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Constraints, Constrained Motion and Degree of freedom

EOM : $m_i \ddot{\vec{r}}_i = \vec{F}_i^{(e)} + \sum_{\substack{i,j \\ i \neq j}} \vec{F}_{ji}$

- Constrained Motion
- Classification of Constraints
- Forces of Constraint
- Degree of freedom
- Generalised Coordinates
- Notations (Generalised)

$f(\vec{r}_1, \vec{r}_2, \vec{r}_3, \dots, \vec{r}_{3N}; t) = 0$ Holonomic Constraints

Rewrite $f(x_1, y_1, z_1, x_2 y_2 z_2, \dots, x_N y_N z_N ; t) = 0$

Non-holonomic Constraints : Inequality

$f(\vec{r}_1, \vec{r}_2, \vec{r}_3, \dots \vec{r}_{3N}; t) \gtrless 0$

$f(\vec{r}, \dot{\vec{r}}, t) = 0$ Bilateral Constraints

≥ 0 Unilateral Constraint

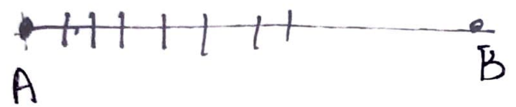
$f(\vec{r}, \ddot{\vec{r}}, t)$

] Differential Constraint / Kinematic Constraint

$f(\vec{r}, t) = 0$ or $f(\vec{r}) \rightarrow$ Geometric Constraint

Constraint eq^n has explicit dependence upon time : Rheonomous Constraint

" no " " " : Scleronomous | stationary constraints



system of N particles subjected to K (say) independent constraints such that they can be expressed as,

$$g_1(r_1, r_2, r_3, \dots, r_{3N}; t) = a_1$$

$$g_2(r_1, r_2, r_3, \dots, r_{3N}; t) = a_2$$

$$\vdots$$

$$g_K(r_1, r_2, r_3, \dots, r_{3N}; t) = a_K$$

$f = 3N - K$

EOM : $\sum_{i,j=1}^{N} \vec{F}_{ij} \overset{\text{modified}}{\Longrightarrow} \sum_{i,j=1}^{f} \vec{F}_{ij}$

Generalised Coord^n : q's, $q_1, q_2, q_3, \dots, q_f$

$q_1 = r$

$q_2 = \theta$

$q_3 = \phi$

① Generalised Displacement : $\delta \vec{r}_i$ [virtual arbitrary Displacement]

held t fixed

$$\vec{r}_i = \vec{r}_i(q_1, q_2, q_3, \dots, q_{3N}; t)$$

Euler's theorem: $X = X(x_1, x_2, x_3)$

$$dX = \frac{\partial X}{\partial x_1} dx_1 + \frac{\partial X}{\partial x_2} dx_2 + \frac{\partial X}{\partial x_3} dx_3$$

$$\boxed{\delta \vec{r}_i = \sum_{j=1}^{3N} \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j}$$

t = fixed.

② Generalised Velocity : $\dot{\vec{r}}_i$

$$\vec{r}_i = \vec{r}_i(q_1, q_2, q_3, \dots, q_{3N}; t)$$

$$\boxed{\dot{\vec{r}}_i = \sum_{j=1}^{3N} \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \vec{r}_i}{\partial t}} \quad (A)$$

$q_j \to x$ or y or z $\quad \dot{q}_j$ = linear velocity

$q_j \to$ linear Momentum $\quad \dot{q}_j$ = force.

③ Generalised Acceleration :- $\ddot{\vec{r}}_i$

$$\ddot{\vec{r}}_i = \frac{d}{dt}\left(\dot{\vec{r}}_i\right)$$

using (A)

$$\ddot{\vec{r}}_i = \frac{d}{dt}\left(\dot{\vec{r}}_i\right) = \frac{d}{dt}\left[\sum_{j=1}^{3N} \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \vec{r}_i}{\partial t}\right] = \sum_{j=1}^{3N} \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)\dot{q}_j + \sum_{j=1}^{3N} \frac{\partial \vec{r}_i}{\partial q_j}\ddot{q}_j + \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial t}\right)$$

$$\Rightarrow \sum_{j=1}^{3N} \frac{\partial \dot{\vec{r}}_i}{\partial q_j}\dot{q}_j + \sum_{j=1}^{3N} \frac{\partial \vec{r}_i}{\partial q_j}\ddot{q}_j + \frac{\partial \dot{\vec{r}}_i}{\partial t}.$$

