Question Based On Canonical Transformation

Key Concepts:

The transformation of old coordinates (p, q) into new coordinates $P = P(p, q)$

and $Q = Q(p, q)$

are canonical only if

$$\boxed{[Q, P]_{q,p} = \left(\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} \right) = 1}$$

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Let (p, q) and (P, Q) be two pair of canonical variables. The Transformation

$Q = q^\alpha \cos \beta p$, $P = q^\alpha \sin \beta p$ is canonical for

what values of α and β :

Solution

Transformations are

$$Q = q^{\alpha}\cos(\beta p) \quad , \quad P = q^{\alpha}\sin(\beta p)$$

Condition for transformation to be canonical is

$$[Q, P]_{q,p} = \left(\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}\right) = 1 \quad —(1)$$

Now, $\dfrac{\partial Q}{\partial q} = \dfrac{\partial}{\partial q} q^{\alpha}\cos(\beta p)$

$$= \cos(\beta p)\cdot\frac{\partial}{\partial q} q^{\alpha}$$

$$= \cos(\beta p)\cdot\left(\alpha q^{\alpha-1}\right)$$

$$\frac{\partial Q}{\partial q} = \alpha q^{\alpha-1} \cos \beta p$$

and $\dfrac{\partial P}{\partial p} = \dfrac{\partial}{\partial p} q^{\alpha} \sin \beta p$

$$= q^{\alpha} \frac{\partial}{\partial p} \sin \beta p$$

$$= q^{\alpha} \frac{\partial \sin x}{\partial p} \qquad ; \; x = \beta p$$

$$= q^{\alpha} \frac{\partial \sin x}{\partial x} \frac{\partial x}{\partial p}$$

$$= q^{\alpha} (\cos x) \frac{\partial}{\partial p}(\beta p)$$

$$= q^{\alpha} . \cos \beta p . \; \beta \frac{\partial p}{\partial p}$$

$$= \beta q^{\alpha} \cos \beta p$$

$$\frac{\partial Q}{\partial p} = \frac{\partial}{\partial p}\left(q^{\alpha} \cos \beta p\right)$$

$$= q^{\alpha} \frac{\partial}{\partial p} \cos \beta p$$

$$= q^{\alpha} \frac{\partial}{\partial p} \cos y \qquad \because y = \beta p$$

$$= q^{\alpha} \frac{\partial \cos y}{\partial y} \cdot \frac{\partial y}{\partial p}$$

$$= q^{\alpha} \left(-\sin y\right) \frac{\partial \beta p}{\partial p}$$

$$= q^{\alpha} \left(-\sin \beta p\right) \beta \frac{\partial p}{\partial p}$$

$$= -q^{\alpha} \sin \beta p \cdot \beta$$

$$= -\beta q^{\alpha} \sin \beta p$$

$$\frac{\partial p}{\partial q} = \frac{\partial}{\partial q} q^{\alpha} \sin(\beta p)$$

$$= \sin(\beta p) \cdot \frac{\partial}{\partial q} q^{\alpha}$$

$$= \sin(\beta p) \cdot \alpha q^{\alpha-1}$$

$$= \alpha q^{\alpha-1} \sin(\beta p)$$

Putting the values back in eq (i)

$$\alpha q^{\alpha-1} \cos\beta p \cdot \beta q^{\alpha} \cos\beta p - (-\beta q^{\alpha} \sin\beta p)(\alpha q^{\alpha-1} \sin\beta p) = 1$$

$$\alpha\beta q^{\alpha-1+\alpha} \cos^2\beta p + \alpha\beta q^{\alpha+\alpha-1} \sin^2\beta p = 1$$

$$\alpha\beta q^{2\alpha-1} \cos^2\beta p + \alpha\beta q^{2\alpha-1} \sin^2\beta p = 1$$

$\alpha\beta q^{2\alpha-1}[\cos^2\beta p + \sin^2\beta p] = 1$

$\alpha\beta q^{2\alpha-1}(1) = 1$

$\alpha\beta q^{2\alpha-1} = 1$ ——— (2)

This eq. can be satified if

$2\alpha - 1 = 0$ so that $q^0 = 1$

$\Rightarrow \alpha = \frac{1}{2}$

and if $\alpha = \frac{1}{2}$ then $\beta = 2$ then only eq (2) will be satified.

Lets check

$\left(\frac{1}{2}\right)(2) q^{2\cdot\frac{1}{2}-1} = (1) q^{1-1} = q^0 = 1$