Mathematical Physics

Tensor Analysis

Scaler → Tensor of rank 0
vector → Tensor of rank 1 (comes under Tensor)
Dyadic → Tensor of rank 2 (magnitude & 2 dir^n)

* Rank of Tensor

No. of indics attached to a Physical quantity

eg. → only 1 subscript ∴ 1 Rank

σ_{ij} → Rank 2

$3^2 \rightarrow 9$ components

→ Tensors are originated or developed by G. Ricci

→ Tensor is a quantity which is invariant under coordinate Transformation and which obey certain ~~transformation~~ transformation laws

if u, v are vectors

if v is a linear form of u

$$\begin{cases} v_x = a_{xx}u_x + a_{xy}u_y + a_{xz}u_z \\ v_y = a_{yx}u_x + a_{yy}u_y + a_{yz}u_z \\ v_z = a_{zx}u_x + a_{zy}u_y + a_{zz}u_z \end{cases}$$

in matrix form

$$\begin{bmatrix} U_x \\ U_y \\ U_z \end{bmatrix} = \begin{bmatrix} a_{xx} & a_{xy} & a_{xz} \\ a_{yx} & a_{yy} & a_{yz} \\ a_{zx} & a_{zy} & a_{zz} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$$

$U = \Phi u \rightarrow$ Tensor dyadic

$x^{-1} = x^{-1}(x, x_1, x_2 \ldots x_n)$

$x^{-2} = x^{-2}(x^1, x^2, x^3, \ldots x^n)$

$\vdots$

$x^{-n} = x^{-n}(x^1, x^2, \ldots x^n)$

$x_1 \rightarrow 1-D$
$x_2 \rightarrow 2-D$
$x_3 \rightarrow 3-D$
$x_4 \rightarrow n-D$

Example : J, E , $J = \sigma E$

isotropic → Properties doesn't vary w.r.t. dir^n
anisotropic → diff Properties in diff. dir^n

isotropic →

$J_1 = \sigma E_1$
$J_2 = \sigma E_2$
$J_3 = \sigma E_3$

$J_1 = \sigma_{11} E_1 + \sigma_{12} E_2 + \sigma_{13} E_3$
$J_2 = \sigma_{21} E_1 + \sigma_{22} E_2 + \sigma_{23} E_3$
$J_3 = \sigma_{33} E_1 + \sigma_{32} E_2 + \sigma_{33} E_3$

$$\begin{bmatrix} J_1 \\ J_2 \\ J_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}$$

↓

Tensor of Rank 2 (as we have 2 subscripts) ∴ has 9 complements

These 9 comb. with electric is trafoming into current density

in this case we have 3 dimensions & Rank 2
∴ complements $= (3)^2 = 9$

$$\boxed{\text{Components} = (\text{dimensions})^{\text{Rank}}}$$

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* Einstein's Summation convention

(S = Sum of series) $S = a_1 x^1 + a_2 x^2 + \cdots + a_n x^n$

$S = \sum_{i=1}^{n} a_i x^i$

if suffix occurs twice in a term then, the suffic implys sum over the define range.

* **Dummy index & Real index** :- in which index is repeating and it can be replace with another index

eg $a_i x^i = a_j x^j$

Real index (free index) : In which index cannot be Replaced.

$a_i x^i \neq a_j x^j$

+ Kronecker Delta

$\delta^i_j = 1$ if $i = j$

$= 0$ if $i \neq j$

Properties :

① If $x^1, x^2, \ldots x^n$ are independent coordinates or variables then $\delta^i_j = \frac{\partial x^i}{\partial x^j}$ or $\frac{\partial x^i}{\partial x^k} \cdot \frac{\partial x^k}{\partial x^j} = \delta^k_j$

② $\delta^j_k \cdot a^j = a^k$

Proof. if $k = 2$, $a^i \delta^i_k = a_1 \delta^1_2 + a^2 \delta^2_2 + \cdots a^n \delta^n_2$

$= 0 + a^2(1) + \cdots 0$

$= a^2$

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(3) if we are dealing with n dimension

$$\delta^i_i = n$$

Proof: $\delta^1_1 + \delta^2_2 + \delta^3_3 + \cdots \delta^n_n$ $\quad \left[\delta^1_1 = 1 = \delta^2_2 = \delta^3_3 = 1\right.$
$= 1+1+1+ \cdots n$ $\quad \left.\delta^2_1 = \delta^3_1 = 0\right]$
$= n$

(4) $\delta^i_j \delta^j_k = \delta^i_k$

Proof: $\frac{\partial x^i}{\partial x^j} \cdot \frac{\partial x^j}{\partial x^k} = \frac{\partial x^i}{\partial x^k} = \delta^i_k$

Ques: $a_{ij} x^i x^j = 0$, where a_{ij} = Constant then show that $a_{ij} + a_{ji} = 0$

Sol

$$a_{ij} x^i x^j = 0$$

Similarly $a_{lm} x^l x^m = 0$ (∵ use of dummy index)

differentiate w.r.t. x_i

$$\frac{\partial}{\partial x_i}(a_{lm} x^l x^m) = 0$$

$$a_{lm} \frac{\partial}{\partial x_i}(x^l x^m) = 0$$

$$a_{lm} \frac{\partial x^l}{\partial x_i} \cdot x^m + a_{lm} \frac{\partial x^m}{\partial x_i} \cdot x^l = 0$$

$$a_{lm} \delta^l_i x^m + a_{lm} \delta^m_i x^l = 0 \quad \left(\text{using } \delta^l_i = \frac{\partial x^l}{\partial x_i}\right)$$

$$a_{im} x^m + a_{li} x^l = 0 \quad (\delta^j_k \cdot a^j = a^k)$$

again differentiate w.r.t j

$$a_{im} \frac{\partial x^m}{\partial x_j} + a_{li} \frac{\partial x^l}{\partial x_j} = 0$$

$\Rightarrow a_{im}\delta_j^m + a_{il}\delta_j^l = 0$

$$\boxed{a_{ij} + a_{ji} = 0}$$

(using Property ② $\delta_k^j \cdot a^j = a^k$)

* Contravarient & Covariant Tensors

Contravarient → Always super script

Covariant → Always subscript

If n quantities $A^1, A^2, \ldots A^n$ in a coordinate sys. $(x^1, x^2, \ldots x^n)$ are relates to n other quantities $A^{-1}, A^{-2}, \ldots A^{-n}$ in another coordinate system. $x^{-1}, x^{-2}, \ldots, x^{-n}$, Then Acc. to Transformation law

$$A^{-p} = \sum_{q=1}^{n} \frac{\partial x^{-p}}{\partial x^q} \cdot A^q \quad (p = 1, 2, \ldots n) \quad — ①$$

A^q → components of contravairent vectors.

