$$= 4\left[85.33 + 16\left(16 + \frac{d^2}{4} + 4d\right)\right] + 2\left[6.67 + 20\left(1 + \frac{d^2}{4} + d\right)\right]$$

$$= 341.32 + 1024 + 16d^2 + 256d + 13.34 + 40 + 5d^2 + 40d$$

$$= 21d^2 + 296d + 1418.66$$

But it is given that $I_{xx} = I_{yy}$

$$\therefore \quad 21d^2 + 296d + 1418.66 = 2658.65$$

$$\text{or} \quad 21d^2 + 296d - 1240 = 0$$

$$\text{or} \quad d^2 + 14.09d - 59.05 = 0$$

Solution of the quadratic equation gives

$$d = \frac{-14.09 \pm \sqrt{(14.09)^2 - 4 \times 1(-59.05)}}{2 \times 1} = \frac{-14.09 \pm 20.85}{2}$$

$$= 3.28 \text{ cm}$$

(Taking + *ve* velue)

Hence the distance between the two shannel sections is 3.28 cm.

8.3. MASS MOMENT OF INERTIA

The mass moment of inertia of a body about a particular axis is defined as *"the product of the mass and the square of the distance between the mass centre of the body and the axis"*.

The mass moment of inertia is an important term for the study of the rotational motion of a rigid body. It gives a measure of the resistance that the body offers to change in angular velocity.

The body can be considered to be split up into small masses. Let

$m_1, m_2 \ldots m_n$ be the masses of the various elements of the body

and $r_1, r_2 \ldots r_n$ be the distances of the above mentioned elements from the axis about which mass moment of inertia is to be determined.

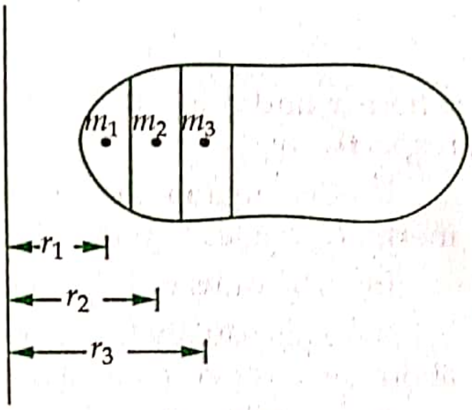

Fig. 8.30

The mass moment of inertia of the body can be written as

$$I = m_1 r_1^2 + m_2 r_2^2 + \ldots m_n r_n^2$$

$$= \Sigma m r^2$$

The summation of a large number of terms in the above expression can be replaced by integration. Consider a small mass dm rotating about and at distance r from the axis of rotation, then

$$I = \int r^2 dm$$

The radius of gyration k of the body with respect to the prescribed axis is defined by the relation

$$I = k^2 M; \quad k = \sqrt{\frac{I}{M}}$$

where M is the mass of the body.

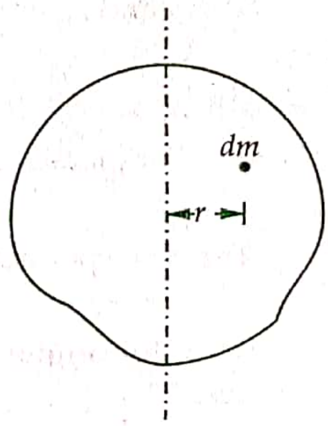

Fig. 8.31

The radius of gyration gives a measure of the distance at which the entire mass of the body is assumed to be concentrated.

Unit of mass moment of inertia is kg-m^2 and that of radius of gyration is m.

Parallel axis theorem : The mass moment of inertia of a body about an axis parallel to the centroidal axis is equal to the moment of inertia about the centroidal axis plus the product of mass and the square of the shortest distance between centre of gravity and the axis concerned. With reference to Fig. 8.32.

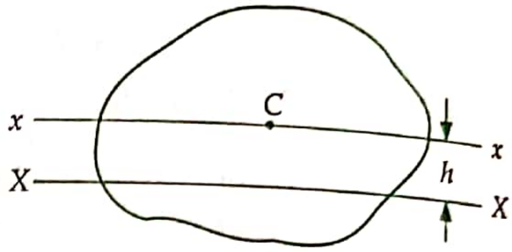

Fig. 8.32

$$I_{XX} = I_{xx} + mh^2$$

where m is the mass of the body and h is the distance between the two parallel axis.

Comparison of mass moment of inertia and area moment of inertia :

(*i*) The mass moment of inertia is mathematically defined as :

$$I_M = \int x^2 \, dm$$

where dm is the mass of a small element of the body at normal distance x from the line or axis about which moment of inertia is desired.

The area moment of inertia I_A is the second moment of area, and is mathematically defined as

$$I_x = \int y^2 \, dA \quad \text{and} \quad I_y = \int x^2 \, dA$$

where x and y are the perpendicular distances of small elemental area dA from y and x-axis respectively.

(*ii*) The area moment of inertia I_A is related to area of plane figures, and the mass moment of inertia I_M relates to bodies having a bulk and hence mass.

(*iii*) The units of I_M are kgm^2. Being the second moment of area, I_A has the units m^4.

(*iv*) I_A is usually taken about an axis in the plane of the figure under consideration. I_M is taken about an axis perpendicular to the plane of the figure.

(*v*) I_A is a useful parameter for the analysis of structures in static conditions. I_M has applications in structural dynamics.

(*vi*) I_A and I_M are related to each other by the expression

$$I_M = \rho \, t \, I_A$$

where ρ and t denote the density and thickness of the material.

(*vii*) For plane figures, the parallel axis theorem is specified as :

$$(I_A)_{AA} = (I_A)_{xx} + A\,h^2$$

The corresponding expression for solid bodies is

$$(I_M)_{AA} = (I_M)_{xx} + M\,h^2$$

(*viii*) The radius of gyration for a plane figure and a solid figure is worked out from the following identities

$$k_A = \sqrt{\frac{I_A}{A}} \quad \text{and} \quad k_M = \sqrt{\frac{I_M}{M}}$$

8.4. MASS MOMENT OF INERTIA FOR SPECIFIED CASES

8.4.1. Thin uniform rod

Fig. 8.33 shows a thin uniform rod of length l. Consider a small elemental length dx located at distance x from the centroidal axis y-y normal to the rod.

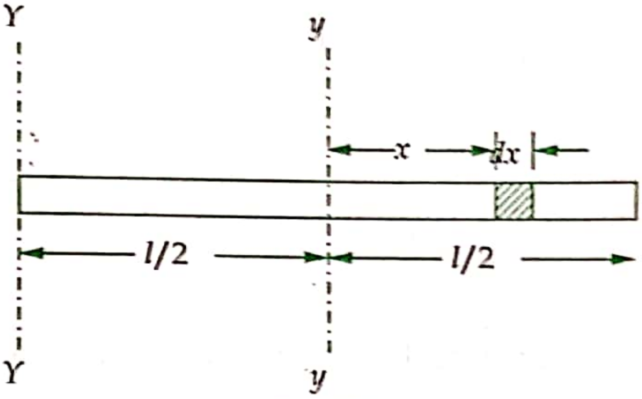

Fig. 8.33

Mass of the segmental element

$$dm = m\,dx$$

where m is the mass per unit of length.

Mass moment of inertia of the element about axis y-y

$$= dm\,x^2 = mx^2\,dx$$

The mass moment of inertia of the entire rod about axis y-y can be worked out by integrating the above expression between the limits $-\frac{l}{2}$ to $\frac{l}{2}$. That is

$$I_{yy} = m\int_{-\frac{l}{2}}^{\frac{l}{2}} x^2\,dx = m\left|\frac{x^3}{3}\right|_{-\frac{l}{2}}^{\frac{l}{2}} = m\left(\frac{l^3}{24}+\frac{l^3}{24}\right)$$

$$= \frac{ml^3}{12} = \frac{Ml^2}{12}$$

where $M = ml$ is the mass of the whole rod.

If it is required to determine the mass moment of inertia of the rod about axis YY at the left end of the rod, we can use the parallel axis theorem

$$I_{YY} = I_{yy} + Mh^2$$

where h is the distance between the axis YY and the centroidal axis yy.

$$I_{YY} = \frac{Ml^2}{12} + M\left(\frac{l}{2}\right)^2 = \frac{Ml^2}{3}$$

8.4.2. Rectangular plate

Figure 8.34, shows a rectangular plate of width b, depth d and uniform thickness t. If ρ is the density of the plate material, then mass of the plate

$$M = \text{density} \times \text{volume}$$
$$= \rho\,b\,t\,d$$

Consider an elemental strip of depth dy located at distance y from the centroidal axis xx

mass of the elementa strip $dm = \rho\,b\,t\,dy$

mass moment of inertial of the strip about axis xx

$$= dm\,y^2 = \rho\,b\,t\,y^2 dy$$

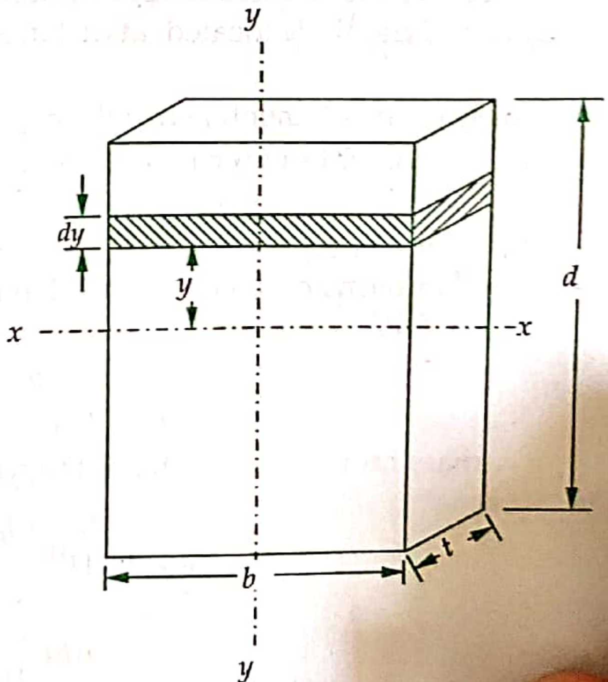

Fig. 8.34

The moment of inertia for the entire mass of plate about axis xx can be worked out by integrating the above expression between the limits $-\frac{d}{2}$ to $\frac{d}{2}$. That is

$$I_{xx} = \rho bt \int_{-\frac{d}{2}}^{\frac{d}{2}} y^2 dy = \rho bt \left| \frac{y^3}{3} \right|_{-\frac{d}{2}}^{\frac{d}{2}}$$

$$= \rho bt \left(\frac{d^3}{24} + \frac{d^3}{24} \right) = (\rho bt d) \times \frac{d^2}{12}$$

$$= \frac{1}{12} Md^2 \qquad \text{...(8.23 a)}$$

Likewise the mass moment of inertia of the rectangular plate about the centroidal axis yy is

$$I_{yy} = \frac{1}{12} Mb^2 \qquad \text{...(8.23 b)}$$

From perpendicular axis theorem, the moment of inertia about axis zz is

$$I_{zz} = I_{xx} + I_{yy}$$

$$= \frac{1}{12} Md^2 + \frac{1}{12} Mb^2 = \frac{1}{12} M(d^2 + b^2) \qquad \text{...(8.24)}$$

