Chapter 11

Virtual Work

Learning Objectives

After completing this chapter, you will be able to

LO 11.1 Relate and apply the concept of virtual work on various structural systems.
LO 11.2 Underline the principle of virtual work and apply it to analyse systems consisting of interconnected rigid members.
LO 11.3 Define elastic potential energy and apply the virtual-work principle to elastic bodies.
LO 11.4 Examine the stability of a system.

LO 11.1

Relate and apply the concept of virtual work on various structural systems.

11.1 ❑ INTRODUCTION

While dealing with equilibrium of rigid bodies, the strategy so far has been to first divide the system into suitable subsystems and then solve them by using equations of equilibrium. This process becomes tedious as the system size grows, thereby increasing the number of subsystems and forces. Moreover, the analysis of the structures consisting of interconnected rigid bodies which can move relative to each other, is very tedious if we use conventional equations of equilibrium.

Can there be a method to directly consider the equilibrium of a whole system without breaking it into many subsystems? The concept of ***virtual work*** affords us just to do that. It is an alternative approach to analyse the bodies in static equilibrium, involving only some, not all, unknown forces acting on the system.

11.1.1 Work

The ***work*** done is defined as the *effort* made by some forces (or couples) to change the position of a body in terms of its displacement (or rotation).

Work Done by a Force Let us consider a block being pushed by a force F acting at an angle α with the direction of displacement of the body along the surface, as shown in Fig. 11.1*a*. The change in the position of the block from A to A′ is represented by the vector **s**, known as *displacement*. By definition, *the work done is the product of the displacement and rectangular component of force in the direction of displacement*. Hence, the work done,

$$U = (F \cos \alpha) s \tag{11.1}$$

The same equation is obtained if the magnitude of force F is multiplied by the rectangular component of the displacement along the line of action of force, as shown in Fig 11.1*b*. Thus,

$$U = F(s \cos \alpha) \tag{11.2}$$

Note that Eqs. (11.1) and (11.2) are effectively the same. It shows that work is a *scalar quantity* as its value is same whether the force is resolved along the displacement or vice versa.

Work done by a force is positive if its working component is in the same direction as the displacement (Fig. 11.1). However, it becomes ***negative*** if the working component of the force is in a direction opposite to the direction of displacement (Fig. 11.2). Thus,

Worthy of Note

You may wonder how the direction of displacement can be opposite to the direction of the working component of the applied force. Yes, it is possible because other forces may also be working on the body.

$$U = -(F \cos \theta)s \tag{11.3}$$

Aletrnatively, we can say that the work done by a force F in displacing a body by a displacement s is given as $U = Fs \cos \alpha$, where α is the angle between the force vector and displacement vector. If α is less than 90°, work done is *positive*; and if α is greater than 90°, work done is *negative*. When $\alpha = 90°$, work done is zero.

(*a*) Component of force along displacement (*b*) Component of displacement along force

Fig. 11.1 *Two ways of calculating work done by a force.*

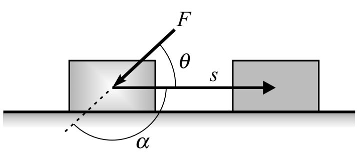


Fig. 11.2 *Force opposite to the displacement.*

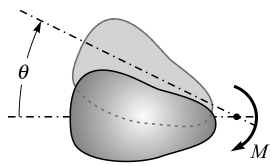


Fig. 11.3 *Body rotated by M through angle θ.*

Work Done by a Couple Work is also done by couples or moments of forces causing angular displacement (i.e., rotation). Figure 11.3 shows a body rotated by a moment M through an angle θ. The work done on the body is simply the product of M and θ. Thus,

$$U = M\theta \tag{11.4}$$

The work done by a couple is positive if M has the same sense as that of rotation, and negative if M has an opposite sense. The SI unit of work is the joule (J).

➢ **Caution**

Take angle θ in radians, not in degrees.

11.1.2 Virtual Work and Equilibrium

Before introducing the concept of virtual work, let us first define ***virtual displacement***. Consider a particle in static equilibrium under the action of a system of forces. If the particle is displaced by an assumed and arbitrary *small displacement* $\delta\mathbf{d}$ away from its natural position conforming to the system constraints, the particle is said to be *virtually displaced*. Such displacement, $\delta\mathbf{d}$ is called the *virtual displacement*. The work done by a force **F** acting on the particle through the virtual displacement is known as the ***virtual work*** and can be written as

$$\delta U = \mathbf{F}\cdot\delta\mathbf{d} = F\delta d\cos\alpha \tag{11.5}$$

Here, F is the magnitude of the force **F**, δd is the magnitude of the displacement $\delta\mathbf{d}$ and α is the angle between **F** and $\delta\mathbf{d}$. Similarly, the virtual work done by a couple M through a small virtual rotation $\delta\theta$ is given as

$$\delta U = M\,\delta\theta \tag{11.6}$$

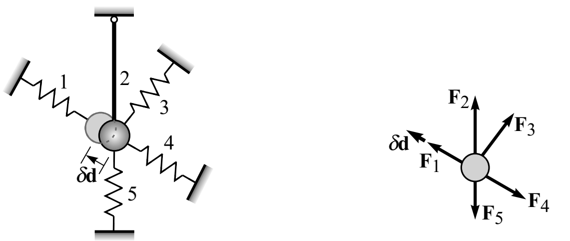
Equilibrium A body is said to be in equilibrium when the sum of all the forces and moments acting on the body is zero.

Equilibrium of a Particle Consider a particle of negligible mass in equilibrium under a system of forces $\mathbf{F_1}$, $\mathbf{F_2}$, $\mathbf{F_3}$, …, $\mathbf{F_n}$ as shown in Fig. 11.4. Let us give a small virtual displacement $\delta\mathbf{d}$ to the particle. The total virtual work done by all the forces is given as

$$\delta U = \Sigma\mathbf{F}\cdot\delta\mathbf{d} = \mathbf{F_1}\cdot\delta\mathbf{d} + \mathbf{F_2}\cdot\delta\mathbf{d} + \cdots + \mathbf{F_n}\cdot\delta\mathbf{d} \tag{11.7}$$

This equation can be written in terms of the scalar components of the virtual displacement and forces as

$$\begin{aligned}\delta U &= \Sigma\mathbf{F}\cdot\delta\mathbf{d} = (\mathbf{i}\Sigma F_x + \mathbf{j}\Sigma F_y + \mathbf{k}\Sigma F_z)\cdot(\mathbf{i}\,\delta x + \mathbf{j}\,\delta y + \mathbf{k}\,\delta z)\\ &= \Sigma F_x\,\delta x + \Sigma F_y\,\delta y + \Sigma F_z\,\delta z\end{aligned} \tag{11.8}$$



(*a*) Particle displaced (*b*) Free body diagram

Fig. 11.4 *Particle in equilibrium.*

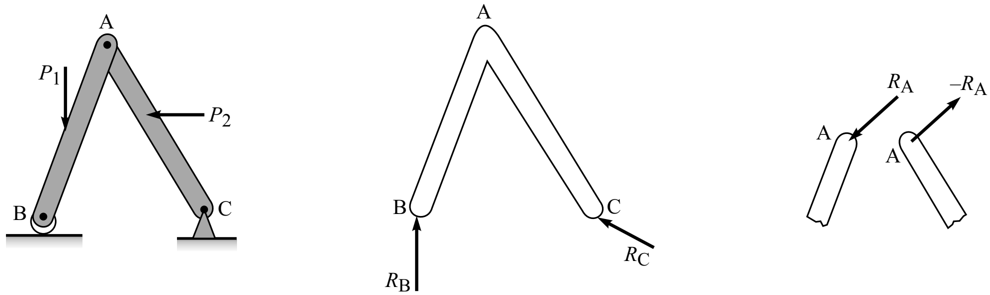
For a particle to be in equilibrium, we must have $\Sigma\mathbf{F} = 0$. This implies that the sum of the scalar components of all the forces along the three axes are all individually zero. That is, $\Sigma F_x = \Sigma F_y = \Sigma F_z = 0$. Therefore, Eq. (11.8) reduces to

$$\delta U = 0 \tag{11.9}$$

Equation (11.9) implies that if the virtual work done by the forces through a virtual displacement is zero, the body has to be in equilibrium. This is known as the ***principle of virtual work***. This principle, however, is not of much help in simplifying the problems related to equilibrium of a particle, since for a particle, both $\delta U = 0$ and $\Sigma\mathbf{F} = 0$ provide the same information.

Equilibrium of a Rigid Body A rigid body can be considered as a system of particles rigidly attached to one another. Since the virtual work done on a particle in equilibrium is zero, it follows that the total virtual work done on a rigid body is also zero.

Equilibrium of Interconnected Rigid Bodies Let us consider a system composed of two (or more) rigid members linked together and supported by frictionless hinges, as shown in Fig. 11.5*a*, such that the relative motion between the two members CA and AB is possible.



(*a*) Active forces (*b*) Reactive forces (*c*) Internal forces

Fig. 11.5 *Interconnected rigid bodies in equilibrium*

This system is in equilibrium under external forces P_1 and P_2, which are called ***active forces*** because they are responsible for virtual work done through any possible virtual displacements. The total virtual work done by the active forces must be zero for the system to be in equilibrium.

Myth

Virtual-work principle is applicable to all bodies whether at rest or in motion.

Fact!

No. It is valid only for the bodies which are in static equilibrium.

As far as the ***reactive forces***, R_B and R_C (Fig. 11.5*b)*, offered by the supports, are concerned, these do not contribute to work done unless the supports themselves are given virtual displacements. Another category of forces is ***internal forces***, which develop always in pairs. They exist as a pair of equal and opposite components, for example in Fig. 11.5*c*, the forces R_A and $-R_A$ at the connection A. Therefore, the net virtual work done by the internal forces cancel each other, which verifies the state of equilibrium.

LO 11.2

Underline the principle of virtual work and apply it to analyse systems consisting of interconnected rigid members.