Multiply eq ① by $\frac{\partial x^k}{\partial x^{-p}}$

$$\frac{\partial x^k}{\partial x^{-p}} \cdot A^{-p} = \frac{\partial x^k}{\partial x^{-p}} \cdot \frac{\partial x^{-p}}{\partial x^q} \cdot A^q$$

$$= \frac{\partial x^k}{\partial x^q} A^q$$

$$\frac{\partial x^k}{\partial x^{-p}} A^{-p} = \delta_q^k A^q \quad (\because k = q \ \therefore \delta_k^k = 1)$$

$$\frac{\partial x^q}{\partial x^{-p}} A^{-p} = A^q$$

By applying summation covention to eq ①

$$A^{-p} = \frac{\partial x^{-p}}{\partial x^q} A^q$$

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Covariant Tensors

If n quantities $A_1, A_2, \ldots A_n$ in a coordinate system $x_1, x_2, \ldots x_n$ are related to n other quantities $\bar{A}_1, \bar{A}_2, \bar{A}_3, \ldots \bar{A}_n$ in another coordinate system $\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots \bar{x}_n$, Then

Covariant Tensors → $\boxed{\bar{A}_{pr} = \frac{\partial x^q}{\partial \bar{x}^p} \frac{\partial x^s}{\partial \bar{x}^r} A_{qs}}$

→ Mixed tensors (Min rank for mixed tensors should be 2)

If n^2 quantities of A^q_s in a coordinate system $x^1, x^2, \ldots x^n$ are related to other n^2 quantities $\bar{A}^p_r$ in another coordinate system $(\bar{x}^1, \bar{x}^2, \ldots \bar{x}^n)$

Acc. to law of Transformation

$$\bar{A}^p_r = \frac{\partial \bar{x}^p}{\partial x^q} \cdot \frac{\partial x^s}{\partial \bar{x}^r} \cdot A^q_s$$

$$\bar{A}^{prm}_{ij} = \frac{\partial \bar{x}^p}{\partial x^q} \cdot \frac{\partial \bar{x}^r}{\partial x^s} \frac{\partial \bar{x}^m}{\partial x^t} \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} \cdot A^{qst}_{kl}$$

first write contravariant terms, then covariant

Ques. Prove that Kronecker Delta is a mixed tensor of rank 2

Sol

$$\delta^i_j = 0 \quad \text{if } i \neq j$$
$$= 1 \quad \text{if } i = j$$

$$\bar{\delta}^k_j = \frac{\partial \bar{x}^k}{\partial \bar{x}^j} = \frac{\partial \bar{x}^k}{\partial x^i} \cdot \frac{\partial x^j}{\partial \bar{x}^l} \frac{\partial x^i}{\partial x^j}$$

$$\boxed{\bar{\delta}^k_j = \frac{\partial \bar{x}^k}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^j} \delta^i_j}$$

Acc. to transformation law

$$\bar{A}_p = \sum_{q=1}^{n} \frac{\partial x^q}{\partial \bar{x}^p} A_q \quad \text{—} (1) \qquad (p = 1, 2, \ldots n)$$

$$\bar{A}_p = \frac{\partial x^q}{\partial \bar{x}^p} \cdot A_q$$

Multiplying by $\frac{\partial \bar{x}^k}{\partial x^q}$

$$\frac{\partial \bar{x}^k}{\partial x^q} \bar{A}_p = \frac{\partial x^q}{\partial \bar{x}^p} \frac{\partial \bar{x}^k}{\partial x^q} A_q$$
$$= \frac{\partial \bar{x}^k}{\partial \bar{x}^p} A_q$$
$$= \delta^k_p A_q \quad (\because k = p \therefore \delta^k_p = 1)$$
$$= A_q$$

* 2nd Rank Tensors

Transformation laws

(1) $$\bar{A}^{pr} = \sum_{s=1}^{n} \sum_{q=1}^{n} \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial \bar{x}^r}{\partial x^s} A^{qs}$$

Contravariant Tensors

If n^2 quantities A^{qs} in a coordinate system $x^1, x^2, x^3 \ldots$ related to $\bar{A}^{pr}$ in another coordinate system $\bar{x}^1, \bar{x}^2, \ldots$ then acc. to transformation law

$$\bar{A}^{pr} = \sum_{s=1}^{n} \sum_{q=1}^{n} \frac{\partial \bar{x}^p}{\partial x^q}$$

$$\cancel{\bar{A}^{pr} = \frac{\partial x^q}{\partial \bar{x}^p} \cdot \frac{\partial x^s}{\partial \bar{x}^r} A_{qs}}$$

* Equality of Tensors

Two tensors $A^{i_1 \cdots i_p}_{j_1 \cdots j_q}$ and $B^{i_1 \cdots i_p}_{j_1 \cdots j_q}$ equal if and only if they have same contravariant rank ~~k~~ & same covariant rank & every component of ~~tensors~~ one is equal to the corresponding component of the other

Then,

$$\boxed{A^{i_1 \cdots i_p}_{j_1 \cdots j_q} = B^{i_1 \cdots i_p}_{j_1 \cdots j_q}}$$

* Addition & Subtraction of Tensors

Addition → The sum of two or more tensor of same rank & same type is also a tensor of same ~~&~~ rank & type.

eg → $\boxed{A_{ik} + B_{ik} = C_{ik}}$

Subtraction → The differences of two or more tenser of the same rank & type is also a tensor of same rank & type.

eg → $\boxed{A_{ik} - B_{ik} = D_{ik}}$

$$A_{ik} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad B_{ik} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$A_{ik} + B_{ik} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

:

* Contraction

$$A^{pqr}_{st} = A^{pq}_{s} \quad (r=t)$$

The algebric operation by which rank of a matrix tensor is reduce by 2.

The Process of getting a tensor of lower order by Putting a covariant index = Contravariant index is known as contraction.

$$\alpha \rightarrow \alpha - 2$$

eg

$$\bar{A}^{ijk}_{lm} = \frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial \bar{x}^j}{\partial x^q} \frac{\partial \bar{x}^k}{\partial x^r} \frac{\partial x^s}{\partial \bar{x}^l} \frac{\partial x^t}{\partial \bar{x}^m} A^{pqr}_{st}$$

$$= \frac{\partial \bar{x}^i}{\partial x^p} \cdot \frac{\partial \bar{x}^j}{\partial x^q} \cdot \frac{\partial \bar{x}^k}{\partial x^r} \cdot \frac{\partial x^s}{\partial \bar{x}^l} \frac{\partial x^r}{\partial \bar{x}^m} \cdot A^{pqr}_{sr} \quad (r=t)$$

$$= \frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial \bar{x}^j}{\partial x^q} \frac{\partial \bar{x}^k}{\partial \bar{x}^m} \cdot \frac{\partial x^s}{\partial \bar{x}^l} A^{pq}_{s}$$

$$\bar{A}^{ijk}_{lm} = \frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial \bar{x}^j}{\partial x^q} \delta^{k}_{m} \frac{\partial x^s}{\partial \bar{x}^l} A^{pq}_{s}$$

$$\bar{A}^{ij}_{l} = \frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial \bar{x}^j}{\partial x^q} \frac{\partial x^s}{\partial \bar{x}^l} A^{pq}_{s} \quad \left(\text{if } k=m \;\; \delta^{k}_{m} = 1\right)$$

Ques If A^i_j is a mixed tensor of rank 2 & B^{kl}_m is a mixed tensor of rank 3. Prove that $A^i_j B^{kl}_m$ is a mixed tensor of rank 3

OR

Show that any inner Product of the tensors A^i_j & B^{kl}_m is a tensor of rank 3.

