Root finding

$$I = A^2 \frac{\sin^2\alpha}{\alpha^2} \quad ; \quad \alpha = \frac{\pi e \sin\theta}{\lambda}$$

$\frac{dI}{d\alpha} = 0$ (Secondary Maxima)

$$\boxed{\alpha = \tan\alpha}$$

$\alpha - \tan\alpha = 0$; $f(\alpha) = 0$

$\downarrow$

α - roots of the equation

Roots

Analytic (Direct)

Numerical Method

Direct Method :-

- $a_1 x + a_0 = 0$

$$\boxed{x = -a_0 / a_1}$$

- $ax^2 + bx + c = 0$

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

- Polynomial equation

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x + a_0$$

$a_n \neq 0$; n^{th} order polynomial.

* $(n+1)$; a's values
* "n" no. of roots

roots $\rightarrow$ $\alpha_1, \alpha_2, \alpha_3 \ldots, \alpha_n$

$$\sum \alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_n = -\frac{a_{n-1}}{a_n}$$

$$\sum \alpha_1 \alpha_2 = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \cdots = \frac{a_{n-2}}{a_n}$$

$$\vdots$$

$$\alpha_1 \alpha_2 \alpha_3 \cdots \alpha_n = \frac{(-1)^n a_0}{a_n}$$

Complex root :- $(a + ib)$ is one root of any $f(x) = 0$

then $(a-ib)$ will also be root of $f(x)=0$

✓
* $(a+ib)$ — $(a-ib)$

→ polynomial (order : odd) ⇒ at least one of root will be a real no.

Question :- $f(x) = 3x^3 - 4x^2 + x + 88$ (order : 3)

one root of this function → $(2+\sqrt{7}\, i)$

find out other roots

Solution

$\alpha_1 = 2+\sqrt{7}\, i$ ✓

$\alpha_2 = 2-\sqrt{7}\, i$ ✓

$\alpha_3 = -\frac{8}{3}$ ✓

Numerical Methods

1. Bisection Method
2. Method of false position (Regula-falsi method)
3. Secant Method
4. Newton Raphson Method
5. Multiple roots by N-R method
6. Convergence (Rate) of these method

Approximate the polynomial with line $(n=1)$ "first order approx. method"

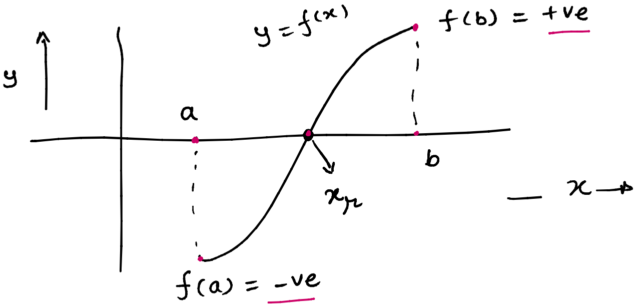
Bisection Method

$f(x)=0$; $[a \quad b]$

function $f(x)$ is continuous in given limit & satisfied the condition — $\boxed{f(a)\,f(b) < 0}$

↓

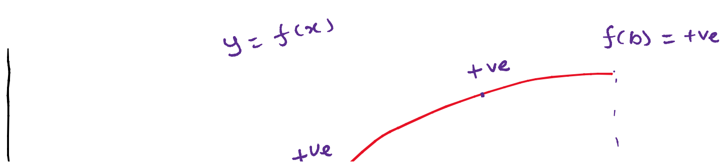
there is atleast one root in the given interval.



$[a \quad b]$

↓

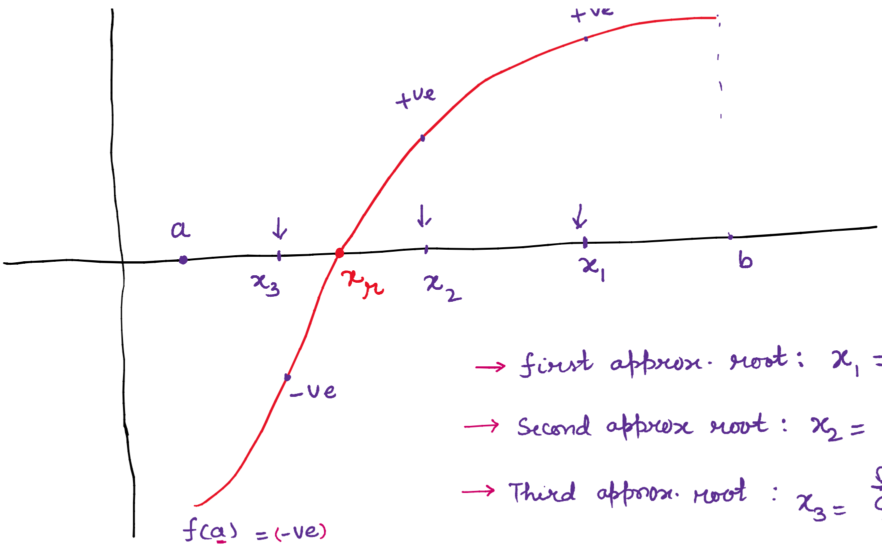
$f(a)\,f(b) < 0$



$f(x)=0$

$[a \quad b]$

$\boxed{f(a)\,f(b) < 0}$



[a b]

