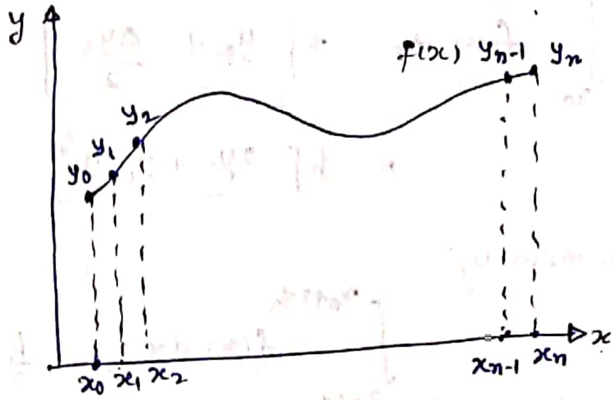
Numerical Integration

Newton's cotes's formula :-



$$I = \int_{x_0}^{x_n} f(x)\,dx$$

where $x_n = x_0 + nh$

(divided into n equispaced data points)

$$I = \int_{x_0}^{x_0+nh} P_n(x)\,dx$$

→ nth order polynomial

$$I = \int_{x_0}^{x_0+nh} \left[y_0 + p\,\Delta y_0 + \frac{\Delta^2 y_0\, p(p-1)}{2!} + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots \right] dx$$

forward difference formula

$$x = x_0 + ph$$

$$dx = h\,dp$$

$$I = h\int_0^n \left[y_0 + p\,.\Delta y_0 + \frac{(p^2-p)}{2!}\Delta^2 y_0 + \frac{(p^3-3p^2+2p)}{3!}\Delta^3 y_0 + \cdots \right] dp$$

$$I = h\left[y_0 p + \frac{p^2}{2}\Delta y_0 + \left(\frac{p^3}{3} - \frac{p^2}{2}\right)\frac{1}{2}\Delta^2 y_0 + \left(\frac{p^4}{4} - p^3 + p^2\right)\frac{1}{3!}\Delta^3 y_0 + \cdots \right]_0^n$$

$$\boxed{I = h\left[y_0 n + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{p^4}{4} - p^3 + p^2\right)\Delta^3 y_0 + \cdots \right]}$$

Newton's cotes formula.

$n = 1$, Trapezoidal formula

$n = 2$ Simpson one-third formula

$n = 3$ Simpson three-eight formula

$n = 4$ Boole's formula

$n = 6$ Weddles formula

Trapezoidal formula

$n=1$, collecting the two consecative points with straight line
$(\Delta^2 y_0 = \Delta^3 y_0 = \cdots = 0)$

$$\int_{x_0}^{x_0+h} f(x)\,dx = h\left[y_0 + \frac{\Delta y_0}{2}\right] \qquad \Delta y_0 = y_1 - y_0$$

$$= h\left[\frac{2y_0 + y_1 - y_0}{2}\right] = \frac{h}{2}\left[y_0 + y_1\right]$$

similarly.

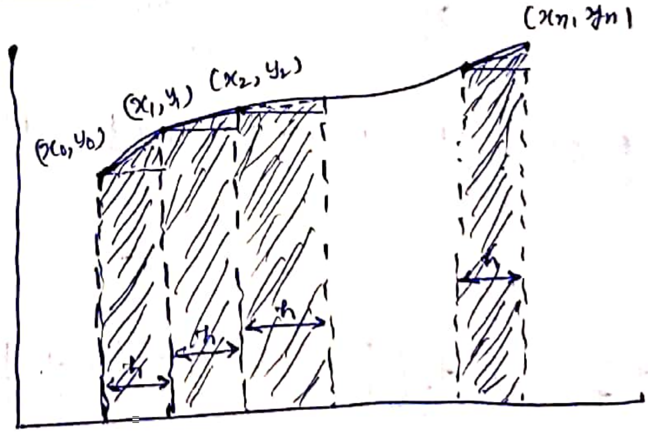
$$\int_{x_0+h}^{x_0+2h} f(x)\,dx = \frac{h}{2}\left[y_1 + y_2\right]$$

$$\vdots$$

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[y_{n-1} + y_n\right]$$

$$\boxed{\int_{x_0}^{x_0+nh=x_n} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+y_3+\cdots+y_{n-1})\right]}$$

