10

NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

10.1. (1) INTRODUCTION

A number of problems in science and technology can be formulated into differential equations. The analytical methods of solving differential equations are applicable only to a limited class of equations. Quite often differential equations appearing in physical problems do not belong to any of these familiar types and one is obliged to resort to numerical methods. These methods are of even greater importance when we realise that computing machines are now readily available which reduce numerical work considerably.

(2) Solution of a differential equation. The solution of an ordinary differential equation means finding an explicit expression for y in terms of a finite number of elementary functions of x. Such a solution of a differential equation is known as the *closed* or *finite form of solution*. In the absence of such a solution, we have recourse to numerical methods of solution.

Let us consider the first order differential equation

$$dy/dx = f(x, y), \text{ given } y(x_0) = y_0, \quad \text{...(1)}$$

to study the various numerical methods of solving such equations. In most of these methods, we replace the differential equation by a difference equation and then solve it. These methods yield solutions *either* as a power series in x from which the values of y can be found by direct substitution, *or* a set of values of x and y. The methods of Picard and Taylor series belong to the former class of solutions. In these methods, y in (1) is approximated by a truncated series, each term of which is a function of x. The information about the curve at one point is utilized and the solution is not iterated. As such, these are referred to as *single-step methods*. The methods of Euler, Runge-Kutta, Milne, Adams-Bashforth etc. belong to the latter class of solutions. In these methods, the next point on the curve is evaluated in short steps ahead, by performing iterations till sufficient accuracy is achieved. As such, these methods are called *step-by-step methods*.

Euler and Runga-Kutta methods are used for computing y over a limited range of x-values whereas Milne and Adams methods may be applied for finding y over a wider range of x-values. Therefore Milne and Adams methods require starting values which are found by Picard's Taylor series or Runge-Kutta methods.

(3) Initial and boundary conditions. An ordinary differential equation of the nth order is of the form

$$F(x, y, dy/dx, d^2y/dx^2, \ldots\ldots, d^ny/dx^n) = 0 \qquad \ldots(2)$$

Its general solution contains n arbitrary constants and is of the form

$$\phi(x, y, c_1, c_2, \ldots\ldots, c_n) = 0 \qquad \ldots(3)$$

To obtain its particular solution, n conditions must be given so that the constants c_1, c_2, ..., c_n can be determined. If these conditions are prescribed at one point only (say : x_0), then the differential equation together with the conditions constitute an *initial value problem* of the nth order. If the conditions are prescribed at two or more points, then the problem is termed as *boundary value problem*.

In this chapter, we shall first describe methods for solving initial value problems and then explain **finite difference method** and **shooting method** for solving boundary value problems.

10.2. PICARD'S METHOD

Consider the first order equation $\dfrac{dy}{dx} = f(x, y)$ ...(1)

It is required to find that particular solution of (1) which assumes the value y_0 when $x = x_0$. Integrating (1) between limits, we get

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)\, dx \quad \text{or} \quad y = y_0 + \int_{x_0}^{x} f(x, y)\, dx \qquad \ldots(2)$$

This is an integral equation equivalent to (1), for it contains the unknown y under the integral sign.

As a first approximation y_1 to the solution, we put $y = y_0$ in $f(x, y)$ and integrate (2), giving

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

For a second approximation y_2, we put $y = y_1$ in $f(x, y)$ and integrate (2), giving

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx.$$

Similarly, *a third approximation is*

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\, dx.$$

Continuing this process, we obtain $y_4, y_5, \ldots\ldots y_n$ where

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx$$

Hence this method gives a sequence of approximations $y_1, y_2, y_3 \ldots\ldots$ each giving a better result than the preceding one.

> **Obs.** Picard's method is of considerable theoretical value, but can be applied only to a limited class of equations in which the successive integrations can be performed easily. The method can be extended to simultaneous equations and equations of higher order (See § 10.11 and § 10.12).

Example 10.1. *Using Picard's process of successive approximations, obtain a solution upto the fifth approximation of the equation $dy/dx = y + x$, such that $y = 1$ when $x = 0$. Check your answer by finding the exact particular solution.*

Sol. (*i*) We have $y = 1 + \int_0^x (y + x)\, dx$.

First approximation. Put $y = 1$ in $y + x$, giving

$$y_1 = 1 + \int_0^x (1 + x)\, dx = 1 + x + x^2/2.$$

$x+1+x+\frac{x^2}{2}$.

Second approximation. Put $y = 1 + x + x^2/2$ in $y + x$, giving

$$y_2 = 1 + \int_0^x (1 + 2x + x^2/2)\, dx = 1 + x + x^2 + x^3/6.$$

Third approximation. Put $y = 1 + x + x^2 + x^3/6$ in $y + x$, giving

$$y_3 = 1 + \int_0^x (1 + 2x + x^2 + x^3/6)\, dx = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}.$$

Fourth approximation. Put $y = y_3$ in $y + x$, giving

$$y_4 = 1 + \int_0^x \left(1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}\right) dx$$

$$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}.$$

Fifth approximation. Put $y = y_4$ in $y + x$, giving

$$y_5 = 1 + \int_0^x \left(1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}\right) dx$$

$$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{720} \qquad \ldots(1)$$

(*ii*) Given equation :

$$\frac{dy}{dx} - y = x \text{ is a Leibnitzs linear in } x.$$

Its I.F. being e^{-x}, the solution is

$$ye^{-x} = \int xe^{-x} dx + c \qquad \text{[Integrate by parts}$$

$$= -xe^{-x} - \int (-e^{-x})\, dx + c = -xe^{-x} - e^{-x} + c$$

$\therefore$ $$y = ce^{x} - x - 1.$$

Since $y = 1$, when $x = 0$, $\therefore$ $c = 2$.

Thus the desired particular solution is

$$y = 2e^{x} - x - 1 \qquad \text{...(2)}$$

Or using the series : $e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots \infty$,

we get $$y = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{360} + \ldots\ldots \infty \qquad \text{...(3)}$$

Comparing (1) and (3), it is clear that (1), approximates to the exact particular solution (3) upto the term in x^5.

> **Obs.** At $x = 1$, the fourth approximation $y_4 = 3.433$ and the fifth approximation $y_5 = 3.434$ whereas exact value is 3.44.

