CAPITAL GROWTH AND THE MEAN-VARIANCE APPROACH TO PORTFOLIO SELECTION

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Three main approaches to the problem of portfolio selection may be discerned in the literature. The first of these is the mean-variance approach, pioneered by Markowitz [21], [22], and Tobin [30]. The second approach is that of chance-constrained programming, apparently initiated by Naslund and Whinston [26]. The third approach, Latané [19] and Breiman [6], [7], has its origin in capital growth considerations. The purpose of this paper is to contrast the mean-variance model, by far the most well-known and most developed model of portfolio selection, with the capital growth model, undoubtedly the least known. In so doing, we shall find the mean-variance model to be severely compromised by the capital growth model in several significant respects.

The standard portfolio problem, minus preference assumptions, is formulated in Section I. Section II shows that long-run growth of capital is obtained only for policies such that the expected logarithm of 1 plus the return is persistently positive in the various periods. When it is persistently negative, ultimate ruin is certain. The optimal policy of the capital growth model is derived in Section III.

In the case of a simple example involving two risky assets, it is found in Section IV that: (1) the optimal growth portfolio is not even close to being mean-variance efficient; (2) the only feasible portfolios which lead to ruin in the long run are mean-variance efficient; and (3) the worst portfolio from a mean-variance point of view will in the long run do better than most of the efficient portfolios. In Section VI, it is demonstrated by means of a three-asset example that the "graphic" mean-variance approach is basically inconsistent with the notion of stochastic dominance as well: portfolios on the most <u>in</u>efficient frontier are found to be absolutely preferred to portfolios on the efficient frontier.

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* *The author is indebted to Professor Myron Gordon for valuable comments and to Tien-Ching Liu for writing the necessary computer programs.*

In Section VII, the solvency constraint is added to the standard portfolio model. The previous three-asset example is again considered, with borrowing now a possibility, in Section VIII. It is found that every portfolio on the efficient frontier is either dominated by some inefficient portfolio or risking insolvency with positive probability, or both.

In Section IX, the "no-easy-money condition" is introduced as an equilibrium property of the capital market. In an example considered in Section X, the efficient frontier is now found to be free from dominated portfolios; however, most efficient portfolios still risk insolvency and the optimal growth portfolio is actually closer to the most inefficient frontier than to the efficient frontier.

Section XI compares the distribution of capital after six periods under the growth-optimal policy with those achieved under selected policies chosen from the solvency feasible efficient frontier. This comparison is quite favorable to the growth-optimal policy. A brief review of the general properties of the mean-variance model and of the capital growth model is then given in Section XII. The paper concludes with a discussion of growth fund policy implications in Section XIII and some final remarks in Section XIV.

I. The Standard Portfolio Problem

Aside from the assumptions concerning preferences, the premises employed in portfolio theory are highly standardized. A positive risk-free interest rate at which funds can be both borrowed and lent is usually assumed. The existence of one or more risky opportunities with stochastically constant returns to scale is a standard postulate. Furthermore, perfect liquidity and divisibility of the assets at each (fixed) decision point and absence of transaction costs, withdrawals, capital additions, and taxes are also implicitly assumed in most instances, along with stochastic independence of returns over time and the opportunity to make short sales. In this paper, these assumptions will again be employed along with the following notation:

x_j = the amount of investment capital at decision point j (the beginning of the j^{th} period) ($x_1 > 0$) .

M_j = the number of investment opportunities available in period j.

S_j = the subset of investment opportunities which is possible to sell short in period j.

$r_j - 1$ = rate of interest in period j, where $r_j > 1$.

β_{ij} = proceeds per unit of capital invested in opportunity i, where $i=2,\dots,M_j$, in the j^{th} period (random variable); i.e., if we invest an amount θ in i at the beginning of the period, we will obtain $\beta_{ij}\theta$ at the end of that period.

z_{1j} = the amount lent in period j (negative z_{1j} indicate borrowing) (decision variable).

z_{ij} = the amount invested in opportunity i, $i=2,\dots,M_j$, at the beginning of the j^{th} period (decision variable).

$z^*_{ij}(x_j)$ = an optimal lending strategy at decision point j.

$z^*_{ij}(x_j)$ = an optimal investment strategy for opportunity i, $i=2,\dots,M_j$, at decision point j.

$$F_j(y_2, y_3, \dots, y_{M_j}) \equiv \Pr\{\beta_{2j} \le y_2, \beta_{3j} \le y_3 \dots, \beta_{M_j j} \le y_{M_j}\}$$

$$v_{ij} \equiv \begin{cases} \dfrac{z_{ij}}{x_j} & x_j \ne 0 \\ 0 & x_j = 0 \end{cases} \qquad i=1,\dots,M_j \; .$$

$$\bar{z}_j \equiv (z_{2j}, \dots, z_{M_j j}) \; .$$

$$\bar{v}_j \equiv (v_{2j}, \dots, v_{M_j j}) \text{ , and}$$

$$\langle \bar{v}_j \rangle \equiv \bar{v}_1, \bar{v}_2, \dots, \bar{v}_j \; .$$

v_{ij} clearly denotes the proportion of capital x_j invested in opportunity i at the beginning of period j.

Since the end-of-period capital position is given by the proceeds from current savings, or the negative of the repayment of current debt plus interest, plus the proceeds from current risky investments, we have

$$x_{j+1} = r_j z_{1j} + \sum_{i=2}^{M_j} \beta_{ij} z_{ij} \qquad j=1,2,\dots \tag{1}$$

where

$$\text{(2)} \qquad \sum_{i=1}^{M_j} z_{ij} = x_j \qquad j=1,2,\ldots .$$

Combining (1) and (2), we obtain

$$\text{(3)} \qquad x_{j+1} = \sum_{i=2}^{M_j} (\beta_{ij} - r_j) z_{ij} + r_j x_j \qquad j=1,2,\ldots$$

$$= x_j R_j(\overline{v}_j) \qquad j=1,2,\ldots$$

where

$$\text{(4)} \qquad R_j(\overline{v}_j) \equiv \sum_{i=2}^{M_j} (\beta_{ij} - r_j) v_{ij} + r_j .$$

Clearly, $R_j(\overline{v}_j)$ is 1 plus the return on the whole portfolio $\overline{v}_j$ in period j. The portfolio problem at decision point j is now seen to be one of choosing the vector of risky investments $\overline{z}_j \equiv (z_{2j}, \ldots, z_{M_j j})$ (proportions $\overline{v}_j$) to produce the most favorable distribution of end-of-period capital x_{j+1} (or $R_j(\overline{v}_j)$). Implicit in every choice of $\overline{z}_j$ is a choice of saving (borrowing) z_{1j} in view of (2). Clearly, $v_{1j} = 1 - \sum_{i=2}^{M_j} v_{ij}$.

The mean-variance school, in essence, takes the position that only the mean and variance of x_{j+1} (or $R_j(\overline{v}_j)$) is relevant in the evaluation of the possible portfolio choices in order to find the best one(s). We shall return to this approach later; for the moment, we shall keep an open mind as to how one might evaluate the different possible distributions of x_{j+1} as of decision point j. In view of the fact that current decisions have an effect on *all* future capital positions, not only the next one, we shall instead express those future capital positions in terms of previous decisions in order to determine whether any particular one-period decision(s) might be more sensible than others. The only assumption we make with respect to preferences is that more money is always preferred to less at all times; this criterion, of course, is in general capable of inducing only a partial ordering of wealth distributions at any given point.

II. Policies Which Lead to Ultimate Ruin

Solving (3) recursively, we obtain

(5) $$x_{j+1} = x_1 \prod_{n=1}^{j} R_n(\bar{v}_n) \qquad j=1,2,\ldots .$$

When

(6) $$\Pr\{R_n(\bar{v}_n) \geq 0\} = 1 \qquad n=1,\ldots,j,$$

(5) may be written

(7) $$x_{j+1} = x_1 e^{\sum_{n=1}^{j} \log R_n(\bar{v}_n)} \qquad j=1,2,\ldots .$$

Letting

(8) $$G_j(\langle\bar{v}_j\rangle) \equiv \frac{\sum_{n=1}^{j} \log R_n(\bar{v}_n)}{j} \qquad j=1,2,\ldots ,$$

It follows from the law of large numbers -- since the distribution functions F_j are independent -- that

(9) $$\Pr\{|G_j - E[G_j]| > \varepsilon\} \to 0 \qquad \text{as } j \to \infty$$

for every $\varepsilon > 0$ whenever

(10) $$-\infty < E[\log R_j(\bar{v}_j)] < \infty \qquad j=1,2,\ldots$$

and

(11) $$\mathrm{Var}[\log R_j(\bar{v}_j)] < \infty \qquad j=1,2,\ldots .$$

Using (8), (7) now becomes

(12) $$x_{j+1} = x_1 \left(e^{G_j}\right)^j \qquad j=1,2,\ldots$$

so that whenever for some number N and some $\delta > 0$

(13) $$E[G_j] < -\delta \qquad j \geq N ,$$

we obtain from (9)

$$(14) \qquad \Pr\{x_{j+1} = 0\} \to 1 \qquad j \to \infty ,$$

while if

$$(15) \qquad E[G_j] > \delta \qquad j \geq N,$$

(9) gives[1]

$$(16) \qquad \Pr\{x_{j+1} = \infty\} \to 1 \qquad \text{as } j \to \infty .$$

Note that if (10) and (11) hold and for some number I

$$(17) \qquad E[\log R_j(\overline{v}_j)] < 0 \qquad j \geq I$$

or

$$(18) \qquad E[\log R_j(\overline{v}_j)] > 0 \qquad j \geq I,$$

where (17) and (18) are bounded away from zero, then we see from (8) that (17) implies (13) and (18) implies (15) for some N.

Note also that (14) holds if $\Pr\{R_j(\overline{v}_j) = 0\} > 0$ for all j, even though (10) and (11) are violated, because in this case $\Pr\{x_{j+1} > 0\} \leq (1 - \varepsilon)^j$ where ε is a positive number, which tends to zero as $j \to \infty$.

From the preceding it is clear that *persistent portfolio choices such that the expected logarithm of 1 + the return is negative lead to ultimate ruin.* From a long-run point of view, the expected logarithm of 1 + the return, $R_j(\overline{v}_j)$, should clearly be prevalently positive in the various periods. What (14), (16), and (16a) tell us, in fact, is that portfolio choices $\overline{v}_j$ such that

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[1]The only other case of interest besides (13) and (15) is that for which $\lim_{j \to \infty} E[G_j] = 0$. In this case, (9) gives

$$(16a) \qquad \sqrt[j]{x_{j+1}/x_1} \to 1 \qquad \text{as } j \to \infty .$$

$$E[\log R_j(\bar{v}_j)] > 0, \qquad j=1,2,\ldots,$$

tend, in the very long run, to be *infinitely better* than choices $\bar{v}_j'$ such that

$$E[\log R_j(\bar{v}_j')] \leq 0, \qquad j=1,2,\ldots .$$

At this point, a pair of illustrations may be useful. A portfolio consisting entirely of savings clearly corresponds to $\bar{v}_j = (0,0,\ldots,0)$ so that by (4)

$$E[\log R_j(0,\ldots,0)] = \log r_j > 0$$

since $r_j > 1$ by assumption. Thus, this portfolio will in time grow to infinity. Now consider a portfolio consisting entirely of one stock (opportunity 2, say) which in each period yields, with equal probability, a profit of 100 per cent or a loss of 60 per cent. This portfolio is then represented by $\bar{v}_j = (1,0,0,\ldots,0)$ so that by (4),

$$E[\log R_j(1,0,\ldots,0)] = 1/2\ (\log\ (0.40) + \log 2) \approx -0.1116.$$

Since

$$\text{(19)} \qquad \mu \equiv e^{E[G_j]} - 1 \qquad j=1,2,\ldots$$

may be interpreted as the average expected growth rate of the portfolio over the first j periods in view of (9) and (12), we obtain, since $e^{-0.1116} \approx 0.8945$, that capital will on the average *decline by approximately 10.55 per cent per period* by putting all assets into opportunity 2, even though this investment has an *expected return of 20 per cent* $(1/2 \cdot 100\% + 1/2 \cdot (-60\%))$. In intuitive Wall Street language, you have to figure on losing half the time and winning half the time; if you win "first," your capital goes to 200 per cent--but the 60 per cent loss then takes your capital, after two periods, to 80 per cent of what you started with. An expected decline rate of 20 per cent per double period is equal to an average expected decline rate of about 10.55 per cent per single period and will, as noted in (14), in the very long run wipe out all capital. Thus, an investment policy which achieves only 1 per cent (for sure) per period is, in the very long run, almost surely infinitely better then one which risks, in each period, 60 per cent of your capital for an equal chance at a 100 per cent profit.

