Jacobs University Bremen

Natural Science Laboratory
Signals and Systems Lab

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Lab Experiment 1 – RLC-Circuits - Transient Response

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1. Introduction and Theory

An electrical circuit consisting of a resistor (R), an inductor (L), and a capacitor (C), connected in series or in parallel is an RLC circuit. An RLC circuit has a property of resonance at certain frequency. Damping in the circuits is caused by the resistance in the circuit. The ability of the circuit to resonate naturally (without a driving source) is determined through damping. Circuits that will resonate in this way are described as underdamped and those that will not are overdamped. Additionally, the special case with the circuit that is just on the border of oscillation is critically damped.

In this experiment the transient response of RLC circuit for the under-damped, over-damped, and critically damped conditions is studied. The resistance value was varied in the circuit to change the course of oscillation of the output voltage which was then measured with an Oscilloscope. After determining the nature of damping of the respective circuit, general differential equations were established to represent the total response of the system. The visualization of the response can be seen through the plot in MATLAB.

Some of the important concepts needed to follow through the experiment include the following:

RLC circuit:

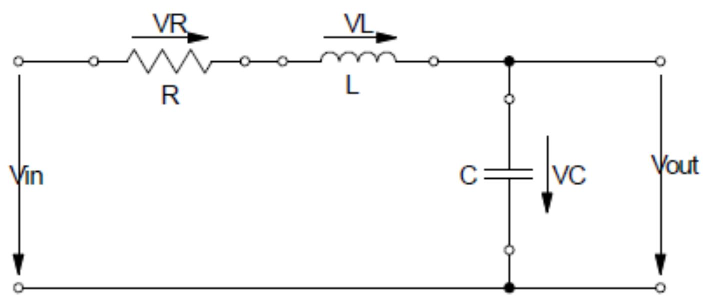

Figure 1: A series RLC circuit with output taken from capacitor

The response of the system with elements such as capacitor and inductor is described by a second-order differential equation which has the form of:

$$a\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + cy(t) = x(t) \qquad (1)$$

where, y(t) is the response of the circuit's system to an applied input x(t). a, b, and c are the system coefficients parameters.

Changing this in form of a linear constant coefficient non homogeneous differential equation becomes:

$$\frac{d^2y(t)}{dt^2} + 2\cdot\zeta\cdot\omega_n\frac{dy(t)}{dt} + {\omega_n}^2y(t) = K\cdot{\omega_n}^2\,x(t) \qquad (2)$$

By comparing equations (1) and (2), the parameters can be written as:

$\zeta = \frac{b}{2\sqrt{a\cdot c}}$ ζ is the damping ratio. (3)

$\omega_n = \sqrt{\frac{c}{a}}$ ω_n is the natural frequency (4)

$K = \frac{1}{c}$ K is the gain of the system. (5)

- The transient response:

Whenever a circuit with energy elements is introduced to an input, the system responds to the circuit, and a change in the equilibrium can be observed. A transient response is the response of a system to a change from an equilibrium or a steady state. The transient response can be classified into three cases on the basis of damping:

i) Over-Damped Case:

An overdamped response is the response of the circuit which do not oscillate around the steady value but takes longer time to reach to the steady value. A system is overdamped if the damping ratio (ζ) is greater than one, i.e., $\zeta > 1$. The general solution of an over-damped system is presented below:

$$y(t) = C_1 \cdot e^{\left(-\zeta + \sqrt{\zeta^2 - 1}\right) \cdot \omega_n \cdot t} + C_2 \cdot e^{\left(-\zeta - \sqrt{\zeta^2 - 1}\right) \cdot \omega_n \cdot t} \quad (6)$$

where, C_1 and C_2 are unknown coefficients derived from initial conditions.

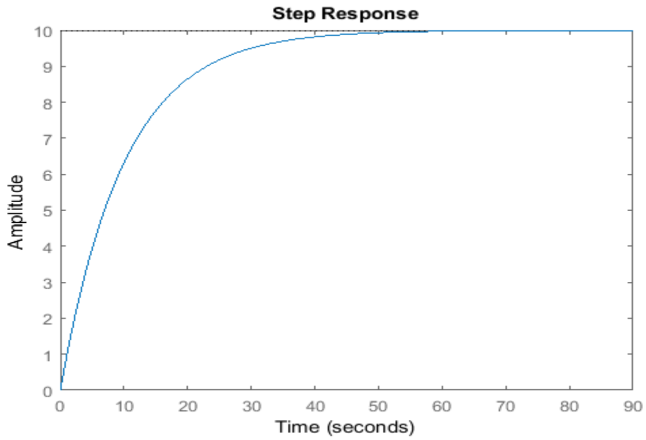

Figure 2: Overdamped response of a circuit (MATLAB)

ii) Critically Damped Case:

In this case, the circuit of the system reaches to the steady state in the fastest possible manner without oscillations. The damping ratio is equal to one, i.e., $\boldsymbol{\zeta} = 1$, with the general solution: (7)

$$y(t) = C_1 \cdot e^{-\zeta \cdot \omega_n \cdot t} + C_2 \cdot t\, e^{-\zeta \cdot \omega_n \cdot t}$$

where, C_1 and C_2 are unknown coefficients derived from initial conditions.

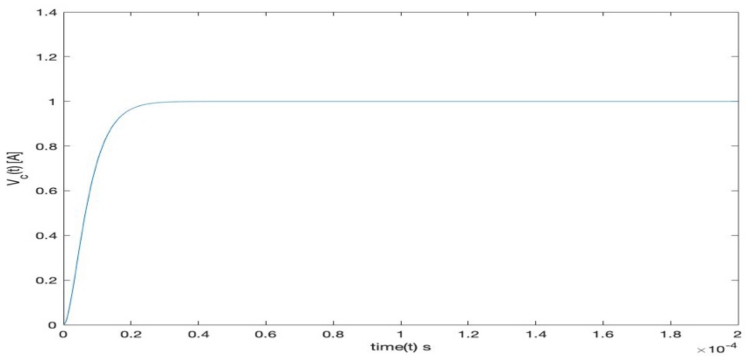

Figure 3: Critically damped response

iii) Under damped condition:

In this case, the circuit reaches to the steady state with oscillations, and gradually decreasing amplitude. The system crosses to equilibrium position multiple time while decaying. Here, it has the damping ratio $0 < \zeta < 1$ and the general solution is:

$$y(t) \; = \; \boldsymbol{e^{-\zeta \cdot \omega_n \cdot t}} \; \left(\boldsymbol{C_1} \cdot \mathbf{cos}\,(\boldsymbol{\omega_d} \cdot \boldsymbol{t}) + \boldsymbol{C_2} \cdot \mathbf{sin}\,(\boldsymbol{\omega_d} \cdot \boldsymbol{t})\,\right) \tag{8}$$

where, C_1 and C_2 are unknown coefficients derived from initial conditions, and

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} \tag{9}$$

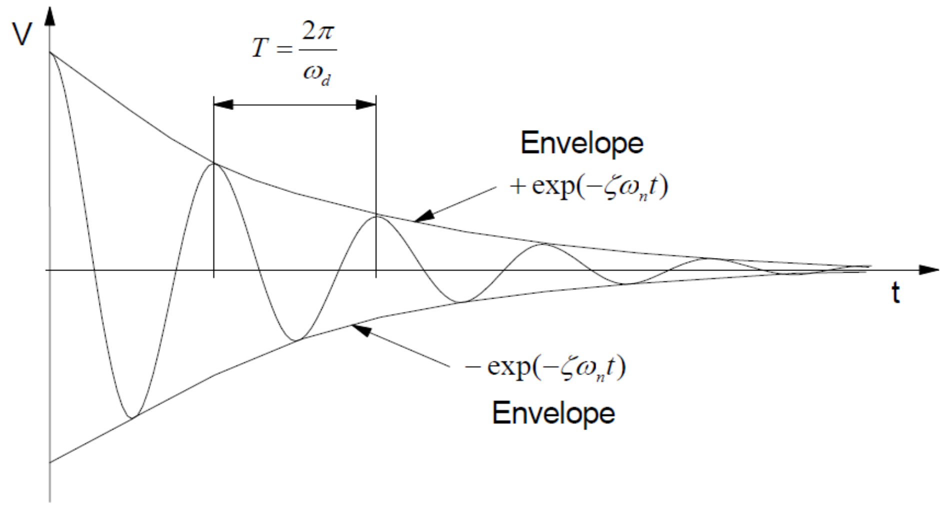

Figure 4: Under-damped second-order homogeneous system

- Forced/Particular Solution:

The forced solution of the non-homogeneous second-order differential equation is given when the input parameter is also considered. The forced response is same in nature as the input response. It is required for the complete solution of the second order differential equation, which is obtained by summation of forced solution and transient solution.

- Solution to a second-order differential system:

From the Figure 1, it can be deduced using the Kirchhoff's Voltage law that: The input voltage is the sum of output voltage across inductor, resistor and capacitor.
So,

$$V_{in} = V_R + V_L + V_C \tag{10}$$

The current in the circuit is 'I_C'. It is the same current that flows through the capacitor
Then,

$$I_C = C \cdot \frac{dV_{out}}{dt} \tag{11}$$

The voltage drop across the resistor is given by:

$$V_R = I_C \cdot R = RC \cdot \frac{dV_{out}}{dt} \tag{12}$$

The Voltage drop across the inductor is given by:

$$V_L = L \cdot \frac{dI_C}{dt} = LC \cdot \frac{d^2V_{out}}{dt^2} \tag{13}$$

Combining equations (10), (11), (12), and (13)

$$LC \cdot \frac{d^2V_{out}}{dt^2} + RC \cdot \frac{dV_{out}}{dt} + V_{out} = V_{in} \tag{14}$$

The above equation (14) is the general 2nd order O.D.E for a series RLC circuit from which the following parameters can be derived as:

$$\omega_n = \frac{1}{\sqrt{LC}} \tag{15}$$

$$\zeta = \frac{R}{2} \cdot \sqrt{\frac{C}{L}} \tag{16}$$

$$K = 1$$

It is evident that the voltage across the capacitor cannot changed instantaneously:

$V_c\ (0^-) = V_c\ (0^+)$

Similarly, for inductor the current cannot changed instantaneously:

$I_c\ (0^-) = I_c\ (0^+)$

- Step Response:
  The Step Response is the response of the system upon applying an input signal in the form of a step function.
- Steady State Value:
  The value (magnitude) of voltage or current after it has reached its stability is called steady state value.
- Ringing:
  Ringing is the oscillation phenomenon that occurs when the system is under-damped.

- The Complete Response:
  To obtain the complete response for a second order differential equation, following steps can be followed:
  a) Using KVL/KCL, Ohm's law and other mathematical tools, a second order non homogeneous differential equation can be obtained
  b) Then, the corresponding homogeneous equation should be solved to obtain a transient response.
  c) Considering the input response, the forced/ particular response can be obtained.
  d) The complete response can be obtained by adding the forced solution and homogeneous solution $f(t) = f_{steady\ state} + f_{particular}$
  e) Using the initial conditions, the values of C_1 and C_2 can be determined.

2. Execution

2.1 Experimental Setup

Workbench Number 8

- Breadboard
- Signal Generator
- Oscilloscope
- BNC Cable
- Tools from workbench
- 1n5F Capacitor
- 10mH Inductor
- Resistor Decade

2.1.1 Experimental Part 1 – Overview and Setup

- The R-Decade, 1.5nF Capacitor and the 10mH inductor are assembled as a series RLC circuit.
- A square wave signal with peak-to-peak voltage of 1V, an offset of 0.5V with a frequency of 100Hz is generated from signal generator.
- The output voltage across the capacitor is observed in the oscilloscope and the damped frequency is measured.
- The type of damping and the damping frequency were calculated for different values of the resistances.

