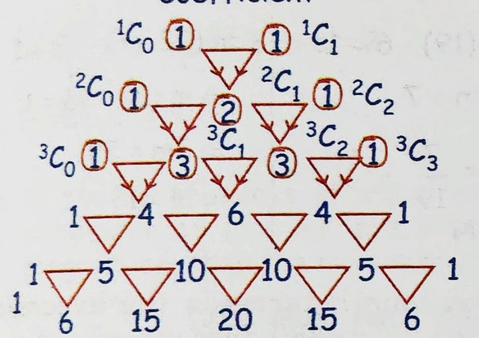
Chapter 8

Binomial Theorem

PASCAL'S TRIANGLE

$[x+y]^1$	$x+y$
$[x+y]^2$	$x^2+2xy+y^2$
$[x+y]^3$	$x^3+3x^2y+3xy^2+y^3$
$[x+y]^4$	$x^4+4x^3y+6x^2y^2+4xy^3+y^4$
$[x+y]^5$	$x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$
$[x+y]^6$	$x^6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6$

Coefficient



BINOMIAL THEOREM (FOR NATURAL INDEX)

$(x+y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + \dots {}^nC_ny^n$

Some Important Expansions

(i) $(x-y)^n = {}^nC_0\,x^n - {}^nC_1\,x^{n-1} + C_2\,x^{n-2}y^2 - \dots + (-1)^n\,{}^nC_ny^n$

(ii) $(1+x)^n = {}^nC_0 + {}^nC_1\,x + {}^nC_2\,x^2 + \dots + {}^nC_nx^n$

(iii) $(1-x)^n = {}^nC_0 - {}^nC_1\,x + {}^nC_2\,x^2 + \dots + (-1)^n.\,{}^nC_n\,x^n$

ex $\to$ 1. $(1+x)^{20} = {}^{20}C_0 + {}^{20}C_1\,x + {}^{20}C_2\,x^2 + {}^{20}C_3x^3 + \dots + {}^{20}C_{19}\,x^{19} + {}^{20}C_{20}\,x^{20}$

ex $\to$ 2. $(1-3x)^{15} = {}^{15}C_0 - {}^{15}C_1\,(3x) + {}^{15}C_2\,(3x)^2 - {}^{15}C_2(3x)^3 + \dots - {}^{15}C_{15}\,(3x)^{15}$

3. $(x+1)^{20} = {}^{20}C_0x^{20} + {}^{20}C_1\,x^{19} + {}^{20}C_2\,x^{18} + {}^{20}C_3x^{17} + \dots + {}^{20}C_{20}$

Important Note

Note 1: Number of terms in the expansion of $(x+y)^n$ is $(n+1)$ i e.

Note 2: Binomial Expansion is a Homogeneous expansion i.e the sum of indices of x and y for each and every terms is n.

Note 3: The binomial coefficients of the terms $({}^nC_0, {}^nC_1 \dots)$ equidistant from the beginning and the end are equal i.e. ${}^nC_r = {}^nC_{r-1}$

Ques. Using Theorem, Find the value of

$$\left(\sqrt{2}+1\right)^6 + \left(\sqrt{2}-1\right)^6$$

Ans.
$$\left(\sqrt{2}+1\right)^6 = {}^6C_0\left(\sqrt{2}\right)^6 + {}^6C_1\left(\sqrt{2}\right)^5 + {}^6C_2\left(\sqrt{2}\right)^4 + {}^6C_3\left(\sqrt{2}\right)^3 + {}^6C_4\left(\sqrt{2}\right)^2 + {}^6C_5\left(\sqrt{2}\right)^1 + {}^6C_6$$

$$\left(\sqrt{2}-1\right)^6 = {}^6C_0\left(\sqrt{2}\right)^6 - {}^6C_1\left(\sqrt{2}\right)^5 + {}^6C_2\left(\sqrt{2}\right)^4 - {}^6C_3\left(\sqrt{2}\right)^3 \dots + {}^6C_6$$

$$\left(\sqrt{2}+1\right)^6 + \left(\sqrt{2}-1\right)^6 = 2[{}^6C_0\left(\sqrt{2}\right)^6 + {}^6C_2\left(\sqrt{2}\right)^4 + {}^6C_4\left(\sqrt{2}\right)^2 + {}^6C_6]$$

$$= 2\,[2^3 + 15.2^2 + 15.(2) + 1]$$

$$= 2\,[8 + 60 + 30 + 1]$$

$$= 99 \times 2 = 198$$

Ques. If the 4th term in the expansion $\left(px + \frac{1}{x}\right)^m$ is $\frac{5}{2}\ \forall x \varepsilon R$ then find the value of m and p.

Ans.
$$\left(px + \frac{1}{x}\right)^m = \underbrace{{}^mC_0\ (px)^m}_{T_1} + \underbrace{{}^mC_1\ (px)^{m-1}\left(\frac{1}{x}\right)}_{T_2} + \underbrace{{}^mC_2\,(px)^{m-2}\left(\frac{1}{x}\right)^2}_{T_3} + \underbrace{{}^mC_3\,(px)^{m-3}\left(\frac{1}{x}\right)^3}_{T_4}$$

$T_4 = {}^mC_3\,(px)^{m-3}\,\frac{1}{x^3} = \frac{5}{2}\ \forall x \in R$

$\Rightarrow {}^mC_3\,p^{m-3}\,\frac{x^{m-3}}{x^3} = \frac{5}{2}$ for all $x \in R$

$\Rightarrow m - 3 = 3$

$\Rightarrow m = 6$

${}^mC_3\,(p)^{m-3} = \frac{5}{2}$

$\Rightarrow {}^6C_3\,(p)^{6-3} = \frac{5}{2}$

$\Rightarrow \frac{6 \times 5 \times 4}{3!}\,(p)^3 = \frac{5}{2}$

$\Rightarrow 5 \times 4\,p^3 = \frac{5}{2}$

$\Rightarrow p^3 = \frac{1}{8} \Rightarrow p = \frac{1}{2}$

GENERAL TERM

The general term or the $(r + 1)^{th}$ in the expansion of $(x + y)^n$ is given by $T_{r+1} = {}^nC_r x^{n-r} y^r$

General terms for the expansion of $(1 + x)^n$ is given by $T_{r+1} = {}^nC_r\,x^r$

Ex $\rightarrow \left(2x + \frac{7}{y}\right)^{10}$ Find 5^{th} terms

$T_{r-1} = {}^{10}C_r\,(2x)^{10-r}\left(\frac{7}{y}\right)^r$

Put $r = 4$

$T_5 = {}^{10}C_4\,(2x)^6\left(\frac{7}{y}\right)^4$

Ques. The coefficient Of x^{-6} in the expansion of $\left(\frac{4x}{5} + \frac{5}{2x^2}\right)^9$, is ....

[JEE Main 31 Jan. 2023 Shift II]

Ans. $T_{r+1} = {}^nC_r\,(x)^{n-r}\,(y)^r$

$= {}^9C_r\left(\frac{4x}{5}\right)^{9-r}\left(\frac{5}{2x^2}\right)^r$

$= {}^9C_r\left(\frac{4}{5}\right)^{9-r}\left(\frac{5}{2}\right)^r\frac{x^{9-r}}{x^{2r}}$

$T_6 = {}^9C_5\left(\frac{4}{5}\right)^4\frac{5^5}{2^5}\,.\,x^{9-3.5}$

put $9 - 3r = -6$

$\Rightarrow 15 = 3r \Rightarrow r = 5$

Coeff $= {}^9C_5\left(\frac{4}{5}\right)^4\left(\frac{5}{2}\right)^5$

$= \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2}\cdot\left(\frac{4}{5}\right)^4\frac{5^5}{2^5}$

$= 9 \times 7 \times 2 \times 5\,.\,\frac{2^8}{2^5} = 9 \times 7 \times 10 \times 2^3$

$= 63 \times 10 \times 8 = 5040$

Ques. The natural number m, for which the cofficent of x in binomial expansion of $\left(x^m + \frac{1}{x^2}\right)^{22}$ is 1540 is ......

[JEE Main Sept. 2020]

Ans. $T_{r+1} = {}^{22}C_r\,(x^m)^{22-r}\left(\frac{1}{x^2}\right)^r$

$= {}^{22}C_r\,\frac{(x)^{m(22-r)}}{x^{2r}}$

$= {}^{22}C_r\,x^{m(22-r)-2r}$

$\because 1540 = {}^{22}C_r$

$\Rightarrow {}^{22}C_3 = {}^{22}C_r$

and

$m(22 - r) - 2r = 1$

$r = 3$

$m(22 - r) - 2r = 1$

$\Rightarrow m(19) - 6 = 1$

$\Rightarrow 19m = 7$

$\Rightarrow m = \frac{7}{19}$ (✗)

$m \in N$

or $r = 19$

$m(22 - r) - 2r = 1$

$\Rightarrow m(3) - 38 = 1$

$\Rightarrow 3m = 39$

$\Rightarrow m = 13$

Ques. If the fourth terms in the expansion of $(x + x\log_2 x)^7$ is 4480, then the value of x where $x \in N$ is equal to: **[JEE Main March 2021]**

✓(a) 2 (b) 4 (c) 3 (d) 1

Ans. $T_{r+1} = {}^7C_r\,(x)^{7-r}\,[x\log_2 x]^r$

$r = 3,\ T_4 = {}^7C_3\,(x)^4\,[x\log_2 x]^3$

$\Rightarrow 35\,(x)^4\,x^{3\log_2 x} = 4480$

$\Rightarrow (x)^4\,x^{3\log_2 x} = 128 = 2^7$

$\Rightarrow x^4\,x^{3\log_2 x} = 2^4\,.\,2^3$

$\Rightarrow x = 2$

TERM INDEPENDENT OF x

Term independent of x means that there is a term present which has no exponent of x.

