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Vector Analysis
Complex variables
Partial differential eq^n
Numerical Analysis

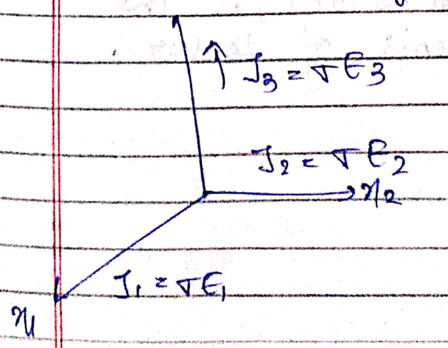
Tensor is a quantity which is invariant under ~~quadrant~~ coordinate transformation & which obeys certain transformation laws.

Let 'V' is linear f^n of 'u'

$$\left.\begin{aligned} v_x &= a_{xx} u_x + a_{xy} u_y + a_{xz} u_z \\ v_y &= a_{yx} u_x + a_{yy} u_y + a_{yz} u_z \\ v_z &= a_{zx} u_x + a_{zy} u_y + a_{zz} u_z \end{aligned}\right\}$$

total 9 components
$3^2 = 9$
$2 \rightarrow$ rank

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \underbrace{\begin{bmatrix} a_{xx} & a_{xy} & a_{xz} \\ a_{yx} & a_{yy} & a_{yz} \\ a_{zx} & a_{zy} & a_{zz} \end{bmatrix}}_{\text{Coefficients of tensor}} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$$



for isotropic material two physical quantities will be same -

$$\boxed{J = \sigma E}$$

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$$J_1 = \sigma_{11} E_1 + \sigma_{12} E_2 + \sigma_{13} E_3$$
$$J_2 = \sigma_{21} E_1 + \sigma_{22} E_2 + \sigma_{23} E_3$$
$$J_3 = \sigma_{31} E_1 + \sigma_{32} E_2 + \sigma_{33} E_3$$

} Anisotropic material

if $E_2 = E_3 = 0 \Rightarrow$
$$J_1 = \sigma_{11} E_1$$
$$J_2 = \sigma_{21} E_1$$
$$J_3 = \sigma_{31} E_1$$

$$\begin{bmatrix} J_1 \\ J_2 \\ J_3 \end{bmatrix} = \underbrace{\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}}_{\text{coeff. of tensor}} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}$$

Rank of a tensor :- The rank of the tensor is equal to the number of suffixes or indices attached to it.

OR

The Rank of a tensor when raised to a power to the dimensions, is the number of components of tensor.

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☆ **Einstein's Summation Convention :-**

$$S = a_1 x^1 + a_2 x^2 + \dots + a_n x^n$$
$$= \sum_{i=1}^{n} a_i x^i = a_i x^i$$

If any index is repeating two times in an expression, there is no need to write the summation symbol.

Dummy & real index :-

↳ umbral , dextral index

$$a_i x^i = a_1 x^1 + a_2 x^2 + \dots + a_n x^n$$
$$a_j x^j = a_1 x^1 + a_2 x^2 + \dots + a_n x^n$$
$$a_i x^i = a_j x^j$$

$\boxed{a_i^j x^i} \Rightarrow$ $i \rightarrow$ dummy index
$j \rightarrow$ real index

☆ **Kronecker Delta (δ) =**

$$\delta^i_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

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(i) $x^1, x^2, \ldots, x^n$ are independent coordinates,

$$\delta^i_j = \frac{\partial x^i}{\partial x^j} = \frac{\partial x^i}{\partial x^k}\frac{\partial x^k}{\partial x^j}$$

(ii) $\delta^j_k \cdot a^j = a^k$

proof:

$$\delta^j_k a^j = a^1\delta^1_k + a^2\delta^2_k + a^3\delta^3_k + \cdots + a^k\delta^k_k + \cdots + a^n\delta^n_2$$

$$\delta^j_k a^j = a^k$$

(iii) $\delta^i_i = n$

proof:

$$\delta^i_i = \delta^1_1 + \delta^2_2 + \delta^3_3 + \cdots + \delta^n_n$$
$$= 1 + 1 + 1 + \cdots + 1$$
$$= n$$

(iv) $\delta^i_j \delta^j_k = \delta^i_k$

proof:

$$\delta^i_j \delta^j_k = \delta^i_1\delta^1_k + \delta^i_2\delta^2_k + \delta^i_3\delta^3_k + \cdots + \delta^i_i\delta^i_k + \cdots + \delta^i_n\delta^n_k$$

$$\delta^i_j \delta^j_k = \delta^i_k$$

Q. Expand $a_{ij}x^i x^j$

$$a_{ij}x^i x^j = a_{1i}x^{i+1} + a_{2i}x^{i+2} + a_{3i}x^{i+3} + \cdots + a_{ni}x^{i+n}$$

$$= \cancel{a_1 x^2 + a_2 x^3 + a_3 x^4 + \cdots + a_n x^{n+1}}$$

$$= a_{11}x^2 + a_{22}(x^2)^2 + a_{33}(x^3)^2 + \cdots + a_{nn}(x^n)^2$$

Q. If $a_{ij}x^i x^j = 0$, $a_{ij} \to$ constant, show that $a_{ij} + a_{ji} = 0$.

$$a_{ij}x^i x^j = 0$$

$$a_{lm}x^l x^m = 0$$

$$\frac{\partial}{\partial x^i}(a_{lm}x^l x^m) = 0$$

$$a_{lm}\left(\frac{\partial x^l}{\partial x^i}\right)x^m + a_{lm}\left(\frac{\partial x^m}{\partial x^i}\right)x^l = 0$$

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$$a_{lm}\,\delta^l_i\,x^m + a_{lm}\,\delta^m_i\,x^l = 0$$

$$a_{im}\,x^m + a_{li}\,x^l = 0$$

diff. w.r.t. x_j

$$a_{im}\frac{\partial x^m}{\partial x_j} + a_{li}\frac{\partial x^l}{\partial x_j} = 0$$

$$a_{im}\,\delta^m_j + a_{li}\,\delta^l_j = 0$$

$$\boxed{a_{ij} + a_{ji} = 0}$$

→ index is written as superscript

Contravariant tensor :-

n quantities → $A^1, A^2, A^3, \ldots, A^n \rightarrow (x^1, x^2, \ldots, x^n)$

$\bar{A}^1, \bar{A}^2, \bar{A}^3, \ldots, \bar{A}^n \rightarrow (\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n)$

$$\bar{A}^p = \sum_{n=1}^{n} \frac{\partial \bar{x}^p}{\partial x^n} \cdot A^n \quad (p = 1, 2, \ldots, n)$$

Components of contravariant vector

$$\boxed{\bar{A}^p = \frac{\partial \bar{x}^p}{\partial x^n} A^n}$$

m

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multiplying $\frac{\partial x^k}{\partial \bar{x}^p}$ both sides

$$\frac{\partial x^k}{\partial \bar{x}^p} \cdot \bar{A}^p = \frac{\partial x^k}{\partial \bar{x}^p} \cdot \frac{\partial \bar{x}^p}{\partial x^n} \cdot A^n$$

$$= \frac{\partial x^k}{\partial x^n} \cdot A^n$$

$$\boxed{A^n = \frac{\partial x^n}{\partial \bar{x}^p} \cdot \bar{A}^p} \quad \text{if } (k = n)$$

If 'n' quantities $A^1, A^2, \ldots, A^n$ in a coordinate system $x^1, x^2, \ldots, x^n$ are related to other quantities $\bar{A}^1, \bar{A}^2, \bar{A}^3, \ldots, \bar{A}^n$ in another coordinate system $\bar{x}^1, \bar{x}^2, \bar{x}^3, \ldots, \bar{x}^n$, then if the quantities obey transformation law, then

Covariant Tensors :- (index is written as subscript)

$$\bar{A}_p = \sum_{n=1}^{n} \frac{\partial x^n}{\partial \bar{x}^p} \cdot A_n$$

$$\boxed{\bar{A}_p = \frac{\partial x^n}{\partial \bar{x}^p} \cdot A_n}$$

multiplying both sides by $\frac{\partial \bar{x}^k}{\partial x^m}$

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$$\frac{\partial \bar{x}^k}{\partial x^n} \bar{A}_p = \frac{\partial x^n}{\partial \bar{x}^p} \cdot \frac{\partial \bar{x}^k}{\partial x^n} A_m$$

$$= \delta^k_p A_m$$

$$\boxed{A_n = \frac{\partial \bar{x}^p}{\partial x^n} \cdot \bar{A}_p}$$

n^2 quantities – A^{ms} (m & s 1 to n) $\rightarrow x^1, x^2, \ldots x^n$
n^2 " – $\bar{A}^{pr}$ (p & r 1 to n) $\rightarrow \bar{x}^1, \bar{x}^2, \ldots \bar{x}^n$

$$\bar{A}^{pr} = \cancel{\frac{\partial \bar{x}^p}{\partial x^m}} \frac{\partial \bar{x}^p}{\partial x^n} \frac{\partial \bar{x}^r}{\partial x^s} A^{ms}$$

$$\bar{A}_{pr} = \frac{\partial x^n}{\partial \bar{x}^p} \cdot \frac{\partial x^s}{\partial \bar{x}^r} \cdot A_{ms}$$

$$\bar{A}^p_r = \frac{\partial \bar{x}^p}{\partial x^m} \frac{\partial x^s}{\partial \bar{x}^r} \cdot A^m_s$$

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$$A^{mst}_{kl}$$

$$\bar{A}^{prm}_{ij} = \frac{\partial \bar{x}^p}{\partial x^n} \frac{\partial \bar{x}^r}{\partial x^s} \frac{\partial \bar{x}^m}{\partial x^t} \cdot A$$

Q. prove that kronecker delta is a mixed tensor of rank 2.

$$\delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$\delta^i_j = \frac{\partial x^i}{\partial x_j}$$

$$\bar{\delta}^k_l = \frac{\partial \bar{x}^k}{\partial \bar{x}^l} = \frac{\partial \bar{x}^k}{\partial x^i} \cdot \frac{\partial x^i}{\partial \bar{x}^l} \cdot \frac{\partial x^i}{\partial x^j}$$

$$\bar{\delta}^k_l = \frac{\partial \bar{x}^k}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^l} \delta^i_j$$

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☆ Equality of Tensors :-

$$A^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q} = B^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}$$

$$A^{\mu\nu}_{\sigma\lambda} = B^{\mu\nu}_{\sigma\lambda}$$

☆ Addition -

$$A_{ik} + B_{ik} = C_{ik}$$

$$A_{ik} - B_{ik} = D_{ik}$$

⊙ Note :- ~~If~~ two tensors of same rank & same type are said to be equal if their components are 1 to 1 equal.

OR

Two tensors are said to be equal, if & only if they have the same contravariant rank & covariant rank & every component of one is equal to the corresponding component of the other.

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⊙ The sum of two tensor of same rank & same type is also a tensor of same rank & type.

