8

NUMERICAL DIFFERENTIATION & INTEGRATION

8.1. NUMERICAL DIFFERENTIATION

It is the process of calculating the value of the derivative of a function at some assigned value of x from the given set of values (x_i, y_i). To compute dy/dx, we first replace the exact relation $y = f(x)$ by the best interpolating polynomial $y = \phi(x)$ and then differentiate the latter as many times as we desire. The choice of the interpolation formula to be used, will depend on the assigned value of x at which dy/dx is desired.

If the values of x are equi-spaced and dy/dx is required near the beginning of the table, we employ Newton's forward formula. If it is required near the end of the table, we use Newton's backward formula. For values near the middle of the table, dy/dx is calculated by means of Stirling's or Bessel's formula.

If the values of x are not equi-spaced, we use Newton's divided difference formula to represent the function.

Hence corresponding to each of the interpolation formulae, we can derive a formula for finding the derivative.

Obs. While using these formulae, it must be observed that the table of values defines the function at these points only and does not completely define the function and the function may not be differentiable at all. As such, *the process of numerical differentiation should be used only if the tabulated values are such that the differences of some order are constants.* Otherwise, errors are bound to creep in which go on increasing as derivatives of higher order are found. This is due to the fact that the difference between $f(x)$ and the approximating polynomial $\phi(x)$ may be small at the data points but $f'(x) - \phi'(x)$ may be large.

8.2. FORMULAE FOR DERIVATIVES

Consider the function $y = f(x)$ which is tabulated for the values $x_i (= x_0 + ih)$, $i = 0, 1, 2, \ldots n$.

(1) Derivatives using forward difference formula. Newton's forward interpolation formula (p. 152) is

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots$$

Differentiating both sides w.r.t. p, we have

$$\frac{dy}{dp} = \Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \dots$$

Since $p = \frac{(x-x_0)}{h}$, therefore $\frac{dp}{dx} = \frac{1}{h}$.

Now
$$\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \frac{4p^3-18p^2+22p-6}{4!}\Delta^4 y_0 + \dots\right] \quad \dots(1)$$

At $x = x_0$, $p = 0$. Hence putting $p = 0$;

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 + \dots\right] \quad \dots(2)$$

Again differentiating (1) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dp}\left(\frac{dy}{dp}\right)\frac{dp}{dx}$$

$$= \frac{1}{h}\left[\frac{2}{2!}\Delta^2 y_0 + \frac{6p-6}{3!}\Delta^3 y_0 + \frac{12p^2-36p+22}{4!}\Delta^4 y_0 + \dots\right]\frac{1}{h}$$

Putting $p = 0$, we obtain

$$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 + \dots\right] \quad \dots(3)$$

which are the same ... respectively.

(2) Derivatives using backward difference formula. Newton's backward interpolation formula (p. 153) is

$$y = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \ldots$$

Differentiating both sides w.r.t. p, we get

$$\frac{dy}{dx} = \nabla y_n + \frac{2p+1}{2!}\nabla^2 y_n + \frac{3p^2+6p+2}{3!}\nabla^3 y_n + \ldots$$

Since $p = \dfrac{x - x_n}{h}$, therefore $\dfrac{dp}{dx} = \dfrac{1}{h}$.

Now
$$\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2p+1}{2!}\nabla^2 y_n + \frac{3p^3+6p+2}{3!}\nabla^3 y_n + \ldots\right] \quad \ldots(5)$$

At $x = x_n$, $p = 0$. Hence putting $p = 0$, we get

$$\left(\frac{dy}{dx}\right)_{x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n + \frac{1}{6}\nabla^6 y_n + \ldots\right] \quad \ldots(6)$$

Again differentiating (5) w.r.t. x, we have

$$\frac{d^2y}{dx^2} = \frac{d}{dp}\left(\frac{dy}{dx}\right)\frac{dp}{dx} = \frac{1}{h}\left[\nabla^2 y_n + \frac{6p+6}{3!}\nabla^3 y_n + \frac{6p^2+18p+11}{12}\nabla^4 y_n + \ldots\right]$$

Putting $p = 0$, we obtain

$$\left(\frac{d^2y}{dx^2}\right)_{x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n + \ldots\right] \quad \ldots(7)$$

