de Broglie's Hypothesis

Particles behave like waves ⇒ Electrons, protons etc.
Wave is denoted by wave function ψ

$$\lambda = \frac{h}{p}, \quad k = \frac{2\pi}{\lambda} \quad \text{and} \quad \hbar = \frac{h}{2\pi}$$

k = wave vector.

$\Rightarrow \; p = \hbar k$.

$\Rightarrow \; E = \hbar \omega$ where $\omega = 2\pi\nu$

$\Psi(r,t)$ = wave function

Probability of finding the particle

Conditions of wave function

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\psi|^2 \, dx\, dy\, dz = 1$$ Normalization condition.

The Simplest type of wave is a plane monochromatic wave described by the wave function

$$\boxed{\Psi(r,t) = A e^{i(\vec{k}\cdot\vec{r} - \omega t)}}$$

Phase velocity $= \frac{\omega}{k}$

Group velocity $= \frac{d\omega}{dk}$.

For along x axis

$$\Psi(x,t) = A e^{i(kx-\omega t)}$$

$$\Psi(x,t) = A e^{\frac{i}{\hbar}(px-Et)}$$

$$\frac{\partial \Psi}{\partial t} = -\frac{i}{\hbar} E A e^{\frac{i}{\hbar}(px-Et)}$$

$$\frac{\partial \Psi}{\partial t} = -\frac{i}{\hbar} E \Psi$$

$$i\hbar \frac{\partial \Psi}{\partial t} = E\Psi$$

$$\boxed{i\hbar \frac{\partial \Psi}{\partial t} = E\Psi}$$

$$\frac{\partial \Psi}{\partial x} = \frac{i}{\hbar} p \Psi$$

$$\frac{\partial^2 \Psi}{\partial x^2} = \frac{i}{\hbar} p \cdot \frac{i}{\hbar} p \Psi = \frac{-p^2}{\hbar^2} \Psi$$

$$\frac{\partial^2 \Psi}{\partial x^2} = -\frac{p^2}{\hbar^2} \Psi$$

$$-\hbar^2 \frac{\partial^2 \Psi}{\partial x^2} = p^2 \Psi$$

For free particle
$E = \frac{p^2}{2m}$

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} = \frac{p^2}{2m} \Psi = E\Psi$$

$$\boxed{-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} = i\hbar \frac{\partial \Psi}{\partial t}}$$

1D
Time
dependent
Schrödinger
Equation

$$E = i\hbar \frac{\partial}{\partial t}$$

$$p = -i\hbar \frac{\partial}{\partial x}$$

$$\Psi(r,t) = A \exp\left[\frac{i}{\hbar}(p \cdot r - Et)\right]$$

$$= A \exp\left[\frac{i}{\hbar}(p_x \cdot x + p_y \cdot y + p_z \cdot z - Et\right]$$

$$E = \frac{p^2}{2m} = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2).$$

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^2 \Psi}{\partial z^2}\right)$$

$$= \frac{-\hbar^2}{2m} \nabla^2 \Psi$$

$$\boxed{\begin{array}{l} \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \\ p_x = -i\hbar \frac{\partial}{\partial x} \\ p_y = -i\hbar \frac{\partial}{\partial y} \\ p_z = -i\hbar \frac{\partial}{\partial z} \end{array}}$$

$$E\Psi = \frac{p^2}{2m}\Psi$$

$$\Rightarrow E\Psi = \frac{p_x^2 + p_y^2 + p_z^2}{2m} = \frac{p^2}{2m}\Psi$$

Particle in field

$$E = \frac{p^2}{2m} + V(r,t)$$

$$i\hbar \frac{\partial \Psi}{\partial t} = \left[\frac{p^2}{2m} + V(r,t)\right]\Psi = \frac{-\hbar^2}{2m}\nabla^2\Psi.$$

$\Downarrow$

$H =$ Hamiltonian.

$$i\hbar \frac{\partial \Psi}{\partial t} = H\Psi$$

$$E\Psi = H\Psi \text{ or } \boxed{H\Psi = E\Psi}$$

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V(x,t)\Psi$$

Single Dimension or 1D.

$$\boxed{i\hbar \frac{\partial \Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V(x,t)\Psi}$$

1-D, Time dependent Schrödinger Equation

$$E\Psi(x,t) = H\Psi$$

Method of Separation of variables

Time independent Schrödinger Equation

$$\Psi(x,t) = \psi(x).\ T(t).$$

Substituting and dividing by $\Psi(x,t)$

$$i\hbar \frac{\psi(x)}{\Psi(x,t)}\frac{\partial T(t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{T(t)}{\Psi(x,t)}\frac{\partial^2\psi}{\partial x^2} + \frac{V(x,t)\psi(x)T(t)}{\Psi(x,t)},$$

$$\frac{i\hbar}{T(t)}\frac{\partial T(t)}{\partial t} = -\frac{\hbar^2}{2m}\cdot\frac{1}{\psi(x)}\cdot\frac{\partial^2\psi}{\partial x^2} + V(x,t)$$

