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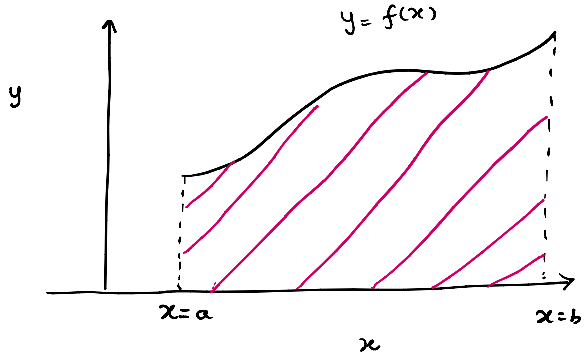
Integration

$$I = \int_a^b y(x)\,dx \rightarrow \text{area under curve } y(x) \text{ between } a \text{ to } b$$

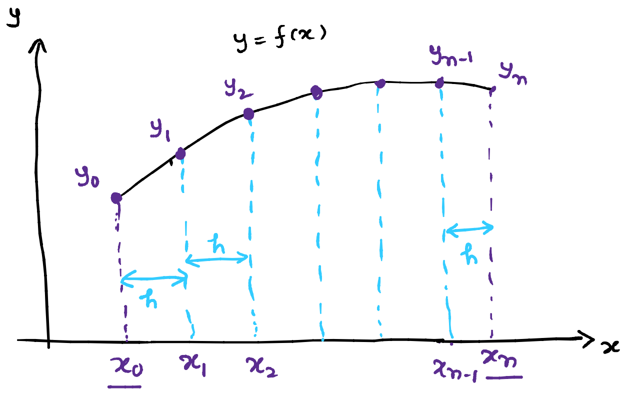


Numerical Method

- Newton cote's formula
  - Trapezoidal method
  - Simpson - one third
  - Simpson - three eight
  - Boole's formula
  - Weddles formula

Newton cote's formula

$$I = \int_{x_0}^{x_n} y(x)\,dx \quad \text{——} \quad (1)$$



$x_1 = x_0 + h$; $x_2 = x_0 + 2h$ .... $x_n = x_0 + nh$

$[x_0 \;\; x_n]$ — n- equal subinterval of width "h"

↓

$(n+1)$ data points

↓

max. order of polynomial "n" $\Rightarrow y = P_n(x)$

$$I = \int_{x_0}^{x_0+nh} P_n(x)\,dx = \int_{x_0}^{x_0+nh} \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots \right] dx$$

$x = x_0 + ph$; $dx = h\,dp$

$x = x_0$; $p = 0$

$x = x_0 + nh$ $\;\; p = n$

$x = x_0 ;\ p = 0$

$x = x_0 + nh \quad p = n$

$$I = h \int_0^n \left[y_0 + p\Delta y_0 + \frac{(p^2 - p)}{2!} \Delta^2 y_0 + \frac{(p^3 - 3p^2 + 2p)}{3!} \Delta^3 y_0 + \cdots \right] dp$$

$$I = h \left[\int_0^n y_0\, dp + \int_0^n p\Delta y_0\, dp + \int_0^n \frac{(p^2 - p)}{2!} \Delta^2 y_0\, dp + \int_0^n \frac{(p^3 - 3p^2 + 2p)}{3!} \Delta^3 y_0\, dp + \cdots \right]$$

$$\boxed{I = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{3!} + \cdots \right]}$$

$\int_{x_0}^{x_0+nh}$ — $n=1$ $\int_{x_0}^{x_0+h}$

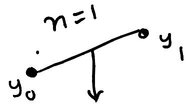
Newton cote's formula

$\Delta \quad \Delta^2 \quad \Delta^3$

— y_0

— y_1

$n=1$



$\int_{x_0}^{x_0+h} y(x)\, dx = ??$

Special Case

1. Trapezoidal Method

$n=1$ (Connect two alternate points with first order polynomial and calculate area under curve)

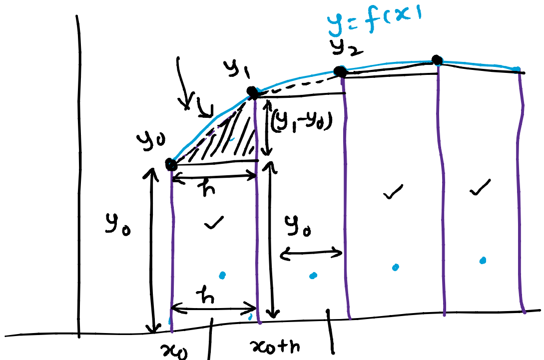
Δ area $= \frac{1}{2}(y_1 - y_0) h$

area of rectangle $= hy_0$

total area of first sub-interval

$= \frac{1}{2}(y_1 - y_0)h + hy_0$

$= \frac{h}{2}(y_1 - y_0 + 2y_0) = \frac{(y_0 + y_1)h}{2}$



$\frac{h}{2}(y_0 + y_1)$ $\quad \frac{h}{2}(y_1 + y_2)$

Total area $= \frac{h}{2}\left[(y_0 + y_1) + (y_1 + y_2) + (y_2 + y_3) + \cdots + (y_{n-1} + y_n) \right]$

