Rayleigh's Dissipation function :

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j$$

$$L = T - \underbrace{V}_{\text{conservative forces}}$$

F_{fx} = x - component of frictional force $\vec{F}_f = -\left(k_x v_x \hat{i} + k_y v_y \hat{j} + k_z v_z \hat{k}\right)$

$$= -k_x v_x$$

$$\mathcal{F} = \frac{1}{2} \sum_i \left[k_x v_{xi}^2 + k_y v_{yi}^2 + k_z v_{zi}^2\right]$$

$\downarrow$

Rayleigh Dissipation function

$$F_f = -\vec{\nabla}_v \mathcal{F}$$

$$\vec{\nabla}_v = \hat{i}\frac{\partial}{\partial v_x} + \hat{j}\frac{\partial}{\partial v_y} + \hat{k}\frac{\partial}{\partial v_z}$$

$$dW_f = -\vec{F}_f \cdot d\vec{r} = -\vec{F}_f \cdot \vec{v}\,dt$$

$$= \left(k_x v_x^2 + k_y v_y^2 + k_z v_z^2\right) dt$$

$$\frac{dW_f}{dt} = 2\mathcal{F}$$ = Rate of dissipation of energy by friction

$$Q_j = \sum_i \vec{F}_{fi} \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\sum_i \vec{\nabla}_v \mathcal{F} \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

$$Q_j = -\sum_i \nabla_v \mathcal{F} \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j}$$

as $\frac{\partial \vec{v}_i}{\partial \dot{q}_j} = \frac{\partial \vec{r}_i}{\partial q_j}$

$$\boxed{Q_j = -\frac{\partial \mathcal{F}}{\partial \dot{q}_j}}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = -\frac{\partial \mathcal{F}}{\partial \dot{q}_j}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} + \frac{\partial \mathcal{F}}{\partial \dot{q}_j} = 0$$

$$\boxed{L \qquad \mathcal{F}}$$

$$v_x \frac{\partial A_x}{\partial x} + v_y \frac{\partial A_y}{\partial x} + v_z \frac{\partial A_z}{\partial x} - \underbrace{v_x \frac{\partial A_x}{\partial x} - v_y \frac{\partial A_x}{\partial y} - v_z \frac{\partial A_x}{\partial z}}_{\text{'A'}}$$

Time derivative $\frac{d\vec{A_x}}{dt} = \frac{d}{dt}\left(A_x(x, y, z, t) \right)$

$$\frac{dA_x}{dt} = \frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}\underbrace{\frac{dx}{dt}}_{v_x} + \frac{\partial A_x}{\partial y}\underbrace{\frac{dy}{dt}}_{v_y} + \frac{\partial A_x}{\partial z}\underbrace{\frac{dz}{dt}}_{v_z}$$

$$\underline{\frac{dA_x}{dt} - \frac{\partial A_x}{\partial t}} = \underline{v_x \frac{\partial A_x}{\partial x} + v_y \frac{\partial A_x}{\partial y} + v_z \frac{\partial A_x}{\partial z}} \rightarrow (A)$$

$$\vec{v}\times(\vec{\nabla}\times\vec{A})\Big|_x = \frac{\partial}{\partial x}(\vec{v}\cdot\vec{A}) - \frac{dA_x}{dt} + \frac{\partial A_x}{\partial t}$$

$$F_x = q\left[-\frac{\partial \phi}{\partial x} - \cancel{\frac{\partial A_x}{\partial t}} + \frac{\partial}{\partial x}(\vec{v}\cdot\vec{A}) - \frac{dA_x}{dt} + \cancel{\frac{\partial A_x}{\partial t}}\right]$$

$$= q\left[-\frac{\partial}{\partial x}(\phi - \vec{v}\cdot\vec{A}) - \frac{dA_x}{dt}\right]$$

$$= q\left[-\frac{\partial}{\partial x}(\phi - \vec{v}\cdot\vec{A}) - \frac{d}{dt}\left[\frac{\partial}{\partial v_x}(\vec{A}\cdot\vec{v})\right]\right]$$

$$= q\left[-\frac{\partial}{\partial x}(\phi - \vec{v}\cdot\vec{A}) - \frac{d}{dt}\frac{\partial}{\partial v_x}(\phi - \vec{v}\cdot\vec{A})\right]$$

as $\partial/\partial v_x(\phi) = 0$

Velocity dependent Potential
or
generalised Potential

Put $U = q(\phi - \vec{v}\cdot\vec{A})$

$$F_x = \underline{-\frac{\partial U}{\partial x} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{x}}\right)} = Q_j \quad -(B)$$

$L = T - V$

$= \frac{1}{2}mv^2 - q(\phi - \vec{v}\cdot\vec{A})$

$L = \frac{1}{2}m\dot{x}^2 - q\phi + q(\vec{v}\cdot\vec{A})$

$$\underline{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)} - \underline{\frac{\partial T}{\partial q_j}} = \underline{Q_j} \quad \text{from (B)}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = \frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right)$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0}$$

$\underline{L = T - U}$

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Lagrangian for a charged particle in an electromagnetic field

$$\vec{F} = q(\vec{E} + \vec{v}\times\vec{B}) \qquad \text{Lorentz Force}$$

Maxwell's eqⁿs.

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \;; \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \;; \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla}\times\vec{E} \neq 0 \qquad \text{but } \vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{B} = \underbrace{\vec{\nabla}\times\vec{A}}_{\text{Magnetic vector Potential}}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A}) = -\nabla\times\frac{\partial}{\partial t}\vec{A}$$

$$\vec{\nabla}\times\left(\vec{E} + \frac{\partial\vec{A}}{\partial t}\right) = 0 \Rightarrow \vec{E} + \frac{\partial\vec{A}}{\partial t} = -\underbrace{\vec{\nabla}\phi}_{\text{scaler Potential}}$$

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}$$

$$\vec{F} = q\left[-\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t} + (\vec{v}\times\vec{B})\right]$$

$$F_x = q\left[-\frac{\partial\phi}{\partial x} - \frac{\partial \vec{A}_x}{\partial t} + (\vec{v}\times\vec{B})_x\right]$$

$$\vec{v}\times\vec{B} = \begin{pmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{pmatrix}$$

$$v\times(\vec{\nabla}\times\vec{A})\Big|_x =$$

$$(\vec{\nabla}\times\vec{A})\;\hat{i}\left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right] - \hat{j}\left[\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right] + \hat{k}\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]$$

$$\vec{v}\times(\vec{\nabla}\times\vec{A})\Big|_x = \begin{pmatrix} i & j & k \\ v_x & v_y & v_z \\ \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} & -\frac{\partial A_z}{\partial x} + \frac{\partial A_x}{\partial z} & \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \end{pmatrix}$$

$$\cancel{\left(-\frac{\partial A_z}{\partial x} + \frac{\partial A_x}{\partial z}\right)}$$

$$\vec{v}\times(\vec{\nabla}\times\vec{A})\Big|_x = v_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - v_z\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)$$

$$= v_y\frac{\partial A_y}{\partial x} - v_y\frac{\partial A_x}{\partial y} - v_z\frac{\partial A_x}{\partial z} + v_z\frac{\partial A_z}{\partial x}$$

Add & subtract, $v_x\frac{\partial A_x}{\partial x}$

$v_x \partial A$