Guass forward Method for differentation

$$y_p(x) = y_0 + G_1 \Delta y_0 + G_2 \Delta^2 y_{-1} + G_3 \Delta^3 y_{-1} + G_4 \Delta^4 y_{-2} + \cdots$$

$$G_1 = p$$

$$G_2 = \frac{p(p-1)}{2!} = \frac{(p^2-p)}{2!}$$

$$G_3 = \frac{(p+1)p(p-1)}{3!} = \frac{(p^3-p)}{3!}$$

$$G_4 = \frac{(p+1)p(p-1)(p-2)}{4!} = \frac{(p^4-2p^3-p^2+2p)}{4!}$$

$$\frac{dy}{dx} = \frac{1}{h}\left(\frac{dy}{dp}\right) = \frac{1}{h}\left[\frac{d}{dp}\left(y_0 + G_1 \Delta y_0 + G_2 \Delta^2 y_{-1} + G_3 \Delta^3 y_{-1} + G_4 \Delta^4 y_{-2} + \cdots\right)\right.$$

$$\frac{dy}{dx} = \frac{1}{h}\frac{d}{dp}\left[y_0 + p\Delta y_0 + \frac{(p^2-p)}{2!}\Delta^2 y_0 + \frac{(p^3-p)}{3!}\Delta^3 y_{-1} + \frac{(p^4-2p^3-p^2+2p)}{4!}\Delta^4 y_{-2} + \cdots\right.$$

$$\boxed{\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{(2p-1)}{2!}\Delta^2 y_0 + \frac{(3p^2-1)}{3!}\Delta^3 y_{-1} + \frac{(4p^3-6p^2-2p+2)}{4!}\Delta^4 y_{-2} - \right.}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + \frac{\Delta^3 y_{-1}(6p)}{3!} + \frac{(12p^2-12p-2)}{4!}\Delta^4 y_{-2} + \cdots\right]}$$

at $x = x_0, \quad p = 0$

$$\boxed{\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_{-1}}{3} + \frac{\Delta^4 y_{-2}}{12} \cdots\right]}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 - \frac{\Delta^4 y_{-2}}{12} \cdots\cdots\right]}$$

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(3) Derivatives using central difference formulae. Stirling's formula (p. 161) is

$$y_p = y_0 + \frac{p}{1!}\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1^2)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{p^2(p^2-1^2)}{4!}\Delta^4 y_{-2} + \ldots$$

Differentiating both sides w.r.t. p, we get

$$\frac{dy}{dp} = \left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{2p}{2!}\Delta^2 y_{-1} + \frac{3p^2-1}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{4p^3-2p}{4!}\Delta^4 y_{-2} + \ldots$$

Since $p = \dfrac{x - x_0}{h}$, $\quad \therefore \quad \dfrac{dp}{dx} = \dfrac{1}{h}$.

$$\frac{d^2y}{dp^2} = \left[\Delta^2 y_{-1} + \frac{p}{2}\left(\Delta^3 y_{-1} + \Delta^3 y_{-2}\right) + \frac{(6p^2-1)}{12}\Delta^4 y_{-2}\right]$$

Now $$\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx} = \frac{1}{h}\left[\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + p\Delta^2 y_{-1} + \frac{3p^2-1}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{2p^3-p}{12}\Delta^4 y_{-2} + \ldots\right]$$

At $x = x_0$, $p = 0$. Hence putting $p = 0$, we get

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} - \frac{1}{6}\,\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} + \frac{1}{30}\,\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2} + \ldots\right] \quad \ldots(9)$$

Similarly $$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-3} - \ldots\right] \quad \ldots(10)$$

Example 8.1. *Given that*

x :	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y :	7.989	8.403	8.781	9.129	9.451	9.750	10.031,

find $\dfrac{dy}{dx}$ *and* $\dfrac{d^2y}{dx^2}$ *at* (*a*) $x = 1.1$ (*Madras B.E., 2003 S*) (*b*) $x = 1.6$.

