UNIT – III

Kinematics of Particles

PART A: KINEMATICS

14.1 INTRODUCTION TO DYNAMICS

In statics we considered the bodies at rest. Now we shall begin with the study of dynamics. *Dynamics is the part of mechanics that deals with the analysis of bodies in motion.* While the study of statics is very old science, dynamics is a comparatively new one. The first significant contribution to dynamics was made by Galileo (1564-1642). Newton later (1642-1727), formulated his fundamental laws of motion.

For convenience, dynamics is divided into two branches called *kinematics* and *kinetics*.

Kinematics is the study of the relationships between displacement, velocity, acceleration and time of a given motion, without considering the forces that cause the motion.

Kinetics is the study of the relationships between the forces acting on a body, the mass of the body and the motion of the body. Kinetics therefore, can be used to predict the motion caused by a given forces or to determine the forces required to produce a prescribed motion.

The science of dynamics is based on the natural laws governing the motion of a particle. *The term particle is a convenient idealization of the physical objects which need not be small in size. In this idealization, the mass of the body is assumed to be concentrated at a point and the motion of the body is considered as the motion of an entire unit neglecting any rotation about its own mass centre.* In case where such rotation is not negligible, then the body cannot be considered as a particle.

Types of Motion : When a particle moves in space it describes a curve, called path. This path can be straight or curved.

(i) **Rectilinear Motion.** When a particle moves along a path which is a straight line, it is called rectilinear motion.

(ii) **Curvilinear Motion.** When a particle moves along a curved path it is called curvilinear motion. *If the curved path lies in a plane it is called plane curvilinear motion.*

In this chapter, we shall discuss the rectilinear motion of a particle. The kinematics and kinetics of which are separated into parts *A* and *B*.

Rectilinear motion

Let us consider the motion of a particle along a straight line and explain the above terms.

Displacement. A particle in rectilinear motion, at any instant of time will occupy a certain position on the straight line. To define this position P of the particle, we have to choose some convenient reference point O, called the origin. The distance x of the particle at any time t, is called the displacement of the particle at that time. The displacement is assumed to be positive to the right of the origin and negative to the left (Fig. 14.1).

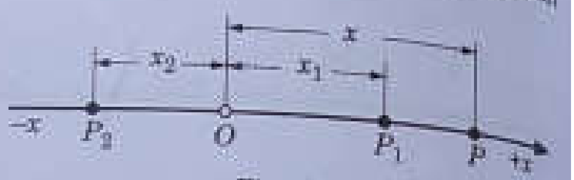
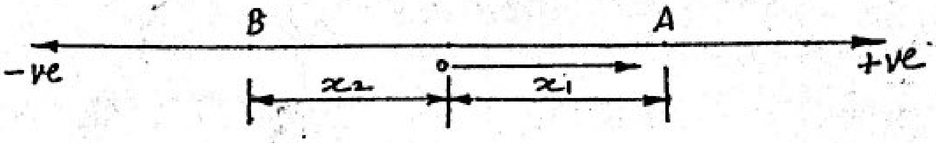


Fig. 14.1

Distance Travelled. The distance travelled by a particle, however, is different that its displacement from the origin. For example, if a particle moves from O to positions P_1 and then to position P_2, its displacement at the position P_2 is $-x_2$ from the origin but, the distance travelled by the particle is $2x_1 + x_2$ (Fig. 14.1).

DISTANCE: The total distance covered by the particle or body along the path is known as distance.



Let a body move from reference point O to position A and then from A to position B.