Sol
$$\bar{A}^p_q = \frac{\partial \bar{x}^p}{\partial x^i}\frac{\partial x^j}{\partial \bar{x}^q} A^i_j \quad -(1)$$

$$\bar{B}^{rs}_t = \frac{\partial \bar{x}^r}{\partial x^k}\frac{\partial \bar{x}^s}{\partial x^l}\frac{\partial x^m}{\partial \bar{x}^t} B^{kl}_m \quad -(2)$$

Multiplying both eq.

$$\bar{A}^p_q \bar{B}^{rs}_t = \frac{\partial \bar{x}^p}{\partial x^i}\frac{\partial x^j}{\partial \bar{x}^q}\cdot\frac{\partial \bar{x}^r}{\partial x^k}\cdot\frac{\partial \bar{x}^s}{\partial x^l}\frac{\partial x^m}{\partial \bar{x}^t}\cdot A^i_j B^{kl}_m$$

$$\bar{C}^{prs}_{qt} = \frac{\partial \bar{x}^p}{\partial x^i}\frac{\partial \bar{x}^r}{\partial x^k}\frac{\partial \bar{x}^s}{\partial x^l}\cdot\frac{\partial x^m}{\partial \bar{x}^t} C^{ikl}_{jm}$$

$$\bar{C}^{prs}_{qs} = \frac{\partial \bar{x}^p}{\partial x^i}\frac{\partial \bar{x}^r}{\partial x^k}\frac{\partial \bar{x}^s}{\partial \bar{x}^t}\frac{\partial x^j}{\partial \bar{x}^q} C^{ik}_j \quad (l = m)$$

$$\bar{C}^{prs}_{qs} = \frac{\partial \bar{x}^p}{\partial x^i}\frac{\partial \bar{x}^r}{\partial x^k}\frac{\partial x^j}{\partial \bar{x}^q} C^{ik}_j$$

* Quotient law:

If $A \times$ tensor $\rightarrow$ tensor,
then A is also a tensor

Ques If $A^i B_{jk}$ is a tensor of all contravariant tensor A^i & B_{jk} is also a tensor

Sol
$$\bar{A}^\alpha \bar{B}_{\beta\gamma} = \frac{\partial x^j}{\partial \bar{x}^\beta}\frac{\partial x^k}{\partial \bar{x}^\gamma}\frac{\partial \bar{x}^\alpha}{\partial x^i} A^i B_{jk}$$

$$\bar{A}^\alpha \bar{B}_{\beta\gamma} = \frac{\partial x^j}{\partial \bar{x}}\frac{\partial x^k}{\partial \bar{x}^\gamma}\frac{\partial \bar{x}^\alpha}{\partial x^i} A^i B_{jk}$$

$$\boxed{\bar{A}^\alpha = \frac{\partial \bar{x}^\alpha}{\partial x^i} A^i}$$

$$A^{\alpha}\bar{B}_{\beta\gamma} = \frac{\partial x^{l}}{\partial \bar{x}^{\beta}}\frac{\partial x^{k}}{\partial \bar{x}^{\gamma}} A^{-\alpha} B_{lk}$$

$$A^{-\alpha}\left[\bar{B}_{\beta\gamma} - B_{lk}\frac{\partial x^{l}}{\partial \bar{x}^{\beta}}\frac{\partial x^{k}}{\partial \bar{x}^{\gamma}}\right] = 0$$

$$\because A^{-\alpha} \neq 0$$

$$\bar{B}_{\beta\gamma} = \frac{\partial x^{l}}{\partial \bar{x}^{\beta}}\frac{\partial x^{k}}{\partial \bar{x}^{\gamma}} B_{lk}$$

$$A^{-s}_{t} = \frac{\partial \bar{x}^{s}}{\partial x^{p}}\cdot\frac{\partial x^{t}}{\partial \bar{x}^{q}}\cdot A^{p}_{q}$$

* Multiplication

→ outer Product :- The outer Product of two tensors is a tensor whose rank is the sum of the ranks of two multiplied tensor

eg. $A^{pr}_{q} B^{m}_{s} = C^{prm}_{qs}$ — 5 Rank

3 Rank 2 Rank

$$A^{-ij}_{k} = \frac{\partial \bar{x}^{i}}{\partial x^{p}}\cdot\frac{\partial \bar{x}^{j}}{\partial x^{r}}\frac{\partial x^{q}}{\partial \bar{x}^{k}} A^{pr}_{q} \quad \text{——} (1)$$

$$B^{-\alpha}_{\beta} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{m}}\cdot\frac{\partial x^{s}}{\partial \bar{x}^{\beta}}\cdot B^{m}_{s} \quad \text{——} (2)$$

Multiplying (1) & (2)

$$A^{-ij}_{k}\cdot B^{-\alpha}_{\beta} = \frac{\partial \bar{x}^{i}}{\partial x^{p}}\cdot\frac{\partial \bar{x}^{j}}{\partial x^{r}}\cdot\frac{\partial x^{q}}{\partial \bar{x}^{k}}\cdot\frac{\partial \bar{x}^{\alpha}}{\partial x^{m}}\cdot\frac{\partial x^{s}}{\partial \bar{x}^{\beta}}\cdot A^{pq}_{r}\cdot B^{m}_{s}$$

$$C^{-ij\alpha}_{k\beta} = \frac{\partial \bar{x}^{i}}{\partial x^{p}}\frac{\partial \bar{x}^{j}}{\partial x^{r}}\frac{\partial x^{q}}{\partial \bar{x}^{k}}\cdot\frac{\partial \bar{x}^{\alpha}}{\partial x^{m}}\cdot\frac{\partial x^{s}}{\partial \bar{x}^{\beta}}\cdot C^{pqm}_{rs}$$

→ Inner Product : The outer Product of two tensors followed by results in a new tensor called an inner Product of two tensor. Sometimes this Product is called

$$A^{mp}_{q} : B^{r}_{s} = C^{mpr}_{qs} \rightarrow \text{outer Product}$$

$$C^{mpr}_{qs} = C^{mp}_{q} \quad (\text{when } r = s)$$

* <u>Symmetric Tensor</u> : A tensor is called symmetric w.r.t 2 contravariant or covariant indics if its components remain unaltered upon interchange of indics

eg. $A^{ij} = A^{ji}$ → symmetric w.r.t ij

$A^{pqr} = A^{qpr}$ → Symmetric w.r.t p & q only

eg. $\left.\begin{array}{l} A^{ij} = -A^{ji} \\ A^{pqr} = -A^{qpr} \end{array}\right\}$ Skew symmetric

* <u>Skew Symmetric</u> : A tensor is called Skew symm. w.r.t 2 contravariant or too covariant if it components change sign apart interchange the indices.

eg $A^{ij} = -A^{ji}$

Q Show that every tensor can be express as the sum of two tensor, one of which is symmatric & the other is skew sym.

OR

Show that every tensor of 2 Rank can be result into symmatric & antisymmatric tensor.

