Mathematical Physics EP-203

Assignment - I

Review of Vector Analysis

Ques - 1

Gauss's Divergence Theorem: If E is a closed and bounded region in space whose boundary is piecewise smooth surface than. Let $\vec{F}$ be a vector function which is continuous and has continuous first order partial derivatives then,

$$\iiint_E \text{div}\,\vec{F}\,dV = \iint_S \vec{F}\cdot\hat{N}\,dS$$

, where $\hat{N}$ is the outward unit normal vector

Proof: Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ and α, β, γ be angles which outward unit normal vector $\hat{N}$ make with the x, y, z axes.

Then, $\hat{N} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$

∴ we have -

$$\iiint_E \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)dx\,dy\,dz = \iint_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)\,dS$$

∴ To prove divergence we need to prove,

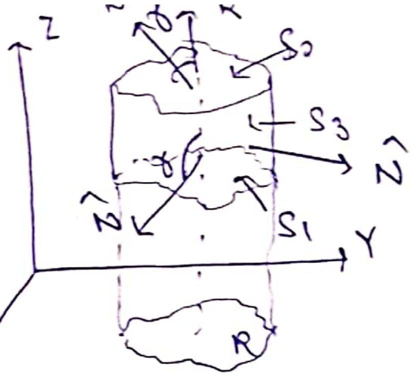
① $\displaystyle\iiint_E \frac{\partial F_1}{\partial x}\,dx\,dy\,dz = \iint_S F_1\cos\alpha\,dS$ ② $\displaystyle\iiint_E \frac{\partial F_2}{\partial y}\,dx\,dy\,dz = \iint_S F_2\cos\beta\,dS$

and ③ $\displaystyle\iiint_E \frac{\partial F_3}{\partial z}\,dx\,dy\,dz = \iint_S F_3\cos\gamma\,dS$

→ proving any one is sufficient as the remaining ones are similar just different axis.

→ Let E be a special region bounded by piecewise orientable surface S that has the property of straight line ∥ to z axis cutting at 2 points only.

Let 'R' be orthogonal projection of 'S' in x-y plane



S_1 : $z = h(x,y)$ $(x,y) \in R$;

S_2 : $z = g(x,y)$ $(x,y) \in R$;

& S_3 : $h(x,y) \leq z \leq g(x,y)$; $(x,y) \in R$

$$\therefore \iiint_E \frac{\partial F_3}{\partial z}\, dx\,dy\,dz = \iint_R \left(\int_{h(x,y)}^{g(x,y)} \frac{\partial F_3}{\partial z}\, dz \right)$$

$$= \iint_R [F_3(x,y,g(x,y)) - F_3(x,y,h(x,y))]\, dx\,dy$$

$$= \iint_R F_3(x,y,g)\, dx\,dy - \iint_R F_3(x,y,h)\, dx\,dy$$

On lateral portion S_3 of S we have $\gamma = \pi/2$ thus $\cos\gamma = 0$
Hence S_3 doesn't contribute to surface integral in (3)

Thus, $\iint_S F_3 \cos\gamma\, dS = \iint_{S_1} F_3 \cos\gamma\, dS + \iint_{S_2} F_3 \cos\gamma\, dS$

On S_1, $\hat{N}$ to S makes obtuse angle with $\hat{k}$ $\therefore dx\,dy = -\cos\gamma\, dS$
and on S_2, $\hat{N}$ to S makes acute angle thus $dx\,dy = \cos\gamma\, dS$

$$\therefore \boxed{\iiint_E \frac{\partial F_3}{\partial z}\, dx\,dy\,dz = \iint_{S_1} F_3 \cos\gamma\, dS + \iint_{S_2} F_3 \cos\gamma\, dS}$$

$\because \iint_R F_3(x,y,h)\, dx\,dy = -\iint_{S_1} F_3 \cos\gamma\, dS$

and $\iint_R F_3(x,y,g)\, dx\,dy = \iint_{S_1} F_3 \cos\gamma\, dS$

This is same as (3)

Hence proved same steps can be taken for, x and y axis ues-2

Stokes Theorem : If S is a piecewise smooth open surface bounded by a piecewise smooth closed curve C then for a vector $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ (continuous differentiable vector):

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{N}\,dS,$$

$\hat{N}$ is the outward normal at any point of S and C transversed in the direction.

