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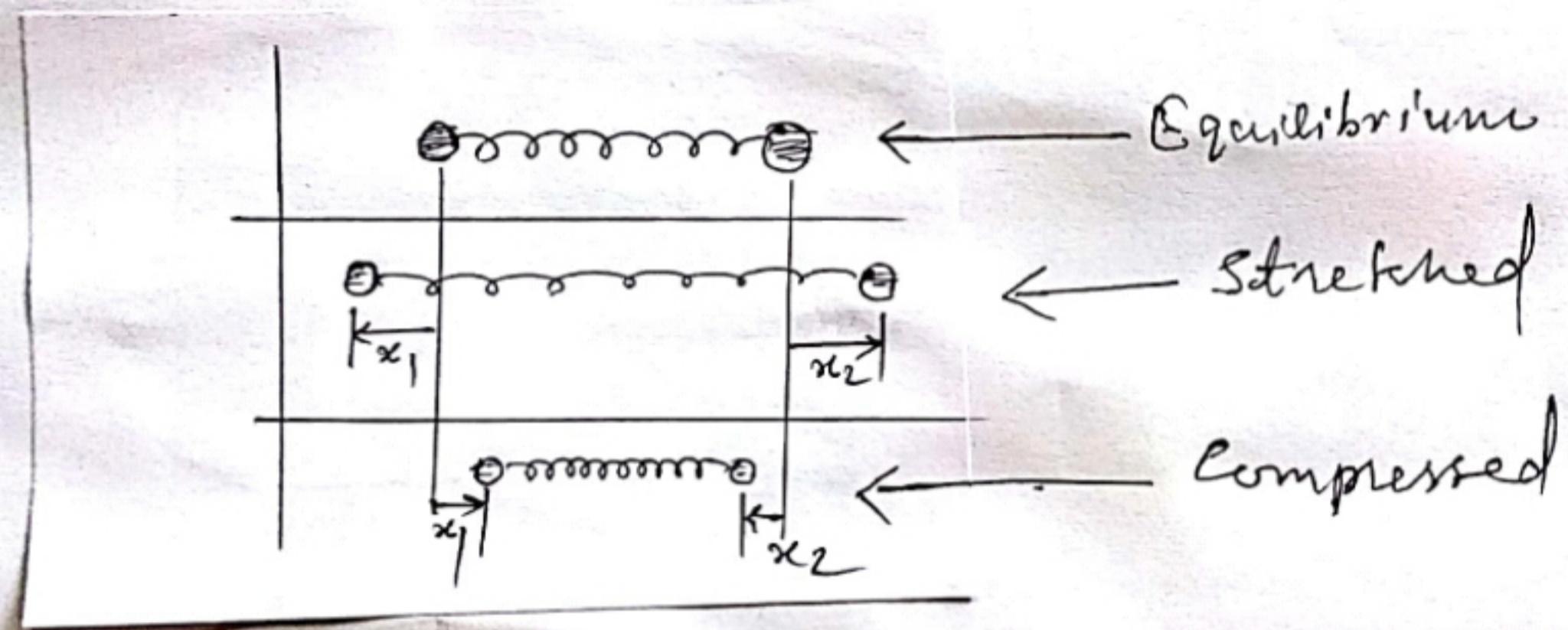
Question Based On

"Frequency of Normal Mode"

Key Concepts:-

if $V = V(x_1, x_2)$ (Potential Energy of System)

$T = \frac{1}{2} T(\dot{x}_1, \dot{x}_2)$ (Kinetic Energy of System)



Normal Modes: Different ways by which system
can vibrate

example



Normal frequency: frequency of normal modes
is called Normal frequencies. (ω)

