8.6. ERRORS IN QUADRATURE FORMULAE

The error in the quadrature formulae is given by

$$E = \int_a^b y\,dx - \int_a^b P(x)\,dx$$

where $P(x)$ is the polynomial representing the function $y = f(x)$, in the interval $[a, b]$.

(1) Error in Trapezoidal rule. Expanding $y = f(x)$ around $x = x_0$ by Taylor's series, we get

$$y = y_0 + (x - x_0)y_0' + \frac{(x - x_0)^2}{2!}\,y_0'' + \ldots \qquad \ldots(1)$$

$x - x_0 = t$
$dx = dt$ (limit $0 - h$)

$$\therefore \quad \int_{x_0}^{x_0+h} y\,dx = \int_{x_0}^{x_0+h} \left[y_0 + (x - x_0)y_0' + \frac{(x - x_0)^2}{2!}\,y_0'' + \ldots\right] dx$$

$$= y_0 h + \frac{h^2}{2!}\,y_0' + \frac{h^3}{3!}\,y_0'' + \ldots \qquad \ldots(2)$$

Also A_1 = area of the first trapezium in the interval $[x_0, x_1] = \frac{1}{2}h(y_0 + y_1)$ ...(3)

Putting $x = x_0 + h$ and $y = y_1$ in (1), we get $\quad y_1 = y_0 + hy_0' + \frac{h^2}{2!}\,y_0'' + \ldots$

Substituting this value of y_1 in (3), we get

$$A_1 = \frac{1}{2}h\left[y_0 + y_0 + hy_0' + \frac{h^2}{2!}\,y_0'' + \ldots\right]$$

$$= hy_0 + \frac{h^2}{2}\,y_0' + \frac{h^3}{2.2!}\,y_0'' + \ldots \qquad \ldots(4)$$

$\therefore$ Error in the interval $[x_0, x_1] = \int_{x_0}^{x_1} y\,dx - A_1$ $\qquad$ [(2) – (4)]

$$= \left(\frac{1}{3!} - \frac{1}{2.2!}\right)h^3 y_0'' + \ldots = -\frac{h^3}{12}\,y_0'' + \ldots$$

i.e. Principal part of the error in $[x_0, x_1] = -\frac{h^3}{12} y_0''$

Similarly principal part of the error in $[x_1, x_2] = -\frac{h^3}{12} y_1''$ and so on.

Hence the total error $E = -\frac{h^3}{12} [y_0'' + y_1'' + \dots + y''_{n-1}]$

Assuming that $y''(X)$ is the largest of the n quantities $y_0'', y_1'', \dots, y''_{n-1}$, we obtain

$$E < -\frac{nh^3}{12} y''(X) = -\frac{(b-a)h^2}{12} y''(X) \qquad [\because \; nh = b - a \qquad \dots(5)$$

Hence the error in the trapezoidal rule is of the order h^2.

(2) Error in Simpson's $\frac{1}{3}$-rule. Expanding $y = f(x)$ around $x = x_0$ by Taylor's series, we get (1).

take $x - x_0 = t$
$dx = dt$

$\therefore$ Over the first doubt strip, we get

$$\int_{x_0}^{x_2} y\,dx = \int_{x_0}^{x_0+2h} \left[y_0 + (x - x_0) y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \dots \right] dx$$

$$= 2hy_0 + \frac{4h^2}{2!} y_0' + \frac{8h^3}{3!} y_0'' + \frac{16h^4}{4!} y_0''' + \frac{32h^5}{5!} y_0^{iv} + \dots \qquad \dots(6)$$

Also A_1 = area over the first doubt strip by Simpson's $\frac{1}{3}$-rule

$$= \frac{1}{3} h (y_0 + 4y_1 + y_2) \qquad \dots(7)$$

Putting $x = x_0 + h$ and $y = y_1$ in (1), we get

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y'' + \frac{h^3}{3!} y_0''' + \dots\dots$$

Again putting $x = x_0 + 2h$ and $y = y_2$ in (1), we have

$$y_2 = y_0 + 2hy_0' + \frac{4h^2}{2!} y_0'' + \frac{8h^3}{3!} y_0''' + \dots$$

Substituting these values of y_1 and y_2 in (7), we get

$$A_1 = \frac{h}{3} \left[y_0 + 4 \left(y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \dots \right) \right.$$

$\frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{iv}$

$$\left. + \left(y_0 + 2hy_0' + \frac{4h^2}{2!} y_0'' + \frac{8h^3}{3!} y_0''' + \dots \right) \right]$$

$\frac{16}{4!} y_0^{iv}$

$$= \cancel{2}hy_0 + 2\cancel{h}^2 y_0' + \cancel{\frac{4h^3}{3}} y_0'' + \cancel{\frac{2h^4}{3}} y_0''' + \frac{5h^5}{18} y_0^{iv} + \dots \qquad \dots(8)$$

$\therefore$ Error in the interval $[x_0, x_2]$

$$= \int_{x_0}^{x_2} y\,dx - A_1 = \left(\frac{4}{15} - \frac{5}{18} \right) h^5 y_0^{iv} + \dots \qquad [(6) - (8)$$

i.e. Principal part of the error in $[x_0, x_2]$

$$= \left(\frac{4}{15} - \frac{5}{18}\right) h^5 y_0^{iv} = -\frac{h^5}{90} y_0^{iv}$$

Similarly principal part of the error in $[x_2, x_4] = -\frac{h^5}{90} y_2^{iv}$ and so on.

