Total acceleration

$$a = \sqrt{(a_x)^2 + (a_y)^2}$$

$$a = \sqrt{(0.365)^2 + (0.341)^2}$$

$$a = 0.5 \text{ m/s}^2 \quad \textit{Ans.}$$

Direction of acceleration a,

$$\phi = \tan^{-1}\frac{a_y}{a_x} = \frac{0.341}{0.365} = 42.97° \quad \textit{Ans.}$$

Components in tangential and normal directions.
Velocity

$$v_n = 0$$

$$v_t = v = 10 \text{ m/s}$$

Acceleration. Component of the acceleration in the direction of normal to the path.

$$a_n = \frac{v^2}{r} = \frac{10 \times 10}{200}$$

$$a_n = 0.5 \text{ m/s}^2 \quad \textit{Ans.}$$

Component of the acceleration in the direction tangent to the path

$$a_t = \frac{dv}{dt} = 0 \quad \textit{Ans.}$$

Angular position θ of the car after 15 seconds

$$= \omega t = \frac{1}{20} \times \frac{180}{\pi} \times 15$$

$$\theta = 42.97°$$

Note: The components of acceleration are shown in the Fig. 15.9.

The total acceleration a of the car acts along the normal to the path as shown, $\phi = \theta = 42.97°$, proves that resultant of a_x and a_y lies along the normal to the path.

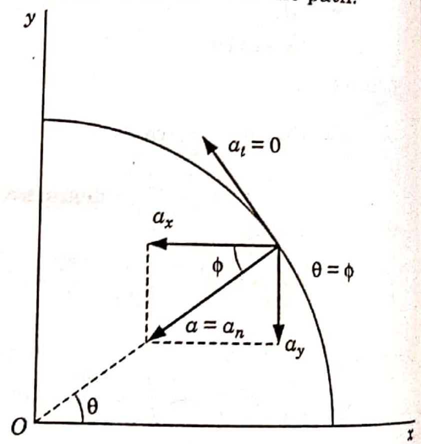

Fig. 15.9

15.4(B) COMPONENTS OF MOTION: RADIAL AND TRANSVERSE COMPONENTS

We saw that the curvilinear motion of a particle can be expressed in terms of rectangular components and also in terms of components directed along the tangent and the normal to the path of the particle.

In certain problems the position of a particle is more conveniently described by its polar coordinates. In that case then it is much simpler to resolve the velocity and acceleration of the particle into components which are parallel and perpendicular to the position vector **r** *of the particle. These components are called radial and transverse components.*

Consider a collar P sliding outward along a straight rod OA which itself is rotating about the fixed point O. It is much convenient to define the position of the collar P (Fig. 15.10) *at any instant* in terms of its distance r from the point O and the angular position θ of the rod OA with respect to some fixed axis OX. The polar coordinates of the point P thus are (r, θ).

It can be shown that the radial and transverse components of the velocity are,

$v_r = \dot{r}$ (Radial component of velocity is directed along the position vector **r**)

$v_\theta = r\dot{\theta}$ (Transverse component of velocity is directed along the normal to the position vector **r**)

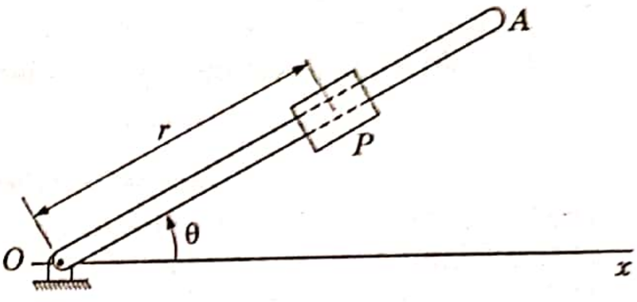

Fig. 15.10

Total velocity **v** [Fig. 15.11(*a*)].

