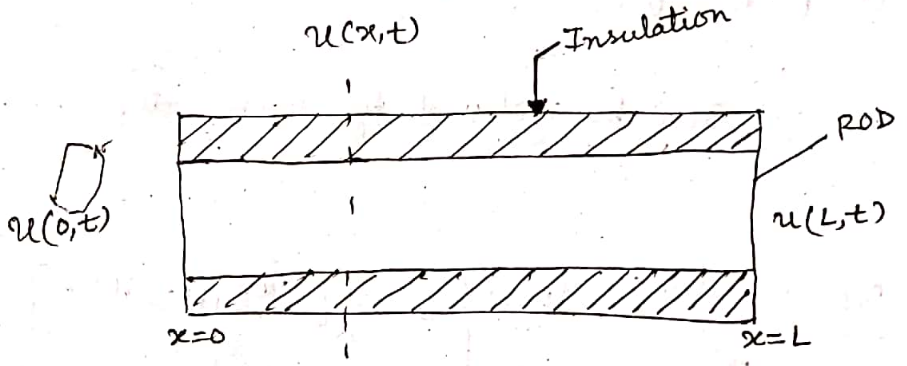
Parabolic equation

elliptic equations are time independent. They describes only problem that are time independent. Such problem are known as steady state problems.

Parabolic equations for which $B^2 - 4AC = 0$ describes the problem that depend upon space & time variables.

A famous parabolic type of equation is the study of heat flow in one-dimentional direction in an insulated rod.

Such problems are governed by both initial & boundary conditions.



$u(x,t)$ represents the temp of rod at any position x at time t.

Heat equation

$$K \frac{\partial^2 u}{\partial x^2} = c\rho \frac{\partial u}{\partial t} \quad — (1)$$

K = Thermal conductivity

c = specific heat of material

ρ = density of material.

$$\frac{K}{c\rho} \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad — (2)$$

let $\boxed{\frac{K}{c\rho} = \alpha^2} \quad — (3)$

equation (2) (heat equation) can be written as.

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

$$\boxed{\alpha^2 u_{xx} = u_t(x,t)} \quad -(4)$$

initial condition

$$u(x,0) = u(x), \quad 0 \le x \le L$$

temperature at all points along the rod.

Boundary conditions (

$$\left.\begin{array}{l} u(0,t) = c_1 \\ u(L,t) = c_2 \end{array}\right\} \quad 0 \le t \le \infty$$

Assuming that temp at both ends points remain constant with time.

equation (4) can be solved by using difference formula

$$\alpha^2 \left[\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} \right] = \frac{1}{k}\left[u_{i,j+1} - u_{i,j} \right]$$

$$\frac{k\alpha^2}{h^2}\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j} \right] = u_{i,j+1} - u_{i,j}$$

let $\boxed{r = \frac{k\alpha^2}{h^2}}$ $-(5)$

$$r\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j} \right] = u_{i,j+1} - u_{i,j}$$

$$\boxed{u_{i,j+1} = r\, u_{i+1,j} + (1-2r)\, u_{i,j} + r\, u_{i-1,j}} \quad -(6)$$

This formula is called Schmidt explicit formula.

for given "h" if choose "k" such that

$$1-2r=0$$
$$r=\frac{1}{2}$$

under this condition.

$$r=\frac{k\alpha^2}{h^2}=\frac{1}{2}$$

$$\boxed{k=\frac{h^2}{2\alpha^2}}$$

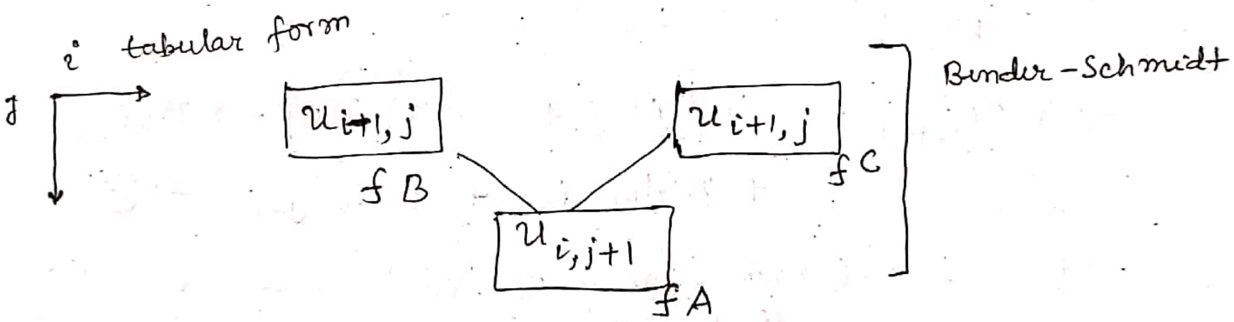
equation (6) reduces to.