Normal frequencies are obtained by solving
Secular Equation

$$|\bar{V} - \omega^2 \bar{T}| = 0$$

Solving $\omega = \omega_1, \omega_2$

Here \bar{V} = potential Energy matrix

$$\bar{V} = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}$$

$V_{11} = \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_1} (V)$

$$V_{12} = \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} (V)$$

$$V_{21} = \frac{\partial}{\partial x_2} \frac{\partial}{\partial x_1} (V)$$

$$V_{22} = \frac{\partial}{\partial x_2} \frac{\partial}{\partial x_2} (V)$$

\bar{T} = kinetic Energy Matrix

$$\bar{T} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

$$T_{11} = \frac{\partial}{\partial \dot{x}_1} \frac{\partial}{\partial \dot{x}_1} (\bar{T})$$

$$T_{12} = \frac{\partial}{\partial \dot{x}_1} \frac{\partial}{\partial \dot{x}_2} (\bar{T})$$

$$T_{21} = \frac{\partial}{\partial \dot{x}_2} \frac{\partial}{\partial \dot{x}_1} (\bar{T})$$

$$T_{22} = \frac{\partial}{\partial \dot{x}_2} \frac{\partial}{\partial \dot{x}_2} (\bar{T})$$

(P-6)

You can also find $T_{11}, T_{12}, T_{21}, T_{22}$

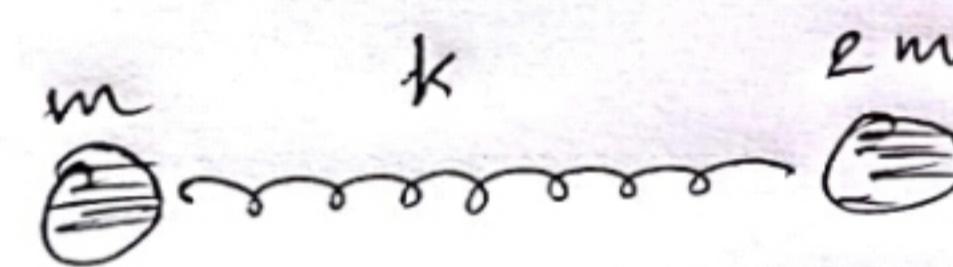
by comparing $\mathbf{q}, \mathbf{K}, \mathbf{E}$. of system with

$$T = \frac{1}{2} [T_{11} \dot{x}_1^2 + T_{12} \dot{x}_1 \dot{x}_2 + T_{21} \dot{x}_2 \dot{x}_1 + T_{22} \dot{x}_2^2]$$