$$\mathbf{v} = \mathbf{v}_r + \mathbf{v}_\theta \text{ (vector sum)}$$

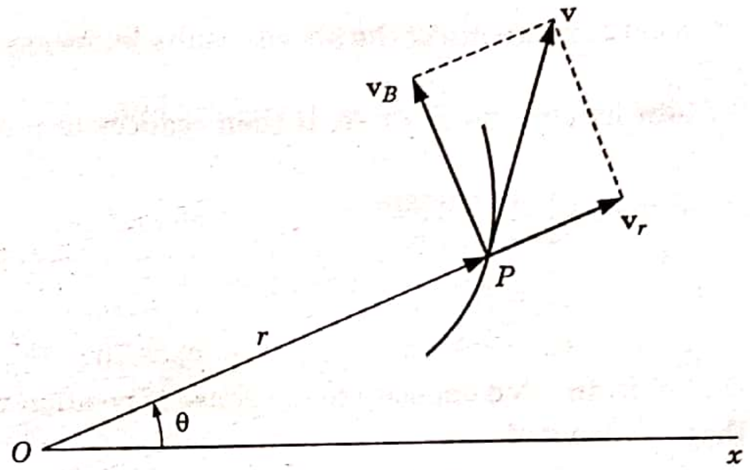

Fig. 15.11(*a*)

The radial and transverse components of acceleration are,

$a_r = \ddot{r} - r(\dot{\theta})^2$ (Radial component of acceleration is directed along the position vector **r**)

$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$ (Transverse component of acceleration is directed along the normal to the position vector **r**)

Total acceleration **a** [Fig. 15.11(*b*)]

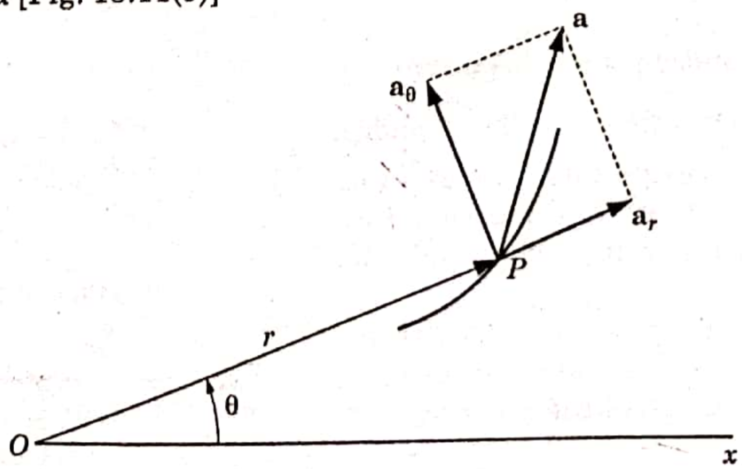

Fig. 15.11 (*b*)

$$\mathbf{a} = \mathbf{a}_r + \mathbf{a}_\theta \text{ (vector sum)}$$

The components of velocity and acceleration are related as

$$a_r = \dot{v}_r - v_\theta\dot{\theta}$$

$$a_\theta = \dot{v}_\theta + v_r\dot{\theta}$$

From the above relations it can be seen that is general,

a_r is not equal to $\dot{v}_r$.

and

a_θ is not equal $\dot{v}_\theta$.

It should be noted that radial components of velocity and acceleration are taken to be positive in the same sense as of position vector **r**.

Transverse components of velocity and acceleration are taken to be positive if pointing towards the increasing values of θ.

To understand the physical significance of the above results let us assume the following two situations.

(*i*) That, r is of constant length and θ varies. It then reduces to rotation along a circular path.

$$r = \text{constant}$$

$$\dot{r} = \ddot{r} = 0$$

$$v_r = 0, \qquad v_\theta = r\dot{\theta}$$

$$a_r = -r(\dot{\theta})^2 \qquad a_\theta = r\ddot{\theta}$$

(–ve sign indicates that a_r is directed opposite to the sense of position vector **r** or towards O)

It may be recalled that we denoted

$$\dot{\theta} = \omega \text{ and } \ddot{\theta} = \alpha$$

(*ii*) That, only r varies and θ is constant.

It then reduces a rectilinear motion along a fixed direction θ.

$$\theta = \text{constant}$$

$$\dot{\theta} = \ddot{\theta} = 0$$

$$v_r = \dot{r} \qquad v_\theta = 0$$

$$a_r = \ddot{r} \qquad a_\theta = 0$$

Example 15.6. The rotation of rod OA is defined by the relation $\theta = \frac{\pi}{2}(4t - 3t^2)$. A collar P slides along this rod in such a way that its distance from O is given by $r = 1.25t^2 - 0.9t^3$. In these relations, θ is expressed in radians, r in metres and t in seconds.

