# Classical AND Quantum Mechanics

## * Schrodinger Equation

Schrodinger Equation → Independent / Steady state
Schrodinger Equation → Time dependent

Q Metal wave associated with the moving particles travels with a speed more than velocity of light?

$\lambda = \frac{h}{p}$ ← De-Broglie eq

Ans It is true,

$$\lambda = \frac{h}{mv}$$

$v$ = Particle velocity

$$v_p = \nu\lambda$$

$$= \frac{mc^2}{\cancel{h}} \times \frac{\cancel{h}}{mv}$$

$$= \frac{c^2}{vv} > c$$

$$h\nu = mc^2$$

$$\nu = \frac{mc^2}{h}$$

Wave Packet → Superpossition of all monochromatic wave.

$$\lambda \pm d\lambda$$

Need of SE :-

→ apply on $II^{nd}$ law of Newton ($F = ma$)

→ valid for all Particle whose velocity less than moving instant of time.

$$\frac{d^2y}{dt^2} = v_p^2 \frac{d^2y}{dx^2}$$

wave eq$^n$

$$\psi(x,t) = y = a\exp[i(kx - \omega t)]$$

$k$ = wave no. $= \frac{2\pi}{\lambda}$

$\omega$ = angular frequency

$y$ → fun. of variation of Pressure & density

$$\psi(x,t) = Ae^{i(kx-\omega t)} \Rightarrow \frac{d\psi(x,t)}{dx} = Ae^{i(kx-\omega t)} \cdot ik$$

(Put $Ae^{i(kx-\omega t)} = \psi(x,t)$)

$$\Rightarrow \frac{d\psi(x,t)}{dx} = ik\,\psi(x,t)$$

$$= i\frac{p}{\hbar}\psi(x,t)$$

$$-i\hbar\frac{\partial}{\partial x}\psi(x,t) = p\psi(x,t)$$

$$\boxed{-i\hbar\frac{\partial}{\partial x} = \hat{p}} \rightarrow \text{Momentum operator}$$

K.E. of operator

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + V(x)\psi(x,t)$$

Energy operator

* Wave Function
1. Complex function
2. operator
3. Expectation value
4. Hamilation operator - $H\psi(x,t) = E\psi(x,t)$
Homilonian

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial \psi^2} + \hat{V} \qquad \left[H = \frac{p^2}{2m} + V\right]$$

5. wave fun. should be in a boundary condition

Particle is moving free in space $V=0$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = i\hbar\frac{\partial\psi}{\partial t}$$

---

$$E = \frac{me^4}{8\epsilon_0^2 h^2}$$

* Schrodinger equation : (steady - state form)

$\Delta x \, \Delta p_x \geq \frac{\hbar}{2}$ (uncertainity Heisenberg principle) $\left(\hbar = \frac{h}{2\pi}\right)$

$\Delta E \, \Delta t \geq h$

$\Delta E = \frac{h}{\Delta t}$

$$E\psi(x,t) = \frac{p^2}{2m}\psi(x,t) + V(x)\psi(x,t)$$

$\hat{G}\psi(x,t) = g\,\psi(x,t)$

$g$ – eigen value of operator $\hat{G}$

Q. $\hat{G} = \frac{d^2}{dx^2} \cdot e^{2x} = 4e^{2x}$

$\therefore \psi(x,t) = e^{2x}$

(eigen value) $g = 4$

$$\psi(x,t) = A \cdot e^{i(kx-\omega t)}$$

$$\frac{d\psi(x,t)}{dx} = ik \cdot \psi(x,t)$$

$$= \frac{ip}{\hbar}\psi(x,t)$$

$p = \frac{h}{\lambda}$

$p = \frac{h \cdot 2\pi}{\lambda \cdot 2\pi}$ $\quad \hbar = \frac{h}{2\pi}, \; k = \frac{2\pi}{\lambda}$

$p = \hbar k$

$k = \frac{p}{\hbar}$

$$-i\hbar \frac{d(\psi(x,t))}{dx} = p\,\psi(x,t)$$

$$-i\hbar \frac{d}{dx} = \hat{p}$$

$$\hat{E}\,\psi(x,t) = \frac{\hat{p}}{2m}\psi(x,t) + \hat{U}(x)\psi(x,t)$$

$$i\hbar \frac{d}{dt}\psi(x,t) = -\frac{\hbar^2}{2m}\frac{d^2\psi(x,t)}{dx^2} + U(x)\,\psi(x,t)$$

$E = i\hbar \frac{\partial}{\partial t}$

$p \to -i\hbar \frac{\partial}{\partial x}$

$H = \frac{p^2}{2m} + V$

$$\hat{H}\psi(x,t) = E\,\psi(x,t)$$

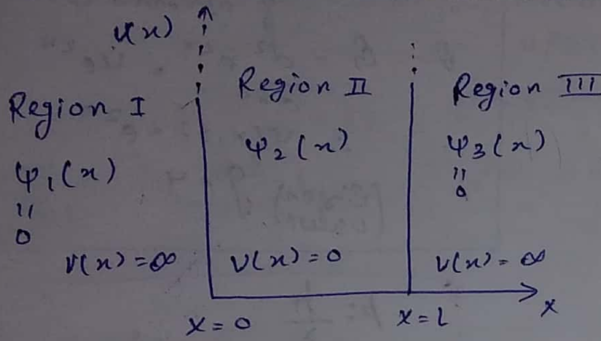
$$\hat{H} = -\frac{\hbar^2}{2m} \cdot \frac{d^2}{dx^2} + \hat{U}$$

Probability in a small volume dv $= |\Psi(x,t)|^2 dv$

Total Prob. of find Particle in universe $= \int_{-\infty}^{+\infty} \Psi(x,t)^2 dv = 1$

impenetrable

## * Particle trapped in 1-D box with impenetable walls:




$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}$$

$$i\hbar \frac{d\psi(x,t)}{dt} = -\frac{\hbar^2}{2m} \frac{d^2\psi(x,t)}{dx^2} + V(x)\,\Psi(x,t)$$ (Time dependent form)

$$\psi(x,t) = A e^{(ikx - i\omega t)}$$

$$\psi(x,t) = \Psi(x).e^{-\omega t}$$

$$\Psi(x) = A e^{ikx}$$

$$\frac{d^2\psi(x)}{dx^2} + \frac{2m}{\hbar^2}(E - V(x))\,\psi(x) = 0$$ (Steady form)

Region I

$$\frac{d^2\psi_1}{dx^2} + \frac{2m}{\hbar^2}(E-\infty)\,\psi_1 = 0$$

Region II

$$\frac{d^2\psi_2}{dx^2} + \frac{2m}{\hbar^2}(E-0)\,\psi_2 = 0$$

Region III

$$\frac{d^2\psi_3}{dx^2} + \frac{2m}{\hbar^2}(E-\infty)\,\psi_3 = 0$$

Scanned by CamScanner

$$\frac{d^2\psi}{dx^2} + k^2\psi = 0 \quad (\text{Region II}) \quad \left(k^2 = \frac{2mE}{\hbar^2}\right)$$

$$\psi(x) = A\sin kx + B\cos kx$$

Boundary conditions → wave function should be finite
→ " " " , single value
→ " " " continous

For Boundary condition ① $\psi(x) \rightarrow$ finite ✓

② at $x = 0$ $\psi(x) = 0$

$x = L$ $\psi(x) = 0$

$A\sin 0 + B\cos 0 = 0$
$\therefore B = 0$

$A\sin kx = 0$
$A \neq 0 \Rightarrow \sin kL = 0$
or $kL = n\pi$
$\boxed{k = \frac{n\pi}{L}}$

$$\psi(x) = A\sin\left(\frac{n\pi}{L}\right)x$$

$$\psi_n(x) = A\sin\frac{n\pi}{L}x$$

$$\int_{-\infty}^{+\infty} |\psi_n(x)|^2 dx = 1$$

$$\int_{-\infty}^{0} \psi_1(x)^2 dx + \int_0^L \psi_2(x)^2 dx + \int_L^{\infty} \psi_3(x)^2 dx = 1$$

→ ① $A^2\int_0^L \sin^2 kx\, dx = 1$

$\frac{A^2}{2}\int_0^L (1 - \cos 2kx)dx = 1$

$\frac{A^2}{2}[L - \sin 2kL] = 1$

$\frac{A^2}{2}\left[L - \sin 2\left(\frac{n\pi}{L}\right).L\right] = 1$

$A^2 = \frac{2}{L}$

$\boxed{A = \sqrt{\frac{2}{L}}}$

$k = \frac{n\pi}{L}$

$k^2 = \frac{2mE}{\hbar^2}$

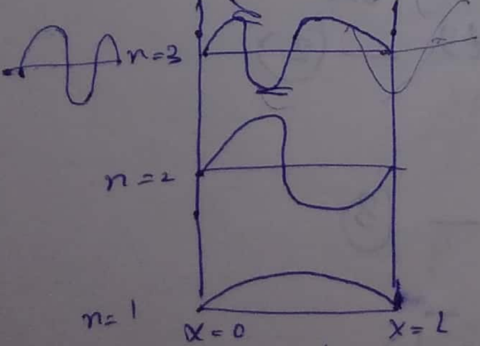
$\frac{2mE}{\hbar^2} = \frac{n^2\pi^2}{L^2}$

$E = \frac{n^2\pi^2\hbar^2}{2mL}$

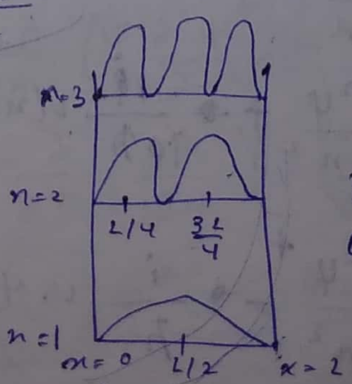
1. Plot 1st 3 wave fun.
2. $|\psi_n(x)|^2$
3. Momentum values
4. Energy Eigen values

$$\psi_n(x) = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots$$

$$|\psi_n(x)|^2 = \frac{2}{L}\sin^2\frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots$$




wave fun.




$|\psi_n(x)|^2$

$|\psi_3(x)|^2$

$|\psi_2(x)|^2$

$E_2 = \frac{4h^2}{8mL^2}$

3D schrodinger eq. –

$$-\frac{\hbar^2}{2m}\nabla^2\psi(x,t)+v(x)\psi(x,t)=i\hbar\frac{\partial\psi(x,t)}{\partial t}$$

* Probability Density and current density

$$P(x,t)=\psi^*(x,t)\,\psi(x,t)$$

$$S(x,t)=\frac{i\hbar}{2m}\left[\psi\frac{\partial\psi^*}{\partial x}-\psi^*\frac{d\psi}{dx}\right] \quad \text{— 1-D}$$

$$=\frac{i\hbar}{2m}\left(\psi\nabla\psi^*-\psi^*\nabla\psi\right) \quad \text{- 3D}$$

$$\psi=\psi(r,t)$$

Ques Show that change in Probability density in a region of space is equal to the ~~region of space is~~ net change in Probability current density into that region.

$$\frac{d}{dt}\int_{x_1}^{x_2}P\,dx=S(x_1,t)-S(x_2,t)$$

Sol

$$\int_{x_1}^{x_2}P\,dx=\int_{x_1}^{x_2}\psi^*\psi\,dx$$

$$\frac{d}{dt}\int_{x_1}^{x_2}\psi^*\psi\,dx=\int_{x_1}^{x_2}\frac{d\psi^*}{dt}\psi\,dx+\int_{x_1}^{x_2}\psi^*\frac{d\psi}{dt}\,dx \quad \text{— (1)}$$

Schroding eq. $-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x}+v(x)\psi(x,t)=i\hbar\frac{\partial\psi(x,t)}{\partial t}$

$$\frac{d\psi}{dt}=\frac{i\hbar}{2m}\frac{\partial^2\psi}{\partial x^2}+\frac{1}{i\hbar}v(x).\psi(x,t) \quad \text{— (2)}$$

$$\frac{\partial\psi^*}{dt}=-\frac{i\hbar}{2m}\frac{\partial^2\psi^*}{\partial x^2}+\frac{i}{\hbar}v.\psi^* \quad \text{— (3)}$$

Padd the eq. (2) & (3) in eq (1)

$$\frac{d}{dt}\int_{x_1}^{x_2}\psi^*\psi\,dx = \int_{x_1}^{x_2}\frac{-i\hbar}{2m}\frac{\partial^2\psi^*}{\partial x^2}\psi + \frac{i}{\hbar}V\psi^*\,dx +$$

$$\left[\int_{x_1}^{x_2}\psi^* + \frac{\hbar}{2mi}\frac{\partial^2\psi}{\partial x^2} + \frac{1}{i\hbar}V(x)\psi(x,t)\,dx\right]$$

$$\frac{d}{dt}\int_{x_1}^{x_2}\psi^*\psi\,dx = \frac{i\hbar}{2m}\int_{x_1}^{x_2}\left(\psi^*\frac{\partial^2\psi}{\partial x^2} - \psi\frac{\partial^2\psi^*}{\partial x^2}\right)dx$$

$$= \frac{i\hbar}{2m}\int_{x_1}^{x_2}\frac{\partial}{\partial x}\left(\psi^*\frac{\partial\psi}{\partial x} - \psi\frac{\partial\psi^*}{\partial x}\right)dx$$

$$= -S(x_2,t) + S(x_1,t)$$

Hence Proved

$\frac{d}{dt}\int_{x_1}^{x_2}P\,dx$ It is the Probability density b/w the $x_1$, $x_2$. It indicating the total Particle Present in $x_1$ & $x_2$. $\frac{d}{dt}\int_{x_1}^{x_2}P\,dx$ = Time of rate change of Probability b/w two Points $x_1$ & $x_2$ at any instant of time is equal to the change in Probability current density.

