CHAPTER – 1

PRELIMINARIES

1.1 *n*-DIMENSIONAL SPACE

In three dimensional rectangular space, the coordinates of a point are (x, y, z). It is convenient to write (x^1, x^2, x^3) for (x, y, z). The coordinates of a point in four dimensional space are given by (x^1, x^2, x^3, x^4). In general, the coordinates of a point in n-dimensional space are given by $(x^1, x^2, x^3, \ldots, x^n)$ such n-dimensional space is denoted by V_n.

1.2 SUPERSCRIPT AND SUBSCRIPT

In the symbol A^{ij}_{kl} , the indices i, j written in the upper position are called *superscripts* and k, l written in the lower position are called *subscripts.*

1.3 THE EINSTEIN'S SUMMATION CONVENTION

Consider the sum of the series $S = a_1 x^1 + a_2 x^2 + \ldots + a_n x^n = \sum_{i=1}^{n} a_i x^i$. By using summation convention, drop the sigma sign and write convention as

$$\sum_{i=1}^{n} a_i x^i = a_i x^i$$

This convention is called *Einstein's Summation Convention* and stated as

"If a suffix occurs twice in a term, once in the lower position and once in the upper position then that suffix implies sum over defined range."

If the range is not given, then assume that the range is from 1 to n.

1.4 DUMMY INDEX

Any index which is repeated in a given term is called a *dummy index* or *dummy suffix*. This is also called *Umbral* or *Dextral Index*.

e.g. Consider the expression $a_i x^i$ where i is dummy index; then

$$a_i x^i = a_1 x^1 + a_2 x^2 + \cdots + a_n x^n$$

and $$a_j x^j = a_1 x^1 + a_2 x^2 + \cdots + a_n x^n$$

These two equations prove that

$$a_i x^i = a_j x^j$$

So, any dummy index can be replaced by any other index ranging the same numbers.

1.5 FREE INDEX

Any index occurring only once in a given term is called a *Free Index.*

e.g. Consider the expression $a_i^j x^i$ where j is free index.

1.6 KRÖNECKER DELTA

The symbol δ^i_j, called Krönecker Delta (a German mathematician Leopold Krönecker, 1823-91 A.D.) is defined by

$$\delta^i_j = \begin{cases} 1 & \text{if } \ i = j \\ 0 & \text{if } \ i \neq j \end{cases}$$

Similarly δ_{ij} and δ^{ij} are defined as

$$\delta^{ij} = \begin{cases} 1 & \text{if } \ i = j \\ 0 & \text{if } \ i \neq j \end{cases}$$

and $$\delta_{ij} = \begin{cases} 1 & \text{if } \ i = j \\ 0 & \text{if } \ i \neq j \end{cases}$$

Properties

1. If $x^1, x^2, \ldots x^n$ are independent coordinates, then

$$\frac{\partial x^i}{\partial x^j} = 0 \quad \text{if} \quad i \neq j$$

$$\frac{\partial x^i}{\partial x^j} = 1 \quad \text{if} \quad i = j$$

This implies that

$$\frac{\partial x^i}{\partial x^j} = \delta^i_j$$

It is also written as $\dfrac{\partial x^i}{\partial x^k}\dfrac{\partial x^k}{\partial x^j} = \delta^i_j$.

2. $\delta^i_i = \delta^1_1 + \delta^2_2 + \delta^3_3 + \cdots + \delta^n_n$ (by summation convention)

$\delta^i_i = 1 + 1 + 1 + \cdots + 1$

$\delta^i_i = n$

3. $a^{ij}\delta^j_k = a^{ik}$

Since $a^{3j}\delta^j_2 = a^{31}\delta^1_2 + a^{32}\delta^2_2 + a^{33}\delta^3_2 + \cdots + a^{3n}\delta^n_2$ (as j is dummy index)

$$= a^{32} \quad (\text{as } \text{ä}_2^1 = \text{ä}_2^3 = \cdots = \text{ä}_2^n = 0 \text{ and } \text{ä}_2^2 = 1)$$

In general,

$$a^{ij}\delta_k^j = a^{i1}\delta_k^1 + a^{i2}\delta_k^2 + a^{i3}\delta_k^3 + \cdots + a^{ik}\delta_k^k + \cdots + a^{in}\delta_k^n$$

$$a^{ij}\delta_k^j = a^{ik} \quad (\text{as } \delta_k^1 = \delta_k^2 = \cdots = \delta_k^n = 0 \text{ and } \delta_k^k = 1)$$

4. $$\delta_j^i\delta_k^j = \delta_k^i$$

$$\delta_j^i\delta_k^j = \delta_1^i\delta_k^1 + \delta_2^i\delta_k^2 + \delta_3^i\delta_k^3 + \cdots + \delta_i^i\delta_k^i + \cdots + \delta_n^i\delta_k^n$$

$$= \delta_k^i \quad (\text{as } \text{ä}_1^i = \delta_2^i = \delta_3^i = \cdots = \delta_n^i = 0 \text{ and } \delta_i^i = 1)$$

EXAMPLE 1

Write $\dfrac{d\phi}{dt} = \dfrac{\partial\phi}{\partial x^1}\dfrac{dx^1}{dt} + \dfrac{\partial\phi}{\partial x^2}\dfrac{dx^2}{dt} + \cdots + \dfrac{\partial\phi}{\partial x^n}\dfrac{dx^n}{dt}$ using summation convention.

Solution

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial x^1}\frac{dx^1}{dt} + \frac{\partial\phi}{\partial x^2}\frac{dx^2}{dt} + \cdots + \frac{\partial\phi}{\partial x^n}\frac{dx^n}{dt}$$

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial x^i}\frac{dx^i}{dt}$$

EXAMPLE 2

Expand: (*i*) $a_{ij}x^ix^j$; (*ii*) $g_{lm}\, g_{mp}$

Solution

(*i*)
$$a_{ij}x^ix^j = a_{1j}x^1x^j + a_{2j}x^2x^j + \cdots + a_{nj}x^nx^j$$
$$= a_{11}x^1x^1 + a_{22}x^2x^2 + \cdots + a_{nn}x^nx^n$$
$$a_{ij}x^ix^j = a_{11}(x^1)^2 + a_{22}(x^2)^2 + \cdots + a_{nn}(x^n)^2$$

(as i and j are dummy indices)

(*ii*)
$$g_{lm}g_{mp} = g_{l1}g_{1p} + g_{l2}g_{2p} + \cdots + g_{ln}g_{np}, \text{ as } m \text{ is dummy index.}$$

EXAMPLE 3

If a_{ij} are constant and $a_{ij} = a_{ji}$, calculate:

(*i*) $\dfrac{\partial}{\partial x_k}(a_{ij}x_ix_j)$ (*ii*) $\dfrac{\partial}{\partial x_k\partial x_l}(a_{ij}x_ix_j)$

Solution

(*i*)
$$\frac{\partial}{\partial x_k}(a_{ij}x_ix_j) = a_{ij}\frac{\partial}{\partial x_k}(x_ix_j)$$

$$= a_{ij}x_i \frac{\partial x_j}{\partial x_k} + a_{ij}x_j \frac{\partial x_i}{\partial x_k}$$

$$= a_{ij}x_i\delta_{jk} + a_{ij}x_j\delta_{ik}, \qquad \text{as} \qquad \frac{\partial x_j}{\partial x_k} = \delta_{jk}$$

$$= (a_{ij}\delta_{jk})x_i + (a_{ij}\delta_{ik})x_j$$

$$= a_{ik}x_i + a_{kj}x_j \qquad \text{as } a_{ij}\,\delta_{jk} = a_{ik}$$

$$= a_{ik}x_i + a_{ki}x_i \qquad \text{as } j \text{ is dummy index}$$

$$\frac{\partial(a_{ij}x_i x_j)}{\partial_{x_k}} = 2a_{ik}x_i \qquad \text{as given } a_{ik} = a_{ki}$$

(*ii*) $$\frac{\partial(a_{ij}x_i x_j)}{\partial x_k} = 2a_{ik}x_i$$

Differentiating it w.r.t. x_l :

$$\frac{\partial^2(a_{ij}x_i x_j)}{\partial x_k \partial x_l} = 2a_{ik}\frac{\partial x_i}{\partial x_l}$$

$$= 2a_{ik}\delta_l^i$$

$$\frac{\partial^2(a_{ij}x_i x_j)}{\partial x_k \partial x_l} = 2a_{lk} \qquad \text{as } a_{ik}\delta_l^i = a_{lk}.$$

EXAMPLE 4

If $$a_{ij}\,x^i\,x^j = 0$$

where a_{ij} are constant then show that

$$a_{ij} + a_{ji} = 0$$

Solution

Given

$$a_{ij}x^i x^j = 0$$

$\Rightarrow$ $$a_{lm}x^l x^m = 0 \qquad \text{since } i \text{ and } j \text{ are dummy indices}$$

Differentiating it w.r.t. x^i partially,

$$\frac{\partial}{\partial x_i}(a_{lm}x^l x^m) = 0$$

$$a_{lm}\frac{\partial}{\partial x_i}(x^l x^m) = 0$$

$$a_{lm}\frac{\partial x^l}{\partial x_i}x^m + a_{lm}\frac{\partial x^m}{\partial x_i}x^l = 0$$

Since $$\frac{\partial x^l}{\partial x_i} = \delta_i^l \quad \text{and} \quad \frac{\partial x^m}{\partial x_i} = \delta_i^m$$

$$a_{lm}\delta_i^l x^m + a_{lm}\delta_i^m x^l = 0$$

$$a_{im}x^m + a_{li}x^l = 0$$

as $a_{lm}\delta_i^l = a_{im}$ and $a_{lm}\delta_i^m = a_{li}$.

Differentiating it w.r.t. x_j partially

$$a_{im}\frac{\partial x^m}{\partial x_j} + a_{li}\frac{\partial x^l}{\partial x_j} = 0$$

$$a_{im}\delta_j^m + a_{li}\delta_j^l = 0$$

$$a_{ij} + a_{ji} = 0$$ **Proved.**

EXERCISES

1. Write the following using the summation convention.
   (*i*) $(x^1)^2 + (x^2)^2 + (x^3)^2 + \cdots + (x^n)^2$
   (*ii*) $ds^2 = g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + \cdots + g_{nn}(dx^n)^2$
   (*iii*) $a_1x^1x^3 + a_2x^2x^3 + \cdots + a_nx^nx^3$
2. Expand the following:
   (*i*) $a_{ij}x^j$ (*ii*) $\frac{\partial}{\partial x^i}(\sqrt{g}a^i)$ (*iii*) $A_i^k B^i$
3. Evaluate:
   (*i*) $x^j\delta_j^i$ (*ii*) $\delta_j^i\delta_k^j\delta_l^k$ (*iii*) $\delta_j^i\delta_i^j$
4. Express $b^{ij}y_iy_j$ in the terms of x variables where $y_i = c_{ij}x_j$ and $b^{ij}c_{ik} = \delta_k^i$.