11.2 ❑ PRINCIPLE OF VIRTUAL WORK

It states that *for a system of rigid bodies connected through frictionless joints, in static equilibrium, the algebraic sum of virtual work done by all the active forces through virtual displacement consistent with geometrical conditions is zero.*

Myth

Virtual-work method can be applied to any kind of problem.

Fact!

No. This method is not valid for a system in which internal forces perform significant amount of virtual work, which cannot be calculated. For example, friction at hinges does negative work, but it is impossible to measure that amount of work done.

Advantages of Using Virtual Work Principle

1. There is no need to dismember the system in order to write the conventional equations of equilibrium.
2. Relationship between active forces may be determined without reference to the reactive forces.
3. The equilibrium of the system is expressed in terms of fewer unknown forces acting on the system, thereby making the analysis easier.

Analysis Procedure

1. Draw a free-body diagram of the entire system showing only the active forces. Such a diagram is known as *active-force diagram*.
2. Give virtual displacement(s) (rotation or deflection) to the system at carefully chosen point(s). The displacements should move the points of unknown forces and these should be consistent with the geometrical conditions of the system.
3. Determine the total virtual work done by all the forces through corresponding virtual displacements.
4. Take into account the virtual work done by the reactions only if the supports themselves are displaced virtually.
5. Equate the total virtual work done to zero and determine the unknown components of moments or forces.

Myth

Work is done only by forces.

Fact!

No. Moments and couples also do work while causing rotation of the body.

Sign Convention The virtual work done by a force/reaction is *positive* if the direction of the force/reaction is same as that of the virtual displacement. If the directions of force/reaction are opposite to the virtual displacement, the virtual work done is *negative*. Similarly, the work done is positive if the direction of moment is same as that of virtual rotation.

Example 11.1 Determine the reactions at the supports A and B of a beam of negligible mass carrying two point loads at the points C and D as shown in Fig. 11.6*a*.

[LO 11.2]

Solution Keeping the support B stationary, we lift the support A upwards by a small virtual displacement δa, as shown in Fig. 11.6*b*. Let the corresponding displacements at points C and D be δc and δd, respectively. The displacement at the point B is zero. Equating the total virtual work done by the forces acting on the beam to zero, we get

$$R_A \times \delta a - 12 \times \delta c - 10 \times \delta d + R_B \times 0 = 0 \qquad (i)$$

Here, the work done by displacements δc and δd are written with negative sign because these displacements are opposite to the direction of corresponding forces.

From similar triangles AA′B, CC′B and DD′B,

$$\frac{\delta a}{10} = \frac{\delta c}{7} = \frac{\delta d}{2} \quad \Rightarrow \quad \delta a = 5\delta d \quad \text{and} \quad \delta c = 3.5\delta d$$

Substituting the values of δa and δc into Eq. (*i*), we get

$$R_A \times 5\delta d - 12 \times 3.5\delta d - 10 \times \delta d + R_B \times 0 = 0$$

or $$R_A \times 5 - 42 - 10 = 0 \quad \Rightarrow \quad R_A = \mathbf{10.4\ kN}$$

Now, to determine R_B, let us lift the support B upwards by virtual displacement $\delta b'$, keeping the support A stationary. The virtual displacements at C and D are $\delta c'$ and $\delta d'$, respectively (Fig. 11.6*c*). Again, equating the total virtual work done by the forces to zero, we get

$$R_B \times \delta b' + 12 \times (-\delta c') + 10 \times (-\delta d') + R_A \times 0 = 0 \qquad (ii)$$

From similar triangles, ABB″, ADD″ and ACC″,

$$\frac{\delta b'}{10} = \frac{\delta d'}{8} = \frac{\delta c'}{3} \quad \Rightarrow \quad \delta b' = 1.25\delta d' \quad \text{and} \quad \delta c' = 0.375\delta d'$$

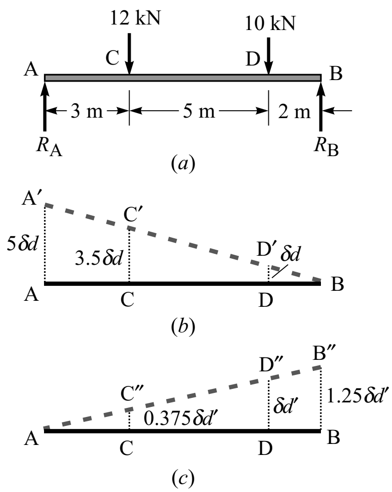


Fig. 11.6

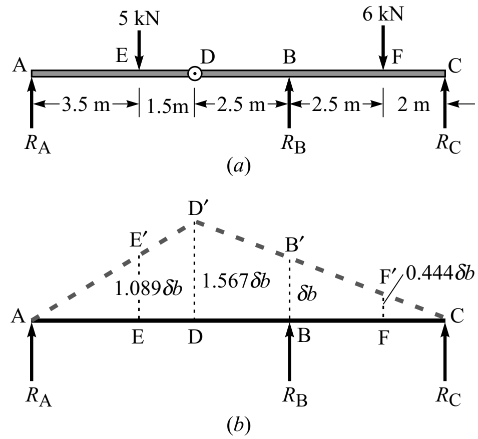


Fig. 11.7

Replacing $\delta b'$ and $\delta c'$ in terms of $\delta d'$ in Eq. (*ii*), we get

$$R_B \times 1.25\delta d' + 12 \times (-0.375\delta d') + 10 \times (-\delta d') + R_A \times 0 = 0$$

or $$R_B \times 1.25 - 4.5 - 10 = 0 \quad \Rightarrow \quad R_B = \mathbf{11.6\ kN}$$

Check: Writing the equation of equilibrium,

$$\Sigma F_y = 0: \quad R_A - 12 - 10 + R_B = 0 \quad \Rightarrow \quad R_A + R_B = 10 + 12 = 22 \text{ kN}$$

Calculated values: $R_A + R_B = 10.4 + 11.6 = 22$ kN (the two values match)

Example 11.2 Two simply supported light beams AD and DC are hinged internally at D. Find the reaction at the support B using the principle of virtual work, if the beams are loaded as shown in Fig. 11.7*a*.

[LO 11.2]

Solution Keeping the supports A and C fixed, lift the support B upwards through a virtual displacement δb so that virtual displacements at the hinge D, points E and F are δd, δe and δf, respectively (Fig. 11.7*b*). From similar triangles DD′C, BB′C and FF′C, we get

$$\frac{\delta d}{7} = \frac{\delta b}{4.5} = \frac{\delta f}{2} \quad \Rightarrow \quad \delta d = \frac{7}{4.5}\delta b \quad \text{and} \quad \delta f = \frac{2}{4.5}\delta b = 0.444\delta b \qquad (i)$$

From similar triangles AEE′ and ADD′,

$$\frac{\delta e}{3.5} = \frac{\delta d}{5} \quad \Rightarrow \quad \delta e = \frac{3.5}{5}\delta d = \frac{3.5}{5} \times \frac{7}{4.5}\delta b = 1.089\delta b \qquad (ii)$$

Using the principle of virtual work, we have

$$R_A \times 0 - 5 \times \delta e + R_B \times \delta b - 6 \times \delta f + R_D \times 0 = 0$$

or $$-5 \times (1.089\delta b) + R_B \times \delta b - 6 \times (0.444\delta b) = 0 \quad \Rightarrow \quad R_B = \mathbf{8.12\ kN}$$

Example 11.3 Determine the reactions R_A and R_B developed in the simply supported overhanging beam shown in Fig. 11.8*a*. Neglect the self-weight of the beam.

[LO 11.2]

Solution Let a virtual displacement δe be given to the free end E of the beam keeping the support A stationary (Fig. 11.8*b*).

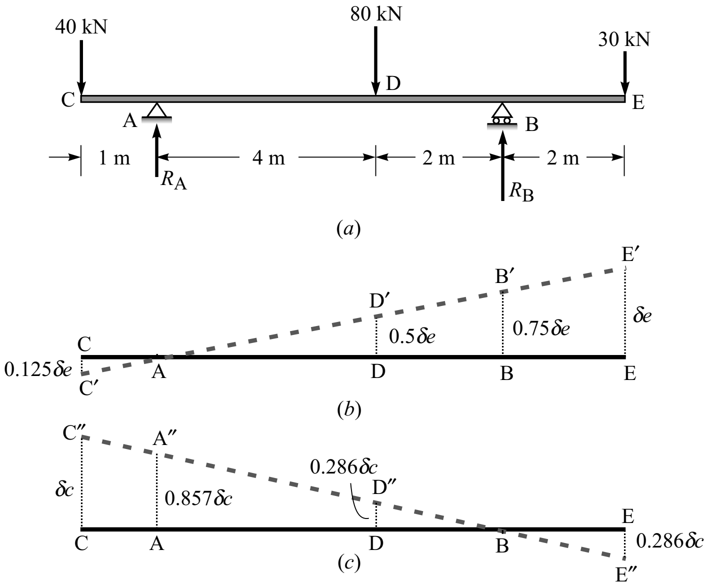


Fig. 11.8

Comparing similar triangles BB′A, DD′A, CC′A and EE′A, the virtual displacements of other points are determined as follows.

$$\frac{\delta b}{6} = \frac{\delta d}{4} = \frac{\delta c}{1} = \frac{\delta e}{8} \quad \Rightarrow \quad \delta b = 0.75\delta e \quad \delta d = 0.5\delta e \quad \delta c = 0.125\delta e$$

Applying the principle of virtual work,

$$40 \times 0.125\delta e + 80 \times (-0.5\delta e) + R_B \times 0.75\delta e + 30 \times (-\delta e) = 0$$

$$\Rightarrow \quad 5 - 40 + R_B \times 0.75 - 30 = 0 \quad \Rightarrow \quad R_B = \mathbf{86.67\ kN}$$

Now, let us give a virtual displacement δc to the free end C (Fig. 11.8*c*). Comparing similar triangles AA″B, DD″B, EE″B and CC″B, the virtual displacements of other points are determined as follows.

$$\frac{\delta a}{6} = \frac{\delta d}{2} = \frac{\delta e}{2} = \frac{\delta c}{7} \quad \Rightarrow \quad \delta a = 0.857\delta c \quad \delta d = 0.286\delta c \quad \delta e = 0.286\delta c$$

Applying the principle of virtual work,

$$40 \times (-\delta c) + R_A \times 0.857\delta c + 80 \times (-0.286\delta c) + 30 \times 0.286\delta c = 0$$

$$\Rightarrow \quad -40 + R_A \times 0.857 - 22.88 + 8.58 = 0 \quad \Rightarrow \quad R_A = \mathbf{63.33\ kN}$$

***Check*:** Applying equation of equilibrium,

$$\Sigma F_y = 0; -40 + R_A - 80 + R_B - 30 = 0 \quad \Rightarrow \quad R_A + R_B = 40 + 80 + 30 = 150 \text{ kN}$$

Calculated values: $R_A + R_B = 86.67 + 63.33 = 150$ kN (the two values match)

Example 11.4 A uniform ladder of 4-m length and 350-N weight rests against rough floor and wall, as shown in Fig. 11.9, making an angle of 50° with the horizontal. If motion of the ladder at its lower end is impending, calculate the forces of friction between the ladder and the surfaces using the principle of virtual work. Take coefficient of friction between ladder and both the surfaces as 0.3.

