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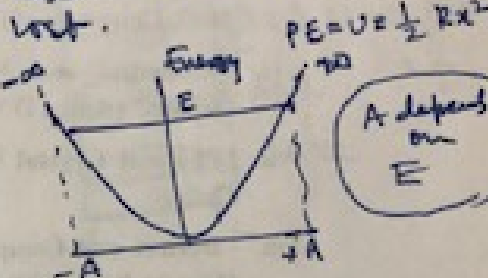
Harmonic Oscillator

Vibration about an equilibrium. System Oscillates Indefinitely if no energy is lost.



A depends on E

$$F = -kx$$
$$F = ma$$
$$-kx = m\frac{d^2x}{dt^2}$$
$$\Rightarrow \frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$
$$x = A\cos(2\pi\nu t + \phi) = A\cos(\omega t + \phi)$$
$$\nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

When x is small about mean position $\Rightarrow$ Simple Harmonic Oscillator

Restoring Force F

$$F(x) = F_{x=0} + \left(\frac{dF}{dx}\right)_{x=0} x + \frac{1}{2}\left(\frac{d^2F}{dx^2}\right)_{x=0} \cdot x^2 + \frac{1}{6}\left(\frac{d^3F}{dx^3}\right)_{x=0} \cdot x^3 + \cdots$$

Maclaurin's Series.

For small value of x

$$F(x) = \left(\frac{dF}{dx}\right)_{x=0} \cdot x$$

If $\frac{dF}{dx}$ is negative – true for all restoring force

Work needed to bring a particle from $x=0$ to $x=x$

$$U(x) = -\int_0^x F(x)\,dx = -k\int_0^x x\,dx = \frac{1}{2}kx^2$$

Schrödinger Equation for Harmonic Oscillator

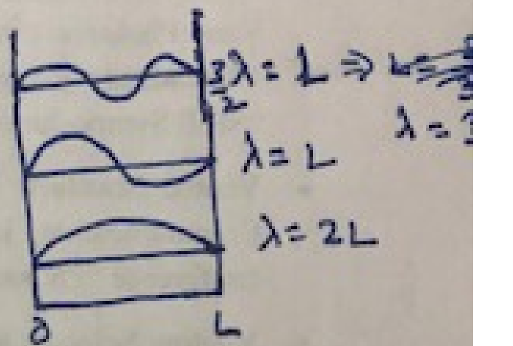
$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}\left(E - \frac{1}{2}kx^2\right)\psi = 0$$

$H_n(y)$

n	$H_n(y)$
0	1
1	$2y$
2	$4y^2 - 2$

$$E_n = \left(n + \frac{1}{2}\right)h\nu$$
$$\psi_n = \left(\frac{2m}{h}\right)^{1/4}\left(2^n \cdot \lfloor n\right)^{-1/2} \cdot H_n(y)\, e^{-\frac{y^2}{2}}$$

De Broglie wavelength of particle trapped in potential well



$$\lambda_n = \frac{2L}{n}$$

$$p = \frac{h}{\lambda_n}$$

$$p = \frac{h \cdot n}{2L} = \frac{h \cdot n}{2L}$$

KE = E = for free Particle Total Energy

$$E_n = \frac{p_n^2}{2m} = \frac{n^2 \cdot h^2}{4L^2 \times 2m} = \frac{n^2 h^2}{8mL^2}$$

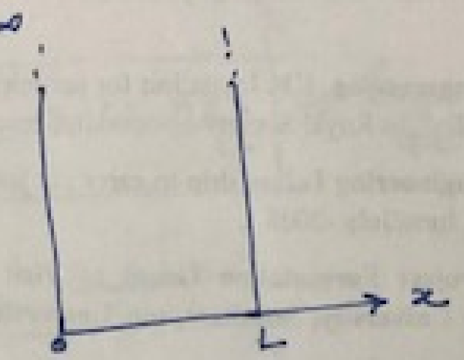
$$\boxed{E_n = \frac{n^2 h^2}{8mL^2}}$$

n = quantum number

En = Energy level.

