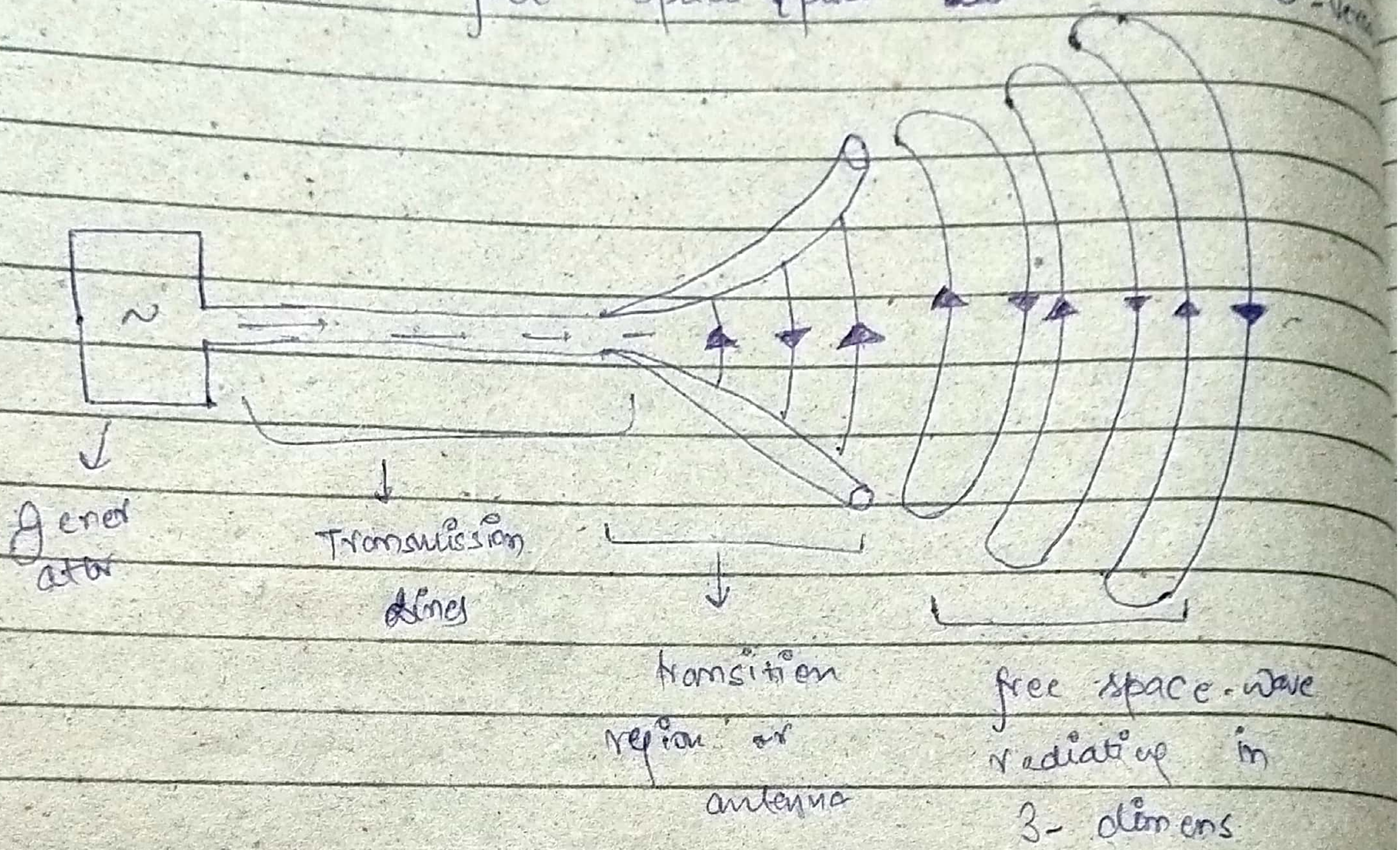
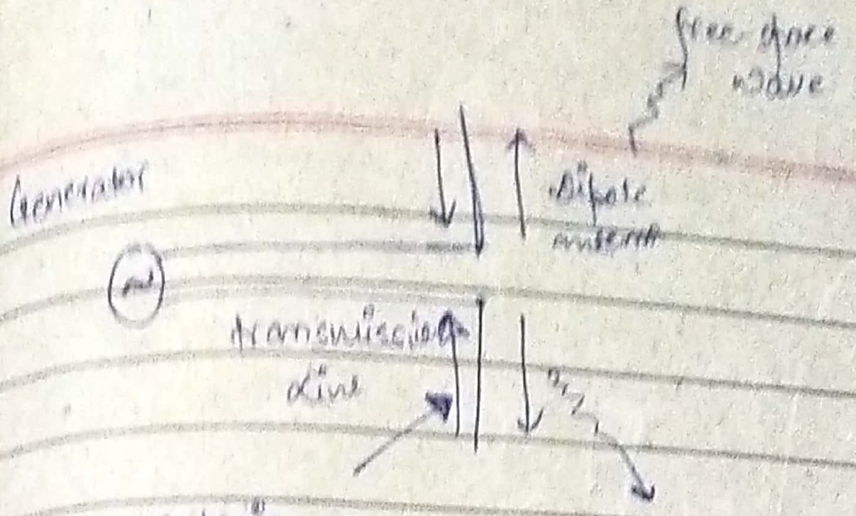
Antenna :-

* **Radio Antenna :** structure associated with the region of transition b/w a guided wave & free space space wave or vice-versa



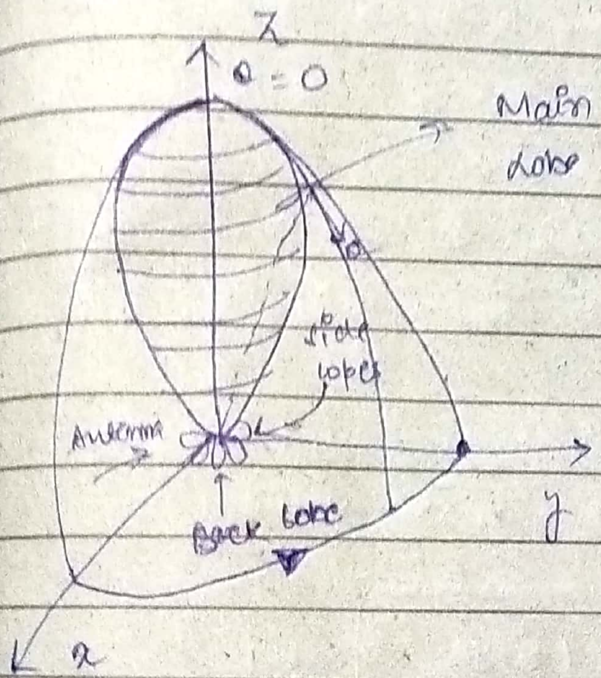
* Antenna is a transformation device converting electromagnetic photons into circuit currents & also convert energy from circuit into photons radiated into space.



Outgoing & reflected waves on antenna.

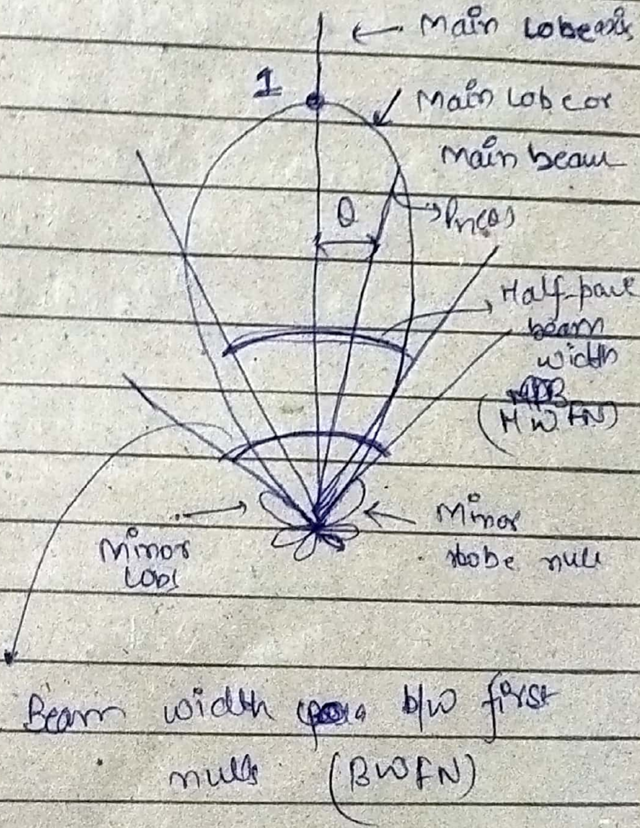
* Patterns :-

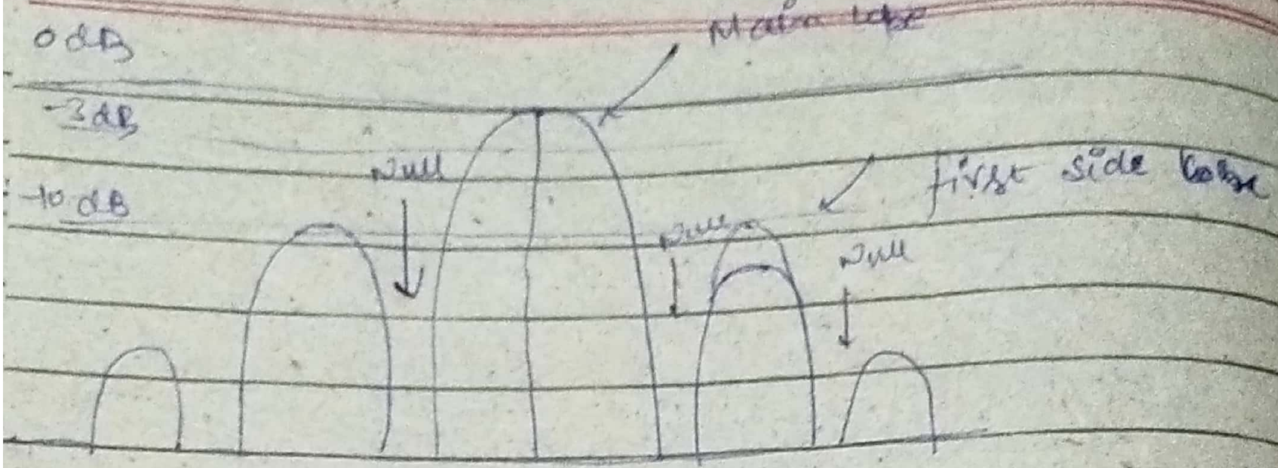
field pattern



Cartesian

Power pattern





To completely specify the radiation pattern wrt field intensity & polarization requires 3 patterns:-

1. The θ component of E as an fn. of angles θ and ϕ or $E_\theta(\theta, \phi)$ (V/m).
2. The ϕ comp. of E as a fn. of angles θ and ϕ or $E_\phi(\theta, \phi)$ (V/m)
3. The phases of these fields as a fn. of angles θ and ϕ or $\delta_\theta(\theta, \phi)$ and $\delta_\phi(\theta, \phi)$ (rad/deg)

Normalized field pattern:- divide a field comp. by its max value.

$$\boxed{E_\theta(\theta, \phi)_n = \frac{E_\theta(\theta, \phi)}{E_\theta(\theta, \phi)_{max}}} \leq 1$$

* Patterns may also be expressed in terms of the power per unit area [or Poynting vector $S(\theta,\phi)$].

Normalised power pattern :

$$\boxed{P_n(\theta,\phi) = \frac{S(\theta,\phi)}{S(\theta,\phi)_{max}}}$$

$$S(\theta,\phi) = \frac{[E_\theta^2(\theta,\phi) + E_\phi^2(\theta,\phi)]}{\underbrace{Z_0}_{376.7\,\Omega}}$$

* Beam Area :- (Beam Solid Angle)

$$dA = (r\sin\theta\, d\phi)(r\, d\theta)$$

$$\Rightarrow r^2 \sin\theta\, d\theta\, d\phi = r^2 d\Omega$$

↳ solid angle subtended by area dA.

Beam Area

$$\boxed{\Omega_A = \int_0^{2\pi}\int_0^{\pi} P_n(\theta,\phi)\, d\Omega}$$

$$d\Omega = \sin\theta\, d\theta\, d\phi$$

This solid angle can often be described approximately in terms of the angles subtended by the half power points of the main lobe in the 2 principle planes as given by

$$\boxed{\Omega_A \approx \theta_{HP}\, \phi_{HP}}$$

$\theta_{HP}, \phi_{HP} \rightarrow$ Half-power beam widths in the 2 principal planes, minor lobes being neglected.