ANSWERS

1. (*i*) x^ix^j (*ii*) $ds^2 = g_{ij}\,dx^idx^j$ (*iii*) $a_ix^ix^3$.
2. (*i*) $a_{i1}x^1 + a_{i2}x^2 + a_{i3}x^3 + \cdots + a_{in}x^n$
   (*ii*) $\frac{\partial}{\partial x^1}(\sqrt{g}a^1) + \frac{\partial}{\partial x^2}(\sqrt{g}a^2) + \cdots + \frac{\partial}{\partial x^n}(\sqrt{g}a^n)$
   (*iii*) $A_1^kB^1 + A_2^kB^2 + \ldots + A_n^kB^n$
3. (*i*) x^i (*ii*) δ_l^i (*iii*) n
4. $C_{ij}x_ix_j$

CHAPTER – 2

TENSOR ALGEBRA

2.1 INTRODUCTION

A scalar (density, pressure, temperature, etc.) is a quantity whose specification (in any coordinate system) requires just one number. On the other hand, a vector (displacement, acceleration, force, etc.) is a quantity whose specification requires three numbers, namely its components with respect to some basis. Scalers and vectors are both special cases of a more general object called a tensor of order n whose specification in any coordinate system requires 3^n numbers, called the components of tensor. In fact, scalars are tensors of order zero with $3^\circ = 1$ component. Vectors are tensors of order one with $3^1 = 3$ components.

2.2 TRANSFORMATION OF COORDINATES

In three dimensional rectangular space, the coordinates of a point are (x, y, z) where x, y, z are real numbers. It is convenient to write (x^1, x^2, x^3) for (x, y, z) or simply x^i where $i = 1, 2, 3$. Similarly in n- dimensional space, the coordinate of a point are n-independent variables $(x^1, x^2, \ldots, x^n)$ in X-coordinate system. Let $(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n)$ be coordinate of the same point in Y-coordinate system.

Let $\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n$ be independent single valued function of $x^1, x^2, \ldots., x^n$, so that,

$$\begin{aligned}
\bar{x}^1 &= \bar{x}^1(x^1, x^2, \ldots., x^n) \\
\bar{x}^2 &= \bar{x}^2(x^1, x^2, \ldots., x^n) \\
\bar{x}^3 &= \bar{x}^3(x^1, x^2, \ldots., x^n) \\
&\vdots \qquad\qquad \vdots \\
\bar{x}^n &= \bar{x}^n(x^1, x^2, \ldots, x^n)
\end{aligned}$$

or

$$\bar{x}^i = \bar{x}^i(x^1, x^2, \ldots, x^n)\,; \quad i = 1, 2, \ldots, n \qquad \ldots(1)$$

Solving these equations and expressing x^i as functions of $\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n$, so that

$$x^i = x^i(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n); \qquad i = 1, 2, \ldots, n$$

The equations (1) and (2) are said to be a transformation of the coordinates from one coordinate system to another

2.3 COVARIANT AND CONTRAVARIANT VECTORS (TENSOR OF RANK ONE)

Let $(x^1, x^2, \ldots, x^n)$ or x^i be coordinates of a point in X-coordinate system and $(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n)$ or $\bar{x}^i$ be coordinates of the same point in the Y-coordinate system.

Let A^i, $i = 1, 2, \ldots, n$ (or $A^1, A^2, \ldots, A^n$) be n functions of coordinates $x^1, x^2, \ldots, x^n$ in X-coordinate system. If the quantities A^i are transformed to $\bar{A}^i$ in Y-coordinate system then according to the law of transformation

$$\bar{A}^i = \frac{\partial \bar{x}^i}{\partial x^j} A^j \quad \text{or} \quad A^j = \frac{\partial x^j}{\partial \bar{x}^i} \bar{A}^i$$

Then A^i are called components of contravariant vector.

Let A_i, $i = 1, 2, \ldots, n$ (or $A_1, A_2, \ldots, A_n$) be n functions of the coordinates $x^1, x^2, \ldots, x^n$ *in* X-coordinate system. If the quantities A_i are transformed to $\bar{A}_i$ in Y-coordinate system then according to the law of transformation

$$\bar{A}_i = \frac{\partial x^j}{\partial \bar{x}^i} A_j \quad \text{or} \quad A_j = \frac{\partial \bar{x}^i}{\partial x^j} \bar{A}_i$$

Then A^i are called components of covariant vector.

The contravariant (or covariant) vector is also called a contravariant (or covariant) tensor of rank one.

Note: A superscript is always used to indicate contravariant component and a subscript is always used to indicate covariant component.

EXAMPLE 1

If x^i be the coordinate of a point in n-dimensional space show that dx^i are component of a contravariant vector.

Solution

Let $x^1, x^2, \ldots, x^n$ or x^i are coordinates in X-coordinate system and $\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n$ or $\bar{x}^i$ are coordinates in Y-coordinate system.

If

$$\bar{x}^i = \bar{x}^i(x^1, x^2, \ldots, x^n)$$

$$d\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^1} dx^1 + \frac{\partial \bar{x}^i}{\partial x^2} dx^2 + \cdots + \frac{\partial \bar{x}^i}{\partial x^n} dx^n$$

$$d\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^j} dx^j$$

It is law of transformation of contravariant vector. So, dx^i are components of a contravariant vector.

EXAMPLE 2

Show that $\dfrac{\partial \phi}{\partial x^i}$ is a covariant vector where ϕ is a scalar function.

Solution

Let $x^1, x^2, \ldots, x^n$ or x^i are coordinates in X-coordinate system and $\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n$ or $\bar{x}^i$ are coordinates in Y-coordinate system.

Consider $\phi(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n) = \phi(x^1, x^2, \ldots, x^n)$

$$\partial\phi = \frac{\partial\phi}{\partial x^1}\partial x^1 + \frac{\partial\phi}{\partial x^2}\partial x^2 + \cdots + \frac{\partial\phi}{\partial x^n}\partial x^n$$

$$\frac{\partial\phi}{\partial \bar{x}^i} = \frac{\partial\phi}{\partial x^1}\frac{\partial x^1}{\partial \bar{x}^i} + \frac{\partial\phi}{\partial x^2}\frac{\partial x^2}{\partial \bar{x}^i} + \cdots + \frac{\partial\phi}{\partial x^n}\frac{\partial x^n}{\partial \bar{x}^i}$$

$$\frac{\partial\phi}{\partial \bar{x}^i} = \frac{\partial\phi}{\partial x^j}\frac{\partial x^j}{\partial \bar{x}^i}$$

or

$$\frac{\partial\phi}{\partial \bar{x}^i} = \frac{\partial x^j}{\partial \bar{x}^i}\frac{\partial\phi}{\partial x^j}$$

It is law of transformation of component of covariant vector. So, $\dfrac{\partial\phi}{\partial x^i}$ is component of covariant vector.

EXAMPLE 3

Show that the velocity of fluid at any point is a component of contravariant vector

or

Show that the component of tangent vector on the curve in n-dimensional space are component of contravariant vector.

Solution

Let $\dfrac{dx^1}{dt}, \dfrac{dx^2}{dt}, \ldots, \dfrac{dx^n}{dt}$ be the component of the tangent vector of the point $(x^1, x^2, \ldots, x^n)$ *i.e.*, $\dfrac{dx^i}{dt}$ be the component of the tangent vector in X-coordinate system. Let the component of tangent vector of the point $(\bar{x}^1, \bar{x}^2, ..., \bar{x}^n)$ in Y-coordinate system are $\frac{d\bar{x}^i}{dt}$. Then $\bar{x}^1$, $\bar{x}^2$, ..., $\bar{x}^n$ or $\bar{x}^i$ being a function of $x^1, x^2, ..., x^n$ which is a function of t. So,

$$\frac{d\bar{x}^i}{dt} = \frac{\partial\bar{x}^i}{dt^1}\frac{dx^1}{dt} + \frac{\partial\bar{x}^i}{dx^2}\frac{dx^2}{dt} + \cdots + \frac{\partial\bar{x}^i}{dx^n}\frac{dx^n}{dt}$$

$$\frac{d\bar{x}^i}{dt} = \frac{\partial\bar{x}^i}{dx^j}\frac{dx^j}{dt}$$

It is law of transformation of component of contravariant vector. So, $\frac{dx^i}{dt}$ is component of contravariant vector.

i.e. the component of tangent vector on the curve in n-dimensional space are component of contravariant vector.

2.4 CONTRAVARIANT TENSOR OF RANK TWO

Let A^{ij} $(i, j = 1, 2, ..., n)$ be n^2 functions of coordinates x^1, x^2, ..., x^n in X-coordinate system. If the quantities A^{ij} are transformed to $\bar{A}^{ij}$ in Y-coordinate system having coordinates $\bar{x}^1, \bar{x}^2, ..., \bar{x}^n$. Then according to the law of transformation

$$\bar{A}^{ij} = \frac{\partial\bar{x}^i}{\partial x^k}\frac{\partial\bar{x}^j}{\partial x^l}A^{kl}$$

Then A^{ij} are called components of Contravariant Tensor of rank two.

2.5 COVARIANT TENSOR OF RANK TWO

Let A_{ij} $(i, j = 1, 2, ..., n)$ be n^2 functions of coordinates x^1, x^2, ..., x^n in X-coordinate system. If the quantities A_{ij} are transformed to A_{ij} in Y-coordinate system having coordinates $\bar{x}^1, \bar{x}^2, ..., \bar{x}^n$, then according to the law of transformation,

$$\bar{A}_{ij} = \frac{\partial x^k}{\partial\bar{x}^i}\frac{\partial x^l}{\partial\bar{x}^j}A_{kl}$$

Then A_{ij} called components of covariant tensor of rank two.

2.6 MIXED TENSOR OF RANK TWO

Let A^i_j $(i, j = 1, 2, ..., n)$ be n^2 functions of coordinates x^1, x^2, ..., x^n in X-coordinate system. If the quantities A^i_j are transformed to $\bar{A}^i_j$ in Y-coordinate system having coordinates $\bar{x}^1, \bar{x}^2, ..., \bar{x}^n$, then according to the law of transformation

$$\bar{A}^i_j = \frac{\partial\bar{x}^i}{\partial x^k}\frac{\partial x^l}{\partial\bar{x}}A^k_l$$

Then A^i_j are called components of mixed tensor of rank two.