III. Long-Run Optimal Policies

In the preceding section we observed that the set of investment policy sequences, $\bar{v}_1, \bar{v}_2, \ldots$, for which $E[\log R_j(\bar{v}_j)] > 0$ for all j holds is clearly the best. But since this set will in general contain infinitely many strategies, the question arises regarding whether we can meaningfully distinguish between policies *within* this set. At first appearance, the answer would seem to be negative; however, intuition also suggests that the "larger" G_j is, the "faster," $x_{j+1} = x_1 e^{jG_j}$ would approach infinity--in the sense that a 5 per cent certain return would lead to more rapid growth than a 4 per cent certain return. In the following paragraph, we shall use a slight generalization of an argument due to Brown [8].

Assume that each $E[G_j]$, $j=1,2,\ldots$, has a maximum on the set of feasible $\langle\bar{v}_j\rangle$ and denote by $\langle\bar{v}^*_j\rangle \equiv \bar{v}^*_1, \bar{v}^*_2, \ldots, \bar{v}^*_j$ any sequential investment policy which achieves the maximum and by $\langle\bar{v}'_j\rangle$ any policy $\bar{v}'_1, \bar{v}'_2, \ldots, \bar{v}'_j$ which differs from $\langle\bar{v}^*_j\rangle$ in almost all, with the exception of a finite number, of time periods. (Conditions for the existence of optimal policies will be given in Section X.) Let μ, where $\mu > 1$, be a discount rate and consider the present value at decision point 1 of x_{j+1}, i.e., $\mu^{-j}x_{j+1}$. Equation (12) gives

$$\mu^{-j}x_{j+1} = x_1\left(\frac{e^{G_j(\langle\bar{v}_j\rangle)}}{\mu}\right)^j \qquad j=1,2,\ldots \tag{20}$$

By the definition of $\langle\bar{v}^*_j\rangle$ and $\langle\bar{v}'_j\rangle$, there exists a number J such that (denoting $G_j(\langle\bar{v}^*_j\rangle)$ by G^*_j and $G_j(\langle v'_j\rangle)$ by G'_j)

$$e^{E[G^*_j]} > e^{E[G'_j]} \qquad j \geq J\,. \tag{21}$$

Thus, there also exists a sequence of discount rates $\mu^*_1, \mu^*_2, \ldots$ such that

$$e^{E[G'_j]} < \mu^*_j < e^{E[G^*_j]} \qquad j \geq J. \tag{22}$$

Since by (22) and the law of large numbers (9) we now obtain, as $j \to \infty$,

$$\Pr\left\{x_1\left(e^{G'_j}\Big/\mu^*_j\right)^j = 0\right\} \to 1, \tag{23}$$

$$(24) \qquad \Pr\left\{x_1 \left(e^{G_j^*} \Big/ \mu_j^*\right)^j = \infty\right\} \to 1,$$

and

$$(25) \qquad \Pr\left\{x_1 \left(e^{G_j(<\bar{v}_j>} \Big/ \mu_j\right)^j = 0\right\} \to 1, \quad \mu_j > e^{E[G_j^*]}, \; j \geq N, \text{ all } <\bar{v}_j>,$$

we find that the discount rates μ_j^*, $j=1,2,\ldots$, have the property that the present value of x_{j+1}, discounted at μ_j^*, goes to zero with probability 1 as $j \to \infty$ for every policy not asymptotically close to the policy $\bar{v}_1^*, \bar{v}_2^*, \ldots$, for which the long-run present value is infinite with the same probability. Thus, in a sense which should appeal to scholars and businessmen alike, we can say that the investment policy which maximizes $E[G_j(<\bar{v}_j>]$ for all j is, in the very long run, almost surely infinitely better than any other policy.

Due to the independence of the distribution functions F_j, it follows from (8) that it is necessary and sufficient, in order to maximize $E[G_1], E[G_2], \ldots$, to maximize $E[\log R_j(\bar{v}_j)]$ at each decision point j. Thus, to behave optimally in the long run, the investor should act at each decision point *as if* he had a short-run logarithmic utility function of 1 plus the return on his portfolio. Note that this result was obtained with no assumption whatsoever regarding the investor's preferences, except the innocuous premise that he prefers more capital to less at all times.[2]

The portfolio $\bar{v}_j$ which maximizes

$$(26) \qquad E\left[\log\left(\sum_{i=2}^{M_j} (\beta_{ij} - r_j)v_{ij} + r_j\right)\right]$$

subject to the constraints operating on $\bar{v}_j$ in period j will in the remainder of the paper be called the growth-optimal portfolio in period j and will be denoted $\bar{v}_j^*$. Thus, v_{ij}^*, $i=2,\ldots,M_j$ represents the optimal proportion of capital x_j to invest in risky opportunity i at decision point j, and $1 - \sum_{i=2}^{M_j} v_{ij}^*$ expresses the optimal proportion to lend (if negative, $\sum_{i=2}^{M_j} v_{ij}^* - 1$ gives the optimal proportion to borrow) under a optimal-growth policy.

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[2]This premise, of course, is of little value in the earlier periods when the capital positions have nondegenerate probability distributions; its applicability is therefore essentially restricted to (degenerate) limiting distributions such as those in (23) and (24).

IV. Example I: No Lending or Borrowing

In comparing the distributions of 1 + the return, $R_j(\bar{v}_j)$ (or equivalently, of the return $R_j(\bar{v}_j) - 1$), of various portfolios $\bar{v}_j$, the mean-variance school, as mentioned earlier, takes the position that the mean and the variance of $R_j(\bar{v}_j)$ capture all that is relevant about the distribution for decision-making purposes. This implies, of course, that all distributions $R_j(\bar{v}_j)$ with the same mean and variance are regarded as equivalent, irrespective of skewness, etc. In addition, if

$$(27) \qquad E[R_j(\bar{v}_j')] > E[R_j(\bar{v}_j'')] \text{ and } Var[R_j(\bar{v}_j')] \leq Var[R_j(\bar{v}_j'')]$$

or

$$(28) \qquad E[R_j(\bar{v}_j')] \geq E[R_j(\bar{v}_j'')] \text{ and } Var[R_j(\bar{v}_j')] < Var[R_j(\bar{v}_j'')],$$

then portfolio $\bar{v}_j'$ is assumed to be preferred to (or dominate) portfolio $\bar{v}_j''$. All portfolios $\bar{v}_j''$ for which there exists no other portfolio $\bar{v}_j'$ such that (27) or (28) hold are said to be efficient. In most situations, the number of efficient portfolios is infinite (at least under the perfect divisibility assumption). While the "optimal" mean-variance portfolio is always an efficient portfolio, its final selection is partly shrouded in mystery. This is because the selection procedure, usually discussed in terms of tradeoffs between mean (expected return) and variance (interpreted as a risk measure) is often left outside the model. When it is formally incorporated into the model, it almost always reflects a linear[3] tradeoff between the mean and the variance or else specifies minimal requirements on either the mean or the variance. In any case, *every* efficient portfolio has an opportunity to be chosen as "optimal."

The question that now arises is whether the growth-optimal portfolio will in general be mean-variance efficient. A priori there is no reason why it should be, although intuitively it is hard to imagine that it would not be at least close to the efficient frontier. We shall consider this question in terms of some examples --these, in turn, will raise other questions connected with the standard portfolio model per se which will be dealt with in subsequent sections.

Let us assume that at each decision point there are one safe and two risky investment opportunities, i.e., $M_j = 3$, as follows:

3 This form is inconsistent with the notion of stochastic dominance (see Section V), as shown by Massé [24, pp. 212-213], and the expected utility theorem (see Section XII). See also [15].

$$S_j = \{1\} ,$$

$$r_j = 1.05 ,$$

$$\beta_{2j} = \begin{cases} 0 & \text{with probability } 0.1 \\ 1.50 & \text{with probability } 0.9 \end{cases} \qquad \begin{aligned} &\Pr\{\beta_{2j} = 0, \beta_{3j} = 1.15\} = 0.1 \\ &\Pr\{\beta_{2j} = 0, \beta_{3j} = 2.65\} = 0 \end{aligned}$$

$$\beta_{3j} = \begin{cases} 1.15 & \text{with probability } 0.9 \\ 2.65 & \text{with probability } 0.1 \end{cases} \qquad \begin{aligned} &\Pr\{\beta_{2j} = 1.50, \beta_{3j} = 1.15\} = 0.8 \\ &\Pr\{\beta_{2j} = 1.50, \beta_{3j} = 2.65\} = 0.1 . \end{aligned}$$

$S_j = \{1\}$, of course, means that short sales are ruled out, i.e., the constraint

$$(29) \qquad v_{ij} \geq 0 \qquad i=2,3, \text{ all } j$$

must be observed.

We shall first look at the simple case when all assets must be allocated to opportunities 2 and 3 and no borrowing is permitted, i.e., v_{2j} and v_{3j} are constrained by

$$(30) \qquad v_{2j} + v_{3j} = 1 \qquad \text{all } j.$$

The mean and standard deviation of $R_j(\bar{v}_j)$ for portfolio $\bar{v}_j$ now become

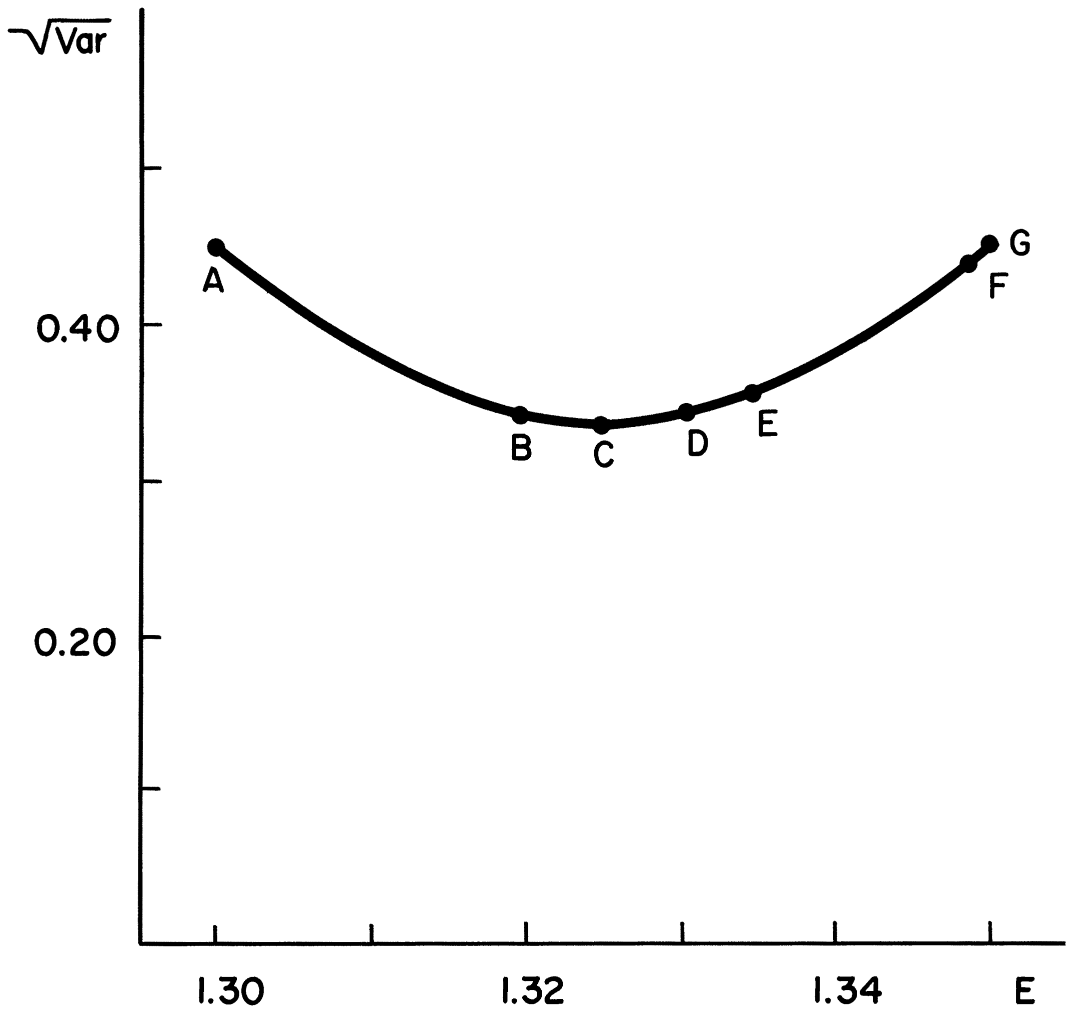
$$(31) \qquad E[R_j(\bar{v}_j)] = E[\beta_{2j}v_{2j} + \beta_{3j}v_{3j}] = 1.35v_{2j} + 1.30v_{3j} ,$$