Sol. (*a*) The difference table is :

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
1.0	7.989						
		0.414					
1.1	8.403		– 0.036				
		0.378		0.006			
1.2	8.781		– 0.030		– 0.002		
		0.348		0.004		0.002	
1.3	9.129		– 0.026		0.000		– 0.003
		0.322		0.004		– 0.001	
1.4	9.451		– 0.023		– 0.001		
		0.299		0.005			
1.5	9.750		– 0.018				
		0.281					
1.6	10.031						

We have

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 + \ldots\right] \qquad \ldots(i)$$

and

$$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 - \ldots\right] \qquad \ldots(ii)$$

Here $h = 0.1$, $x_0 = 1.1$, $\Delta y_0 = 0.378$, $\Delta^2 y_0 = -0.03$ etc.

Substituting these values in (*i*) and (*ii*), we get

$$\left(\frac{dy}{dx}\right)_{1.1} = \frac{1}{0.1}\left[0.378 - \frac{1}{2}(-0.03) + \frac{1}{3}(0.004) - \frac{1}{4}(0) + \frac{1}{5}(-0.001) - \frac{1}{6}(-0.003)\right] = 3.946$$

$$\left(\frac{d^2y}{dx^2}\right)_{1.1} = \frac{1}{(0.1)^2}\left[-0.03 - (0.004) + \frac{11}{12}(0) - \frac{5}{6}(-0.001) + \frac{137}{180}(-0.003)\right] = -3.545.$$

(*b*) We use the above difference table and the backward difference operator ∇ instead of Δ.

$$\left(\frac{dy}{dx}\right)_{x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n + \frac{1}{6}\nabla^6 y_n + \ldots\right] \qquad \ldots(i)$$

and

$$\left(\frac{d^2y}{dx^2}\right)_{x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n + \ldots\right] \qquad \ldots(ii)$$

Here $h = 0.1$, $x_n = 1.6$, $\nabla y_n = 0.281$, $\nabla^2 y_n = -0.018$ etc.

Putting these values in (*i*) and (*ii*), we get

$$\left(\frac{dy}{dx}\right)_{1.6} = \frac{1}{0.1}\left[0.281 + \frac{1}{2}(-0.018) + \frac{1}{3}(0.005) + \frac{1}{4}(-0.001) + \frac{1}{5}(-0.001) + \frac{1}{6}(-0.003)\right] = 2.727$$

$$\left(\frac{d^2y}{dx^2}\right)_{1.6} = \frac{1}{(0.1)^2}\left[-0.018 + 0.005 + \frac{11}{12}(-0.001) + \frac{5}{6}(-0.001) + \frac{137}{180}(-0.003)\right] = -1.703.$$

Example 8.2. *A slider in a machine moves along a fixed straight rod. Its distance x cm. along the rod is given below for various values of the time t seconds. Find the velocity of the slider and its acceleration when t = 0.3 second.*

$t =$	0	0.1	0.2	0.3	0.4	0.5	0.6
$x =$	30.13	31.62	32.87	33.64	33.95	33.81	33.24

Sol. The difference table is :

t	x	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
0	30.13						
		1.49					
0.1	31.62		−0.24				
		1.25 1/2		−0.24 1/12		1/60	] : $\frac{dy}{dx}$
0.2	32.87		−0.48		0.26		
		0.77		0.02		−0.27	
0.3	33.64		−0.46		−0.01		0.29
		0.31		0.01		0.02	
0.4	33.95		−0.45		0.01		
		−0.14		0.02			
0.5	33.81		−0.43		−1/12		1/90] $\frac{d^2y}{dx^2}$
		−0.57					
0.6	33.24						

As the derivatives are required near the middle of the table, we use Stirling's formulae :

$$\left(\frac{dx}{dt}\right)_{t_0} = \frac{1}{h}\left(\frac{\Delta x_0 + \Delta x_{-1}}{2}\right) - \frac{1}{6}\left(\frac{\Delta^3 x_{-1} + \Delta^3 x_{-2}}{2}\right) + \frac{1}{30}\left(\frac{\Delta^5 x_{-2} + \Delta^5 x_{-3}}{2}\right) + \ldots \quad \ldots(i)$$

$$\left(\frac{d^2x}{dt^2}\right)_{t_0} = \frac{1}{h^2}\left[\Delta^2 x_{-1} - \frac{1}{12}\Delta^4 x_{-2} + \frac{1}{90}\Delta^6 x_{-3} \ldots\right] \quad \ldots(ii)$$

Here $h = 0.1$, $t_0 = 0.3$, $\Delta x_0 = 0.31$, $\Delta x_{-1} = 0.77$, $\Delta^2 x_{-1} = -0.46$ etc.

