WKB- Approximation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$ - Schrödinger Equation

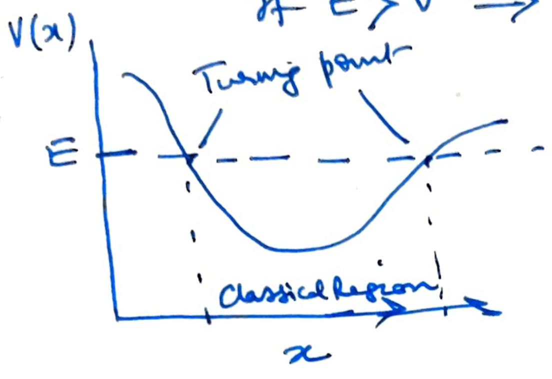
$$\Rightarrow \frac{d^2\psi}{dx^2} = -\frac{p^2}{\hbar^2}\psi \qquad \text{— (1)}$$

where $p(x) = \sqrt{2m[E - V(x)]}$

Classical formula for the momentum of a particle with total energy E and potential energy $V(x)$

If $E > V \Rightarrow p(x)$ is real — Classical region

i.e. particle is confined to this region of x



In general ψ is some complex function and can be expressed in terms of Amplitude $A(x)$ and its phase $\phi(x)$ both of which are real.

$$\psi(x) = A(x)\, e^{i\phi(x)}$$

$$\frac{d\psi}{dx} = (A' + iA\phi')\, e^{i\phi}$$

$$\frac{d^2\psi}{dx^2} = [A'' + 2iA'\phi' + iA\phi'' - A\phi'^2]\, e^{i\phi} \quad \text{-(2)}$$

(2) into (1)

$$A'' + 2iA'\phi' + iA\phi'' - A\phi'^2 = -\frac{p^2}{\hbar^2}A$$

$$\Rightarrow \quad A'' - A\phi'^2 = -\frac{p^2}{\hbar^2}A \Rightarrow A'' = A\left[\phi'^2 - \frac{p^2}{\hbar^2}\right] \quad \text{-(3)}$$

and

$$i[2A'\phi' + A\phi''] = 0 \quad \text{— (4)}$$

$$\cancel{(A^2\phi')' = 0} \Rightarrow (A^2\phi')' = 0 \Rightarrow A^2\phi' = C_1^2 \Rightarrow \boxed{A = \frac{C_1}{\sqrt{\phi'}}}$$

$\cancel{\frac{d}{dx}(A^2\phi')}$

$\cancel{A^2\frac{d\phi'}{dx} + \phi' 2A}$

$\cancel{A^2\phi'' + \phi'\frac{d}{dx}A^2}$

A varies slowly $\Rightarrow A''$ term is negligible

$\Rightarrow \frac{A''}{A}$ is much more smaller so that

$$\phi'^2 = \frac{p^2}{\hbar^2} \Rightarrow \left(\frac{d\phi}{dx}\right)^2 = \frac{p^2}{\hbar^2}$$

$$\frac{d\phi}{dx} = \pm\frac{p}{\hbar} \Rightarrow \boxed{\phi(x) = \pm\frac{1}{\hbar}\int p(x)\,dx}$$

$$\psi(x) = A(x)\, e^{i\phi(x)}$$
$$= A \cdot e^{i\phi}.$$

General sol^n will be linear combination of two

$$\boxed{\psi(x) \simeq \frac{c_1}{\sqrt{\phi'}} \cdot e^{\pm \frac{i}{\hbar}\int p(x)\,dx}}$$

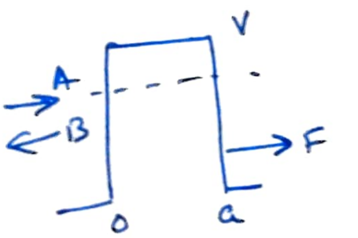
where $\phi' = \frac{p}{\hbar}$

$$|\psi(x)|^2 = \psi^*(x)\,\psi(x)$$
$$= \frac{|c_1|^2}{\phi'}$$

$$|\psi(x)|^2 \simeq \frac{|c|^2}{p(x)} = \frac{|c|^2}{p(x)}$$

Probability of finding the particle at point x is inversely proportional to its classical momentum & hence its velocity at that point. That is particle does not spend long in the places where it is moving rapidly, so ~~that~~ the probability of getting caught at that point is very small

Tunneling - $E > V$ - so $p(x)$ is real
when $E < V$ - non classical region where $p(x)$ is imaginary.

