Hamiltonian formulation Prof. R. Sherma 5th October 2020

$L(q_j, \dot{q}_j, t)$ Configuration space

$n q_j$'s & $n \dot{q}_j$

Basis set $\rightarrow \overset{3N}{q_j}, \overset{3N}{p_j}, t$

$H(q_j, p_j, t)$ $\quad p_j = \frac{\partial L}{\partial \dot{q}_j}$

6N-D vector space $\rightarrow$ Phase space

$$H = \sum_j \dot{q}_j p_j - L(q_j, \dot{q}_j) \rightarrow (A)$$

Hamilton's Canonical Equations :-

$$H = H(q_j, p_j, t)$$

$$dH = \frac{\partial H}{\partial q_j} dq_j + \frac{\partial H}{\partial p_j} dp_j + \frac{\partial H}{\partial t} dt \rightarrow (B)$$

$$dH = \sum_j \dot{q}_j dp_j + \sum_j p_j d\dot{q}_j - dL.$$

Where

$$dL = \frac{\partial L}{\partial q_j} dq_j + \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j + \frac{\partial L}{\partial t} dt$$

$$dH = \sum_j \dot{q}_j dp_j + \sum_j \cancel{p_j d\dot{q}_j} - \frac{\partial L}{\partial q_j} dq_j - \cancel{p_j d\dot{q}_j} - \frac{\partial L}{\partial t} dt$$

where $\frac{\partial L}{\partial \dot{q}_j} = p_j$

$$dH = \sum_j \dot{q}_j dp_j - \sum_j \frac{\partial L}{\partial q_j} dq_j - \sum_j \frac{\partial L}{\partial t} dt$$

$$\frac{\partial L}{\partial q_j} = \dot{p}_j$$

$$dH = \sum_j \dot{q}_j dp_j - \sum_j \dot{p}_j dq_j - \sum_j \frac{\partial L}{\partial t} dt \text{ --- } (C)$$

Compare eqn (B) and eqn (C)

$$\frac{\partial H}{\partial q_j} = -\dot{p}_j \quad \Big| \quad \frac{\partial H}{\partial p_j} = \dot{q}_j$$

$$\frac{\partial H}{\partial t} = \frac{\partial L}{\partial t}.$$ Hamilton's Canonical EOM.

Physical Significance of Hamiltonian 'H'

Restrictions imposed for $H = E$

- Conservative i.e. $V(q_j)$ and not on velocity
- $\sum_j p_j \dot{q}_j = 2T$

ex-1 Linear Harmonic oscillator

$$L = K - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

$$H = \sum_j \dot{q}_j p_j - L \qquad p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$H = \dot{x}p_x - L = \dot{x}p_x - \left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2\right)$$

$$= m\dot{x}^2 - \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

$$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = K + V$$

$$\dot{x} = \frac{\partial H}{\partial p_x} \; ; \quad \dot{p}_x = -\frac{\partial H}{\partial x}$$

$$\dot{x} = \frac{p_x}{m} \; ; \quad \dot{p}_x = -kx$$

$$\cancel{m\ddot{x} = \dot{p}} \qquad m\ddot{x} = -kx$$

or $\boxed{m\ddot{x} + kx = 0}$

Simple Pendulum

$$L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta)$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

$$H = \sum \dot{\theta}p_\theta - L = \dot{\theta}(ml^2\dot{\theta}) - \frac{1}{2}ml^2\dot{\theta}^2 + mgl(1-\cos\theta)$$

$$H = \frac{1}{2}ml^2\dot{\theta}^2 + mgl(1-\cos\theta) = T + V$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = ml^2\ddot{\theta} \qquad -\dot{p}_\theta = \frac{\partial H}{\partial \theta} = mgl\sin\theta.$$

$$\boxed{\ddot{\theta} + \frac{g}{l}\theta = 0}$$

Motion of Particle in a central force field.

$$f = \frac{-k}{r^2} , \quad V(r) = -\frac{k}{r}$$

$$L = T - V(r) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{k}{r}$$

HW

Derive the Lagrange's eqn of motion using Euler - Lagrange's dynamics in differential calculus

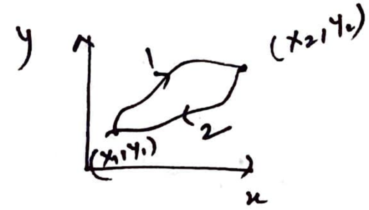
$y = y(x)$ in 1-D betn x_1 and x_2, such that the line integral of some fn $f(y, y', x)$ is an extremum

$$\downarrow$$

$$y(\dot{x})$$

$$J = \int_{x_1}^{x_2} f(y, y', x)\, dx = \text{extremum}$$

Hamilton's Variational Principle



$$I = \int_{t_1}^{t_2} L\, dt \quad \text{is an extremum}$$

$$\delta I = \int_{t_1}^{t_2} \delta L\, dt = 0$$

$$L = L(q_j, \dot{q}_j, t)$$

$$\Delta V = \underbrace{(\Delta x\, \Delta y\, \Delta z)}_{\text{spatial}} \; \underbrace{\Delta p_x\, \Delta p_y\, \Delta p_z}$$

$$\Delta V = \underbrace{\Delta x \Delta p_x}_{\sim h} \; \underbrace{\Delta y \Delta p_y}_{\sim h} \; \underbrace{\Delta z \Delta p_z}_{\sim h}$$

$$\Delta V = h^3 = \text{cell}$$