Putting these values in (*i*) and (*ii*), we get

$$\left(\frac{dx}{dt}\right)_{0.3} = \frac{1}{0.1}\left[\frac{0.31 + 0.77}{2} - \frac{1}{6}\left(\frac{0.01 + 0.02}{2}\right) + \frac{1}{30}\left(\frac{0.02 - 0.27}{2}\right) - \ldots\ldots\right] = 5.33$$

$$\left(\frac{d^2x}{dt^2}\right)_{0.3} = \frac{1}{(0.1)^2}\left[-0.46 - \frac{1}{12}(-0.01) + \frac{1}{90}(0.29) - \ldots\ldots\right] = -45.6$$

Hence the required velocity is 5.33 cm/sec and acceleration is – 45.6 cm/sec².

Example 8.3. *Using Bessel's formula, find f′(7.5) from the following table :*

x :	7.47	7.48	7.49	7.50	7.51	7.52	7.53
f(*x*) :	0.193	0.195	0.198	0.201	0.203	0.206	0.208

Sol. Taking $x_0 = 7.50$, $h = 0.1$, we have $p = \dfrac{x - x_0}{h} = \dfrac{x - 7.50}{0.01}$

The difference table is

x	p	y_p	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
7.47	– 3	0.193						
			0.002					
7.48	– 2	0.195		0.001				
			0.003		– 0.001			
7.49	– 1	0.198		0.000		0.000		
			0.003		– 0.001		0.003	
7.50	0	0.201		– 0.001		0.003		– 0.01
			0.002		0.002		– 0.007	
7.51	1	0.203		0.001		– 0.004		
			0.003		– 0.002			
7.52	2	0.206		– 0.001				
			0.002					
7.53	3	0.208						

Bessel's formula (p. 162) is

$$y_p = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\cdot\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{(p-\frac{1}{2})\,p\,(p-1)}{3!}\cdot\Delta^3 y_{-1}$$

$$+ \frac{(p+1)\,p\,(p-1)(p-2)}{4!}\cdot\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \frac{(p-\frac{1}{2})(p+1)\,p\,(p-1)(p-2)}{5!}\cdot\Delta^5 y_{-2}$$

$$+ \frac{(p+2)(p+1)\,p\,(p-1)(p-2)(p-3)}{6!}\cdot\frac{\Delta^6 y_{-3} + \Delta^6 y_{-2}}{2} + \ldots\ldots \qquad \ldots(i)$$

Since $\quad p = \dfrac{x - x_0}{h}$, $\therefore \quad \dfrac{dp}{dx} = \dfrac{1}{h}$ and $\dfrac{dy}{dx} = \dfrac{dy}{dp}\cdot\dfrac{dp}{dx} = \dfrac{1}{h}\dfrac{dy}{dp}$

Differentiating (*i*) w.r.t. p and putting $p = 0$, we get

$$\left(\frac{dy}{dx}\right) = \frac{1}{h}\left(\frac{dy}{dp}\right)_{p=0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{4}(\Delta^2 y_{-1} + \Delta^2 y_0) + \frac{1}{12}\Delta^3 y_{-1} + \frac{1}{24}(\Delta^4 y_{-2} + \Delta^4 y_{-1}) - \frac{1}{120}\Delta^5 y_{-2} - \frac{1}{240}(\Delta^6 y_{-3} + \Delta^6 y_{-2})\right]$$

$$\left(\frac{dy}{dx}\right)_{7.5} = \frac{1}{0.01}\left[0.002 - \frac{1}{4}(-0.001 + 0.001) + \frac{1}{12}(0.002)\right.$$
$$\left. + \frac{1}{24}(-0.004 + 0.003) - \frac{1}{120}(-0.007) - \frac{1}{240}(-0.010 + 0)\right]$$
$$[\because \quad \Delta^6 y_{-2} = 0$$
$$= 0.2 + 0 + 0.01666 - 0.0416 + 0.00583 + 0.00416 = 0.223.$$