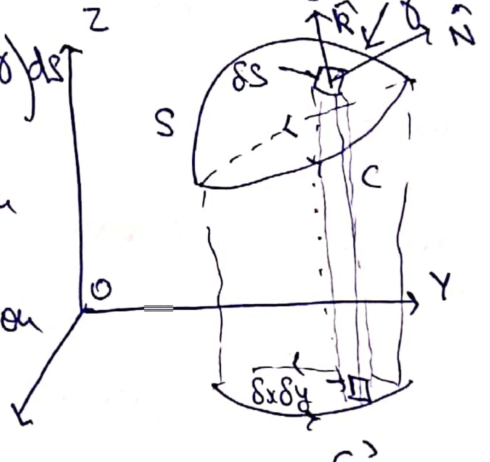
Proof: $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$ then:

$$\oint (F_1dx + F_2dy + F_3dz) = \iint_S \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\cos\gamma\right]dS$$

→ Result for surface S can be represented in forms :-
a) $z = g(x,y)$ b) $x = h(y,z)$ c) $y = k(z,x)$

like in Gauss Theorem we have $\hat{N} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$

first we prove : $\oint_C F_1dx = \iint_S \left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right)dS$



→ consider when equation of surface S is written in form $z = g(x,y)$ and projection of S on xy plane is region E. projection of C on xy plane is C'.

$$\oint_C F_1(x,y,z)\,dx = \oint_{C'} F_1[x,y,g(x,y)]\,dx$$

$$= \oint_{C'} F_1[x,y,g(x,y)]\,dx + 0\,dy$$

$$= -\iint_E \frac{\partial}{\partial y}F_1(x,y,g)\,dx\,dy$$

using Green's Theorem we have !-

$$\rightarrow \int_C F_1\,dx = -\iint_E \left[\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial g}{\partial y}\right] dx\,dy$$

→ The direction ratio's normal to the surface $z = g(x,y)$ are: $\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, -1$

hence,
$$\frac{\cos\alpha}{-\frac{\partial g}{\partial x}} = \frac{\cos\beta}{-\frac{\partial g}{\partial y}} = \frac{\cos\gamma}{1} \Rightarrow \boxed{\frac{\partial g}{\partial y} = \frac{-\cos\beta}{\cos\gamma}}$$

also $dx\,dy$, the projection of dS on xy-plane is $\cos\gamma\, dS$

$$\therefore \oint_C F_1\,dx = -\iint_S \left[\frac{\partial F_1}{\partial y} - \frac{\partial F_1}{\partial z}\frac{\cos\beta}{\cos\gamma}\right]\cos\gamma\, dS$$
$$= \iint_S \left[\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right] dS$$

Hence proved. Illy we can prove for others by taking surfaces 'h' and 'k' as mentioned.

Ques-3

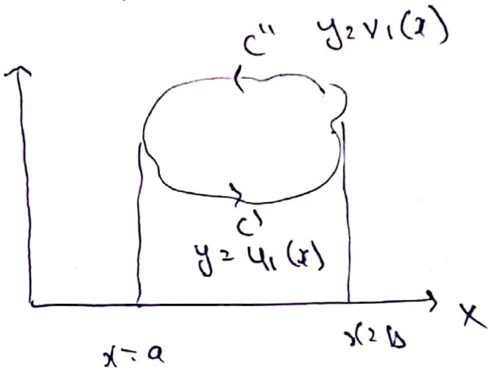
Green's Theorem: If E is a plane region in xy-plane bounded by a closed curve C. and $f(x,y)$, $g(x,y)$, $\frac{\partial f}{\partial y}$ and $\frac{\partial g}{\partial x}$ are continuous on E, then,

$$\iint_E \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy = \oint_C \left[f(x,y)\,dx + g(x,y)\,dy\right]$$

proof: we prove this by taking a special region E bounded by a closed curve C which is cut by any line ∥ to the axes at the most in 2 points.

s-3 cont..