Note: (i) The rank of the tensor is defined as the total number of indices per component.

(ii) Instead of saying that " A^{ij} are the components of a tensor of rank two" we shall often say " A^{ij} is a tensor of rank two."

THEOREM 2.1 *To show that the Krönecker delta is a mixed tensor of rank two.*

Solution

Let X and Y be two coordinate systems. Let the component of Kronecker delta in X-coordinate system δ^i_j and component of Krönecker delta in Y-coordinate be $\overline{\delta}^i_j$, then according to the law of transformation

$$\overline{\delta}^i_j = \frac{\partial \overline{x}^i}{\partial \overline{x}^j} = \frac{\partial \overline{x}^i}{\partial x^k}\frac{\partial x^l}{\partial \overline{x}^j}\frac{\partial x^k}{\partial x^l}$$

$$\overline{\delta}^i_j = \frac{\partial \overline{x}^i}{\partial x^k}\frac{\partial x^l}{\partial \overline{x}^j}\delta^k_l$$

This shows that Krönecker δ^i_j is mixed tensor of rank two.

EXAMPLE 4

If A_i is a covariant tensor, then prove that $\dfrac{\partial A_i}{\partial x^j}$ do not form a tensor.

Solution

Let X and Y be two coordinate systems. As given A_i is a covariant tensor. Then

$$\overline{A}_i = \frac{\partial x^k}{\partial \overline{x}^i} A_k$$

Differentiating it w.r.t. $\overline{x}^j$

$$\frac{\partial \overline{A}_i}{\partial \overline{x}^j} = \frac{\partial}{\partial \overline{x}^j}\left(\frac{\partial x^k}{\partial \overline{x}^i} A_k\right)$$

$$\frac{\partial \overline{A}_i}{\partial \overline{x}^j} = \frac{\partial x^k}{\partial \overline{x}^i}\frac{\partial A_k}{\partial \overline{x}^j} + A_k \frac{\partial^2 x^k}{\partial \overline{x}^i \partial \overline{x}^j} \qquad \ldots(1)$$

It is not any law of transformation of tensor due to presence of second term. So, $\dfrac{\partial A_i}{\partial x^j}$ is not a tensor.

THEOREM 2.2 *To show that* δ^i_j *is an invariant i.e., it has same components in every coordinate system.*

Proof: Since δ^i_j is a mixed tensor of rank two, then

$$\overline{\delta}^i_j = \frac{\partial \overline{x}^i}{\partial x^k}\frac{\partial x^l}{\partial \overline{x}^j}\delta^k_l$$

$$= \frac{\partial \bar{x}^i}{\partial x^k}\left(\frac{\partial x^l}{\partial \bar{x}^j}\delta_l^k\right)$$

$$= \frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial x^k}{\partial \bar{x}^j}\text{, as } \frac{\partial x^l}{\partial \bar{x}^j}\delta_l^k = \frac{\partial x^k}{\partial \bar{x}^j}$$

$$\bar{\delta}^i_j = \frac{\partial \bar{x}^i}{\partial \bar{x}^j} = \delta^i_j \text{ , as } \frac{\partial \bar{x}^i}{\partial \bar{x}^j} = \delta^i_j$$

So, δ^i_j is an invariant.

THEOREM 2.3 *Prove that the transformation of a contravariant vector is transitive.*

or

Prove that the transformation of a contravariant vector form a group.

Proof: Let A^i be a contravariant vector in a coordinate system $x^i (i = 1, 2, \ldots, n)$. Let the coordinates x^i be transformed to the coordinate system $\bar{x}^i$ and $\bar{x}^i$ be transformed to $\bar{\bar{x}}^i$.

When coordinate x^i be transformed to $\bar{x}^i$, the law of transformation of a contravariant vector is

$$\bar{A}^p = \frac{\partial \bar{x}^p}{\partial x^q}A^q \qquad \ldots (1)$$

When coordinate $\bar{x}^i$ be transformed to $\bar{\bar{x}}^i$, the law of transformation of contravariant vector is

$$\bar{\bar{A}}^i = \frac{\partial \bar{\bar{x}}^i}{\partial \bar{x}^p}\bar{A}^p$$

$$\bar{\bar{A}}^i = \frac{\partial \bar{\bar{x}}^i}{\partial \bar{x}^i}\frac{\partial \bar{x}^p}{\partial x^q}A^q \text{ from (1)}$$

$$\bar{\bar{A}}^i = \frac{\partial \bar{\bar{x}}^i}{\partial x^q}A^q$$

This shows that if we make direct transformation from x^i to $\bar{\bar{x}}^i$, we get same law of transformation. This property is called that transformation of contravariant vectors is transitive or form a group.

THEOREM 2.4 *Prove that the transformation of a covariant vector is transitive.*

or

Prove that the transformation of a covariant vector form a group.

Proof: Let A_i be a covariant vector in a coordinate system $x^i (i = 1, 2, \ldots, n)$. Let the coordinates x^i be transformed to the coordinate system $\bar{x}^i$ and $\bar{x}^i$ be transformed to $\bar{\bar{x}}^i$.

When coordinate x^i be transformed to $\bar{x}^i$, the law of transformation of a covariant vector is

$$\bar{A}_p = \frac{\partial x^q}{\partial \bar{x}^p}A_q \qquad \ldots (1)$$

When coordinate $\bar{x}^i$ be transformed to $\bar{\bar{x}}^i$, the law of transformation of a covariant vector is

$$\bar{\bar{A}}_i = \frac{\partial \bar{x}^p}{\partial \bar{\bar{x}}^i}\bar{A}_p$$

$$\bar{\bar{A}}_i = \frac{\partial \bar{x}^p}{\partial \bar{\bar{x}}^i} \frac{\partial x^q}{\partial \bar{x}^p} A_q$$

$$\bar{\bar{A}}_i = \frac{\partial x^q}{\partial \bar{\bar{x}}^i} A_q$$

This shows that if we make direct transformation from x^i to $\bar{\bar{x}}^i$, we get same law of transformation. This property is called that transformation of covariant vectors is transitive or form a group.

THEOREM 2.5 *Prove that the transformations of tensors form a group*

or

Prove that the equations of transformation a tensor (Mixed tensor) posses the group property.

Proof: Let A^i_j be a mixed tensor of rank two in a coordinate system $x^i \, (i = 1, 2, \ldots, n)$. Let the coordinates x^i be transformed to the coordinate system $\bar{x}^i$ and $\bar{x}^i$ be transformed to $\bar{\bar{x}}^i$.

When coordinate x^i be transformed to $\bar{x}^i$, the transformation of a mixed tensor of rank two is

$$\bar{A}^p_q = \frac{\partial \bar{x}^p}{\partial x^r} \frac{\partial x^s}{\partial \bar{x}^q} A^r_s \qquad \ldots (1)$$

When coordinate $\bar{x}^i$ be transformed to $\bar{\bar{x}}^i$, the law of transformation of a mixed tensor of rank two is

$$\bar{\bar{A}}^i_j = \frac{\partial \bar{\bar{x}}^i}{\partial \bar{x}^p} \frac{\partial \bar{x}^q}{\partial \bar{\bar{x}}^j} \bar{A}^p_q$$

$$= \frac{\partial \bar{\bar{x}}^i}{\partial \bar{x}^p} \frac{\partial \bar{x}^q}{\partial \bar{\bar{x}}^j} \frac{\partial \bar{x}^p}{\partial x^r} \frac{\partial x^s}{\partial \bar{x}^q} A^r_s \quad \text{from (1)}$$

$$\bar{\bar{A}}^i_j = \frac{\partial \bar{\bar{x}}^i}{\partial x^r} \frac{\partial x^s}{\partial \bar{\bar{x}}^j} A^r_s$$

This shows that if we make direct transformation from x^i to $\bar{\bar{x}}^i$, we get same law of transformation. This property is called that transformation of tensors form a group.

THEOREM 2.6 *There is no distinction between contravariant and covariant vectors when we restrict ourselves to rectangular Cartesian transformation of coordinates.*

Proof: Let $P(x, y)$ be a point with respect to the rectangular Cartesian axes X and Y. Let $(\bar{x}, \bar{y})$ be the coordinate of the same point P in another rectangular cartesian axes $\bar{X}$ and $\bar{Y}$, Let (l_1, m_1) and (l_2, m_2) be the direction cosines of the axes $\bar{X}$, $\bar{Y}$ respectively. Then the transformation relations are given by

$$\left.\begin{aligned} \bar{x} &= l_1 x + m_1 y \\ \bar{y} &= l_2 x + m_2 y \end{aligned}\right\} \qquad \ldots (1)$$

and solving these equations, we have

$$\left.\begin{aligned} x &= l_1 \bar{x} + l_2 \bar{y} \\ y &= m_1 \bar{x} + m_2 \bar{y} \end{aligned}\right\} \qquad \ldots (2)$$

put $x = x^1$, $y = x^2$, $\bar{x} = \bar{x}^1$, $\bar{y} = \bar{x}^2$

Consider the contravariant transformation

$$\overline{A}^i = \frac{\partial \overline{x}^i}{\partial x^j} A^j; \quad j = 1,2$$

$$\overline{A}^i = \frac{\partial \overline{x}^i}{\partial x^1} A^1 + \frac{\partial \overline{x}^i}{\partial x^2} A^2$$

for $i = 1,2$.

$$\overline{A}^1 = \frac{\partial \overline{x}^1}{\partial x^1} A^1 + \frac{\partial \overline{x}^1}{\partial x^2} A^2$$

$$\overline{A}^2 = \frac{\partial \overline{x}^2}{\partial x^1} A^1 + \frac{\partial \overline{x}^2}{\partial x^2} A^2$$

From (1) $\frac{\partial \overline{x}}{\partial x} = l_1$, but $x = x^1$, $y = x^2$, $\overline{x} = \overline{x}^1$, $\overline{y} = \overline{x}^2$

Then

$$\frac{\partial \overline{x}}{\partial x} = \frac{\partial \overline{x}^1}{\partial x^1} = l_1 .$$

Similarly,

$$\left.\begin{aligned} &\frac{\partial \overline{x}}{\partial y} = m_1 = \frac{\partial \overline{x}^1}{\partial x^2}; \\ &\frac{\partial \overline{y}}{\partial x} = l_2 = \frac{\partial \overline{x}^2}{\partial x^1}; \quad \frac{\partial \overline{y}}{\partial y} = m_2 = \frac{\partial \overline{x}^2}{\partial x^2} \end{aligned}\right\} \qquad \text{...(3)}$$

So, we have

$$\left.\begin{aligned} \overline{A}^1 &= l_1 A^1 + m_1 A^2 \\ \overline{A}^2 &= l_2 A^1 + m_2 A^2 \end{aligned}\right\} \qquad \text{..(4)}$$

Consider the covariant transformation

$$\overline{A}_i = \frac{\partial x^j}{\partial \overline{x}^i} A_j; \quad j = 1,2$$

$$\overline{A}_i = \frac{\partial x^1}{\partial \overline{x}^i} A_1 + \frac{\partial x^2}{\partial \overline{x}^1} A_2$$

for $i = 1, 2$.

$$\overline{A}_1 = \frac{\partial x^1}{\partial \overline{x}^1} A_1 + \frac{\partial x^2}{\partial \overline{x}^1} A_2$$

$$\overline{A}_2 = \frac{\partial \overline{x}^1}{\partial \overline{x}^2} A_1 + \frac{\partial x^2}{\partial \overline{x}^2} A_2$$

From (3)

$$\left.\begin{aligned} \overline{A}_1 &= l_1 A_1 + m_1 A_2 \\ \overline{A}_2 &= l_2 A_1 + m_2 A_2 \end{aligned}\right\} \qquad \text{...(5)}$$

So, from (4) and (5), we have

$$\overline{A}^1 = \overline{A}_1 \text{ and } \overline{A}^2 = \overline{A}_2$$

Hence the theorem is proved.