8.4.3. Triangular plate

Figure 8.35 shows a triangular plate of base width b, height h and uniform thickness t. If ρ is the density of the plate material, then

mass of the plate,

$$M = \text{density} \times \text{volume}$$

$$= \text{density} \times (\text{area} \times \text{thickness})$$

$$= \rho \times \frac{1}{2} bht$$

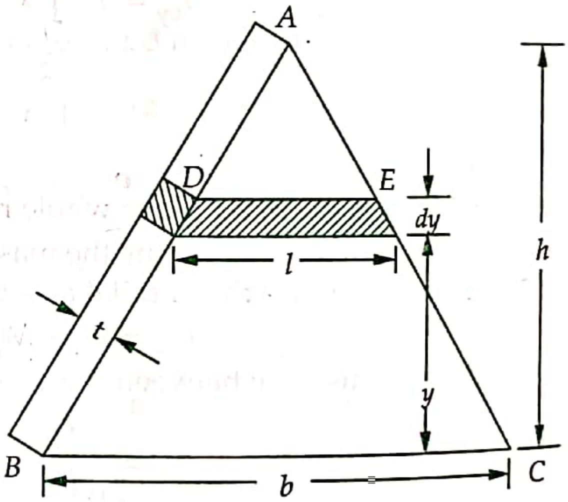

Fig. 8.35

Consider an elemental strip (assumed rectangle) of width l and depth dy located at distance y from the base line.

mass of the elemental strip $dm = \rho l t\, dy$

mass moment of inertia of the strip about base

$$= dm\, y^2 = \rho l t y^2 dy$$

Since the integration is to be done with respect to y within the limits 0 to h, it is necessary to express l in terms of y. For that we have the following correlation from the similarity of triangles ADE and ABC,

$$\frac{l}{b} = \frac{h-y}{h}, \quad l = b\,\frac{h-y}{h}$$

$\therefore$ mass moment of inertia of the triangular plate about base line

$$I_{base} = \int_0^h \rho b \left(\frac{h-y}{h} \right) t\, y^2 dy$$

$$= \frac{\rho bt}{h} \int_0^h (h-y) y^2 dy$$

$$= \frac{\rho b t}{h}\left|\frac{h y^3}{3} - \frac{y^4}{4}\right|_0^h = \frac{\rho b t}{h}\left(\frac{h^4}{3} - \frac{h^4}{4}\right) = \frac{\rho b t h^3}{12}$$

$$= \frac{\rho b h t}{2} \times \frac{h^2}{6} = \frac{1}{6} M h^2 \qquad \text{...(8.25)}$$

8.4.4. Circular lamina

Figure 8.36 shows a thin circular plate of radius R and uniform thickness t. If ρ is the density of the plate material, then

mass of the plate M = density × volume

= density × (area × thickness) = $\rho \pi R^2 t$

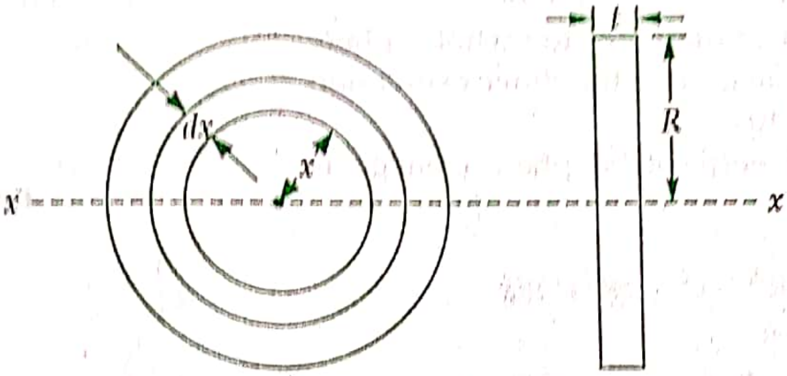

Fig. 8.36

Consider an elementary ring of radius r and width dr.

mass of elemental ring $dm = \rho \{\pi(r + dr)^2 - \pi r^2\} t$

$= \rho (2\pi r\, dr)\, t = 2\pi t \rho r\, dr$

Mass moment of inertia of this elementary ring about the polar axis zz

$= dm\, r^2 = 2\pi t \rho r^3 dr$

Mass moment of inertia of the circular plate about polar axis zz

$$= 2\pi t \rho \int_0^R r^3\, dr = 2\pi t \rho \left|\frac{r^4}{4}\right|_0^R$$

$$= \rho \pi R^2 t \times \frac{R^2}{2} = \frac{1}{2} M R^2 \qquad \text{...(8.26)}$$

where $M = \rho \pi R^2 t$ is the mass of the circular lamina

Invoking the theorem of perpendicular axis, the mass moment of inertia of a circular lamina about xx or yy axis is

$$I_{xx} = I_{yy} = \frac{I_{zz}}{2} = \frac{1}{4} M R^2 \qquad \text{...(8.27)}$$

8.4.5. Solid sphere

Figure 8.37 shows a solid sphere of radius R with O as centre. If ρ is the density of the material of the sphere, then

mass of the sphere = density × volume

$= \rho \times \frac{4}{3} \pi R^3$

Let attention be focussed on a thin disc AB of thickness dx and at radius x from the centre.

radius of the disc $y = \sqrt{R^2 - x^2}$

mass of the disc dm

$$= \rho \times \pi y^2 dx$$
$$= \rho\pi (R^2 - x^2)\, dx$$

Mass moment of inertia of this elementary disc about the polar axis zz

$$dm\, y^2 = \rho\pi (R^2 - x^2)\, dx \times (R^2 - x^2)$$
$$= \rho\pi (R^2 - x^2)^2\, dx$$
$$= \rho\pi (R^4 + x^4 - 2R^2x^2)\, dx$$

The mass moment of inertia of the whole sphere can be worked out by integrating the above expression between the limits $-R$ to R.

∴ Mass moment of inertia of the sphere about polar axis zz

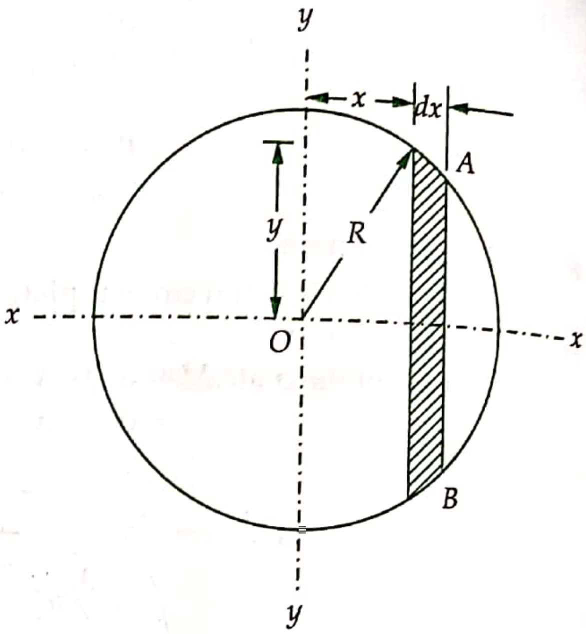

Fig. 8.37

$$I_{zz} = \rho\pi \int_{-R}^{R} (R^4 + x^4 - 2R^2x^2)\, dx$$

$$I_{zz} = \rho\pi \left| R^4 x + \frac{x^5}{5} - 2R^2 \frac{x^3}{3} \right|_{-R}^{R} = \frac{16\rho\pi R^5}{15} = \frac{4}{5} MR^2 \qquad \text{...(8.28)}$$

where $M = \frac{4}{3}\rho\pi R^3$ is the mass of the solid sphere

Invoking the theorem of perpendicular axis, the mass moment of inertia of a solid sphere about xx or yy axis is,

$$I_{xx} = I_{yy} = \frac{I_{zz}}{2} = \frac{2}{5} MR^2$$

8.4.6. Solid cylinder

Figure 8.38 shows a solid cylinder of radius R and height h. If ρ is the density of the material of the cylinder, then

$$\text{mass of the cylinder} = \text{density} \times \text{volume}$$
$$M = \rho \times \pi R^2 h$$

Consider a thin disc of thickness dy located at distance y from the centroidal axis xx.

mass of the elemental disc, $dm = \rho \times \pi R^2 dy$

It may be recalled that mass moment of inertia of a circular lamina about its diametral axis is given by

$$= \frac{1}{4} MR^2$$

∴ mass moment of inertia of the elemental disc about its diameteral axis is

$$I_{dia} = \frac{1}{4} dm\, R^2$$

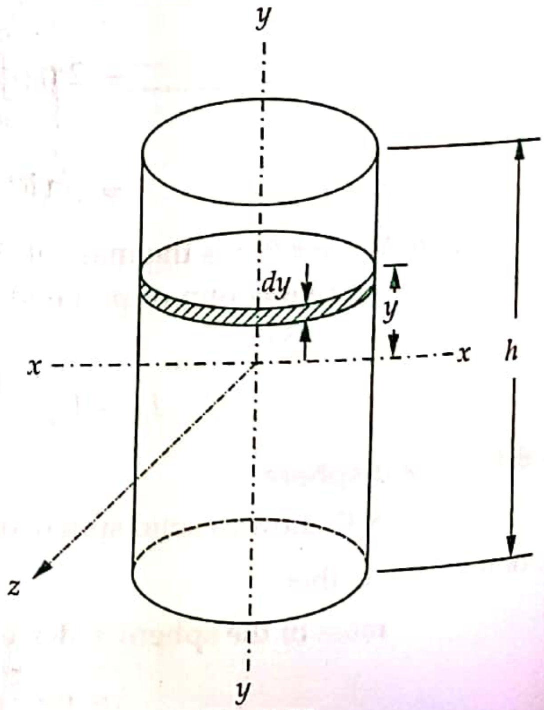

Fig. 8.38

Invoking parallel axis theorem, the mass moment of inertia of elemental disc about axis zz is

$$I_{xx} = I_{disc} + (dm)\, y^2$$

$$= \frac{1}{4} dm R^2 + dm\, y^2$$

$$= \frac{1}{4}(\rho\pi R^2 dy)R^2 + (\rho\pi R^2 dy)y^2$$

$$= \frac{1}{4}\rho\pi R^4 dy + \rho\pi R^2 y^2 dy$$

The mass moment of inertia of the entire solid cylinder can be worked out by integrating the above expression between the limits $-\frac{h}{2}$ to $\frac{h}{2}$. Thus,

$$I_{xx} = \frac{1}{4}\rho\pi R^4 \int_{-\frac{h}{2}}^{\frac{h}{2}} dy + \rho\pi R^2 \int_{-\frac{h}{2}}^{\frac{h}{2}} y^2 dy$$

$$= \frac{1}{4}\rho\pi R^4 h + \frac{1}{12}\rho\pi R^2 h^3$$

$$= \rho\pi R^2 h\left(\frac{R^2}{4} + \frac{h^2}{12}\right) = M\left(\frac{R^2}{4} + \frac{h^2}{12}\right) = \frac{1}{12} M(3R^2 + h^2) \quad ...(8.29\ a)$$

where $M = \rho\pi R^2 h$ is the mass of the cylinder.