Hence, to find the terms independent of x, find the value of r for which the combined exponent of x is zero.

Ques. If the term independent of x in the expansion of $\left(\sqrt{a}x^2 + \frac{1}{2x^3}\right)^{10}$ is 105, then a^2 is equal to:

[JEE Main April 2024 Shift II]

(a) 2 (b) 4 (c) 6 (d) 9

Ans. $T_{r+1} = {}^{10}C_r\left(\sqrt{a}x^2\right)^{10-r}\left(\frac{1}{2x^3}\right)^r$

$$= {}^{10}C_r \cdot \frac{\left(\sqrt{a}\right)^{10-r}}{2^r} \cdot \frac{x^{20-2r}}{x^{3r}}$$

$$= \frac{{}^{10}C_r\left(\sqrt{a}\right)^{10-r}}{2^r} x^{20-5r} \xrightarrow{} 0 \Rightarrow 20 = 5r \Rightarrow r = 4$$

$$T_5 = \frac{{}^{10}C_4\left(\sqrt{a}\right)^6}{2^4} = \frac{10\times9\times8\times7}{4\times3\times2\times2^4}\cdot(a)^3 = 105$$

$$\Rightarrow \frac{10\times21\times a^3}{16} = 105 \Rightarrow \frac{10a^3}{16} = 5$$

$$\Rightarrow \frac{2a^3}{16} = 1 \Rightarrow a^3 = 8 \Rightarrow a = 2$$

Hence, $a^2 = 4$

Ques. If the coefficients of x^4, x^5 and x^6 in the expansion of $(1 + x)^n$ are the arithmetic progression, then the maximum value of n is:

[JEE Main April 2024 Shift II]

(a) 7 (b) 21 (c) 28 (d) 14

Ans. $(1 + x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + \text{---} + {}^nC_nx^n$

coeff of x^4: ${}^nC_4$
coeff of x^5: ${}^nC_5$ → AP
coeff of x^6: ${}^nC_6$

$$\Rightarrow 2\,{}^nC_5 = {}^nC_4 + {}^nC_6$$

$$\Rightarrow 2\frac{n!}{5!(n-5)!} = \frac{n!}{4!(n-4)!} + \frac{n!}{6!(n-6)!}$$

$$\Rightarrow \frac{2}{5(n-5)} = \frac{1}{(n-4)(n-5)} + \frac{1}{6\times5}$$

$$\Rightarrow \frac{2}{5(n-5)} = \frac{30+(n-4)(n-5)}{30(n-4)(n-5)}$$

$$\Rightarrow 2\times6(n-4) = 30 + n^2 - 9n + 20$$

$$\Rightarrow 12n - 48 = 50 + n^2 - 9n$$

$$\Rightarrow n^2 - 21n + 98 = 0 \Rightarrow n = 14, 7$$

Brain Teaser

Let the coefficients of three consecutive terms in the binomial expansion of $(1 + 2x)^n$ be the ratio 2:5:8. Then the coefficient of the term, which is in the middle of these three terms, is

Ans. $T_{r+1} = {}^nC_r(2x)^r$

$$= {}^nC_r2^r(x)^r$$

$${}^nC_{r-1}2^{r-1},\ {}^nC_r2^r,\ {}^nC_{r+1}2^{r+1}$$

2 : 5 : 8

$$\frac{{}^nC_r2^r}{{}^nC_{r-1}2^{r-1}} = \frac{5}{2}$$

$$\Rightarrow \frac{(n-r+1)}{r}.2^1 = \frac{5}{2}$$

$$\Rightarrow \frac{(n-r+1)2}{r} = \frac{5}{2}$$

$$\Rightarrow 4(n-r+1) = 5r$$

$$\Rightarrow 4n + 4 = 9r \quad \text{...(i)}$$

$$\frac{{}^nC_{r+1}2^{r+1}}{{}^nC_r2^r} = \frac{8}{5}$$

$$\Rightarrow \left(\frac{n-r}{r+1}\right).2^1 = \frac{8}{5}$$

$$\Rightarrow \frac{n-r}{r+1} = \frac{4}{5}$$

$$\Rightarrow 5n - 5r = 4r + 4$$

$$\Rightarrow 5n = 9r + 4 \quad \text{...(ii)}$$

By eqn (i) and (ii)

$$\Rightarrow n = 8,\ r = 4$$

MIDDLE TERM

Theorem (i) If n is even, there is only one middle term which is given by $T_{\left(\frac{n}{2}+1\right)}$

Theorem (ii) If n is odd, there are two middle terms which are $T_{(n+1)/2}$ and $T_{[(n+1)/2]+1}$

Ques. Find the middle term is the expansion of

$$\left(2x^2 - \frac{1}{3x}\right)^{\boxed{11}} \xrightarrow{} \text{odd}$$

Ans. $T_{r+1} = {}^{11}C_r(2x^2)^{11-r}\left(-\frac{1}{3x}\right)^r$

for T_6 Put r = 5

$$T_6 = {}^{11}C_5(2x^2)^6\left(-\frac{1}{3x}\right)^5$$

For T_7 Put r = 6

$$T_7 = {}^{11}C_6(2x^2)^5\left(-\frac{1}{3x}\right)^6$$

2 middle terms

$$T_{\frac{n+1}{2}} \ \&\ T_{\frac{n+3}{2}}$$

$$= T_{\frac{11+1}{2}},\ T_{\frac{11+3}{2}}$$

$$= T_6 \ \&\ T_7$$

Ques. The middle term in the expansion of $\left(x-\frac{1}{x}\right)^{18}$ is

(a) $^{18}C_9$ (b) $-^{18}C_9$ (c) $^{18}C_{10}$ (d) $-^{18}C_{10}$

Ans. $T_{r+1} = {}^{18}C_r (x)^{18-r}\left(-\frac{1}{x}\right)^r$

1 middle terms

$T_{\left(\frac{18}{2}+1\right)} = T_{10}$

for T_{10} put $r = 9$

$$T_{10} = {}^{18}C_9 (x)^9\left(-\frac{1}{x}\right)^9 = -{}^{18}C_9\, x^9\,\frac{1}{x^9} = -{}^{18}C_9$$

Ques. 5^{th} terms the end in the expansion of $\left(\frac{x^3}{2}-\frac{2}{x^2}\right)^{12}$ is

(a) $-7920x^{-4}$ (b) $x+7920x^4$

(c) $7920x^{-4}$ (d) $x-7920x^4$

Ans. r^{th} term from end for $(x+y)^n$ is same as r^{th} term from beginning for $(y+x)^n$

$$5^{th}\text{ Terms from end} = {}^{12}C_4\left(\frac{-2}{x^2}\right)^{12-4}\left(\frac{x^3}{2}\right)^4$$

$$= {}^{12}C_4\frac{(-2)^8}{x^{16}}\frac{x^{12}}{2^4} = \frac{{}^{12}C_4\, 2^8}{x^4 . 2^4} = \frac{{}^{12}C_4\, 2^4}{x^4} = 7920x^{-4}$$

Ques. If in the expansion of $\left(\sqrt[3]{2}+\frac{1}{\sqrt[3]{3}}\right)^n$, the ratio of the 7^{th} term from the beginning to the 7^{th} terms from end is equal to 1/6 then n is equqal to

Ans. $T_{r+1} = {}^nC_r\left(2^{1/3}\right)^{n-r}\left(\frac{1}{\sqrt[3]{3}}\right)^r$

put $r = 6$

$$T_7 = {}^nC_6\left(2^{1/3}\right)^{n-6}\left(\frac{1}{\sqrt[3]{3}}\right)^6 \Rightarrow T_7 = {}^nC_6\left(2^{\frac{1}{3}}\right)^{n-6}\frac{1}{3^2}$$

$$T'_{r+1} = {}^nC_r\left(\frac{1}{\sqrt[3]{3}}\right)^{n-r}\left(\sqrt[3]{2}\right)^r$$

put $r = 6$

$$T^1_7 = {}^nC_6\left(\frac{1}{\sqrt[3]{3}}\right)^{n-6}\left(\sqrt[3]{2}\right)^6 \Rightarrow T^1_7 = {}^nC_6\left(\frac{1}{\sqrt[3]{3}}\right)^{n-6}.2^2$$

$$\frac{T_7}{T^1_7} = \frac{1}{6} \Rightarrow \frac{{}^nC_6\left(2^{1/3}\right)^{n-6}}{3^2\,{}^nC_6\,2^2}\left(\sqrt[3]{3}\right)^{n-6} = \frac{1}{6}$$

$$\Rightarrow \frac{\left(2^{\frac{1}{3}}\right)^{n-6}\left(3^{\frac{1}{3}}\right)^{n-6}}{3^2\ 2^2} = \frac{1}{3\times 2}$$

$$\Rightarrow \left(2^{\frac{1}{3}}.3^{\frac{1}{3}}\right)^{n-6} = 6 \Rightarrow (2.3)^{\frac{n-6}{3}} = 6^1$$

$$\Rightarrow (6)^{\frac{n-6}{3}} = 6^1 \Rightarrow \frac{n-6}{3} = 1 \Rightarrow n = 9$$

Important Note

Note 1: The r^{th} term from end is $(T_{n-r+2})^{th}$ term from beginning.