$$(32) \qquad \sqrt{\mathrm{Var}[R_j(\bar{v}_j)]} = \sqrt{v_{2j}^2\mathrm{Var}[\beta_{2j}] + v_{3j}^2\mathrm{Var}[\beta_{3j}] + 2v_{2j}v_{3j}\mathrm{Cov}[\beta_{2j}\beta_{3j}]}$$

$$= \sqrt{0.2025(v_{2j}^2 + v_{3j}^2) + 0.045v_{2j}v_{3j}} .$$

Plotting the set of mean-standard deviation combinations which are feasible under (29) and (30), we obtain curve ABCDEFG in Figure I. Point A represents the portfolio (0, 1) and point C the portfolio (0.5, 0.5), while point G represents the portfolio (1, 0). In other words, point G represents 1 + the expected return and the standard deviation of the return resulting from an

FIGURE I



investment of all assets in opportunity 2. Clearly, all portfolios which give rise to the curve CDEFG are efficient, i.e., all portfolios with 50 per cent or more of their assets in opportunity 2.

To obtain the growth-optimal portfolio, we maximize

$$E[\log R_j(\bar{v}_j)] = E[\log(\beta_{2j}v_{2j} + \beta_{3j}v_{3j})]$$

subject to (29) and (30). The unique solution $\bar{v}_j^*$ is approximately (0.394, 0.606); this portfolio is represented by point B in Figure I. It is immediately clear that the growth-optimal portfolio is *not efficient in a mean-variance sense*. Thus, the expected growth rate of this portfolio (approximately 27.76 per cent; see (19)) is higher than that of any mean-variance efficient portfolio. Put differently, at any discount rate strictly between 27.51 per cent (the highest expected growth rate of an efficient portfolio) and 27.76 per cent, the present value of the capital resulting from repeated selections of the portfolio represented by B is in the very long run infinitely higher than the present value of the capital resulting from repeated choices of mean-variance efficient portfolios. The interpretation of this is that in the very long run, the growth-optimal portfolio will do as well as a portfolio which returns 27.76 per cent *for sure* in each period.

Note that of all the feasible portfolios (those on the curve ABCDEFG), only those on the curve FG have a negative expected growth rate. Thus, the only feasible portfolios which lead to ultimate ruin with probability 1 in this example are mean-variance efficient.

In terms of the basic mean-variance criterion, portfolio B is dominated by every portfolio between B and D on curve BCD since these portfolios have both higher expected return *and* a lower variance of return than portfolio B. Yet, these portfolios will, as we have shown, in the long run be outperformed by portfolio B.

The *worst* portfolio under the mean-variance criterion is portfolio A since it has the smallest expected return and the highest variance of return. Yet, this portfolio has an expected growth rate of approximately 25.01 per cent, greater than that of any portfolio on curve EFG. In other words, the *most inefficient* portfolio will, in the long run, give better results than *most* of the mean-variance efficient portfolios!

While the preceding conclusions are based on a very simple example, they are devastating enough with respect to the mean-variance approach to suggest further study of that method. We shall next introduce lending as a "true" alternative, but first we shall review the notion of stochastic dominance.

V. Stochastic Dominance

The assumption that more money is preferred to less at any point in time j is equivalent to the assumption that $R_j(\overline{v}_j)$ is preferred to $R_j(\overline{v}_j')$, whenever R_j has a degenerate distribution, if and only if $R_j(\overline{v}_j) > R_j(\overline{v}_j')$. In essence, $R_j(\overline{v}_j)$ is preferred to $R_j(\overline{v}_j)$ because the former "dominates" the latter. This concept of deterministic dominance also has a counterpart in the stochastic case. Let

(33)
$$H_j(y) \equiv \Pr\{R_j(\overline{v}_j) \leq y\} ,$$

i.e., let $H_j(y)$ be the distribution function of $R_j(\overline{v}_j)$. Then, $H_j(y)$ is said to be absolutely preferred to $H_j'(y)$, or to stochastically dominate $H_j'(y)$, if and only if

(34)
$$H_j(y) \leq H_j'(y) \qquad \text{all } y$$
$$H_j(y) < H_j'(y) \qquad \text{some } y .$$

Thus, if for every possible return the probability of getting more than that return for one prospect is at least as high as for another prospect, and for some returns it is actually higher, then the first prospect would be preferred. As a result, anyone who prefers more to less (exhibits deterministic dominance) would clearly be expected to always exhibit absolute preference as well--or else be considered pathological.

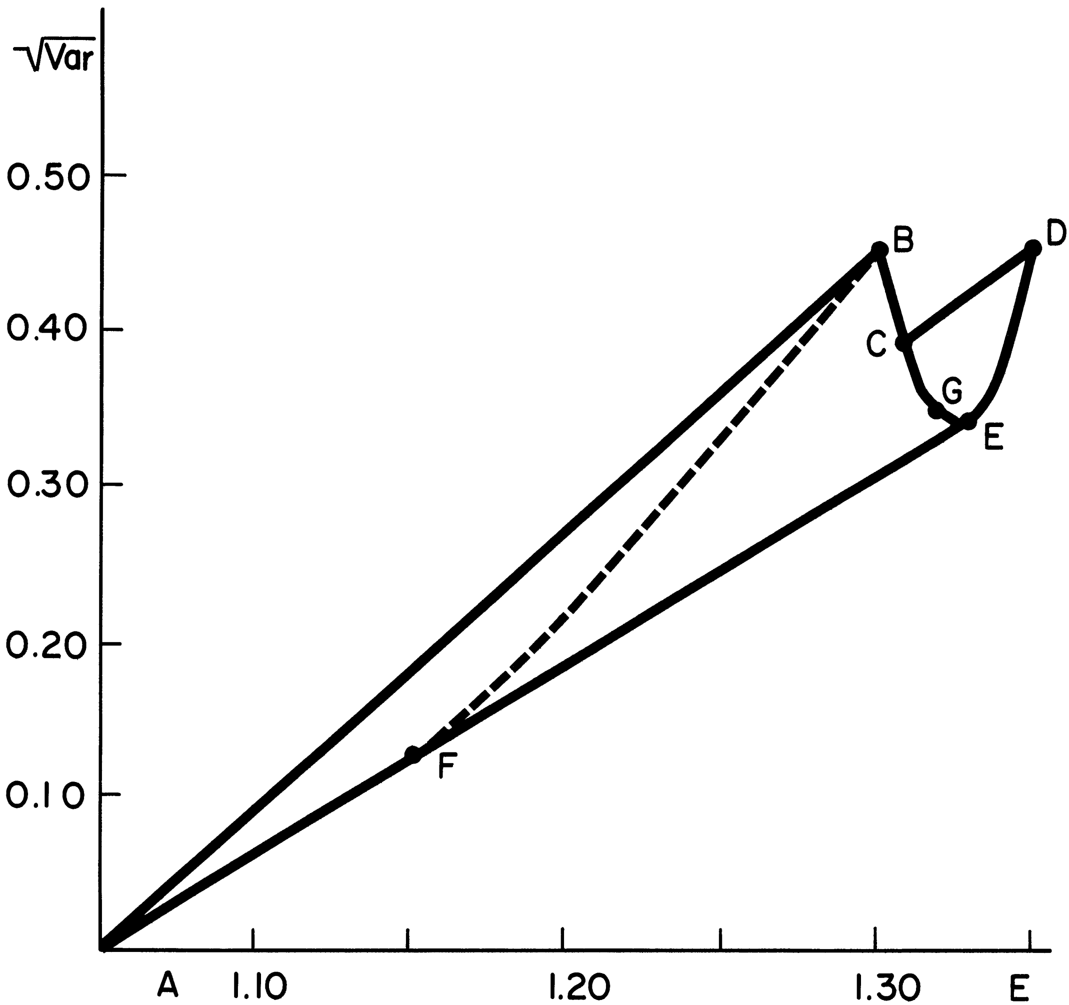
VI. Example II: Lending But No Borrowing

We now retain the restriction against borrowing and short sales of Section IV, but replace constraint (30) with

(35)
$$v_{2j} + v_{3j} \leq 1.$$

In other words, we now permit non-negative investment combinations of lending, opportunity 2, and opportunity 3.

FIGURE II



The set of feasible mean-standard deviation combinations is now considerably expanded, from those on curve BCGED in Figure II to the set of all points bounded by ABCDEFA. The portfolios on the curve AFED are now the efficient ones; they include (0, 0) (point A) and, as before, (1, 0) (point D). The growth optimal portfolio is also the same as before, i.e., approximately (0.394, 0.606) (point G); it is still some distance from the efficient frontier.

In Example I it is readily determined that no feasible portfolio is absolutely preferred to any other portfolio. However, in Example II, point B, for example, clearly dominates point A since $\Pr\{\beta_{2j} > r_j\} = 1$. It is easily shown that all investment results $R_j(\bar{v}_j)$ in the region bounded by ABFA, and no others, are stochastically dominated by some investment result(s) in the whole region. Specifically, all the points in region ABFA (except B) are stochastically dominated by point B. Common sense, then, would limit the portfolio choice to a selection among those portfolios whose investment results fall in the region FBCDEF, since the optimal portfolio must presumably come from that set.

Since point B dominates all points on curve AF (which is part of the efficient frontier), we see that the mean-variance approach is inconsistent with the notion of stochastic dominance. This is due to the fact that if an individual were sufficiently risk-averse (in a mean-variance sense), no one would presumably argue with a portfolio choice on the segment AF of the efficient frontier. At least no one, with the possible exception of Baumol has done so in the literature.[4] Perhaps even more significant is the fact that *every* point on the *most inefficient* frontier AB (except point A) is absolutely preferred to some of the points on the efficient curve AF.

