Question Based On

"Lagrangian"

Basic Concept:

1)

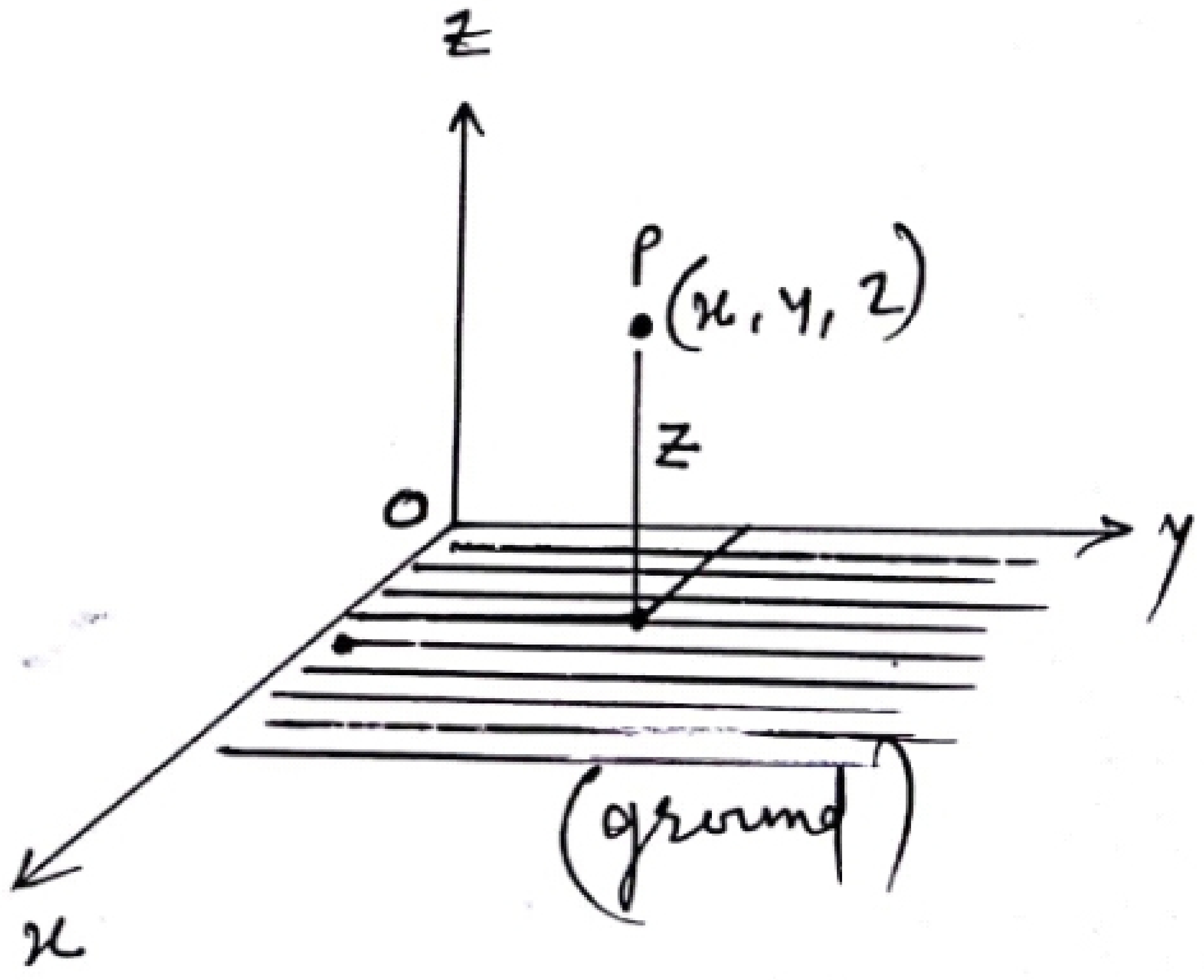
Lagrangian = Kinetic Energy − Potential Energy.

