Partial Differential Equation

example of partial differential equation in physics & engineering.

- wave equation
- Poisson's equation
- Laplace equation
- Heat equation

Classification of second-order partial differential equation

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + Fu = G$$

$$A U_{xx} + B U_x U_y + C U_{yy} + D U_x + E U_y + FU = G$$

where A, B, C, D, E, F & G are the ~~coeffici~~ function of x & y.

The above equation is said to be

(i) elliptic at a point (x, y) in the place if $B^2 - 4AC < 0$

(ii) parabolic if $B^2 - 4AC = 0$

(iii) Hyperbolic if $B^2 - 4AC > 0$

Example:-

Elliptic type

1. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (Laplace eq^n in two dim.)

2. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$ (Poisson's equation)

Parabolic type :-

$$\frac{\partial^2 u}{\partial x^2} + \frac{1}{\alpha^2}\frac{\partial u}{\partial t} = 0 \quad (\text{one dim. heat equation})$$

Hyperbolic type

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial^2 u}{\partial t^2} \quad (\text{one dim. wave equation})$$

Example :- Classify the following equations

1. $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$

2. $x^2 u_{xx} + (1-y^2) u_{yy} = 0$

1. $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$

$A = 1, \quad B = 2, \quad C = 1$

$B^2 - 4AC = 4 - 4(1 \times 1) = 0$ for all values of x & y.

Hence equation is parabolic at all points

2. $x^2 u_{xx} + (1-y^2) u_{yy} = 0$

$A = x^2, \quad B = 0, \quad C = (1-y^2)$

$B^2 - 4AC = 0 - 4x^2(1-y^2)$

$= 4x^2(y^2-1)$

$-1 < y < 1 ; \quad y^2 - 1 = -ve .$

so $B^2 - 4AC < 0$ if $-1 < y < 1, \; x \neq 0$
(equation is elliptic)

- for $-\infty < x < \infty$; $x \neq 0$, $y < -1$ or $y > 1$, the equation is hyperbolic.

- for $x = 0$ & for any value of y. $4x^2(y^2-1) = 0$ equation is parabolic

→ $y = \pm 1$ for any value of x; $4x^2(y^2-1) = 0$. equation is parabolic.

<u>Example:-</u> classify the partial differential equation.

$$x u_{xx} + y u_{yy} = 0 \; ; \quad x > 0, \; y > 0$$

<u>solution :-</u>

$$x u_{xx} + y u_{yy} = 0 \; ; \text{ for } x > 0, y > 0$$

$$A = x, \; B = 0, \; C = y$$

$$B^2 - 4AC = 0 - 4xy = -4xy$$

for $x > 0$, $y > 0$

$B^2 - 4AC = -4xy$ is always negative.

Thus the given partial differential equation is elliptic for $x > 0, y > 0$.

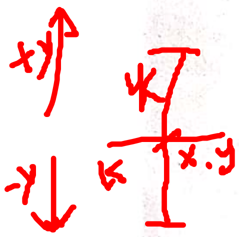
Finite difference Approximations to partial derivatives

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

similarly

$$\frac{\partial u}{\partial x} = \frac{u(x+h, y) - u(x, y)}{h} + O(h) = \frac{u(x, y) - u(x-h, y)}{h} + O(h)$$

writing $u(x,y) = u(ih, jk)$ as simply $u_{i,j}$, the above approximation can be rewritten as



$$u_x = \frac{u_{i+1,j} - u_{i,j}}{h} = \frac{u_{i,j} - u_{i-1,j}}{h}$$ (~~forward differ~~