* <u>Radiation</u> <u>Intensity</u> :- U (watt/sr)

Power radiated from an antenna per unit solid angle.

independent of distance — depends on distance $\propto \frac{1}{r^2}$

$$P_n(\theta,\phi) = \frac{U(\theta,\phi)}{U(\theta,\phi)_{max}} = \frac{S(\theta,\phi)}{S(\theta,\phi)_{max}}$$

* <u>Beam</u> <u>Efficiency</u> :

the (total) beam area Ω_A (or beam solid angle) consists of the main beam area Ω_M plus the minor-lobe area Ω_m

$$\Omega_A = \Omega_M + \Omega_m$$

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Beam efficiency ε_m,

$$\boxed{\varepsilon_m = \frac{\Omega_m}{\Omega_A}}$$

* stray factor :

$$\boxed{\varepsilon_m = \frac{\Omega_m}{\Omega_A}}$$

$$\therefore \boxed{\varepsilon_M + \varepsilon_m = 1}$$

* Directivity :- D → Ratio of Max. Radiation intensity to the average radiation intensity (U_{av})

or

Ratio of max. Poynting vector to av. Poynting vector

$$\boxed{D = \frac{U(\theta,\phi)_{max}}{U_{av}} = \frac{S(\theta,\phi)_{max}}{S_{av}}}$$

Av. Poynting vector

$$S(\theta,\phi) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} S(\theta,\phi)\, d\Omega \quad (W/m^2)$$

$\cancel{4\pi} = 41,000$

$$\boxed{4\pi \ sr = 41253^0} \text{ sq. degrees}$$

$$D = \frac{S(\theta,\phi)_{max}}{\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} S(\theta,\phi)\, d\Omega}$$

$$D = \frac{1}{\frac{1}{4\pi}\iint \frac{S(\theta,\phi)}{S(\theta,\phi)_{max}}\, d\Omega}$$

$$D = \frac{1}{\frac{1}{4\pi}} \frac{1}{\iint P_n(\theta,\phi)\, d\Omega}$$

$$D = \frac{4\pi}{\Omega_A}$$

$$\boxed{D = \frac{4\pi}{\Omega_A}}$$

$$\Rightarrow \boxed{D \geq 1}$$

e.g. If an antenna has a main lobe with both half-power beam widths (HPBWs = 20°), it

$$D = \frac{4\pi\ (sr)}{\Omega_A\ (sr)} = \frac{41,000(\text{deg}^2)}{\theta^\circ_{HP}\ \phi^\circ_{HP}} = \frac{41,000}{20 \times 20}$$

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$\simeq 103 \simeq 20$ dBi (dB above isotropic)

* Gain: Depends on its D & efficiency.

$$\boxed{G = kD}$$

$k \rightarrow$ efficiency factor of antenna

$0 \leq k \leq 1$

* Resolution: $\left(\frac{BWFN}{2}\right)^2$

* Directivity is equal to the no. of beam areas into which the antenna pattern can subdivide the sky & D is equal to no. of pt. sources in the sky that the antenna can resolve under the assumed ideal conditions. of a uniform source distribution

* Significance of area:

for a loseless Antenna

$$P = \frac{V^2}{4R_r}$$ ← Radiation resistance

$$A_{em} = \frac{V^2}{4 R_r S}$$

$A_{em} \rightarrow$ Max effective aperture

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Aperture efficiency :-

$$\varepsilon_{ap} = \frac{A_e}{A_p}$$

$A_e \rightarrow$ eff. aperture

$A_p \rightarrow$ physical aperture

Power

$$\boxed{P = \frac{E_a^2}{Z_0} A_e} \text{ watt}$$

$Z_0 = 377$

Aperture beam Area

$$\lambda^2 = A_e \Omega_A$$

Directivity

$$D = \frac{4\pi A_e}{\lambda^2}$$

* Determine the length of Antenna operating at frequency of 500 kHz.

$$\boxed{l = \frac{c}{f} \times 0.950}$$

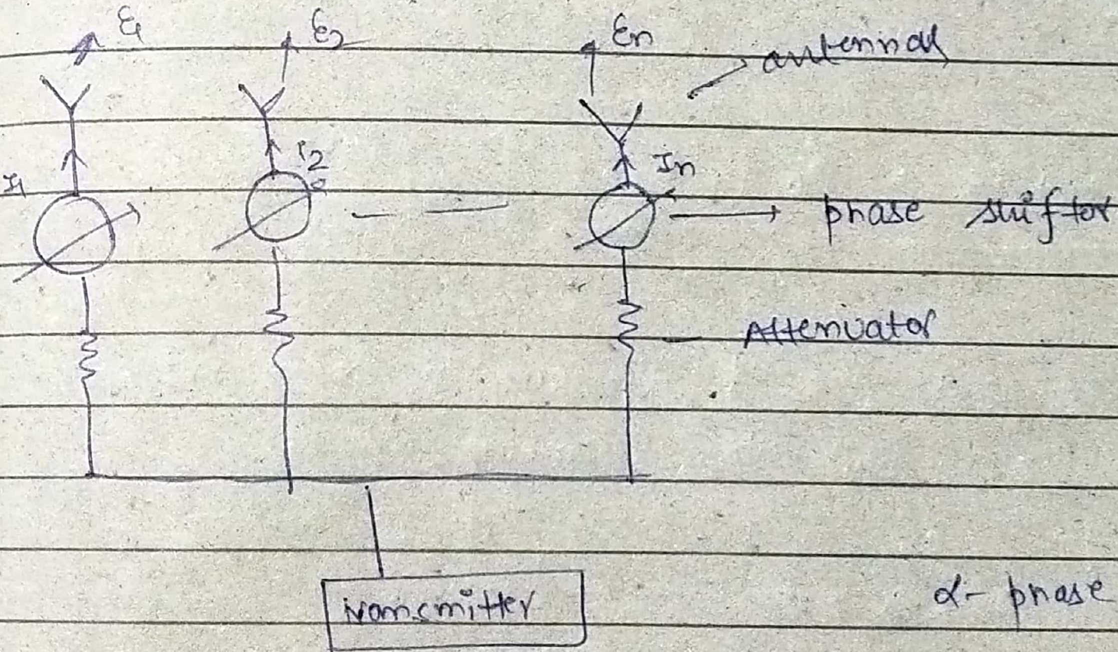
* Antenna Arrays :-

In wireless comm., we need to have narrow beam for large distance comm. So, it is possible by 2 ways.

i) Incr. the size of antenna

ii) using antenna array.

* Antenna formed by multiple elements of antenna is antenna array.

* 

α - phase of curr.

$\rightarrow \vec{I_1} = I_1 e^{j\alpha_1}$, $\vec{I_2} = I_2 e^{j\alpha_2}$, $\vec{I_n} = I_n e^{j\alpha_n}$

$\Rightarrow \vec{E} = \vec{E_1} + \vec{E_2} + \text{———} \vec{E_n}$

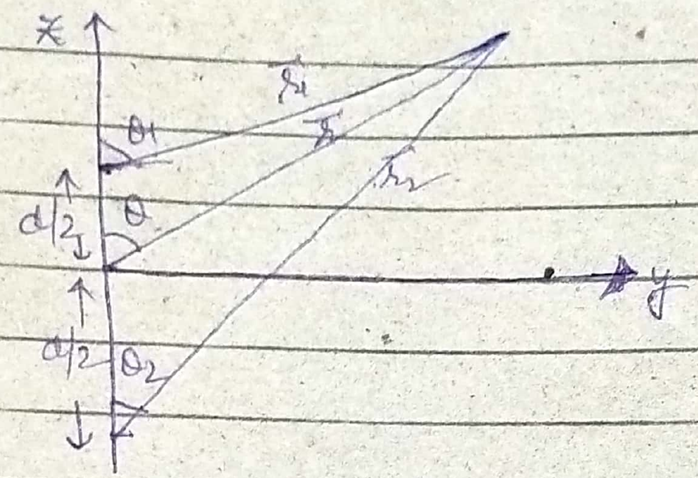
$\Rightarrow E_1 e^{j\psi_1} + E_2 e^{j\psi_2} + \text{———} E_n e^{j\psi_n}$

$\boxed{\psi = \beta d + \alpha}$ $\quad$ $\left(\beta = \frac{2\pi}{\lambda} \times d\right)$

d - spacing b/w elements

$\alpha \rightarrow$ initial phase

* **2 - element Arrays :-**



$$E_t = E_1 + E_2 = \hat{a}_\theta \, j\eta \frac{k I_0 l}{4\pi} \left[\frac{e^{-j(kr_1 - \frac{\beta}{2})}}{r_1} \cos\theta_1 + \frac{e^{-j(k r_2}}{r_2} \right.$$

$\beta \rightarrow$ the difference in phase excition b/w the elements

Let

$$\left.\begin{aligned} \theta_1 &\approx \theta_2 \approx \theta \\ r_1 &\simeq r - \frac{d}{2}\cos\theta \\ r_2 &\simeq r + \frac{d}{2}\cos\theta \end{aligned}\right\} \text{for phase variation}$$

$r_1 \approx r_2 \approx r$ for amp. variation

So,

$$E_t = \hat{a}_\theta \, j\eta \frac{K I_0 l}{4\pi r} e^{-jkr} \cos\theta \left[\frac{e^{j(kd\cos\theta + \beta)}}{2} + \frac{e^{-j(kd\cos\theta + \beta)}}{2} \right]$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$E_t = a_\theta \, j\eta \frac{K I_0 l}{4\pi r} e^{-jkr} \cos\theta \; \underbrace{2\cos\left(\frac{1}{2} kd\cos\theta + \beta\right)}_{Af} \quad -②$$

$\eta \rightarrow$ intrinsic impedance of medium

$$Af \text{ (array factor)} = 2\cos\left(\frac{1}{2} kd\cos\theta + \beta\right)$$

Normalised form

$$(Af)_n = \cos\left[\frac{1}{2}\left(kd\cos\theta + \beta\right)\right]$$

E (total) = [E (single element at reference pt.)] x Array factor

→ the array factor, in general is the function of the no. of elements, their geometrical arrangement, their relative magnitude, their relative phase & their spacing

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* 2 Ques

* N - element array :-

Uniform amplitude & spacing :-

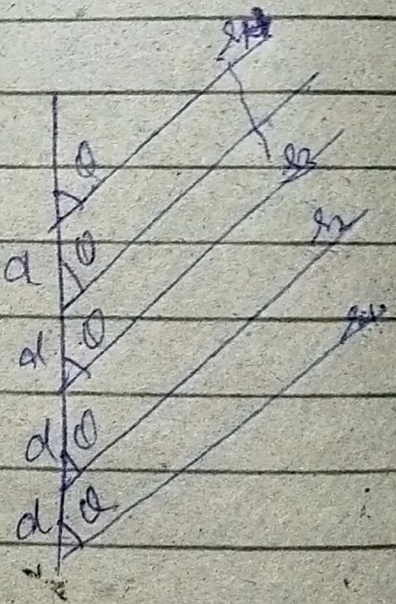
N - element arrays
β - progressive phase

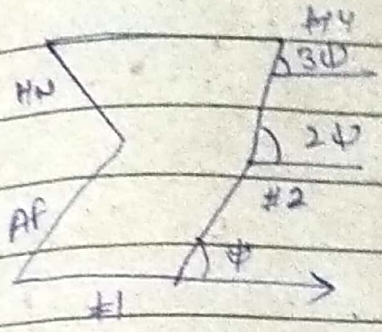
$$Af = 1 + e^{j(kd\cos\theta + \beta)} + e^{j2(kd\cos\theta+\beta)} + e^{j3(\ldots)} + \ldots + e^{j(N-1)[kd\cos\theta+\beta]}$$

$$\boxed{Af = \sum_{n=1}^{N} e^{j(n-1)\underbrace{(kd\cos\theta+\beta)}_{\psi}}} \quad -① $$

$$Af = \sum_{n=1}^{N} e^{j(n-1)\psi}$$

②





(b) Phase diagram

Multiplying eqn (2) by $e^{j\psi}$ both sides, we ge

$$Af \cdot e^{j\psi} = e^{j\psi} + e^{j2\psi} + e^{j3\psi} \ldots\ldots e^{j\psi N} \quad -(3)$$

eqn (3) − eqn (2)

$$Af\,(e^{j\psi}-1) = (-1 + e^{jN\psi}) \quad -(4)$$

$$\boxed{Af = \left(\frac{e^{j\psi N}-1}{e^{j\psi}-1}\right)}$$

$$\Rightarrow \frac{e^{j(N-1)\psi}}{2}\left[\frac{e^{j\frac{N}{2}\psi} - e^{-j\frac{N\psi}{2}}}{e^{\frac{j\psi}{2}} - e^{-\frac{j\psi}{2}}}\right]$$

$$Af = \boxed{e^{j\frac{(N-1)\psi}{2}} \; \frac{\sin\left(\frac{N\psi}{2}\right)}{\sin\left(\frac{\psi}{2}\right)}} \quad -(5)$$

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consider reference point

$N = 1$

$$AF = \frac{\sin \frac{N\psi}{2}}{\sin \frac{\psi}{2}} \quad \text{---} (6)$$

for small value of ψ

$$AF = \frac{\sin \frac{N\psi}{2}}{\frac{\psi}{2}} \quad - (7)$$

$\cancel{180 \times 28}$ / $\cancel{16}$ 4

30×13 / 9

300×41

$\cancel{800} \times 90$ 8 / $\cancel{100}$ 720

Max. value of eqn (6) & (7)

eqn $(AF)_{max} = \frac{\frac{N\psi}{2}}{\psi/2} = N$

eqn (7) $(AF)_{max} = \frac{\frac{N\psi}{2}}{\psi/2} = N$

so, normalize array factor

$$(AF)_n = \frac{1}{N} \frac{\sin \left(\frac{N\psi}{2}\right)}{(\psi/2)} \quad - (8)$$

$$(AF)_n \approx \frac{\sin\left(\frac{N\psi}{2}\right)}{\left(\frac{N\psi}{2}\right)} \quad \text{— (9)}$$

to find the nulls of the array

$\sin \frac{N\psi}{2} = 0 \qquad \left.\frac{N\psi}{2}\right|_{\theta=\theta_n} = n\pi$

$$\theta_n = \cos^{-1}\left[\frac{\lambda}{2d\pi}\left(-\beta + \frac{2n\pi}{N}\right)\right] \quad \text{— (12)}$$

$\psi = \frac{2n\pi}{N} \qquad \psi = kd\cos\theta + \beta$

$\beta = d$

$\pi = 0$

$d = \beta$

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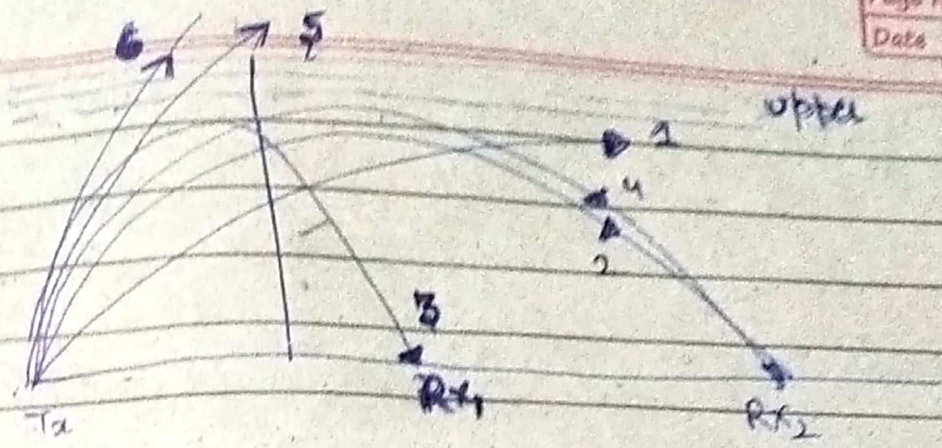
* Critical frequency & skip distance :-

Skip distance :- is the distance a radio wave travels, usually including a hop in the ionosphere.
A skip distance is a distance on the Earth's surface b/w the 2 pts. where radio waves from a transmitter, refracted downwards by diff. layers of ionosphere, fall.

