7.E: Quantum Mechanics (Exercises)

Conceptual Questions

7.1 Wave Functions

1. What is the physical unit of a wave function, $\Psi(x, t)$? What is the physical unit of the square of this wave function?

2. Can the magnitude of a wave function $(\Psi * (x, t)\Psi(x, t))$ be a negative number? Explain.

3. What kind of physical quantity does a wave function of an electron represent?

4. What is the physical meaning of a wave function of a particle?

5. What is the meaning of the expression "expectation value?" Explain.

7.2 The Heisenberg Uncertainty Principle

6. If the formalism of quantum mechanics is 'more exact' than that of classical mechanics, why don't we use quantum mechanics to describe the motion of a leaping frog? Explain.

7. Can the de Broglie wavelength of a particle be known precisely? Can the position of a particle be known precisely?

8. Can we measure the energy of a free localized particle with complete precision?

9. Can we measure both the position and momentum of a particle with complete precision?

7.3 The Schrödinger Equation

10. What is the difference between a wave function $\psi(x, y, z)$ and a wave function $\Psi(x, y, z, t)$ for the same particle?

11. If a quantum particle is in a stationary state, does it mean that it does not move?

12. Explain the difference between time-dependent and -independent Schrödinger's equations.

13. Suppose a wave function is discontinuous at some point. Can this function represent a quantum state of some physical particle? Why? Why not?

7.4 The Quantum Particle in a Box

14. Using the quantum particle in a box model, describe how the possible energies of the particle are related to the size of the box.

15. Is it possible that when we measure the energy of a quantum particle in a box, the measurement may return a smaller value than the ground state energy? What is the highest value of the energy that we can measure for this particle?

16. For a quantum particle in a box, the first excited state (Ψ_2) has zero value at the midpoint position in the box, so that the probability density of finding a particle at this point is exactly zero. Explain what is wrong with the following reasoning: "If the probability of finding a quantum particle at the midpoint is zero, the particle is never at this point, right? How does it come then that the particle can cross this point on its way from the left side to the right side of the box?

7.5 The Quantum Harmonic Oscillator

17. Is it possible to measure energy of $0.75\hbar\omega$ for a quantum harmonic oscillator? Why? Why not? Explain.

18. Explain the connection between Planck's hypothesis of energy quanta and the energies of the quantum harmonic oscillator.

19. If a classical harmonic oscillator can be at rest, why can the quantum harmonic oscillator never be at rest? Does this violate Bohr's

correspondence principle?

20. Use an example of a quantum particle in a box or a quantum oscillator to explain the physical meaning of Bohr's correspondence principle.

21. Can we simultaneously measure position and energy of a quantum oscillator? Why? Why not?

22. When an electron and a proton of the same kinetic energy encounter a potential barrier of the same height and width, which one of them will

tunnel through the barrier more easily? Why?

23. What decreases the tunneling probability most: doubling the barrier width or halving the kinetic energy of the incident particle?

24. Explain the difference between a box-potential and a potential of a quantum dot.

25. Can a quantum particle 'escape' from an infinite potential well like that in a box? Why? Why not?

26. A tunnel diode and a resonant-tunneling diode both utilize the same physics principle of quantum tunneling. In what important way are they different?

Problems

7.1 Wave Functions

27. Compute $|\Psi(x,t)|^2$ for the function $\Psi(x,t) = \psi(x)sin\omega t$, where ω is a real constant.

28. Given the complex-valued function $f(x,y) = (x-iy)/(x+iy)$, calculate $|f(x,y)|^2$.

29. Which one of the following functions, and why, qualifies to be a wave function of a particle that can move along the entire real axis?

(a) $\psi(x) = Ae^{-x^2}$;

(b) $\psi(x) = Ae^{-x}$;

(c) $\psi(x) = Atanx$;

(d) $\psi(x) = A(sinx)/x$;

(e) $\psi(x) = Ae^{-|x|}$.