Difference b/w the Prob. current density $x$ & Prob. following out at $x_2$

Q

$$\frac{dP}{dt} = \frac{d}{dt}(\psi^*\psi)$$

$$\frac{dP}{dt} = \frac{d\psi^*}{dt}\psi + \psi^*\frac{d\psi}{dt}$$

$$\frac{dP}{dt} = \frac{-i\hbar}{2m}\left(\psi\frac{\partial^2\psi^*}{\partial x^2} - \psi^*\frac{\partial^2\psi}{\partial x^2}\right)$$

$$\frac{dP}{dt} = -\frac{\partial}{\partial x}\left[\frac{i\hbar}{2m}\left(\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x}\right)\right]$$

$$P = \frac{i\hbar}{2m}\left(\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x}\right)$$

$$P = S(x,t)$$

$$\frac{d\rho}{dt} = -\frac{\partial}{\partial x} J(x,t)$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial J(x,t)}{\partial x} = 0 \qquad - \text{1D}$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}.\vec{J}(x,t) = 0 \qquad \text{3-D}$$

Q Show that the Probability density current for a Plan wave in a medium is equal to the Prob. density multiplyied by the velocity of Particle in that region.

$$J(x,t) = v(\psi^* \psi)$$

Sol)
$$\psi(x,t) = A.e^{i(kx-\omega t)}$$

$$\frac{d\psi}{dx} = A.e^{i(kx-\omega t)}.k$$

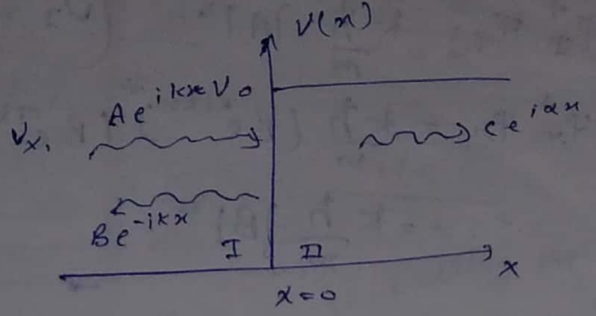
$$J(x,t) = \frac{i\hbar}{2m}\left(\psi \frac{\partial \psi^*}{\partial x} - \psi^* \frac{\partial \psi}{\partial x}\right)$$

$$= \frac{i\hbar}{2m}\left[\psi \{Ake^{i(kx-\omega t)} - \psi^* Ake^{i(kx-\omega t)}\right]$$

$$= \frac{i\hbar k}{m}(\psi^* \psi)$$

$$= v_{particle}(\psi^* \psi)$$

# Step Potential




$$V(x) = 0 \quad x<0$$
$$= V_0 \quad x>0.$$

**Case I** $E > V_0$

Region I

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E-0)\psi_1 = 0$$

Put $k^2 = \frac{2mE}{\hbar^2}$

$$\therefore \frac{d^2\psi_1}{dx^2} + k^2\psi_1 = 0$$

$$\psi_1(x) = \underbrace{Ae^{ikx}}_{\text{incident Beam}} + \underbrace{Be^{-ikx}}_{\text{Reflected wave}}$$

$$(D^2 + k^2)\psi_1 = 0$$
$$D = \pm ik$$

Region II

$$\frac{d^2\psi_2}{dx^2} + \frac{2m}{\hbar^2}(E-V_0)\psi_2 = 0$$

let $\alpha^2 = \frac{2m}{\hbar^2}(E-V_0)$

$$\therefore \frac{d^2\psi_2}{dx^2} + \alpha^2\psi_2 = 0$$

$$\psi_2 = \underbrace{Ce^{i\alpha x}}_{\text{Transmitted wave}} + De^{-i\alpha x}$$

$De^{-i\alpha x} = 0$
There is no boundary in reg. 2, so that there is no Reflected wave.

Reflectance / Reflected current density ($J_r$)
Transmittance / Transmitted current density ($J_t$)

$E > V_0$

To Prove : $J_r + J_t = J_I$ ($J_I \to$ incident current density)

$$R + T = 1. \quad \text{— (I)}$$

$$\frac{J_r}{J_I} + \frac{J_t}{J_I} = 1$$

$J(x,t) = v\psi^*\psi$

$J_I = v_I \psi_I^* \psi_I = \frac{k\hbar}{m}|A|^2$ $\quad \left(v_I = \frac{k\hbar}{m}\right)$

$J_r = v_I \psi_r^* \psi_r = \frac{k\hbar}{m}(Be^{-ikx})^*(Be^{-ikx})$ $\quad (\psi_r = Be^{-ikx})$

$= \frac{k\hbar}{m}|B|^2$

$J_t = v_I \psi_t^* \cdot \psi_t = \frac{\alpha\hbar}{m}(Ce^{i\alpha x})^*(Ce^{i\alpha x})$ $\quad (\psi_t^* = Ce^{-i\alpha x})$

$= \frac{\alpha\hbar}{m} \cdot |C|^2$ $\quad \left(v = \frac{\alpha\hbar}{m}\right)$

$R = \frac{J_r}{J_I} = \frac{|B|^2}{|A|^2} = ?$ , $T = \frac{J_t}{J_I} = \frac{\alpha}{k}\frac{|C|^2}{|A|^2}$

Boundary condition,

→ $x = \pm\infty$, finite everywhere → $Ae^{ikx} + Be^{-ikx} = Ce^{ikx}$

Putting $x=0$

→ $x = 0$ → $\psi_1(x) = \psi_2(x)$ ⇒ $A + B = C$

$\frac{d\psi_1}{dx} = \frac{d\psi_2}{dx}$ ⇒ $ik(A-B) = i\alpha C$

$A + B = C$ — ①

$A - B = \frac{\alpha}{k}C$ — ②

Solving eq. ① & ②

$A - (C - A) = \frac{\alpha}{k}C$

$A - C + A = \frac{\alpha}{k}C \Rightarrow 2A - C = \frac{\alpha}{k}C$

$\frac{\alpha}{k} + C = 2A$

$C\left(\frac{\alpha + k}{k}\right) = 2A$

$C = 2A\left(\frac{k}{\alpha + k}\right)$

Similarly

$T =$

Case II

$k^2 = \frac{2mE}{\hbar^2}$

$\frac{d^2\psi_1}{dx^2}$

$\frac{d^2\psi_1}{dx^2}$

$\psi_1(x) =$

$J_I$

$J_r =$

$J_t$

Similarly, $B = \left(\frac{k-\alpha}{k+\alpha}\right) A$

$$\therefore \boxed{R = \frac{J_r}{J_I} = \frac{|B|^2}{|A|^2} = \left(\frac{k-\alpha}{k+\alpha}\right)^2}$$

Similiarly
ok ok

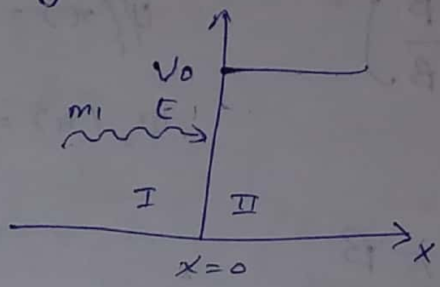
$$\boxed{T = \frac{J_t}{J_I} = \frac{\alpha |C|^2}{k |A|^2} = \frac{4\alpha k}{(k+\alpha)^2}}$$

By Putting the value of R & T in eq I we Proved the eq

Case II $\quad E < V_0$

$k^2 = \frac{2mE}{\hbar^2}$, $\alpha^2 = \frac{2m(E-V_0)}{\hbar^2}$,

$\beta^2 = \frac{2m}{\hbar^2}(V_0 - E)$




$$V(x) = 0 \quad x<0$$
$$= V_0 \quad x>0$$

$$\frac{d^2\psi_1}{dx^2} + \frac{2m}{\hbar^2}(E-0)\psi_1 = 0$$

$$\frac{d^2\psi_1}{dx^2} + k^2\psi_1 = 0$$

$$\psi_1(x) = Ae^{ikx} + Be^{-ikx}$$

$$\frac{d^2\psi_2}{dx^2} - \frac{2m}{\hbar^2}(V_0 - E)\psi_2 = 0$$

$$\frac{d^2\psi_2}{dx^2} - \beta^2\psi_2 = 0$$

$$\psi_2(x) = Ce^{\beta x} + De^{-\beta x}$$

$$D^2 + \beta^2 = 0$$
$$D = \pm \beta$$

$$J_I = v_I \psi_I^* \psi \Rightarrow \frac{k\hbar}{m}(Ae^{ikx})^* (Ae^{ikx}) \Rightarrow \frac{k\hbar}{m}|A|^2$$

$$J_r = v_I \psi_r^* \psi_r \Rightarrow \frac{k\hbar}{m}(Be^{ikx})^* (Be^{ikx}) \Rightarrow \frac{k\hbar}{m}|B|^2$$

$$J_t = v_t \psi_t^* \psi_t \Rightarrow \frac{\beta\hbar}{m}(Ce^{\beta x})^* (Ce^{\beta x}) \Rightarrow \frac{\beta\hbar}{m}|C|^2$$

Scanned by CamScanner

Boundary Condition,

$x = \pm\infty$ , finite everywhere

at $x = 0$ , $\psi_1 = \psi_2$

$\frac{\partial \psi_1}{\partial x} = \frac{\partial \psi_2}{\partial x}$

$A + B = D$

$A - B = \frac{i\beta D}{K}$

$R = \frac{v_1}{v_1}\frac{|B|^2}{|A|^2} = \frac{|B|^2}{|A|^2}$, $T = \frac{v_2}{v_1}\frac{|C|^2}{|A|^2} = \frac{\beta |C|^2}{K|A|^2}$

$R = \left(\frac{ik-\beta}{ik+\beta}\right)^2$

$\psi_1(x) = A e^{ikx} + A\left(\frac{ik-\beta}{ik+\beta}\right) \cdot e^{ikx}$

$\psi_2(x) = A\left(\frac{2ik}{ik+\beta}\right) \cdot e^{-\beta x}$

$T = \frac{4k\beta}{(ik+\beta)^2}$

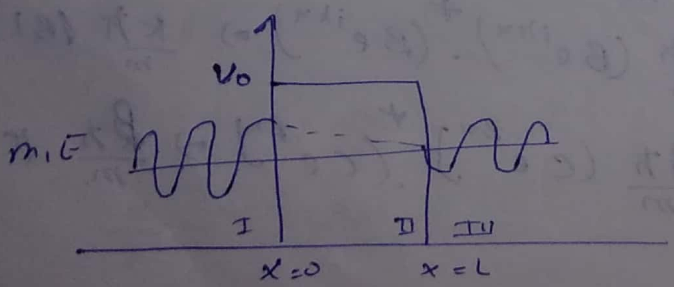
Penetration depth:

$\psi_2(\text{at } x = \Delta x) = \frac{1}{e}\,\psi_2(x=0)$

$C e^{-\beta \Delta x} = \frac{1}{e} \cdot C$

$\Delta x = \frac{1}{\beta} = \frac{\hbar}{\sqrt{2m(V_0 - E)}}$

## * Rectangular Barrier : Quantum Mechanical Tunneling




$U(x) = 0 \quad x < 0 \ , \ x > L$

$= V_0 \quad 0 \le x \le L$

$$\frac{d^2\psi_1}{dx^2} + \frac{2m(E-0)}{\hbar^2}\psi_1 = 0 \Rightarrow \frac{d^2\psi_1}{dx^2} + \alpha^2\psi_1 = 0$$

$$\frac{d^2\psi_2}{dx^2} - \frac{2m(V_0-E)}{\hbar^2}\psi_2 = 0 \Rightarrow \frac{d^2\psi_2}{dx^2} + \beta^2\psi_2 = 0$$

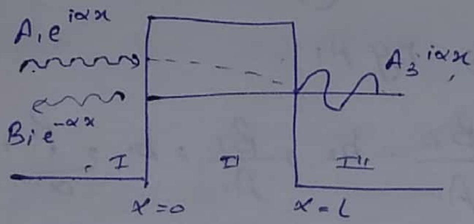
$$\frac{d^2\psi_3}{dx^2} + \frac{2m(E-0)}{\hbar^2}\psi_3 = 0 \Rightarrow \frac{d^2\psi_3}{dx^2} + \alpha^2\psi_3 = 0$$

$$\psi_1(x) = A_1 e^{i\alpha x} + B_1 e^{-\alpha x}$$

$$\psi_2(x) = A_2 e^{\beta x} + B_2^{-\beta x}$$

$$\psi_3(x) = A_3 e^{i\alpha x} + B_3 e^{-\alpha x} = A_3 e^{i\alpha x}$$

$B_3 e^{-\alpha x} = 0$ bcoz, in III region, there is no boundary




$$T = \frac{|A_3|^2}{|A_1|^2} \quad \text{(find out)}$$

Boundary con. $x = \pm\infty$, finite everywhere

at $x = 0$

$$\psi_1(x) = \psi_2(x) \qquad A_1 + B_1 = A_2 + B_2$$

$$\frac{d\psi_1}{dx} = \frac{d\psi_2}{dx} \qquad i\alpha(A_1 - B_1) = \beta(A_2 - B_2)$$

at $x = L$

$$\psi_2(x) = \psi_3(x)$$

$$\frac{d\psi_2}{dx} = \frac{d\psi_3}{dx}$$

at $x = 0$

$$\psi_1(x) = \psi_2(x)$$

$$\frac{d\psi_1}{dx} = \frac{d\psi_2}{dx}$$

$$\frac{d}{dx}\left(A_1 e^{i\alpha x} + B_1 e^{-\alpha i x}\right) = \frac{d}{dx}\left(A_2 e^{\beta x} + B_2 e^{-\beta x}\right)$$

$$A_1 e^{i\alpha x} \cdot i\alpha - B_1 e^{-\alpha i x} \cdot i\alpha = A_2 e^{\beta x} \cdot \beta - B_2 e^{-\beta x} \cdot \beta$$

$$i\alpha \left(A_1 e^{i\alpha x} - B_1 e^{-i\alpha x}\right) = \beta \left(A_2 e^{\beta x} - B_2 e^{-\beta x}\right)$$

$$\text{or}, \; i\alpha (A_1 - B_1) = \beta (A_2 - B_2) \quad \text{— (1) (Putting } x = 0)$$

at $x = L$

$$\frac{d\psi_2}{dx} = \frac{d\psi_3}{dx}$$

$$\frac{d}{dx}\left(A_2 e^{\beta x} + B_2 e^{-\beta x}\right) = \frac{d}{dx}\left(A_3 e^{i\alpha x}\right)$$

$$A_2 \cdot e^{\beta x} \cdot \beta - B_2 e^{-\beta x} \cdot \beta = A_3 \cdot e^{i\alpha x} \cdot i\alpha$$

$$\beta \left(A_2 e^{\beta x} - B_2 e^{-\beta x}\right) = A_3 \cdot e^{i\alpha x} \cdot i\alpha$$

$$\beta \left(A_2 e^{\beta L} - B_2 e^{-\beta L}\right) = i\alpha A_3 \cdot e^{i\alpha L} \quad \text{— (2) (Putting } x = L)$$

Dividing ~~by $A_1$~~ eq. (1) & (2) by $A_1$

$$\frac{A_2}{A_1} = a_2, \quad \frac{A_3}{A_1} = a_3, \quad \frac{B_2}{A_1} = b_2, \quad \frac{B_1}{A_1} = b_1, \quad \frac{\beta}{\alpha} = n$$

Putting values of $a_2, a_3, b_1, b_2$ & $n$ in eq. (1) & (2)

$$1 + b_1 = a_2 + b_2$$

$$1 - b_1 = in (a_2 - b_2) \quad \text{— (3)}$$

$$a_2 e^{\beta L} + b_2 \cdot e^{-\beta L} = a_3 e^{i\alpha L}$$

$$a_2 e^{\beta L} - b_2 e^{-\beta L} = \frac{i}{n} a_3 e^{i\alpha L} \quad \text{— (4)}$$

By solving (3) & (4)

$$a_3 = \frac{4ni e^{i\alpha L}}{(n+i)^2 e^{-\beta L} - (n-i)^2 \cdot e^{\beta L}}$$

$$\because \beta >> 1$$

$$\therefore e^{-\beta L} << e^{\beta L}$$

$$\therefore a_3 \doteq \frac{-4ni e^{i\alpha L}}{(n-i)^2} \cdot e^{-\beta L}$$

Scanned by CamScanner

$$T = a_3^* a_3 = |a_3|^2$$

$$a_3 = \frac{4ni e^{-i\alpha L} \cdot e^{-\beta L}}{(n+i)^2}$$

$$T = \frac{4ni e^{-i\alpha L} \cdot e^{-\beta L}}{(n+i)^2} \times \frac{-4ni e^{i\alpha L} \cdot e^{-\beta L}}{(n-i)^2}$$

$$T = \frac{16 n^2 e^{-2\beta L}}{(n^2+1)^2} \to \text{order of } 1$$

$$T \approx e^{-2\beta L}$$

$$\approx \exp\left[-2\left(\sqrt{\frac{2m(V_0-E)}{\hbar^2}}\right) L\right]$$

The wider the length, transmitting Probability is reduce.