Area (shadded) is the area calculated by Trapezoidal method. ⟵



Simpson - One-third Rule

Choose $n=2$ ((x_0, y_0), (x_1, y_1), (x_2, y_2) by parabola)

choose two strips at a time. $\Delta^3 y_0 = \Delta^4 y_0 = \cdots = 0$

$$I_1 = \int_{x_0}^{x_0+2h} f(x)\,dx = h\left[2y_0 + 2\Delta y_0 + \frac{1}{2}\left(\frac{8}{3} - 2\right)\Delta^2 y_0\right]$$

$$= h\left[2y_0 + 2(y_1 - y_0) + \frac{1}{3}(y_2 - 2y_1 + y_0)\right]$$

$$= \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]$$

$$\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta(y_1 - y_0) = \Delta y_1 - \Delta y_0 = y_2 - y_1 - y_1 + y_0 = (y_2 - 2y_1 + y_0)$$

$$\Delta^3 y_0 = \Delta(y_2 - 2y_1 + y_0) = \Delta y_2 - 2\Delta y_1 + \Delta y_0 = y_3 - y_2 - 2y_2 + 2y_1 + y_1 - y_0 = (y_3 - 3y_2 + 3y_1 - y_0)$$

$$I_2 = \int_{x_0+2h}^{x_0+4h} f(x)\,dx = \frac{h}{3}\left[y_2 + 4y_3 + y_4\right]$$

$$\vdots$$

$$\int_{x_0+(n-2)h}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left[y_{n-2} + 4y_{n-1} + y_n\right]$$

$$\boxed{I = \int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\Big[(y_0 + y_n) + 2(y_2 + y_4 + y_6 + \cdots + y_{n-2}) + 4(y_1 + y_3 + y_5 + \cdots + y_{n-1})\Big]}$$

for $n = 3$, (Simpson three eight formula)

(y_0, y_1, y_2, y_3), $\Delta^4 y_0 = 0$

$$I_1 = \int_{x_0}^{x_0+3h} y(x)\,dx = h\left[3y_0 + \frac{9}{2}\Delta y_0 + \left(9 - \frac{9}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{81}{4} - 27 + 9\right)\frac{\Delta^3 y_0}{3!}\right]$$

$$= h\left[3y_0 + \frac{9}{2}(y_1 - y_0) + \frac{9}{4}(y_2 - 2y_1 + y_0) + \frac{(\cancel{81}^{9} - \cancel{27}^{3}\times 4 + \cancel{9}\times 4)}{4\times 2}\frac{\Delta^3 y_0}{\cancel{3}}\right]$$

$= \cancel{\frac{3h}{8}[8y_0}$

$$= h\left[3y_0 + \frac{9}{2}(y_1 - y_0) + \frac{9}{4}(y_2 - 2y_1 + y_0) + \frac{(9 - 12 + 4)\times 3}{4\times 2}\Delta^3 y_0\right]$$

$$= h\left[3y_0 + \frac{9}{2}(y_1 - y_0) + \frac{9}{4}(y_2 - 2y_1 + y_0) + \frac{3}{4\times 2}(\Delta^3 y_0)\right]$$

$$= \frac{h}{4}\left[12y_0 + 18(y_1 - y_0) + 9(y_2 - 2y_1 + y_0) + \frac{3}{2}\Delta^3 y_0\right]$$

$$\Delta^3 y_0 = \Delta(\Delta^2 y_0) = \Delta(y_2 - 2y_1 + y_0) = (y_3 - y_2) - 2(y_2 - y_1) + y_1 - y_0$$

$$= (y_3 - y_2 - 2y_2 + 2y_1 + y_1 - y_0) = (y_3 - 3y_2 + 3y_1 - y_0)$$

$$= \frac{h}{4}\left[12y_0 + 18(y_1 - y_0) + 9(y_2 - 2y_1 + y_0) + \frac{3}{2}(y_3 - 3y_2 + 3y_1 - y_0)\right]$$

$$= \frac{h}{2\times 4}\left[24y_0 + 36y_1 - 36y_0 + 18y_2 - 36y_1 + 18y_0 + 3y_3 - 9y_2 + 9y_1 - 3y_0\right]$$

$$= \frac{h}{8}\left[3y_0 + 9y_1 + 9y_2 + 3y_3\right] = \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + y_3\right]$$