Determine (*i*) the velocity of the collar (*ii*) the total acceleration of the collar and (*iii*) the acceleration of the collar relative to the rod when t = 1 s.

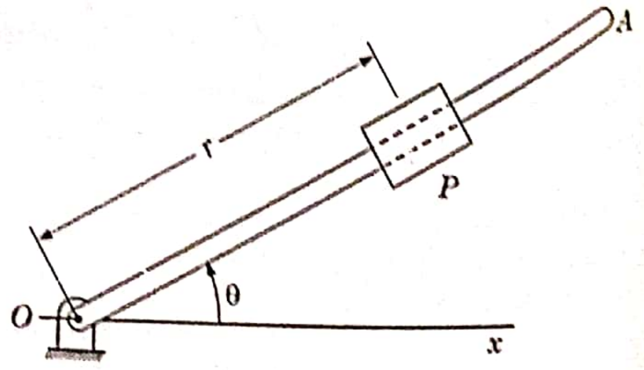

Fig. 15.12 (*a*)

Solution: Equations of motions are

$$\theta = \frac{\pi}{2}(4t - 3t^2)$$

and
$$r = 1.25t^2 - 0.9t^3$$

Evaluating θ, $\dot{\theta}$, $\ddot{\theta}$, $\dot{r}$ and $\ddot{r}$ at $t = 1$ s

$$\theta = \frac{\pi}{2}(4-3) = \frac{\pi}{2}\text{rad} \qquad r = 1.25 - 0.9 = 0.35 \text{ m}$$

$$\dot{\theta} = \frac{\pi}{2}(4 - 6t) \qquad \dot{r} = 1.25(2t) - 0.9(3)t^2$$

$$\dot{\theta} = \frac{\pi}{2}(4-6) = \pi \text{ rad/s} \qquad \dot{r} = 2.5 - 2.7 = -0.2 \text{ m/s}$$

$$\ddot{\theta} = \frac{\pi}{2}(0-6) \qquad \ddot{r} = 2.5 - 2.7(2)\,t$$

$$\ddot{\theta} = -3\pi \text{ rad/s}^2 \qquad \ddot{r} = 2.5 - 5.4 = -2.9 \text{ m/s}^2$$

(*i*) Velocity of collar P

$$v_r = \dot{r} = -0.2 \text{ m/s} \qquad v_\theta = r\dot{\theta} = 0.35 \times (-\pi) = -1.1 \text{ m/s}$$

$$v = \sqrt{(v_r)^2 + (v_\theta)^2} = 1.118 \text{ m/s}$$

$$\tan\alpha = \frac{V_\theta}{V_r} = \frac{-1.1}{-0.2}$$

$$\alpha = 79.70°$$

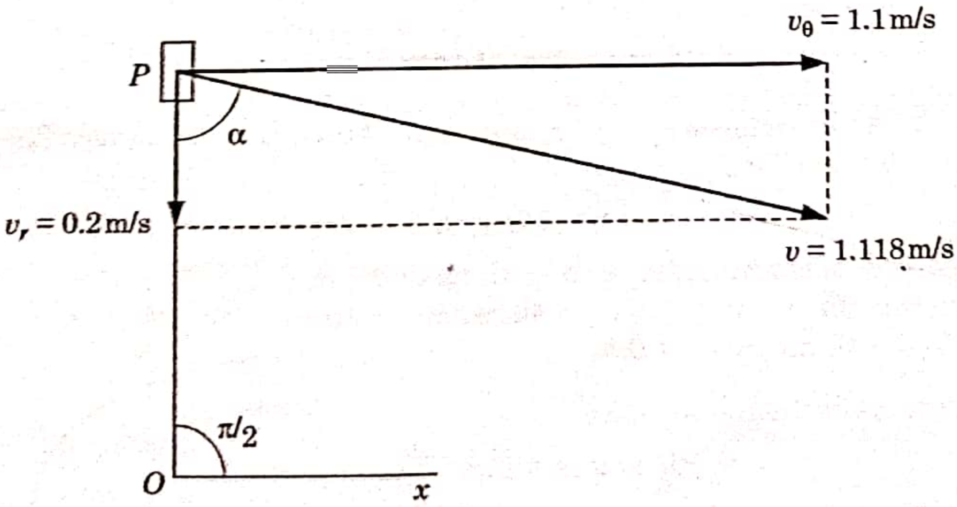

Fig. 15.12 (*b*)