Displacement of body = $-x_2$ 2

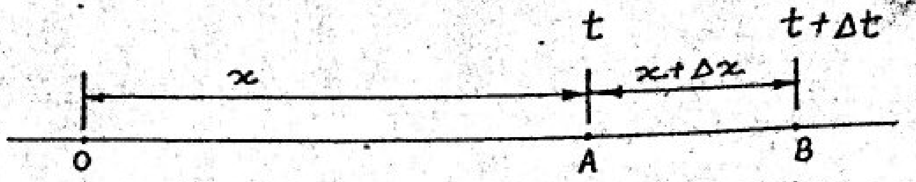
Distance of body = $x_1 + x_1 + x_2$

$= 2x_1 + x_2$

	DISPLACEMENT	DISTANCE
1.	Shortest distance covered by the particle from reference point.	Total distance covered by the particle along the path.
2.	It is taken - positive to right of reference point and negative to its left.	It is always taken as positive.
3.	It has both magnitude and direction.	It has magnitude only.
4.	It is a vector quantity.	It is a scalar quantity.

VELOCITY: The rate of change of position of a body with respect to time is called velocity.

The position A of a body at time t is at a distance x from reference point O. At time $(t+\Delta t)$, the body moves to position B at a distance $x+\Delta x$ from O.



Now, the average velocity of body over time interval Δt is

$$V_{av} = \frac{\Delta x}{\Delta t}$$

INSTANTANEOUS VELOCITY : It is defined as the velocity at a particular instant of time.

Taking time interval Δt and displacement Δx to be very very small.

$$\text{Instantaneous velocity}, v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

$$= \frac{dx}{dt}$$

1) Velocity is a measure of rate of change of position in a particular direction and hence is a vector quantity.

2) Velocity is positive if displacement is increasing and moving in positive direction.

ACCELERATION : The rate of change of velocity of a body with respect to time.

Let 'v' be the velocity at time t and later the velocity becomes $(v+dv)$ at time $(t+\Delta t)$.

$$\text{Average acceleration}, a = \frac{\Delta v}{\Delta t}$$

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If time interval Δt and change in velocity Δv is taken to be very very small.

$$\text{Acceleration}, \ a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$$

$$= \frac{dv}{dt}$$

Acceleration is positive if velocity is increasing and body is moving in positive direction.

$$a = \frac{dv}{dt} \ , \ v = \frac{dx}{dt}$$

$$\therefore \quad a = \frac{d^2x}{dt^2}$$

Also $\quad a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt}$

$$\boxed{a = v\frac{dv}{dx}}$$

UNIFORM MOTION : When a body moves with constant velocity (i.e when acceleration is zero), the body is said to have uniform motion.

UNIFORMLY ACCELERATED MOTION : A body having a constant acceleration is referred to as uniformly accelerated motion.

GRAPHICAL REPRESENTATION

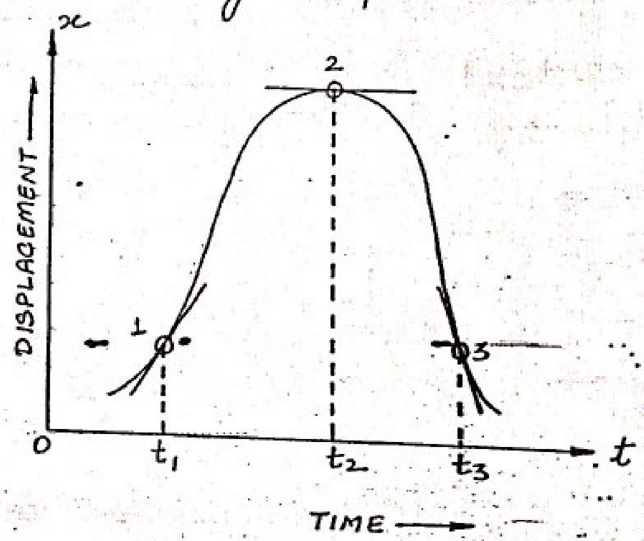
1.) DISPLACEMENT - TIME GRAPH (x-t CURVE)

Displacement of the body is plotted with respect to time, displacement lies along y axis and time along x axis.

As $v = \frac{dx}{dt}$.

the slope of x-t curve at any instant gives the velocity of body at that instant.

a) At time t_1, the curve has a positive slope and hence the velocity is positive.



b) At time t_2, the slope is zero and hence the velocity of body is zero. The body is at rest.

c) At time t_3, the curve has negative slope and hence the velocity is negative.