VII. Solvency

Creditors are primarily interested in two things with respect to debtors: (1) ability to pay and (2) willingness to pay. It may be some time before the second criterion becomes a part of formal models in finance. The absence of the first consideration from models of portfolio selection is more surprising, however, considering how readily the notion is dealt with and the significance of its role in shaping borrowing limits and restrictions on short sales. Borrowing

[4] On the basis of normally distributed portfolio returns, Baumol formulated a criterion which, in effect, rules out a portion of the lower efficient frontier when efficient portfolios with high mean returns have only a small chance of doing worse than efficient portfolios with low mean returns [3].

limits and rules governing short sales are, of course, primarily, but not exclusively, operational surrogates for attempts to insure solvency. In theory, however, it seems intuitively more satisfactory to deal with the solvency problem directly. One reason for this is that solvency and survival in the long run are intimately linked, especially under the standard assumption that there are no capital additions at future decision points. In this case, insolvency at one point implies insolvency from that point on--unless someone is willing to lend the insolvent investor additional funds for investment.

Another reason why the absence of the solvency notion from the standard portfolio model is surprising is that the same model usually assumes equal rates for borrowing and lending and that lending is perfectly safe. Therefore, the lending investor is implicitly assumed to be repaid both principal and interest with probability 1. But lending investors are not ruled out as a source of funds to borrowing investors, either by differences in interest rates or otherwise, which seems inconsistent with the lack of requirements on repayment ability (solvency with probability 1) on the part of borrowers.

To require solvency at the end of period j is to require that

$$\Pr\{x_{j+1} \geq 0\} = 1 \tag{36}$$

holds. Usually, this imposes an additional restriction on the set of feasible portfolios $\bar{v}_j$ at decision point j.

VIII. Example III: Borrowing and Lending

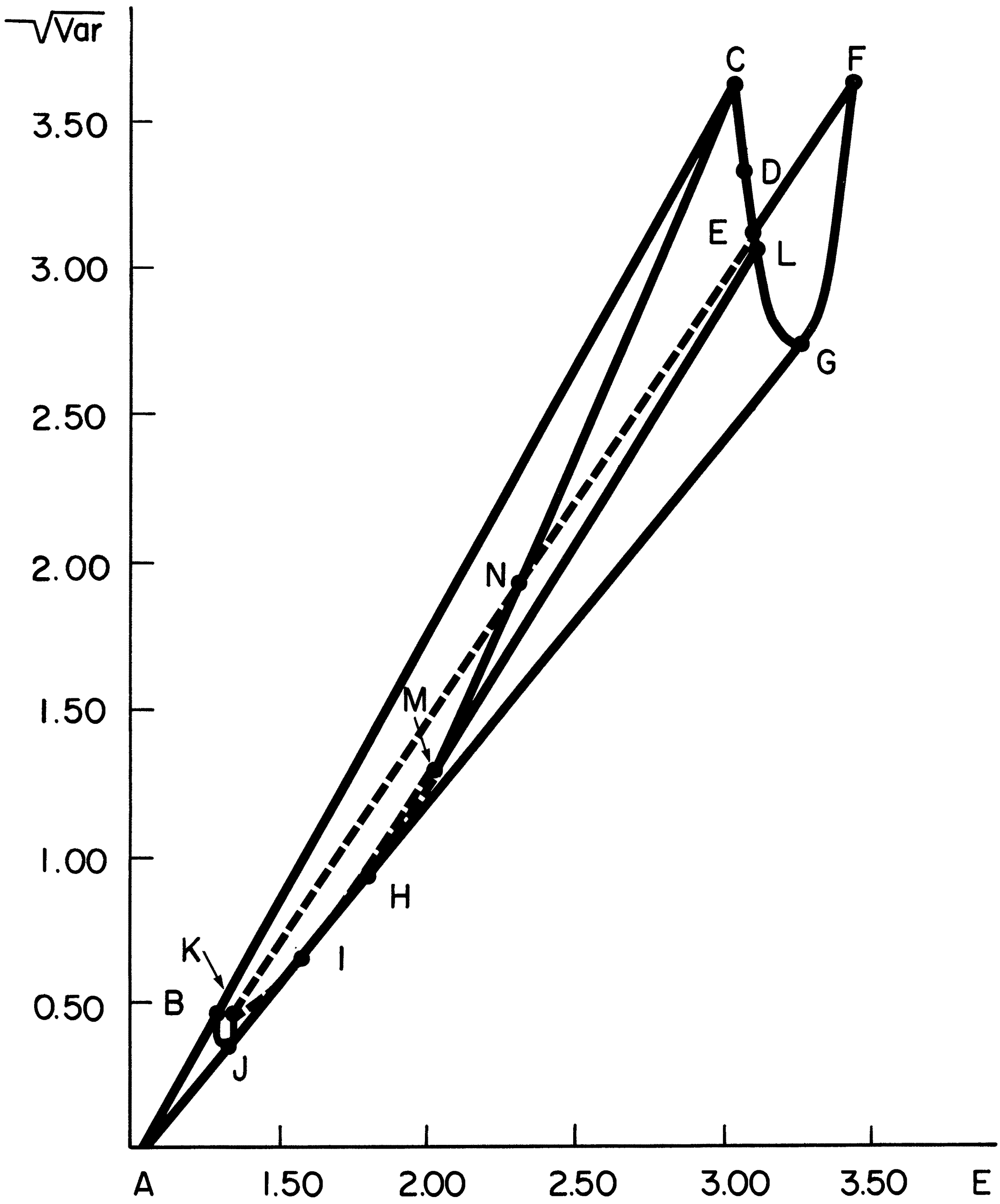
We shall now relax the prohibition against borrowing in Example II, at least partly, by assuming a margin requirement of 12 1/2 per cent, i.e., a borrowing limit of seven times the investor's cash contribution. Since short sales will still be ruled out, the set of feasible choices in Example III are those portfolios $\bar{v}_j$ which satisfy

$$v_{ij} \geq 0 \qquad i=2,3, \text{ all } j, \text{ and} \tag{29}$$

$$v_{2j} + v_{3j} \leq 8 \qquad \text{all } j. \tag{37}$$

The set of investment results $R_j(\bar{v}_j)$ achievable with these portfolios in the mean-standard deviation space is that bounded by ABCDEFGHIJA in Figure III.

FIGURE III



√Var
3.50
3.00
2.50
2.00
1.50
1.00
0.50
A
1.50
2.00
2.50
3.00
3.50
E
C
F
D
E
L
G
N
M
H
I
K
B
J

Note that the curve ABCDEFG in Figure I is the same as curve BJK in Figure III. The efficient frontier is given by the curve AJIHGF.

As in Example II, some of the feasible investment results in Figure III are dominated by other feasible investment results. Specifically, the segment ABCNMHIJA (except point C) is dominated by point C, including the portion AJIH of the efficient curve as well as the whole efficient frontier AJK of Example II.

Let us now turn to the question of solvency. It is easily verified that none of the investment results in the segment IMLEFGHI satisfies the solvency condition (36). For example, point F is obtained by the portfolio (8, 0) which means that

$$x_{j+1} = \begin{cases} 4.65x_j & \text{with probability } 0.9 \\ -7.35x_j & \text{with probability } 0.1, \end{cases}$$

a clear violation of (36). Furthermore, *some* of the portfolios in segment KNELMIK violate (36), though not all.[5] Consequently, the set of stochastically undominated portfolios which are consistent with solvency is represented by the space MNCDELM. Of these, however, only the portfolios in region NCDEN guarantee solvency.

We have now reached the startling conclusion that *every* capital distribution $R_j(\bar{v}_j)$ on the efficient mean-variance frontier is either stochastically dominated by inefficient distributions or leads to insolvency with positive probability. Only a small portion of the mean-variance feasible portfolios (those which satisfy (29) and (37)) are also solvency feasible and stochastically undominated by other feasible (and solvent) portfolios.

To obtain the growth-optimal portfolio, we again maximize (26) with respect to $\bar{v}_j$, this time subject to (29) and (37). The unique solution is ($\approx$0.79, $\approx$7.21), which gives the capital distribution represented by point D in Figure III. Since this point lies in the space NCDEN, it is both undominated and guarantees solvency.

IX. The "No-Easy-Money Condition"

In view of the findings in Sections VI and VIII, we shall now analyze the role of stochastic dominance and solvency in the portfolio selection model more

[5] While point C, for example, is obtainable only with portfolio (0, 8), point M, for example, is achievable with an infinite number of portfolios.

closely. Our concern at this point will be to ascertain whether dominating portfolios can be expected to exist, and, if so, what kinds of such portfolios are likely to be found.

Consider points A and B (Figure II) in Example II. As noted, point B (resulting from investment of all assets in opportunity 3 clearly dominates point A (resulting from investment of all assets in lending). Is this possible, or to be expected, when the capital asset market is in equilibrium? In Example II, which corresponds to the case of 100 per cent margin requirements, the answer is yes. In Example III, with limited borrowing, the answer is also yes, but less clearly so. This is because in Example II there can be no upward pressure on the interest rate due to investor demand for funds to invest in opportunity 3, a pressure which would raise the interest rate sufficiently to prevent the portfolio represented by point B from dominating that represented by point A. In Example III, a limited demand for borrowing funds exists but this demand is not necessarily sufficient, due to the borrowing constraint, to bring the interest rate high enough to prevent point A from being dominated by point B in Figure III.

In the absence of a borrowing constraint in Example III, one could guarantee an infinite profit by borrowing an infinite amount and investing it in opportunity 3. However, it is implausible that this demand for funds could be met without a rise in the interest rate. However, as long as the interest rate remained at 15 per cent or less, demand would still be infinite. Since lendable funds, while perhaps large, are usually finite, a rise in the interest rate to above 15 per cent could be expected. This would also suddenly make demand for both borrowing and opportunity 3 finite if solvency were of any concern, since the return on the investment would no longer cover the interest rate with probability 1. Nondominance of the interest rate by the return on any risky portfolio would appear to be a necessary condition which the prices of capital assets must satisfy in equilibrium. To paraphrase, when the prices of capital assets are in equilibrium, we would expect no opportunity for "easy money."

Extending the preceding notion to the case of several assets as well as short sales, the "no-easy-money condition" [16] may be written: the distribution functions F_j are such that

$$\Pr\left\{\sum_{i=2}^{M_j} (\beta_{ij} - r_j)\theta_i < 0\right\} > 0 \tag{38}$$

for all j and all finite θ_i such that $\theta_i \geq 0$ for all $i \notin S_j$ and $\theta_i \neq 0$ for at least one i. In essence, this condition states: (1) that no combination of risky investments exists which provides, with probability 1, a return at least as high as the (borrowing) rate of interest; (2) that no combination of short sales exists for which the probability is zero that a loss will exceed the (lending) rate of interest; and (3) that no combination of risky investments made from the proceeds of any combination of short sales can guarantee against loss.

Note that (38) does not rule out the possibility of stochastic dominance among *risky* assets. All it requires is that for some return on a given investment there is a positive probability that another investment will have a smaller return, and that for the same return or some other return, the second investment will have a greater return with positive probability. Expression (38) would appear to be the weakest condition that must hold in equilibrium.

It would also seem reasonable to assume that

$$\text{(39)} \qquad \Pr\{0 \leq \beta_{ij} < \infty\} = 1 \qquad \text{all } i \text{ and } j,$$

i.e., that one can, at most, lose one's investment in a long position (limited liability) and that a finite investment will always bring a finite return over a finite time period.

On the basis of the preceding, we obtain the following significant consequence:

<u>Theorem</u>: Let r_j be greater than 1 and finite and let F_j be such that (38) and (39) hold. Moreover, let the function $u(y)$ be monotone increasing and strictly concave for $y \geq 0$. Then, the function $h_j(\bar{v}_j)$ given by

$$\text{(40)} \qquad h_j(\bar{v}_j) \equiv E[u(R_j(\bar{v}_j))]$$

subject to

$$\text{(29)} \qquad v_{ij} \geq 0 \qquad i \notin S_j$$

and

$$\text{(6)} \qquad \Pr\{R_j(\bar{v}_j) \geq 0\} = 1$$

has a maximum, and the maximizing $\bar{v}_j$ is finite and unique for each j.