## * Orthogonality and orthonormality conditions

Schrodinger eq. for free Particle

$$\frac{d^2\psi(x)}{dx^2} + \frac{2m}{\hbar^2}(E)\psi(x) = 0$$

$$\psi(x) = Ae^{ikx} + Be^{-ikx}$$

$$\int_{-\infty}^{+\infty} \psi_m^*(x,t)\,\psi_n(x,t)\,dx = 0 \quad \text{(orthogonality)}$$

$$\int_{-\infty}^{+\infty} \psi_n^*(x,t)\,\psi_n(x,t)\,dx = 1$$

$$\int_{-\infty}^{+\infty} \psi_m^*(x,t)\,\psi_m(x,t)\,dx = 1$$

[orthonormality]

## Orthogonality condition for wave function:-

if $\psi_m(x,t)$ and $\psi_n(x,t)$ represents two different Quantum States of a system then, the integral of the Product of wave fun. and complex conjugate of the other wave fun. over a common domain is equal zero.

$$\int_{-\infty}^{+\infty} \psi_m^* \psi_n = 0$$

## Orthogonality condition for Stationary State:

Or for Time dependent eq.

$$\psi_n(x,t) = \psi_n(x) \cdot e^{-iE_n t}$$

$$\psi_m(x,t) = \psi_m(x) e^{iE_m t}$$

$$\int_{-\infty}^{+\infty} \psi_m^*(x) e^{iE_m t} \psi_n(x) e^{\frac{-iE_n t}{\hbar}} dx = 0$$

$$e^{i\frac{(E_m - E_n)t}{\hbar}} \int_{-\infty}^{+\infty} \psi_m^*(x) \psi_n(x) dx = 0$$

as $E_m \neq E_n$

$$\therefore \int_{-\infty}^{+\infty} \psi_m^*(x) \psi_n(x) dx = 0$$

## Orthonomality condition

$$\int_{-\infty}^{+\infty} \psi_m^*(x,t) \psi_n(x,t) dx = \delta_{mn}$$

Kronecker delta fun. $\delta_{mn} = 0 \quad m \neq n$
$\quad = 1 \quad m = n$

Q Prove that $\int_{-\infty}^{+\infty} \psi_m^* \psi_n dx = 0$, where $\psi_m, \psi_n$ are the sol. of 1D Time independent schrodinger eq. for energy values $E_m$ & $E_n$. $E_m \neq E_n$

OR

Scanned by CamScanner

Q Prove the orthogonal properties of energy eign fun. for 1D case.

Sol Time indepented schrodinger eq.

$$i\hbar \frac{\delta \psi}{\delta t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + U\psi \qquad \left(\psi = \frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2}\right)$$

$$= -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + U(x)\psi$$

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi_m^*(x)}{\partial x^2} + U(x)\psi_m^*(x) = E_m \psi_m^*(x) \quad \text{— (1)}$$

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi_n(x)}{dx^2} + U(x)\psi_n(x) = E_n \psi_n(x) \quad \text{— (2)}$$

Multiply eq. (1) by $\psi_n$ & eq (2) by $\psi_m^*$

$$-\frac{\hbar^2}{2m} \psi_n \frac{\partial^2 \psi_m^*(x)}{\partial x^2} + U(x)\psi_m^*(x)\psi_n = E_m \psi_m^* \psi_n \quad \text{— (3)}$$

$$-\frac{\hbar^2}{2m} \psi_m^* \frac{\partial^2 \psi_n(x)}{\partial x^2} + U(x)\psi_n(x)\psi_m^* = E_n \psi_n(x)\psi_m^* \quad \text{— (4)}$$

Sub. (4) from (3)

$$\frac{\hbar^2}{2m}\left(\psi_m^* \frac{\partial^2 \psi_n}{\partial x^2} - \psi_n \frac{\partial^2 \psi_m^*}{\partial x^2}\right) = (E_m - E_n)\psi_m^* \psi_n$$

$$\psi_m^* \frac{\partial^2 \psi_n}{\partial x^2} - \psi_n \frac{\partial^2 \psi_m^*}{\partial x^2} = \frac{2m}{\hbar^2}(E_m - E_n)\psi_m^* \psi_n$$

$$\frac{\partial}{\partial x}\left[\psi_m^* \frac{\partial \psi_n}{\partial x} - \psi_n \frac{\partial \psi_m^*}{\partial x}\right] = \frac{2m}{\hbar^2}(E_m - E_n)\psi_m^* \psi_n$$

$$\int_{-\infty}^{+\infty} \frac{\partial}{\partial x}\left(\psi_m^* \frac{\partial \psi_n}{\partial x} - \psi_n \frac{\partial \psi_m^*}{\partial x}\right) dx = \frac{2m}{\hbar^2}(E_m - E_n)\int_{-\infty}^{+\infty} \psi_m^* \psi_n \, dx$$

$$\boxed{\left[\psi_m^* \frac{\partial \psi_n}{\partial x} - \psi_n \frac{\partial \psi_m}{\partial x}\right]_{-\infty}^{+\infty}} = \frac{2m}{\hbar^2}(E_m - E_n) \int_{-\infty}^{+\infty} \psi_m^* \psi_n \, dx$$

$$\underbrace{\phantom{xx}}_{= 0}$$

$$\underbrace{\frac{2m}{\hbar^2}(E_m - E_n)}_{\neq 0} \int_{-\infty}^{+\infty} \psi_m^* \psi_n \, dx = 0$$

bcoz $E_m \neq E_n$

$$\therefore \boxed{\int_{-\infty}^{+\infty} \psi_m^* \psi_n \, dx = 0}$$ Hence proved

Q If $\psi_1(x,t)$ & $\psi_2(x,t)$ are both sol. of the Schordinger wave eq. for a given Potential $V(x)$ Show that $\Psi(x,t) = a_1\psi_1(x,t) + a_2\psi_2(x,t)$

($\Psi \rightarrow$ Total wave fun.)

Sol
$$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi_1}{\partial x^2} + V(x)\psi_1(x,t) = i\hbar \frac{\partial \psi_1}{\partial t} \quad \times a_1 \psi_2$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi_2}{\partial x^2} + V(x)\psi_2 = i\hbar \frac{\partial \psi_2}{\partial t} \quad \times a_2 \psi_1$$

Super Position Principle says that the linear Combination/vector sum of all the wave fun. representing a system multiply by a Constant c will results in a wave fun. which also represent the same system.

$$\Psi(x) = \sum_i C_i \psi_i(x)$$

## Commulation operators & Hermition operators

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$ (Commulation operators)

$$\int_{-\infty}^{+\infty} f^*(\hat{A} g)\, dx = \int_{-\infty}^{+\infty} (\hat{A} f)^* g \, dx$$ (Hermition operators)

eg (1) $[\hat{x}, \hat{p}_x]\psi(x) = -i\hbar\left[\dfrac{x\,\partial\psi(x)}{\partial x} - \dfrac{\partial(x\psi(x))}{\partial x}\right]$

$[\hat{x}, \hat{p}_x]\psi(x) = -i\hbar\left[x\dfrac{\partial\psi}{\partial x} - \cancel{x\dfrac{\partial\psi}{\partial x}} - \psi(x)\right]$

$[\hat{x}, \hat{p}_x]\psi(x) = i\hbar\psi(x)$ $\boxed{\hat{p}_x = -i\hbar\dfrac{\partial}{\partial x}}$

$[\hat{x}, \hat{p}_x] = i\hbar$

$[\hat{p}_x, \hat{x}] = -i\hbar$

(2) $[\hat{y}, \hat{p}_x] = (y p_x - p_x y)$

$[\hat{y}, \hat{p}_x]\psi(x) = -i\hbar\left(y\dfrac{\partial\psi(x)}{\partial x} - \dfrac{\partial}{\partial x}(y\psi)\right)$

$[\hat{y}, \hat{p}_x]\psi(x) = -i\hbar\left[y\dfrac{\partial\psi(x)}{\partial x} - y\dfrac{\partial\psi}{\partial x} - 0\right]$

$[\hat{y}, \hat{p}_x] = 0$ (becoz, wave fun is depend on x not on y)

(3) $[\hat{t}, \hat{E}] = (tE - Et) = -i\hbar$ $\left(E = i\hbar\dfrac{\partial}{\partial t}\right)$

(4) $\left[\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial t}\right]$

Scanned by CamScanner

Commutation Rel's :-

(1) $[\hat{x}, \hat{p}_x] = i\hbar$

(2) $[\hat{t}, \hat{E}] = -i\hbar$ $\quad [\hat{t}, \hat{E}]\psi = [\hat{t}, i\hbar\frac{\partial}{\partial t}]\psi$

(3) $\left[\frac{\hat{\partial}}{\partial x}, \frac{\hat{\partial}}{\partial t}\right] = \left[\frac{\partial}{\partial t}\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\frac{\partial}{\partial x}\right]\psi = 0$

(4) $\left[\frac{\hat{\partial}}{\partial x}, \frac{\hat{\partial}^2}{\partial x^2}\right] = \left[\frac{\partial}{\partial x}\cdot\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial\cdot}\right]\psi = \left[\frac{\partial^3}{\partial} - \frac{\partial^3}{\partial}\right]\psi = 0$

(5) $[\hat{x}, [\hat{x}, \hat{H}]] = \left(\frac{-\hbar^2}{m}\right)$ $\quad , \hat{H} = \frac{\hat{p}^2_{x}}{2m} + \hat{V}(x)$

(6) $[\hat{x}, \hat{p}_{x^2}]$

$$[\hat{x}, \hat{H}] = x\cdot\frac{-i\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + \cancel{V\psi}x + \frac{i\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}\cancel{x} - \cancel{V\psi}x$$
$$= -\frac{i\hbar}{2m}$$

(7) $\left[\frac{\hat{\partial}}{\partial x}, \hat{f}(x)\right]$

# Properties of a linear operator :-

1. $\hat{A}\alpha g = \alpha\hat{A}g$ , $\alpha$ = Constant
   $g$ = function $g(x)$

2. $\hat{A}(f+g) = \hat{A}f + \hat{A}g$

3. $[\hat{A}+\hat{B}, \hat{C}] = [\hat{A}, \hat{C}] + [\hat{B}, \hat{C}]$

4. $[\hat{A}, \hat{B}+\hat{C}] = [\hat{A}, \hat{B}] + [\hat{A}, \hat{C}]$

5. $[\hat{A}\cdot\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$

6. $[\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}]$

7. $[\hat{A}, [\hat{B}, \hat{C}]] + [\hat{C}, [\hat{A}, \hat{B}]] + [\hat{B}, [\hat{C}, \hat{A}]] = 0$

Commutation rel's

(4) $[\hat{x}, \hat{p}_x^2] = [\hat{x}, \hat{p}_x \hat{p}_x] = [\hat{x}, \hat{p}_x]\hat{p}_x + \hat{p}_x[\hat{x}, \hat{p}_x]$

$x \hat{p}_x \hat{p}_x x$ $\quad = -i\hbar\left[x \cdot \frac{\partial \psi}{\partial x}(x) - \frac{\partial (x\psi)}{\partial x}\right]\hat{p}_x + \hat{p}_x\left[x \cdot \frac{\partial \psi}{\partial x}(x) - \frac{\partial (x\psi)}{\partial x}\right]$

$= -i\hbar\left[x \cdot \frac{\partial \psi}{\partial x} - x\frac{\partial \psi}{\partial x} - \psi\right]\hat{p}_x + \hat{p}_x\left[x \cdot \frac{\partial \psi}{\partial x} - \frac{x\partial \psi}{\partial x} - \psi\right]$

$= -i\hbar\left([-\psi]\hat{p}_x + \hat{p}_x[-\psi]\right)$

$[\hat{x}, \hat{p}_x^2]\psi = -i\hbar[$

(5) $[\hat{x}, [\hat{x}, \hat{H}]] + \hat{H}[\hat{x}, \hat{x}] + (\hat{x}, [\hat{H}, \hat{x}]) = 0$

$\hat{H} \leftarrow \psi^2$

$[\hat{x}, [\hat{x}, \hat{H} - \hat{H}\hat{x}]] + [$

$[\hat{x}, [\hat{x}, \hat{H} - \hat{H}\hat{x}]] = \hat{x}, x$

$[\hat{x}, \hat{H}] = \hat{x}\hat{H} - \hat{H}\hat{x}$

$= \hat{x}\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V\right)\psi - \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V\right)\hat{x}\psi$

$+V\psi - V\psi)$ $\quad = \hat{x}\left(-\frac{\hbar^2}{2m} \cdot \frac{\partial^2}{\partial x^2} + \psi\right)\psi - \left(-\frac{\hbar^2}{2m} \cdot \frac{\partial^2 \psi}{\partial x^2}\right)\hat{x} + \frac{\hbar^2}{2m}\frac{\partial^2 (x\psi)}{\partial x^2}$

$= -\hat{x}\frac{\hbar^2}{2m} \cdot \frac{\partial^2 \psi}{\partial x^2} + \frac{\hbar^2}{2m}\frac{\partial}{\partial x}\left[x\frac{\partial \psi}{\partial x} + \psi\right] \Rightarrow$

$= -x\frac{\hbar^2}{2m} \cdot \frac{\partial^2 \psi}{\partial x^2} + \frac{\hbar^2}{2m}\left[x\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial x}\right] \Rightarrow -x\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + \frac{x\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + \frac{\hbar^2}{m}\frac{\partial \psi}{\partial x}$

$= \hbar^2/m \cdot \frac{\partial \psi}{\partial x}$

Ques Find the condition in order that $\psi(x,t)$ be a simuntonous eigen function of the operators A & B. Calculate the product A & B ($\hat{A}\hat{B}\psi(x,t) - \hat{A}\hat{B}\psi = \alpha\beta$

$$\hat{A}\psi(x,t) = \alpha\psi(x,t)$$

$$\hat{B}\psi(x,t) = \beta\psi(x,t)$$

Calculate $\hat{B}\hat{A}\psi(x,t) = \beta\alpha\psi(x,t)$

$$(\hat{A}\hat{B} - \hat{B}\hat{A})\psi(x,t) = (\alpha\beta - \beta\alpha)\psi(x,t)$$

$$[\hat{A};\hat{B}] = 0$$

Nessasity condition so $\psi$ be a simuntonous function of both operator. If $\psi$ is a simuntonous function of two operators, then this condition should be nesscessry for it.

