18 February 2021 12:04

Interpolation

1. Difference operator

   - forward difference operator (Δ)
   - Backward difference operator (∇)
   - Central difference (δ)
   - Averaging operator (μ)
   - Shifting operator (fundamental operator) (E)

2.

x	y
x_0	y_0
→ x	← $y(x)$
x_1	y_1
x_2	y_2
x_3	y_3
⋮	⋮
x_n	y_n

Numerical method to find out unknown functional value at some intermediate point x.

$$y = y(x)$$

Type of data

1. equally spaced data

$$\boxed{x_i - x_{i-1} = h} \quad (\text{constant})$$

2. unequally space data

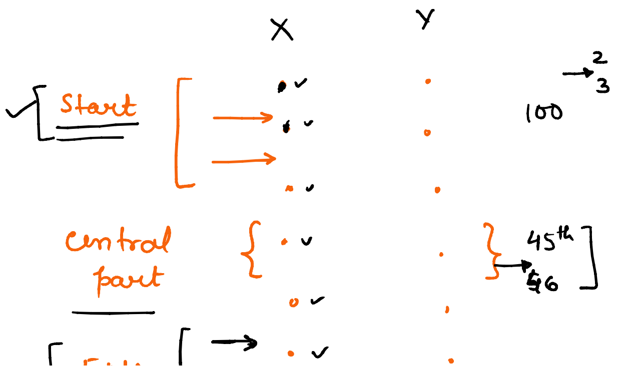
$$x_i - x_{i-1} \neq h \quad (\text{always})$$

when data is having un-equal interval

1. Lagrange's method
2. Newton's finite divided difference method

when data is having equal interval

1. Newton's forward Method
2. Newton's Backward Method
3. Gauss forward method



3. Gauss forward method

4. Gauss Backward method

5. Stirling method

6. Bessel's method

7. Everett's method

part

[END

1. create a difference table $\Rightarrow$
2. Use of formula $\Rightarrow$

forward difference operator (Δ)

X	Y	Δ	Δ^2	Δ^3
x_0	y_0			
		$y_1 - y_0 = \Delta y_0$		
x_1	y_1		$\Delta y_1 - \Delta y_0 = \Delta^2 y_0$	
		$y_2 - y_1 = \Delta y_1$		$\Delta^2 y_1 - \Delta^2 y_0 = \Delta^3 y_0$
x_2	y_2		$\Delta y_2 - \Delta y_1 = \Delta^2 y_1$	
		$y_3 - y_2 = \Delta y_2$		
x_3	y_3			

$$\Delta y_0 = y_1 - y_0$$

$$\Delta y_1 = y_2 - y_1$$

$$\vdots$$

$$\boxed{\Delta y_n = y_{n+1} - y_n}$$

$$\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta(y_1 - y_0) = \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0)$$

$$\boxed{\Delta^2 y_0 = y_2 - 2y_1 + y_0}$$

$$\vdots$$

$$\boxed{\Delta^n y_0 = y_n - {}^nc_1\, y_{n-1} + {}^nc_2\, y_{n-2} - {}^nc_3\, y_{n-3} \cdots\cdots}$$

$n=1$; $\Delta y_0 = y_1 - {}^1c_1\, y_0 = y_1 - y_0$ ✓

$n=2$; $\Delta^2 y_0 = y_2 - {}^2c_1\, y_1 + {}^2c_2\, y_0 = \boxed{y_2 - 2y_1 + y_0}$

Backward difference operator (∇)

X	Y	∇
x_0	y_0	
		$y_1 - y_0 = \nabla y_1$
x_1	y_1	
		$y_2 - y_1 = \nabla y_2$
x_2	y_2	
		$y_3 - y_2 = \nabla y_3$
x_3	y_3	

$$\boxed{\nabla y_n = y_n - y_{n-1}}$$

Central difference operator (δ)

X	Y	δ
x_0	y_0	
		$y_1 - y_0 = \delta y_{1/2}$
x_1	y_1	
		$y_2 - y_1 = \delta y_{3/2}$
x_2	y_2	
		$y_3 - y_2 = \delta y_{5/2}$
x_3	y_3	

$$\boxed{\delta y_n = y_{n+\frac{1}{2}} - y_{n-\frac{1}{2}}}$$

Averaging operator (μ)

$$\boxed{\mu y_n = \left(\frac{y_{n+\frac{1}{2}} + y_{n-\frac{1}{2}}}{2}\right)}$$

Shifting operator (E) (fundamenta operator)

$$x_i - x_{i-1} = h$$

$$x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = h$$

X	Y
x_0	y_0
x_1	y_1

$$\begin{tikzcd} x_0 \arrow[d] & E \rightarrow y_0 \arrow[d, bend left] \\ x_1 \arrow[r] & y_1 \arrow[u, bend left] \end{tikzcd}$$

E, E^{-1}

$$y = f(x) \quad ; \quad E y_0 = f(x_0 + h)$$

$$= f(x_1) = y_1$$

$$E y = f(x+h)$$

$$E^2 y = f(x+2h) \qquad \boxed{E^2 y_0 = y_2}$$

$x_1 \xrightarrow{E} y_1$, $\quad y_1 \xrightarrow{E^{-1}} x_1$

$x_2 \qquad y_2$

$x_3 \qquad y_3$

$$E\,y = f(x+h)$$

$$E^2 y = f(x+2h) \qquad \boxed{E^2 y_0 = y_2}$$

$$E^{-1} y_1 = y_0$$

$$\boxed{E^n y = f(x+nh)} \qquad \boxed{E^{-1} y_2 = y_1}$$

$$\boxed{E^{-n} y = f(x-nh)}$$

* $$\Delta y_n = y_{n+1} - y_n$$

$$= E y_n - y_n$$

$$\boxed{\Delta y_n = (E-1) y_n} \qquad \boxed{\Delta = E - 1 \;; \qquad E = 1 + \Delta}$$

* $$\nabla y_n = y_n - y_{n-1}$$

$$E^{1/2} y_0 =$$

$$= y_n - E^{-1} y_n$$

$$= (1 - E^{-1}) y_n$$

$$\nabla = (1 - E^{-1}) \;; \qquad \boxed{E = (1-\nabla)^{-1}}$$

* $$\delta y_n = y_{n+\frac{1}{2}} - y_{n-\frac{1}{2}}$$

* $$\mu y_n = \frac{\left(y_{n+\frac{1}{2}} + y_{n-\frac{1}{2}}\right)}{2}$$

$$= E^{1/2} y_n - E^{-1/2} y_n$$

$$\boxed{\mu = \frac{E^{1/2} + E^{-1/2}}{2}}$$

$$= (E^{1/2} - E^{-1/2}) y_n$$

$$\boxed{\delta = E^{1/2} - E^{-1/2}}$$

Q. $E\nabla = \Delta = \nabla E$

Sol.

