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2K19/EP/005

Class Test - 2

[1] for irrotational, curl$(\vec{A}) = 0$.

$$\vec{A} = (x^2 + xy^2)\hat{i} + (y^2 + x^2y)\hat{j}$$

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

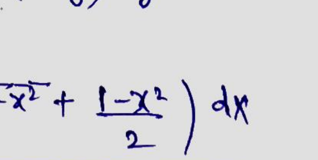
$$= \hat{i}\left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(y^2 + x^2y)\right] - \hat{j}\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(x^2 + xy^2)\right]$$
$$+ \hat{k}\left(\frac{\partial}{\partial x}(y^2 + x^2y) - \frac{\partial}{\partial y}(x^2 + xy^2)\right]$$

$$= \hat{i}[0] - \hat{j}(0) + \hat{k}[2xy - 2xy] = 0.$$

As $\nabla \times \vec{A}$ is zero, $\therefore \vec{A}$ is irrotational.

[2] By Green's Theorem, we have $\oint_C (\phi\, dx + \psi\, dy) = \iint_S \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\, dy$

$$= \int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} \left(\frac{\partial}{\partial x}(x^2+y^2) - \frac{\partial}{\partial y}(2x^2 - y^2)\right) dx\, dy$$



$$= \int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (2x + 2y)\, dx\, dy = 2\int_{-1}^{1} dx \int_{0}^{\sqrt{1-x^2}} (x+y)\, dy.$$

$$= 2\int_{-1}^{1} dx \left(xy + \frac{y^2}{2}\right]_{0}^{\sqrt{1-x^2}} = 2\int_{-1}^{1}\left(x\sqrt{1-x^2} + \frac{1-x^2}{2}\right) dx$$

$$= 2\int_{-1}^{1} x\sqrt{1-x^2}\, dx + \int_{-1}^{1} 1 - x^2\, dx$$

(even)

$$\boxed{\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx \text{ ; odd} \\ = 0 \text{ , even}}$$

$$= 0 + 2\int_{0}^{1} 1 - x^2\, dx = 2\left(x - \frac{x^3}{3}\right)_0^1 = \boxed{\frac{4}{3}}$$

(3) Laplace Eq^n is :

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Now let, $u(x,y) = X(x) . Y(y)$

Differentiating

$$X''(x)Y(y) + X(x)Y''(y) = 0.$$

Seperating variables, $\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = K$ (say)

$$\frac{X''}{X} = KX \quad -(i) \qquad Y'' = -KY = 0 \quad -(ii)$$

Case I if $K>0$, $K = p^2$

$$\frac{d^2X}{dx^2} - p^2X = 0 \qquad \frac{d^2Y}{dy^2} + p^2Y = 0$$

A.E
$$m^2 - p^2 = 0 \qquad m^2 + p^2 = 0$$
$$m = \pm p \qquad m = \pm pi$$

$$X(x) = c_1 e^{px} + c_2 e^{-px} \qquad Y(y) = c_3 \cos py + c_4 \sin py$$

$$u(x,y) = XY$$
$$= (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$$

Case 2 if $K=0$,

$$\frac{d^2X}{dx^2} = 0, \qquad \frac{d^2Y}{dy^2} = 0$$

$$X(x) = c_5 x + c_6, \qquad Y(y) = c_7 y + c_8$$

$$u(x,y) = (C_5 x + C_6)(C_7 y + C_8)$$

Case ~~#~~ 3 when $K<0$, $K=-p^2$

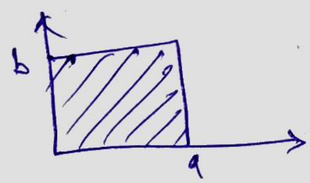
$$\frac{d^2X}{dx^2} + p^2 X = 0 \quad , \quad \frac{d^2Y}{dy^2} - p^2 Y = 0$$

soln are resp.,

$$X(x) = C_9 \cos px + C_{10} \sin px = 0$$

$$Y(y) = C_{11} e^{py} + C_{12} e^{-py} .$$

$\boxed{4}$



$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\frac{\partial u}{\partial t}$$

differentiating,

$$YT \frac{\partial^2 X}{\partial x^2} + XT \frac{\partial^2 Y}{\partial y^2} = \frac{XY}{c^2}\frac{dT}{dt}$$

Separating,

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = \frac{1}{c^2 T}\frac{dT}{dt}$$

accepted solution will be,

$$\frac{1}{X}\cdot\frac{d^2X}{dx^2} = -K_1^2 \quad , \quad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -K_2^2$$

$$\frac{1}{c^2 T}\frac{\partial T}{dt} = -K^2 \quad , \quad K = K_1^2 + K_2^2 .$$

$X = A\cos K_1 x + B\sin K_1 x$, $Y = C\cos K_2 y + D\sin K_2 y$.

$T = E e^{-c^2 K t}$

$$u = XYT = (A\cos K_1 x + B\sin K_1 x)(C\cos K_2 y + D\sin K_2 y) E e^{-c^2 K t}$$

— (1)

$\rightarrow$ $u=0,\ x=0$ in ①,

$$0 = A(C\cos k_2 y + D\sin k_2 y) E e^{-c^2 kT}$$

$$\Rightarrow \boxed{A=0}$$

$$u = (B\sin k_1 x)(C\cos k_2 y + D\sin k_2 y)\, e^{-c^2 kT} \quad — ②$$

$\rightarrow$ $u=0,\ x=a$ in ②,

$$0 = B\sin k_1 a, \quad \sin k_1 a = 0 = \sin n\pi$$

$$k_1 a = n\pi \qquad \boxed{k_1 = \frac{n\pi}{a}}.$$

$$u = \left(B\sin \frac{n\pi}{a}x\right)(C\cos k_2 y + D\sin k_2 y)\, e^{-c^2 kt} \quad — ③$$

$\rightarrow$ $u=0,\ y=0$ in ③,

$$0 = \left(B\sin \frac{n\pi}{a}x\right) C e^{-c^2 kT} \rightarrow \boxed{C=0}$$

$$u = \left(B\sin \frac{n\pi}{a} x\right)(D\sin k_2 y)\, e^{-c^2 kt} \quad — ④$$

$\rightarrow$ $u=0,\ y=b$ in ④,

$$0 = \left(B\sin \frac{n\pi}{a}x\right)(D\sin k_2 b)\, e^{-c^2 kT}$$

$$\sin k_2 b = 0 = \sin m\pi \qquad \boxed{k_2 = \frac{m\pi}{b}}$$

$$u = \left(B\sin \frac{n\pi}{a}x\right)\left(D\sin \frac{m\pi}{b}y\right) e^{-c^2 kt}$$

$$u = A_{mn} \sin \frac{n\pi}{a}x \sin \frac{m\pi}{b}y \cdot e^{-c^2 kt}$$

$$\Rightarrow \quad k^2 = k_1^2 + k_2^2 = \pi^2\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right)$$

$$u = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn} \sin\frac{n\pi}{a}x \sin\frac{m\pi}{b}y \, e^{-c^2 K_{mn} t}$$

On applying initial condition, $u = f(x,y)$, $t=0$

$$f(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn} \sin\frac{n\pi}{a}x \sin\frac{m\pi}{b}y$$

where $$A_{mn} = \frac{2}{a}\frac{2}{b}\int_{x=0}^{a}\int_{y=0}^{b} \sin\frac{n\pi}{a}x \sin\frac{m\pi}{b}y \, f(x,y)\, dx\, dy .$$