* Hermitian operators :-

$\hat{A}\hat{B} - \hat{B}\hat{A}$

$$[\hat{x},(\hat{x},\hat{H})] + [\hat{H}(\hat{x},\hat{x})] + [\hat{x},(\hat{H},\hat{x})] = 0$$

$$\Rightarrow \left[x\left(-\frac{\hat{x}\hbar^2}{2m}\cdot\frac{\partial^2\psi}{\partial x^2}\cdot\hat{x} + \frac{\hbar^2}{2m}\right]\right] + \left[\frac{\hat{p}^n}{2m} + \hat{v}\left(-\frac{\hat{x}\cdot\hbar^2}{2m}\cdot\frac{\partial^2\psi}{\partial x^2}\cdot\hat{x} + \frac{\hbar^2}{2m}\right)\right]$$

$$\Rightarrow \left[x\left(-\frac{x\cdot\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}\hat{x} + \frac{\hbar^2}{2m}\right) - \left(-\frac{\hat{x}\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}\hat{x} + \frac{\hbar^2}{2m}\right)x\right]$$

$$+\left[\frac{\hat{p}}{2m} + \hat{v}\left(-\hat{x}\cdot\frac{\hbar^2}{2m}\cdot\frac{\partial^2\psi}{\partial x^2}\cdot\hat{x} + \frac{\hbar^2}{2m}\right) - \left(-\hat{x}\cdot\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}\cdot\hat{x} + \frac{\hbar^2}{2m}\right)\left(\frac{p}{2m} + v\right)\right]$$

$$+\left[x\left(-\frac{x\cdot\hbar^2}{2m}\cdot\frac{\partial^2\psi}{\partial x^2}\hat{x} + \frac{\hbar^2}{2m}\right) - \left(+\frac{\hat{x}\cdot\hbar^2}{2m}\cdot\frac{\partial^2\psi}{\partial x^2}\hat{x} + \frac{\hbar^2}{2m}\right)x\right].$$

$$\Rightarrow -\frac{\hat{x}^3\hbar^2}{2m}\cdot\frac{\partial^2\psi}{\partial x^2} + \frac{\hat{x}^3\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + \frac{\hbar^2 x}{2m} +$$

$$[\hat{x},\hat{H}]\psi = -\frac{x\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + \frac{\hbar^2}{2m}\left(2\frac{\partial\psi}{\partial x} + \frac{\partial^2\psi}{\partial x^2}\right)$$

$$= -\frac{x\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + \frac{x\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + \frac{\hbar^2}{2m}\frac{\partial\psi}{\partial x}$$

$$\frac{x\hbar^2}{2m}\frac{\partial}{\partial x} + \frac{x\hbar^2}{}$$

$$[\hat{x},\hat{H}] = \frac{\hbar^2}{m}\frac{\partial}{\partial x}$$

$$\cdot\,[\hat{x},\hat{H}]$$

$$[\hat{x},[\hat{x},\hat{H}]]\psi = \left[\hat{x},\frac{\hbar^2}{m}\frac{\partial}{\partial x}\right]\psi$$

$$= x\left[\frac{\hbar^2}{m}\frac{\partial\psi}{\partial x}\right] - \frac{\hbar^2}{m}\frac{\partial}{\partial x}(x.\psi)$$

$$[\hat{x},[\hat{x},\hat{H}]]\psi = x\frac{\hbar^2}{m}\cancel{\frac{\partial\psi}{\partial x}} - x\frac{\hbar^2}{m}\cancel{\frac{\partial\psi}{\partial x}} - \frac{\hbar^2}{m}\psi = -\frac{\hbar^2}{m}\psi$$

$$[\hat{x},[\hat{x},\hat{H}]] = -\frac{\hbar^2}{m}$$

---

* Hermition operator.

→ The expectation of a dynamical quantity represented by Hermition operator is always real

→ The two eigen fun. of a Hermitian operator belonging to diff. eigen states are always orthogonal.

Condition for Hermitian operator $\hat{A}$

$$\int_{-\infty}^{+\infty} f^*(\hat{A}g)\,dx = \int_{-\infty}^{+\infty}(\hat{A}f)^*\,g\,dx$$

$$(f,\hat{A}g) = (\hat{A}f,g)$$

$\langle \hat{A} \rangle = \int_{-\infty}^{\infty} \psi^* \hat{A} \psi \, dx = (\psi ; \hat{A}\psi)$

$\hat{A}\psi(x) = \lambda \psi(x)$

$(\psi, \hat{A}\psi) = \lambda(\psi, \psi)$ —— ①

$(\hat{A}\psi, \psi) = \lambda^*(\psi, \psi)$ —— ②

$(\psi, \hat{A}\psi) = (\hat{A}\psi, \psi)$

$\int_{-\infty}^{\infty} \psi^* (\hat{A}, \psi) dx = \lambda \cdot \int_{-\infty}^{\infty} \psi^* \psi \, dx$

$\int_{-\infty}^{-\infty} (\hat{A}\psi)^* \psi \, dx = \lambda^* \int_{-\infty}^{\infty} \psi^* \psi \, dx$

① = ②

it means $\lambda = \lambda^*$

$\psi_m$ ~~end~~ and $\psi_n$

$\hat{A}\psi_m = \lambda_m \psi_m$

$\hat{A}\psi_n = \lambda_n \psi_n$

To Prove $\int_{-\infty}^{+\infty} \psi_m^* \psi_n \, dx = 0$

$(\psi_m \cdot \psi_n) = 0$

if $\hat{A}$ is hermitian then,

$(\psi_m, \hat{A}\psi_n) = (\hat{A}\psi_m, \psi_n)$ —— ①

$\lambda_m(\psi_m, \psi_n) = \lambda_m^*(\psi_m, \psi_n)$

∴ eigenvalues of Hermitian operator

∴ $\lambda_n(\psi_m, \psi_n) = \lambda_m(\psi_m, \psi_n)$

$$(\lambda_n - \lambda_m)(\psi_m, \psi_n) = 0$$

$\because \lambda_n \neq \lambda_m$

$\therefore (\psi_m, \psi_n) = 0$

Test ~~Ass.~~ Ques

Ques Show that the average motion of a wave packet satisfy the ~~term~~ condition ① $\frac{d}{dt}\langle x \rangle = \frac{\langle p_x \rangle}{m}$

② $\frac{d\langle p_x \rangle}{dt} = \langle f_x \rangle = -\left\langle \frac{dv}{dx} \right\rangle$

Ehrenfest Theorem

→ It State that the Quantum mechanics gives the same result as classical mechanics for motion of a wave packet associated with a moving particle if we use the expectation value or average value of dynamical ~~ident~~ quantities involved.

① $\frac{d\langle x \rangle}{dt} = \frac{d}{dt}\int_{-\infty}^{+\infty} \psi^*(x,t)\, \hat{x}\, \psi(x,t)\, dx$ — $\hat{x}$ independent of time

$$= \int_{-\infty}^{+\infty} \frac{\partial \psi^*}{\partial t} x \psi\, dx + \int_{-\infty}^{+\infty} \psi^* x \frac{\partial \psi}{\partial t}\, dx$$

TDSE $-\frac{\hbar^2}{2m}\frac{\partial^2 \psi(x,t)}{\partial x^2} + v(x)\psi(x,t) = i\hbar \frac{\partial \psi}{\partial t}(x,t)$

$$\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \cdot \frac{1}{i\hbar} \frac{\partial^2 \psi}{\partial x^2} - \frac{i}{\hbar} v \psi$$

# Classical Mechanics

Book → Gupta kumar Sharma
gold stream

* <u>Constrained Motion</u> : It is a motion which can't processed arbitrary in any manner.

* <u>Forces of Constraint</u> : with the strained by vertue of which they can restricte the motion of system of particle.

$$f = f(r_1, r_2, \ldots r_{3N}, t)$$

$$f = (x, y, z\ ;\ x_2\ y_2\ z_2\ ;\ \ldots\ ;\ x_N\ y_N\ z_N, t)$$

K constraints — have forces, dise, 3D

$$f_1 = f_1(x, y, z, \ldots z_N) = a_1$$

$$f_2 = f_2(x, y, z, \ldots z_n) = a_2$$

$$\vdots$$

$$f_K = f_K(x, y, z, \ldots z_n) = a_K$$

3N. K → Degree of freedom.

* <u>Degree of freedom</u> : It is the no. of independent ways in which the mechanical system can move without violting forces of constrained acting on the system

* **Generalised Coordinates :** The set of independent coordinate which describe the configration of the system.

$q_i$'s

$$r_i = r_i(q_1, q_2, \ldots q_{3N})$$

$$q_i = q_i(r_1, r_2, \ldots r_{3N})$$

## * Generalised coordinates

**Constraints :-**

① Holonomic $f = f(r_1, r_2, --- r_{3N}, t) = 0$ (simply the sys.)

② non-Holonomic $f(r_1, r_2, --- r_{3N}, t) \geq 0$

③ Unilateral in the form of inequalities $f(\vec{r}, \dot{\vec{r}}, t) \geq 0$

④ Bilateral in the form of equation $f(\vec{r}, \dot{\vec{r}}, t) = 0$

(if have displacement, Time derivatives velocity and then, it is unilateral or Bilateral constraints)

Kinemetic constraint

⑤ Differential or Kinematic constraints $f(\vec{r}, \dot{\vec{r}}, t) \geq 0$

⑥ Geometric Constraints $f(\vec{r}, t) = 0$

$f(\vec{r}) = 0$

⑦ Rheonomous constraints : explicit Time Dependence

⑧ Scleronomous " : No explicit Time "

eg → ① $r^2 - a^2 \geq 0$ — ②, ③, ⑥, ⑧ Type

→ ② simple Pendulum

$r^2 = x^2 + y^2 + z^2 = l^2 \rightarrow$ ②, ⑥, ⑧

→ ③ A spherical container of fixed radius r with a gas

The constraints eq in this case $|r_i|^2 \leq R^2$

Type of Constraints → ②, ⑥, ⑥

Generalised Displacement  Configurance ~~Phase~~ space
" velocity
" Acceleration
" K.E
" Potential
" Momentum

## Generalised Displacement

$\vec{r_i} = r_i(q_1, q_2, \ldots q_{3N}; t)$ (i → Real index, i → 1 to N)

$\delta\vec{r_i}$ = Arbitary displacement

we can consider a small displacement of N Particles $\delta\vec{r_i}$ = Arbitary displacement

$$\delta\vec{r_i} = \sum_{j=1}^{3N} \frac{\partial \vec{r_i}}{\partial q_j} \delta q_j$$

Euler's Theorem
$X(x_1, x_2, x_3)$
$dx = \frac{\partial x}{\partial x_1} dx_1 + \frac{\partial x}{\partial x_2} dx_2 + \frac{\partial x}{\partial x_3} dx_3$

$\partial q_j \rightarrow$ Arbitary generalised displacement

## Generalised velocity  $v = \frac{D}{T}$

$$\dot{\vec{r_i}} = \frac{d(\vec{r_i})}{dt} = \sum_j \frac{\partial \vec{r_i}}{\partial q_j} \dot{q_j} + \frac{\partial \vec{r_i}}{\partial t}$$

$$= \sum_j \frac{\partial \vec{r_i}}{\partial q_j} \dot{q_j} + 0$$

Time is taken 0, because b/w transiting stage all state are equilibrium so we assume that there is no time change

## Generalised Acceleration

a = velocity × Time

$$\ddot{\vec{r_i}} = \frac{d}{dt}(\dot{\vec{r_i}})$$

$$= \frac{d}{dt}\left[\sum_{j=1}^{3N} \frac{\partial \vec{r_i}}{\partial q_j} \cdot \dot{q_j} + \frac{\partial \vec{r_i}}{\partial t}\right]$$

$$= \sum_{j=1}^{3N}\left[\frac{d}{dt}\left(\frac{\partial \vec{r_i}}{\partial q_j}\right)\dot{q_j} + \frac{\partial \vec{r_i}}{\partial q_j}\frac{d}{dt}(\dot{q_j}) + \frac{\partial}{\partial t}\left(\frac{\partial \vec{r_i}}{\partial t}\right)\right]$$

$$\dot{\vec{r}}_i = \sum_j \left[ \frac{\partial \vec{r}_i}{\partial q_j} \cdot \dot{q}_j + \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \dot{\vec{r}}_i}{\partial t} \right] \quad \left[ \frac{\partial}{\partial t} \to \cdot \right]$$

$$\ddot{\vec{r}}_i = \sum_j \frac{\partial}{\partial q_j} \left( \frac{\partial \vec{r}_i}{\partial q_j} \cdot \dot{q}_j + \frac{\partial \vec{r}_i}{\partial t} \right) \dot{q}_j + \sum_j \frac{\partial r_i}{\partial q_j} \ddot{q}_j + \frac{\partial}{\partial t} \left[ \sum_{k=1}^{3N} \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \vec{r}_i}{\partial t} \right]$$

$$\ddot{\vec{r}}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j} \ddot{q}_j + \sum_k \sum_j \frac{\partial^2 r_i}{\partial q_j^2} \dot{q}_j \dot{q}_k + 2 \sum_j \frac{\partial^2 \vec{r}_i}{\partial q_j \partial t} \dot{q}_j + \frac{\partial^2 \vec{r}_i}{\partial t^2}$$

quadratic terms in generalised acceleration.