⊙ The difference of two tensors of same rank & same type is also a tensor of same rank & type.

eg -

$$A_{ik} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$B_{ik} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$A_{ik} + B_{ik} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} = C_{ik}$$

Q. If A^{pq}_{r}, B^{pq}_{r} are tensors, prove that there their sum & difference are tensors of same rank & same type.

$$A^{pq}_{r} + B^{pq}_{r} = C^{pq}_{r}$$

$$\bar{A}^{jk}_{l} = \frac{\partial \bar{x}^j}{\partial x^p} \cdot \frac{\partial \bar{x}^k}{\partial x^q} \cdot \frac{\partial x^r}{\partial \bar{x}^l} \cdot A^{pq}_{r}$$

$$\bar{B}^{jk}_{l} = \frac{\partial \bar{x}^j}{\partial x^p} \cdot \frac{\partial \bar{x}^k}{\partial x^q} \frac{\partial x^r}{\partial \bar{x}^l} \cdot B^{pq}_{r}$$

$$\bar{A}^{jk}_{l} + \bar{B}^{jk}_{l} = \left(A^{pq}_{r} + B^{pq}_{r}\right) \frac{\partial \bar{x}^j}{\partial x^p} \cdot \frac{\partial \bar{x}^k}{\partial x^q} \cdot \frac{\partial x^r}{\partial \bar{x}^l}$$

$$\bar{C}^{jk}_{l} = \frac{\partial \bar{x}^j}{\partial x^p} \frac{\partial \bar{x}^k}{\partial x^q} \frac{\partial x^r}{\partial \bar{x}^l} C^{pq}_{r}$$

& $$A^{pq}_{r} - B^{pq}_{r} = C^{pq}_{r}$$

$$\bar{A}^{jk}_{l} = \frac{\partial \bar{x}^j}{\partial x^p} \frac{\partial \bar{x}^k}{\partial x^q} \frac{\partial x^r}{\partial \bar{x}^l} A^{pq}_{r}$$

$$\bar{B}^{jk}_{l} = \frac{\partial \bar{x}^j}{\partial x^p} \frac{\partial \bar{x}^k}{\partial x^q} \frac{\partial x^r}{\partial \bar{x}^l} B^{pq}_{r}$$

$$\bar{A}^{jk}_{l} - \bar{B}^{jk}_{l} = \frac{\partial \bar{x}^j}{\partial x^p} \frac{\partial \bar{x}^k}{\partial x^q} \frac{\partial x^r}{\partial \bar{x}^l} \left(A^{pq}_{r} - B^{pq}_{r}\right)$$

$$\bar{C}^{jk}_{l} = \frac{\partial \bar{x}^j}{\partial x^p} \cdot \frac{\partial \bar{x}^k}{\partial x^q} \cdot \frac{\partial x^r}{\partial \bar{x}^l} C^{pq}_{r}$$

Contraction: The algebraic operation by which we can reduce the rank of a tensor by 2, is called as contraction.

$$\bar{A}^{ijk}_{lm} = \frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial \bar{x}^j}{\partial x^q} \frac{\partial \bar{x}^k}{\partial x^r} \frac{\partial x^s}{\partial \bar{x}^l} \frac{\partial x^t}{\partial \bar{x}^m} A^{pqr}_{st}$$

rank = 5

$r = t$

$$\bar{A}^{ijk}_{lm} = \frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial \bar{x}^j}{\partial x^q} \frac{\partial \bar{x}^k}{\partial x^r} \frac{\partial x^s}{\partial \bar{x}^l} \frac{\partial x^r}{\partial \bar{x}^m} A^{pqr}_{sr}$$

$$\bar{A}^{ijk}_{lm} = \frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial \bar{x}^j}{\partial x^q} \bar{\delta}^k_m \frac{\partial x^s}{\partial \bar{x}^l} A^{pqr}_{sr}$$

if $k = m$

$$\bar{A}^{ijk}_{lk} = \frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial \bar{x}^j}{\partial x^q} \cdot \frac{\partial x^s}{\partial \bar{x}^l} \cdot A^{pqr}_{sr}$$

$$\boxed{\bar{A}^{ij}_{l} = \frac{\partial \bar{x}^i}{\partial x^p} \cdot \frac{\partial \bar{x}^j}{\partial x^q} \frac{\partial x^s}{\partial \bar{x}^l} A^{pq}_{s}} \rightarrow \text{rank} = 3$$

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Q. Apply contraction on A^p_q.

Multiplication of tensors :-

- Outerproduct : The outerproduct of two tensor is a tensor whose rank is the sum of the ranks of two multiplied tensors.

$$A^{pr}_q B^m_s = C^{prm}_{qs}$$

$$A^{mn} B^{pq} = C^{mnpq}$$

$$\bar{A}^{ij}_k = \frac{\partial \bar{x}^i}{\partial x^p} \cdot \frac{\partial \bar{x}^j}{\partial x^r} \cdot \frac{\partial x^q}{\partial \bar{x}^k} \cdot A^{pr}_q$$

$$\bar{B}^a_b = \frac{\partial \bar{x}^a}{\partial x^m} \frac{\partial x^s}{\partial \bar{x}^b} B^m_s$$

$$\bar{A}^{ij}_k \cdot \bar{B}^a_b = \frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial \bar{x}^j}{\partial x^r} \frac{\partial x^q}{\partial \bar{x}^k} \cdot \frac{\partial \bar{x}^a}{\partial x^m} \cdot \frac{\partial x^s}{\partial \bar{x}^b} \cdot B^m_s \cdot A^{pr}_q$$

$$\bar{C}^{ija}_{kb} = \frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial \bar{x}^j}{\partial x^r} \cdot \frac{\partial x^q}{\partial \bar{x}^k} \cdot \frac{\partial \bar{x}^a}{\partial x^m} \frac{\partial x^s}{\partial \bar{x}^b} C^{prm}_{sq}$$

- Innerproduct : The outerproduct of two tensors followed by contraction results in a new tensor called innerproduct of two tensors.

$$A^{mp}_q \cdot B^r_s = C^{mpr}_{qs}$$

set $r = s$

$$A^{mp}_q \cdot B^r_s = C^{mp}_q$$

Q. If A^i_j is mixed tensor of rank 2 & B^{kl}_m is a mixed tensor of rank 3. Prove that $A^i_j B^{kl}_m$ is a mixed tensor of rank 3.

OR

Show that any innerproduct of the tensor A^i_j & B^{kl}_m is a tensor of rank 3.

Quotient Law :-

If $A^i B_{ek}$ is a tensor for all contravariant tensors A^i, then B_{ek} is also a tensor.

$$\bar{A}^\alpha \bar{B}_{\beta\gamma} = \frac{\partial \bar{x}^\alpha}{\partial x^c} \frac{\partial x^e}{\partial \bar{x}^\beta} \frac{\partial x^k}{\partial \bar{x}^\gamma} A^i B_{ek}$$

$\because A^i$ is a tensor

$$\bar{A}^\alpha = \frac{\partial \bar{x}^\alpha}{\partial x^i} A^i$$

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$$\bar{A}^{\alpha}\bar{B}_{\beta\gamma} = \frac{\partial x^{l}}{\partial \bar{x}^{\beta}}\frac{\partial x^{k}}{\partial \bar{x}^{\gamma}}\bar{A}^{\alpha} B_{lk}$$

$$\bar{A}^{\alpha}\left[\bar{B}_{\beta\gamma} - \frac{\partial x^{l}}{\partial \bar{x}^{\beta}}\frac{\partial x^{k}}{\partial \bar{x}^{\gamma}} B_{lk}\right] = 0$$

$\Rightarrow \because \bar{A}^{\alpha} \neq 0$

$$\Rightarrow \boxed{\bar{B}_{\beta\gamma} = \frac{\partial x^{l}}{\partial \bar{x}^{\beta}}\frac{\partial x^{k}}{\partial \bar{x}^{\gamma}} B_{lk}}$$

$\Rightarrow B_{lk}$ is also a tensor.

Symmetric tensor - A tensor is called symmetric w.r.t. 2 contravariant or 2 covariant indices if its ~~compound~~ components remain unaltered ~~after~~ upon interchange of indices.

$$A^{ij} = A^{ji}$$

$\Rightarrow A^{ij}$ is symmetric w.r.t. i,j.

$$A^{mpr}_{qs} = A^{pmr}_{qs}$$

$\Rightarrow A^{mpr}_{qs}$ is symmetric w.r.t. m & p.

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Skew-Symmetric Tensor - A tensor is called skew-symmetric w.r.t. 2 contravariant or 2 covariant indices if its components changes sign upon interchange of indices.

$$A^{ij} = -A^{ji}$$

$$A^{mpr}_{qs} = -A^{pmr}_{qs}$$

Q. Show that every tensor can be expressed as the sum of two tensors one of which is symmetric & the other is skew-symmetric in a pair of covariant & contravariant indices.

Let B^{pq} is a tensor.

$$B^{pq} = \frac{1}{2}\left(B^{pq} + B^{qp}\right) + \frac{1}{2}\left(B^{pq} - B^{qp}\right)$$

$$= C^{pq} + D^{pq}$$

$$C^{pq} = \frac{1}{2}\left(B^{pq} + B^{qp}\right)$$

$$= \frac{1}{2}\left(B^{qp} + B^{pq}\right)$$

$C^{pq} = C^{qp} \rightarrow$ Symmetric

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$$D^{pq} = \frac{1}{2}\left(B^{pq} - B^{qp}\right)$$

$$D^{pq} = -\frac{1}{2}\left(B^{qp} - B^{pq}\right)$$

$$D^{pq} = -D^{qp} \longrightarrow \text{skew-symmetric}$$

If a tensor B^{pq} is symmetric w.r.t. indices p & q in one coordinate system. Show that it remain symmetric w.r.t. p & q in any coordinate system.

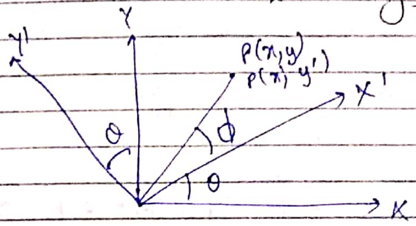
$$\bar{B}^{jk} = \frac{\partial \bar{x}^j}{\partial x^p}\frac{\partial \bar{x}^k}{\partial x^q} B^{pq}$$

if B^{pq} is symmetric, $B^{pq} = B^{qp}$

$$\bar{B}^{jk} = \frac{\partial \bar{x}^k}{\partial x^q}\frac{\partial \bar{x}^j}{\partial x^p} B^{qp}$$

$$\bar{B}^{jk} = \bar{B}^{kj}$$

Q. Show that the array $A = \begin{bmatrix} -xy & -y^2 \\ x^2 & xy \end{bmatrix}$ is a tensor & $B = \begin{bmatrix} -xy & -y^2 \\ x^2 & -xy \end{bmatrix}$ is not a tensor.



$$\boxed{\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \end{aligned}}$$

$$\boxed{\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ y &= x'\sin\theta + y'\cos\theta \end{aligned}}$$

for A -

$$A = \begin{bmatrix} -xy & -y^2 \\ x^2 & xy \end{bmatrix}$$

$A^{11} = -xy$, $A^{12} = -y^2$, $A^{21} = x^2$, $A^{22} = xy$

$$\frac{\partial x'}{\partial x} = \cos\theta, \quad \frac{\partial x'}{\partial y} = \sin\theta, \quad \frac{\partial y'}{\partial x} = -\sin\theta,$$

$$\frac{\partial y'}{\partial y} = \cos\theta$$

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↳ for A

$$-x'y' = (A^{11})' = \sum_{\alpha,\beta=1}^{2} \frac{\partial x_1'}{\partial x^\alpha}\frac{\partial x_1'}{\partial x^\beta} A^{\alpha\beta}$$

$$-x'y' = \frac{\partial x'}{\partial x}\frac{\partial x'}{\partial x}A^{11} + \frac{\partial x'}{\partial x}\frac{\partial x'}{\partial y}A^{12} + \frac{\partial x'}{\partial y}\frac{\partial x'}{\partial x}A^{21} + \frac{\partial x'}{\partial y}\frac{\partial x'}{\partial y}A^{22}$$

$$-x'y' = A^{11}\cos^2\theta + A^{12}\sin\theta\cos\theta + A^{21}\sin\theta\cos\theta + A^{22}\sin^2\theta$$

$$-x'y' = -xy\cos^2\theta - y^2\sin\theta\cos\theta + x^2\sin\theta\cos\theta + xy\sin^2\theta$$

$$-x'y' = -y\cos\theta(x\cos\theta + y\sin\theta) + x\sin\theta(x\cos\theta + y\sin\theta)$$

$$-x'y' = (x\sin\theta - y\cos\theta)(x\cos\theta + y\sin\theta)$$

$$-x'y' = (-y')x'$$

$$-x'y' = -x'y'$$

For B -

$$B = \begin{bmatrix} -xy & -y^2 \\ x^2 & -xy \end{bmatrix}$$

$B^{11} = -xy$, $B^{12} = -y^2$, $B^{21} = x^2$ & $B^{22} = -xy$

$\frac{\partial x'}{\partial x} = \cos\theta$, $\frac{\partial x'}{\partial y} = \sin\theta$, $\frac{\partial y'}{\partial x} = -\sin\theta$, $\frac{\partial y'}{\partial y} = \cos\theta$

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↳

$$-x'y' = (B^{11})' = \sum_{\alpha\beta=1}^{2} \frac{\partial x_1'}{\partial x^\alpha}\frac{\partial x_1'}{\partial x^\beta} B^{\alpha\beta}$$

$$-x'y' = \frac{\partial x'}{\partial x}\frac{\partial x'}{\partial x}B^{11} + \frac{\partial x'}{\partial x}\frac{\partial x'}{\partial y}B^{12} + \frac{\partial x'}{\partial y}\frac{\partial x'}{\partial x}B^{21} + \frac{\partial x'}{\partial y}\frac{\partial x'}{\partial y}B^{22}$$

$$-x'y' = (-xy)\cos^2\theta + (-y^2)\sin\theta\cos\theta + x^2\sin\theta\cos\theta - xy\sin^2\theta$$

$$+(x\cos\theta + y\sin\theta)(+x\sin\theta - y\cos\theta) = -xy\cos^2\theta - y^2\sin\theta\cos\theta + x^2\sin\theta\cos\theta - xy\sin^2\theta$$

$$x^2\sin\theta\cos\theta - xy\cos^2\theta + xy\sin^2\theta - y^2\sin\theta\cos\theta = -xy\cos^2\theta - y^2\sin\theta\cos\theta + x^2\sin\theta\cos\theta - xy\sin^2\theta$$

$$\Rightarrow xy\sin^2\theta \neq -xy\sin^2\theta$$

Q. Show that (i) a symmetric tensor of 2nd order has only $\frac{1}{2}n(n+1)$ different components.
(ii) a skew-symmetric tensor of 2nd order has only $\frac{1}{2}(n)(n-1)$ different non-zero components.