(*ii*) Acceleration of the collar P

$$a_r = \ddot{r} - r(\dot{\theta})^2 \qquad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

$$a_r = -2.9 - 0.35\,(-\pi)^2 \qquad a_\theta = 0.35(-3\pi) + 2(-0.2)(-\pi)$$

$$a_r = -6.35 \text{ m/s}^2 \qquad a_\theta = -2.04 \text{ m/s}^2$$

$$a = \sqrt{(a_r)^2 + (a_\theta)^2} = 6.667 \text{ m/s}^2$$

$$\tan\beta = \frac{a_\theta}{a_r} = \frac{-2.04}{-6.35}$$

$$\beta = 17.8°.$$

(*iii*) Acceleration of the collar P relative to the rod.

Motion of the collar with respect to the rod is rectilinear. Hence

$$a_{r/OA} = \ddot{r} = -2.9 \text{ m/s (towards } O)$$

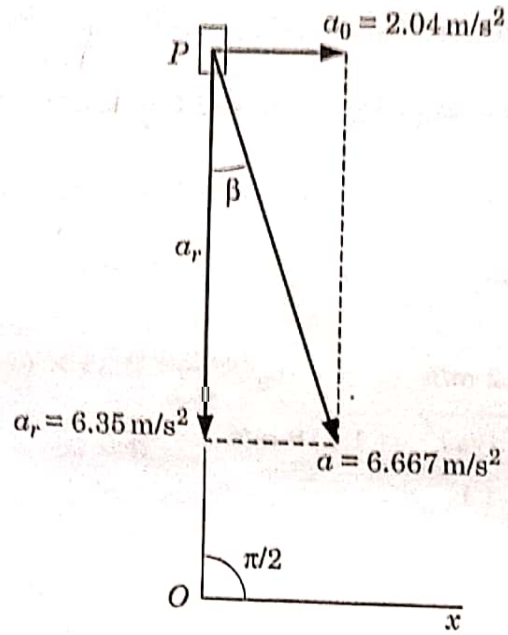

Fig. 15.12 (c)

Example 15.7 A particle moves along the spiral as shown [Fig. 15.13(*a*)]. The motion of the particle is defined by relations

$$r = 10t \text{ and } \theta = 2\pi t$$

where r is expressed in centimetres, θ in radians and t in seconds. Determine the velocity and acceleration of the particle when (*i*) $t = 0$ and (*ii*) $t = 0.3$s

Solution: Equations of the motions are

$$r = 10t \text{ and } \theta = 2\pi t$$

Evaluating $\dot{\theta}$, $\ddot{\theta}$, θ, r, $\dot{r}$ and $\ddot{r}$ at

$t = 0$ and $t = 0.3$ s

$\theta = 2\pi t \qquad r = 10t$

$\dot{\theta} = 2\pi \qquad \dot{r} = 10$

$\ddot{\theta} = 0 \qquad \ddot{r} = 0$

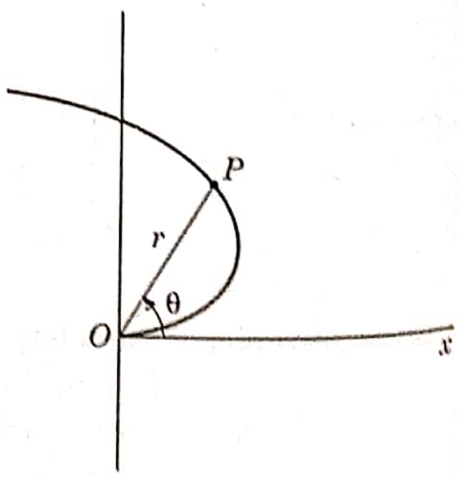

Fig. 15.13 (a)