u_x = (forward difference) FD — (Backward difference) BD

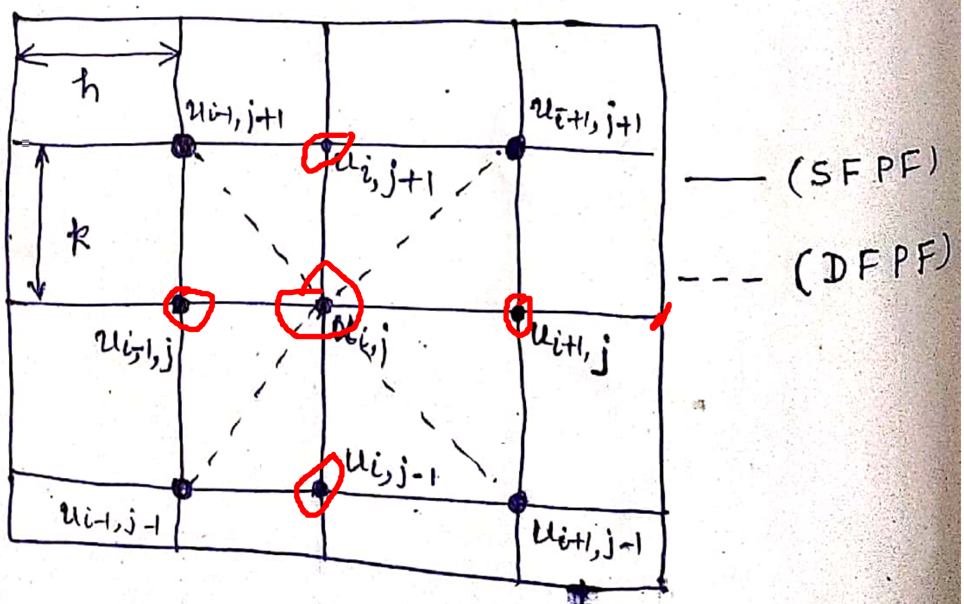
similarly

$$u_y = \frac{(u_{i,j+1} - u_{i,j})}{k} = \frac{u_{i,j} - u_{i,j-1}}{k}$$

(forward difference) FD — (Backward difference) BD

h - const. diff. along x-axis.

k - const diff. along y-axis.



$$u_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left[\frac{u_{i+1,j} - u_{i,j}}{h}\right] = \frac{1}{h}\left[(u_{i+1,j})_{BD} - (u_{i,j})_{BD}\right]$$

$$u_{xx} = \frac{1}{h}\left[\frac{(u_{i+1,j} - u_{i,j})}{h} - \frac{(u_{i,j} - u_{i-1,j})}{h}\right]$$

$$\boxed{u_{xx} = \frac{1}{h^2}\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right]} + O(h^2)$$

similarly.

$$\boxed{u_{yy} = \frac{1}{k^2}\left[u_{i,j+1} - 2u_{i,j} + u_{i,j-1}\right]} + O(k^2)$$

Elliptic equations

Consider Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \; ; \text{ i.e. } \nabla^2 u = 0, \text{ or } u_{xx} + u_{yy} = 0$$

Replacing above equation with its derivative

$$\frac{(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})}{h^2} + \frac{(u_{i,j+1} - 2u_{i,j} + u_{i,j-1})}{k^2} = 0$$

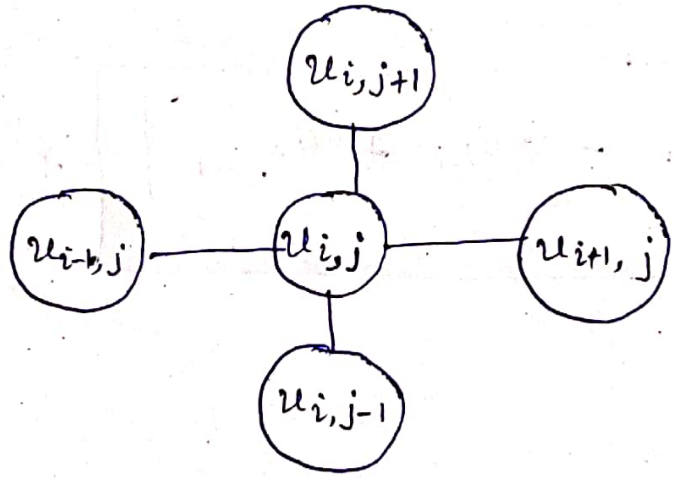
in case of square mesh $h = k$.