Repeation: Simplest Potential well

How boundary conditions and normalization determine the wave function



$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2} E\psi = 0$$

$$\frac{d^2\psi}{dx^2} + k^2\psi = 0 \quad \text{where } k^2 = \frac{2mE}{\hbar^2}$$

$\psi = A \sin kx$ is one solution

$\psi = B \cos kx$ is also another solution

$\psi = A \sin kx + B\cos kx$

$\psi = A \sin kx + B \cos kx$ – General solⁿ.

when $x = 0$ $\psi \neq 0$, as $B \cos kx \neq 0$ at $x = 0$

Therefore $B = 0$

$$\boxed{\psi = A \sin kx}$$

now at $x = L$, $A \sin kL = 0$ when it is $\sin n\pi$

$\Rightarrow$ $kL = n\pi$, $n = 1, 2, 3 \ldots$ — E_n

$k^2 L^2 = n^2\pi^2 \Rightarrow \frac{2mE_n}{\hbar^2} \cdot L^2 = n^2\pi^2$

$$\boxed{E_n = \frac{n^2 h^2}{8mL^2}}$$

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Wave function

$$\psi_n = A\sin kx$$

$$= A\sin\frac{\sqrt{2mE_n}}{\hbar}x \quad \text{where } E_n = \frac{n^2h^2}{8mL^2}$$

$$\boxed{\psi_n = A\sin\frac{n\pi x}{L}}$$ ⇒ Eigenfunction corresponding to every value E_n

Normalization condition

$$\int_{-\infty}^{\infty}|\psi_n|^2 dx = \int_0^L |\psi_n|^2 dx = A^2\int_0^L \sin^2\left(\frac{n\pi x}{L}\right)dx$$

$$\Rightarrow 1 = \frac{A^2}{2}\left[\int_0^L dx - \int_0^L \cos\frac{2n\pi x}{L}\,dx\right]$$

$$\Rightarrow \frac{A^2}{2}\left[x - \left(\frac{L}{2n\pi}\right)\sin\frac{2n\pi x}{L}\right]_0^L = 1$$

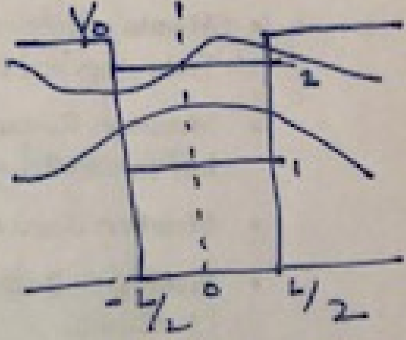
$$\Rightarrow \frac{A^2}{2}L = 1$$

$$\Rightarrow A^2 = \frac{2}{L}$$

$$\Rightarrow A = \sqrt{\frac{2}{L}}$$

$$\boxed{\psi_n = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}}$$

Particle in One-D potential well of finite depth



$V(x) = 0$ for $-\frac{L}{2} < x < \frac{L}{2}$

$= V_0$ for $|x| > \frac{a}{2}$

$$\frac{d^2\psi(x)}{dx^2} + \frac{2mE}{\hbar^2}\psi(x) = 0 \text{ for } |x| < \frac{a}{2} \quad (1)$$

$$\frac{d^2\psi}{dx^2} + \frac{2m(E-V_0)}{\hbar^2}\psi(x) = 0 \text{ for } |x| > \frac{L}{2}$$

Here $E < V_0$

$$\frac{d^2\psi}{dx^2} - \frac{2m}{\hbar^2}(V_0 - E)\psi(x) = 0 \quad \text{for } |x| > \frac{L}{2}$$

$$\frac{d^2\psi}{dx^2} + k_1^2\psi(x) = 0 \text{ for } |x| < \frac{L}{2}$$

$$\frac{d^2\psi}{dx^2} - k_2^2\psi(x) = 0 \text{ for } |x| > \frac{L}{2}$$

where $k_1^2 = \frac{2mE}{\hbar^2}$ & $k_2^2 = \frac{2m}{\hbar^2}(V_0 - E)$

For Symmetric $V(-x) = V(x)$

$$\psi(x) = A\cos k_1 x, \quad |x| < \frac{L}{2}$$

$$= B\exp(-k_2|x|), \quad |x| > \frac{L}{2}$$

We have rejected $\sin k_1 x$ & $\exp k_2|x|$ because $\to \infty$ at $|x| \to \infty$