Sol

$$B^{pq} = \frac{1}{2}(B^{pq} + B^{qp}) + \frac{1}{2}(B^{pq} - B^{qp})$$

$$= C^{pq} + D^{pq}$$

$$C^{pq} = C^{qp} \rightarrow \text{Symmetric}$$

$$D^{qp} = \frac{1}{2}(B^{qp} - B^{pq}) = -\frac{1}{2}(B^{pq} - B^{qp})$$

$$D^{qp} = -D^{pq} \rightarrow \text{Skew sym.}$$

Q if a tensor B^{pq} is a symmetric w.r.t indices $p\,q$ in 1 coordinate sys. Show that it remain symmetric w.r.t p & q in any coordinate system.

Q Show that a symmetric tensor of the
(i) 2 oder as only $\frac{1}{2}n(n+1)$ different components
(ii) A skew sys tensor of the 2nd order tensor as only $\frac{1}{2}n(n-1)$ diff. comp.

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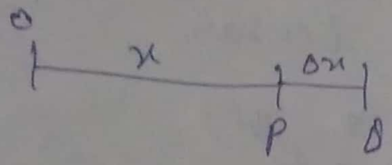
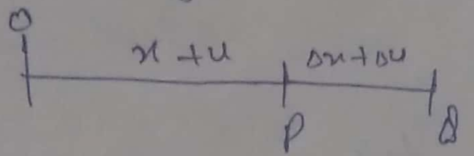
* Application of Tensor

→ 1 D String → Strain

Strain type – Extension
– Compression
– change angle



After applying force



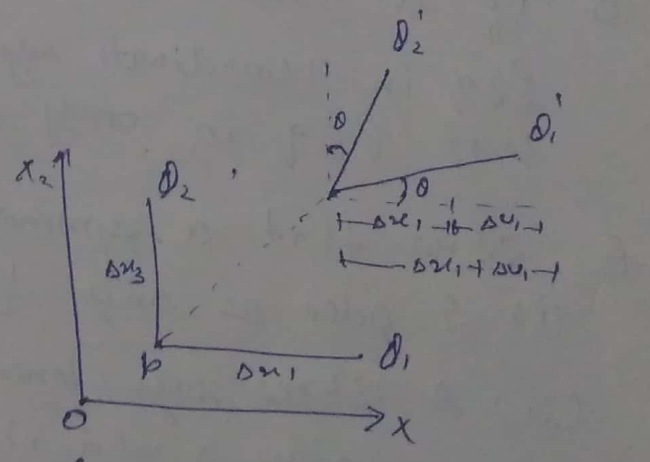
$$\text{Strain} = e = \frac{\Delta u}{\Delta x} = \frac{\text{increase in length}}{\text{original length}}$$

$$\boxed{e = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \frac{du}{dx}}$$ Component of Strain

→ 2 D

e_{11} (e along x_1) $= \dfrac{\partial u_1}{\partial x_1}$

$e_{22} = \dfrac{\partial u_2}{\partial x_2}$



(extension in x or y dir^n)

clockwise

anticlock wise

$e_{12} = \dfrac{\partial u_1}{\partial x_2}$, $e_{21} = \dfrac{\partial u_2}{\partial x_1}$

Represents change in Rotation

(extension along x_1 dir^n)

(change in u_1 w.r.t. x_2)

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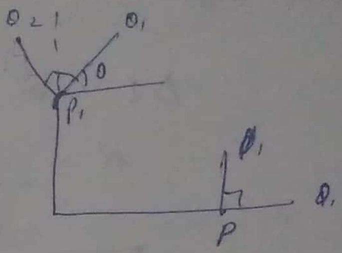
$$\tan\theta = \frac{\delta u_2}{\delta x_1 + \delta u_1} = \frac{\delta u_2}{\delta x_1}$$ (δu_1 is very small $\therefore$ it is neglected)

$$\theta = \tan\theta = \frac{\delta u_2}{\delta x_1} = \frac{\partial u_2}{\partial x_1} = e_{21}$$



$$\begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix}$$

$$e_{ij} = \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix}$$

(if there is no change in compression & expension $e_{ij} = 0$ (if $i=j$) eg ($e_{11}, e_{22}, e_{33} \dots = 0$))

$$e_{ij} = \frac{e_{ij} + e_{ji}}{2} + \frac{e_{ij} - e_{ji}}{2}$$

$$\begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} = \begin{bmatrix} e_{11} & \frac{e_{12}+e_{21}}{2} & \frac{e_{13}+e_{31}}{2} \\ \frac{e_{21}+e_{12}}{2} & e_{22} & \frac{e_{23}+e_{32}}{2} \\ \frac{e_{31}+e_{13}}{2} & \frac{e_{32}+e_{23}}{2} & e_{33} \end{bmatrix} + \begin{bmatrix} 0 & \frac{e_{12}-e_{21}}{2} & \frac{e_{13}-e_{31}}{2} \\ \frac{e_{21}-e_{12}}{2} & 0 & \frac{e_{23}-e_{32}}{2} \\ \frac{e_{31}+e_{13}}{2} & \frac{e_{32}-e_{23}}{2} & 0 \end{bmatrix}$$

let $e_{12} = -e_{21}$, $e_{13} = -e_{31}$, $e_{21} = -e_{12}$ ---- so on

$$= \begin{bmatrix} e_{11} & 0 & 0 \\ 0 & e_{22} & 0 \\ 0 & 0 & e_{33} \end{bmatrix} + \begin{bmatrix} 0 & e_{12} & -e_{31} \\ -e_{21} & 0 & e_{32} \\ e_{31} & -e_{23} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{32} \\ e_{31} & e_{23} & e_{33} \end{bmatrix}$$

= RHS

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* Thermal Expansion

Coeff. of (α) Thermal exp. = $\dfrac{\text{increase in length}}{\text{initial length} \times \text{rise in temp.}}$

$$\Delta L = \alpha L \Delta T$$

$$\alpha = \frac{l_2 - l_1}{l_1 \times \Delta T} = \frac{\varepsilon}{\Delta T}$$

$$\boxed{\varepsilon_{ij} = \alpha_{ij} \Delta T}$$

Relation b/w Strain & Temp.

When we write Matrix from Tensor we use dim. Notation

Matrix		Tensor
11	$\rightarrow$	1
22	$\rightarrow$	2
33	$\rightarrow$	3
23, 32	$\rightarrow$	4
31, 13	$\rightarrow$	5
12, 21	$\rightarrow$	6

$1 + \alpha \Delta T$

$1 + \alpha_1 \Delta T \rightarrow$ along x dir^n

$1 + \alpha_2 \Delta T \rightarrow$ " y "

$1 + \alpha_3 \Delta T \rightarrow$ " z "

$$\delta \Delta = l^3 (1+\alpha_1)(1+\alpha_2)(1+\alpha_3) - l^3$$

$$= V(\alpha_1 + \alpha_2 + \alpha_3) \qquad (l^3 = V \text{ volume})$$

$$\boxed{\frac{\delta V}{V} = \alpha_v = \alpha_1 + \alpha_2 + \alpha_3} \text{ — Thermal expansion}$$

* Piezoelecticity

Develpment of dipolo by applying stress in a crystal

$$P = d\sigma$$

$d \to$ Piezoelectric Modular

The direct Piezoelectric is a devolpment of electric dipole in a crystal when a stress is applying.