$$\boxed{u_{i,j+1}=\frac{1}{2}\left(u_{i+1,j}+u_{i-1,j}\right)} \quad (7)$$

This is called Bender-Schmidth recurrence formula.

equation (7) is valid (stable) if and only if

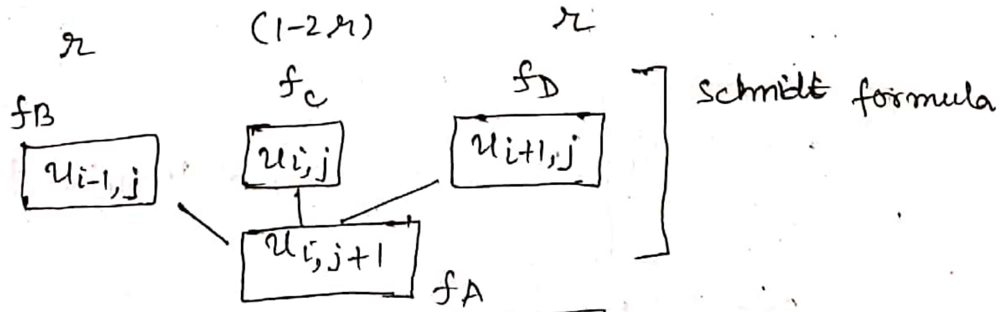
$$0\le r\le\frac{1}{2}$$



equation 7 can be represented as above.

$$\boxed{f_A=\left(\frac{f_B+f_C}{2}\right)}$$

similarly equation (6) can be written as



$$\boxed{f_A=rf_B+(1-2r)f_c+rf_D}$$

Problem of this formula is its lengthy computation process. Crank-Nicolson proposed the another method by replacing u_{xx} - by average value on the j^{th} and $(j+1)^{th}$ rows.

equation (4) can be written as

$$\frac{\alpha^2}{2}\left[\frac{(u_{i+1,j}-2u_{i,j}+u_{i-1,j})}{h^2}+\frac{(u_{i+1,j+1}-2u_{i,j+1}+u_{i-1,j+1})}{h^2}\right]$$

$$=\frac{1}{k}\left[u_{i,j+1}-u_{i,j}\right]$$

$$\underbrace{\frac{k\alpha^2}{h^2}}_{r}\left[u_{i+1,j}-2u_{i,j}+u_{i-1,j}+u_{i+1,j+1}-2u_{i,j+1}+u_{i-1,j+1}\right]$$

$$=2u_{i,j+1}-2u_{i,j}$$

let $r=\frac{k\alpha^2}{h^2}$

$$2(1+r)\,u_{i,j+1}=r\,u_{i+1,j}+2(1-r)\,u_{i,j}+r\,u_{i-1,j}+r\,u_{i+1,j+1}+r\,u_{i-1,j+1}\quad -(8)$$

if $r=1$ (choose value of "k" for given h such that $r=1$)

$$\frac{k\alpha^2}{h^2}=1\qquad \boxed{k=\frac{h^2}{\alpha^2}}\quad -(9)$$

for $r=1$, equation (8) can be written as,

$$4u_{i,j+1}=u_{i+1,j}+u_{i-1,j}+u_{i+1,j+1}+u_{i-1,j+1}$$

$$\boxed{u_{i,j+1}=\frac{1}{4}\left[u_{i+1,j}+u_{i-1,j}+u_{i+1,j+1}+u_{i-1,j+1}\right]}$$

Crank-Nicolson formula. $-(10)$

$$\begin{array}{ccccc}
f_C \; \boxed{u_{i-1,j}} & & & & \boxed{u_{i+1,j}} \; f_E \\
& & \boxed{u_{i,j+1}} & & \\
\underset{f_B}{\boxed{u_{i-1,j+1}}} & \text{———} & \underset{f_A}{} & \text{———} & \boxed{u_{i+1,j+1}} \; f_D
\end{array}$$

$$\boxed{f_A = \frac{(f_B + f_C + f_D + f_E)}{4}}$$

Example: solve $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$; given $u(0,t) = u(4,t) = 0$

$u(x,0) = x(4-x)$ assuming $h = k = 1$. Find the value of u upto $t = 5$.

solution :- $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$; $\alpha^2 = 1$

$h = 1, \; k = 1$

$r = \frac{k\alpha^2}{h^2} = 1$

$0 \le x \le 4$
$0 \le t \le 5$
$u(0,t) = u(4,t) = 0$
$u(x,0) = x(4-x)$

since $r = 1$ (Bendir Schmidt recurrence eqn can not used).