Similarly

$$I_{yy} = \frac{1}{12} M(3R^2 + h^2) \quad ...(8.29\ b)$$

and

$$I_{zz} = I_{xx} + I_{yy} = \frac{1}{6} M(3R^2 + h^2) \quad ...(8.30)$$

Note : For a thin cylinder, $R = 0$. That gives :

$$I_{xx} = I_{yy} = \frac{1}{12} Mh^2 = \frac{1}{12} Ml^2$$

For a thin disc, $h = 0$. That gives :

$$I_{xx} = I_{yy} = \frac{1}{4} MR^2 \quad \text{and} \quad I_{zz} = \frac{1}{2} MR^2$$

8.4.7. Right circular cone

Consider a solid cone of height h and radius R. If ρ is the density of the material of the cone, then

mass of the cone M = density × volume

$$= \rho \times \frac{1}{3}\pi R^2 h$$

Consider an element of thickness dy and radius r at distance y from the apex A

mass of the elemental strip, $dm = \rho\pi r^2 dy$

mass moment of inertia of the elemental strip about axis yy

$$= \frac{1}{2} \times \text{mass moment of inertia about polar axis}$$

$$= \frac{1}{2} dm\, r^2 = \frac{1}{2} (\rho\pi r^2 dy) r^2$$

$$= \frac{1}{2} \rho\pi r^4 dy$$

Since the integration is to be done with respect to y within the limits 0 to h, it is necessary to express r in terms of y. For that we have the following correlation from the similarity of triangles ADE and ABC

$$\frac{r}{R} = \frac{y}{h}, \quad r = R\frac{y}{h}$$

∴ mass moment of inertia of the cone about axis yy

$$I_{yy} = \int_0^h \frac{1}{2} \rho\pi \left(R\frac{y}{h}\right)^4 dy$$

$$= \frac{\rho\pi R^4}{2h^4} \left|\frac{y^5}{5}\right|_0^h$$

$$= \frac{\rho\pi R^4 h}{10} = \frac{\rho\pi R^2 h}{3} \times \frac{3}{10} R^2 = \frac{3}{10} MR^2 \quad ...(8.31)$$

where $M = \frac{1}{3}\pi\rho R^2 h$ is the mass of the right circular cone.

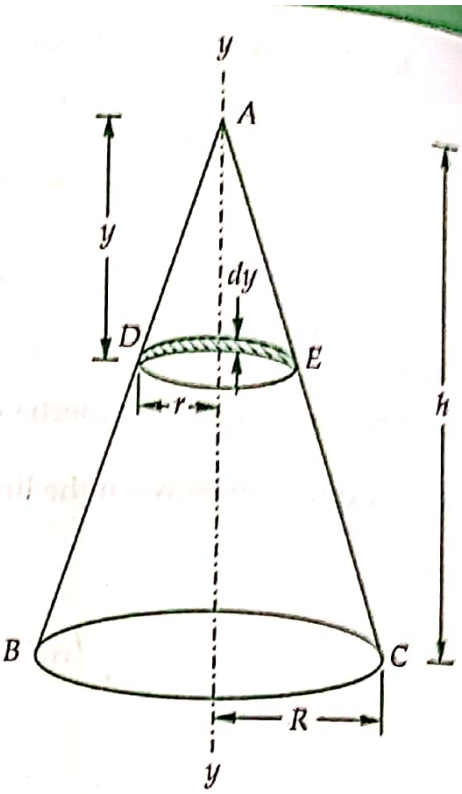

Fig. 8.39

EXAMPLE 8.15

Would you imagine that the moment of inertia of the earth around its own axis is negligible fraction of its moment of inertia about the axis of rotation around the sun? Take mean radius of the earth as 6,371 km and the mean radius of rotation around the sun as 149.7×10^6 km.

Solution : (b) Moment of inertia of the earth about its axis,

$$I_1 = \frac{2}{5} MR^2 = \frac{2}{5} M(6371)^2 = 16.23 \times 10^6 M$$

Moment of inertia of the earth about the axis of rotation around the sun,

$$I_2 = I_1 + Md^2 = 16.23 \times 10^6 M + M \times (149.7 \times 10^6)^2$$
$$= 16.23 \times 10^6 M + 22410.09 \times 10^{12} M$$
$$= 22410.09002 \times 10^{12} M$$

$$\text{Ratio } \frac{I_1}{I_2} = \frac{16.23 \times 10^6}{22410.09002 \times 10^{12}} = 7.24 \times 10^{-10}$$

Since the ratio is negligible, the moment of inertia of the earth around its own axis can be imagined to be a negligible fraction of its moment of inertia about the axis of rotation around the sun.

EXAMPLE 8.16

(a) *Set up a relation between the mass moment of inertia and the moment of inertia of a plate about an axis lying in the plane of the plate.*

(b) *Determine the mass moment of inertia of a steel rectangular plate 8 cm × 16 cm and thickness 1 cm about the centroidal axis parallel to the 8 cm side. Take mass density of steel as 8000 kg/m³.*

(c) *Determine the mass moment of inertia of a triangular plate of base 50 cm and height 100 cm about its base. The plate has a uniform thickness of 1 cm and density of the plate material is 7500 kg/m³.*

Solution : Figure 8.40 shows a thin plate of uniform thickness t and homogeneous mass density ρ. Consider an elemental area dA located at distance r from the axis xx

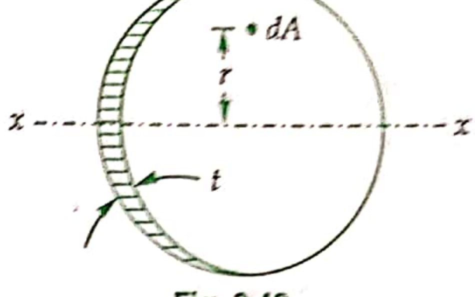

Fig. 8.40

mass of the elemental area, dm = density × volume

$$= \rho \times (dA \times t)$$

$$= \rho t\, dA$$

mass moment of inertia of this elemental area

$$= dm \times r^2 = \rho t r^2 dA$$

For the entire plate,

$$(I_{xx})_{mass} = \int \rho t r^2\, dA = \rho t \int r^2\, dA$$

The integral $\int r^2 dA$ is the moment of inertia of the area of the plate about the axis xx

$$\therefore \quad (I_{xx})_{mass} = \rho t\, (I_{xx})_{area}$$

(b) Refer Fig. 8.41 for the configuration of the rectangular plate

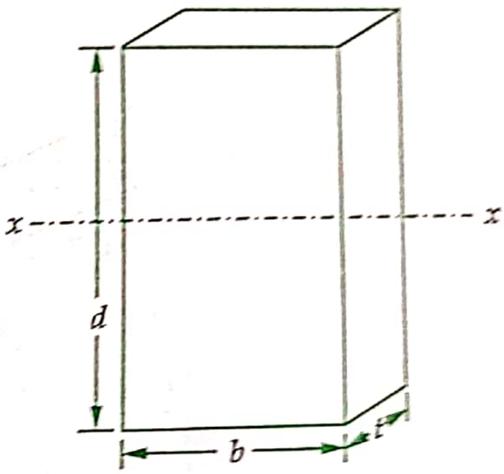

Fig. 8.41

$$b = 8 \text{ cm}; \quad d = 16 \text{ cm} \quad \text{and} \quad t = 1 \text{ cm}$$

mass of the plate M= density × volume

$$= 8000 \times \left(\frac{8 \times 16 \times 1}{10^6}\right)$$

$$= 1.024 \text{ kg}$$

Mass moment of inertia about axis xx is

$$I_{xx} = \frac{1}{12} M d^2 = \frac{1}{12} \times 1.024 \times \left(\frac{16}{100}\right)^2 = 2.184 \times 10^{-3} \text{ kg m}^2$$

(c) For the triangular plate,

base width b = 50 cm; height h = 100 cm and thickness t = 1 cm

mass of the triangular plate, M = density × volume

$$= \text{density} \times (\text{area} \times \text{thickness})$$

$$= 7500 \times \left(\frac{1}{2} \times 0.5 \times 1\right) \times 0.01 = 18.75 \text{ kg}$$

Mass moment of inertia of a triangular plate about its base is

$$= \frac{1}{6} M h^2 = \frac{1}{6} \times 18.75 \times 1^2 = 3.125 \text{ kg m}^2$$

EXAMPLE 8.17

A rectangular column of section 30 cm × 50 cm and height 4 m is centrally cast over a concrete bed which measures 3 m × 5 m and thickness 40 cm. If the mass density of concrete is 2500 kg/m³, make calculations for the mass moment of inertia of the column and bed combination about a pole.

Solution : Mass of column M_c = density × volume

$$= 2500 \times (0.3 \times 0.5 \times 4) = 1500 \text{ kg}$$

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mass moment of inertia of the column about the pole

$$= \frac{1}{2} M(d^2 + b^2) = \frac{1}{12} \times 1500 \times (0.3^2 + 0.5^2) = 42.5 \text{ kg m}^2$$

Mass of the concrete bed = 2500 × (3 × 5 × 0.4) = 15000 kg

mass moment of inertia of concrete bed about the pole

$$= \frac{1}{12} \times 15000 \times (3^2 + 5^2) = 42500 \text{ kg m}^2$$

The mass moment of inertia of the composite body is given by the sum of mass moment of inertia of the column and the mass moment of inertia of the bed.

Mass moment of inertia of composite section

$$= 42.5 + 42500 = 42542.5 \text{ kg m}^2$$

EXAMPLE 8.18

Compute I_{xx}, I_{yy} and I_{zz} for the homogeneous rectangular parallelopiped shown below in Fig. 8.42.

