EMT

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Assignment - 1

Ques. 1.

i) What are the Cavity resonator? Using rectangular cavity resonators, obtain the expression for resonant frequencies :-

$$f_r = \frac{1}{2\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2+\left(\frac{p}{d}\right)^2} \quad \text{for } TE_{mnp}, TM_{mnp} ?$$

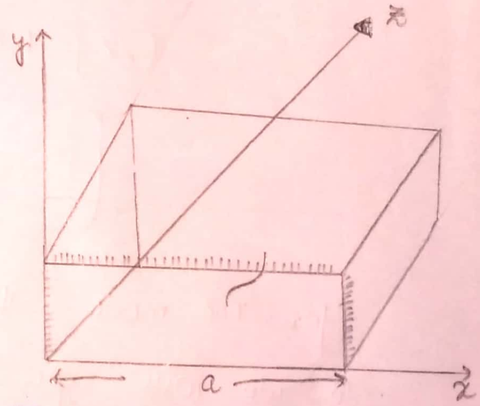
Ans. Cavity resonator is a metallic enclosure that confines the EM wave energy. The stored electric & magnetic energy determine its equivalent inductance & capacitance. The energy dissipated by its finite conductivity of the cavity wall's determine its resistance. Theoretically, a cavity has ∞ resonant modes, each corresponding to a particular resonant frequency. When amplitude is maximum, then the frequency is called resonant frequency, the lowest frequency at which amp is maximum is known as dominant mode.

Now,

$$\nabla^2\psi = \gamma^2\psi$$

where,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$



So,

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = \gamma^2\psi \quad -(1)$$

Now let $\psi = X(x)Y(y)Z(z)$ putting in (1)

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = \gamma^2 = -(k_x^2+k_y^2+k_z^2) \quad -(2)$$

Now, using seperation of variables on eqn (2)

$X = A \sin(k_x x) + B\cos(k_x x)$

$Y = C \sin(k_y y) + D\cos(k_y y)$

$Z = E \sin(k_z z) + F\cos(k_z z)$

So,

$$\psi = XYZ = [A\sin(k_x x) + B\cos(k_x x)][C\sin(k_y y) + D\cos(k_y y)][E\sin(k_z z) + F\cos(k_z z)]$$

Now, $\psi = 0$ at $x = 0,\ x = a$

$y = 0,\ y = b$

$z = 0,\ z = c$

Putting these boundaries conditions in ψ, we get

$$k_x = \frac{n\pi}{a}, \quad k_y = \frac{m\pi}{b} \quad k_z = \frac{p\pi}{c}$$

Now,

$$\gamma^2 = \left(\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2} + \frac{p^2\pi^2}{c^2}\right)$$

Now, as there is no propagation

$$\gamma^2 = -\omega^2\mu\epsilon$$

$$\Rightarrow \gamma^2 = -\omega^2\mu\epsilon = -(k_x^2 + k_y^2 + k_z^2)$$

$$(2\pi f_r)^2\mu\epsilon = \frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2} + \frac{p^2\pi^2}{c^2}$$

$$\boxed{f_r = \frac{1}{2\sqrt{\mu\epsilon}}\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2} + \frac{p^2}{c^2}}}$$

for TM mode $\psi = \vec{E}$ and putting conditions as $H_z = 0 \Rightarrow \frac{\partial}{\partial z} = 0$

so, we will get,

$$\vec{E_x} = E_{ox}[\sin(k_x x) + \cos(k_x x)]\sin(k_y y)\sin(k_z z)$$

$$\vec{E_y} = E_{oy}[\sin(k_x x)][\sin(k_y y) + \cos(k_y y)]\sin(k_z z)$$

$$\vec{E_z} = E_{oz}[\sin(k_x x)][\sin(k_y y)][\sin(k_z z) + \cos(k_z z)]$$

Now putting conditions on x, y, z

$$\vec{E_x} = E_{ox} \cos(k_x x) \sin(k_y y) \sin(k_z z)$$

$$\vec{E_y} = E_{oy} \sin(k_x x) \cos(k_y y) \sin(k_z z)$$

$$\vec{E_z} = E_{oz} \sin(k_x x) \sin(k_x x) \cos(k_z z)$$

Similarly in TE mode $E_z = 0$

$$\vec{H_x} = H_{ox} \sin(k_x x) \sin(k_y y) \cos(k_z z)$$

$$\vec{H_y} = H_{oy} \cos(k_x x) \sin(k_y y) \sin(k_z z)$$

$$\vec{H_z} = H_{oz} \sin(k_x x) \cos(k_y y) \sin(k_z z)$$

ii) What are the applications of cavity resonator? Obtain the field expressions for rectangular cavity resonator e.g for TM waves & for TE waves.

