

## 4 Fourth-rank Tensors: Elasticity

### 4.1 Introduction

We will jump from second-rank tensors to fourth-rank tensors. We make this leap partly because properties involving third-rank tensors tend to arise less often, but mainly because the fourth-rank tensor for elasticity enables us conveniently to consider the techniques available for handling more complicated tensors. Elasticity is a physical property which is relatively easy to comprehend despite the large number of coefficients which can be involved. We can understand that if we apply forces (compressive or tensile) to a crystalline solid, there will be resulting deformation and that this deformation will not only depend on the directions of the forces but also on the ease with which the crystal deforms in particular directions due to underlying directional variations of structure. The physical property of elasticity, being represented by a fourth-rank tensor, relates two second-rank tensors, namely stress (i.e. a series of applied forces) and strain (the resulting deformation). Therefore we must first introduce these two second-rank tensors.



### 4.2 Strain

Strain is defined as the change in dimensions of a body as a result of subjecting the body to a system of forces which are in equilibrium. Strain may consist of an extension or a compression or alternatively it may be a shear.

For simplicity, we first consider linear extension in a thin

wire. Let the wire have a length  $x$  between points O and P and a further small length  $\Delta x$  between P and Q (figure 4.1). We can then see what happens to the wire when we add a force to the far end of the wire keeping the other end (origin) fixed. Length  $x$  between O and P becomes  $x + u$  where  $u$  is a small extension and measures the movement of P in a direction away from the origin O. The incremental length  $\Delta x$  between P and Q now becomes  $\Delta x + \Delta u$  where  $\Delta u$  will be very small. Provided stretching of the wire is homogeneous, the extension  $u$  at any point P distance  $x$  along the wire will increase linearly with  $x$ . The strain in the particular length PQ is given by

$$\frac{\text{increase in length}}{\text{original length}} = \frac{\Delta u}{\Delta x}$$

More particularly, the strain at P is given by

$$(\Delta u / \Delta x)_{x \rightarrow 0} = du/dx = e \quad (4.1)$$

where  $e$  is extension per unit length (relative change of length).

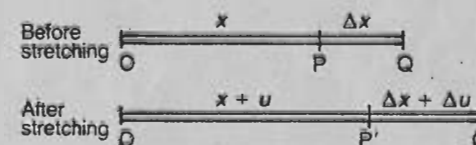


Figure 4.1 Extension of a thin wire.

Extending to a two-dimensional body, i.e. a plate, the situation immediately increases in complexity and we require four components. When we apply a system of forces to the plate, it will distort, and point P moves to P' and the initial infinitesimal lengths  $PQ_1$  and  $PQ_2$  before deformation become the infinitesimal lengths  $P'Q'_1$  and  $P'Q'_2$  (figure 4.2). Lengths  $PQ_1$  and  $PQ_2$  were taken orthogonal to each other and with their directions parallel to the  $x_1$  and  $x_2$  axes respectively. After the deformation, this will no longer be the case as can be seen in the figure. Whereas  $PQ_1$  originally had length  $\Delta x_1$  in the  $x_1$  direction, after the deformation it has a resolved component of length  $\Delta x_1 + \Delta u_1$  in the  $x_1$  direction and a resolved component of length  $\Delta u_2$  in the  $x_2$  direction. It has rotated clockwise an

angle  $\theta$  relative to its original orientation. Depending on the forces deforming the plate, movement of  $PQ_2$  will be somewhat different and independent of the movement of  $PQ_1$ , as shown in the figure.

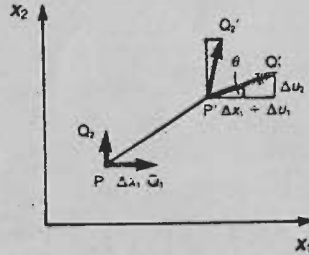


Figure 4.2 Distortion within a plate when subjected to a system of forces.