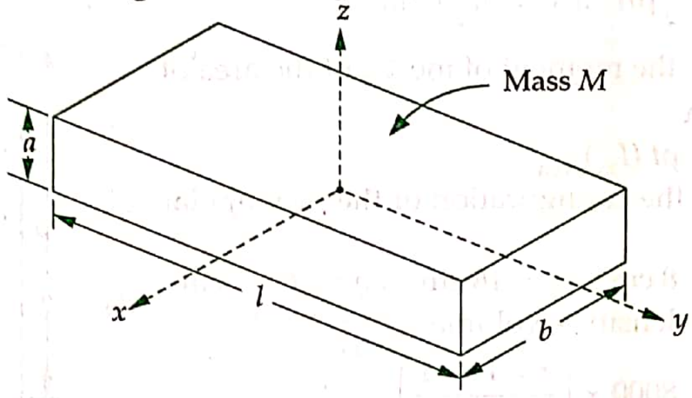

Fig. 8.42

Solution :

$$I_{xx} = \rho \iiint_V (y^2 + z^2)\,dv$$

$$= \rho \int (y^2 + z^2)\,dx\,dy\,dz \qquad \text{where } dv = dx\,dy\,dz$$

$$= \rho \int_{x=-b/2}^{b/2} \int_{y=-l/2}^{l/2} \int_{z=-a/2}^{a/2} (y^2 + z^2)\,dx\,dy\,dz$$

$$= \rho \int_{y=-l/2}^{l/2} \int_{z=-a/2}^{a/2} (y^2 + z^2)\,dy\,dz \left| \frac{x}{1} \right|_{-b/2}^{b/2}$$

$$= \rho b \int_{y=-l/2}^{l/2} \int_{z=-a/2}^{a/2} (y^2 + z^2)\,dy\,dz$$

$$= \rho b \int_{z=-a/2}^{a/2} \left| \frac{y^3}{3} + z^2 y \right|_{-l/2}^{l/2} dz = \rho b \int_{z=-a/2}^{a/2} \left(\frac{l^3}{12} + z^2 l \right) dz$$

$$= \rho b \left| \frac{l^3}{12} z + \frac{z^3}{3} l \right|_{-a/2}^{a/2} = \frac{\rho a b l}{12}(l^3 + a^2) = \frac{M}{12}(l^2 + a^2)$$

where $M = \rho\,(abl) = \rho V$ is the mass of the parallelopiped.

Following the same steps, I_{yy} and I_{zz} can be worked out as

$$I_{yy} = \rho \iiint_V (x^2 + z^2)\,dV = \frac{M}{12}(b^2 + a^2)$$

$$\text{and} \quad I_{zz} = \rho \iiint_V (x^2 + y^2)\,dV = \frac{M}{12}(b^2 + l^2)$$

EXAMPLE 8.19

The flat surface of hemisphere of radius R is connected to one flat surface of a cylinder of radius R and length l and made of same material. If the total mass be M, show that the moment of inertia of combination about the axis of cylinder is

$$\frac{MR^2(l/2 + 4R/15)}{l + 2R/3}$$

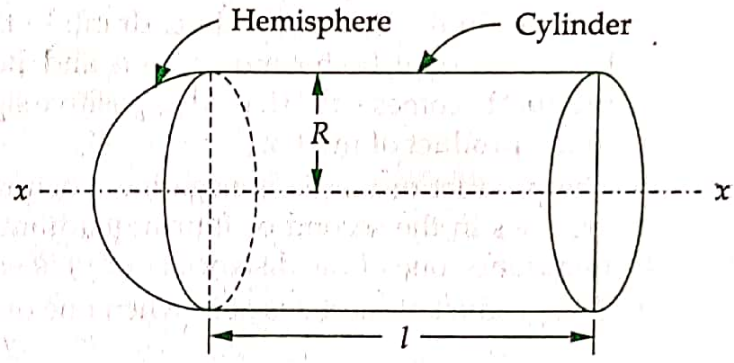

Fig. 8.43

Solution : Refer Fig. 8.43, for the arrangement of the hemisphere and the cylinder.

Let ρ be the density of the material of the given solids. Then mass of the system

$$M = \frac{2}{3}\pi R^2 \rho + \pi R^2 l \rho$$

$$\text{or} \quad \rho = \frac{M}{\pi R^2 (l + 2R/3)}$$

Moment of inertia of the combination,

$$I_{xx} = I_{xx} \text{ of hemisphere} + I_{xx} \text{ of solid cylinder}$$

For a cylinder of mass m and radius r : $I = \frac{1}{2}mr^2$ and

For a sphere of mass m and radius r : $I = \frac{2}{5}mr^2$

$$\therefore \quad I_{xx} = \frac{2}{5}\left(\frac{2}{3}\pi R^3 \rho\right) R^2 + \frac{1}{2}\left(\pi R^2 \rho l\right) R^2 = \left(\frac{4}{15}\pi R^5 + \frac{1}{2}\pi R^4 l\right)\rho$$

$$= \left(\frac{4}{15}\pi R^5 + \frac{1}{2}\pi R^4 l\right) \times \frac{M}{\pi R^2 (l + 2R/3)} = \left[\frac{4}{15}MR^3 + \frac{1}{2}MR^2 l\right] \times \frac{1}{l + 2R/3}$$

$$= \frac{MR^2(4R/15 + l/2)}{(l + 2R/3)}$$ which is the required expression.

8.5. PRODUCT OF INERTIA

Consider an elemental area dA which represents one of the small areas that comprise the plane area A. Let this elemental area be at a distance x and y from y-axis and x-axis respectively.

The moment of this area about x-axis is $y\,dA$. Further, the moment of $y dA$ about y-axis is $xy\,dA$. Then the term $xy\,dA$ is called the *product of inertia* of area dA with respect to x-axis and y-axis. The integral $\int xy\,dA$ is called the product inertia of the entire area A about the x and y axis. The product of inertia is denoted by I_{xy}. Thus,

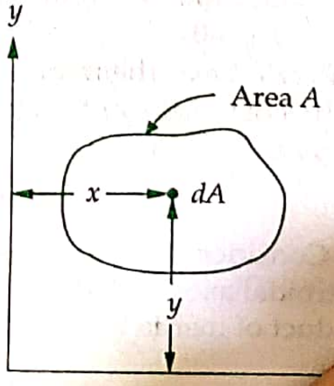

Fig. 8.44

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$$I_{xy} = \int xy\,dA \qquad ...(8.32)$$

The unit for product of inertia is m^4.

Salient features :

- The moment of inertia I_{xx}, I_{yy} is always positive.
- The product of inertia I_{xy} may be either positive or negative or it may be zero depending on the location of the area relative to the axis.
- The product of inertia is positive when the area lies in the first or third quadrant. In the third quardrant, both x and y are $-ve$ and their product becomes $+ve$. That gives positive sign to the product of inertia.

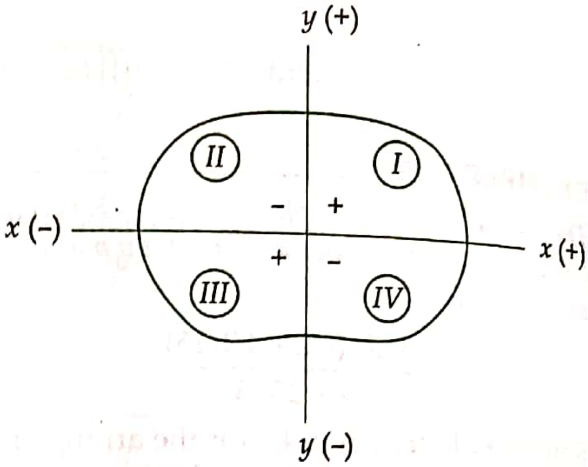

Fig. 8.45

- The product of inertia is negative when the area lies in the second or fourth quadrant. This aspect stems form the fact that in these quadrants, one of the distance (x or y) is $-ve$ and that makes the product xy negative.
- The product of inertia is zero when one or both of the x and y are axis of symmetry.

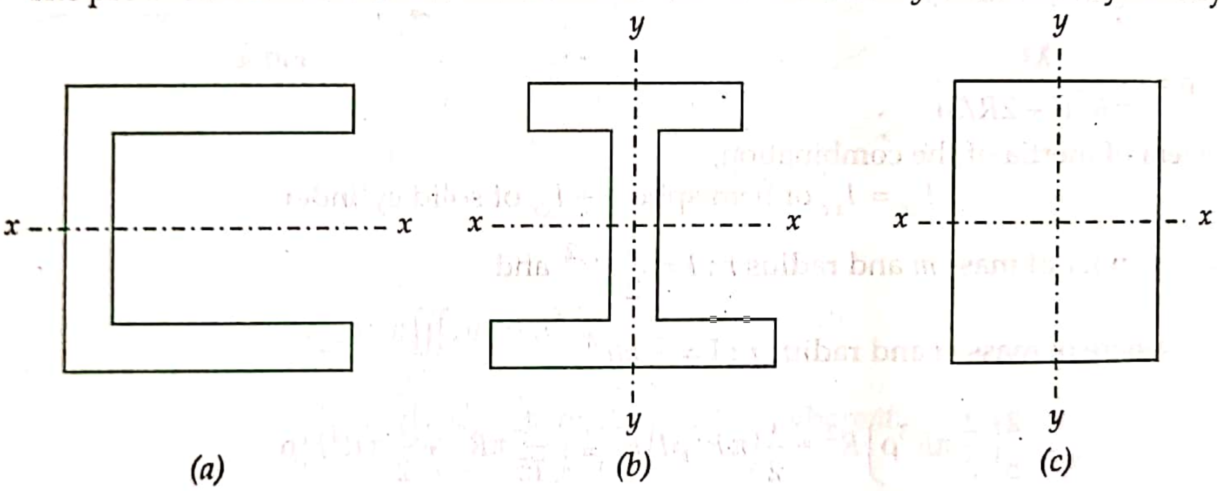

Fig. 8.46

With reference to Fig. 8.46 :

(a) Section is symmetrical about x-axis and $I_{xy} = 0$

(b) Section is symmetrical about y-axis and $I_{xy} = 0$

(c) Section is symmetrical about both the axes and $I_{xy} = 0$

Parallel axis theorem : With reference to Fig. 8.47, the product of inertia for the elemental area about axis xx and axis yy is

$$I_{xy} = \int xy\,dA$$

Consider any two axes $x'x'$ and $y'y'$ parallel to the centroidal axes xx and yy. With respect to these axes, the product of inertia is

$$I_{x'y'} = \int x'y'\,dA$$

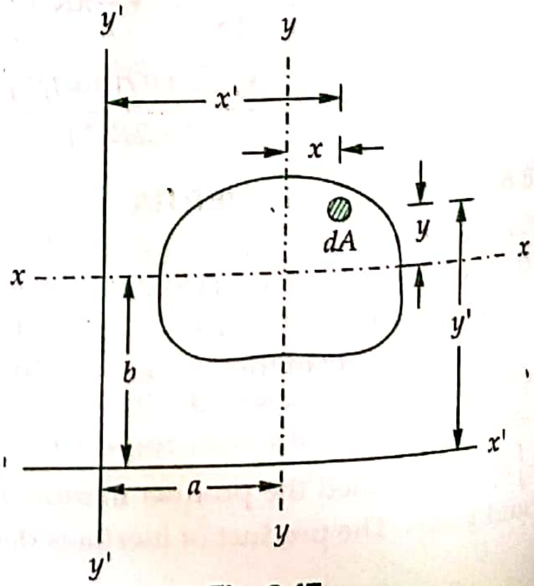

Fig. 8.47

It may be noted from Fig. 8.47, $x' = x + a$ and $y' = y + b$

$$\therefore \quad I_{x'y'} = \int (x+a)(y+b)\,dA$$
$$= \int (xy + xb + ay + ab)\,dA$$
$$= \int xy\,dA + b\int x\,dA + a\int y\,dA + ab\int dA$$

The terms $\int x\,dA$ and $\int y\,dA$ represent the first moment of area about the centroidal horizontal axis and vertical axis respectively and each equals zero. That gives

$$I_{x'y'} = \int xy\,dA + ab\int dA$$
$$= I_{xy} + abA \qquad ...(8.34)$$

The above identity is called the *transfer formula* for the product of inertia.