Critical frequency :- is the highest magnitude of frequency above which the waves penetrates the ionosphere & below which the waves are reflected back from the ionosphere. Denoted by fc.

Max. usable freq (MUF) :- is highest radio freq that can be used for transmission b/w 2 pt. via reflection from the ionosphere (skywave or "skip" propagation) at a specified time, independent of transmitted power



ϕ_0 : Angle of incidence

refractive index $\eta = \sin\phi_0$

$\phi_0 = \phi_c$ (critical angle)

Max. ~~usable~~ freq.

$\phi_0 \rightarrow$ angle of incidence

$\phi_i \rightarrow$ angle of incident

$\phi_r \rightarrow$ angle of refraction

$$\eta = \frac{\sin\phi_i}{\sin\phi_2} = \frac{\sin\phi}{\sin\phi_0} \Rightarrow \sin\phi_i = \frac{\sqrt{1 - 81 N_{max}}}{f_{MUF}}$$

$$\sin^2\phi_i = 1 - \frac{81 N_{max}}{f^2_{MUF}}$$

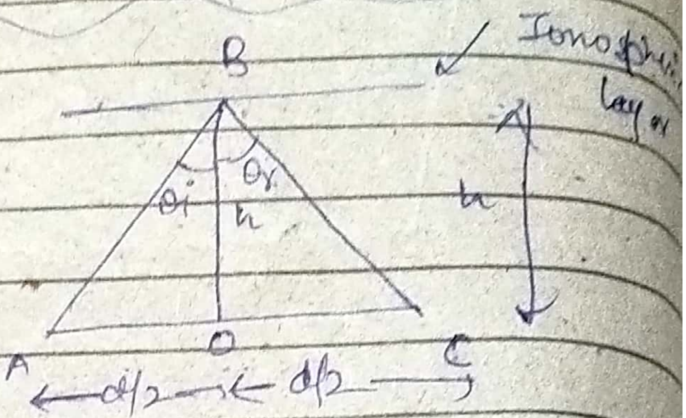
or $$\frac{81 N_{max}}{f^2_{MUF}} = \cos^2\phi_i$$

$$\boxed{f_{MUF} = f_c \sec\phi_i}$$

$$\{f_c^2 = 81 N_{max}\}$$

* Relation b/w MUF & skip distance :-

For flat Earth case



$$\cos\theta_i = \frac{OB}{AB}$$

$$\cos\theta_i = \frac{h}{\sqrt{h^2 + \frac{d^2}{4}}}$$

$$\cos^2\theta_i = \frac{4h^2}{4h^2 + d^2}$$

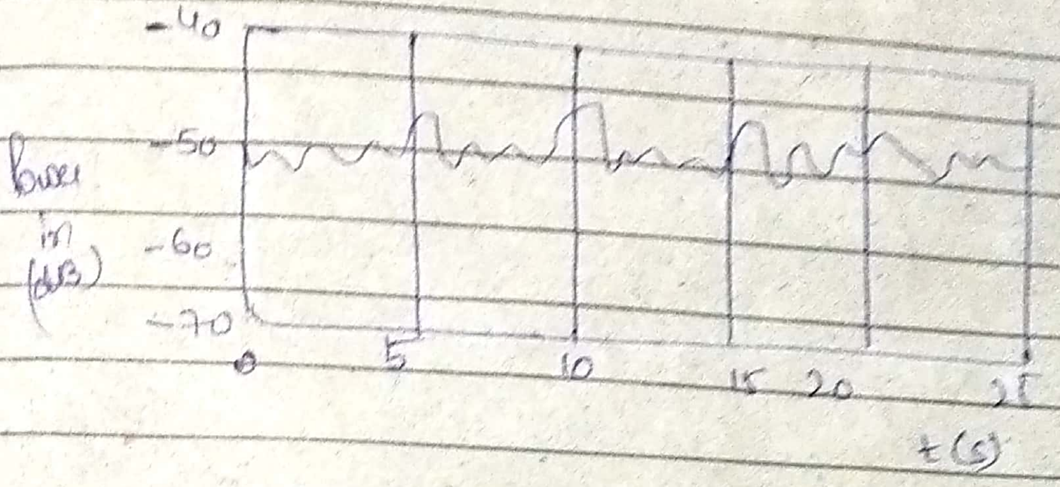
$$\frac{f_c^2}{f_{muf}^2} = \cos^2\theta_i = \frac{4h^2}{4h^2+d^2}$$

$$\boxed{f_{muf} = f_c\left(1 + \frac{d^2}{4h^2}\right)^{1/2}}$$

or

$$d = 2h\left(\frac{f_{muf}^2}{f_c^2} - 1\right)^{1/2}$$

* **Fading :-** Fading is variation of attenuation of a signal with various variables. These variables include time, geographical position & radio frequency.



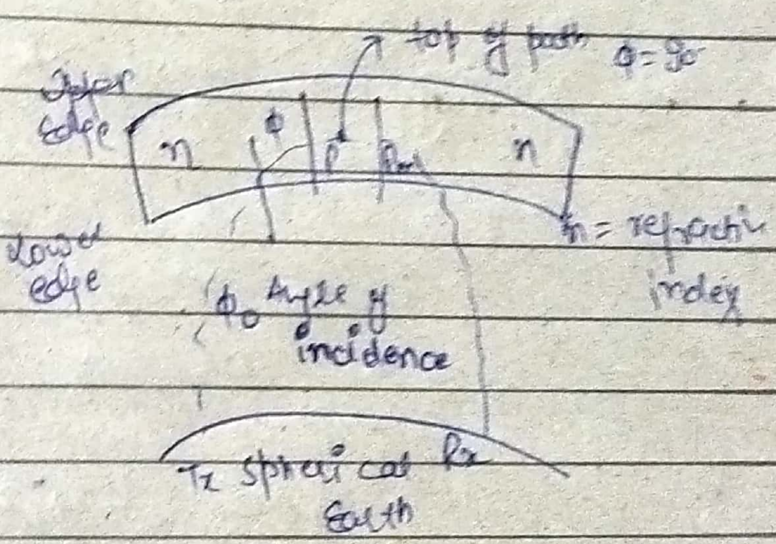
* **Refraction & reflection of sky wave by ionosphere :-**

$$n = \sqrt{\epsilon} = \sqrt{1 - \frac{81N}{f^2}}$$

$$n \sin\phi = \sin\phi_0$$

$$v_p = c/n$$

$N \rightarrow$ no. of e^-/cm^3



Bending of wave from an ionospheric layer

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A <u>plane wave</u>

$$E = E_x a_x$$
$$\downarrow$$
$$E \sin\omega t$$

$$H = H_y a_y$$

$$F_x = -e E_x$$

$$m\frac{d^2x}{dt^2} = -eE\sin\omega t$$

$$\frac{dx}{dt} = \frac{+eE}{m\omega}\cos\omega t$$

So, if there are N e^-/m^3 in space, each carrying a charge $-e$ & mass m, So

$$J = -Ne\frac{dx}{dt} = -\frac{Ne^2E}{m\omega}\cos\omega t$$

also $J = \sigma E$

$$\boxed{\sigma = -\frac{Ne^2E}{m\omega}}$$

Maxwell

$$\nabla \times H = J + \frac{\partial D}{\partial t} = \sigma E + \epsilon \frac{\partial E}{\partial t}$$

$$\nabla \times E = -\frac{\partial B}{\partial t} = -\mu \frac{\partial H}{\partial t}$$

$$J + \epsilon_0 \frac{\partial E_x}{dt} = -\frac{\partial H_y}{\partial z}$$

$$\mu_0 \frac{\partial H_y}{\partial t} = -\frac{\partial E_x}{\partial z}$$

$$\sigma E + \epsilon_0 \frac{\partial E_x}{\partial t} = -\frac{\partial H_y}{\partial x}$$

$$-\frac{Ne^2 E}{m\omega^2} \cos \omega t + \epsilon_0 \frac{\partial}{\partial t}(E \sin \omega t) = -\frac{\partial H_y}{\partial x}$$

$$\left(\epsilon_0 - \frac{Ne^2}{m\omega^2}\right) \omega E \cos \omega t = \epsilon_0 \left(1 - \frac{Ne^2}{m\omega^2 \epsilon_0}\right)$$

$\epsilon_0 \left(\cancel{1 - Ne^2} \right.$

$$\epsilon_0 \epsilon_r \omega E \cos \omega t = -\frac{\partial H_y}{\partial H_y}$$

$$\epsilon_r = \left(1 - \frac{Ne^2}{m\omega^2 \epsilon_0}\right)$$

$$n = \sqrt{\epsilon_r}$$

$$n = \sqrt{1 - \frac{Ne^2}{m\omega^2\epsilon_0}}$$

$$n = \sqrt{1 - \frac{Ne^2}{m\epsilon_0 (2\pi f)^2}}$$

$$n = \left(1 - \frac{81N}{f^2}\right)^{1/2}$$

phase velocity

$$v_p = \frac{c}{\sqrt{\epsilon_r}} = \frac{c}{\left(1 - \frac{81N}{f^2}\right)^{1/2}} \quad \mu r = 1$$

Group velocity

$$v_g = c \left(1 - \frac{81N}{f^2}\right)^{1/2}$$

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Attenuation :-

Power attenuation

$$\alpha_p = 10 \log_{10} \frac{P_1}{P_2} = 10 \log_{10} \frac{P_t / 4\pi r_1^2}{P_t / 4\pi r_2^2}$$

$$\Rightarrow 10 \log \left(\frac{r_2}{r_1}\right)^2$$

$$\boxed{\alpha_p = 20 \log \frac{r_2}{r_1}}$$

field intensity attenuation

$$\alpha_E = 20 \log \sqrt{\frac{30 P_t / r_1}{30 P_t / r_2}}$$

$$\boxed{\alpha_E = 20 \log \frac{r_2}{r_1}}$$

Refraction

$$\text{Gain } (P_r) = \frac{P_t G_1 G_2 d_0}{(4\pi S)^2}$$

$$P_r = \frac{P_t G^2 d_0}{(4\pi S)^2}$$

$$G = 4\pi S \sqrt{\frac{P_r}{P_t d_0}}$$

Gain in dB

$10 \log G$

* Ground waves :-

distance b/w transmitted & receiver

$$d = \frac{50}{(f_{MHz})^{1/3}}$$ in miles

* Space waves :-

field strength at a distance

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$\int^{\infty}$

$E = \dfrac{120\pi h_t I}{\lambda d}$ in ~~V~~ V/m

$E = \dfrac{\sqrt{30 P_t}}{d}$

h_t → eff. height of transmitting antenna

h_r → receiving antenna

d → distance

MUF = 10 MHz

h = 300 km

μ = 0.9

$\mu = \left(1 - \dfrac{81 N}{f^2}\right)$

1. The desired wave equations are written in the form of either rectangular or cylindrical coordinate systems suitable to the problem at hand.
2. The boundary conditions are then applied to the wave equations set up in step 1.
3. The resultant equations usually are in the form of partial differential equations in either time or frequency domain. They can be solved by using the proper method.

4-1 RECTANGULAR WAVEGUIDES

A rectangular waveguide is a hollow metallic tube with a rectangular cross section. The conducting walls of the guide confine the electromagnetic fields and thereby guide the electromagnetic wave. A number of distinct field configurations or modes can exist in waveguides. When the waves travel longitudinally down the guide, the plane waves are reflected from wall to wall. This process results in a component of either electric or magnetic field in the direction of propagation of the resultant wave; therefore the wave is no longer a *transverse electromagnetic* (TEM) wave. Figure 4-1-1 shows that any uniform plane wave in a lossless guide may be resolved into TE and TM waves.

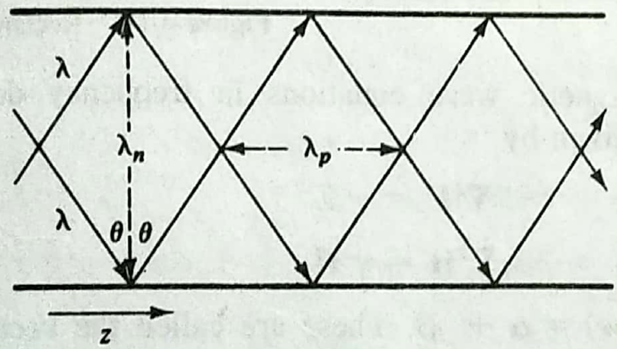

Figure 4-1-1 Plane wave reflected in a waveguide.

It is clear that when the wavelength λ is in the direction of propagation of the incident wave, there will be one component λ_n in the direction normal to the reflecting plane and another λ_p parallel to the plane. These components are

$$\boxed{\lambda_n = \frac{\lambda}{\cos\theta}} \tag{4-1-1}$$

$$\boxed{\lambda_p = \frac{\lambda}{\sin\theta}} \tag{4-1-2}$$

where θ = angle of incidence
λ = wavelength of the impressed signal in unbounded medium

A plane wave in a waveguide resolves into two components: one standing wave in the direction normal to the reflecting walls of the guide and one traveling wave in the direction parallel to the reflecting walls. In lossless waveguides the modes may be classified as either *transverse electric* (TE) mode or *transverse magnetic* (TM) mode. In rectangular guides the modes are designated TE_{mn} or TM_{mn}. The integer m

denotes the number of half waves of electric or magnetic intensity in the x direction and n is the number of half waves in the y direction if the propagation of the wave is assumed in the positive z direction.

4-1-1 Solutions of Wave Equations in Rectangular Coordinates

As stated previously, there are time-domain and frequency-domain solutions for each wave equation. However, for the simplicity of the solution to the wave equation in three dimensions plus a time-varying variable, only the sinusoidal steady-state or the frequency-domain solution will be given. A rectangular coordinate system is shown in Fig. 4-1-2.

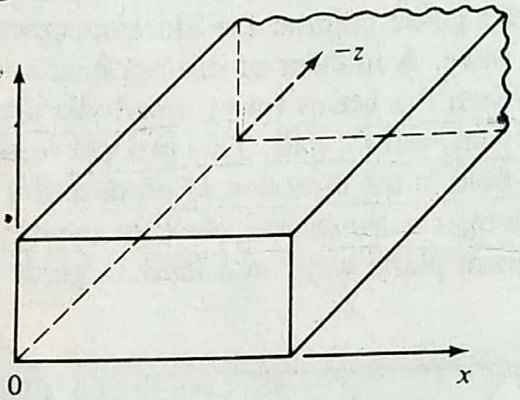

Figure 4-1-2 Rectangular coordinates.