$\boxed{f(a)\,f(b) < 0}$

$f(a) = -ve$

$f(b) = +ve$ ✓

→ first approx. root : $x_1 = \dfrac{a+b}{2}$; $f(x_1) = +ve$

→ Second approx root : $x_2 = \dfrac{a+x_1}{2}$; $f(x_2) = +ve$

→ Third approx. root : $x_3 = \dfrac{a+x_2}{2}$; $f(x_3) = -ve$

forth approx root : $x_4 = \dfrac{x_3+x_2}{2}$

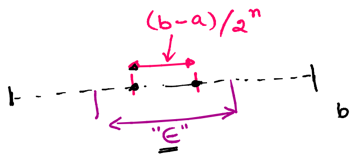
iteration

Minimum no. steps to find out roots within "ϵ" accuracy in given [a b] by bisection method.

$f(x) = 0 \qquad [a \quad b] \qquad f(a)\,f(b) < 0$

$\epsilon = 10^{-3} \qquad \epsilon = 10^{-7}$

$$\left(\frac{b-a}{2}\right) \longrightarrow \frac{(b-a)}{4} \longrightarrow \frac{(b-a)}{8} \longrightarrow \frac{(b-a)}{2^n}$$



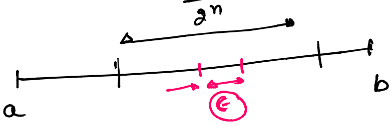
desired accuracy of root :- ϵ

$$\boxed{\frac{(b-a)}{2^n} \le \epsilon}$$

$\dfrac{b-a}{2^n} = 0.0001$

$\epsilon = 0.001$

$\dfrac{b-a}{2^n} = 0.01$



$\epsilon = 0.0001$

$$\log(b-a) - n\log 2 \le \log \epsilon$$

$$n\log 2 \geq \log(b-a) - \log\epsilon$$

$$\boxed{n \geq \frac{(\log(b-a) - \log\epsilon)}{\log 2}}$$

Root with certain decimal place accuracy

desired accuracy : 5th place of decimal

✓ $x_1 = 2.5$ compare these results

$x_1 = 2.5$

$x_2 = 2.305261$

$x_3 = 2.340256$

$x_4 = 2.\boxed{34021}6$

$x_5 = 2.\boxed{34021}3$

$.000001$

compare these results

$$\boxed{abs\,(x_i - x_{i-1}) \leq 10^{-6}}$$

$a = 2.3\times10^{-10}\,m$

$E \sim 2.2\,eV$

$E \sim 2.234895\,eV$

$\langle\psi_1\rangle = \boxed{1.15102367332} \times 10^{-10}\,m$

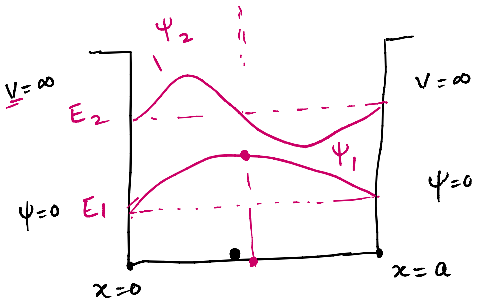
$= 115.10 \ldots \times 10^{-12}$

$[10^{-12}]$

$a \sim 10^{-10}\,m$

1.25 m

1.2505 m



$$\boxed{\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E-V)\,\psi = 0}$$

eigen value

$$\boxed{E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}}$$

$n = 1, 2, 3, \ldots$

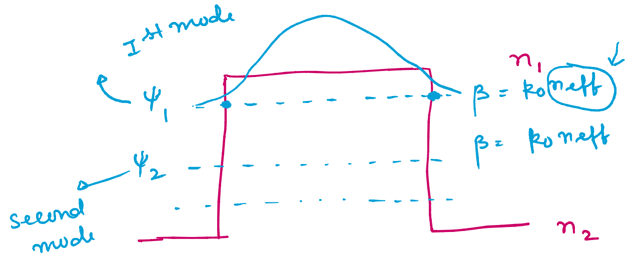
$$\boxed{\psi_n = \sqrt{\frac{2}{a}}\,\sin\left(\frac{\pi n x}{a}\right)}$$