2.7 TENSORS OF HIGHER ORDER

(*a*) *Contravariant tensor of rank r*

Let $A^{i_1 i_2 .. i_r}$ be n^r function of coordinates $x^1, x^2, ..., x^n$ in X-coordinates system. If the quantities $A^{i_i i_2 .. i_r}$ are transformed to $\overline{A}^{i_1 i_2 .. i_r}$ in Y-coordinate system having coordinates $\overline{x}^1, \overline{x}^2, ..., \overline{x}^n$. Then according to the law of transformation

$$\overline{A}^{i_1 i_2 .. i_r} = \frac{\partial \overline{x}^{i_1}}{\partial x^{p_1}} \frac{\partial \overline{x}^{i_2}}{\partial x^{p_2}} \cdots \frac{\partial \overline{x}^{i_r}}{\partial x^{p_r}} \quad A^{p_1 p_2 .. p_r}$$

Then $A^{i_i i_2 .. i_r}$ are called components of contravariant tensor of rank r.

(*b*) *Covariant tensor of rank s*

Let $A_{j_1 j_2 .. j_s}$ be n^s functions of coordinates $x^1, x^2, ..., x^n$ in X-coordinate system. If the quantities $A_{j_1 j_2 .. j_s}$ are transformed to $\overline{A}_{j_1 j_2 .. j_s}$ in Y- coordinate system having coordinates $\overline{x}^1, \overline{x}^2, ..., \overline{x}^n$. Then according to the law of transformation

$$\overline{A}_{j_1 j_2 .. j_s} = \frac{\partial x^{q_1}}{\partial \overline{x}^{j_1}} \frac{\partial x^{q_2}}{\partial \overline{x}^{j_2}} \cdots \frac{\partial x^{q_s}}{\partial \overline{x}^{j_s}} \quad A_{q_1, q_2 ... q_s}$$

Then $A_{j_1 j_2 .. j_s}$ are called the components of covariant tensor of rank s.

(*c*) *Mixed tensor of rank r + s*

Let $A^{i_1 i_2 .. i_r}_{j_1 j_2 .. j_s}$ be n^{r+s} functions of coordinates $x^1, x^2, ..., x^n$ in X-coordinate system. If the quantities $A^{i_1 i_2 .. i_r}_{j_1 j_2 .. j_s}$ are transformed to $\overline{A}^{i_1 i_2 .. i_r}_{j_1 j_2 .. j_s}$ in Y-coordinate system having coordinates $\overline{x}^1, \overline{x}^2, ..., \overline{x}^n$. Then according to the law of transformation

$$\overline{A}^{i_1 i_2 .. i_r}_{j_1 j_2 .. j_s} = \frac{\partial \overline{x}^{i_1}}{\partial x^{p_1}} \frac{\partial \overline{x}^{i_2}}{\partial x^{p_2}} \frac{\partial \overline{x}^{i_r}}{\partial x^{p_r}} \quad \frac{\partial x^{q_1}}{\partial \overline{x}^{j_1}} \frac{\partial x^{q_2}}{\partial \overline{x}^{j_2}} \cdots \frac{\partial x^{q_s}}{\partial \overline{x}^{j_s}} \quad A^{p_1 p_2 .. p_r}_{q_1 q_2 .. q_s}$$

Then $A^{i_1 i_2 .. i_r}_{j_1 j_2 .. j_s}$ are called component of mixed tensor of rank $r + s$.

A tensor of type $A^{j_1 j_2 .. j_s}_{i_1 i_2 .. i_r}$ is known as tensor of type (r, s), In (r,s), the first component r indicates the rank of contravariant tensor and the second component s indicates the rank of covariant tensor.

Thus the tensors A_{ij} and A^{ij} are type (0, 2) and (2, 0) respectively while tensor A^i_j is type (1, 1).

EXAMPLE

A^{ijk}_{lm} is a mixed tensor of type (3, 2) in which contravariant tensor of rank three and covariant tensor of rank two. Then according to the law of transformation

$$\overline{A}^{ijk}_{lm} = \frac{\partial \overline{x}^i}{\partial x^\alpha}\frac{\partial \overline{x}^j}{\partial x^\beta}\frac{\partial \overline{x}^k}{\partial x^\gamma}\frac{\partial x^a}{\partial \overline{x}^l}\frac{\partial x^b}{\partial \overline{x}^m} \quad A^{\alpha\beta\gamma}_{ab}$$

2.8 SCALAR OR INVARIANT

A function $\phi\ (x^1, x^2, \ldots, x^n)$ is called Scalar or an invariant if its original value does not change upon transformation of coordinates from $x^1, x^2, \ldots, x^n$ to $\overline{x}^1, \overline{x}^2, \ldots, \overline{x}^n$. *i.e.*

$$\phi(x^1, x^2, \ldots, x^n) = \overline{\phi}(\overline{x}^1, \overline{x}^2, \ldots, \overline{x}^n)$$

Scalar is also called tensor of rank zero.

For example, $A^i B_i$ is scalar.

2.9 ADDITION AND SUBTRACTION OF TENSORS

THEOREM 2.7 *The sum (or difference) of two tensors which have same number of covariant and the same contravariant indices is again a tensor of the same rank and type as the given tensors.*

Proof: Consider two tensors $A^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s}$ and $B^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s}$ of the same rank and type (*i.e.*, covariant tensor of rank s and contravariant tensor of rank r). Then according to the law of transformation

$$\overline{A}^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s} = \frac{\partial \overline{x}^{i_1}}{\partial x^{p_1}}\frac{\partial \overline{x}^{i_2}}{\partial x^{p_2}}\cdots\frac{\partial \overline{x}^{i_r}}{\partial x^{p_r}}\frac{\partial x^{q_1}}{\partial \overline{x}^{j_1}}\frac{\partial x^{q_2}}{\partial \overline{x}^{j_2}}\cdots\frac{\partial x^{q_s}}{\partial \overline{x}^{j_s}}\cdots A^{p_1 p_2 \ldots p_r}_{q_1 q_2 \ldots q_s}$$

and

$$\overline{B}^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s} = \frac{\partial \overline{x}^{i_1}}{\partial x^{p_1}}\frac{\partial \overline{x}^{i_2}}{\partial x^{p_2}}\cdots\frac{\partial \overline{x}^{i_r}}{\partial x^{p_r}}\frac{\partial x^{q_1}}{\partial \overline{x}^{j_1}}\frac{\partial x^{q_2}}{\partial \overline{x}^{j_2}}\cdots\frac{\partial x^{q_s}}{\partial \overline{x}^{j_s}}\cdots B^{p_1 p_2 \ldots p_r}_{q_1 q_2 \ldots q_s}$$

Then

$$\overline{A}^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s} \pm \overline{B}^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s} = \frac{\partial \overline{x}^{i_1}}{\partial x^{p_1}}\frac{\partial \overline{x}^{i_2}}{\partial x^{p_2}}\cdots\frac{\partial \overline{x}^{i_r}}{\partial x^{p_r}}\frac{\partial x^{q_1}}{\partial \overline{x}^{j_1}}\frac{\partial x^{q_2}}{\partial \overline{x}^{j_2}}\cdots\frac{\partial x^{q_s}}{\partial \overline{x}^{j_s}}\left(A^{p_1 p_2 \ldots p_r}_{q_1 q_2 \ldots q_s} \pm B^{p_1 p_2 \ldots p_r}_{q_1 q_2 \ldots q_s}\right)$$

If

$$\overline{A}^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s} \pm \overline{B}^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s} = \overline{C}^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s}$$

and

$$A^{p_1 p_2 \ldots p_r}_{q_1 q_2 \ldots q_s} \pm B^{p_1 p_2 \ldots p_r}_{q_1 q_2 \ldots q_s} = C^{p_1 p_2 \ldots p_r}_{q_1 q_2 \ldots q_s}$$

So,

$$\overline{C}^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s} = \frac{\partial \overline{x}^{i_1}}{\partial x^{p_1}}\frac{\partial \overline{x}^{i_2}}{\partial x^{p_2}}\cdots\frac{\partial \overline{x}^{i_r}}{\partial x^{p_r}}\frac{\partial x^{q_1}}{\partial \overline{x}^{j_1}}\frac{\partial x^{q_2}}{\partial \overline{x}^{j_2}}\cdots\frac{\partial x^{q_s}}{\partial \overline{x}^{j_s}}\ C^{p_1, p_2, \ldots, p_r}_{q_1, q_2, \ldots, q_s}$$

This is law of transformation of a mixed tensor of rank $r+s$. So, $\overline{C}^{i_1, i_2, \ldots, i_r}_{j_1, j_2, \ldots, j_s}$ is a mixed tensor of rank $r+s$ or of type (r, s).

EXAMPLE 5

If A_k^{ij} and B_n^{lm} are tensors then their sum and difference are tensors of the same rank and type.

