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Numerical Differentiation

$$\left.\begin{matrix} \rightarrow x_0 & y_0 \\ \rightarrow x_1 & y_1 \\ \vdots & \\ \rightarrow x_n & y_n \end{matrix}\right\}$$

$(x_0, y_0)\ (x_1, y_1)\ (x_2, y_2)\ \text{------}\ (x_n, y_n)$

$\left(\frac{dy}{dx}\right)_x$ $\quad x - x_i\ (i = 0, 1, 2, 3, \dots)$

$\left(\frac{d^2y}{dx^2}\right)_x$ $\quad x - \text{intermediate point}$

1. Newton's forward difference formula

Newton's forward difference formula –

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots \qquad —(1)$$

$$\left.\frac{dy}{dx}\right|_x = ?? \qquad \left.\frac{d^2y}{dx^2}\right|_x = ??$$

$(p^2 - 3p + 2)$

$$x = x_0 + ph$$

$$dx = h\,dp$$

$$\frac{dp}{dx} = \frac{1}{h} \quad —(2)$$

$$\frac{dy}{dx} = \frac{dy}{dp}\frac{dp}{dx} = \frac{1}{h}\frac{dy}{dp} \quad —(3)\ ; \quad \frac{d^2y}{dx^2} = \frac{1}{h^2}\frac{d^2y}{dp^2} \quad —(4)$$

from eqⁿ (1)

$$y_p = y_0 + p\Delta y_0 + \frac{(p^2 - p)}{2!}\Delta^2 y_0 + \frac{(p^3 - 3p^2 + 2p)}{3!}\Delta^3 y_0 + \dots$$

$$\frac{dy}{dp} = \Delta y_0 + \frac{(2p-1)}{2!}\Delta^2 y_0 + \frac{(3p^2 - 6p + 2)}{3!}\Delta^3 y_0 + \dots \qquad —(5)$$

$$\frac{d^2y}{dp^2} = \Delta^2 y_0 + (p-1)\Delta^3 y_0 + \frac{(12p^2 - 36p + 22)}{4!}\Delta^4 y_0 + \dots \qquad —(6)$$

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{(2p-1)}{2!}\Delta^2 y_0 + \frac{(3p^2-6p+2)}{3!}\Delta^3 y_0 + \cdots\cdots\right] \text{ — NF}$$

$$\nabla y_n + \frac{(2p+1)}{2}\nabla^2 y_n + \frac{(3p^2+6p+2)}{3!}\nabla^3 y_n \text{ — NB}$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (p-1)\Delta^3 y_0 + \frac{(12p^2-36p+22)}{4!}\Delta^4 y_0 + \cdots\cdots\right] \quad \text{— (7)}$$

Special case :- $x = x_0$; $p = 0$

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \frac{\Delta^5 y_0}{5} \cdots\cdots\right] \quad \text{— (8)}$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 \cdots\cdots\right]$$

Newton's Backward difference formula :-

$$y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n + \cdots$$

$$y_p = y_n + p\nabla y_n + \frac{(p^2+p)}{2!}\nabla^2 y_n + \frac{(p^3+3p^2+2p)}{3!}\nabla^3 y_n + \frac{(p^4+6p^3+11p^2+6p)}{4!}\nabla^4 y_n + \cdots\cdots$$

$$\left.\begin{aligned}\frac{dy}{dp} &= \nabla y_n + \frac{(2p+1)}{2!}\nabla^2 y_n + \frac{(3p^2+6p+2)}{3!}\nabla^3 y_n + \frac{(4p^3+18p^2+22p+6)}{4!}\nabla^4 y_n + \cdots \\ \frac{d^2y}{dp^2} &= \nabla^2 y_n + (p+1)\nabla^3 y_n + \frac{(12p^2+36p+22)}{4!}\nabla^4 y_n + \cdots\cdots\end{aligned}\right\}$$

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{(2p+1)}{2!}\nabla^2 y_n + \frac{(3p^2+6p+2)}{3!}\nabla^3 y_n + \frac{(4p^3+18p^2+22p+6)}{4!}\nabla^4 y_n + \cdots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + (p+1)\nabla^3 y_n + \frac{(12p^2+36p+22)}{4!}\nabla^4 y_n + \cdots\cdots\right] \quad \text{— (9)}$$

Special case :- $x = x_n$; $p = 0$ $(x = x_n + ph)$

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \frac{\nabla^4 y_n}{4} + \cdots\cdots\right]$$

$$\frac{d^2y}{dx^2} \ldots \left[\nabla^2 \ldots \nabla^3 \ldots \nabla^4 \ldots \cdots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots\cdots\right]$$

Q.

x:	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y:	7.989	8.403	8.781	9.129	9.451	9.750	10.031

$\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ at $x = 1.1$ & 1.6

Sol.

