

Question Based On

"Normal frequencies"

and

"Theory of small oscillations"

Key Concepts:

According to the Theory of small oscillations

frequencies of Normal Mode are obtained by

solving the secular eq.

$$|\bar{V} - \omega^2 \bar{T}| = 0$$

where \bar{V} = Potential Energy matrix

, \bar{T} = Kinetic Energy matrix

ω = Normal frequencies .

Question: GATE 2009.

The Lagrangian of a diatomic molecule is given by

$$L = \frac{m}{2} (\dot{x}_1^2 + \dot{x}_2^2) - \frac{k}{2} x_1 x_2$$

where m is the mass of each atom and x_1, x_2 are displacement from equilibrium position. and $k > 0$. The normal frequencies

are.

Solution \rightarrow

$$\text{Given } L = \frac{m}{2} (\dot{x}_1^2 + \dot{x}_2^2) - \frac{k}{2} x_1 x_2$$

$$\text{But } L = T - V$$

$$\therefore T = \frac{m}{2} (\dot{x}_1^2 + \dot{x}_2^2)$$

$$V = \frac{k}{2} x_1 x_2$$

Since here there are two generalised coordinates x_1 and x_2

$\therefore \bar{V}$ and \bar{T} will be 2×2 matrix

$$\bar{V} = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$$

$$V_{11} = \frac{\partial^2 V}{\partial x_1^2}$$

$$V_{21} = \frac{\partial^2 V}{\partial x_2 \partial x_1}$$

$$V_{12} = \frac{\partial^2 V}{\partial x_1 \partial x_2}$$

$$V_{22} = \frac{\partial^2 V}{\partial x_2^2}$$

Solving.

$$V_{11} = \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_1} (V) = \frac{\partial}{\partial x_1} \left\{ \frac{\partial}{\partial x_1} \left(\frac{k}{2} x_1 x_2 \right) \right\}$$

$$= \frac{\partial}{\partial x_1} \left\{ \frac{k}{2} x_2 \frac{\partial x_1}{\partial x_1} \right\} = \frac{\partial}{\partial x_1} \left(\frac{k}{2} x_2 \right)$$

$$= 0$$

$$V_{12} = \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} \left(\frac{k}{2} x_1 x_2 \right) = \frac{\partial}{\partial x_1} \left\{ \frac{k}{2} x_1 \right\} = \frac{k}{2}$$

$$V_{21} = \frac{\partial}{\partial x_2} \frac{\partial}{\partial x_1} \left(\frac{k}{2} x_1 x_2 \right)$$

$$= \frac{\partial}{\partial x_2} \left(\frac{k}{2} x_2 \right)$$

$$= \frac{k}{2}$$

$$V_{22} = \frac{\partial}{\partial x_2} \frac{\partial}{\partial x_2} \left(\frac{k}{2} x_1 x_2 \right)$$

$$= \frac{\partial}{\partial x_2} \left(\frac{k}{2} x_1 \right)$$

$$= 0$$

$$\therefore \bar{V} = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} = \begin{pmatrix} 0 & k/2 \\ k/2 & 0 \end{pmatrix}$$

Now $\bar{T} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$

$$T_{11} = \frac{\partial^2}{\partial \dot{x}_1^2} T \quad T_{12} = \frac{\partial^2}{\partial \dot{x}_1 \partial \dot{x}_2}$$

$$T_{21} = \frac{\partial^2}{\partial \dot{x}_2 \partial \dot{x}_1} \quad [T_{22} = \frac{\partial^2}{\partial \dot{x}_2^2} T]$$

$$T_{11} = \frac{\partial \dot{x}_1}{\partial \dot{x}_1} \left(\frac{m}{2} (\dot{x}_1^2 + \dot{x}_2^2) \right)$$

$$= \frac{m}{2} \frac{\partial}{\partial \dot{x}_1} \left[\frac{\partial}{\partial \dot{x}_1} (\dot{x}_1^2 + \dot{x}_2^2) \right]$$