Ans. The main application of cavity resonator is storing EM energy.

Now,

TE mode in Rectangular Cavity:

$E_z = 0$ so, $\nabla^2 H_z = \gamma^2 H_z$ — ①

H_z = solution of ①

using maxwell's eqn

$$\nabla \times \vec{E} = -j\omega\mu\vec{H}$$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = -j\omega\mu\left[iH_x + iH_y + iH_z\right]$$

$$\left.\begin{aligned} \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} &= -j\omega\mu H_x \\ -\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right) &= -j\omega\mu H_y \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= -j\omega\mu H_z \end{aligned}\right\} \text{ - ②}$$

Also,

$$\nabla \times \vec{H} = j\omega\epsilon\vec{E}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = j\omega\epsilon\left[E_x + E_y + E_z\right]$$

$$\left.\begin{aligned} \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} &= j\omega\epsilon E_x \\ -\left(\frac{\partial H_z}{\partial x} - \frac{\partial H_x}{\partial z}\right) &= j\omega\epsilon E_y \\ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} &= j\omega\epsilon E_z \end{aligned}\right\} - (3)$$

for TE mode

$E_z = 0$ in (2) and (3)

$$\left.\begin{aligned} -\frac{\partial}{\partial z} + E_y &= -j\omega\mu H_x \\ \frac{\partial E_x}{\partial z} &= j\omega\mu H_y \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= -j\omega\mu H_z \\ \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} &= j\omega\epsilon E_x \\ -\left(\frac{\partial H_z}{\partial x} - \frac{\partial H_x}{\partial z}\right) &= j\omega\epsilon E_y \\ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} &= 0 \end{aligned}\right\} - (4)$$

Now putting in conditions & solving the eqn (4)

$$\vec{E_x} = E_{0x} \cos(k_x x) \sin(k_y y) \sin(k_z z)$$

$$\vec{E_y} = E_{0y} \sin(k_x x) \cos(k_y y) \sin(k_z z)$$

$$\vec{E_z} = 0$$

Now as $\vec{E_x} + \vec{E_y} + \vec{E_z} = 0$

and using eqn (4)

$$H_x = H_{0x} \sin(k_x x)\cos(k_y y)\cos(k_z z)$$
$$H_y = H_{0y} \cos(k_x x)\sin(k_y y)\cos(k_z z)$$
$$H_z = H_{0z} \cos(k_x x)\cos(k_y y)\sin(k_z z)$$

for TM mode, $H_z = 0$, so putting in ② & ③

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -j\omega\mu H_x$$
$$-\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right) = -j\omega\mu H_y$$
$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0$$
$$-\frac{\partial H_y}{\partial z} = j\omega\epsilon E_x$$
$$+\frac{\partial H_x}{\partial z} = j\omega\epsilon E_y$$
$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\epsilon E_z$$

in solving the equations and using TM conditions $H_z = 0$

and $E_x = E_{0x} \sin(k_x x)\sin(k_y y)\cos(k_z z)$

$$E_y = E_{0y} \sin(k_x x)\cos(k_y y)\sin(k_z z)$$
$$E_z = E_{0z} \cos(k_x x)\sin(k_y y)\sin(k_z z)$$

Now, $H_y = \frac{1}{j\omega\mu}\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right)$

$$= H_{0y} \cos(k_x x)\sin(k_y y)\cos(k_z z)$$

$$H_x = H_{0x} \sin(k_x x)\cos(k_y y)\cos(k_z z)$$

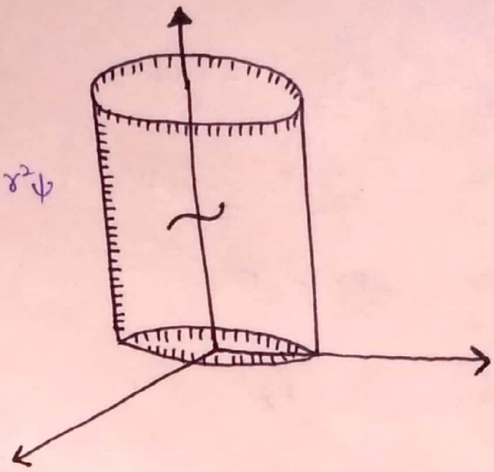
Ques. 2.