Solution

As given A_k^{ij} and B_k^{ij} are tensors. Then according to the law of transformation

$$\overline{A}_k^{ij} = \frac{\partial \overline{x}^i}{\partial x^p}\frac{\partial \overline{x}^j}{\partial x^q}\frac{\partial x^r}{\partial \overline{x}^k}A_r^{pq}$$

and

$$\overline{B}_k^{ij} = \frac{\partial \overline{x}^i}{\partial x^p}\frac{\partial \overline{x}^j}{\partial x^q}\frac{\partial x^r}{\partial \overline{x}^k}B_r^{pq}$$

then

$$\overline{A}_k^{ij} \pm \overline{B}_k^{ij} = \frac{\partial \overline{x}^i}{\partial x^p}\frac{\partial \overline{x}^j}{\partial x^q}\frac{\partial x^r}{\partial \overline{x}^k}\left(A_r^{pq} \pm B_r^{pq}\right)$$

If

$$\overline{A}_k^{ij} \pm \overline{B}_k^{ij} = \overline{C}_k^{ij} \text{ and } A_r^{pq} \pm B_r^{pq} = C_r^{pq}$$

So,

$$\overline{C}_k^{ij} = \frac{\partial \overline{x}^i}{\partial x^p}\frac{\partial \overline{x}^j}{\partial x^q}\frac{\partial x^r}{\partial \overline{x}^k}C_r^{pq}$$

The shows that C_k^{ij} is a tensor of same rank and type as A_k^{ij} and B_k^{ij}.

2.10 MULTIPLICATION OF TENSORS (OUTER PRODUCT OF TENSOR)

THEOREM 2.8 *The multiplication of two tensors is a tensor whose rank is the sum of the ranks of two tensors.*

Proof: Consider two tensors $A_{j_1 j_2 \ldots j_s}^{i_1 i_2 \ldots i_r}$ (which is covariant tensor of rank s and contravariant tensor of rank r) and $B_{l_1 l_2 \ldots l_n}^{k_1 k_2 \ldots k_m}$ (which is covariant tensor of rank m and contravariant tensor of rank n). Then according to the law of transformation.

$$\overline{A}_{j_1 j_2 \ldots j_s}^{i_1 i_2 \ldots i_r} = \frac{\partial \overline{x}^{i_1}}{\partial x^{p_1}}\frac{\partial \overline{x}^{i_2}}{\partial x^{p_2}}\cdots\frac{\partial \overline{x}^{i_r}}{\partial x^{p_r}}\frac{\partial x^{q_1}}{\partial \overline{x}^{j_1}}\frac{\partial x^{q_2}}{\partial \overline{x}^{j_2}}\cdots\frac{\partial x^{q_r}}{\partial \overline{x}^{j_s}}\, A_{q_1 q_2 \ldots q_s}^{p_1 p_2 \ldots p_r}$$

and

$$\overline{B}_{l_1 l_2 \ldots l_n}^{k_1 k_2 \ldots k_m} = \frac{\partial \overline{x}^{k_1}}{\partial x^{\alpha_1}}\frac{\partial \overline{x}^{k_2}}{\partial x^{\alpha_2}}\cdots\frac{\partial \overline{x}^{k_m}}{\partial x^{\alpha_m}}\frac{\partial x^{\beta_1}}{\partial \overline{x}^{l_1}}\frac{\partial x^{\beta_2}}{\partial \overline{x}^{l_2}}\cdots\frac{\partial x^{\beta_n}}{\partial \overline{x}^{l_n}}\, B_{\beta_1 \beta_2 \ldots \beta_n}^{\alpha_1 \alpha_2 \ldots \alpha_m}$$

Then their product is

$$\overline{A}_{j_1 j_2 \ldots j_s}^{i_1 i_2 \ldots i_r}\; \overline{B}_{l_1 l_2 \ldots l_n}^{k_1 k_2 \ldots k_m} = \frac{\partial \overline{x}^{i_1}}{\partial x^{p_1}}\cdots\frac{\partial \overline{x}^{i_r}}{\partial x^{p_r}}\frac{\partial x^{q_1}}{\partial \overline{x}^{j_1}}\cdots\frac{\partial x^{q_s}}{\partial \overline{x}^{j_s}}\frac{\partial \overline{x}^{k_1}}{\partial x^{\alpha_1}}\cdots\frac{\partial \overline{x}^{k_m}}{\partial x^{\alpha_m}}\cdot\frac{\partial x^{\beta_1}}{\partial \overline{x}^{l_1}}\cdots\frac{\partial x^{\beta_n}}{\partial \overline{x}^{l_m}}$$

$$A_{q_1 q_2 \ldots q_s}^{p_1 p_2 \ldots p_r}\; B_{\beta_1 \beta_2 \ldots \beta_n}^{\alpha_1 \alpha_2 \ldots \alpha_m}$$

If

$$\overline{C}^{i_1 i_2 \cdots i_r k_1 k_2 \cdots k_m}_{j_1 j_2 \cdots j_s l_1 l_2 \cdots l_n} = \overline{A}^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s} \overline{B}^{k_1 k_2 \cdots k_n}_{l_1 l_2 \cdots l_n}$$

and

$$C^{p_1 p_2 \cdots p_r \alpha_1 \alpha_2 \cdots \alpha_m}_{q_1 q_2 \cdots q_s \beta_1 \beta_2 \cdots \beta_n} = A^{p_1 p_2 \cdots p_r}_{q_1 q_2 \cdots q_s} B^{\alpha_1 \alpha_2 \cdots \alpha_m}_{\beta_1 \beta_2 \cdots \beta_n}$$

So,

$$\overline{C}^{i_1 i_2 \cdots i_r k_1 k_2 \cdots k_m}_{j_1 j_2 \cdots j_s l_1 l_2 \cdots l_n} = \frac{\partial \overline{x}^{i_1}}{\partial x^{p_1}} \cdots \frac{\partial \overline{x}^{i_r}}{\partial x^{p_r}} \cdot \frac{\partial \overline{x}^{q_1}}{\partial \overline{x}^{j_1}} \cdots \frac{\partial x^{q_s}}{\partial \overline{x}^{j_s}} \frac{\partial \overline{x}^{k_1}}{\partial x^{\alpha_1}}$$
$$\cdots \frac{\partial \overline{x}^{k_m}}{\partial x^{\alpha_m}} \cdot \frac{\partial x^{\beta_1}}{\partial \overline{x}^{l_1}} \cdots \frac{\partial x^{\beta_n}}{\partial \overline{x}^{l_n}} \quad C^{p_1 p_2 \cdots p_r \alpha_1 \alpha_2 \cdots \alpha_h}_{q_1 q_2 \cdots q_s \beta_1 \beta_2 \cdots \beta_n}$$

This is law of transformation of a mixed tensor of rank $r+m+s+n.$. So, $\overline{C}^{i_1 i_2 \cdots i_r k_1 k_2 \cdots k_m}_{j_1 j_2 \cdots j_s l_1 l_2 \cdots l_n}$ is a mixed tensor of rank $r+m+s+n$. or of type $(r+m, s+n)$. Such product is called outer product or open proudct of two tensors.

THEOREM 2.9 *If A^i and B_j are the components of a contravariant and covariant tensors of rank one then prove that $A^i B_j$ are components of a mixed tensor of rank two.*

Proof: As A^i is contravariant tensor of rank one and B_j is covariant tensor of rank one. Then according to the law of transformation

$$\overline{A}^i = \frac{\partial \overline{x}^i}{\partial x^k} A^k \qquad ...(1)$$

and

$$\overline{B}_j = \frac{\partial x^l}{\partial \overline{x}^j} B_l \qquad ...(2)$$

Multiply (1) and (2), we get

$$\overline{A}^i \overline{B}_j = \frac{\partial \overline{x}^i}{\partial x^k} \frac{\partial x^l}{\partial \overline{x}^j} A^k B_l$$

This is law of transformation of tensor of rank two. So, $A^i B_j$ are mixed tensor of rank two. Such product is called outer product of two tensors.

EXAMPLE 6

Show that the product of two tensors A^i_j and B^{kl}_m is a tensor of rank five.

Solution

As A^i_j and B^{kl}_m are tensors. Then by law of transformation

$$\overline{A}^i_j = \frac{\partial \overline{x}^i}{\partial x^p} \frac{\partial x^q}{\partial \overline{x}^j} A^p_q \quad \text{and} \quad \overline{B}^{kl}_m = \frac{\partial \overline{x}^k}{\partial x^r} \frac{\partial \overline{x}^l}{\partial x^s} \frac{\partial x^t}{\partial \overline{x}^m} B^{rs}_t$$

Multiplying these, we get

$$\overline{A}^i_j \overline{B}^{kl}_m = \frac{\partial \overline{x}^i}{\partial x^p} \frac{\partial x^q}{\partial \overline{x}^j} \frac{\partial \overline{x}^k}{\partial x^r} \frac{\partial \overline{x}^l}{\partial x^s} \frac{\partial x^t}{\partial \overline{x}^m} A^p_q B^{rs}_t$$

This is law of transformation of tensor of rank five. So, $A^i_j B^{kl}_m$ is a tensor of rank five.

2.11 CONTRACTION OF A TENSOR

The process of getting a tensor of lower order (reduced by 2) by putting a covariant index equal to a contravariant index and performing the summation indicated is known as *Contraction.*

In other words, if in a tensor we put one contravariant and one covariant indices equal, the process is called contraction of a tensor.

For example, consider a mixed tensor A^{ijk}_{lm} of order five. Then by law of transformation,

$$\overline{A}^{ijk}_{lm} = \frac{\partial \overline{x}^i}{\partial x^p} \frac{\partial \overline{x}^j}{\partial x^q} \frac{\partial \overline{x}^k}{\partial x^r} \frac{\partial x^s}{\partial \overline{x}^l} \frac{\partial x^t}{\partial \overline{x}^m} A^{pqr}_{st}$$

Put the covariant index l = contravariant index i, so that

$$\overline{A}^{ijk}_{im} = \frac{\partial \overline{x}^i}{\partial x^p} \frac{\partial \overline{x}^j}{\partial x^q} \frac{\partial \overline{x}^k}{\partial x^r} \frac{\partial x^s}{\partial \overline{x}^i} \frac{\partial x^t}{\partial \overline{x}^m} A^{pqr}_{st}$$

$$= \frac{\partial \overline{x}^j}{\partial x^q} \frac{\partial \overline{x}^k}{\partial x^r} \frac{\partial x^s}{\partial x^p} \frac{\partial x^t}{\partial \overline{x}^m} A^{pqr}_{st}$$

$$= \frac{\partial \overline{x}^j}{\partial x^q} \frac{\partial \overline{x}^k}{\partial x^r} \frac{\partial x^t}{\partial \overline{x}^m} \delta^s_p A^{pqr}_{st} \qquad \text{Since } \frac{\partial x^s}{\partial x^p} = \delta^s_p$$

$$\overline{A}^{ijk}_{im} = \frac{\partial \overline{x}^j}{\partial x^q} \frac{\partial \overline{x}^k}{\partial x^r} \frac{\partial x^t}{\partial \overline{x}^m} A^{pqr}_{pt}$$

This is law of transformation of tensor of rank 3. So, A^{ijk}_{im} is a tensor of rank 3 and type (1, 2) while A^{ijk}_{lm} is a tensor of rank 5 and type (2, 3). It means that contraction reduces rank of tensor by two.