$$(E\nabla) y_n = E(\nabla y_n)$$

$$= E(y_n - y_{n-1})$$

$$= E y_n - E y_{n-1}$$

$$(\nabla E) y_n = \nabla (E y_n)$$

$$= \nabla (y_{n+1})$$

$$= y_{n+1} - y_n$$

$$= \Delta y_n$$

$= E y_n - E y_{n-1}$

$= y_{n+1} - y_n$

$= \Delta y_n$

$\boxed{E \nabla = \Delta}$ — ①

$= \Delta y_n$

$\boxed{\nabla E = \Delta}$ — ②

$\boxed{E \nabla = \Delta = \nabla E}$

$\boxed{\nabla \Delta = \Delta - \nabla = \delta^2}$

$\boxed{\Delta^3 y_2 = \nabla^3 y_5}$

Q. $\Delta^2 (ab^x)$; $\Delta^n e^x$; $h = 1$

$\Delta f(x) = f(x+h) - f(x)$

$\Delta y_0 = y_1 - y_0$

Sol. $\Delta^2 (ab^x) = ??$

$\Delta^2 (ab^x) = \Delta [\Delta (ab^x)]$ — ①

$\Delta (ab^x) = a \Delta b^x = a(b^{x+1} - b^x) = ab^x (b-1)$ — ②

from ① & ②

$\Delta^2 (ab^x) = \Delta (a(b-1) b^x) = a(b-1) \Delta b^x$

$= a(b-1)(b^{x+1} - b^x) = a(b-1)^2 b^x$

$\boxed{\Delta^2 (ab^x) = a(b-1)^2 b^x}$

(b) $\Delta^n e^x = ??$

$\Delta e^x = e^{x+1} - e^x = e^x (e-1)$

$\Delta^2 e^x = \Delta(\Delta e^x) = (e-1) \Delta e^x = (e-1)^2 e^x$

$\Delta^3 e^x = (e-1)^3 e^x$

$\vdots$

$\boxed{\Delta^n e^x = (e-1)^n e^x}$

* $f(x)$ — n^{th} order polynomial

$$\Delta^n f(x) = c \quad \text{— Constant}$$

$$\Delta^{n+1} f(x) = \Delta^{n+2} f(x) \cdots = 0$$

nth difference of a polynomial of nth degree are constant and all higher order differences are zero.

$$f(x) = ax^n + bx^{n-1} + cx^{n-2} + \cdots + kx + l \qquad a \neq 0$$

$$\Delta f(x) = f(x+h) - f(x)$$

$$= [a(x+h)^n + b(x+h)^{n-1} + c(x+h)^{n-2} + \cdots + k(x+h) + l] - [a(x^n) + bx^{n-1} + cx^{n-2} + \cdots + kx + l]$$

$$= \left[a\{(x+h)^n - x^n\} + b\{(x+h)^{n-1} - x^{n-1}\} + \cdots kh\right]$$

$$= \left[a\{(\cancel{x^n} + nx^{n-1}h + \cdots) - \cancel{x^n}\} + b\{(\cancel{x^{n-1}} + (n-1)x^{n-2}h + \cdots) - \cancel{x^{n-1}}\} + \cdots + kh\right]$$

$$= \left[a\left\{nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \cdots\right\} + b\{(n-1)x^{n-2}h + \cdots\} + kh\right]$$

$$= anh\,x^{n-1} + b'x^{n-2} + c'x^{n-3} + \cdots + kh$$

$$\boxed{\Delta f(x) = han\,x^{n-1} + b'x^{n-2} + c'x^{n-3} + \cdots + kh}$$

$$\Delta^2 f(x) = an(n-1)h^2 x^{n-2} + b''x^{n-3} + c''x^{n-4} + \cdots + k''$$

$$\vdots$$

$$\Delta^n f(x) = a\,n(n-1)(n-2)(n-3)\cdots 2\cdot 1\; h^n$$

$$\boxed{\Delta^n f(x) = a\,n!\,h^n} \qquad a\,n!\,h^n = \text{Constant}$$

$$\boxed{\Delta^{n+1} f(x) = \Delta^{n+2} f(x) = 0}$$

Q.

x :	45	50	55	60	65
y :	3.0		2.0	??	−2.4

y :	3.0	—	2.0	$-??$	-2.4
	$\downarrow y_0$	$\downarrow$	y_2		y_4

Sol.

$y_0 = 3.0,\ y_2 = 2.0,\ y_4 = -2.4$ are three given value. With these value function y can be represented by 2^{ND} order polynomial.

$$\Delta = E - 1$$

$$\Delta^3 y_0 = 0$$

$$(E-1)^3 y_0 = 0$$

$$(E^3 - {}^3c_1 E^2 + {}^3c_2 E - {}^3c_3)\ y_0 = 0$$

$$(E^3 - 3E^2 + 3E - 1)\ y_0 = 0$$

$$y_3 - 3y_2 + 3y_1 - y_0 = 0$$

$$y_3 - 3(2) + 3y_1 - 3.0 = 0$$

$$y_3 + 3y_1 = 9 \quad \text{— (1)}$$

$$\Delta^3 y_1 = 0$$

$$(E^3 - 3E^2 + 3E - 1)\ y_1 = 0$$

$$y_4 - 3y_3 + 3y_2 - y_1 = 0$$

$$-2.4 - 3y_3 + 6 - y_1 = 0$$

$$3y_3 + y_1 = 3.6 \quad \text{— (2)}$$

from eqn (1) & (2)

$$\boxed{y_1 = 2.925,\quad y_3 = 0.225}$$

Effect of error in difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0	y_0					
		Δy_0				
x_1	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0 + \epsilon$		
x_2	y_2		$\Delta^2 y_1 + \epsilon$		$\Delta^4 y_0 - 4\epsilon$	
		$\Delta y_2 + \epsilon$		$\Delta^3 y_1 - 3\epsilon$		$\Delta^5 y_0 + 10\epsilon$
x_3	$y_3 + \epsilon$		$\Delta^2 y_2 - 2\epsilon$		$\Delta^4 y_1 + 6\epsilon$	
		$\Delta y_3 - \epsilon$		$\Delta^3 y_2 + 3\epsilon$		$\Delta^5 y_1 - 10\epsilon$
x_4	y_4		$\Delta^2 y_3 + \epsilon$		$\Delta^4 y_2 - 4\epsilon$	
		Δy_4		$\Delta^3 y_3 - \epsilon$		$\Delta^5 y_2 + 5\epsilon$
x_5	$y_5 + \epsilon$		$\Delta^2 y_4$		$\Delta^4 y_3 + \epsilon$	
		Δy_5		$\Delta^3 y_4$		
x_6	y_6		$\Delta^2 y_5$			
		Δy_6				
x_7	y_7					
x_8						

error increases with order of difference

04 March 2021 12:03

Newton's forward Method

X	Y
x_0	y_0
x	$y(x) = ??$
x_1	y_1
x_2	y_2
x_3	y_3
$\vdots$	
x	$y(x) = ??$
x_n	y_n

$x_i = x_0 + ih$; $i = 0, 1, 2, 3, \ldots$

$x = x_0 + ph$

$y(x) = ??$

$\Delta = E - 1$

$E = (1+\Delta)$

$$y(x_0 + ph) = E^p y(x_0)$$
$$= E^p y_0$$
$$= (1+\Delta)^p y_0$$
$$= \left(1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \ldots\right) y_0$$

$$\boxed{y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots}$$

Newton Backward formula

$x_i = x_0 + ih$

$E = (1-\nabla)^{-1}$

$x = x_n + ph$

$$y(x) = y(x_n + ph)$$
$$= E^p y(x_n)$$
$$= (1-\nabla)^{-p} y_n$$

$y_0 \rightarrow y_n$
$- \rightarrow +$
$\Delta \rightarrow \nabla$
NB method

$y(x) = y_p(x) = y_p$

$$= \left(1 + p\nabla + \frac{p(p+1)}{2!}\nabla^2 + \frac{p(p+1)(p+2)}{3!}\nabla^3 + \ldots\right) y_n$$

$$\boxed{y(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \ldots}$$

Q.