36

24
18
42
39
3

$$= \frac{3h}{8}\left[y_0 + 3(y_1 + y_2) + y_3\right]$$

$$I_2 = \frac{3h}{8}\left[y_3 + 3(y_4 + y_5) + y_6\right] \ldots$$

$$I = I_1 + I_2 + I_3 + \cdots = \frac{3h}{8}\left[(y_0 + y_n) + 2(y_3 + y_6 + y_9 + y_{12} \cdots) + 3(y_1 + y_2 + y_4 + y_5 + y_7 + y_8 + \cdots)\right]$$

Adding all such expressions from x_0 to $x_0 + nh$, where n is a multiple of 3, we obtain

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \ldots + y_{n-1}) \right.$$
$$\left. + 2(y_3 + y_6 + \ldots + y_{n-3})\right] \qquad \ldots(4)$$

which is known as *Simpson's three-eighth rule.*

Obs. While applying (4), *the number of sub-intervals should be taken as multiple of 3.*

IV. Boole's rule. Putting $n = 4$ in (1) above and neglecting all differences above the fourth, we obtain

$$\int_{x_0}^{x_0+4h} f(x)\,dx = 4h\left(y_0 + 2\Delta y_0 + \frac{5}{3}\Delta^2 y_0 + \frac{2}{3}\Delta^3 y_0 + \frac{7}{90}\Delta^4 y_0\right)$$
$$= \frac{2h}{45}(7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4)$$

Similarly

$$\int_{x_0+4h}^{x_0+8h} f(x)\,dx = \frac{2h}{45}(7y_4 + 32y_5 + 12y_6 + 32y_7 + 7y_8) \text{ and so on.}$$

Adding all these integrals from x_0 to $x_0 + nh$, where n is a multiple of 4, we get

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{2h}{45}(7y_0 + 32y_1 + 12y_2 + 32y_3 + 14y_4$$
$$+ 32y_5 + 12y_6 + 32y_7 + 14y_8 + \ldots \qquad \ldots(5)$$

This is known as *Boole's rule.*

Obs. While applying (5), the number of sub-intervals should be taken as a multiple of 4.

V. Weddle's rule. Putting $n = 6$ in (1) above and neglecting all differences above the sixth, we obtain

$$\int_{x_0}^{x_0+6h} f(x)\,dx = 6h\left(y_0 + 3\Delta y_0 + \frac{9}{2}\Delta^2 y_0 + 4\Delta^3 y_0 + \frac{123}{60}\Delta^4 y_0 + \frac{11}{20}\Delta^5 x_0 + \frac{1}{6}\cdot\frac{41}{140}\Delta^6 y_0\right)$$

If we replace $\frac{41}{140}\Delta^6 y_0$ by $\frac{3}{10}\Delta^6 y_0$, the error made will be negligible.

$$\therefore \quad \int_{x_0}^{x_0+6h} f(x)\,dx = \frac{3h}{10}(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6)$$

Similarly

$$\int_{x_0+6h}^{x_0+12h} f(x)\,dx = \frac{3h}{10}(y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}) \text{ and so on.}$$

Adding all these integrals from x_0 to $x_0 + nh$, where n is a multiple of 6, we get

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{10}(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + y_8 + \ldots) \qquad \ldots(6)$$

n should be multiple of 6.

Example 8.6. *Evaluate* $\int_0^6 \frac{dx}{1+x^2}$ *by using (i) Trapezoidal rule,*

(ii) Simpson's 1/3 rule, *(Madras, B.E., 2003)*

(iii) Simpson's 3/8 rule,

(iv) Weddle's rule and compare the results with its actual value. *(Rohtak, B.E., 2003)*

Sol. Divide the interval (0, 6) into six parts each of width $h = 1$. The values of $f(x) = \frac{1}{1+x^2}$ are given below :

x	0	1	2	3	4	5	6
$f(x)$ $= y$	1 y_0	0.5 y_1	0.2 y_2	0.1 y_3	0.0588 y_4	0.0385 y_5	0.027 y_6