$$\boxed{I = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1}) \right]}$$

$$I = \int_{x_0}^{x_0+nh} f(x)\, dx = \int_{x_0}^{x_0+h} f(x)\, dx + \int_{x_0+h}^{x_0+2h} f(x)\, dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)\, dx$$

for $n=1$ $(\Delta^2 y_0 = \Delta^3 y_0 \cdots = 0)$

$y \quad \Delta \quad \Delta^2 \quad \Delta^3 \quad \Delta^4$

0

$$\int_{x_0}^{x_0+h} f(x)\,dx = h\left[1\,y_0 + \frac{1}{2}\Delta y_0 + 0 + 0\right]$$

$$= h\left[y_0 + \frac{(y_1-y_0)}{2}\right] = h\left[\frac{2y_0 + y_1 - y_0}{2}\right] = \underline{\frac{h}{2}(y_0+y_1)}$$

similarly. $\int_{x_0+h}^{x_0+2h} f(x)\,dx = \frac{h}{2}(y_1+y_2)$ ..... $\int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx = \frac{h}{2}(y_{n-1}+y_n)$

$$I = \frac{h}{2}(y_0+\underline{y_1}) + \frac{h}{2}(\underline{y_1}+y_2) + \cdots\cdots + \frac{h}{2}(y_{n-1}+y_n)$$

$$\boxed{I = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+y_3+\cdots\cdots+y_{n-1})\right]}$$

- $[x_0\ x_n]$ can be divided into any no. (odd/even) of sub-interval

Simpson One-third Method

$n=2$ (connect three data points at once with 2^{nd} order polynomial)

- No of sub-interval should be even.

$$I = \int_{x_0}^{x_0+nh} f(x)dx = \underbrace{\int_{x_0}^{x_0+2h} f(x)\,dx}_{I_1} + \underbrace{\int_{x_0+2h}^{x_0+4h} f(x)\,dx}_{I_2} + \cdots\cdots + \int_{x_0+(n-2)h}^{x_0+nh} f(x)dx$$

$\int_{x_0}^{x_0+2h} f(x)\,dx = ??$

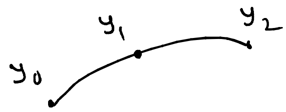
- $n = 2$
- $\Delta^3 y_0 = \Delta^4 y_0$
  $= 0$

	Δ	Δ^2	$\Delta^3 = \Delta^4$	
y_0				
	.			
y_1		.	0	0
	.			
y_2				

$\Delta^2 y_0 = y_2 - 2c_1 y_1 + 2c_2 y_0$

$$\int_{x_0}^{x_0+2h} f(x)dx = h\left[n y_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 y_0 + 0\right]$$

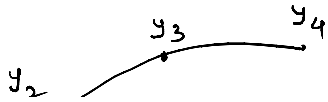
$$= h\left[2y_0 + 2(y_1-y_0) + \frac{1}{2}\left(\frac{8}{3}-2\right)\Delta^2 y_0\right]$$



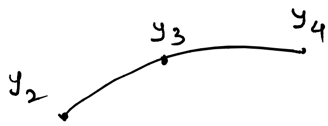
$$= h\left[\cancel{2y_0} + 2y_1 - \cancel{2y_0} + \frac{1}{3}(y_2 - 2y_1 + y_0)\right]$$

$$= \frac{h}{3}\left[6y_1 + y_2 - 2y_1 + y_0\right] = \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]$$

$$\int_{x_0}^{x_0+2h} f(x)\,dx = \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]$$



$$\int_{x_0} f(x)\,dx = \frac{h}{3}\left[y_0 + 4y_1 + y_2 \right]$$

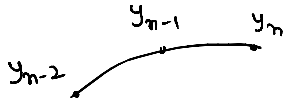


similarly $\left[\underset{x_2}{\underline{x_0+2h}} \quad \underset{x_4}{\underline{x_0+4h}} \right]$

$$\int_{x_0+2h}^{x_0+4h} f(x)\,dx = \frac{h}{3}\left[y_2 + 4y_3 + y_4 \right]$$

$\vdots$

$$\int_{x_0+(n-2)h}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left[y_{n-2} + 4y_{n-1} + y_n \right]$$



$$I = \frac{h}{3}\left[(y_0 + 4y_1 + \underline{y_2}) + (\underline{y_2} + 4y_3 + y_4) + (\underline{y_4} + 4y_5 + y_6) + \cdots \right]$$

$$\boxed{I = \frac{h}{3}\left[(y_0 + y_n) + 2(y_2 + y_4 + y_6 + y_8 + \cdots) + 4(y_1 + y_3 + y_5 + \cdots) \right]}$$

Simpson three eight Method

$n = 3$ (four points with 3RD order polynomial)

$$I = \int_{x_0}^{x_0+3h} f(x) + \int_{x_0+3h}^{x_0+6h} f(x)\,dx + \cdots + \int_{x_0+(n-3)h}^{x_0+nh} f(x)\,dx$$

$\Delta^3 y_0 = y_3 - {}^3C_1 y_2 + {}^3C_2 y_1 - y_0$

- No. of sub-intervals should be multiple of three.