Corollary: When $u(y)$ has no lower bound for $y \geq 0$, the solvency constraint (6) is not binding.

The reader is referred to [16] for the proofs. Note that when $x_j > 0$, (36) holds if and only if (6) holds.

Since $\log y$ is monotone increasing and strictly concave, we see that an growth-optimal portfolio always exists which is both unique and finite--irrespective of any borrowing constraints--when the "no-easy-money condition" holds and solvency is required. Thus, since (6) is not binding (by the corollary), which implies (10) and (11), there is a single portfolio in each period which in the very long run, in the language of Section III, is infinitely better than any other.

X. Example IV: No Easy Money

We shall now modify Example III to satisfy the "no-easy-money condition." Thus, we assume

(1) $M_j = 3$,

(2) $S_j = \{1\}$,

(3) $r_j = 1.05$,

(4) $\beta_{2j} = \begin{cases} 0 & \text{with probability } 0.1 \\ 1.50 & \text{with probability } 0.9 \end{cases}$

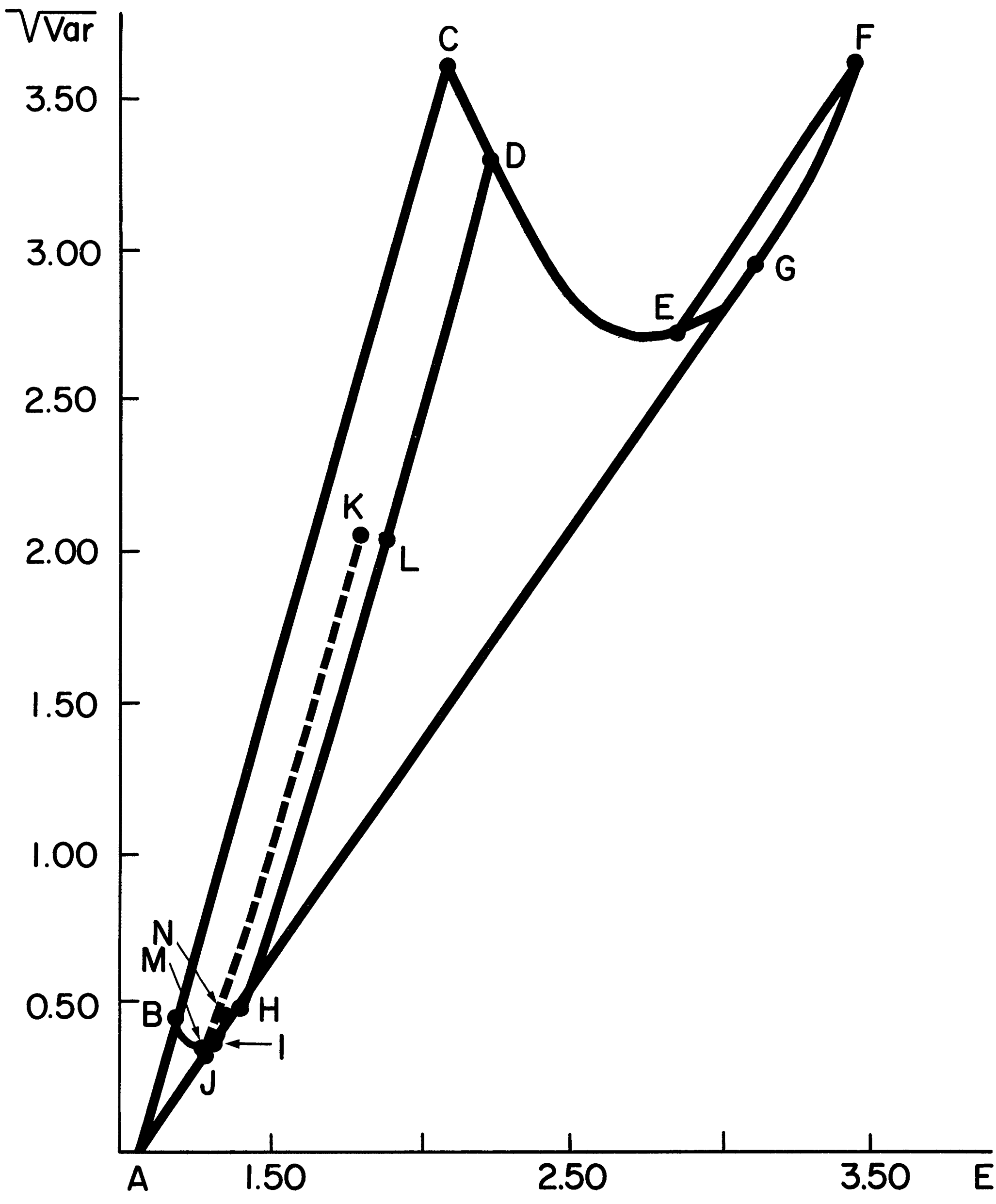
(5) $\beta_{3j} = \begin{cases} 1.03 & \text{with probability } 0.9 \\ 2.53 & \text{with probability } 0.1 \end{cases}$

$$\begin{cases} \Pr\{\beta_{2j} = 0,\ \beta_{3j} = 1.03\} = 0.1 \\ \Pr\{\beta_{2j} = 0,\ \beta_{3j} = 2.53\} = 0 \\ \Pr\{\beta_{2j} = 1.50,\ \beta_{3j} = 1.03\} = 0.8 \\ \Pr\{\beta_{2j} = 1.50,\ \beta_{3j} = 2.53\} = 0.1 \ . \end{cases}$$

The set of feasible investment results in the mean-standard deviation space is now the space bounded by ABCDEFGHIJA in Figure IV. The efficient frontier is given by the boundary AJIHGF. Since F_j satisfies (38), point A is not stochastically dominated by any other point. In this example, all feasible investment results $R_j(\bar{v}_j)$, in fact, are nondominated.

The portfolios which lead to insolvency with a positive probability in Example IV are those bounded by HLDEFGH in Figure IV. In addition, some of the investment results in a small area near curve NH may lead to insolvency. Thus, with these exceptions, solvency is guaranteed only by those portfolios which give rise to investment results in region ABCDLHIJA. Most of the portfolios which

FIGURE IV



are mean-variance efficient are seen to be risk insolvent.

In this case, the portfolio (≈ 0.582, ≈ 4.436) is the growth-optimal one. Represented by point K in the mean-standard deviation space, it is clearly much closer to the *most inefficient* frontier than to the efficient frontier. Introduction of the "no-easy-money condition," therefore, does not eliminate the difficulties encountered in previous sections with respect to the mean-variance approach.

XI. The Intermediate Run

The properties of the growth-optimal portfolio were isolated by means of an analysis of the very long run; by use, in fact, of infinity as a time limit. While the long run is of interest to most individuals and firms, the very long run is probably not. An important question, then, is whether the growth-optimal portfolio has any desirable properties in the short and/or intermediate run. We shall concentrate on the intermediate run in this section and save short-run considerations for the next section. However, before this topic is tackled, it should be noted that Breiman [7] has shown that if the objective is to achieve a certain level of capital as soon as possible, then the optimal-growth portfolio is asymptotically optimal in that it minimizes the expected time to reach the given level. The mean-variance model has been examined in a sequential context by Tobin [32].

If a growth-optimal investment policy is superior in the very long run, one might expect that it should also do quite well in the intermediate run compared to other policies. To examine this question, we shall compare the investment results obtained after six periods on the basis of a growth-optimal policy with those obtained over the same span by use of several mean-variance efficient portfolios. Thus, we shall compare the probability distributions of x_7, where, by (5),

$$x_7 = x_1 \prod_{j=1}^{6} R_j(\bar{v}_j).$$

Initial capital x_1 will be assumed to be \$1,000 and the investment opportunities are assumed to be the same as in Example IV in each period.[6] The probability

[6]These opportunities, which are given at the beginning of Section X, may be given a real-world interpretation. The 5 per cent interest rate and the slight positive correlation between the risky opportunities 2 and 3 seem reasonable enough. Opportunity 2, which has a 0.9 probability of a 50 per cent gain and a 0.1 probability of a 100 per cent loss, may be thought of as the stock of a

that x_7 exceeds or equals x under policy $\bar{v}$ will be denoted $B(x; \bar{v})$, i.e.,

$$B(x; \bar{v}) \equiv \Pr\{x_7 \geq x | \bar{v}\} = 1 - \Pr\{x_7 < x | \bar{v}\}.$$

Since the investment returns β_{ij} are discrete for all i, $B(x; \bar{v})$ will be a decreasing step function.

Tables 1 and 2 give the step functions $B(x; \bar{v})$ for the portfolios $\bar{v}$ which lead to the (one-period) investment results A, J, H, L, and D on the solvency feasible efficient frontier in Figure IV as well as for the growth-optimal portfolio (point K). Point A, of course, is obtained through the riskless portfolio (0,0). Point J is the investment result of portfolio (0.678, 0.230) which is the portfolio with the highest expected growth rate among the portfolios on the solvency feasible efficient frontier AJIHLD. Investment result H is the point on the "original" efficient frontier AJIHGF with the highest expectation consistent with solvency; it is obtained via portfolio (0.994, 0.336). L (portfolio (0.918, 4.305)) is the point on the solvency feasible efficient frontier AJIHLD which is "closest" to the growth-optimal investment result K. Finally, point D denotes the outcome of the portfolio with the highest expected return consistent with solvency (portfolio (0.864, 7.136)).

From Tables 1 and 2, we see that the probability of being broke after six periods is about 0.47 on the basis of the portfolio policies associated with points H, L, and D. The reason for this, of course, is that these policies risk losing all assets with probability 0.1 in each period. Since the probability of having positive assets after n periods, therefore, is $(0.9)^n$, we see that these policies lead to ultimate ruin for sure in the very long run. The tables

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promising but nondiversified new company completely dependent on a large government contract. Opportunity 3 may be viewed as the convertible bond of a solid company paying 2.4 per cent and selling at \$80, which presently could be exchanged for \$60 in stock, i.e., the bond sells for a \$20 premium over its conversion value. The probability that the common stock, which sells for \$24, say, will remain pretty much where it is, leaving the bond price unchanged and therefore producing a 3 per cent total return (in the form of interest) on the bond investment, is 0.9. However, there is also a 0.1 probability that the stock will move to \$80, producing a 150 per cent capital gain on the bond in addition to the interest, for a total return of 153 per cent.

Since the growth-optimal portfolio is approximately (0.582, 4.436), an investor with \$10,000 on hand seeking such a portfolio would commit \$5,820 to the (opportunity 2) stock and \$44,360 to the convertible bond, borrowing \$40,180 to complete the transaction. He may then be viewed as acquiring the stock for cash and the convertible bond with 90 per cent financing.