* <u>Generalised K.E.</u>

$$T = \frac{1}{2} \sum_{j=1}^{N} m_i (\dot{\vec{r}}_i)^2 = \frac{1}{2} \sum_{i=1}^{N} m_i (\dot{\vec{r}}_i \cdot \dot{\vec{r}}_i)$$

$$\dot{\vec{r}}_i \cdot \dot{\vec{r}}_i = \left[ \sum_{j=1}^{3N} \frac{\partial \vec{r}_i}{\partial q_j} \cdot \dot{q}_j + \frac{\partial \vec{r}_i}{\partial t} \right] \left[ \sum_{k=1}^{3N} \frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial r_i}{\partial t} \right]$$

$$= \sum_k \sum_j \frac{\partial \vec{r}_i}{\partial q_j} \cdot \frac{\partial \vec{r}_i}{\partial q_k} \cdot \dot{q}_j \dot{q}_k + 2 \sum_k \frac{\partial^2 r_i}{\partial t \, \partial q_k} \cdot \dot{q}_k + \left( \frac{\partial \vec{r}_i}{\partial t} \right)^2$$

$$T = \sum_{i=1}^{N} \left[ \frac{1}{2} \sum_{j,k} \left( \frac{\partial \vec{r}_i}{\partial q_j} \right) \left( \frac{\partial \vec{r}_i}{\partial q_k} \right) \dot{q}_j \dot{q}_k + \sum_k \frac{\partial^2 \vec{r}_i}{\partial t \, \partial q_k} \dot{q}_k + \frac{1}{2} \left( \frac{\partial \vec{r}_i}{\partial t} \right)^2 \right]$$

$$T^{(2)} + T^{(1)} + T^{(0)}$$

if there is No Time dependent

$$\therefore T = \sum_{i=1}^{N} \left[ \frac{1}{2} \sum_{j,k} \left( \frac{\partial \vec{r}_i}{\partial q_i} \right) \left( \frac{\partial \vec{r}_i}{\partial q_k} \right) \dot{q}_i \dot{q}_k \right]$$

* Generalised Kinetic Energy

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i (\dot{\vec{r}_i} \cdot \dot{\vec{r}_i}) = \sum_i m_i \left[ \frac{1}{2} \sum_{j,k} \left(\frac{\partial \vec{r}_i}{\partial q_j}\right)\left(\frac{\partial \vec{r}_i}{\partial q_k}\right) \dot{q}_j \dot{q}_k + \sum_j \frac{\partial \vec{r}_i}{\partial q_j \partial t} \dot{q}_j + \frac{1}{2}\left(\frac{\partial \vec{r}_i}{\partial t}\right)^2 \right]$$

Q ~~Ex~~ justify the statement generalised energy of a system is a quadric fun. of generalised velocity

$$\dot{\vec{r}_i} \cdot \dot{\vec{r}_i} = \left( \sum_{j=1}^{3N} \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \vec{r}_i}{\partial t} \right)\left( \sum_{k=1}^{3N} \frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \vec{r}_i}{\partial t} \right)$$

$$T = T^{(2)} + T^{(1)} + T^{(0)}$$

if no Explicit Time dependence $\quad T = \sum_{j,k} a_{jk} \dot{q}_j \dot{q}_k \;, \quad a_{jk} = \sum_i \frac{1}{2} m_i \left(\frac{\partial \vec{r}_i}{\partial q_j}\right)\left(\frac{\partial \vec{r}_i}{\partial q_k}\right)$

$$\left(\frac{\partial \vec{r}_i}{\partial q_j}\right) \cdot \left(\frac{\partial \vec{r}_i}{\partial q_k}\right) = 0 \quad (j \neq k)$$

Then it is orthogonal System.

---

lang Circular

q r, θ

$$\vec{r} = x\hat{i} + y\hat{j}$$

$x = r\cos\theta \Rightarrow \frac{\partial x}{\partial r} = \cos\theta$ | $\frac{\partial x}{\partial \theta} = -r\sin\theta$

$y = r\sin\theta \rightarrow \frac{\partial y}{\partial r} = \sin\theta$ | $\frac{\partial y}{\partial \theta} = r\cos\theta$

$$T = \frac{1}{2} m (\dot{\vec{r}} \cdot \dot{\vec{r}})^2$$

$$= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$T = \sum_{i,j,k} \frac{1}{2} m_i \left(\frac{\partial \vec{r_i}}{\partial q_j}\right)\cdot\left(\frac{\partial \vec{r_i}}{\partial q_k}\right) \dot{q}_j \dot{q}_k$$

$$T = \frac{1}{2} m \sum_{j=1}^{2} \sum_{k=1}^{2} \left(\frac{\partial \vec{r_i}}{\partial q_j}\right)\left(\frac{\partial \vec{r_i}}{\partial q_k}\right) \dot{q}_j \dot{q}_k$$

$$= \frac{1}{2} m \left[\left(\frac{\partial \vec{r_i}}{\partial r}\right)\cdot\left(\frac{\partial \vec{r_i}}{\partial r}\right) \dot{r}\dot{r} + \left(\frac{\partial r_i}{\partial \theta}\right)\left(\frac{\partial r_i}{\partial \theta}\right)\dot{\theta}\dot{\theta} + 2\left(\frac{\partial \vec{r_i}}{\partial r}\right)\cdot\left(\frac{\partial \vec{r_i}}{\partial \theta}\right)\dot{r}\cdot\dot{\theta}\right]$$

$=$

1st term $= \left(\frac{\partial \vec{r_i}}{\partial r}\right)\cdot\left(\frac{\partial \vec{r_i}}{\partial r}\right)\dot{r}\dot{r}$

$$= \left(\frac{\partial(x\hat{i}+y\hat{j})}{\partial r}\right)\left(\frac{\partial(x\hat{i}+y\hat{j})}{\partial r}\right)\dot{r}\dot{r}$$

$$= \left(\frac{\partial x}{\partial r}\hat{i} + \frac{\partial y}{\partial r}\hat{j}\right)\left(\frac{\partial x}{\partial r}\hat{i} + \frac{\partial y}{\partial r}j\right)\dot{r}\dot{r}$$

$$= (\cos^2\theta + \sin^2\theta)\dot{r}^2$$

$$= \dot{r}^2$$

2nd term $= \left(\frac{\partial r_i}{\partial \theta}\right)\cdot\left(\frac{\partial r_i}{\partial \theta}\right)\dot{\theta}\dot{\theta}$

$$= \left(\frac{\partial(x\hat{i}+y\hat{j})}{\partial \theta}\right)\cdot\left(\frac{\partial(x\hat{i}+y\hat{j})}{\partial \theta}\right)\dot{\theta}\dot{\theta}$$

$$= \left(\frac{\partial x}{\partial \theta}\hat{i} + \frac{\partial y}{\partial \theta}\hat{j}\right)\cdot\left(\frac{\partial x}{\partial \theta}\hat{i} + \frac{\partial y}{\partial \theta}\hat{j}\right)\dot{\theta}\dot{\theta}$$

$$= \cancel{r^2\sin^2\theta + r^2\cos\theta}$$

$$= (-r\sin\theta + r\cos\theta)\cdot(-r\sin\theta + r\cos\theta)\dot{\theta}^2$$

$= (r^2\sin^2\theta - r^2\sin\theta\cos\theta - r^2\sin\theta\cos\theta + r^2\cos\theta)\dot\theta^2$

$= (r^2 - 2r^2\sin\cos\theta)\dot\theta^2 = r^2.\theta^2$

$(\cos^2\theta + \sin^2\theta)\dot r^2 + (-r\sin\theta + r\cos^2\theta)\theta^2$, $r^2\theta^2$

3rd Term : same as 1 (Also $r^2\theta^2$)

$$\boxed{T = \frac{1}{2}m(\dot r^2 + r^2.\dot\theta^2)}$$

## * Generalised Momentum :-

$$p_{x_i} = \frac{\partial T}{\partial x_i} \qquad (P = \text{mass} \times \text{velocity})$$

$mgh$

$m \cdot \rho \times d \times h$

$$T = \frac{1}{2} m_i \dot x_i^2$$

$$p_{x_i} = \frac{\partial T}{\partial \dot x_i} = m_i \dot x_i$$

$$p_k = \frac{\partial T}{\partial \dot q_k}$$

$$p_j = \sum_{i,j,k} \frac{1}{2} m_i \left(\frac{\partial \vec{r_i}}{\partial q_j}\right)\cdot\left(\frac{\partial \vec{r_i}}{\partial q_k}\right)\dot q_k + \sum_i m_i\left(\frac{\partial r_i}{\partial t}\right)$$

## * Generalised force :-

if $\vec{F_i}$ is the external force acting on the system and causes a arbitary displacement $\partial\vec{r_i}$ of a system then acc. to work energy theorem, the work done by the system will be equal to

$$\delta W = \sum_{i=1}^{N} \vec{F_i}\cdot\delta\vec{r_i}$$

$$= \sum_{i=1}^{N} \vec{F_i}\cdot\sum_j \frac{\partial\vec{r_i}}{\partial q_j}\delta q_j$$

$$= \sum_{j=1}^{3N} Q_j\cdot\delta q_j \left(\text{where, } Q_j = \sum_{i=1}^{N}\vec{F_i}\frac{\partial\vec{r_i}}{\partial q_j}\right)$$

Que What will be the expression for generalised force in circular polar combination.

$$\vec{F}_i = -\vec{\nabla}_i v_i \qquad v_i = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$$

$$Q_j = \sum_{i=1}^{N} -\nabla_i v_i \left(\frac{\partial \vec{r}_i}{\partial q_j}\right)$$

$$\partial\vec{r}_i = \partial x_i \hat{i} + \partial y_i \hat{j} + \partial z_i \hat{k}$$

$$Q_j = -\frac{\partial v}{\partial q_j}$$

## D'Alembert's Principle

→ Virtual displacement ($\delta\vec{r}_i$)
→ work done by forces of constraints ($f_i$)
→ Work done in equil. on $i^{th}$ Particle ($\delta W$) in static & Dynamic equilibrium.

Virtual Displacement :-
it is an imaginary displacement which a system of $L$ Particle undergoes in configurence space consisting with the forces of constraint acting on a system at any instant (Without changing the time or time is (or fixed).

$$\delta W = \sum_i \vec{F_i} \, \delta \vec{r_i} = 0$$

$\vec{F_i} = \vec{F_i}^{(a)}$ external app. forces (– frictional forces)

$+$

$f_i$ forces of constraint

virtual displacement at fixed time

$$\delta W = \sum_i \vec{F_i}^{(a)} \cdot \delta \vec{r_i} + \underbrace{\sum_i f_i \cdot \delta \vec{r_i}}_{0} = 0$$

The Virtual work done by the applied forces ($F_i^{(a)}$) acting on the system consisting under the condition of static equilibrium will be zero provided no ~~frequent for~~ frictional forces are present

$$\vec{F_i}^{(a)} = \dot{\vec{p_i}}$$

$$\Rightarrow \vec{F_i}^{(a)} - \underbrace{\dot{\vec{p_i}}}_{\text{effective force or Kinetic Reaction}} = 0$$

$$\boxed{\delta W = \sum_i \left( F_i^{(a)} - \dot{\vec{p_i}} \right) \delta \vec{r_i} = 0}$$

D'Alembert's No. in dynamic equilibrium

## * Lagrange's Equation by D'Alembert's Principle

$$\delta W = \sum_i (F_i^{(a)} - \dot{\vec{p}}_i)\, \delta \vec{r}_i = 0$$ Total work done

$$= \sum_i (F_i^{(a)} - \dot{\vec{p}}_i) \sum_j \frac{\partial \vec{r}_i}{\partial q_j} . \delta q_j = 0 \quad \text{--- (1)}$$

1st term $= \sum_i F_i^{(a)} \sum_j \frac{\partial \vec{r}_i}{\partial q_j} . \delta q_j = 0$

$$= \sum_j Q_j \, \delta q_j \quad \text{--- (A)}$$

2nd term $= \sum_i \dot{\vec{p}}_i \sum_j \frac{\partial \vec{r}_i}{\partial q_j} . \delta q_j$

$$= \sum_i m_i \ddot{\vec{r}}_i \sum_j \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j$$

(Rewrite) $\rightarrow$ $= \sum_i \sum_j m_i \frac{d}{dt}\left[\dot{\vec{r}}_i \frac{\partial \vec{r}_i}{\partial q_j}\right] \delta q_j - \sum_{i,j} m_i \dot{\vec{r}}_i \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right) \delta q_j$

$$= \sum_i \sum_j m_i \frac{d}{dt}\left[\dot{\vec{r}}_i \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j}\right] \delta q_j - \sum_{i,j} m_i \dot{\vec{r}}_i \frac{\partial \dot{\vec{r}}_i}{\partial q_j} \delta q_j$$

$$= \sum_i \sum_j \left[ m_i \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2}\dot{\vec{r}}_i . \dot{\vec{r}}_i\right)\right] - m_i \frac{\partial}{\partial q_j}\left(\frac{1}{2}\dot{\vec{r}}_i . \dot{\vec{r}}_i\right)\right] \delta q_j$$

$$\left( \frac{\partial}{\partial q}(\dot{\vec{r}}_i . \dot{\vec{r}}_i) = \frac{\partial \dot{\vec{r}}_i}{\partial q_j} . \dot{\vec{r}}_i + \dot{\vec{r}}_i . \frac{\partial \dot{\vec{r}}_i}{\partial q_j} = 2\left(\frac{\partial \dot{\vec{r}}_i}{\partial q_j} . \dot{\vec{r}}_i\right) \right)$$

$$= \sum_j \left[\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j} T\right) - \frac{\partial}{\partial q_j} T\right] \delta q_j \quad \text{--- (B)}$$

Put the value eq. (A) & (B) in eq (1)

Scanned by CamScanner

$$\delta W = \sum_j F_i^{(a)} \frac{\partial \vec{r_i}}{\partial q_j} \cdot \delta q_j = \dot{p}_i \sum_j \frac{\partial \vec{r_i}}{\partial q_j} \cdot \delta q_j$$

$$\Rightarrow Q_j \, \delta q_j = \left[ \frac{d}{dt}\left( \frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right] \delta q_j$$

$$\Rightarrow Q_j = \frac{d}{dt}\left( \frac{dT}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} \right) \quad \cancel{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0} \quad L = T - V$$

→ Calculate lagrange's eq. for conservative system

$$Q_j = -\frac{\partial V}{\partial q_j}$$

$$\frac{d}{dt}\left( \frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0$$

$$L = T - V$$

$$\frac{\partial}{\partial t}\left( \frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

$$\frac{\partial}{\partial t}\left( \frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = 0$$

$$\frac{d}{dt}\left( \frac{\partial (T - V)}{\partial \dot{q}_j} \right) - \frac{\partial}{\partial q_j}(T - V) = 0 \qquad \left( \because \frac{\partial V}{\partial \dot{q}_j} = 0 \text{ in conservative forces} \right)$$

$$\boxed{\frac{d}{dt}\left( \frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0}$$

→ for Non conservative system

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left( \frac{\partial U}{\partial \dot{q}_j} \right)$$

$$\frac{d}{dt}\left( \frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} + \frac{\partial U}{\partial q_j} - \frac{d}{dt}\left( \frac{\partial U}{\partial \dot{q}_j} \right) = 0$$

$$\frac{d}{dt}\left( \frac{\partial (T - U)}{\partial \dot{q}_j} \right) - \frac{\partial (T - U)}{\partial q_j} = 0$$

$$\boxed{\frac{d}{dt}\left( \frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0}$$

* Lagrangian formulation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j$$

$9 \times 3$

$27$

$(3\sqrt{3})^2$

Conservation forces $\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$

$(L = T - U)$

① Particle moving in space ; $\vec{F} = f_x \hat{i} + f_y \hat{j} + f_z \hat{k}$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

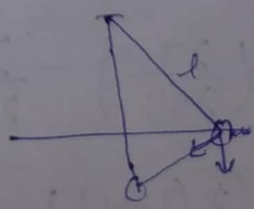
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} = f_x \Rightarrow m\ddot{x} \Rightarrow f_x = \frac{dp_x}{dt}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) - \frac{\partial T}{\partial y} = f_y \Rightarrow m\ddot{y} \Rightarrow f_y = \frac{dp_y}{dt}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}}\right) - \frac{\partial T}{\partial z} = f_z \Rightarrow m\ddot{z} \Rightarrow f_z = \frac{dp_z}{dt}$$

② Simple Pendulum

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j$$




$$\frac{d}{dt} \quad \frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2} m l^2 2\dot{\theta} = ml^2\dot{\theta}$$

$$\frac{\partial L}{\partial \theta} = mgl\sin\theta \qquad T = \frac{1}{2} m (l\dot{\theta})^2$$

$$U = mgl(1 - \cos\theta)$$

$$L = T - U \Rightarrow L = \frac{1}{2} ml^2\dot{\theta}^2 - mgl(1 - \cos\theta)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = 0 \Rightarrow \frac{d}{dt}(ml^2\dot\theta) + mgl\sin\theta \Rightarrow ml^2\ddot\theta + mgl\sin\theta = 0$$

$$\Rightarrow ml^2\ddot\theta = mgl\sin\theta$$

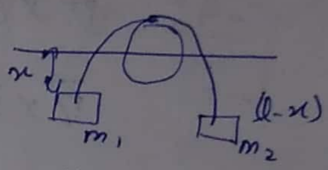
$$\ddot\theta + \left(\frac{g}{l}\right)\theta = 0 \quad \frac{g}{l} \to \omega^2$$

$$\ddot\theta = \frac{g}{l}\theta \quad (\sin\theta = \theta \text{ for small } \theta)$$

$$\ddot\theta + \frac{g}{l}\theta = 0$$

Q ~~consider~~ a Consider a frictional pulley device.