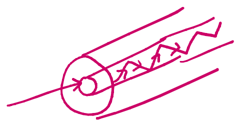
$(E-V) \rightarrow (n-E)$

Newton law

→ Maxwell eqn →

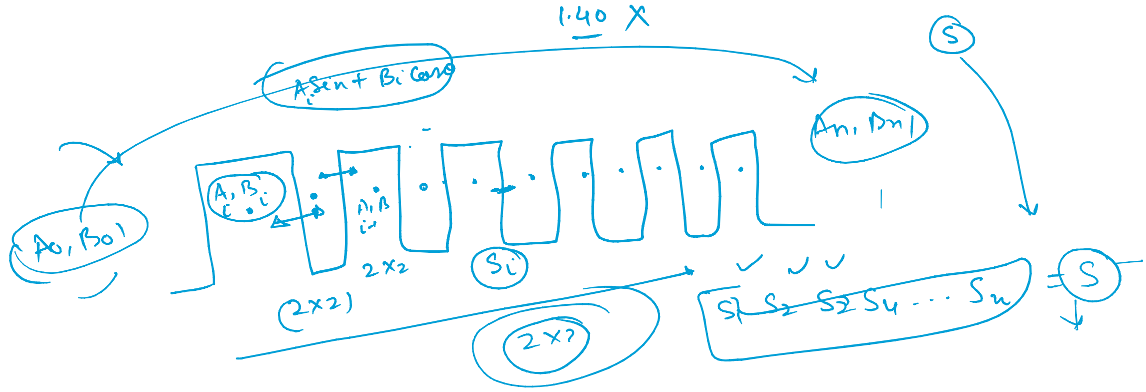
Schrodinger law





$n_{eff} = 1.4023456$

1.40 ✗



04 February 2021 12:04

Q.1. $f(x) = x^3 - 4x - 9 = 0$; $[2, 3]$; accuracy of root upto 3^{rd} place of decimal.

Bisection method.

Sol.

$$f(x) = x^3 - 4x - 9$$

$$\left.\begin{array}{l} f(2) = 8 - 8 - 9 = -9 = -ve \\ f(3) = 27 - 12 - 9 = +6 = +ve \end{array}\right]$$

$$x_1 = \frac{\overset{-}{2} + \overset{+}{3}}{2} = \underline{2.5}; \quad f(\overline{2.5}) = -3.375$$

$$x_2 = \frac{\overset{-}{2.5} + \overset{+}{3}}{2} = \underline{2.75}; \quad f(\overset{+}{2.75}) = 0.7969$$

$$x_3 = \frac{\overset{-}{2.5} + \overset{+}{2.75}}{2} = \underline{2.625}; \quad f(\overline{2.625}) = -1.4121$$

$$x_4 = \frac{\overset{-}{2.625} + \overset{+}{2.75}}{2} = \underline{2.6875}$$

$$x_5 = 2.7188$$

$$x_6 = 2.7031$$

$$x_7 = 2.7109$$

$$x_8 = 2.7070$$

$$x_9 = 2.7051$$

$$x_{10} = \underline{2.7060}$$

$$x_{11} = \underline{2.7065}$$

$$\boxed{x_{root} = 2.706}$$

Q.2. $f(x) = x\sin x - 1 = 0$; $[\underbrace{1 \quad 1.5}_{\text{radian}}]$; itaration = 5 (Bisection method)

Sol. :-

$$f(\overline{1}) = -0.15849; \quad f(\overset{+}{1.5}) = 0.49625$$

$$x_1 = \frac{\overset{-}{1} + \overset{+}{1.5}}{2} = 1.25; \quad f(\overset{+}{1.25}) = 0.18627$$

$$x_2 = \frac{\overset{-}{1} + \overset{+}{1.25}}{2} = 1.125; \quad f(\overset{+}{1.125}) = 0.01509$$

$$x_3 = \frac{\overset{-}{1} + \overset{+}{1.125}}{2} = 1.0625; \quad f(\overline{1.0625}) = -0.0718$$

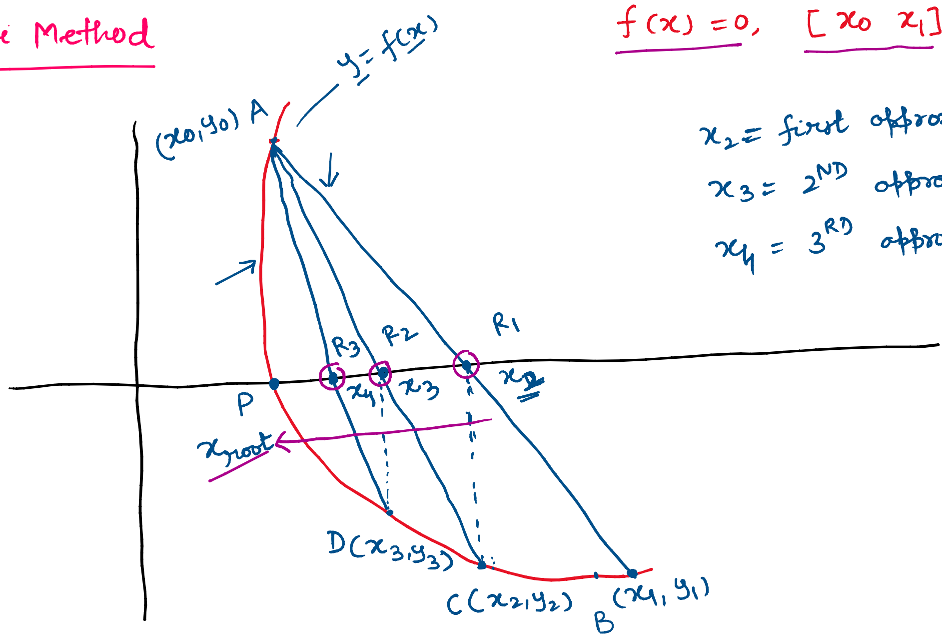
$$x_4 = \frac{\overset{-}{1.0625} + \overset{+}{1.125}}{2} = 1.09375; \quad f(\overline{1.09375}) = -0.02836$$