At, $t = 0$

$\theta = 0$ $\qquad r = 0$

$\dot{\theta} = 2\pi$ rad/s $\qquad \dot{r} = 10$ cm/s

$\ddot{\theta} = 0$ $\qquad \ddot{r} = 0$

At, $t = 0.3$ s

$\theta = 2\pi \times 0.3 = 1.88$ rad $\qquad r = 10 \times 0.3 = 3$ cm

$\dot{\theta} = 2\pi$ rad/s $\qquad \dot{r} = 10 = 10$ cm/s

$\ddot{\theta} = 0$ $\qquad \ddot{r} = 0$

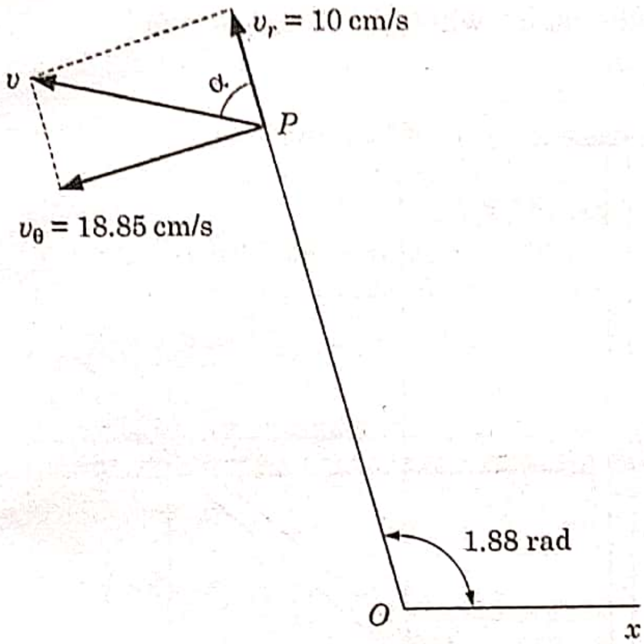

Fig. 15.13 (b)

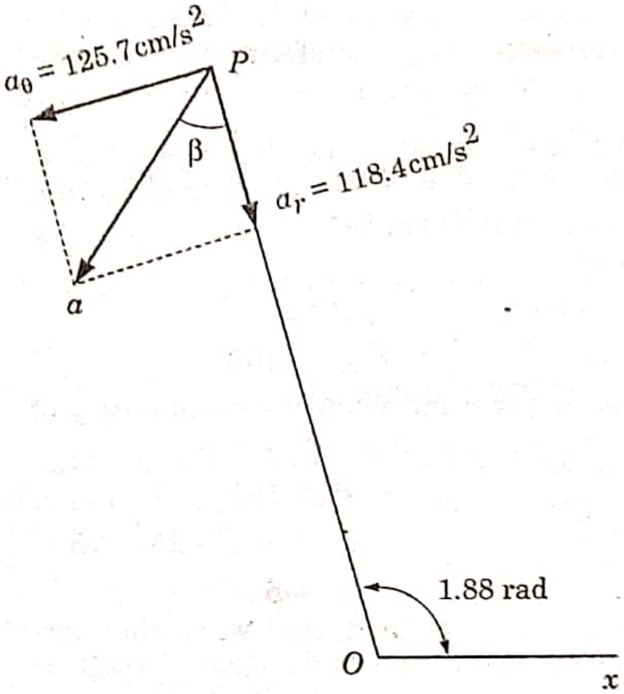

Fig. 15.13 (c)

(*i*) Velocity

$v_r = \dot{r}$ $\qquad v_\theta = r\dot{\theta}$

$v_r = 10$ cm/s $\qquad v_\theta = 0$

$v = \sqrt{(v_r)^2 + (v_\theta)^2} = 10$ cm/s (along v_r)

Acceleration

$a_r = \ddot{r} - r(\dot{\theta})^2$ $\qquad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$

$a_r = 0 - 0(2\pi)^2$ $\qquad a_\theta = 0(0) + 2(10)(2\pi)$

$a_r = 0$ $\qquad a_\theta = +\ 125.6$ cm/s^2

$a = \sqrt{(a_r)^2 + (a_\theta)^2} = 125.6$ cm/s^2 (along a_θ)