$$\rightarrow \sum_{j=1}^{3N} \frac{\partial}{\partial q_j}\left(\sum_k \frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \vec{r}_i}{\partial t}\right)\dot{q}_j + \sum_{j=1}^{3N} \frac{\partial \vec{r}_i}{\partial q_j}\ddot{q}_j + \cancel{\frac{\partial \vec{r}_i}{\partial t}}\frac{\partial}{\partial t}\left[\sum_k \frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \vec{r}_i}{\partial t}\right]$$

$$= \sum_{j=1}^{3N}\sum_{k=1}^{3N} \frac{\partial^2 \vec{r}_i}{\partial q_j \partial q_k}\dot{q}_k\dot{q}_j + 2\sum_{j=1}^{3N} \frac{\partial^2 \vec{r}_i}{\partial q_j \partial t}\dot{q}_j + \sum_{j=1}^{3N} \frac{\partial \vec{r}_i}{\partial q_j}\ddot{q}_j + \sum_{k=1}^{3N} \cancel{\frac{\partial^2 \vec{r}_i}{\partial q_k \partial t}}\dot{q}_k +$$

$$\frac{\partial^2 \vec{r}_i}{\partial t^2}$$

$$= \sum_{j=1}^{3N}\sum_{k=1}^{3N} \frac{\partial^2 \vec{r}_i}{\partial q_j \partial q_k}\dot{q}_k\dot{q}_j + \underline{\sum_{j=1}^{3N} \frac{\partial \vec{r}_i}{\partial q_j}\ddot{q}_j} + 2\sum_{j=1}^{3N} \frac{\partial^2 \vec{r}_i}{\partial q_j \partial t}\dot{q}_j + \frac{\partial^2 \vec{r}_i}{\partial t^2}$$

Generalised Kinetic Energy $\quad T = \frac{1}{2} m_i \dot{\vec{r}}_i^{\,2} = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i$

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i \cdot \left(\sum_{j=1}^{3N} \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \vec{r}_i}{\partial t}\right)\cdot\left(\sum_{k=1}^{3N} \frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \vec{r}_i}{\partial t}\right)$$

$$T = \sum_{i=1}^{N}\sum_{j=1}^{3N}\sum_{k=1}^{3N} \frac{1}{2} m_i \left[\frac{\partial \vec{r}_i}{\partial q_j}\frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_j\dot{q}_k\right] + \sum_{i=1}^{N}\sum_{k} \frac{1}{2} m_i \frac{\partial \vec{r}_i}{\partial t}\frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_k + \sum_{i=1}^{N}\sum_{j=1}^{3N}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)\left(\frac{\partial \vec{r}_i}{\partial t}\right)\dot{q}_j$$

$$+ \sum_{i} \frac{1}{2} m_i \left(\frac{\partial \vec{r}_i}{\partial t}\right)^2$$

In 2nd and 3rd $\underline{\underline{j=k}}$

$$T = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{3N}\sum_{k=1}^{3N} m_i \left(\frac{\partial \vec{r}_i}{\partial q_j}\right)\left(\frac{\partial \vec{r}_i}{\partial q_k}\right)\dot{q}_j\dot{q}_k + 2\times\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{3N} m_i \left(\frac{\partial \vec{r}_i}{\partial q_j}\right)\cdot\dot{q}_j\left(\frac{\partial \vec{r}_i}{\partial t}\right) + \underbrace{\frac{1}{2}\sum_{i=1}^{N} m_i \left(\frac{\partial \vec{r}_i}{\partial t}\right)^2}$$

$$= \underbrace{T^{(2)}}_{\text{if no explicit dependance on time}} + T^{(1)} + T^{(0)}$$

1st term

$$T^{(2)} = \frac{1}{2}\sum_{j,k} m_i \left(\frac{\partial \vec{r}_i}{\partial q_j}\right)\left(\frac{\partial \vec{r}_i}{\partial q_k}\right)\dot{q}_j\dot{q}_k = \sum_{j,k} a_{jk}\dot{q}_j\dot{q}_k$$

$$a_{jk} = \frac{1}{2}\sum_{i=1}^{N} m_i \underbrace{\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)\cdot\left(\frac{\partial \vec{r}_i}{\partial q_k}\right)}.$$

~~wherever~~ whenever $\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)\cdot\left(\frac{\partial \vec{r}_i}{\partial q_k}\right) = 0$ ORTHOGONAL SYSTEM

Generalised Momentum :- $p_j = \frac{\partial T}{\partial \dot{q}_j}$. MLT^{-1} not necessary

Cartesian system, $p_{x_i} = \frac{\partial T}{\partial \dot{x}_i} = m_i \dot{x}_i$