(*i*) By Trapezoidal rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{2} [(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \tfrac{1}{2} [(1 + 0.027) + 2(0.5 + 0.2 + 0.1 + 0.0588 + 0.0385)] = \mathbf{1.4108}.$$

(*ii*) By Simpson's 1/3 rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{3} [(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{1}{3} [(1 + 0.027) + 4(0.5 + 0.1 + 0.0385) + 2(0.2 + 0.0588)] = \mathbf{1.3662}.$$

(*iii*) By Simpson's 3/8 rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{8} [(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3}{8} [(1 + 0.027) + 3(0.5 + 0.2 + 0.0588 + 0.0385) + 2(0.1)] = \mathbf{1.3571}.$$

(*iv*) By Weddle's rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{10} [y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6]$$

$$= 0.3[1 + 5(0.5) + 0.2 + 6(0.1) + 0.0588 + 5(0.0385) + 0.027] = \mathbf{1.3735}$$

Also $\int_0^6 \frac{dx}{1+x^2} = \left| \tan^{-1} x \right|_0^6 = \tan^{-1} 6 = \mathbf{1.4056}$

This shows that the value of the integral found by Weddle's rule is the nearest to the actual value followed by its value given by Simpson's 1/3 rule.

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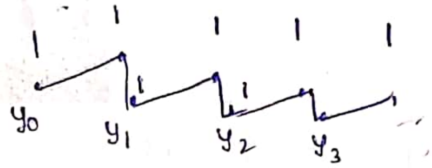
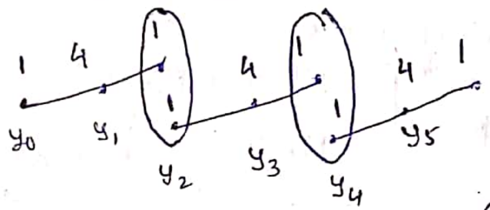
(1) Trap. $(n=1)$



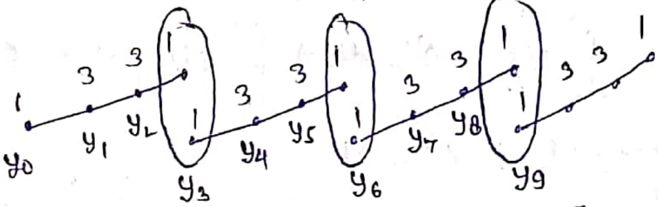
$$I = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+y_3+\dots y_{n-1})\right]$$

(2) $(n=2)$ Simp (1/3)



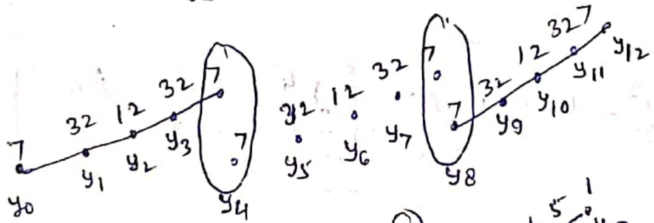
$$I = \frac{h}{3}\left[(y_0+y_n)+2(y_2+y_4+y_6+\dots) + 4(y_1+y_3+y_5+\dots)\right]$$

(3) $(n=3)$ Simp(3/8)



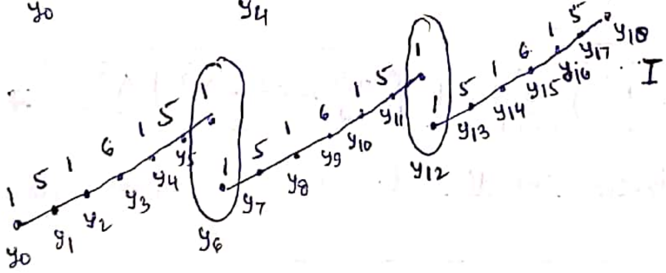
$$I = \frac{3h}{8}\left[(y_0+y_n)+2(y_3+y_6+y_9+\dots) + 3(y_1+y_2+y_4+y_5+y_7+y_8+\dots\right.$$

(4) $(n=4)$ (Boole's)



$$I = \frac{2h}{45}\left[7(y_0+y_n)+14(y_4+y_8+y_{12}+\dots) + 32(y_1+y_3+y_5+y_7+y_9+y_{11}+\dots) + 12(y_2+y_6+y_{10}+\dots)\right]$$