Year (x)	1951	1961	1971	1981	1991
population (thousands)	46	66	81	93	101

$y(1955) = ??$ $\quad y(1985) = ??$

Sol. X Y Δ Δ^2 Δ^3 Δ^4

Newton forward

x	y	Δ	Δ²	Δ³	Δ⁴
1951	46 (y_0)				
		20 (Δy_0)			
1961	66		-5 ($\Delta^2 y_0$)		
		15		2 ($\Delta^3 y_0$)	
1971	81		-3		-3 ($\Delta^4 y_0$, $\nabla^4 y_n$)
		12		-1 ($\nabla^3 y_n$)	
1981	93		-4 ($\nabla^2 y_n$)		
		08 (∇y_n)			
1991	101 (y_n)				

Newton Backward

$y(1955) = ??$; $x_0 = 1951$; $x = 1955$, $x = x_0 + ph$; $h = 10$

$$p = \frac{(x - x_0)}{h} = \frac{1955 - 1951}{10} = 0.4$$

$$y(1955) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$= 46 + 0.4 \times 20 + \frac{0.4(-0.6)}{2!}(-5) + \frac{0.4(-0.6)(-1.6)}{3!} \times 2 + \frac{0.4(-0.6)(-1.6)(-2.6)}{4!} \times (-3)$$

$y(1955) = 54.85$ thousands.

$y(1985) = ??$ $x_n = 1991$, $x = 1985$, $h = 10$. $x = x_n + ph$

$$p = \frac{(x - x_n)}{h} = \frac{1985 - 1991}{10} = -0.6$$

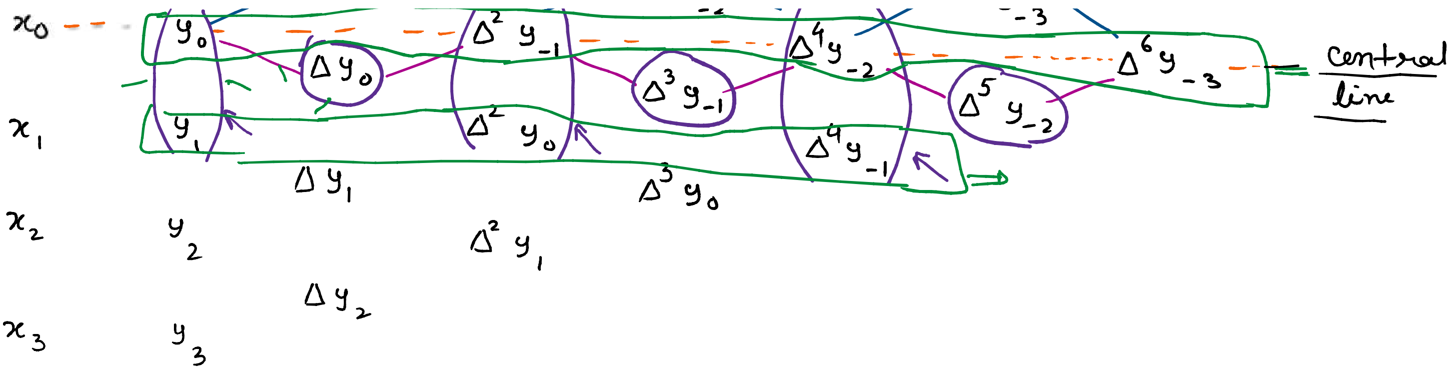
$$y(1985) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n$$

$= 96.84$ thousands

Central difference Method

X	Y	Δ	Δ²	Δ³	Δ⁴	Δ⁵	Δ⁶
x_{-3}	y_{-3}						
		Δy_{-3}					
x_{-2}	y_{-2}		$\Delta^2 y_{-3}$				
		Δy_{-2}		$\Delta^3 y_{-3}$			
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		
		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$	
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$
		Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$	

central line



$$\Delta^n y_{-1} = \Delta^n y_0 - \Delta^{n+1} y_0 + \Delta^{n+2} y_0 - \Delta^{n+3} y_0 + \cdots\cdots$$

$$\Delta^n y_{-2} = \Delta^n y_0 - 2\Delta^{n+1} y_0 + 3\Delta^{n+2} y_0 - 4\Delta^{n+3} y_0 + \cdots\cdots$$

$\Delta = E - 1$

$E = (1+\Delta)$

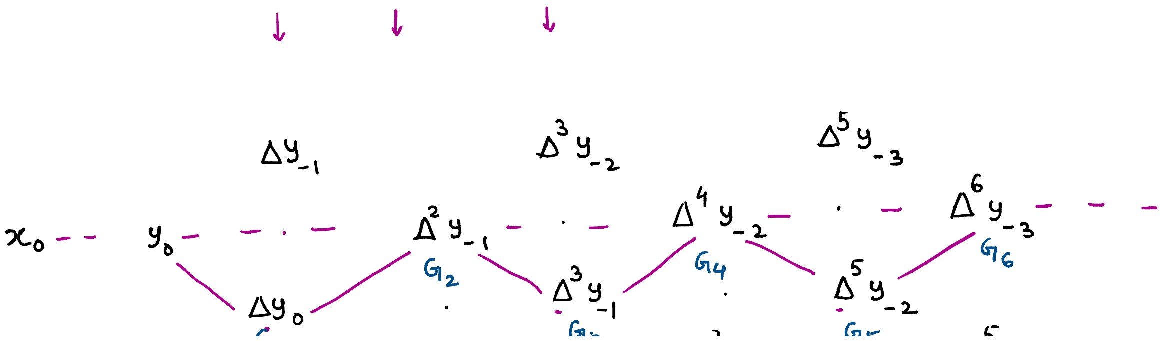
$$\begin{aligned}
\Delta^n y_{-1} &= \Delta^n E^{-1} y_0 \\
&= \Delta^n (1+\Delta)^{-1} y_0 \\
&= \Delta^n (1 - \Delta + \Delta^2 - \Delta^3 + \Delta^4 + \cdots) y_0 \\
&= (\Delta^n - \Delta^{n+1} + \Delta^{n+2} - \Delta^{n+3} \cdots) y_0
\end{aligned}$$

$$\boxed{\Delta^n y_{-1} = \Delta^n y_0 - \Delta^{n+1} y_0 + \Delta^{n+2} y_0 - \Delta^{n+3} y_0 \cdots}$$

$$\begin{aligned}
\Delta^n y_{-2} &= \Delta^n E^{-2} y_0 \\
&= \Delta^n (1+\Delta)^{-2} y_0 \\
&= \Delta^n (1 - 2\Delta + 3\Delta^2 - 4\Delta^3 \cdots) y_0 \\
&= (\Delta^n - 2\Delta^{n+1} + 3\Delta^{n+2} - 4\Delta^{n+3} \cdots) y_0
\end{aligned}$$

$$\boxed{\Delta^n y_{-2} = \Delta^n y_0 - 2\Delta^{n+1} y_0 + 3\Delta^{n+2} y_0 - 4\Delta^{n+3} y_0 \cdots}$$