$$= \frac{m}{2} \frac{\partial}{\partial \dot{x}_1} [2\dot{x}_1 + 0]$$

$$= \frac{m}{2} \cdot 2 \frac{\partial \dot{x}_1}{\partial \dot{x}_1} = m$$

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$$T_{12} = \frac{\partial}{\partial \dot{x}_1} \cdot \frac{\partial}{\partial \dot{x}_2} \left[\frac{m}{2} (\dot{x}_1^2 + \dot{x}_2^2) \right]$$

$$= \frac{m}{2} \frac{\partial}{\partial \dot{x}_1} \left(\frac{\partial}{\partial \dot{x}_2} (\dot{x}_1^2 + \dot{x}_2^2) \right)$$

$$= \frac{m}{2} \frac{\partial}{\partial \dot{x}_1} [0 + 2\dot{x}_2]$$

$$= \frac{m}{2} \cdot 2 \frac{\partial \dot{x}_2}{\partial \dot{x}_1}$$

$$\approx m \times 0 = 0$$

$$T_{21} = \frac{\partial}{\partial \dot{x}_2} \cdot \frac{\partial}{\partial \dot{x}_1} \frac{m}{2} (\dot{x}_1^2 + \dot{x}_2^2)$$

$$= 0 \quad (\text{on solving}).$$

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$$T_{22} = \frac{\partial}{\partial \dot{x}_2} \frac{\partial}{\partial \dot{x}_2} \left[\frac{m}{2} (\dot{x}_1^2 + \dot{x}_2^2) \right]$$

$$= \frac{m}{2} \frac{\partial}{\partial \dot{x}_2} \left[\frac{\partial}{\partial \dot{x}_2} (\dot{x}_1^2 + \dot{x}_2^2) \right]$$

$$= \frac{m}{2} \frac{\partial}{\partial \dot{x}_2} (0 + 2\dot{x}_2)$$

$$= \frac{m}{2} \cdot 2 \frac{\partial}{\partial \dot{x}_2} \dot{x}_2$$

$$= m$$

$$\therefore \bar{T} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$$

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putting the values

$$\left| \bar{V} - \omega^2 \bar{T} \right| = 0$$

$$\left| \begin{pmatrix} 0 & \frac{k}{2} \\ \frac{k}{2} & 0 \end{pmatrix} - \omega^2 \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \right| = 0$$

$$\left| \begin{pmatrix} 0 & \frac{k}{2} \\ \frac{k}{2} & 0 \end{pmatrix} - \begin{pmatrix} \omega^2 m & 0 \\ 0 & \omega^2 m \end{pmatrix} \right| = 0$$

$$\left| \begin{pmatrix} -\omega^2 m & \frac{k}{2} \\ \frac{k}{2} & -\omega^2 m \end{pmatrix} \right| = 0$$

$$(-\omega_{n0})(-\omega_m) - \left(\frac{k}{2}\right)^2 = 0 \quad (11)$$

$$\omega^4 m^2 - \frac{k^2}{4} = 0$$

$$(\omega_m)^2 - \left(\frac{k}{2}\right)^2 = 0$$

$$\left(\omega_m + \frac{k}{2}\right)\left(\omega_m - \frac{k}{2}\right) = 0 \quad [\because a^2 - b^2 = (a+b)(a-b)]$$

$$\omega_m + \frac{k}{2} = 0 \quad ; \quad \omega_m - \frac{k}{2} = 0$$

$$\omega^2 = -\frac{k}{2m} \Rightarrow \omega = \pm \sqrt{-\frac{k}{2m}} = \pm \frac{i\sqrt{k}}{\sqrt{2m}}$$

neglected \because frequencies
cannot be imaginary
or negative

$$\text{and } \omega_m = \frac{k}{2} \Rightarrow \omega^2 = \frac{k}{2m} \Rightarrow \omega = \sqrt{\frac{k}{2m}}$$

$\omega = + \sqrt{\frac{k}{2m}}$ because frequency cannot be negative.