Let E be representated by $u_1(x) \le y \le v_1(x)$, $a \le x \le b$



$$\iint_E \frac{\partial f}{\partial y} \, dx \, dy = \int_a^b \left[\int_{u_1(x)}^{v_1(x)} \frac{\partial f}{\partial y} \, dy \right] dx$$

$$= \int_a^b \left[f(x,y) \right]_{u_1(x)}^{v_1(x)} dx$$

$$= \int_a^b \left[f(x, v_1(x)) - f(x, u_1(x)) \right] dx = - \int_b^a f(x, v_1(x)) \, dx - \int_a^b f(x, u_1(x)) \, dx$$

C'' is representated by $y = v_1(x)$ illarly

C' is " " $y = u_1(x)$

$$\iint_E \frac{\partial f}{\partial y} \, dx \, dy = - \int_{C''} f(x,y) \, dx - \int_{C'} f(x,y) \, dx$$

$$= - \oint f(x,y) \, dx$$

in illas terms E represented by $u_2(y) \le x \le v_2(y)$ and $c \le y \le d$

will give for $\frac{\partial g}{\partial x}$

Hence proved.

Ques-4 Evaluate $\iint_S (x^3-yz)\,dydz - 2x^2y\,dzdx + z\,dxdy$ over coordinate planes and planes $x=y=z=a$

$\rightarrow \vec{f} = (x^3-yz)\vec{i} + (-2x^2y)\vec{j} + (3)\vec{k}$

using divergence theorem :

$$\iint_S f_1\,dydz + f_2\,dzdx + f_3\,dxdy = \iiint \text{div}\,\vec{f}\,dV$$

$$= \iiint_V \left(\frac{\partial}{\partial x}(x^3-yz) + \frac{\partial}{\partial y}(-2x^2y) + \frac{\partial}{\partial z}(3)\right) dV$$

$$= \iiint_V 3x^2 - 2x^2 + 1\,dV = \iiint_V x^2+1\,dV$$

$$= \int_{z=0}^{a}\int_{y=0}^{a}\int_{x=0}^{a} x^2+1\,dx\,dy\,dz$$

$$= \int_{z=0}^{a}\int_{y=0}^{a} \left(\frac{a^3}{3}+a\right) dy\,dz$$

$$= \frac{a^5}{3} + a$$

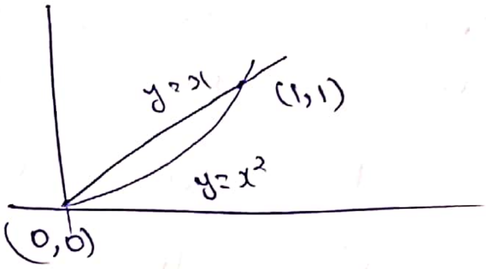
Hence

$$\boxed{\iint_S (x^3-yz)\,dydz - 2x^2y\,dzdx + z\,dxdy = \frac{a^5}{3} + a}$$

Ques-5 Verify Greens for $\int_C (xy+y^2)\,dx + x^2\,dy$

'c' is closed curve bounded by $y=x$ and $y=x^2$

Green's Theorem :-



$$\int_C f\,dx + g\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy$$

for line integral : $\int_C (xy+y^2)\,dx + x^2\,dy = A$

$$A = \underbrace{\int_{y=x^2} (xy+y^2)\,dx + x^2\,dy}_{} + \underbrace{\int_{y=x} (xy+y^2)\,dx + x^2\,dy}_{}$$

$y=x^2 \to dy = 2x\,dx$ $\qquad$ $y=x$ $\quad dy = dx$

$x=0, \to x=1$ $\qquad$ $x=1, \to x=0$

$$A = \int_0^1 x^3 + x^4\,dx + 2x^3\,dx + \int_1^0 x^2 + x^2 + x^2\,dx$$

$$= \frac{1}{4} + \frac{1}{5} + \frac{1}{2} + (-1) = \boxed{\frac{-1}{20} = A}$$

$f = xy + y^2$ and $g = x^2$

$$A = \iint_S (2x - x - 2y)\,dx\,dy = \int_{x=0}^{x=1}\int_{y=x^2}^{y=x} (x-2y)\,dx\,dy$$

$$= \int_0^1 \left[xy - y^2\right]_{y=x^2}^{y=x} dx = -\int_0^1 x^3 - x^4\,dx$$

$$\boxed{A = -\frac{1}{4} + \frac{1}{5} = \frac{-1}{20}}$$

Hence verified.