Assuming potential energy is constant)

$$= \frac{1}{\psi(x)}\left[-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi(x)\right.$$

$$= \frac{1}{\psi(x)}\left[-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi(x)\right].$$

$$\frac{i\hbar}{T(t)}\frac{\partial T(t)}{\partial t} = E$$

$$T(t) = e^{-\frac{iEt}{\hbar}}$$

$$\frac{\partial T(t)}{\partial t} = -\frac{i}{\hbar}E\,T(t)$$

$$\frac{i\hbar}{T(t)}\frac{\partial T}{\partial t} = E$$

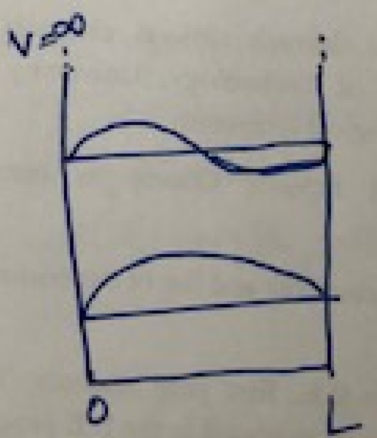
$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\,\psi(x) = E\,\psi(x)$$

$$\boxed{\frac{\partial^2\psi}{\partial x^2} + \frac{2m}{\hbar^2}(E-V)\,\psi(x) = 0}$$

$\Rightarrow$ Time independent Schrödinger Equation

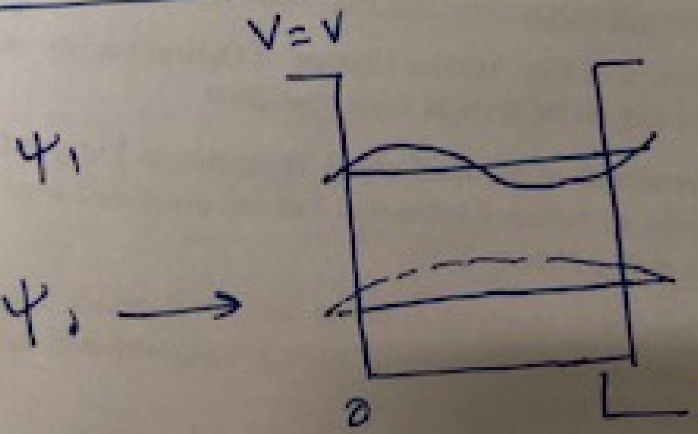
$$\Rightarrow \quad \frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E-V)\,\psi(x) = 0$$

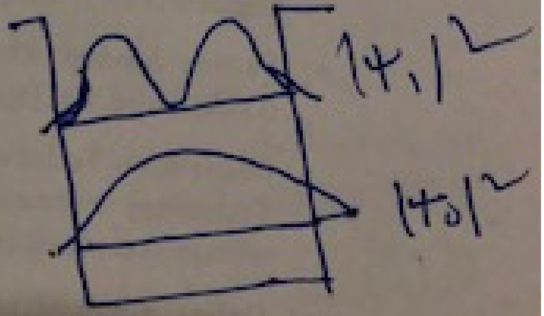
Boundary and Continuity Condition



Infinite potential well.

Finite potential well





Summary

Using De Broglie's concept of matter & wave

Time dependent
&
Time independent Schrödinger Equation

Question

(1) A particle of 'm' is confined in finite & Infinite potential well. wavelength associated with this particle of mass 'm' in infinite potential well is λ_I and for finite potential well, wavelength is λ_F. Find the correct answer

(a) $\lambda_I = \lambda_F$

(b) $\lambda_I > \lambda_F$

(c) $\lambda_F > \lambda_I$

(d) No relationship

(2) A function $\psi(x) = A e^{-2kx}$ where A and k are constants is known to be a solution to the time independent one dimensional Schrödinger Equation with Energy E. which one is correct potential function $V(x)$? Assume 'm' is mass of particle

(a) $V(x) = \frac{\hbar^2 k^2}{2m}$

(b) $V(x) = E + \frac{\hbar^2 k^2}{2m}$

(c) $V(x) = -\frac{\hbar^2}{2m} k^2$

(d) $V(x) = E - \frac{\hbar^2 k^2}{2m}$