[LO 11.2]

Solution From Fig 11.9, it may be written that,

$$h = \text{AG} \sin\theta = (l/2)\sin\theta, \quad a = \text{AC} = \text{AB}\cos\theta = l\cos\theta, \quad b = \text{BC} = \text{AB}\sin\theta = l\sin\theta \qquad (i)$$

where l is the length of the ladder. Let the end A be given a leftward virtual displacement δa along the floor. Simultaneously, the centre of gravity G and the top end B of the ladder move downward by displacements δh and δb, respectively. Differentiating the expressions in Eq. (i), we get

$$\delta h = (l/2)\cos\theta\cdot\delta\theta, \quad \delta a = -l\sin\theta\cdot\delta\theta \quad \text{and} \quad \delta b = l\cos\theta\cdot\delta\theta$$

➢ **Note**

The *positive* or *negative* sign of the differentials above, indicates whether the quantity increases or decreases with an increase in θ. Therefore, while writing the virtual work equation we consider only the magnitude of the change in a quantity and give *positive* or *negative* sign to the work done depending upon whether the displacement is along the force or opposite to it.

Equating the total virtual work done by all the forces acting on the ladder to zero, we get

$$-F_A\times|\delta a| + R_A\times 0 - F_B\times|\delta b| + R_B\times 0 + W\times|\delta h| = 0$$

$$\Rightarrow \quad -F_A\times(l\sin\theta\cdot\delta\theta) - F_B\times(l\cos\theta\cdot\delta\theta) + W\times\{(l/2)\cos\theta\cdot\delta\theta\} = 0$$

$$\Rightarrow \quad -F_A\times 4\sin 50° - F_B\times 4\cos 50° + 350\times(4/2)\cos 50° = 0$$

$$\Rightarrow \quad -3.06F_A - 2.57F_B = -450 \quad\Rightarrow\quad 3.06F_A + 2.57F_B = 450 \qquad (ii)$$

Applying $\Sigma F_y = 0$ on the ladder, we may write

$$R_A + F_B = 350 \quad\Rightarrow\quad \frac{F_A}{\mu} + F_B = 350 \quad\Rightarrow\quad F_A + 0.3F_B = 105 \qquad (iii)$$

($\because$ $F_A = \mu R_A$ for impending motion at end A)

Solving Eqs. (ii) and (iii), we get F_A = **81.63 N** and F_B = **77.91 N**

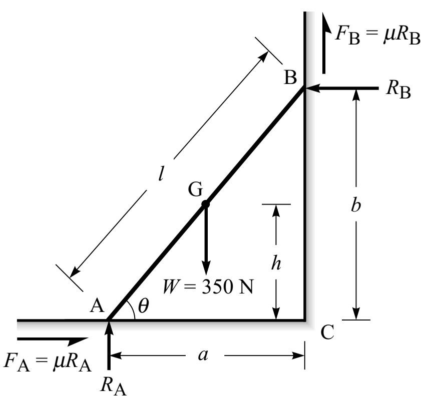


Fig. 11.9

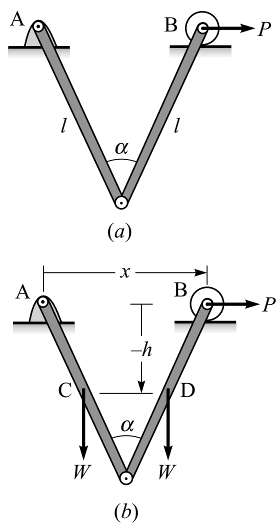


Fig. 11.10

Example 11.5 Two bars of mass m each are connected to each other and supported as shown in Fig. 11.10*a*. Determine the value of α for which the system is in equilibrium under the force P.

[LO 11.2]

Solution Let the roller at B be moved by a small virtual displacement δx in the direction of the force P as shown in Fig. 11.10*b*. Simultaneously, the centres of gravity C and D of the bars move up by a displacement δh each. Equating the virtual work done by the forces to zero, we get

$$P \times \delta x - 2 \times mg \times \delta h = 0 \qquad (i)$$

Both the gravity forces, mg do *negative* work as the displacement δh is upwards.

Now taking A as the origin, we may write from geometry,

$$x = 2l\sin\frac{\alpha}{2} \quad \Rightarrow \quad \delta x = l\left(\cos\frac{\alpha}{2}\right)\delta\alpha \quad \text{and} \quad h = -\frac{l}{2}\cos\frac{\alpha}{2} \quad \Rightarrow \quad \delta h = \frac{l}{4}\left(\sin\frac{\alpha}{2}\right)\delta\alpha$$

Substituting δx and δh into Eq. (i), we get

$$P \times l\left(\cos\frac{\alpha}{2}\right)\delta\alpha - 2 \times mg \times \frac{l}{4}\left(\sin\frac{\alpha}{2}\right)\delta\alpha = 0$$

Thus, $$\tan\frac{\alpha}{2} = \frac{2P}{mg} \quad \text{or} \quad \boldsymbol{\alpha = 2\tan^{-1}\frac{2P}{mg}}$$

LO 11.3

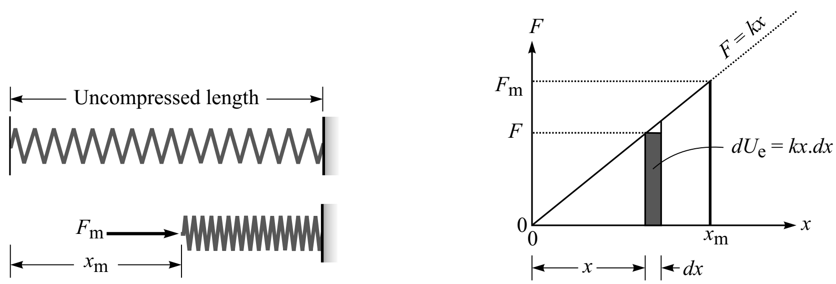
Define elastic potential energy and apply the virtual-work principle to elastic bodies.

11.3 ❑ POTENTIAL ENERGY

Let us now consider the systems involving elastic elements like springs. These elastic elements are capable of being compressed or elongated when an external force is applied. The work done by the external force is stored in elastic element as energy, known as ***elastic potential energy***. On removing the external force, the system tends to lose this energy, which is used in the work done by the restoring forces.

11.3.1 Elastic Potential Energy

An elastic body under the action of forces deforms, and the work done by forces is stored in the body itself, which is known as ***elastic potential energy***, U_e. The body acquires the ability to dissipate this energy in any form like doing work on other bodies. Consider a spring compressed by displacement x_m with a force F_m as shown in Fig. 11.11*a*. The applied force F is varied from 0 to the maximum value F_m. The spring is linearly elastic, i.e., the force F at all stages is directly proportional to x and can be expressed as $F \propto x$ or $F = kx$. Here, the constant of proportionality k is known as ***stiffness*** of the spring or ***spring constant***.



(a) Spring compressed by force F_{max} (b) Load-deflection curve

Fig. 11.11

Spring Constant or Stiffness In the equation, $F = kx$, if x is kept unity, k becomes equal to F. Thus, *stiffness may be defined as the force required to deflect a spring by unit displacement*.

Now, let us determine the work done on the spring. For a displacement x, the applied force $F = kx$. For an additional incremental displacement dx, we can assume the force F to remain constant. Thus, the work done through this displacement is given as

$$dU_e = F \cdot dx = kx \cdot dx$$

This work done is represented by the area of dark strip in Fig. 11.11*b*. The total work done through displacement x_m may be calculated by integrating the above.

$$U_e = \int_0^{x_m} kx \cdot dx = \tfrac{1}{2}kx_m^2 = \tfrac{1}{2}(kx_m)x_m = \tfrac{1}{2}F_m x_m \tag{11.10}$$

Thus, the area under load-deflection curve shown in Fig. 11.11*b* represents the total work done in compressing the spring; this work done is stored in the spring as potential energy.

➢ Note

The potential energy U_e is always positive, whether the spring is compressed or expanded from its natural length.

The internal restoring force (say, P) developed in the spring is always equal to the applied force F, i.e., $P = -F$. When F is increased from 0 to F_m, P also increases from 0 to P_m. If the spring is allowed to return to its original position, it does an equal amount of work by releasing all its potential energy. Thus,

$$U = \tfrac{1}{2}P_m x_m = -\tfrac{1}{2}F_m x_m \tag{11.11}$$

In other words, Eq. (11.11) reveals that the work done by the spring is equal to the *negative* potential energy *released*. If compression in the spring is relaxed by a small virtual displacement δx, then the decrease in potential energy is given as

$$\delta U_e = F \cdot (-\delta x) = -kx \cdot \delta x \tag{11.12}$$

Equation (11.12) also gives the amount of work that the spring does, while releasing its potential energy through displacement δx.

In case of *torsional spring*, which resists the twisting of a shaft or other members, the potential energy stored for an angle of twist θ is obtained as

$$U_e = \int_0^{\theta} T \cdot d\theta = \int_0^{\theta} K\theta \cdot d\theta = \tfrac{1}{2}K\theta^2 \tag{11.13}$$

where, T is the torsional moment or torque, $d\theta$ is a small differential twist, and K is the torsional stiffness.