Ques-6 using Greens in space evaluate $\iint_S 4xz\,dydz - y^2 dzdx + yz\,dxdy$

where S is surface of cube bounded by $x=0=y=z$, $x=y=z=1$.

→ Green's in plane is special case of Stoke's theorem. Clearly Green's in space is special case of Gauss's theorem ∴

$$\vec{F} = 4xz\,\hat{i} + -y^2\,\hat{j} + yz\,\hat{k}$$

$$\text{div}\,\vec{F} = 4z - 2y + y = 4z - y$$

∴ by Divergence theorem!

$$\iint_S \vec{F}.\hat{N}\,dS = \iiint_V \text{div}\,F\,dV$$

$$= \int_0^1\int_0^1\int_0^1 4z - y \; dx\,dy\,dz$$

$$= \int_0^1\int_0^1 4z - y \; dy\,dz$$

$$= \int_0^1 4z - \frac{1}{2} \; dz$$

$$= \left[2z^2 - \frac{z}{2}\right]_0^1$$

$$= 2 - \frac{1}{2}$$

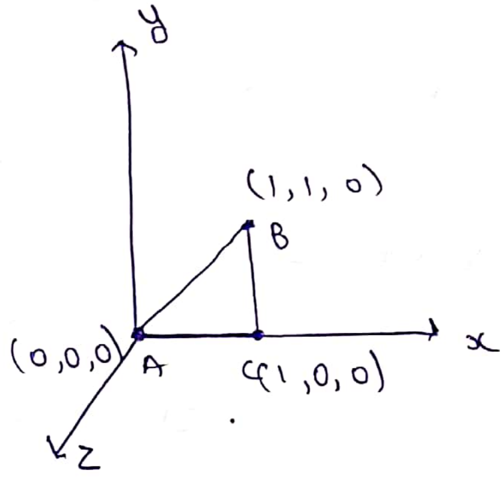
$$\boxed{\iint_S \vec{F}.\hat{N}\,dS = \frac{3}{2}}$$

Thus, $\boxed{\iint_S 4xz\,dydz - y^2 dzdx + yz\,dxdy = \frac{3}{2}}$

over $x=0, y=0, z=0, x=1, y=1, z=1$

Ques-7 Evaluate $\int_C \vec{f}\cdot d\vec{r}$ by stokes $\vec{f} = y^2\hat{i} + x^2\hat{j} - (x+2)\hat{k}$

C is the boundary of Δ with vertices at $(0,0,0)$, $(1,0,0)$ and $(1,1,0)$



Stokes:

$$\int_C \vec{f}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{N}\,dS$$

$$\text{curl}\,F = \nabla\times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+2) \end{vmatrix}$$

$$\text{curl}\,F = -\hat{j}(-1) + \hat{k}(2x - 2y) = \hat{j} + (2x-2y)\hat{k}$$

Δ lies in x-y plane $\therefore \hat{n} = \hat{k}$

$$\therefore \text{curl}\,\vec{F}\cdot\vec{n} = 2(x-y)$$

equation of AB $= y = x$

$\therefore$ by stokes:

$$\int_C \vec{f}\cdot d\vec{r} = \int_{x=0}^{x=1}\int_{y=0}^{y=x} 2(x-y)\,dx\,dy$$

$$= \int_0^1 2xy - y^2\,dx$$

$$= \int_0^1 x^2\,dx$$

$$\boxed{\oint_C \vec{f}\cdot d\vec{r} = \frac{1}{3}}$$

Ques-8 Find curl F

$\vec{F} = \nabla (x^3 + y^3 + z^3 - 3xyz)$

$\vec{F} = (3x^2 - 3yz)\hat{i} + (3y^2 - 3xz)\hat{j} + (3z^2 - 3yx)\hat{k}$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 - 3yz & 3y^2 - 3xz & 3z^2 - 3yx \end{vmatrix}$$

$= (\hat{i})(-3x + 3x) + (-\hat{j})(-3y + 3y) + (\hat{k})(-3z + 3z)$

$\boxed{\nabla \times \vec{F} = \vec{0}}$

Ques-9 Find div F

$F = \nabla (x^3 + y^3 + z^3 - 3xyz)$

$= 3(x^2 - yz)\hat{i} + 3(y^2 - xz)\hat{j} + (3z^2 - 3xy)\hat{k}$

$\nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(3(x^2 - yz)\hat{i} + 3(y^2 - xz)\hat{j} + \hat{k}3(z^2 - xy)\right)$