Example 10.2. *Find the value of y for x = 0.1 by Picard's method, given that*

$$\frac{dy}{dx} = \frac{y - x}{y + x},\ y(0) = 1. \qquad \text{(P.T.U. B.E., 2002)}$$

Sol. We have $$y = 1 + \int_0^x \frac{y - x}{y + x}\, dx$$

First approximation. Put $y = 1$ in the integrand, giving

$$y_1 = 1 + \int_0^x \frac{1 - x}{1 + x}\, dx = 1 + \int_0^x \left(-1 + \frac{2}{1 + x}\right) dx$$

$$= 1 + [-x + 2 \log (1 + x)]_0^x = 1 - x + 2 \log (1 + x) \qquad \text{...}(i)$$

Second approximation. Put $y = 1 - x + 2 \log (1 + x)$ in the integrand, giving

$$y_2 = 1 + \int_0^x \frac{1 - x + 2 \log (1 + x) - x}{1 - x + 2 \log (1 + x) + x}\, dx$$

$$= 1 + \int_0^x \left[1 - \frac{2x}{1 + 2 \log (1 + x)}\right] dx$$

which is very difficult to integrate.

Hence we use the first approximation and taking $x = 0.1$ in (i) we obtain

$$y(0.1) = 1 - (0.1) + 2 \log 1.1 = 0.9828.$$

10.3. TAYLOR'S SERIES METHOD

Consider the first order equation $\frac{dy}{dx} = f(x, y)$ ...(1)

Differentiating (1), we have

$$\frac{d^2y}{dx^2} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} \quad i.e. \quad y'' = f_x + f_y f' \qquad ...(2)$$

Differentiating this successively, we can get y''', y^{iv} etc. Putting $x = x_0$ and $y = 0$, the values of $(y')_0$, $(y'')_0$, $(y''')_0$ can be obtained. Hence the Taylor's series

$$y = y_0 + (x - x_0)(y')_0 + \frac{(x-x_0)^2}{2!}(y'')_0 + \frac{(x-x_0)^3}{3!}(y''')_0 + \ldots \qquad ...(3)$$

gives the values of y for every value of x for which (3) converges.

On finding the value y_1 for $x = x_1$ from (3), y', y'' etc. can be evaluated at $x = x_1$ by means of (1), (2) etc. Then y can be expanded about $x = x_1$. In this way, the solution can be extended beyond the range of convergence of series (3).

> **Obs.** This is a single step method and works well so long as the successive derivatives can be calculated easily. If (x, y) is somewhat complicated and the calculation of higher order derivatives becomes tedious, then Taylor's method cannot be used gainfully. This is the main drawback of this method and therefore, has little application for computer programmes. However, it is useful for finding starting values for the application of powerful methods like Runga-Kutta, Milne and Adams-Bashforth which will be described in the subsequent sections.

Example 10.3. *Find by Taylor's series method, the values of y at x = 0.1 and x = 0.2 to five places of decimals from* $dy/dx = x^2y - 1$, $y(0) = 1$.

Sol. Here $(y_0) = 1$.

$\therefore$ Differentiating successively and substituting, we get

$y' = x^2y - 1,$ $\qquad (y')_0 = -1$

$y'' = 2xy + x^2y',$ $\qquad (y'')_0 = 0$ $\qquad 2xy + x^2\frac{dy}{dx}$

$y''' = 2y + 4xy' + x^2y'',$ $\qquad (y''')_0 = 2$

$y^{iv} = 6y' + 6xy'' + x^2y''',$ $\qquad (y^{iv})_0 = -6$, etc.

Putting these values in the Taylor's series, we have

$$y = 1 + x(-1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \ldots$$

$$= 1 - x + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

Hence $\qquad y(0.1) = 0.90033$ and $y(0.2) = 0.80227$.

Example 10.4. *Employ Taylor's method to obtain approximate value of y at x = 0.2 for the differential equation* $dy/dx = 2y + 3e^x$, $y(0) = 0$. *Compare the numerical solution obtained with the exact solution.* *(V.T.U., B.E., 2004)*

Sol. (*a*) We have $y' = 2y + 3e^x$; $y'(0) = 2y(0) + 3e^0 = 3$.

Example 13.1

Use the Taylor method to solve the equation

$$y' = x^2 + y^2$$

for $x = 0.25$ and $x = 0.5$ given $y(0) = 1$

The solution of this equation is given by Eq. (13.15). That is,

$$y(x) = 1 + x + x^2 + 8\frac{x^3}{3!} + \dots$$

Therefore,

$$y(0.25) = 1 + 0.25 + (0.25)^2 + \frac{8}{6}(0.25)^3 + \dots$$

$$= 1.33333$$

Similarly,

$$y(0.5) = 1 + 0.5 + 0.5^2 + \frac{8}{6}(0.5)^3 + \dots$$

$$= 1.81667$$

Improving Accuracy

The error in Taylor method is in the order of $(x - x_0)^{n+1}$. If $|x - x_0|$ is large, the error can also become large. Therefore, the result of this method in the interval (x_0, b) when $(b - x_0)$ is large, is often found unsatisfactory.

The accuracy can be improved by dividing the entire interval into subintervals (x_0, x_1), (x_1, x_2), (x_2, x_3), ... of equal length and computing $y(x_i)$, $i = 1, 2, ..., n$ successively, using the Taylor series expansion. Here, $y(x_i)$ is used as an initial condition for computing $y(x_{i+1})$. Thus,

$$y(x_{i+1}) = y(x_i) + \frac{y(x_i)}{1!}(x_{i+1} - x_i) + \frac{y''(x_i)}{2!}(x_{i+1} - x_i)^2 + \ldots$$

$$\ldots + \frac{y^{(m)}(x_i)}{m!}(x_{i+1} - x_i)^m \tag{13.16}$$

If we denote the size of each subinterval as h, then,

$$x_{i+1} - x_i = h \qquad \text{for } i = 0, 1, ..., n - 1$$

and Eq. (13.16) becomes

$$\boxed{y_{i+1} = y_i + \frac{y_i'}{1!}h + \frac{y_i''}{2!}h^2 + \ldots + \frac{y_i^{(m)}}{m!}h^m} \tag{13.17}$$

The derivatives $y_i^{(k)}$ are determined using Eq. (13.12), (13.13) and (13.14) at $x = x_i$ and $y = y_i$. This formula can be used recursively to obtain y_i values.