11.3.2 Gravitational Potential Energy

To move a body of mass m upwards through a height h, one has to apply a force equal and opposite to the gravitational force ($= mg$) on the body. During this process, the work done on the body,

$$U = \text{Force} \times \text{Displacement} = mgh \tag{11.14}$$

The work done U is stored in the body as gravitational potential energy given as

$$U_g = U = mgh \tag{11.15}$$

If the body is lowered by a height h, the gravitational potential energy is released and the body does a work equal to mgh. The gravitational potential energy U_g of a body is measured with respect to an arbitrary level, called *reference level* or *datum*. If the body is raised above the datum, its energy U_g is positive; if lowered below datum, it is negative.

Virtual Work Principle for Elastic Bodies *The virtual work done by all the external active forces (excluding gravitational and spring forces) on a mechanical system in equilibrium is equal to the corresponding change in the total elastic and gravitational potential energy of the system for any or all virtual displacements consistent with all the constraints.*

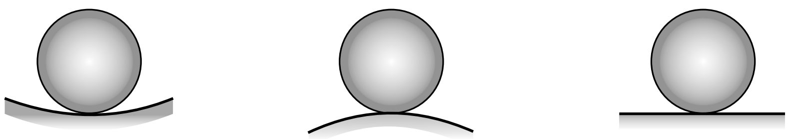
LO 11.4
Examine the stability of a system.

11.4 ❑ STABILITY OF EQUILIBRIUM

A system when left to itself, always tends to possess a state of equilibrium in which its potential energy acquires a stationary value. Mathematically, we may state it as

$$\frac{dU}{dx} = 0 \qquad (11.16)$$

Here, x is the variable on which potential energy depends. For example, in a spring, x is the extension or compression in the spring. There are three kinds of static equilibrium.



(*a*) Stable (*b*) Unstable (*c*) Neutral

Fig. 11.12 *Three states of equilibrium*

1. *Stable Equilibrium* If the potential energy is minimum, the equilibrium of the system is stable. A roller placed in a concave surface describes the stable equilibrium (Fig. 11.12*a*). A small displacement of the roller causes its potential energy to increase. It then tends to return to its original position.

2. *Unstable Equilibrium* If the potential energy is maximum, the equilibrium of the system is unstable. Figure 11.12*b* shows a roller placed on the crest of a convex surface. Even a small displacement causes its potential energy to reduce. The displacement further increases as the roller tends to acquire a state of lower energy.

3. *Neutral Equilibrium* If the potential energy remains constant on slightly displacing the body, the equilibrium is said to be neutral. Figure 11.12*c* shows a roller placed on a horizontal flat surface. A displacement of the roller does not cause any change in its potential energy. It has no tendency to move either way.

Mathematically, the three kinds of equilibrium may be expressed as

(a) Stable Equilibrium.

$$\frac{dU}{dx} = 0 \quad \text{and} \quad \frac{d^2U}{dx^2} > 0 \quad \Rightarrow \quad U \text{ is minimum and its second derivative is +ve}$$

(b) Unstable Equilibrium.

$$\frac{dU}{dx} = 0 \quad \text{and} \quad \frac{d^2U}{dx^2} < 0 \quad \Rightarrow \quad U \text{ is maximum and its second derivative is –ve}$$

(c) Neutral Equilibrium.

$$\frac{dU}{dx} = 0 \quad \text{and} \quad \frac{d^2U}{dx^2} = 0 \quad \Rightarrow \quad U \text{ is constant}$$

Example 11.6 Figure 11.13 shows a 10-kg cylinder suspended at the lower end of a spring. Calculate the extension in the spring and potential energy in the system.

[LO 11.4]

Solution Given: $m = 10$ kg, $k = 2$ kN/m.

Let x be the extension in the spring. Its elastic potential energy is $\frac{1}{2}kx^2$ and the gravitational potential energy of the mass is $-mgx$. Thus, the total potential energy of the system,

$$U = \tfrac{1}{2}kx^2 - mgx \tag{i}$$

For equilibrium, $\dfrac{dU}{dx} = 0 \quad \Rightarrow \quad kx - mg = 0 \quad \Rightarrow \quad x = \dfrac{mg}{k}$

Putting the values of k and m, $x = \dfrac{mg}{k} = \dfrac{10 \times 9.81}{2000} = 0.04905 \text{ m} = \mathbf{49.05 \text{ mm}}$

Putting $x = 0.04905$ m in Eq. (i), the potential energy,

$$U = \tfrac{1}{2} \times 2000 \times (0.04905)^2 - 10 \times 9.81 \times 0.04905 = -2.41 \text{ Nm} = -\mathbf{2.14 \text{ J}}$$

➢ **Note**

From Eq. (i), $\dfrac{d^2U}{dx^2} = k > 0$. It means that the total potential energy is the minimum, and the system is in ***stable*** equilibrium.

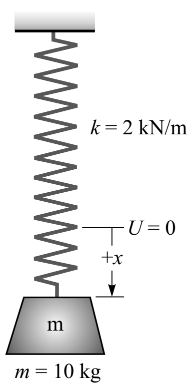


Fig. 11.13

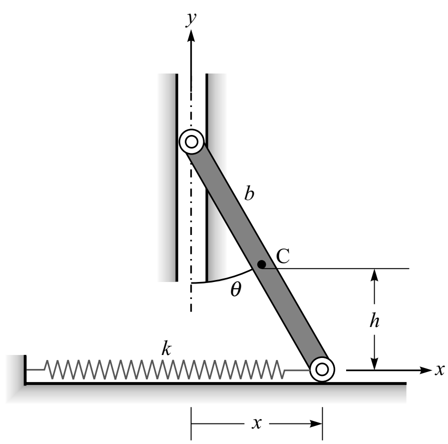


Fig. 11.14

Example 11.7 The ends of a uniform bar of mass m and length b slide freely in the horizontal and vertical guides as shown in Fig. 11.14. Examine the stability conditions for the positions of equilibrium. The spring of stiffness k is undeformed when the bar is vertical.

[LO 11.4]

Solution Let x be the extention in the spring and θ be the rotation of the bar with the vertical. Let h be the height of the centroid of the bar from spring level (assumed as the datum line). From geometry, $x = b\sin\theta$, and $h = \frac{1}{2}b\cos\theta$.

Elastic and gravitational potential energies are, respectively

$$U_e = \tfrac{1}{2}kx^2 = \tfrac{1}{2}kb^2\sin^2\theta \quad \text{and} \quad U_g = mgh = mg\left(\tfrac{1}{2}b\cos\theta\right)$$

Total potential energy of the system,

$$U = \tfrac{1}{2}kb^2\sin^2\theta + \tfrac{1}{2}mgb\cos\theta \tag{i}$$

For equilibrium, $\dfrac{dU}{d\theta} = 0 \quad \Rightarrow \quad kb^2\sin\theta\cos\theta - \tfrac{1}{2}mgb\sin\theta = 0$

$$\Rightarrow \quad \left(kb^2\cos\theta - \tfrac{1}{2}mgb\right)\cdot\sin\theta = 0 \tag{ii}$$

This implies that

(i) $\sin\theta = 0 \quad \Rightarrow \quad \theta = 0°$ or

(ii) $kb^2\cos\theta - \frac{1}{2}mgb = 0 \quad \Rightarrow \quad \cos\theta = \dfrac{mg}{2kb}$ or $\theta = \cos^{-1}\dfrac{mg}{2kb}$

In order to examine the stability, we need to find the second derivative of U,

$$\frac{d^2U}{d\theta^2} = kb^2(\cos^2\theta - \sin^2\theta) - \tfrac{1}{2}mgb\cos\theta = kb^2(2\cos^2\theta - 1) - \tfrac{1}{2}mgb\cos\theta$$

Case-1: $\theta = 0°$,

$$\frac{d^2U}{d\theta^2} = kb^2(2-1) - \tfrac{1}{2}mgb = kb^2\left(1 - \frac{mg}{2kb}\right)$$

$$= +\text{ve (stable)} \quad \text{if} \quad \frac{mg}{2kb} < 1 \quad \Rightarrow \quad k > \frac{mg}{2b}$$

$$= -\text{ve (unstable) if} \quad \frac{mg}{2kb} > 1 \quad \Rightarrow \quad k < \frac{mg}{2b}$$

Case-2: $\theta = \cos^{-1}\dfrac{mg}{2kb}$,

$$\frac{d^2U}{d\theta^2} = kb^2\left[2\left(\frac{mg}{2kb}\right)^2 - 1\right] - \frac{1}{2}mgb\left(\frac{mg}{2kb}\right) = \frac{m^2g^2}{2k} - kb^2 - \frac{m^2g^2}{4k}$$

$$= kb^2\left[\left(\frac{mg}{2kb}\right)^2 - 1\right] = kb^2\left[\cos^2\theta - 1\right]$$

Since cosine of an angle cannot exceed unity, so $\dfrac{d^2U}{d\theta^2} < 0$ i.e., the equilibrium is *unstable.* Hence, the equilibrium for Case-2 is never stable.

Thus, the system can be in **stable equilibrium** only when $\theta = 0°$, and $k > \dfrac{mg}{2b}$.

ADDITIONAL SOLVED EXAMPLES

Example 11.8 The differential pulley block in Fig. 11.15 has the radii of its outer and inner pulleys as 20 cm and 10 cm, respectively. Determine the maximum load that can be lifted by an effort of 500 N applied on the outer pulley.

[LO 11.2]

Solution If the differential pulley is given a *clockwise* virtual rotation of $\delta\theta$ radians, the displacement of 500-N force (and the rope A) is $20 \times \delta\theta$ (in cm). Rope B also has $20 \times \delta\theta$ displacement upwards. However, the rope C has downward displacement of $10 \times \delta\theta$. Therefore, the movable pulley and hence the load W moves *upwards* by a virtual displacement given by $(20 \times \delta\theta - 10 \times \delta\theta) = 10\delta\theta$.

