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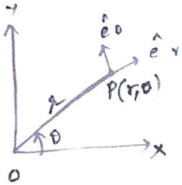
Central forces :

$\vec{F} = f_r \hat{e}_r$ $n=2$ Inverse square force

$n=-1$ SHM

$$\left[\vec{F} = \frac{K}{r^n} \hat{e}_r \right]$$

Polar Coord's. $\vec{r} = r\hat{e}_r$



$\vec{v} = v_r \hat{e}_r + v_\theta \hat{e}_\theta$

$= \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$

$\vec{a} = a_r \hat{e}_r + \underline{a_\theta \hat{e}_\theta}$

$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$$

$$a_\theta = 2\left(\frac{dr}{dt}\right)\left(\frac{d\theta}{dt}\right) + r\frac{d^2\theta}{dt^2}$$

as forces are radial in nature

$\therefore a_\theta = 0$

$$2\left(\frac{dr}{dt}\right)\left(\frac{d\theta}{dt}\right) + r\frac{d^2\theta}{dt^2} = 0$$

$\times$ by r

$$2r\left(\frac{dr}{dt}\right)\left(\frac{d\theta}{dt}\right) + r^2\frac{d^2\theta}{dt^2} = 0$$

$$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0$$

$$\boxed{r^2\frac{d\theta}{dt} = \text{constant}} \longrightarrow (1)$$

$\vec{L} = \vec{r} \times \vec{p}$

$= r\hat{e}_r \times m\vec{v}$

$= r\hat{e}_r \times m\left[\frac{dr}{dt}\hat{e}_r + r\frac{d\theta}{dt}\hat{e}_\theta\right]$

$$= mr^2\frac{d\theta}{dt}\underbrace{(\hat{e}_r \times \hat{e}_\theta)}_{\hat{n}}$$

$\hat{n} \rightarrow$ unit vector normal to (r,θ)

$$\vec{L} = mr^2\frac{d\theta}{dt}\hat{n}$$

$$\left|\frac{L}{m}\right| = r^2\frac{d\theta}{dt}$$

$\vec{L} = \vec{r} \times \vec{p}$ Alternate Method

$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

$= \vec{v} \times m\vec{v} + \vec{r} \times \vec{F}_{ext}$

$= 0 + \vec{r} \times \vec{F}(r)$

$= 0$

Hence

$\vec{L}$ = conserved

Areal Velocity :-

$\vec{r} \to \vec{r} + d\vec{r}$ in $\underline{dt}$ time

$d\vec{s} = \frac{1}{2} \vec{r} \times d\vec{r} = \frac{1}{2} r (r d\theta) \hat{n}$

Areal velocity $\frac{d\vec{s}}{dt} = \frac{1}{2} \underbrace{r^2 \frac{d\theta}{dt}} \hat{n}$

$\frac{d\vec{s}}{dt} = \frac{\vec{L}}{2m} =$ constant

Equation of Orbit :- Inverse Square force

$F_r = -\frac{c}{r^2}$ (say) $c > 0$ $= -cu^2$

$F_r = m\left(\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)$ — (2)

(1) $\vec{L}$ is conserved

$\therefore \; mr^2 \frac{d\theta}{dt} =$ constant $= L$

$\Rightarrow \frac{d\theta}{dt} = \frac{L}{mr^2} = \frac{K}{r^2}$

(2) let $u = \frac{1}{r}$

$\frac{d\theta}{dt} = Ku^2$

$L = T + V$

$= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - V(r)$

$m\ddot{r} - mr\dot{\theta}^2 + \frac{dV}{dr} = 0$ → Lagrange's EOM for $\vec{r}$, generalised

$\vec{F}_r = -\frac{dV}{dr} = m\ddot{r} - mr\dot{\theta}^2$

$K = \frac{L}{m}$

$\frac{dr}{dt} = \frac{d}{dt}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2} \cdot \frac{du}{d\theta} \cdot \frac{d\theta}{dt}$

$\frac{dr}{dt} = -\frac{1}{u^2} \cdot Ku^2 \cdot \frac{du}{d\theta} = -K\frac{du}{d\theta}$

$\frac{d^2 r}{dt^2} = \frac{d}{dt}\left(-K\frac{du}{d\theta}\right) = -K\frac{d}{dt}\left(\frac{du}{d\theta}\right) = -K^2 u^2 \frac{d^2 u}{d\theta^2}$

$= -K \frac{d}{d\theta}\left(\frac{du}{d\theta}\right) \cdot \frac{d\theta}{dt}$

Put in eq^n (2)

$-cu^2 = m\left(-K^2 u^2 \frac{d^2 u}{d\theta^2} - \frac{1}{u}(Ku^2)^2\right)$

$\frac{c}{mK^2} = \left(\frac{d^2 u}{d\theta^2} + u\right)$

let $\frac{c}{mK^2} = A$ (say)

$$\frac{d^2u}{d\theta^2} + u = A$$

$$\frac{d^2u}{d\theta^2} + (u-A) = 0$$

Rewriting this eq^n,

$$\frac{d^2(u-A)}{d\theta^2} + (u-A) = 0$$

as $\frac{d^2A}{d\theta^2} = 0$

$$(u-A) = B\cos(\theta-\theta_0)$$

$$u = A + B\cos(\theta-\theta_0)$$

$$\frac{1}{r} = A\left[1 + \frac{B}{A}\cos(\theta-\theta_0)\right]$$

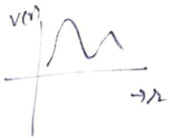
Put $p = \frac{1}{A}$ and $e = \frac{B}{A}$

$$\frac{p}{r} = 1 + e\cos(\theta-\theta_0)$$

Std. eq^n of Conic section

$$p = \text{orbital parameter} = \frac{1}{A} = \frac{Mk^2}{C} = \frac{mL^2}{m^2C} = \frac{L^2}{mC}$$

$r(t)$, $\theta(t)$



gravitational force $C = GMm$ (say).

$$p = \frac{L^2}{GMm^2} \qquad e = \frac{B}{A} = \frac{BL^2}{GMm^2}$$

$e > 1$	Hyperbola	unbounded Motions
$e = 1$	Parabola	
$e = 0$	Circle	Bounded Motions.
$e < 1$	Ellipse	

Perigee and Apogee

$$\frac{p}{r} = 1 + e\cos(\theta-\theta_0)$$

$\theta = 0.$

$$\frac{p}{r_{min}} = 1 + e$$

$\theta = \pi$

$$\frac{p}{r_{max}} = 1 - e$$

$$\frac{r_{max}}{r_{min}} = \frac{1+e}{1-e}$$

$$e = \frac{r_{max} - r_{min}}{r_{max} + r_{min}}$$

$$p = \frac{2\, r_{max} r_{min}}{r_{max} + r_{min}}$$