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
1.0	7.989						
		0.414					
$x_0 \to$ 1.1	y_0 = 8.403		−0.036				
		0.378 (Δy_0)		0.006			
1.2	8.781		−0.030 ($\Delta^2 y_0$)		−0.002		
		0.348		0.004 ($\Delta^3 y_0$)		0.002	
1.3	9.129		−0.026		0.000 ($\Delta^4 y_0$)		−0.001 ($\nabla^6 y_n$)
		0.322		0.004		0.001 ($\Delta^5 y_0$, $\nabla^5 y_n$)	
1.4	9.451		−0.023		+0.001 ($\nabla^4 y_n$)		
		0.299		0.005 ($\nabla^3 y_n$)			
1.5	9.750		−0.018 ($\nabla^2 y_n$)				
		0.281 (∇y_n)					
x_n = 1.6	10.031 (y_n)						

$h = 0.1$.

$\boxed{h = 0.1,\ x_0 = 1.1}$

$$\frac{dy}{dx} = \frac{1}{0.1}\left[0.378 - \frac{(-0.030)}{2} + \frac{(0.004)}{3} - 0 + \frac{(0.001)}{5}\right] = 3.945$$

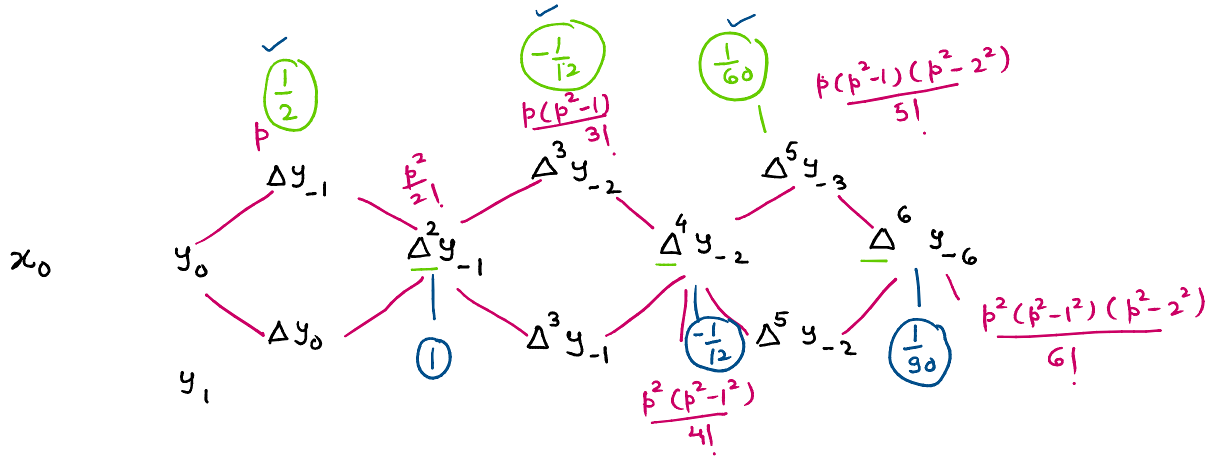
$$\frac{d^2y}{dx^2} = \frac{1}{(0.1)^2}\left[-0.030 - 0.004 + \frac{11}{12}\times 000\right] = -3.4$$

$h = 0.1,\ x_n = 1.6$

$$\frac{dy}{dx} = \frac{1}{0.1}\left[0.281 + \frac{(-0.018)}{2} + \frac{(0.005)}{3} + \frac{(0.001)}{4} + \frac{(0.001)}{5} + \frac{(-0.001)}{6}\right] = 2.7395$$

$$\frac{d^2y}{dx^2} = \frac{1}{0.01}\left[-0.018 + 0.005 + \frac{11}{12}(0.001)\right] = -1.208$$

Stirling's Method



$$y_p = y_0 + p\frac{(\Delta y_{-1}+\Delta y_0)}{2} + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\left(\frac{\Delta^3 y_{-2}+\Delta^3 y_{-1}}{2}\right) + \frac{p^2(p^2-1^2)}{4!}\Delta^4 y_{-2} +$$
$$+ \frac{p\overbrace{(p^2-1^2)(p^2-2^2)}^{(p^4-5p^2+4)}}{5!}\left(\frac{\Delta^5 y_{-3}+\Delta^5 y_{-2}}{2}\right) + \frac{p^2(p^2-1^2)(p^2-2^2)}{6!}\Delta^6 y_{-6}$$