As per the principle of virtual work, equating the total virtual work done to zero, we get

$$500 \times (20 \times \delta\theta) - W \times (10\delta\theta) = 0$$

$$\Rightarrow \quad 10000 - W \times 10 = 0 \quad \Rightarrow \quad W = \mathbf{1000\ N}$$

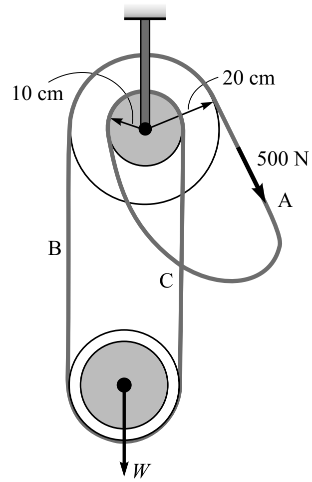


Fig. 11.15

Example 11.9 In Fig. 11.16*a*, calculate the value of load W required to keep the system in equilibrium when a horizontal force of 1000 N acts at the point F. Given, $\theta = 60°$.

[LO 10.2]

Solution Let us virtually move the load W upwards so that the arms AD and BF rotate in counterclockwise direction by a virtual angle $\delta\theta$, as shown in Fig. 11.16*b*. In order to determine virtual displacement of W in vertical direction, we differentiate the expression of height y of the point D above the floor. Thus,

$$y = \text{AD} \sin\theta = 1.25 \sin\theta \quad \Rightarrow \quad \delta y = 1.25 \cos\theta\, \delta\theta$$

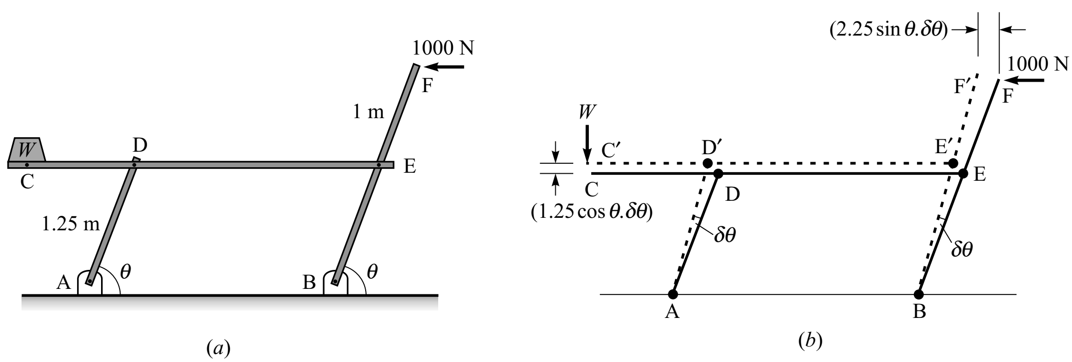


Fig. 11.16

Similarly, the virtual displacement of the 1000-N force in horizontal is obtained by differenting the horizontal distance of the point F from B,

$$x = \text{BF}\cos\theta = 2.25\cos\theta \quad \Rightarrow \quad \delta x = -2.25\sin\theta\,\delta\theta$$

Equating the total virtual work done by W and 1000-N force to zero, we get

$$-W \times |\delta y| + 1000 \times |\delta x| = 0 \quad \Rightarrow \quad -W \times 1.25\cos\theta\,\delta\theta + 1000 \times (2.25\sin\theta\,\delta\theta) = 0$$

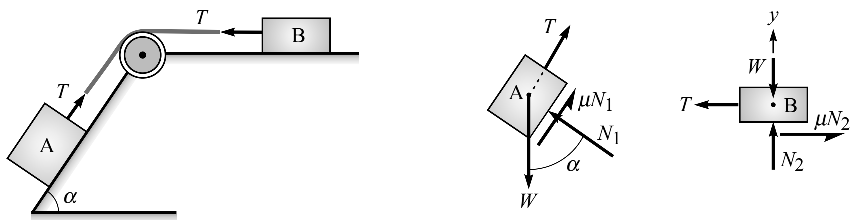
$$\Rightarrow \qquad W = \frac{1000 \times 2.25}{1.25} \cdot \frac{\sin\theta}{\cos\theta} = 1800\tan 60° = \mathbf{3117.69\ N}$$

Example 11.10 Two rectangular blocks of weights W_1 and W_2 are connected by a flexible cord and rest upon an inclined plane and horizontal plane, respectively, with the cord passing over a pulley as shown in Fig. 11.17*a*. In a particular case, where $W_1 = W_2$ and the coefficient of static friction μ is same for the continuous surface, find the angle α of inclination of an inclined plane at which the motion of the system will impend. Neglect friction in the pulley.

[LO 11.2]

➢ **Note**

The same problem is solved in Example 7.17 by considering the equilibrium of blocks A and B, individually.



(*a*) (*b*) (*c*)

Fig. 11.17

Solution Free-body diagrams of the two blocks are shown in Fig. 11.17*b* and *c*. The frictional forces μN_1 and μN_2 act opposite to the direction of impending motion. Let the block A be given a small virtual displacement δ down the slope. The block B also moves through the same displacement. Reactions are $N_1 = W\cos\alpha$ and $N_2 = W$.

The virtual work done by the block A is the sum of work done by component of its weight along the plane and the frictional force μN_1. Thus, $U_A = W\sin\alpha \times \delta - \mu N_1 \times \delta$. However, the virtual work done by the block B consists of the work done by frictional force μN_2 only; its weight does zero work because the displacement is perpendicular to the weight. Thus, $U_B = -\mu N_2 \times \delta$. Equating the total work done by the system to zero, we get

$$W\sin\alpha \times \delta - \mu N_1 \times \delta - \mu N_2 \times \delta = 0 \quad \Rightarrow \quad W\delta\sin\alpha - \mu W\delta\cos\alpha - \mu W\delta = 0$$

$$\Rightarrow \qquad \mu = \frac{\sin\alpha}{(1+\cos\alpha)} = \frac{2\sin(\alpha/2)\cos(\alpha/2)}{2\cos^2(\alpha/2)} = \tan(\alpha/2) \quad \Rightarrow \quad a = \mathbf{2\tan^{-1}}\mu$$

Example 11.11 Figure 11.18*a* shows a simply supported beam carrying transverse loading. The beam is also subjected to an anticlockwise bending moment of 40 kNm about the point C. Using the principle of virtual work, determine the reaction at A.

[LO 11.2]

Solution Let us lift the support A through a small virtual displacement δa, keeping the support D stationary. Let δp, δb, δc and δe be the corresponding virtual displacements at points P, B, C and E, respectively, as shown in Fig. 11.18*b*. The virtual rotation of the beam at the point D is $\delta\theta$.

From similar triangles with the common corner D, we have

$$\frac{\delta a}{7.5} = \frac{\delta p}{6} = \frac{\delta b}{4} = \frac{\delta e}{2.5} \quad \Rightarrow \quad \delta p = 0.8\delta a \quad \delta b = 0.53\delta a \quad \delta e = 0.33\delta a \quad \delta\theta = \frac{\delta a}{7.5} = 0.133\delta a$$

Applying the principle of virtual work, we have

$$R_A \times \delta a + 12 \times (-\delta p) + 20 \times 3.5 \times [-(\delta a + \delta b)/2] + 40 \times (-\delta\theta) + 30 \times \delta e = 0 \qquad (i)$$

Here, $\delta\theta$ is taken as *negative* since it is opposite in direction to the 40-kNm moment at C. Putting all the displacements and rotation in terms of δa in Eq. (*i*), we get

$$R_A \times \delta a + 12 \times (-0.8\delta a) + 20 \times 3.5 \times [-(\delta a + 0.53\ \delta a)/2]$$
$$+ 40 \times (-0.133\delta a) + 30 \times 0.33\delta a = 0$$

$$\Rightarrow \qquad R_A - 9.6 - 53.55 - 5.32 + 10 = 0 \quad \Rightarrow \quad R_A = \mathbf{58.47\ kN}$$

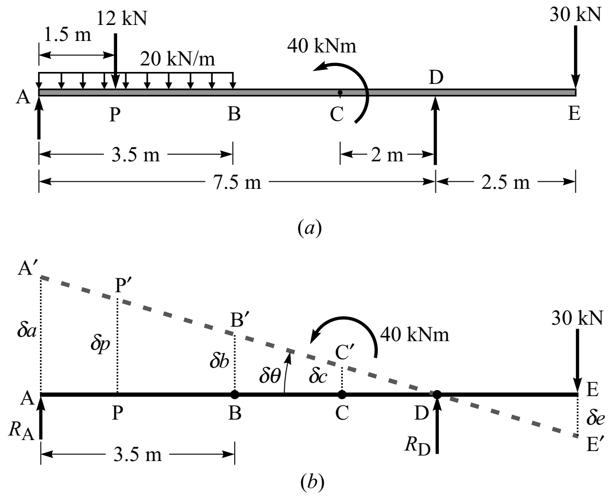


Fig. 11.18

Example 11.12 Two rollers C and D produce vertical forces P and Q on the horizontal beam AB, as shown in Fig. 11.19*a*. Determine the distance x of the load P from the support A if the reaction at A is twice the reaction at B. The weight of the beam is to be neglected. Given that P = 18 kN, Q = 9 kN, l = 3.6 m, c = 0.9 m.