Gauss forward Method



x_0 $\quad y_0$ $\quad\quad \Delta y_{-1}$

$\Delta y_0 \quad G_2 \quad \Delta^3 y_{-1} \quad G_4 \quad \Delta^5 y_{-2} \quad G_6$

$G_1 \quad\quad\quad G_3 \quad\quad\quad G_5$

$x_1 \quad y_1 \quad\quad \Delta^2 y_0 \quad\quad \Delta^3 y_{-1} \quad\quad \Delta^5 y_{-2}$

Gauss forward

$$y_p = y_0 + G_1 \Delta y_0 + G_2 \Delta^2 y_{-1} + G_3 \Delta^3 y_{-1} + G_4 \Delta^4 y_{-2} + G_5 \Delta^5 y_{-2} + G_6 \Delta^6 y_{-3} \quad \text{— (1)}$$

Newton forward

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots\cdots \quad (2)$$

from eq^n (1)

$$y_p = y_0 + G_1 \Delta y_0 + G_2\left[\Delta^2 y_0 - \Delta^3 y_0 + \Delta^4 y_0 \cdots\right] + G_3\left[\Delta^3 y_0 - \Delta^4 y_0 + \Delta^5 y_0 \cdots\right] + G_4\left[\Delta^4 y_0 - 2\Delta^5 y_0 + 3\Delta^6 y_0 \cdots\right]$$

$$+ G_5\left[\Delta^5 y_0 - 2\Delta^6 y_0 + 3\Delta^7 y_0 \cdots\right] + G_6 \Delta^6 y_{-3} \quad \text{— (3)}$$

Comparing eq^n (2) with eq^n (1)

Δy_0: $\boxed{G_1 = p}$ $\quad$ $\Delta^2 y_0$: $\boxed{G_2 = \frac{p(p-1)}{2!}}$ $\quad$ $\Delta^3 y_0$: $-G_2 + G_3 = \frac{p(p-1)(p-2)}{3!}$

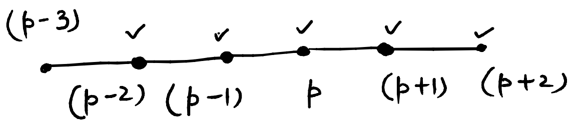
$$G_3 = \frac{p(p-1)(p-2)}{3!} + \frac{p(p-1)}{2!} = \frac{p(p-1)}{3!}\Big[\underbrace{(p-2)+3}_{(p+1)}\Big] = \boxed{\frac{p(p-1)(p+1)}{3!} = G_3}$$

$\Delta^4 y_0$: $G_2 - G_3 + G_4 = \frac{p(p-1)(p-2)(p-3)}{4!}$; $\quad G_4 = \frac{p(p-1)(p-2)(p-3)}{4!} + (-G_2 + G_3)$

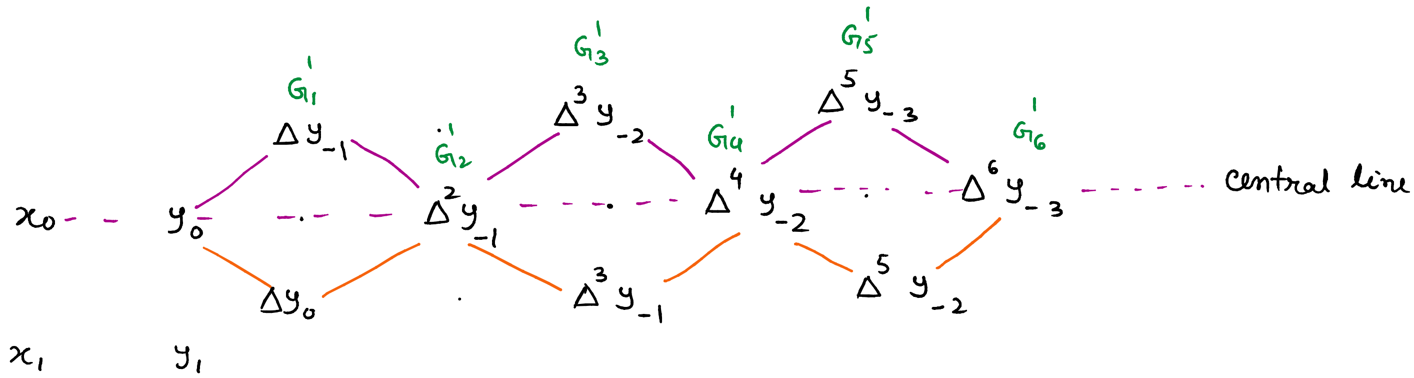
$$G_4 = \frac{p(p-1)(p-2)(p-3)}{4!} + \frac{p(p-1)(p-2)}{3!} = \frac{p(p-1)(p-2)}{4!}\Big[\underbrace{(p-3)+4}_{(p+1)}\Big] = \frac{p(p-1)(p+1)(p-2)}{4!}$$

$$\boxed{G_4 = \frac{p(p-1)(p+1)(p-2)}{4!}}$$

$$\boxed{\begin{aligned}
G_1 &= p \\
G_2 &= \frac{p(p-1)}{2!} \\
G_3 &= \frac{p(p-1)(p+1)}{3!} \\
G_4 &= \frac{p(p-1)(p+1)(p-2)}{4!} \\
G_5 &= \frac{p(p-1)(p+1)(p-2)(p+2)}{5!} \\
G_6 &= \frac{p(p-1)(p+1)(p-2)(p+2)(p-3)}{6!}
\end{aligned}}$$



Gauss' Backward Method



Gauss Backward

$$y_p = y_0 + G_1' \Delta y_{-1} + G_2' \Delta^2 y_{-1} + G_3' \Delta^3 y_{-2} + G_4' \Delta^4 y_{-2} + G_5' \Delta^5 y_{-3} + G_6' \Delta^6 y_{-3} \quad \text{— (1)}$$

Newton's forward

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \cdots \quad (2)$$

from eq^n (1) —

$$y_p = y_0 + G_1'[\Delta y_0 - \Delta^2 y_0 + \Delta^3 y_0 \cdots] + G_2'[\Delta^2 y_0 - \Delta^3 y_0 + \Delta^4 y_0 \cdots] + G_3'[\Delta^3 y_0 - 2\Delta^4 y_0 + 3\Delta^5 y_0 \cdots]$$
$$+ \cdots \quad -(3)$$