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(i) Let the tensor be of n^{th} dimension.

$\therefore$ Total no. of elements (components) $= n^2$

Let the diagonal element are different.

$\therefore$ Remaining elements $= n^2 - n = n(n-1)$

$\because$ the tensor is symmetric (Let A^{ij} be the tensor) $\Rightarrow A^{ij} = A^{ji}$

$\therefore A^{12} = A^{21},\ A^{13} = A^{31}, \ldots$ (excluding diagonal)

$\therefore$ different components $= \dfrac{n(n-1)}{2}$

total different components $= \dfrac{n^2-n}{2} + n$

$= \dfrac{n(n+1)}{2}$

(ii) Total no. of elements $= n^2$

Diagonal element are A^{ii}

$\because A^{ij}$ is a skew-symmetric tensor

$A^{ij} = -A^{ji}$

$\therefore A^{ii} = -A^{ii}$

$\therefore A^{ii} = 0$

$\therefore$ The diagonal elements are zero.

$\therefore$ Total non-zero components $= n^2 - n$

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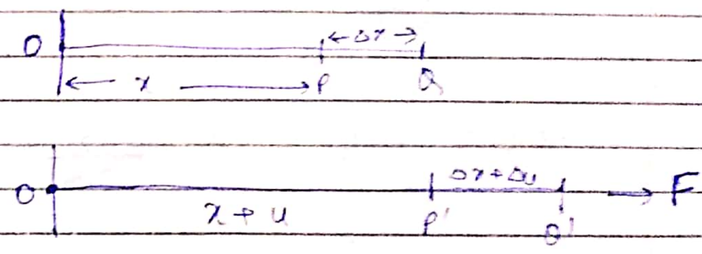
now, $A^{12} = -A^{21},\ A^{13} = -A^{31}, \ldots$

$\therefore$ total different non-zero elements $= \dfrac{n(n-1)}{2}$

Strain :-

1-D strain

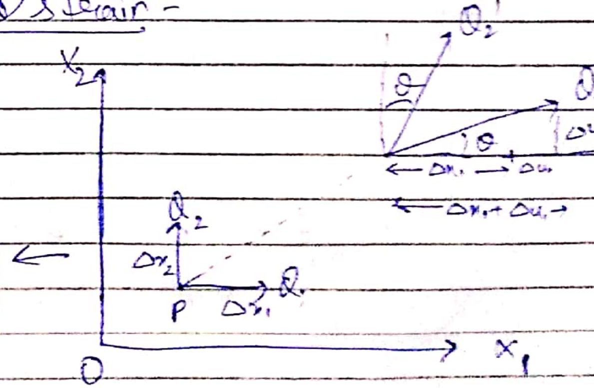


strain $= \dfrac{\Delta u}{\Delta x}$

strain at a point, $e = \lim\limits_{\Delta x \to 0} \dfrac{\Delta u}{\Delta x} = \dfrac{du}{dx}$

2-D strain -

$$\begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix}$$



Since it is 2-D tensor, we need 4 components —

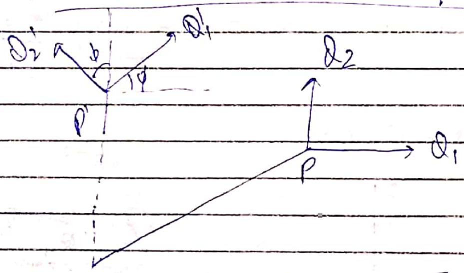
$$e_{11} = \frac{\partial u_1}{\partial x_1}, \quad e_{22} = \frac{\partial u_2}{\partial x_2}$$

$$e_{12} = \frac{\partial u_1}{\partial x_2}, \quad e_{21} = \frac{\partial u_2}{\partial x_1}$$

$$\tan\theta = \frac{\Delta u_2}{\Delta x_1 + \Delta u_1} = \frac{\Delta u_2}{\Delta x_1}$$

$$\tan\theta = \frac{\partial u_2}{\partial x_1} = e_{21}$$

Rotation without extension / compression



$$e = \begin{bmatrix} 0 & -\phi \\ \phi & 0 \end{bmatrix}$$

$$e_{ij} = \frac{(e_{ij} + e_{ji})}{2} + \frac{(e_{ij} - e_{ji})}{2}$$

$$e = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix}$$

$$= \begin{bmatrix} e_{11} & \frac{(e_{12}+e_{21})}{2} & \frac{(e_{13}+e_{31})}{2} \\ \frac{e_{21}+e_{12}}{2} & e_{22} & \frac{e_{23}+e_{32}}{2} \\ \frac{e_{31}+e_{13}}{2} & \frac{e_{32}+e_{23}}{2} & e_{33} \end{bmatrix} + \begin{bmatrix} 0 & \frac{e_{12}-e_{21}}{2} & \frac{e_{13}-e_{31}}{2} \\ \frac{e_{21}-e_{12}}{2} & 0 & \frac{e_{23}-e_{32}}{2} \\ \frac{e_{31}-e_{13}}{2} & \frac{e_{32}-e_{23}}{2} & 0 \end{bmatrix}$$

substitute $e_{12} = -e_{21}$, $e_{13} = -e_{31}$, ...

$$e = \begin{bmatrix} e_{11} & 0 & 0 \\ 0 & e_{22} & 0 \\ 0 & 0 & e_{33} \end{bmatrix} + \begin{bmatrix} 0 & e_{12} & -e_{13} \\ -e_{21} & 0 & e_{23} \\ e_{31} & -e_{32} & 0 \end{bmatrix}$$

$$e = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix}$$

Thermal Expansion :-

$$\epsilon_{ij} \propto \Delta T$$

$$\epsilon_{ij} = \alpha \Delta T$$

↳ coefficient of thermal expansion

$$\alpha = \frac{L_2 - L_1}{L_1 (T_2 - T_1)} = \frac{\text{strain}}{\Delta T}$$

$$L_2 = L_1 + L_1 \alpha \Delta T$$

$$L_2 = L_1 (1 + \alpha \Delta T)$$

$$\epsilon_{ij} = \alpha_{ij} \Delta T$$

$$\epsilon_1 = \alpha_1 \Delta T$$
$$\epsilon_2 = \alpha_2 \Delta T$$
$$\epsilon_3 = \alpha_3 \Delta T$$

$$\alpha_1 x_1^2 + \alpha_2 x_2^2 + \alpha_3 x_3^2 = 1$$

$$\delta V = L^3(1+\alpha_1)(1+\alpha_2)(1+\alpha_3) - L^3$$

Volume expansivity, $\frac{\delta V}{V} = \alpha_1 + \alpha_2 + \alpha_3$

Peizoelectric effect :-

Peizoelectricity → electricity developed from Pressure.

$$P \propto \sigma$$
$$P = d\sigma$$

↓

Peizoelectric Modulus

$$P_i = d_{ijk} \sigma_{jk} \quad (i, j, k = 1, 2, 3)$$

↳ 27 components

Stress is a symmetric tensor

$\therefore \sigma_{jk} = \sigma_{kj}$

∴ 27 components of 'd_{ijk}' will be reduced to 18 components

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$P_i = d_{in} \sigma_n \quad \left(\begin{matrix} i = 1,2,3 \\ n = 1,2,3,4,5,6 \end{matrix} \right)$

$d_{ijk} = d_{in} \quad (n = 1,2,3)$

$2 d_{ijk} = d_{in} \quad (n = 4,5,6)$

Note-

Tensor →	11	22	33	23, 32	31, 13	12, 21
Matrix →	1	2	3	4	5	6

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} & d_{13} & d_{14} & d_{15} & d_{16} \\ d_{21} & d_{22} & d_{23} & d_{24} & d_{25} & d_{26} \\ d_{31} & d_{32} & d_{33} & d_{34} & d_{35} & d_{36} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix}$$

Converse - Peizoelectric effect :-

$E \rightarrow$ electric field vector

$\varepsilon_{jk} \propto E_i$

$\varepsilon_{jk} = d_{ijk} E_i$

$\varepsilon = (d)^T E$

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} d_{11} & d_{21} & d_{31} \\ d_{12} & d_{22} & d_{32} \\ d_{13} & d_{23} & d_{33} \\ d_{14} & d_{24} & d_{34} \\ d_{15} & d_{25} & d_{35} \\ d_{16} & d_{26} & d_{36} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}$$

		ε_1 / σ_1	ε_2 / σ_2	ε_3 / σ_3	ε_4 / σ_4	ε_5 / σ_5	ε_6 / σ_6
E_1	P_1	d_{11}	d_{12}	d_{13}	d_{14}	d_{15}	d_{16}
E_2	P_2	d_{21}	d_{22}	d_{23}	d_{24}	d_{25}	d_{26}
E_3	P_3	d_{31}	d_{32}	d_{33}	d_{34}	d_{35}	d_{36}

$P = d\sigma \quad \& \quad \varepsilon = dE$

	Tensor	Matrix $\left(\begin{matrix} i = 1,2,3 \\ j = 1,2,3,4,5,6 \end{matrix} \right.$
Direct	$P_i = d_{ijk} \sigma_{jk}$	$P_i = d_{ij} \sigma_j$
Converse	$\varepsilon_{jk} = d_{ijk} E_i$	$\varepsilon_j = d_{ij} E_i$

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PARTIAL DIFFERENTIAL EQ^n's

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$z = z(x,y)$

$$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}$$

$$x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = z \rightarrow \text{1st order}$$

$$\frac{\partial^2 u}{\partial x^2} + u^2 \frac{\partial u}{\partial x} = f(x,y) \rightarrow 2^{nd} \text{ order}$$

$$\frac{\partial^2 z}{\partial u^2} + \left(\frac{\partial^2 z}{\partial u \partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^6 = 0$$

$\rightarrow 2^{nd}$ order

$\rightarrow$ Degree 2

Partial differential eq^n – An eq^n involving one or more partial derivatives of an unknown f^n of two or more independent variables.

(i) $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2} \rightarrow$ one-dimensional wave eq^n

(ii) $\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2} \rightarrow$ 1-D heat flow eq^n

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(iii) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \rightarrow$ 2-D Laplace eq^n

(iv) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y) \rightarrow$ 2-D Poisson's eq^n

(v) $\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \rightarrow$ 2-D wave eq^n

(vi) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \rightarrow$ 3-D Laplace eq^n

Q. $u = x^2 \sin t$

$\frac{\partial u}{\partial x} = 2x \sin t, \quad \frac{\partial u}{\partial t} = x^2 \cos t$

Method of Separation of Variables -

Q. $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$

Assume ~~$z = X(x) \cdot Y(y)$~~

$Z = X(x) \cdot Y(y)$

$\frac{\partial z}{\partial x} = X'(x) \cdot Y(y)$

$\frac{\partial^2 z}{\partial x^2} = X''(x) \cdot Y(y)$

$\frac{\partial z}{\partial y} = X(x) \cdot Y'(y)$

$\therefore \left[X''(x) - 2X(x)\right] Y(y) + X(x) \cdot Y'(y) = 0$

$\left[X''(x) - 2X(x)\right] Y(y) = -X(x) \cdot Y'(y)$

$\frac{X''(x) - 2X(x)}{X(x)} = -\frac{Y'(y)}{Y(y)} = p^2$

$\therefore X''(x) - 2X(x) - p^2 X(x) = 0$

$m^2 - 2m - p^2 = 0$

$m = \frac{2 \pm \sqrt{4 + 4p^2}}{2}$

~~$m = 1 \pm \sqrt{1+p^2}$~~

$m = 1 \pm \sqrt{1+p^2}$

~~$X(x) = C_1 \cos(\sqrt{1+p^2}\,x) + C_2 \sin(\sqrt{1+p^2}\,x)$~~

$X(x) = C_1 e^{(1+\sqrt{1+p^2})x} + C_2 e^{(1-\sqrt{1+p^2})x}$

& $\frac{-Y'(y)}{Y(y)} = p^2$

$p^2 Y(y) + Y'(y) = 0$

$Y'(y) + p^2 Y(y) = 0$

$m + p^2 = 0 \Rightarrow$ ~~$m = -p$~~ $m = -p^2$

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$$Y(y) = c_3 \cos py + c_4 \sin py$$

$$\therefore z = X(x) \cdot Y(y)$$

$$= \left[c_1 e^{(1+\sqrt{1+p^2})x} + c_2 e^{(1-\sqrt{1+p^2})x}\right] \cdot \left[c_3 \cos py + c_4 \sin py\right]$$

$$Y(y) = c_3 e^{-p^2 y}$$

$$z = \left[c_1 e^{(1+\sqrt{1+p^2})x} + c_2 e^{(1-\sqrt{1+p^2})x}\right] \cdot \left[c_3 e^{-p^2 y}\right]$$

$$z = \left[c_4 e^{(1+\sqrt{1+p^2})x} + c_5 e^{(1-\sqrt{1+p^2})x}\right]\left(e^{-p^2 y}\right)$$