The extension per unit length of  $P'Q'_1$  parallel to the  $x_1$  direction is given by

$$e_{11} = \partial u_1 / \partial x_1.$$

Similarly the extension per unit length parallel to the  $x_2$  direction is given by

$$e_{22} = \partial u_2 / \partial x_2.$$

We can also write down further terms of the form

$$e_{12} = \partial u_1 / \partial x_2 \quad \text{and} \quad e_{21} = \partial u_2 / \partial x_1.$$

These terms represent rotations. Referring to figure 4.2, we can see that

$$\begin{aligned} \tan \theta &= \Delta u_2 / (\Delta x_1 + \Delta u_1) \\ &\approx \Delta u_2 / \Delta x_1 \end{aligned}$$

and so if displacements are small

$$\tan \theta = e_{21}.$$

$e_{12}$  can also be expressed as an angle. But let us consider a pure rotation of our plate as a rigid sheet. The relative orientation of  $PQ_1$  and  $PQ_2$  remains  $90^\circ$  but  $PQ_1$  and  $PQ_2$  have

rotated as a whole by angle  $\phi$  (figure 4.3).  $e_{11}$  and  $e_{22}$  must be zero as there are no extensions. However,  $e_{21}$  is  $\phi$  and  $e_{12}$  is  $-\phi$  so that we have

$$[e_{ij}] = \begin{bmatrix} 0 & -\phi \\ \phi & 0 \end{bmatrix}. \quad (4.2)$$

This represents the pure rotation of the sheet and these components should not be part of the strain tensor. What we do is split our tensor  $[e_{ij}]$  up into symmetrical and antisymmetrical parts.

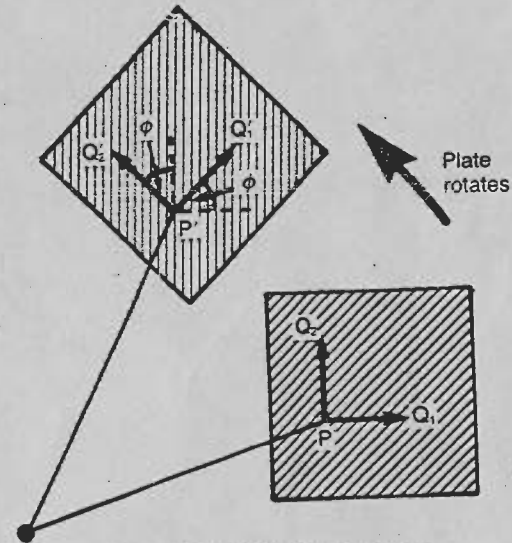


Figure 4.3 Rotation of a plate.

### 4.3 Symmetrical and Antisymmetrical Tensors

Any tensor can be divided up into the sum of two tensors, one symmetrical and the other antisymmetrical. Any single component of the original tensor can be put in the form

$$e_{ij} = \underbrace{(e_{ij} + e_{ji})/2}_{\text{symmetrical part}} + \underbrace{(e_{ij} - e_{ji})/2}_{\text{antisymmetrical part}}. \quad (4.3)$$

Thus tensor  $[e_{ij}]$  is given by

$$\begin{aligned}
 & \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} \\
 &= \begin{bmatrix} e_{11} & (e_{12} + e_{21})/2 & (e_{13} + e_{31})/2 \\ (e_{12} + e_{21})/2 & e_{22} & (e_{23} + e_{32})/2 \\ (e_{13} + e_{31})/2 & (e_{23} + e_{32})/2 & e_{33} \end{bmatrix} \\
 &+ \begin{bmatrix} 0 & (e_{12} - e_{21})/2 & (e_{13} - e_{31})/2 \\ (e_{21} - e_{12})/2 & 0 & (e_{23} - e_{32})/2 \\ (e_{31} - e_{13})/2 & (e_{32} - e_{23})/2 & 0 \end{bmatrix}. \quad (4.4)
 \end{aligned}$$

Sometimes it is useful to rotate the reference axes so as to make them coincide with the principal axes of the symmetrical part such that there are only the three diagonal terms in the symmetrical part. Hence  $e_{12}$  must equal  $-e_{21}$  and so on. Doing this we get

$$\begin{aligned}
 & \begin{bmatrix} e_{11} & 0 & 0 \\ 0 & e_{22} & 0 \\ 0 & 0 & e_{33} \end{bmatrix} + \begin{bmatrix} 0 & e_{12} & -e_{31} \\ -e_{12} & 0 & e_{23} \\ e_{31} & -e_{23} & 0 \end{bmatrix} \\
 & \text{symmetrical part} \quad \text{antisymmetrical} \\
 & \text{referred to} \quad \text{part} \\
 & \text{principal axes} \\
 &= \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ -e_{12} & e_{22} & e_{23} \\ -e_{13} & -e_{23} & e_{33} \end{bmatrix} \\
 &= \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix}. \quad (4.5)
 \end{aligned}$$

We have already referred to Appendix 2 for a discussion of diagonalisation.