30. A particle with mass m moving along the x-axis and its quantum state is represented by the following wave function:
$$\Psi(x,t) = \begin{cases} 0 & x < 0 \\ Axe^{-\alpha x}e^{-iEt/\hbar} & ,x \geq 0 \end{cases}, \text{ where } \alpha = 2.0 \times 10^{10}m^{-1}.$$

(a) Find the normalization constant.

(b) Find the probability that the particle can be found on the interval $0 \leq x \leq L$.

(c) Find the expectation value of position.

(d) Find the expectation value of kinetic energy.

31. A wave function of a particle with mass m is given by $\psi(x) = \begin{cases} Acos\alpha x & -\frac{\pi}{2\alpha} \leq x \leq +\frac{\pi}{2\alpha} \\ 0 & otherwise \end{cases}$, where $\alpha = 1.00 \times 10^{10}/m$.

(a) Find the normalization constant.

(b) Find the probability that the particle can be found on the interval $0 \leq x \leq 0.5 \times 10^{-10}m$.

(c) Find the particle's average position.

(d) Find its average momentum.

(e) Find its average kinetic energy $-0.5 \times 10^{-10}m \leq x \leq +0.5 \times 10^{-10}m$.

7.2 The Heisenberg Uncertainty Principle

32. A velocity measurement of an α-particle has been performed with a precision of 0.02 mm/s. What is the minimum uncertainty in its position?

33. A gas of helium atoms at 273 K is in a cubical container with 25.0 cm on a side.

(a) What is the minimum uncertainty in momentum components of helium atoms?

(b) What is the minimum uncertainty in velocity components?

(c) Find the ratio of the uncertainties in

(b) to the mean speed of an atom in each direction.

34. If the uncertainty in the y-component of a proton's position is 2.0 pm, find the minimum uncertainty in the simultaneous measurement of the proton's y-component of velocity. What is the minimum uncertainty in the simultaneous measurement of the proton's xx-component of velocity?

35. Some unstable elementary particle has a rest energy of 80.41 GeV and an uncertainty in rest energy of 2.06 GeV. Estimate the lifetime of this particle.

36. An atom in a metastable state has a lifetime of 5.2 ms. Find the minimum uncertainty in the measurement of energy of the excited state.

37. Measurements indicate that an atom remains in an excited state for an average time of 50.0 ns before making a transition to the ground state with the simultaneous emission of a 2.1-eV photon.

(a) Estimate the uncertainty in the frequency of the photon.

(b) What fraction of the photon's average frequency is this?

38. Suppose an electron is confined to a region of length 0.1 nm (of the order of the size of a hydrogen atom).

(a) What is the minimum uncertainty of its momentum?

(b) What would the uncertainty in momentum be if the confined length region doubled to 0.2 nm?

7.3 The Schrödinger Equation

39. Combine Equation 7.17 and Equation 7.18 to show $k^2 = \dfrac{\omega^2}{c^2}$.

40. Show that $\Psi(x,t) = Ae^{i(kx-\omega t)}$ is a valid solution to Schrödinger's time-dependent equation.

41. Show that $\Psi(x,t) = A\sin(kx - \omega t)$ and $\Psi(x,t) = A\cos(kx - \omega t)$ do not obey Schrödinger's time-dependent equation.

42. Show that when $\Psi_1(x,t)$ and $\Psi_2(x,t)$ are solutions to the time-dependent Schrödinger equation and A,B are numbers, then a function $\Psi(x,t)$ that is a superposition of these functions is also a solution: $\Psi(x,t) = A\Psi_1(x,t) + B\Psi_1(x,t)$.

43. A particle with mass m is described by the following wave function: $\psi(x) = A\cos kx + B\sin kx$, where A, B, and k are constants. Assuming that the particle is free, show that this function is the solution of the stationary Schrödinger equation for this particle and find the energy that the particle has in this state.

44. Find the expectation value of the kinetic energy for the particle in the state, $\Psi(x,t) = Ae^{i(kx-\omega t)}$. What conclusion can you draw from your solution?