TABLE 1

CAPITAL DISTRIBUTIONS AFTER SIX PERIODS--
PORTFOLIOS K, A, H, L, AND D

	$\Pr\{x_7 \geq x \mid \overline{v}\}$				
	$\overline{v}$				
	(0.582, 4.436)	(0, 0)	(0.994, 0.336)	(0.918, 4.305)	(0.864, 7.136)
x	K	A	H	L	D
0.00	1.000000	1.000000	1.000000	1.000000	1.000000
0.01	1.000000	1.000000	0.531441	0.531441	0.531441
1.84	1.000000	1.000000	0.531441	0.531441	0.531441
6.44	0.999999	1.000000	0.531441	0.531441	0.531441
22.50	0.999951	1.000000	0.531441	0.531441	0.531441
41.48	0.998991	1.000000	0.531441	0.531441	0.531441
78.59	0.998985	1.000000	0.531441	0.531441	0.531441
144.89	0.988745	1.000000	0.531441	0.531441	0.531441
274.50	0.988505	1.000000	0.531441	0.531441	0.531441
506.09	0.927065	1.000000	0.531441	0.531441	0.531441
933.05	0.923225	1.000000	0.531441	0.531441	0.531441
958.83	0.923210	1.000000	0.531441	0.531441	0.531441
1,340.10	0.726602	1.000000	0.531441	0.531441	0.531441
1,767.77	0.726602	0	0.531441	0.531441	0.531441
3,259.16	0.695882	0	0.531441	0.531441	0.531441
3,349.21	0.695402	0	0.531441	0.531441	0.531441
4,740.14	0.433258	0	0.531441	0.531441	0.531441
6,174.81	0.433258	0	0.531441	0.531441	0.269297
6,817.16	0.310378	0	0.531441	0.531441	0.269297
10,968.11	0.310378	0	0.531441	0.269297	0.269297
11,384.27	0.310378	0	0.269297	0.269297	0.269297
14,676.68	0.304618	0	0.269297	0.269297	0.269297
19,639.21	0.304618	0	0.072689	0.269297	0.269297
20,988.74	0.304618	0	0.011299	0.269297	0.269297
21,568.63	0.304598	0	0.011299	0.269297	0.269297
26,279.69	0.107990	0	0.011299	0.269297	0.269297
35,165.47	0.107990	0	0.001009	0.269297	0.269297
38,786.54	0.107990	0	0.000049	0.269297	0.269297
39,765.28	0.107990	0	0.000049	0.072689	0.269297
43,887.73	0.077270	0	0.000049	0.072689	0.269297
47,055.73	0.077270	0	0.000049	0.072689	0.072689
62,966.38	0.077270	0	0.000001	0.072689	0.072689
73,313.75	0.077270	0	0	0.072689	0.072689
138,900.00	0.076790	0	0	0.072689	0.072689
220,678.00	0.015350	0	0	0.072689	0.072689
256,085.00	0.015350	0	0	0.011249	0.072689
405,345.00	0.011510	0	0	0.011249	0.072689
472,135.00	0.011510	0	0	0.011249	0.011249
894,506.00	0.011495	0	0	0.011249	0.011249

TABLE 1 (CONT.)

x	$\Pr\{x_7 \geq x \mid \bar{v}\}$				
	$\bar{v}$				
	(0.582, 4.436)	(0, 0)	(0.994, 0.336)	(0.918, 4.305)	(0.864, 7.136)
	K	A	H	L	D
1,255,555.00	0.001255	0	0	0.011249	0.011249
1,649,168.00	0.001255	0	0	0.001009	0.011249
3,762,251.00	0.001015	0	0	0.001009	0.011249
5,760,548.00	0.001015	0	0	0.001009	0.001009
7,143,532.00	0.000055	0	0	0.001009	0.001009
10,620,506.00	0.000055	0	0	0.000049	0.001009
34,833,741.00	0.000049	0	0	0.000049	0.001009
37,097,461.00	0.000049	0	0	0.000049	0.000049
40,643,431.00	0.000001	0	0	0.000049	0.000049
231,242,528.00	0.000001	0	0	0.000001	0.000049
238,904,640.00	0.000001	0	0	0	0.000049
322,516,868.00	0	0	0	0	0.000049
2,986,102,816.00	0	0	0	0	0.000001
2,986,102,817.00	0	0	0	0	0

TABLE 2

CAPITAL DISTRIBUTIONS AFTER SIX PERIODS
PORTFOLIOS K AND J

	$Pr\{x_7 \geq x\|\bar{v}\}$			$Pr\{x_7 \geq x\|\bar{v}\}$	
	$\bar{v}$			$\bar{v}$	
	(0.582, 4.436)	(0.678 0.230)		(0.582, 4.436)	(0.678, 0.230)
x	K	J	x	K	J
1.37	1.000000	1.000000	2,361.43	0.695882	0.566247
1.84	1.000000	0.999999	2,964.68	0.695882	0.535527
5.57	0.999999	0.999999	3,259.16	0.695882	0.531687
6.44	0.999999	0.999951	3,349.21	0.695402	0.531687
6.99	0.999951	0.999951	3,722.04	0.433258	0.531687
22.50	0.999951	0.999945	4,672.88	0.433258	0.531447
22.56	0.998991	0.999945	6,066.91	0.433258	0.531441
28.33	0.998991	0.998985	6,174.81	0.433258	0.269297
35.56	0.998991	0.998745	7,616.77	0.310378	0.269297
41.48	0.998991	0.998730	9,562.55	0.310378	0.072689
78.59	0.998985	0.998730	11,384.27	0.310378	0.011249
91.36	0.988745	0.998730	12,005.41	0.304618	0.011249
114.70	0.988745	0.988490	15,072.33	0.304618	0.001009
144.01	0.988745	0.984650	18,922.72	0.304618	0.000049
144.89	0.988745	0.984170	20,988.74	0.304618	0.000001
180.79	0.988505	0.984170	21,568.63	0.304598	0.000001
274.50	0.988505	0.984150	23,756.73	0.107990	0.000001
369.97	0.927065	0.984150	39,765.28	0.107990	0
464.49	0.927065	0.922710	73,313.75	0.077270	0
506.09	0.927065	0.891990	138,900.00	0.076790	0
583.14	0.923225	0.891990	256,085.00	0.015350	0
732.12	0.923225	0.886230	472,135.00	0.011510	0
919.14	0.923225	0.885750	894,506.00	0.011495	0
933.05	0.923225	0.885735	1,649,168.00	0.001255	0
958.83	0.923210	0.885735	5,760,548.00	0.001015	0
1,498.20	0.726602	0.885735	10,620,506.00	0.000055	0
1,767.77	0.726602	0.689127	37,097,460.00	0.000049	0
1,880.93	0.695882	0.689127	238,904,640.00	0.000001	0
			238,904,641.00	0	0

also show that the portfolio policies associated with points A, J, and K guarantee positive assets with probability one after six periods (in fact, infinitely long).

Tables 1 and 2 also reveal that portfolio J has a probability of approximately 0.89 of doing at least as well as the certain return portfolio A, which is up to \$1,340.10 after six periods. The corresponding probability for the growth-optimal portfolio is about 0.73. However, for this portfolio, the probability of losing more than \$41.17 is only 0.08 against 0.11 for portfolio J. Table 2 also reveals that the growth-optimal portfolio has a probability of about 0.30 of exceeding \$21,500 while portfolio J has only a 1 per cent chance of reaching \$12,000. In addition, portfolio J has no chance of producing as much as \$24,000 whereas the probability that the growth-optimal portfolio will do at least as well is about 0.11. In fact, there is a 0.08 chance that it will top \$138,000 and one chance in a million that it will return more than \$238 million. The highest possible return is exhibited by portfolio D, which has one chance in a million to reach nearly \$3 billion.

A graphic comparison of the step function $B(x; \bar{v}^*)$ (that of the growth-optimal portfolio) with those of portfolios A, J, H, L, and D is given in Figures V-VIII. Two things stand out. First, it is remarkable how close $B(x; \bar{v}^*)$ comes to dominating each of the other step functions after only six periods. Second, the growth-optimal portfolio offers as many opportunities for large gains as the portfolios (H, L, and D) which risk everything in pursuit of such gains--while offering very limited exposure to loss.

XII. Other Considerations

For completeness, we shall now briefly review some of the general properties of the mean-variance model and the capital growth model.

Tobin [30] was the first to note that the mean-variance model is consistent with the von Neumann-Morgenstern postulates of rational behavior [33] if the utility of money is quadratic, i.e., if the cardinal utility u of money x is given by