- Test Circuit:

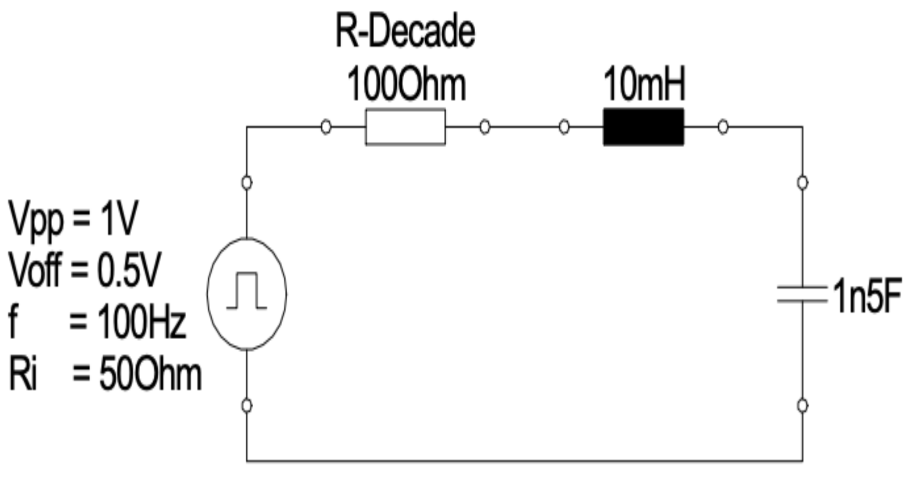

Figure 5: Test RLC Circuit Diagram

2.1.2 Experimental Part 1 – Execution and Results

- The resistance of the R-decade is changed to 100Ω and the damped frequency is measured.

Measured Parameter	Magnitude [kHz]
Damped Frequency f_d	39.24

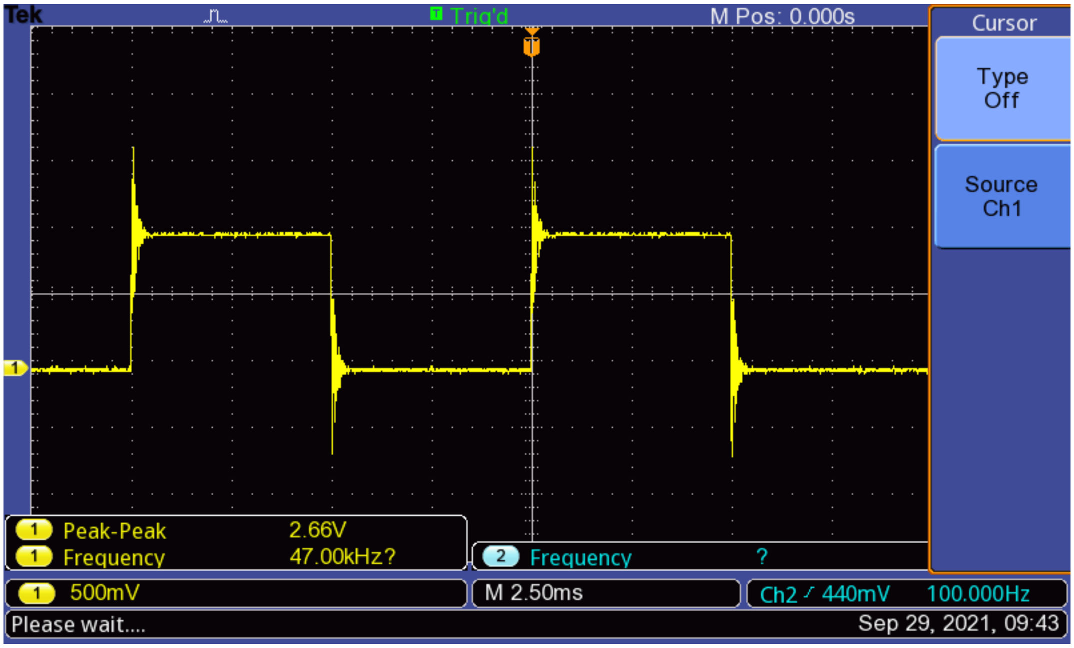

Figure 6: Hardcopy of one generated signal period

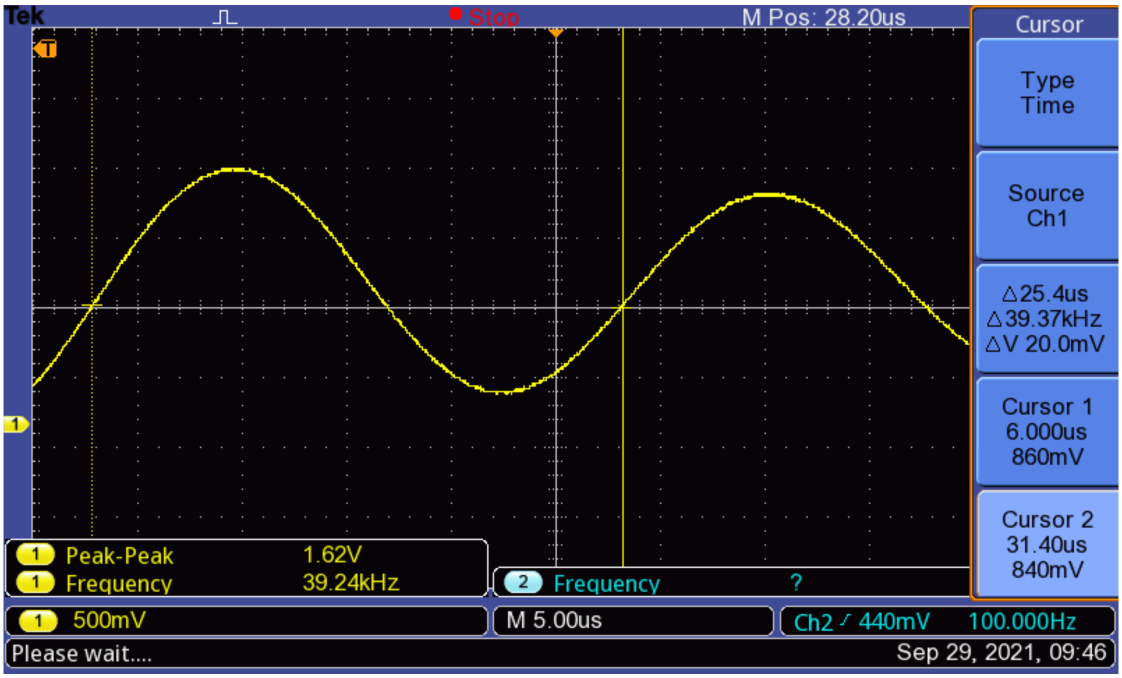

Figure 7: Hardcopy of the ringing phenomenon

- Calculating damped radian frequency ω_d:

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \qquad \text{and} \qquad \zeta = \frac{R}{2}\cdot\sqrt{\frac{C}{L}}$$

$$\omega_d = \frac{1}{\sqrt{LC}}\sqrt{1-(\frac{R}{2}\cdot\sqrt{\frac{C}{L}})^2}$$

$$\omega_d = \frac{1}{\sqrt{10\cdot 10^{-3}\cdot 1.5\cdot 10^{-9}}}\cdot\sqrt{1-\left(\frac{150}{2}\cdot\sqrt{\frac{1.5\cdot 10^{-9}}{10\cdot 10^{-3}}}\right)^2} = 258089.94\, rads^{-1}$$

$$\therefore \omega_d = 258089.9\, rads^{-1}$$

Now,

$$f_d = \frac{\omega_d}{2\pi} = \frac{258089.9}{2\pi} = 41.08\, kHz$$

The measured value and the calculated value of f_d are consistent with each other with the calculated value of 41.08 kHz being 4.7% greater than the measured value of 39.24 kHz.

- Calculating the resistance for critically damped circuit:
  For a circuit to be critically damped the value of $\zeta = 1$
  Now since, $\zeta = \frac{R}{2}\cdot\sqrt{\frac{C}{L}}$
  For $\zeta = 1$, $1 = \frac{R}{2}\cdot\sqrt{\frac{1.5\cdot 10^{-9}}{10\cdot 10^{-3}}}$
  $R = 5164\,\Omega$
  As the internal resistance of the signal generator contributes 50Ω resistance, the resistance R of the R-Decade should be:
  $R = 5164 - 50 = 5114\Omega$

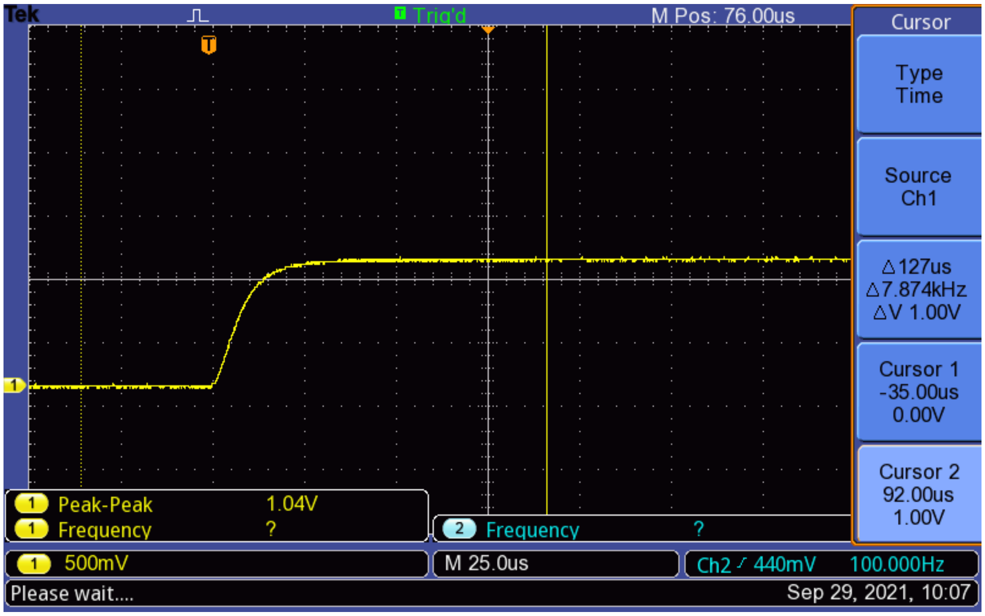

Figure 8: Hardcopy of Critically Damped Circuit with R=5114Ω

- Practically, the signal became critically damped for R = 4000Ω

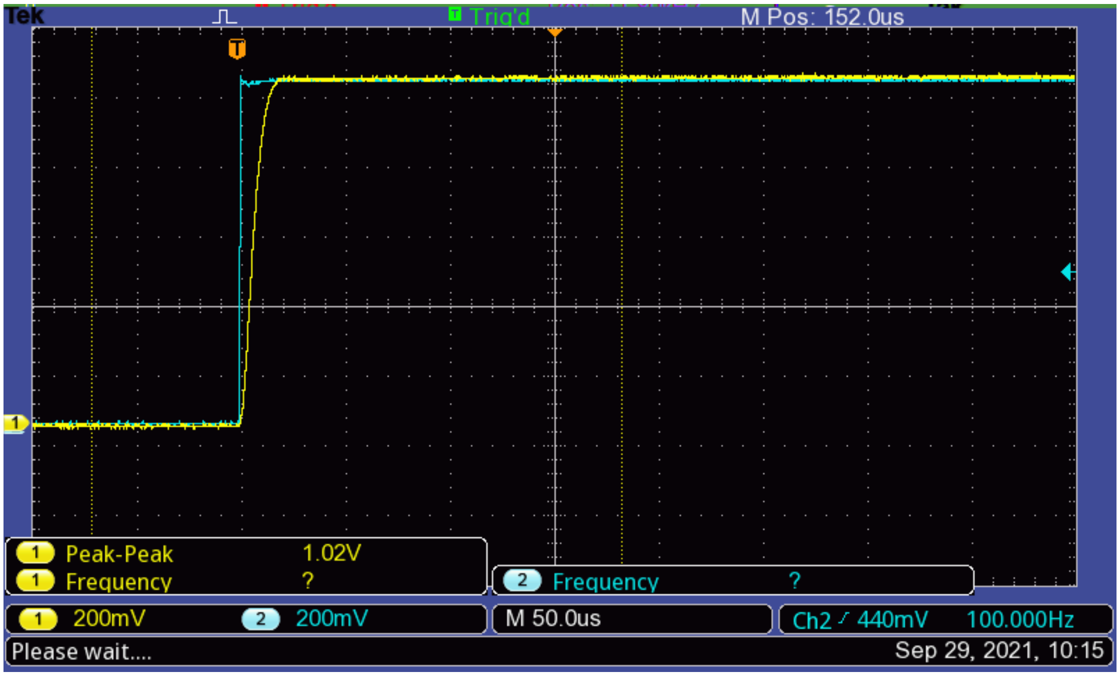

Figure 9: Hardcopy of the practically critical damped circuit at R = 4000Ω

- The R-Decade was set to 30kΩ to make the circuit overdamped:

Figure 10: Hardcopy of overdamped circuit

3 Evaluation

3.1 Evaluation Experiment Part 1

- Obtaining the differential equation for Voltage $v_c(t)$ when R = 100Ω:

  $$\zeta = \frac{R}{2} \cdot \sqrt{\frac{C}{L}} = \frac{100}{2} \cdot \sqrt{\frac{1.5 \cdot 10^{-9}}{10 \cdot 10^{-3}}} = 0.019$$

  Since, $0 < \zeta < 1$, the circuit is underdamped.

$$\omega_n = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10 \cdot 10^{-3} \cdot 1.5 \cdot 10^{-9}}} = 258198.9\, rads^{-1}$$

$$\omega_d = \omega_n \sqrt{1-\zeta^2} = 81649.7 \cdot \sqrt{1-(0.019)^2} = 258150.5\, rads^{-1}$$

The general equation of second-order homogenous differential equation for under-damped case is:

$$v_c(t) = e^{-\zeta \cdot \omega_n \cdot t}\ (C_1 \cdot \cos(\omega_d t) + C_2 \cdot \sin(\omega_d t))$$

In the RLC circuit: $V_{pp} = 1V$, So $V_{in} = \frac{1}{2} \cdot 1V = 0.5V$. However, since $V_{off} = 0.5V$, the final value of $V_{in} = 1V$

The voltage across the capacitor will be:

$v_c(0^-) = v_c(0^+) = 0V$

The total response $v_c(t) = v_{steady\ state} + v_{particular}$

$v_{steady\ state} = v_c(\infty) = 1V$

The general solution then becomes:

$$v_c(t) = 1 + e^{-0.019 \cdot 258198.9 \cdot t}\ (C_1 \cdot \cos(258150.5 \cdot t) + C_2 \cdot \sin(258150.5 \cdot t))$$

For t = 0:

$v_c(0) = 1 + e^0\ (C_1 \cdot \cos(0) + C_2 \cdot \sin(0))$

$0 = 1 + C_1$

$C_1 = -1$

Similarly, $I_c(0^-) = I_c(0^+) = 0A$

$$I_c(0) = C \cdot \frac{dV_c(0)}{dt}$$

$$\frac{dV_c(0)}{dt} = 0$$

Differentiating the general equation:

$\frac{dv_c(0)}{dt} = -4905.8 \cdot C_1 + 258150.5 \cdot C_2$

Thus,

$C_2 = -0.019$

So final general equation is,

$$v_c(t) = 1 + e^{-4905.8 \cdot t}\ (-1 \cdot \cos(258150.5t) - 0.019 \cdot \sin(258150.5t)$$

- Plot of the $v_c(t)$ in Matlab:

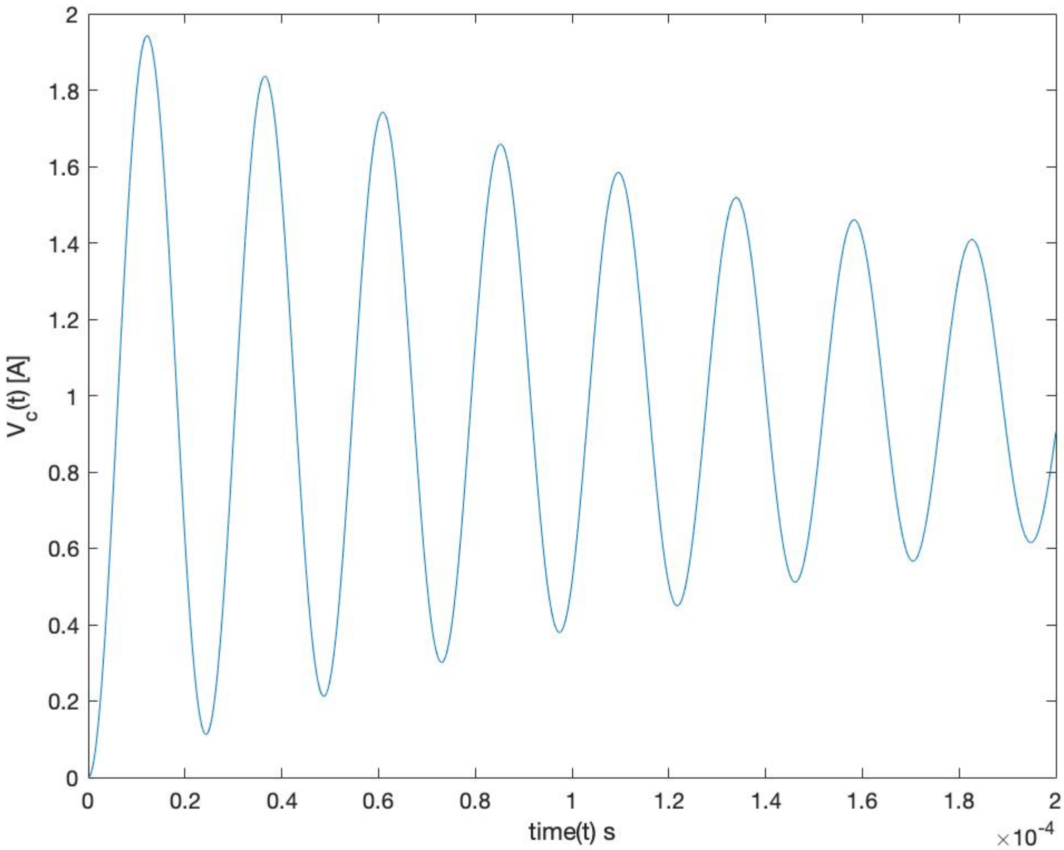

Figure 11 : MATLAB Plot of $v_c(t)$, the underdamped response

The MATLAB code used is:

```
zeta = 4905.8;
omega = 258150.5;
t = 0:0.0000002:0.0002;
A = -1;% coefficient C1
B = -0.019;%coefficient C2
y = 1 + exp(-zeta*t)*A.*cos(omega*t)+ B.*sin(omega*t);% General
Solution for under damped Response
plot(t,y);
ylim([0 2]);
xlim([0 2*10^-4]);
xlabel("time(t) / s");
ylabel("V_c(t) / [V]");
```