Δy_0: $\boxed{G_1' = p}$ $\quad \Delta^2 y_0$: $-G_1' + G_2' = \frac{p(p-1)}{2!} \Rightarrow G_2' = \frac{p(p-1)}{2!} + p = \frac{p}{2!}(\underbrace{p-1-2}_{(p+1)}) = \boxed{\frac{p(p+1)}{2!}}$

$\Delta^3 y_0$: $G_1' - G_2' + G_3' = \frac{p(p-1)(p-2)}{3!}$; $G_3' = \frac{p(p-1)(p-2)}{3!} + (-G_1' + G_2')$

$$G_3' = \frac{p(p-1)(p-2)}{3!} + \frac{p(p-1)}{2!} = \frac{p(p-1)}{\_\_}\left[(p-2)+3\right] = \boxed{p(p+1)(p-1)}$$

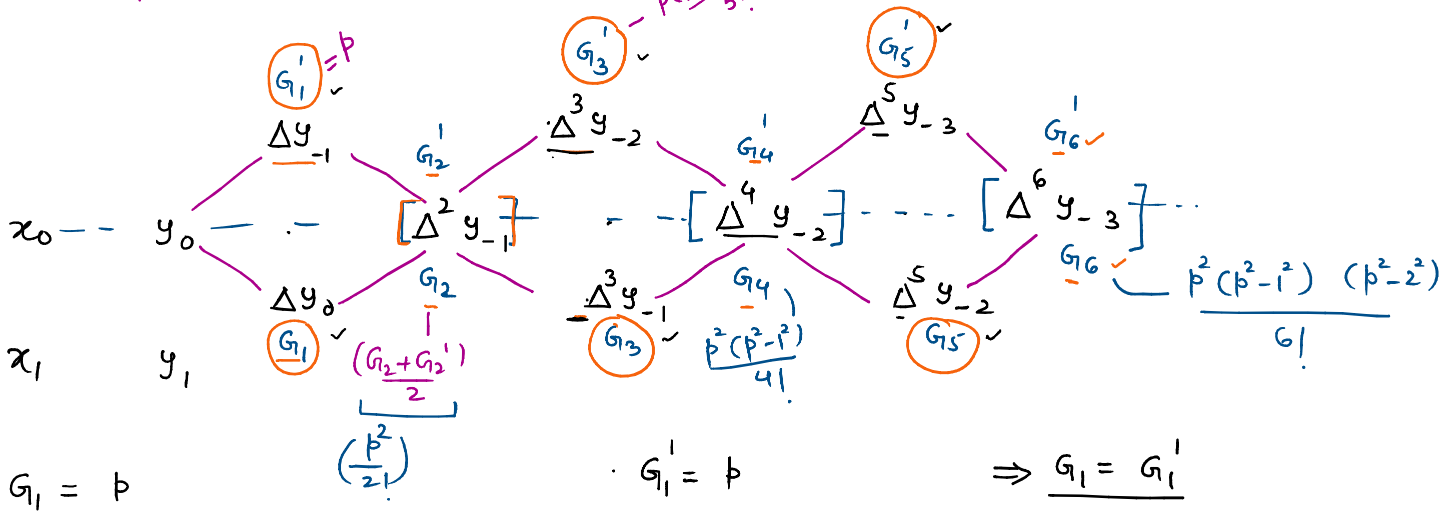
$$G_3' = \frac{p(p-1)(p-2)}{3!} + \frac{p(p-1)}{2!} = \frac{p(p-1)}{3!}\left[\underbrace{(p-2)+3}_{(p+1)}\right] = \boxed{\frac{p(p+1)(p-1)}{3!}}$$

$$\boxed{\begin{aligned}
G_1' &= p \\
G_2' &= \frac{p(p+1)}{2!} \\
G_3' &= \frac{p(p+1)(p-1)}{3!} \\
G_4' &= \frac{p(p+1)(p-1)(p+2)}{4!} \\
G_5' &= \frac{p(p+1)(p-1)(p+2)(p-2)}{5!} \\
G_6' &= \frac{p(p+1)(p-1)(p+2)(p-2)(p+3)}{6!}
\end{aligned}}$$

$(p-1) \leftarrow p \rightarrow (p+1) \quad (p+2)$

$$y_p = y_0 + \frac{G_1}{2}(\Delta y_{-1} + \Delta y_0) + \boxed{\frac{(G_2+G_2')}{2}}\Delta^2 y_{-1} + \frac{G_3}{2}(\Delta^3 y_{-2} + \Delta^3 y_{-1}) + \boxed{\frac{(G_4+G_4')}{2}}\Delta^4 y_{-1} + \cdots$$

Stirling Method



$G_1 = p$ $\qquad G_1' = p \qquad \Rightarrow \underline{G_1 = G_1'}$

$G_2 = \frac{p(p-1)}{2!} \qquad G_2' = \frac{p(p+1)}{2!} \qquad G_2 \neq G_2'$

$G_3 = \frac{p(p-1)(p+1)}{3!} \qquad G_3' = \frac{p(p+1)(p-1)}{3!} \qquad \underline{G_3 = G_3'}$

$G_4 = \frac{p(p-1)(p+1)(p-2)}{4!} \qquad G_4' = \frac{p(p+1)(p-1)(p+2)}{4!} \qquad G_4 \neq G_4'$

$G_5 = \frac{p(p-1)(p+1)(p-2)(p+2)}{5!} \qquad G_5' = \frac{p(p+1)(p-1)(p+2)(p-2)}{5!} \qquad \underline{G_5 = G_5'}$

$$\underline{\frac{G_2+G_2'}{2}} = \frac{1}{2}\left[\frac{p(p-1)}{2!} + \frac{p(p+1)}{2!}\right] = \frac{p}{2\times 2}\left[p - \cancel{1} + p + \cancel{1}\right] = \frac{2p^2}{4} = \frac{p^2}{2!}$$

- $$\frac{G_2 + G_2'}{2} = \frac{1}{2}\left[\frac{p(p-1)}{2!} + \frac{p(p+1)}{2!}\right] = \frac{p}{2\times 2}\left[p - \cancel{1} + p + \cancel{1}\right] = \frac{2p^2}{4} = \frac{p^2}{2!}$$

$$\boxed{\frac{G_2 + G_2'}{2} = \frac{p^2}{2!}}^{*}$$

- $$\frac{G_4 + G_4'}{2} = \frac{1}{2}\left[\frac{p(p-1)(p+1)(p-2)}{4!} + \frac{p(p+1)(p-1)(p+2)}{4!}\right]$$

$$= \frac{1}{2\times 4!}\, p(p-1)(p+1)\left[(p-\cancel{2}) + (p+\cancel{2})\right] = \frac{p^2(p^2-1^2)^{*}}{4!}$$

$$\boxed{\frac{G_4 + G_4'}{2} = \frac{p^2(p^2-1^2)}{4!}}$$

$$\boxed{\frac{G_6 + G_6'}{2} = \frac{p^2(p^2-1^2)(p^2-2^2)}{6!}}^{*}$$

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Q.