$$u(x) = ax - x^2 \qquad a > 0. \tag{41}$$

Later, Borch [5] noted that this is the only form $u(x)$ can have if it is to be

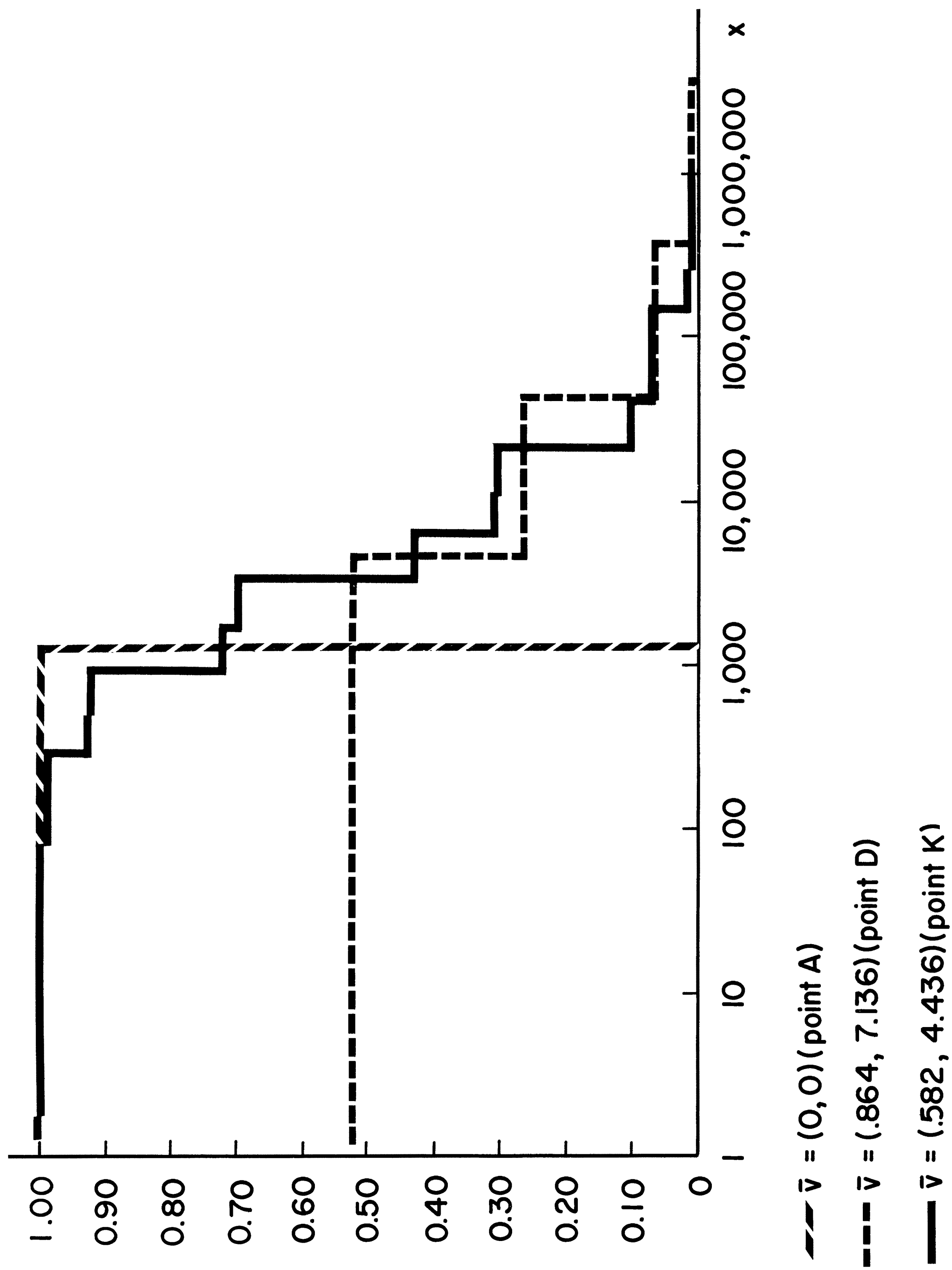
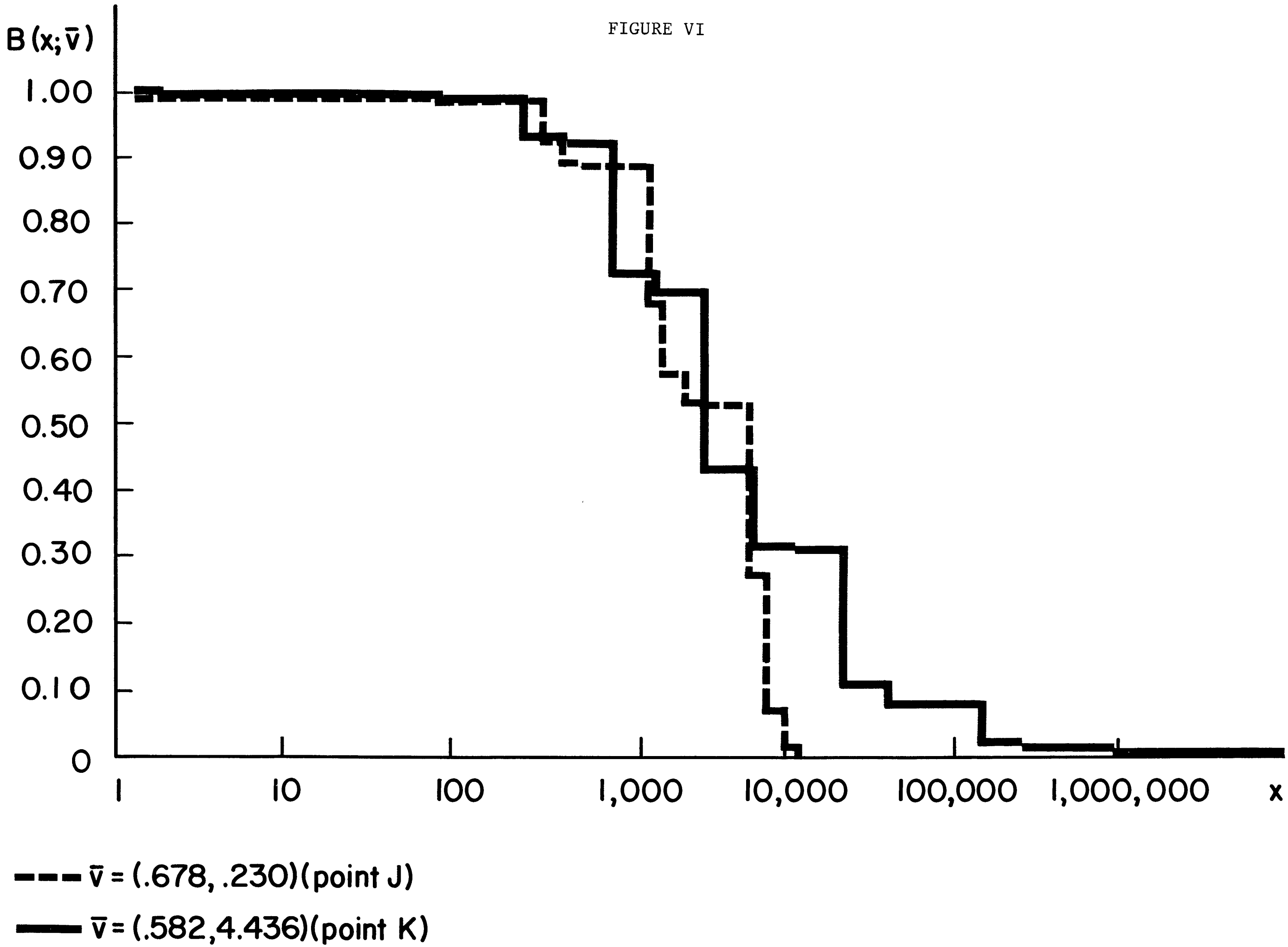
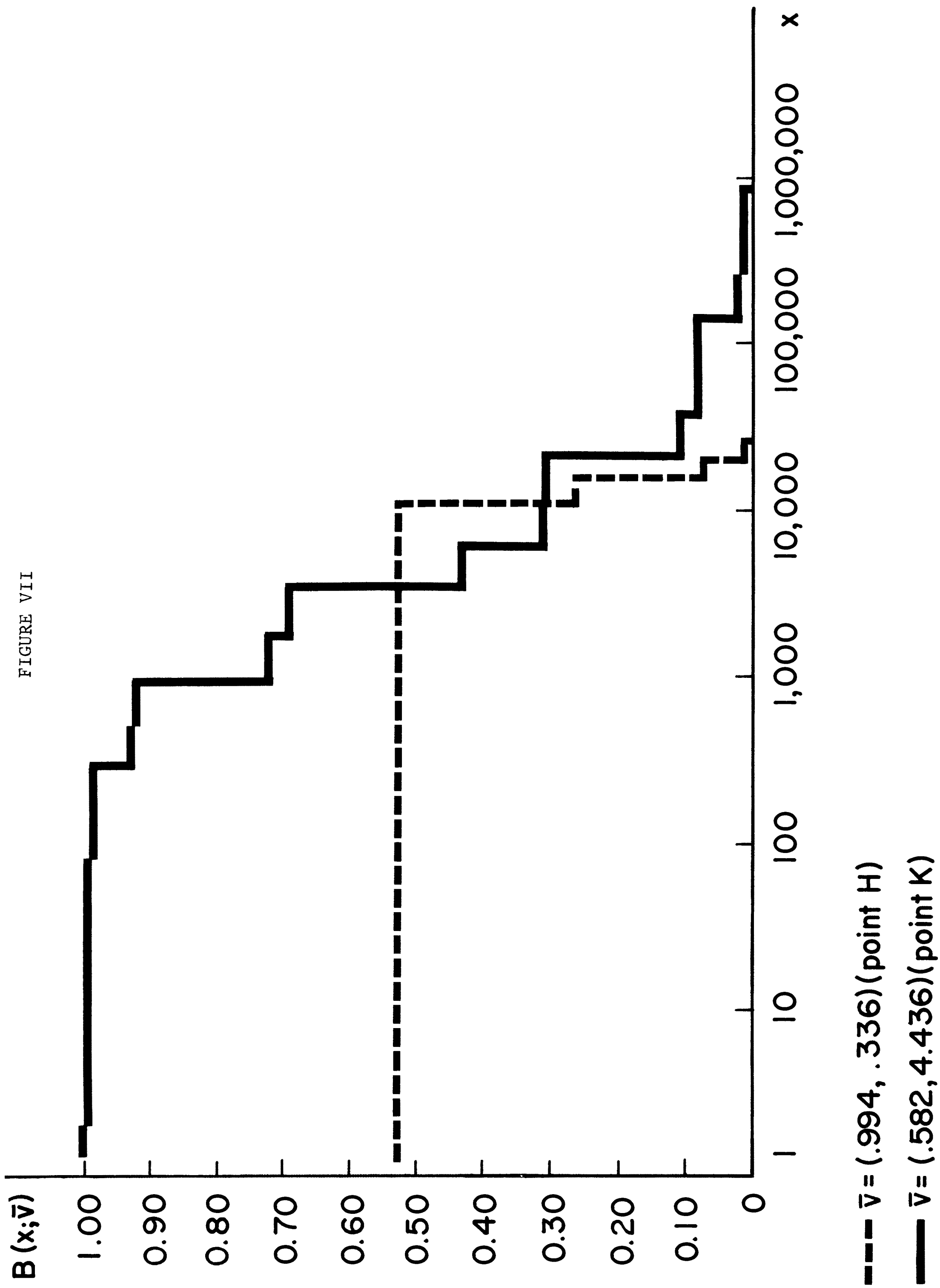

1.00
0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0
1
10
100
1,000
10,000
100,000
1,000,000
x
v̄ = (0,0) (point A)
v̄ = (.864, 7.136) (point D)
v̄ = (.582, 4.436) (point K)

FIGURE VI



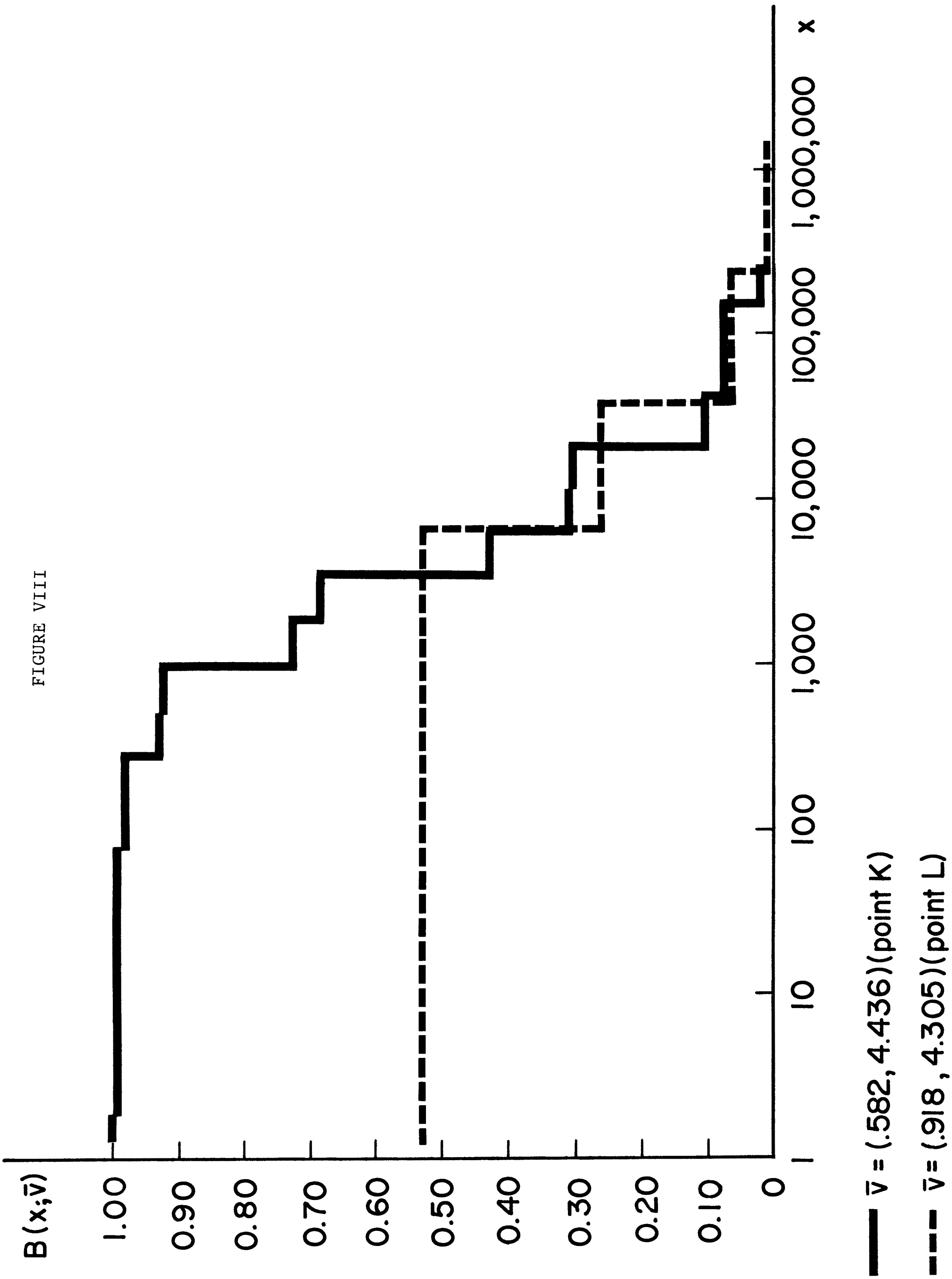
B (x; v̄)
1.00
0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0
1
10
100
1,000
10,000
100,000
1,000,000
x
v̄ = (.678, .230)(point J)
v̄ = (.582, 4.436)(point K)

FIGURE VII



B (x;$\bar{v}$)
1.00
0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0
1
10
100
1,000
10,000
100,000
1,000,000
x
$\bar{v}$ = (.994, .336)(point H)
$\bar{v}$ = (.582, 4.436)(point K)

FIGURE VIII



B(x;v̄)
1.00
0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0
1
10
100
1,000
10,000
100,000
1,000,000
x
v̄ = (.582, 4.436)(point K)
v̄ = (.918, 4.305)(point L)

consistent with (27) and (28) or express risk aversion $(u''(x) < 0)$.[7] Unfortunately, (41) is increasing only for $x \leq a/2$, as noted by several authors, which means that (41) can only be used to evaluate portfolios for which

$$\Pr\left\{x_{j+1} \leq \frac{a}{2}\right\} = 1$$

without violating the common-sense premise that more is preferred to less. While the von Neumann-Morgenstern postulates are not sacrosanct, this is a significant drawback.