- For the circuit to be critically damped $\zeta = 1$:
  So,
  $$\zeta = \frac{R}{2} \cdot \sqrt{\frac{C}{L}}$$
  $$1 = \frac{R}{2} \cdot \sqrt{\frac{1.5 \cdot 10^{-9}}{10 \cdot 10^{-3}}}$$
  $R = 5164\,\Omega$

  As the internal resistance of the signal generator contributes 50Ω resistance, the resistance R of the R-Decade should be:
  $R = 5164 - 50 = 5114\Omega$

  The general solution for a critically damped circuit is:
  $\mathrm{v_c(t)} = C_1 e^{-\zeta \cdot \omega_n \cdot t} + C_2 \cdot t e^{-\zeta \cdot \omega_n \cdot t}$

  The total response $v_c(t) = v_{steady\ state} + v_{particular}$

  $v_{steady\ state} = 1V$ and $v_c(0) = 0V$ and $\frac{dV_c(0)}{dt} = 0$

  $$\omega_n = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10 \cdot 10^{-3} \cdot 1.5 \cdot 10^{-9}}} = 258198.9\ rads^{-1}$$

  $\mathrm{v_c(t)} = 1 + C_1 e^{-1 \cdot 258198.9 \cdot t} + C_2 \cdot t e^{-1 \cdot 258198.9 \cdot t}$
  The initial condition:
  $\mathrm{v_c(0)} = 1 + C_1$
  $C_1 = -1$

$\frac{dv_c(t)}{dt} = -258198.9 \cdot C_1 e^{-258198.9 \cdot t} - 258198.9 \cdot C_2 \cdot t e^{-258198.9 \cdot t} + C_2 e^{-258198.9 \cdot t}$
  $\frac{dv_c(0)}{dt} = -258198.9 \cdot C_1 + C_2$
  $C_2 = -258198.9$

The final general solution of $\mathrm{v_c(t)}$ of a critically damped circuit is:

$v_c(t) = 1 - e^{-258198.9 \cdot t} - 258198.9 \cdot t e^{-258198.9 \cdot t}$

- The plot of the $v_c(t)$ for critically damped circuit:

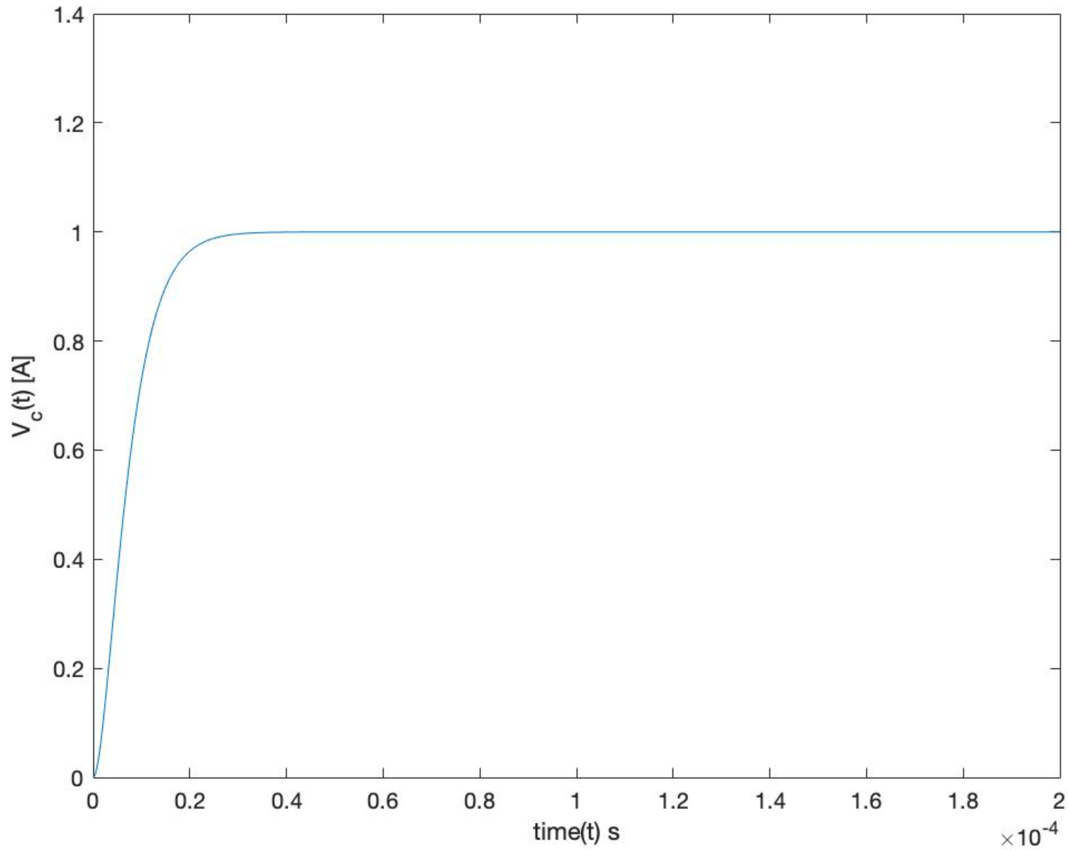

Figure 12 : MATLAB Plot of $v_c(t)$, the critically damped response

The MATLAB code used is:

```
omega = 258198.9;
t= 0:0.0000002:0.0003;
A = -1;% coefficient C1
B = -258198.9;%coefficient C2
y = 1 + exp(-omega*t).*(A + B*t);% General Solution for critically
damped Response
plot(t,y);
ylim([0 1.4]);
xlim([0 2*10^-4]);
xlabel("time(t) [s]");
ylabel("V_c(t) [V]");
```

- Comparison of experimental results with the calculations:

  From the calculations, the value of resistance comes out as 5114Ω for the circuit to be critically damped. However, the experimental results from the oscilloscope show that the circuit reaches the point of critical damping at 4000Ω. The reasons for this deviation of the value in resistance could be:

  The accuracy of the R-Decade used has a certain degree of accuracy. The error in the value of the resistance can be $\pm 1\%$ of the read value.
  Similarly, in the experiment the value of the capacitor as well as the inductor are not accurate (within $\pm 10\%$ of the given values) as the given value which was used in the calculation.
  In addition, the wrapped-up coil of the inductor in itself also contributed to some resistance which is not taken into account during calculation.