2.) VELOCITY- TIME GRAPH (v-t CURVE)

The velocity is plotted as a function of time.

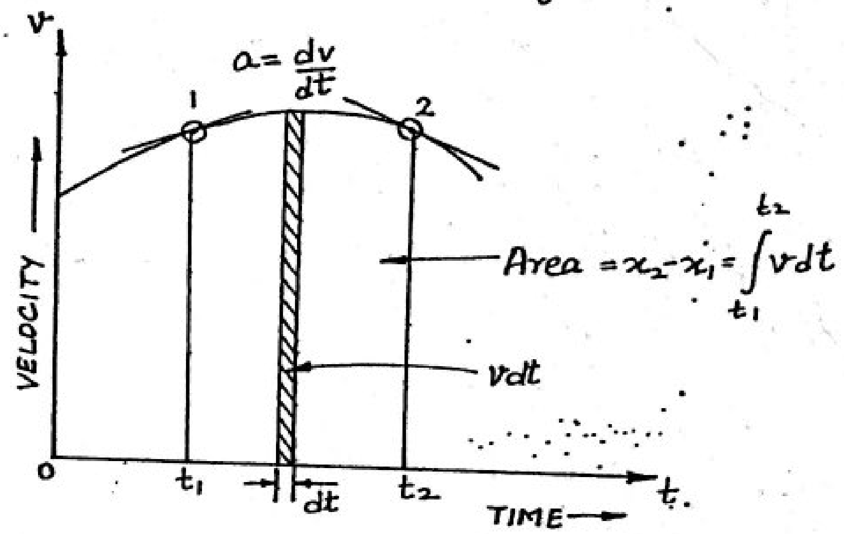
As $a = \frac{dv}{dt}$, the slope of v-t curve gives acceleration at any instant.

We know, $v = \frac{dx}{dt}$

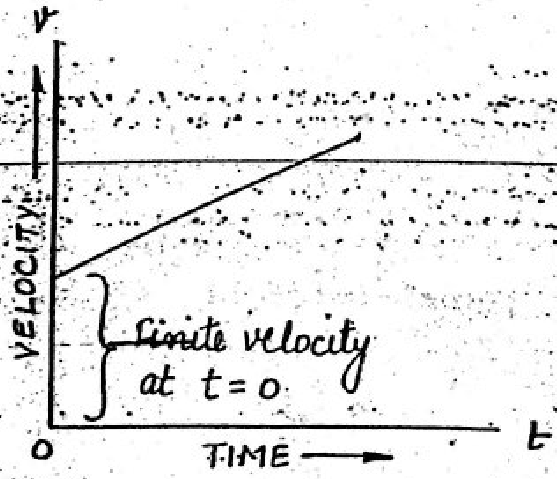
$$\int dx = \int v\,dt$$

$$x_2 - x_1 = \int v\,dt$$

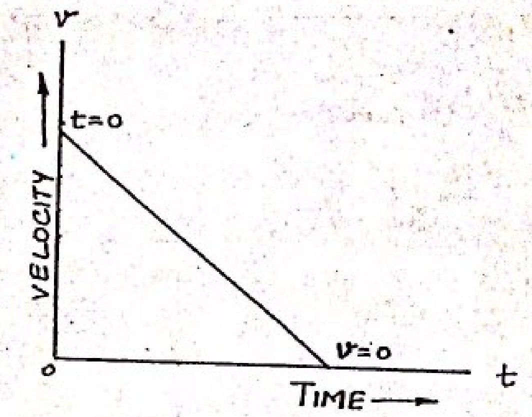
The above expression implies that area under v-t curve in a given time interval gives change in displacement or distance travelled during same interval



a) The velocity increases linearly with time.
As v-t curve has constant slope, the velocity increases by equal amounts at equal intervals of time. Hence the body moves with constant acceleration.



b) The body has finite velocity and with passage of time velocity decreases linearly with time. Finally velocity becomes zero and body comes to rest. The body moves with constant retardation (negative acceleration).