at $x = -L/2$

$$A\cos\frac{k_1 L}{2} = B\exp\left(-\frac{k_2 L}{2}\right)$$

$$-Ak_1\sin\frac{k_1 L}{2} = -Bk_2\exp\left(-\frac{k_2 L}{2}\right)$$

$$\Rightarrow \tan\left(\frac{k_1 L}{2}\right) = \frac{k_2}{k_1} = \left(\frac{V_0 - E}{E}\right)^{1/2}$$

(6)

$$\tan\left(\frac{k_1 L}{2}\right) = \left(\frac{V_0 - E}{E}\right)^{1/2}$$

For $V_0 \gg E$ $\quad \tan\frac{k_1 L}{2} \approx \tan\left(p+\frac{1}{2}\right)\pi \quad p = 0, 1, 2 \ldots$

$$\Rightarrow \frac{k_1 L}{2} = \left(p+\frac{1}{2}\right)\pi = \left(\frac{2p+1}{2}\right)\pi$$

$$\Rightarrow k_1 = \frac{\cancel{2}}{L}\left(\frac{2p+1}{\cancel{2}}\right)\pi = \frac{(2p+1)\pi}{L}$$

$$k_1^2 = \frac{(2p+1)^2\pi^2}{L^2}$$

$$\frac{2mE}{\hbar^2} = \frac{(2p+1)\pi^2}{L^2} =$$

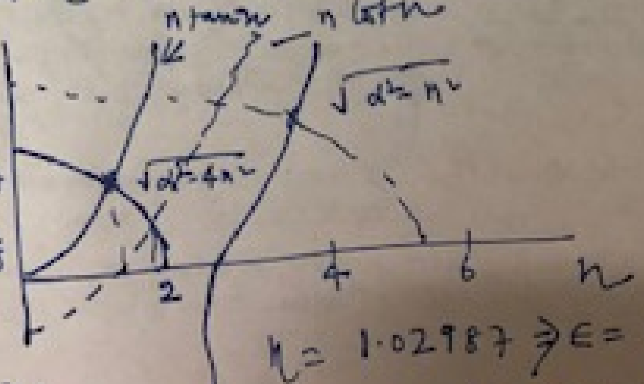
$$\Rightarrow \boxed{E = \frac{(2p+1)\pi^2\hbar^2}{2mL^2}} \;\; \left[\frac{\pi^2 \cdot \frac{h^2}{4\pi^2}}{2mL^2} = \frac{h^2}{8mL^2}\right.$$

if $\eta = \frac{kL}{2}$ $\quad \boxed{\eta \tan\eta = (\alpha^2 - \eta^2)^{1/2}}$ Symmetric

$$\alpha = \frac{2mV_0L^2}{4\hbar^2}$$

$$(\alpha^2 - \eta^2)^{1/2} = \left[\frac{4m^2V_0^2L^4}{16\hbar^4} - \frac{k_1^2L^2}{4}\right]^{1/2}$$

$$-\eta\cot\eta = (\alpha^2 - \eta^2)^{1/2} \quad \text{Anti Symmetric}$$



Example

Proton $m = 1.67 \times 10^{-27}$

$V_0 \simeq 25\ \text{MeV} = 4 \times 10^{-12}\ \text{J}$

$L = 3.65 \times 10^{-15}\ \text{m}$

$$\alpha = \frac{2mV_0L^2}{4\hbar^2} \approx 2.0$$

i.e. $\alpha < \pi$

$\eta = 1.02987 \Rightarrow E = 6.63\ \text{MeV}$ Sym

$\eta = 1.8549 \Rightarrow E = 22.45\ \text{MeV}$