  Therefore, the deviation of the resistances between 4000Ω and 5114Ω for the circuit to be critically damped could be seen.

- Solving the problem of the following transient circuit when switch is opened at t = 0:

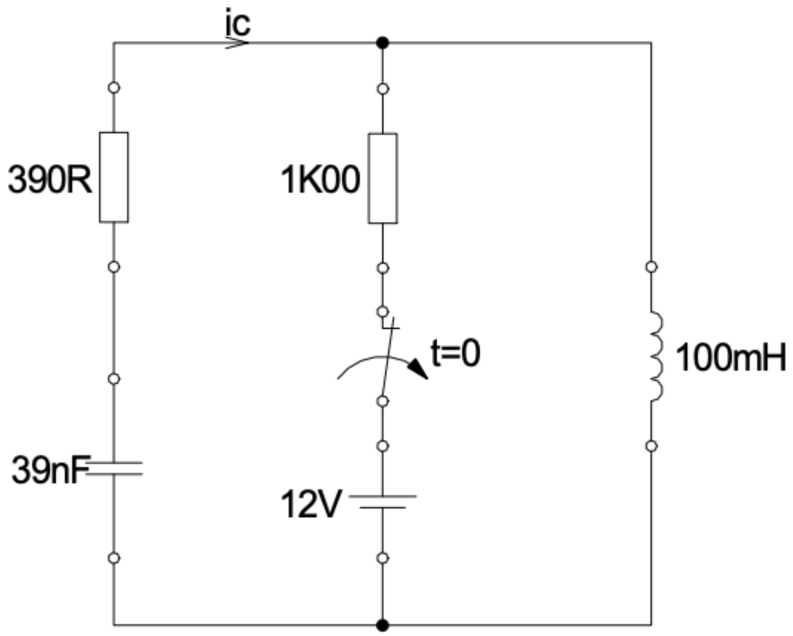

Figure 13: RLC Circuit (From Lab manual)

- Obtaining the differential equation for the current $i_c(t)$ through the capacitor and identifying the damping nature of the circuit and determine the values for the coefficients C_1 and C_2.

When time (t) > 0, the circuit will change to the following:

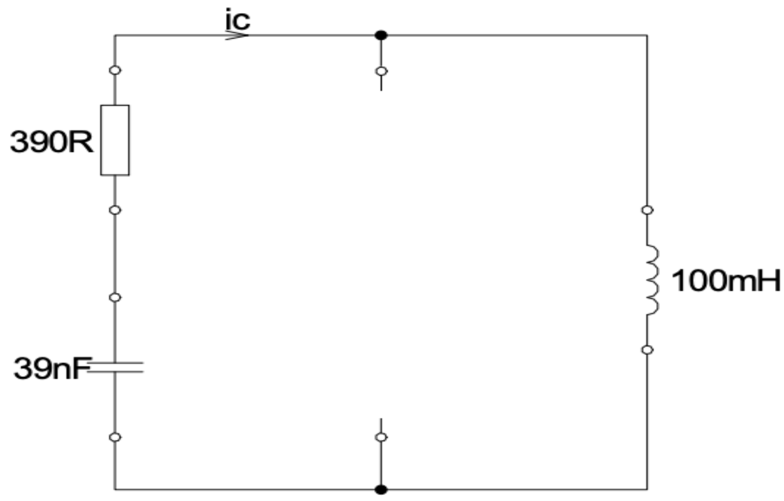

Figure 14: RLC Circuit for t > 0

Using Kirchhoff's Voltage Law:
$V_R + V_L + V_C = 0$
$i_c \cdot R + L \cdot \frac{di_c}{dt} + \frac{1}{C}\int i_c \, dt = 0$
Differentiating both sides with respect to t:
$R \cdot \frac{di_c}{dt} + L \cdot \frac{d^2 i_c}{dt^2} + \frac{1}{C} i_c = 0$
Multiplying both sides by C:
$LC \cdot \frac{d^2 i_c}{dt^2} + RC \cdot \frac{di_c}{dt} + i_c = 0$
Since $\zeta = \frac{R}{2} \cdot \sqrt{\frac{C}{L}}$, for R = 390Ω, C = 39nF and L = 100mH

$\zeta = \frac{390}{2} \cdot \sqrt{\frac{39 \cdot 10^{-9}}{100 \cdot 10^{-3}}} = 0.122$
$\because 0 < \zeta < 1$, the circuit is Underdamped.
Now,
$\omega_n = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{100 \cdot 10^{-3} \cdot 39 \cdot 10^{-9}}} = 16012.8\, rads^{-1}$
$\omega_d = \omega_n \sqrt{1 - \zeta^2} = 16012.8 \cdot \sqrt{1 - (0.122)^2} = 15893.2\, rads^{-1}$
The general equation of second-order homogenous differential equation for under-damped case is:
$f(t) = e^{-\zeta \cdot \omega_n \cdot t} \left(C_1 \cdot \cos(\omega_d t) + C_2 \cdot \sin(\omega_d t)\right)$

In this circuit, the current does not change instantaneously in the inductor,
$i_c(0^-) = i_c(0^+) = \frac{12}{1000} = 0.012$ A

The complete response of the system is:
$i_c(\mathrm{t}) = i_{steady\ state} + i_{particular}$
Here, $i_{c,\ steady\ state} = i_c(\infty) = 0\ A$

Thus, the equation changes to:
i_c(t) = $0 \ + \ e^{-0.122 \cdot 16012.8 \cdot t} \ (C_1 \cdot \cos(15893.2 \cdot t) + C_2 \cdot \sin(15893.2 \cdot t))$
i_c(t) = $0 \ + \ e^{-1953.6 \cdot t} \ (C_1 \cdot \cos(15893.2 \cdot t) + C_2 \cdot \sin(15893.2 \cdot t))$