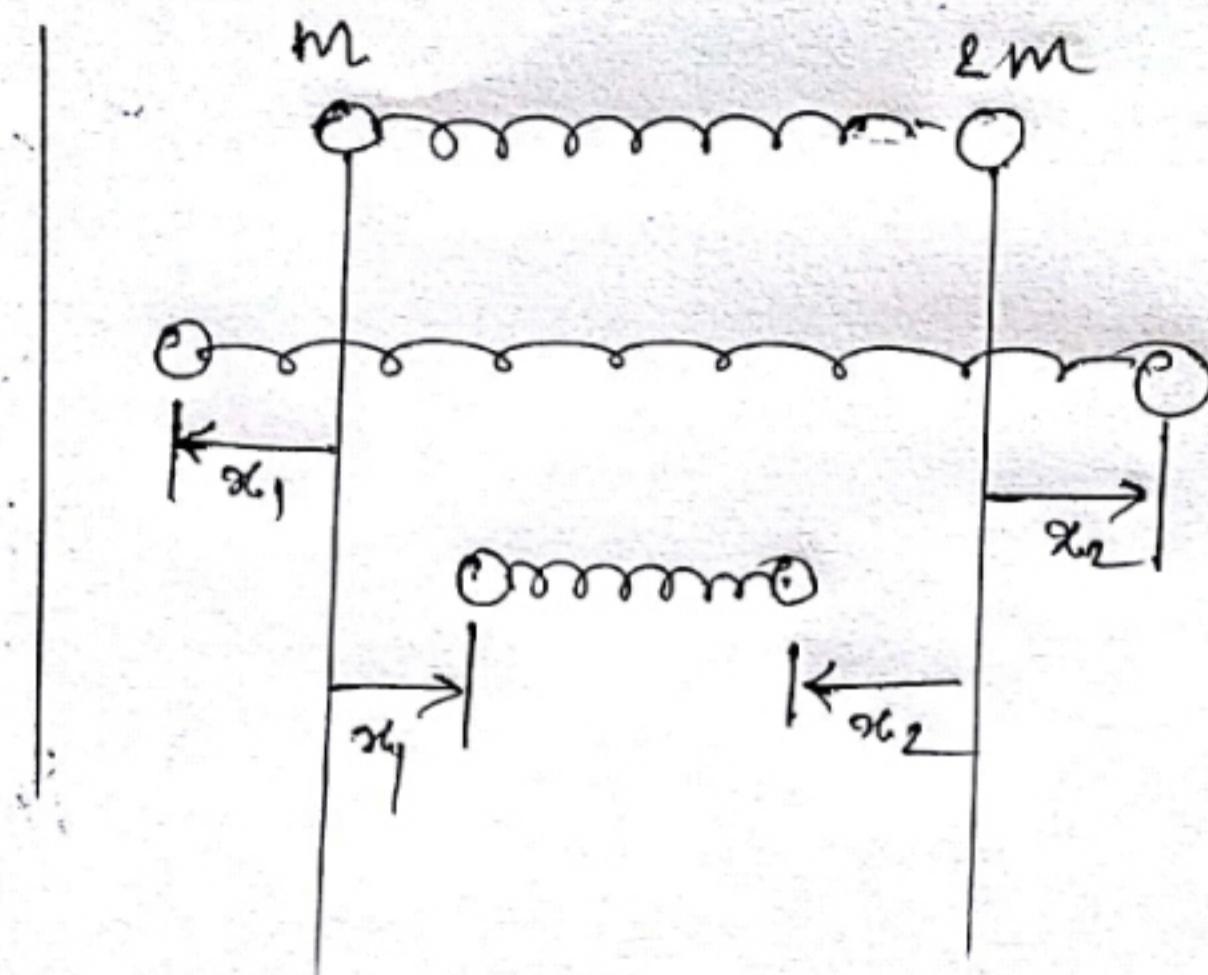
[for two masses]

Question: GATE 2011

Two bodies of masses m and $2m$ are connected by a spring constant, the frequency of normal mode is



Solution



$$T = \frac{1}{2} m v_1^2 + \frac{1}{2} (2m) v_2^2$$

$$= \frac{1}{2} m \left(\frac{dx_1}{dt} \right)^2 + \frac{1}{2} (2m) \left(\frac{dx_2}{dt} \right)^2$$

$$= \frac{1}{2} m (\dot{x}_1)^2 + \frac{1}{2} (2m) (\dot{x}_2)^2$$

$$V = \frac{1}{2} k (x_2 - x_1)^2 \quad (\text{Potential Energy}) \quad (P-9)$$

$x_2 - x_1$ = Increase in length of spring or decrease in length of spring.

Solving secular eq.

$$|\bar{V} - \omega^2 \bar{T}| = 0 \quad \text{to find '}\omega\text{' (normal freq)}$$

where $\bar{V} = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}$

$$V_{11} = \frac{\partial^2}{\partial x_1^2} V$$

$$V_{11} = \frac{\partial^2}{\partial x_1^2} \left\{ \frac{1}{2} k (x_2 - x_1)^2 \right\}$$

$$V_{11} = \frac{1}{2} k \frac{\partial^2}{\partial x_1^2} (x_2 - x_1)^2$$

$$V_{11} = \frac{1}{2} k \frac{\partial}{\partial x_1} \left[\frac{\partial}{\partial x_1} (x_2 - x_1)^2 \right]$$

$$= \frac{1}{2} k \frac{\partial}{\partial x_1} \left[2(x_2 - x_1) \cdot \frac{\partial}{\partial x_1} (x_2 - x_1) \right]$$

$$= \frac{1}{2} k \frac{\partial}{\partial x_1} \left[2(x_2 - x_1) \cdot \left(0 - \frac{\partial x_1}{\partial x_1} \right) \right]$$

