_3 = Fe^{ikx} \longrightarrow$ Transmitted wave

In region I. Ae^{ikx} = Incident wave

Be^{-ikx} = Reflected wave.

In region III Fe^{ikx} = Transmitted wave

Amplitude of Incident wave = A

How to find A ?

$$|\psi_{incident}(x)|^2 = \psi_{in}^*(x)\,\psi_{in}(x) = \left(Ae^{+ikx}\right)^* Ae^{+ikx}$$

$$= Ae^{-ikx} \cdot Ae^{ikx}$$

$$= |A|^2 \quad (9)$$

$$|\psi_{transmitted}(x)|^2 = |Fe^{ikx}|^2 \quad (10)$$

$$= |F|^2$$

Transmission Probability or Tunneling Probability is $\frac{|F|^2}{|A|^2}$

$$T(L, E) = \left|\frac{\psi_{trans}(x)}{\psi_{in}(x)}\right|^2 = \left(\frac{F}{A}\right)^2 \quad (11)$$

T depends $\begin{cases} L \text{ is width of the barrier } \& \\ E \text{ is the total energy of the particle.} \\ + \text{ Barrier Height} \end{cases}$

<u>In region II</u>

$$\frac{d^2\psi_2}{dx^2} + \frac{2m}{\hbar^2}(E - V_0)\psi_2 = 0$$

Since $E < V_0$

$$\frac{d^2\psi_2}{dx^2} - \frac{2m}{\hbar^2}(V_0 - E)\psi_2 = 0 \quad (12)$$

$$\frac{d^2\psi_2}{dx^2} - k_2^2\psi_2 = 0$$

Where $k_2^2 = \frac{2mE}{\hbar^2}(V_0 - E)$ (13)

$$\frac{F}{A} = \frac{e^{-ik_2L}}{\cosh(k_2L) + i\frac{\gamma}{2}\sinh(k_2L)}$$

where $\cosh y = \dfrac{e^{y} + e^{-y}}{2}$

$\sinh y = \dfrac{e^{y} - e^{-y}}{2}$

$$\gamma = \frac{k_2}{k} - \frac{k}{k_2}$$

$$T(L,E) = \left(\frac{F}{A}\right)^{*}\frac{F}{A}$$

$$T(L,E) = \frac{1}{\cosh^2(k_2L) + \left(\frac{\gamma}{2}\right)^2 \sinh^2(k_2L)} \qquad (24)$$

where $\left(\dfrac{\gamma}{2}\right)^2 = \dfrac{1}{4}\left(\dfrac{1-\frac{E}{V_0}}{E/V_0} + \dfrac{\frac{E}{V_0}}{1-\frac{E}{V_0}} - 2\right)$

For wide & high barrier that transmits poorly

Thick Barrier $\boxed{T(L,E) = 16\frac{E}{V_0}\left(1-\frac{E}{V_0}\right)e^{-2k_2L}}$ (25)

Adjust V_0 and L to design nano-devices with desirable transmission coefficient.

Thin Barrier $\boxed{T_{thin} \approx e^{-2k_2L}}$ (26)

when $k_2L \ll 1$

1

$$\left.\frac{d\psi_2(x)}{dx}\right|_{x=L} = \left.\frac{d\psi_3(x)}{dx}\right|_{x=L}$$

$$\psi_2 = Ce^{-k_2 x} + De^{k_2 x}$$

$$\psi_3 = Fe^{ikx}$$

$$\Rightarrow Ce^{-k_2 L} + De^{k_2 L} = Fe^{ikL}$$

$$\Rightarrow e^{-k_2 L}(-k_2 C + k_2 D) = ik\,F\,e^{ikL}$$

$$k_2 e^{-k_2 L}\cdot(D - C) = ikFe^{ikL} \quad \rightarrow (22)$$

<u>Thus</u>

$$A + B = (C + D) \quad \text{from } (17)$$

$$ik(A-B) = k_2(D-C) \quad \text{from } (21)$$

$$k_2 e^{-k_2 L}(D-C) = ikFe^{ikL} \quad \text{from } (22)$$

$$Ce^{-k_2 L} + De^{k_2 L} = Fe^{ikL} \quad - (23)$$

$$\frac{\cancel{k_2} e^{-k_2 L}\cdot \cancel{ik}(A-B)}{\cancel{k_2}} = \cancel{ik}Fe^{ikL}$$

$$Fe^{ikL} = e^{-k_2 L}(A-B) = (A-B)e^{-k_2 L}$$

Now using (5) Boundary Condition that
$\psi_1(0) = \psi_2(0)$ in region I & II

$$A + B = C + D \longrightarrow (17)$$

Now using (6) Boundary condition at $x = L$

$$\psi_2(L) = \psi_3(L)$$

$$Ce^{-k_2L} + De^{k_2L} = Fe^{ikL} \quad (18)$$

Now using (7), First derivative Should be Continuous at $x = 0$

$$\psi_1 = Ae^{ikx} + Be^{-ikx} \quad \text{in region I}$$

$$\therefore \left.\frac{d\psi_1}{dx}\right|_{x=0} = ikA - ikB = ik(A - B) \quad (19)$$

$$\psi_2 = Ce^{-k_2x} + De^{k_2x}$$

$$\left.\frac{d\psi_2}{dx}\right|_{x=0} = -k_2C + k_2D = k_2(D - C) \quad (20)$$

Now $\frac{d\psi_1}{dx} = \frac{d\psi_2}{dx}$ at $x = 0$

So

$$ik(A - B) = k_2(D - C) \quad (21)$$

ψ_2 =

$$\frac{d^2\psi_2(x)}{dx^2} - k_2^2\,\psi_2(x) = 0 \qquad (14)$$

$$\psi_2 = Ce^{-k_2 x} + De^{+k_2 x} \qquad (15)$$

Oscillatory Behaviour in region I & III where particle moves freely and exponential decay behavior in region II where the particle moves in potential V_0

~~From Equation~~

$$\left.\begin{aligned}\psi_1 &= Ae^{ikx} + Be^{-ikx}\\ \psi_2 &= Ce^{-k_2 x} + De^{k_2 x}\\ \psi_3 &= Fe^{ikx} + \cancel{Ge^{-ikx}} \text{ — Because No Reflection}\\ &= Fe^{ikx}.\end{aligned}\right\} \qquad (16)$$