$r = 1$ $\quad (1-2\times1) = -1$ $\quad r$

$\underset{f_B}{\boxed{u_{i-1,j}}} \quad \boxed{u_{i,j}} \; f_C \quad \boxed{u_{i+1,j}} \; f_D$ } Schmidt formula

$\underset{f_A}{\boxed{u_{i,j+1}}}$

$\boxed{f_A = f_B - f_C + f_D}$ for $r = 1$

j	$i =$	0	1	2	3	4
	$x = ih$ / $t = jk$	0	1	2	3	4
0	0	0	3	4	3	0
1	1	0	1	2	1	0
2	2	0	1	0	1	0
3	3	0	-1	2	-1	0
4	4	0	3	-4	3	0
5	5	0	-7	10	-7	0

Example :- solve $u_{xx} = u_t$, subject to $u(0,t) = u(1,t) = 0$

$u(x,0) = \sin \pi x$; $0 < x < 1$ by Bender schmidt method for three time steps

Solution:- $u_{xx} = u_t$

$\alpha^2 = 1, \quad 0 < x < 1$

let $h = 0.25 = 1/4$

$r = \frac{k\alpha^2}{h^2}$; Bender schmidt method $r = \frac{1}{2}$

this give $\frac{k\alpha^2}{h^2} = \frac{1}{2}$. $k = \frac{h^2}{2\alpha^2} = \frac{1}{32}$

j \ i	$t = jk$ \ $x = ih$	0	1	2	3	4
		0	$\frac{1}{4} = 0.25$	$\frac{2}{4} = 0.5$	0.75	1.0
0	0	0	0.7071	1	0.7071	0
1	$\frac{1}{32}$	0	0.5	0.7071	0.5	0
2	$\frac{2}{32}$	0	0.3536	0.5	0.3536	0
3	$\frac{3}{32}$	0	0.25	0.3536	0.25	0

Scanned with CamScanner

Example :- Using Crank-Nicolson method, solve $u_{xx} = 16u_t$

$0 < x < 1$, given $u(x,0) = u(0,t) = 0$, $u(1,t) = 100t$.

Compute u for one step in t direction taking $h = 1/4$.

solution:- $u_{xx} = 16u_t$: $\frac{1}{16}u_{xx} = u_t$

$\alpha^2 = 1/16$, $h = 1/4$

for method $r = 1$, $r = \frac{k\alpha^2}{h^2}$

$k = \frac{h^2}{\alpha^2} = \frac{1}{16} \times 16 = 1$

$x = ih$: $i = \frac{x}{h} = \frac{1}{1/4} = 4$

j	i →	0	1	2	3	4
	$x = ih$ / $t = jk$	0	0.25	0.50	0.75	1.0
0	0	0	0	0	0	0
1	1	0	u_1	u_2	u_3	100

$$u_1 = \frac{(0+0+0+u_2)}{4} = \frac{u_2}{4}$$

$$u_2 = \frac{(u_1 + u_2)}{4}$$

$$u_3 = \frac{(100 + u_2)}{4}$$

solving these equation we get

$u_1 = 1.7857$

$u_2 = 7.1429$

$u_3 = 26.7857$

Hyperbolic Equation

example:- Vibration of structures such as buildings, beams & machines.

if take an example of vibrating string fixed at both ends, lateral displacement varies with time t and distance x along the string.

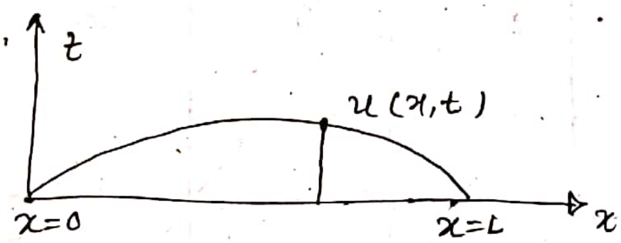
wave equation under this condition can be written as.

$$T\frac{\partial^2 u}{\partial x^2} = \rho\frac{\partial^2 u}{\partial t^2}$$ here $B^2-4AC>0$