In view of (26), the growth-optimal model is clearly consistent with the expected utility theorem since it induces, to be precise, a short-run cardinal utility function of money which is logarithmic.[8] This function has a longer history than any other in the theory of capital risk, being introduced over 200 years ago by Daniel Bernoulli, who argued for it on purely intuitive grounds [4]. A more recent proponent of this utility function is Savage, who wrote that "to this day, no other function has been suggested as a better prototype of Everyman's utility function" [29, p. 94]. In contrast to the quadratic utility (41), $u(x) = \log x$ is increasing throughout for non-negative capital values.

Two intuitively meaningful measures of the risk disposition reflected in a (cardinal) utility function were proposed by Arrow [2] and Pratt [27]. The first is

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[7] In mean-variance (E-V) terms, the objective function now becomes $aE - E^2 - V$; thus, the commonly used linear form $aE - V$ is inconsistent with the von Neumann-Morgenstern postulates [5].

When the distribution functions F_j are normal, $u(x)$ may have any form (as long as it is defined for negative wealth levels) [31]. But the normal distribution does not have empirical validity [10], nor does it satisfy the range requirement (39).

[8] Strictly speaking, the logarithmic function, being unbounded, is not completely consistent with a von Neumann-Morgenstern utility function, which is bounded [1]. For the same reason, while consistent with the notion of stochastic dominance for all distributions $R_j(v_j)$ such that $E[\log R_j(\overline{v}_j)]$ exists, the logarithmic utility is inconsistent with the notion of stochastic dominance whenever $\Pr\{R_j(\overline{v}_j) = 0\} > 0$. However, since $\Pr\{x_{j+1} = 0\} \to 1$ as $j \to \infty$ when $\Pr\{R_j(\overline{v}_j) = 0\} > 0$ for all j, as shown in Section II, it could also be argued that a sensible utility function should at least be unbounded from below or have infinite marginal utility at zero wealth.

$$- \frac{u''(x)}{u'(x)}$$

and is known as the absolute risk aversion index at wealth level x. The second measure is the relative risk aversion index which is given by

$$- \frac{xu''(x)}{u'(x)} .$$

It has been demonstrated convincingly by Arrow [1] and others that any reasonable utility function of money must exhibit decreasing absolute risk aversion. Since the quadratic function (41) displays *increasing* absolute risk aversion, it must be considered empirically implausible, as many writers have indeed noted. On the other hand, the absolute risk aversion index of the logarithmic utility is $1/x$, i.e., decreasing; thus, this function clearly satisfies the plausibility criterion.

What the relative risk aversion index would look like for a meaningful utility function is less clear, although Arrow has hypothesized that it should be (slightly) increasing [1]. The quadratic utility has this property but this is of little comfort in view of its absolute index. It is readily seen that the relative index of the logarithmic utility is constant at 1. In view of Arrow's conclusion that "...broadly speaking, the relative risk aversion must hover around 1, being, if anything, somewhat less for low wealths and somewhat higher for high wealths" [1, p. 37], the optimal-growth model seems to be on safe ground.

Tobin [30] appears to be the first to have discovered the so-called separation property of the mean-variance model, i.e., that the optimal mix of risky assets is independent of wealth. Pye [28] noted that this property also holds for the logarithmic function; actually, the optimal mix of *all* assets is independent of wealth in the latter case. However, in the presence of the solvency constraint (6), the separation property does not hold for low levels of wealth in the mean-variance (quadratic utility) model, as shown by Hakansson [18].

A single-period decision rule is said to be myopic when it is independent of the future (beyond the current period) in a sequential decision problem. It is implicit in both the Latané [19] and Breiman [6], [7] papers that the growth-optimal policy has this property. However, it remained for Mossin [25] to demonstrate this explicitly and to show that only when the utility of distant wealth is logarithmic or is a power function is the single-period decision rule in fact myopic. In view of the difficulty of estimating the future returns, this is a most important property which, again, the mean-variance model lacks.

Latané [19] also noted that in the optimal-growth model, the payment of a dividend which is a proportion (not necessarily stationary) of capital at each decision point does not change the optimal policy $\bar{v}_j^*$. However, the expected growth rate is clearly affected. The preceding is not true in the mean-variance model, although the optimal mix of risky assets will remain unchanged unless the solvency constraint is binding or becomes binding as a result of the dividend. The growth-optimal policy, applied to the nonconsumed portion of capital plus the present value of the (certain) noncapital income stream, is also optimal for individuals who maximize expected utility of consumption sequentially when the utility function of consumption is additive and logaritimic [16], [17].

In recent years, a great deal of effort has been spent on finding efficient algorithms for solving the quadratic programming problem (and thus the mean-variance portfolio problem). In fact, Markowitz became an early contributor to this literature through his interest in the portfolio problem [23]. So far, these efforts have been crowned with considerable success. The computational aspects of the capital growth model are much less advanced.

XIII. Growth Funds

By the separation property, it is clear that all growth-oriented investors who have the same probability assessments will choose the same portfolio $\bar{v}_j^*$ (but not $\bar{z}_j^*$) regardless of wealth. Thus, the formation of a no-load, open-end mutual fund by these individuals seems eminently sensible. In the absence of borrowing constraints, the fund's objective at each decision point j should clearly be to maximize the expected logarithm of end-of-period capital subject to $v_{ij} \geq 0$, $i \notin S_j$, i.e., to maximize *its* expected growth rate--which is also achieved by policy $\bar{v}_j^*$. In Example IV, this policy is approximately (0.582, 4.436), i.e., the fund should put ≈11.6 per cent of its own and the borrowed funds in opportunity 2 and ≈88.4 per cent in opportunity 3.

Now suppose that the margin requirement for both funds and individuals is 100 per cent. It is easily shown that the growth-optimal portfolio is now approximately (0.58, 0.42) (58 per cent in opportunity 2, 42 per cent in opportunity 3), leading to investment result M in Figure IV. Similarly, when the margin requirement is 50 per cent, the optimal portfolio is (≈0.60, ≈1.40), i.e., 30 per cent of *total* funds should be committed to opportunity 2 and 70 per cent to opportunity 3. As the borrowing limit changes from 0 per cent to 402 per cent of assets and beyond, the optimal investment result moves from point M to point K in Figure IV.

Now suppose that individual investors face no margin requirement (except the solvency constraint) but that the fund has a self-imposed borrowing limit of, say, 100 per cent of assets (most real-world funds have such limits). In Example IV, the portfolio desired by the shareholders is clearly (≈ 0.582, ≈ 4.436). However, as indicated in the preceding paragraph, the optimal-growth portfolio from the point of view of the *fund* is (≈ 0.60, ≈ 1.40). Thus, when the fund's borrowing limit is smaller than that of the shareholders, it cannot maximize *both* its own expected growth rate and that of the shareholders' capital at the same time. In the preceding example, it should choose the portfolio (0.232, 1.768) (11.6 per cent of total capital invested in opportunity 2, 88.4 per cent in opportunity 3). This way, by borrowing 150.9 per cent of his own capital privately and investing it along with this capital in the mutual fund, the shareholder's portfolio becomes $2.509 \cdot (0.232, 1.768) = (0.582, 4.436)$, which is the desired one.

XIV. Concluding Remarks

The mean-variance approach to portfolio selection is intrinsically based on a single-period model, while the capital growth model represents by definition a sequential portfolio approach. A comparison of the two models partly in terms of long-run results may therefore seem unfair to the mean-variance approach. However, both models are ultimately *portfolio* models which claim to offer guidance to sensible portfolio choices at any given decision point -- they must therefore stand or fall on the merits of this guidance. It should also be noted that the sequential mean-variance policy used for comparison in Section XI is consistent with a multiperiod adaption of the graphic mean-variance model since preferences in this case are usually stated in terms of means and variances of *returns* (R_j-1), making them implicitly independent of wealth. To derive an explicit sequential policy based on wealth preferences, it first becomes necessary to use the quadratic function (41), which is partially myopic [25]. But this only makes matters worse since (if and) when wealth becomes sufficiently large, the investor will deliberately attempt to *lose* money by following the optimal policy due to the negative marginal utility of wealth above a certain level.

In the examples, we considered a three-asset case in which the returns of the two risky assets were highly skewed. It is doubtful that the results would be as dramatic when a larger number of risky assets are available or the return distributions are symmetric. Real-world returns, however, are not likely to be symmetric; studies of listed securities indicate that return distributions are substantially skewed [11]. The effect of skewness per se on the optimal holding of a single risky asset in the logarithmic case has been studied by Freimer and Gordon [13]; their findings are consistent with the present results. At any rate, the present study would seem to offer a significant counter-example to the proposition that the best portfolio should be chosen from the efficient set.

Some of the "formal" deficiencies of the mean-variance approach mentioned in Section XII have been known for some time while others are of more recent origin. However, these deficiencies have apparently done very little to slow enthusiasm for the mean-variance model, especially among those who rely on graphic analysis in the mean-standard deviation space. One purpose of this paper has been to examine further, by graphic means, the soundness of the mean-variance model. As we have seen, this analysis has exposed further serious deficiencies in the mean-variance model itself as well as the standard portfolio problem formulation. These drawbacks are perhaps best illustrated by the two portfolios leading to investment results K and L, which are very close together, in Figure IV. Quite close in composition also (0.582, 4.436) vs (0.918, 4.305), the two portfolios have means and standard deviations of 1.80 and 2.04, and 1.89 and 2.03, respectively. Thus, from a mean-variance point of view, L *dominates* K. But the portfolio which gives L leads to ultimate ruin for sure while the portfolio which produces K will, in the very long run, have grown at an average rate of 30.03 per cent, compounded each period.

In view of these findings it is not surprising that empirical studies concerned (in part) with the question of where the investment results of professionally managed portfolios are in relation to the efficient frontier have reached different conclusions (quite aside from the hypotheses employed). Farrar [12] found mutual fund portfolios to give results quite close to the efficient frontier. While Friend and Vickers [14] did not concern themselves with the efficient frontier explicitly, actual portfolios clearly show no tendency to be close to it. Cohen and Pogue [9] reach the Farrar conclusion for mutual funds with low actual returns (less than 15 per cent) and the conclusion imputed to Friend and Vickers for funds with high actual returns. On the basis of the current paper, of course, one would not expect an astute fund manager to see any significant relationship between "good" portfolios and efficient portfolios.[9]

[9] It is perhaps worth noting that despite several man-years spent by a major computer manufacturer on the development of a computer program for portfolio selection (based on the mean-variance model), not a single user of this program had been found as of a recent point in time.

The most inexplicable aspect of portfolio theory to date is the absence of a formal recognition of the "survival" or solvency motive of investors and fund managers. As we have seen, this omission is disastrous for the mean-variance model. In the optimal-growth model, ironically, the survival criterion, represented by the solvency constraint (6), is automatically satisfied, and, as noted, never binding when the "no-easy-money condition" holds (see Corollary). Thus, one of the strengths of the logarithmic utility function is that an "air-tight survival motive" is automatically built in.

A second aspect of portfolio theory which seems difficult to explain is why the capital growth model has been virtually ignored. This is especially so in view not only of its extraordinary properties and its strong position vis-a-vis the mean-variance model, but because of the recent emphasis in financial circles on portfolio growth.

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