Q Derive the eq. of sys. using Lagrangian formulation




Q A Particle of mass m moves on a Plane in the given forces by $\vec{F} = -\hat{r}.rk\cos\theta$, where k is a Constant and r is the radian unit vector. obtain the differential eq. of the orbit of the Particle.

Q A beam slide on the wire in the shape of cycloid describe by the eq's $x = a(\theta - \sin\theta)$, $y = a(1+\cos\theta)$, $0 \le \theta \le 2\pi$. Find the Lagrangian fun. of fun. (l) & find

Q Find the lagrangian eq. for the electric circuit with inductance l and capistance c. The Conductor is charge to q Adlumns & the ~~circuit~~ current following in the circuit is 2 Amh.

$$ME = \frac{1}{2}Li^2 = \frac{1}{2}L\dot q^2$$

$$EE = \frac{1}{2}\frac{q^2}{C}$$

$$L = T - V \Rightarrow L = ME - EE$$

## * Lagrangian of a charged Particle in Electromagnetic field :-

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$ (Lorentz force eq)

$$\vec{\nabla} \cdot \vec{E} = \rho / \epsilon_0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \mu_0 \vec{J} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0 = B = -(\vec{\nabla} \times \vec{A}) \quad \text{——— (A)}$$

where A → Magnetic vector Potential

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

rewrite, $\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{A})$

$$\vec{\nabla} \times \vec{E} = -\vec{\nabla} \times \left(\frac{\partial \vec{A}}{\partial t}\right)$$

$$\Rightarrow \vec{\nabla} \times \left[E + \frac{\partial \vec{A}}{\partial t}\right] = 0$$

$$\therefore \vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}\phi$$

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \quad \text{——— (B)}$$

Put (A) & (B) in Lorentz eq force eq.

$$\therefore \vec{F} = q\left[-\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} + \vec{v} \times (\vec{\nabla} \times \vec{A})\right]$$

x component of this eq.

$$F_x = q\left[-\frac{\partial \phi}{\partial x} - \frac{\partial A_x}{\partial t} + \vec{v}\times(\vec{\nabla}\times\vec{A})|_x\right]$$

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right] - \hat{j}\left[\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right] + \hat{k}\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]$$

$$(\vec{v}\times\vec{\nabla}\times\vec{A})_x = \begin{pmatrix} i & j & k \\ v_x & v_y & v_z \\ \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) & \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) & \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \end{pmatrix}$$

$$= \hat{i}\left[v_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - v_z\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right]$$

$$\Rightarrow v_y\frac{\partial A_y}{\partial x} + v_z\frac{\partial A_z}{\partial x} - v_y\frac{\partial A_x}{\partial y} - v_z\frac{\partial A_z}{\partial z}$$

add and subtract ; $v_x\frac{\partial A_x}{\partial x}$

$$\vec{v}\times(\vec{\nabla}\times\vec{A})|_x = \underbrace{v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_y}{\partial x} + v_z\frac{\partial A_z}{\partial x}}_{\frac{\partial}{\partial x}(\vec{v}\cdot\vec{A})} - v_x\frac{\partial A_x}{\partial x} - v_y\frac{\partial A_x}{\partial y} - v_z\frac{\partial A_x}{\partial z}$$

$$dA_x = \frac{\partial A_x}{\partial x}dx + \frac{\partial A_x}{\partial y}dy + \frac{\partial A_x}{\partial z}dz + \frac{\partial A_x}{\partial t}dt$$

$$\frac{dA_x}{dt} = \frac{\partial A_x}{dt} + \left[\frac{\partial A_x}{\partial x}v_x + \frac{\partial A_x}{\partial y}v_y + \frac{\partial A_x}{\partial z}v_z\right]$$

$$\frac{dA_x}{dt} - \frac{\partial A_x}{dt} = \frac{\partial A_x}{\partial x}v_x + \frac{\partial A_x}{\partial y}v_y + \frac{\partial A_x}{\partial z}v_z.$$

$$F_x = q\left[-\frac{\partial \phi}{\partial x} - \frac{\partial A_x}{\partial t} + \frac{\partial}{\partial x}(\vec{v}.\vec{A}) - \frac{dA_x}{dt} + \frac{\partial A_x}{\partial t}\right]$$

$$= q\left[-\frac{\partial \phi}{\partial x} + \frac{\partial}{\partial x}(\vec{v}.\vec{A}) - \frac{dA_x}{dt}\right]$$

$$= q\left[-\frac{\partial(\phi - \vec{v}.\vec{A})}{\partial x} - \frac{\partial A_x}{\partial t}\right]$$

where,

$$= q\left[-\frac{\partial}{\partial x}(\phi - \vec{v}.\vec{A}) - \frac{d}{dt}\left(\frac{\partial(\vec{v}.\vec{A})}{\partial v_x}\right)\right], \left(A_x = \frac{\partial(\vec{v}.\vec{A})}{\partial v_x}\right)$$

if $\frac{d}{dt}\left(\frac{\partial \phi}{\partial v_x}\right) = 0$

$$= q\left[-\frac{\partial}{\partial x}(\phi - \vec{v}.\vec{A}) + \frac{d}{dt}\left(\frac{\partial}{\partial v_x}(\phi - \vec{v}.\vec{A})\right)\right]$$

Put $q(\phi - \vec{v}.\vec{A}) = u$

$$F_x = -\frac{\partial u}{\partial x} + \frac{d}{dt}\left(\frac{\partial u}{\partial x}\right)$$

write this eq. in the form of lagrangion eq

$$\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q}\right] = Q_j = \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right) - \frac{\partial U}{\partial q_j}$$

~~rewrite this,~~ if there is non conservation ; then it is equal $Q_j$ , otherwise it is equal 0.

rewrite above eq.

$$\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j}\right] = Q_j$$

$$L = T - V$$

$$L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - q(\phi - \vec{v}\cdot\vec{A})$$

For friction force,

$$F_{fx} = -k_x v_x$$

$$\vec{F}_f = -[k_x v_x \hat{i} + k_y v_y \hat{j} + k_z v_z \hat{k}]$$

* Rayleigh's dissipation function ($F$)

$$F = \frac{1}{2} \sum_i [k_x v_{x_i} + k_y v_{y_i} + k_z v_{z_i}]$$

$$\vec{F}_f = -\vec{\nabla}_v \cdot F$$

$\nabla_v$ = velocity dependent differential

$$= \hat{i}\frac{\partial}{\partial v_x} + \hat{j}\frac{\partial}{\partial v_y} + \hat{k}\frac{\partial}{\partial v_z}$$

work energy done $dW = \vec{F}_f \cdot dr$

$$= \vec{F}_f \cdot \vec{v}\,dt$$

$$= -(k_x v_x^2 + k_y v_y^2 + k_z v_z^2)\,dt$$

$$dW = 2F$$

The work done by the non conservative frictional force is twice of Rayleigh's dissipation function

## generalized force for Rayleigh's eq.

$$Q_j = \sum_i \vec{F_i} \frac{\partial \vec{r_i}}{\partial q_j}$$

$$Q_j = -\sum_i \vec{\nabla}_v F \frac{\partial \dot{r}_i}{\partial \dot{q}_j}$$

$$= - \frac{\partial F}{\partial \dot{q}_j}$$

$$\left( \frac{\partial \vec{r_i}}{\partial q_j} = \frac{\partial \dot{\vec{r_i}}}{\partial \dot{q}_j} \right)$$

Cyclic → if coordinate is not Present in lagran eq.

$$\left[ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} \right] = Q_j$$

if $q_j$ is absent

$$\frac{\partial L}{\partial q_j} = 0$$

$$\therefore \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0 \Rightarrow \frac{\partial L}{\partial \dot{q}_j} = \text{Constant} = p_j$$

$$\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i} - \frac{\partial V}{\partial \dot{x}_i}$$

$$T = \frac{1}{2} m \dot{x}^2$$

Scanned by CamScanner

## * Conversation laws :-

① Conservation of linear momentum

② Conservation of Angular Momentum

③ Conservation of Energy

① if a coordinate corresponding to a displacement is cyclic, then the translation of the sys. has no effect i.e., the discription of motion of such system remains ~~if the time does not~~ invarient and the corresponding momentum is conserved.

if $q_j$ is cyclic

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

$$L = T - V$$

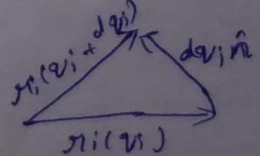
as $T \neq T(q_j)$

$V \neq V(\dot{q}_j)$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) + \frac{\partial V}{\partial q_j} = 0$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) = -\frac{\partial V}{\partial q_j} = Q_j} \text{ — (A)} \quad \left(Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right)$$

if $Q_j$ is in 1 dir$^n$ (x dir$^n$)




$$\partial \vec{r}_i = \partial q_j \hat{n}$$

$$\frac{\partial \vec{r}_i}{\partial q_j} = \hat{n}$$

$$\therefore \; Q_j = \sum_i \vec{F}_i \cdot \hat{n} \quad \text{—— ①}$$

$$T = \sum_i \frac{1}{2} m_i \dot{\vec{r_i}}^2$$

$$\frac{\partial T}{\partial \dot{q}_j} = m_i \dot{\vec{r_i}} \cdot \frac{\partial \vec{r_i}}{\partial q_j}$$

$$= m_i \dot{\vec{r_i}} \cdot \hat{n}$$

$$\frac{\partial T}{\partial \dot{q}_j} = p_j \cdot \hat{n} \quad — (2)$$

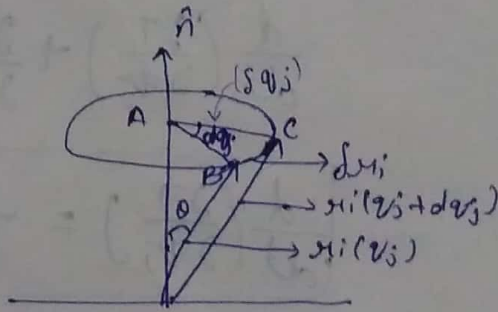
Put eq. (1) & (2) in eq. (A)

$$\frac{d}{dt}(p_j) = -\frac{\partial V}{\partial q_j} = Q_j = \vec{F_i} \cdot \hat{n}$$

$q_j$ cyclic, $\frac{\partial V}{\partial q_j} = 0$ or $\vec{F_i} \cdot \hat{n} = 0$ or $p_j = \text{const.}$

Case II

(2) if the system is not in the dir^n of the generalized force.




$$\delta \vec{r_i} = AB \, dq_j$$

$$\delta \vec{r_i} = r_i \sin\theta \, \delta q_j$$

$$\frac{\delta \vec{r_i}}{\delta q_j} = \vec{r_i} \times \hat{n}$$

$$T = \sum_i \frac{1}{2} m_i \dot{\vec{r_i}}^2$$

$$\frac{\partial T}{\partial \dot{q}_j} = m_i \dot{\vec{r_i}} \frac{\partial \vec{r_i}}{\partial q_j}$$

$$\frac{\partial T}{\partial \dot{q}_j} = m_i \dot{\vec{r_i}} \cdot \vec{r_i} \times \hat{n}$$

$$\frac{\partial T}{\partial \dot{q}_j} = m_i \dot{\vec{r_i}} \cdot (\vec{r_i} \times \hat{n}) \Rightarrow p_j \cdot \vec{r_i} \times \hat{n}$$

$$Q_j = \sum_i \vec{F_i} \cdot \frac{\partial \vec{r_i}}{\partial q_j}$$

$$Q_j = \sum_i \vec{F_i} \cdot (\vec{r_i} \times \hat{n})$$

$$Q_j = \hat{n} \cdot (\vec{r_i} \times \vec{F_i})$$

($\vec{r_i} \times \vec{F_i}$) represents Torque acting on a sys.

$$Q_j = \hat{n} \cdot \vec{N_i}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) = \frac{d}{dt}(p_j \cdot \vec{r_i} \times \hat{n})$$

$$= \frac{d}{dt}(\hat{n} \cdot \vec{r_i} \times \vec{p_i})$$

$$= \frac{d}{dt}(\hat{n} \cdot \vec{L})$$

($\vec{r_i} \times \vec{p_i}$) represents Angular momentum associated with Particle)

if $q_j$ is cyclic

$$\hat{n} \cdot \vec{L} = \text{Const.}$$

if a coordinate corresponding to a rotational is cyclic, then the rotational of the sys. has no effect i.e., discription of motion of such system remains invarient and corresponding angular momentum is constant.

lagrangian for central force

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2)$$

$$V = V(r)$$

$$L = T - V$$

$$= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - V$$

$\theta$ = cyclic (angular mom. in dir. of $\theta$ is conserved)

Conservation of energy

(1) The lagrange of a sys. is given by $T - V$ where, $T$ is a fun. of $\dot{q}_j$ and $V$ is a fun of $(q_j)$ — $T = T(\dot{q}_j)$, $V = V(q_j)$

(2) The constraints of a sys. does not change with time, i.e., the transformation eq. does not depend upon the time explicity.

$$L(\dot{q}_j, q_j) = T(\dot{q}_j) - V(q_j)$$

$$\frac{dL}{dt} = \frac{\partial L}{\partial \dot{q}_j} \frac{d\dot{q}_j}{dt} + \frac{\partial L}{\partial q_j} \frac{dq_j}{dt}$$

Lagrangian eq. for motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial q_j}$$

$$\frac{dL}{dt} = \frac{\partial L}{\partial q_j}\frac{dq_j}{dt} + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\frac{dq_j}{dt} \qquad \left(\frac{\partial L}{\partial \dot{q}_j} = p_j\right)$$

$$= \frac{d}{dt}\left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right)$$

$$\frac{dL}{dt} = \frac{d}{dt}(\dot{q}_j p_j)$$

$$\frac{d}{dt}(L - \dot{q}_j p_j) = 0$$

$$\frac{d}{dt}(T - V - \dot{q}_j p_j) = 0$$

Euler's theorem state that there is function $f$ which is a homogenous fun. of order $n$ depending upon variable $q_j$, then Euler's theorem states

$$\sum_j q_j \frac{\partial f}{\partial q_j} = nf$$

$$T = \sum_{jk} a_{jk}\dot{q}_j \dot{q}_k$$

$$\sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = nT$$

$$\sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = 2T$$

$$\sum_j \dot{q}_j p_j = 2T$$

$$\therefore \frac{d}{dt}(T - V - \dot{q}_j - p_j) = 0$$

$$\frac{d}{dt}(T - V - 2T) = 0$$

$T + V$ = Const = Hamiltonian $(H)$

if coordinates corresponding to time is cyclic,

$$H = (L - \dot{q}_j p_j) = T + V$$

$$H(q_j, p_j, t) = L(q_j, \dot{q}_j) - \dot{q}_j p_j$$

Phase space (6-D)

Hamiltonian eq.

