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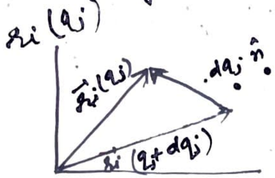
Conservation Theorems :- Cyclic Coord^{n}s

Lagrange's Differential eq^n.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \quad \text{— (A)}$$

Cons. of Linear Momentum



$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

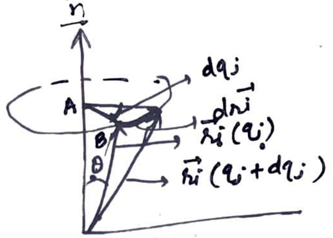
$$\frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j} = 0$$

$$L(q_j, \dot{q}_j, t) = \underset{(\dot{q}_j, t)}{T} - \underset{(q_j)}{V}$$

n degrees of freedom → n differential eq^{n}s to be solved

Cons. of Angular Momentum

Cyclic coord^n ⇒ Any coord^n q_j is absent in the L,



$d\vec{r}_i = AB\, dq_j$

$d\vec{r}_i = r_i \sin\theta\, dq_j$

$\frac{dr_i}{dq_j} = r_i \sin\theta$

eq^n(A) $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0 \Rightarrow \frac{\partial L}{\partial \dot{q}_j} = \text{constant}$ — (B)

as $\frac{\partial L}{\partial q_j} = 0$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) + \frac{\partial V}{\partial q_j} = 0$$

$$\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial (T-V)}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j} - \underset{0}{\frac{\partial V}{\partial \dot{q}_j}}$$

$\because \frac{\partial V}{\partial \dot{q}_j} = 0$ and $\frac{\partial T}{\partial q_j} = 0$

$$\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j} = p_j$$

eq^n (B) $p_j = \text{Constant}$ $\quad q_j = 0$

1st Integral of the motion

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) = -\frac{\partial V}{\partial q_j} = Q_j$$

$$\boxed{\dot{p}_j = Q_j} \rightarrow (C)$$

$$\underline{\dot{p}_j} = \dot{Q}_j = \vec{F}_i \cdot \hat{n}$$

$$Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

$$\partial \vec{r}_i = \hat{n}\, \partial q_j$$

$$\boxed{Q_j = \sum_i \vec{F}_i \cdot \hat{n}}$$

$$T = \frac{1}{2} \sum_i m_i \dot{\vec{r}}_i^{\,2}$$

$$p_j = \frac{\partial T}{\partial \dot{q}_j} = \sum_i m_i \vec{v}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i m_i \vec{v}_i \cdot \hat{n}$$

$$\sum_i m_i \dot{\vec{v}}_i \cdot \hat{n} = \vec{F}_i \cdot \hat{n}$$

If q_j is cyclic in nature

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) = -\frac{\partial V}{\partial q_j} = 0$$

$$\frac{d}{dt}(p_j) = 0$$

$$\boxed{p_j = \text{constant}}$$

Conservation of Angular Momentum

$$\left|\frac{\partial \vec{r}_i}{\partial q_j}\right| = \vec{r}_i \sin\theta = \vec{r}_i \times \hat{n}$$

$$= \vec{r}_i \times \hat{n}$$

$$|d\vec{r}_i| = AB\, dq_j$$

$$|d\vec{r}_i| = r_i \sin\theta\, dq_j$$

$$\underline{\dot{p}_j} = Q_j$$

$$Q_j = \sum_i \vec{F}_i \cdot (\hat{n} \times \vec{r}_i)$$

$$= \sum_i \hat{n} \cdot (\vec{r}_i \times \vec{F}_i)$$

$$Q_j = \sum_i \hat{n} \cdot \underbrace{\tau_i}_{N}$$

$$Q_j = \sum_i \hat{n} \cdot \vec{N}$$

$$p_j = \frac{\partial T}{\partial \dot{q}_j} = \sum_i m_i \vec{v}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

$$= \sum_i m_i v_i \cdot (\hat{n} \times \vec{r}_i)$$

$$= \hat{n} \sum_i m_i (\vec{r}_i \times \vec{v}_i)$$

$$= \hat{n} \sum_i \vec{r}_i \times m_i \vec{v}_i$$

$$= \hat{n} \sum_i \vec{r}_i \times \vec{p}_i = \hat{n} \cdot \vec{L}$$

Component of total Angular mom. along the dirn of Rotation

$\dot{p}_j = Q_j$

$\frac{d}{dt}(\hat{n}\cdot\vec{L}) = \hat{n}\cdot\vec{N}$

$Q_j = -\frac{\partial V}{\partial q_j}$ if $\underline{q_j}$ is cyclic

$\dot{p}_j = 0 \Rightarrow \frac{d}{dt}(\hat{n}\cdot\vec{L}) = 0$

or $\hat{n}\cdot\vec{L} =$ constant

If q_j is cyclic then the Q_j which is the component of Torque along $\hat{n}$ vanishes and the component of $\vec{L}$ along $\hat{n}$ is conserved.

Conservation of Energy :-

(i) conservative system, $V(q_j)$

(ii) Constraints do not change with time,

$\vec{r}_i = \vec{r}_i(q_1, q_2, \ldots, q_{3N})$ Donot involve time explicitly.

(iii) $L = L(q_j, \dot{q}_j)$

$\because \frac{\partial V}{\partial \dot{q}_j} = 0 \quad \therefore \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j}$

Lagrangian for a particle subject to Central force (Inverse square force)

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{k}{r}$$

$\underline{r, \theta}$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$ $\quad$ $\underline{\theta \text{ is cyclic coord}^n}$

$\frac{d}{dt}(mr^2\dot{\theta}) = 0$

$\underline{mr^2\dot{\theta}} =$ constant

Ang. Mom.

$$\frac{dL}{dt} = \sum_j \frac{\partial L}{\partial q_j}\frac{dq_j}{dt} + \sum_j \frac{\partial L}{\partial \dot{q}_j}\frac{d\dot{q}_j}{dt}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial q_j}$$

$$\therefore \frac{dL}{dt} = \sum_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j}\frac{d\dot{q}_j}{dt}$$

$$\frac{dL}{dt} = \sum_j \frac{d}{dt}\left(\dot{q}_j\frac{\partial L}{\partial \dot{q}_j}\right) = \sum_j \frac{d}{dt}\left(\dot{q}_j\frac{\partial T}{\partial \dot{q}_j}\right)$$

$$\frac{dL}{dt} - \sum_j \frac{d}{dt}\left(\dot{q}_j p_j\right) = 0$$

$$\because \frac{\partial T}{\partial \dot{q}_j} = p_j$$

$$\frac{d}{dt}\left[L - \sum_j (\dot{q}_j p_j)\right] = 0$$

or $$\frac{d}{dt}\left[\sum_j (\dot{q}_j p_j) - L\right] = 0$$

$$\Rightarrow \boxed{\sum_j \dot{q}_j p_j - L = \text{constant}}$$

$= H.$

= Hamiltonian of the system

$H(q_j, p_j, t)$

$H(q, p, t)$

$$T = \sum_{j,k} a_{jk} \dot{q}_j \dot{q}_k$$

Euler's theorem which states

$$\sum_j q_j \frac{\partial f}{\partial q_j} = nf$$

$f = T, \quad q_j = \dot{q}_j$

$\therefore n = 2$

$$\sum_j q_j \frac{\partial f}{\partial q_j} = nf$$

$$\sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = 2T.$$

$$\sum_j \dot{q}_j p_j = 2T$$

$$2T - L = \text{constant}$$

$$2T - (T - V) = H$$

$$2T - T + V = H$$

$$\boxed{T + V = H}$$