EXAMPLE 8.20

(a) Set up a relation for the product of inertia of a rectangle of width b and height h with respect to axes shown in Fig. 8.48.

(b) A right angled triangle has width b and height h. Show that its product of inertia with respect to x and y axes shown in Fig. 8.49 is given by $b^2h^2/2y$.

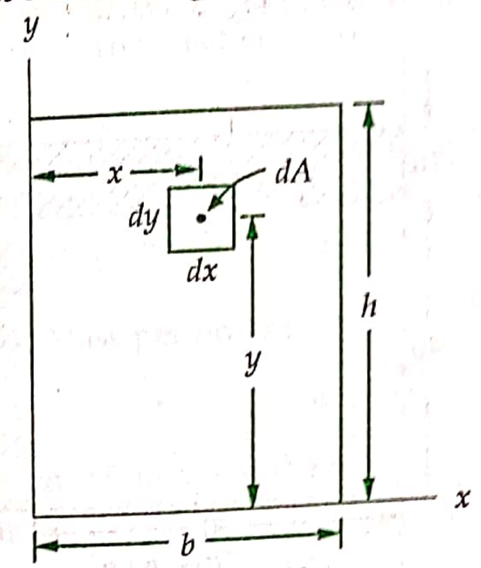

Fig. 8.48

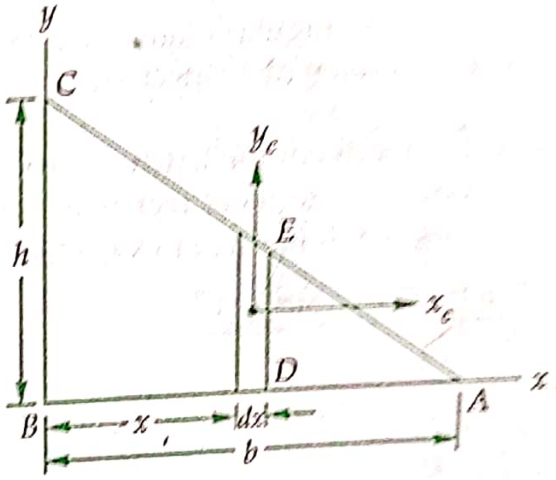

Fig. 8.49

Solution : Consider an elemental area $dA = dx\,dy$ of the rectangle with coordinates x and y.

Product of inertia of the element $= xy\,dA = xy\,dx\,dy$

Product of inertia of the whole rectangle

$$= \int_0^b\int_0^h xy\,dx\,dy = \int_0^b x\,dx \times \int_0^h y\,dy$$

$$= \left|\frac{x^2}{2}\right|_0^b \times \left|\frac{y^2}{2}\right|_0^h = \frac{b^2}{2} \times \frac{h^2}{2} = \frac{1}{4}b^2h^2$$

(*b*) Consider an elemental area $dA = y\,dx$ of the triangle. For this element x_c and y_c are the axes of symmetry. Then from the parallel axis theorem

$$dI_{xy} = dI_{x_cy_c} + x_cy_c\,dA$$
$$= 0 + \left(x \times \frac{y}{2}\right)(y\,dx) = \frac{xy^2}{2}\,dx$$

Since the integration is to be done with respect to x within the limit 0 to b, it is necessary to express y in terms of x. For that we have the following correlation from the similarity of triangles ADE and ABC.

$$\frac{y}{h} = \frac{b-x}{b}; \quad y = h\left(1 - \frac{x}{b}\right)$$

Then
$$dI_{xy} = \frac{x}{2} h^2 \left(1 - \frac{x}{b}\right)^2 dx = \frac{h^2}{2}\left(x + \frac{x^3}{b^2} - \frac{2x^2}{b}\right) dx$$

∴
$$I_{xy} = \frac{h^2}{2}\int_0^b \left(x + \frac{x^3}{b^2} - \frac{2x^2}{b}\right) dx$$

$$= \frac{h^2}{2}\left|\frac{x^2}{2} + \frac{x^4}{4b^2} - \frac{2x^3}{3b}\right|_0^b$$

$$= \frac{h^2}{2}\left(\frac{b^2}{2} + \frac{b^2}{4} - \frac{2b^2}{3}\right) = \frac{1}{24} b^2 h^2$$

EXAMPLE 8.21

Determine the product of inertia of the sectioned area about xy axes shown in Fig. 8.50. All dimensions are in mm.

Solution : For a rectangular plate of width b and height h, the product of inertia with respect to x and y axes is $b^2h^2/4$. Then

product of inertia of the hatched area

= product of inertia of rectangle 60 mm × 50 mm

– product of inertia of rectangle 50 mm × 40 mm

$$= \frac{60^2 \times 50^2}{4} - \frac{50^2 \times 40^2}{4}$$

$$= \frac{9 \times 10^6}{4} - \frac{4 \times 10^6}{4}$$

$$= \mathbf{1.25 \times 10^6\ mm^4}$$

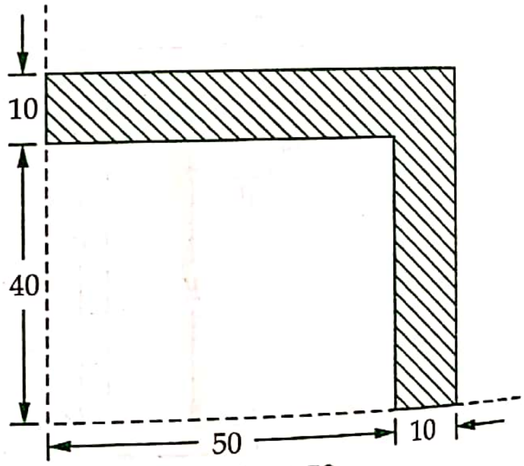

Fig. 8.50

EXAMPLE 8.22

Determine the product of inertia of the channel section (Fig. 8.51) with respect to its centroidal axis. All dimensions are in mm. Comment on the result.

Solution : The section consists of three rectangular segments which have been numbered as 1, 2 and 3

Rectangle 1 :

$$a_1 = 40 \times 10 = 400\ \text{mm}^2; \quad x_1 = \frac{40}{2} = 20\ \text{mm}$$

Rectangle 2 :

$$a_2 = 100 \times 10 = 1000\ \text{mm}^2; \quad x_2 = \frac{10}{2} = 5\ \text{mm}$$

Rectangle 3 :

$$a_3 = 40 \times 10 = 400\ \text{mm}^2; \quad x_3 = \frac{40}{2} = 5\ \text{mm}$$

Then $$\bar{x} = \frac{a_1x_1 + a_2x_2 + a_3x_3}{a_1 + a_2 + a_3}$$

$$= \frac{400 \times 20 + 1000 \times 5 + 400 \times 20}{400 + 1000 + 400}$$

$$= \frac{21000}{1800} = 11.67 \text{ mm}$$

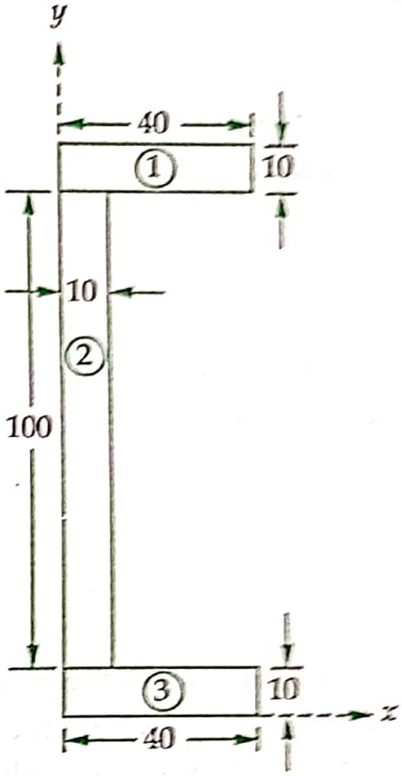

Fig. 8.51

Further the given channel section is symmetrical about x-axis and therefore

$$\bar{y} = \frac{100 + 10 + 10}{2} = 60 \text{ mm}$$

Product of inertia about centroidal axis

$= a_1(x_1 - \bar{x})(y_1 - \bar{y}) + a_2(x_2 - \bar{x})(y_2 - \bar{y}) + a_3(x_3 - \bar{x})(y_3 - \bar{y})$

$= 400(20 - 11.67)(115 - 60) + 1000(5 - 11.67)(60 - 60) + 400(20 - 11.67)(5 - 60)$

$= 183260 + 0 - 183260 = 0$

Comment : The section is symmetrical about x-axis. Hence the product of inertia has to be zero.

EXAMPLE 8.23

Find the inertia components for the thin plate shown in Fig. 8.52 given below. The weight of the plate is 0.002 N/mm² and for the top edge $y = 2\sqrt{x}$ where dimensions x and y are in mm.