Note 2: Better Approach r^{th} term from end in $(x+y)^n$ is same as r^{th} from beginning in $(y+x)^n$

Ques. In the expansion of $(5^{1/2}+7^{1/8})^{1024}$ the number if integral term is

(a) 128 (b) 129 (c) 130 (d) 131

Ans. $T^{r+1} = {}^{1024}C_r\left(5^{\frac{1}{2}}\right)^{1024-r}\left(7^{\frac{1}{8}}\right)^r$

for integral terms, 1024 - r is even & r is a multiple of 8

$\Rightarrow$ r = multiple of 8

no of terms, r = 0, 8, 16, 24,---, 1024

1024 = 0 + (n - 1).8

$$\Rightarrow \frac{1024}{8} = n-1 \Rightarrow 128 = n-1 \Rightarrow n = 129$$

Ques. The number of integral terms in the expansion of $(3^{1/2}+5^{1/4})^{680}$ is equal to

[JEE Main April 2023 Shift 1]

Ans. $T_{r+1} = {}^{680}C_r\left(3^{\frac{1}{2}}\right)^{680-r}\left(5^{\frac{1}{4}}\right)^r$

680 - r = even & r $\rightarrow$ multiple of 4

$\Rightarrow$ r = even & r$\rightarrow$multiple of 4

$\Rightarrow$ r = 4k

r = 0 4, 8, 12,---- 680 (680 = 4 × 170)

Hence, total terms = 171

Important Note

Note 1: $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$

Note 2: $\frac{{}^nC_{r+1}}{{}^nC_r} = \frac{n-r}{r+1}$

Note 3: In the expansion $(1+x)^n$

$$\frac{T_{r+1}}{T_r} = \frac{n-r+1}{r}|x| \Rightarrow \frac{T_{r+1}}{T_r} = \frac{n-r+1}{r}|x|$$

NUMERICALLY GREATEST TERM

Ques. Find numerically greatest term in the expansion of $(1+4x)^8$ if $x=\frac{1}{3}$.

Ans. $(1+4x)^8 = {}^8C_0 + {}^8C_1\, 4x + {}^8C_2\,(4x)^2 + \dots + {}^8C_8\,(4x)^8$

Consider:

$$\frac{T_{r+1}}{T_r} = \frac{n-r+1}{r}|4x|$$

$$x=\frac{1}{3},\ n=8$$

$$\Rightarrow \frac{T_{r+1}}{T_r} = \frac{8-r+1}{r}\cdot\left(\frac{4}{3}\right)$$

$$\Rightarrow \frac{T_{r+1}}{T_r} = \frac{4(9-r)}{3r}$$

For Numerically greatest term,

$$\frac{T_{r+1}}{T_r} = \frac{4(9-r)}{3r} \ge 1$$

$$\Rightarrow 36-4r \ge 3r \Rightarrow 36 \ge 7r \Rightarrow 7r \le 36 \Rightarrow r \le \frac{36}{7}$$

$r_{max} = 5$

Numerically greatest term = $T_{5+1} = T_6$

$$= {}^8C_5\left(\frac{4}{3}\right)^5$$

Ques. Find the greatest term in the expansion of $(3-2x)^9$ when $x = 1$.

Ans. $$\frac{T_{r+1}}{T_r} = \frac{n-r+1}{r}\left|\frac{-2x}{3}\right|$$

$\because n = 9 \ \& \ x = 1$

$$\frac{T_{r+1}}{T_r} = \left(\frac{9-r+1}{r}\right)\left(\frac{2}{3}\right)$$

$$\Rightarrow \frac{T_{r+1}}{T_r} = \frac{2}{3}\left(\frac{10-r}{r}\right) \ge 1$$

$$\Rightarrow 2(10-r) \ge 3r \Rightarrow 20 \ge 5r \Rightarrow r \le 4$$

$$r=4,\ \frac{T_5}{T_4} = \frac{2}{3}\left(\frac{6}{4}\right) = 1$$

$$\Rightarrow \frac{T_5}{T_4} = 1 \Rightarrow T_5 = T_4$$

$$r=5,\ \frac{T_6}{T_5} < 1 \Rightarrow T_6 < T_5$$

$T_1 - T_2 + T_3 - T_4 + T_5 \dots$

Numerically greatest → $T_4 = T_5$ → both

Greatest term (with sign) → T_5 only

Ques. The largest terms in the expansion of $(3+2x)^{50}$, where, $x = \frac{1}{5}$.

Ans. $(3+2x)50 = 3^{50}\left[1+\frac{2x}{3}\right]^{50}$

Considers $\frac{T_{r+1}}{T_r} = \frac{n-r+1}{r}\cdot\left|\frac{2x}{3}\right|$

$\because n = 50,\ x = 1/5$

$$\Rightarrow \frac{T_{r+1}}{T_r} = \left(\frac{51-r}{r}\right)\frac{2}{15} \ge 1$$

$$\Rightarrow 2(51-r) \ge 15r \Rightarrow 102 \ge 17r$$

$$\Rightarrow r \le \frac{102}{17} \Rightarrow r \le 6$$

$$r=6,\ \frac{T_7}{T_6} = 1 \Rightarrow T_7 = T_6$$

$$T_{r+1} = {}^{50}C_r(3)^{50-r}(2x)^r$$

2 largest terms, T^6 and T_7

$$T_6 = T_7 = {}^{50}C_6\,(3)^{44}\left(\frac{2}{5}\right)^6$$

Ques. Find the remainder when

1. 5^{99} is divided by 8

Ans. $5^{99} = (1+4)^{99}$

$$= {}^{99}C_0 + {}^{99}C_1(4) + {}^{99}C_2(4)^2 + {}^{99}C_3(4)^3 + \dots {}^{99}C_{99}4^{99}$$

$$\frac{1+99\times4+16I}{8} = \frac{1+99\times4}{8} = \frac{397}{8} = \frac{49\times8+5}{8}$$

Remainder = 5

2. 5^{99} is divided by 13

Ans. $5^{99} = 5.5^{98} = 5.(25)^{49} = 5\,[26-1]^{49}$

$$= 5[{}^{49}C_0(26)^{49} - {}^{49}C_1(26)^{48} + \dots + 1]$$

$$= \frac{5(26I-1)}{13} = \frac{5\times26I-5}{13}$$

Remainder = - 5 + 13 = 8

3. 3^{2023} is divided by 28

Ans. $3^{2023} = 3 \times 3^{2022} = 3 \times (3^3)^{674} = 3 \times (27)^{674}$

$$= \frac{3(28-1)^{674}}{28} = \frac{3[28I+1]}{28} = \frac{3 \times 28I + 3}{28}$$

Remainder = 3

Ques. Find last 2 digits of $(3)^{400}$

Ans. $(1 \pm 10k)^n \rightarrow$ Break

$(3^2)^{200} = (9)^{200} = (10-1)^{200}$

$= {}^{200}C_0(10)^{200} - {}^{200}C_1(10)^{199} + {}^{200}C_2(10)^{198} -$

$\ldots - 200 \times 10\, C_{199}(10)^1 + {}^{200}C_{200}(1)$

$= 1000I + 1$

Hence, last 2 digits = 01

Problem on GIF & Fractional Function

Add or Subtract the two equations depending on how irrational terms get cancelled.

Ques. If $X = \left(3+\sqrt{7}\right)^n$, $n \in N$. Show that $[X]$ is always an odd Integer & hence prove that $X(1 - \{X\}) = 2^n$

Ans. $X = \left(3+\sqrt{7}\right)^n \begin{smallmatrix} \nearrow I \\ \searrow f \end{smallmatrix}$

Consider $I + f = \left(3+\sqrt{7}\right)^n$

$= {}^nC_0 3^n + {}^nC_1 3^{n-1}\left(\sqrt{7}\right)^1 + {}^nC_2 3^{n-2}\left(\sqrt{7}\right)^2 + \ldots$ ....(i)

$\left(3-\sqrt{7}\right)^n \approx (0.4)^n \Rightarrow 0 < \left(3-\sqrt{7}\right)^n < 1 \Rightarrow 0 < f' < 1$

$f' = \left(3-\sqrt{7}\right)^n$

$= {}^nC_0 3^n - {}^nC_1 3^{n-1}\left(\sqrt{7}\right)^1 + {}^nC_2 3^{n-2}\left(\sqrt{7}\right)^2 - {}^nC_3 3^{n-3}\left(\sqrt{7}\right)^3 + \ldots$ ....(ii)

Equation (i) + (ii),

$I + f + f' = 2[{}^nC_0 3^n + {}^nC_2 3^{n-2}\left(\sqrt{7}\right)^2 + {}^nC_4 3^{n-4}\left(\sqrt{7}\right)^4 + \ldots]$

$\Rightarrow I + f + f' =$ Even Integer

$\Rightarrow f + f' =$ Even Integer $- I \Rightarrow f + f' =$ Integer

We know

$0 < f' < 1$

$0 < f < 1$

$\overline{0 < f + f' < 2}$

$\Rightarrow f + f' = 1 \Rightarrow f' = 1 - f$

$I + f + f' =$ Even

$\Rightarrow I + 1 =$ Even

$\Rightarrow I =$ odd, Hence proved

To Prove: $X(1 - \{x\}) = 2^n$

LHS: $(I + f)(1 - f)$

$= (I + f)f'$

$= \left(3+\sqrt{7}\right)^n \cdot \left(3-\sqrt{7}\right)^n$

$= \left[\left(3+\sqrt{7}\right)\left(3-\sqrt{7}\right)\right]^n$

$= \left[3^2 - \left(\sqrt{7}\right)^2\right]^n$

$= (2)^n$

Ques. Let $R = \left(5\sqrt{5}+11\right)^{2n+1}$ and $f = R - [R]$ where [.] denotes the greatest integer function. The value of R.f is 4^{2n+1}

Ans. $I + f = \left(5\sqrt{5}+11\right)^{2n+1} = {}^{2n+1}C_0\left(5\sqrt{5}\right)^{2n+1} + {}^{2n+1}C_1\left(5\sqrt{5}\right)^{2n}.(11)^1 + {}^{2n+1}C_2\left(5\sqrt{5}\right)^{2n-1}(11)^2 + \ldots$

