Lecture -3

$T = \sum_{jk} a_{jk} \dot{q}_j \dot{q}_k$ where $a_{jk} = \sum_i \frac{1}{2} m_i \left(\frac{\partial \vec{r}_i}{\partial q_j}\right)\left(\frac{\partial \vec{r}_i}{\partial q_k}\right)$

$j = 2, \; k = 2$

(r, θ)

$q_1 = r \quad j=1 \quad k=1$

$q_2 = \theta \quad j=2 \quad k=2$

$(r, \theta) \qquad \vec{r} = x\hat{i} + y\hat{j} \qquad x = r\cos\theta, \quad y = r\sin\theta$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m \dot{r}^2 = \frac{1}{2} m (\dot{r} \cdot \dot{r})$$

$$\cancel{T} = \cancel{\tfrac{1}{2}m} \quad a_{jk} = \frac{1}{2} m_i \left[\underbrace{\left(\frac{\partial \vec{r}}{\partial r}\right)\cdot\left(\frac{\partial \vec{r}}{\partial r}\right)}_{j=1,\,k=1} + \underbrace{\left(\frac{\partial \vec{r}}{\partial \theta}\right)\cdot\left(\frac{\partial \vec{r}}{\partial \theta}\right)}_{j=2,\,k=2} + \underbrace{2\left(\frac{\partial \vec{r}}{\partial r}\right)\cdot\left(\frac{\partial \vec{r}}{\partial \theta}\right)}_{j=1,\,k=2\;/\;j=2,\,k=1} \right]$$

$$a_{jk} = \frac{1}{2} m_i \left[\left\{ \hat{i}\overbrace{\left(\frac{\partial x}{\partial r}\right)}^{r\cos\theta} + \hat{j}\overbrace{\left(\frac{\partial y}{\partial r}\right)}^{r\sin\theta} \right\} \cdot \left\{ \hat{i}\frac{\partial x}{\partial r} + \hat{j}\frac{\partial y}{\partial r} \right\} + \left(\hat{i}\frac{\partial x}{\partial \theta} + \hat{j}\frac{\partial y}{\partial \theta}\right)\cdot\left(\hat{i}\frac{\partial x}{\partial \theta} + \hat{j}\frac{\partial y}{\partial \theta}\right) + 2\left(\frac{\partial x}{\partial r}\hat{i} + \hat{j}\frac{\partial y}{\partial r}\right)\cdot\left(\frac{\partial x}{\partial \theta}\hat{i} + \frac{\partial y}{\partial \theta}\hat{j}\right) \right]$$

$$a_{jk} = \frac{1}{2} m_i \left[(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}) \cdot (\cos\theta\,\hat{i} + \sin\theta\,\hat{j}) + \right.$$

$$T = \frac{1}{2} m_i \left[\cancel{\cos\theta}\,(\underbrace{\cos^2\theta + \sin^2\theta}_{1})\,\dot{r}\dot{r} + \left(-\sin\theta \cdot r\,\hat{i} + r\cos\theta\,\hat{j}\right)\cdot\left(-\sin\theta\, r\,\hat{i} + r\cos\theta\,\hat{j}\right)\dot{\theta}\dot{\theta} + 2(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})\cdot r(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})\,\dot{r}\dot{\theta} \right]$$

$$T = \frac{1}{2} m_i \left[\dot{r}\dot{r} + r^2 \dot{\theta}\dot{\theta} \right] = \frac{1}{2} m \left[\dot{r}^2 + r^2\dot{\theta}^2 \right] \qquad i = 1$$

$p_r = \frac{\partial T}{\partial \dot{r}} = m\dot{r}$ (linear Momentum) $\qquad p_\theta = \frac{\partial T}{\partial \dot{\theta}} = \cancel{\tfrac{1}{2}} \cdot m r^2 \dot{\theta} \rightarrow$ Angular Momentum.

Generalised Force :- $\sum_i F_i$, $\sum_i \delta r_i = \sum_j \frac{\partial r_i}{\partial q_j} \cdot \delta q_j$

$$\delta W = \sum_{i=1}^{N} \vec{F}_i \cdot \delta \vec{r}_i = \sum_{i=1}^{N} \sum_{j=1}^{3N} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \cdot \delta q_j = \sum_{j=1}^{3N} Q_j \cdot \delta q_j .$$

generalised force $Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$

Generalised Potential : scalar V $\quad \vec{F}_i = -\vec{\nabla}_i V$

$$Q_j = \sum_i -\vec{\nabla}_i V \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\sum_j \frac{\partial V}{\partial q_j}$$

D' Alemberts Principle :- Pr. of Virtual Work.

Virtual displacement $\quad \delta x_i = dx_i \big|_{at\ dt=0} \quad q_j = (q_1, q_2, \dots q_n)$

$\downarrow$

do not represent the actual displacements of particles

Virtual displacement $\delta \vec{r}_i$ $\quad$ STATIC EQUILIBRIUM $\rightarrow$ Total force : $\vec{F}_i = \vec{F}_i^{(a)} + \vec{f}_i = 0$

$\vec{F}_i^{(a)} + \vec{f}_i = 0$

$$\delta W = \vec{F}_i \cdot \delta \vec{r}_i = 0 = \sum_i \left(\vec{F}_i^{(a)} + \vec{f}_i \right) \cdot \delta \vec{r}_i = 0$$

$$= \sum_i \vec{F}_i^{(a)} \cdot \delta \vec{r}_i + \underbrace{\sum_i \vec{f}_i \cdot \delta \vec{r}_i}_{0} = 0 \Rightarrow \boxed{\sum_{i=1}^{N} F_i^{(a)} \cdot \delta \vec{r}_i = 0} \quad N=10$$

$\vec{F}_i = \dot{\vec{p}}_i$ Newton 2nd law of motion

$\boxed{\vec{F}_i - \dot{\vec{p}}_i = 0}$ Dynamic system appears to be in equil.

$\vec{F}_i \rightarrow$ app. force; $\dot{\vec{p}}_i \rightarrow$ kinetic Reaction / effective force

$$\delta W \sum_i \vec{F}_i \cdot \delta\vec{r}_i = 0 \Rightarrow \sum_i (\vec{F}_i - \dot{\vec{p}}_i) \cdot \delta\vec{r}_i = 0 = \sum_i (\vec{F}_i^{(a)} - \dot{\vec{p}}_i) \cdot \delta\vec{r}_i + \sum_i \underset{0}{\underbrace{\vec{f}_i \cdot \delta\vec{r}_i}} = 0$$

$$\boxed{\sum_i (\vec{F}_i^{(a)} - \dot{\vec{p}}_i) \cdot \underbrace{\delta\vec{r}_i}_{\mathrm{I}} = 0}$$

$$\delta\vec{r}_i = \sum_{j=1}^{3N} \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j$$