Q. $3u_x + 2u_y = 0$

$$3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$$

$$u = X(x) \cdot Y(y)$$

$$\frac{\partial u}{\partial x} = X'(x) \cdot Y(y)$$

$$\frac{\partial u}{\partial y} = X(x) \cdot Y'(y)$$

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$$3X'(x)Y(y) + 2X(x) \cdot Y'(y) = 0$$

$$3X'(x)Y(y) = -2X(x) \cdot Y'(y)$$

$$\frac{3X'(x)}{X(x)} = \frac{-2Y'(y)}{Y(y)} = p$$

Now, $3X'(x) - pX(x) = 0$

$$3m - p = 0$$

$$m = p/3$$

$$X(x) = c_1 e^{\frac{p}{3}x}$$

&

$$-2Y'(y) - pY(y) = 0$$

$$Y'(y) + \frac{p}{2}Y(y) = 0$$

$$m = -p/2$$

$$\therefore Y(y) = c_2 e^{-\frac{p}{2}y}$$

$$\therefore u = c_3 e^{\left(\frac{p}{3}x - \frac{p}{2}y\right)}$$

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$$V = C_4 e^{-\frac{n^2\pi^2 t}{\ell^2}} \cdot \sin\left(\frac{n\pi x}{\ell}\right)$$

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Q. $\frac{\partial^2 V}{\partial x^2} = \frac{\partial V}{\partial t}$ $V = 0$ when $t \to \infty$
$V = 0$ at $x = 0$ & $x = \ell$

~~$V = X(x) \cdot T(t)$~~

$X''(x)\, T(t) = X(x)\, T'(t)$

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} = -k^2$$

$X''(x) - k^2 X(x) = 0$

$X(x) = C_1 e^{ikx} + C_2 e^{-ikx}$

& $T'(t) - k^2 T(t) = 0$

$T(t) = C_3 e^{-k^2 t}$

~~$V = (C_1 e^{kx} + C_2 e^{-kx}) \cdot (C_3 e^{k^2 t})$~~

~~$V = C_4 e^{kx + k^2 t} + C_5 e^{-kx + k^2 t}$~~

~~$V = 0$ at $t \to \infty \Rightarrow C_4 = 0$~~

$V = (C_1 \cos kx + C_2 \sin kx)(C_3 e^{-k^2 t})$

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Q. Find the possible sol^n for Laplace eq^n in 2-D.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$u = X(x) \cdot Y(y)$

$X''(x) \cdot Y(y) + X(x) \cdot Y''(y) = 0$

(i) $\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = p^2$ (+ve)

$X''(x) - p^2 X(x) = 0$

$X(x) = C_1 e^{px} + C_2 e^{-px}$

& $-Y''(y) - p^2 Y(y) = 0$

$Y''(y) + p^2 Y(y) = 0$

$Y(y) = C_3 \cos py + C_4 \sin py$

$\therefore u = (C_1 e^{px} + C_2 e^{-px})(C_3 \cos py + C_4 \sin py)$

(ii) $\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -p^2$ (-ve)

$X(x) = C_5 \cos px + C_6 \sin px$

$Y(y) = C_7 e^{py} + C_8 e^{-py}$

(iii) $\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = 0$

$X(x) = c_9 x + c_{10}$

$Y(y) = c_{11} y + c_{12}$

☆ D'Alembert's sol.^{n} for wave eq.^{n} —

Q. Show that sol.^{n} of the wave eq.^{n} $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ can be expressed in the form $y(x,t) = f_1(x+ct) + f_2(x-ct)$.

If $y(x,0) = f(x)$ & $\left.\frac{\partial y}{\partial t}\right|_{(x,0)} = 0$,

Show that $y(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right]$

OR

Transform the eq.^{n} $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ to its normal form using the transformation $u = x+ct$, $v = x-ct$ & solve it. Show that the sol.^{n} $y(x,t) = \frac{1}{2}\{f(x+ct) + f(x-ct)\}$ by assuming the initial conditions $y|_{t=0} = f(x)$ & $\left.\frac{\partial y}{\partial t}\right|_{t=0} = 0$

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \quad \text{— (1)}$$

Let $u = x+ct$, $v = x-ct$ — (2)

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial x} = 1 \\ \frac{\partial u}{\partial t} &= \frac{-\partial v}{\partial t} = c \end{aligned}\right\} \text{— (3)}$$

$y = z(u,v)$

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v} = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)y \quad \text{— (4)}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \quad \text{— (5)}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right]y$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial x^2} + 2\frac{\partial^2 y}{\partial x\cdot\partial y} + \frac{\partial^2 y}{\partial y^2} \quad \text{— (6)}$$

$$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial t}$$

$$= c\left(\frac{\partial y}{\partial u} - \frac{\partial y}{\partial v}\right) \quad \text{— (7)}$$

$$\cancel{\frac{\partial^2 y}{\partial t^2} = c\left[\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial u}\right) - \frac{\partial}{\partial t}\left(\frac{\partial y}{\partial v}\right)\right]}$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial u} - \frac{\partial}{\partial v} \quad \text{--- (8)}$$

$$\frac{\partial^2 y}{\partial t^2} = c\,\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial u} - \frac{\partial y}{\partial v}\right)$$

$$= c^2\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)\left(\frac{\partial y}{\partial u} - \frac{\partial y}{\partial v}\right)$$

$$= c^2\left[\frac{\partial^2 y}{\partial u^2} - 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}\right] \quad \text{--- (9)}$$

now, $\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$

using (6) & (9)

$$\cancel{\frac{\partial^2 y}{\partial u^2}} + 2\frac{\partial^2 y}{\partial u \partial v} + \cancel{\frac{\partial^2 y}{\partial v^2}} = \cancel{\frac{\partial^2 y}{\partial u^2}} - 2\frac{\partial^2 y}{\partial u \partial v} + \cancel{\frac{\partial^2 y}{\partial v^2}}$$

$$\frac{\partial^2 y}{\partial u \partial v} = 0$$

$$\Rightarrow \frac{\partial}{\partial u}\left(\frac{\partial y}{\partial v}\right) = 0 \quad \text{--- (10)}$$

integrating w.r.t. u

$$\frac{\partial y}{\partial v} = f(v) \quad \text{--- (11)}$$

integrating w.r.t. v

$$y = f_1(u) + \int f(v)\,dv$$

$$y = f_1(u) + f_2(v) \quad \left[\because f_2(v) = \int f(v)\,dv\right]$$

$$y = f_1(x+ct) + f_2(x-ct) \quad \text{--- (13)}$$

using initial conditions

$$y(x,0) = f_1(x) + f_2(x) = f(x)$$

$$f_1(x) + f_2(x) = f(x)$$

$$\frac{\partial y}{\partial t} = c f_1'(x+ct) - c f_2'(x-ct)$$

$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = c\left(f_1'(x) - f_2'(x)\right) = 0$$

$$f_1'(x) - f_2'(x) = 0$$

$$\& \quad f_1'(x) + f_2'(x) = f'(x)$$

$$2f_1'(x) = f'(x)$$

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$$F_1'(x) = \frac{1}{2} f'(x)$$

& $$F_2'(x) = \frac{1}{2} f'(x)$$

on integrating -

$$F_1(x) = \frac{1}{2}\left[f(x) + b\right]$$

$$F_2(x) = \frac{1}{2}\left[f(x) - b\right]$$

$$y(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right]$$

Solⁿ of 2-D heat flow in a thin rectangular plate :

A thin rectangular plate whose surface is impervious to heat flow has arbitrary distribution of temperature $f(x,y)\big|_{t=0}$, its 4 edges $x=0, x=a, y=0, y=b$ are kept at zero temp. Determine the subsequent temp. of the plate after time t.

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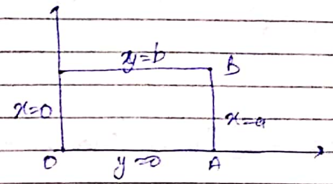
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{h^2}\frac{\partial u}{\partial t} \qquad \left(h^2 = \frac{K}{s\rho}\right)$$

↳ diffusivity



Intial conditions -

(i) $u=0$ at $x=0$
(ii) $u=0$ at $x=a$
(iii) $u=0$ at $y=0$
(iv) $u=0$ at $y=b$
$u(x,y) = f(x,y)$ at $t=0$

$$u = X(x)\cdot Y(y)\cdot T(t)$$

$$X''(x)\,Y(y)\,T(t) + X(x)\,T(t)\,Y''(y) = \frac{1}{h^2} X Y T'(t)$$

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = \frac{1}{h^2}\frac{T'(t)}{T(t)} = -k^2$$

Let $k^2 = k_1^2 + k_2^2$

$$\frac{X''(x)}{X(x)} = -k_1^2, \quad \frac{Y''(y)}{Y(y)} = -k_2^2, \quad \frac{1}{h^2}\frac{T'(t)}{T(t)} = -k^2$$

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$X(x) = C_1 \cos k_1 x + C_2 \sin k_1 x$
$Y(y) = C_3 \cos k_2 y + C_4 \sin k_2 y$
$T(t) = C_5 e^{-k^2 h^2 t}$

$$u = (C_1 \cos k_1 x + C_2 \sin k_1 x)(C_3 \cos k_2 y + C_4 \sin k_2 y)\, C_5 e^{-k^2 h^2 t}$$

Applying initial conditions

$u = 0 \big|_{x=0} \Rightarrow C_1 = 0$

$u = 0 \big|_{x=a} \Rightarrow C_2 \sin k_1 a = 0$
$k_1 = \frac{n\pi}{a}$

$u = 0 \big|_{y=0} \Rightarrow C_3 = 0$

$u = 0 \big|_{y=b} \Rightarrow k_2 = \frac{m\pi}{b}$

$$u = \overset{C}{C_2 C_4 C_5} \left(\sin \frac{n\pi x}{a}\right)\left(\sin \frac{n\pi y}{b}\right) e^{-k^2 h^2 t}$$

$\cancel{u(x,y) = f(x,y)} \big|_{t=0}$

$\therefore \cancel{f(x,y) = C} \left(\sin \frac{n\pi x}{a}\right)\left(\sin \frac{m\pi x}{b}\right)$

also, $k^2 = \frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}$

$$k^2 = \pi^2\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right)$$

$$u(x,y,t) = \sum_{n=1}^{N} \sum_{m=1}^{N} A_{mn} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} e^{-h^2 k_{mn}^2 t}$$

$u(x,y,t) = f(x,y) \big|_{t=0}$

$$\therefore f(x,y) = \sum_{n=1}^{N} \sum_{m=1}^{N} A_{mn} \sin \frac{n\pi x}{a} \cdot \sin \frac{m\pi y}{b}$$

multiplying both sides by $\left(\sin \frac{n\pi x}{a} \sin \frac{m\pi x}{b}\right)$

$$\int_0^a \int_0^b f(x,y) \sin \frac{n\pi x}{a} \cdot \sin \frac{m\pi x}{b}\, dx\, dy = \sum_n^N \sum_m^N A_{mn} \int_0^a \sin \frac{n\pi x}{a} \sin \frac{n\pi x}{a}\, dx \times \int_0^b \sin \frac{m\pi y}{b} \sin \frac{m\pi y}{b}\, dy$$

now $\displaystyle\int_0^1 \sin\frac{n\pi x}{1} \cdot \sin\frac{m\pi x}{1}\, dx = 0$ if $n \neq m$

$\frac{1}{2}$ if $n = m$

$$\therefore A_{mn} \times \frac{a}{2} \times \frac{b}{2} = \int_0^a \int_0^b f(x,y) \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b}\, dx\, dy$$

$$\therefore \boxed{A_{mn} = \frac{4}{ab} \int_0^a \int_0^b f(x,y) \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b}\, dx\, dy}$$

$$\boxed{u(x,y) = \sum_n \sum_m A_{mn} \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b}\, e^{-\lambda^2 K_{mn}^2 t}}$$

Rectangular Membrane –

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{\gamma^2} \frac{\partial^2 u}{\partial t^2}$$

