Double Integration

$$I = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

x range — a to b

y range — c to d

Trapezoidal Method

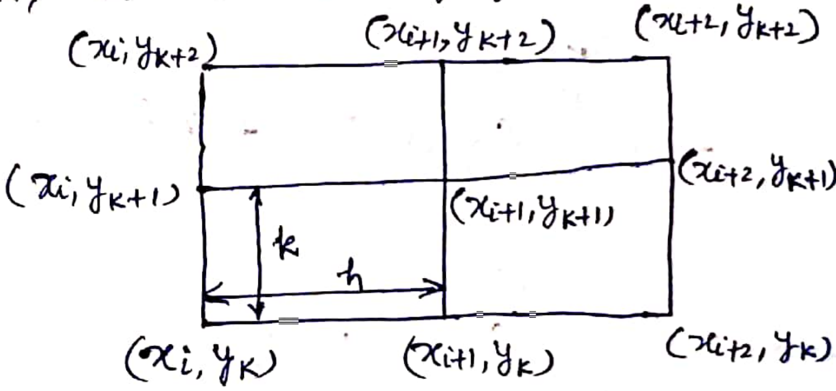
Take a simple case x range divided into '2' interval and y range divided into '2' interval.

$a = x_i$, $b = x_i + 2h$; h interval along x

$c = y_k$; $d = y_k + 2k$; k interval along y

$x_{i+1} = x_i + h$

$y_{k+1} = y_k + k$



$$I = \int_{y_k}^{y_{k+2}} \int_{x_i}^{x_{i+2}} f(x,y)\,dx\,dy$$

$$= \int_{y_k}^{y_{k+2}} \left[\frac{h}{2} \left(f(x_i,y) + 2 f(x_{i+1},y) + f(x_{i+2},y) \right) \right] dy$$

$$I = \frac{h}{2} \left[\underbrace{\int_{y_k}^{y_{k+2}} f(x_i,y)\,dy}_{= I_1} + 2 \underbrace{\int_{y_k}^{y_{k+2}} f(x_{i+1},y_k)\,dy}_{= I_2} + \underbrace{\int_{y_k}^{y_{k+2}} f(x_{i+2},y)\,dy}_{= I_3} \right]$$

$$I = \frac{h}{2} \left[I_1 + 2 I_2 + I_3 \right] \quad \text{— ①}$$

$$I_1 = \int_{y_K}^{y_{K+2}} f(x_{i\cancel{+1}}, y)\, dy$$

$$= \frac{k}{2}\left[f(x_{i\cancel{+1}}, y_K) + 2f(x_{i\cancel{+1}}, y_{K+1}) + f(x_{i\cancel{+1}}, y_{K+2}) \right]$$

$$= \frac{k}{2}\left[f(x_i, y_K) + 2f(x_i, y_{K+1}) + f(x_i, y_{K+2}) \right] \quad \text{—} \ (2)$$

$$I_2 = \int_{y_K}^{y_{K+2}} f(x_{i+1}, y)\, dy$$

$$= \frac{k}{2}\left[f(x_{i+1}, y_K) + 2f(x_{i+1}, y_{K+1}) + f(x_{i+1}, y_{K+2}) \right] \text{—} (3)$$

$$I_3 = \int_{y_K}^{y_{K+2}} f(x_{i+2}, y)\, dy$$

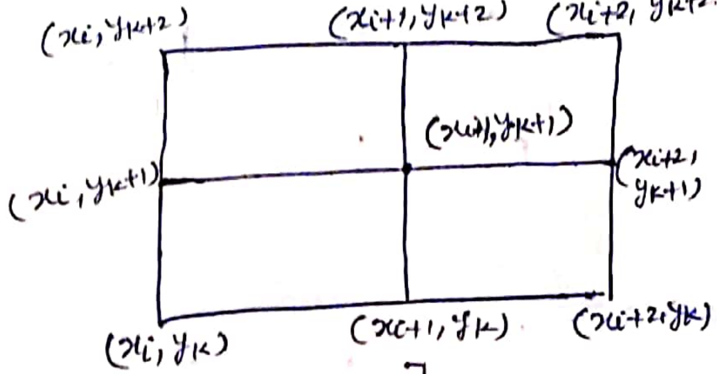
$$= \frac{k}{2}\left[f(x_{i+2}, y_K) + 2f(x_{i+2}, y_{K+1}) + f(x_{i+2}, y_{K+2}) \right] \text{—} (4)$$

from eq (1), (2), (3) & (4)

$$I = \frac{hk}{4}\Big[f(x_i, y_K) + 2f(x_i, y_{K+1}) + f(x_i, y_{K+2}) +$$
$$2f(x_{i+1}, y_K) + 4f(x_{i+1}, y_{K+1}) + 2f(x_{i+1}, y_{K+2}) +$$
$$f(x_{i+2}, y_K) + 2f(x_{i+2}, y_{K+1}) + f(x_{i+2}, y_{K+2}) \Big]$$

$$\boxed{\begin{aligned} I = \frac{hk}{4}\Big[&\{ f(x_i, y_K) + f(x_{i+2}, y_K) + f(x_{i+2}, y_{K+2}) + f(x_i, y_{K+2}) \} \\ &+ 2\{ f(x_{i+1}, y_K) + f(x_{i+2}, y_{K+1}) + f(x_{i+1}, y_{K+2}) + f(x_i, y_{K+1}) \} \\ &+ 4f(x_{i+1}, y_{K+1}) \Big] \end{aligned}}$$