$n = 3$ & $\Delta^4 y_0 = \Delta^5 y_0 \ldots\ldots = 0$

$$\int_{x_0}^{x_0+3h} f(x)\,dx = h\left[n y_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2} + \frac{1}{6}\left(\frac{n^4}{4} - n^3 + n^2\right)\Delta^3 y_0 + 0 \right]$$

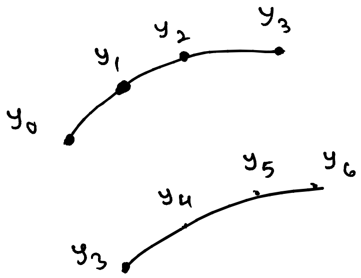
$$= h\left[3y_0 + \frac{9}{2}(y_1 - y_0) + \frac{9}{2}\left(1 - \frac{1}{2}\right)(y_2 - 2y_1 + y_0) + \frac{9}{6}\left(\frac{9}{4} - 3 + 1\right)(y_3 - 3y_2 + 3y_1 - y_0) \right]$$

$$= h\left[3y_0 + \frac{9}{2}(y_1 - y_0) + \frac{9}{4}(y_2 - 2y_1 + y_0) + \frac{\cancel{9}^{3}}{\cancel{6}_{2}} \times \frac{1}{4}(y_3 - 3y_2 + 3y_1 - y_0) \right]$$

$$= \frac{3}{8}h\left[\underline{8y_0} + 12(y_1 - \underline{y_0}) + 6(y_2 - 2y_1 + \underline{y_0}) + (y_3 - 3y_2 + 3y_1 - y_0) \right]$$

$$= \frac{3}{8}h\left[8y_0 + 12(y_1 - y_0) + 6(y_2 - 2y_1 + y_0) + (y_3 - 3y_2 + 3y_1 - y_0)\right]$$

$$= \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + y_3\right]$$



$$\int_{x_0}^{x_0+3h} f(x)\,dx = \frac{3h}{8}\left[y_0 + 3(y_1 + y_2) + y_3\right]$$

$$\int_{x_0+3h}^{x_0+6h} f(x)\,dx = \frac{3h}{8}\left[y_3 + 3(y_4 + y_5) + y_6\right]$$

$\vdots$

$$\int_{x_0+(n-3)h}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\left[y_{n-3} + 3(y_{n-2} + y_{n-1}) + y_n\right]$$

$$\boxed{I = \frac{3h}{8}\left[(y_0 + y_n) + 2(y_3 + y_6 + y_9 + y_{12} + \cdots\cdots) + 3(y_1 + y_2 + y_4 + y_5 + y_7 + y_8 \cdots\cdots)\right]}$$

Boole's formula

$n = 4$ (five points connected by 4^{th} order polynomial)

$\Delta^5 y_0 = \Delta^6 y_0 = \cdots\cdots = 0$

- $[x_0 \;\; x_n]$ — No. of subintervals should be multiple of four.

$$\boxed{I = \frac{2h}{45}\left[7(y_0 + y_n) + 32(y_1 + y_3 + y_5 + y_7 + \cdots\cdots) + 12(y_2 + y_6 + y_{10} + y_{14}) + 14(y_4 + y_8 + y_{12} + \cdots\cdots)\right]}$$

Weddle's formula

$n = 6$

- No. of sub interval should be multiple of six

$$\boxed{I = \frac{3h}{10}\left[(y_0 + y_n) + 2(y_6 + y_{12} + y_{18} + \cdots\cdots) + 5(y_1 + y_5 + y_7 + y_{11} + \cdots\cdots) + (y_2 + y_4 + y_8 + y_{10} + \cdots\cdots)\right.}$$

$$I = \frac{3h}{10}\Big[(y_0+y_n) + 2(y_6+y_{12}+y_{18}+\cdots) + 5(y_1+y_5+y_7+y_{11}+\cdots) + (y_2+y_4+y_8+y_{10}+\cdots) + 6(y_3+y_9+y_{15}+\cdots)\Big]$$

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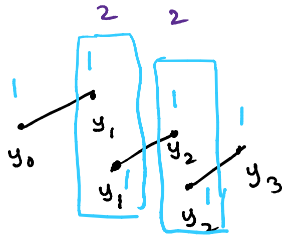
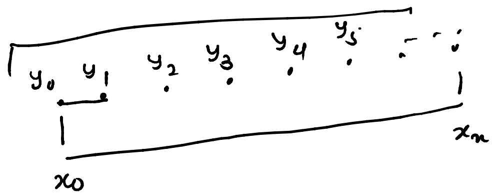
Trapezoidal Method