#### 4.4 The Strain Tensor

We can compare equations (4.2) and (4.3) and see that the rotation in (4.2) corresponds to the antisymmetrical part in (4.3). If we take out the antisymmetrical part and keep the

symmetrical part, we eliminate the pure rotation and retain the strain only. We write the symmetrical part representing the strain in a plate in the form

$$[\varepsilon_{ij}] = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{bmatrix} = \begin{bmatrix} e_{11} & (e_{12} + e_{21})/2 \\ (e_{12} + e_{21})/2 & e_{22} \end{bmatrix}.$$

We can look at  $\varepsilon_{12}$  and  $\varepsilon_{21}$  and see that we have no rotation:

$$\begin{aligned}
 \varepsilon_{12} &= (\partial u_1 / \partial x_2 + \partial u_2 / \partial x_1) / 2 \\
 &= \varepsilon_{21} \\
 &= 0.
 \end{aligned}$$

We can go a step further if we wish and diagonalise the symmetrical part of the tensor.

In an analogous way, we can go on to define the strain tensor  $[\varepsilon_{ij}]$  for a three-dimensional body as the symmetrical part of  $[e_{ij}]$  in three dimensions where  $i$  and  $j = 1, 2, 3$ .

$$\begin{aligned}
 [\varepsilon_{ij}] &= \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} \\
 &= \begin{bmatrix} e_{11} & (e_{12} + e_{21})/2 & (e_{13} + e_{31})/2 \\ (e_{12} + e_{21})/2 & e_{22} & (e_{23} + e_{32})/2 \\ (e_{13} + e_{31})/2 & (e_{23} + e_{32})/2 & e_{33} \end{bmatrix}. \quad (4.6)
 \end{aligned}$$

The three diagonal components are the tensile strains. If they have negative magnitude they become compressive strains. The other off-diagonal components measure the shear strains. As the principal strains can be either positive or negative, one cannot always represent strain by a real ellipsoid, something we could always do for resistivity and conductivity.

Strain is the response of the material, more particularly for our interests a crystal, to an influence. If this influence arises from some change within the crystal, such as a thermal expansion due to a uniform change of temperature in the crystal, then the influence itself will have no orientation. Hence the resulting strain must conform to the crystal symmetry. The strain will have directional properties because the thermal expansion coefficient will be different for different directions but Neumann's principle (p.31) must be obeyed. (Thermal expansion which will be considered briefly in Chapter 7 is a second-rank tensor.) In general, however, the strain will arise from

external forces which can be applied with arbitrary magnitudes and directions. Whereas the resulting strain will be affected by the underlying crystal structure, the variation in strain will not arise purely from the underlying crystal structure. The strain tensor, and also the stress tensor, are examples of *field tensors* which are so-called because they do not represent crystal properties. Strictly, they are not encompassed by the title of this book. The resistivity and conductivity tensors, which we considered previously, do represent crystal properties and, as we have seen, conform to Neumann's principle and to the crystal properties. They are so-called *matter tensors*.

#### 4.5 Stress

Any body which is acted on externally by a system of forces is in a state of stress. To consider the stress (i.e. the force per unit area) within a body it is usual to consider an element of volume, and for convenience we will consider a cube of material within our body. Provided the stress is homogeneous, the forces acting on the surfaces of this element will be independent of the position of this element within the body. We shall assume this. The assumption is not, of course, the same as assuming that the stresses are isotropic, and in fact anisotropic forces will be allowed for.

Figure 4.4(a) shows a cubic element within our body. The normal and shear components of the stresses are shown for three of the six faces.  $\sigma_{11}$ ,  $\sigma_{22}$  and  $\sigma_{33}$  are the normal components and  $\sigma_{12}$ ,  $\sigma_{23}$ , etc. are the shear components. The sign convention used is that if  $\sigma_{11}$ , etc. are positive, the stress is tensile, whereas if  $\sigma_{11}$ , etc. are negative, the stress is compressive. We assume that our body is in dynamic equilibrium, i.e. it is stationary with no rotation. For this to be so the forces on opposite faces must be equal and opposite. Also, if we look at figure 4.4(b), which takes a cross section in the  $x_2x_3$  plane, we can see that for no turning moment  $\sigma_{32} = \sigma_{23}$  and in general  $\sigma_{ij} = \sigma_{ji}$ . It is possible to prove that  $\sigma_{ij}$  is a second-rank tensor (see for instance Nye 1957, p.87) relating resultant stress with direction. We shall assume this.