Example 8.4. *Find f '(10) from the following data :*

x :	*3*	*5*	*11*	*27*	*34*
f(x) :	*- 13*	*23*	*899*	*17315*	*35606.*

Sol. As the values of x are not equi-spaced, we shall use Newton's divided difference formula. The divided difference table is

x	$f(x)$	*1st div. diff.*	*2nd div. diff.*	*3rd div. diff.*	*4th div. diff.*
3	– 13				
		18			
5	23		16		
		146		0.998	
11	899		39.96		0.0002
		1025		1.003	
27	17315		69.04		
		2613			
34	35606				

Fifth differences being zero, Newton's divided difference formula is

$$f(x) = f(x_0) + (x - x_0)f(x_0, x_1) + (x - x_0)(x - x_1)f(x_0, x_1, x_2)$$
$$+ (x - x_0)(x - x_1)(x - x_2)f(x_0, x_1, x_2, x_3) + (x - x_0)(x - x_1)$$
$$\times (x - x_2)(x - x_3)f(x_0, x_1, x_2, x_3, x_4)$$

Differentiating it w.r.t. x, we get

$$f'(x) = f(x_0, x_1) + (2x - x_0 - x_1)f(x_0, x_1, x_2)$$
$$+ [3x^2 - 2x(x_0 + x_1 + x_2) + x_0x_1 + x_1x_2 + x_2x_0)] \times f(x_0, x_1, x_2, x_3)$$
$$+ [4x^3 - 3x^2(x_0 + x_1 + x_2 + x_3) + 2x(x_0x_1 + x_1x_2 + x_2x_3 + x_3x_0 + x_1x_3 + x_0x_2)$$
$$- (x_0x_1x_2 + x_1x_2x_3 + x_2x_3x_0 + x_0x_1x_3)]\, f(x_0, x_1, x_2, x_3, x_4)$$

Putting $x_0 = 3, x_1 = 5, x_2 = 11, x_3 = 27$ and $x = 10$, we obtain

$$f'(0) = 18 + 12 \times 16 + 23 \times 0.998 - 426 \times 0.0002 = 232.869.$$

8.3. MAXIMA AND MINIMA OF A TABULATED FUNCTION

Newton's forward interpolation formula is

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots..$$

Differentiating it w.r.t. p, we get

$$\frac{dy}{dp} = \Delta y_0 + \frac{2p-1}{2}\Delta^2 y_0 + \frac{3p^2 - 6p + 2}{6}\Delta^3 y_0 + \ldots... \quad ...(1)$$

For maxima or minima, $dy/dp = 0$. Hence equating the righthand side of (1) to zero and retaining only upto third differences, we obtain

$$\Delta y_0 + \frac{2p-1}{2}\Delta^2 y_0 + \frac{3p^2-6p+2}{6}\Delta^3 y_0 = 0$$

i.e. $$(\tfrac{1}{2}\Delta^3 y_0)p^2 + (\Delta^2 y_0 - \Delta^3 y_0)p + (\Delta y_0 - \tfrac{1}{2}\Delta^2 y_0 + \tfrac{1}{3}\Delta^3 y_0) = 0.$$

Substituting the values of $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0$ from the difference table, we solve this quadratic for p. Then the corresponding values of x are given by $x = x_0 + ph$ at which y is maximum or minimum.

Example 8.5. *From the table below, for what value of x, y is minimum ? Also find this value of y.*

x :	3	4	5	6	7	8
y :	0.205	0.240	0.259	0.262	0.250	0.224.

Sol. The difference table is

x	y	Δ	Δ^2	Δ^3
3	0.205			
		0.035		
4	0.240		– 0.016	
		0.019		0.000
5	0.259		– 0.016	
		0.003		0.001
6	0.262		– 0.015	
		– 0.012		0.001
7	0.250		– 0.014	
		– 0.026		
8	0.224			

Taking $x_0 = 3$, we have $y_0 = 0.205$, $\Delta y_0 = 0.035$, $\Delta^2 y_0 = -0.016$ and $\Delta^3 y_0 = 0$.

$\therefore$ Newton's forward difference formula gives

$$y = 0.205 + p(0.035) + \frac{p(p-1)}{2}(-0.016) \qquad \text{...}(i)$$

Differentiating it w.r.t. p, we have

$$\frac{dy}{dp} = 0.035 + \frac{2p-1}{2}(-0.016)$$

For y to be minimum, $\quad dy/dp = 0 \quad \therefore \quad 0.035 - 0.008(2p-1) = 0$

which gives $p = 2.6875$

$\therefore \quad x = x_0 + ph = 3 + 2.6875 \times 1 = 5.6875.$

Hence y is minimum when $x = 5.6875$.