$n=1 \quad (1, 1)$

$\frac{h}{2}$



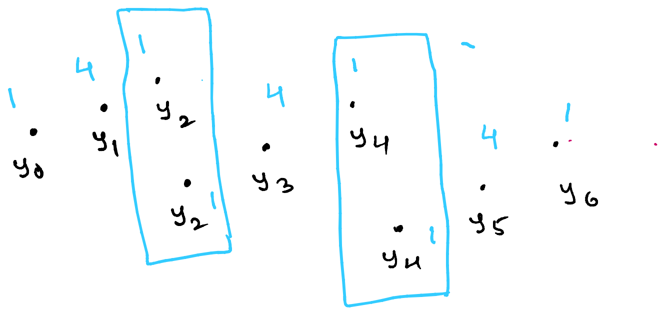


$$I = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+y_3+y_4+\cdots+y_{n-1})\right]$$

Simpson one third

$n=2 \quad (1, 4, 1)$

$\frac{h}{3}$

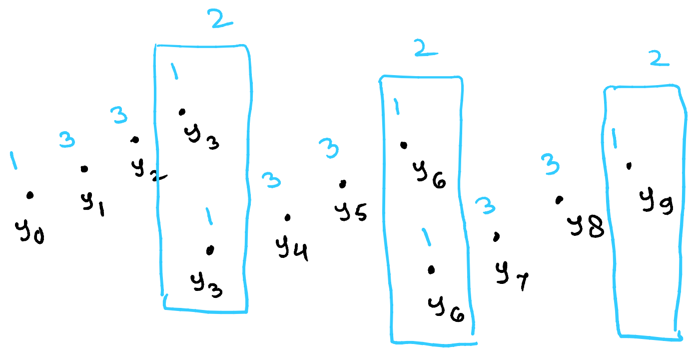


$$I = \frac{h}{3}\left[(y_0+y_n) + 2(y_2+y_4+y_6+\cdots) + 4(y_1+y_3+y_5+y_7+\cdots)\right]$$

Simpson-three eight

$n=3 \quad (1 \quad 3 \quad 3 \quad 1)$

$\frac{3h}{8}$

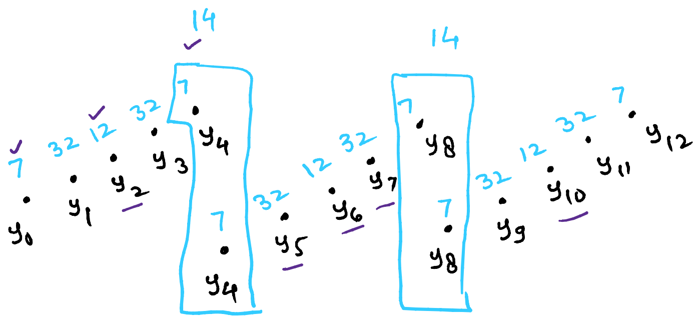


$$I = \frac{3h}{8}\left[(y_0+y_n) + 2(y_3+y_6+y_9+y_{12}+\cdots) + 3(y_1+y_2+y_4+y_5+y_7+y_8+\cdots)\right]$$

Boole's Method

$n=4 \quad (7 \quad 32 \quad 12 \quad 32 \quad 7)$

$\frac{2h}{45}$

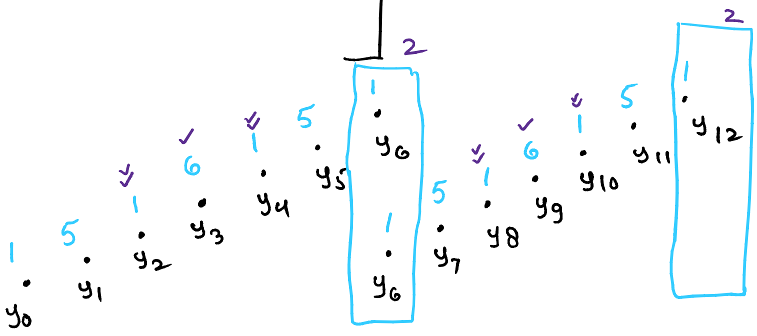


$$I = \frac{2h}{45}\left[7(y_0+y_n) + 14(y_4+y_8+y_{12}+\cdots) + 12(y_2+y_6+y_{10}+\cdots) + 32(y_1+y_3+y_5+y_7+y_9+y_{11}+\cdots)\right]$$