$P_i = d_{ijk}\, \sigma_{ik} \longrightarrow$ 2 dir^n (where $i, j, k = 1, 2, 3 \ldots$)

$d_{ijk} \to$ 3 dir^n in 3D

$\sigma_{jk} = \sigma_{kj}$

$d_{ijk} = d_{ikj}$

d_{ijk} is symmetric w.r.t d_{ikj}

if it is symmetric 27 components.

$P_i = d_{in}\, \sigma_n \quad \left(\begin{array}{l} i = 1, 2, 3 \\ n = 1, 2, \ldots, 6 \end{array} \right)$

(To convert Tensor to matrix) Tensor to Matrix Rule

$d_{ijk} = d_{in} \quad , \quad n = 1, 2, 3$

$2\, d_{ijk} = d_{in} \quad , \quad n = 4, 5, 6$

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} & d_{13} & d_{14} & d_{15} & d_{16} \\ d_{21} & d_{22} & d_{23} & d_{24} & d_{25} & d_{26} \\ d_{31} & d_{32} & d_{33} & d_{34} & d_{35} & d_{36} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix}$$

$\sigma_1, \sigma_2, \sigma_3 \to$ Compression, Extension

$\sigma_5, \sigma_4, \sigma_6 \to$ Rotation

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2 Conversion Piezoelectric effect

$$\varepsilon_{jk} = d_{ijk} E_i$$

$$\varepsilon = (d)_{transverse} \cdot E$$ (Relation b/w strain & Electric field)

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} d_{11} & d_{21} & d_{31} \\ d_{12} & d_{22} & d_{32} \\ d_{13} & d_{23} & d_{33} \\ d_{14} & d_{24} & d_{34} \\ d_{15} & d_{25} & d_{35} \\ d_{16} & d_{26} & d_{36} \end{bmatrix} = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}$$

$$\begin{array}{ccc} & & \begin{matrix} \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & \varepsilon_5 & \varepsilon_6 \end{matrix} \\ E_1 & \cdot P_1 & \\ E_2 & \cdot P_2 & \begin{bmatrix} d_{11} & d_{12} & d_{13} & d_{14} & d_{15} & d_{16} \\ d_{21} & d_{22} & d_{23} & d_{24} & d_{25} & d_{26} \\ d_{31} & d_{32} & d_{33} & d_{34} & d_{35} & d_{36} \end{bmatrix} \\ E_3 & \cdot P_3 & \end{array}$$

	$(i\,j\,k = 1,2,3)$ Tensor	$\left(\begin{matrix} i = 1,2,3 \\ j = 1,2,\ldots 6 \end{matrix}\right)$ Matrix
Direct	$P_i = d_{ijk} \sigma_{jk}$	$P_i = d_{ij} \sigma_i$
Conversion	$\varepsilon_{ij} = d_{ijk} E_i$	$\varepsilon_j = d_{ij} E_i$

strain field

31-Aug-16

Partial Diffentiation

→ An equation involving one or more Partial derivatives of an unknown fun. of two or more variables is called Partial differential equation.

$$p = \frac{\partial z}{\partial x} \qquad q = \frac{\partial z}{\partial y}$$

$$\frac{\partial^2 z}{\partial u^2} + \left(\frac{\partial^2 z}{\partial u \partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^6 = 0$$

higher degree = 2

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{dx^2} \longrightarrow \text{1D wave eq.}$$

$$\frac{\partial u}{dt} = c^2 \frac{\partial^2 u}{\partial x^2} \longrightarrow \text{1D heat flow eq.}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \longrightarrow \text{2D laplace eq.}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y) \longrightarrow \text{2D Paisson eq.}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \longrightarrow \text{2D wave eq.}$$

$$u = x^2 \sin t$$

$$\frac{\partial u}{\partial x} = x^2 \cos t \frac{\partial t}{\partial x} + \sin t \, 2x$$

$$\frac{\partial u}{\partial t} = x^2 \cos t + \sin 2t \, 2x$$

$$u = (x^2+t^2)^2 + e^x$$

$$u = e^4 + t^4 + 2x^2t^2 + e^x$$

$$\frac{\partial u}{\partial x} = 4x^3 + 4t^2x + e^x = 4x(x^2+t^2) + e^x$$

$$\frac{\partial u}{\partial t} = 4t^3 + 4tx^2 \Rightarrow 4t(t^2+x^2)$$

Method of Solving Partial diff. eq.

* Method of Separation of variable

Q Apply the method of separation to solve

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0 \quad — (1)$$

Sol Assuming sol $z = X(x)\, Y(y)$ — (2)

Substitute eq (2) in eq (1)

$$X''Y - 2X'Y + XY' = 0$$

$$Y(X'' - 2X') = -XY'$$

$$\frac{X'' - 2X'}{X} = \frac{-Y'}{Y}$$

$$\frac{X'' - 2X'}{X} = \frac{-Y'}{Y} = k$$

$X'' - 2X' = kX$, $-Y' = kY$

$X'' - 2X' - kX = 0$ $\qquad -Y' - kY = 0$

$m^2 - 2m - k = 0$ $\qquad m + k = 0 \Rightarrow m = -k$

$m = \frac{2 \pm \sqrt{4+4k}}{2}$ $\qquad Y = C_3 e^{-ky}$

$m = 1 \pm \sqrt{1+k}$

$$X = C_1 e^{(1+\sqrt{1+k})x} + C_2 e^{(1-\sqrt{1+k})x}$$

$\therefore$ $z = c_1 e^{(1+\sqrt{1+k})x} + c_2 e^{(1-\sqrt{1+k})x} \cdot c_3 e^{-ky}$

$z = c_4 e^{(1+\sqrt{1+k})x} + c_5 e^{(1-\sqrt{1+k})x} \cdot e^{-ky}$

Q Apply the method of separation. Find the sol for

$3u_x + 2u_y = 0$

$3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$

sol

$u = X(x)\,Y(y)$

$3X'Y + 2XY' = 0$

$3X'Y = -2XY'$

$\frac{3X'}{X} = -\frac{2Y'}{Y} = k$

$3X' - kx = 0$

$3m - k = 0$

$m = \frac{k}{3}$

$X = c_1 e^{\frac{k}{3}x}$

$-2Y' - kY = 0 \Rightarrow 2Y' + k = 0$

$2m + k = 0$

$m = -\frac{k}{2}$

$Y = c_2 e^{-\frac{k}{2}y}$

$u = c_1 e^{\frac{k}{3}x} \cdot c_2 e^{-\frac{k}{2}y} \Rightarrow c_1 c_2 e^{k(x/3 - y/2)}$

$\boxed{u = Ae^{k(x/3 - y/2)}}$ $(A = c_1 c_2)$

Q using a no. of separation variables

$\frac{\partial u}{\partial x} = \frac{2\partial u}{\partial t} + u$ $\quad u(x,0) = 6e^{-3x}$

sol

Q using the method of Separation method of Variable to Solve the eq. $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$.
Given that $U=0$ when $t \to \infty$ and $U=0$ when at $x=0$ & $x=l$