$= 6x\hat{i} + 6y\hat{j} + 6z\hat{k}$

$\boxed{\nabla \cdot \vec{F} = 6(x\hat{i} + y\hat{j} + z\hat{k})}$

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Ques-10 show $\vec{v} = (yz)\hat{i} + (zx)\hat{j} + (xy)\hat{k}$ is irrotational.

a vector is said to be irrotational if curl of that vector is 0.

∴ to prove → $\nabla \times \vec{v} = \vec{0}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix} = (\hat{i})(x-x) - (\hat{j})(y-y) + \hat{k}(z-z)$$

$$\boxed{\nabla \times \vec{v} = \vec{0}}$$

Thus $\vec{v}$ is irrotational

Ques-11 $\vec{v} = \dfrac{x\hat{i} + y\hat{j} + 2\hat{k}}{\sqrt{x^2+y^2+2^2}}$ find curl $\vec{v}$ and div $\vec{v}$

$$\boxed{\text{curl } \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{|\vec{v}|} & \frac{y}{|\vec{v}|} & \frac{2}{|\vec{v}|} \end{vmatrix} = \vec{0}}$$

$$\text{div } \vec{v} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(\frac{x\hat{i} + y\hat{j} + 2\hat{k}}{|\vec{v}|}\right)$$

$$\boxed{\text{div } \vec{v} = \frac{2}{\sqrt{x^2+y^2+4}}}$$

Ques-12 Prove $(y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}$

for vector to be solenoidal $\operatorname{div} \vec{F} = 0$:

$\rightarrow$ $\operatorname{div} \vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot (\vec{F})$

$= -2 + 2x + -2x + 2$

$\operatorname{div} \vec{F} = 0$

$\therefore$ given vector is solenoidal

For a vector to be irrotation $\operatorname{curl} \vec{F} = 0$

$\rightarrow$ $\operatorname{curl} \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial i} & \frac{\partial}{\partial j} & \frac{\partial}{\partial k} \\ y^2 - z^2 + 3yz - 2x & 3xz + 2xy & 3xy - 2xz + 2z \end{vmatrix}$

$= (\hat{i})(3x - 3x) - (\hat{j})(3y - 2z - (-2z + 3y)) + (\hat{k})(3z + 2y - (2y + 3z))$

$\boxed{\operatorname{curl} \vec{F} = 0}$

Hence $\vec{F}$ is both solenoidal and irrotational.

Ques-13 Show $\vec{x} = \vec{a} - 2\vec{b} + 3\vec{c}$, $\vec{y} = -2\vec{a} - 3\vec{b} + 4\vec{c}$
$\vec{z} = -\vec{b} + 2\vec{c}$

for 3 vectors to be coplanar $\therefore$ scalar triple product must be 0.

i.e. $\vec{x}\cdot[\vec{y}\times\vec{z}] = 0$

$$[\vec{x}\ \vec{y}\ \vec{z}] = \begin{vmatrix} \vec{a} & -2\vec{b} & 3\vec{c} \\ -2\vec{a} & -3\vec{b} & 4\vec{c} \\ 0 & -\vec{b} & 2\vec{c} \end{vmatrix}$$

$$= \vec{a}(-6\vec{b}\vec{c} + 4\vec{b}\vec{c}) + 2\vec{b}(-4\vec{a}\vec{c}) + 3\vec{c}(+2\vec{a}\vec{b})$$

$$= 2\vec{a}\vec{b}\vec{c} - 8\vec{a}\vec{b}\vec{c} + 6\vec{a}\vec{b}\vec{c}$$

$$\boxed{[\vec{x}\ \vec{y}\ \vec{z}] = 0}$$

Ques-14 $\vec{a} = 2\hat{i} - 3\hat{j}$ $\vec{b} = \hat{i} + \hat{j} - \hat{k}$, $\vec{c} = 3\hat{i} - \hat{k}$
construct $\vec{v}$ $\perp$ to $\vec{a}$ and $\vec{b}$ and having unit scalar product with $\vec{c}$.