2.12 INNER PRODUCT OF TWO TENSORS

Consider the tensors A^{ij}_k and B^l_{mn} if we first form their outer product $A^{ij}_k B^l_{mn}$ and contract this by putting $l = k$ then the result is $A^{ij}_k B^k_{mn}$ which is also a tensor, called the inner product of the given tensors.

Hence the inner product of two tensors is obtained by first taking outer product and then contracting it.

EXAMPLE 7

If A^i and B_i are the components of a contravariant and covariant tensors of rank are respectively then prove that A^iB_i is scalar or invariant.

Solution

As A^i and B_i are the components of a contravariant and covariant tensor of rank one respectively, then according to the law of the transformation

$$\overline{A}^i = \frac{\partial \overline{x}^i}{\partial x^p} A^p \text{ and } \overline{B}_i = \frac{\partial x^q}{\partial \overline{x}^i} B_q$$

Multiplying these, we get

$$\begin{aligned}
\overline{A}^i \overline{B}^i &= \frac{\partial \overline{x}^i}{\partial x^p} \frac{\partial x^q}{\partial \overline{x}^i} A^p B_q \\
&= \frac{\partial x^q}{\partial x^p} A^p B_q, \quad \text{since } \frac{\partial x^q}{\partial x^p} = \delta^q_p \\
&= \delta^q_p A^p B_q \\
\overline{A}^i \overline{B}_i &= A^p B_p
\end{aligned}$$

This shows that $A^i B_i$ is scalar or Invariant.

EXAMPLE 8

If A^i_j is mixed tensor of rank 2 and B^{kl}_m is mixed tensor of rank 3. Prove that $A^i_j B^{jl}_m$ is a mixed tensor of rank 3.

Solution

As A^i_j is mixed tensor of rank 2 and B^{kl}_m is mixed tensor of rank 3. Then by law of transformation

$$\overline{A}^i_j = \frac{\partial \overline{x}^i}{\partial x^p} \frac{\partial x^q}{\partial \overline{x}^j} A^p_q \text{ and } \overline{B}^{kl}_m = \frac{\partial \overline{x}^k}{\partial x^r} \frac{\partial \overline{x}^l}{\partial x^s} \frac{\partial x^t}{\partial \overline{x}^m} B^{rs}_t \qquad ...(1)$$

Put $k = j$ then

$$\overline{B}^{jl}_m = \frac{\partial \overline{x}^j}{\partial x^r} \frac{\partial \overline{x}^l}{\partial x^s} \frac{\partial x^t}{\partial \overline{x}^m} B^{rs}_t \qquad ...(2)$$

Multiplying (1) & (2) we get

$$\begin{aligned}
\overline{A}^i_j B^{jl}_m &= \frac{\partial \overline{x}^i}{\partial x^p} \frac{\partial x^q}{\partial \overline{x}^j} \frac{\partial \overline{x}^j}{\partial x^r} \frac{\partial \overline{x}^l}{\partial x^s} \frac{\partial x^t}{\partial \overline{x}^m} A^p_q B^{rs}_t \\
&= \frac{\partial \overline{x}^i}{\partial x^p} \frac{\partial \overline{x}^l}{\partial x^s} \frac{\partial x^t}{\partial \overline{x}^m} \delta^q_r A^p_q B^{rs}_t \qquad \text{since} \quad \frac{\partial x^q}{\partial \overline{x}^j} \frac{\partial \overline{x}^j}{\partial x^r} = \frac{\partial x^q}{\partial x^r} = \delta^q_r \\
\overline{A}^i_j \overline{B}^{jl}_m &= \frac{\partial \overline{x}^i}{\partial x^p} \frac{\partial \overline{x}^l}{\partial x^s} \frac{\partial x^t}{\partial \overline{x}^m} A^p_q B^{qs}_t \qquad \text{since} \quad \delta^q_r B^{rs}_t = B^{qs}_t
\end{aligned}$$

This is the law of transformation of a mixed tensor of rank three. Hence $A^i_j B^{jl}_m$ is a mixed tensor of rank three.

2.13 SYMMETRIC TENSORS

A tensor is said to be symmetric with respect to two contravariant (or two covariant) indices if its components remain unchanged on an interchange of the two indices.

EXAMPLE

(1) The tensor A^{ij} is symmetric if $A^{ij} = A^{ji}$

(2) The tensor A^{ijk}_{lm} is symmetric if $A^{ijk}_{lm} = A^{jik}_{lm}$

THEOREM 2.10 *A symmetric tensor of rank two has only* $\frac{1}{2}n(n+1)$ *different components in n-dimensional space.*

Proof: Let A^{ij} be a symmetric tensor of rank two. So that $A^{ij} = A^{ji}$.

$$\text{The component of } A^{ij} \text{ are } \begin{bmatrix} A^{11} & A^{12} & A^{13} & \cdots & A^{1n} \\ A^{21} & A^{22} & A^{23} & \cdots & A^{2n} \\ A^{31} & A^{32} & A^{33} & \cdots & A^{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ A^{n1} & A^{n2} & A^{n3} & \cdots & A^{nn} \end{bmatrix}$$

i.e., A^{ij} will have n^2 components. Out of these n^2 components, n components $A^{11}, A^{22}, A^{33}, \ldots, A^{nn}$ are different. Thus remaining components are $(n^2 - n)$. In which $A^{12} = A^{21}$, $A^{23} = A^{32}$ etc. due to symmetry.

So, the remaining different components are $\frac{1}{2}(n^2 - n)$. Hence the total number of different components

$$= n + \frac{1}{2}(n^2 - n) = \frac{1}{2}n(n+1)$$

2.14 SKEW-SYMMETRIC TENSOR

A tensor is said to be skew-symmetric with respect to two contravariant (or two covariant) indices if its components change sign on interchange of the two indices.

EXAMPLE

(*i*) The tensor A^{ij} is Skew-symmetric of $A^{ij} = -A^{ji}$

(*ii*) The tensor A^{ijk}_{lm} is Skew-symmetric if $A^{ijk}_{lm} = -A^{jik}_{lm}$

THEOREM 2.11 *A Skew symmetric tensor of second order has only* $\frac{1}{2}n(n-1)$ *different non-zero components.*

Proof: Let A^{ij} be a skew-symmetric tensor of order two. Then $A^{ij} = -A^{ji}$.

The components of A^{ij} are $\begin{bmatrix} 0 & A^{12} & A^{13} & \cdots & A^{1n} \\ A^{21} & 0 & A^{23} & \cdots & A^{2n} \\ A^{31} & A^{32} & 0 & \cdots & A^{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ A^{n1} & A^{n2} & A^{n3} & \cdots & 0 \end{bmatrix}$

$$\left[\text{Since } A^{ii} = -A^{ii} \Rightarrow 2A^{ii} = 0 \Rightarrow A^{ii} = 0 \Rightarrow A^{11} = A^{22} = \cdots A^{nn} = 0\right]$$

i.e., A^{ij} will have n^2 components. Out of these n^2 components, n components A^{11}, A^{22}, A^{33}, ..., A^{nn} are zero. Omitting there, then the remaining components are $n^2 - n$. In which $A^{12} = -A^{21}$, $A^{13} = -A^{31}$ etc. Ignoring the sign. Their remaining the different components are $\frac{1}{2}(n^2 - n)$.

Hence the total number of different non-zero components $= \frac{1}{2}n(n-1)$

Note: Skew-symmetric tensor is also called anti-symmetric tensor.

THEOREM 2.12 *A covariant or contravariant tensor of rank two say A_{ij} can always be written as the sum of a symmetric and skew-symmetric tensor.*

Proof: Consider a covariant tensor A_{ij}. We can write A_{ij} as

$$A_{ij} = \frac{1}{2}(A_{ij} + A_{ji}) + \frac{1}{2}(A_{ij} - A_{ji})$$

$$A_{ij} = S_{ij} + T_{ij}$$

where $$S_{ij} = \frac{1}{2}(A_{ij} + A_{ji}) \text{ and } T_{ij} = \frac{1}{2}(A_{ij} - A_{ji})$$

Now,

$$S_{ji} = \frac{1}{2}(A_{ji} + A_{ij})$$

$$S_{ji} = S_{ij}$$

So, S_{ij} is symmetric tensor.

and

$$T_{ij} = \frac{1}{2}(A_{ij} + A_{ji})$$

$$T_{ji} = \frac{1}{2}(A_{ji} - A_{ij})$$

$$= -\frac{1}{2}(A_{ij} - A_{ji})$$

$$T_{ji} = -T_{ij}$$

or $$T_{ij} = -T_{ji}$$

So, T_{ij} is Skew-symmetric Tensor.

EXAMPLE 9

If $\phi = a_{jk} A^j A^k$. Show that we can always write $\phi = b_{jk} A^j A^k$ where b_{jk} is symmetric.

Solution

As given

$$\phi = a_{jk} A^j A^k \qquad ...(1)$$

Interchange the indices i and j

$$\phi = a_{kj} A^k A^j \qquad ...(2)$$

Adding (1) and (2),

$$2\phi = (a_{jk} + a_{kj}) A^j A^k$$

$$\phi = \frac{1}{2}(a_{jk} + a_{kj}) A^j A^k$$

$$\phi = b_{jk} A^j A^k$$

where $b_{jk} = \frac{1}{2}(a_{jk} + a_{kj})$

To show that b_{jk} is symmetric.

Since

$$b_{jk} = \frac{1}{2}(a_{jk} + a_{kj})$$

$$b_{kj} = \frac{1}{2}(a_{kj} + a_{jk})$$

$$= \frac{1}{2}(a_{jk} + a_{kj})$$

$$b_{kj} = b_{jk}$$

So, b_{jk} is Symmetric.

EXAMPLE 10

If T_i be the component of a covariant vector show that $\left(\frac{\partial T_i}{\partial x^j} - \frac{\partial T_j}{\partial x^i}\right)$ are component of a Skew-symmetric covariant tensor of rank two.