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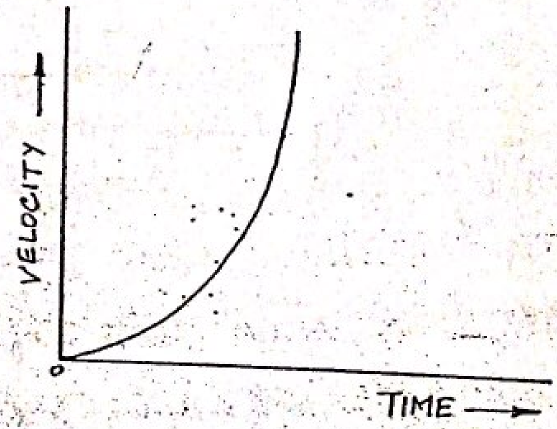
c) The slope of v-t curve is different at different times. No uniform acceleration is there as the velocity is not changing at constant interval of time.



3.) ACCELERATION-TIME GRAPH (a-t CURVE)

The acceleration of the body is plotted as the function of time.

We know,

$$a = \frac{dv}{dt}$$

$$\int dv = \int a\,dt$$

$$v_2 - v_1 = \int a\,dt$$

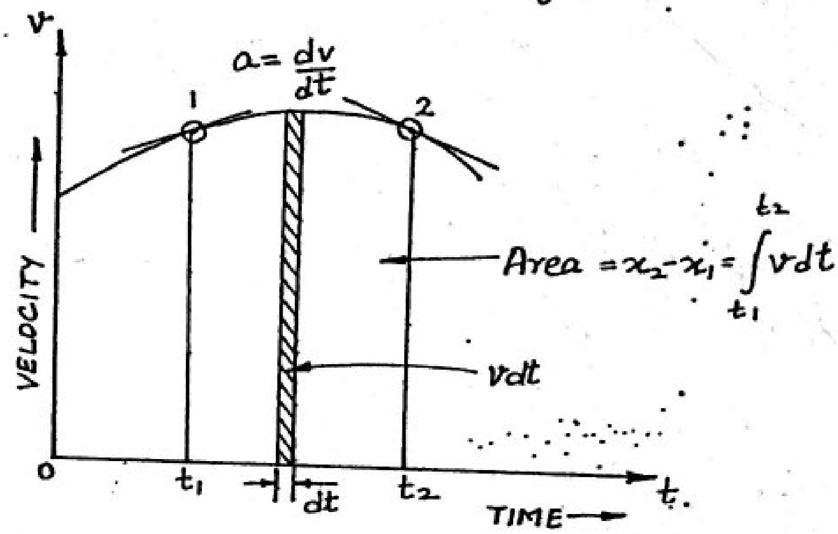
acceleration at any instant.