The electric and magnetic wave equations in frequency domain in Eqs. (2-1-20) and (2-1-21) are given by

$$\nabla^2 \mathbf{E} = \gamma^2 \mathbf{E} \tag{4-1-3}$$

$$\nabla^2 \mathbf{H} = \gamma^2 \mathbf{H} \tag{4-1-4}$$

where $\gamma = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)} = \alpha + j\beta$. These are called the *vector wave equations*.

Rectangular coordinates are the usual right-hand system. The rectangular components of **E** or **H** satisfy the complex scalar wave equation or Helmholtz equation

$$\nabla^2 \psi = \gamma^2 \psi \tag{4-1-5}$$

The Helmholtz equation in rectangular coordinates is

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \gamma^2 \psi \tag{4-1-6}$$

This is a linear and inhomogeneous partial differential equation in three dimensions. By the method of separation of variables, the solution is assumed in the form of

$$\psi = X(x)Y(y)Z(z) \tag{4-1-7}$$

where $X(x)$ = a function of the x coordinate only
$Y(y)$ = a function of the y coordinate only
$Z(z)$ = a function of the z coordinate only

Substitution of Eq. (4-1-7) in Eq. (4-1-6) and division of the resultant by Eq. (4-1-7) yield

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = \gamma^2 \tag{4-1-8}$$

Since the sum of the three terms on the left-hand side is a constant and each term is independently variable, it follows that each term must be equal to a constant.

Let the three terms be k_x^2, k_y^2, and k_z^2, respectively; then the separation equation is given by

$$-k_x^2 - k_y^2 - k_z^2 = \gamma^2 \tag{4-1-9}$$

The general solution of each differential equation in Eq. (4-1-8)

$$\frac{d^2X}{dx^2} = -k_x^2X \tag{4-1-10}$$

$$\frac{d^2Y}{dy^2} = -k_y^2Y \tag{4-1-11}$$

$$\frac{d^2Z}{dz^2} = -k_z^2Z \tag{4-1-12}$$

will be in the form of

$$X = A\sin(k_xx) + B\cos(k_xx) \tag{4-1-13}$$

$$Y = C\sin(k_yy) + D\cos(k_yy) \tag{4-1-14}$$

$$Z = E\sin(k_zz) + F\cos(k_zz) \tag{4-1-15}$$

The total solution of the Helmholtz equation in rectangular coordinates is

$$\psi = [A\sin(k_xx) + B\cos(k_xx)][C\sin(k_yy) + D\cos(k_yy)] \times [E\sin(k_zz) + F\cos(k_zz)] \tag{4-1-16}$$

The propagation of the wave in the guide is conventionally assumed in the positive z direction. It should be noted that the propagation constant γ_g in the guide differs from the intrinsic propagation constant γ of the dielectric. Let

$$\gamma_g^2 = \gamma^2 + k_x^2 + k_y^2 = \gamma^2 + k_c^2 \tag{4-1-17}$$

where $k_c = \sqrt{k_x^2 + k_y^2}$ is usually called the *cutoff wave number*. For a lossless dielectric, $\gamma^2 = -\omega^2\mu\epsilon$. Then

$$\gamma_g = \pm\sqrt{\omega^2\mu\epsilon - k_c^2} \tag{4-1-18}$$??

There are three cases for the propagation constant γ_g in the waveguide.

Case I. There will be no wave propagation (evanescence) in the guide if $\omega_c^2\mu\epsilon = k_c^2$ and $\gamma_g = 0$. This is the critical condition for cutoff propagation. The cutoff frequency is expressed as

$\gamma_g^2\ \omega^2\mu\epsilon - \gamma_g^2$

$$f_c = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{k_x^2 + k_y^2} = \frac{c}{2\pi}\sqrt{k_x^2+k_y^2} \qquad (4\text{-}1\text{-}19)$$

Case II. The wave will be propagating in the guide if $\omega^2\mu\epsilon > k_c^2$ and

$$\gamma_g = \pm j\beta_g = \pm j\omega\sqrt{\mu\epsilon}\sqrt{1 - \left(\frac{f_c}{f}\right)^2} \qquad (4\text{-}1\text{-}20)$$

This means that the operating frequency must be above the cutoff frequency in order for a wave to propagate in the guide.

Case III. The wave will be attenuated if $\omega^2\mu\epsilon < k_c^2$ and

$$\gamma_g = \pm\alpha_g = \pm\omega\sqrt{\mu\epsilon}\sqrt{\left(\frac{f_c}{f}\right)^2 - 1} \qquad (4\text{-}1\text{-}21)$$

This means thaf if the operating frequency is below the cutoff frequency, the wave will decay exponentially with respect to a factor of $-\alpha_g z$ and there will be no wave propagation because the propagation constant is a real quantity. Therefore the solution to the Helmholtz equation in rectangular coordinates is given by

$$\psi = [A\sin(k_x x) + B\cos(k_x x)][C\sin(k_y y) + D\cos(k_y y)]e^{-j\beta_g z} \qquad (4\text{-}1\text{-}22)$$

4-1-2 TE Modes in Rectangular Waveguides

to dominant mode

$E_x = 0$ at $y = 0, b$

$E_y = 0$ at $x = 0, a$

It has been previously assumed that the waves are propagating in the positive z direction in the waveguide. Figure 4-1-3 shows the coordinates of a rectangular waveguide.

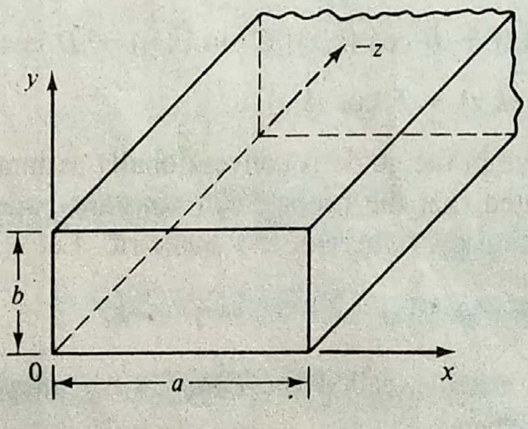

Figure 4-1-3 Coordinates of a rectangular guide.

The TE$_{mn}$ modes in a rectangular guide are characterized by $E_z = 0$. In other words, the z component of the magnetic field, H_z, must exist in order to have energy transmission in the guide. Consequently, from a given Helmholtz equation,

$$\nabla^2 H_z = \gamma^2 H_z \qquad (4\text{-}1\text{-}23)$$

a solution in the form of

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$$H_z = \left[A_m \sin\left(\frac{m\pi x}{a}\right) + B_m \cos\left(\frac{m\pi x}{a}\right)\right] \times \left[C_n \sin\left(\frac{n\pi y}{b}\right) + D_n \cos\left(\frac{n\pi y}{b}\right)\right] e^{-j\beta_g z}$$

(4-1-24)

will be determined in accordance with the given boundary conditions, where $k_x = m\pi/a$ and $k_y = n\pi/b$ are replaced. For a lossless dielectric, Maxwell's curl equations in frequency domain are

$$\nabla \times \mathbf{E} = -j\omega\mu \mathbf{H} \tag{4-1-25}$$

$$\nabla \times \mathbf{H} = j\omega\epsilon \mathbf{E} \tag{4-1-26}$$

In rectangular coordinates, their components are

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -j\omega\mu H_x \tag{4-1-27}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y \tag{4-1-28}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z \tag{4-1-29}$$

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = j\omega\epsilon E_x \tag{4-1-30}$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\omega\epsilon E_y \tag{4-1-31}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\epsilon E_z \tag{4-1-32}$$

With the substitution $\partial/\partial z = -j\beta_g$ and $E_z = 0$, the foregoing equations are simplified to

$$\beta_g E_y = -\omega\mu H_x \tag{4-1-33}$$

$$\beta_g E_x = \omega\mu H_y \tag{4-1-34}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z \tag{4-1-35}$$

$$\frac{\partial H_z}{\partial y} + j\beta_g H_y = j\omega\epsilon E_x \tag{4-1-36}$$

$$-j\beta_g H_x - \frac{\partial H_z}{\partial x} = j\omega\epsilon E_y \tag{4-1-37}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = 0 \tag{4-1-38}$$

Solving these six equations for E_x, E_y, H_x, and H_y in terms of H_z will give the TE-mode field equations in rectangular waveguides as

$$E_x = \frac{-j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial y} \quad (4\text{-}1\text{-}39)$$

$$E_y = \frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial x} \quad (4\text{-}1\text{-}40)$$

$$E_z = 0 \quad (4\text{-}1\text{-}41)$$

$$H_x = \frac{-j\beta_g}{k_c^2}\frac{\partial H_z}{\partial x} \quad (4\text{-}1\text{-}42)$$

$$H_y = \frac{-j\beta_g}{k_c^2}\frac{\partial H_z}{\partial y} \quad (4\text{-}1\text{-}43)$$

$$H_z = \text{Eq. (4-1-24)} \quad (4\text{-}1\text{-}44)$$

where $k_c^2 = \omega^2\mu\epsilon - \beta_g^2$ has been replaced.

Differentiating Eq. (4-1-24) with respect to x and y and then substituting the results in Eqs. (4-1-39) through (4-1-43) yield a set of field equations. The boundary conditions are applied to the newly found field equations in such a manner that either the tangent **E** field or the normal **H** field vanishes at the surface of the conductor. Since $E_x = 0$, then $\partial H_z/\partial y = 0$ at $y = 0, b$. Hence $C_n = 0$. Since $E_y = 0$, then $\partial H_z/\partial x = 0$ at $x = 0, a$. Hence $A_m = 0$.

It is generally concluded that the normal derivative of H_z must vanish at the conducting surfaces—that is,

$$\frac{\partial H_z}{\partial n} = 0 \quad (4\text{-}1\text{-}45)$$

at the guide walls. Therefore the magnetic field in the positive z direction is given by

$$H_z = H_{0z}\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-j\beta_g z} \quad (4\text{-}1\text{-}46)$$

where H_{0z} is the amplitude constant.

Substitution of Eq. (4-1-46) in Eqs. (4-1-39) through (4-1-43) yields the TE field equations in rectangular waveguides as

$$E_x = E_{0x}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-j\beta_g z} \quad (4\text{-}1\text{-}47)$$

$$E_y = E_{0y}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-j\beta_g z} \quad (4\text{-}1\text{-}48)$$

$$E_z = 0 \quad (4\text{-}1\text{-}49)$$

$$H_x = H_{0x}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-j\beta_g z} \quad (4\text{-}1\text{-}50)$$

$$H_y = H_{0y}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-j\beta_g z} \quad (4\text{-}1\text{-}51)$$

$$H_z = \text{Eq. (4-1-46)}$$

where $m = 0, 1, 2, \ldots$ (4-1-52)
$n = 0, 1, 2, \ldots$
$m = n = 0$ excepted

The cutoff wave number k_c, as defined by Eq. (4-1-17) for the TE_{mn} modes, is given by

$$k_c = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} = \omega_c\sqrt{\mu\epsilon} \tag{4-1-53}$$

where a and b are in meters. The cutoff frequency, as defined in Eq. (4-1-19) for the TE_{mn} modes, is

$$f_c = \frac{1}{2\sqrt{\mu\epsilon}}\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \tag{4-1-54}$$

$f_c = \frac{k_c c}{2\pi}$ $f_c = \frac{c}{\lambda}$

The propagation constant (or the phase constant here) β_g, as defined in Eq. (4-1-18), is expressed by

$$\beta_g = \omega\sqrt{\mu\epsilon}\sqrt{1 - \left(\frac{f_c}{f}\right)^2} \tag{4-1-55}$$

$\beta_g = k\sqrt{1 - \left(\frac{f_c}{f}\right)^2}$

The phase velocity in the positive z direction for the TE_{mn} modes is shown as

$$v_g = \frac{\omega}{\beta_g} = \frac{v_p}{\sqrt{1 - (f_c/f)^2}} \tag{4-1-56}$$

where $v_p = 1/\sqrt{\mu\epsilon}$ is the phase velocity in an unbounded dielectric.

The characteristic wave impedance of TE_{mn} modes in the guide can be derived from Eqs. (4-1-33) and (4-1-34):

$$Z_g = \frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{\omega\mu}{\beta_g} = \frac{\eta}{\sqrt{1 - (f_c/f)^2}} \tag{4-1-57}$$

where $\eta = \sqrt{\mu/\epsilon}$ is the intrinsic impedance in an unbounded dielectric. The wavelength λ_g in the guide for the TE_{mn} modes is given by

$$\lambda_g = \frac{\lambda}{\sqrt{1 - (f_c/f)^2}} \tag{4-1-58}$$

where $\lambda = v_p/f$ is the wavelength in an unbounded dielectric.

Since the cutoff frequency shown in Eq. (4-1-54) is a function of the modes and guide dimensions, the physical size of the waveguide will determine the propagation of the modes. Table 4-1-1 tabulates the ratio of cutoff frequency of some modes with respect to that of the dominant mode in terms of the physical dimension.

Whenever two or more modes have the same cutoff frequency, they are said to be *degenerate modes*. In a rectangular guide the corresponding TE_{mn} and TM_{mn} modes are always degenerate. In a square guide the TE_{mn}, TE_{nm}, TM_{mn}, and TM_{nm} modes form a foursome of degeneracy. Rectangular guides ordinarily have dimensions of $a = 2b$ ratio. The mode with the lowest cutoff frequency in a particular

TABLE 4-1-1 MODES OF $(f_c)_{mn}/f_c$ FOR $a \geq b$

Modes f/f_{10} a/b	TE_{10}	TE_{01}	TE_{11} TM_{11}	TE_{20}	TE_{02}	TE_{21} TM_{21}	TE_{12} TM_{12}	TE_{22} TM_{22}	TE
1	1	1	1.414	2	2	2.236	2.236	2.828	3
1.5	1	1.5	1.803	2	3	2.500	3.162	3.606	3
2	1	2	2.236	2	4	2.828	4.123	4.472	3
3	1	3	3.162	2	6	3.606	6.083	6.325	3
∞	1	∞	∞	2	∞	∞	∞	∞	3

guide is called the *dominant mode*. The dominant mode in a rectangular guide with $a > b$ is the TE_{10} mode. Each mode has a specific mode pattern (or field pattern).

It is normal for all modes to exist simultaneously in a given waveguide. The situation is not very serious, however. Actually, only the dominant mode propagates, and the higher modes near the sources or discontinuities decay very fast.