45. Find the expectation value of the square of the momentum squared for the particle in the state, $\Psi(x,t) = Ae^{i(kx-\omega t)}$. What conclusion can you draw from your solution?

46. A free proton has a wave function given by $\Psi(x,t) = Ae^{i(5.02\times10^{11}x - 8.00\times10^{15}t)}$. The coefficient of x is inverse meters (m^{-1}) and the coefficient on t is inverse seconds (s^{-1}). Find its momentum and energy.

7.4 The Quantum Particle in a Box

47. Assume that an electron in an atom can be treated as if it were confined to a box of width 2.0Å. What is the ground state energy of the electron? Compare your result to the ground state kinetic energy of the hydrogen atom in the Bohr's model of the hydrogen atom.

48. Assume that a proton in a nucleus can be treated as if it were confined to a one-dimensional box of width 10.0 fm.

(a) What are the energies of the proton when it is in the states corresponding to $n = 1, n = 2,$ and $n = 3$?

(b) What are the energies of the photons emitted when the proton makes the transitions from the first and second excited states to the ground state?

49. An electron confined to a box has the ground state energy of 2.5 eV. What is the width of the box?

50. What is the ground state energy (in eV) of a proton confined to a one-dimensional box the size of the uranium nucleus that has a radius of approximately 15.0 fm?

51. What is the ground state energy (in eV) of an αα-particle confined to a one-dimensional box the size of the uranium nucleus that has a radius of approximately 15.0 fm?

52. To excite an electron in a one-dimensional box from its first excited state to its third excited state requires 20.0 eV. What is the width of the box?

53. An electron confined to a box of width 0.15 nm by infinite potential energy barriers emits a photon when it makes a transition from the first excited state to the ground state. Find the wavelength of the emitted photon.

54. If the energy of the first excited state of the electron in the box is 25.0 eV, what is the width of the box?

55. Suppose an electron confined to a box emits photons. The longest wavelength that is registered is 500.0 nm. What is the width of the box?

56. Hydrogen H_2 molecules are kept at 300.0 K in a cubical container with a side length of 20.0 cm. Assume that you can treat the molecules as though they were moving in a one-dimensional box.

(a) Find the ground state energy of the hydrogen molecule in the container.

(b) Assume that the molecule has a thermal energy given by $k_B T/2$ and find the corresponding quantum number n of the quantum state that would correspond to this thermal energy.

57. An electron is confined to a box of width 0.25 nm.

(a) Draw an energy-level diagram representing the first five states of the electron.

(b) Calculate the wavelengths of the emitted photons when the electron makes transitions between the fourth and the second excited states, between the second excited state and the ground state, and between the third and the second excited states.

58. An electron in a box is in the ground state with energy 2.0 eV.

(a) Find the width of the box.

(b) How much energy is needed to excite the electron to its first excited state?

(c) If the electron makes a transition from an excited state to the ground state with the simultaneous emission of 30.0-eV photon, find the quantum number of the excited state?

7.5 The Quantum Harmonic Oscillator

59. Show that the two lowest energy states of the simple harmonic oscillator, $\psi_0(x)$ and $\psi_1(x)$ from Equation 7.57, satisfy Equation 7.55.

60. If the ground state energy of a simple harmonic oscillator is 1.25 eV, what is the frequency of its motion?

61. When a quantum harmonic oscillator makes a transition from the $(n+1)$ state to the n state and emits a 450-nm photon, what is its frequency?

62. Vibrations of the hydrogen molecule H_2 can be modeled as a simple harmonic oscillator with the spring constant $k = 1.13 \times 10^3 N/m$ and mass $m = 1.67 \times 10^{-27} kg$.

(a) What is the vibrational frequency of this molecule?

(b) What are the energy and the wavelength of the emitted photon when the molecule makes transition between its third and second excited states?

63. A particle with mass 0.030 kg oscillates back-and-forth on a spring with frequency 4.0 Hz. At the equilibrium position, it has a speed of 0.60 m/s. If the particle is in a state of definite energy, find its energy quantum number.