(5) $(n=6)$ (Weddle's)



$$I = \frac{3h}{10}\left[(\cancel{y_0+y_n})+2(y_6+y_{12}+y_{18}+\dots) + (y_0+y_2+y_4+y_8+y_{10}+y_{14}+y_{16}+\dots) + 5(y_1+y_5+y_7+y_{11}+y_{13}+y_{17}+\dots) + 6(y_3+y_9+y_{15}+\dots)\right]$$

Romberg Integration :-

Modification in quadrature formula followed by (by incorporating) finite difference.

$$I = \int_{x_0}^{x_n} f(x)\, dx$$

Suppose :- (Trapezoidal rule)

h_1 h_2 (h_1, h_2) ⇒ result will be difference

(h) $(h/2)$

suppose $I_1 = I(h)$

$I_2 = I(h/2)$

$I_3 = I(h/4)$

then $I(h, h/2) = \left(\dfrac{4I(h/2) - I(h)}{3}\right)$

$I(h)$

$I(h, h/2)$

$I(h/2)$

$I(h, h/2, h/4)$

$I(h, h/2, h/4, h$

$I(h/2, h/4)$

$I(h/4)$

$I(h/2, h/4, h/8)$

$I(h/4, h/8)$

$I(h/8)$

Example:- Use Romberg's method to compute $\int_0^1 \frac{dx}{(1+x^2)}$ by taking $h = 0.5,\ 0.25,\ 0.125.$

Solution:- Trapezoidal rule

$x:$	0	0.5	1.0
$y:$	1	0.8	0.5

$I(h):$ $I(0.5) \Rightarrow$

$= 0.775$

x	0	0.25	0.50	0.75	1.00
y	1	0.9412	0.80	0.64	0.5

$I(0.25)$:

$I(0.25) = 0.7828$

$I(0.125) = 0.7848$

h	$I(h)$	$I(h, h/2)$	---
0.5	0.775		
		0.7854	
			0.7855
0.25	0.7828		
		0.7855	
0.125	0.7848		

Euler's Maclaurin formula

$$I = \int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})\right] - \underbrace{\frac{h^2}{12}(y_n' - y_0') + \frac{h^4}{720}(y_n''' - y_0''')}_{\text{correction term}}$$

Example:-

$$\int_0^1 \frac{1}{(1+x^{\cancel{2}})}\,dx \qquad h = 0.1,\ n = 10$$

$$y = \frac{1}{(1+x^{\cancel{2}})}, \quad y' = \frac{-\cancel{2x}}{(1+x^2)^2} \;??, \quad y''' = \frac{-6}{(1+x^2)^4} \;??$$

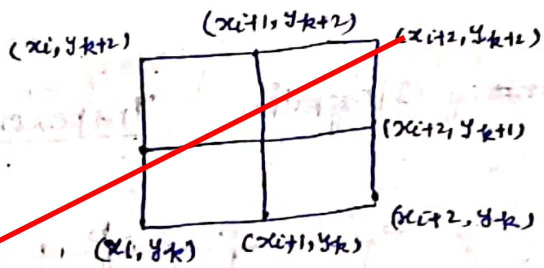
$I(h)$ by trapezoidal method $= 0.693773$

$= 0.$

$I = 0.693773 - 0.000625 + 0.000001$

$= 0.693149$

Simpson One third Method (Double Integral)



$$I = \int_k^{k+2}\int_i^{i+2} f(x,y)\,dx\,dy \qquad \left.\begin{matrix} x_{i+1} = x_i + h \\ y_{k+1} = y_k + h \end{matrix}\right\}$$

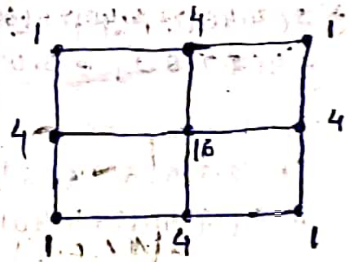
$$I = \int_k^{k+2}\left[\int_i^{i+2} f(x,y)\,dx\right] dy$$

$$= \int_k^{k+2}\left[\frac{h}{3}\left\{ f(x_i, y_k) + 4f(x_{i+1}, y_k) + f(x_{i+2}, y_k)\right\}\right] dy$$