Example 4-1-1: TE_{10} in Rectangular Waveguide

An air-filled rectangular waveguide of inside dimensions 7 × 3.5 cm operates in the dominant TE_{10} mode as shown in Fig. 4-1-4.

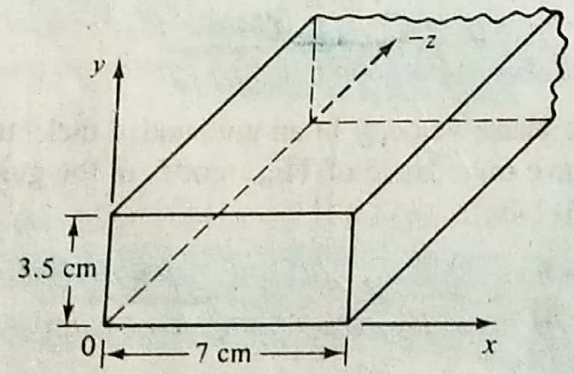


Figure 4-1-4 Rectangular waveguide for Example 4-1-1.

a. Find the cutoff frequency.
b. Determine the phase velocity of the wave in the guide at a frequency of 3.5 GHz.
c. Determine the guided wavelength at the same frequency.

Solution

$$\textbf{a. } f_c = \frac{c}{2a} = \frac{3 \times 10^8}{2 \times 7 \times 10^{-2}} = 2.14 \text{ GHz}$$

$$\textbf{b. } v_g = \frac{c}{\sqrt{1-(f_c/f)^2}} = \frac{3 \times 10^8}{\sqrt{1-(2.14/3.5)^2}} = 3.78 \times 10^8 \text{ m/s}$$

$$\textbf{c. } \lambda_g = \frac{\lambda_0}{\sqrt{1-(f_c/f)^2}} = \frac{3 \times 10^8/(3.5 \times 10^9)}{\sqrt{1-(2.14/3.5)^2}} = 10.8 \text{ cm}$$

$E_z = 0$ at $x = 0, a$
$E_z = 0$ at $y = 0, b$.

4.1-3 TM Modes in Rectangular Waveguides

The TM_{mn} modes in a rectangular guide are characterized by $H_z = 0$. In other words, the z component of an electric field E must exist in order to have energy transmission in the guide. Consequently, the Helmholtz equation for E in the rectangular coordinates is given by

$$\nabla^2 E_z = \gamma^2 E_z \tag{4-1-59}$$

A solution of the Helmholtz equation is in the form of

$$E_z = \left[A_m \sin\left(\frac{m\pi x}{a}\right) + B_m \cos\left(\frac{m\pi x}{a}\right)\right]\left[C_n \sin\left(\frac{n\pi y}{b}\right) + D_n \cos\left(\frac{n\pi y}{b}\right)\right]e^{-j\beta_g z} \tag{4-1-60}$$

which must be determined according to the given boundary conditions. The procedures for doing so are similar to those used in finding the TE-mode wave.

The boundary conditions on E_z require that the field vanishes at the waveguide walls, since the tangent component of the electric field E_z is zero on the conducting surface. This requirement is that for $E_z = 0$ at $x = 0$, a, then $B_m = 0$, and for $E_z = 0$ at $y = 0$, b, then $D_n = 0$. Thus the solution as shown in Eq. (4-1-60) reduces to

$$E_z = E_{0z} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-j\beta_g z} \tag{4-1-61}$$

where $m = 1, 2, 3, \ldots$
$n = 1, 2, 3, \ldots$

If either $m = 0$ or $n = 0$, the field intensities all vanish. So there is no TM_{01} or TM_{10} mode in a rectangular waveguide, which means that TE_{10} is the dominant mode in a rectangular waveguide for $a > b$. For $H_z = 0$, the field equations, after expanding $\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E}$, are given by

& $\nabla \times E = -j\omega\mu H$

$H_z = 0$
$\frac{\partial}{\partial z} = -j\beta_g$

for getting 6 eq^{n}s
from $\nabla \times H = j\omega\epsilon E$
& $\nabla \times E = -j\omega\mu H$
[illegible] $H_z = 0$
& $\frac{\partial}{\partial z} = -j\beta g$

$$\frac{\partial E_z}{\partial y} + j\beta_g E_y = -j\omega\mu H_x \tag{4-1-62}$$

$$j\beta_g E_x + \frac{\partial E_z}{\partial x} = j\omega\mu H_y \tag{4-1-63}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0 \tag{4-1-64}$$

$$\beta_g H_y = \omega\epsilon E_x \tag{4-1-65}$$

$$-\beta_g H_x = \omega\epsilon E_y \tag{4-1-66}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\epsilon E_z \tag{4-1-67}$$

These equations can be solved simultaneously for E_x, E_y, H_x, and H_y in terms of E_z.

The resultant field equations for TM modes are

$$E_x = \frac{-j\beta_g}{k_c^2}\frac{\partial E_z}{\partial x} \tag{4-1-68}$$

$$E_y = \frac{-j\beta_g}{k_c^2}\frac{\partial E_z}{\partial y} \tag{4-1-69}$$

$$E_z = \text{Eq. (4-1-61)} \tag{4-1-70}$$

$$H_x = \frac{j\omega\epsilon}{k_c^2}\frac{\partial E_z}{\partial y} \tag{4-1-71}$$

$$H_y = \frac{-j\omega\epsilon}{k_c^2}\frac{\partial E_z}{\partial x} \tag{4-1-72}$$

$$H_z = 0 \tag{4-1-73}$$

where $\beta_g^2 - \omega^2\mu\epsilon = -k_c^2$ is replaced.

Differentiating Eq. (4-1-61) with respect to x or y and substituting the results in Eqs. (4-1-68) through (4-1-73) yield a new set of field equations. The TM_{mn} mode field equations in rectangular waveguides are

$$E_x = E_{0x}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-j\beta_g Z} \tag{4-1-74}$$

$$E_y = E_{0y}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-j\beta_g Z} \tag{4-1-75}$$

$$E_z = \text{Eq. (4-1-61)} \tag{4-1-76}$$

$$H_x = H_{0x}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-j\beta_g Z} \tag{4-1-77}$$

$$H_y = H_{0y}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-j\beta_g Z} \tag{4-1-78}$$

$$H_z = 0 \tag{4-1-79}$$

Some of the TM-mode characteristic equations are identical to those of the TE modes, but some are different. For convenience, all are shown here:

$$f_c = \frac{1}{2\sqrt{\mu\epsilon}}\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}} \tag{4-1-8}$$

$$\beta_g = \omega\sqrt{\mu\epsilon}\sqrt{1-\left(\frac{f_c}{f}\right)^2} \tag{4-1-}$$

$$\lambda_g = \frac{\lambda}{\sqrt{1-(f_c/f)^2}} \tag{4-1-}$$

$$v_g = \frac{v_p}{\sqrt{1-(f_c/f)^2}} \tag{4-1-}$$

$$Z_g = \frac{\beta_g}{\omega\epsilon} = \eta\sqrt{1 - \left(\frac{f_c}{f}\right)^2} \tag{4-1-84}$$

4-1-4 Power Transmission in Rectangular Waveguides

The power transmitted through a waveguide and the power loss in the guide walls can be calculated by means of the complex Poynting theorem described in Chapter 2. It is assumed that the guide is terminated in such a way that there is no reflection from the receiving end or that the guide is infinitely long compared with the wavelength. From the Poynting theorem in Section 2-2, the power transmitted through a guide is given by

$$P_{tr} = \oint \mathbf{p} \cdot d\mathbf{s} = \oint \frac{1}{2}(\mathbf{E} \times \mathbf{H}^*) \cdot d\mathbf{s} \tag{4-1-85}$$

For a lossless dielectric, the time-average power flow through a rectangular guide is given by

$$P_{tr} = \frac{1}{2Z_g}\int_a |E|^2\, da = \frac{Z_g}{2}\int_a |H|^2\, da \tag{4-1-86}$$

where $Z_g = \dfrac{E_x}{H_y} = -\dfrac{E_y}{H_x}$

$|E|^2 = |E_x|^2 + |E_y|^2$

$|H|^2 = |H_x|^2 + |H_y|^2$

For TE_{mn} modes, the average power transmitted through a rectangular waveguide is given by

$$P_{tr} = \frac{\sqrt{1 - (f_c/f)^2}}{2\eta}\int_0^b\int_0^a (|E_x|^2 + |E_y|^2)\, dx\, dy \tag{4-1-87}$$

For TM_{mn} modes, the average power transmitted through a rectangular waveguide is given by

$$P_{tr} = \frac{1}{2\eta\sqrt{1 - (f_c/f)^2}}\int_0^b\int_0^a (|E_x|^2 + |E_y|^2)\, dx\, dy \tag{4-1-88}$$

where $\eta = \sqrt{\mu/\epsilon}$ is the intrinsic impedance in an unbounded dielectric.

4-1-5 Power Losses in Rectangular Waveguides

There are two types of power losses in a rectangular waveguide:

1. Losses in the dielectric
2. Losses in the guide walls

First we shall consider power losses caused by dielectric attenuation. In a low-loss dielectric (that is, $\sigma \ll \mu\epsilon$), the propagation constant for a plane wave travel[ing] in an unbounded lossy dielectric is given in Eq. (2-5-20) by

$$\alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}} = \frac{\eta\sigma}{2} \qquad \text{(4-1-8[9])}$$

The attenuation caused by the low-loss dielectric in the rectangular waveguide [for] the TE_{mn} or TM_{mn} modes is given by

$$\alpha_g = \frac{\sigma\eta}{2\sqrt{1-(f_c/f)^2}} \quad \text{for TE mode} \qquad \text{(4-1-9[0])}$$

$$\alpha_g = \frac{\sigma\eta}{2}\sqrt{1-(f_c/f)^2} \quad \text{for TM mode} \qquad \text{(4-1-90a)}$$

As $f \gg f_c$, the attenuation constant in the guide approaches that for the unbounde[d] dielectric given by Eq. (4-1-89). However, if the operating frequency is way belo[w] the cutoff frequency, $f \ll f_c$, the attenuation constant becomes very large and no[n]propagation occurs.

Now we shall consider power losses caused by the guide walls. When the elec[-]tric and magnetic intensities propagate through a lossy waveguide, their magnitude[s] may be written

$$|E| = |E_{0z}|e^{-\alpha_g z} \qquad \text{(4-1-9[1])}$$

$$|H| = |H_{0z}|e^{-\alpha_g z} \qquad \text{(4-1-9[2])}$$

where E_{0z} and H_{0z} are the field intensities at $z = 0$. It is interesting to note that, for [a] low-loss guide, the time-average power flow decreases proportionally to $e^{-2\alpha_g z}$. Hence

$$P_{tr} = (P_{tr} + P_{loss})e^{-2\alpha_g z} \qquad \text{(4-1-9[3])}$$

For $P_{loss} \ll P_{tr}$ and $2\alpha_g z \ll 1$,

$$\frac{P_{loss}}{P_{tr}} + 1 = 1 + 2\alpha_g z \qquad \text{(4-1-9[4])}$$

Finally,

$$\alpha_g = \frac{P_L}{2P_{tr}} \qquad \text{(4-1-9[5])}$$

where P_L is the power loss per unit length. Consequently, the attenuation constant [of] the guide walls is equal to the ratio of the power loss per unit length to twice [the] power transmitted through the guide.

Since the electric and magnetic field intensities established at the surface of [a] low-loss guide wall decay exponentially with respect to the skin depth while [the] waves progress into the walls, it is better to define a surface resistance of the gu[ide] walls as

$$R_s \equiv \frac{\rho}{\delta} = \frac{1}{\sigma\delta} = \frac{\alpha_g}{\sigma} = \sqrt{\frac{\pi f\mu}{\sigma}} \quad \Omega/\text{square} \qquad \text{(4-1-9[6])}$$

where ρ = resistivity of the conducting wall in ohms-meter
σ = conductivity in mhos per meter
δ = skin depth or depth of penetration in meters

The power loss per unit length of guide is obtained by integrating the power density over the surface of the conductor corresponding to the unit length of the guide. This is

$$P_L = \frac{R_s}{2}\int_s |H_t|^2\, ds \qquad \text{W/unit length} \tag{4-1-97}$$

where H_t is the tangential component of magnetic intensity at the guide walls.

Substitution of Eqs. (4-1-86) and (4-1-97) in Eq. (4-1-95) yields

$$\alpha_g = \frac{R_s \int_s |H_t|^2\, ds}{2Z_g \int_a |H|^2\, da} \tag{4-1-98}$$

where

$$|H|^2 = |H_z|^2 + |H_y|^2 \tag{4-1-99}$$

$$|H_t|^2 = |H_{tx}|^2 + |H_{ty}|^2 \tag{4-1-100}$$

Example 4-1-2: TE_{10} Mode in Rectangular Waveguide

An airfilled waveguide with a cross section 2 × 1 cm transports energy in the TE_{10} mode at the rate of 0.5 hp. The impressed frequency is 30 GHz. What is the peak value of the electric field occurring in the guide? (Refer to Fig. 4-1-5.)

Solution The field components of the dominant mode TE_{10} can be obtained by substituting $m = 1$ and $n = 0$ in Eqs. (4-1-47) through (4-1-52). Then

$$E_x = 0 \qquad H_x = \frac{E_{0y}}{Z_g}\sin\left(\frac{\pi x}{a}\right)e^{-j\beta_g z}$$

$$E_y = E_{0y}\sin\left(\frac{\pi x}{a}\right)e^{-j\beta_g z} \qquad H_y = 0$$

$$E_z = 0 \qquad H_z = H_{0z}\cos\left(\frac{\pi x}{a}\right)e^{-j\beta_g z}$$

where $Z_g = \omega\mu_0/\beta_g$.

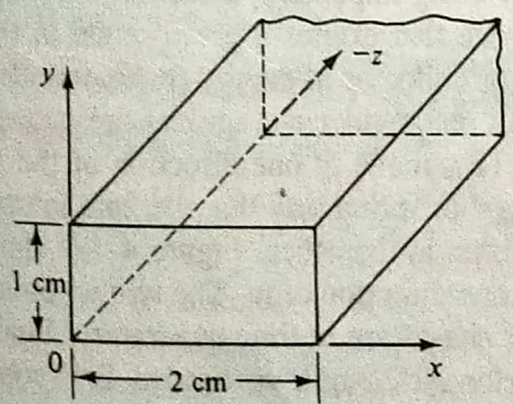


Figure 4-1-5 Rectangular waveguide for Example 4-1-2.