Weddle's Method

$n = 6 \quad (1 \quad 5 \quad 1 \quad 6 \quad 1 \quad 5 \quad 1)$

$\frac{3h}{10}$



$$I = \frac{3h}{10}\Big[(y_0 + y_n) + 2(y_6 + y_{12} + y_{18} + \cdots) + 6(y_3 + y_9 + y_{15} + \cdots) + (y_2 + y_4 + y_8 + y_{10} + \cdots) + 5(y_1 + y_5 + y_7 + y_{11} + \cdots)\Big]$$

example $I = \int_0^6 \frac{dx}{(1+x^2)}$. divide $(0, 6)$ into equal six subinterval and calculate "I" using possible method.

solution $h = 1$

x	0	1	2	3	4	5	6
$y = \frac{1}{1+x^2}$	1	0.5	0.2	0.1	0.0588	0.0385	0.027
	y_0	y_1	y_2	y_3	y_4	y_5	y_6
	1	4	2	4	2	4	1
	1	3	3	2	3	3	1
	1	5	1	6	1	5	1

(i) Trapezoidal

$$I = \frac{1}{2}\Big[(1 + 0.027) + 2(0.5 + 0.2 + 0.1 + 0.0588 + 0.385)\Big]$$
$$= 1.4108$$

(ii) Simpson's 1/3 rule

$$I = \frac{1}{3}\Big[(1 + 0.027) + 2(0.2 + 0.588) + 4(0.5 + 0.1 + 0.0385)\Big] = 1.3662$$

(iii) Simpson 3/8 rule

$$I = \frac{3}{8}\Big[(1 + 0.027) + 2(0.1) + 3(0.5 + 0.2 + 0.0588 + 0.0385)\Big] = 1.3571$$

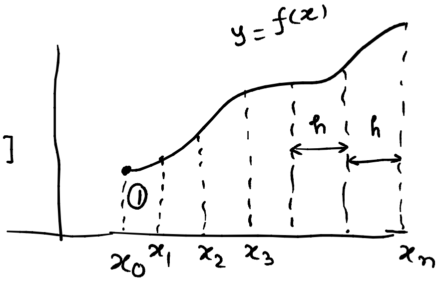
(iv) Weddle's rule : $I = \frac{3}{10}\Big[(1 + 0.2 + 0.0588 + 0.027) + 5(0.5 + 0.0385) + 6(0.1)\Big]$

$$= 1.3735$$

Error in the Quadrature formulae

error $E = \int_a^b y(x)\,dx - \int_a^b P_n(x)\,dx$

error in Trapezoidal Method



$[x_0 \;\; x_n] \to [x_0 \;\; x_1] \;\; [x_1 \;\; x_2] \; \dots \; [x_{n-1} \;\; x_n]$

expanding $y=f(x)$ around $x=x_0$

$$y = y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \dots \quad \text{— (1)}$$

$[x_0 \;\; x_1]$ $(x_0 - x_0+h)$

$$\int_{x_0}^{x_0+h} y(x)\,dx = \int_{x_0}^{x_0+h}\left[y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \dots\right]dx$$

take $x-x_0=t$; $dx=dt$

$x=x_0 \to t=0$

$x=x_0+h \quad t=h$

$$\int_{x_0}^{x_0+h} y(x)\,dx = \int_0^h\left[y_0 + t\,y_0' + \frac{t^2}{2!}y_0'' + \dots\right]dt = \left[y_0h + \frac{h^2}{2}y_0' + \frac{h^3}{3!}y_0'' + \dots\right] \quad \text{— (2)}$$

from trapezoidal formula; $\int_{x_0}^{x_0+h} P_n(x)\,dx = \frac{h}{2}(y_0+y_1)$ — (3)

from eq^n (1) put $x=x_1=x_0+h$; $y(x=x_1)=y_1$

$$y_1 = y_0 + (x_1-x_0)y_0' + \frac{(x_1-x_0)^2}{2!}y_0'' + \frac{(x_1-x_0)^3}{3!}y_0''' + \dots$$

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \dots \quad (4)$$

from eq^n (3) & (4)

$$\int_{x_0}^{x_0+h} P_n(x) = \frac{h}{2}\left[2y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \dots\right] \quad (5)$$

error in $[x_0 \;\; x_1]$ $(x_0 - x_0+h)$

$$E_1 = \int_{x_0}^{x_0+h} y(x)\,dx - \int_{x_0}^{x_0+h} P_n(x) \quad [\text{eq}^n\ (2) - \text{eq}^n\ (5)]$$

$$E_1 = \left\{ \cancel{y_0 h} + \cancel{\frac{h^2}{2!} y_0'} + \frac{h^3}{3!} y_0'' + \cdots \right\} - \left\{ \cancel{y_0 h} + \cancel{\frac{h^2}{2} y_0'} + \frac{h^3}{2\times 2!} y_0'' + \frac{h^4}{2\times 3!} y_0''' + \cdots \right\}$$

$$\underline{E_1} = \left(\frac{1}{3!} - \frac{1}{2\times 2!}\right) h^3 y_0'' + \cdots \quad = \quad -\frac{h^3}{12} y_0'' + \cdots\cdots$$

principal part of error in $[x_0 \; x_1] = -\frac{h^3}{12} y_0''$

similarly principal part of error $[x_1, x_2] = -\frac{h^3}{12} y_1''$

Hence total error $= E_1 + E_2 + \cdots\cdots$

$$E = -\frac{h^3}{12}\left(\underline{y_0''} + \underline{y_1''} + \underline{y_2''} + \cdots\cdots + \underline{y_{n-1}''} \right)$$