Sol

$$U = X(x)\,T(t)$$

$$X''T = XT'$$

$$\frac{X''}{X} = \frac{T'}{T} = -p^2$$

$X'' + p^2 X = 0$

$m^2 + p = 0$

$m = -p$

$\cancel{X = C_1 e}$

$X = C_2 \cos px + C_3 \sin px$

$T' + p^2 T = 0$

$m = -p^2$

$T = C_1 e^{-p^2 t}$

$T = C_1 e^{-p^2 t}$

using

$(e^{-\theta} = \cos\theta + \sin\theta)$

$$U = C_1 e^{-p^2} + (C_2 \cos px + C_3 \sin px)$$

$\rightarrow$ $X=0$, $U=0$ (given condition)

$0 = C_1 e^{-p^2} \cdot C_2$

$C_2 = 0$, $C_1 \neq 0$

$U = C_1 e^{-p^2} + C_3 \sin px$

$\rightarrow$ $U=0$ at $x=l$ (given condition)

$0 = C_1 e^{-p^2} + C_3 \sin pl$

$\sin pl = 0$

$pl = n\pi$

$p = \frac{n\pi}{l}$

$$v = c_1 c_3 \, e^{\left(-\frac{n^2\pi^2 t}{l^2}\right)} . \sin\left(\frac{n\pi x}{l}\right)$$

$$v = p_n \, e^{\left(-\frac{n^2\pi^2 t}{l^2}\right)} . \sin\left(\frac{n\pi x}{l}\right) \qquad (p_n = c_1 c_3)$$

Solving
Q Laplace eq. by the method of separation

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

sol

$$V = XY$$

$$X''Y + XY'' = 0$$

$$\frac{X''}{X} = \frac{-Y''}{Y} = k^2$$

$$X'' - k^2 X = 0$$

$$m^2 - k^2 = 0$$

$$m = \pm k$$

$$X = c_1 e^{kx} + c_2 e^{-kx}$$

$$-Y'' - k^2 Y = 0$$

$$Y'' + k^2 Y = 0$$

$$m^2 + k^2 = 0$$

$$m = -k$$

$$Y = c_3 e^{-ky}$$

$$= c_3 \cos ky + c_4 \sin ky$$

$$V = c_1 e^{kx} . c_2 e^{-ky}$$

$$k = -p^2 , \quad k = p^2 , \quad k = 0$$

$$k = -p^2 , \quad X = c_1 \cos px + c_2 \sin px$$

$$Y = c_3 e^{py} + c_4 e^{-py}$$

$$k = p^2 , \quad X = c_1 e^{px} + c_2 e^{-px}$$

$$Y = c_3 \cos py + c_4 \sin py$$

$$k = 0 , \quad X = c_1 x + c_2$$

$$Y = c_3 y + c_4$$

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Ques D'Alembert's solution for the wave eqⁿ

OR

Solution of wave eq. by D'Alembert's

OR

Show that solution of wave eq $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ can be express in the form of $y(x,t) = f_1(x+ct) + f_2(x-ct)$

if $y(x,0) = f(x)$ and $\frac{\partial y}{\partial t}(x,0) = 0$

Show that $y(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)]$

OR

Transform the eq. $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ to its normal form using the transformation $u = x+ct$, $v = x-ct$. Show that the solution $y = \frac{1}{2}[f(x+ct) + f(x-ct)]$ by assuming the initial condition $y = f(x)$ & $\frac{\partial y}{\partial t} = 0$ $|$ at $t=0$

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{—(1)}$$

Sol

$$u = x + ct, \quad v = x - ct \quad \text{—(2)}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 1 \qquad \frac{\partial u}{\partial t} = -\frac{\partial v}{\partial t} = c \quad \text{—(3)}$$

$$y = y(u,v) \quad \text{—(4)}$$

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial x}$$

$$= \frac{\partial y}{\partial u}\cdot 1 + \frac{\partial y}{\partial v}\cdot 1$$

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \quad \text{—(5)}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right)$$

$$= \frac{\partial^2 y}{\partial u^2} + \frac{2\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2} \quad — (6)$$

$$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial t}$$

$$= \frac{\partial y}{\partial u} c + \frac{\partial y}{\partial v}(-c)$$

$$\frac{\partial y}{\partial t} = c\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right) y$$

$$\frac{\partial}{\partial t} = c\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right) \quad — (7)$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}\right)$$

$$= c\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right) c\left(\frac{\partial y}{\partial u} - \frac{\partial y}{\partial v}\right)$$

$$\frac{\partial^2 y}{\partial t^2} = c^2\left(\frac{\partial^2 y}{\partial u^2} - \frac{\partial^2 y}{\partial v^2}\right) \quad — (8)$$

Putting eq (8) & (6) in eq (1)

$$\cancel{\frac{\partial^2 y}{\partial u^2}} + \frac{2\partial^2 y}{\partial u . \partial v} + \cancel{\frac{\partial^2 y}{\partial v^2}} = \frac{1}{\cancel{c^2}} \cancel{c^2}\left(\cancel{\frac{\partial^2 y}{\partial u^2}} + \cancel{\frac{\partial^2 y}{\partial v^2}} - \frac{2\partial^2 y}{\partial u \partial v}\right)$$

$$\frac{4\partial^2 y}{\partial u \partial v} = 0$$

$$\frac{\partial}{\partial u}\left(\frac{\partial y}{\partial v}\right) = 0 \quad — (9)$$

integration w.r.t u

$$\frac{\partial y}{\partial v} = f(v) \quad — (10)$$

Again integrate w.r.t v

$$y = f_1(u) + \int f(v)\,dv \qquad \left(f_2(v) = \int f(v)\,dv \right)$$
$$= f_1(u) + f_2(v) \quad \text{—} \quad (11)$$

$$y = f_1(x+ct) + f_2(x-ct)$$

$$y(x,0) = f(x), \quad \frac{\partial y}{\partial t} = 0 \quad \text{when } t=0$$

diff w.r.t y'

$$\frac{dy}{dt} = cf_1'(x+ct) - cf_2'(x-ct)$$

$$\frac{dy}{dt} = 0, \quad \text{when } t=0$$

$$0 = cf_1'(x) - cf_2'(x)$$
$$f_1'(x) = f_2'(x)$$

$$f_1(x) = f_2(x) + b$$
$$y = f(x) \quad \text{and } t \neq 0$$
$$f(x) = f_1(x) + f_2(x)$$
$$f(x) = [f_2(x) + b] + f_2(x)$$
$$f(x) = 2f_2(x) + b$$
$$f_2(x) = \frac{1}{2}[f(x) - b] \qquad \therefore f_1 = \frac{1}{2}[f(x) + b]$$

$$\therefore y(x,t) = f_1(x+ct) + f_2(x-ct)$$
$$y(x,t) = \frac{1}{2}[f(x+ct) + b] + \frac{1}{2}[f(x-ct) - b]$$
$$= \frac{1}{2}[f(x+ct) + f(x-ct)]$$

Solution of a 2D heat flow in a thin Rectangular Plate.