Example 13.2

Use the Taylor method recursively to solve the equation

$$y' = x^2 + y^2, \qquad y(0) = 0$$

for the interval (0, 0.4) using two subintervals of size 0.2.

The derivatives of y are given by

$$y' = x^2 + y^2$$
$$y'' = 2x + 2yy'$$
$$y''' = 2 + 2(y')^2 + 2yy''$$
$$y^{(4)} = 6y'\,y'' + 2yy'''$$

Iteration 1

$$y_1 = y_0 + \frac{y_0'}{1!}h + \frac{y_0''}{2!}h^2 + \frac{y_0'''}{3!}h^3 + \frac{y^{(4)}}{4!}h^4 + \ldots$$

$$h = 0.2,\ y_0 = y(0) = 0$$

$$y_0' = y'(0) = 0 + y(0)^2 = 0$$

$$y_0'' = y''(0) = 2 \times 0 + 2 \times y(0) \times y'(0) = 0$$

Similarly,

$$y_0''' = 2$$

$$y_0^{(4)} = 0$$

Therefore,

$$y_1 = 0 + 0 + 0 + \frac{2}{3!}(0.2)^3 + 0$$

$$= 0.002667 \qquad (\text{at } x = 0.2)$$

Iteration 2

$$x_1 = 0.2$$

$$y_1 = 0.002667$$

$$y_1' = x_1^2 + y_1^2 = (0.2)^2 + (0.002667)^2 = 0.04$$

$$y_1'' = 2x_1 + 2y_1 y_1'$$

$$= 2(0.2) + 2(0.002667)(0.04)$$

$$= 0.400213$$

$$y_1''' = 2 + 2(y_1')^2 + 2y_1 y_1''$$

$$= 2 + 2(0.04)^2 + 2(0.002667)(0.400213)$$

$$= 2.005335$$

$$y_1^{(4)} = 6y_1' y_1'' + 2y_1 y_1'''$$

$$= 6(0.04)(0.400213) + 2(0.002667)(2.005335)$$

$$= 0.106748$$

$$y_2 = y_1 + y_1' h + \frac{y_1''}{2}h^2 + \frac{y_1'''}{6}h^3 + \frac{y^{(4)}}{24}h^4$$

$$= 0.002667 + 0.04(0.2) + \frac{0.400213}{2}(0.2)^2$$

$$+ \frac{2.005335}{6}(0.2)^3 + \frac{0.106748}{24}(0.2)^4$$

$$= 0.021352$$

That is

$$y(0.4) = 0.021352$$

If we use $h = (b - x_0) = 0.4$ (without subdividing), we obtain

$$y(0.4) = \frac{2}{6}(0.4)^3 = 0.021333$$

The correct answer to the accuracy shown is $y(0.4) = 0.021359$. It shows that the accuracy has been improved by using subintervals. The accuracy can be further improved by reducing h further, say, $h = 0.1$.

Differentiating successively and substituting $x = 0$, $y = 0$ we get

$$y'' = 2y' + 3e^x, \qquad y''(0) = 2y'(0) + 3 = 9$$
$$y''' = 2y'' + 3e^x, \qquad y'''(0) = 2y''(0) + 3 = 21$$
$$y^{iv} = 2y''' + 3e^x, \qquad y^{iv}(0) = 2y'''(0) + 3 = 45 \text{ etc.}$$

Putting these values in the Taylor's series, we have

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{iv}(0) + \ldots\ldots$$

$$= 0 + 3x + \frac{9}{2}x^2 + \frac{21}{6}x^3 + \frac{45}{24}x^4 + \ldots\ldots$$

$$= 3x + \frac{9}{2}x^2 + \frac{7}{2}x^3 + \frac{15}{8}x^4 + \ldots\ldots$$

Hence $y(0.2) = 3(0.2) + 4.5(0.2)^2 + 3.5(0.2)^3 + 1.875(0.4)^4 + \ldots\ldots = 0.8110$ ...(*i*)

(*b*) Now $\frac{dy}{dx} - 2y = 3e^x$ is a Leibnitz's linear in x.

Its I.F. being e^{-2x}, the solution is

$$ye^{-2x} = \int 3e^x \cdot e^{-2x}\, dx + c = -3e^{-x} + c \quad \text{or} \quad y = -3e^x + ce^{2x}$$

Since $y = 0$ when $x = 0$, $\therefore$ $c = 3$.

Thus the exact solution is $y = 3(e^{2x} - e^x)$

When $x = 0.2$, $y = 3(e^{0.4} - e^{0.2}) = 0.8112$ ...(*ii*)

Comparing (*i*) and (*ii*), it is clear that (*i*) approximates to the exact value upto 3 decimal places.

10.4. EULER'S METHOD

Consider the equation $\dfrac{dy}{dx} = f(x, y)$ ...(1)

given that $y(x_0) = y_0$. Its curve of solution through $P(x_0, y_0)$ is shown dotted in Fig. 10.1. Now we have to find the ordinate of any other point Q on this curve.

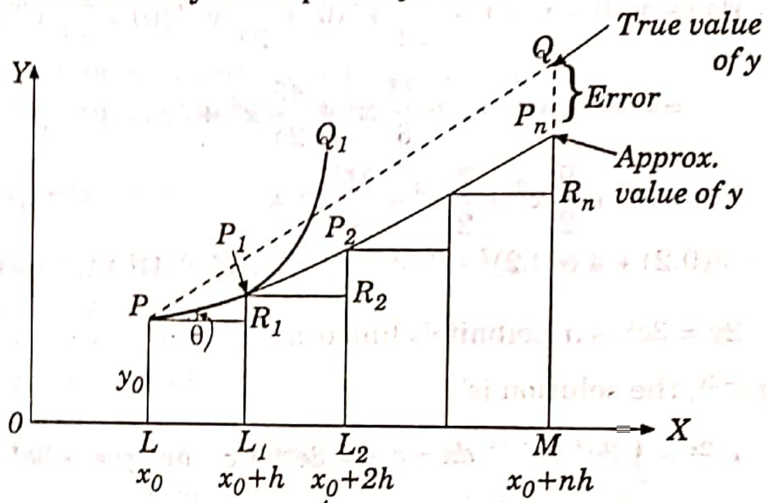

Fig. 10.1

Let us divide LM into n sub-intervals each of width h at L_1, L_2 ...... so that h is quite small.