Solution

As T_i is covariant vector. Then by the law of transformation

$$\overline{T}_i = \frac{\partial x^k}{\partial \overline{x}^i} T_k$$

Differentiating it w.r.t. to $\overline{x}^j$ partially,

$$\frac{\partial \overline{T}_i}{\partial \overline{x}^j} = \frac{\partial}{\partial \overline{x}^j}\left(\frac{\partial x^k}{\partial \overline{x}^i} T_k\right)$$

$$= \frac{\partial^2 x^k}{\partial \overline{x}^j \partial \overline{x}^i} T_k + \frac{\partial x^k}{\partial \overline{x}^i}\frac{\partial T_k}{\partial \overline{x}^j}$$

$$\frac{\partial \overline{T}_j}{\partial \overline{x}^j} = \frac{\partial^2 x^k}{\partial \overline{x}^j \partial \overline{x}^i} T_k + \frac{\partial x^k}{\partial \overline{x}^i}\frac{\partial x^l}{\partial \overline{x}^j}\frac{\partial T_k}{\partial x^l} \qquad ...(1)$$

Similarly,

$$\frac{\partial \overline{T}_j}{\partial \overline{x}^i} = \frac{\partial^2 x^k}{\partial \overline{x}^i \partial \overline{x}^j} T_k + \frac{\partial x^k}{\partial \overline{x}^j}\frac{\partial x^l}{\partial \overline{x}^i}\frac{\partial T_k}{\partial x^l}$$

Interchanging the dummy indices *k & l*

$$\frac{\partial \overline{T}_j}{\partial \overline{x}^i} = \frac{\partial^2 x^k}{\partial \overline{x}^i \partial \overline{x}^j} T_k + \frac{\partial x^k}{\partial \overline{x}^i}\frac{\partial x^l}{\partial \overline{x}^j}\frac{\partial T_l}{\partial x^k} \qquad ...(2)$$

Substituting (1) and (2), we get

$$\frac{\partial \overline{T}_i}{\partial \overline{x}^j} - \frac{\partial \overline{T}_j}{\partial \overline{x}^i} = \frac{\partial x^k}{\partial \overline{x}^i}\frac{\partial x^l}{\partial \overline{x}^j}\left(\frac{\partial T_k}{\partial x^l} - \frac{\partial T_l}{\partial x^k}\right)$$

This is law of transformation of covariant tensor of rank two. So, $\frac{\partial T_i}{\partial x^j} - \frac{\partial T_j}{\partial x^i}$ are component of a covariant tensor of rank two.

To show that $\frac{\partial T_i}{\partial x^j} - \frac{\partial T_j}{\partial x^i}$ is Skew-symmetric tensor.

Let

$$A_{ij} = \frac{\partial T_i}{\partial x^j} - \frac{\partial T_j}{\partial x^i}$$

$$A_{ji} = \frac{\partial T_j}{\partial x^i} - \frac{\partial T_i}{\partial x^j}$$

$$= -\left(\frac{\partial T_i}{\partial x^j} - \frac{\partial T_j}{\partial x^i}\right)$$

$$A_{ji} = -A_{ij}$$

or

$$A_{ij} = -A_{ji}$$

So, $A_{ij} = \frac{\partial T_i}{\partial x^j} - \frac{\partial T_j}{\partial x^i}$ is Skew-symmetric.

So, $\frac{\partial T_i}{\partial x^j} - \frac{\partial T_j}{\partial x^i}$ are component of a Skew-symmetric covariant tensor of rank two.

2.15 QUOTIENT LAW

By this law, we can test a given quantity is a tensor or not. Suppose given quantity be A and we do not know that A is a tensor or not. To test $A,$ we take inner product of A with an arbitrary tensor, if this inner product is a tensor then A is also a tensor.

Statement

If the inner product of a set of functions with an atbitrary tensor is a tensor then these set of functions are the components of a tensor.

The proof of this law is given by the following examples.

EXAMPLE 11

Show that the expression $A(i,j,k)$ is a covariant tensor of rank three if $A(i,j,k)B^k$ is covariant tensor of rank two and B^k is contravariant vector

Solution

Let X and Y be two coordinate systems.

As given $A\ (i, j,\ k)B^k$ is covariant tensor of rank two then

$$\overline{A}(i,j,k)\overline{B}^k = \frac{\partial x^p}{\partial \overline{x}^i}\frac{\partial x^q}{\partial \overline{x}^j}A(p,q,r)B^r \qquad ...(1)$$

Since B^k is contravariant vector. Then

$$\overline{B}^k = \frac{\partial \overline{x}^k}{\partial x^r}B^r \qquad \text{or} \qquad B^r = \frac{\partial x^r}{\partial \overline{x}^k}\overline{B}^k$$

So, from (1)

$$\overline{A}(i,j,k)\overline{B}^k = \frac{\partial x^p}{\partial \overline{x}^i}\frac{\partial x^q}{\partial \overline{x}^j}A(p,q,r)\frac{\partial x^r}{\partial \overline{x}^k}\overline{B}^k$$

$$\overline{A}(i,j,k)\overline{B}^k = \frac{\partial x^p}{\partial \overline{x}^i}\frac{\partial x^q}{\partial \overline{x}^j}\frac{\partial x^r}{\partial \overline{x}^k}A(p,q,r)\overline{B}^k$$

$$\overline{A}(i,j,k) = \frac{\partial x^p}{\partial \overline{x}^i}\frac{\partial x^q}{\partial \overline{x}^i}\frac{\partial x^r}{\partial \overline{x}^k}A(p,q,r)$$

As $\overline{B}^k$ is arbitrary.

So, $A(i,j,k)$ is covariant tensor of rank three.

EXAMPLE 12

If $A\ (i, j,\ k)A^iB^jC_k$ is a scalar for arbitrary vectors A^i, B^j, C_k. Show that $A(i, j,\ k)$ is a tensor of type (1, 2).

Solution

Let X and Y be two coordinate systems. As given $A(i,j,k)A^iB^jC_k$ is scalar. Then

$$\overline{A}(i,j,k)\,\overline{A}^i\overline{B}^j\overline{C}_k = A(p,q,r)A^pB^qC_r \qquad ...(1)$$

Since A^i, B^i and C_k are vectors. Then

$$\overline{A}^i = \frac{\partial \overline{x}^i}{\partial x^p} A^p \quad \text{or} \quad A^p = \frac{\partial x^p}{\partial \overline{x}^i} \overline{A}^i$$

$$\overline{B}^j = \frac{\partial \overline{x}^j}{\partial x^q} B^q \quad \text{or} \quad B^q = \frac{\partial x^q}{\partial \overline{x}^j} \overline{B}^j$$

$$\overline{C}^k = \frac{\partial \overline{x}^k}{\partial x^r} C^r \quad \text{or} \quad C^r = \frac{\partial x^r}{\partial \overline{x}^k} \overline{C}^k$$

So, from (1)

$$\overline{A}(i,j,k)\, \overline{A}^i \overline{B}^j \overline{C}_k = A(p,q,r) \frac{\partial x^p}{\partial \overline{x}^i} \frac{\partial x^q}{\partial \overline{x}^j} \frac{\partial \overline{x}^k}{\partial x^r} \overline{A}^i \overline{B}^j \overline{C}_k$$

As $\overline{A}^i, \overline{B}^j, \overline{C}_k$ are arbitrary.

Then

$$\overline{A}(i,j,k) = \frac{\partial x^p}{\partial \overline{x}^i} \frac{\partial x^q}{\partial \overline{x}^j} \frac{\partial \overline{x}^k}{\partial x^r} A(p,q,r)$$

So, $A(i, j, k)$ is tensor of type (1, 2).

2.16 CONJUGATE (OR RECIPROCAL) SYMMETRIC TENSOR

Consider a covariant symmetric tensor A_{ij} of rank two. Let d denote the determinant $|A_{ij}|$ with the elements A_{ij} *i.e.*, $d = |A_{ij}|$ and $d \neq 0$.

Now, define A^{ij} by

$$A^{ij} = \frac{\text{Cofactor of } A_{ij} \text{ is the determinant} |A_{ij}|}{d}$$

A^{ij} is a contravariant symmetric tensor of rank two which is called conjugate (or Reciprocal) tensor of A_{ij}.

THEOREM 2.13 *If* B_{ij} *is the cofactor of* A_{ij} *in the determinant* $d = |A_{ij}| \neq 0$ *and* A^{ij} *defined as*

$$A^{ij} = \frac{B_{ij}}{d}$$

Then prove that $A_{ij} A^{kj} = \delta_i^k$.

Proof: From the properties of the determinants, we have two results.

(*i*) $A_{ij} B_{ij} = d$

$$\Rightarrow \qquad A_{ij} \frac{B_{ij}}{d} = 1$$

$$A_{ij} A^{ij} = 1, \qquad \text{given} \qquad A^{ij} = \frac{B_{ij}}{d}$$

(*ii*) $A_{ij}B_{kj} = 0$

$$A_{ij}\frac{B_{kj}}{d} = 0, \qquad d \neq 0$$

$$A_{ij}A^{kj} = 0 \qquad \text{if } i \neq k$$

from (*i*) & (*ii*)

$$A_{ij}A^{kj} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

i.e.,
$$A_{ij}A^{kj} = \delta_i^k$$

2.17 RELATIVE TENSOR

If the components of a tensor $A^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s}$ transform according to the equation

$$\bar{A}^{k_1 k_2 \ldots k_r}_{l_1 l_2 \ldots l_s} = \left|\frac{\partial x}{\partial \bar{x}}\right|^{\omega} A^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s} \quad \frac{\partial \bar{x}^{k_1}}{\partial x^{i_1}}\frac{\partial \bar{x}^{k_2}}{\partial x^{i_2}}\cdots\frac{\partial \bar{x}^{k_r}}{\partial x^{i_r}}\cdot\frac{\partial x^{j_1}}{\partial \bar{x}^{l_1}}\frac{\partial x^{j_2}}{\partial \bar{x}^{l_2}}\cdots\frac{\partial x^{j_s}}{\partial \bar{x}^{l_s}}$$

Hence $A^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_r}$ is called a relative tensor of weight ω, where $\left|\frac{\partial x}{\partial \bar{x}}\right|$ is the Jacobian of transformation. If $\omega = 1,$ the relative tensor is called a tensor density. If $w = 0$ then tensor is said to be absolute.