[LO 11.2]

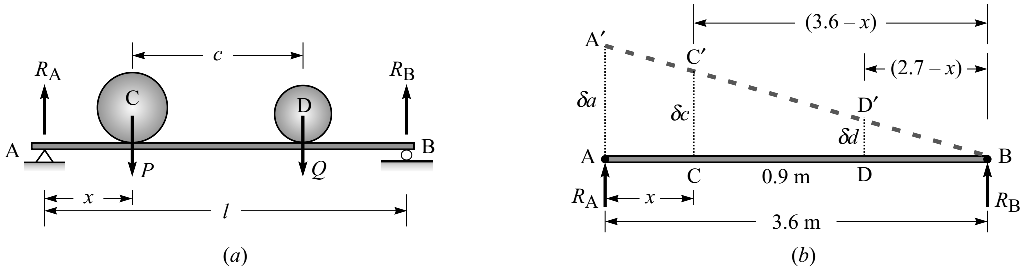


Fig. 11.19

Solution Applying equation of equilibrium in vertical direction,

$$\Sigma F_x = 0: \quad R_A + R_B = P + Q = 27 \text{ kN}$$

Since, $R_B = 0.5R_A$, hence $R_A + 0.5R_A = 27 \quad \Rightarrow \quad R_A = 18$ kN

Now, let us lift the support A by a virtual displacement δa so that points C and D undergo upward displacements δc and δd, respectively, as shown in Fig. 11.19*b*. From similar triangles with the common corner B, we have

$$\frac{\delta a}{3.6} = \frac{\delta c}{3.6 - x} = \frac{\delta d}{2.7 - x} \quad \Rightarrow \quad \delta c = \frac{(3.6 - x)\delta a}{3.6} \quad \text{and} \quad \delta d = \frac{(2.7 - x)\delta a}{3.6}.$$

Applying the principle of virtual work, we get

$$R_A \times \delta a + P \times (-\delta c) + Q \times (-\delta d) = 0$$

$$\Rightarrow \quad 18 \times \delta a - 18 \times \frac{(3.6 - x)\delta a}{3.6} - 9 \times \frac{(2.7 - x)\delta a}{3.6} = 0$$

$$\Rightarrow \quad 18 - 5(3.6 - x) - 2.5(2.7 - x) = 0 \quad \Rightarrow \quad x = \mathbf{0.9\ m}$$

Example 11.13 An assembly is made of bars joined with hinges forming four identical rhombuses as shown in Fig. 11.20. Using the principle of virtual work, determine the force P to keep the system in equilibrium.

[LO 11.2]

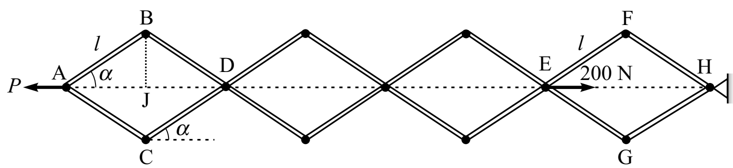


Fig. 11.20

Solution From geometry, we have AH = a = 8 × AJ = $8l\cos\alpha$ and EH = e = $2l\cos\alpha$. If the angle α is increased by a small virtual angle $\delta\alpha$, the virtual displacements of points A and E are given by differentiating the expressions for a and e with respect to α,

$$\delta a = -8l\sin\alpha\,\delta\alpha \quad \text{and} \quad \delta e = -2l\sin\alpha\,\delta\alpha.$$

The negative sign for δa and δe indicates that on increasing angle α by $\delta\alpha$, the distances a and e decrease by δa and δe, respectively. That is, the displacement of points A and E are rightward.

$$\therefore \quad |\delta a| = 8l\sin\alpha\,\delta\alpha \quad \text{and} \quad |\delta e| = 2l\sin\alpha\,\delta\alpha$$

As per the principle of virtual work, equating the total work done by forces P and 200 N to zero,

$$-P \times |\delta a| + 200 \times |\delta e| = 0 \quad \Rightarrow \quad -P \times (8l \sin \alpha \delta\alpha) + 200 \times (2l \sin \alpha \delta\alpha) = 0$$

$$\Rightarrow \qquad P = \frac{200}{4} = \mathbf{50\,N}$$

Example 11.14 A uniform ladder AB whose weight is 200 N and length 6 m rests against a smooth vertical wall making with it an angle of 30°. The other end rests on the ground surface. Find the reactions given to the ladder by the wall, the floor, and their inclination to the vertical, when a man weighing 700 N climbs the ladder by a 1.5-m distance along the length of the ladder.

[LO 11.2]

Solution Given: AB = 6 m, AG = 3 m, AM = 1.5 m, $\theta = 60°$.

From Fig. 11.21, we have

$$a = \text{AC} = \text{AB} \cos\theta = 6\cos\theta \qquad h_1 = \text{AG}\sin\theta = 3\sin\theta \qquad \text{and} \qquad h_2 = \text{AM}\sin\theta = 1.5\sin\theta$$

where, h_1 and h_2 are heights of G and M above ground respectively.

Let the end A be given a leftward virtual displacement δa along the floor. The point G (at which the ladder weight works) and point M carrying the man's weight move downward by displacements δh_1 and δh_2, respectively. These virtual displacements are given by

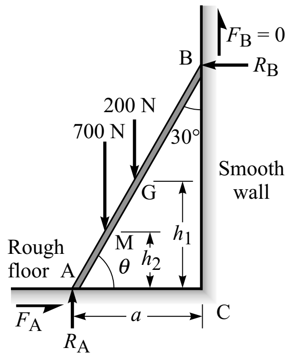


Fig. 11.21

$$\delta a = -6\sin\theta\,\delta\theta \qquad \delta h_1 = 3\cos\theta\,\delta\theta \qquad \delta h_2 = 1.5\cos\theta\,\delta\theta$$

As per principle of virtual work, equating the total virtual work done to zero,

$$-F_A \times |\delta a| + 200 \times |\delta h_1| + 700 \times |\delta h_2| = 0$$

$$\Rightarrow \qquad -F_A \times (6\sin\theta\,\delta\theta) + 200 \times 3\cos\theta\,\delta\theta + 700 \times 1.5\cos\theta\,\delta\theta = 0$$

$$\Rightarrow \qquad F_A = \frac{600\cos 60° + 1050\cos 60°}{6\sin 60°} = 158.77\text{ N}$$

Applying equation of equilibrium in horizontal and vertical directions,

$$\Sigma F_x = 0: \quad F_A - R_B = 0 \quad \Rightarrow \quad R_B = F_A = \mathbf{158.77\ N} \text{ (horizontally leftward)}$$

$$\Sigma F_y = 0: \quad R_A - 200 - 700 = 0 \quad \Rightarrow \quad R_A = 900\text{ N}$$

Resultant reaction at the point A on floor, $R = \sqrt{R_A^2 + F_A^2} = \sqrt{900^2 + 158.77^2} = \mathbf{913.9\ N}$

The angle made by R with the vertical, $\theta_y = \tan^{-1}\dfrac{F_A}{R_A} = \tan^{-1}\dfrac{158.77}{900} = \mathbf{10°}$

Example 11.15 A truss shown in Fig 11.22*a*, has all its members of equal length. Determine the force in the top member BD using the virtual-work method.

[LO 11.2]

Solution From symmetry of the truss and the loading we may write, $R_A = R_B = 100$ N. Let us remove the member BD and replace it by an equivalent axial force F (assumed compressive), as shown in Fig. 11.22*b*. The system is then made movable by imagining the portion DCE to rotate through an angle $\delta\theta$. Then D gets displaced to D′ and E to E′ (as shown in Fig. 11.22*c*).

Displacements of points D and E = DD′ = EE′ = $l\,\delta\theta$

In ΔDOD′ and ΔCDE, DD′ $\perp$ CD and OD′ $\perp$ CE, hence $\angle$DD′O = $\angle$DCE = 60°. Therefore, from ΔDOD′ (in Fig. 11.22*d*), the horizontal and vertical displacements at D are

$$DO = DD'\sin 60° = l\,\delta\theta \times 0.866 \qquad D'O = DD'\cos 60° = l\,\delta\theta \times 0.5$$

Work done by axial force $F = -F \times DO = -0.866 F l\,\delta\theta$

Work done by the load of 100 N at the point D $= -100 \times D'O = -50 l\,\delta\theta$

Vertical displacement of the point E $= EE' = l \sin\delta\theta \approx l\delta\theta$

Work done by 100-N reaction at the point E $= 100 \times EE' = 100 l\,\delta\theta$

Total work done $= -0.866 F l\,\delta\theta - 50 l\,\delta\theta + 100 l\,\delta\theta$

According to principle of virtual work, we have

$$-0.866 F l\delta\theta - 50 l\delta\theta + 100 l\delta\theta = 0 \quad\Rightarrow\quad 0.866F = 50 \quad\Rightarrow\quad F = \mathbf{57.74\ N}$$

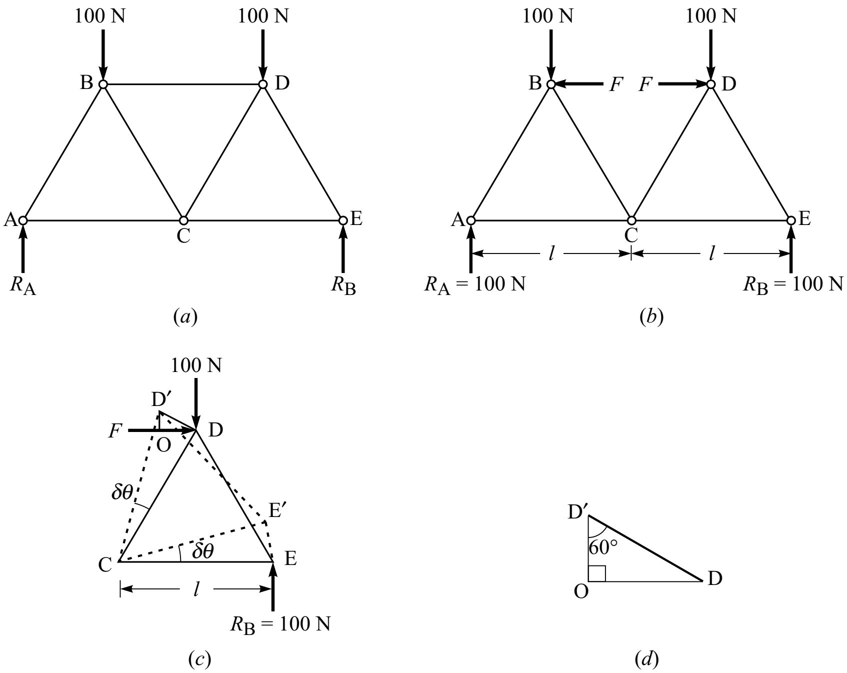


Fig. 11.22

Example 11.16 A roller of radius r = 0.3 m and weight Q = 2000 N is to be pulled over a kerb of height h = 0.15 m by a horizontal force P applied to the end of a string wound around the circumference of the roller, as shown in Fig. 11.23*a*. Find the magnitude of P required to start the roller over the kerb.