Simpson's One third Method



$$I = \int_{y_k}^{y_{k+2}} \int_{x_i}^{x_{i+2}} f(x, y)\, dx\, dy$$

$$I = \int_{y_k}^{y_{k+2}} \frac{h}{3}\left[f(x_i, y) + 4 f(x_{i+1}, y) + f(x_{i+2}, y) \right] dy$$

$$I = \frac{h}{3}\left[\underbrace{\int_{y_k}^{y_{k+2}} f(x_i, y)\, dy}_{= I_1} + 4 \underbrace{\int_{y_k}^{y_{k+2}} f(x_{i+1}, y)\, dy}_{= I_2} + \underbrace{\int_{y_k}^{y_{k+2}} f(x_{i+2}, y)\, dy}_{= I_3} \right]$$

$$I = \frac{h}{3}\left[I_1 + 4 I_2 + I_3 \right] \quad \text{—— (1)}$$

$$I_1 = \int_{y_k}^{y_{k+2}} f(x_i, y)\, dy = \frac{k}{3}\left[f(x_i, y_k) + 4 f(x_i, y_{k+1}) + f(x_i, y_{k+2}) \right] \quad \text{— (2)}$$

$$I_2 = \int_{y_k}^{y_{k+2}} f(x_{i+1}, y)\, dy = \frac{k}{3}\left[f(x_{i+1}, y_k) + 4 f(x_{i+1}, y_{k+1}) + f(x_{i+1}, y_{k+2}) \right] \quad \text{— (3)}$$

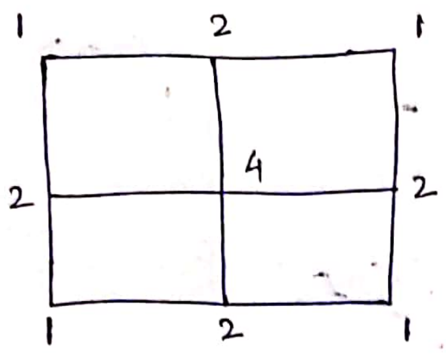
$$I_3 = \int_{y_k}^{y_{k+2}} f(x_{i+2}, y)\, dy = \frac{k}{3}\left[f(x_{i+2}, y_k) + 4 f(x_{i+2}, y_{k+1}) + f(x_{i+2}, y_{k+2}) \right] \quad \text{— (4)}$$

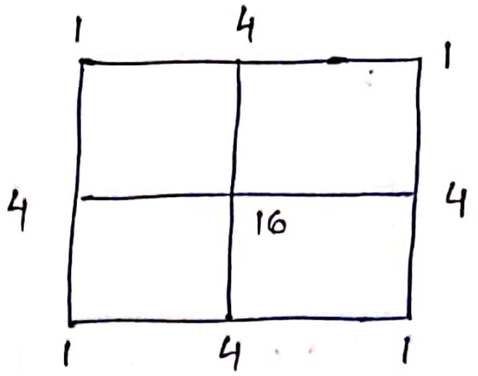
from equation (1 – 4)

$$I = \frac{hk}{9}\Big[f(x_i, y_k) + 4 f(x_i, y_{k+1}) + f(x_i, y_{k+2}) + 4 f(x_{i+1}, y_k) + 16 f(x_{i+1}, y_{k+1}) + 4 f(x_{i+1}, y_{k+2}) + f(x_{i+2}, y_k) + 4 f(x_{i+2}, y_{k+1}) + f(x_{i+2}, y_{k+2}) \Big]$$

$$I = \frac{hk}{9}\Big[f(x_i, y_k) + f(x_{i+2}, y_k) + f(x_{i+2}, y_{k+2}) + f(x_i, y_{k+2}) + 4\{ f(x_{i+1}, y_k) + f(x_{i+2}, y_{k+1}) + f(x_{i+1}, y_{k+2}) + f(x_i, y_{k+1}) \} + 16 f(x_{i+1}, y_{k+1}) \Big]$$