$f' = \left(5\sqrt{5}-11\right)^{2n+1} = {}^{2n+1}C_0\left(5\sqrt{5}\right)^{2n+1} - {}^{2n+1}C_1\left(5\sqrt{5}\right)^{2n}(11)^1 + {}^{2n+1}C_2\left(5\sqrt{5}\right)^{2n-1}(11)^2 - \ldots$

$\ominus$

$I + f - f' =$ Even

$\Rightarrow f - f' =$ Even Integer - I

$\Rightarrow f - f' =$ Integer

we know,

$0 < f < 1$,

$0 < f' < 1 \Rightarrow -1 < -f' < 0$

$\Rightarrow -1 < f - f' < 1$

$\Rightarrow f - f' = 0 \Rightarrow f = f' \Rightarrow I =$ Even Integer

Now, the value of Rf =

$\left(5\sqrt{5}+11\right)^{2n+1} . \left(5\sqrt{5}-11\right)^{2n+1}$

$= (125 - 121)^{2n+1} = 4^{2n+1}$

Important Note

Note 1: If we add I + f & f' to cancel out irrational terms then always I + f + f' = even & f + f' = 1

Note 2: If we subtract I + f & f' to cancel out the irrational terms then always I + f - f' = even & f - f' = 0

Ques. Let $x = \left(2+\sqrt{3}\right)^n$, then the value of $x - x^2 + x[x]$, where [.] denotes the greatest integer function, is equal to

(a) 1 (b) 2

(c) 2^{2n} (d) 2^n

Ans. $I + f = \left(2+\sqrt{3}\right)^n = {}^nC_0\, 2^n + {}^nC_1\, 2^{n-1}\left(\sqrt{3}\right)^1 + \ldots$

$f' = \left(2-\sqrt{3}\right)^n = {}^nC_0\, 2^n - {}^nC_1\, 2^{n-1}\left(\sqrt{3}\right)^1 + \ldots$

+

$I + f + f' = \text{Even}$ & $f + f' = 1$

Now, $x - x^2 + x[x]$

$= x - x(x - [x]) = x - xf$

$= x(1 - f)$

$= \left(2+\sqrt{3}\right)^n\left(2-\sqrt{3}\right)^n = (4-3)^n = 1$

Ques. If $\left(9+\sqrt{80}\right)^n = I + f$ where I, n are Natural numbers and $0 < f < 1$, then

(a) I is an odd integer

(b) I is an even integer

(c) $(I + f)(1 - f) = 1$

(d) $1 - f = \left(9-\sqrt{80}\right)^n$

Ans. $I + f = \left(9+\sqrt{80}\right)^n$

$f' = \left(9-\sqrt{80}\right)^n$

+

$I + f + f' = \text{Even}$ & $f + f' = 1$

$\Rightarrow I = \text{odd}$

$\Rightarrow I + f = \left(90+\sqrt{80}\right)^n$

and $1 - f = f' = \left(90-\sqrt{80}\right)^n$

Hence, $(I + f)(1 - f) = (81 - 80)^n = 1$

Ques. Let $x = \left(8\sqrt{3}+\sqrt{13}\right)^{13}$, and $y = \left(7\sqrt{2}+\sqrt{13}\right)^9$. If [t] denotes the greatest integer $\le t$, then

[JEE Main Jan. 2023 Shift II]

(a) [x] + [y] is even

(b) [x] is odd but [y] is even

(c) [x] is even but [y] is odd

(d) [x] and [y] are both odd

Ans. For x:

$I + f = \left(8\sqrt{3}+\sqrt{13}\right)^{13}$

$\ominus\ f' = \left(8\sqrt{3}-\sqrt{13}\right)^{13}$

$I + f - f' = \text{Even}$ & $f - f' = 0$

$\Rightarrow I = \text{Even} \Rightarrow [x] = \text{Even}$

For y:

$I + f = \left(7\sqrt{2}+9\right)^9$

$f' = \left(7\sqrt{2}-9\right)^9$

−

$I + f - f' = \text{Even}$ & $f - f' = 0$

$\Rightarrow I = \text{Even}$

Hence, [x] + [y] is even

Brain Teaser

Let $x = \left(3\sqrt{6}+7\right)^{89}$. If {x} denotes the fractional part of 'x' then find the remainder when $x\{x\} + (x\{x\})^2 + (x\{x\})^3$ is divided by 31.

Ans. $I + f = \left(3\sqrt{6}+7\right)^{89}$

$f' = \left(3\sqrt{6}-7\right)^{89}$

−

$I + f - f' = \text{Even}$ & $f - f' = 0$

$\Rightarrow I = \text{Even}$ & $f = f'$

$xf = \left(3\sqrt{6}+7\right)^{89}\left(3\sqrt{6}-7\right)^{89}$

$= (54 - 49)^{89} = (5)^{89}$

Now, $xf + (xf)^2 + (xf)^3$

$= (5)^{89} + (5^{89})^2 + (5^{89})^3$

$= \dfrac{5^{89} + 5^{178} + 5^{267}}{31}$

$= \dfrac{5^2.125^{29} + 5.125^{59}. + (125)^{89}}{31}$

$= \dfrac{25(1+31I)+5(1+31I)+(1+31I}{31}$

$= \dfrac{25+5+1}{31} = 1$

Hence, Remainder is 0.

PROPERTIES OF BINOMIAL COEFFICIENTS

1. Sum of all the binomial coefficients in $(1 + x)^n$ is equal 2^n i.e.

$$C_0 + C_1 + C_2 + C_3 \ldots\ldots C_n = 2^n$$

Proof: $\rightarrow (1 + x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + \ldots\ldots + {}^nC_nx^n$

Put $x = 1$, $2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots\ldots + {}^nC_n$

2. Sum of Even coefficient equals the sum of Odd coefficient and is equal to 2^{n-1}

$$C_0 + C_2 + C_4 + \ldots\ldots = C_1 + C_3 + C_5 + \ldots\ldots = 2^{n-1}$$

Proof:

Put $x = -1$

$(1 - 1)^n = {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + {}^nC_4 \ldots\ldots = 0$

${}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots\ldots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots\ldots = x$

$\Rightarrow x + x = 2^n \Rightarrow x = \frac{2n}{2} = 2^{n-1}$

Ex: Sum of all coefficient

$(1 - 3x)^n = {}^{10}C_0 - {}^{10}C_1(3x) + {}^{10}C_2(3x)^2 - {}^{10}C_3(3x)^3$ ......

Put $x = 1$

Sum $= [1 - 3 \times 1]^{10} = (-2)^{10} = (2)^{10}$

Ex: $(2x - 3y)^{20} = {}^{20}C_0(2x)^{20} - {}^{20}C_1(2x)^{19}(3y)^1 + \ldots$

Put $x = 1, y = -1$

$\Rightarrow$ Sum of all coeff $= (2 + 3)^{20} = (5)^{20}$

> To obtain sum of all coeff Put all variables equal to 1

Ques. The value of ${}^{10}C_2 + {}^{10}C_4 + {}^{10}C_6 + {}^{10}C_8 + {}^{10}C_{10}$ is.

(a) 2^{10} (b) 2^9 (c) 2^8 (d) 511

Ans. $[{}^{10}C_0 + {}^{10}C_2 + {}^{10}C_4 + \ldots + {}^{10}C_{10}] - {}^{10}C_0$

$= 2^{n-1} - 1 = 2^{10-1} - 1 = 2^9 - 1 = 512 - 1 = 511$

Ques. The value of $\frac{1}{1!50!} + \frac{1}{3!48!} + \frac{1}{5!46!} + \ldots + \frac{1}{49!2!} + \frac{1}{51!0!}$ is

[JEE Main Feb. 2023 Shift II]

(a) $\frac{2^{50}}{50!}$ (b) $\frac{2^{50}}{51!}$ (c) $\frac{2^{51}}{51!}$ (d) $\frac{2^{51}}{50!}$

Ans. $\frac{1}{1!50!} + \frac{1}{3!48!} + \frac{1}{5!46!} + \ldots + \frac{1}{49!2!} + \frac{1}{51!0!}$

$= \frac{1}{51!}\left[\frac{51!}{1!50!} + \frac{51!}{3!48!} + \frac{51!}{5!46!} + \ldots + \frac{51!}{51!0!}\right]$

$= \frac{1}{51!}\left[{}^{51}C_1 + {}^{51}C_3 + {}^{51}C_5 + \ldots + {}^{51}C_{51}\right]$

$= \frac{1}{51!}\left[2^{51-1}\right] = \frac{1}{51!} \cdot 2^{50}$

1 + 1

Ques. If $1 + [2 + {}^{49}C_1 + {}^{49}C_2 + \ldots\ldots {}^{49}C_{49}][{}^{50}C_2 + {}^{50}C_4 + \ldots\ldots {}^{50}C_{50}]$ is equal to $2^n \cdot m$, where m is odd, then n + m is equal to ________.