For t = 0:
$i_c(0) = 0 \ + \ e^{0} \ (C_1 \cdot \cos(0) + C_2 \cdot \sin(0))$
$C_1 = 0.012$
For t > 0^+
$V_R + V_L + V_C = 0$
$i_c(0^+) \cdot R + L \cdot \frac{di_c(0^+)}{dt} + 0 = 0$
$0.012 \cdot 390 + 100 \cdot 10^{-3} \cdot \frac{di_c(0^+)}{dt} + 0 = 0$
$\frac{di_c(0^+)}{dt} = -46.8$
Differentiating the general equation:
$\frac{di_c(0)}{dt} = -1953.6 \cdot C_1 + 15893.2 \cdot C_2$
Thus,
$C_2 = 0.00147$
So,
i_c(t) = $e^{-1953.6 \cdot t} \ (0.012 \cdot \cos(15893.2 \cdot t) + 0.00147 \cdot \sin(15893.2 \cdot t))$

The plot of $i_c(t)$ is:

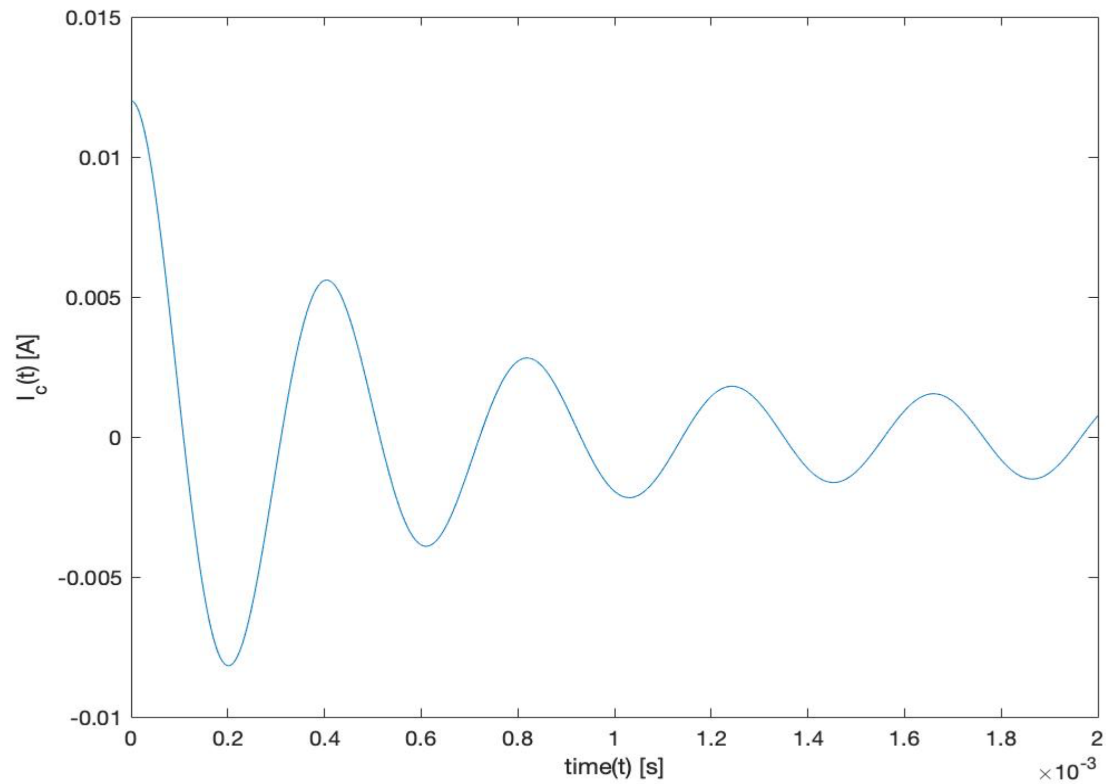

Figure 15: MATLAB Plot of $i_c(t)$ for underdamped case

The MATLAB code used is:

```
zeta = 1953.6;
omega = 15893.6;
t= 0:0.0000002:0.0020;
A = 0.012;% coefficient C1
B = 0.00147;%coefficient C2
y = exp(-zeta*t)*A.*cos(omega*t)+ B.*sin(omega*t);% General Solution
for under damped Response
plot(t,y);
xlabel("time(t) [s]");
ylabel("I_c(t) [A]");
```

4 Conclusion

The experiments conducted, focused on the concept and understanding of the Transient Analysis of an RLC circuit. The series of experiments dealt with the case of underdamped, critically damped and overdamped response of the circuit by the change of resistance value through R-decade to a square wave as input.

For the first part of the experiment, the value of the frequency of the output voltage response was measured through which it was determined to be an underdamped response. The general equation of the response was determined as the values of the coefficients were solved through initial conditions.

For the second part of the experiment general equation of the response of critically damped circuit was determined theoretically and then compared with the resistance value needed for the critical damp response through the experiment. In this process it was observed that the value had some difference mainly due to the resistance offered by the components used in the experiment which was not taken into account. Finally, the resistance was again altered to observe the overdamped response of the circuit.

Throughout the experiments the parameters R, L and C were used to determine the nature of damping and the damping frequency. The transient behavior of RLC circuit was observed and the general differential equations were calculated and plotted in the MATLAB for visual evaluation.

It was observed that the damping ratio helps to find out the nature of transient response. The damping ratio depends only on resistance, capacitance and inductance of the elements used.

- Error Analysis
  The experiment had fair amount of errors contributed through different components. The signal generator value of V_{pp} does not actually correspond with the value provided, thus the value had to be checked from the oscilloscope and the value could not be set to the exact requested values. However, the errors in the value of the resistance required for critical damping response were contributed due to the resistance provided by the components like R-decade with accuracy $\pm 1\%$, capacitor and inductor with $\pm 10\%$ range, with the resistance of the coils of inductor also not taken into account. These values could have added to the error in the value which was observed as there was noticeable difference between calculated and measured value.

All in all, the series of experiments fulfilled its objective of making us understand the analysis of a circuit with transient response through second order differential general equation solving. The difference between different type of damping in the oscillation was also visually acknowledged through their plots.

5 References

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