We know, $v = \frac{dx}{dt}$

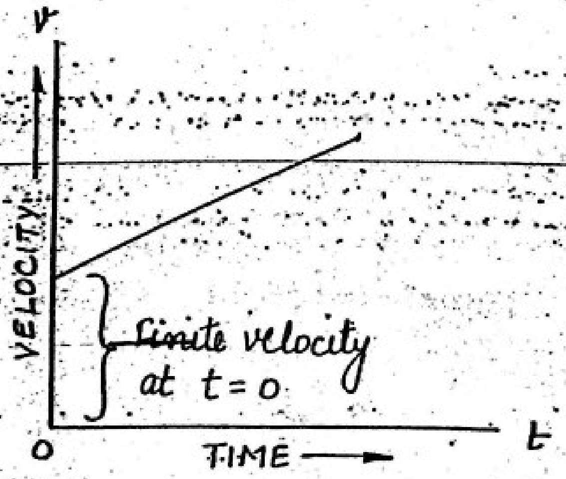
$$\int dx = \int v\,dt$$

$$x_2 - x_1 = \int v\,dt$$

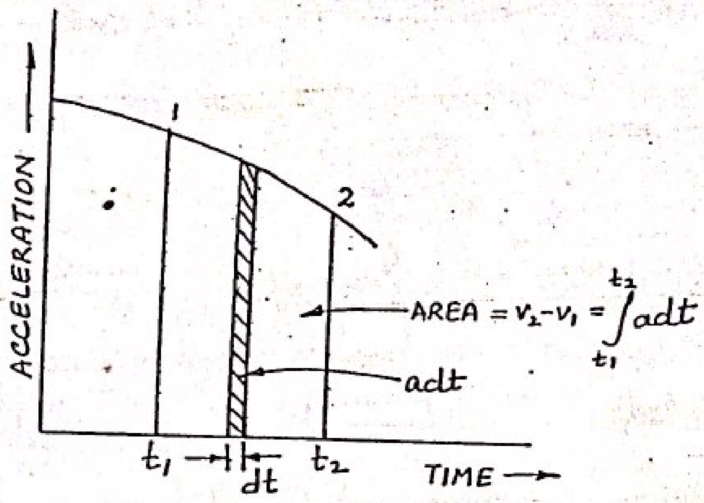
The above expression implies that area under v-t curve in a given time interval gives change in displacement or distance travelled during same interval



a) The velocity increases linearly with time.
As v-t curve has constant slope, the velocity increases by equal amounts at equal intervals of time. Hence the body moves with constant acceleration.



The above expression implies that the area under v-t curve in a given time interval gives change in velocity during same time interval.



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Motion With Uniform Acceleration

1.) Acceleration, a = Constant

We know, $$a = \frac{dv}{dt}$$

$$dv = a\,dt$$

$$\int_u^v dv = \int_0^t a\,dt$$

$$v - u = at$$

$$\boxed{v = u + at}$$

Also a = Rate of change of velocity

$$= \frac{\text{Change in velocity}}{\text{Time}}$$

$$= \frac{v-u}{t}$$

$$v - u = at$$

$$\boxed{v = u + at}$$

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2.) Distance travelled = Average velocity × Time

$$s = \frac{u+v}{2} \times t$$

$$= \frac{u+u+at}{2} \times t \qquad \left[\because v = u+at\right]$$

$$= \left[u + \frac{at}{2}\right]t$$

$$\boxed{s = ut + \frac{1}{2}at^2}$$

Also

$$v = \frac{dx}{dt}$$

$$dx = v\,dt$$

$$dx = (u+at)\,dt$$

$$\int_0^x dx = \int_0^t (u+at)\,dt$$

$$x = ut + \frac{at^2}{2}$$

$$\boxed{x = ut + \frac{1}{2}at^2}$$

3.) Distance = Average velocity × Time

$$s = \frac{u+v}{2} \times t \qquad \left[\because v = u+at,\ t = \frac{v-u}{a}\right]$$

$$= \frac{u+v}{2} \times \frac{v-u}{a}$$

$$= \frac{v^2-u^2}{2a}$$

$$\boxed{v^2 - u^2 = 2as}$$

Also, $$a = v\frac{dv}{dx}$$

$$a\,dx = v\,dv$$

$$\int_0^x a\,dx = \int_u^v v\,dv$$

$$ax = \left[\frac{v^2}{2}\right]_u^v$$

$$ax = \frac{v^2-u^2}{2}$$

$$\boxed{v^2-u^2 = 2ax}$$

4) Distance covered in n seconds

$$S_n = un + \frac{1}{2}an^2$$

Distance covered in (n-1) seconds

$$S_{n-1} = u(n-1) + \frac{1}{2}a(n-1)^2$$

$$= u(n-1) + \frac{1}{2}a(n^2+1-2n)$$

∴ Distance covered in n^{th} second

$$S_{nth} = S_n - S_{n-1}$$

$$= un + \frac{1}{2}an^2 - u(n-1) - \frac{1}{2}a(n^2+1-2n)$$

$$= un + \frac{1}{2}an^2 - un + u - \frac{1}{2}an^2 - \frac{1}{2}a + an$$

$$= u - \frac{a}{2} + an$$

$$S_{nth} = u - \frac{a + 2an}{2}$$

$$\boxed{S_{nth} = u + (2n-1)\frac{a}{2}}$$

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17.5. DISTANCE TRAVELLED IN THE *n*th SECOND

Fig. 17.4. Distance travelled in *n*th second.