i) What is the circular cavity resonator? Obtain the expression for resonant frequency

$$f_r = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{\left(\frac{X_{np}}{a}\right)^2 + \left(\frac{q\pi}{d}\right)^2} \quad \text{for TM mode}$$

Now as wave is trapped inside resonator

$$\nabla^2\psi = \gamma^2\psi$$



$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2} = \gamma^2\psi$$

Now let $\psi = R(r)\,\Phi(\phi)\,Z(z)$

Now putting in above eqn

and let $\gamma^2 = -(k_r^2 + k_\phi^2 + k_z^2)$

Now,

$$\frac{1}{r^2}\frac{\partial}{\partial r}(rR') + \frac{1}{r^2}\Phi'' + \frac{Z''}{Z} = (-k_r^2 - k_\phi^2 - k_z^2)$$

Now in z- direction

$$\frac{Z''}{Z} = -k_z^2$$

So, $Z = A\sin(k_z z) + B\cos(k_z z)$ — (2)

Now at $z=0$ at $z=c$

$$k_z = \frac{2\pi}{c}$$

Now for ϕ putting in

$$\frac{1}{r^2}\frac{\Phi''}{\Phi} = k_\Phi^2 \Rightarrow \frac{\Phi''}{\Phi} = k_\Phi^2 r^2 = n^2$$

So, $\Phi = A\cos(n\phi) + B\sin(n\phi) = C\cos\left(n\phi + \tan^{-1}\left(\frac{A}{B}\right)\right)$

$\Rightarrow C\cos(n\phi)$ — (3)

Now putting (2) and (3) in 1

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{rR'}{R}\right) + \frac{1}{r^2}n^2 + (-k_z^2) = \gamma^2$$

$$\left[\because k_r^2 + k_\phi^2 = k_c^2\right]$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(rR') + \frac{1}{r^2}(n^2 - k_c^2 r^2)R = 0$$

$$\frac{\partial}{\partial r}(rR') - (n^2 - k_c^2 r^2)R = 0$$

This is bessel's eqn in $k_c^2 r^2$ and R

thus $R = D J_n(k_c r) + E\, Y_n(k_c r)$

as at $r = 0$ $\quad Y_n(k_c r) \rightarrow \infty$

thus $E = 0$

$$R = D J_n(k_c r)$$

$$\Rightarrow J_n(k_c r) \times D$$

So,

$$\psi = \psi_0 J_n(k_c r)\cos(n\phi)\left[A \sin(k_z z) + B\cos(k_z z)\right]$$

in TM mode

Now at $r = 0$ and $r = a$

$$J_n(k_c r) = 0$$

So, $\quad J_n(k_c r)\Big|_{r=a} = 0$

$$k_c r = X_{np}$$

$$k_c a = X_{np}$$

$$k_c = \frac{X_{np}}{a}$$

So, now $\quad \gamma^2 = -(k_c^2 + k_z^2)$

So, $\quad +\omega_r^2 \mu\epsilon = +\left(\frac{X_{np}^2}{a^2} + \frac{q^2\pi^2}{c^2}\right)$

$$\boxed{f_r = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{\frac{X_{np}^2}{a^2} + \frac{q^2\pi^2}{c^2}}}$$

Ques. 2
ii) Derive the field expression for TE_{npq} & TM_{npq} modes in a circular waveguide.