$\underline{y''(X)}$ is the largest in the given values of $y_0'', y_1'', y_2'' \cdots y_{n-1}''$

$$E < -\frac{h^3}{12} n y''(X) \qquad\qquad b - a = nh$$

$$\boxed{E < -(b-a)\, h^2\, y''(X)}$$

error in trapezoidal method is of the order h^2.

<u>Simpson-</u> <u>1/3 method</u>

$$E < -\frac{(b-a)}{180} h^4 y^{IV}(X)$$

error in simpson 1/3 rule is order of h^4

error in simpson 3/8 rule is of order h^4

error in (Boole's / Weddle's) method is of order h^7

<u>Correction</u> <u>formula</u>

1. Romberg's method
2. Euler's - Maclaurin formethod

<u>Romberg's</u> <u>Method</u>

$$\underline{I = \int_a^b y(x)\,dx} \qquad \left. \begin{array}{l} h_1 \longrightarrow I_1 \\ h_2 \longrightarrow I_2 \end{array} \right] \qquad \left. I = \frac{I_1 h_2^2 - I_2 h_1^2}{h_2^2 - h_1^2} \right] \quad \text{Richardson method}$$

$h_1 = h; \quad h_2 = h/2$

$$\boxed{I = \frac{1}{3}\left[4I(h/2) - I(h)\right]}$$

Example :-

$$I = \int_0^1 \frac{1}{(1+x^2)}\,dx$$

solution :- $h = 0.5, \quad 0.25, \quad 0.125$

(1) $h = 0.5$;

x	0	0.5	1.0
y	1	0.8	0.5

$$I(h=0.5) = \frac{0.5}{2}\left[1 + 0.5 + 2\times 0.8\right] = 0.775$$

(2) $h = 0.25$

x:	0	0.25	0.50	0.75	1.0
y:	1	0.9412	0.80	0.64	0.50

$$I(h=0.25) = \frac{0.25}{2}\left[(1+0.5) + 2(0.9412 + 0.80 + 0.64)\right] = 0.7828$$

(3) $h = 0.125$; $\quad I(h=0.125) = 0.7848$

$I(h)$ 0.5

$I(h, h/2)$

$I(h/2)$ 0.25 $\qquad I(h, h/2, h/4)$

$I(h/2, h/4)$

$I(h/4)$ 0.125

$$I(h, h/2) = \frac{1}{3}\left[4I(h/2) - I(h)\right]$$

$$I(0.5, 0.25) = \frac{1}{3}\left[4(0.7828) - 0.7750\right] = 0.7854$$

$$I(0.25, 0.125) = \frac{1}{3}\left[4(0.7848) - 0.7828\right] = 0.7855$$

0.7750

0.7854

0.7828 $\qquad \boxed{0.7855}$

0.7855

$$I(0.5, 0.25, 0.125) = \frac{1}{3}\left[4(0.7855) - 0.7854\right]$$

$$\left[\begin{array}{l} 0.7828 \\ \\ 0.7848 \end{array}\right] \quad 0.7855 \quad 0.7855$$

$$I(0.5, 0.25, 0.125) = \frac{1}{3}\left[4(0.7855) - 0.7854\right]$$

$$= 0.7855$$

$$\boxed{\text{Romberg's method} = 0.7855}$$

Euler's Maclaurin formula

$$\int_{x_0}^{x_n} y(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1})\right] - \frac{h^2}{12}(y_n' - y_0') + \frac{h^4}{720}(y_n''' - y_0''')$$

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Double Integration

$$I = \int_{y_0}^{y_n}\int_{x_0}^{x_n} f(x,y)\,dx\,dy$$

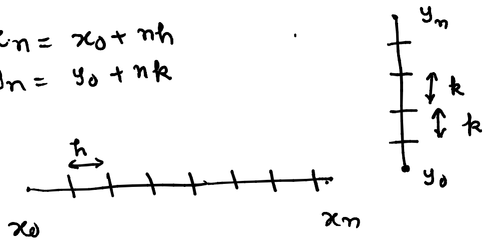
x limit – $(x_0 \quad x_n)$

y limit – $(y_0, \; y_n)$

$y = f(x, y)$

$x_n = x_0 + nh$

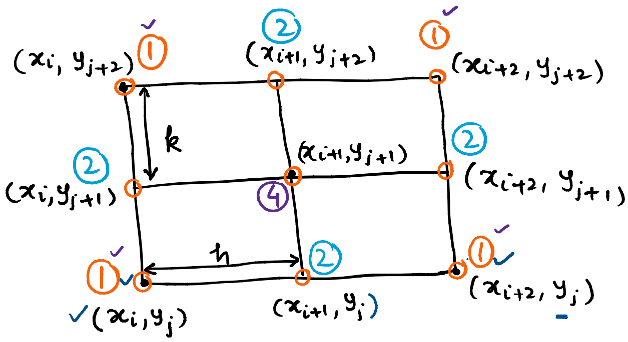
$y_n = y_0 + nk$



Trapezoidal Method

$$I = \int_{y_j}^{y_{j+2}}\int_{x_i}^{x_{i+2}} f(x,y)\,dx\,dy \quad —— \quad (1)$$