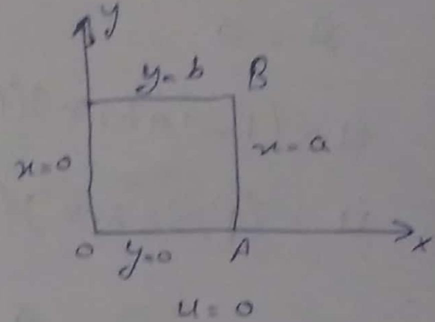
Ques A Thin rect. Plate two surfaces impervious 2 heat flow has arbitary distribution of temp. $f(x,y)$ at $t=0$. It's four edges $x=0, x=a, y=0, y=b$ are kept at 0 temp. Determine the subsequent temp of Plate after time.

Sol) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} = \frac{1}{h^2}\frac{\partial u}{\partial t} \quad \text{(2 D heat eq., } h^2 = \frac{k}{s\rho}\text{)} \quad \text{— (1)}$$

$t=0, \; u = f(x,y)$

$u = X(x)\,Y(y)\,T(t)$ — (2)

$X''YT + XY''T = \frac{1}{h^2}XYT'$



Dividing by XYT

$$\frac{X''}{X} + \frac{Y''}{Y} = \frac{1}{Th^2}T' \quad \text{— (3)}$$

let $\frac{1}{Th^2}T' = -k^2$ $\quad (k^2 = k_1^2 + k_2^2)$

$m + k^2Th^2 = 0$

$m = -k^2Th^2$

$m = Ce^{-k^2h^2T}$

$$\frac{X''}{X} + \frac{Y''}{Y} = -k^2$$

$X'' + Xk_1^2 = 0$

$m^2 + k_1^2 = 0$

$m = -k_1$

$Y'' + k_2^2 Y = 0$

$m^2 + k_2^2 = 0$

$m = -k_2$

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$X = A\cos k_1 x + B\sin k_1 x$

$Y = C\cos k_2 y + D\sin k_2 y$

$T = E e^{-h^2k^2t}$

$u = (A\cos k_1 x + B\sin k_1 x)(C\cos k_2 y + B\sin k_2 y)(E e^{-h^2k^2t})$

Boundary condition

(i) $u = 0, \quad x = 0$

$0 = A(C\cos k_2 y + B\sin k_2 y) E e^{-h^2k^2t}$

$A = 0$

$u = (B\sin k_1 x)(C\cos k_2 y + D\sin k_2 y)(E e^{-h^2k^2t})$

$u = (F\sin k_1 x)(C\cos k_2 y + D\sin k_2 y)(e^{-h^2k^2t})$

(ii) $u = 0, \quad x = a$

$0 = (F\sin k_1 a)(C\cos k_2 y + D\sin k_2 y)(e^{-h^2k^2T})$

$\sin k_1 a = 0$

$k_1 a = n\pi \Rightarrow k_1 = \frac{n\pi}{a}$

$u = \left(F\sin \frac{n\pi x}{a}\right)(C\cos k_2 y + D\sin k_2 y)(e^{-h^2k^2T})$

(iii) $u = 0, \quad y = 0$

$0 = \left(F\sin \frac{n\pi x}{a}\right) C e^{-h^2k^2t}$

$C = 0$

$u = \left(F\sin \frac{n\pi x}{a}\right)(D\sin k_2 y) e^{-h^2k^2t}$

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(iv) $u=0$, $y=b$

$$0 = \left(F\sin\frac{n\pi x}{a}\right)(D\sin k_2 b)e^{-h^2k^2t}$$

$$\sin k_2 b = 0$$

$$k_2 b = m\pi \Rightarrow k_2 = \frac{m\pi}{b}$$

$$u = \left(F\sin\frac{n\pi x}{a}\right)\left(D\sin\frac{m\pi y}{b}\right)e^{-h^2k^2t}$$

$$\boxed{\begin{aligned} k^2 &= k_1^2 + k_1^2 \\ &= \frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2} \end{aligned}}$$

$$u = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}.e^{-h^2k^2t}$$

<u>initial cond.</u> $u = f(x,y)$ at $t=0$

$$f(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}$$

$$\boxed{\text{formula used} \rightarrow \int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) = 0 \quad m \neq n \qquad = \frac{L}{2} \quad m=n}$$

$$\int_0^a\int_0^b f(x,y)\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\,dx\,dy$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\int_0^a \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi x}{a}\right)dx.\int_0^b \sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{m\pi y}{b}\right)dy$$

$$= A_{mn}\frac{a}{2}.\frac{b}{2}$$

$$= A_{mn}\frac{ab}{4}$$

$$A_{mn} = \frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)dx\,dy$$

Ques Solve the sol. of a eq. of vibrating Membrane

Ans
$$\frac{\partial^2 u}{\partial t^2} = v^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 y}{\partial y^2}\right)$$

$$\frac{\partial^2 u}{\partial t^2}\left(\frac{1}{v^2}\right) = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$\frac{\partial u}{\partial t} = 0$, $t = 0$ (intial cond.)

$x = 0$, $x = a$, $y = 0$, $y = b$ (Bound. cond.)

Sol

$$u = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$\left(A_{mn}\cos Pt + B_{mn}\sin Pt \quad P = \pi v\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}\right)$$

$\frac{\partial u}{\partial t} = 0$, $t = 0$

$$\frac{\partial u}{\partial t} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\left(-PA_{mn}\sin Pt + PB_{mn}\cos Pt\right)$$

$$0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\left(PB_{mn}\right)$$

$B_{mn} = 0$

$u = f(x,y)$ at $t = 0$

$$f(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

$$\int_0^a \int_0^b f(x,y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\, dx\, dy = \frac{ab}{4} A_{mn}$$

$$A_{mn} = \frac{4}{ab} \int_0^a \int_0^b f(x,y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\, dy\, dx$$

$$\boxed{u(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos pt}$$

$$\left(p = \pi v \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \right)$$

III UNIT **Numerical Analysis**

$$y = f(x)$$

indepen(d)ent — argument

y — entry

$x = x_0, x_0 + h, x_0 + 2h + \dots x_0 + nh$

$y = f(x)$

$y_0 = f(x_0), \; y_1 = f(x_0 + h), \; y_2 = f(x_0 + 2h) \dots y_n \; f_n(x_0 + nh)$

~~forward~~ ~~first~~ ~~difference~~

$y_1 - y_0, \; y_2 - y_1, \dots y_n - y_{n-1}$ (final difference)

$\Delta y_0, \quad \Delta y_1, \quad \Delta y_{n-1}$ (forward differance(Δ))

Δ - forward diff.

∇ - backward "

$y_1 - y_0 = \Delta y_0$ (first order diff.)