$\gamma^2 \to A^2$ (let)

$u(x,0) = u(x,b) = 0$

$u(0,y) = u(a,y) = 0$

$u(x,y,0) = f(x,y)$

$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x,y)$

$u = X(x) \cdot Y(y) \cdot T(t)$

$X''(x)\, Y(y)\, T(t) + X(x) \cdot Y''(y)\, T(t) = \frac{1}{A^2} X Y T''(t)$

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = \frac{1}{A^2}\frac{T''(t)}{T(t)} = -K^2 = -K_1^2 - K_2^2$$

$\therefore X(x) = C_1 \cos K_1 x + C_2 \sin K_1 x$

$Y(y) = C_3 \cos K_2 y + C_4 \sin K_2 y$

$T(t) = C_5 \cos AKt + C_6 \sin AKt$

using initial condition –

$C_1 = 0,\ C_3 = 0,\ K_1 = \frac{n\pi}{a},\ K_2 = \frac{m\pi}{b}$

$K^2 = K_1^2 + K_2^2$

$\cancel{K^2 = n^2\pi^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right)}$

$$K_{mn}^2 = \frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}$$

$$\therefore u = \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b}\left[P_{mn} \cos AKt + Q_{mn} \sin AKt\right]$$

$\cancel{u(x,y,0) = f(x,y)}$

$$u = \sum_n \sum_m \sin\frac{n\pi x}{a} \cdot \sin\frac{m\pi y}{b}\left[P_{mn} \cos AKt + Q_{mn} \sin AKt\right]$$

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$$u(x,y,0) = f(x,y)$$

$$\iint \sum_m \sum_n P_{mn} \sin\frac{n\pi x}{a} \cdot \sin\frac{m\pi y}{a} = \int_0^a\int_0^b f(x,y)\, dy\, dx$$

$\cancel{P_{mn} \times \frac{a}{2} \times \frac{b}{2}}$

$$\iint \sum_m \sum_n P_{mn} \sin\frac{n\pi x}{a} \sin\frac{n\pi x}{a} \cdot \sin\frac{m\pi y}{b} \sin\frac{m\pi y}{b}\, dx\, dy$$

$$= \int_0^a\int_0^b f(x,y) \sin\frac{n\pi x}{a} \cdot \sin\frac{m\pi y}{b}\, dx\, dy$$

$$P_{mn} \times \frac{a}{2} \times \frac{b}{2} = \int_0^a\int_0^b f(x,y) \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b}$$

$$\boxed{P_{mn} = \frac{4}{ab} \int_0^a\int_0^b f(x,y) \sin\frac{n\pi x}{a} \cdot \sin\frac{m\pi y}{b}\, dx\, dy}$$

& $\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x,y)$

$$\sum_m \sum_n Q_{mn} \times A_R \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b} = g(x,y)$$

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$$\therefore \boxed{Q_{mn} = \frac{4}{ab\, A_R} \int_0^a\int_0^b \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b}\, g(x,y)\, dx\, dy}$$

Complex Variables

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Multiplicative Inverse -

$$z = x + iy \;,\; z' = ?$$

$$z \cdot z' = (1,0)$$

$$(x,y)\cdot(x,y)^{-1} = (1,0)$$

$$\text{Let } (x,y)^{-1} = (x',y')$$

now, $z_1 z_2 = (xx' - yy') + i(xy' + yx') = (1,0)$ $1+0i$

$$\therefore \; xx' - yy' = 1$$
$$xy' + yx' = 0$$

$$\therefore \; \boxed{x' = \frac{x}{x^2+y^2} \;,\; y' = \frac{-y}{x^2+y^2}}$$

Q. $z_1 = 2-3i$, $z_2 = -5+i$. Calculate z_1/z_2.

$$\frac{z_1}{z_2} = \frac{2-3i}{-5+i} \times \left(\frac{-5-i}{-5-i}\right)$$

$$= \frac{-10-3+13i}{26}$$

$$= \frac{-13+13i}{26} = \frac{-1}{2} + \frac{1}{2}i$$

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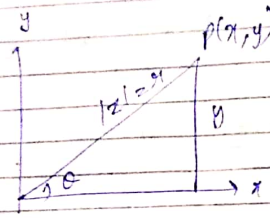
* Polar form :-



$$z = x + iy$$

$$|z| = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}(y/x) = \arg(z)$$

in polar form, $x = r\cos\theta$, $y = r\sin\theta$

$$\therefore z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

Q. Separate $\log_e(z)$ into real & imaginary parts.

Let $z = x + iy$

$x = r\cos\theta$

$y = r\sin\theta$

$$\log_e\left(r(e^{i\theta})\right)$$

$$= \log_e r + i\theta$$

$$= \frac{1}{2}\log_e(x^2+y^2) + i\tan^{-1}(y/x)$$

Q Prove that the value of $\log(1+i) + \log(1-i) = \log 2 + 4n\pi i$

$$\log(1+i) + \log(1-i)$$

$$= \log e^{i\pi/4} + \log e^{-i\pi/4} + 2\log\sqrt{2}$$

$$= i\frac{\pi}{4} + i\left(\frac{-\pi}{4}\right) + \log 2$$

$$= \log 2 + i\left(2n\pi + \frac{\pi}{4}\right) + i\left(2n\pi - \frac{\pi}{4}\right)$$

$$= \log 2 + i4n\pi$$

Q $\tan\left(i\log\left(\frac{a-ib}{a+ib}\right)\right) = \frac{2ab}{a^2-b^2}$

$$\cancel{\tan\left(i\log\left(\frac{a-ib}{a+ib}\times\frac{a-ib}{a-ib}\right)\right)}$$

$$= \cancel{\tan}$$

$a = r\cos\theta$

$b = r\sin\theta$

$$\tan\left(i\log\left(\frac{re^{-i\theta}}{re^{i\theta}}\right)\right) = \frac{2r^2\sin\theta\cos\theta}{\cancel{r^2}[\cos^2\theta - \sin^2\theta]}$$

$$\tan\left(i\times(-2i\theta)\right) = \frac{\sin 2\theta}{\cos 2\theta}$$

$$\tan(2\theta) = \tan 2\theta$$

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$\tan\theta = b/a$ $\cos\theta = \frac{a}{\sqrt{a^2+b^2}}$

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Q. $\cos\left(i \log\left(\frac{a-ib}{a+ib}\right)\right) = ?$

$\cos(i(-2i\theta)) = \cos 2\theta$

$= 2\cos^2\theta - 1$

$= \frac{2a^2}{(a^2+b^2)} - 1$

$= \frac{a^2-b^2}{(a^2+b^2)}$

Q. Find $Re(z)$ & $Im(z)$ for $\frac{z+2}{z-1}$

$z = x+iy$

$\frac{z+2}{z-1} = \frac{(x+2)+iy}{(x-1)+iy} \times \frac{(x-1)-iy}{(x-1)-iy}$

$= \frac{(x+2)(x-1)+y^2 + i((x+2)(-y)+y(x-1))}{(x-1)^2+y^2}$

$= \frac{x^2+x+y^2-2}{(x-1)^2+y^2} + \frac{i(-3y)}{(x-1)^2+y^2}$

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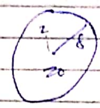
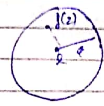
Set of points - Any collection of points in a complex is called set of points.

Neighbourhood of a point - Let z_0 be a point in the Argand diagram. A set of all points 'z' such that $|z - z_0| < \varepsilon$, then called as neighbourhood of z_0.

Limits - $f(z) = l$
$z \to z_0$





$\lim_{z\to z_0} f(z) = l$

Continuity - A f^n $f(z)$ is said to be continuous at $z = z_0$ if

(i) $\lim_{z\to z_0} f(z)$ exists uniquely

(ii) $\lim_{z\to z_0} f(z) = f(z_0)$

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Differentiability – A f^n $f(z)$ is said to be differentiable at $z = z_0$ if

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

Q. A f^n $f(z) = z^2$, check differentiability.

$$\frac{d}{dz}(z^2) = \lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\Delta z^2 + 2z\Delta z}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \Delta z + 2z$$

$$= 2z$$

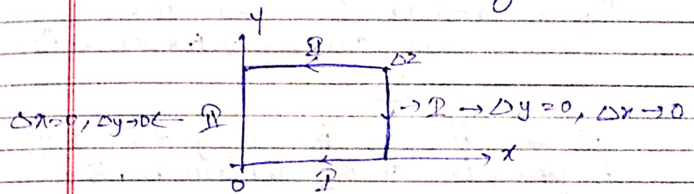
Analytic f^n – A f^n $f(z)$ is said to be analytic at a point $z = z_0$ if it is a single valued & has a derivative at every point in some neighbourhood of z_0.

Q. A f^n $f(z) = \omega = x - iy$ is not differentiable anywhere. Prove.

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(x+\Delta x) - i(y + \Delta y) - (x - iy)}{\Delta x + i\Delta y}$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$



Using path I –

$$f'(z) = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

Using path II –

$$f'(z) = \lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = -1$$

(C-R)

Cauchy - Riemann equations :-

The necessary & sufficient conditions for $f(z)$ to be analytic.

OR

A necessary condition that $f(z)$, $f(z) = u(x,y) + i\,v(x,y)$ be analytic in a domain D, then u & v satisfy the cauchy Riemann eq^ns.

OR

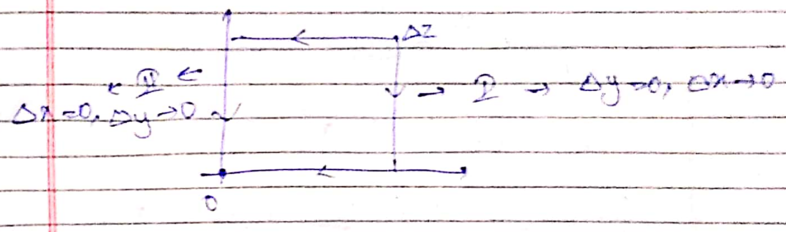
If the f^n $f(z) = u(x,y) + i\,v(x,y)$ is analytic in the domain D, then $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ are satisfied in the domain D.

$w = f(z) = u(x,y) + i\,v(x,y)$

$z = x+iy$, $\Delta z = \Delta x + i\Delta y$

$$f'(z) = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \left[\frac{u(x+\Delta x, y+\Delta y) + i\,v(x+\Delta x, y+\Delta y) - u(x,y) - i\,v(x,y)}{\Delta x + i\Delta y}\right]$$

$$= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{[u(x+\Delta x, y+\Delta y) - u(x,y)] + i\,[v(x+\Delta x, y+\Delta y) - v(x,y)]}{\Delta x + i\Delta y}$$



using I path -

$$f'(z) = \lim_{\Delta x\to 0} \frac{[u(x+\Delta x, y) - u(x,y)] + i\,[v(x+\Delta x, y) - v(x,y)]}{\Delta x}$$

$$\boxed{f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}} \longrightarrow (1)$$

using II path -

$$f'(z) = \lim_{\Delta y\to 0} \frac{[u(x, y+\Delta y) - u(x,y)] + i\,[v(x,y+\Delta y) - v(x,y)]}{i\Delta y}$$

$$f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \quad — (2)$$

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from 1 & 2

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = 0$$

on comparing both sides

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}, \quad \boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

→ Cauchy Riemann eq.^{n}.

Q. Using Cauchy Riemann eq^{n}s show that $f(z) = z^3$ is analytic in the entire z-plane.

~~$f(z) = u(x,y) + iv(x,y)$~~

$$f(z) = z^3 = (x+iy)^3 = x^3 + i^3y^3 + i3x^2y - 3xy^2$$

$$= (x^3 - 3xy^2) + i(y^3 + 3x^2y)$$

$$u(x,y) = x^3 - 3xy^2$$
$$v(x,y) = y^3 + 3x^2y$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2$$

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$$\frac{\partial u}{\partial y} = -6xy$$

$$\frac{\partial v}{\partial x} = 6xy$$

$$\frac{\partial v}{\partial y} = 3y^2 + 3x^2$$

Using C-R eq^{n}s -

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow 3x^2 - 3y^2 = 3y^2 + 3x^2$$

$$\& \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \Rightarrow 6xy = 6xy$$

Q. Show that the f.^{n} $e^x(\cos y + i\sin y)$ is analytic f.^{n}. & find its derivative.

$$f(z) = e^x(\cos y + i\sin y) = e^{x+iy}$$

$$= e^x\cos y + i\,e^x\sin y$$

$$u(x,y) = e^x\cos y$$
$$v(x,y) = e^x\sin y$$

$$\frac{\partial u}{\partial x} = e^x\cos y, \quad \frac{\partial u}{\partial y} = -e^x\sin y$$

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$$\frac{\partial v}{\partial x} = e^x \sin y \quad , \quad \frac{\partial v}{\partial y} = e^x \cos y$$

Sufficient conditions:

If $f(z) = u(x,y) + v(x,y)$ is defined in a domain D & the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ & $\frac{\partial v}{\partial y}$ are continuous & satisfy C-R eq's, then $f(z)$ is analytic.