[JEE Main Jul. 2022 Shift II]

Ans. $1 + [1 + {}^{49}C_0 + {}^{49}C_1 + {}^{49}C_2 + \ldots + {}^{49}C_{49}]$
$[{}^{50}C_0 + {}^{50}C_2 + \ldots + {}^{50}C_{50} - 1]$

$= 1 + (1 + 2^{49})(2^{49} - 1)$

$= 1 + (2^{49})^2 - 1^2 = 2^{98} = 2^n.m$

$\because n = 98, m = 1 \Rightarrow n + m = 99$

3. ${}^{n+1}C_{r+1} = \left(\frac{n+1}{r+1}\right){}^nC_r$

proof: ${}^{n+1}C_{r+1} =$

$$\frac{(n+1)!}{(r+1)!(n-r)!} = \frac{(n+1)\,n!}{(r+1)r!(n-r)!} = \frac{n+1}{r+1}{}^nC_r$$

Example 1: ${}^{100}C_{20} = \frac{100}{20}\,{}^{99}C_{19}$

Example 2: ${}^{200}C_{40} = \frac{200}{40}\,{}^{199}C_{39}$

4. ${}^{n+1}C_{r+1} = \left(\frac{n+1}{r+1}\right)\left(\frac{n}{r}\right){}^{n-1}C_{r-1}$

Example: ${}^{100}C_{20} = \frac{100}{20}\,{}^{99}C_{19} = \frac{100}{20}\cdot\frac{99}{19}\cdot{}^{98}C_{18}$

Ques. If A denotes the sum of all the coefficients in the expansion of $(1 - 3x + 10x^2)^n$ and B denotes the sum of all the coefficients in the expansion of $(1 + x^2)^n$, then:

[JEE Main Jan. 2024 Shift I]

(a) $A = B^3$ (b) $3A = B$

(c) $B = 3^3$ (d) $A = 3B$

Ans. $A = (1 - 3 + 10)^n = 8^n = 2^{3n}$

$B = (2)^n$

$A = 2^{3n} = (2^n)^3 \Rightarrow A = B^3$

TYPES OF BINOMIAL SERIES

Type 1

Series in which the coefficients of binomial Coefficients are in AP

Type 2

Series in which the coefficients of binomial Coefficients are in HP

Type 3

Series in which the each term is a product of 2 or more binomial Coefficients.

Type 4

Series in which the binomial Coefficients occur with fixed gaps.

Binomial Series Type-1

When coefficients of binomial Coefficients are in A.P.

Ques. Prove that $1.C_1 + 2.C_2 + 3.C_3 + ... + n.C_n = n.2^{n-1}$

Method-1

Ans.

$S = 0.\,C_0 + 1.\,C_1 + 2.\,C_2 + 3.\,C_3 + ... + nC_n$

$S = n.\,C_0 + (n-1)\,C_1 + (n-2)\,C_2 + ... + 0\,.\,C_n$

$+$

$\Rightarrow 2S = n.C_0 + n.C_1 + n.C_2 + n.C_3 + ... + nC_n$

$\Rightarrow 2S = n\,[C_0 + C_1 + C_2 + C_3 + ... + C_n]$

$\Rightarrow 2S = n\,.\,2^n$

$\Rightarrow S = \dfrac{n.2^n}{2^1} = n.2^{n-1}$

Method-2

By differentiation

$(1 + x)^n = {}^nC_0 + {}^nC_1\,x + {}^nC_2\,x^2 + ... + {}^nC_n\,x^n$

diff wr.t 'x'

$\Rightarrow n\,(1 + x)^{n-1} = 0 + {}^nC_1.1 + {}^nC_2.2x + ... + {}^nC_n.n\,x^{n-1}$

Put $x = 1$

$n\,2^{n-1} = 1.C_1 + 2C_2 + 3\,.\,C_3 + ...\,nC_n$

Ques. 1. $C_0 + 2.C_1 + 3.C_2 + ... + (n+1)\,C_n = 2^{n-1}\,(n+2).$

Ans.

$S = 1.\,C_0 + 2.\,C_1 + 3.\,C_2 + ... + (n+1)\,C_n$

$S = (n+1)\,C_0 + n\,.\,C_1 + {}^{(n-2)}C_2 + ... + 1\,.\,C_n$

$+$

$\Rightarrow 2S = C_0\,[n+2] + C_1\,[n+2] + C_2\,(n+2) + ... + (n+2)\,C_n$

$\Rightarrow 2S = (n+2)\,[C_0 + C_1 + C_2 + ... + C_n]$

$\Rightarrow 2S = (n+2)\,.\,2^n$

$\Rightarrow S = (n+2)\,2^{n-1}$

Ques. 1. $C_0 + 3.C_1 + 5.C_2 + 7.C_3 + ... + [2n+1].C_n = [n+1]2^n.$

Ans. Consider

$(1 + x)^n = {}^nC_0 + {}^nC_1\,x + {}^nC_2\,x^2 + ... + {}^nC_n\,x^n$

In general c.d = d

$x \rightarrow x^d$

In given series c.d. = 2

$x \rightarrow x^2$

$\Rightarrow (1 + x^2)^n = {}^nC_0\,x + {}^nC_1\,x^2 + {}^nC_2\,x^4 + ... + {}^nC_n\,x^{2n}$

Multipy with x

$\Rightarrow x\,(1 + x^2)^n = {}^nC_0 + {}^nC_1\,x^3 + {}^nC_2\,x^5 + ... + {}^nC_n\,x^{2n+1}$

diff. wrt x

$\Rightarrow x\,(1 + x^2)^n\,.\,1 + x.n\,(1 + x^2)^{n-1}\,.\,2x = 1\,.\,C_0 + 3\,.\,C_1\,x^2 + 5\,.\,C_2\,x^4 + ... + (2n+1).C_n\,x^{2n}$

Put $x = 1$

$\Rightarrow 2^n + n.2^{n-1}.2 = 1.C_0 + 3.C_1 + 5\,.\,C_2 + ... + (2n+1).C_n$

$= 2^n\,[1 + n] = \text{RHS}$

Ques. 1. 2. $C_1 + 2.3.C_2 + 3.4.C_3 + ... + n[n+1]\,C_n = n[n+3]\,2^{n-2}.$

Ans. $(1 + x)^n = {}^nC_0 + {}^nC_1\,x + {}^nC_2\,x^2 + ... + {}^nC_n\,x^n$

diff. wrt x

$\Rightarrow n\,(1 + x)^{n-1} = 0 + 1.\,{}^nC_1 + 2.\,{}^nC_2\,x + 3.\,{}^nC_3\,x^2 + ... + n.\,{}^nC_n\,x^{n-1}$

Multipy with x^2

$\Rightarrow n\,x^2\,(1 + x)^{n-1} = 1.\,{}^nC_1\,x^2 + 2.\,{}^nC_2\,x^3 + 3.\,C_3\,x^4 + ... + n.\,C_n\,x^{n+1}$

diff. wrt x

$\Rightarrow n\,2x\,(1 + x)^{n-1} + n.\,x^2\,(n-1)\,(1 + x)^{n-2} = 1.2.C_1\,x + 2.3.C_2\,x^2 + 3.4\,.\,C_3\,x^3 + ... + n\,(n+1)\,C_n\,x^n$

Put $x = 1$

$\Rightarrow n.\,2\,.\,2^{n-1} + n.\,(n-1)\,.\,2^{n-2} = S$

$\Rightarrow n\,.\,2^{n-2}\,[2.2 + n - 1] = S$

$\Rightarrow n\,.\,2^{n-2}\,(n + 3) = S$

Ques. $3C_0 - 8C_1 + 13C_2 - 18C_3 + ...$ upto $[n+1]$ terms $= 0$.

Ans. $(1+x^5)^n = {}^nC_0 + {}^nC_1 x^5 + {}^nC_2 x^{10} + {}^nC_3 x^{15} + ... + {}^nC_n x^{5n}$

Multiply with x^3

$\Rightarrow x^3(1+x^5)^n = {}^nC_0 x^3 + {}^nC_1 x^8 + {}^nC_2 x^{13} + ...$

differentiate wrt x

$\Rightarrow 3x^2(1+x^5)^n + x^3 n(1+x^5)^{n-1} 5x^4 = 3C_0 x^2 + 8C_1 x^7 + 13C_2 x^{12} + ...$

Put $x = -1$

$\Rightarrow 0 + 0 = 3.C_0 - 8C_1 + 13.C_2 - 18C_3 ...$

$\Rightarrow 0 = S$

Binomial Series Type-2

When coefficients of Binomial Coefficients are in H.P.

Ques. Show that: $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + ... + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$

Ans. $(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + ... + {}^nC_n x^n$

$$\int_0^1 (1+x)^n dx = \int_0^1 C_0 dx + \int_0^1 C_1 x dx + \int_0^1 C_2 x^2 dx + ... + \int_0^1 C_n x^n dx$$

$$\Rightarrow \int_0^1 (1+x)^n dx = C_0 x\Big|_0^1 + C_1\frac{x^2}{2}\Big|_0^1 + C_2\frac{x^3}{3}\Big|_0^1 + ... C_n\frac{x^{n+1}}{n+1}\Big|_0^1$$

$$\Rightarrow \frac{(1+x)^{n+1}}{1(n+1)}\Big|_0^1 = \underbrace{C_0 + \frac{C_1}{2} + \frac{C_2}{3} + ... + \frac{C_n}{n+1}}_{S}$$

$$\Rightarrow S = \frac{2^{n+1}}{n+1} - \frac{1^{n+1}}{n+1}$$

Ques. Show that:

$$\frac{C_0}{2} + \frac{C_1}{3} + \frac{C_2}{4} + ... + \frac{C_n}{n+2} = \frac{n.2^{n+1}+1}{(n+1)(n+2)}$$

Ans. $(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + ... + {}^nC_n x^n$

$$\int_0^1 x(1+x)^n = \underbrace{\int_0^1 C_0 x + \int_0^1 C_1 x^2 + ... + \int_0^1 C_n x^{n+1}}_{S}$$

$$\Rightarrow \int_0^1 x(1+x)^n dx = S \Rightarrow \int_0^1 (x+1-1)(1+x)^n dx = S$$

$$\Rightarrow \int_1^0 (1+x)^{n+1} - \int_0^1 (1+x)^n = S$$

$$\Rightarrow \frac{(1+x)^{n+2}}{n+2}\Big|_0^1 - \frac{(1+x)^{n+1}}{n+1}\Big|_0^1 = S$$

$$\Rightarrow S = \left(\frac{2^{n+2}-1}{n+2}\right) - \left(\frac{2^{n+1}-1}{n+1}\right)$$

$$\Rightarrow s = \frac{n.2^{n+1}+1}{(n+1)(n+2)}$$

Ques. Show that:

$$\frac{C_0}{1} + \frac{C_2}{3} + \frac{C_4}{5} + ... = \frac{2^n}{n+1}$$

Ans.

