Question Based on

Lagrangian and

Hamiltonian Equations.

Key Concepts:

Lagrangian $L = T - V$

Generalised Momenta $p_k = \frac{\partial L}{\partial \dot{q}_k}$

Hamilton's Equation $\dot{q}_k = \frac{\partial H}{\partial p_k}$, $\dot{p}_k = \frac{-\partial H}{\partial \dot{q}_k}$, $\frac{\partial L}{\partial t} = \frac{-\partial H}{\partial t}$

Hamiltonian $H = \sum_{k=1}^{n} p_k \dot{q}_k - L$

Force or time derivative of p_k, $\dot{p}_k = \frac{\partial L}{\partial q_k}$

Linked

Question Gate 2010.

The Lagrangian for a simple pendulum is given by

$$L = \frac{1}{2} m l^2 \dot{\theta}^2 - mg(1-\cos\theta)$$

Q1. Hamilton's Equations are then given by

(a) $\dot{p}_\theta = -mgl\sin\theta$, $\dot{\theta} = \frac{p_\theta}{ml^2}$

(b) $\dot{p}_\theta = mgl\sin\theta$, $\dot{\theta} = \frac{p_\theta}{ml^2}$

Q2 The poission bracket $[\theta, \dot{\theta}]_{\theta, p_\theta}$ is

Solution of Q1.

Given: $L = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl(1-\cos\theta)$ —— (1)

$$P_\theta = \frac{\partial L}{\partial \dot{\theta}}$$

$$P_\theta = \frac{\partial}{\partial \dot{\theta}} \left\{ \frac{1}{2} m l^2 \dot{\theta}^2 - mgl(1-\cos\theta) \right\}$$

$$P_\theta = \frac{1}{2} m l^2 \frac{\partial}{\partial \dot{\theta}} \dot{\theta}^2 - 0$$

$$P_\theta = \frac{1}{2} m l^2 \cdot 2\dot{\theta}$$

$$P_\theta = m l^2 \dot{\theta}$$

$$\Rightarrow \quad \dot{\theta} = \frac{P_\theta}{m l^2} \quad \text{—— (2)}$$

and $\dot{p}_\theta = \frac{\partial L}{\partial \theta}$

$$\dot{p}_\theta = \frac{\partial}{\partial \theta}\left\{\frac{1}{2} m l^2 \dot{\theta}^2 - mgl(1-\cos\theta)\right\}$$

$$\dot{p}_\theta = 0 - \frac{\partial}{\partial \theta}\left(mgl(1-\cos\theta)\right)$$

$$\dot{p}_\theta = -mgl \frac{\partial}{\partial \theta}(1-\cos\theta)$$

$$\dot{p}_\theta = -mgl\left(0 - \frac{\partial}{\partial \theta}\cos\theta\right)$$

$$\dot{p}_\theta = -mgl\left(-(-\sin\theta)\right)$$

$$\dot{p}_\theta = -mgl\sin\theta \quad \text{--- (3)}$$

By eq (2) & (3)

$$\dot{\theta} = \frac{p_\theta}{md^2}$$

$$\dot{p}_\theta = -mgd\sin\theta$$

$\therefore$ (a) is correct objective.

Solution of Q2.

$$[\theta, \dot{\theta}]_{\theta, p_\theta} = \left(\frac{\partial \theta}{\partial \theta} \cdot \frac{\partial \dot{\theta}}{\partial p_\theta} - \frac{\partial \theta}{\partial p_\theta} \cdot \frac{\partial \dot{\theta}}{\partial \theta} \right)$$

$$= \left(1 \cdot \frac{\partial}{\partial p_\theta}\left(\frac{p_\theta}{md^2}\right) - 0 \right) = \frac{1}{md^2} \cdot \frac{\partial p_\theta}{\partial p_\theta} = \frac{1}{md^2}$$

(Ans)