TE and TM modes in circular cavity resonators

$$\nabla \times H = j\omega\epsilon\vec{E} \quad \text{and} \quad \nabla\times\vec{E} = -j\omega\mu\vec{H}$$

$$\frac{1}{r}\frac{\partial}{\partial\phi}\vec{E_z} - \frac{\partial}{\partial z}\vec{E_\phi} = -j\omega \vec{H_r}\mu$$

$$\frac{\partial}{\partial z}\vec{E_r} - \frac{\partial}{\partial r}\vec{E_z} = -j\omega\mu\vec{H_\phi} \qquad (1)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\vec{E_\phi}) - \frac{1}{r}\frac{\partial}{\partial\phi}\vec{E_r} = -j\omega\mu\vec{H_z}$$

$$\frac{1}{r}\frac{\partial\vec{H_z}}{\partial\phi} - \frac{\partial}{\partial z}\vec{H_\phi} = j\omega\epsilon\vec{E_r}$$

$$\frac{\partial}{\partial z}\vec{H_r} - \frac{\partial}{\partial r}\vec{H_z} = j\omega\epsilon\vec{E_\phi} \qquad (1)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r(\phi)) - \frac{1}{r}\frac{\partial H_r}{\partial\phi} = j\omega\epsilon\vec{E_z}$$

Now in TE mode $E_z = 0$, So

$$-\frac{\partial}{\partial z}\vec{E_\phi} = -j\omega\mu\vec{H_r}$$

$$\frac{\partial}{\partial z}\vec{E_r} = -j\omega\mu\vec{H_\phi}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\vec{E_\phi}) - \frac{1}{r}\frac{\partial}{\partial\phi}\vec{E_r} = -j\omega\mu\vec{H_z}$$

$$\frac{1}{r}\left(\frac{\partial\vec{H_z}}{\partial\phi}\right) - \frac{\partial}{\partial z}\vec{H_\phi} = j\omega\epsilon\vec{E_r} \qquad (2)$$

$$\frac{\partial\vec{H_r}}{\partial z} - \frac{\partial\vec{H_z}}{\partial r} = j\omega\epsilon\vec{E_\phi}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rH_\phi) - \frac{1}{r}\frac{\partial}{\partial\phi}H_r = 0$$

Now putting $H_z = H_{oz} J_n(k_c r)\cos(n\phi)[A\sin(k_z z) + B\cos(k_z z)]$

Now $H_z = 0$ at $z = 0$, $z = c$

So, $H_z = H_{oz} J_n(k_c r)\cos(n\phi)\sin(k_z z)$

Now in (2) putting H_z

$E_z = 0$

$E_r = E_{or} J_n(k_c r)\cos(n\phi)\sin(k_z z)$

$E_\phi = E_{o\phi} J_n'(k_c r)\cos n\phi \sin k_z z$

$H_z = H_{oz} J_n(k_c r)\cos n\phi \sin k_z z$

$H_r = H_{or} J_n'(k_c r)\cos n\phi \sin k_z z$

$H_\phi = H_{o\phi} J_n'(k_c r)\cos n\phi \sin(k_z z)$

In TM mode $H_z = 0$ $E_z = E_{0z} J_n(K_c r) \cos n\phi \cos(K_z z)$

Now in (1)

$$\left.\begin{aligned}
&\frac{1}{r}\frac{\partial}{\partial \phi}\vec{E_z} - \frac{\partial}{\partial z}\vec{E_\phi} = -j\omega\mu\vec{H_r} \\
&\frac{\partial}{\partial z}\vec{H_r} = j\omega\epsilon\vec{E_\phi} \\
&\frac{1}{r}\frac{\partial}{\partial r}(r\vec{H_\phi}) - \frac{1}{r}\frac{\partial}{\partial \phi}\vec{H_r} = j\omega\epsilon\vec{E_z} \\
&\frac{\partial}{\partial z}\vec{E_r} - \frac{\partial}{\partial r}\vec{E_z} = -j\omega\mu H_\phi \\
&\frac{1}{r}\frac{\partial}{\partial r}(rE_\phi) - \frac{1}{r}\frac{\partial}{\partial \phi}E_r = 0
\end{aligned}\right\} \quad (3)$$

using (3)

$E_z = E_{0z} J_n(K_c r) \cos n\phi \cos(K_z z)$

$E_r = E_{0r} J_n'(K_c r) \cos n\phi \sin(K_z z)$

$E_\phi = E_{0\phi} J_n(K_c r) \sin n\phi \sin(K_z z)$

$H_z = 0$

$H_r = H_{0r} J_n(K_c r) \sin n\phi \cos(K_z z)$

$H_\phi = H_{0\phi} J_n'(K_c r) \cos n\phi \cos(K_z z)$

Ques. 3. find the resonant frequencies of first five lowest mode of an air filled rectangular cavity of dimensions 5cm × 4cm × 2.5 cm. List then in ascending order.