Putting $p = 2.6875$ in (*i*), the minimum value of y

$$= 0.205 + 2.6875 \times 0.035 + \tfrac{1}{2}(2.6875 \times 1.6875)(-0.016) = 0.2628.$$

All the fomula at one place (Numerical differentiation)

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{(2p-1)}{2!}\Delta^2 y_0 + \frac{(3p^2-6p+2)}{3!}\Delta^3 y_0 + \cdots\right] \quad (1)$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + \frac{(p-1)}{1}\Delta^3 y_0 + \left(\frac{6p^2-18p+11}{12}\right)\Delta^4 y_0 + \cdots\right] \quad (2)$$

forward Method

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{(2p+1)}{2!}\nabla^2 y_n + \frac{(3p^2+6p+2)}{3!}\nabla^3 y_n + \cdots\right] \quad (3)$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + (p+1)\nabla^3 y_n + \frac{(6p^2+18p+11)}{12}\nabla^4 y_n + \cdots\right] \quad (4)$$

~~forward~~ Backward difference Method

$$\frac{dy}{dx} = \frac{1}{h}\left[\frac{1}{2}(\Delta y_0 + \Delta y_{-1}) + p\,\Delta^2 y_{-1} + \frac{(3p^2-1)}{12}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) + \cdots\right] \quad (5)$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_{-1} + \frac{6p}{12}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) + \left(\frac{6p^2-1}{12}\right)\Delta^4 y_{-2} + \cdots\right] \quad (6)$$

Strilling formula (central diff)

un-equal spacing (Divided difference formula)

$$y_p = y_0 + (x-x_0)[x_0, x_1] + (x-x_0)(x-x_1)[x_0, x_1, x_2] + (x-x_0)(x-x_1)(x-x_2)[x_0, x_1, x_2, x_3] + \cdots$$

$$y_p = y_0 + (x-x_0)[x_0, x_1] + (x^2-(x_0+x_1)x + x_0x_1)[x_0, x_1, x_2] + \{x^3 - (x_0+x_1+x_2)x^2 + (x_0x_1 + x_1x_2 + x_0x_2)x - x_0x_1x_2\}[x_0, x_1, x_2, x_3] + \cdots$$

$$\boxed{\frac{dy_p}{dx} = [x_0, x_1] + \{2x - (x_0+x_1)\}[x_0, x_1, x_2] + \{3x^2 - 2(x_0+x_1+x_2)x + (x_0x_1 + x_1x_2 + x_0x_2)\}[x_0, x_1, x_2, x_3] + \cdots \quad (7)}$$

$$\boxed{\frac{d^2y}{dx^2} = 2[x_0, x_1, x_2] + \{6x - 2(x_0+x_1+x_2)\}[x_0, x_1, x_2, x_3] + \cdots \quad (8)}$$

Equation (1), (3), and (5) gives position maxima & minima

Question :- (1)

x	1.0	1.1	1.2	1.3	1.4
$V(y)$	43.1	47.7	52.1	56.4	60.8

$\left(\frac{dy}{dx}\right)_{1.1}$

Ans:-

49.917

equal differences, use the newton divided difference formula directly. (see the equation NO. (6 & 7))

Example:- (Maxima & minima finding)

* you have to see in which region you have (peak / dip).
* Accordling apply the numerical differential formula. ($\frac{dy}{dp} = 0$).
* Calculate $\frac{dy}{dp} = 0 \Rightarrow p$ will give the desired result.

* for non-equispaced data; use Newton's finite difference method.

Question:- Find minimum value of f(x) which has values

x	0	2	4	6
y	3	3	11	27

$$\frac{dy}{dp} = 0 = \left[\Delta y_0 + \frac{(2p-1)}{2!} \Delta^2 y_0 + \frac{(3p^2-2)}{3!} \Delta^3 y_0 + \dots \right] = 0$$

$\Rightarrow \quad p = \frac{1}{2} \qquad x = x_0 + ph = 0 + \frac{1}{2} \times 2 = 1$

$p = \frac{1}{2}; x = 1$. so

$$y = y_0 + p \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \dots$$

$= 2.$ ans

* if continuous increasing / decreasing (no dip / peak), roots will be imaginary.