$(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + ... + {}^nC_n x^n$

$(1-x)^n = C_0 - C_1 x + C_2 x^2 - C_3 x^3 + ...$

$+$

$$\Rightarrow \int_0^1 \frac{(1+x)^n + (1-x)^n}{2} = \left(\underbrace{\int_0^1 C_0 + \int_0^1 C_2 x^2 + \int_0^1 C_4 x^4 + \int_0^1 C_6 x^6 + ...}_{S}\right)$$

$$\Rightarrow S = \frac{1}{2}\int_0^1 \left[(1+x)^n + (1+x)^n\right] dx$$

$$\Rightarrow S = \frac{1}{2}\left\{\int_0^1 (1+x)^n dx + \int_0^1 (1+x)^n dx\right\}$$

$$\Rightarrow S = \frac{1}{2}\left[\frac{(1+x)^{n+1}}{n+1}\Big|_0^1 + \frac{(1-x)^{n+1}}{(-1)(n+1)}\Big|_0^1\right]$$

$$\Rightarrow S = \frac{1}{2}\left[\frac{2^{n+1}-1}{n+1} - \left\{\frac{0-1}{n+1}\right\}\right]$$

$$\Rightarrow S = \frac{1}{2}\left[\frac{2^{n+1}-1+1}{n+1}\right] = \frac{1}{2}\frac{2^{n+1}}{n+1} = \frac{2^n}{n+1}$$

Ques. If $\frac{1}{n+1}{}^nC_n + \frac{1}{n}{}^nC_{n-1} + ... + \frac{1}{2}{}^nC_1 + {}^nC_0 = \frac{1023}{10}$ then n is equal to

(a) 9 (b) 8 (c) 7 (d) 6

Ans. ${}^nC_0 + \frac{{}^nC_1}{2} + \frac{{}^nC_2}{3} + ... + \frac{{}^nC_n}{n+1} = \frac{2^{n+1}-1}{n+1} = \frac{2^{10}-1}{10}$

$\Rightarrow n + 1 = 10 \Rightarrow n = 9$

Concluding Question

If $S = 3.\frac{{}^{10}C_0}{1} + 3^2.\frac{{}^{10}C_1}{2} + 3^3.\frac{{}^{10}C_2}{3} + \dots + 3^{11}\frac{{}^{10}C_{10}}{11}$.

Find S.

Ans. $\int_0^3 (1+x)^{10} = \int_0^3 {}^{10}C_0 + \int_0^3 {}^{10}C_1 x + \int_0^3 {}^{10}C_2 x^2 + \int_0^3 {}^{10}C_3 x^3$

$+ \dots + \int_0^3 {}^{10}C_{10} x^{10}$

$= {}^{10}C_0 x \Big|_0^3 + \frac{{}^{10}C_1 x^2}{2}\Big|_0^3 + \frac{{}^{10}C_2 x^3}{3}\Big|_0^3 + \dots + \frac{{}^{10}C_{10} x^{11}}{11}\Big|_0^3$

$\Rightarrow \frac{(1+x)^{11}}{11}\Big|_0^3 = {}^{10}C_0.3^1 + \frac{{}^{10}C_1 3^2}{2} + \frac{{}^{10}C_2 3^3}{3} + \dots + \frac{{}^{10}C_{10} 3^{11}}{11}$

$\Rightarrow \frac{4^{11}-1}{11} = S$

Method-2

Ques. Prove that $1.C_1 + 2.C_2 + 3.C_3 + \dots + n.C_n = n.2^{n-1}$

Ans. $T_r = r.{}^nC_r$

$\Rightarrow S_n = \sum_{r=1}^{n} T_r \quad (S_n = \sum_{r=1}^{n} T_r = T_1 + T_2 + T_3 + \dots + T_n)$

$= \sum_{r=1}^{n} r\,{}^nC_r$

$= \sum_{r=1}^{n} r.\frac{n}{r}\,{}^{n-1}C_{r-1} \qquad {}^nC_r = \frac{n}{r}\,{}^{n-1}C_{r-1}$

$= \sum_{r=1}^{n} n\,{}^{n-1}C_{r-1}$

$= n\sum_{r=1}^{n} {}^{n-1}C_{r-1}$

$= n[{}^{n-1}C_0 + {}^{n-1}C_1 + {}^{n-1}C_2 + \dots + {}^{n-1}C_{n-1}]$

$= n.2^{n-1}$

Ques. $1.C_0 + 2.C_1 + 3C_2 + \dots + (n+1)\,C_n = 2^{n-1}(n+2)$.

Ans. $T_r = r.C_{r-1}$

$r \to 1$ to $n+1$

$S_n = \sum_{r=1}^{n+1} T_r = \sum_{r=1}^{n} r\,C_{r-1}$

$= \sum((r-1)+1)\,C_{r-1}$

$\Rightarrow S_n = \sum_{r=1}^{n+1} [(r-1)\,{}^nC_{r-1} + C_{r-1}]$

$= \sum\left[(r-1)\frac{n}{(r-1)}\,{}^{n-1}C_{r-2} + {}^nC_{r-1}\right]$

$= \sum[n.{}^{n-1}C_{r-2} + {}^nC_{r-1}]$

$= \sum_{r=2}^{n+1} n\,{}^{n-1}C_{r-2} + \sum_{r=1}^{n+1} {}^nC_{r-1}$

$= n\sum_{r=2}^{n+1} {}^{n-1}C_{r-2} + 2^n$

$= n[{}^{n-1}C_0 + C_1 + {}^{n-1}C_2 + \dots {}^{n-1}C_{n-1}] + 2^n$

$= (n\,2^{n-1} + 2^n)$

Method-3

Ques. Show that: $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$

Ans. $S_n = \sum_{r=0}^{n} \frac{{}^nC_r}{(r+1)}$

$\because \frac{1}{(n+1)}\,{}^{n+1}C_{r+1} = \frac{1}{r+1}.{}^nC_r$

$= \frac{1}{n+1}\sum_{r=0}^{n} {}^{n+1}C_{r+1}$

$= \frac{1}{n+1}[{}^{n+1}C_1 + {}^{n+1}C_2 + {}^{n+1}C_3 + \dots {}^{n+1}C_{n+1}]$

$= \frac{1}{n+1}(2^{n+1}-1)$

Binomial Series Type-3

When each term f the series is expressed as a product of two or more Binomial Coefficent.

TYPE-3 (A) When difference of lower suffixes is constant in each terms.

Step 1 write $(1+x)^n$

Step 2 replce $x \to \frac{1}{x}$

Step 3 coeff of x^d in product $(1+x)^n\left(1+\frac{1}{x}\right)$

Ques. $C_0^2 + C_1^2 + C_2^2 + ... + C_n^2 = \frac{(2n)!}{n!n!} = {}^{2n}C_n$

Ans. $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + ---+C_nx^n$

$$\left(1+\frac{1}{x}\right)^n = C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \frac{C_3}{x^3} + ---+\frac{C_n}{x^n}$$

$(1+x)^n\left(1+\frac{1}{x}\right)^n \rightarrow$ coeff of x^0 in this product is our Ans

Ans: coeff of x^0 is $\frac{(1+x)^{2n}}{x^n}$

= coeff of x^n in $(1+x)^{2n}$

$= {}^{2n}C_n$

Ques. $C_0\,C_2 + C_1\,C_3 + C_2C_4 + ... + C_{n-2}\,C_n = \frac{(2n!)}{(n+2)!(n-2)!}$

$\boxed{d = 2}$

Ans. Coeff of x^2 in the product $\frac{(1+x)^{2n}}{x^n}$ is our Ans

$\Rightarrow$ Coeff of x^{n+2} in $(1+x)^{2n}$

$= {}^{2n}C_{n+2}$ or ${}^{2n}C_{n-2}$

TYPE-3(B) When sum of lower suffxes is constant in each term.

Trick method:

${}^mC_r{}^nC_0 + {}^mC_{r-1}{}^nC_1 + {}^mC_{r-2}{}^nC_2 + .. {}^mC_0{}^nC_r = {}^{m+n}C_r$

m boys
m girls $\rightarrow$ r student select ${}^{m+n}C_r$

Ques. Prove that:

$${}^mC_r{}^nC_0 + {}^mC_{r-1}{}^nC_1 + {}^mC_{r-2} + ...+ {}^mC_0{}^nC_r = {}^{m+n}C_r$$

Ans. $\boxed{\text{sum} = r}$

$(1+x)^m = {}^mC_0 + {}^mC_1x + {}^mC_2x^2 + ...\ {}^mC_r\,x^r + ...$
$+ {}^mC_{m-1}x^{m-1} + {}^mC_mx^m$

$(1+x)^n = {}^nC_0 + {}^nC_1\,x + {}^nC_1\,x + {}^nC_2\,x^2 + ... + {}^4C_rx^r$
$+ ...\ {}^nC_nx^n$

coeff of x^r in the product $(1+x)^m\,(1+x)^n$

= coeff of x^r in $(1+x)^{m+n} = {}^{m+n}C_r$

Ques. The value of $\sum_{r=0}^{6} ({}^6C_r.{}^6C_{6-r})$ is equal to:

(a) 1124 (b) 1324 (c) 1024 (d) 924 ✓

Ans. ${}^6C_0.{}^6C_6 + {}^6C_1{}^6C_5 + {}^6C_2{}^6C_4 + --- + {}^6C_6{}^6C_0$

sum = 6

6 boys
6 girls $\rightarrow$ Select 6 students

$= {}^{12}C_6 = 924$

Ques. The value $\sum_{r=0}^{22} {}^{22}C_r{}^{23}C_r$ is

[JEE Main Jan. 2023 Shift I]