$\rightarrow$ vector orthogonal to $\vec{a}$ and $\vec{b}$:

$$\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & 0 \\ 1 & 1 & -1 \end{vmatrix} = (\hat{i})(3) + (\hat{j})(2) + \hat{k}(5)$$

$$\boxed{\vec{v} = 3\hat{i} + 2\hat{j} + 5\hat{k}}$$

for unit scalar product checking $\vec{v}.\vec{c}$:

$\vec{v}.\vec{c} = 9 - 5 = 4$

Thus $\boxed{\vec{v} = \frac{3}{4}\hat{i} + \frac{1}{2}\hat{j} + \frac{5}{4}\hat{k}}$ is vector

orthogonal to $\vec{a}$ and $\vec{b}$ and has unit scalar product with $\vec{c}$.

Ques-15 A is a vector point function ϕ is a scalar point function.

a) Prove (i) div Curl A = 0

$\Rightarrow \nabla . \nabla \times A$. let $A = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$

$$\Rightarrow \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) . \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$\Rightarrow \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) . \left((\hat{i})\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) - (\hat{j})\left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right) + (\hat{k})\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\right)$$

$$\Rightarrow \frac{\partial^2 A_3}{\partial x \partial y} - \frac{\partial^2 A_2}{\partial z \partial x} + \frac{\partial^2 A_1}{\partial y \partial z} - \frac{\partial^2 A_3}{\partial y \partial x} + \frac{\partial^2 A_2}{\partial x \partial z} - \frac{\partial^2 A_1}{\partial y \partial z}$$

$\Rightarrow 0$ hence, $\boxed{\text{div curl } A = 0}$

(ii) curl grad $\phi = 0$

$\Rightarrow \nabla \times \nabla\phi = 0$ $\quad \nabla\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix} = (\hat{i})\left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial y \partial z}\right) - (\hat{j})\left(\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial z \partial x}\right) + (\hat{k})\left(\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x}\right)$$

$\Rightarrow 0 \Rightarrow$ Hence $\boxed{\text{curl grad } \phi = 0}$

Ques-15 ϕ and ψ are scalar point functions.

b) Prove that $\text{curl}(\phi \text{ grad } \psi) = \nabla\phi \times \nabla\psi = -\text{curl}(\psi \text{ grad } \phi)$

$\text{curl}(\phi \text{ grad } \psi)$: $\nabla \times (\phi \nabla \psi)$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi\frac{\partial \psi}{\partial x} & \phi\frac{\partial \psi}{\partial y} & \phi\frac{\partial \psi}{\partial z} \end{vmatrix} = (\hat{i})\left(\frac{\partial \phi}{\partial y}\frac{\partial \psi}{\partial z} + \cancel{\phi\frac{\partial^2 \psi}{\partial y \partial z}} - \frac{\partial \phi}{\partial z}\frac{\partial \psi}{\partial y} - \cancel{\phi\frac{\partial^2 \psi}{\partial z \partial y}}\right) - \hat{j}$$

$$\left(\cancel{\phi\frac{\partial^2 \psi}{\partial z \partial x}} + \frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial z} - \cancel{\phi\frac{\partial^2 \psi}{\partial x \partial z}} - \frac{\partial \phi}{\partial z}\frac{\partial \psi}{\partial x}\right) + \hat{k}\left(\cancel{\phi\frac{\partial^2 \psi}{\partial x \partial y}} + \frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial y} - \cancel{\phi\frac{\partial^2 \psi}{\partial y \partial x}} - \frac{\partial \phi}{\partial y}\frac{\partial \psi}{\partial x}\right)$$

$$\therefore \boxed{\text{curl}(\phi \text{ grad } \psi) = (\hat{i})\left(\frac{\partial \phi}{\partial y}\frac{\partial \psi}{\partial z} - \frac{\partial \phi}{\partial z}\frac{\partial \psi}{\partial y}\right) + (-\hat{j})\left(\frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial z} - \frac{\partial \phi}{\partial z}\frac{\partial \psi}{\partial x}\right) + (\hat{k})\left(\frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial y} - \frac{\partial \phi}{\partial y}\frac{\partial \psi}{\partial x}\right)} \quad -(1)$$