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
1.5	37.9					
		208.3				
2.0	246.2		-45.2			
		163.1		10.0		
2.5	409.3 (y_0)		-35.2		-3.6	
		127.9		6.4		6.5
3.0	537.2		-28.8		2.9	
		99.1		9.3		
3.5	636.3		-19.5			
		79.6				
4.0	715.9					

$y(x=2.7) = ??$; Gauss-forward formula ; $x = x_0 + ph$; $x = 2.7$, $x_0 = 2.5$, $h = 0.5$

$$p = \frac{2.7-2.5}{0.5} = \frac{0.2}{0.5} = 0.4$$

$$y(x) = y_0 + \overset{G_1}{p}(127.9) + \overset{G_2}{\frac{p(p-1)}{2!}}(-35.2) + \overset{G_3}{\frac{p(p-1)(p+1)}{3!}}(6.4) + \frac{p(p-1)(p+1)(p-2)}{4!}(-3.6)$$
$$+ \frac{p(p-1)(p+1)(p-2)(p+2)}{5!}(6.5)$$

$$y(x) = 409.3 + 0.4 \times 127.9 + (0.4)(0.4-1)\frac{(-35.2)}{2!} + 0.4\frac{(0.16-1)(6.4)}{3!} + 0.4\frac{(0.16-1)(0.4-2)(-3.6)}{4!}$$
$$+ 0.4\frac{(0.16-1)(0.16-4)(6.5)}{5!}$$

$$y(x) = 409.3 + 51.16 + 4.224 - 0.3584 - 0.08064 + 0.06989 = \boxed{464.31}$$

Q. Gauss-Backward y = population (thousands); x = year ; $y(x=1974)$

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
1939	12					
		3				
1949	15		2			
		5		0		
1959	20		2		3	
		7		3		-10
1969	27		5		-7	
		12		-4		
1979	39		1			
		13				
1989	52					

$x_0 = 1969$; $x = 1974$

$h = 10$

$x = x_0 + ph$

$$p = \frac{(1974-1969)}{10} = \frac{5}{10}$$

$p = 0.5$

$$y(x) = 27 + 0.5\times 7 + \frac{0.5(0.5+1)}{2!}5 + \frac{(0.5)(0.5+1)(0.5-1)}{3!}3 + \frac{0.5(0.5+1)(0.5-1)(0.5+2)}{4!}(-7)$$

$$+ \frac{0.5(0.25-1)(0.25-2)}{5!}(-10)$$

$$= 27 + 3.5 + 1.875 + (-0.1875) + (0.2743) - 0.1172 = \boxed{32.345 \text{ thousand}}$$

Q. Stirling's method

$y(x=0.543) = ??$

$p = \frac{0.543-0.54}{0.01}$

$p = 0.3$

x	y	Δ	Δ^2	Δ^2	Δ^3	Δ^4	Δ^5
0.51	0.529						
		0.009					
0.52	0.538		−0.001				
		0.008		0.002		$\frac{p(p^2-1^2)(p^2-2^2)}{5!}$	
0.53	0.546	p	0.001	$\frac{p(p^2-1^2)}{3!}$	−0.004		
		0.009	$\frac{p^2}{2!}$	−0.002	$\frac{p^2(p^2-1^2)}{4!}$	0.008	$\frac{p^2(p^2-1^2)(p^2-2^2)}{6!}$
0.54	0.555		−0.001		0.004		−0.016
		0.008		0.002		−0.008	
0.55	0.563		0.001		−0.004		
		0.009		−0.002			
0.56	0.572		−0.001				
		0.008					
0.57	0.580						

$$y = 0.555 + \frac{p}{2}(0.009+0.008) + \frac{p^2}{2!}(-0.001) + \frac{p(p^2-1^2)}{3!}\underbrace{\left(\frac{\cancel{0.002 - 0.002}}{2}\right)}_{=0} + \frac{p^2(p^2-1^2)}{4!}(0.004) +$$

$$\frac{p(p^2-1^2)(p^2-2^2)}{5!}\underbrace{\left(\frac{\cancel{0.008 - 0.008}}{2}\right)}_{=0} + \frac{p^2(p^2-1^2)(p^2-2^2)}{6!}(-0.016)$$

$$y = 0.555 + \frac{0.3}{2}(0.009+0.008) + \frac{0.09}{2}(-0.001) + \frac{0.09(0.09-1)}{4!}(0.004) + \frac{0.09(0.09-1)(0.09-4)}{6} \times (-0.016)$$

$$y = 0.5575$$

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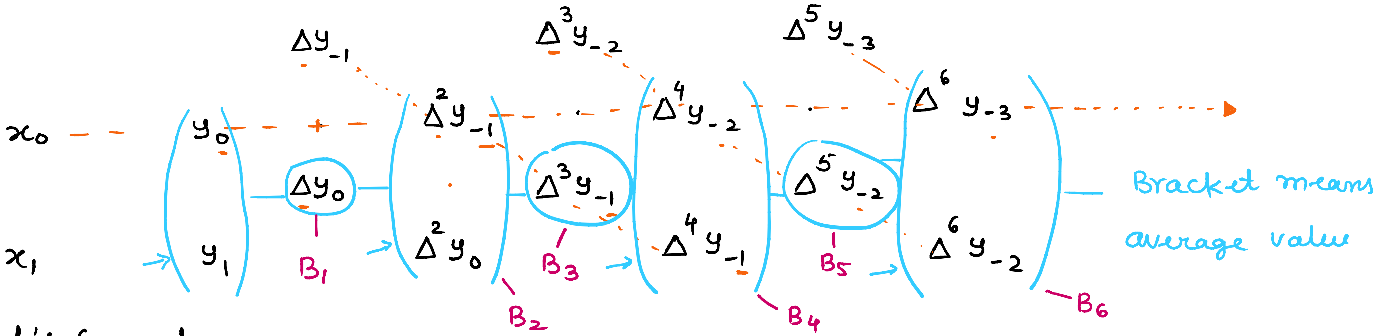
Bessel's formula



Bessel's formula

$$y = \frac{(y_0+y_1)}{2} + B_1\Delta y_0 + \frac{B_2}{2}\left(\Delta^2 y_{-1} + \Delta^2 y_0\right) + B_3\Delta^3 y_{-1} + \frac{B_4}{2}\left(\Delta^4 y_{-2} + \Delta^4 y_{-1}\right) + \cdots \quad -(1)$$

$\Delta y_0 = y_1 - y_0$; $y_1 = y_0 + \Delta y_0$

$$y = \frac{(y_0 + y_0 + \Delta y_0)}{2} + B_1\Delta y_0 + \frac{B_2}{2}\left[\Delta^2 y_0 + \Delta^2 y_0 - \Delta^3 y_0 + \cdots\right] + B_3\left[\Delta^3 y_0 - \Delta^4 y_0 + \Delta^5 y_0 \cdots\right] +$$

$$\frac{B_4}{2}\left[\left\{\Delta^4 y_0 - 2\Delta^5 y_0 + 3\Delta^6 y_0 \cdots\right\} + \left\{\Delta^4 y_0 - \Delta^5 y_0 + \Delta^6 y_0 \cdots\right\}\right] \quad -(2)$$

$$y = y_0 + \left(B_1 + \tfrac{1}{2}\right)\Delta y_0 + \cdots$$

Nf formula :- $y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \cdots \quad (3)$