$$\left.\begin{aligned} x_{i+1} &= x_i + h \; ; \quad x_{i+2} = x_i + 2h \\ y_{j+1} &= y_j + k \; ; \quad y_{j+2} = y_j + 2k \end{aligned}\right\} \quad (2)$$



$$I = \int_{y_j}^{y_{j+2}} \frac{h}{2}\Big[f(x_i, y) + 2f(x_{i+1}, y) + f(x_{i+2}, y) \Big]\,dy$$

$$I = \frac{h}{2}\Big[\underbrace{\int_{y_j}^{y_{j+2}} f(x_i, y)\,dy}_{I_1} + \underbrace{2\int_{y_j}^{y_{j+2}} f(x_{i+1}, y)\,dy}_{I_2} + \underbrace{\int_{y_j}^{y_{j+2}} f(x_{i+2}, y)\,dy}_{I_3} \Big] \quad — \quad (3)$$

$$I_1 = \int_{y_j}^{y_{j+2}} f(x_i, y)\,dy = \frac{k}{2}\Big[f(x_i, y_j) + f(x_i, y_{j+2}) + 2f(x_i, y_{j+1}) \Big] \quad — \quad (4)$$

$$I_2 = 2\int_{y_j}^{y_{j+2}} f(x_{i+1}, y)\,dy = 2\,\frac{k}{2}\Big[f(x_{i+1}, y_j) + f(x_{i+1}, y_{j+2}) + 2f(x_{i+1}, y_{j+1}) \Big] \quad — \quad (5)$$

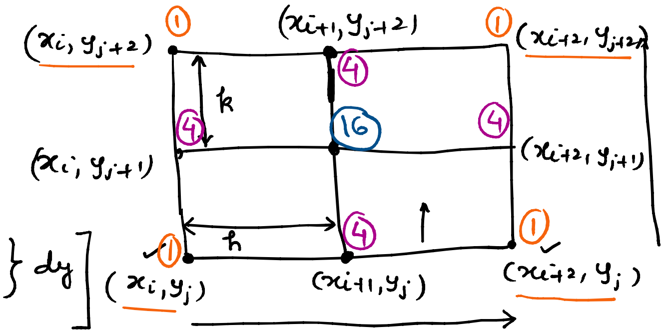
$$I_3 = \int_{y_j}^{y_{j+2}} f(x_{i+2}, y)\,dy = \frac{k}{2}\Big[f(x_{i+2}, y_j) + f(x_{i+2}, y_{j+2}) + 2f(x_{i+2}, y_{j+1}) \Big] \quad — \quad (6)$$

$$I = \frac{h}{2}\Big[I_1 + I_2 + I_3 \Big] = \frac{hk}{4}\Big[f(x_i, y_j) + f(x_i, y_{j+2}) + f(x_{i+2}, y_j) + f(x_{i+2}, y_{j+2}) + 2\{ f(x_i, y_{j+1}) + f(x_{i+1}, y_{j}) + f(x_{i+2}, y_{j+1}) \} + 4f(x_{i+1}, y_{j+1}) \Big]$$

$$I = \frac{h}{2}\left[I_1 + I_2 + I_3 \right] = \frac{hk}{4}\Big[f(x_i,y_j) + f(x_i,y_{j+2}) + f(x_{i+2},y_j) + f(x_{i+2},y_{j+2}) + 2\{ f(x_i,y_{j+1}) + f(x_{i+1},y_{j+2}) + f(x_{i+2},y_{j+2})\} + 4 f(x_{i+1},y_{j+1}) \Big]$$