In the interval LL_1, we approximate the curve by the tangent at P. If the ordinate through L_1 meets this tangent in $P_1(x_0 + h, y_1)$, then

$$y_1 = L_1P_1 = LP + R_1P_1 = y_0 + PR_1 \tan\theta$$

$$= y_0 + h\left(\frac{dy}{dx}\right)_P = y_0 + h\, f(x_0, y_0)$$

Let P_1Q_1 be the curve of solution of (1) through P_1 and let its tangent at P_1 meet the ordinate through L_2 in $P_2(x_0 + 2h, y_2)$. Then

$$y_2 = y_1 + hf(x_0 + h, y_1) \quad ...(1)$$

Repeating this process n times, we finally reach on an approximation MP_n of MQ given by

$$y_n = y_{n-1} + hf(x_0 + \overline{n-1}\,h, y_{n-1})$$

This is *Euler's method* of finding an approximate solution of (1).

Obs. In Euler's method, we approximate the curve of solution by the tangent in each interval, *i.e.* by a sequence of short lines. Unless h is small, the error is bound to be quite significant. This sequence of lines may also deviate considerably from the curve of solution. As such, the method is very slow and hence there is a modification of this method which is given in the next section.

Example 10.5. *Using Euler's method, find an approximate value of y corresponding to x = 1, given that dy/dx = x + y and y = 1 when x = 0.*

(Calicut, B. Tech., 1999)

Sol. We take $n = 10$ and $h = 0.1$ which is sufficiently small. The various calculations are arranged as follows :

x	y	$x + y = dy/dx$	Old y + 0.1 (dy/dx) = new y
0.0	1.00	1.00	1.00 + 0.1 (1.00) = 1.10
0.1	1.10	1.20	1.10 + 0.1 (1.20) = 1.22
0.2	1.22	1.42	1.22 + 0.1 (1.42) = 1.36
0.3	1.36	1.66	1.36 + 0.1 (1.66) = 1.53
0.4	1.53	1.93	1.53 + 0.1 (1.93) = 1.72
0.5	1.72	2.22	1.72 + 0.1 (2.22) = 1.94
0.6	1.94	2.54	1.94 + 0.1 (2.54) = 2.19
0.7	2.19	2.89	2.19 + 0.1 (2.89) = 2.48
0.8	2.48	3.29	2.48 + 0.1 (3.29) = 2.81
0.9	2.81	3.71	2.81 + 0.1 (3.71) = 3.18
1.0	3.18		

Thus the required approximate value of $y = 3.18$.

Obs. In example 10.1 (Obs.), we obtained the true values of y from its exact solution to be 3.44 whereas by Euler's method $y = 3.18$ and by Picard's method $y = 3.434$. In above solution, had we chosen $n = 20$, the accuracy would have been considerably increased but at the expense of double the labour of computation. Euler's method is no doubt very simple but cannot be considered as one of the best.

Example 10.6. *Given* $\dfrac{dy}{dx} = \dfrac{y - x}{y + x}$ *with initial condition* $y = 1$ *at* $x = 0$ *; find* y *for* $x = 0.1$ *by Euler's method.* *(P.T.U., B.E., 2001)*

Sol. We divide the interval (0, 0.1) into five steps *i.e.* we take $n = 5$ and $h = 0.02$. The various calculations are arranged as follows :

x	y	dy/dx	Old y + 0.02 (dy/dx) = new y
0.00	1.0000	1.0000	1.0000 + 0.02(1.0000) = 1.0200
0.02	1.0200	0.9615	1.0200 + 0.02(0.9615) = 1.0392
0.04	1.0392	0.926	1.0392 + 0.02(0.926) = 1.0577
0.06	1.0577	0.893	1.0577 + 0.02(0.893) = 1.0756
0.08	1.0756	0.862	1.0756 + 0.02(0.862) = 1.0928
0.10	1.0928		

Hence the required approximate value of $y = 1.0928$.

10.5. MODIFIED EULER'S METHOD

In the Euler's method, the curve of solution in the interval LL_1 is approximated by the tangent at P (Fig. 10.1) such that at P_1, we have

$$y_1 = y_0 + hf(x_0, y_0) \quad ...(1)$$

Then the slope of the curve of solution through P_1

[*i.e.* $(dy/dx)P_1 = f(x_0 + h, y_1)$]

is computed and the tangent at P_1 to P_1Q_1 is drawn meeting the ordinate through L_2 in $P_2(x_0 + 2h, y_2)$.

Now we find a better approximation $y_1^{(1)}$ of $y(x_0 + h)$ by taking the slope of the curve as the mean of the slopes of the tangents at P and P_1, *i.e.*

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_0 + h, y_1)] \quad ...(2)$$

As the slope of the tangent at P_1 is not known, we take y_1 as found in (1) by Euler's method and insert it on R.H.S. of (2) to obtain the first modified value $y_1^{(1)}$.

Again (2) is applied and we find a still better value $y_1^{(2)}$ corresponding to L_1 as

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_0 + h, y_1^{(1)})]$$

We repeat this step, till two consecutive values of y agree. This is then taken as the starting point for the next interval L_1L_2.

Once y_1 is obtained to desired degree of accuracy, y corresponding to L_2 is found from (1).

$$y_2 = y_1 + hf(x_0 + h, y_1)$$

and a better approximation $y_2^{(1)}$ is obtained from (2)

$$y_2^{(1)} = y_1 + \frac{h}{2}[(f(x_0 + h, y_1) + f(x_0 + 2h, y_2)]$$

We repeat this step until y_2 becomes stationary. Then we proceed to calculate y_3 as above and so on.

This is the *modified Euler's method* which gives great improvement in accuracy over the original method.

