Class Test (QM)

Aditya Singh
2K19/EP/005

(1)

$$E_n^{(0)} = \frac{\pi^2 \hbar n^2}{2 m a^2}, \quad \psi_n^{(0)} = \sqrt{\frac{2}{a}} \sin \frac{\pi x}{a}.$$

$E_n^{(1)}$ (First Order Correction)

$$= \langle \psi_n | H' | \psi_n \rangle \qquad \text{where } H' = V_0 \cos\left(\frac{\pi x}{a}\right)$$

$$= \int_0^{a/2} \frac{2}{a} \sin^2\left(\frac{\pi x}{a}\right) V_0 \cos\left(\frac{\pi x}{a}\right) dx.$$

$$= \frac{2V_0}{a} \int_0^{a/2} \sin^2\left(\frac{\pi x}{a}\right) \cos\left(\frac{\pi x}{a}\right) dx$$

$$= \frac{2V_0}{a} \sin^3\left(\frac{\pi x}{a}\right)\Big]_0^{a/2} \Rightarrow \cancel{\frac{2V_0}{a}} \; \Delta E = 2V_0$$

(3) Perturbation theory is used to calculate small disturbance in Hamiltonian of the system.

$H = H_0 + H'$ where $H \rightarrow$ perturbed Hamiltonian.
$H_0 \rightarrow$ unperturbed

* For non-degenerate case,

Lets take $u = a + \varepsilon u^2$

if ε is small,

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots$$

Substitute series in eq(1).

$$u = a + \epsilon a^2 + 2\epsilon^2 a^3 + \cdots$$

exact soluth

$$u = \frac{1}{2\epsilon}\left[1 - (1 - 4a\epsilon)^{1/2}\right]$$

$$= a + \epsilon a^2 + 2\epsilon^2 a^3 + \cdots$$

$$\boxed{H = H_0 + gH'}$$

$$\psi_n = \psi_n^{(0)} + g\,\psi_n^{(1)} + g^2\,\psi_n^{(2)} + \cdots$$

$$W_n = W_n^{(0)} + g\,W_n^{(1)} + g^2\,W_n^{(2)} + \cdots$$

$$(H_0 + gH')(\psi_n^{(0)} + g\,\psi_n^{(1)} + g_2\,\psi_n^{(2)} + \cdots)$$

$$= \left[W_n^{(0)} + g\,W_n^{(1)} + g^2\,W_n^{(2)} + \cdots\right]\left(\psi_n^{(0)} + g\,\psi_n^{(1)} + g^2\,\psi_n^{(2)}\right)$$

$$\hookrightarrow \qquad H_0\psi_n^{(0)} = W_n^{(0)}\,\psi_n^{(0)}$$

$$H_0\psi_n^{(1)} + H'\psi_n^{(0)} = W_n^{(0)}\psi_n^{(1)} + \psi_n^{(0)}W_n^{(1)}$$

$$H_0\psi_n^{(2)} + H'\psi_n^{(1)} = W_n^{(0)}\psi_n^{(2)} + W_n^{(1)}\psi_n^{(1)} + W_n^{(2)}\psi_n^{(0)}$$

First Order

$$\psi_n^{(1)} = \sum_m a_m^{(1)}\,u_m.$$

$$\sum_m a_m^{(1)} E_m u_m + H'u_n = E_n\sum_m a_m^{(1)} u_m + W_n^{(1)} u_n.$$

$$a_k^{(1)}(E_n - E_k) + W_n^{(1)}\delta_{kn} = H'_{kn}$$

k=n

$$W_n^{(1)} = H'_{nn} = \int u_k^* H' u_n\, d\tau = \langle n|H'|n\rangle$$

$$a_k^{(1)} = \left(\frac{H'_{kn}}{E_n - E_k}\right), \qquad k \neq n$$

Second order

$$\psi_n^{(2)} = \sum_m a_m^{(2)} \psi_m$$

$$\sum_m a_m^{(2)} E_m \delta_{km} + \sum_m a_m^{(1)} H'_{km} = E_n \sum_m a_m^{(2)} \delta_{km} + W_n^{(2)} \delta_{kn} + H'_{nn} \sum_m a_m^{(1)} \delta_{km}$$

$k=n$

$$W_n^{(2)} = \sum_m a_m^{(1)} - a_n^{(1)} H'_{nn}$$

$$= \sum_{m\neq n} a_m^{(1)} H'_{nm} = \sum_{m\neq n} \frac{|H'_{mn}|^2}{E_n - E_m}$$