[LO 11.2]

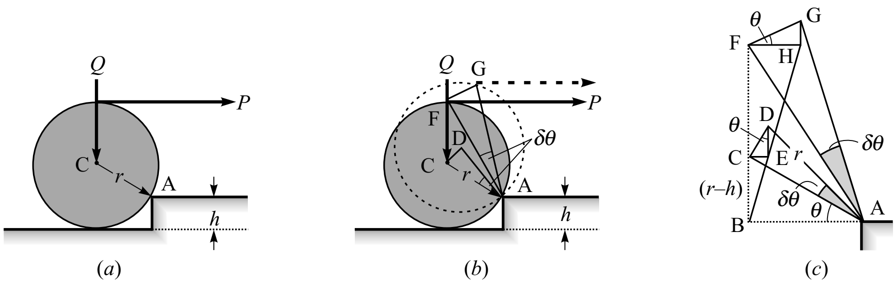


Fig. 11.23

Solution Given: $r = 0.3$ m, $h = 0.15$ m, $Q = 2000$ N.

Let the roller be virtually rotated about the point A by an infinitesimal angle $\delta\theta$, as shown in Fig. 11.23*b*. The centre C of the roller moves to the point D and the point F of application of the force P moves to the point G. The exploded view of the displacements is shown in Fig. 11.23*c*. From ΔABC, we get

$$\sin\theta = \frac{\text{CB}}{\text{CA}} = \frac{(r-h)}{r} = \frac{0.3-0.15}{0.3} = 0.5 \quad \Rightarrow \quad \theta = 30°$$

Weight Q does work through the vertical displacement DE of the point C, and the force P does work through the horizontal displacement FH of the point F. Taking the point A as reference, let us express the vertical height y and horizontal distance x of the point C in terms of angle θ,

$$y = r\sin\theta \quad \text{and} \quad x = r\cos\theta$$

When θ is given a small increment $\delta\theta$, the corresponding changes in y and x are determined by differentiating the above expressions,

$$\delta y = r\cos\theta\,\delta\theta \quad \text{and} \quad \delta x = r(-\sin\theta)\delta\theta = -r\sin\theta\,\delta\theta$$

The negative value of δx shows that x decreases with increase in θ.

Now, let us determine the horizontal displacement FH ($= \delta x_1$) of the point F in terms of the horizontal displacement CE ($= \delta x$) of the point C. When the roller is rotated by an angle $\delta\theta$ about the point A, all points of the roller rotate by the same angle $\delta\theta$ about the point A. The horizontal displacement of a point is proportional to its height from the base line AB. From similar triangles ΔBFH and ΔBCE, we get

$$\frac{\text{FH}}{\text{CE}} = \frac{\text{BF}}{\text{BC}} = \frac{\text{BC}+\text{CF}}{\text{BC}} = \frac{0.15+0.3}{0.15} = 3 \quad \Rightarrow \quad \text{FH} = 3\times\text{CE} \quad \Rightarrow \quad \delta x_1 = 3\delta x$$

$$\Rightarrow \qquad \delta x_1 = -3r\sin\theta\,\delta\theta$$

Applying the principle of virtual work,

$$-Q\times|\delta y| + P\times|\delta x_1| = 0 \quad \Rightarrow \quad -2000\times r\cos\theta\,\delta\theta + P\times 3r\sin\theta\,\delta\theta = 0$$

$$\Rightarrow \qquad P = \frac{2000\times\cos\theta}{3\sin\theta} = \frac{2000\times\cos 30°}{3\sin 30°} = \mathbf{1154.7\ N}$$

***Alternatively*,** from Fig. 11.23*c*, the displacement CD $= r\,\delta\theta$. Since $\delta\theta$ is very small, ΔCED is considered to be right-angled at E and ∠CDE $= \theta$. Therefore, the vertical component of the displacement CD,

$$\delta y = \text{DE} = \text{CD}\cos\theta = r\,\delta\theta\cos 30° = \left(\sqrt{3}/2\right) r\,\delta\theta$$

The ΔACF is an isosceles triangle because its two sides AC and AF are equal. In ΔACF, $\angle$FCA $= 180° - (90° - \theta) = 120°$. Therefore, $\angle$CFA $= (180° - 120°)/2 = 30°$. Thus, from geometry, AF $= 2r\cos 30° = \sqrt{3}r$. Now, since FG$\perp$AF, in ΔFGH, $\angle$HFG $= \angle$CFA $= 30°$. Thus, the displacement FG $=$ AF$\delta\theta = \sqrt{3}r\delta\theta$. Therefore, the horizontal component of the displacement FH,

$$\delta x_1 = \text{FH} = \text{FG}\cos\theta = (\text{AF}\delta\theta)\cos 30° = \left(\sqrt{3}r\right)\left(\sqrt{3}/2\right)\delta\theta = 1.5r\delta\theta.$$

Now, applying the principle of virtual work,

$$-Q \times |\delta y| + P \times |\delta x_1| = 0 \quad \Rightarrow \quad -2000 \times \left(\sqrt{3}/2\right) r\delta\theta + P \times 1.5r\delta\theta = 0$$

$$\Rightarrow \qquad P = \frac{2000 \times \left(\sqrt{3}/2\right)}{1.5} = \mathbf{1154.7\ N}$$

Example 11.17 Four uniform rods, each of length a and weight W are hinged together at their ends to form a rhombus. A bar joins the ends of the shorter diagonal having length b. If the lowermost rod is supported in horizontal position, show that the compression F in the bar is

$$F = \frac{2W(2a^2 - b^2)}{a\sqrt{4a^2 - b^2}}$$

[LO 11.2]

Solution Figure 11.24 shows the rhombus formed by the four uniform rods. The centre of gravity G of four rods is the point of intersection of the diagonals AC and BD. We can assume that the total weight of the four rods is concentrated at G. If the angle $\angle$BCD is θ, heights of the point C and G above level of AB are, respectively

$$y_C = \text{CE} = \text{BC}\sin\theta = a\sin\theta$$

and $\quad y_G = \text{G}'\text{E} = (1/2)\text{CE} = (a/2)\sin\theta$

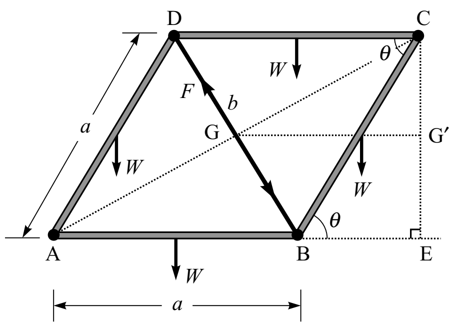


Fig. 11.24

Maintaining the rod AB in the horizontal position let us give a small virtual displacement to the point C so that angle θ increases by $\delta\theta$. Then, the virtual displacement of the point G in the vertical direction is

$$\delta y_G = (a/2)\cos\theta \cdot \delta\theta \quad \text{(upward)} \qquad (i)$$

From geometry, the length of bar BD, $b = 2\text{BG} = 2\text{BC}\sin(\theta/2) = 2a\sin(\theta/2)$.

Therefore, the virtual displacement of the point D (along force F), $\delta b = a\cos(\theta/2)\delta\theta$.

As per the principle of virtual work, equating the algebraic sum of the virtual work done by the weights of the four rods and force F in the bar BD to zero, we get

$$-4W \times \delta y_G + F \times \delta b = 0 \quad \Rightarrow \quad F = 4W \times \frac{\delta y_G}{\delta b} = 4W \times \frac{(a/2)\cos\theta\,\delta\theta}{a\cos(\theta/2)\,\delta\theta} = \frac{2W\cos\theta}{\cos(\theta/2)} \qquad (ii)$$

From the geometry of the rhombus, $\sin\dfrac{\theta}{2} = \dfrac{\text{BG}}{\text{BC}} = \dfrac{(b/2)}{a} = \dfrac{b}{2a}$

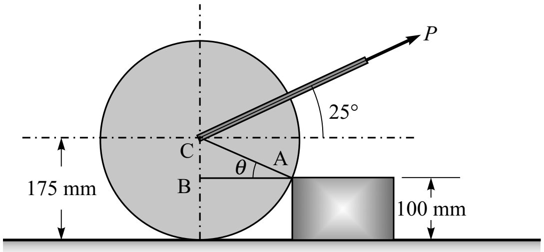
$$\therefore \qquad \cos\frac{\theta}{2} = \sqrt{1-\sin^2\frac{\theta}{2}} = \sqrt{1-\frac{b^2}{4a^2}} = \sqrt{\frac{4a^2-b^2}{4a^2}} = \frac{\sqrt{4a^2-b^2}}{2a}$$

$$\text{and} \qquad \cos\theta = 1 - 2\sin^2\frac{\theta}{2} = 1 - 2\frac{b^2}{4a^2} = \frac{2a^2-b^2}{2a^2}$$

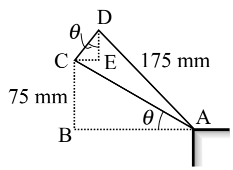
Equation (*ii*) becomes, $T = \dfrac{2W\left(2a^2-b^2\right)/2a^2}{\left(\sqrt{4a^2-b^2}\right)/2a} = \boldsymbol{\dfrac{2W(2a^2-b^2)}{a\sqrt{4a^2-b^2}}}$

Example 11.18 A roller shown in Fig. 11.25*a* is of mass 150 kg. What force *P* is necessary to start the roller over the block placed as shown?