$$L = T - V$$

2) P.V. $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\vec{V} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

$$\vec{V} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}$$



comparing with

$$\vec{V} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$

$$v_x = \dot{x} \quad , \quad v_y = \dot{y} \quad , \quad v_z = \dot{z}$$

$$\therefore \quad T = \frac{1}{2} m v^2$$

$$= \frac{1}{2} m \left(\sqrt{v_x^2 + v_y^2 + v_z^2}\right)^2$$

$$= \frac{1}{2} m \left(v_x^2 + v_y^2 + v_z^2\right)$$

$$= \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

$V = mgh$; h = height above ground

$V = mgz$

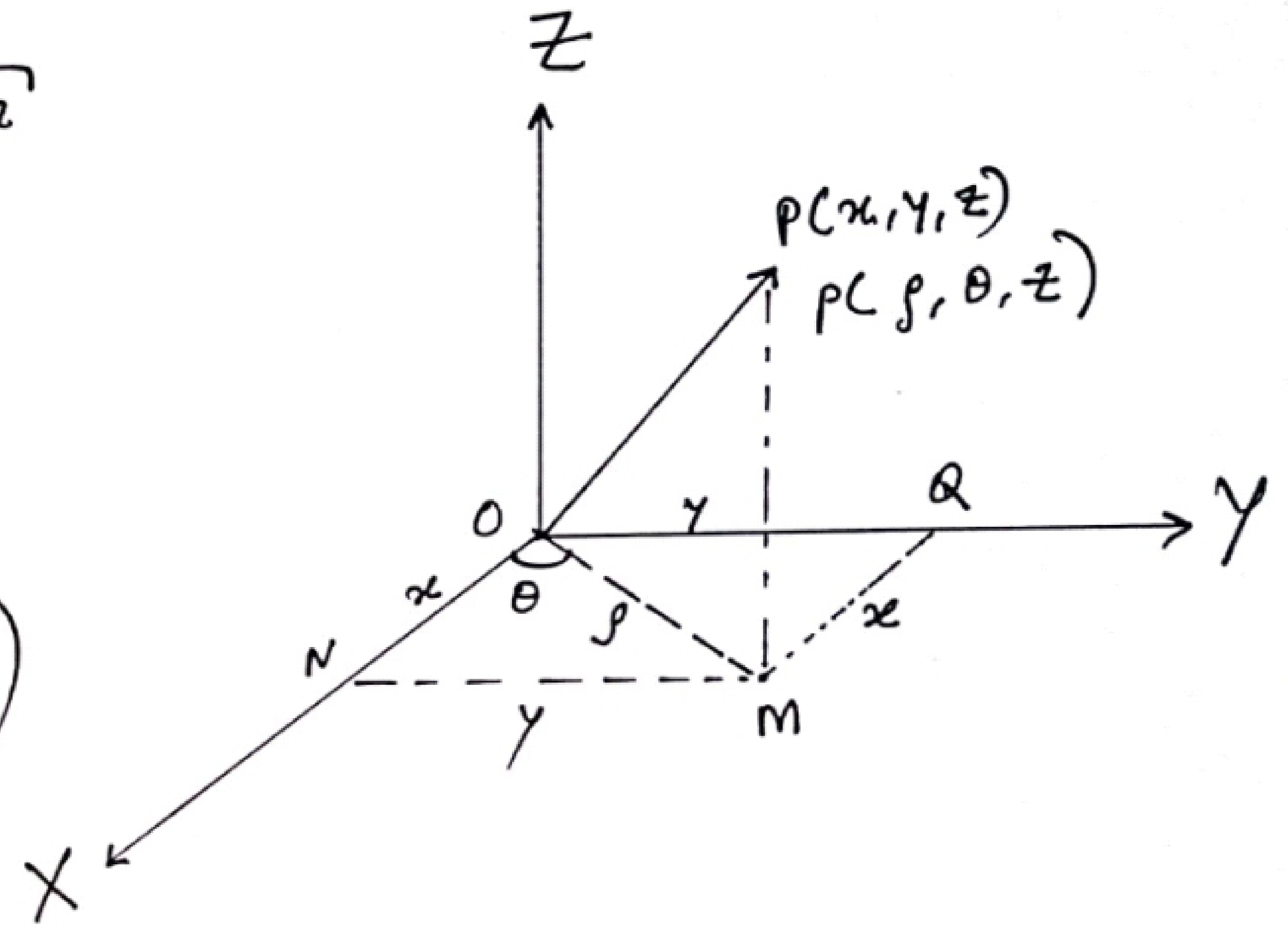
(3). Cylindrical Coordinate system

$$\rho^2 = x^2+y^2 \Rightarrow \rho = \sqrt{x^2+y^2}$$

$$\cos\theta = \frac{x}{\rho} \Rightarrow x = \rho\cos\theta$$

$$\sin\theta = \frac{y}{\rho} \Rightarrow y = \rho\sin\theta$$

$$\tan\theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$$



Sometimes instead of 'ρ', we use 'r'

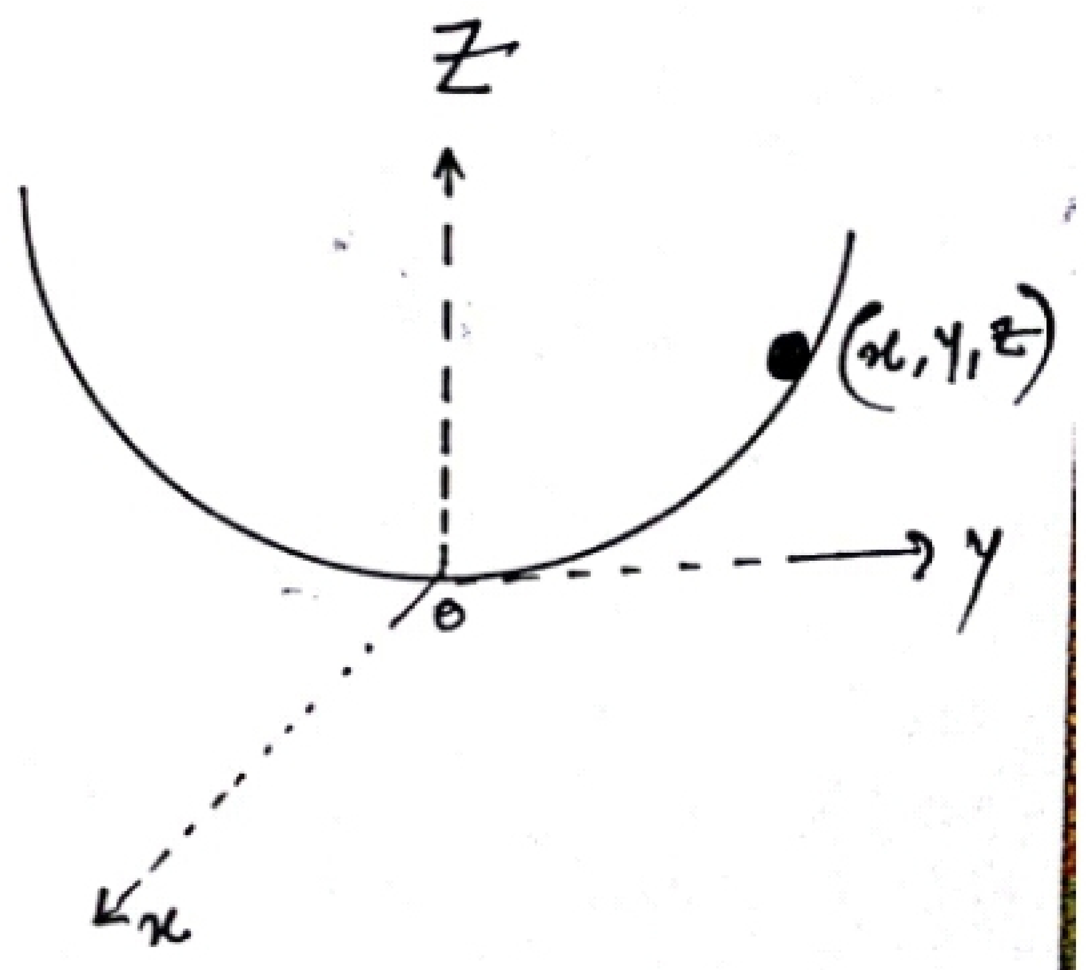
Question: CSIR UGC NET Dec 2011

A particle of mass m moves inside a bowl. If the surface of bowl is given by the equation $z = \frac{1}{2} a (x^2+y^2)$ where 'a' is a constant. The Lagrangian of the particle is ———