Example 10.7. *Using modified Euler's method, find an approximate value of y when x = 0.3, given that dy/dx = x + y and y = 1 when x = 0.* (*Delhi, B.E., 2002*)

Sol. Taking $h = 0.1$, the various calculations are arranged as follows :

x	$x + y = y'$	*Mean slope*	*Old y + 0.1 (mean slope) = new y*
0.0	0 + 1	—	1.00 + 0.1 (1.00) = 1.10
0.1	0.1 + 1.1	$\frac{1}{2}$(1 + 1.2)	1.00 + 0.1 (1.1) = 1.11
0.1	0.1 + 1.11	$\frac{1}{2}$ (1 + 1.21)	1.00 + 0.1 (1.105) = 1.1105
0.1	0.1 + 1.1105	$\frac{1}{2}$ (1 + 1.2105)	1.00 + 0.1 (1.1052) = 1.1105
0.1	1.2105	—	1.1105 + 0.1 (1.2105) = 1.2316
0.2	0.2 + 1.2316	$\frac{1}{2}$ (1.12105 + 1.4316)	1.1105 + 0.1 (1.3211) = 1.2426
0.2	0.2 + 1.2426	$\frac{1}{2}$(1.2105 + 1.4426)	1.1105 + 0.1 (1.3266) = 1.2432
0.2	0.2 + 1.2432	$\frac{1}{2}$ (1.2105 + 1.4432)	1.1105 + 0.1 (1.3268) = 1.2432

0.2	1.4432	—	1.2432 + 0.1 (1.4432) = 1.3875
0.3	0.3 + 1.3875	$\frac{1}{2}$ (1.4432 + 1.6875)	1.2432 + 0.1 (1.5654) = 1.3997
0.3	0.3 + 1.3997	$\frac{1}{2}$ (1.4432 + 1.6997)	1.2432 + 0.1 (1.5715) = 1.4003
0.3	0.3 + 1.4003	$\frac{1}{2}$ (1.4432 + 1.7003)	1.2432 + 0.1 (1.5718) = 1.4004
0.3	0.3 + 1.4004	$\frac{1}{2}$ (1.4432 + 1.7004)	1.2432 + 0.1 (1.5718) = 1.4004

Hence $y(0.3) = 1.4004$ approximately.

Obs. In example 10.5, we obtained the approximate value of y for $x = 0.3$ to be 1.53 whereas by modified Euler's method the corresponding value is 1.4003 which is nearer its true value 1.3997, obtained from its exact solution $y = 2e^x - x - 1$ by putting $x = 0.3$.

Example 10.8. *Solve the following by Euler's modified method :*

$$\frac{dy}{dx} = \log (x + y),\ y(0) = 2$$

at $x = 1.2$ and 1.4 with $h = 0.2$.

Sol. The various calculations are arranged as follows :

x	*log (x + y) = y′*	*Mean slope*	*Old y + 0.2 (mean slope) = new y*
0.0	log (0 + 2)	—	2 + 0.2(0.301) = 2.0602
0.2	log (0.2 + 2.0602)	$\frac{1}{2}$ (0.310 = 0.3541)	2 + 0.2 (0.3276) = 2.0655
0.2	log (0.2 + 2.0655)	$\frac{1}{2}$ (0.301 + 0.3552)	2 + 0.2 (0.3281) = 2.0656
0.2	0.3552	—	2.0656 + 0.2 (0.3552) = 2.1366
0.4	log (0.4 + 2.1366)	$\frac{1}{2}$ (0.3552 + 0.4042)	2.0656 + 0.2 (0.3797) = 2.1415
0.4	log (0.4 + 2.1415)	$\frac{1}{2}$ (0.3552 + 0.4051)	2.0656 + 0.2 (0.3801) = 2.1416
0.4	0.4051	—	2.1416+ 0.2 (0.4051) = 2.2226
0.6	log (0.6 + 2.2226)	$\frac{1}{2}$ (0.4051 + 0.4506)	2.1416 + 0.2 (0.4279) = 2.2272
0.6	log (0.6 + 2.2272)	$\frac{1}{2}$ (0.4051 + 0.4514)	2.1416 + 0.2 (0.4282) = 2.2272
0.6	0.4514	—	2.2272 + 0.2 (0.4514) = 2.3175
0.8	log (0.8 + 2.3175)	$\frac{1}{2}$ (0.4514 + 0.4938)	2.2272 + 0.2 (0.4726) = 2.3217
0.8	log (0.8 + 2.3217)	$\frac{1}{2}$ (0.4514 + 0.4943)	2.2272 + 0.2 (0.4727) = 2.3217
0.8	0.4943	—	2.3217 + 0.2 (0.4943) = 2.4206
1.0	log (1 + 2.4206)	$\frac{1}{2}$ (0.4943 + 0.5341)	2.3217 + 0.2 (0.5142) = 2.4245
1.0	log (1 + 2.4245)	$\frac{1}{2}$ (0.4943 + 0.5346)	2.3217 + 0.2 (0.5144) = 2.4245

1.0	0.5346	—	2.4245 + 0.2 (0.5346) = 2.5314
1.2	log (1.2 + 2.5314)	$\frac{1}{2}$ (0.5346 + 0.5719)	2.4245 + 0.2 (0.5532) = 2.5351
1.2	log (1.2 + 2.5351)	$\frac{1}{2}$ (0.5346 + 0.5723)	2.4245 + 0.2 (0.5534) = 2.5351
1.2	0.5723	—	2.5351 + 0.2 (0.5723) = 2.6496
1.4	log (1.4 + 2.6496)	$\frac{1}{2}$ (0.5723 + 0.6074)	2.5351 + 0.2 (0.5898) = 2.6531
1.4	log (1.4 + 2.6531)	$\frac{1}{2}$ (0.5723 + 0.6078)	2.5351 + 0.2 (0.5900) = 2.6531

Hence $y(1.2) = 2.5351$ and $y(1.4) = 2.6531$ approximately.

Example 10.9. *Using Euler's modified method, obtain a solution of the equation*

$$dy/dx = x + |\sqrt{y}|,$$

with initial conditions $y = 1$ at $x = 0$, for the range $0 \le x \le 0.6$ in steps of 0.2.