$$u_{i+1,j} - 2u_{i,j} + u_{i-1,j} + u_{i,j+1} - 2u_{i,j} + u_{i,j-1} = 0$$

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = 0$$

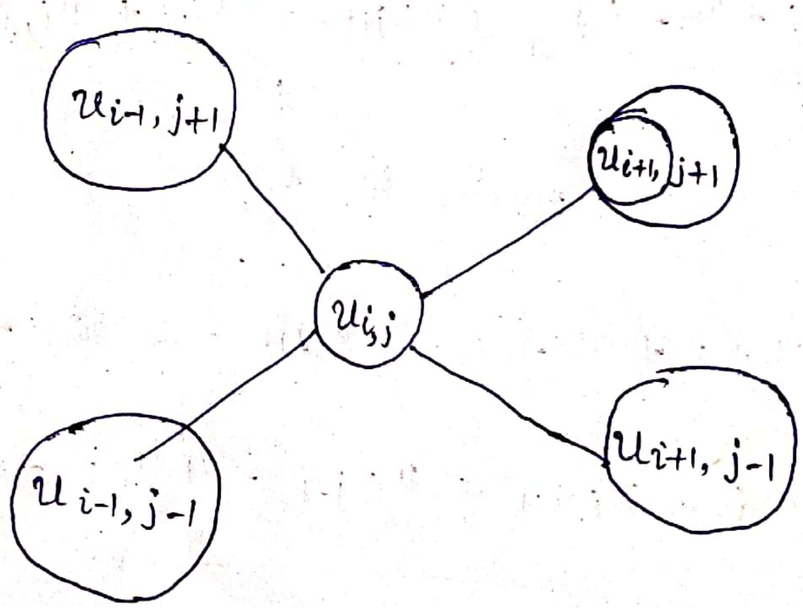
$$\boxed{u_{i,j} = \frac{1}{4}\left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}\right)}$$

the value of "u" at any interior point is the arithmetic mean of the values of "u" at the four lattice points. This is called standard five points formula. (SFPF)



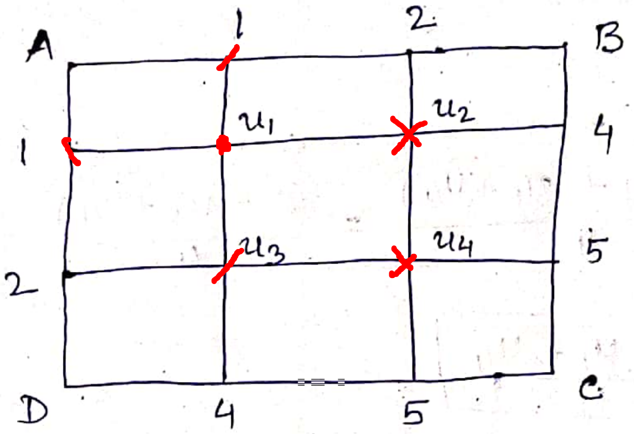
central value = average of four other values

A less accurate sometimes can be used which is the average of four neighbouring mesh points and called Diagonal five point formula (DFPF).



Solution of Laplace's Equation (using Gauss-siedal formula) iteration method

Example:- solve $u_{xx} + u_{yy} = 0$ for the following square mesh with boundary values as shown in figure below:



solution. Boundary values are symmetrical about diagonal AC. By symmetry we can write

$$u_2 = u_3$$

take $u_2 = 0$, $\Rightarrow u_3 = 0$

Rough values:

$$u_1 = \frac{(1+1+0+0)}{4} = \frac{2}{4} = 0.5 \text{ (SFPF)}$$

$$u_2 = 0$$

$$u_3 = 0$$

$$u_4 = \frac{5+5+0+0}{4} = \frac{10}{4} = 2.5 \text{ (SFPF)}$$

—(1)

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Using SFPF, we have the following equations.

$$u_1 = \frac{(1+1+u_3+u_2)}{4} = \frac{(2+2u_2)}{4} = \frac{(1+u_2)}{2}$$
(as $u_2 = u_3$)

$$\boxed{u_1 = \frac{1+u_2}{2}}$$

$$u_2 = \frac{(2+4+u_1+u_4)}{4} = \frac{(6+u_1+u_4)}{4}$$

$$\boxed{u_2 = \frac{6+u_1+u_4}{4}}$$

$$\boxed{u_3 = u_2}$$

$$u_4 = \frac{5+5+u_2+u_3}{4} = \frac{10+2u_2}{4} = \frac{(5+u_2)}{2}$$
(as $u_2 = u_3$)