MISCELLANEOUS EXAMPLES

1. Show that there is no distinction between contravariant and covariant vectors when we restrict ourselves to transformation of the type

$$\bar{x}^i = a^i_m x^m + b^i;$$

where *a*'s and *b*'s are constants such that

$$a^i_r a^i_m = \delta^r_m$$

Solution

Given that

$$\bar{x}^i = a^i_m x^m + b^i \qquad \text{...(1)}$$

or
$$a^i_m x^m = \bar{x}^i - b^i \qquad \text{...(2)}$$

Multiplying both sides (2) by $a^i_r,$ we get

$$a^i_r a^i_m x^m = a^i_r \bar{x}^i - b^i a^i_r$$

$$\delta^r_m x^m = a^i_r \bar{x}^i - b^i a^i_r \quad \text{as given } a^i_r a^i_m = \delta^r_m$$

$$x^r = a^i_r \bar{x}^i - b^i a^i_r \quad \text{as } \delta^r_m x^m = x^r$$

or
$$x^s = a^i_s \bar{x}^i - b^i a^i_s$$

Differentiating Partially it w.r.t. to $\bar{x}^i$

$$\frac{\partial x^s}{\partial \bar{x}^i} = a^i_s \qquad \text{..(3)}$$

Now, from (1)

$$\bar{x}^i = a^i_s x^s + b^i$$

$$\frac{\partial \bar{x}^i}{\partial x^s} = a^i_s \qquad ...(4)$$

The contravariant transformation is

$$\bar{A}^i = \frac{\partial \bar{x}^i}{\partial x^s} A^s = a^i_s A^s \qquad ...(5)$$

The covariant transformation is

$$\bar{A}_i = \frac{\partial x^s}{\partial \bar{x}^i} A_s = a^i_s A_s \qquad ...(6)$$

Thus from (5) and (6), it shows that there is no distinction between contravariant and covariant tensor law of transformation

2. If the tensors a_{ij} and g_{ij} are symmetric and u^i, v^i are components of contravariant vectors satisfying the equations

$$(a_{ij} - kg_{ij})u^i = 0, \; i, j = 1,2,\ldots,n$$

$$(a_{ij} - k'g_{ij})v^i = 0, \; k \neq k'.$$

Prove that $g_{ij}u^i v^j = 0$, $a_{ij}u^i v^j = 0$.

Solution

The equations are

$$(a_{ij} - kg_{ij})u^i = 0 \qquad ...(1)$$

$$(a_{ij} - k'g_{ij})v^i = 0 \qquad ...(2)$$

Multiplying (1) and (2) by u^j and v^j respectively and subtracting, we get

$$a_{ij}u^i v^j - a_{ij}v^i u^j - kg_{ij}u^i v^j + k'g_{ij}u^j v^i = 0$$

Interchanging i and j in the second and fourth terms,

$$a_{ij}u^i v^j - a_{ji}v^j u^i - kg_{ij}u^i v^j + k'g_{ji}u^i v^j = 0$$

As a_{ij} and g_{ij} is symmetric *i.e.,* $a_{ij} = a_{ji}$ & $g_{ij} = g_{ji}$

$$-kg_{ij}v^j u^i + k'g_{ij}u^i v^j = 0$$

$$(k' - k)g_{ij}u^i v^j = 0$$

$$g_{ij}u^i v^j = 0 \text{ since } k \neq k' \Rightarrow k - k' \neq 0$$

Multiplying (1) by v^j, we get

$$a_{ij}v^j u^i - kg_{ij}u^i v^j = 0$$

$$a_{ij}u^i v^j = 0 \text{ as } g_{ij}u^i v^j = 0.$$

Proved.

3. If A_{ij} is a Skew-Symmetric tensor prove that

$$(\delta^i_j\delta^k_l + \delta^i_l\delta^k_j)A_{ik} = 0$$

Solution

Given A_{ij} is a Skew-symmetric tensor then $A_{ij} = -A_{ji}$.

Now,

$$\begin{aligned}(\delta^i_j\delta^k_l + \delta^i_l\delta^k_j)A_{ik} &= \delta^i_j\delta^k_l A_{ik} + \delta^i_l\delta^k_j A_{ik}\\ &= \delta^i_j A_{il} + \delta^i_l A_{ij}\\ &= A_{jl} + A_{lj}\\ (\delta^i_j\delta^k_l + \delta^i_l\delta^k_j)A_{ik} &= 0 \quad \text{as } A_{jl} = -A_{lj}\end{aligned}$$

4. If a_{ij} is symmetric tensor and b_i is a vector and $a_{ij}b_k + a_{jk}b_i + a_{ki}b_j = 0$ then prove that $a_{ij} = 0$ or $b_k = 0$.

Solution

The equation is

$$a_{ij}b_k + a_{jk}b_i + a_{ki}b_j = 0$$

$$\Rightarrow \quad \overline{a}_{ij}\overline{b}_k + \overline{a}_{jk}\overline{b}_i + \overline{a}_{ki}\overline{b}_j = 0$$

By tensor law of transformation, we have

$$a_{pq}\frac{\partial x^p}{\partial \overline{x}^i}\frac{\partial x^q}{\partial \overline{x}^j}b_r\frac{\partial x^r}{\partial \overline{x}^k} + a_{pq}\frac{\partial x^p}{\partial \overline{x}^j}\frac{\partial x^q}{\partial \overline{x}^k}b_r\frac{\partial x^r}{\partial \overline{x}^i} + a_{pq}\frac{\partial x^p}{\partial \overline{x}^k}\frac{\partial x^q}{\partial \overline{x}^i}b_r\frac{\partial x^r}{\partial \overline{x}^j} = 0$$

$$a_{pq}b_r\left[\frac{\partial x^p}{\partial \overline{x}^i}\frac{\partial x^q}{\partial \overline{x}^j}\frac{\partial x^r}{\partial \overline{x}^k} + \frac{\partial x^p}{\partial \overline{x}^j}\frac{\partial x^q}{\partial \overline{x}^k}\frac{\partial x^r}{\partial \overline{x}^i} + \frac{\partial x^p}{\partial \overline{x}^k}\frac{\partial x^q}{\partial \overline{x}^i}\frac{\partial x^r}{\partial \overline{x}^j}\right] = 0$$

$$\Rightarrow \quad a_{pq}b_r = 0 \Rightarrow a_{pq} = 0 \text{ or } b_r = 0$$

$$\Rightarrow a_{ij} = 0 \text{ or } b_k = 0$$

5. If $a_{mn}x^m x^n = b_{mn}x^m x^n$ for arbitrary values of x^r, show that $a_{(mn)} = b_{(mn)}$ *i.e.*, $a_{mn} + a_{nm} = b_{mn} + b_{nm}$

If a_{mn} and b_{mn} are symmetric tensors then further show the $a_{mn} = b_{mn}$.

Solution

Given

$$a_{mn}x^m x^n = b_{mn}x^m x^n$$

$$(a_{mn} - b_{mn})x^m x^n = 0$$

Differentiating w.r.t. x^i partially

$$(a_{in} - b_{in})x^n + (a_{mi} - b_{mi})x^m = 0$$

Differentiating again w.r.t. x^j partially

$$(a_{ij} - b_{ij}) + (a_{ji} - b_{ji}) = 0$$

$$a_{ij} + a_{ji} = b_{ij} + b_{ji}$$

or $$a_{mn} + a_{nm} = b_{mn} + b_{nm} \text{ or } a_{(mn)} = b_{(mn)}$$

Also, since a_{mn} and b_{mn} are symmetric then $a_{mn} = a_{nm}$, $b_{mn} = b_{nm}$.

So,

$$2a_{mn} = 2b_{mn}$$

$$a_{mn} = b_{mn}$$

EXERCISES

1. Write down the law of transformation for the tensors
   (*i*) A_{ij}
   (*ii*) B_k^{ij}
   (*iii*) C_{lm}^{ijk}
2. If A_r^{pq} and B_t^s are tensors then prove that $A_r^{pq} B_t^s$ is also a tensor.
3. If A^{ij} is a contravariant tensor and B_i is covariant vector then prove that $A^{ij}B_k$ is a tensor of rank three and $A^{ij}B_j$ is a tensor of rank one.
4. If A^i is an arbitrary contravariant vector and $C_{ij}A^iA^j$ is an invariant show that $C_{ij} + C_{ji}$ is a covariant tensor of the second order.
5. Show that every tensor can be expressed in the terms of symmetric and skew-symmetric tensor.
6. Prove that in n-dimensional space, symmetric and skew-symmetric tensor have $\frac{n}{2}(n+1)$ and $\frac{n}{2}(n-1)$ independent components respectively.
7. If $U_{ij} \neq 0$ are components of a tensor of the type (0, 2) and if the equation $fU_{ij} + gU_{ji} = 0$ holds w.r.t to a basis then prove that either $f = g$ and U_{ij} is skew-symmetric or $f = -g$ and U_{ij} is symmetric.
8. If A_{ij} is skew-symmetric then $(B_j^i B_l^k + B_l^i B_j^k)A_{ik} = 0$.
9. Explain the process of contraction of tensors. Show that $a_{ij}a^{ij} = \delta_j^i$.

10. If A_r^{pq} is a tensor of rank three. Show that A_r^{pr} is a contravariant tensor of rank one.

11. If $a_k^{ij}\lambda_i\mu_j\gamma^k$ is a scalar or invariant, $\lambda_i, \mu_j, \gamma^k$ are vectors then a_k^{ij} is a mixed tensor of type (2, 1).

12. Show that if $a_{hijk}\lambda^h\mu^i\lambda^h\mu^k = 0$ where λ^i and μ^i are components of two arbitrary vectors then

    $a_{hijk} + a_{hkji} + a_{jihk} + a_{jkhi} = 0$

13. Prove that $A_{ij}B^iC^j$ is invariant if B^i and C^j are vector and A_{ij} is tensor of rank two.

14. If $A(r, s, t)$ be a function of the coordinates in n-dimensional space such that for an arbitrary vector B^r of the type indicated by the index a $A(r, s, t)B^r$ is equal to the component C^{st} of a contravariant tensor of order two. Prove that $A(r, s, t)$ are the components of a tensor of the form A_r^{st}.

15. If A^{ij} and A_{ij} are components of symmetric relative tensors of weight w. show that

$$\left|\overline{A}^{ij}\right| = \left|A^{ij}\right|\left|\frac{\partial x}{\partial \overline{x}}\right|^{w-2} \quad \text{and} \quad \left|\overline{A}_{ij}\right| = \left|A_{ij}\right|\left|\frac{\partial x}{\partial \overline{x}}\right|^{w+2}$$

16. Prove that the scalar product of a relative covariant vector of weight w and a relative contravariant vector of weight w' is a relative scalar of weight $w + w'$.