[LO 11.2]



(*a*)



(*b*)

Fig. 11.25

Solution Weight of the roller, $W = 150 \times 9.81 = 1471.5$ N. Let the roller be rotated virtually about the point A so that the centre of the roller moves from C to D as shown in Fig. 11.25*b*. It can be seen that the centre C moves by distance CE in horizontal and by DE in vertical direction and so do the force *P* and weight *W*. From ΔABC,

$$\sin\theta = \frac{\text{BC}}{\text{AC}} = \frac{75}{175} = 0.428 \quad \Rightarrow \quad \theta = 25.38°$$

Since DE⊥AB and CD⊥AC, therefore, ∠CDE = ∠BAC = θ = 25.38°. In ΔCDE,

$$\text{CE} = \text{CD}\sin\theta = 0.428\text{CD} \quad \text{and} \quad \text{DE} = \text{CD}\cos\theta = 0.903\text{CD}$$

Horizontal and vertical components of the force *P*, i.e., *P* cos 25° and *P* sin 25° move through CE and DE, respectively. The reaction at the point A does no work as it does not move. Now, applying the virtual-work principle,

$$P\cos 25° \times \text{CE} + P\sin 25° \times \text{DE} - W \times \text{DE} = 0$$

$$\Rightarrow \qquad P\cos 25° \times 0.428\text{CD} + P\sin 25° \times 0.903\text{CD} - W \times 0.903\text{CD} = 0$$

$$\Rightarrow \qquad P = \frac{0.903W}{(\cos 25° \times 0.428 + \sin 25° \times 0.903)} = \mathbf{1726.56\ N}$$

Example 11.19 Two rollers of weights P and Q are connected by a string DE and rest on two mutually perpendicular planes AB and BC, as shown in Fig. 11.26*a*. Find the tension T in the string and the angle θ that it makes with the horizontal when the system is in equilibrium. The following numerical data are given, $P = 270$ N, $Q = 450$ N, $\alpha = 30°$. Assume that the string is inextensible and passes freely through slots in the smooth inclined planes AB and BC.

[LO 11.2]

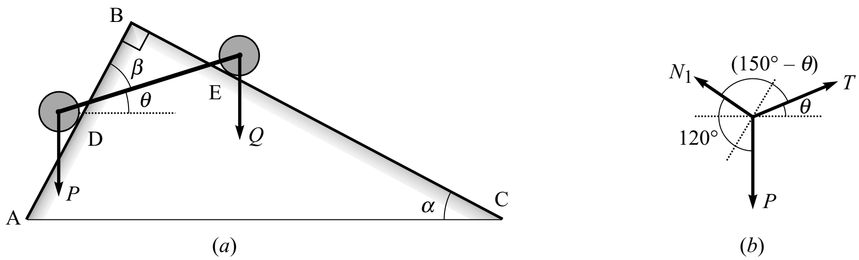


Fig. 11.26

Solution Since $\alpha = 30°$ and $\angle BAC = 60°$, $\angle BDE = (60° – \theta) = \beta$ (say).

If l is length of the string, DB = $l \cos \beta$ and BE = $l \sin \beta$

Now, if a small virtual rotation (clockwise) is given to string DE so that angle β increases by $\delta\beta$, the changes in distances DB and BE are given as

$$\delta_{DB} = -l \sin \beta \, \delta\beta \quad \text{and} \quad \delta_{BE} = l \cos \beta \, \delta\beta.$$

The resolved parts of the weights along the planes are $P \sin 60°$ and $Q \sin 30°$. Only these resolved parts contribute to the work done. Thus,

Work done at the point D = $-(P \sin 60°) \times |\delta_{DB}| = -233.8 l \sin \beta \, \delta\beta$

Work done at the point E = $(Q \sin 30°) \times |\delta_{BE}| = 225 l \cos \beta \, \delta\beta$

Equating total virtual work to zero,

$$-233.8 l \sin \beta \, \delta\beta + 225 l \cos \beta \, \delta\beta = 0 \quad \Rightarrow \quad \tan \beta = \frac{225}{233.8} = 0.962 \quad \Rightarrow \quad \beta = 43.9°$$

$$\Rightarrow \qquad \theta = 60° – \beta = \mathbf{16.1°}$$

The tension T in the string can be obtained by applying Lami's theorem on any of the rollers. Considering free body diagram of the roller at D (Fig. 11.26*b*), we may write

$$\frac{P}{\sin(150° - \theta)} = \frac{T}{\sin 120°} \quad \Rightarrow \quad T = \frac{270 \times \sin 120°}{\sin(150° - 16.1°)} = \mathbf{324.5\ N}$$

Example 11.20 A homogeneous block of mass m rests on the top surface of a semi-cylinder as shown in Fig. 11.27*a*. Show that, if $h > 2R$, the block is in unstable equilibrium.

[LO 11.2]

Solution Let us take the base of the semi-cylinder as datum. To examine the stability of equilibrium of the block, let us roll the block by a small angle θ, as shown in Fig. 11.27*b*. The potential energy of the block in this position is, $U = mg \times y$, where y is the height of centre of gravity of the block above datum. From Fig. 11.27*b*,

$$y = \left(R + \frac{h}{2}\right)\cos\theta + R\theta\sin\theta.$$

$$\therefore \qquad U = mg \times \left[\left(R + \frac{h}{2}\right)\cos\theta + R\theta\sin\theta\right]$$

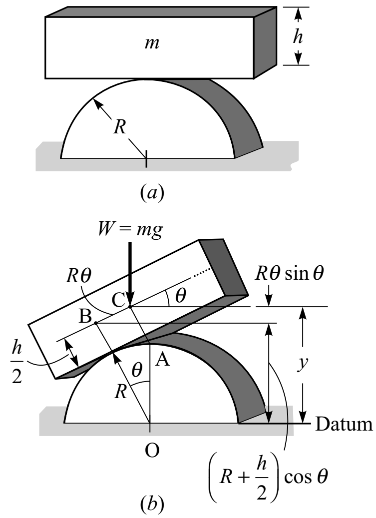


Fig. 11.27

For equilibrium, we must have

$$\frac{dU}{d\theta} = mg\left(-R\sin\theta - \frac{h}{2}\sin\theta + R\sin\theta + R\theta\cos\theta\right) = 0$$

$$\Rightarrow \qquad \frac{DU}{d\theta} = mg\left(-\frac{h}{2}\sin\theta + R\theta\cos\theta\right)$$

The above condition is satisfied for $\theta = 0°$. Thus, the condition for equilibrium of the system is $\theta = 0°$.

Now, for examining the stability of the system, we determine the second derivative of U at $\theta = 0°$,

$$\left.\frac{d^2U}{d\theta^2}\right|_{\theta=0°} = mg\left(-\frac{h}{2}\cos\theta + R\cos\theta - R\theta\sin\theta\right)\Bigg|_{\theta=0°} = mg\left(-\frac{h}{2} + R\right)$$

For the block to be in ***unstable*** equilibrium, we must have

$$\left.\frac{d^2U}{d\theta^2}\right|_{\theta=0°} < 0 \quad \Rightarrow \quad -\frac{h}{2} + R < 0 \quad \text{or} \quad \boldsymbol{h > 2R} \qquad \text{(since both } h \text{ and } R \text{ are positive)}$$

SUMMARY

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LO 11.1 Relate and apply the concept of virtual work on various structural systems.

Virtual displacement is the imaginary displacement given to a body at some points without applying any force. ***Virtual work*** is the product of real forces/moments and corresponding virtual displacements.

LO 11.2 Underline the principle of virtual work and apply it to analyse systems consisting of interconnected rigid members.

Principle of virtual work states that for a system in equilibrium, the sum of work done by active forces through virtual displacements is zero. It can be used to determine the unknown forces and moments.

It does not demand breaking of a system into sub-systems; rather the system is analysed as a whole conforming to the constraints, if any.

LO 11.3 Define elastic potential energy and apply the virtual-work principle to elastics bodies.

The work done on an elastic body (such as a spring) is stored as ***elastic potential energy*** in the body. The body tends to lose this energy in doing equal amount of work by the *restoring force*.

Stiffness or ***spring constant*** is defined as the force required to produce unit compression or elongation in the spring.

LO 11.4 Examine the stability of a system.

Principle of minimum potential energy states that every system tends to possess a state in which its potential energy is a minimum. In such a state, the system is in ***stable*** equilibrium and the expression $dU/dx = 0$ holds true.

IMPORTANT FORMULAE

- Force developed in a spring, $P = kx$ **[LO 11.2]**
- Elastic potential energy stored, $U_e = (1/2)\,kx^2$. **[LO 11.3]**
- For ***stable*** equilibrium, $\dfrac{dU}{dx} = 0$ and $\dfrac{d^2U}{dx^2} > 0 \Rightarrow U$ is minimum. **[LO 11.4]**
- For ***unstable*** equilibrium, $\dfrac{dU}{dx} = 0$ and $\dfrac{d^2U}{dx^2} < 0 \Rightarrow U$ is maximum.
- For ***neutral*** equilibrium, $\dfrac{dU}{dx} = 0$ and $\dfrac{d^2U}{dx^2} = 0 \Rightarrow U$ is stationary.

SHORT-ANSWER QUESTIONS

1. **When is the work done on a body negative? Give an example.** **[LO 11.1]**

   Work done is negative if the working component of the force is in a direction opposite to the direction of displacement. For example, work done by gravitational force on a body when it is moved upwards.

2. **What is the limitation of application of the virtual-work principle?** **[LO 11.2]**

   It is not valid for a system in which internal forces perform significant amount of virtual work (due to friction, etc.), which cannot be measured/expressed analytically.

3. **When do we take into account reactive forces in the virtual-work method?** **[LO 11.2]**

   The reactive forces do not contribute to work done unless the supports themselves are given virtual displacements.

4. **Why do internal forces not contribute to the virtual work done?** **[LO 11.2]**

   Internal forces develop always as a pair of equal and opposite components. Therefore, the net virtual work done by the internal forces vanishes.

5. **Define virtual-work principle for elastic bodies.** **[LO 11.3]**

   The virtual work done by all the external active forces (excluding gravitational and spring forces) on a mechanical system in equilibrium is equal to the corresponding change in the total elastic and gravitational potential energy of the system for any or all virtual displacements consistent with all the constraints.

6. **What is the difference between a stable equilibrium and neutral equilibrium?** **[LO 11.4]**

   When a slight displacement of a body in equilibrium causes an increase in its potential energy, it is in stable equilibrium. If on changing its position, its potential energy does not change, the equilibrium is neutral.