Question Based On

"Lagrangian" and

"Lagrange Equation of Motion."

Key Concept

Lagrangian $L = T - V$

Lagrange Eq. $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \left(\frac{\partial L}{\partial q_k}\right) = 0$

where q_k = k^{th} generalised coordinate

Question: Gate 2009

The Lagrangian of a particle of mass 'm' moving in one dimension is

$$L = e^{\alpha t} \cdot \left(\frac{m \dot{x}^2}{2} - \frac{k x^2}{2} \right)$$

$[e = 2.718 = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots \infty]$

where α, k are positive constants.

The equation of motion of the particle is

Solution:

Given $L = e^{\alpha t}\left(\frac{m\dot{x}^2}{2} - \frac{kx^2}{2}\right)$

α, k = positive Constants

$$\frac{\partial L}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}} e^{\alpha t}\left(\frac{m\dot{x}^2}{2} - \frac{kx^2}{2}\right)$$

$$= e^{\alpha t}\left[\frac{\partial}{\partial \dot{x}} \frac{m\dot{x}^2}{2} - \frac{\partial}{\partial \dot{x}}\left(\frac{kx^2}{2}\right)\right]$$

$$= e^{\alpha t}\left[\frac{m}{2}\frac{\partial}{\partial \dot{x}} \dot{x}^2 - 0\right]$$

$$= e^{\alpha t} \cdot \frac{m}{2} \cdot 2\dot{x} = e^{\alpha t} \cdot m\dot{x}$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}e^{\alpha t}$$

$$\frac{\partial L}{\partial x} = \frac{\partial}{\partial x} e^{\alpha t}\left(\frac{m\dot{x}^2}{2} - \frac{kx^2}{2}\right)$$

$$= e^{\alpha t}\left[\frac{\partial}{\partial x}\frac{m\dot{x}^2}{2} - \frac{\partial}{\partial x}\frac{kx^2}{2}\right]$$

$$= e^{\alpha t}\left[0 - \frac{k}{2}\frac{\partial}{\partial x}x^2\right]$$

$$= e^{\alpha t}\left[-\frac{k}{2}\cdot 2x\right]$$

$$\frac{\partial L}{\partial x} = -kx\, e^{\alpha t}$$

putting these values in Lagrange Equation of Motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt}\left(m\dot{x}\, e^{\alpha t}\right) - \left(-kx\, e^{\alpha t}\right) = 0$$

$$\frac{d}{dt}\left(m\dot{x}\, e^{\alpha t}\right) + kx\, e^{\alpha t} = 0$$

$$m\,\frac{d}{dt}\left(\dot{x}\, e^{\alpha t}\right) + kx\, e^{\alpha t} = 0$$

$$m\left[\dot{x}\,\frac{d}{dt} e^{\alpha t} + e^{\alpha t}\,\frac{d}{dt}\dot{x}\right] + kx\, e^{\alpha t} = 0.$$

$$m\left[\dot{x}\left(\alpha e^{\alpha t}\right) + e^{\alpha t}\ddot{x}\right] + kxe^{\alpha t} = 0$$

$$e^{\alpha t}\left[m\dot{x}\alpha + m\ddot{x} + kx\right] = 0$$

$\because$ α = +ve constant

time t = always +ve. and $e = +2.7$

$\therefore$ $e^{\alpha t} \neq 0$

$$\therefore \left[m\dot{x}\alpha + m\ddot{x} + kx\right] = 0.$$

$$\Rightarrow m\ddot{x} + m\alpha\dot{x} + kx = 0$$

dividing both sides by m

$$\frac{m\ddot{x}}{m} + \frac{m\alpha\dot{x}}{m} + \frac{kx}{m} = \frac{0}{m}$$

$$\ddot{x} + \alpha\dot{x} + \frac{k}{m}x = 0$$

This is the required Equation of motion.