The phase constant β_g can be found from

$$\beta_g = \sqrt{\omega^2\mu_0\epsilon_0 - \frac{\pi^2}{a^2}} = \pi\sqrt{\frac{(2f)^2}{c^2} - \frac{1}{a^2}} = \pi\sqrt{\frac{\ldots}{\ldots}}$$

$$= 193.5\pi = 608.81 \text{ rad/m}$$

The power delivered in the z direction by the guide is

$$P = \mathrm{Re}\left[\frac{1}{2}\int_0^b\int_0^a (\mathbf{E} \times \mathbf{H}^*)\cdot dx\,dy\,\mathbf{u}_z\right]$$

$$= \frac{1}{2}\int_0^b\int_0^a \left[\left(E_{0y}\sin\left(\frac{\pi x}{a}\right)e^{-j\beta_g z}\mathbf{u}_y\right) \times \left(\frac{-\beta_g}{\omega\mu_0}E_{0y}\sin\left(\frac{\pi x}{a}\right)e^{j\beta_g z}\mathbf{u}_x\right)\right]\ldots$$

$$= \frac{1}{2}E_{0y}^2\frac{\beta_g}{\omega\mu_0}\int_0^b\int_0^a \left(\sin\left(\frac{\pi x}{a}\right)\right)^2 dx\,dy$$

$$= \frac{1}{4}E_{0y}^2\frac{\beta_g}{\omega\mu_0}ab$$

$$373 = \frac{1}{4}E_{0y}^2\frac{193.5\pi(10^{-2})(2\times 10^{-2})}{2\pi(3\times 10^{9})(4\pi\times 10^{-7})}$$

$$E_{0y} = 53.87 \text{ kV/m}$$

The peak value of the electric intensity is 53.87 kV/m.

4-1-6 Excitations of Modes in Rectangular Waveguides

In general, the field intensities of the desired mode in a waveguide can be established by means of a probe or loop-coupling device. The probe may be called a monopole antenna; the coupling loop, the loop antenna. A probe should be located so as to excite the electric field intensity of the mode, and a coupling loop or slot should be located to generate the magnetic field intensity for the desired mode. If two or more probes or loops are to be used, care must be taken to ensure the proper phase relationship between the currents in the various antennas. This factor can be achieved by using additional lengths of transmission line in one or more of the antenna leads. Impedance matching can be accomplished by varying the position and length of the antenna in the guide or by using impedance-matching stubs on the coaxial line feeding the waveguide. A device that excites a given mode in the guide can also act reciprocally as a receiver or collector of energy for that mode. The methods of excitation for various modes in rectangular waveguides are shown in Fig. 4-1-6.

In order to excite a TE_{10} mode in one direction of the guide, the two coupling antennas should be arranged in such a way that the field intensities cancel each other in one direction and reinforce in the other. Figure 4-1-7 shows an arrangement for launching a TE_{10} mode in one direction only. The two antennas are placed a quarter-wavelength apart and their phases are in time quadrature. Phasing is compensated by the use of an additional quarter-wavelength section of line connected to the antenna

TABLE 4-1-7 CHARACTERISTICS OF STANDARD RECTANGULAR WAVEGUIDES (Cont.)

EIA* designation WR*( )	Physical dimensions				Cutoff frequency for air-filled waveguide in GHz	Recommended frequency range for TE_{10} mode in GHZ
	Inside, in cm (in.)		Outside, in cm (in.)			
	Width	Height	Width	Height		
34	0.864 (0.340)	0.432 (0.170)	1.067 (0.420)	0.635 (0.250)	17.361	21.70–33.00
28	0.711 (0.280)	0.356 (0.140)	0.914 (0.360)	0.559 (0.220)	21.097	26.40–40.00
22	0.569 (0.224)	0.284 (0.112)	0.772 (0.304)	0.488 (0.192)	26.362	32.90–50.10
19	0.478 (0.188)	0.239 (0.094)	0.681 (0.268)	0.442 (0.174)	31.381	39.20–59.60
15	0.376 (0.148)	0.188 (0.074)	0.579 (0.228)	0.391 (0.154)	39.894	49.80–75.80
12	0.310 (0.122)	0.155 (0.061)	0.513 (0.202)	0.358 (0.141)	48.387	60.50–91.90
10	0.254 (0.100)	0.127 (0.050)	0.457 (0.180)	0.330 (0.130)	59.055	73.80–112.00
8	0.203 (0.080)	0.102 (0.040)	0.406 (0.160)	0.305 (0.120)	73.892	92.20–140.00
7	0.165 (0.065)	0.084 (0.033)	0.343 (0.135)	0.262 (0.103)	90.909	114.00–173.00
5	0.130 (0.051)	0.066 (0.026)	0.257 (0.101)	0.193 (0.076)	115.385	145.00–220.00
4	0.109 (0.043)	0.056 (0.022)	0.211 (0.083)	0.157 (0.062)	137.615	172.00–261.00
3	0.086 (0.034)	0.043 (0.017)	0.163 (0.064)	0.119 (0.047)	174.419	217.00–333.00

high, and its inside dimensions are 2.286 cm (0.90 in.) wide and 1.016 cm (0.40 in.) high. Table 4-1-7 tabulates the characteristics of the standard rectangular waveguides.

4-2 CIRCULAR WAVEGUIDES

A circular waveguide is a tubular, circular conductor. A plane wave propagating through a circular waveguide results in a transverse electric (TE) or transerse magnetic (TM) mode. Several other types of waveguides, such as elliptical and reentrant guides, also propagate electromagnetic waves.

4-2-1 Solutions of Wave Equations in Cylindrical Coordinates

As described in Section 4-1 for rectangular waveguides, only a sinusoidal steady-state or frequency-domain solution will be attempted for circular waveguides. A cylindrical coordinate system is shown in Fig. 4-2-1.

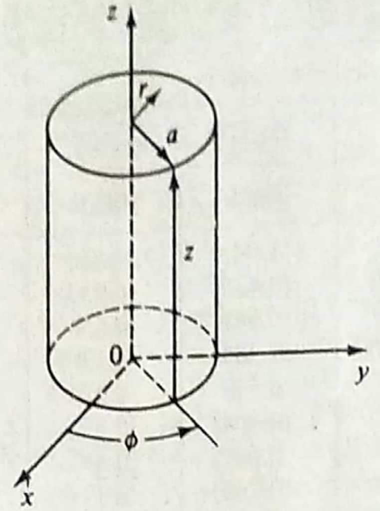


Figure 4-2-1 Cylindrical coordinates.

The scalar Helmholtz equation in cylindrical coordinates is given by

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2} = \gamma^2 \psi \tag{4-2-1}$$

Using the method of separation of variables, the solution is assumed in the form of

$$\Psi = R(r)\Phi(\phi)Z(z) \tag{4-2-2}$$

where $R(r)$ = a function of the r coordinate only
$\Phi(\phi)$ = a function of the ϕ coordinate only
$Z(z)$ = a function of the z coordinate only

Substitution of Eq. (4-2-2) in (4-2-1) and division of the resultant by (4-2-2) yield

$$\frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{r^2\Phi}\frac{d^2\Phi}{d\phi^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = \gamma^2 \tag{4-2-3}$$

Since the sum of the three independent terms is a constant, each of the three terms must be a constant. The third term may be set equal to a constant γ_g^2:

$$\frac{d^2Z}{dz^2} = \gamma_g^2 Z \tag{4-2-4}$$

The solutions of this equation are given by

$$Z = Ae^{-\gamma_g z} + Be^{\gamma_g z} \tag{4-2-5}$$

where γ_g = propagation constant of the wave in the guide.

Inserting γ_g^2 for the third term in the left-hand side of Eq. (4-2-3) and multiplying the resultant by r^2 yield

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} - (\gamma^2 - \gamma_g^2)\, r^2 = 0 \tag{4-2-6}$$

The second term is a function of ϕ only; hence equating the second term to a constant $(-n^2)$ yields

$$\frac{d^2\Phi}{d\phi^2} = -n^2\Phi \tag{4-2-7}$$

The solution of this equation is also a harmonic function:

$$\Phi = A_n \sin(n\phi) + B_n \cos(n\phi) \tag{4-2-8}$$

Replacing the Φ term by $(-n^2)$ in Eq. (4-2-6) and multiplying through by R, we have

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + [(k_c r)^2 - n^2]R = 0 \tag{4-2-9}$$

This is Bessel's equation of order n in which

$$\boxed{k_c^2 + \gamma^2 = \gamma_g^2} \tag{4-2-10}$$

This equation is called the *characteristic equation* of Bessel's equation. For a lossless guide, the characteristic equation reduces to

$$\boxed{\beta_g = \pm\sqrt{\omega^2\mu\epsilon - k_c^2}} \tag{4-2-11}$$

The solutions of Bessel's equation are

$$\boxed{R = C_n J_n(k_c r) + D_n N_n(k_c r)} \tag{4-2-12}$$

where $J_n(k_c r)$ is the nth-order Bessel function of the first kind, representing a standing wave of $\cos(k_c r)$ for $r < a$ as shown in Fig. 4-2-2. $N_n(k_c r)$ is the nth-order Bessel function of the second kind, representing a standing wave of $\sin(k_c r)$ for $r > a$ as shown in Fig. 4-2-3.

Therefore the total solution of the Helmholtz equation in cylindrical coordinates is given by

$$\Psi = \overbrace{[C_n J_n(k_c r) + D_n N_n(k_c r)]}^{\Psi_0}[A_n \sin(n\phi) + B_n \cos(n\phi)]e^{\pm j\beta_g z} \tag{4-2-13}$$

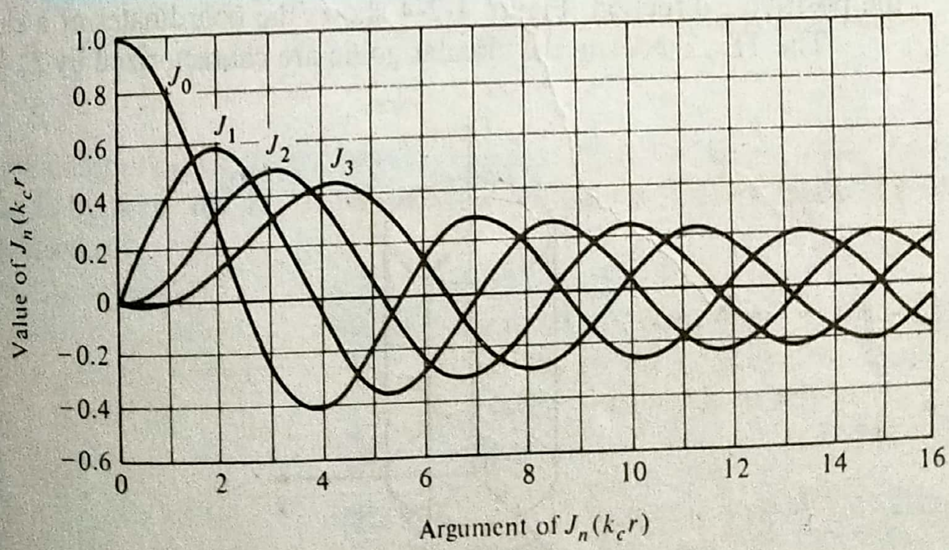


Figure 4-2-2 Bessel functions of the first kind.

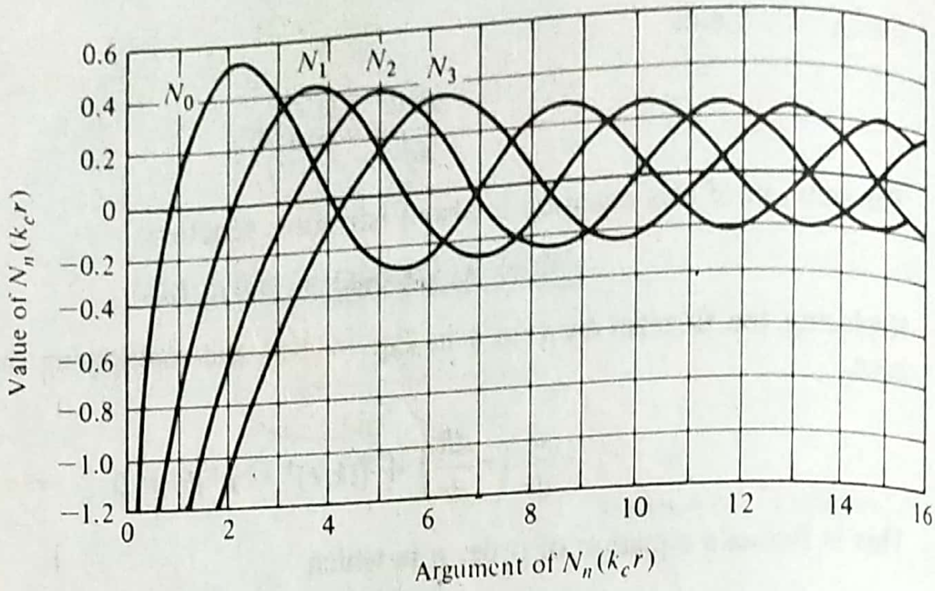


Figure 4-2-3 Bessel functions of the second kind.

At $r = 0$, however, $k_c r = 0$; then the function N_n approaches infinity, so $D_n = 0$. This means that at $r = 0$ on the z axis, the field must be finite. Also, by use of trigonometric manipulations, the two sinusoidal terms become

$$A_n \sin(n\phi) + B_n \cos(n\phi) = \sqrt{A_n^2 + B_n^2} \cos\left[n\phi + \tan^{-1}\left(\frac{A_n}{B_n}\right)\right]$$

$$= F_n \cos(n\phi) \quad (4\text{-}2\text{-}14)$$

Finally, the solution of the Helmholtz equation is reduced to

$$\Psi = \Psi_0 J_n(k_c r) \cos(n\phi) e^{-j\beta_g z} \quad (4\text{-}2\text{-}15)$$

4-2-2 TE Modes in Circular Waveguides

dominant mode

It is commonly assumed that the waves in a circular waveguide are propagating in the positive z direction. Figure 4-2-4 shows the coordinates of a circular guide.