(a) ✓ ${}^{45}C_{23}$ (b) ${}^{44}C_{23}$ (c) ${}^{45}C_{24}$ (d) ${}^{44}C_{22}$

Ans. $\sum_{r=0}^{22} {}^{22}C_r{}^{23}C_r = \sum_{r=0}^{22} {}^{22}C_r{}^{23}C_{23-r}$

$= {}^{22}C_0{}^{23}C_{23} + {}^{22}C_1{}^{23}C_{22} + {}^{22}C_2{}^{23}C_{21} + ... + {}^{22}C_{22}{}^{23}C_1$

22 boys
23 girls $\rightarrow$ Select 23 students

$= {}^{45}C_{23}$

Ques. If $\sum_{k=1}^{31}({}^{31}C_k)({}^{31}C_{k-1}) - \sum_{k=1}^{31}({}^{30}C_k)({}^{30}C_{k-1}) = \frac{\alpha(60!)}{(30!)(31!)}$

Where $\alpha \in R$. Then the value of 16α is equal to

[JEE Main June 2022 Shift I]

(a) ✓ 1411 (b) 1320 (c) 1615 (d) 1855

Ans. $\sum_{k=1}^{31} {}^{31}C_k\,{}^{31}C_{32-k}$ $(\because {}^nC_r = {}^nC_{n-r})$

$= {}^{31}C_1\,{}^{31}C_{31} + {}^{31}C_2\,{}^{31}C_{30} + ... + {}^{31}C_{31}\,{}^{31}C_1 = {}^{62}C_{32}$

31 B
31 G $\rightarrow$ 32 students

$\sum_{k=1}^{30} {}^{30}C_k\,{}^{30}C_{31-k} = {}^{60}C_{31}$

30 B
30 G $\rightarrow$ 31 students

$= {}^{62}C_{32} - {}^{60}C_{31}$

$$= \frac{62!}{32!30!} - \frac{60!}{31!29!}$$

$$= \frac{60!}{31!}\left[\frac{62\times 61}{32.30!} - \frac{30.}{30.29!}\right]$$

$$= \frac{60!}{31!30!}\left[\frac{31\times 61}{16}-30\right] = \frac{\alpha(60!)}{(30!)(31!)}$$

$$\alpha = \left[\frac{31\times 61}{16}-30\right] \Rightarrow \alpha = \frac{31\times 61-480}{16}$$

$\Rightarrow 16\alpha = 31 \times 61 - 480$

$\Rightarrow 16\alpha = 1411$

Misc. Series

Ques. $C_1^2+2C_2^2+3.C_3^2+\ldots+n.C_n^2 = \dfrac{(2n-1)!}{\{(n-1)!\}^2}$

Method-1

Ans. Consider, $(1 + x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots + C_nx^n$

Differentiate w.r.t. x, we get

$n(1 + x)^{n-1} = 1.C_1 + 2.C_2x + 3.C_3x^2 + \ldots + n.C_nx^{n-1}$

$$x \to \frac{1}{x}\left(1+\frac{1}{x}\right)^n = C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \frac{C_3}{x^3} + \ldots + \frac{C_1}{x^n}$$

$1.C_1^2+2.C_2^2+3.C_3^2+\ldots+n.C_n^2$

$= \text{coefficient of } x^{-1} \text{ in } n(1+x)^{n-1}\left(1+\frac{1}{x}\right)^n$

$= \text{coefficient of } x^{-1}$ in

$$\frac{n(1+x)^{n-1}.(1+x)^n}{x^n} = \frac{n(1+x)^{2n-1}}{x^n}$$

$= \text{coefficient of } x^{-1}\ \dfrac{n(1+x)^{2n-1}}{x^n}$

$= \text{coefficient of } x^{n-1} \text{ in } n(1+x)^{2n-1}$

$$= n.\,{}^{2n-1}C_{n-1} = n.\frac{(2n-1)!}{(n-1)!n!} = \frac{(2n-1)!}{(n-1)(n-1)!}$$

Method-2

Ans. $\Sigma r(C_r)^2 = \sum_{r=1}^{n} r\,{}^nC_r.{}^nC_r$

$= \sum r\frac{n}{r}\,{}^{n-1}C_{r-1}\,{}^nC_r$

$= n\sum {}^{n-1}C_{r-1}\,{}^nC_r$

$= n\sum_{r=1}^{n} {}^{n-1}C_{n-r}\,{}^nC_r$

${}^{n-1}C_{n-1}\,{}^nC_1 + {}^{n-1}C_{n-2}\,{}^nC_2 + \ldots$

(n-1) boys
n girls
→ Select n students

$= n.{}^{2n-1}C_n$

$$= n.\frac{(2n-1)!}{(n-1)!n!} = \frac{(2n-1)!}{(n-1)!(n-1)!}$$

Ques. Suppose $\sum_{r=0}^{2023} r^2\,{}^{2023}C_r = 2023\times\alpha\times 2^{2022}$. Then the value of α is ....

[JEE Main Jan. 2023 Shift I]

Ans. $\sum r^2\frac{2023}{r}\,{}^{2022}C_{r-1} = 2023\sum(r-1+1)\,{}^{2022}C_{r-1}$

$$= 2023\left\{\sum_{r=1}^{2023}(r-1)\,{}^{2022}C_{r-1} + \sum_{r=1}^{2023}{}^{2022}C_{r-1}\right\}$$

$$= 2023\left\{\sum_{r=2}^{2023}\frac{2022}{r-1}(r-1)\,{}^{2021}C_{r-2} + 2^{2022}\right\}$$

$$= 2023\left[2022\sum_{r=2}^{2023}{}^{2021}C_{r-2} + 2^{2022}\right]$$

$$= 2023\left[2022\times 2^{2021} + 2^{2022}\right]$$

$$= 2023\left[1011\times 2^{2022} + 2^{2022}\right]$$

$$= 2023.2^{2022}\left[1011+1\right] = 2023.2^{2022}(1012)$$

Hence, $\alpha = 1012$

Ques. If $\dfrac{1}{n+1}\,{}^nC_n + \dfrac{1}{n}\,{}^nC_{n-1} + \ldots + \dfrac{1}{2}\,{}^nC_1 + {}^nC_0 = \dfrac{1023}{10}$ then n is equal to

[JEE Main April 2023 Shift I]

(a) 9 (b) 8 (c) 7 (d) 6

Ans. $\sum_{r=0}^{n}\dfrac{{}^nC_r}{r+1}$

$$= \sum_{r=0}^{n}\frac{{}^{n+1}C_{r+1}}{n+1} = \frac{1}{n+1}\left(\sum_{r=0}^{n}{}^{n+1}C_{r+1}\right)$$

$$= \frac{1}{n+1}\left[{}^{n+1}C_1 + {}^{n+1}C_2 + \ldots + {}^{n+1}C_{n+1}\right]$$

$$= \frac{1}{n+1}\left(2^{n+1}-1\right) = \left(\frac{2^{10}-1}{10}\right) \Rightarrow n+1 = 10$$

Ques. The value of $\binom{50}{0}\binom{50}{1}+\binom{50}{1}\binom{50}{2}+....+\binom{50}{49}\binom{50}{50}$ is, where ${}^nC_r=\binom{n}{r}$

(a) $\binom{100}{50}$ (b) $\binom{100}{51}$ (c) $\binom{50}{25}$ (d) $\binom{50}{25}^2$

Ans. ${}^{50}C_0 \cdot {}^{50}C_1 + {}^{50}C_1 \cdot {}^{50}C_2 + ... + {}^{50}C_{49} \cdot {}^{50}C_{50}$

$= \sum_{r=0}^{49} {}^{50}C_r\ {}^{50}C_{r+1}$

$= \sum_{r=0}^{49} {}^{50}C_r\ {}^{50}C_{49-r}$

50B, 50G → 49 students

$= {}^{100}C_{51}$ or ${}^{100}C_{49}$

Binomial Series Type-5

When binomial Coefficients occur with fixed gaps

Examples:

$C_0+C_3+C_6+C_9+...$
$C_1+C_4+C_7+C_{10}+...$ → Gap of 3 (use cube root of unity)

$C_0+C_4+C_8+C_{12}+...$
$C_1+C_5+C_9+C_{13}+...$ → Gap of 4 (use 4^{th} root of unity (i))

$$1+i=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$$

Ques. Find the value of

(i) $C_1+C_5+C_9$ .....

(ii) $C_0+C_4+C_8$ .....

Ans. $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + ... + C_nx^n$

Put $x = i$

$(1+i)^n = C_0 + C_1 i + C_2 i^2 + C_3 i^3 + C_4 i^4 + ... + C_n i^n$

$\Rightarrow (1+i)^n = C_0 + C_1 i - C_2 - iC_3 + C_4 + ...$

$\Rightarrow (1+i)^n = (C_0-C_2+C_4-C_6...) + i(C_1-C_3+C_5-C_7...)$

$\Rightarrow \left[\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]^n$

$= (C_0-C_2+C_4-C_6 .....) + i(C_1-C_3+C_5-C_7 .....)$

$\Rightarrow \left(\sqrt{2}\right)^n\left[\cos\frac{n\pi}{4}+i\sin\frac{n\pi}{4}\right]$

$= (C_0-C_2+C_4-C_6 .....) + i(C_1-C_3+C_5-C_7 .....)$

(i) $\left(\sqrt{2}\right)^n\sin\frac{n\pi}{4} = C_1-C_3+C_5-C_7+C_9 ......$

$\oplus$ $2^{n-1} = C_1+C_3+C_5+C_7+C_9 ......$

$\frac{\left(\sqrt{2}\right)^n\sin\left(\frac{n\pi}{4}\right)+2^{n-1}}{2} = C_1+C_5+C_9+ .....$

$\left(\sqrt{2}\right)^n\cos\frac{n\pi}{4} = C_0-C_2+C_4-C_6 ......$

(ii) $\oplus$ $2^{n-1} = C_0+C_2+C_4+C_6 ......$

$\frac{\left(\sqrt{2}\right)^n\cos\frac{n\pi}{4}+2^{n-1}}{2} = [C_0+C_4+C_8+C_{12}+ ....]$

$\Rightarrow \frac{2^{\frac{n}{2}}\cos n\frac{\pi}{4}+2^{n-1}}{2} = [C_0+C_4+C_8+C_{12}+ ....]$

Ques. Find the value of $C_0+C_3+C_6+$ ....