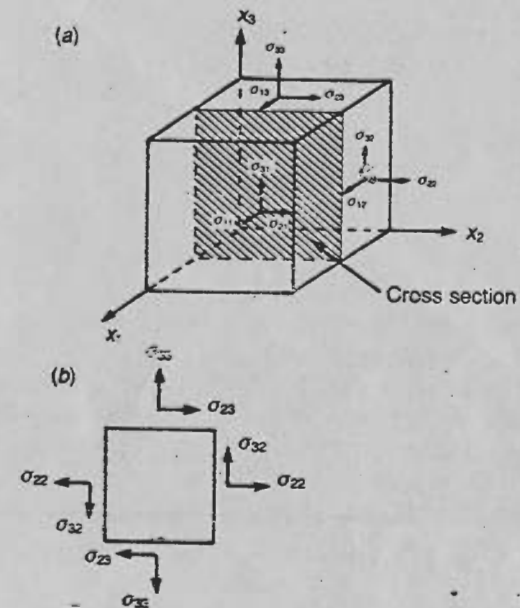


Figure 4.4 (a) Normal and shear components of stress on a cubic element. (b) Cross section of the cube in the  $x_2x_3$  plane. For no turning moment  $\sigma_{ij} = \sigma_{ji}$ .

#### 4.6 Elasticity

We start with the well known law that for a solid body the strain is proportional to the magnitude of the stress, provided that the magnitude of the stress is below that corresponding to the elastic limit of the material such that the strain is recoverable. This is Hooke's law, usually expressed for an isotropic medium in the form

$$\sigma = c\varepsilon$$

where  $c$  is Young's modulus. We shall refer to  $c$  as elastic stiffness. Alternatively, we can write

$$\varepsilon = s\sigma$$

where  $s$  is elastic compliance. It is unfortunate that the convention used for stiffness and compliance involves symbols  $c$  and  $s$  which are the reverse of their initial letters.



## 7 Further Tensor Applications

### 7.1 Introduction

The topics included in this chapter are very varied in their nature. They illustrate further features of the application of tensors to physical properties. They are included in this separate chapter as none of the examples should be regarded as central to a basic understanding of the subject. Yet each example demonstrates an interesting point. We start with a relatively straightforward example, thermal expansion, but we go on to consider higher-order effects which require tensors of greater complexity. Some of these effects are of increasing importance with the current interest in nonlinear phenomena, notably in optics.

### 7.2 Thermal Expansion

When we consider heating of crystals, this process can give rise to expansions and sometimes contractions of different magnitudes in different directions. The expansions in three dimensions in a crystal constitute strain and the thermal expansion tensor is a second-rank tensor which relates the strain to the change in temperature. We considered the strain tensor in Chapter 4 and demonstrated its symmetrical second-rank nature. Measurement of change of dimensions of crystals with temperature in different directions can be made most easily by finding the variation of lattice spacings by crystallographic techniques. Having measured the strain for a number of different directions, the principal

coefficients of expansion can be found. These will have values specific to particular crystals and the expansion tensor is a matter tensor as mentioned on p.56. The thermal expansion tensor is essentially a strain tensor which arises from the internal forces consequent on a change of temperature. This is in contrast with the strain tensor used to describe change of dimensions with load, where the tensor is a field tensor.

Let us assume that the temperature of our crystal increases uniformly by a small amount  $\Delta T$  and that this produces strain components

$$\varepsilon_{ij} = \alpha_{ij} \Delta T \quad (7.1)$$

where  $\alpha_{ij}$  are components relating the strain to the temperature rise in our particular crystal.  $[\varepsilon_{ij}]$  is the strain tensor and is symmetrical. Therefore  $[\alpha_{ij}]$  must be a second-rank symmetrical tensor.