Advantage

(1)

(2) $O_j$ is 1st diff. order.

Scanned by CamScanner

Hamiltonian formulation :- Hamilton's EOM

$$H(q_j, p_j, t) = \sum_j \dot{q}_j p_j - L(q_j, \dot{q}_j, t)$$

$$dH = \frac{\partial H}{\partial q_j} dq_j + \frac{\partial H}{\partial p_j} dp_j + \frac{\partial H}{\partial t} dt$$

$$dL = \frac{\partial L}{\partial \dot{q}_j} dq_j + \frac{\partial L}{\partial q_j} d\dot{q}_j + \frac{\partial L}{\partial t} dt$$

$$d \sum_j \dot{q}_j p_j = \sum_j d\dot{q}_j p_j + \sum_j \dot{q}_j dp_j$$

$$\left( \frac{\partial L}{\partial \dot{q}_j} = p_j \ , \frac{\partial L}{\partial q_j} \cdot \dot{p}_j \right)$$

$$\frac{\partial H}{\partial q_j} = -\dot{p}_j \ , \quad \frac{\partial H}{\partial p_j} = \dot{q}_j \ , \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

Cylinderical coordinates

$(r, \phi, z)$

$x = r\cos\phi$

$y = r\sin\phi$

$z = z$

$L = T - V$

$= \frac{1}{2} m(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2) - V$

momentum ass. with $(r, \phi, z)$

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \qquad p_r = \frac{\partial L}{\partial \dot{r}}$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \quad \left[\dot{r} = \frac{p_r}{m} \Rightarrow \dot{r}^2 = \frac{p_r^2}{m^2}\right]$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi} \quad \left[r^2\dot{\phi} = \frac{p_\phi}{m} \Rightarrow r^4\dot{\phi}^2 = \frac{p_\phi^2}{m^2} \Rightarrow r^2\dot{\phi}^2 = \frac{p_\phi^2}{m^2 r^2}\right]$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \quad \left[\dot{z} = \frac{p_z}{m} \Rightarrow \dot{z}^2 = \frac{p_z^2}{m^2}\right]$$

$$L = \frac{1}{2m}\left[p_r^2 + \frac{p_\phi^2}{r^2} + p_z^2\right] - V$$

$$H = \frac{1}{2m}\left[p_r^2 + \frac{p_\phi^2}{r^2} + p_z^2\right] + V$$

$$\frac{\partial H}{\partial r} = \frac{p_r}{m} - \frac{p_\phi^2}{mr^3} \qquad \frac{\partial H}{\partial p_r} = \frac{p_r}{m}$$

$$\frac{\partial H}{\partial \phi} = \frac{p_\phi}{m \cdot r^2} \qquad \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{m \cdot r^2}$$

$$\frac{\partial H}{\partial z} = \frac{p_z}{m} \qquad \frac{\partial H}{\partial z} = \frac{p_z}{m}$$

* Central forces

$$F(r) = \frac{K}{r^{n+1}} \cdot \hat{e}_r \rightarrow \text{General form}$$

K = Constant

if $n = -1$ $\quad F = Kr$ (Simble forces)

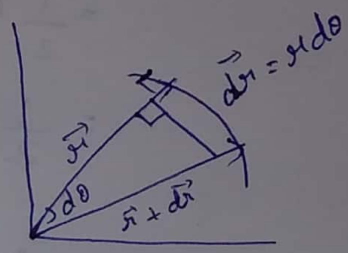
$n = 2$

## Properties of central forces

(1) conservation of energy
(2) conser. of angular momentum
(3) conser. of aerial velocities.

---

### * central forces




(1) Conservation of angular momentum

$$\vec{L} = \vec{r} \times \vec{p}$$
$$= \vec{r} \times m\vec{v}$$

using, $\vec{v} = \frac{dr}{dt}\hat{e}_r + r\frac{d\theta}{dt}\hat{e}_\theta$

$$\Rightarrow \vec{r} \times m(\dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta)$$
$$= r\hat{e}_r \times m(\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta)$$
$$= mr^2\dot{\theta}[\hat{e}_r \times \hat{e}_\theta] \quad \text{as } \hat{e}_r \times \hat{e}_r$$
$$\vec{L} = mr^2\dot{\theta}.\hat{n}$$

$n \rightarrow$ unit vector which is $\perp$r to plane containing $\hat{e}_r$ & $\hat{e}_\theta$ respectively

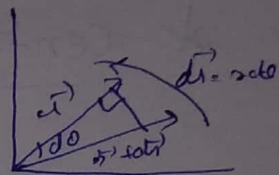
Torque, $\vec{N} = \vec{r} \times \vec{F} = \frac{d\vec{L}}{dt} \Rightarrow \vec{L} = \text{const.} = mr^2\dot{\theta}\,\hat{n}$

$$= r\hat{e}_r \times f r\hat{e}_r$$
$$= 0$$

## (2) Conservation of areal velocity

Areal velocity is a term is to denote the surface area swept out by the radial vector per unit time during the planar motion of particle under the influence of a central force

Surface Area $d\vec{A} = \frac{1}{2}(\vec{r} \times d\vec{r})$ ($\Delta$ law)




$$= \frac{1}{2}(r\hat{e}_r \times r\,d\theta\,\hat{e}_\theta)$$

$$= \frac{1}{2} r^2 d\theta\,\hat{n}$$

$$d\vec{A} = \frac{1}{2} r^2 d\theta\,\hat{n}$$

$$\frac{d\vec{A}}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}\,\hat{n}$$

$$\frac{d\vec{A}}{dt} = \frac{\vec{L}}{2m} \quad \text{or} \quad \frac{dA}{dt} = \frac{L}{2m} = \text{Constant}$$

* Langrange eq. for central force

$$L = T - V$$

$$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

$$\left(\frac{\partial L}{\partial \dot{r}} = \frac{1}{2}m \cdot 2\dot{r} = m\dot{r}, \quad \frac{\partial L}{\partial r} = \frac{1}{2}m \cdot 2r\dot{\theta}^2 = mr\dot{\theta}^2\right.$$

$$\left.\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}m \cdot r^2 \cdot 2\dot{\theta} = mr^2\dot{\theta}\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

$$\vec{r} \rightarrow \frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0$$

Scanned by CamScanner

$\theta \to \dfrac{d}{dt}(mr^2\dot\theta) = 0$

$\therefore\ mr^2\dot\theta = \text{Constant}$ (which is angular momentum)

$r(t)$ and $\theta(t)$

$l = mr^2\dot\theta$ (in terms of $\theta$)

$\dot\theta = \dfrac{l}{mr^2(t)}$

$l = mr^2\dot\theta$

$mr\dot\theta^2 = \dfrac{l}{r}$

$$\int_{\theta_0}^{\theta} d\theta = \int_0^t \frac{l}{mr^2(t)}\,dt \quad \text{—— (A)}$$

→ in terms of $r$

$$m\ddot r = mr\dot\theta^2 - \frac{\partial V}{\partial r}$$

$$m\ddot r = -\frac{\partial V}{\partial r} + mr\dot\theta^2$$

$l = mr^2\dot\theta\hat n$

$mr = \dfrac{l}{r\dot\theta\hat n}$

$$m\ddot r = -\frac{\partial V}{\partial r} + \frac{l^2}{mr^3} \quad \text{—— (B)}$$

$$m\ddot r\dot r = -\frac{\partial}{\partial r}\left[V + \frac{l^2}{2mr^2}\right]\frac{\partial r}{dt}$$

$$\frac{d}{dt}\left(\frac{1}{2}m\dot r^2\right) = \frac{-\partial}{\partial t}\left[V + \frac{l^2}{2mr^2}\right]$$

or $$\frac{\partial}{\partial t}\left[\frac{1}{2}m\dot r^2 + V + \frac{l^2}{2mr^2}\right] = 0$$

accelaration produced in the particle

$$= \frac{1}{2} m\dot{r}^2 + V + \frac{L^2}{2mr^2} = \text{Const.} = E$$

$$\Rightarrow \frac{1}{2} m\dot{r}^2 + V' = E$$

$$\dot{r}^2 = \frac{2}{m}(E - V')$$

$$\int_{r_0}^{r} dr = \pm \int_0^t \sqrt{\frac{2}{m}(E-V')}\, dt$$

- if $E > V'$, motion is possible
- but if $E < V'$, then motion is not poss.
- $E = V'$, it is on equilibrium

Force $\frac{L^2}{mr^3} = \frac{(mr^2\dot{\theta})^2}{mr^3}$

$$= \frac{m^2 r^4 \dot{\theta}^2}{mr^3}$$

$$= \frac{m(r\dot{\theta})^2}{r}$$

$$= \frac{m v_0^2}{r}$$

in classical mechanics, equilibrium points → Turning points

Radial or tangential points then eq in terms of $r(t)$ & $\theta(t)$

A ~~body~~ particle of mass $m$ which is moving in the influence of a spherical symmetrically inverse square force is given by

$$F_r = -\frac{C}{r^2} \quad (C > 0)$$

$$ma_r = F_r = -\frac{C}{r^2}$$

from Lagrange's Radial eq.,

$$m\ddot{r} - mr\dot{\theta}^2 = -\frac{C}{r^2} = -\frac{\partial V}{\partial r} \quad \text{——} \quad (1)$$

$$m\ddot{r} - \frac{l^2}{mr^3} = -\frac{c}{r^2} \quad \text{(2nd order diff. eq)}$$

Replace $r = \frac{1}{u}$

$$L = mr^2\dot{\theta}$$

$$\dot{\theta} = \frac{L}{mr^2} = \frac{Lu^2}{m} = ku^2 \quad \left(k = \frac{L}{m} = \text{const.}\right)$$

$$\frac{dr}{dt} = \frac{d}{dt}\left(\frac{1}{u}\right) = \frac{-1}{u^2}\frac{du}{dt} = \frac{-1}{u^2}\frac{du}{d\theta}\cdot\frac{d\theta}{dt}$$

$$= -\frac{1}{u^2}ku^2\frac{du}{d\theta} = -k\frac{du}{d\theta}$$

$$\frac{d}{dt}\left(\frac{dr}{dt}\right) = \frac{d}{dt}\left(-k\frac{du}{d\theta}\right)$$

$$= \frac{d}{d\theta}\left(-k\frac{du}{d\theta}\right)\frac{d\theta}{dt}$$

$$= -k\frac{d^2u}{d\theta^2}ku^2$$

$$= -k^2u^2\frac{d^2u}{d\theta^2}$$

Put values of $m\ddot{r}$ in eq. (1)

$$-mk^2u^2\frac{d^2u}{d\theta^2} - \frac{m}{u}(ku^2)^2 = -cu^2$$

$$-mk^2u^2\frac{d^2u}{d\theta^2} - mk^2u^3 = -cu^2$$

divided by $m k^2 u^2$

$$\frac{d^2u}{d\theta^2} + u = \frac{Cu^2}{m k^2 u^2} = \frac{C}{m k^2} = A \text{ (say)}$$

$$\left( A = \frac{C}{m k^2} \quad f_r = \frac{-C}{r^2} \right)$$

$C = GM_s m$

$$\frac{d^2u}{d\theta^2} + u = A$$

$$\frac{d^2u}{d\theta^2} + (u - A) = 0$$

or

$$\frac{d^2(u-A)}{d\theta^2} + (u - A) = 0$$

as

$$\frac{d^2A}{d\theta^2} = 0$$

$$u - A = B\cos(\theta - \theta_0)$$

$$u = A + B\cos(\theta - \theta_0)$$

$$\frac{1}{r} = A + B\cos(\theta - \theta_0)$$

$$\frac{1}{Ar} = 1 + \frac{B}{A}\cos(\theta - \theta_0)$$

$$\boxed{\frac{p}{r} = 1 + e\cos(\theta - \theta_0)} \quad \left( \frac{1}{A} = p, \ \frac{B}{A} = e \right)$$

if • $e = 0$ circle ] Bounded motion

$e < 1$ elliptical

$e = 1$ Parabola ] unbounded Motion

$e > 1$ hyperbola

Perigee → Distance of closest approach

Perigee, $\frac{p}{r_{min}} = 1 + e\cos(\theta - \theta_0)$

$\theta = 0$ $\boxed{\frac{p}{r_{min}} = 1 + e}$ $\boxed{\frac{p}{r_{max}} = 1 - e}$

Passian bracket in Quantum mech. with its Commutation relation in classical mech.

$$\boxed{p = \frac{r_{max} - r_{min}}{r_{max} + r_{min}}}$$

$$\boxed{e = \frac{2 r_{max} r_{min}}{r_{max} + r_{min}}}$$

# Approximation Techniques.

* **Hilbert space** → $\psi / \psi^*$

Any wave fun. & Particle, vector is represent by
BRA vector — KET vector

KET vector → $|\psi\rangle$ → $\psi(x,t) / \psi(x)$
BRA vector → $\langle\psi|$ → $\psi^*(x,t) / \psi^*(x)$

$$P = \int_{-\infty}^{+\infty} \psi^* \psi \, dx = (\psi, \psi)$$

$$= \langle \psi | \psi \rangle \quad \text{(in hilbert space)}$$

$$\langle x \rangle = \int_{-\infty}^{\infty} \psi^* x \psi \, dx = (\psi, x\psi)$$

$$= \langle \psi | x | \psi \rangle \quad \text{(in hilber space)}$$

---

$$\underbrace{H_0}_{\text{Hamiltonian of exactly solved sys.}} + \underbrace{H'}_{\text{Slight deviation}}$$