Trapezoidal $\underline{p = q = 2}$ Simpson one third





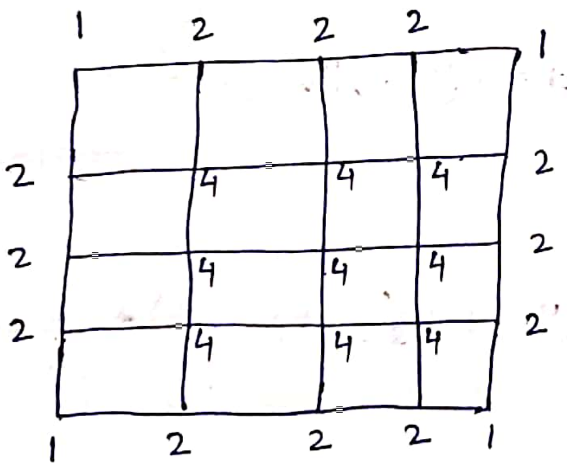
$p = q =$ # of division along x- & y axes

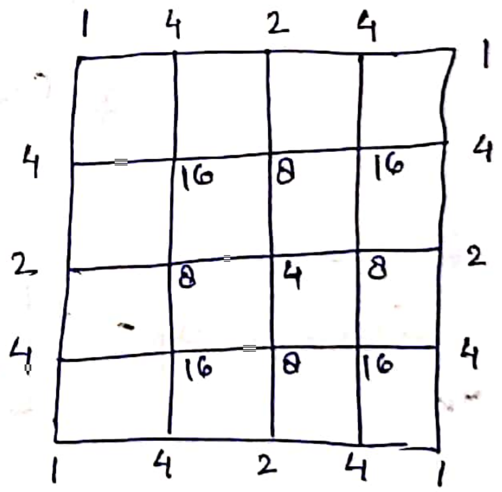
* Formula of double integration depends on no. of interval along x & y axis.

* Formula of double integration depends on methods of integration

Formula of Double integration when $p = q = 4$

<u>Trapezoidal Method</u> <u>Simpson one third Method</u>





* no. of division along x- & y-axis (i.e. p & q) should be according to method of integration.

$$I = \int_{2.0}^{2.6} \int_{4.0}^{4.4} \frac{dx\,dy}{(xy)}$$

$m = n = 2$

$h = 0.2, \quad k = 0.3$

Simpson 1/3 Method

y \ x	4.0	4.2	4.4
2.0	1 0.125	4 0.119	1 0.1136
2.3	4 0.1087	16 0.1035	4 0.0980
2.6	1 0.0962	4 0.0916	1 0.0874

$\frac{hk}{9}\left[\cancel{f(2,4)+f(2.0,4.4)+f(2.6,4.}\right.$

$$\frac{hk}{9}\Big[\{f(4,2.0)+f(4.4,2.0)+f(4.4,2.6)+f(4.0,2.6)\}+4\{f(4.2,2.0)+f(f(4.4,2.3)+f(4.2,2.6)+f(4.0,2.3)\}+16f(4.2,2.3)\Big]$$

$= 0.025$

$$\frac{hk}{9}\left[1(0.4222)+4(0.4181)+16(0.1035)\right] = \frac{0.2\times0.3}{9}\left[3.7506\right] = 0.025$$

$\boxed{\text{Do it for Trapezoidal Method}}$

x	4.0	4.2	4.4
2.0	1	2	1
2.3	2	4	2
2.6	1	2	1

repeated term

$$\frac{hk}{4}\Big[f(4.0,2.0)+f(4.4,2.0)+f(4.4,2.6)+f(4.0,2.6)+2\{f(4.2,2.0)+f(4.4,2.3)+f(4.2,2.6)+f(4.0,2.3)\}+4f(4.2,2.3)\Big]$$

$$\frac{hk}{4}[\ ] = \frac{0.2\times0.3}{4}\left[1(0.4222)+2(0.4181)+4(0.1035)\right]$$

$$= \frac{0.06}{4}\left[1.6724\right] = \underline{0.0251}$$

$$f(x,y) = e^{x+y} \; ; \quad h = k = 0.5$$

Example:- Trapezoidal Rule:-

$y \backslash x$	0.0	0.5	1.0
0.0	1 1 1	1.6487 4 2	2.7183 1 1
0.5	1.6487 4 2	2.7183 16 4	4.4817 2 4
1.0	2.7183 1 1	4.4817 2 4	7.3851 1 1

$$I = \frac{hk}{4}\Big[(1 + 2.7183 + 7.3851 + 2.7183) + 2(4.4817 + 4.4817 + 1.6487 + 1.6487) + 4(2.7183)\Big] = \underline{3.0763}$$

$$I = \frac{hk}{9}\Big[(1 + 2.7183 + 7.3851 + 2.7183) + 4(4.4817 + 4.4817 + 1.6487 + 1.6487) + 16(2.7183)\Big]$$

$$= \underline{2.9544}$$

Calculate the same for choosing $h = 0.25$, $k = 0.25$

$n = 4$

$y \backslash x$	x_0	x_0+h	x_0+2h	x_0+3h	x_0+4h
y_0	1 1	4 2	2 2	4 2	1 1
y_0+h	4 2	16 4	8 4	4 16	4 2
y_0+2h	2 2	8 4	4 4	4 8	2 2
y_0+3h	4 2	16 4	4 8	4 16	4 2
y_0+4h	1 1	4 2	2 2	2 4	1 1