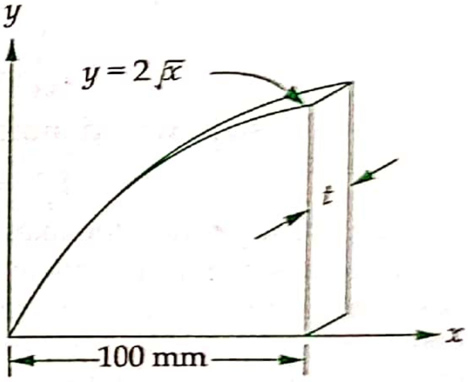

Fig. 8.52

Solution : Mass per unit area,

$$\rho t = \frac{0.002}{9.81} = 0.000204 \text{ kg/mm}^2$$

Moment of inertia of the plate about x-axis,

$$(I_{xx})_{\text{area}} = \int_0^{100} \int_{y=0}^{y=2\sqrt{x}} y^2 dy\, dx$$

$$= \int_0^{100} \left| \frac{y^3}{3} \right|_0^{2\sqrt{x}} dx = \int_0^{100} \frac{8}{3} x^{3/2} dx$$

$$= \frac{8}{3} \left| \frac{x^{5/2}}{5/2} \right|_0^{100} = \frac{8}{3} \times \frac{2}{5} (100)^{5/2} = 1.067 \times 10^5 \text{ mm}^4$$

Moment of inertia of the plate about y-axis,

$$(I_{yy})_{\text{area}} = \int_0^{100} \int_{y=0}^{y=2\sqrt{x}} x^2 dy\, dx = \int_0^{100} x^2 \left| \frac{y}{1} \right|_0^{2\sqrt{x}} dx = \int_0^{100} x^2\, 2\sqrt{x}\, dx$$

$$= \int_0^{100} 2\, x^{5/2}\, dx = 2 \left| \frac{x^{7/2}}{7/2} \right|_0^{100}$$

$$= 2 \times \frac{2}{7} \times (100)^{7/2} = 5.71 \times 10^6 \text{ mm}^4$$

Product of inertia, $$I_{xy} = \int_0^{100} \int_{y=0}^{y=2\sqrt{x}} xy \, dx \, dy = \int_0^{100} x \times \left|\frac{y^2}{2}\right|_0^{2\sqrt{x}} dx$$

$$= \int_0^{100} 2x^2 \, dx = 2\left|\frac{x^3}{3}\right|_0^{100} = 6.67 \times 10^5 \text{ mm}^4$$

The mass moment of inertia and the moment of inertia of a plate about an axis lying in the plane of the plate are related as

$$(I_{xx})_{mass} = \rho t \, (I_{xx})_{area}; \quad (I_{yy})_{mass} = \rho t \, (I_{yy})_{area} \quad \text{and} \quad (I_{xy})_{mass} = \rho t \, (I_{xy})_{area}$$

$$\therefore \quad (I_{xx})_{mass} = 0.000204 \times (1.067 \times 10^5) = \mathbf{21.77 \text{ kg mm}^2}$$

$$(I_{yy})_{mass} = 0.000204 \times (5.71 \times 10^6) = \mathbf{1164.84 \text{ kg mm}^2}$$

$$(I_{xy})_{mass} = 0.000204 \times (6.67 \times 10^5) = \mathbf{136.1 \text{ kg mm}^2}$$

8.6. PRINCIPAL AXIS AND PRINCIPAL MOMENT OF INERTIA

Consider an area A and the co-ordinate axis xx and yy passing through the centroid C. An elemental area dA has the co-ordinates x and y with respect to these axes. The product of inertia of the plane area is then given by

$$I_{xy} = \int xy \, dA$$

When the axes are rotated through 90 degree anti-clockwise, then elemental area shifts to the second quadrant and its ordinates take up the positions x' and y'. The product of inertia then becomes

$$I_{x'y'} = \int x' y' dA$$

A little insight would indicate that $x' = y$ and $y' = -x$. That gives

$$I_{x'y'} = -\int xy \, dA = -I_{xy}$$

Apparently when the axes are rotated, the product of inertia for the plane area changes sign and becomes negative. This implies that there are certain directions of axes for which the product of inertia is zero. Such axes are referred to as the *principal axes* of the area. When the plane figure is symmetrical about one or both the axes, the product of inertia about its centroidal axes is zero. Obviously the centroidal axes are the principal axes.

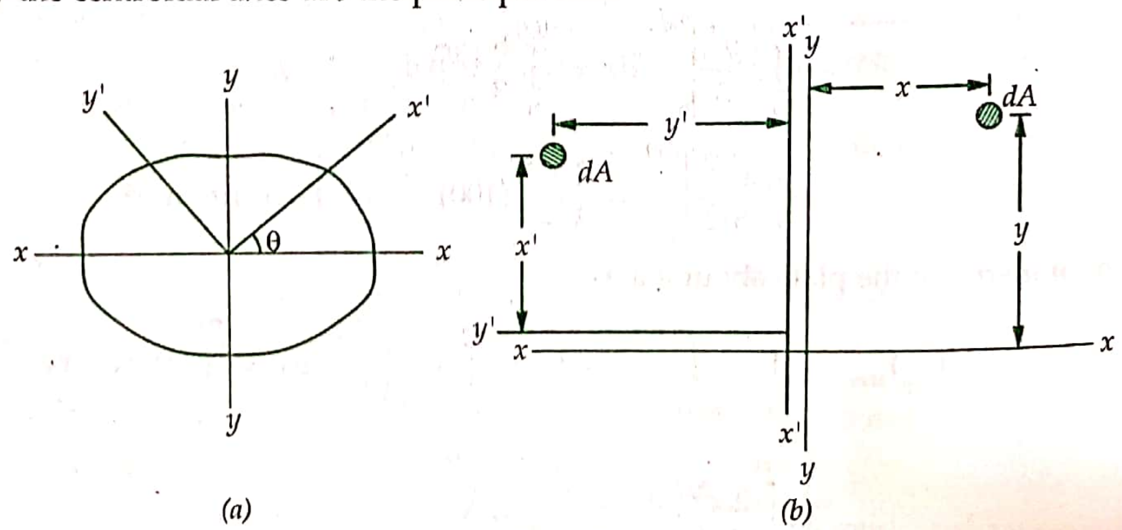

Fig. 8.53

There will always be two principal axes at a given point in the area and they will be mutually perpendicular to each other. The moment of inertia about the principal axes is called the *principal moment of inertia*. The moment of inertia about one of the axis will be maximum and the other will be minimum. The maximum moment of inertia is called the major principal moment of inertia, and the minimum moment of inertia is called the minor principal moment of inertia.

8.6.1. Principal moment of inertias and the location of principal axes

With reference to Fig. 8.54,

- xx and yy are the axes through the centroid C of the plane area A
- $x'x'$ and $y'y'$ are another coordinate axes inclined at angle θ to the axes xx and yy
- dA is an elemental area with coordinates (x, y) with respect to axes xx and yy, and coordinates (x', y') with respect to axes $x'x'$ and $y'y'$.

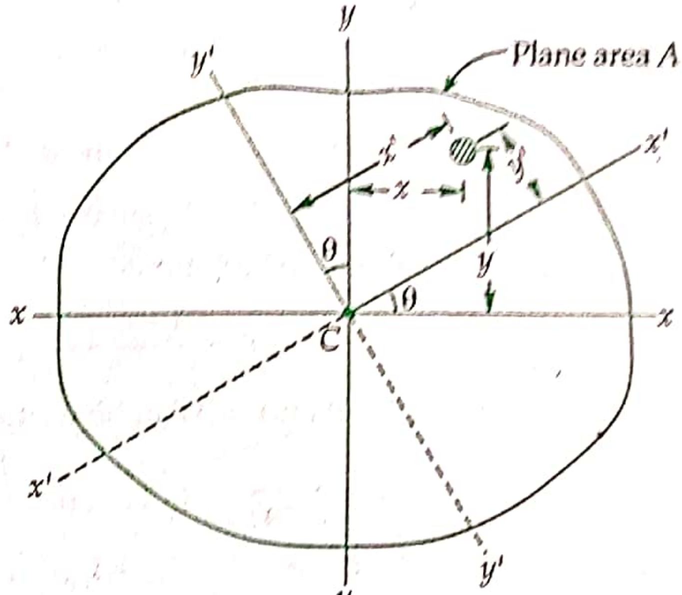

Fig. 8.54.

The following correlations apply to x, y, x', y' and angle of rotation θ (Fig. 8.55)

$$x' = EB + BP = CF + BP = y\sin\theta + x\cos\theta$$

$$y' = CE = BF = AF - AB = y\cos\theta - x\sin\theta$$

We know:

$$I_{x'y'} = \int x'y'\,dA = \int (x\cos\theta + y\sin\theta)(y\cos\theta - x\sin\theta)\,dA$$

$$= \int xy\cos^2\theta\,dA - \int x^2\sin\theta\cos\theta\,dA + \int y^2\sin\theta\cos\theta\,dA - \int xy\sin^2\theta\,dA$$

Since $\int y^2\,dA = I_{xx}$; $\int x^2\,dA = I_{yy}$ and $\int xy\,dA = I_{xy}$

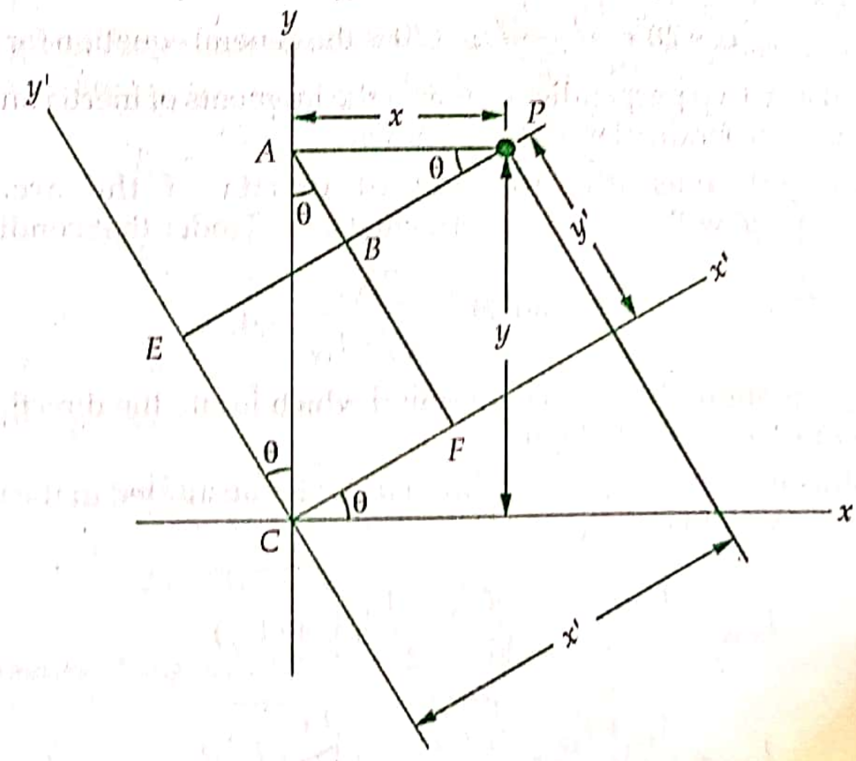

Fig. 8.55

we may write

$$I_{x'y'} = I_{xy}\cos^2\theta - I_{yy}\sin\theta\cos\theta + I_{xx}\sin\theta\cos\theta - I_{xy}\sin^2\theta \quad ...(i)$$

and

$$I_{x'x'} = \int (y')^2 dA = \int (y\cos\theta - x\sin\theta)^2 dA$$

$$= \int y^2\cos^2\theta\, dA + \int x^2\sin^2\theta\, dA - \int 2\,xy\sin\theta\cos\theta\, dA$$

$$= I_{xx}\cos^2\theta + I_{yy}\sin^2\theta - 2I_{xy}\sin\theta\cos\theta \quad ...(ii)$$

and

$$I_{y'y'} = \int (x')^2 dA = \int (y\sin\theta - x\cos\theta)^2 dA$$

$$= \int y^2\sin^2\theta\, dA + \int x^2\cos^2\theta\, dA - \int 2xy\sin\theta\cos\theta\, dA$$

$$= I_{xx}\sin\theta + I_{yy}\cos^2\theta + 2I_{xy}\sin\theta\cos\theta \quad ...(iii)$$

Using the trigonometric relations

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}\ ; \quad \sin^2\theta = \frac{1-\cos 2\theta}{2} \quad \text{and} \quad 2\sin\theta\cos\theta = \sin 2\theta,$$

the identities (*i*), (*ii*) and (*iii*) can be written as

$$I_{x'y'} = I_{xy}\cos 2\theta + \frac{I_{xx} - I_{yy}}{2}\sin 2\theta \quad ...(8.34)$$

$$I_{x'x'} = \frac{I_{xx} + I_{yy}}{2} + \frac{I_{xx} - I_{yy}}{2}\cos 2\theta - I_{xy}\sin 2\theta \quad ...(8.35)$$

and

$$I_{y'y'} = \frac{I_{xx} + I_{yy}}{2} - \frac{I_{xx} - I_{yy}}{2}\cos 2\theta + I_{xy}\sin 2\theta \quad ...(8.36)$$

- It is observed that $I_{x'x'} + I_{y'y'} = I_{xx} + I_{yy}$. Apparently the sum of moments of inertia about any two perpendicular axes remains constant. This characteristic is called the *invariant property*.
- The identity $I_{x'y'} = I_{xy}\cos 2\theta + \dfrac{I_{xx} - I_{yy}}{2}\sin 2\theta$ is the general equation for finding the product of inertia about any two perpendicular axis if the moments of inertia and product of inertia about two other perpendicular axis are known.