(Two way to calculate the differentiation depending upon point of reference)

from following table of values of x & y, obtain

$\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ for $x = 1.2,\ 2.0$ and 1.6

x	1.0	1.2	1.4	1.6	1.8	2.0	2.2
y	2.7183	3.3201	4.0552	4.9530	6.0496	7.3891	9.0250

Solution :-

Difference table

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
1.0	2.7183						
		0.6018					
1.2	3.3201		0.1333	(1)			
		0.7351		0.0294			
1.4	4.0552		0.1627		0.0067		
		0.8978		0.0361		0.0013	
1.6	4.9530		0.1988		0.0080		0.0001
		1.0966		0.0441		0.0014	
1.8	6.0496		0.2429		0.0094		
		1.3395		0.0535			
2.0	7.3891		0.2964				
		1.6359					
2.2	9.0250						

(Newton forward method)

choose :- $x_0 = 1.0$, $h = 0.2$, $p = 1$, $x = 1.2$ use eqn (i)

$$\left(\frac{dy}{dx}\right)_{x=1.2} = \frac{1}{0.2}\left[0.6018 + \frac{0.1333}{2} + \frac{1}{6}\times(0.0294) + \frac{1}{12}(0.0067) - \frac{1}{20}(0.0013)\right]$$

$$= 3.3205$$

choose $x_0 = 1.2$ then $p = 0$; $\left.\frac{dy}{dx}\right| = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \frac{\Delta^5 y_0}{5} - \cdots\right]$ 6

here $\Delta y_0 = 0.7351$, $\Delta^2 y_0 = 0.1627$, $\Delta^3 = 0.0361$, ...

$\Delta^4 y_0 = 0.0080$, $\Delta^5 y_0 = 0.0014$

$$= \frac{0.6640}{0.2} = 3.3203$$

Newton Backward Method

$x = 2.0$

Choose :- $y_n = 2.2$, $h = 0.2$, $x = x_n + ph$, $\Rightarrow p = -1$

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n - \frac{1}{2}\nabla^2 y_n - \frac{1}{6}\nabla^3 y_n - - - \cdot\right] = 7.3896$$

here $\nabla y_n = 1.6359$
$\nabla^2 y_n = 0.2964$
$\nabla^3 y_n = 0.0535$
$\nabla^4 y_n = 0.0094$

Choose:- $x_n = 2.0$, $p = 0$ so eq^n reduces to.
$h = 0.20$

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \frac{\nabla^4 y_n}{4} + - - - - -\right] = 7.3896$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + - - - - - -\right] =$$

here $\nabla y_n = 1.33395$
$\nabla^2 y_n = 0.2429$
$\nabla^3 y_n = 0.0441$
$\nabla^4 y_n = 0.0080$

Central difference formula :-

Choose:- $x = x_0$, $p = 0$, $\frac{dy}{dx} = \frac{1}{h}\left[\frac{1}{2}(\Delta y_0 + \Delta y_{-1}) + \frac{1}{12}(\Delta^2 y_{-1} + \Delta^3 y_{-2}) + \frac{1}{60}(- - - -)\right]$

$h = 0.20$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_{-1} + \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-3}\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{0.04}\left[0.1988 - \frac{0.0080}{12} + \frac{0.0001}{90}\right]$$
$= 4.9533$

$$\boxed{\frac{dy}{dx} = \frac{1}{h}[c_1, c_2, c_3] \quad \begin{cases} c_1 = \frac{1}{2} \\ c_2 = -\frac{1}{12} \\ c_3 = \frac{1}{60} \end{cases} \qquad \frac{d^2y}{dx^2} = \frac{1}{h^2}[c_1, c_2, c_3] = \begin{cases} c_1 = 1, \\ c_2 = -\frac{1}{12} \\ c_3 = \frac{1}{90} \end{cases}}$$

$$\frac{dy}{dx} = \frac{1}{0.20}\left[\frac{1}{2}(0.8978 + 1.0966) - \frac{1}{12}(0.0361 + 0.0441) + \frac{1}{60}(0.0013 + 0.0014)\right]$$
$$= \frac{1}{0.20}\left[0.9972 - 0.00683 + 0.000045\right]$$
$$= 4.952075$$

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