$$x_5 = \frac{1.09375 + 1.125}{2} = 1.10937$$

$$\boxed{x_{root} = 1.10937}$$

Regula-Falsi Method

$f(x) = 0$, $[x_0 \ x_1]$ (A, B); $f(a)\,f(b) < 0$



$x_2 =$ first approx.
$x_3 = 2^{ND}$ approx
$x_4 = 3^{RD}$ approx
} root

equation of chord AB passing through (x_0 & x_1)

$$y - y_0 = \left(\frac{y_1 - y_0}{x_1 - x_0}\right)(x - x_0) \quad \text{—— (1)}$$

if chord AB intercept the x-axis at $x = x_2$; $y(x_2) = 0$

$$0 - y_0 = \left(\frac{y_1 - y_0}{x_1 - x_0}\right)(x_2 - x_0)$$

$$\boxed{x_2 = x_0 - \frac{(x_1 - x_0)}{(y_1 - y_0)}\, y_0} \qquad \boxed{x_2 = x_0 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)}\, f(x_0)}$$

→ This method always converges.

Q.1. $f(x) = x^3 - 2x - 5 = 0$; $[2 \ 3]$; root correct to three decimal place by Regula-falsi method.

Sol. $f(x) = x^3 - 2x - 5$

$f(2) = -1$ (−ve), $f(3) = 16$ (+ve) $\qquad x_0 = 2$, $x_1 = 3$

Root interval → $[2 \ 3]$ (− +) $\qquad f(x_0) = -1$, $f(x_1) = 16$

$$x_2 = x_0 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)}\, f(x_0)$$

$$x_2 = 2 - \frac{(3-2)}{16-(-1)}(-1) = 2 + \frac{1}{17} = 2.0588.$$

$$f(\underline{2.0588}) = \underline{-0.3908}$$

$\rightarrow$ Root interval $[\overset{-}{2.0588} \quad \overset{+}{3}]$

$\downarrow x_0 \qquad \downarrow x_1$

$$x_3 = x_0 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_0)$$

$$x_3 = 2.0588 - \frac{(3 - 2.0588)}{(16 - (-0.3908))}(-0.3908) = 2.0588 + \frac{0.9412}{16.3908} \times 0.3908$$

$$= 2.0813$$

$x_4 = 2.0862$

$x_5 = 2.0915$

$x_6 = 2.0934$

$x_7 = \underline{2.094}1$

$x_8 = \underline{2.094}3$

$$\boxed{x_{root} = 2.094}$$

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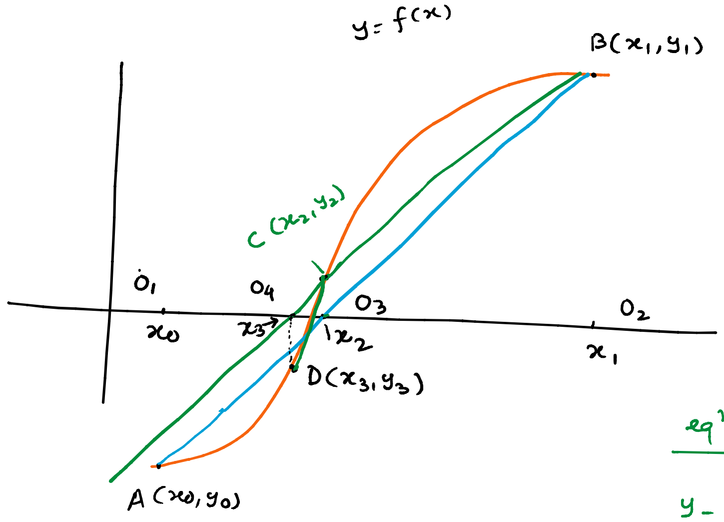
Secant Method

- Improved Regula-falsi method
- We use two previous approximated root to find out next root
- $f(x)\,f(y) < 0$ this condition is not essential

$y = f(x)\ ;\ [x_0, x_1]$



step 1. approximated $y = f(x)$ with first order polynomial by joing chord AB.

- this has given you first approximated root "x_2".

eqn of chord AB

$$y - y_1 = \frac{(y_1 - y_0)}{(x_1 - x_0)}(x - x_1) \quad \text{— } 1$$

at $x = x_2$ (AB is intersecting x-axis)

$y = 0$

$$0 - y_1 = \frac{y_1 - y_0}{(x_1 - x_0)}(x_2 - x_1)$$

I st approximated root by secant method

$$\boxed{x_2 = x_1 - \frac{(x_1 - x_0)}{(y_1 - y_0)}\,y_1} \quad \text{— } [x_0, x_1] \quad \text{— } 2$$

for second approximated root $[x_1, x_2]$ — two recent approximation

eqn of chord BC —

$$y - y_2 = \frac{(y_2 - y_1)}{(x_2 - x_1)}(x - x_2) \quad \text{— } 3$$

this chord passes through x-axis at $x = x_3$, $y = 0$

$$0 - y_2 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x_3 - x_2)$$

2nd approximation root by secant method

$$\boxed{x_3 = x_2 - \frac{(x_2 - x_1)}{(y_2 - y_1)} \cdot y_2} \quad \text{— } 4$$

in general.