If we refer our tensors to their principal axes we obtain

$$\varepsilon_1 = \alpha_1 \Delta T$$

$$\varepsilon_2 = \alpha_2 \Delta T$$

$$\varepsilon_3 = \alpha_3 \Delta T$$

where  $\alpha_1$ ,  $\alpha_2$  and  $\alpha_3$  are the principal expansion coefficients. They are the components required to measure the thermal expansion of the crystal and the directions of the principal axes will be related to the directions of the crystallographic axes. Thus, if we write out the representation quadric for thermal expansion, referred to the principal axes, we obtain

$$\alpha_1 x_1^2 + \alpha_2 x_2^2 + \alpha_3 x_3^2 = 1. \quad (7.2)$$

The shape, as well as the orientation, of the quadric will be restricted by the crystal symmetry, i.e. by Neumann's principle (see p.31). Thermal expansion will not cancel any symmetry elements provided the crystal does not go through a phase transformation.

It is important to realise that equation (7.2) represents the crystal expansion but does not represent the shape of the expanded crystalline material. Suppose we take a spherical sample of crystal (we could take a spherical volume within an arbitrarily shaped crystal). After a temperature increase  $\Delta T$  the

crystal volume becomes ellipsoidal in shape with axes  $(1 + \alpha_1 \Delta T)$ ,  $(1 + \alpha_2 \Delta T)$  and  $(1 + \alpha_3 \Delta T)$ . Only along the principal axes will the change be restricted to a change in length. Any line not parallel to a principal axis will exhibit a rotation relative to the system of axes.

The expansion coefficients  $\alpha_1$ ,  $\alpha_2$  and  $\alpha_3$  are usually all positive, so that the representation quadric is ellipsoidal. However, a number of materials exhibit negative coefficients so that the thermal expansion quadric is a further example which can illustrate the various alternatives described on p.29.

The change in overall volume of a cube of side  $L$  for a 1 K temperature change will be given by

$$\begin{aligned}\delta V &= L^3(1 + \alpha_1)(1 + \alpha_2)(1 + \alpha_3) - L^3 \\ &= V(\alpha_1 + \alpha_2 + \alpha_3)\end{aligned}$$

to first order. Hence the volume expansivity,  $\alpha_v = \delta V/V$ , is given by

$$\alpha_v = (\alpha_1 + \alpha_2 + \alpha_3). \quad (7.3)$$

### 7.3 The Pyroelectric Effect

In the pyroelectric effect, a change in temperature leads to a release or movement of charge and the rate of spontaneous polarisation or change of electrical dipole moment,  $\Delta P_i$ , is related to the temperature change  $\Delta T$  by the pyroelectric coefficients  $p_i$  which are components of a vector which represents the crystal property pyroelectricity.  $\Delta T$  is a small uniform temperature change within the crystal. Thus

$$\Delta P_i = p_i \Delta T. \quad (7.4)$$

The magnitude of the effect will differ according to whether the crystal is clamped during heating, so that it can change neither its shape nor its size, or whether it is free to expand. When the pyroelectric effect is observed under the first condition it is called primary pyroelectricity and when it is observed under the second condition (free expansion) there is an *additional* effect called secondary pyroelectricity. No crystal showing a vectorial property can possess a centre of symmetry (something which is

self-evident), so that the only crystals which can exhibit the pyroelectric effect are 1, 2, m, 2mm, 4, 4mm, 3, 3m, 6 and 6mm.

In the most general case, the triclinic class 1, the direction of  $p$  is not fixed with respect to the crystal structure. In the monoclinic classes,  $p$  must lie in the mirror plane for class m and parallel to the twofold rotational axis for class 2. In all the other cases there is a unique crystallographic axis to which  $p$  must lie parallel.

All pyroelectric materials are also piezoelectric (although the reverse is not true) and this effect will be considered in the next section.

### 7.4 Piezoelectricity

This is an example of a third-rank tensor. The *direct piezoelectric effect* is the development of an electric dipole moment  $P$  in a crystal when a stress  $\sigma$  is applied to the crystal. The magnitude of the electric moment is proportional to the stress, so we can write

$$P = d\sigma$$

where  $d$  is the piezoelectric modulus. A tensile stress gives rise to a positive value for the modulus and a compressive stress gives rise to a negative value.

As we have seen, in general the stress can be represented by a second-rank tensor, and the polarisation can be in any direction and is a vector. So we may write

$$P_i = d_{ijk} \sigma_{jk} \quad (7.5)$$

and  $d_{ijk}$  are now the piezoelectric moduli.