(a) $\frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 - gar^2)$ (b) $\frac{1}{2} m [(1+a^2r^2)\dot{r}^2 + r^2\dot{\theta}^2 - gar^2]$

Solution.

Given $z = \frac{a}{2}(x^2+y^2)$



Here It will be more convenient to use cylindrical coordinates

$\therefore \; r^2 = x^2+y^2$

$\therefore \; z = \frac{a}{2} r^2$

$$\frac{dz}{dt} = \frac{a}{2}\frac{d}{dt}r^2$$

$$\dot{z} = \frac{a}{2}\frac{dr^2}{dr}\cdot\frac{dr}{dt}$$

$$\dot{z} = \frac{a}{2}2r^{2-1}\dot{r}$$

$$\dot{z} = ar\dot{r}$$

als $x = r\cos\theta$

$$\Rightarrow \frac{dx}{dt} = \frac{d}{dt}r\cos\theta$$

$$= r\frac{d\cos\theta}{dt} + \cos\theta\frac{dr}{dt}$$

$$= r(-\sin\theta)\frac{d\theta}{dt} + \cos\theta\,\dot{r}$$

$$= -r\sin\theta\,\dot{\theta} + \dot{r}\cos\theta$$

and $y = r\sin\theta$

$$\frac{dy}{dt} = \frac{d}{dt}(r\sin\theta)$$

$$\dot{y} = r\frac{d}{dt}\sin\theta + \sin\theta\frac{dr}{dt}$$

$$\dot{y} = r\cos\theta \cdot \dot{\theta} + \sin\theta\,\dot{r}$$

$$\dot{y} = r\cos\theta \cdot \dot{\theta} + \dot{r}\sin\theta$$

Now : $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

$$T = \frac{1}{2}m\left[(-r\dot{\theta}\sin\theta + \dot{r}\cos\theta)^2 + (r\dot{\theta}\cos\theta + \dot{r}\sin\theta)^2 + (a\dot{r})^2\right]$$

$$T = \frac{1}{2}m\Big[r^2\dot{\theta}^2\sin^2\theta + \dot{r}^2\cos^2\theta + 2(-r\dot{\theta}\sin\theta)(\dot{r}\cos\theta) + r^2\dot{\theta}^2\cos^2\theta + \dot{r}^2\sin^2\theta + 2(r\dot{\theta}\cos\theta)(\dot{r}\sin\theta) + a^2r^2\dot{r}^2\Big]$$

$$T = \frac{1}{2}m\left[r^2\dot{\theta}^2[\sin^2\theta + \cos^2\theta] + \dot{r}^2\{\cos^2\theta + \sin^2\theta\} + a^2r^2\dot{r}^2\right]$$

$$T = \frac{1}{2}m\left[r^2\dot{\theta}^2 + \dot{r}^2 + a^2r^2\dot{r}^2\right]$$

$$T = \frac{1}{2}m\left[r^2\dot{\theta}^2 + (1 + a^2r^2)\dot{r}^2\right]$$

Now $V = mgh$

$$V = mgz$$

$$V = mg\left(\frac{a}{2}r^2\right) \qquad \because z = \frac{ar^2}{2}$$

$$V = \frac{amgr^2}{2}$$

So, $L = T - V$

$$L = \frac{1}{2} m\left[r^2\dot{\theta}^2 + (1+a^2r^2)\dot{r}^2\right] - \frac{amgr^2}{2}$$

$$L = \frac{1}{2} m\left[r^2\dot{\theta}^2 + (1+a^2r^2)\dot{r}^2 - agr^2\right]$$ (Ans)