$\nabla\phi \times \nabla\psi$:

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \\ \frac{\partial \psi}{\partial x} & \frac{\partial \psi}{\partial y} & \frac{\partial \psi}{\partial z} \end{vmatrix} = (\hat{i})\left(\frac{\partial \phi}{\partial y}\frac{\partial \psi}{\partial z} - \frac{\partial \phi}{\partial z}\frac{\partial \psi}{\partial y}\right) + (-\hat{j})\left(\frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial z} - \frac{\partial \phi}{\partial z}\frac{\partial \psi}{\partial x}\right) + (\hat{k})\left(\frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial y} - \frac{\partial \phi}{\partial y}\frac{\partial \psi}{\partial x}\right)$$

Hence $\boxed{\text{curl}(\phi \text{ grad } \psi) = \nabla\phi \times \nabla\psi}$

replacing $\phi \to \psi$ and $\psi \to \phi$ gives us $\text{curl}(\psi \text{ grad } \phi)$ in (1) :-

$$\text{curl}(\psi \text{ grad } \phi) = (\hat{i})\left(\frac{\partial \phi}{\partial z}\frac{\partial \psi}{\partial y} - \frac{\partial \phi}{\partial y}\frac{\partial \psi}{\partial z}\right) + (-\hat{j})\left(\frac{\partial \phi}{\partial z}\frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial z}\frac{\partial \phi}{\partial x}\right) + (\hat{k})\left(\frac{\partial \phi}{\partial y}\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial y}\right)$$

clearly $\text{curl}(\psi \text{ grad } \phi) = -\text{curl}(\phi \text{ grad } \psi)$

thus $\boxed{\text{curl}(\phi \text{ grad } \psi) = \nabla\phi \times \nabla\psi = -\text{curl}(\psi \text{ grad } \phi)}$

Ques-16 Evaluate $\iint_S (x^3\,dy\,dz + y^3\,dz\,dx + z^3\,dx\,dy)$

Ques-17 stokes verify for : $\vec{A} = (2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$ over upper half of the surface of sphere $x^2+y^2+z^2=1$

Stokes: $\oint \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{N}\,dS$

$$\text{curl}\,F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix} = (\hat{i})(-2yz+2yz) - (\hat{j})(0-0) + (\hat{k})(0-(-1)) = \hat{k}$$

assuming circle in x-y plane and upper half of sphere in $\hat{k}$. ⇒ thus $\text{curl}\,\vec{F}\cdot\hat{N} = \hat{k}\cdot\hat{N}$

also $dS = \frac{dx\,dy}{|\hat{k}\cdot\hat{N}|}$ ⇒ $\iint_S \text{curl}\,\vec{F}\cdot\hat{N}\,dS = 4\iint \frac{\cancel{\hat{k}\cdot\hat{N}}\,dx\,dy}{\cancel{|\hat{k}\cdot\hat{N}|}}$

$$\iint_S \text{curl}\,\vec{F}\cdot\hat{N}\,dS = 4\int_{x=0}^{1}\int_{y=0}^{y=\sqrt{1-x^2}} dx\,dy = 4\int_0^1 \sqrt{1-x^2}\,dx$$

$$= 4\left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}x\right]_0^1 = (2)\frac{\pi}{2} = \pi$$

$$\Rightarrow \boxed{\iint_S \text{curl}\,\vec{F}\cdot\hat{N}\,dS = \pi}$$

$$\oint \vec{F}\cdot d\vec{r} = \oint_C (2x-y)\,dx - yz^2\,dy - y^2z\,dz$$

taking parametric : $x = \cos\theta$ $y = \sin\theta$ and $z = 0$ $\theta \in [0, 2\pi]$

$$= \int_0^{2\pi} (2\cos\theta - \sin\theta)(-\sin\theta)\,d\theta = \int_0^{2\pi} -\sin 2\theta + \sin^2\theta\,d\theta$$

$$\boxed{\oint \vec{F}\cdot d\vec{r} = \left[\frac{\cos 2\theta}{2} + \frac{\theta}{2} - \frac{\sin 2\theta}{2}\right]_0^{2\pi} = \pi}$$