About the principal axes, the product of inertia of the area is zero. When $I_{x'y'} = 0$, the axes $x'x'$ and $y'y'$ will become the principal axes. Under that condition

$$I_{xy}\cos 2\theta + \frac{I_{xx} - I_{yy}}{2}\sin 2\theta = 0 \quad \text{or} \quad \tan 2\theta = \frac{2I_{xy}}{I_{yy} - I_{xx}} \quad ...(8.37)$$

Solution to above equation gives two values of θ which locate the directions of the principal axes. The two values of θ differ by 90 degree.

- When the value of θ as defined by equation (8.37) is substituted in the identities (8.35) and (8.36), we obtain we obtain

$$I_{max} = \frac{I_{xx} + I_{yy}}{2} + \sqrt{\left(\frac{I_{xx} - I_{yy}}{2}\right)^2 + (I_{xy})^2}$$

$$I_{min} = \frac{I_{xx} + I_{yy}}{2} - \sqrt{\left(\frac{I_{xx} - I_{yy}}{2}\right)^2 + (I_{xy})^2} \quad ...(8.38)$$

where I_{max} and I_{min} are the major and minor principal moment of inertias.

8.6.2. Mohr's circle for principle moments of inertia

With reference to Fig. 8.56, the following procedure is adopted for the construction of Mohr's circle for principal moments of inertia :

(i) Taking O as origin, measure $OL = I_{yy}$ and $OM = I_{xx}$ (according to some scale) along the horizontal I-axis.

(ii) At point L, draw upward perpendicular if I_{xy} is positive and measure $LP = I_{xy}$. If I_{xy} is negative, draw the perpendicular downward.

(iii) Bisect LM at N.

(iv) With N as centre and NP as radius, draw a circle meeting OL and M extended at points A and B respectively.

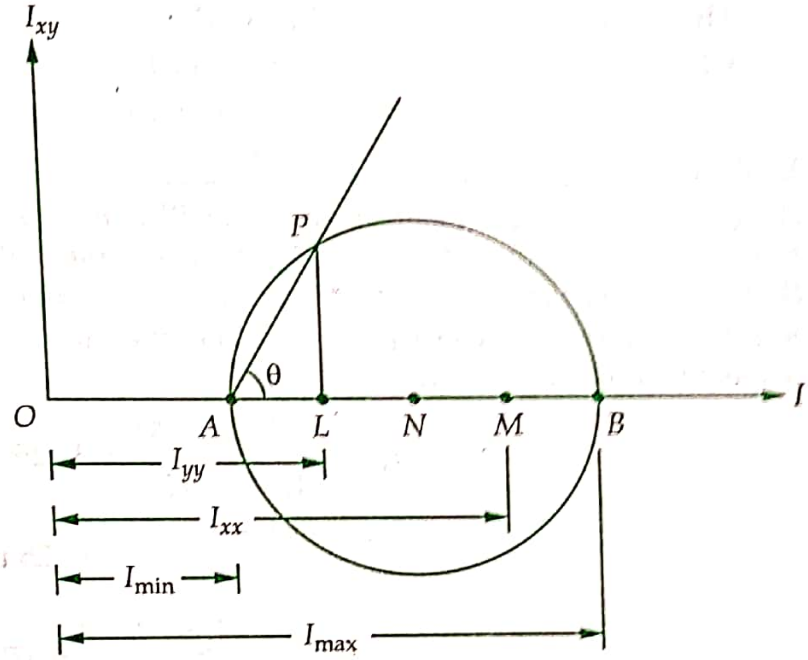

Fig. 8.56

Then OA gives the minor principal moment of inertia and OB gives major principal moment of inertia. The direction of minor principal axis lies along AP and the direction of major principal axis is at right angle to it.

EXAMPLE 8.24

Find the product of inertia of the rectangular section shown in Fig. 8.57, with respect to x and y axes. Proceed to calculate the location of the principal axes and the values of the principal moments of inertia of the section about point A.

Solution : For the given rectangular plate, the moments of inertia and the product of inertia with respect to x and y axes are :

$$I_{xx} = \frac{bh^3}{3} = \frac{2 \times 4^3}{3} = 42.67 \text{ m}^4$$

$$I_{yy} = \frac{hb^3}{3} = \frac{4 \times 2^3}{3} = 10.67 \text{ m}^4$$

$$I_{xy} = \frac{b^2h^2}{4} = \frac{2^2 \times 4^2}{4} = 16 \text{ m}^4$$

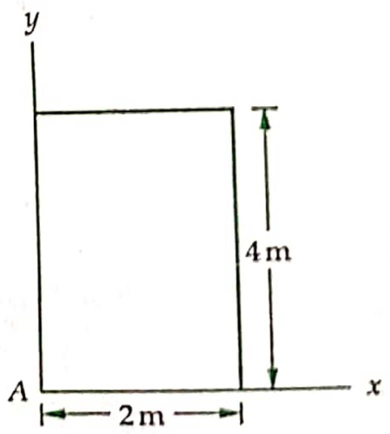

Fig. 8.57

Location of principal axes :

$$\tan 2\theta = \frac{2I_{xy}}{I_{yy} - I_{xx}} = \frac{2 \times 16}{10.67 - 42.67} = -1.0$$

$$2\theta = 315^\circ\,; \quad \theta_1 = 157.5^\circ$$

$$\theta_2 = 157.5^\circ - 90^\circ = 67.5^\circ$$

Principal moments of inertia :

$$I_{max}, I_{min} = \frac{I_{xx} + I_{yy}}{2} \pm \sqrt{\left(\frac{I_{xx} - I_{yy}}{2}\right)^2 + \left(I_{xy}\right)^2}$$

$$= \frac{42.67 + 10.67}{2} \pm \sqrt{\left(\frac{42.67 - 10.67}{2}\right)^2 + 16^2}$$

$$= 26.67 \pm 22.63$$

That gives : $I_{max} = \mathbf{49.3\ m^4}$ and $I_{min} = \mathbf{4.04\ m^4}$

Check : $I_{max} + I_{min} = 49.3 + 4.04 = 53.34\ cm^4$

$I_{xx} + I_{yy} = 42.67 + 10.67 = 53.34\ cm^4$

EXAMPLE 8.25

A right angled triangular plate has a base width 3 m and height 5 m. Find the product of inertia with respect to x and y axes shown in Fig. 8.58. Proceed to determine the location of the principal axes and the values of the principal moments of inertia of the section about point A.

Solution : For the given triangular plate, the moments of inertia and the product of inertia with respect to x and y axes are :

$$I_{xx} = \frac{bh^3}{12} = \frac{3 \times 5^3}{12} = 31.25\ m^4$$

$$I_{yy} = \frac{hb^3}{12} = \frac{5 \times 3^3}{12} = 11.25\ m^4$$

$$I_{xy} = \frac{b^2h^2}{24} = \frac{3^2 \times 5^2}{24} = 9.375\ m^4$$

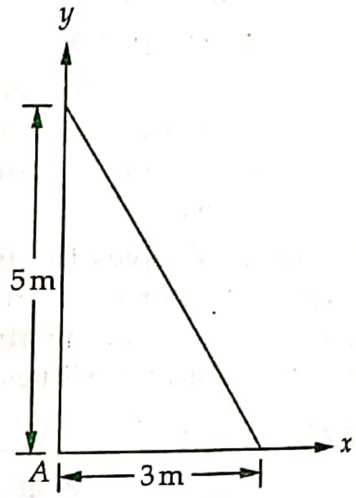

Fig. 8.58

Location of principal axes :

$$\tan 2\theta = \frac{2 I_{xy}}{I_{yy} - I_{xx}} = \frac{2 \times 9.375}{11.25 - 31.25} = -0.9375$$

$$2\theta = 316.85° ; \quad \theta_1 = \mathbf{158.42°}$$

$$\theta_2 = 158.42° - 90° = \mathbf{68.42°}$$

Principal moments of inertia :

$$I_{max}, I_{min} = \frac{I_{xx} + I_{yy}}{2} \pm \sqrt{\left(\frac{I_{xx} - I_{yy}}{2}\right)^2 + (I_{xy})^2}$$

$$= \frac{31.25 + 11.25}{2} \pm \sqrt{\left(\frac{31.25 - 11.25}{2}\right)^2 + (9.375)^2}$$

$$= 21.25 \pm 13.71$$

That gives : $I_{max} = \mathbf{34.96\ m^4}$ and $I_{min} = \mathbf{7.54\ m^4}$

Check : $I_{max} + I_{min} = 34.96 + 7.54 = 52.50\ cm^4$

$I_{xx} + I_{yy} = 31.25 + 11.25 = 52.50\ cm^4$

EXAMPLE 8.26

For the unequal angle section shown in Fig. 8.59, determine the location of principal axes and the principal moments of inertia.

Solution : The given section is split into two parts marked 1 and 2.