— $[x_n, x_{n-1}]$

in general.

$$\boxed{x_{n+1} = x_n - \frac{(x_n - x_{n-1})}{(y_n - y_{n-1})}\, y_n}$$ — $[x_n, x_{n-1}]$

Convergence rate = 1.618

* if $y_n = y_{n-1}$; under this condition, method will not converge.

Ex. $f(x) = \cos x - x e^x$; $[0, 1] \rightarrow x_0 = 0, x_1 = 1$;

root correct upto 4 decimal place by secant method

sol.

- $x_0 = 0;\ y_0 = 1$; $x_1 = 1,\ y_1 = -2.17798$

$$x_2 = x_1 - \frac{(x_1 - x_0)}{(y_1 - y_0)}\, y_1$$

$$x_2 = 1 - \frac{(1-0)}{(-2.17798 - 1)}(-2.17798) = 0.31467;\quad y_2 = -0.51987$$

$$x_3 = x_2 - \frac{(x_2 - x_1)}{(y_2 - y_1)}\, y_2$$

$$= 0.31467 - \frac{(0.31467 - 1)}{(-0.51987 + 2.17798)}(-0.51987)$$

$$= 0.44673\ ;\quad y_3 = 0.20354$$

$x_4 = 0.53171$

$x_5 = 0.51690$

$x_6 = 0.51775$

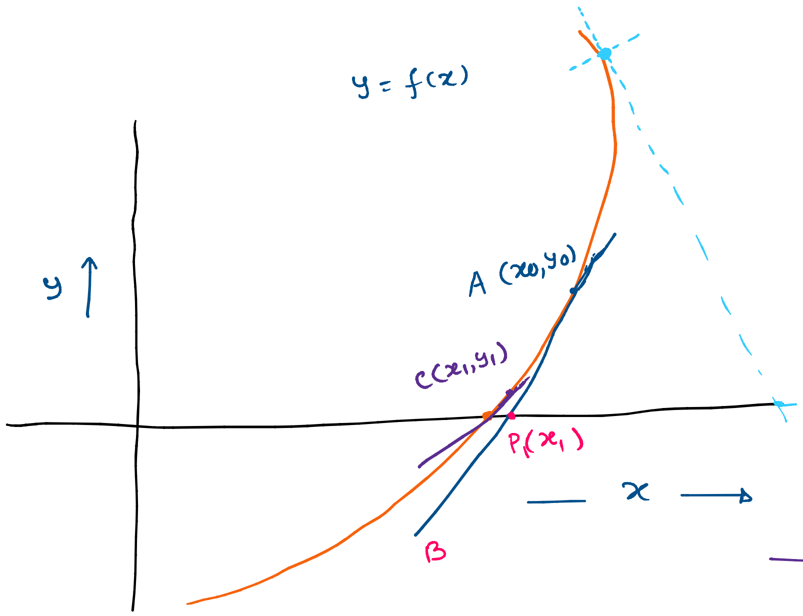
$x_7 = 0.51776$

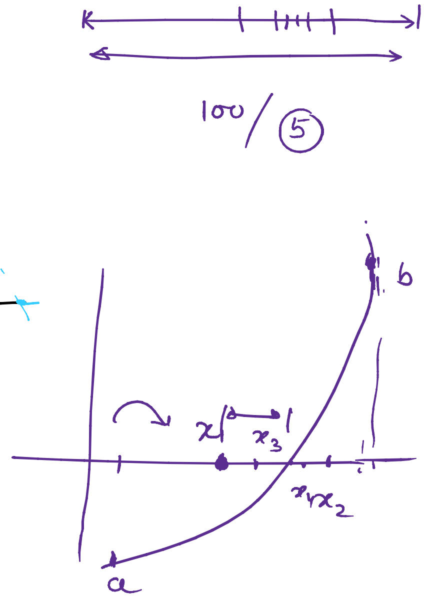
$x_{root} = 0.5177$

Newton-Raphson Method

- fastest converging method
- rate of convergence = 2 ✓
- change of sign condition is not essential
- we replace polynomial with straight line by using slope at given guess value.

- this method also not converges always.
- Convergence of this method is going to depend on choice of guess value.





equation of chord AB

$$y - y_0 = f'(x_0)(x - x_0)$$

at point P_1 : $x = x_1, \quad y = 0$

$$0 - y_0 = f'(x_0)(x_1 - x_0)$$

$$x_1 = x_0 - \frac{y_0}{f'(x_0)} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}} \quad \text{— I}^{\text{st}} \text{ approx. root}$$

$$\boxed{x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}} \quad \text{— 2}^{\text{ND}} \text{ approx. root}$$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

* $f'(x_n) = 0 \Rightarrow$ this method will not converge.