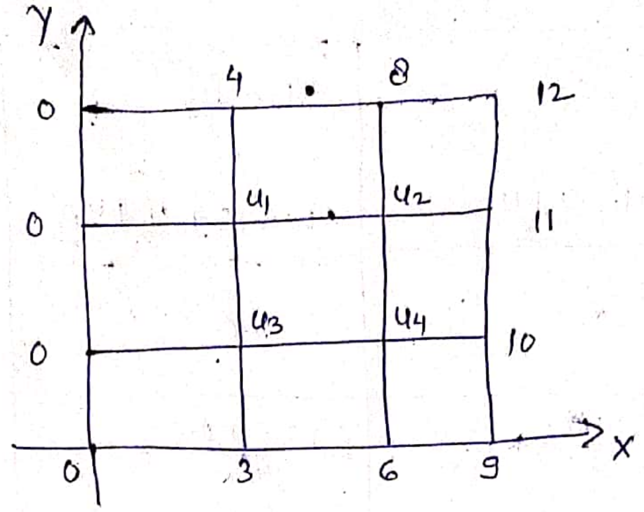
$$\boxed{u_4 = \frac{5+u_2}{2}}$$

Iteration n	u_1 $u_1 = \frac{1+u_2}{2}$	$u_2 = u_3$ $u_2 = \left(\frac{6+u_1+u_4}{4}\right)$	u_4 $u_4 = \frac{5+u_2}{2}$
0	0.5	0.0	2.5
1	$\frac{1+0}{2} = 0.5$	$\frac{6+0.5+2.5}{4} = 2.25$	$\frac{5+2.25}{2} = 3.6250$
2	1.6250	2.8125	3.9063
3	1.9063	2.9532	3.9766
4	1.9766	2.9883	3.9942
5.	1.9942	2.9971	3.9985
6.	1.9986	2.9993	3.9996
7	1.9997	2.9998	3.9999
8	1.9999	3.0000	4.0000
9.	2.0	3.0000	4.0000
10	2.0	3.0000	4.0000

solution — $u_1 = 2.0000$; $u_2 = u_3 = 3.0000$

$u_4 = 4.0000$

Example :-



solution : There is no symmetry.

let us take $u_4 = 0$

rough value

$$u_1 = \frac{0+0+8+0}{4} = 2 \quad (DFPF)$$

$$u_2 = \frac{2+8+11+0}{4} = \frac{21}{4} = 5.2500 \; (SFPF)$$

$$u_3 = \frac{(0+0+2+3)}{4} = \frac{5}{4} = 1.2500 \; (SFPF)$$

$$u_4 = 0$$

— (1)

using SFPF we can write following equation

$$u_1 = \frac{0+4+u_2+u_3}{4} = \frac{(4+u_2+u_3)}{4}$$

$$\boxed{u_1 = \frac{4+u_2+u_3}{4}} \quad — (2)$$

$$u_2 = \left(\frac{11+8+u_1+u_4}{4}\right) = \frac{(19+u_1+u_4)}{4}$$

$$\boxed{u_2 = \frac{19+u_1+u_4}{4}} \quad — (3)$$

$$u_3 = \left(\frac{0+3+u_1+u_4}{4}\right) = \left(\frac{3+u_1+u_4}{4}\right)$$

$$\boxed{u_3 = \left(\frac{3+u_1+u_4}{4}\right)} \quad — (4)$$

$$u_4 = \frac{10+6+u_2+u_3}{4} = \frac{16+u_2+u_3}{4}$$

$$\boxed{u_4 = (16+u_3+u_2)/4} \quad — (5)$$

Iteration n	$u_1 = \left(\frac{4+u_2+u_3}{4}\right)$	$u_2 = \left(\frac{19+u_1+u_4}{4}\right)$	$u_3 = \frac{3+u_1+u_4}{4}$	$u_4 = \left(16+u_3+u_2\right)/4$
0	2	5.2500	1.2500	0
1	2.6250	5.4063	1.4063	5.7031
2	2.7032	6.0516	2.0516	6.4258
3	3.4258	7.2129	3.2129	6.6065
4	3.6065	7.3032	3.3033	6.6516
5	3.6516	7.3258	3.3258	6.6629
6	3.6629	7.3315	3.3315	6.6657
7	3.6658	7.3329	3.3329	6.6664
8	3.6665	7.3332	3.3332	6.6666
9	3.6666	7.3333	3.3333	6.6667

$u_1 = 3.666$, $u_2 = 7.333$, $u_3 = 3.333$, $u_4 = 6.666$

correct upto 3^{RD} place of decimal.