Sol. The various calculations are arranged as follows :

x	$x + \|\sqrt{y}\| = y'$	*Mean slope*	*Old y + 0.2 (mean slope) = new y*
0.0	0 + 1 = 1	—	1 + 0.2 (1) = 1.2
0.2	$0.2 + \|\sqrt{(1.2)}\|$ = 1.2954	$\frac{1}{2}$(1 + 1.2954) = 1.1477	1 + 0.2 (1.1477) = 1.2295
0.2	$0.2 + \|\sqrt{(1.2295)}$ = 1.3088	$\frac{1}{2}$ (1 + 1.3088) = 1.1544	1 + 0.2 (1.1544) = 1.2309
0.2	$0.2 + \|\sqrt{(1.2309)}\|$ = 1.3094	$\frac{1}{2}$(1 + 1.3094) = 1.1547	1 + 0.2 (1.1547) = 1.2309
0.2	1.3094	—	1.2309 + 0.2 (1.3094) = 1.4927
0.4	$0.4 + \|\sqrt{(1.4927)}\|$ = 1.6218	$\frac{1}{2}$(1.3094 + 1.6218) = 1.4654	1.2309 + 0.2 (1.4654) = 1.5240
0.4	$0.4 + \|\sqrt{(1.524)}\|$ = 1.6345	$\frac{1}{2}$(1.3094 + 1.6345) = 1.4718	1.2309 + 0.2 (1.4718) = 1.5253
0.4	$0.4 + \|\sqrt{(1.5253)}\|$ = 1.6350	$\frac{1}{2}$ (1.3094 + 1.6350) = 1.4721	1.2309 + 0.2 (1.4721) = 1.5253

0.4	1.6350	—	1.5253 + 0.2 (1.635) = 1.8523
0.6	$0.6 + \| \sqrt{(1.8523)} \|$ = 1.9610	$\frac{1}{2}(1.635 + 1.961)$ = 1.798	1.5253 + 0.2 (1.798) = 1.8849
0.6	$0.6 + \| \sqrt{(1.8849)} \|$ = 1.9729	$\frac{1}{2}(1.635 + 1.9729)$ = 1.8040	1.5253 + 0.2 (1.804) = 1.8861
0.6	$0.6 + \| \sqrt{(1.8861)} \|$ = 1.9734	$\frac{1}{2}(1.635 + 1.9734)$ = 1.8042	1.5253 + 0.2 (1.8042) = 1.8861

Hence $y(0.6) = 1.8861$ approximately.

is as follows :

Calculate successively $k_1 = hf(x_0, y_0)$, $k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) \text{ and } k_4 = hf(x_0 + h, y_0 + k_3)$$

Finally compute $k = \frac{1}{6} + (k_1 + 2k_2 + 2k_3 + k_4)$

which gives the required approximate value as $y_1 = y_0 + k$.

(Note that k is the weighted mean of k_1, k_2, k_3 *and* k_4*).*

> **Obs.** One of the advantages of these methods is that the operation is identical whether the differential equation is linear or non-linear.

Example 10.11. *Apply Runge-Kutta fourth order method to find an approximate value of y when x = 0.2 given that dy/dx = x + y and y = 1 when x = 0.* *(Bhopal, B.E., 2002)*

Sol. Here $x_0 = 0, y_0 = 1, h = 0.2, f(x_0, y_0) = 1$

$\therefore$ $k_1 = hf(x_0, y_0) = 0.2 \times 1 \qquad = 0.2000$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = 0.2 \times f(0.1, 1.1) \qquad = 0.2400$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = 0.2 \times f(0.1, 1.12) \qquad = 0.2440$$

and $k_4 = hf(x_0 + h, y_0 + k_3) = 0.2 \times f(0.2, 1.244) \qquad = 0.2888$

$$\therefore \quad k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = \frac{1}{6}(0.2000 + 0.4800 + 0.4880 + 0.2888)$$

$$= \frac{1}{6} \times (1.4568) = 0.2428.$$

Hence the required approximate value of y is 1.2428.

Example 10.12. *Using Runge-Kutta method of fourth order, solve* $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ *with y(0) = 1 at x = 0.2, 0.4.* *(Madras, B.E., 2001 S)*

Sol. We have $f(x, y) = \frac{y^2 - x^2}{y^2 + x^2}$

To find y(0.2) :

Here $x_0 = 0, y_0 = 1, h = 0.2$

$k_1 = h\, f(x_0, y_0) = 0.2\, f(0, 1) \qquad = 0.2000$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = 0.2\, f(0.1, 1.1) \qquad = 0.19672$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = 0.2f(0.1, 1.09836) \qquad = 0.1967$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2f(0.2, 1.1967) \qquad = 0.1891$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[0.2 + 2(0.19672) + 2(0.1967) + 0.1891] \qquad = 0.19599$$

Hence $y(0.2) = y_0 + k = 1.196$.

To find y(0.4) :

Here $x_1 = 0.2$, $y_1 = 1.196$, $h = 0.2$.

$$k_1 = h\,f(x_1, y_1) \qquad = 0.1891$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = 0.2\,f(0.3, 1.2906) \qquad = 0.1795$$

$$k_3 = h\,f\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = 0.2\,f(0.3, 1.2858) \qquad = 0.1793$$

$$k_4 = h\,f(x_1 + h, y_1 + k_3) = 0.2\,f(0.4, 1.3753) \qquad = 0.1688$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[0.1891 + 2(0.1795) + 2(0.1793) + 0.1688] \qquad = 0.1792$$

Hence $\qquad y(0.4) = y_1 + k = 1.196 + 0.1792 = 1.3752.$

Example 10.13. *Apply Runge-Kutta method to find approximate value of y for $x = 0.2$, in steps of 0.1, if $dy/dx = x + y^2$, given that $y = 1$ where $x = 0$.* (Osmania, B.E., 2002)

Sol. *Given* $f(x, y) = x + y^2$.

Here we take $h = 0.1$ and carry out the calculations in two steps.

Step I. $x_0 = 0$, $y_0 = 1$, $h = 0.1$

$$\therefore \quad k_1 = hf(x_0, y_0) = 0.1\,f(0, 1) \qquad = 0.1000$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = 0.1\,f(0.05, 1.1) \qquad = 0.1152$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = 0.1\,f(0.05, 1.1152) \qquad = 0.1168$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1\,f(0.1, 1.1168) \qquad = 0.1347$$

$$\therefore \quad k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.1000 + 0.2304 + 0.2336 + 0.1347) \qquad = 0.1165$$

giving $\qquad y(0.1) = y_0 + k = 1.1165.$

Step II. $x_1 = x_0 + h = 0.1$, $y_1 = 1.1165$, $h = 0.1$

$$\therefore \quad k_1 = hf(x_1, y_1) = 0.1\,f(0.1, 1.1165) \qquad = 0.1347$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = 0.1\,f(0.15, 1.1838) \qquad = 0.1551$$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = 0.1\,f(0.15, 1.194) \qquad = 0.1576$$

$$k_4 = hf(x_1 + h, y_2 + k_3) = 0.1\, f(0.2, 1.1576) \qquad = 0.1823$$

$$\therefore \qquad k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad = 0.1571$$

Hence $y(0.2) = y_1 + k = 1.2736$.

