Dirac Notation

How to define Quantum States

$\Rightarrow \psi(r, t)$
$\Rightarrow \psi(x, y, z, t)$
$\Rightarrow \psi(q_1, q_2 \cdots q_n \cdots t)$.

Bra-Ket notation is a standard notation for describing Quantum States of dynamical System. in Hilbert Space.

A Hilbert Space is a linear vector Space with some additional properties

P. A. M. Dirac

electron $\Rightarrow \uparrow \downarrow$ — States.
Bra-Ket.

$\langle \quad \rangle$

Ket vector $|\psi\rangle$

Bra vector $\langle\,|$

col^m

Ket vector $|\psi\rangle = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{pmatrix}$

Complex Conjugate
$a + ib$
$a - ib$

Each Ket will be having bra vector

$\langle\psi| = (a_1^*, a_2^* \; a_3^* \cdots a_n)$

A= $|\psi\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\langle\psi| = [1, 0]$

$\langle\psi|\psi\rangle = [1, 0]\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1$

Inner Product

Vector – Vector Space

3 D
4 D (space-time)

Infinite Diamension vector $|A\rangle$ ket vector

$\langle A|$

Properties of Bra & Ket

① $|\psi\rangle$ — $\langle\psi|$

② $|a\psi\rangle = a|\psi\rangle$

Imp —— ③ $\langle a\psi| = a^* \langle\psi| \Rightarrow \langle a\psi| = a^* \langle\psi|$

Every ket has a corresponding bra
There is one to one correspondence between bra & kets

$$a|\psi\rangle + b|\phi\rangle = a^* \langle\psi| + b^* \langle\phi|.$$

④ **Properties of Scalar product**
Scalar product is a complex number,
the ordering matters.

Example $\langle\psi|\phi\rangle \neq \langle\phi|\psi\rangle$

$$\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^*$$

Proof

$$\langle\phi|\psi\rangle^* = \left(\int \phi^* \psi \, d^3r\right)^* = \int \psi^* \phi \, d^3r$$

$$= \langle\psi|\phi\rangle$$

$|\psi|^2 = \int \psi^* \psi \cdot d^3r$

Proove

$$\langle \psi | \psi \rangle \geq 0$$

will be zero when $\psi = 0$

(5) Orthogonal State

Two kets $|\psi\rangle$ and $|\phi\rangle$ are said to be orthogonal if they have vanishing scalar product

$$\langle \psi | \phi \rangle = 0$$

(6) Orthonormal state

Conditions

$$\langle \psi | \phi \rangle = 0 \quad \text{and } \langle \psi | \psi \rangle = 1$$
$$\text{and } \langle \phi | \phi \rangle = 1$$

(7) $\langle \psi | \phi_1 \rangle = \langle \psi | \phi_2 \rangle$ for any $\langle \psi |$

then $|\phi_1\rangle = |\phi_2\rangle$

and $\langle \phi_1 | \psi \rangle = \langle \phi_2 | \psi \rangle$ for any $|\psi\rangle$

$$\langle \phi_1 | = \langle \phi_2 | .$$

Operator

An operator $\hat{A}$ is a mathematical rule that when applied to $|\psi\rangle$, transforms it to another ket $|\psi'\rangle$ of the same space and when it acts on bra $\langle\phi|$ transforms it to another bra $\langle\phi'|$

$$\hat{A}|\psi\rangle = |\psi'\rangle$$

$$\langle\phi|\hat{A} = \langle\phi'|$$

Unity operator $\hat{I}|\psi\rangle = |\psi\rangle$

Gradient Operator

$$\nabla\psi(r) = \left(\frac{\partial}{\partial x}\psi(r)\right)i + \left(\frac{\partial}{\partial y}(\psi r)\right)j + \left(\frac{\partial(\psi r)}{\partial z}\right)k.$$

Linear Operator

$$\hat{A}|\psi\rangle = |\phi\rangle$$

– Distributive

(i) $\hat{A}\left[|\psi\rangle + |\phi\rangle\right] = \hat{A}|\psi\rangle + \hat{A}|\phi\rangle$

(ii) $\hat{A}\left[c_1|\psi_1\rangle + c_2|\psi_2\rangle + \cdots\right] =$

$$c_1\hat{A}|\psi_1\rangle + c_2\hat{A}|\psi_2\rangle$$

Two operators $\hat{A}$ & $\hat{B}$ are equal if

$$\langle\psi|\hat{A}|\psi\rangle = \langle\psi|\hat{B}|\psi\rangle$$

Eigen value Equation: $\left\{ \hat{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \psi = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