Consider the motion of a particle, starting from O and moving along OX as shown in Fig. 17.4.

Let

u = Initial velocity of the particle,

v = Final velocity of the particle

a = Constant positive acceleration,

s_n = Distance (OQ) travelled in n sec,

s_{n-1} = Distance (OP) travelled in $(n-1)$ sec,

$s = (s_n - s_{n-1})$ = Distane (PQ) travelled in nth sec,

n = No. of second.

Substituting the values of $t = n$ and $t = (n-1)$ in the general equation of motion,

$$s_n = un + \frac{1}{2}a(n)^2 \qquad \text{...}(i)$$

and

$$s_{n-1} = u(n-1) + \frac{1}{2}a(n-1)^2 \qquad \text{...}(ii)$$

∴ Distance travelled in the nth sec,

$$s = s_n - s_{n-1}$$

$$= \left[un + \frac{1}{2}a(n)^2\right] - \left[u(n-1) + \frac{1}{2}a(n-1)^2\right]$$

$$= un + \frac{1}{2}an^2 - un + u - \frac{1}{2}a(n^2 + 1 - 2n)$$

$$= \frac{1}{2}an^2 + u - \frac{1}{2}an^2 - \frac{1}{2}a + an$$

$$= u - \frac{1}{2}a + an = u + a\left(n - \frac{1}{2}\right) = u + \frac{a}{2}(2n-1)$$

Q: A body is moving with uniform acceleration and covers 15 m in fifth second and 25 m in 10th second. Determine :

1) Initial velocity of the body

2) Acceleration of the body.

SolN: Distance covered in 5th second = 15 m

Distance covered in 10th second = 25 m 6

Let $u \rightarrow$ Initial velocity

$a \rightarrow$ Acceleration

Distance covered in 5th second

$$15 = u + \frac{a}{2}(2 \times 5 - 1)$$

$$15 = u + \frac{9a}{2} \quad \text{------} \ (1)$$

Distance covered in 10th second

$$25 = u + \frac{a}{2}(2 \times 10 - 1)$$

$$25 = u + \frac{19a}{2} \quad \text{------} \ (2)$$

Solving (1) and (2)

$$15 = u + \frac{9a}{2}$$

$$25 = u + \frac{19a}{2}$$

$$-10 = \left(\frac{9a}{2} - \frac{19a}{2}\right)$$

$$-10 = \frac{-10a}{2}$$

$$\boxed{a = 2 \text{ m/sec}^2}$$

$$15 = u + \frac{9 \times 2}{2}$$

$$u = 15 - 9$$

$$\boxed{u = 6 \text{ m/sec}}$$

Example 17.5. *A motor car takes 10 seconds to cover 30 meters and12 seconds to cover 42 meters. Find the uniform acceleration of the car and its velocity at the end of 15 seconds.*

Solution. Given : When t = 10 seconds, s = 30 m and when t = 12 seconds, s = 42 m.

Uniform acceleration

Let u = Initial velocity of the car, and

a = Uniform acceleration.

We know that the distance travelled by the car in 10 seconds,

$$30 = ut + \frac{1}{2}at^2 = u \times 10 + \frac{1}{2} \times a\,(10)^2 = 10u + 50\,a$$

Multiplying the above equation by 6,

$$180 = 60u + 300a \qquad ...(i)$$

Similarly, distance travelled by the car in 12 seconds,

$$42 = u \times 12 + \frac{1}{2} \times a\,(12)^2 = 12u + 72a$$

Mulitiplying the above equation by 5,

$$210 = 60u + 360a \qquad ...(ii)$$

Subtracting equation (i) from (ii),

$$30 = 60a \quad \text{or} \quad a = \frac{30}{60} = 0.5 \text{ m/s}^2 \textbf{ Ans.}$$