(4) $$\langle E \rangle = \frac{-\hbar^2}{2m} \langle \psi | \frac{\partial^2}{\partial x^2} | \psi \rangle + g \langle \psi | |x| | \psi \rangle$$

as $$\psi(x) = \begin{cases} \sqrt{\frac{c}{a^5}} (a^2 - x^2), & x < |a| . \\ 0, & x \geq |a| . \end{cases}$$

$$\langle \psi | \frac{\partial^2}{\partial x^2} | \psi \rangle = \int_{-a}^{a} \sqrt{\frac{c}{a^5}} (a^2 - x^2) \frac{\partial^2}{\partial x^2} \left(\sqrt{\frac{c}{a^5}} (a^2 - x^2) \right) dx .$$

$$= \frac{c}{a^5} \int_{-a}^{a} (a^2 - x^2)(-2)\, dx$$

$$= \frac{-4c}{a^5} \left[a^2 x - \frac{x^3}{3} \right]_0^a$$

$$= \frac{-4c}{a^5} \left[\frac{2a^3}{3} \right] = \frac{-8c}{3a^2} .$$

$$\Rightarrow \quad -\frac{\hbar^2}{2m} \langle \psi | \frac{d^2}{dx^2} | \psi \rangle = \frac{4c\hbar^2}{3ma^2} .$$

$$g\langle\varphi|\,|x|\,|\varphi\rangle = \frac{c}{a^5}\int_{-a}^{a}(a^2-x^2)^2\,|x|\,dx.$$

$$= \frac{c}{a^5}\left[\int_{-a}^{0}(-x)(a^2-x^2)^2\,dx + \int_{0}^{a} x\,(a^2-x^2)^2\,dx\right]$$

$$= \frac{2c}{a^5}\int_0^a x(a^2-x^2)^2\,dx$$

$$= \frac{2c}{a^5}\int_0^a (a^4x + x^5 - 2a^2x^3)\,dx$$

$$= \frac{2c}{a^5}\left[\frac{6a^6 - 2a^6 - 6a^6}{12}\right] = \frac{ac}{3}g.$$

~~EX~~ Normalization $\int_{-a}^{a}\varphi^*\varphi = 1$

$$\Rightarrow \frac{c}{a^5}\int_{-a}^{a}(a^2-x^2)^2\,dx = 1$$

$$\frac{2c}{a^5}\left[\frac{8a^5}{15}\right] = 1 \qquad \boxed{c = \frac{15}{16}}$$

$$\langle E\rangle = \frac{4\hbar^2}{3ma^2}\left(\frac{15}{16}\right) + \frac{ga}{3}\left(\frac{15}{16}\right)$$

$$\frac{d\langle E\rangle}{da} = 0\,; \qquad -\frac{5}{4}\frac{\hbar^2}{M}\cdot 2a^{-3} + \frac{5g}{16} = 0$$

$$a^3 = \frac{8\hbar^2}{mg} \qquad = \qquad a = \left(\frac{8\hbar^2}{mg}\right)^{1/3}$$

$$\langle E\rangle = \frac{5\hbar^2}{4m}\left(\frac{mg}{8\hbar^2}\right) + \frac{5ga}{16} = \frac{15ag}{16}\left(\frac{3}{2}\right)\left(\frac{8\hbar}{mg}\right)^{1/3}$$

$$= \frac{15}{16}\left(\frac{\hbar^2 g^2}{m}\right)^{1/3}.$$

(2) $E_n^{(1)} = \langle n|H'|n\rangle = 0$

$$E_n^{(2)} = \sum_{m\neq n}\frac{|\langle m|H'|n\rangle|^2}{E_n - E_m} = \frac{\hbar\lambda^2}{2m\omega}\sum\frac{|\langle m|a^\dagger + a|n\rangle|^2}{E_n - E_m}$$

$$= \frac{\hbar\lambda^2}{2m\omega}\frac{|\sqrt{n+1}\langle m|n+1\rangle + \sqrt{n}\langle m|n-1\rangle|^2}{E_n - E_m}$$

$$= (\,''\,)\frac{|\sqrt{n+1}\,\delta_{m,n+1} + \sqrt{n}\,\delta_{m,n-1}|^2}{E_n - E_m}$$

$$= \frac{\hbar\lambda^2}{2m\omega}\frac{1}{\hbar\omega}\left[-(n+1) + n\right] = \frac{-\lambda^2}{2m\omega^2}.$$

$$|\psi_n^{(1)}\rangle = \sum_{m\neq n}\frac{\langle m|H'|n\rangle}{E_n - E_m}|m\rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\,\lambda\left(\sum\frac{\sqrt{n+1}\,\delta_{m,n+1}}{E_n - E_m}|m\rangle + \sum\frac{\sqrt{n}\,\delta_{m,n-1}}{E_n - E_m}|m\rangle\right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\,\lambda\left(\frac{\sqrt{n+1}}{-\hbar\omega}|n+1\rangle + \frac{\sqrt{n}}{\hbar\omega}|n-1\rangle\right).$$