Question Based On

"Normal frequencies"

and

"Theory of Small Oscillations"

Key Concepts:

According to the Theory of small oscillations frequencies of Normal Mode are obtained by solving the secular eq.

$$\left|\bar{V} - \omega^2 \bar{T}\right| = 0$$

where $\bar{V}$ = Potential Energy matrix

, $\bar{T}$ = Kinetic Energy matrix

ω = Normal frequencies.

Question: Gate 2009.

The Lagrangian of a diatomic molecule is given by

$L = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2) - \frac{k}{2} x_1 x_2$. where m is the mass of each atom and x_1, x_2 are displacement from equilibrium position. and $k > 0$. The normal frequencies are.

Solution :-

Given $L = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2) - \frac{K}{2} x_1 x_2$

But $L = T - V$

$\therefore T = \frac{m}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right)$

$V = \frac{K}{2} x_1 x_2$

Since here there are two generalised coordinates x_1 and x_2

$\therefore \bar{V}$ and $\bar{T}$ will be 2×2 matrix

$$\bar{V} = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} ; \quad V_{11} = \frac{\partial^2 V}{\partial x_1^2} \quad V_{12} = \frac{\partial^2 V}{\partial x_1 \partial x_2}$$

$$\boxed{V_{21}} = \frac{\partial^2 V}{\partial x_2 x_1} \quad V_{22} = \frac{\partial^2 V}{\partial x_2 \partial x_2}$$

Solving.

$$V_{11} = \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_1}(V) = \frac{\partial}{\partial x_1}\left\{ \frac{\partial}{\partial x_1}\left(\frac{k}{2} x_1 x_2 \right) \right\}$$

$$= \frac{\partial}{\partial x_1}\left\{ \frac{k}{2} x_2 \frac{\partial x_1}{\partial x_1} \right\} = \frac{\partial}{\partial x_1}\left(\frac{k}{2} x_2 \right)$$

$$= 0$$

$$V_{12} = \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2}\left(\frac{k}{2} x_1 x_2 \right) = \frac{\partial}{\partial x_1}\left\{ \frac{k}{2} x_1 \right\} = \frac{k}{2}$$

$$V_{21} = \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1}\left(\frac{k}{2}x_1 x_2\right)$$

$$= \frac{\partial}{\partial x_2}\left(\frac{k}{2}x_2\right)$$

$$= \frac{k}{2}$$

$$V_{22} = \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_2}\left(\frac{k}{2}x_1 x_2\right)$$

$$= \frac{\partial}{\partial x_2}\left(\frac{k}{2}x_1\right)$$

$$= 0$$

$$\therefore \quad \bar{V} = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} = \begin{pmatrix} 0 & K/2 \\ K/2 & 0 \end{pmatrix}$$

Now $\bar{T} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$; $T_{11} = \dfrac{\partial^2}{\partial \dot{x}_1^2} T$ $\quad T_{12} = \dfrac{\partial^2 T}{\partial \dot{x}_1 \partial \dot{x}_2}$

$T_{21} = \dfrac{\partial^2 T}{\partial \dot{x}_2 \partial \dot{x}_1}$ $\quad \boxed{T_{22}} = \dfrac{\partial^2 T}{\partial \dot{x}_2 \partial \dot{x}_2}$

$$T_{11} = \frac{\partial \, \partial}{\partial \dot{x}_1 \partial \dot{x}_1} \left[\frac{m}{2} (\dot{x}_1^2 + \dot{x}_2^2) \right]$$

$$= \frac{m}{2} \frac{\partial}{\partial \dot{x}_1} \left[\frac{\partial}{\partial \dot{x}_1} (\dot{x}_1^2 + \dot{x}_2^2) \right]$$

$$= \frac{m}{2} \frac{\partial}{\partial \dot{x}_1} \left[2\dot{x}_1 + 0 \right]$$

$$= \frac{m}{2} \cdot 2 \frac{\partial \dot{x}_1}{\partial \dot{x}_1} = m$$

$$T_{12} = \frac{\partial}{\partial \dot{x}_1} \cdot \frac{\partial}{\partial \dot{x}_2} \left[\frac{m}{2} \left(\dot{x}_1^2 + \dot{x}_2^2 \right) \right]$$

$$= \frac{m}{2} \frac{\partial}{\partial \dot{x}_1} \left[\frac{\partial}{\partial \dot{x}_2} \left(\dot{x}_1^2 + \dot{x}_2^2 \right) \right]$$

$$= \frac{m}{2} \frac{\partial}{\partial \dot{x}_1} \left[0 + 2\dot{x}_2 \right]$$

$$= \frac{m}{2} \cdot 2 \frac{\partial \dot{x}_2}{\partial \dot{x}_1}$$

$$= m \times 0 = 0$$

$$T_{21} = \frac{\partial}{\partial \dot{x}_2} \cdot \frac{\partial}{\partial \dot{x}_1} \frac{m}{2} \left(\dot{x}_1^2 + \dot{x}_2^2 \right)$$

$$= 0 \quad \text{(on solving)}.$$

$$T_{22} = \frac{\partial}{\partial \dot{x}_2}\frac{\partial}{\partial \dot{x}_2}\left[\frac{m}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right)\right]$$

$$= \frac{m}{2}\frac{\partial}{\partial \dot{x}_2}\left[\frac{\partial}{\partial \dot{x}_2}\left(\dot{x}_1^2 + \dot{x}_2^2\right)\right]$$

$$= \frac{m}{2}\frac{\partial}{\partial \dot{x}_2}\left(0 + 2\dot{x}_2\right)$$

$$= \frac{m}{2} \cdot 2 \frac{\partial}{\partial \dot{x}_2}\dot{x}_2$$

$$= m$$

$$\therefore \; \bar{T} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$$

putting the values

$$\left| \bar{V} - \omega^2 \bar{T} \right| = 0$$

$$\left| \begin{pmatrix} 0 & \frac{k}{2} \\ \frac{k}{2} & 0 \end{pmatrix} - \omega^2 \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \right| = 0$$

$$\left| \begin{pmatrix} 0 & \frac{k}{2} \\ \frac{k}{2} & 0 \end{pmatrix} - \begin{pmatrix} \omega^2 m & 0 \\ 0 & \omega^2 m \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} -\omega^2 m & k/2 \\ \frac{k}{2} & -\omega^2 m \end{vmatrix} = 0$$

$$(-\omega^2 m)(-\omega^2 m) - \left(\frac{k}{2}\right)^2 = 0$$

$$\omega^4 m^2 - \frac{k^2}{4} = 0$$

$$(\omega^2 m)^2 - \left(\frac{k}{2}\right)^2 = 0$$

$$\left(\omega^2 m + \frac{k}{2}\right)\left(\omega^2 m - \frac{k}{2}\right) = 0 \qquad \left[\because a^2 - b^2 = (a+b)(a-b)\right]$$

$$\omega^2 m + \frac{k}{2} = 0 \quad ; \quad \omega^2 m - \frac{k}{2} = 0$$

$$\omega^2 = \frac{-k}{2m} \Rightarrow \omega = \pm\sqrt{\frac{-k}{2m}} = \pm\frac{ik}{2m}$$

neglected $\because$ frequencies cannot be Imaginary or negative

and $\omega^2 m = \frac{k}{2} \Rightarrow \omega^2 = \frac{k}{2m} \Rightarrow \omega = \pm\sqrt{\frac{k}{2m}}$

$\omega = + \sqrt{\frac{K}{2m}}$ because frequency cannot be negative.