Perturbtion Theory — Time dep.
Perturbtion Theory — Time independent

# * Time dependent Perturbation Theory

$$H_0 \psi = E_0 \psi$$

, $H_0 \rightarrow$ uperturbed hamiltonian

H is taken only as single order (1st order derivative)

$$H = H_0 + \lambda H' + \lambda^2 H'' + \cdots$$

$\lambda \rightarrow$ degeneracy Parameter, which vary from 0 to 1

Hamiltonian eq. for the system

$$H\psi = E\psi$$

$$\psi = \psi_n^{(0)} + \lambda \psi_n^{(1)} + \lambda^2 \psi_n^{(2)} + \cdots$$

$$E = E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \cdots$$

$$H\psi = E\psi$$

$$(H_0 + \lambda H')(\psi_n^{(0)} + \lambda \psi_n^{(1)} + \lambda^2 \psi_n^{(2)} + \cdots) = (E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \cdots)(\psi_n^{(0)} + \lambda \psi_n^{(1)} + \lambda^2 \psi_n^{(2)} + \cdots)$$

$\lambda^0 \Rightarrow H_0 \psi_n^{(0)} = E_n^{(0)} \psi_n^{(0)}$ (unperturber hamiltonian eq)

$$\lambda^1 \Rightarrow H_0 \psi_n^{(1)} + H' \psi_n^{(0)} = E_n^{(0)} \psi_n^{(1)} + E_n^{(1)} \psi_n^{(0)}$$

$$\lambda^2 \Rightarrow H_0 \psi_n^{(2)} + H' \psi_n^{(1)} = E_n^{(0)} \psi_n^{(2)} + E_n^{(1)} \psi_n^{(1)} + E^{(2)} \psi_n^{(0)}$$

In General

$$\lambda^j \Rightarrow H_0 \psi_n^j + H' \psi_n^{j-1} = E_n^{(0)} \psi_n^j + E_n^{(1)} \psi_n^{j-1} + \cdots + E_n^{(j)} \psi_n^{(0)}$$

Correction in the energies → Permultiply by $\psi_n^{(0)*}$ and integrating are space

for $\lambda^1$, $\int_{-\infty}^{+\infty} \psi_n^{(0)} H_0 \psi_n^{(0)} dx = E_n \int_{-\infty}^{+\infty} \psi_n^{(0)*} \psi_n^{(0)} dx$

$$H_0 \psi_n^{(0)} = E_n^{(0)} \psi_n^{(0)}$$

in Bra-Kit Notation, $\boxed{\int_{-\infty}^{+\infty} \psi_n^{(0)} H_0 \psi_n^{(0)} dx = \langle \psi_n^{(0)} | H_0 | \psi_n^{(0)} \rangle}$

$$[H_0 - E_n^{(0)}] \psi_n^{(1)} + [H' - E_n^{(1)}] \psi_n^{(0)} = 0$$

$$\langle \psi_n^{(0)} | H_0 - E_n^{(0)} | \psi_n^{(1)} \rangle + \langle \psi_n^{(0)} | H' - E_n^{(1)} | \psi_n^{(0)} \rangle = 0$$

as $H_0$ is Harmition,

$\therefore$ Ist term → $\langle \psi_n^{(1)} | H_0 - E_n^{(0)} | \psi_n^{(0)} \rangle^* = \langle \psi_n^{(1)} | E_n^{(0)} - E_n^{(0)} | \psi_n^{(0)} \rangle$

$$= 0$$

2nd term → $\langle \psi_n^{(0)} | H' - E_n^{(1)} | \psi_n^{(0)} \rangle$.

$$= \underbrace{\langle \psi_n^{(0)} | H' | \psi_n^{(0)} \rangle}_{H'_{nn} = \int_{-\infty}^{+\infty} \psi_n^{(0)*} H' \psi_n^{(0)} dx} - \underbrace{\langle \psi_n^{(0)} | E_n^{(1)} | \psi_n^{(0)} \rangle}_{\int_{-\infty}^{+\infty} \psi_n^{*} E_n^{(1)} \psi_n^{(0)} dx = E_n^{(0)} \int_{-\infty}^{+\infty} \psi_n^{(0)} \psi_n^{(0)} dx} = 0$$

$$\therefore \quad H'_{nn} - E_n^{(1)} \delta_{nm} = 0$$

$$\boxed{E_n^{(1)} = H'_{nn}} = \langle \psi_n^{(0)} | H' | \psi_n^{(0)} \rangle$$

in Partical in box,

1st order correction in energy, $E_n^{(1)} = \int_{-\infty}^{+\infty} \sqrt{\frac{2}{L}} \sin \frac{\pi}{L} n . V_0 \sqrt{\frac{2}{L}} \sin \frac{\pi x}{L} dx$

$$E_n = E_n^{(0)} + E_n^{(1)}$$

## * Time Indebendent Perturbation Theory

$$H_0 \psi_n^{(1)} + H' \psi_n^{(0)} = E_n^{(0)} \psi_n^{(1)} + E_n^{(1)} \psi_n^{(0)}$$

$$E_n^{(1)} = H'_{nn} = \langle \psi_n^{(0)} | H' | \psi_n^{(0)} \rangle$$

$$\int_{-\infty}^{+\infty} \psi_n^{(0)*} H' \psi_n^{(0)} dx$$

2nd order Perturbation eq.,

$$H_0 \psi_n^{(2)} + H' \psi_n^{(1)} = E_n^{(0)} \psi_n^{(2)} + E_n^{(1)} \psi_n^{(1)} + E_n^{(2)} \psi_n$$

$$(H_0 - E_n^{(0)}) \psi_n^{(2)} + (H' - E_n^{(1)}) \psi_n^{(1)} = E_n^{(2)} \psi_n^{(0)}$$

$$H = \underbrace{\frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2} k x^2}_{H.O} + b x^3$$

$$\underbrace{H'_{nn} = \int_{-\infty}^{+\infty} \psi_n^{(0)*} b x^3 \psi_n^{(0)} dx}_{\text{Stdand integral}}$$

$\times \psi_n^{(0)*}$ and $\int_{-\infty}^{+\infty} (\;) \, dx$

$$\langle \psi_n^{(0)} | \underbrace{H_0 - E_n^{(0)}}_{=0} | \psi_2^{(2)} \rangle + \langle \psi_n | H' - E_n^{(1)} | \psi_n^{(1)} \rangle = E_n^{(2)} \underbrace{\langle \psi_n^{(0)} | \psi_n^{(0)} \rangle}_{=1}$$

$$E_n^{(2)} = \langle \psi_n^{(0)} | H' - E_n^{(1)} | \psi_n^{(1)} \rangle$$

Q Third order energy eign value

To find Correction in wave fun. we will used rarely Schordinger ~~Lagrange's~~ techniques in which the Perturb wave fun is expanded as a basies set of the unperturb wave function.

$$\psi_n^{(1)} = \sum_K a_{nk}^{(1)} \psi_k^{(0)}$$

1st order ~~wave fun.~~ Perturbation eq. →

$$H_0 \psi_n^{(1)} + H' \psi_n^{(0)} = E_n^{(0)} \psi_n^{(1)} + E_n^{(1)} \psi_n^{(0)}$$

$$(H_0 - E_n^{(0)}) \psi_n^{(1)} + (H' - E_n^{(1)}) \psi_n^{(0)} = 0$$

* Permultiply by $\psi_l^{(0)*}$ and $\int_{-\infty}^{+\infty} (\;) \, dx$

$$\Rightarrow \langle \psi_l^{(0)} | H_0 - E_n^{(0)} | \psi_n^{(1)} \rangle + \langle \psi_l^{(0)} | H' - E_n^{(1)} | \psi_n^{(0)} \rangle = 0$$

$$\langle \psi_l^{(0)} | H_0 | \psi_n^{(1)} \rangle - \langle \psi_l^{(0)} | E_n^{(0)} | \psi_n^{(1)}$$

Scanned by CamScanner

$\Rightarrow \langle \psi_l^{(0)} | H_0 | \sum_k a_{nk}^{(1)} \psi_k^{(0)} \rangle - \langle \psi_l^{(0)} | E_n^{(0)} | \sum_k a_{nk}^{(0)} \psi_k^{(0)} \rangle = 0$

$H_0 | \psi_k^{(0)} \rangle = E_k^{(0)}$

$\int_{-\infty}^{+\infty} \psi_l^{(0)} H_0 a_{nk}^{(1)} \psi_k^{(0)} dx = \int \psi_l^{(0)} \psi_k dx = \delta_{lk} = 1 \quad l = k$

$= 0 \quad l \neq k$

$\therefore = E_l^{(0)} a_{nl}^{(1)}$

1st term

$\langle \psi_l^{(0)} | H_0 - E_n^{(0)} | \psi_n^{(1)} \rangle = (E_l^{(0)} - E_n^{(0)}) a_{nl}^{(1)}$

2nd term.

$\langle \psi_l^{(0)} | H' - E_n^{(1)} | \psi_n^{(0)} \rangle = 0 \Rightarrow E_n^{(1)} \delta_{ln} = 0$

$\boxed{(E_l^{(0)} - E_n^{(0)}) a_{nl}^{(1)} + H'_{ln} - E_n^{(1)} \delta_{ln} = 0}$

$\boxed{\int_{-\infty}^{+\infty} \psi^{(0)*} H' \psi_n^{(0)} dx}$

if $\delta_{ln} = 1$, $l = n$

$H'_{nn} = E_n^{(1)}$

if $l \neq n$, $\delta_{ln} = 0$

$(E_l^{(0)} - E_n^{(0)}) a_{nl}^{(1)} = - H'_{ln}$

$a_{nl}^{(1)} = \frac{- H'_{ln}}{E_l^{(0)} - E_n^{(0)}} = \frac{H'_{ln}}{E_n^{(0)} - E_l^{(0)}}$

1st order correction,

$\psi_n = \psi_n^{(0)} + \lambda \psi_n^{(1)}$

$= \psi_n^{(0)} + \sum_l \lambda \frac{H'_{ln}}{E_n^{(0)} - E_l^{(0)}} \psi_l^{(0)}$

using

$\boxed{\psi_n^{(1)} = \sum_k a_{nk}^{(1)} \psi_k^{(0)}}$

if $\lambda = 1$

$$\psi_n = \psi_n^{(0)} + \sum_l \frac{H'_{ln}}{E_n^{(0)} - E_l^{(0)}} \cdot \psi_l^{(0)}$$

$$H_0\psi_n^{(2)} + H'\psi_n^{(1)} = E_n^{(0)}\psi_n^{(2)} + E_n^{(1)}\psi_n^{(1)} + E_n^{(2)}\psi_n^{(0)}$$

$$(H_0 - E_n^{(0)})\psi_n^{(2)} + (H' - E_n)\psi_n^{(1)} = E_n^{(2)}\psi_n^{(0)}$$

$$\langle \psi_l^{(0)} | H_0 - E_n^{(0)} | \psi_n^{(2)} \rangle + \langle \psi_l^{(0)} | H' - E_n^{(1)} | \psi_n^{(1)} \rangle = E_n^{(2)} \langle \psi_l^{(0)} | \psi_n^{(0)} \rangle$$

$$= E_n^{(2)} \delta_{ln}$$

$$\psi_n^{(2)} = \sum_k a_{nk}^{(2)} \psi_k^{(0)} \quad ; \quad \psi_n^{(1)} = \sum_k a_{nk}^{(1)} \psi_k^{(0)}$$

$$\langle \psi_l^{(0)} | H_0 - E_n^{(0)} | \sum_k a_{nk}^{(2)} \psi_k^{(0)} \rangle + \langle \psi_l^{(0)} | H' - E_n^{(1)} | \sum_k a_{nk}^{(1)} \psi_k^{(0)} \rangle$$

$$= E_n^{(2)} \delta_{ln}$$

$$\left[E_l^{(0)} - E_n^{(0)}\right] a_{nl}^{(2)} + H'_{lk} a_{nk}^{(1)} - E_n^{(1)} a_{nl} = E_n^{(2)} \delta_{ln}$$

using!

$$a_{nl} = \frac{H'_{ln}}{E_n^{(0)} - E_l^{(0)}}$$

$$\psi_n^{(1)} = \sum_k a_{nk}^{(1)} \psi_k^{(0)}$$

if $n = l$

$$E_n^{(2)} = \sum_k H'_{nk} a_{nk}^{(1)} - \underbrace{E_n^{(1)}}_{= H'_{nn}} a_{nn}$$

$$E_n^{(2)} = \sum_{n \neq k} H'_{nk} a_{nk}^{(1)}$$

$$E_n^{(2)} = \sum_{n \neq k} H'_{nk} \cdot \frac{H'_{kn}}{(E_n^{(0)} - E_k^{(0)})}$$

Case II if $n \neq l$

$$[E_l^{(0)} - E_n^{(0)}] a_{nl}^{(2)} + H'_{lk} a_{nk}^{(1)} - E_n^{(1)} a_{nl}^{(1)} = 0$$

$$[E_l^{(0)} - E_n^{(0)}] a_{nl}^{(2)} = H'_{nn} a_{nl}^{(1)} - H'_{lk} a_{nk}^{(1)} \qquad (E_n^{(1)} = H'_{nn})$$

$$= H'_{nn} \cdot \frac{H'_{ln}}{E_n^{(0)} - E_l^{(0)}} - H'_{lk} \sum_k \frac{H'_{nk}}{E_n^{(0)} - E_k^{(0)}}$$

$$a_{nl}^{(2)} = \frac{1}{E_l^{(0)} - E_n^{(0)}} \sum \left( \qquad \right) - \frac{H'_{lk} H'_{nk}}{(E_n^{(0)} - E_k^{(0)})(E_l^{(0)} - E_n^{(0)})}$$

$$\psi_n = \psi_n^{(0)} + \lambda \psi_n^{(1)} + \lambda^2 \psi_n^{(2)}$$

$$H' - E_n^{(1)} | \sum_k a_{nk}^{(1)} \psi_k^{(0)} \rangle = E_n^{(2)} \delta_{nl}$$

$$\psi_n = \psi_n^{(0)} + \lambda \sum_k a_{nk}^{(1)} \psi_k^{(0)} + \lambda^2 \sum_k a_{nk}^{(2)} \psi_k^{(0)}$$

$$E_n = E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} \qquad \text{if } \lambda = 1$$

---

$$E_0 \leq \langle H_0 \rangle$$

$$H_0 = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x)$$

$$\phi(x) = e^{-\alpha x^2}$$

$$\langle H \rangle = \frac{\langle \phi | H | \phi \rangle}{\langle \phi | \phi \rangle}$$

$$\frac{\partial \langle H \rangle}{\partial \alpha} = 0$$

$$\langle H \rangle_{min} = E_0$$