Δy_0 :- $B_1 + \frac{1}{2} = p$; $\boxed{B_1 = p - \frac{1}{2}}$

$\Delta^2 y_0$: $\boxed{B_2 = \frac{p(p-1)}{2!}}$

$\Delta^3 y_0$: $-\frac{B_2}{2} + B_3 = \frac{p(p-1)(p-2)}{3!}$;

$$B_3 = \frac{p(p-1)(p-2)}{3!} + \frac{p(p-1)}{2\times 2} = \frac{p(p-1)}{3!}\left[p - 2 + \frac{3}{2}\right] = \frac{p(p-1)(p-\frac{1}{2})}{3!}$$

$$\boxed{B_3 = \frac{p(p-\frac{1}{2})(p-1)}{3!}}$$

$\Delta^4 y_0$: $\frac{B_2}{2} - B_3 + B_4 = \frac{p(p-1)(p-2)(p-3)}{4!}$;

$$B_4 = \frac{p(p-1)(p-2)(p-3)}{4!} + \frac{p(p-1)(p-2)}{3!} = \frac{p(p-1)(p-2)}{4!}(p-3+4) = \frac{p(p-1)(p+1)(p-2)}{4!}$$

$$\boxed{B_4 = \frac{p(p-1)(p+1)(p-2)}{4!}}$$

Everett's Method

	E_0	E_2	E_4	E_6
x_0	y_0	$\Delta^2 y_{-1}$	$\Delta^4 y_{-2}$	$\Delta^6 y_{-3}$

	E_0	E_2	E_4	E_6
x_0	y_0	$\Delta^2 y_{-1}$	$\Delta^4 y_{-2}$	$\Delta^6 y_{-3}$
x_1	y_1	$\Delta^2 y_0$	$\Delta^4 y_{-1}$	$\Delta^6 y_{-2}$
	F_0	F_2	F_4	F_6

$$y = \left.\begin{array}{l} E_0 y_0 + E_2 \Delta^2 y_{-1} + E_4 \Delta^4 y_{-2} + E_6 \Delta^6 y_{-3} + \\ F_0 y_1 + F_2 \Delta^2 y_0 + F_4 \Delta^4 y_{-1} + F_6 \Delta^6 y_{-2} \end{array}\right] \quad -(1)$$

$$y = \left.\begin{array}{l} E_0 y_0 + E_2\{\Delta^2 y_0 - \Delta^3 y_0 + \Delta^4 y_0 + \ldots\} + E_4\{\Delta^4 y_0 - 2\Delta^5 y_0 + 3\Delta^6 y_0 \cdots\} + \cdots\cdots \\ F_0(y_0 + \Delta y_0) + F_2 \Delta^2 y_0 + F_4\{\Delta^4 y_0 - \Delta^5 y_0 + \Delta^6 y_0 \cdots\} + \cdots\cdots \end{array}\right] \quad -(2)$$

comparing eq^n (2) with Newton's forward formula.

y_0: $E_0 + F_0 = 1$; $\boxed{E_0 = 1 - F_0}$

Δy_0: $\boxed{F_0 = p}$ $\boxed{E_0 = 1 - p = q}$ $(p = 1 - q)$

$\Delta^2 y_0$: $E_2 + F_2 = \dfrac{p(p-1)}{2!}$

$\Delta^3 y_0$: $-E_2 = \dfrac{p(p-1)(p-2)}{3!}$; $-E_2 = \dfrac{(1-q)(\cancel{1}-q-\cancel{1})(1-q-2)}{3!} = \dfrac{-q(1-q)(-1-q)}{3!} = \dfrac{-q(q-1)(q+1)}{3!}$

$$F_2 = \frac{p(p-1)}{2!} + \frac{p(p-1)(p-2)}{3!} = \frac{p(p-1)}{3!}(3+p-2) = \frac{p(p-1)(p+1)}{3!} = \frac{p(p^2-1^2)}{3!}$$

$$\boxed{F_2 = \frac{p(p^2-1^2)}{3!} \qquad E_2 = \frac{q(q^2-1^2)}{3!}}$$

$\boxed{q = 1 - p}$

$F_0 = p$	$E_0 = q$
$F_2 = \dfrac{p(p^2-1^2)}{3!}$	$E_2 = \dfrac{q(q^2-1^2)}{3!}$
$F_4 = \dfrac{p(p^2-1^2)(p^2-2^2)}{5!}$	$E_4 = \dfrac{q(q^2-1^2)(q^2-2^2)}{5!}$

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Q: $\cos(0.17) = ??$ Using method Bessel's method

Sol.

x	y $(\cos x)$	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
0	1						
		-0.0012					
0.05	0.9988		-0.0026				
		-0.0038		0.0002			
0.10	0.9950		-0.0024		-0.0003		
		-0.0062		-0.0001		0.0004	
0.15	0.9888		-0.0025		0.0001		-0.0004
		-0.0087		0		0	
0.20	0.9801		-0.0025		0.0001		
		-0.0112		0.0001			
0.25	0.9689		-0.0024				
		-0.0136					
0.30	0.9553						

$\left(p-\frac{1}{2}\right)$ $\quad \frac{p(p-1)}{2!}$ $\quad \frac{p(p-1)(p+1)(p-2)}{4!}$

$$x = x_0 + ph; \quad p = \frac{(x - x_0)}{h} = \frac{0.17 - 0.15}{0.05} = 0.4$$

$$y(x) = \frac{(0.9888 + 0.9801)}{2} + (0.4 - 0.5)(-0.0087) + \frac{0.4(0.4-1)}{2!}\left(\frac{-0.0025 - 0.0025}{2}\right)$$

$$+ B_2 \times 0 + \frac{0.4(0.4-1)(0.4+1)(0.4-2)}{4!}\left(\frac{0.0001 + 0.0001}{2}\right) + 0$$

$$y(x) = 0.98445 + 0.00087 + 0.0003 + 0.00000224$$

$$= 0.9856$$

Q. Everett's method : $\cos(12.5)$

	x	y $(\cos x)$	Δ	Δ^2	Δ^3	Δ^4
	0°	1.000				
			-0.0038			
	5°	0.9962		-0.0076		
		E_0	-0.0114	E_2	0.0001	E_4
x_0	10°	0.9848		-0.0075		0.0001
			-0.0189		0.0002	
x_1	15°	0.9659		-0.0073		
		F_0	-0.0262	F_2		
	20°	0.9397				

$$x = x_0 + ph; \quad p = \frac{12.5 - 10}{5} = \frac{2.5}{5} = 0.5 \qquad \underline{p = 0.5}, \quad q = 1 - p = \underline{0.5}$$

$$y(x) = q(0.9848) + \frac{q(q^2-1^2)}{3!}(-0.0075) + \frac{q(q^2-1^2)(q^2-2^2)}{5!}(0.0001) +$$