PROBLEMS 10.3

1. Use Runge's method to approximate y when $x = 1.1$, given that $y = 1.2$ when $x = 1$ and $dy/dx = 3x + y^2$.
2. Given that $dy/dx = (y^2 - 2x)/(y^2 + x)$ and $y = 1$ as $x = 0$; find y for $x = 0.1, 0.2, 0.3, 0.4$ and 0.5. *(Delhi, B.E., 2002)*
3. Using Runge-Kutta method of order 4, compute $y(0.2)$ and $y(0.4)$ from $10\dfrac{dy}{dx} = x^2 + y^2$, $y(0) = 1$, taking $h = 0.1$. *(Rohtak, B.E., 2003)*
4. Using Runge-Kutta method of order 4, find $y(0.2)$ given that $dy/dx = 3x + \dfrac{1}{2}y$, $y(0) = 1$, taking $h = 0.1$. *(V.T.U., B.E. 2004)*
5. Find by Runge-Kutta method an approximate value of y for $x = 0.8$, given that $y = 0.41$ when $x = 0.4$ and $dy/dx = \sqrt{(x + y)}$.
6. Using Runge-Kutta method of order 4, find $y(0.2)$ for the equation $\dfrac{dy}{dx} = \dfrac{y - x}{y + x}$, $y(0) = 1$. Take $h = 0.2$. *(V.T.U., B.E., 2003)*
7. Using Runge-Kutta method of fourth order, solve for y at $x = 1.2, 1.4$ from $\dfrac{dy}{dx} = \dfrac{2xy + e^x}{x^2 + xe^x}$ with $x_0 = 1, y_0 = 0$.

$\frac{dy}{dx} = f(x, y, z)$. $\frac{dz}{dx} = \phi(x, y, z)$

with initial conditions $y(x_0) = y_0$ and $z(x_0) = z_0$ can be solved by the methods discussed in the preceding sections, especially Picard's or Runge-Kutta methods.

(i) *Picard's method gives*

$$y_1 = y_0 + \int f(x, y_0, z_0)\,dx,\; z_1 = z_0 + \int \phi(x, y_0, z_0)\,dx$$

$$y_2 = y_0 + \int f(x, y_1, z_1)\,dx,\; z_2 = z_0 + \int \phi(x, y_1, z_1)\,dx$$

$$y_3 = y_0 + \int f(x, y_2, z_2)\,dx,\; z_3 = z_0 + \int \phi(x, y_2, z_2)\,dx$$

and so on.

(ii) *Taylor's series method* is used as follows :

If h be the step-size, $y_1 = y(x_0 + h)$ and $z_1 = z(x_0 + h)$. Then Taylor's algorithm for (1) and (2) gives

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \ldots \quad \ldots(3)$$

$$z_1 = z_0 + hz_0' + \frac{h^2}{2!}z_0'' + \frac{h^3}{3!}z_0''' + \ldots \quad \ldots(4)$$

Differentiating (1) and (2) successively, we get y'', z'', etc. So the values $y_0', y_0'', y_0''' \ldots$ and $z_0', z_0'', z_0''' \ldots$ are known. Substituting these in (3) and (4), we obtain y_1, z_1 for the next step.

Similarly, we have the algorithms

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \ldots \quad \ldots(5)$$

$$z_2 = z_1 + hz_1' + \frac{h^2}{2!}z_1'' + \frac{h^3}{3!}z_1''' + \ldots \quad \ldots(6)$$

Since y_1 and z_1 are known, we can calculate $y_1', y_1'', \ldots$ and $z_1', z_1'', \ldots$. Substituting these in (5) and (6), we get y_2 and z_2.

Proceeding further, we can calculate the other values of y and z step by step.

(iii) *Runge-Kutta method* is applied as follows :

Starting at (x_0, y_0, z_0) and taking the step-sizes for x, y, z to be h, k, l respectively, the Runge-Kutta method gives,

$$k_1 = hf(x_0, y_0, z_0)$$
$$l_1 = h\phi(x_0, y_0, z_0)$$
$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$$
$$l_2 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$$
$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right)$$

$$l_3 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$l_4 = h\phi(x_0 + h, y_0 + k_3, z_0 + l_3)$$

Hence $$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

and $$z_1 = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

To compute y_2 and z_2, we simply replace x_0, y_0, z_0 by x_1, y_1, z_1 in the above formulae.

Example 10.18. *Using Picard's method, find approximate values of y and z corresponding to x = 0.1, given that y(0) = 2, z(0) = 1 and*

$$dy/dx = x + z,\ dz/dx = x - y^2.$$

Sol. Here $x_0 = 0, y_0 = 2, z_0 = 1,$

and $$\frac{dy}{dx} = f(x, y, z) = x + z$$

$$\frac{dz}{dx} = \phi(x, y, z) = x - y^2$$

$$\therefore \quad y = y_0 + \int_{x_0}^{x} f(x, y, z)\, dx \quad \text{and} \quad z = z_0 + \int_{x_0}^{x} \phi(x, y, z)\, dx.$$

First approximations

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0, z_0)\, dx = 2 + \int_0^x (x + 1)\, dx = 2 + x + \frac{1}{2}x^2$$

$$z_1 = z_0 + \int_{x_0}^{x} \phi(x, y, z_0)\, dx = 1 + \int_0^x (x - 4)\, dx = 1 - 4x + \frac{1}{2}x^2.$$

Second approximations

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1, z_1)\, dx = 2 + \int_0^x \left(x + 1 - 4x + \frac{1}{2}x^2\right) dx$$

$$= 2 + x - \frac{3}{2}x^2 + \frac{x^3}{6}$$

$$z_2 = z_0 + \int_{x_0}^{x} \phi(x, y_1, z_1)\, dx = 1 + \int_0^x \left[x - \left(2 + x + \frac{1}{2}x^2\right)^2\right] dx$$

$$= 1 - 4x + \frac{3}{2}x^2 - x^3 - \frac{x^4}{4} - \frac{x^5}{20}.$$

Third approximations

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2, z_2)\, dx = 2 + x - \frac{3}{2}x^2 - \frac{1}{2}x^3 - \frac{1}{4}x^4 - \frac{1}{20}x^5 - \frac{1}{120}x^6$$

$$z_3 = z_0 + \int_{x_0}^{x} \phi(x, y_2, z_2)\, dx$$

$$= 1 - 4x - \frac{3}{2}x^2 + \frac{5}{3}x^3 + \frac{7}{12}x^4 - \frac{31}{60}x^5 + \frac{1}{12}x^6 - \frac{1}{252}x^7$$

and so on.