64. Find the expectation value $\langle x^2 \rangle$ of the square of the position for a quantum harmonic oscillator in the ground state. Note: $\int_{-\infty}^{+\infty} dx\, x^2 e^{-ax^2} = \sqrt{\pi}(2a^{3/2})^{-1}$.

65. Determine the expectation value of the potential energy for a quantum harmonic oscillator in the ground state. Use this to calculate the expectation value of the kinetic energy.

66. Verify that $\psi_1(x)$ given by Equation 7.57 is a solution of Schrödinger's equation for the quantum harmonic oscillator.

67. Estimate the ground state energy of the quantum harmonic oscillator by Heisenberg's uncertainty principle. Start by assuming that the product of the uncertainties Δx and Δp is at its minimum. Write Δp in terms of Δx and assume that for the ground state $x \approx \Delta x$ and $p \approx \Delta p$, then write the ground state energy in terms of x. Finally, find the value of x that minimizes the energy and find the minimum of the energy.

68. A mass of 0.250 kg oscillates on a spring with the force constant 110 N/m. Calculate the ground energy level and the separation between the adjacent energy levels. Express the results in joules and in electron-volts. Are quantum effects important?

7.6 The Quantum Tunneling of Particles through Potential Barriers

69. Show that the wave function in

(a) Equation 7.68 satisfies Equation 7.61, and

(b) Equation 7.69 satisfies Equation 7.63.

70. A 6.0-eV electron impacts on a barrier with height 11.0 eV. Find the probability of the electron to tunnel through the barrier if the barrier width is

(a) 0.80 nm and

(b) 0.40 nm.

71. A 5.0-eV electron impacts on a barrier of with 0.60 nm. Find the probability of the electron to tunnel through the barrier if the barrier height is

(a) 7.0 eV;

(b) 9.0 eV; and

(c) 13.0 eV.

72. A 12.0-eV electron encounters a barrier of height 15.0 eV. If the probability of the electron tunneling through the barrier is 2.5 %, find its width.

73. A quantum particle with initial kinetic energy 32.0 eV encounters a square barrier with height 41.0 eV and width 0.25 nm. Find probability that the particle tunnels through this barrier if the particle is

(a) an electron and,

(b) a proton.

74. A simple model of a radioactive nuclear decay assumes that α-particles are trapped inside a well of nuclear potential that walls are the barriers of a finite width 2.0 fm and height 30.0 MeV. Find the tunneling probability across the potential barrier of the wall for αα-particles having kinetic energy

(a) 29.0 MeV and

(b) 20.0 MeV. The mass of the α-particle is $m = 6.64 \times 10^{-27} kg$.

75. A muon, a quantum particle with a mass approximately 200 times that of an electron, is incident on a potential barrier of height 10.0 eV. The kinetic energy of the impacting muon is 5.5 eV and only about 0.10% of the squared amplitude of its incoming wave function filters through the barrier. What is the barrier's width?

76. A grain of sand with mass 1.0 mg and kinetic energy 1.0 J is incident on a potential energy barrier with height 1.000001 J and width 2500 nm. How many grains of sand have to fall on this barrier before, on the average, one passes

through?

Additional Problems

77. Show that if the uncertainty in the position of a particle is on the order of its de Broglie's wavelength, then the uncertainty in its momentum is on the order of the value of its momentum.

78. The mass of a ρ-meson is measured to be $770\,MeV/c^2$ with an uncertainty of $100\,MeV/c^2$. Estimate the lifetime of this meson.

79. A particle of mass m is confined to a box of width L. If the particle is in the first excited state, what are the probabilities of finding the particle in a region of width 0.020 L around the given point x:

    (a) $x = 0.25L$;

    (b) $x = 0.40L$;

    (c) $x = 0.75L$; and

    (d) $x = 0.90L$.