Example :- solve $u_{xx} + u_{yy} = 0$ over the square mesh of side 4 units satisfying the following boundary conditions

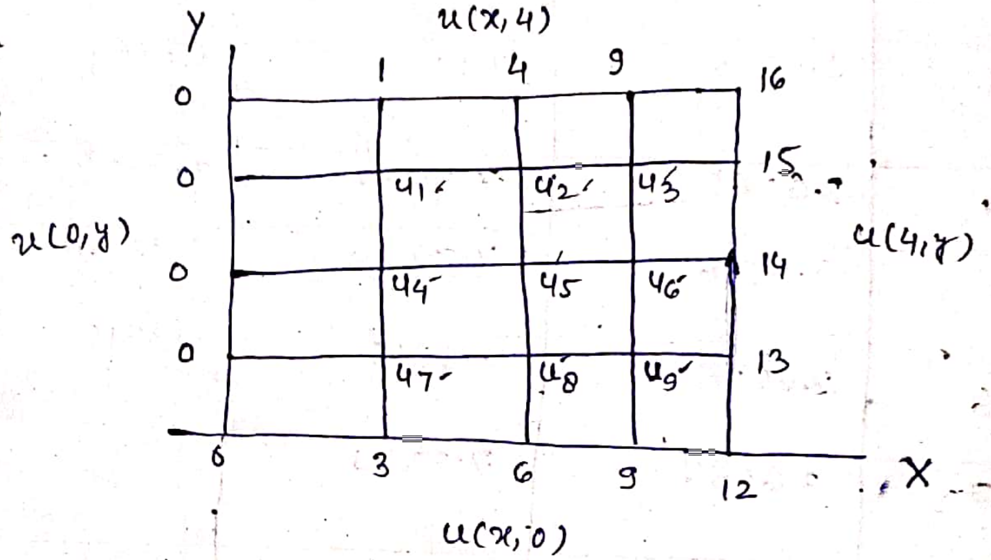
i) $u(0,y) = 0$ for $0 \le y \le 4$

ii) $u(4,y) = 12 + y$ for $0 \le y \le 4$

iii) $u(x,0) = 3x$ for $0 \le x \le 4$

iv) $u(x,4) = x^2$ for $0 \le x \le 4$

Solution



Rough value :-

$$u_5 = \frac{(0+4+14+6)}{4} = 6 \text{ (SFPF)}$$

$$u_1 = \frac{(0+6+0+4)}{4} = 2.5 \text{ (DFPF)}$$

$$u_3 = \frac{(16+6+4+14)}{4} = 10 \text{ (DFPF)}$$

$$u_2 = \frac{2.5+4+10+6}{4} = 5.625 \text{ (SFPF)}$$

$$u_9 = \frac{(6+14+12+6)}{4} = 9.5 \text{ (DFPF)}$$

$$u_6 = \frac{(6+9.5+14+10)}{4} = 9.875 \text{ (SFPF)}$$

$$u_7 = \frac{(0+6+6+0)}{4} = 3.0 \text{ (DFPF)}$$

$$u_4 = \frac{(0+2.5+6.0+3.0)}{4} = 2.8750 \text{ (SFPF)}$$

$$u_8 = \frac{6+3.0+6+9.5}{4} = 6.125 \text{ (SFPF)}$$

(1)

using SFPF following eq^n can be used to improve the rough value of u's by Gauss - Siedel method.

$u_1 = \left(\frac{0+1+u_2+u_4}{4}\right)$; $\boxed{u_1 = \frac{1+u_2+u_4}{4}}$

$u_2 = \frac{4+u_1+u_3+u_5}{4}$; $\boxed{u_2 = \frac{4+u_1+u_3+u_5}{4}}$

$u_3 = \frac{9+15+u_2+u_6}{4}$; $\boxed{u_3 = \frac{24+u_2+u_6}{4}}$

$u_4 = \left(\frac{0+u_1+u_5+u_7}{4}\right)$; $\boxed{u_4 = \frac{u_1+u_5+u_7}{4}}$

$\boxed{u_5 = \frac{(u_2+u_4+u_6+u_8)}{4}}$

$\boxed{u_6 = \frac{(14+u_3+u_5+u_9)}{4}}$

$\boxed{u_7 = \frac{3+u_4+u_8}{4}}$

$\boxed{u_8 = \frac{6+u_5+u_7+u_9}{4}}$

$u_9 = \frac{13+9+u_6+u_8}{4} = \frac{22+u_6+u_8}{4}$; $\boxed{u_9 = \frac{22+u_6+u_8}{4}}$