Simpson. One third formula

$$I = \int_{y_j}^{y_{j+2}} \int_{x_i}^{x_{i+2}} f(x,y)\, dy \quad — \quad (1)$$

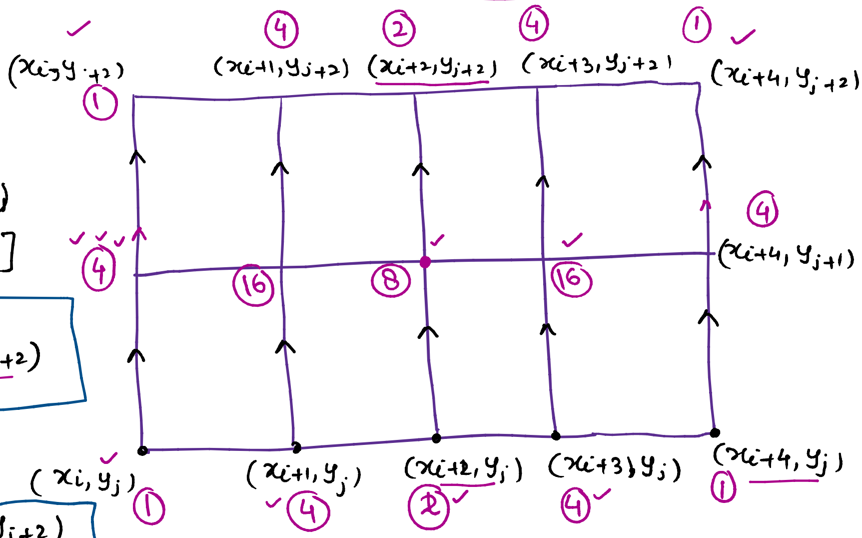


$$I = \frac{h}{3}\left[\int_{y_j}^{y_{j+2}} \{ f(x_i,y) + f(x_{i+2},y) + 4 f(x_{i+1},y) \}\, dy \right]$$

$$I = \frac{hk}{9}\Big[\{ f(x_i,y_j) + f(x_i,y_{j+2}) + 4 f(x_i,y_{j+1}) \} + \{ f(x_{i+2},y_j) + f(x_{i+2},y_{j+2}) + 4 f(x_{i+2},y_{j+1}) \} + 4\{ f(x_{i+1},y_j) + f(x_{i+1},y_{j+2}) + 4 f(x_{i+1},y_{j+1}) \} \Big]$$

$$\boxed{I = \frac{hk}{9}\Big[\{ f(x_i,y_j) + f(x_{i+2},y_j) + f(x_{i+2},y_{j+2}) + f(x_i,y_{j+2}) \} + 16 f(x_{i+1},y_{j+1}) + 4\{ f(x_{i+1},y_j) + f(x_{i+2},y_{j+1}) + f(x_{i+1},y_{j+2}) + f(x_i,y_{j+1}) \} \Big]}$$

$$I = \int_{y_j}^{y_{j+2}} \int_{x_i}^{x_{i+4}} f(x,y)\, dx\, dy$$



$$I = \frac{h}{3}\left[\int_{y_j}^{y_{j+2}} \{ f(x_i,y) + f(x_{i+4},y) + 2 f(x_{i+2},y) + 4 f(x_{i+1},y) + 4 f(x_{i+3},y) \}\, dy \right]$$

$$I = \frac{hk}{9}\Big[\{ f(x_i,y_j) + 4 f(x_i,y_{j+1}) + f(x_i,y_{j+2})$$
$$+ f(x_{i+4},y_j) + 4 f(x_{i+4},y_{j+1}) + f(x_{i+4},y_{j+2})$$
$$+ 2 f(x_{i+2},y_j) + 8 f(x_{i+2},y_{j+1}) + 2 f(x_{i+2},y_{j+2})$$
$$+ 4 f(x_{i+1},y_j) + 16 f(x_{i+1},y_{j+1}) + 4 f(x_{i+1},y_{j+2})$$
$$+ 4 f(x_{i+3},y_j) + 16 f(x_{i+3},y_{j+1}) + 4 f(x_{i+3},y_{j+2}) \Big]$$

(1, 4, 1)

Q. $I = \int_{2.0}^{2.6} \int_{4.0}^{4.4} \frac{dx\, dy}{(xy)} = ??$ $h = 0.2$, $k = 0.3$

Trapezoidal / simpson 1/3 method

solution $f = \frac{1}{(xy)}$

$x \rightarrow$

x	4.0	4.2	4.4

solution

$$f = \frac{1}{(xy)}$$

y \ x	4.0	4.2	4.4
2.0	(1) 0.125 (1)	(2) 0.119 (4)	(1) 0.1136 (1)
2.3	(2) 0.1087 (4)	(4) 0.1035 (16)	(2) 0.0988 (4)
2.6	(1) 0.0962 (1)	(2) 0.0916 (4)	(1) 0.0874 (1)

$$I_{trap} = \frac{0.2 \times 0.3}{4}\Big[(0.0962 + 0.0874 + 0.1136 + 0.125) + 2(0.0916 + 0.0988 + 0.119 + 0.1087) + 4(0.1035)\Big] = \frac{0.06}{4}\Big[0.4222 + 2(0.4181) + 4(0.1035)\Big]$$

$$\simeq 0.0251$$

$$I_{Simp} = \frac{0.2 \times 0.3}{9}\Big[(0.0962 + 0.0874 + 0.1136 + 0.125) + 4(0.0916 + 0.0988 + 0.119 + 0.1087) + 16(0.1035)\Big] \approx \underline{0.025}$$