Location of centroidal axis (with respect to faces AB and BC)

$$\bar{x} = \frac{a_1x_1 + a_2x_2}{a_1 + a_2} = \frac{10 \times 5 + 14 \times 0.5}{10 + 14} = 2.375\ cm$$

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2} = \frac{10 \times 0.5 + 14 \times 8}{10 + 14} = 4.875 \text{ cm}$$

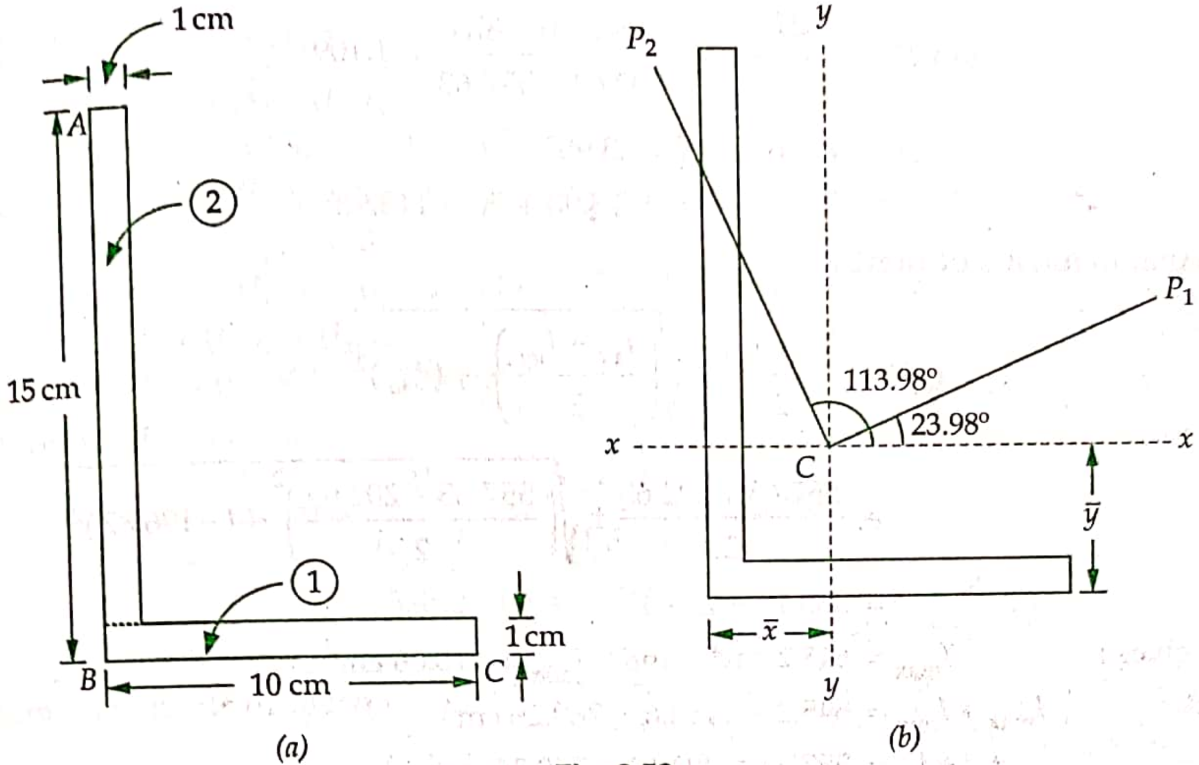

Fig. 8.59

Calculations for moments of inertia and product of inertia

$$(I_{xx})_1 = \frac{bh^3}{12} + A(\bar{y} - y_1)^2$$

$$= \frac{1}{12} \times 10 \times 1^3 + (10 \times 1) \times (4.875 - 0.5)^2 = 192.24 \text{ cm}^4$$

$$(I_{xx})_2 = \frac{1}{12} \times 1 \times 14^3 + (14 \times 1) \times (4.875 - 8)^2 = 365.39 \text{ cm}^4$$

That gives : $I_{xx} = 192.24 + 365.39 = 557.63 \text{ cm}^4$

$$(I_{yy})_1 = \frac{bh^3}{12} + A(\bar{x} - x)^2$$

$$= \frac{1}{12} \times 1 \times 10^3 + (10 \times 1) \times (2.375 - 5)^2 = 152.24 \text{ cm}^4$$

$$(I_{yy})_2 = \frac{1}{12} \times 1^3 \times 14 + (14 \times 1) \times (2.375 - 0.5)^2 = 50.39 \text{ cm}^4$$

That gives : $I_{yy} = 152.24 + 50.39 = 202.63 \text{ cm}^4$

$$(I_{xy})_1 = 0 + (10 \times 1) \times (5 - 2.375)(0.5 - 4.875) = -114.84 \text{ cm}^4$$

$$(I_{xy})_2 = 0 + (14 \times 1) \times (8 - 4.875) \times (0.5 - 2.375) = -82.03 \text{ cm}^4$$

The above results for product of inertia have been written by using parallel axis theorem and noting that product of inertia vanishes if any one of the axes is the axis of symmetry.

That gives : $I_{xy} = -114.84 - 82.03 = -196.87 \text{ cm}^4$

Location of principal axes

$$\tan 2\theta = \frac{2I_{xy}}{I_{yy} - I_{xx}} = \frac{2 \times (-196.87)}{202.63 - 557.63} = 1.109$$

$$2\theta = 47.96^\circ\ ; \quad \theta_1 = 23.98^\circ$$

$$\theta_2 = 23.98 + 90 = 113.98^\circ$$

Principal moments of inertia

$$I_{max}, I_{min} = \frac{I_{xx} + I_{yy}}{2} \pm \sqrt{\left(\frac{I_{xx} - I_{yy}}{2}\right)^2 + (I_{xy})^2}$$

$$= \frac{557.63 + 202.63}{2} \pm \sqrt{\left(\frac{557.63 - 202.63}{2}\right)^2 + (-196.87)^2}$$

$$= 380.13 \pm 265.07$$

That gives : $I_{max} = \mathbf{645.2\ cm^4}$ and $I_{min} = \mathbf{115.06\ cm^4}$

Check : $I_{max} + I_{min} = 645.2 + 115.06 = 760.26 \text{ cm}^4$

$I_{xx} + I_{yy} = 557.63 + 202.63 = 760.26 \text{ cm}^4$

EXAMPLE 8.27

For the centrally symmetrical z section shown in Fig. 8.60, make calculations for the followings :

(a) product of inertia of the section with respect to the x and y axes,

(b) principal axes and the principal moments of inertia of the section about point P.

All dimensions are in cm

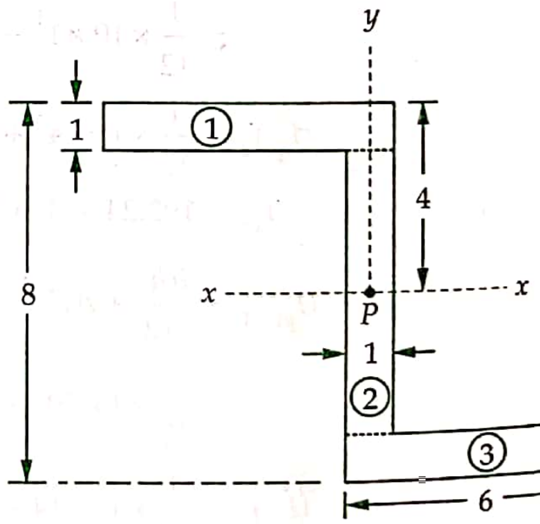

Fig. 8.60

Solution : The section can be split up into three rectangles marked 1, 2 and 3.

Moments of inertia

$$(I_{xx})_1 = \frac{1}{12} \times 60 \times 1^3 + (6 \times 1) \times (4 - 0.5)^2$$
$$= 74 \text{ cm}^4$$

$$(I_{xx})_2 = \frac{1}{12} \times 1 \times 6^3 = 18 \text{ cm}^4$$

$$(I_{xx})_3 = \frac{1}{12} \times 6 \times 1^3 + (6 \times 1) \times (4 - 0.5)^2$$
$$= 74 \text{ cm}^4$$

$$\therefore \quad I_{xx} = (I_{xx})_1 + (I_{xx})_2 + (I_{xx})_3 = 74 + 18 + 74$$
$$= 166 \text{ cm}^4$$

$$(I_{yy})_1 = \frac{1}{12} \times 1 \times 6^3 + (6 \times 1) \times (3 - 0.5)^2 = 55.5 \text{ cm}^4$$

$$(I_{yy})_2 = \frac{1}{12} \times 6 \times 1^3 = 0.5 \text{ cm}^4$$

$$(I_{yy})_3 = \frac{1}{12} \times 1 \times 6^3 + (6+1) \times (3-0.5)^2 = 55.5 \text{ cm}^4$$

$$\therefore \quad I_{yy} = (I_{yy})_1 + (I_{yy})_2 + (I_{yy})_3$$
$$= 55.5 + 0.5 + 55.5 = 111.5 \text{ cm}^2$$

$$(I_{xy})_1 = 0 + (6 \times 1) \times (3 - 5.5)\,(4 - 0.5) = -52.5 \text{ cm}^4$$
$$(I_{xy})_2 = 0$$
$$(I_{xy})_3 = (6 \times 1) \times (5.5 - 3) \times (0.5 - 4) = -52.5 \text{ cm}^4$$

$$\therefore \quad I_{xy} = (I_{xy})_1 + (I_{xy})_2 + (I_{xy})_3$$
$$= -52.5 + 0 - 52.5 = -105 \text{ cm}^4$$

(b) Position of principal axis

$$\tan 2\theta = \frac{2I_{xy}}{I_{yy} - I_{xx}} = \frac{2 \times (-105)}{111.5 - 166} = 3.853$$

$$2\theta = 75.45°; \quad \theta_1 = 37.72°$$
$$\theta_2 = 37.72 + 90° = 127.72°$$

Principal moments of inertia

$$I_{max}, I_{min} = \frac{I_{xx} + I_{yy}}{2} \pm \sqrt{\left(\frac{I_{xx} - I_{yy}}{2}\right)^2 + (I_{xy})^2}$$

$$= \frac{166 + 111.5}{2} \pm \sqrt{\left(\frac{166 - 111.5}{2}\right)^2 + (-105)^2}$$

$$= 138.75 \pm 108.48$$

That gives : $I_{max} = 247.23 \text{ cm}^4$ and $I_{min} = 30.27 \text{ cm}^4$

Check : $I_{max} + I_{min} = 247.23 + 30.27 = 277.5 \text{ cm}^4$

$I_{xx} + I_{yy} = 166 + 111.5 = 277.5 \text{ cm}^4$

REVIEW AND SUMMARY

1. The moment of an area about any axis in its plane is called the first moment of area and the second moment is known as moment of inertia of the area. Mathematically

$$I_{xx} = \int_A x^2 dA \quad \text{and} \quad I_{yy} = \int_A y^2 dA$$

2. The first moment of area about centroidal axis is zero, while the second moment of area (moment of inertia) about centroidal axis has a minimum non-zero value.
3. The mass moment of inertia about an axis is the product of elemental mass and the square of the distance between the mass centre of the elemental mass and the axis.

$$\text{Mass moment of inertia} = \int x^2 dm$$

4. Radius of gyration (k) of a body or a given body (or lamina) is defined as the distance from the axis of reference to where the whole mass (or area) of a body is assumed to be concentrated, and that is not to make any change in the moment of inertia about the given axis.

$$k_x = \sqrt{I_{xx}/A} \quad ; \quad k_y = \sqrt{I_{yy}/A}$$