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n	u_1 $\frac{1+u_2+u_4}{4}$	u_2 $\frac{4+u_1+u_3+u_5}{4}$	u_3 $\frac{24+u_2+u_6}{4}$	u_4 $\frac{u_1+u_5+u_7}{4}$	u_5 $\frac{u_2+u_4+u_6+u_8}{4}$	u_6 $\frac{u_3+u_5+u_9+14}{4}$	u_7 $\frac{3+u_4+u_8}{4}$	u_8 $\frac{6+u_5+u_7+u_9}{4}$	u_9 $\frac{22+u_6+u_8}{4}$
0	2.5	5.625	10	2.8750	6.0	9.8750	3.0	6.125	9.5
1	2.3750	~~5.625~~ 5.5938	~~9.8750~~ 9.8672	~~2.8750~~ 2.8438	6.1094	9.8692	2.9922	6.1504	9.5049
2	2.3594	5.5840	9.8633	2.8653	6.1172	9.8714	3.0039	6.1565	9.5070
3	2.3623	5.5857	9.8643	2.8709	6.1211	9.8731	3.0069	6.1587	9.5080
4	2.3642	5.5874	9.8651	2.8731	6.1231	9.8740	3.0080	6.1598	9.5084

correct upto 2ND place of decimal

$u_1 = 2.36$; $u_2 = 5.58$, $u_3 = 9.86$, $u_4 = 2.87$, $u_5 = 6.12$, $u_6 = 9.87$. $u_7 = 3.00$

$u_8 = 6.15$, $u_9 = 9.50$

Poisson's Equation

$$\nabla^2 u = f(x,y)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$$

for square mesh $h = k$. ; $x = ih$, $y = jk = jh$

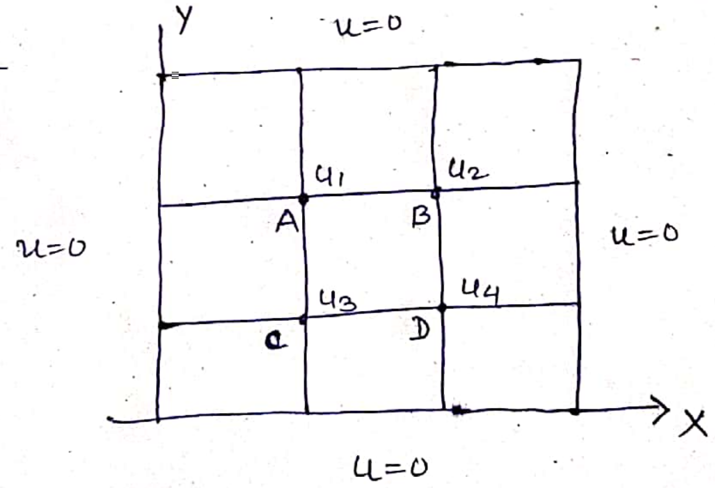
$$\frac{(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})}{h^2} + \frac{(u_{i,j+1} - 2u_{i,j} + u_{i,j-1})}{h^2} = f(ih, jh)$$

$$\boxed{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2 f(ih, jh)}$$

By applying the above formula at each mesh point, we get a system of linear equation.

Example :- solve $\nabla^2 u = -10(x^2+y^2+10)$ over the square mesh with sides $x=0$, $y=0$, $x=3$ & $y=3$ with $u=0$ on the boundary and mesh length of 1 unit.

Solution :-



At point $A(i=1, j=2)$; $h=1$

$$0 + 0 + u_2 + u_3 - 4u_1 = -10 f(ih, jh)$$
$$= -10 f(1,2)$$

$$u_2 + u_3 - 4u_1 = -150$$

$$\boxed{u_1 = \frac{(u_2 + u_3 + 150)}{4}}$$

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Similarly, $B(i=2, j=2)$

$$\boxed{u_2 = (u_1 + u_4 + 180)/4}$$

At $C(i=1, j=1)$

$$\boxed{u_3 = \frac{1}{4}(u_1 + u_4 + 120)}$$

At $D(i=2, j=1)$

$$\boxed{u_4 = \frac{1}{4}(u_2 + u_3 + 150)}$$

from above $\boxed{u_1 = u_4}$

Using Gauss-seidel formula we get

$u_1 = u_4 = 74.999$, $u_2 = 82.499$, $u_3 = 67.499$