A general third-rank tensor would require 27 components but  $\sigma_{jk}$  is symmetric in  $jk$  and hence  $d_{ijk}$  will be symmetric in  $jk$  also. This reduces the number of components to 18 and a triclinic crystal of point group 1 can have 18 independent components. A matrix notation can be used for the piezoelectric moduli in the same way as was used for stiffness and compliance in Chapter 4. The first subscript of the tensor notation remains the first subscript of the matrix notation and the second and

third subscripts of the tensor notation become single matrix subscripts 1 to 6 following the pattern on p.58. Hence the piezoelectric moduli are represented by a  $6 \times 3$  matrix and in subscript notation by

$$P_m = d_{mn} \sigma_n \quad (7.6)$$

where  $m = 1, 2, 3$  and  $n = 1, 2, \dots, 6$ .

As with compliance, it is necessary to introduce factors in the conversion from  $d_{ijk}$  to  $d_{mn}$ .

$$d_{ijk} = d_{mn} \quad \text{for } n = 1, 2, 3$$

$$2d_{ijk} = d_{mn} \quad \text{for } n = 4, 5, 6.$$

The matrix array for the piezoelectric moduli then takes the form

$$\begin{pmatrix} d_{11} & d_{12} & d_{13} & d_{14} & d_{15} & d_{16} \\ d_{21} & d_{22} & d_{23} & d_{24} & d_{25} & d_{26} \\ d_{31} & d_{32} & d_{33} & d_{34} & d_{35} & d_{36} \end{pmatrix}.$$

Remember that the  $d_{mn}$  do not transform as tensors, whereas the  $d_{ijk}$  do.

There is also the converse piezoelectric effect in which the application of an electric field  $E$  produces a strain within the crystal.

$$\epsilon_{jk} = d_{ijk} E_i. \quad (7.7)$$

Here the  $d_{ijk}$  are the same moduli as those in equation (7.3), a fact which can be proved thermodynamically. To change to the matrix form, the same conversions involving factors of  $\frac{1}{2}$  are used in the conversion of  $\epsilon_{jk}$  to  $\epsilon_n$  as were used on p.59 for  $\epsilon$ . This then fits with the use of the factors of  $\frac{1}{2}$  to convert  $d_{ijk}$  to  $d_{mn}$ . Overall we have in matrix form

$$\epsilon_n = d_{mn} E_m \quad (7.8)$$

where  $m = 1, 2, 3$  and  $n = 1, 2, \dots, 6$ . As in the case of pyroelectricity, the piezoelectric effect can be measured under differing conditions, notably under constant stress or under constant strain. The principles of applying tensors to the effect remain the same in each case and the details will not be pursued further.

## 7.5 Photoelasticity

When a stress is applied to a crystal, not only can an electric moment be produced, but there can also be a change of dielectric constant and hence a change of refractive index. This change in the refractive index is called the photoelastic effect. The stress will be directional so an anaxial material will become doubly refracting and a uniaxial or biaxial crystal will show a change of components. The phenomenon has particular application to the representation of large solid objects such as machinery by small Perspex models. These models can be loaded in a similar manner to the larger original objects and changes of refractive index measured by viewing the models whilst they are positioned between crossed polarisers. This illustrates the effect in a non-crystalline material which is isotropic until the stress is applied.

The photoelastic tensor relates the change in the optical impermeability  $\Delta\eta_{ij}$  to the strain tensor  $\sigma_{kl}$  and we write

$$\begin{aligned} \Delta\eta_{ij} &= \Delta(1/n^2)_{ij} \\ &= \Pi_{ijkl} \sigma_{kl}. \end{aligned} \quad (7.9)$$

The  $\Pi_{ijkl}$  are the components of the strain-optic tensor and are dimensionless. Previously we had equation (5.3) for the indicatrix given by

$$x_1^2/n_1^2 + x_2^2/n_2^2 + x_3^2/n_3^2 = 1$$

where  $n_1$ ,  $n_2$  and  $n_3$  are the principal values of the refractive index.

Alternatively, we can write the indicatrix equation in the form

$$\eta_1 x_1^2 + \eta_2 x_2^2 + \eta_3 x_3^2 = 1 \quad (7.10)$$

where we are using principal values of the optical impermeability  $\eta_{ij}$ .

More generally

$$\eta_i x_i x_j = 1. \quad (7.11)$$

When the change of optical impermeability is included, this equality becomes