80. A particle in a box [0;L] is in the third excited state. What are its most probable positions?

81. A 0.20-kg billiard ball bounces back and forth without losing its energy between the cushions of a 1.5 m long table

    (a) If the ball is in its ground state, how many years does it need to get from one cushion to the other? You may compare this time interval to the age of the universe.

    (b) How much energy is required to make the ball go from its ground state to its first excited state? Compare it with the kinetic energy of the ball moving at 2.0 m/s.

82. Find the expectation value of the position squared when the particle in the box is in its third excited state and the length of the box is L.

83. Consider an infinite square well with wall boundaries $x = 0$ and $x = L$. Show that the function $\psi(x) = Asinkx$ is the solution to the stationary Schrödinger equation for the particle in a box only if $k = \sqrt{2mE}/\hbar$. Explain why this is an acceptable wave function only if k is an integer multiple of π/L.

84. Consider an infinite square well with wall boundaries $x = 0$ and $x = L$. Explain why the function $\psi(x) = Acoskx$ is not a solution to the stationary Schrödinger equation for the particle in a box.

85. Atoms in a crystal lattice vibrate in simple harmonic motion. Assuming a lattice atom has a mass of $9.4 \times 10^{-26}kg$, what is the force constant of the lattice if a lattice atom makes a transition from the ground state to first excited state when it absorbs a $525 - \mu m$ photon?

86. A diatomic molecule behaves like a quantum harmonic oscillator with the force constant 12.0 N/m and mass $5.60 \times 10^{-26}kg$.

    (a) What is the wavelength of the emitted photon when the molecule makes the transition from the third excited state to the second excited state?

    (b) Find the ground state energy of vibrations for this diatomic molecule.

87. An electron with kinetic energy 2.0 MeV encounters a potential energy barrier of height 16.0 MeV and width 2.00 nm. What is the probability that the electron emerges on the other side of the barrier?

88. A beam of mono-energetic protons with energy 2.0 MeV falls on a potential energy barrier of height 20.0 MeV and of width 1.5 fm. What percentage of the beam is transmitted through the barrier?

Challenge Problems

89. An electron in a long, organic molecule used in a dye laser behaves approximately like a quantum particle in a box with width 4.18 nm. Find the emitted photon when the electron makes a transition from the first excited state to the ground state and from the second excited state to the first excited state.

90. In STM, an elevation of the tip above the surface being scanned can be determined with a great precision, because the tunneling-electron current between surface atoms and the atoms of the tip is extremely sensitive to the variation of the separation gap between them from point to point along the surface. Assuming that the tunneling-electron current is in direct proportion to the tunneling probability and that the tunneling probability is to a good approximation expressed by the exponential function $e^{-2\beta L}$ with $\beta = 10.0/nm$, determine the ratio of the tunneling current when the tip is 0.500 nm above the surface to the current when the tip is 0.515 nm above the surface.

91. If STM is to detect surface features with local heights of about 0.00200 nm, what percent change in tunneling-electron current must the STM electronics be able to detect? Assume that the tunneling-electron current has characteristics given in the preceding problem.

92. Use Heisenberg's uncertainty principle to estimate the ground state energy of a particle oscillating on an spring with angular frequency, $\omega = \sqrt{k/m}$, where k is the spring constant and m is the mass.

93. Suppose an infinite square well extends from $-L/2$ to $+L/2$. Solve the time-independent Schrödinger's equation to find the allowed energies and stationary states of a particle with mass m that is confined to this well. Then show that these solutions can be obtained by making the coordinate transformation $x' = x - L/2$ for the solutions obtained for the well extending between 0 and L.

94. A particle of mass m confined to a box of width L is in its first excited state $\psi_2(x)$.

   (a) Find its average position (which is the expectation value of the position).

   (b) Where is the particle most likely to be found?

Contributors and Attributions

Samuel J. Ling (Truman State University), Jeff Sanny (Loyola Marymount University), and Bill Moebs with many contributing authors. This work is licensed by OpenStax University Physics under a Creative Commons Attribution License (by 4.0).