The TE_{np} modes in the circular guide are characterized by $E_z = 0$. This means

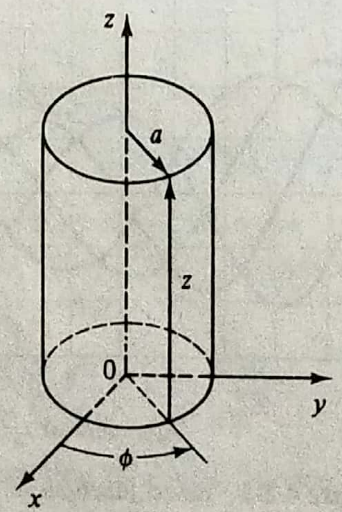


Figure 4-2-4 Coordinates of a circular waveguide.

that the z component of the magnetic field H_z must exist in the guide in order to have electromagnetic energy transmission. A Helmholtz equation for H_z in a circular guide is given by

$$\nabla^2 H_z = \gamma^2 H_z \tag{4-2-16}$$

Its solution is given in Eq. (4-2-15) by

$$H_z = H_{0z} J_n(k_c r) \cos(n\phi) e^{-j\beta_g z} \tag{4-2-17}$$

which is subject to the given boundary conditions.

For a lossless dielectric, Maxwell's curl equations in frequency domain are given by

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \tag{4-2-18}$$

$$\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E} \tag{4-2-19}$$

In cylindrical coordinates, their components are expressed as

$$\frac{1}{r}\frac{\partial E_z}{\partial \phi} - \frac{\partial E_\phi}{\partial z} = -j\omega\mu H_r \tag{4-2-20}$$

$$\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = -j\omega\mu H_\phi \tag{4-2-21}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rE_\phi) - \frac{1}{r}\frac{\partial E_r}{\partial \phi} = -j\omega\mu H_z \tag{4-2-22}$$

$$\frac{1}{r}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z} = j\omega\epsilon E_r \tag{4-2-23}$$

$$-j\beta_g H_r - \frac{\partial H_z}{\partial r} = j\omega\epsilon E_\phi \tag{4-2-24}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rH_\phi) - \frac{1}{r}\frac{\partial H_r}{\partial \phi} = j\omega\epsilon E_z \tag{4-2-25}$$

$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$

$\nabla = \frac{1}{r}\frac{\partial}{\partial r}(r) + \frac{1}{r}\frac{\partial}{\partial \phi} + \frac{\partial}{\partial z}$

When the differentiation $\partial/\partial z$ is replaced by $(-j\beta_g)$ and the z component of electric field E_z by zero, the TE-mode equations in terms of H_z in a circular waveguide are expressed as

$$E_r = -\frac{j\omega\mu}{k_c^2}\frac{1}{r}\frac{\partial H_z}{\partial \phi} \tag{4-2-26}$$

$$E_\phi = \frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial r} \tag{4-2-27}$$

$$E_z = 0 \tag{4-2-28}$$

$$H_r = \frac{-j\beta_g}{k_c^2}\frac{\partial H_z}{\partial r} \tag{4-2-29}$$

$$H_\phi = \frac{-j\beta_g}{k_c^2}\frac{1}{r}\frac{\partial H_z}{\partial \phi} \quad (4\text{-}2\text{-}30)$$

$$H_z = H_{0z} J_n(k_c r) \cos(n\phi) e^{-j\beta_g Z} \quad (4\text{-}2\text{-}31)$$

where $k_c^2 = \omega^2\mu\epsilon - \beta_g^2$ has been replaced.

The boundary conditions require that the ϕ component of the electric field E_ϕ, which is tangential to the inner surface of the circular waveguide at $r = a$, must vanish or that the r component of the magnetic field H_r, which is normal to the inner surface of $r = a$, must vanish. Consequently

$$E_\phi = 0 \text{ at } r = a \qquad \therefore \left.\frac{\partial H_z}{\partial r}\right|_{r=a} = 0$$

or

$$H_r = 0 \text{ at } r = a \qquad \therefore \left.\frac{\partial H_z}{\partial r}\right|_{r=a} = 0$$

This requirement is equivalent to that expressed in Eq. (4-2-17):

$$\left.\frac{\partial H_z}{\partial r}\right|_{r=a} = H_{0z} J_n'(k_c a) \cos(n\phi) e^{-j\beta_g Z} = 0 \quad (4\text{-}2\text{-}32)$$

Hence

$$J_n'(k_c a) = 0 \quad (4\text{-}2\text{-}33)$$

where J_n' indicates the derivative of J_n.

Since the J_n are oscillatory functions, the $J_n'(k_c a)$ are also oscillatory functions. An infinite sequence of values of $(k_c a)$ satisfies Eq. (4-2-32). These points, the roots of Eq. (4-2-32), correspond to the maxima and minima of the curve $J_n'(k_c a)$, as shown in Fig. 4-2-2. Table 4-2-1 tabulates a few roots of $J_n'(k_c a)$ for some lower-order n.

TABLE 4-2-1 pth ZEROS OF $J_n'(K_c a)$ FOR TE_{np} MODES

p	$n =$ 0	1	2	3	4	5
1	3.832	1.841	3.054	4.201	5.317	6.416
2	7.016	5.331	6.706	8.015	9.282	10.520
3	10.173	8.536	9.969	11.346	12.682	13.987
4	13.324	11.706	13.170			

The permissible values of k_c can be written

$$\boxed{k_c = \frac{X_{np}'}{a}} \quad (4\text{-}2\text{-}34)$$

Substitution of Eq. (4-2-17) in Eqs. (4-2-26) through (4-2-31) yields the complete field equations of the TE_{np} modes in circular waveguides:

$$E_r = E_{0r} J_n\left(\frac{X_{np}' r}{a}\right) \sin(n\phi) e^{-j\beta_g Z} \quad (4\text{-}2\text{-}35)$$

$$E_\phi = E_{0\phi} J'_n\left(\frac{X'_{np}r}{a}\right) \cos(n\phi)e^{-j\beta_g z} \tag{4-2-36}$$

$$E_z = 0 \tag{4-2-37}$$

$$H_r = -\frac{E_{0\phi}}{Z_g} J_n\left(\frac{X'_{np}r}{a}\right) \cos(n\phi)e^{-j\beta_g z} \tag{4-2-38}$$

$$H_\phi = \frac{E_{0r}}{Z_g} J_n\left(\frac{X'_{np}r}{a}\right) \sin(n\phi)e^{-j\beta_g z} \tag{4-2-39}$$

$$H_z = H_{0z} J_n\left(\frac{X'_{np}r}{a}\right) \cos(n\phi)e^{-j\beta_g z} \tag{4-2-40}$$

where $Z_g = E_r/H_\phi = -E_\phi/H_r$ has been replaced for the wave impedance in the guide and where $n = 0, 1, 2, 3, \ldots$ and $p = 1, 2, 3, 4, \ldots$.

The first subscript n represents the number of full cycles of field variation in one revolution through 2π rad of ϕ. The second subscript p indicates the number of zeros of E_ϕ—that is, $J'_n(X'_{np}r/a)$ along the radial of a guide, but the zero on the axis is excluded if it exists.

The mode propagation constant is determined by Eqs. (4-2-26) through (4-2-31) and Eq. (4-2-34):

$$\beta_g = \sqrt{\omega^2\mu\epsilon - \left(\frac{X'_{np}}{a}\right)^2} \tag{4-2-41}$$

The cutoff wave number of a mode is that for which the mode propagation constant vanishes. Hence

$$k_c = \frac{X'_{np}}{a} = \omega_c\sqrt{\mu\epsilon} \tag{4-2-42}$$

The cutoff frequency for TE modes in a circular guide is then given by

$$f_c = \frac{X'_{np}}{2\pi a\sqrt{\mu\epsilon}} \tag{4-2-43}$$

and the phase velocity for TE modes is

$$v_g = \frac{\omega}{\beta_g} = \frac{v_p}{\sqrt{1-(f_c/f)^2}} \tag{4-2-44}$$

where $v_p = 1/\sqrt{\mu\epsilon} = c/\sqrt{\mu_r\epsilon_r}$ is the phase velocity in an unbounded dielectric.

The wavelength and wave impedance for TE modes in a circular guide are given, respectively, by

$$\lambda_g = \frac{\lambda}{\sqrt{1-(f_c/f)^2}} \tag{4-2-45}$$

and

$$Z_g = \frac{\omega\mu}{\beta_g} = \frac{\eta}{\sqrt{1-(f_c/f)^2}} \tag{4-2-46}$$

where $\lambda = \frac{v_p}{f}$ = wavelength in an unbounded dielectric

$\eta = \sqrt{\frac{\mu}{\epsilon}}$ = intrinsic impedance in an unbounded dielectric

Example 4-2-1: TE Mode in Circular Waveguide

A TE_{11} mode is propagating through a circular waveguide. The radius of the guide is 5 cm, and the guide contains an air dielectric (refer to Fig. 4-2-5).

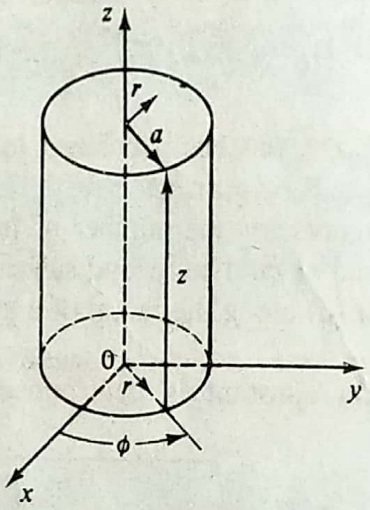


Figure 4-2-5 Diagram for Example 4-2-1.

a. Determine the cutoff frequency.

b. Determine the wavelength λ_g in the guide for an operating frequency of 3 GHz.

c. Determine the wave impedance Z_g in the guide.

Solution

a. From Table 4-2-1 for TE_{11} mode, $n = 1$, $p = 1$, and $X'_{11} = 1.841 = k_c a$. The cutoff wave number is

$$k_c = \frac{1.841}{a} = \frac{1.841}{5 \times 10^{-2}} = 36.82$$

The cutoff frequency is

$$f_c = \frac{k_c}{2\pi\sqrt{\mu_0\epsilon_0}} = \frac{(36.82)(3 \times 10^8)}{2\pi} = 1.758 \times 10^9 \text{ Hz}$$

b. The phase constant in the guide is

$$\beta_g = \sqrt{\omega^2\mu_0\epsilon_0 - k_c^2}$$

$$= \sqrt{(2\pi \times 3 \times 10^9)^2(4\pi \times 10^{-7} \times 8.85 \times 10^{-12}) - (36.82)^2}$$

$$= 50.9 \text{ rads/m}$$

The wavelength in the guide is

$$\lambda_g = \frac{2\pi}{\beta_g} = \frac{6.28}{50.9} = 12.3 \text{ cm}$$

c. The wave impedance in the guide is

$$Z_g = \frac{\omega\mu_0}{\beta_g} = \frac{(2\pi \times 3 \times 10^9)(4\pi \times 10^{-7})}{50.9} = 465\ \Omega$$

✓
4-2-3 TM Modes in Circular Waveguides

The TM_{np} modes in a circular guide are characterized by $H_z = 0$. However, the z component of the electric field E_z must exist in order to have energy transmission in the guide. Consequently, the Helmholtz equation for E_z in a circular waveguide is given by

$$\nabla^2 E_z = \gamma^2 E_z \tag{4-2-47}$$

Its solution is given in Eq. (4-2-15) by

$$E_z = E_{0z} J_n(k_c r) \cos(n\phi) e^{-j\beta_g z} \tag{4-2-48}$$

which is subject to the given boundary conditions.

The boundary condition requires that the tangential component of electric field E_z at $r = a$ vanishes. Consequently,

$$J_n(k_c a) = 0 \tag{4-2-49}$$

Since $J_n(k_c r)$ are oscillatory functions, as shown in Fig. 4-2-2, there are infinite numbers of roots of $J_n(k_c r)$. Table 4-2-2 tabulates a few of them for some lower-order n.

TABLE 4-2-2 pth ZEROS OF $J_n(K_c a)$ FOR TM_{np} MODES

p	$n =$ 0	1	2	3	4	5
1	2.405	3.832	5.136	6.380	7.588	8.771
2	5.520	7.106	8.417	9.761	11.065	12.339
3	8.645	10.173	11.620	13.015	14.372	
4	11.792	13.324	14.796			

For $H_z = 0$ and $\partial/\partial z = -j\beta_g$, the field equations in the circular guide, after expanding $\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$ and $\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E}$, are given by

$$E_r = \frac{-j\beta_g}{k_c^2}\frac{\partial E_z}{\partial r} \tag{4-2-50}$$

$$E_\phi = \frac{-j\beta_g}{k_c^2}\frac{1}{r}\frac{\partial E_z}{\partial \phi} \tag{4-2-51}$$

$$E_z = \text{Eq. (4-2-48)} \tag{4-2-52}$$

$$H_r = \frac{j\omega\epsilon}{k_c^2}\frac{1}{r}\frac{\partial E_z}{\partial \phi} \tag{4-2-53}$$

$$H_\phi = \frac{j\omega\epsilon}{k_c^2}\frac{\partial E_z}{\partial r} \quad (4\text{-}2\text{-}...)$$

$$H_z = 0 \quad (4\text{-}2\text{-}...)$$

where $k_c^2 = \omega^2\mu\epsilon - \beta_g^2$ has been replaced.

Differentiation of Eq. (4-2-48) with respect to z and substitution of the r... in Eqs. (4-2-50) through (4-2-55) yield the field equations of TM_{np} modes in a cir... lar waveguide:

$$E_r = E_{0r}J_n'\left(\frac{X_{np}r}{a}\right)\cos(n\phi)e^{-j\beta_g z} \quad (4\text{-}2\text{-}...)$$

$$E_\phi = E_{0\phi}J_n\left(\frac{X_{np}r}{a}\right)\sin(n\phi)e^{-j\beta_g z} \quad (4\text{-}2\text{-}...)$$

$$E_z = E_{0z}J_n\left(\frac{X_{np}r}{a}\right)\cos(n\phi)e^{-j\beta_g z} \quad (4\text{-}2\text{-}...)$$

$$H_r = \frac{E_{0\phi}}{Z_g}J_n\left(\frac{X_{np}r}{a}\right)\sin(n\phi)e^{-j\beta_g z} \quad (4\text{-}2\text{-}...)$$

$$H_\phi = \frac{E_{0r}}{Z_g}J_n'\left(\frac{X_{np}r}{a}\right)\cos(n\phi)e^{-j\beta_g z} \quad (4\text{-}2\text{-}...)$$

$$H_z = 0 \quad (4\text{-}2\text{-}...)$$

where $Z_g = E_r/H_\phi = -E_\phi/H_r = \beta_g/(\omega\epsilon)$ and $k_c = X_{np}/a$ have been replaced a... where $n = 0, 1, 2, 3, \ldots$ and $p = 1, 2, 3, 4, \ldots$.