$$= \frac{1}{2} k \frac{\partial}{\partial x_1} \left[2(x_2 - x_1)(-1) \right]$$

$$= \frac{1}{2} k (-2) \frac{\partial}{\partial x_1} (x_2 - x_1)$$

$$= \frac{1}{2} k (-2) (-1)$$

$$V_{11} = k$$

Similarly $V_{12} = \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} V$

$$= \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} \left\{ \frac{1}{2} k (x_2 - x_1)^2 \right\}$$

$$= -k \quad (\text{on solving})$$

$$V_{21} = \frac{\partial}{\partial x_2} \frac{\partial}{\partial x_1} (V)$$

$$= \frac{\partial}{\partial x_2} \frac{\partial}{\partial x_1} \left\{ \frac{1}{2} k (x_2 - x_1)^2 \right\}$$

$$= -k$$

$$V_{22} = \frac{\partial}{\partial x_2} \frac{\partial}{\partial x_2} \left\{ \frac{1}{2} k (x_2 - x_1)^2 \right\}$$

$$= k$$

$$\therefore \bar{V} = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$$

Now

$$\bar{T} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

$$T_{11} = \frac{\partial^2}{\partial \dot{x}_1^2} \left(\frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} (m) \dot{x}_2^2 \right)$$

$$= \frac{\partial^2}{\partial \dot{x}_1^2} \left(\frac{1}{2} m \dot{x}_1^2 \right) + \circlearrowleft$$

$$= \frac{1}{2} m \frac{\partial}{\partial \dot{x}_1} \left\{ \frac{\partial}{\partial \dot{x}_1} (\dot{x}_1^2) \right\}$$

$$= \frac{1}{2} m \frac{\partial}{\partial \dot{x}_1} 2 \dot{x}_1 = m \frac{\partial}{\partial \dot{x}_1} \dot{x}_1 = m,$$

$$T_{12} = \frac{\partial}{\partial \dot{x}_1} \frac{\partial}{\partial \dot{x}_2} \left(\frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} (2m) \dot{x}_2^2 \right) = 0$$

$$T_{21} = \frac{\partial}{\partial \dot{x}_2} \frac{\partial}{\partial \dot{x}_1} \left(\frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} (2m) \dot{x}_2^2 \right) = 0$$

$$T_{22} = \frac{\partial}{\partial \dot{x}_2} \frac{\partial}{\partial \dot{x}_2} \left(\frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} (2m) \dot{x}_2^2 \right) = 2m$$

$$\therefore \bar{T} = \begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix}$$

Putting the values in $|\bar{v} - w^2 \bar{T}| = 0$.

$$\left| \begin{pmatrix} k & -k \\ -k & k \end{pmatrix} - \omega^2 \begin{pmatrix} m & 0 \\ 0 & 2m \end{pmatrix} \right| = 0$$

$$\left| \begin{pmatrix} k & -k \\ -k & k \end{pmatrix} - \begin{pmatrix} \omega^2 m & 0 \\ 0 & 2\omega^2 m \end{pmatrix} \right| = 0$$

$$\left| \begin{pmatrix} k-\omega^2 m & -k \\ -k & k-2\omega^2 m \end{pmatrix} \right| = 0.$$

$$(k-\omega^2 m)(k-2\omega^2 m) - (-k)(-k) = 0$$

$$\Rightarrow k(k-2\omega^2 m) - \omega^2 m(k-2\omega^2 m) - k^2 = 0$$

$$\omega^2 - 2\omega^2 mk - \omega^2 mk + 2\omega^4 m^2 - k^2 = 0$$

$$\omega^2 - 3\omega^2 mk + 2(\omega^2 m)^2 - k^2 = 0$$

$$- 3\omega^2 mk + 2\omega^4 m^2 = 0$$

$$\Rightarrow 2\omega^4 m^2 = + 3\omega^2 mk$$

$$\omega^4 = \frac{3}{2} \frac{\omega^2 m k}{m^2}$$

$$\omega^2 = \frac{3}{2} \frac{k}{m}$$

$$\omega = \pm \sqrt{\frac{3}{2} \frac{k}{m}}$$

$$\therefore \omega = + \sqrt{\frac{3}{2} \frac{k}{m}}$$