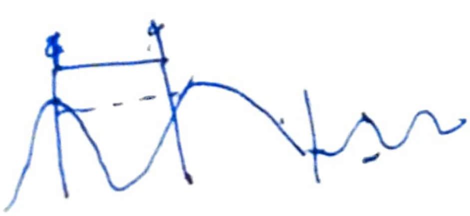


$$\psi(x) = \frac{C}{\sqrt{p(x)}}\, e^{\pm \frac{i}{\hbar}\int p(x)\,dx}$$

$$\psi(x) = A e^{ikx} + B e^{-ikx} \quad \text{for } x < a$$

Where A is incident, B is reflected amplitude and $k = \sqrt{2mE}/\hbar$

For $x > a$, $\psi(x) = F e^{ikx}$



$$T = \frac{|F|^2}{|A|^2} =$$

$$T = e^{-2\gamma} \quad \text{with } \gamma = \frac{1}{\hbar}\int_0^a p(x)\,dx$$

When $V(x)$ is slowly varying then we apply WKB
But when $V(x) = V_0 =$ constant and $E > V_0$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V_0\psi(x) = E\psi(x)$$

$$\Rightarrow \frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}(E-V)\psi(x) = -\frac{p^2}{\hbar^2}\psi(x)$$

where $p = \sqrt{2m(E-V)}$, $k = \frac{2\pi}{\lambda} = \frac{2\pi}{h}p$

$k = \frac{p}{\hbar}$

$$\Rightarrow \frac{d^2\psi}{dx^2} = -k^2\psi(x)$$

$$\Rightarrow \psi(x) = Ae^{\pm ikx}$$

$$\Rightarrow \boxed{\psi(x) = c_1 e^{ikx} + c_2 e^{-ikx}}$$

When $E < V_0$, then $\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}(V_0 - E)$

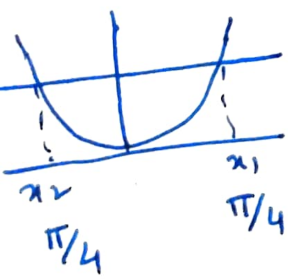
$$\Rightarrow \frac{d^2\psi}{dx^2} = k^2\psi(x).$$

$$\psi(x) = Ae^{\pm kx}$$

$$\boxed{\psi(x) = c_1 e^{kx} + c_2 e^{-kx}}$$

Quantization condition to obtain bound state Energy Eigen value

$$\frac{1}{\hbar}\int_{x_1}^{x_2} p(x)\,dx = \begin{cases} n\pi - (\beta_1+\beta_2) & \text{when } n = 1, 2, 3 \ldots \\ (n+1)\pi - (\beta_1+\beta_2) & \text{when } n = 0, 1, 2, \ldots \end{cases}$$



$$\frac{1}{\hbar}\int_{x_1}^{x_2} \sqrt{2m\{E - V(x)\}}\,dx =$$

x_1 and x_2 are turning point i.e. when K.E. $(T) = 0$

$T = 0 \Rightarrow E - V(x) = 0$ or $E = V(x)$

$\beta =$ phase factor $= 0$ for $V(x) = \infty$ and $\frac{\pi}{4}$ when $V(x) =$ finite

Quantization Condition $\frac{1}{\hbar}\int_{x_1}^{x_1} p(x)\,dx = n\pi - \left(\frac{\pi}{4}+\frac{\pi}{4}\right) = n\left(\pi+\frac{1}{2}\right)$

or $(n+1)\pi - \frac{\pi}{2} = n\left(\pi + \frac{1}{2}\right)$

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