$$= \frac{h}{3}\left[\int_k^{k+2} f(x_i, y_k)\,dy + 4\int_k^{k+2} f(x_{i+1}, y_k)\,dy + \int_k^{k+2} f(x_{i+2}, y_k)\,dy\right]$$

$$= \frac{h}{3}\Big[f(x_i, y_k) + 4f(x_i, y_{k+1}) + f(x_i, y_{k+2}) + 4\{ f(x_{i+1}, y_k) + 4f(x_{i+1}, y_{k+1}) + f(x_{i+1}, y_{k+2})\} + f(x_{i+2}, y_k) + 4f(x_{i+2}, y_{k+1}) + f(x_{i+2}, y_{k+2})\Big]$$

$$= \frac{h}{3}\Big[f(x_i, y_k) + f(x_{i+2}, y_k) + f(x_{i+2}, y_{k+2}) + f(x_i, y_{k+2}) + 4\{ f(x_{i+1}, y_k) + f(x_{i+2}, y_{k+1}) + f(x_{i+1}, y_{k+2}) + f(x_i, y_{k+1})\} + 16 f(x_{i+1}, y_{k+1})\Big]$$



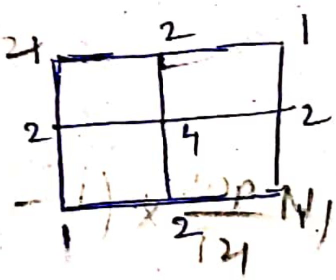
Trapezoidal Method (Double Integral)

$$I = \int_k^{k+2}\int_i^{i+2} f(x,y)\,dx\,dy \qquad x_{i+1} = x_i + h, \quad y_{i+1} = y_i + k$$

$$I = \int_k^{k+2}\left[\frac{h}{2}\left\{ f(x_i, y_k) + 2f(x_{i+1}, y_k) + f(x_{i+2}, y_k)\right\}\right] dy$$

$$I = \frac{h}{2}\int_k^{k+2}\left[f(x_i, y_k) + 2f(x_{i+1}, y_k) + f(x_{i+2}, y_k)\right] dy$$

$$I = \frac{h}{2}\Big[\underbrace{\int_k^{k+2} f(x_i, y_k)\,dy}_{I_1} + 2\underbrace{\int_k^{k+2} f(x_{i+1}, y_k)\,dy}_{I_2} + \underbrace{\int_k^{k+2} f(x_{i+2}, y_k)\,dy}_{I_3}\Big]$$



$$I = \frac{h}{2}\left[I_1 + 2I_2 + I_3\right]$$

$$I_1 = \int_k^{k+2} f(x_i, y_k)\,dy = \frac{k}{2}\left[f(x_i, y_k) + 2f(x_i, y_{k+1}) + f(x_i, y_{k+2})\right]$$

$$I_2 = \int_k^{k+2} f(x_{i+1}, y_k)\,dy = \frac{k}{2}\left[f(x_{i+1}, y_k) + 2f(x_{i+1}, y_{k+1}) + f(x_{i+1}, y_{k+2})\right]$$

$$I_3 = \int_k^{k+2} f(x_{i+2}, y_k)\,dy = \frac{k}{2}\left[f(x_{i+2}, y_k) + 2f(x_{i+2}, y_{k+1}) + f(x_{i+2}, y_{k+2})\right]$$

$$I = \frac{hk}{4}\Big[f(x_i, y_k) + 2f(x_i, y_{k+1}) + f(x_i, y_{k+2}) + 2f(x_{i+1}, y_k) + 4f(x_{i+1}, y_{k+1}) + 2f(x_{i+1}, y_{k+2}) + f(x_{i+2}, y_k) + 2f(x_{i+2}, y_{k+1}) + f(x_{i+2}, y_{k+2})\Big]$$

$$I = \frac{hk}{4}\Big[f(x_i, y_k) + f(x_i, y_{k+2}) + f(x_{i+2}, y_k) + f(x_{i+2}, y_{k+2}) + 2\{ f(x_i, y_{k+1}) + f(x_{i+1}, y_k) + f(x_{i+1}, y_{k+2}) + f(x_{i+2}, y_{k+1})\} + 4f(x_{i+1}, y_{k+1})\Big]$$