$$p(0.9659) + \frac{p(p^2-1^2)}{3!}(-0.0073)$$

$$y(x) = 0.5 \times 0.9848 + \frac{0.5(0.25-1)}{3!}(-0.0075) + \frac{0.5(0.25-1)(0.25-4)}{5!}(0.0001) +$$

$$+ 0.5(0.9659) + \frac{0.5(0.25-1)}{3!}(-0.0073)$$

$$= \underline{0.9763}$$

Choice of formula

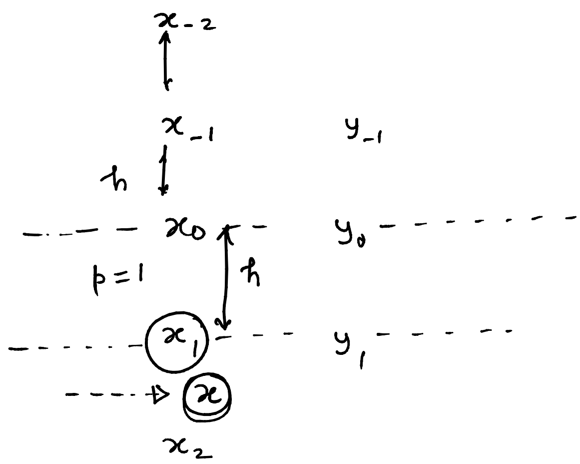
1. Unknown data is in the start of table : Newton's forward method
2. Unknown data is in the end of table : Newton Backward method
3. $0 < p < 1$ (Gauss forward method)

   $-1 < p < 0$ (Gauss Backward method)

   $-\frac{1}{4} < p < \frac{1}{4}$ (Stirling method)

   $\frac{1}{4} < p < \frac{3}{4}$ (Bessel's method / Everett's method)

$x = x_0 + ph$



$\boxed{p > 1}$

for un-equal data –

1. Lagrange's method

2. Newton's finite divided difference method

Lagrange's method

$(x_0, y_0)\ (x_1, y_1)\ (x_2, y_2)\ \ldots\ldots\ (x_n, y_n)$

$(n+1)$: data point $\qquad x_i \neq x_0 + ih$

$$y(x) = P_n(x) = a_0 (x-x_1)(x-x_2)\ldots\ldots(x-x_n) + a_1 (x-x_0)(x-x_2)(x-x_3)\ldots(x-x_n)$$
$$+ \ldots\ldots + a_n (x-x_0)(x-x_1)\ldots\ldots(x-x_{n-1}) \quad \text{—(1)}$$

at $x = x_0$, $y = y_0$

$$y_0 = a_0(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n); \quad a_0 = \frac{y_0}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)} \quad \text{—(2)}$$

$x = x_1$, $y = y_1$

$$y_1 = a_1(x_1-x_0)(x_1-x_2)\ldots\ldots(x_1-x_n); \quad a_1 = \frac{y_1}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)} \quad \text{—(3)}$$

$$a_n = \frac{y_n}{(x_n-x_0)(x_n-x_1)\ldots\ldots(x_n-x_{n-1})} \quad \text{(4)}$$

$$\boxed{y(x) = \frac{(x-x_1)(x-x_2)\ldots\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\ldots\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)} y_1 + \ldots\ldots + \frac{(x-x_0)(x-x_1)\ldots\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})} y_n}$$

Q.

x :	5	7	11	13	17
y :	150	392	1452	2366	5202

$y(x=9)$ – lagrange's method

Sol.

$$y(x) = \frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)}(150) + \frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)}(392) +$$
$$+ \frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)}(1452) + \frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)} \times (2366) +$$

$$+\frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)}(1452)+\frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)}\times(2366)+$$

$$+\frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)}\times(5202)$$

$$= 810$$

Divided difference

$(x_0, y_0) \quad (x_1, y_1) \quad - \; - - - - \quad (x_n, y_n)$

first divided difference

$$[x_0 \quad x_1] = \frac{(y_1 - y_0)}{(x_1 - x_0)} \quad ; \quad [x_2 \quad x_3] = \frac{(y_3 - y_2)}{(x_3 - x_2)}$$

$$[x_0 \quad x_1] = [x_1 \quad x_0]$$

Second divided difference

$$[x_0 \quad x_1 \quad x_2] = \frac{[x_1 \quad x_2] - [x_0 \quad x_1]}{(x_2 - x_0)}$$

third divided difference

$$[x_0 \quad x_1 \quad x_2 \quad x_3] = \frac{[x_1 \quad x_2 \quad x_3] - [x_0 \quad x_1 \quad x_2]}{(x_3 - x_0)}$$

Newton's finite divided difference method

$$[x \quad x_0] = \frac{(y_0 - y)}{(x_0 - x)} \qquad \underline{y = y(x)} \; ; \qquad x - \text{intermediate value}$$

$$y_0 - y = (x_0 - x)\,[x \quad x_0]$$

$$y = y_0 + (x - x_0)\,\underline{[x \quad x_0]} \quad \text{———} \quad (1)$$

$$[x \quad x_0 \quad x_1] = \frac{[x_0 \quad x_1] - [x \quad x_0]}{(x_1 - x)}$$

$$(x_1 - x)[x \; x_0 \; x_1] = [x_0 \; x_1] - [x \; x_0]$$

$$[x \; x_0] = [x_0 \; x_1] + (x - x_1)[x \; x_0 \; x_1] \quad \text{--- (2)}$$

from eq^n (1) & (2)

$$y = y_0 + (x - x_0)\left\{[x_0 \; x_1] + (x - x_1)[x \; x_0 \; x_1]\right\}$$

$$y = y_0 + (x - x_0)[x_0 \; x_1] + (x - x_0)(x - x_1)\underline{[x \; x_0 \; x_1]} \quad \text{--- (3)}$$

similarly —

$$\boxed{y(x) = y_0 + (x - x_0)\underline{[x_0 \; x_1]} + (x - x_0)(x - x_1)\underline{[x_0 \; x_1 \; x_2]} + (x - x_0)(x - x_1)(x - x_2)\underline{[x_0 \; x_1 \; x_2 \; x_3]} + \cdots}$$

Q.

	x	y	I^{st} dd	2^{nd} dd	3^{RD} dd
x_0	5	150 (y_0)			
			$\frac{392-150}{(7-5)} = \underline{121}$ ($[x_0 \; x_1]$)		
x_1	7	392		$\frac{265-121}{(11-5)} = 24$ ($[x_0 \; x_1 \; x_2]$)	
			$\frac{1452-392}{11-7} = \underline{265}$		$\frac{32-24}{13-5} = 1$ ($[x_0 \; x_1 \; x_2 \; x_3]$)
x_2	11	1452		$\frac{457-265}{13-7} = 32$	
			$\frac{2366-1452}{13-11} = \underline{457}$		$\frac{42-32}{17-7} = 1$
x_3	13	2366		$\frac{709-457}{17-11} = 42$	
			$\frac{5202-2366}{17-13} = \underline{709}$		
x_4	17	5202			

$$y(x=9) = 150 + (9-5)(121) + (9-5)(9-7)(24) + (9-5)(9-7)(9-11) \times 1 + 0$$

$$= 150 + 484 + 192 - 16 = \underline{810}$$