Velocity at the end of 15 seconds

Substituting the value of a in equation (i)

$$180 = 60u + (300 \times 0.5) = 60u + 150$$

$$\therefore \quad u = \frac{(180 - 150)}{60} = 0.5 \text{ m/s}$$

We know that the velocity of the car after 15 seconds,

$$v = u + at = 0.5 + (0.5 \times 15) = 8 \text{ m/s} \quad \textbf{Ans.}$$

Example 17.7. *A train is uniformly accelerated and passes successive kilometre stones with velocities of 18 km.p.h. and 36 km.p.h. respectively. Calculate the velocity, when it passes the third kilometre stone. Also find the time taken for each of these two intervals of one kilometre.*

Solution. First of all, consider the motion of the train between the first and second kilometre stones. In this case, distance (s) = 1 km = 1000 m ; initial velocity (u) = 18 km.p.h. = 5 m/s ; and final velocity (v) = 36 km.p.h. = 10 m/s

Velocity with which the train passes the third km stone

Let v = Velocity with which the train passes the third km, and

a = Uniform acceleration.

We know that
$$v^2 = u^2 + 2as$$
$$(10)^2 = (5)^2 + (2a \times 1000) = 25 + 2000\,a$$
$$\therefore \quad a = \frac{100 - 25}{2000} = \frac{75}{2000} = 0.0375 \text{ m/s}^2$$

Now consider the motion of the train between the second and third kilometre stones. In this case, distance (s) = 1 km = 1000 m and initial velocity (u) = 36 km.p.h. = 10 m/s.

We know that
$$v^2 = u^2 + 2as = (10)^2 + (2 \times 0.0375 \times 1000) = 175$$
$$\therefore \quad v = 13.2 \text{ m/s} = 47.5 \text{ km.p.h.} \qquad \textbf{Ans.}$$

Time taken for each of the two intervals of one kilometre

Let t_1 = Time taken by the train to travel the first one kilometre, and

t_2 = Time taken by the train to travel the second kilometre.

We know that velocity of the train after passing the first kilometre *i.e.*, in t_1 seconds (v_1),
$$10 = u + at_1 = 5 + 0.0375\,t_1$$
$$\therefore \quad t_1 = \frac{10 - 5}{0.0375} = 133.3 \text{ s} \quad \textbf{Ans.}$$

Similarly, velocity of the train after passing the second kilometre *i.e.* in t_2 seconds,
$$13.2 = u + at_2 = 10 + 0.0375\,t_2$$
$$\therefore \quad t_2 = \frac{13.2 - 10}{0.0375} = 85.3 \text{ s} \qquad \textbf{Ans.}$$

Example 17.26. *A body was thrown vertically downwards from the top of a tower and traverses a distance of 40 metres during its 4th second of its fall. Find the initial velocity of the body.*

Solution. Given : Distance traversed (s) = 40 m ; No of second (n) = 4 and acceleration (a) = g = 9.8 m/s^2

Let u = Initial velocity of the body.

We know that distance traversed by the body in the 4th second (s),

$$40 = u + \frac{a}{2}(2n - 1) = u + \frac{9.8}{2}(2 \times 4 - 1) = u + 34.3$$

or $u = 40 - 34.3 = 5.7$ m/s **Ans.**

Alternative Method

We know that distance travelled in 3 seconds

$$s_3 = ut + \frac{1}{2}gt^2 = u \times 3 + \frac{1}{2} \times 9.8(3)^2 = 3u + 44.1 \text{ m}$$

and distance travelled in 4 seconds,

$$s_4 = ut + \frac{1}{2}gt^2 = u \times 4 + \frac{1}{2} \times 9.8(4)^2 = 4u + 78.4 \text{ m}$$

$\therefore$ Distance traversed in the 4th second

$$40 = s_4 - s_3 = (4u + 78.4) - (3u + 44.1) = u + 34.3$$

or $u = 40 - 34.3 = 5.7$ m/s **Ans.**