Ques :- $f(x) = x \log_{10} x - 1.2 \quad ; \quad x_0 = 2$

root correct upto four decimal place by Newton-Raphson method

Sol.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f(x_n) = x_n \log_{10} x_n - 1.2 \quad — (1)$$

$$f(x) = x \log_{10} x - 1.2$$

$$f'(x) = \log_{10} x + x \frac{1}{x} \log_{10} e = \log_{10} x + 0.43429$$

$$f'(x_n) = \log_{10} x_n + 0.43429 \quad — (2)$$

$$x_{n+1} = x_n - \frac{(x_n \log_{10} x_n - 1.2)}{(\log_{10} x_n + 0.43429)}$$

$$x_{n+1} = \frac{(\cancel{x_n \log_{10} x_n} + 0.43429\, x_n) - (\cancel{x_n \log_{10} x_n} - 1.2)}{(\log_{10} x_n + 0.43429)}$$

$$\boxed{x_{n+1} = \frac{0.43429\, x_n + 1.2}{\log_{10} x_n + 0.43429}}$$

$x_0 = 2$;

$n=0$ $$x_1 = \frac{0.43429\, x_0 + 1.2}{(\log_{10} x_0 + 0.43429)} = \underline{2.81317}$$

$n=1$ $$x_2 = \frac{0.43429\, x_1 + 1.2}{\log_{10} x_1 + 0.43429} = \underline{2.74111}$$

$n=2$ $x_3 = \underline{2.74065}$

$n=3$ $x_4 = \underline{2.74065}$

$$\boxed{x_{root} = 2.7406}$$

Iterative formula by Newton-Raphson Method

① square root of given no. $\sqrt{N}$

Sol. :- $x = \sqrt{N}$

$x^2 = N$; $\underline{x^2 - N = 0}$

$\underline{f(x)} = x^2 - N$; $\underline{f'(x)} = 2x$

Newton Raphson Method –

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{(x_n^2 - N)}{(2x_n)}$$

$$= \frac{2x_n^2 - (x_n^2 - N)}{2x_n} = \frac{x_n^2 + N}{2x_n}$$

$$\boxed{x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)}$$

example :- $\sqrt{28} \simeq 5$ $N = 28$

$\underline{x_0 = 5}$

$n=0 \Rightarrow x_1 = \frac{1}{2}\left(5 + \frac{28}{5}\right) = 5.3$

$n=1 \Rightarrow x_2 = \frac{1}{2}\left(5.3 + \frac{28}{5.3}\right) = \underline{5.29151}$

$n=2$ $x_3 = \frac{1}{2}\left(5.29151 + \frac{28}{5.29151}\right) = \underline{5.29156}$

② iterative formula for $1/\sqrt{N}$

sol. $x = \frac{1}{\sqrt{N}}$

$x^2 = \frac{1}{N}$; $x^2 - \frac{1}{N} = 0$

$f(x) = x^2 - \frac{1}{N}$; $f'(x) = 2x$

N.R- formula

$(x_n^2 - \frac{1}{\ }$

N.R.- formula

$$x_{n+1} = x_n - \frac{(x_n^2 - \frac{1}{N})}{2x_n} = \frac{2x_n^2 - (x_n^2 - \frac{1}{N})}{2x_n}$$

$$x_{n+1} = (x_n^2 + \tfrac{1}{N}) / 2x_n$$

$$\boxed{x_{n+1} = \frac{1}{2}\left(x_n + \frac{1}{N x_n}\right)}$$

ex. $\frac{1}{\sqrt{14}} = ??$ $N = 14$; if you take $N = 16$, $\frac{1}{\sqrt{16}} = 0.25 \rightarrow x_0$

sol. $x_0 = 0.25$

$n=0$: $x_1 = \frac{1}{2}\left(0.25 + \frac{1}{14 \times 0.25}\right) = 0.26785$

$n=1$: $x_2 = \frac{1}{2}\left(0.26785 + \frac{1}{14 \times 0.26785}\right) = 0.26726$

$n=2$: $x_3 = \frac{1}{2}\left(0.26726 + \frac{1}{14 \times 0.26726}\right) = 0.26726$

3RD place

5th place

$\leftrightarrow$ find out iterative formula for $- \sqrt[k]{N}$

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Convergence

1. Bracking method ✓
2. Open method ✓

$y = f(x)$; $x = x_{root}$; $y = f(x_{root}) = 0 = 1.0e-7$

Method 1	Method 2
2.50	2.4023
2.4423	2.4013
2.4223	2.4012
2.4106	
2.4023	
2.4032	

rate of convergence of method 2 is faster than that of method 1.