When $x = 0.1$

$y_1 = 2.105,\quad y_2 = 2.08517,\quad y_3 = 2.08447$

$z_1 = 0.605,\quad z_2 = 0.58397,\quad z_3 = 0.58672.$

Hence $y(0.1) = 2.0845,\quad z(0.1) = 0.5867$

correct to four decimal places.

10.12. SECOND ORDER DIFFERENTIAL EQUATIONS

Consider the second order differential equation

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right)$$

By writing $dy/dx = z$, it can be reduced to two first order simultaneous differential equations

$$\frac{dy}{dx} = z, \quad \frac{dz}{dx} = f(x, y, z)$$

These equations can be solved as explained above.

Example 10.19. *Find an approximate series solution of the simultaneous equations $dx/dt = xy + 2t$, $dy/dt = 2ty + x$ subject to the initial conditions $x = 1, y = -1, t = 0$.*

Sol. x and y both being functions of t, Taylor's series gives

$$\left.\begin{aligned} x(t) &= x_0 + tx_0' + \frac{t^2}{2!}x_0'' + \frac{t^3}{3!}x_0''' + \ldots\ldots \\ \text{and}\quad y(t) &= y_0 + ty_0' + \frac{t^2}{2!}y_0'' + \frac{t^3}{3!}y_0''' + \ldots\ldots \end{aligned}\right\} \qquad \ldots(i)$$

Differentiating the given equations

$$x' = xy + 2t \quad \ldots(ii) \qquad y' = 2ty + x \quad \ldots(iii)$$

w.r.t. t, we get

$$\begin{array}{l|l} x'' = xy' + x'y + 2 & y'' = 2ty' + 2y + x' \\ x''' = (xy'' + x'y') + x''y + x'y' & y''' = 2ty'' + 2y' + 2y' + x'' \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \end{array} \Bigg\} \qquad \ldots(iv)$$

Putting $x_0 = 1, y_0 = -1, t_0 = 0$ in (ii), (iii) and (iv), we obtain

$$\begin{array}{l|l} x_0' = -1 + 2(0) = -1 & y_0' = 1 \\ x_0'' = x_0y_0' + x_0'y_0 + 2 & y_0'' = 0 + 2y_0 + x_0' \\ \quad = 1.1 + (-1)(-1) + 2 = 4 & \quad = 2(-1) + (-1) = -3 \\ x_0''' = -3 + (-1)(1) + 4(-1) - 1 = -9 & y_0''' = 2 + 2 + 4 = 8 \text{ etc} \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \end{array}$$

Substituting these values in (*i*), we get

$$x(t) = 1 - t + 4\frac{t^2}{2!} + (-9)\frac{t^3}{3!} + \ldots\ldots = 1 - t + 2t^2 - \frac{3}{2}t^3 + \ldots\ldots$$

$$y(t) = 1 + t - 3\frac{t^2}{2!} + 8\frac{t^3}{3!} + \ldots\ldots = 1 + t - \frac{3}{2}t^2 + \frac{4}{3}t^3 + \ldots\ldots$$

Example 10.20. *Using Runge-Kutta method, solve $y'' = xy'^2 - y^2$ for $x = 0.2$ correct to 4 decimal places. Initial conditions are*

$$x = 0, \quad y = 1, \quad y' = 0.$$

Sol. Let $\quad dy/dx = z = f(x, y, z)$

Then $\quad \frac{dz}{dx} = xz^2 - y^2 = \phi(x, y, z)$

We have $x_0 = 0, y_0 = 1, z_0 = 0, h = 0.2.$

$\therefore$ Runge-Kutta formulae become

$$k_1 = hf(x_0, y_0, z_0) = 0.2(0) = 0$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right) = 0.2(-0.1) = -0.02$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right) = 0.2(-0.0999) = -0.02$$

$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3) = 0.2(-0.1958) = -0.0392$$

$$\therefore \quad k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = -0.0199$$

$$l_1 = h\phi(x_0, y_0, z_0) = 0.2(-1) = -0.2$$

$$l_2 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right) = 0.2(-0.999) = -0.1998$$

$$l_3 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right) = 0.2(-0.9791) = -0.1958$$

$$l_4 = h\phi(x_0 + h, y_0 + k_3, z_0 + l_3) = 0.2(0.9527) = -0.1905$$

$$l = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = -0.1970$$

Hence at $x = 0.2$,

$$y = y_0 + k = 1 - 0.0199 = 0.9801$$

and $\quad y' = z = z_0 + l = 0 - 0.1970 = -0.1970.$

The most important aspect of numerical methods is to minimize the errors and obtain the solutions with the least errors. It is usually not possible to follow error development quite closely. We can make only rough estimates. That is why, our treatment of error analysis at times, has to be somewhat intuitive.

In any method, the truncation error can be reduced by taking smaller sub-intervals. The round-off error cannot be controlled easily unless the computer used has the double precision arithmetic facility. In fact, this error has proved to be more elusive than the truncation error.

The truncation error in Euler's method is $\frac{1}{2} h^2 y_n''$ *i.e.* of $O(h^2)$ while that of modified Euler's method is $\frac{1}{2} h^3 y_n'''$ *i.e.* of $O(h^3)$.

Similarly in the fourth order Runge-Kutta method, the truncation error is of $O(h^5)$.

In the Milne's method, the truncation error

due to predictor formula $= \frac{14}{45} y_n^{v} h^5$

and due to corrector formula $= -\frac{1}{90} y_n^{v} h^5$.