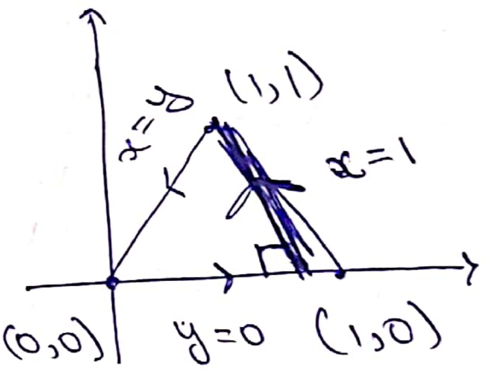
Thus verified

Ques - 18 evaluate $\oint_C x^2y\,dx + x^2\,dy$ where C is bound described counter clockwise of Δ (0,0) (1,0) (1,1).

using Green's : $\iint_E \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy = \oint_C [f(x,y)\,dx + g(x,y)\,dy]$

$f = x^2y$ and $g = x^2$

$$\rightarrow \oint_C x^2y\,dx + x^2\,dy = \iint_E (2x - x^2)\,dx\,dy$$



$$= \int_{y=1}^{y=0}\int_{x=1}^{x=y} (2x - x^2)\,dx\,dy$$

$$= \int_1^0 \left[x^2 - \frac{x^3}{3}\right]_1^y dy$$

$$= \int_1^0 y^2 - \frac{y^3}{3} - \frac{2}{3}\,dy$$

$$= \left[\frac{y^3}{3} - \frac{y^4}{12} - \frac{2}{3}y\right]_1^0$$

$$\boxed{\oint x^2y\,dx + x^2dy = \frac{5}{12}}$$

ues- 19

Equation of continuity in fluids :-

$$\text{div}(\rho u) + \frac{\partial \rho}{\partial t} = 0$$

$\rho \to$ fluid density

$t \to$ time $\quad u \to$ flow velocity vector.

given ; fluid is incompressible $\Rightarrow$ ρ is constant

thus $\nabla . \rho u = 0 \Rightarrow \nabla . u = 0$

thus $\because \nabla . u = 0$

u can be expressed as the curl of any function

$\because \nabla . (\nabla \times \phi) = 0$

$\therefore \nabla . u = \nabla . (\nabla \times \phi)$

$$\boxed{\text{thus 'u' can be expressed as } \nabla \times \phi \text{ or curl } \phi}$$

Ques-20 'ϱ' denotes charge density and J current density due to charges, show $\frac{\partial \varrho}{\partial t} + \text{div } J = 0$ expresses conservation of charge.

we know from Ampere-Maxwell's law:

$$\text{curl } B = \nabla \times B = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

or $$\nabla \times H = J + \frac{\partial D}{\partial t}$$

where H is the magnetic field, J is free current density and D is electric displacement field $\boxed{D = \epsilon_0 E}$

taking divergence on both sides :-

$$\nabla \cdot (\nabla \times H) = \text{div } J + \frac{\partial (\nabla \cdot D)}{\partial t}$$

div curl is 0 :

$$\text{div } J + \frac{\partial (\nabla \cdot D)}{\partial t} = 0$$

$$\Rightarrow \boxed{\text{div } J + \frac{\partial \varrho}{\partial t} = 0} \quad \text{using (1)}$$

→ here if charge is moving out of a differential volume (i.e. div J = +ve) then amount of charge within that volume is going to decrease, so rate of charge density is -ve.

thus ~~continuity~~ this equation amounts to conservation of charge.

By Gauss's law:-

$$\oiint_S E \cdot dS = \frac{Q}{\epsilon_0}$$

by divergence theorem:

$$\iiint_V \nabla \cdot E \, dV = \frac{Q}{\epsilon_0}$$

for any volume 'V' containing charge 'Q'.

$$\frac{Q}{\epsilon_0} = \frac{1}{\epsilon_0} \iiint_V \underbrace{\varrho}_{\text{charge density}} dV$$

$$\therefore \iiint_V \nabla \cdot E \, dV = \iiint_V \frac{\varrho}{\epsilon_0} dV$$

$$\Rightarrow \nabla \cdot E = \frac{\varrho}{\epsilon_0}$$

$$\Rightarrow \boxed{\nabla \cdot D = \varrho} \quad (1)$$