Rate of Convergence

$y = f(x)$; $x = x_{root} \rightarrow$ actual root

$x_1, x_2, x_3, \ldots$ approximated root

$x_1 - x_r = e_1$ (error)

$x_2 - x_r = e_2$ (error in 2ND iteration)

$\vdots$

$x_n - x_r = e_n$ (error in nth iteration)

$x_{n+1} - x_r = e_{n+1}$ (error in $(n+1)^{th}$ iteration)

$$\boxed{|e_{n+1}| \leq A\,|e_n|^p}$$

rate of convergence of method is p.

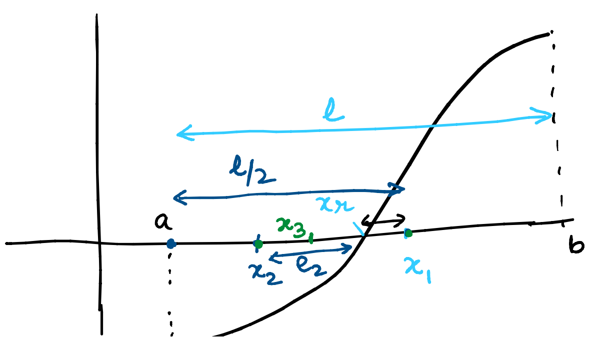
$p = 1$ (linear convergence)

A = asymptotic error constant

$p = 2$ (Quadratic convergence)

Bisection Method

$y = f(x)$, $[a\ b]$



x_1 = Ist approx. root

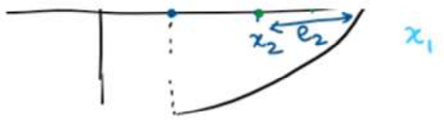
x_r = real root

$e_1 = x_1 - x_r$ (error in root cal in Ist approx)

$e_1 < l$

$e_2 = x_2 - x_r$

$e_2 < \frac{l}{2}$



b

$$e_2 < \frac{l}{2}$$

$$e_3 < \frac{l}{2^2} \quad (e_3 = x_3 - x_r)$$

$$\vdots$$

$$\left.\begin{array}{l} e_n < \dfrac{l}{2^{n-1}} \\ e_{n+1} < \dfrac{l}{2^n} \end{array}\right]$$

$$\frac{e_{n+1}}{e_n} < \frac{1}{2}$$

$$e_{n+1} < \frac{1}{2} e_n$$

$$|e_{n+1}| < \frac{1}{2} |e_n|$$

$p=1$, (rate of convergence of bisection method is 1)

linear convergence (slow convergence)

Secant Method

$y = f(x)$, $[a \quad b]$; x_r = real root

e_n, e_{n+1} are errors in the root calculation in nth & $(n+1)$th iteration.

$$\left.\begin{array}{l} e_n = x_n - x_r \\ e_{n+1} = x_{n+1} - x_r \end{array}\right] \quad (1)$$

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} f(x_n) \quad \text{—— } (2)$$

from eq^n (1) & (2)

$$\cancel{x_r} + e_{n+1} = \cancel{x_r} + e_n - \frac{[\cancel{x_r} + e_n - \cancel{x_r} - e_{n-1}]}{f(x_r + e_n) - f(x_r + e_{n-1})} f(x_r + e_n)$$

$$e_{n+1} = e_n - \frac{[e_n - e_{n-1}]}{[f(x_r + e_n) - f(x_r + e_{n-1})]} f(x_r + e_n)$$

$$e_{n+1} = e_n - \frac{[e_n - e_{n-1}]\left[\overset{=0}{\cancel{f(x_r)}} + e_n f'(x_r) + \frac{e_n^2}{2!} f''(x_r) + \cdots\right]}{\left\{\left[\underset{\shortparallel_0}{f(\cancel{x_r})} + e_n f'(x_r) + \frac{e_n^2}{2} f''(x_r) + \cdots\right] - \left[\underset{\shortparallel_0}{f(\cancel{x_r})} + e_{n-1} f'(x_r) + \frac{e_{n-1}^2}{2} f''(x_r)\right]\right\}}$$

$f(x_r) = 0$

$$e_{n+1} = e_n - \frac{(e_n - e_{n-1})\left(e_n f'(x_r) + \frac{e_n^2}{2} f''(x_r)\right)}{\left[\underbrace{(e_n - e_{n-1}) f'(x_r)} + (e_n^2 - e_{n-1}^2) \frac{f''(x_r)}{2}\right]}$$

$$(e_n - \cancel{e_{n-1}}) e_n f'(\cancel{x_r}) \left(1 + \frac{e_n}{2} \frac{f''(x_r)}{\quad}\right)$$

$$e_{n+1} = e_n - \frac{\cancel{(e_n - e_{n-1})}\, e_n \cancel{f'(x_r)} \left(1 + \frac{e_n}{2}\frac{f''(x_r)}{f'(x_r)}\right)}{\cancel{(e_n - e_{n-1}) f'(x_r)} \left[1 + (e_n + e_{n-1})\frac{f''(x_r)}{2f'(x_r)}\right]}$$

$$e_{n+1} = e_n - \frac{e_n\left(1+\frac{e_n}{2}\frac{f''(x_r)}{f'(x_r)}\right)}{\left(1+(e_n+e_{n-1})\frac{f''(x_r)}{2f'(x_r)}\right)}$$