Ans. $5 \times 4 \times 2.5$ cm^3 assuming $a \times b \times c$

$a = 5$ cm $b = 4$ cm $c = 2.5$ cm

$$f_r = \frac{1}{2\sqrt{\mu_r \epsilon_r}}\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2} + \frac{p^2}{c^2}}$$

$$f_r = \frac{3\times10^8}{2}\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2} + \frac{p^2}{c^2}}$$

So, for 001

$$f_r = \frac{3\times10^8}{2}\sqrt{\frac{1}{c^2}} = 6 \text{ GHz}$$

for 010

$$f_r = \frac{3\times10^8}{2}\sqrt{\frac{1}{b^2}} = 3.75 \text{ GHz}$$

for 100

$$f_r = \frac{3\times10^8}{2}\sqrt{\frac{1}{a^2}} = 5\,GHz$$

for 011

$$f_r = \frac{3\times10^8}{2}\sqrt{\frac{1}{c^2}+\frac{1}{b^2}}$$

$$\Rightarrow 7.07\,GHz$$

for 110

$$f_r = \frac{3\times10^8}{2}\sqrt{\frac{1}{a^2}+\frac{1}{b^2}} = 4.8\,GHz$$

for 101

$$f_r = \frac{3\times10^8}{2}\sqrt{\frac{1}{a^2}+\frac{1}{c^2}} = 6.7\,GHz$$

So, in ascending order

010 → 3.75 GHz
110 → 4.84 GHz
100 → 5 GHz
001 → 6 GHz
101 → 6.7 GHz

Ques. 4. find the resonant frequencies of the five lowest modes of an air filled cylindrical cavity of radius 1.905 cm & length 2.54 cm. List them in ascending order.

$$K_c = \frac{X_{np}}{a} \qquad K_z = \frac{\omega\pi}{c}$$

$$a = 1.905\,cm \qquad c = 2.54\,cm$$

$$TE_{110} \Rightarrow f_r = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{\frac{X_{11}^2}{a^2}+\frac{\pi^2\omega^2}{c^2}}$$

$$\Rightarrow \frac{3\times10^8}{2\pi}\sqrt{(0.9650)^2} \Rightarrow 4.57\,GHz$$

$$TM_{111} \Rightarrow f_r = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{\frac{X_{11}^2}{a^2}+\frac{\pi^2}{c^2}}$$

$$\Rightarrow \frac{3\times10^8}{2\pi}\sqrt{4.04+1.529\times10^8} \Rightarrow 11.2\,GHz$$

$TE_{111} \Rightarrow f_r = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{\frac{x_{11}'^2}{a^2} + \frac{\pi^2}{c^2}}$

$\Rightarrow 0.477 \times 10^8 \sqrt{\left(\frac{1.84}{1.905}\right)^2 + \left(\frac{\pi}{2.54}\right)^2}$

$\Rightarrow 7.54$ GHz

$TM_{110} = f_r = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{\left(\frac{x_{21}}{a}\right)^2 + \left(\frac{\pi(0)}{2.54}\right)^2}$

$\Rightarrow 9.5$ GHz

$TM_{011} = f_r = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{\left(\frac{x_{01}}{a}\right)^2 + \left(\frac{\pi}{2.54}\right)^2}$

$\Rightarrow 0.477 \times 10^8 \sqrt{(1.59) + 1.529}$

$\Rightarrow 8.4$ GHz

$TE_{211} \Longrightarrow f_r \Rightarrow \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{\frac{(x_{21}')^2}{a^2} + \left(\frac{\pi}{c}\right)^2}$

$\Rightarrow 9.64$ GHz

Ascending order

$TE_{110} \rightarrow 4.57$ GHz

$TE_{111} \rightarrow 7.54$ GHz

$TM_{011} \rightarrow 8.4$ GHz

$TM_{110} \rightarrow 9.5$ GHz

$TE_{211} \rightarrow 9.6$ GHz

$TM_{111} \rightarrow 11.2$ GHz