If $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ & $\frac{\partial v}{\partial y}$ are continuous, then according to Mean Value Theorem

$$u(x+\Delta x, y+\Delta y) - u(x,y) = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y$$

$$v(x+\Delta x, y+\Delta y) - v(x,y) = \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y$$

now,

$$f(z+\Delta z) - f(z) =$$

$$= u(x+\Delta x, y+\Delta y) + i\,v(x+\Delta x, y+\Delta y) - u(x,y) - i\,v(x,y)$$

$$= u(x+\Delta x, y+\Delta y) - u(x,y) + i\left[v(x+\Delta x, y+\Delta y) - v(x,y)\right]$$

$$\therefore f(z+\Delta z) - f(z) = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + i\left[\frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y\right]$$

using C-R eq's

$$f(z+\Delta z) - f(z) = \frac{\partial u}{\partial x}\Delta x - \frac{\partial v}{\partial x}\Delta y + i\left[\frac{\partial v}{\partial x}\Delta x + \frac{\partial u}{\partial x}\Delta y\right]$$

$$= \frac{\partial u}{\partial x}(\Delta x + i\Delta y) + i\frac{\partial v}{\partial x}(\Delta x + i\Delta y)$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(\Delta x + i\Delta y)$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta z$$

dividing both sides by Δz

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

↳ derivative is unique.

Since we have taken assumption that $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$ are continuous & using that we have find derivate of z which is continous & unique. Hence our assumption are correct.

Q. Show that the real & imaginary parts of the f^n $w = \log z$ satisfy the C-R eq^n's when $z \neq 0$ & find its derivative.

$$w = \log z = u(x,y) + i\,v(x,y)$$
$$= \log(|z| e^{i\theta})$$

$$w = \log|z| + i\theta = u + iv$$

$$u = \log\sqrt{x^2+y^2}, \quad v = \theta = \tan^{-1} y/x$$

$$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2}, \quad \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2} \quad -(1)$$

$$\frac{\partial v}{\partial x} = \frac{1}{(1+y^2/x^2)}\left(\frac{-y}{x^2}\right), \quad \frac{\partial v}{\partial y} = \frac{1}{(1+y^2/x^2)}\left(\frac{1}{x}\right)$$

$$\frac{\partial v}{\partial x} = \frac{-y}{x^2+y^2}, \quad \frac{\partial v}{\partial y} = \frac{x}{x^2+y^2} \quad -(2)$$

using (1) & (2)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$f'(\omega) = \lim_{\Delta z \to 0} \frac{\log(z+\Delta z) - \log z}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{[(u+\Delta u) + i(v+\Delta v)] - [u+iv]}{\Delta z}$$

$=$

A. Harmonic f^n's:- A f^n $u(x,y)$ or $v(x,y)$ is said to be harmonic if its satisfy the Laplace eq^n.

eg- If the f^n $f(z)$ is analytic, then $u(x,y)$ ~~...~~ & $v(x,y)$ are harmonic f^n's.

$$f(z) = u(x,y) + i\,v(x,y)$$

if $f(z)$ is analytic,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad -(1) \qquad \& \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad -(2)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \qquad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y}$$

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using above eq^{n}s -

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

Similarly, $\boxed{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0}$

then u & v ~~a~~ are conjugate harmonic f^{n}s.

Q. Show that $u(x,y) = \frac{1}{2} \ln(x^2+y^2)$ is harmonic. Find its conjugate harmonic f^n.

$$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2}, \quad \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2}, \quad \frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2) - 2y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \quad \& \quad \frac{\partial^2 u}{\partial y^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = \frac{-y}{x^2+y^2}$$

$$\& \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = \frac{x}{x^2+y^2} \quad \text{— (1)}$$

$$v = \int \frac{-y}{x^2+y^2}\, dx = -\frac{y \times 1}{y} \tan^{-1}(x/y) + g(y)$$

$$= -\tan^{-1}(x/y) + g(y)$$

from eq^n (1)

$$\frac{\partial v}{\partial y} = \frac{x}{x^2+y^2} = -\frac{1}{1+x^2/y^2} \times \left(\frac{-x}{y^2}\right) + g'(y)$$

$$\frac{x}{x^2+y^2} = \frac{x}{x^2+y^2} + g'(y)$$

$$g'(y) = 0$$

$$~~g(y)~~ \Rightarrow g(y) = c$$
↳ constant

$$v = -\tan^{-1}\left(\frac{x}{y}\right) + c$$

now, $u + iv = \frac{1}{2} \ln(x^2+y^2) - i\theta + K$

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* Cauchy's theorem (Cauchy's integral theorem):-

If a f^n $f(z)$ is analytic & $f'(z)$ is continous at all points inside and on a simple closed contour (curve) 'C', then

$$\boxed{\int_C f(z)\,dz = 0}$$

$$z = x + iy$$
$$dz = dx + i\,dy$$

$$f(z) = u + iv$$

$$\int_C f(z)\,dz = \int_C (u+iv)(dx + i\,dy)$$
$$\int_C f(z)\,dz = \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy) \quad \text{--- (1)}$$

if $f'(z)$ is continous then, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ & $\frac{\partial v}{\partial y}$ are also continous.

Applying Green's theorem to eq. (1)

$$\cancel{\int_C f(z)\,dz = -\iint_R \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)dx\,dy + i\iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx\,dy}$$

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$$\int_C f(z)\,dz = -\iint_R \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)dx\,dy - i\iint_R \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right)dx\,dy$$

using CR eq^{n}s -

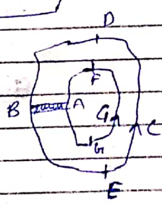
$$\int_C f(z)\,dz = -\iint_R \cancelto{0}{\left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right)}dx\,dy - i\iint_R \cancelto{0}{\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right)}dx\,dy$$

$$\boxed{\int_C f(z)\,dz = 0}$$

* Extension of Cauchy's theorem:-

If $f(z)$ is analytic f^n in region 'R' b/w two simple closed contour 'C' & 'C_1', then -

$$\boxed{\int_C f(z)\,dz = \int_{C_1} f(z)\,dz}$$



$$\int_C f(z)\,dz = 0$$

$$\int_{BEDBAFGAB} f(z)\,dz = 0$$

$$\Rightarrow \int_{BEDB} f(z)\,dz + \int_{BA} f(z)\,dz + \int_{AFGA} f(z)\,dz + \int_{AB} f(z)\,dz = 0$$

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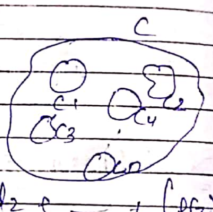
$$\therefore \int_C f(z)\,dz + \int_{BA} f(z)\,dz + (-1)\int_{AGFA} f(z)\,dz + \int_{BA} f(z)\,dz = 0$$

$$\int_C f(z)\,dz - \int_{C_1} f(z)\,dz = 0$$

$$\therefore \boxed{\int_C f(z)\,dz = \int_{C_1} f(z)\,dz}$$

• Generalized -



$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \cdots + \int_{C_n} f(z)\,dz$$

✗ Cauchy's Integral Formula :-

If $f(z)$ is an analytic f.^{n} within and on a closed contour 'c' & 'a' is any point within C, then

$$\boxed{f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz}$$

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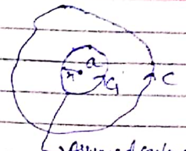

Assumed contour of radius 'r'

$f(z)$ is analytic

↳ $\frac{f(z)}{z-a}$ is also analytic, except for $z=a$

$$\int_C \frac{f(z)}{z-a}\,dz = \int_{C_1} \frac{f(z)}{z-a}\,dz$$

now, $z-a = re^{i\theta}$

$dz = ire^{i\theta}\,d\theta$

$$\int_{C_1} \frac{f(z)}{z-a}\,dz = \int_0^{2\pi} \frac{f(a+re^{i\theta})}{re^{i\theta}}(ire^{i\theta})\,d\theta$$

$$\therefore \int_C \frac{f(z)}{z-a}\,dz = i\int_0^{2\pi} f(a+re^{i\theta})\,d\theta$$

if $r \to 0$

$$\int_C \frac{f(z)}{z-a}\,dz = i\int_0^{2\pi} f(a)\,d\theta$$

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

$$\therefore \boxed{f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz}$$

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also,

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz$$

$$f''(a) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-a)^3}\,dz$$

$$f^n(a) = \frac{n!}{2\pi i}\int \frac{f(z)}{(z-a)^{n+1}}\,dz$$

or

$$\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!}\,f^n(a)$$

Q. Using Cauchy's Integral formula, Evaluate -

$$\int_C \frac{z^2-z+1}{z-1}\,dz \quad \text{where } C \text{ is}$$

(i) $|z| = 2$

(ii) $|z| = 1/2$

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(i) $|z| = 2$

$$\int \frac{f(z)}{z-a}\,dz = 2\pi i\cdot f(a)$$

$$\int \frac{z^2-z+1}{z-1}\,dz = 2\pi i f(1)$$

$$= 2\pi i$$

(ii) $|z| = 1/2$

$$\int_C f(z)\,dz = 0$$

$$\therefore \int \frac{z^2-z+1}{z-1}\,dz = 0$$

Q. $$\int_{|z|=3} \frac{e^{2z}}{(z+1)^4}\,dz = ?$$

$$\int_C \frac{f(z)\,dz}{(z-a)^{n+1}} = \frac{2\pi i}{n!}\,f^n(a)$$

$f(z) = e^{2z}$

$a = -1$

$f(a) = e^{-2}$

~~$f'(a) = ...$~~

$f'(z) = 2e^{2z}$

$f''(z) = 4e^{2z}$

$f'''(z) = 8e^{2z}$

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$$f'''(z) = 8e^{-2}$$

$$\therefore \int_C \frac{e^{2z}}{(z+1)^4}\,dz = \frac{2\pi i}{n!} f^n(a)$$

$$= \frac{2\pi i}{3!} f'''(a)$$

$$= \frac{8}{3}\pi i\, e^{-2}$$

★ **Taylor Series** – If f^n $f(z)$ is analytic at all points inside a circle 'C' with its centre at the point 'A' & radius 'r', then for any z inside the circle 'C', f(z) can be represented as a converged taylor series.

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots$$

$$\cdots + \frac{f^n(a)}{n!}(z-a)^n + \cdots$$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(z-a)^n$$

If $a=0$ → Maclaurin series of $f(z)$

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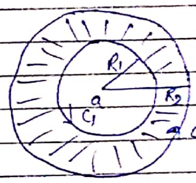
Q. Expand $\sin z$ in the form of taylor series about $z = \pi/4$.

$f(z) = \sin z \Rightarrow f(\pi/4) = 1/\sqrt{2}$
$f'(z) = \cos z \Rightarrow f'(\pi/4) = 1/\sqrt{2}$
$f''(z) = -\sin z \Rightarrow f''(\pi/4) = -1/\sqrt{2}$

$$f(z) = f\left(\frac{\pi}{4}\right) + \left(z-\frac{\pi}{4}\right) f'(\pi/4) + \frac{(z-\pi/4)^2}{2!} f''(\pi/4) + \cdots$$

$$f(z) = \frac{1}{\sqrt{2}}\left[1 + \left(z-\frac{\pi}{4}\right) - \frac{1}{2!}\left(z-\frac{\pi}{4}\right)^2 - \frac{1}{3!}\left(z-\frac{\pi}{4}\right)^3 \cdots\right]$$

★



$$R_1 < |z-a| < R_2$$

Regular Principle

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

$$b_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{-n+1}}\,dz$$

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Q. Expand $\frac{1}{z^2-3z+2}$ in the region –

(i) $1<|z|<2$
(ii) $|z|>2$
(iii) $0<|z-1|<1$

$$f(z) = \frac{1}{z^2-3z+2} = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$$

(i) if $1<|z|<2$

$|z|>1 \Rightarrow \frac{1}{|z|}<1$ & $|z|<2$
$\Rightarrow \left|\frac{z}{2}\right|<1$

$$f(z) = \frac{1}{-2\left(1-\frac{z}{2}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)}$$

$$= -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= -\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\dots\right) - \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\dots\right)$$

$$\therefore f(z) = -\frac{1}{2} - \frac{z}{4} - \frac{z^2}{8} - \frac{z^3}{16} + \dots - \frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{z^4} \dots$$

(ii) $|z|>2$

$$f(z) = \frac{1}{z\left(1-\frac{2}{z}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)}$$

$$f(z) = \frac{1}{z}\left(1+\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+\dots\right) - \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\dots\right)$$

$$= \frac{1}{z}+\frac{2}{z^2}+\frac{4}{z^3}+\frac{8}{z^4}+\dots - \frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{z^4} \dots$$

$$f(z) = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n}$$

★ **Singularities of Analytic f^n's** :- A point at which a f^n $f(z)$ is not analytic is known as singular point or singularity of a f^n.