$$y_p = y_0 + p\left(\frac{\Delta y_{-1}+\Delta y_0}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{(p^3-p)}{3!}\left(\frac{\Delta^3 y_{-2}+\Delta^3 y_{-1}}{2}\right) + \frac{(p^4-p^2)}{4!}\Delta^4 y_{-2} +$$
$$\frac{(p^5-5p^3+4p)}{5!}\left(\frac{\Delta^5 y_{-3}+\Delta^5 y_{-2}}{2}\right) + \frac{(p^6-5p^4+4p^2)}{6!}\Delta^6 y_{-6}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{h}\left[\frac{(\Delta y_{-1}+\Delta y_0)}{2} + p\,\Delta^2 y_{-1} + \frac{(3p^2-1)}{3!}\left(\frac{\Delta^3 y_{-2}+\Delta^3 y_{-1}}{2}\right) + \frac{(4p^3-2p)}{4!}\Delta^4 y_{-2} + \frac{(5p^4-15p^2+4)}{5!}\left(\frac{\Delta^5 y_{-3}+\Delta^5 y_{-2}}{2}\right) + \frac{(6p^5-20p^3+8p)}{6!}\Delta^6 y_{-6}\right]}$$

Special case $(p=0)$ $x=x_0$

$$\boxed{\left.\frac{dy}{dx}\right|_{x=x_0} = \frac{1}{h}\left[\frac{(\Delta y_{-1}+\Delta y_0)}{2} - \frac{1}{12}(\Delta^3 y_{-2}+\Delta^3 y_{-1}) + \frac{1}{60}(\Delta^5 y_{-3}+\Delta^5 y_{-2})\right]}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_{-1} + p\left(\frac{\Delta^3 y_{-2}+\Delta^3 y_{-1}}{2}\right) + \frac{(12p^2-2)}{4!}\Delta^4 y_{-2} + \frac{(20p^3-30p)}{5!}\left(\frac{\Delta^5 y_{-3}+\Delta^5 y_{-2}}{2}\right) + \frac{(30p^4-60p^2+8)}{6!}\Delta^6 y_{-6}\right]}$$

Special case : $p=0;\ x=x_0$

Special case : $p = 0$; ...

$$\boxed{\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-6}\right]}$$

Q.

t (sec)	0.0	0.1	0.2	0.3	0.4	0.5	0.6
x (cm)	30.13	31.62	32.87	33.64	33.95	33.81	33.24

velocity $\left(\frac{dx}{dt}\right)$; Acceleration $\left(\frac{d^2x}{dt^2}\right)\Big|_{t=0.3}$

Sol.

t	x						
0.0	30.13						
		1.49					
0.1	31.62		−0.24				
		1.25		−0.24			
0.2	32.87		−0.48		0.26		
		0.77		0.02		−0.27	
0.3	33.64		−0.46		−0.01		0.29
		0.31		0.01		0.02	
0.4	33.95		−0.45		0.01		
		−0.14		0.02			
0.5	33.81		−0.43				
		−0.57					
0.6	33.24						

$h = 0.1$

$$\frac{dy}{dx}\Big|_{t=0.3} = \frac{1}{0.1}\left[\frac{(0.77+0.31)}{2} - \frac{1}{12}(0.02+0.01) + \frac{1}{60}(-0.27+0.02)\right]$$

$$= 5.333 \text{ cm/sec}$$

$$\frac{d^2y}{dx^2}\Big|_{t=0.3} = \frac{1}{0.01}\left[-0.46 - \frac{1}{12}(-0.01) + \frac{1}{90}(0.29)\right] = -45.594 \text{ cm/sec}^2$$

Q.

x	0	2	3	4	7	9
$y = f(x)$	4	26	58	112	466	922

$f'(6)$; max/minima

Sol.

x	y	Δ	Δ^2	Δ^3	Δ^4

$y(x) = 4 + (x-0)11 + (x-0)(x-2)7 + (x-0)(x-2)(x-3)1 + 0$

$y(x) = 4 + 11x + 7x(x-2) + x(x-2)(x-3)$

x	y	Δ	Δ^2		
$x_0 \to 0$	4 (y_0)				
		$\frac{26-4}{2-0} = 11$			
$x_1 - 2$	26		7		
		32		1	
3	58		11		0
		54		1	
4	112		16		0
		118		1	
7	466		22		
		228			
9	922				

$(x-0)(x-2)(x-3)\,1 + 0$

$y(x) = 4 + 11x + 7x(x-2) + x(x-2)(x-3)$

$\boxed{y(x) = x^3 + 2x^2 + 3x + 4}$

$y'(x) = 3x^2 + 4x + 3$

$y'(6) = 3 \times 36 + 4 \times 6 + 3 = \boxed{135}$

maxima/minima : $y'(x) = 0$

$3x^2 + 4x + 3 = 0$

root are imaginary

No maxima/minima value.