(*ii*) Velocity

$v_r = \dot{r}$ $\qquad v_\theta = r\dot{\theta} = 3 \times 2\pi$

$v_r = 10$ cm/s $\qquad v_\theta = 18.85$ cm/s

$v = \sqrt{(v_r)^2 + (v_\theta)^2} = \sqrt{(10)^2 + (18.85)^2}$

$v = 21.34$ cm/s

Acceleration

$$a_r = \ddot{r} - r(\dot{\theta})^2 \qquad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

$$a_r = 0 - 3(2\pi)^2 \qquad a_\theta = 3(0) + 2(10)2\pi$$

$$a_r = -118.4 \text{ cm/s}^2 \qquad a_\theta = +125.7 \text{ cm/s}^2$$

$$a = \sqrt{(a_r)^2 + (a_\theta)^2} = \sqrt{(-188.4)^2 + (125.7)^2}$$

$$a = 172.65 \text{ cm/s}^2.$$

Example 15.8. Pivoted link OA carries a pin P whose position is controlled by the horizontal slotted bar which can slide along a fixed vertical bar (Fig. 15.14). What are the x-components of the velocity and acceleration of the pin P at the instant when $y = 5$ cm and the slotted bar is moving upward at a constant velocity of 10 cm/s?

Solution: Pin P is moving in a circular path $x^2 + y^2 = r^2 = 100$ guided by the rotating link OA and the slotted bar moving upward.

Equation of the path is

$$x^2 + y^2 = 100$$

At the instant when the coordinate y of the pin is 5 cm, x coordinate is

$$x^2 + y^2 = 100$$

$$x^2 = 100 - 25 = 75$$

$$x = 8.66 \text{ cm.}$$

Further, it is given that when the coordinate of P are (8.66, 5) its upward acceleration $\dfrac{dy}{dt} = \dot{y} = 10$ cm/s and $\dfrac{d^2y}{dt^2} = \ddot{y} = 0.$

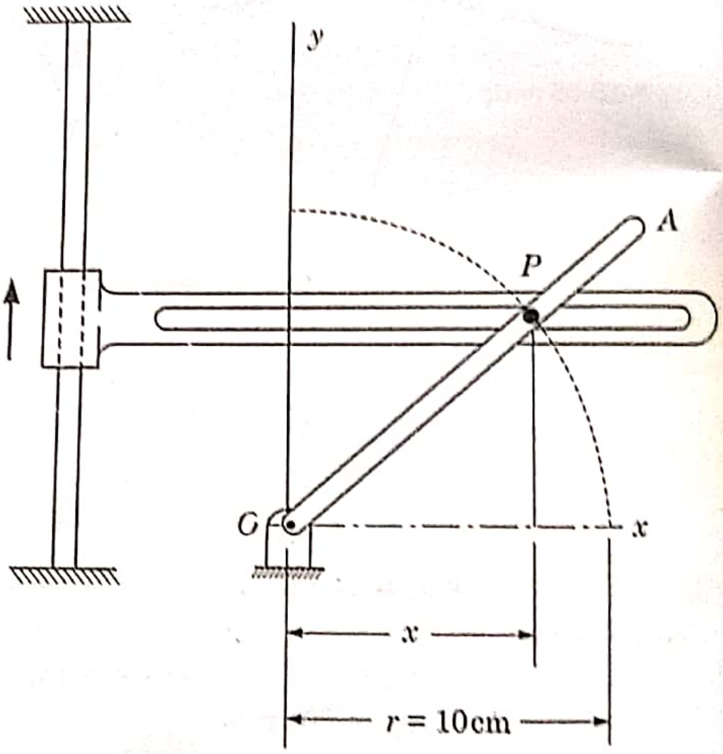

Fig. 15.14

Differentiating equation (*i*)

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \qquad \text{...(ii)}$$

$$x\dot{x} + y\dot{y} = 0$$

Differentiating again

$$x\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 + y\frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^2 = 0$$

$$x\ddot{x} + (\dot{x})^2 + y\ddot{y} + (\dot{y})^2 = 0 \qquad \text{...(iii)}$$

Substituting in equation (*ii*)

$$x = 8.66 \text{ cm}, \qquad y = 5 \text{ cm}$$

$$\dot{y} = 10 \text{ cm/s} \qquad \ddot{y} = 0$$

$$8.66(\dot{x}) + 5(10) = 0$$

$$\dot{x} = -5.77 \text{ cm/s}$$

The x-component of the velocity of the pin = 5.77 cm/s and is directed towards O.