$$e_{n+1} = e_n - e_n\left[\left(1+\frac{e_n}{2}\frac{f''(x_r)}{f'(x_r)}\right)\right]\left[1+(e_n+e_{n-1})\frac{f''(x_r)}{2f'(x_r)}\right]^{-1}$$

$$e_{n+1} = e_n - e_n\left[\left(1+\underline{\frac{e_n}{2}\frac{f''(x_r)}{f'(x_r)}}\right)\left(1-(e_n+e_{n-1})\frac{f''(x_r)}{2f'(x_r)}\right)\right]$$

$$e_{n+1} = e_n - \underline{e_n}\left[1 - (\cancel{e_n}+e_{n-1})\frac{f''(x_r)}{2f'(x_r)} + \frac{e_n f''(x_r)}{2f'(x_r)} + \underbrace{O(e_n^2)}_{\downarrow 0}\right]$$

$$e_{n+1} = e_n - e_n\left[1 - \frac{e_{n-1} f''(x_r)}{2f'(x_r)}\right]$$

$$e_{n+1} = \cancel{e_n} - \cancel{e_n} + e_n e_{n-1}\frac{f''(x_r)}{2f'(x_r)} = c$$

$$\boxed{e_{n+1} = c\, e_n e_{n-1}} \quad — (2)$$

$$e_{n+1} = A\, \underline{e_n}^{p} \quad — (3)$$

$$e_n = A\, e_{n-1}^{p} \Rightarrow e_{n-1}^{p} = A^{-1} e_n\,; \quad \boxed{e_{n-1} = A^{-1/p}\, \underline{e_n}^{1/p}} \quad — (4)$$

from (2), (3) & (4)

$$A\, e_n^{p} = c\, e_n A^{-1/p} e_n^{1/p} \quad — (5)$$

Comparing the power of en in both side.

$$p = 1 + \frac{1}{p}$$

$$p^2 - p - 1 = 0 ; \qquad p = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$\boxed{p = \frac{1+\sqrt{5}}{2} = 1.618}$$

Order of rate of convergence of secant method is 1.618.

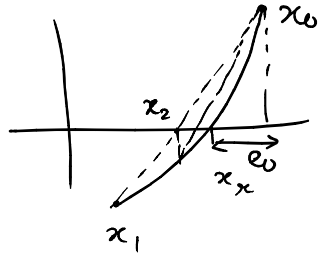
Regula-falsi method

from eq^n 2.

$$e_{n+1} = c\, e_n\, e_{n-1} \qquad C = \frac{1}{2} \frac{f''(x_r)}{f'(x_r)}$$

Regula-falsi method; one of the end remains fixed.

thus $e_{n-1} = e_0 = x_0 - x_r$



$$e_{n+1} = \underline{c\, e_0}\, e_n$$

$$\boxed{e_{n+1} = A\, e_n} \qquad \boxed{A = \frac{1}{2} e_0 \frac{f''(x_r)}{f'(x_r)}}$$

$\Downarrow$

$p = 1$

Newton-Raphson Method

$x_1, x_2, x_3 \ldots\ldots$ approx. root ; x_r = real root

$e_1, e_2, e_3 \ldots$ error in approx. root.

$$e_{n+1} = x_{n+1} - x_r \; ; \quad e_n = x_n - x_r \quad — \; (1)$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad — \; (2)$$

$$\cancel{x_r} + e_{n+1} = \cancel{x_r} + e_n - \frac{f(x_r + e_n)}{f'(x_r + e_n)}$$

$$e_{n+1} = e_n - \left[\underbrace{f(x_r)}_{0} + e_n f'(x_r) + \frac{e_n^2}{2} f''(x_r)\right] \left[f'(x_r) + e_n f''(x_r) + \underbrace{\frac{e_n^2}{2} f''(x_r)}_{\to 0}\right]^{-1}$$

$$e_{n+1} = e_n - \left[e_n f'(x_r) + \frac{e_n^2}{2} f''(x_r)\right] \frac{1}{f'(x_r)} \left[1 + \frac{e_n f''(x_r)}{f'(x_r)}\right]^{-1}$$

$$= e_n - \left[e_n + \frac{e_n^2}{2} \frac{f''(x_r)}{f'(x_r)} \right] \left[1 - \frac{e_n f''(x_r)}{f'(x_r)} \right]$$

$$e_{n+1} = e_n - \left[e_n - \frac{e_n^2 f''(x_r)}{f'(x_r)} + \frac{e_n^2 f''(x_r)}{2 f'(x_r)} + \underset{\downarrow 0}{O(e_n^3)} \right]$$

$$e_{n+1} = \cancel{e_n} - \cancel{e_n} + \frac{e_n^2 f''(x_r)}{2 f'(x_r)}$$

$$e_{n+1} = \underbrace{\left(\frac{1}{2} \frac{f''(x_r)}{f'(x_r)} \right)}_{= A} e_n^2$$

$$\boxed{e_{n+1} = A e_n^2}$$

$p = 2$, (Quadratic convergence)