Some of the TM-mode characteristic equations in the circular guide are ide... cal to those of the TE mode, but some are different. For convenience, all are sho... here:

$$\beta_g = \sqrt{\omega^2\mu\epsilon - \left(\frac{X_{np}}{a}\right)^2} \quad (4\text{-}2\text{-}...)$$

$$k_c = \frac{X_{np}}{a} = \omega_c\sqrt{\mu\epsilon} \quad (4\text{-}2\text{-}...)$$

$$f_c = \frac{X_{np}}{2\pi a\sqrt{\mu\epsilon}} \quad (4\text{-}2\text{-}...)$$

$$v_g = \frac{\omega}{\beta_g} = \frac{v_p}{\sqrt{1-(f_c/f)^2}} \quad (4\text{-}2\text{-}...)$$

$$\lambda_g = \frac{\lambda}{\sqrt{1-(f_c/f)^2}} \quad (4\text{-}2\text{-}...)$$

$$Z_g = \frac{\beta_g}{\omega\epsilon} = \eta\sqrt{1-\left(\frac{f_c}{f}\right)^2} \quad (4\text{-}2\text{-}...)$$

It should be noted that the dominant mode, or the mode of lowest cutoff freque...

in a circular waveguide, is the mode of TE_{11} that has the smallest value of the product, $k_c a = 1.841$, as shown in Tables 4-2-1 and 4-2-2.

Example 4-2-2: Wave Propagation in Circular Waveguide

An air-filled circular waveguide has a radius of 2 cm and is to carry energy at a frequency of 10 GHz. Find all the TE_{np} and TM_{np} modes for which energy transmission is possible.

Solution Since the physical dimension of the guide and the frequency of the wave remain constant, the product of $(k_c a)$ is also constant. Thus

$$k_c a = (\omega_c \sqrt{\mu_0 \epsilon_0})a = \frac{2\pi \times 10^{10}}{3 \times 10^8}(2 \times 10^{-2}) = 4.18$$

Any mode having a product of $(k_c a)$ less than or equal to 4.18 will propagate the wave with a frequency of 10 GHz. This is

$$k_c a \leqq 4.18$$

The possible modes are

$TE_{11}(1.841)$ $TM_{01}(2.405)$

$TE_{21}(3.054)$ $TM_{11}(3.832)$

$TE_{01}(3.832)$

4-2-4 TEM Modes in Circular Waveguides

The transverse electric and transverse magnetic (TEM) modes or transmission-line modes are characterized by

$$E_z = H_z = 0$$

This means that the electric and magnetic fields are completely transverse to the direction of wave propagation. This mode cannot exist in hollow waveguides, since it requires two conductors, such as the coaxial transmission line and two-open-wire line. Analysis of the TEM mode illustrates an excellent analogous relationship between the method of circuit theory and that of the field theory. Figure 4-2-6 shows a coaxial line.

Maxwell's curl equations in cylindrical coordinates

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \tag{4-2-68}$$

$$\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E} \tag{4-2-69}$$

become

$$B_g E_r = \omega\mu H_\phi \tag{4-2-70}$$

$$B_g E_\phi = \omega\mu H_r \tag{4-2-71}$$

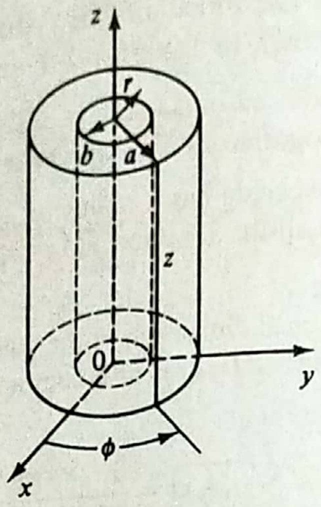


Figure 4-2-6 Coordinates of a co[axial]
line.

$$\frac{\partial}{\partial r}(rE_\phi) - \frac{\partial Er}{\partial \phi} = 0 \quad (4\text{-}$$

$$\beta_g H_r = -\omega\epsilon E_\phi \quad (4\text{-}$$

$$\beta_g H_\phi = \omega\epsilon E_r \quad (4\text{-}$$

$$\frac{\partial}{\partial r}(rH_\phi) - \frac{\partial H_r}{\partial \phi} = 0 \quad (4\text{-}$$

where $\partial/\partial r = -j\beta_g$ and $E_z = H_z = 0$ are replaced.

Substitution of Eq. (4-2-71) in (4-2-73) yields the propagation constant of TEM mode in a coaxial line:

$$\beta_g = \omega\sqrt{\mu\epsilon} \quad (4\text{-}$$

which is the phase constant of the wave in a lossless transmission line with a di[elec]tric.

In comparing the preceding equation with the characteristic equation of Helmholtz equation in cylindrical coordinates as given in Eq. (4-2-11) by

$$\beta_g = \sqrt{\omega^2\mu\epsilon - k_c^2} \quad (4\text{-}$$

it is evident that

$$k_c = 0 \quad (4\text{-}$$

This means that the cutoff frequency of the TEM mode in a coaxial line is [ze]ro, which is the same as in ordinary transmission lines.

The phase velocity of the TEM mode can be expressed from Eq. (4-2-76)

$$v_p = \frac{\omega}{\beta_g} = \frac{1}{\sqrt{\mu\epsilon}} \quad (4\text{-}$$

which is the velocity of light in an unbounded dielectric.

The wave impedance of the TEM mode is found from either Eqs. (4-2-70) and (4-2-73) or Eqs. (4-2-71) and (4-2-74) as

$$\eta(\text{TEM}) = \sqrt{\frac{\mu}{\epsilon}} \tag{4-2-80}$$

which is the wave impedance of a lossless transmission line in a dielectric.

Ampère's law states that the line integral of **H** about any closed path is exactly equal to the current enclosed by that path. This is

$$\oint \mathbf{H} \cdot d\ell = I = I_0 e^{-j\beta_g z} = 2\pi r H_\phi \tag{4-2-81}$$

where I is the complex current that must be supported by the center conductor of a coaxial line. This clearly demonstrates that the TEM mode can only exist in the two-conductor system—not in the hollow waveguide because the center conductor does not exist.

In summary, the properties of TEM modes in a lossless medium are as follows:

1. Its cutoff frequency is zero.
2. Its transmission line is a two-conductor system.
3. Its wave impedance is the impedance in an unbounded dielectric.
4. Its propagation constant is the constant in an unbounded dielectric.
5. Its phase velocity is the velocity of light in an unbounded dielectric.

4-2-5 Power Transmission in Circular Waveguides or Coaxial Lines

In general, the power transmitted through circular waveguides and coaxial lines can be calculated by means of the complex Poynting theorem described in Section 2-2. For a lossless dielectric, the time-average power transmitted through a circular guide can be given by

$$P_{tr} = \frac{1}{2Z_g} \int_0^{2\pi} \int_0^a [|E_r|^2 + |E_\phi|^2] r \, dr \, d\phi \tag{4-2-82}$$

$$P_{tr} = \frac{Z_g}{2} \int_0^{2\pi} \int_0^a [|H_r|^2 + |H_\phi|^2] r \, dr \, d\phi \tag{4-2-83}$$

where $Z_g = \dfrac{E_r}{H_\phi} = -\dfrac{E_\phi}{H_r}$ = wave impedance in the guide

a = radius of the circular guide

Substitution of Z_g for a particular mode in Eq. (4-2-82) yields the power transmitted by that mode through the guide.

For TE_{np} modes, the average power transmitted through a circular guide is given by

$$P_{tr} = \frac{\sqrt{1 - (f_c/f)^2}}{2\eta} \int_0^{2\pi} \int_0^a [|E_r|^2 + |E_\phi|^2] r \, dr \, d\phi \tag{4-2-}$$

where $\eta = \sqrt{\mu/\epsilon}$ is the intrinsic impedance in an unbounded dielectric.

For TM_{np} modes, the average power transmitted through a circular guide is given by

$$P_{tr} = \frac{1}{2\eta\sqrt{1 - (f_c/f)^2}} \int_0^{2\pi} \int_0^a [|E_r|^2 + |E_\phi|^2] r \, dr \, d\phi \tag{4-2-}$$

For TEM modes in coaxial lines, the average power transmitted through a coaxial line or two-open-wire line is given by

$$P_{tr} = \frac{1}{2\eta} \int_0^{2\pi} \int_0^a [|E_r|^2 + |E_\phi|^2] r \, dr \, d\phi \tag{4-2-}$$

If the current carried by the center conductor of a coaxial line is assumed to be

$$I_z = I_0 e^{-j\beta_g z} \tag{4-2-}$$

the magnetic intensity induced by the current around the center conductor is given by Ampère's law as

$$H_\phi = \frac{I_0}{2\pi r} e^{-j\beta_g z} \tag{4-2-}$$

The potential rise from the outer conductor to the center conductor is given by

$$V_r = -\int_b^a E_r dr = -\int_b^a \eta H_\phi dr = \frac{I_0 \eta}{2\pi} \ln\left(\frac{b}{a}\right) e^{-j\beta_g z} \tag{4-2-}$$

The characteristic impedance of a coaxial line is

$$Z_0 = \frac{V}{I} = \frac{\eta}{2\pi} \ln\left(\frac{b}{a}\right) \tag{4-2-}$$

where $\eta = \sqrt{\mu/\epsilon}$ is the intrinsic impedance in an unbounded dielectric.

The power transmitted by TEM modes in a coaxial line can be expressed from Eq. (4-2-86) as

$$P_{tr} = \frac{1}{2\eta} \int_0^{2\pi} \int_a^b |\eta H_\phi|^2 r \, dr \, d\phi = \frac{\eta I_0^2}{4\pi} \ln\left(\frac{b}{a}\right) \tag{4-2-}$$

Substitution of $|V_r|$ from Eq. (4-2-89) into Eq. (4-2-91) yields

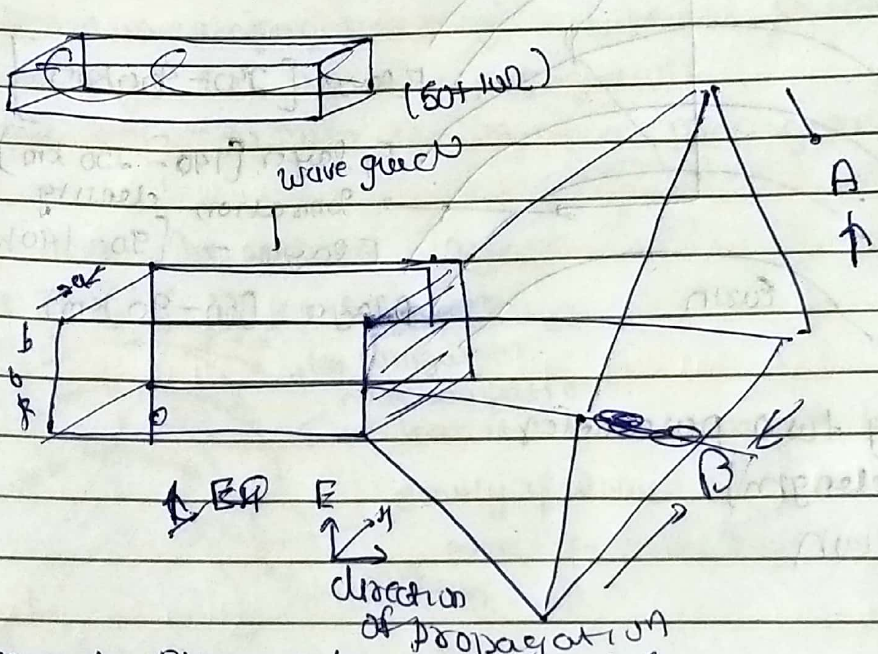
$$P_{tr} = \tfrac{1}{2} V_0 I_0 \tag{4-2-}$$

This shows that the power transmission derived from the Poynting theory is the same as from the circuit theory for an ordinary transmission line.

Horn antenna

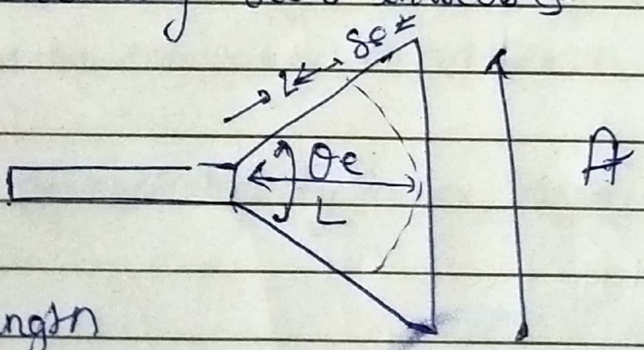
→ Horn antenna are constructed by flaring of wave guide
→ It increases directivity
→ It improves the impedance matching
→ It is directional Antenna, so it can utilized for long distance communication

Structure of Horn Antenna



we provide flaring to wave guid for impedance matching free space

The crosset section area increases from a×b to A×B and Hence Directivity also increases.



δ_e → E plane flaring differe

L - flaring length
θ_e = flaring angle with E plane

→ By pythogorous theorem

$$(L+\delta e)^2 = L^2 + (A/2)^2$$

$$\cancel{L^2} + 2\delta e L + \delta e^2 = \cancel{L^2} + \frac{A^2}{4}$$

$$2\delta e L + \delta e^2 = \frac{A^2}{4}$$

δe is small, so neglecting δe^2

$$2\delta \quad \boxed{L = \frac{A^2}{8\delta e}}$$

$\delta_E < 0.25\lambda$

$\delta_H < 0.4\lambda$

① Sectoral E-plane Horn, $HPBW = \frac{56}{(A/\lambda)}$

Flaring towards the electric field direction

② Sectoral H-plane ~~H~~ Horn $HPBW = \frac{67}{(B/\lambda)}$

Flaring towards the Magnetic field direction

③ Pyramidal horn

Flaring towards both direction

④ Conical horn

$$\text{gain} = \frac{4\pi A e}{\lambda^2} \times \eta$$

Application of Horn antenna

→ Microwave Engineering

Feed for Parabolic Reflector

Short range Radar