★ **Isolated Singularity** – If $z=a$ is singularity of $f(z)$ & if there is no other singularity within a small circle surrounding the point $z=a$, then $z=a$ is said to be an isolated singularity.

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Q. $z=0$ is the non-isolated singularity of $f(z)$ =
$f(z) = \sin(\pi/z)$

$\sin\frac{\pi}{z} = 0$

$\sin\frac{\pi}{z} = \sin n\pi$

$z = \frac{1}{n}$, $n = 1, 2, \ldots$

Types of Singularities :-

(i) Poles
(ii) Essential
(iii) Removal

(i) Pole -

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \ldots + \frac{b_n}{(z-a)^{n+1}}$$

where $b_n \neq 0$ & $b_n = 0$ at all $n > m$.

(ii) Essential - If the principle parts in the Lorentz expansion expands to ∞, then it is called ES.

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(iii) Removal - if principle part is completely Removed

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 \ldots$$

Residue :-

Let $z=a$ be an isolated singularity of $f(z)$, then the coff. of $(z-a)^{-1}$ i.e. b_1, in the Lorentz expansion of $f(z)$. Also b_1 is called residue of $f(z)$ at $z=a$.

$$Res[f(z), a] = b_1 = \frac{1}{2\pi i}\int_C f(z)dz$$

Residue at poles :-

If $z=a$ is a simple pole of $f(z)$, then its Lorentz expansion about $z=a$ is given by

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + b_1 (z-a)^{-1}$$

multiplying both sides by $\lim_{z\to a}(z-a)$

$$\therefore \lim_{z\to a}(z-a) f(z) = b_1$$

for 1st order

$$Res[f(z), a] = \lim_{z\to a}[(z-a) f(z)]$$

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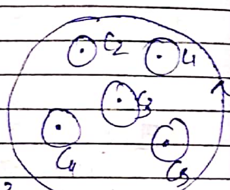
for m^{th} order —

$$\text{Res}(f(z), a) = \frac{1}{(m-1)!} \lim_{z \to a} \left[\frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z) \right] \right]$$

Cauchy Residue Theorem :- If $f(z)$ is analytic in a closed curve 'C', except at a finite number of singular points within 'C' then

$$\oint_C f(z)\,dz = 2\pi i \times (\text{sum of residues at singular points within } C)$$

using extension of cauchy's theorem :



$$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \cdots + \oint_{C_5} f(z)dz$$

$$= 2\pi i \,\text{Res}[f(z), z_1] + 2\pi i \,\text{Res}[f(z), z_2] + \cdots + 2\pi i \,\text{Res}[f(z), z_5]$$

$$= 2\pi i \left[\text{Res}(f(z), z_1) + \text{Res}(f(z), z_2) + \cdots + \text{Res}(f(z), z_5) \right]$$

Q. $\oint_C \frac{z^2}{(z-1)^2 (z+2)} dz$ where $|z| = 3$.

$z = 1 \to$ second order
$z = -2 \to$ first order

$$\text{Res}(f(z), 1) = \lim_{z \to 1} \frac{d}{dz} \left[\frac{\cancel{(z-1)^2} (z^2)}{\cancel{(z-1)^2} (z+2)} \right]$$

$$= \lim_{z \to 1} \frac{(z+2)(2z) - z^2}{(z+2)^2}$$

$$= \lim_{z \to 1} \frac{z^2 + 4z}{(z+2)^2} = 5/9$$

$$\text{Res}(f(z), -2) = \lim_{z \to -2} \frac{\cancel{(z+2)} z^2}{(z-1)^2 \cancel{(z+2)}}$$

$$= 4/9$$

using Cauchy Residue theorem —

$$\oint_C \frac{z^2}{(z-1)^2 (z+2)} dz = 2\pi i \left(5/9 + 4/9 \right) = 2\pi i$$

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(Ans = $-i\pi/2$)

Q $\int_C \frac{1}{z^2(z+1)(z-1)} dz$ where $C: |z| = 3$

* Evaluation of real definite integrals :-

(I). $\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$

substitute -

$r = 1$

$\sin\theta = \frac{z^2-1}{2iz}$, $\cos\theta = \frac{z^2+1}{2z}$

& $dz = i e^{i\theta} d\theta \Rightarrow d\theta = \frac{dz}{iz}$

Q Evaluate $\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta} d\theta$

$\int_0^{2\pi} \frac{4\cos^3\theta - 3\cos\theta\, d\theta}{5 - 4\cos\theta}$

$\oint$ $\cos 3\theta = \frac{1}{2}\left(z^3 + \frac{1}{z^3}\right)$

$\int \frac{\frac{1}{2}\left(z^3 + \frac{1}{z^3}\right)}{z^3\left[5 - 4\left(\frac{z^2+1}{2z}\right)\right]} \frac{dz}{iz}$

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$= \frac{-1}{2i} \int_0^{2\pi} \frac{z^6+1}{z^3(2z-1)(z-2)} dz$

$z = 0 \rightarrow$ 3rd order

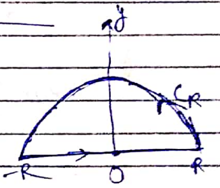
$z = 1/2, 2 \rightarrow$ 1st order

$\int_C f(z)dz = 2\pi i \left[\mathrm{Res}(f(z), 0) + \mathrm{Res}(f(z), 1/2)\right]$

$\mathrm{Res}(f(z),0) \rightarrow 21/8$, $\mathrm{Res}(f(z),1/2) \rightarrow -65/24$

$= \frac{-1}{2i} \times 2\pi i \left[\frac{21}{8} - \frac{65}{24}\right] = \pi/12$

(II). $\int_{-\infty}^{\infty} f(x)\, dx =$



$\int_C f(z)dz$

$|z| = R$

$\int_C f(z)dz = 2\pi i$ [sum of residues of $f(z)$ at the poles within C]

$\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = 2\pi i [\;,\;]$

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$$\lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx = -\lim_{R\to\infty}\cancel{\int_{C_R} f(z)\,dz}^{\,0} + 2\pi i\,[\;\ldots\;]$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i\,[\text{sum of residues within } c]$$

Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx$

$z = \pm i, \quad z = \pm 2i$

$z = +i, \quad z = +2i$ lies with C_R

$$\operatorname{Res}[f(z), i] = \lim_{z\to i} \frac{(z-i)\,z^2}{(z^2+1)(z^2+4)} = \frac{-1}{6i}$$

$$\operatorname{Res}(f(z), 2i) = \frac{-1}{3i}$$

$$\text{sum of residues} = \frac{-1}{6i} + \frac{1}{3i} =$$

$$\therefore \int_C f(z)\,dz = 2\pi i\left[\frac{-1}{6i} + \frac{1}{3i}\right] = \pi/3$$

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$$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \cancel{\int_{C_R} f(z)\,dz}^{\,0}$$

$$= \int_{-\infty}^{\infty} f(x)\,dx = \pi/3$$

Numerical Analysis

entry ← $y = f(x)$

argument ← $x = x_0, x_0+h, x_0+2h, \ldots, x_0+nh$

$x_n = x_0 + nh$

$y_0 = f(x_0)$, $y_1 = f(x_0+h)$, $y_2 = f(x_0+2h)$, $\ldots$, $y_n = f(x_0+nh)$

differences in y, $y_1 - y_0 = \Delta y_0$

↳ forward difference (1st order)

$\Delta y_1 = y_2 - y_1$

$\Delta y_2 = y_3 - y_2$

$\vdots$

$\Delta y_{n-1} = y_n - y_{n-1}$

$$\boxed{\Delta f(x) = f(x+h) - f(x)}$$

$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$

↳ 2nd order forward difference

$\Delta^2 y_0 = y_2 - y_1 - y_1 + y_0 = y_2 - 2y_1 + y_0$

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$$\Delta^2 y_n = \Delta y_{n+1} - \Delta y_n$$

$$\boxed{\Delta^m y_n = \Delta^{m-1} y_{n+1} - \Delta^{m-1} y_n}$$

Argument (x)		Entry (y)	1st difference (Δy)	2nd diff. ($\Delta^2 y$)	3rd diff. ($\Delta^3 y$)
x_0	x_0	y_0	$y_1 - y_0 = \Delta y_0$	$\Delta y_1 - \Delta y_0 = \Delta^2 y_0$	$\Delta^2 y_1 - \Delta^2 y_0 = \boxed{\Delta^3 y_0}$
x_1	x_0+h	y_1	$y_2 - y_1 = \Delta y_1$	$\Delta y_2 - \Delta y_1 = \Delta^2 y_1$	
x_2	x_0+2h	y_2	$y_3 - y_2 = \Delta y_2$		
x_3	x_0+3h	y_3			

$y = f(x)$

$x = x_0, x_0+h, x_0+2h, \text{———}, x_0+nh$

$y = y_0, y_1, y_2, \text{———}, y_n$

$\underline{\nabla y_n} = y_n - y_{n-1}$

backward difference

$\nabla y_1 = y_1 - y_0$

$\nabla y_2 = y_2 - y_1$

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$$\boxed{\nabla f(x) = f(x) - f(x-h)}$$

Argument (x)		Entry (y)	1st diff. (∇y)	2nd diff. ($\nabla^2 y$)	3rd diff. ($\nabla^3 y$)
x_0	x_0	y_0			
x_1	x_0+h	y_1	$y_1 - y_0 = \nabla y_1$		
x_2	x_0+2h	y_2	$y_2 - y_1 = \nabla y_2$	$\nabla y_2 - \nabla y_1 = \nabla^2 y_2$	
x_3	x_0+3h	y_3	$y_3 - y_2 = \nabla y_3$	$\nabla y_3 - \nabla y_2 = \nabla^2 y_3$	$\nabla^2 y_3 - \nabla^2 y_2 = \nabla^3 y_3$

Central Difference operator -

$$\delta f(x) = f(x + h/2) - f(x - h/2)$$

$$\delta y_{1/2} = y_1 - y_0$$

$$\delta y_{3/2} = y_2 - y_1$$

$$\delta y_{5/2} = y_3 - y_2$$

$$\vdots$$

$$\delta y_n = y_{n+1/2} - y_{n-1/2}$$

$$\delta^m y_n = \delta^{m-1} y_{n+1/2} - \delta^{m-1} y_{n-1/2}$$

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Argument (x)		Entry (y)	1st diff. (δy)	2nd diff. $(\delta^2 y)$	3rd diff. $(\delta^3 y)$
x_0	x_0	y_0	$\delta y_{1/2}$		
x_1	x_0+h	y_1	$\delta y_{3/2}$	$\delta^2 y_1$	$\delta^3 y_{3/2}$
x_2	x_0+2h	y_2	$\delta y_{5/2}$	$\delta^2 y_2$	
x_3	x_0+3h	y_3			

Averge (or) Mean operator –

$$\mu f(x) = \frac{1}{2}\left[f(x+h/2) + f(x-h/2)\right]$$

Ex: $\mu y_{1/2} = \frac{1}{2}(y_0 + y_1)$

$$\mu y_{3/2} = \frac{1}{2}(y_2 + y_1)$$

$$\mu y_x = \frac{1}{2}(y_{x+1/2} + y_{x-1/2})$$

Shift Operator (E) –

$$E f(x) = f(x+h)$$

$$E^n f(x) = f(x+nh)$$

$E y_0 = y_1$, $E^2 y_0 = y_2$

$$\boxed{E^r y_x = y_{x+r}}$$

Relation b/w operators :–

(i) $\Delta = E - 1$

(ii) $\nabla = 1 - E^{-1}$

(iii) $\delta = E^{1/2} - E^{-1/2}$

(iv) $\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$ &

$\mu^2 = 1 + \frac{1}{4}\delta^2$

(v) $\Delta = E\nabla = \nabla E = \delta E^{1/2}$

(vi) $E = e^{hD}$ $\left(\because D \equiv d/dx \right)$

proof:

(i) $\Delta f(x) = f(x+h) - f(x)$

$\Delta f(x) = E f(x) - f(x)$

$\Delta f(x) = (E-1) f(x)$

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(ii) $\nabla = 1 - E^{-1}$

$$\nabla f(x) = f(x) - E^{-1} f(x)$$

$$E^{-1} f(x) = (1-\nabla) f(x)$$

$$E^{-1} f(x) = f(x-h)$$

(iii) $\delta f(x) = E^{1/2} f(x) - E^{-1/2} f(x)$

$$\delta f(x) = f(x+h/2) - f(x-h/2)$$

$$= E^{1/2} f(x) - E^{-1/2} f(x)$$

(iv) $\mu f(x) = \frac{1}{2}\left[E^{1/2} f(x) + E^{-1/2} f(x)\right]$

$$\mu f(x) = \frac{1}{2}\left[f(x+h/2) + f(x-h/2)\right]$$

$$\mu f(x) = \frac{1}{2}\left[E^{1/2} f(x) + E^{-1/2} f(x)\right]$$

& $\mu^2 f(x) = 1 + \frac{1}{4}\delta^2$

$$\mu^2 f(x) = \frac{1}{4}\left[E^{1/2} f(x) + E^{-1/2} f(x)\right]^2$$

$$\mu^2 f(x) = \frac{1}{4}\left[E f(x) + E^{-1} f(x) + 2\right]$$

$$= \frac{1}{4}\left[E f(x) + E^{-1} f(x) - 2 + 4\right]$$

$$= \frac{1}{4}\left[\delta^2 f(x) + 4\right]$$

$$\mu^2 f(x) = 1 + \frac{1}{4}\delta^2 f(x)$$

(v) $\Delta f(x) = E\nabla f(x) = \nabla E f(x) = \delta E^{1/2} f(x)$

$$\Delta f(x) = f(x+h) - f(x)$$

$$= E f(x) - f(x)$$

$$= (E-1) f(x)$$

$$\nabla = 1 - E^{-1} = \frac{E-1}{E}$$

$$\nabla f(x) = \frac{\Delta f(x)}{E f(x)}$$

$$\Delta f(x) = \nabla E f(x)$$

$$\delta E^{1/2} f(x) = E f(x) - f(x) = (E-1) f(x) = \Delta f(x)$$

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(vi). $E = e^{hD}$

$D f(x) = f'(x)$

~~D f(x)~~

$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \ldots$

$= f(x) + h D f(x) + \frac{h^2}{2!} D^2 f(x) + \ldots$

$= \left(1 + hD + \frac{h^2}{2!} D^2 + \ldots\right) f(x)$

$E f(x) = e^{hD} f(x)$

Also

$$\boxed{\delta = \frac{\nabla}{\sqrt{1-\nabla}} = \frac{\Delta}{\sqrt{1+\Delta}}}$$

Q. Construct forward diff. table for $f(x) = x^3$, & $x = -3, -2, -1, 0, 1, 2, 3$

$f(0) = 0$, $f(1) = 1$, $f(2) = 8$, $f(3) = 27$,
$f(-1) = -1$, $f(-2) = -8$, $f(-3) = -27$

Arguments		Entry	1st diff.	2nd diff.	3rd diff	4th		
-3	0-3	-27	19					
				-12				
-2	0-2	-8	7		6			
-1	0-1	-1	1	-6	6	0	0	
0	0	0	1	0	6	0	0	0
1	0+1	1	7	6	6	0		
2	0+2	8		12				
			19					
3	0+3	27						

Q. $f(x) = x^3 - 2x^2 + 5x + 7$, $x = 0, 2, 4, 6, 8, 10$

Argument	Entry	1st	2nd	3rd	4th	5th
0	7					
		6				
2	13	30	24	48	0	
4	43	102	72	48		0
6	145	222	120	48	0	
8	367	390	168			
10	757					

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Q. Construct backward diff. table
$f(x) = x^3 - 3x^2 + 5x - 7$, $x = -1, 0, 1, 2, 3, 4, 5$

Interpolation & Extrapolation :-

$x =$	0	4	12	16	20
$f(x)$	5	8	13	17	22

$\underbrace{f(2), f(5), f(19)}_{\text{inter}}$ $\underbrace{f(-1), f(21), f(31)}_{\text{extra}}$

(Newton's) Newton's - Gregorey formula for forward interpolation:-

Known $y_0, y_1, \ldots \to y_n \to$ for $x = x_0, x_0+h, \ldots, x_0+nh$

$x = x_0 + ph$

$f(x_0+ph) = E^p f(x_0) = (1+\Delta)^p y_0$

$= \left[1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \ldots + \frac{p(p-1)\ldots(p-n+1)}{n!}\Delta^n\right] y_0$

$y_{x_0+ph} = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \ldots$

$\ldots + \frac{p(p-1)\ldots(p-n+1)}{n!}\Delta^n y_0$

for Backward interpolation:-

$x = x_n + ph$

$y_p = f(x_n+ph) = E^p y(x) = (E^{-1})^{-p} y_n$

$y_p = (1-\nabla)^{-p} y_n$

$= \left[1 + p\nabla + \frac{p(p+1)}{2!}\nabla^2 + \ldots + \frac{p(p+1)\ldots(p+n-1)}{n!}\nabla^n\right] y_n$

$y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \ldots$

$\ldots + \frac{p(p+1)\ldots(p+n-1)}{2n!}\nabla^n y_n$

Q. Calculate the approximate value of $\sin x$ for $x = 0.54$ & $x = 1.36$ using the following table—

x	0.5	0.7	0.9	1.1	1.3	1.5
$\sin x$	0.47943	0.64422	0.78333	0.89121	0.96356	0.99749

$x_0 = 0.5$, $y_0 = 0.47943$

$x_0 + h = 0.7$

$h = 0.2$

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Forward/backward:-

Arguments (x)	Entry (y)	1st (Δy)	2nd ($\Delta^2 y$)	3rd ($\Delta^3 y$)	4th ($\Delta^4 y$)	5th ($\Delta^5 y$)
x_0 0.5	0.47943	0.16479				
0.7	0.64422	0.13911	-0.02568	-0.00555		
0.9	0.78333	0.10788	-0.03123	-0.0043	+0.00125	0.0016
1.1	0.89121	0.07235	-0.03553	-0.00289	+0.00141	
1.3	0.96356	0.03393	-0.03842			
1.5	0.99749					

$$y_{0.54} = y_{0.5+0.04} = y_{0.5+0.2(0.2)}$$

$$\Rightarrow p = 0.2$$

$$y_{0.54} = y_{0.5} + (0.2)\Delta y_0 + \frac{(0.2)(-0.8)}{2!}\Delta^2 y_0 + 0.2 \ldots$$

$$\ldots + \frac{(0.2)(0.8)(1.8)(2.8)(3.8)}{5!}(0.00016)$$

$$y_{0.54} = 0.514138$$

~~Backward~~ =

(x)	(y)	~~(∇y)~~	~~($\nabla^2 y$)~~	~~($\nabla^3 y$)~~	~~($\nabla^4 y$)~~	~~($\nabla^5 y$)~~
0.5	0.47948					
0.7	0.64422					
0.9	0.78333					
1.1	0.89121					
1.3	0.96356					
1.5	[illegible]					

$$y_{1.36} = y_{1.5-0.14} = y_{1.5+(0.2)(-0.7)}$$

$$= y_{1.5} + (-0.7)\nabla y_n + \frac{(-0.7)(0.3)}{2!}\nabla^2 y_n + \frac{(-0.7)(0.3)(1.3)}{3!}\nabla^3 y_n + \frac{(-0.7)(0.3)(1.3)(2.3)}{4!}\nabla^4 y_n$$

$$+ \frac{(-0.7)(0.3)(1.3)(2.3)(3.3)}{5!}\nabla^5 y_n$$

$$= 0.97742$$

Q. Find the cubic polynomial which takes the following values -

x	0	1	2	3
$f(x)$	1	2	1	10

Find $f(4)$.

x	$f(x)$	1st	2nd	3rd
0	1	1	-2	12
1	2	-1	10	
2	1	9		
3	10			

$x_0 = 0, \quad x_0 + ph = x$

$\Rightarrow \boxed{p = x}$

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$$f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

$$f(x) = 1 + x(1) + \frac{x(x-1)}{2}(-2) + \frac{x(x-1)(x-2)(12)}{6}$$

$$= 1 + x - x(x-1) + 2x(x^2 - 3x + 2)$$

$$= 1 + x - x^2 + x + 2x^3 - 6x^2 + 4x$$

$$f(x) = 2x^3 - 7x^2 + 6x + 1$$

$$f(4) = 128 - 112 + 24 + 1 = 41$$

Q. The population of a town decennial census was as given below

Year	1891	1901	1911	1921	1931
Population in thousands	46	66	81	93	101

Estimate the population for the years 1895 & 1925.

Runge - Kutta Method :-

$$\frac{dy}{dx} = f(x, y) \qquad y(x_0) = y_0$$

$$K_1 = h\, f(x_0, y_0)$$

$$K_2 = h\, f(x_0 + h/2,\ y_0 + K_1/2)$$

$$K_3 = h\, f\left(x_0 + \frac{h}{2},\ y_0 + \frac{K_2}{2}\right)$$

$$K_4 = h\, f(x_0 + h,\ y_0 + K_3)$$

$$K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$y_1 = y_0 + K$$

Q. Solve -

$$\frac{dy}{dx} = x + y \qquad , \left(\text{at } x_0 = 0,\ y_0 = 1\right)$$

from $x = 0$ to $x = 0.4$ with the interval 0.1 (or $h = 0.1$) -

for $x = 0.1$

$K_1 = 0.1(0 + 1) = 0.1$

$K_2 = 0.1[0.05 + 1 + 0.05]$

$= 0.11$

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$K_3 = 0.1\left(0 + 0.05 + 1 + \frac{0.11}{2}\right) = 0.1105$

$K_4 = 0.1(0 + 0.1 + 1 + 0.1105) = 0.12105$

$K = \frac{1}{6}(0.1 + 2(0.11) + 2(0.1105) + 0.12105)$

$= 0.11034$

$\boxed{y_1 = y_0 + K = 1.11034 \quad x_1 = 0.1}$

for $x = 0.2$

$K_1 = 0.1(0.1 + 1.11034) = 0.121034$

$K_2 = 0.1\left(0.1 + 0.05 + 1.11034 + \frac{0.121034}{2}\right)$

$= 0.13205$

$K_3 = 0.13263$

$K_4 = 0.14429$

$\boxed{y_2 = 1.2428 \quad x_2 = 0.2}$

for $x = 0.3$

$K_1 = 0.1(0.2 + 1.2428) = 0.14428$

$K_2 = 0.1\left(0.2 + 0.05 + 1.2428 + \frac{0.14428}{2}\right)$

$= 0.1564941$

$K_3 = 0.1\left(0.2 + 0.05 + 1.2428 + 0.078247\right)$

$= 0.1571047$

$K_4 = 0.1(0.2 + 0.05 + 1.2428 + 0.1571047)$

$= 0.16499047$

$K = \frac{1}{6}[0.14428 + 0.3129882 + 0.3142094 + 0.16499047]$

$y_3 = 1.2428 + K$

$\boxed{y_3 = 1.39887798 \quad x_3 = 0.3}$

for $x = 0.4$

$K_1 = 0.1(0.3 + 1.39887798) = 0.169887798$

$K_2 = 0.1\left(0.3 + 0.05 + 1.4 + \frac{0.17}{2}\right)$

$= 0.1835$

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$K_3 = 0.1\left[0.35 + 1.4 + 0.09175\right]$

$= 0.184175$

$K_4 = 0.1\left[0.35 + 1.4 + 0.184175\right]$

$= 0.1934175$

$K = \frac{1}{6}\left[0.17 + 0.367 + \cancel{68350} \ldots\right]$

$= \frac{1.0986553}{6} =$

$y_4 = y_3 + K$

$= 1.39887798 + \frac{1.0986553}{2}$

$\cancel{y_4 = 2.49753328}$

$\boxed{y_4 = 1.5819872}$

Q. Find y when $x = 0.2$ given that

$\frac{dy}{dx} = x + y$ $\quad \boxed{y(0) = 1}$

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$h = 0.2$

$x_0 = 0, \; y_0 = 1$

$K_1 = 0.2\,[1] = 0.2$

$K_2 = 0.2\left[0 + 0.1 + 1 + 0.1\right]$

$= 0.24$

$K_3 = 0.2\left[0 + 0.1 + 1 + 0.12\right]$

$= 0.244$

$K_4 = 0.2\left[0.2 + 1.244\right] = 0.2888$

$K = \frac{1}{6}\left[0.2 + 0.48 + 0.488 + 0.2888\right]$

$= 0.2428$

$y_{0.2} = y_0 + K$

$= 1.2428$

Population Ques -

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1891	46				
		20			
1901	66		-5		
		15		2	
1911	81		-3		-3
		12		-1	
1921	93		-4		
		8			
1931	101				

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