T = tension in string

ρ = mass per unit length

Hyperbolic problems are also governed by boundary and initial conditions.



solution of Hyperbolic equations

$$T\frac{\partial^2 u}{\partial x^2} = \rho\frac{\partial^2 u}{\partial x^2}$$

$$\frac{T}{\rho}\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} \Rightarrow \boxed{\alpha^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}} \quad \alpha^2 = \frac{T}{\rho} \quad \text{—— (1)}$$

$$\alpha^2\left[\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}\right] = \left[\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2}\right]$$

$$\frac{k^2}{h^2}\alpha^2\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right] = u_{i,j+1} - 2u_{i,j} + u_{i,j-1}$$

let $\lambda = k/h$

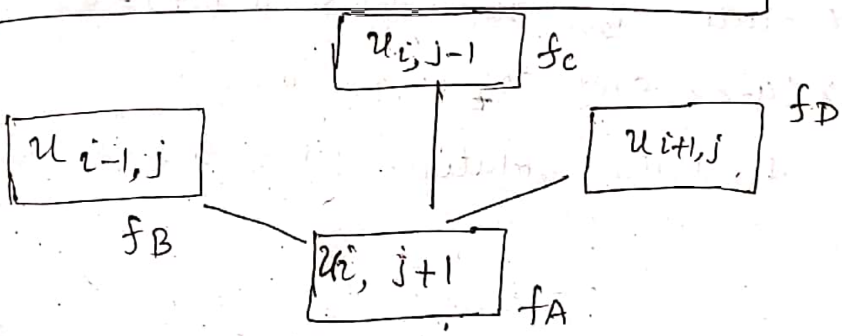
$$\lambda^2 a^2 \left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j} \right] = u_{i,j+1} - 2u_{i,j} + u_{i,j-1}$$

$$u_{i,j+1} = \lambda^2 a^2 u_{i+1,j} + 2(1-\lambda^2 a^2) u_{i,j} + \lambda^2 a^2 u_{i-1,j} - u_{i,j-1}$$

for $\lambda^2 a^2 = 1$; above equation can be rewritten as

$\boxed{\lambda a = 1}$

$$\boxed{u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1}} \quad — ②$$



$$\boxed{f_A = f_B + f_D - f_C}$$

however I^{st} f_C calculation is needed.

In order to find out first f_C. lets use equation ②

write eqn ② for $j=0$

$$u_{i,1} = u_{i+1,0} + u_{i-1,0} - u_{i,-1} \quad — ③$$

given Boundary condition $u_t(x,0)=0$ (Most common case)

$$\left[\frac{u_{i,j+1} - u_{i,j-1}}{2k} \right] = 0$$

$$u_{i,j+1} = u_{i,j-1} \; ; \text{ for } j=0$$

$$u_{i,0} = u_{i,-1} \quad — ④$$

Scanned with CamScanner

from equation (3) & (4) we can write

$$2u_{i,1} = u_{i+1,0} + u_{i-1,0}$$

$$\boxed{u_{i,1} = \frac{(u_{i+1,0} + u_{i-1,0})}{2}} \quad \text{—} \ (5)$$

used to create I^{st} row ($j=1$ & for all i)

equation (5) & (3) helps in calculation.

<u>Example</u>:- Evalute $u_{xx} = u_{tt}$, given $u(0,t) = u(4,t) = 0$

$u(x,0) = \frac{1}{2}x(4-x)$ and $u_t(x,0) = 0$,

Take $h=1$, find the solution upto 5 steps in t-direction

<u>solution</u>:-

$u_{xx} = u_{tt}$

$\alpha^2 = 1$, $h = 1$,

$u_t(x,0) = 0$; gives condition of eq^n 5 to calculate I^{st} row for $j=1$; $0 \le i \le 4$

$\lambda\alpha = 1$; $\frac{k}{h}\alpha = 1$; $k = \frac{h}{\alpha} = \frac{1}{1} \Rightarrow k = 1$

$0 \le x \le 4$

	j \ i		0	1	2	3	4
		$x = ih$ / $t = jk$	0	1	2	3	4
	0	0	0	1.5	2	1.5	0
using eq^n 5	1	1	0	$\frac{0+2}{2} = 1$	1.5	1.0	0
using eq^n 3	2	2	0	0	0	0	0
	3	3	0	-1	-1.5	-1.0	0
	4	4	0	-1.5	-2.0	-1.5	0
	5	5	0	-1.0	-1.5	-1.0	0