$\Delta y_1 - \Delta y_0 = \Delta^2 y_0$ (Second order difference)

general formula for ~~forward diff.~~ 2nd order diff.

$$\boxed{\Delta^2 y_r = \Delta y_{r+1} - \Delta y_r} \quad (r = 0, 1, 2, \dots n)$$

$\Delta f(x) = f(x+h) - f(x)$

$\Delta y_n = y_{n+1} - y_n$

(formula for forward diff. operator)

$$\boxed{\Delta^m y_r = \Delta^{m-1} y_{r+1} - \Delta^{m-1} y_2} \rightarrow \text{General formula for forward diff.}$$

* Construct the forward diff Table

$x_0, x_0+h, x_0+2h \ldots x_0+nh$

	Argument x	Entry y	1st diff Δy	2nd diff $\Delta^2 y$	3rd diff $\Delta^3 y$
x_1	x_0	y_0			
			$y_1 - y_0 = \Delta y_0$		
x_2	x_0+h	y_1		$\Delta y_1 - \Delta y_0 = \Delta^2 y_0$	
			$y_2 - y_1 = \Delta y_1$		$\Delta^2 y_1 - \Delta^2 y_0 = \Delta^3 y_0$
x_3	x_0+2h	y_2		$\Delta y_2 - \Delta y_1 = \Delta^2 y_1$	
			$y_3 - y_2 = \Delta y_2$		
x_4	x_0+3h	y_3			

$+nh)$

* Backward diff Table

$y_1, y_2 \ldots \ldots y_n$

$x_0, x_0+h \ldots \ldots x_0+nh$

$\nabla y_n = y_n - y_{n-1}$

$\nabla f(x) = f(x) - f(x-h)$

$\nabla y_1 = y_1 - y_0 \quad , \nabla y_2 = y_2 - y_1$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
x_0	y_0			
		$y_1 - y_0 = \nabla y_1$		
$x+h$	y_1		$\nabla y_2 - \nabla y_1 = \nabla^2 y_2$	
		$y_2 - y_1 = \nabla y_2$		$\nabla^2 y_3 - \nabla^2 y_2 = \nabla^3 y_3$
$x+2h$	y_2		$\nabla y_3 - \nabla y_2 = \nabla^2 y_3$	
		$y_3 - y_2 = \nabla y_3$		
$x+3h$	y_3			

* Central diff. operator

$$\delta f(x) = f(x + \tfrac{h}{2}) - f(x - h/2)$$

$$\delta y_{1/2} = y_1 - y_0$$

$$\delta y_n = y_{n+\frac{1}{2}} - y_{n-\frac{1}{2}}$$

General formula $\boxed{\delta^m y_n = \delta^{m-1} y_{n+\frac{1}{2}} - \delta^{m-1} y_{n-1}}$

x	y	δy	$\delta^2 y$	$\delta^3 y$
x_0	y_0			
		$\delta y_{\frac{1}{2}}$		
$x_0 + h$	y_1		$\delta^2 y_1$	
		$\delta y_{\frac{3}{2}}$		$\delta^3 y_{3/2}$
$x_0 + 2h$	y_2		$\delta^2 y_2$	
		$\delta y_{5/2}$		
$x_0 + 3h$	y_3			

* μ operator

$\mu \rightarrow$ Average or Mean

$$\mu f(x) = \frac{1}{2}\left(f(x + h/2) + f(x - h/2)\right)$$

$$\mu y_{\frac{1}{2}} = \frac{1}{2}(y_1 + y_0)$$

$$\mu y_{3/2} = \frac{1}{2}(y_2 - y_1)$$

$$\boxed{\mu y_x = \frac{1}{2}\left(y_{x+\frac{1}{2}} + y_{x-\frac{1}{2}}\right)} \longrightarrow \text{General formula}$$

* Shift operator (enlargement & displacement operator)

$$E f(x) = f(x+h)$$

$$\boxed{E^n f(x) = f(x+nh)}$$ General formula

$$E^2 y_0 = y_2$$

* Relations b/w operators

① $\Delta = E - 1$

② $\nabla = 1 - E^{-1}$

③ $\delta = E^{1/2} - E^{-1/2}$, $\delta = \frac{\nabla}{\sqrt{1-\nabla}}$, $\delta = \frac{\Delta}{\sqrt{1+\Delta}}$

④ $\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$ & $\mu^2 = 1 + \frac{1}{4}\delta^2$

⑤ $\Delta = E\nabla = \nabla E = \delta E^{1/2}$

⑥ $E = e^{hD}$ $\left(\because D = \frac{d}{dx}\right)$

① Proof:

$$\Delta f(x) = f(x+h) - f(x)$$
$$= E f(x) - f(x) \quad (E f(x) = f(x+h))$$
$$\Delta f(x) = (E-1) f(x)$$
$$\boxed{\Delta = E - 1}$$

②

$$\nabla = 1 - E^{-1}$$
$$\Rightarrow E^{-1} = 1 - \nabla$$
$$(1-\nabla) f(x) = f(x) - \nabla f(x)$$
$$= f(x) - [f(x) - f(x-h)]$$
$$= f(x-h)$$
$$= E^{-1}$$

(3) $\delta = E^{1/2} - E^{-1/2}$

$= E^{1/2}(f x) - E^{-1/2}(f x)$

$= f(x + \frac{1}{2}h) - f(x - \frac{1}{2}h)$

$\delta = \frac{\Delta}{\sqrt{1+\Delta}}$

$\delta = E^{1/2} - E^{-1/2}$

$= (1+\Delta)^{\frac{1}{2}} - \frac{1}{(1+\Delta)^{\frac{1}{2}}}$ $\left(\begin{matrix}\Delta = E-1 \\ E = \Delta+1\end{matrix}\right)$

$= \frac{1+\Delta-1}{(1+\Delta)^{1/2}}$

$$\boxed{\delta = \frac{\Delta}{\sqrt{1+\Delta}}}$$

$\delta = \frac{\nabla}{\sqrt{1-\nabla}}$

(4) $\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$

$\mu f(x) = \frac{1}{2}\left[f(x + \frac{h}{2}) + f(x - \frac{h}{2})\right]$

$= \frac{1}{2}\left[E^{1/2} f(x) + E^{-1/2} f(x)\right]$

$\mu f(x) = \frac{1}{2} f(x) \left[E^{1/2} + E^{-1/2}\right]$

$\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$

$$u^2 = \frac{1}{4}\left(E^{1/2} + E^{-1/2}\right)^2$$

$$= \frac{1}{4}\left(E + E^{-1} + 2\right)$$

$$= \frac{1}{4}\left(E^{\frac{1}{2}} - E^{-1/2}\right)^2 + 4.$$

$$= \frac{1}{4}\left(\delta^2 + 4\right)$$

$$\boxed{\mu^2 = 1 + \frac{1}{4}\delta^2}$$

(5)

$$\Delta = E\nabla = \delta E^{\frac{1}{2}}$$

$$\Delta = E - 1$$

$$\nabla = 1 - E^{-1}$$

$$= 1 - \frac{1}{E}$$

$$= \frac{E-1}{E}$$

$$\nabla \cancel{\Delta} = \frac{\Delta}{E}$$

$$\Delta = E\nabla$$

for $\Delta = \delta E^{\frac{1}{2}}$

$$\delta = E^{1/2} - E^{-1/2}$$

$$\frac{\delta}{E^{1/2}} = \frac{E^{1/2} - E^{-1/2}}{E^{1/2}}$$

(6)

$$E = e^{hD}$$

$$Df(x) = f'(x)$$

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \cdots$$

$$= f(x) + hDf(x) + \frac{h^2}{2!} D^2 f(x)$$

$$E f(x) = \left(1 + hD + \frac{h^2}{2!}D^2 + \cdots\right) f(x)$$

$$E = 1 + hD + \frac{h^2}{2!}D^2 + \cdots$$

$$= e^{hD}$$

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