Hence the total error $E = -\frac{h^5}{90} [y_0^{iv} + y_2^{iv} + \ldots + y^{iv}_{2(n-1)}]$

Assuming the $y^{iv}(X)$ is the largest of $y_0^{iv}, y_2^{iv}, \ldots, y^{iv}_{2n-2}$, we get

$$E < -\frac{nh^5}{90} y_0^{iv}(X) = -\frac{(b-a)h^4}{180} y^{iv}(X) \qquad [\because\ 2nh = b - a \qquad \ldots(9)$$

i.e. the error in Simpson's $\frac{1}{3}$-rule is of the order h^4.

(3) Error in Simpson's 3/8-rule. Proceeding as above, here the principal part of the error in the interval $[x_0, x_3]$

$$= -\frac{3h^5}{80} y^{iv} \qquad \ldots(10)$$

(4) Error in Boole's rule. In this case, the principal part of the error in the interval $[x_0, x_4]$

$$= -\frac{8h^7}{945} y^{vi} \qquad \ldots(11)$$

(5) Error in Weddle's rule. In this case, principle part of the error in the interval

$[x_0, x_6]$
$$= -\frac{h^7}{140} y_0^{vi}. \qquad \ldots(12)$$

8.7. ROMBERG'S METHOD

In § 8.5, we have derived approximate quadrature formulae with the help of finite differences method. Romberg's method provides a simple modification to these quadrature formulae for finding their better approximations. As an illustration, let us improve upon the value of the integral

$$I = \int_a^b f(x)\, dx,$$

by Trapezoidal rule. If I_1, I_2 be the values of I with sub-intervals of width h_1, h_2 and E_1, E_2 be their corresponding errors respectively, then

$$E_1 = -\frac{(b-a)h_1^2}{12} y''(X), E_2 = -\frac{(b-a)^2 h_2^2}{12} y''(\bar{X})$$

Since $y''(\bar{X})$ is also the largest value of $y''(x)$, we can reasonably assume that $y''(X)$ and $y''(\bar{X})$ are very nearly equal.

$$\therefore \qquad \frac{E_1}{E_2} = \frac{h_1^2}{h_2^2} \quad \text{or} \quad \frac{E_1}{E_2 - E_1} = \frac{h_1^2}{h_2^2 - h_1^2} \qquad \ldots(1)$$

Now since $$I = I_1 + E_1 = I_2 + E_2,$$

$\therefore$ $$E_2 - E_1 = I_1 - I_2 \qquad ...(2)$$

From (1) and (2), we have

$$E_1 = \frac{h_1^2}{h_2^2 - h_1^2}(I_1 - I_2)$$

Hence $$I = I_1 + E_1 = I_1 + \frac{h_1^2}{h_2^2 - h_1^2}(I_1 - I_2) \quad i.e. \quad I = \frac{I_1 h_2^2 - I_2 h_1^2}{h_2^2 - h_1^2} \qquad ...(3)$$

which is a better approximation of I.

To evaluate I systematically, we take $h_1 = h$ and $h_2 = \frac{1}{2}h$

so that (3) gives $$I = \frac{I_1 (h/2)^2 - I_2 h_2^2}{(h/2)^2 - h^2} = \frac{4I_2 - I_1}{3} \quad i.e. \; I(h, h/2) = \tfrac{1}{3}\,[4I(h/2) - I(h)] \qquad ...(4)$$

Now we use the trapezoidal rule several times successively halving h and apply (4) to each pair of values as per the following scheme :

$I(h)$			
	$I(h, h/2)$		
$I(h/2)$		$I(h, h/2, h/4)$	
	$I(h/2, h/4)$		$I(h, h/2, h/4, h/8)$
$I(h/4)$		$I(h/2, h/4, h/8)$	
	$I(h/4, h/8)$		
$I(h/8)$			

The computation is continued till successive values are close to each other. This method is called *Richardson's deferred approach to the limit* and its systematic refinement is called *Romberg's method.*

Example 8.10. *Use Romberg's method to compute* $\int_0^1 \frac{dx}{1+x^2}$ *correct to 4 decimal places.*

(Anna, B.E., 2002)

Sol. We take $h = 0.5$, 0.25 and 0.125 successively and evaluate the given integral using Trapezoidal rule.

(*i*) When $h = 0.5$, the values of $y = (1 + x^2)^{-1}$ are

x :	0	0.5	1.0
y :	1	0.8	0.5

$$\therefore \qquad I = \frac{0.5}{2}\,[1 + 2 \times 0.8 + 0.5) = 0.775$$

(*ii*) When $h = 0.25$, the values of $y = (1 + x^2)^{-1}$ are

x :	0	0.25	0.5	0.75	1.0
y :	1	0.9412	0.8	0.64	0.5

$$\therefore \qquad I = \frac{0.25}{2}\,[1 + 2(0.9412 + 0.8 + 0.64) + 0.5] = 0.7828$$

(iii) When $h = 0.125$, we find that $I = 0.7848$

Thus we have

$$I(h) = 0.7750,\ I(h/2) = 0.7828,\ I(h/4) = 0.7848$$

Now using (4) above, we obtain

$$I(h, h/2) = \tfrac{1}{3}\,[4I(h/2) - I(h)] = \tfrac{1}{3}\,(3.1312 - 0.775) = 0.7854$$

$$I(h/2, h/4) = \tfrac{1}{3}\,[(4I(h/4) - I(h/2)] = \tfrac{1}{3}\,(3.1392 - 0.7828) = 0.7855$$

and $$I(h, h/2, h/4) = \tfrac{1}{3}\,[4I(h/2, h/4) - I(h, h/2)] = \tfrac{1}{3}\,(3.142 - 0.7854) = 0.7855$$

$\therefore$ The table of these values is

0.7750		
	0.7854	
0.7828		0.7855
	0.7855	
0.7848		

Hence the value of the integral = 0.7855.