Substituting values in equation (*iii*),

$$8.66(\ddot{x}) + (-5.77)^2 + 5(0) + (10)^2 = 0$$

$$\ddot{x} = -15.40 \text{ cm/s}^2$$

The x-component of the acceleration of the pin = 15.40 cm/s^2 and is directed towards O.

PROBLEMS

15.1. The motion of a particle is described by the following equations

$$x = t^2 + 8t + 4$$
$$y = t^3 + 3t^2 + 8t + 4$$

Determine (*a*) initial velocity of the particle, (*b*) velocity of the particle at $t = 2$s (*c*) acceleration of the particle at $t = 2$s

[(*a*) 11.31 m/s, 45° (*b*) 34.18 m/s, 69.44° (*c*) 18.11/s^2, 83.66°]

15.2. The distance s travelled by a particle moving along a circular path of radius r is given by the equation

$$s = kt^2, \text{ where } k \text{ is a constant.}$$

If the particle starts from rest, find (*a*) tangential velocity and acceleration, (*b*) the normal velocity and acceleration of the particle.

$$\left[(a)\ 2kt, 2k\quad (b)\ 0, \frac{4k^2t^2}{r}\right]$$

15.3. A ladder AB of length $2l$ has its ends A and B resting against a floor and a wall as shown in Fig. P.15.3. The ladder slips while its ends maintain contact with the floor and the wall.

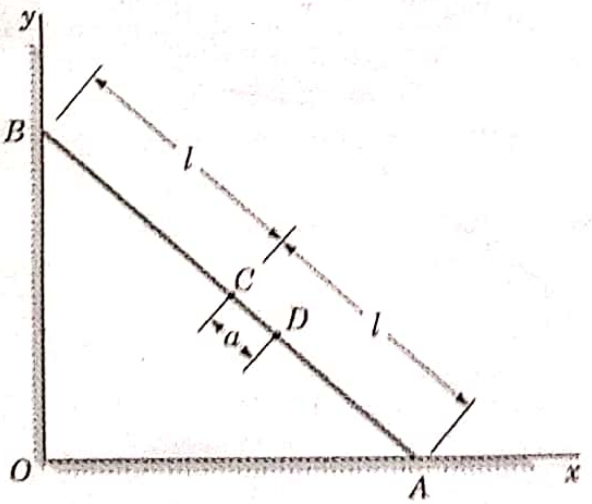

Fig. P.15.3

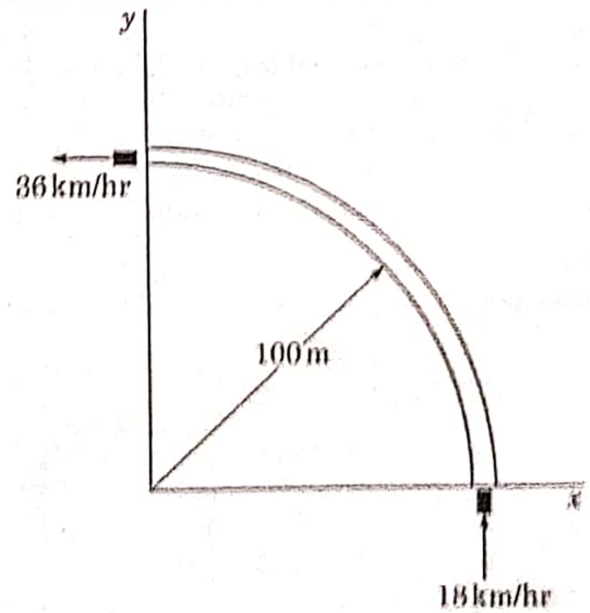

Fig. P.15.5

Show that the mid-point C of the ladder describes a circle of radius l with centre at O. Also show that any other point such as D describes an ellipse.

15.4. A car starts from rest on a curved road of radius 250 m and attains a speed of 18 km/hour at the end of 60 seconds while travelling with a uniform acceleration. Find the tangential and normal accelerations of the car 30 seconds after it started. [0.083 m/s^2, 0.025 m/s^2]