Eigen value Equation

$$\hat{A}|\psi\rangle = c|\psi\rangle$$

If we take $\hat{A}$ as $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $|\psi\rangle$ as $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$A|\psi\rangle \rightarrow \begin{bmatrix} 2 \\ 1 \end{bmatrix} \neq c|\psi\rangle$$

So this example is not Eigen value Equn

$\hat{A}$ as $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $|\psi\rangle$ as $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$A|\psi\rangle = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -1\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$A|\psi\rangle = c|\psi\rangle$$

Eigen Ket

Eigen value Equation

Hermitial Operator

$$\hat{H} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Same
no i

$$\hat{H}^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\hat{H} = H^* \rightarrow$ Hermition again

$\hookrightarrow$ Hermitian Operator

$$\hat{H} = \begin{bmatrix} 1 & 0 \\ +i & 0 \end{bmatrix}$$

$$\hat{H} \begin{bmatrix} 1 & 0 \\ i & 0 \end{bmatrix}$$

$$\hat{H}^* = \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$$

Any Operator Satisfying this condition is Hermitian Operator

$\hat{H} = -H^* \rightarrow$ Skew Hermitian

Two states of a physical system are given by

$$|\psi_1\rangle = (4i|\phi_1\rangle - 12i|\phi_2\rangle)$$

and $$|\psi_2\rangle = (|\phi_1\rangle - 6i|\phi_2\rangle)$$

Here $|\phi_1\rangle$ and $|\phi_2\rangle$ are orthonormal vectors

Determine $|\psi_1+\psi_2\rangle$ and $\langle\psi_1+\psi_2|$

Solⁿ.

$$|\psi_1+\psi_2\rangle = (4i|\phi_1\rangle - 12i|\phi_2\rangle) + (|\phi_1\rangle - 6i|\phi_2\rangle)$$

$$= (1+4i)|\phi_1\rangle - (18i)|\phi_2\rangle$$

Taking Complex Conjugate

$\cancel{\langle\psi_2 \pm \psi_1}$

$$\cancel{(|\psi_1+\psi_2\rangle)^* = \langle\psi_2 \pm \psi_1\ \langle\psi_1+\psi_2|}$$

$$\cancel{\langle\psi_1+\psi_2| = (1-4i)|\phi_1\rangle + (18i)|\phi_2\rangle}$$

$$(|\psi_1+\psi_2\rangle)^* = (1-4i)\langle\phi_1| + (18i)\langle\phi_2|$$

$$\langle\psi_1+\psi_2| = (1-4i)\langle\phi_1| + 18i\langle\phi_2|.$$

Two States of a physical system are Represented by

$$|\psi_1\rangle = 6i|\phi_1\rangle - 3i|\phi_2\rangle$$

$$|\psi_2\rangle = -2|\phi_1\rangle + 4i|\phi_2\rangle$$

$|\phi_1\rangle$ and $|\phi_2\rangle$ are orthonormal. Determine the scalar product $\langle\psi_1|\psi_2\rangle$.

$$|\psi_1\rangle = 6i|\phi_1\rangle - 3i|\phi_2\rangle$$

Since $(|\psi\rangle)^* = \langle\psi|$

$$\langle\psi_1| = (|\psi_1\rangle)^* = -6i\langle\phi_1| + 3i\langle\phi_2|$$

Scalar product $\langle\psi_1|\psi_2\rangle = (-6i\langle\phi_1| + 3i\langle\phi_2|)\cdot(-2|\phi_1\rangle + 4i|\phi_2\rangle)$

$$\langle\psi_1|\psi_2\rangle = (-6i\langle\phi_1|)(-2|\phi_1\rangle) + \cancel{(-6i\langle\phi_1|)(4i|\phi_2\rangle)}$$
$$+ \cancel{(3i\langle\phi_2|)(-2|\phi_1\rangle)} + (3i\langle\phi_2|)(4i|\phi_2\rangle)$$

Orthonormal $\Rightarrow$ $\langle\phi_2|\phi_2\rangle = 1$ and $\langle\phi_1|\phi_2\rangle = \langle\phi_2|\phi_1\rangle = 0$

$$\langle\phi_1|\phi_1\rangle = 1$$

$$= \cancel{+12i\langle\phi_1|}$$

$$\langle\psi_1|\psi_2\rangle = 12i - 12$$