Ans. $(1+x)^n = C_0+C_1x+C_2+C_3x^2+C_3x^3+ .... C_nx^n$

Put $x = 1$, $2^n = C_0+C_1+C_2+C_3+C_4+ .... C_n$

Put $x = w$, $(1+w)^n = C_0+C_1w+C_2w^2+C_3w^3+C_4w^4+ .... C_nw^n$

Put $x = w^2$ $(1+w^2)^n = C_0+C_1w^2+C_2w^4+C_3w^6+C_4w^8+ .... +C_nw^{2n}$

After adding these 3 equations we get,

$2^n+(1+w)^n+(1+w^2)^n = 3[C_0+C_3+C_6+C_9+ ....]$

$\Rightarrow 2^n + 2\,\mathrm{Re}\,(1+w)^n = 3[C_0+C_3+C_6+C_9+ .....]$

$\Rightarrow \frac{2^n+2\cos\frac{n\pi}{3}}{2} = [C_0+C_3+C_6+C_9+ .....]$

BINOMIAL THEOREM FOR RATIONAL INDEX

- $(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + ... \infty$

This expression is valid only if $\left|\frac{y}{x}\right|<1$ & $n \in Q$

- $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ... \infty$

This expression is valid only if $|x| < 1$

- If x is very small, we can ignore higher power of x

$\Rightarrow (1+x)^n \approx 1+nx$

Ques. If $|x|<1$, then $(1+x)^{-2}$ =

(a) $1-2x+3x^2-...$ ✓

(b) $1+2x+3x^2+...$

(c) $1-\frac{2}{x}+\frac{3}{x^2}-...$

(d) $\frac{1}{x^2}-\frac{2}{x^3}+\frac{3}{x^4}-...$

Ans. $(1+x)^n = 1+nx+\frac{(n-1)}{2}x^2+\frac{n(n-1)(n-1)}{3!}x^3+---\infty$

n = - 2

$\Rightarrow (1+x)^{-2} = 1-2x+\frac{-2(-3)}{2}x^2 - ....$

$= 1-2x+3x^2 - \frac{2(-3)(-4)}{3!}x^3 + ...$

$= 1-2x+3x^2-4x^3+5x^4....$

SOME IMPORTANT EXPANSIONS

- $(1+x)^{-1} = 1-x+x^2-x^3+x^4-........\infty$
- $(1-x)^{-1} = 1+x+x^2+x^3+x^4+........\infty$
- $(1+x)^{-2} = 1-2x+3x^2-4x^3+........\infty$
- $(1-x)^{-2} = 1+2x+3x^2+4x^3+.......\infty$
- $(1+x)^{-3} = 1-3x+6x^2-10x^3+........\infty$
- $(1-x)^{-3} = 1+3x+6x^2+10x^3+........\infty$

Ques. The coefficient of x^3 in the expansion of $\frac{(1+3x)^2}{1-2x}$ will be

(a) 8 (b) 32

(c) 50 ✓ (d) None of these

Ans. $(1+3x)^2 (1-2x)^{-1}$

$= (1+9x^2+6x)\left(1+2x+(2x)^2+(2x)^3+.....\infty\right)$

coefficient of $x^3 \Rightarrow 8+18+24 = 50$

Ques. If x is so small that x^2 & higher powers can be neglected and $\frac{\left(1+\frac{2x}{5}\right)^{-5}+\sqrt{4+2x}}{(4+x)^{\frac{3}{2}}} = a+bx.$

Find (a + b)

Ans. $\frac{\left[1+\frac{2x}{5}(-5)\right]+(4+2x)^{\frac{1}{2}}}{(4+x)^{\frac{3}{2}}}$ $(\because (1+x)^n \approx 1+nx)$

$= \frac{(1-2x)+2\left(1+\frac{2x}{4}\right)^{\frac{1}{2}}}{4^{\frac{3}{2}}\left[1+\frac{x}{4}\right]^{\frac{3}{2}}}$

$= \frac{(1-2x)+2\left[1+\frac{2x}{4}\times\frac{1}{2}\right]}{2^3\left[1+\frac{x}{4}.\frac{3}{2}\right]}$

$= \frac{(1-2x)+2\left(1+\frac{x}{4}\right)}{8\left[1+\frac{3x}{8}\right]} = \frac{3-2x+\frac{x}{2}}{8\left[1+\frac{3x}{8}\right]} = \frac{3-\frac{3x}{2}}{8\left[1+\frac{3x}{8}\right]}$

$= \frac{3\left(1-\frac{x}{2}\right)}{8}\left(1+\frac{3x}{8}\right)^{-1}$

$= \frac{3}{8}\left(1-\frac{x}{2}\right)\left(1-\frac{3x}{8}\right)$

$= \frac{3}{8}\left[1-\frac{3x}{8}-\frac{x}{2}+0\right]$

$= \frac{3}{8}\left[1-\frac{3x}{8}-\frac{4x}{8}\right]$

$= \frac{3}{8}\left[1-\frac{7x}{8}\right] = a+bx$

$\therefore a = \frac{3}{8}, b = \frac{-21}{64}$

Hence, $a+b = \frac{3}{8}-\frac{21}{64}$

$= \frac{3}{64}$

Ques. If x is so small that x^3 and higher powers of x may be neglected, then $\frac{(1+x)^{\frac{3}{2}}-\left(1+\frac{1}{2}x\right)^3}{(1-x)^{\frac{1}{2}}}$ may be approximated as

(a) $-\frac{3}{8}x^2$ (b) $\frac{x}{2}-\frac{3}{8}x^2$

(c) $1-\frac{3}{8}x^2$ (d) $3x+\frac{3}{8}x^2$

Ans. $$= \frac{\left[\cancel{1}+\cancel{\frac{3x}{2}}+\frac{\frac{3}{2}\left(\frac{3}{2}-1\right)}{2}x^2\right]-\left[\cancel{1}+\cancel{\frac{3x}{2}}+\frac{3(3-1)}{2}\left(\frac{x}{2}\right)^2\right]}{1-\frac{x}{2}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2}(x)^2}$$

$$=\frac{\frac{3}{8}x^2-\frac{3x^2}{4}}{1-\frac{x}{2}-\frac{1}{8}x^2}=-\frac{3x^2}{8}\left[1-\frac{x}{2}-\frac{x^2}{8}\right]^{-1}$$

$$=\frac{-3x^2}{8}\left(1-\frac{x}{2}\right)^{-1}=\frac{-3x^2}{8}\left[1+\frac{x}{2}\right]=\frac{-3x^2}{8}$$

Ques. Simplify: $S=1-\frac{1}{2}\cdot\frac{1}{2}+\frac{1.3}{2.4.2^2}-\frac{1.3.5}{2.4.6}\cdot\frac{1}{2^3}+.....\infty$

Ans. $(1+x)^n=1+nx+\frac{n(n-1)}{2}x^2+\frac{n(n-1)(n-2)}{3!}x^3+ .... \infty$

Compare RHS

$nx=\frac{-1}{4}\Rightarrow x=\frac{-1}{4n}$ ...(i)

$\frac{n(n-1)x^2}{2}=\frac{3}{32}$

$\Rightarrow \frac{n(n-1)}{2}\frac{1}{16n^2}=\frac{3}{32}$

$\Rightarrow \frac{n-1}{n}=3\Rightarrow n-1=3n$

$\Rightarrow 2n=-1\Rightarrow n=-\frac{1}{2}$

By eqn (i)

$\left(-\frac{1}{2}\right)x=-\frac{1}{4}\Rightarrow x=\frac{1}{2}$

Compare LHS

$s=(1+x)^n$

$=\left(1+\frac{1}{2}\right)^{-\frac{1}{2}}$

$=\left(\frac{3}{2}\right)^{-\frac{1}{2}}=\left(\frac{2}{3}\right)^{\frac{1}{2}}$

$=\sqrt{\frac{2}{3}}$

Ques. $1-\frac{1}{8}+\frac{1}{8}\cdot\frac{3}{16}-\frac{1.3.5}{8.16.24}+....=S$

(a) $\frac{2}{5}$ (b) $\frac{\sqrt{2}}{5}$

(c) $\frac{2}{\sqrt{5}}$ (d) None of these

Ans. $(1+x)^n=1+nx+\frac{n(n-1)}{2}x^2+....$

Compare RHS

$nx=\frac{-1}{8}\Rightarrow x=\frac{-1}{8n}$ ...(i)

$\frac{3}{8.16}=\frac{n(n-1)}{2}x^2$

$\Rightarrow \frac{3}{8.16}=\frac{n(n-1)}{2}\cdot\left(\frac{-1}{8n}\right)^2$

$\Rightarrow \frac{3}{8\times16}=\frac{n(n-1)}{2}\cdot\frac{1}{64n^2}$

$\Rightarrow 3=\frac{n-1}{n}$

$\Rightarrow 3n=n-1$

$\Rightarrow 2n=-1$

$\Rightarrow n=-\frac{1}{2}$

By eqn (i)

$\left(\frac{-1}{2}\right)x=\frac{-1}{8}\Rightarrow x=\frac{1}{4}$

Compare LHS

$S=(1+x)^n$

$=\left(1+\frac{1}{4}\right)^{-\frac{1}{2}}$

$=\left(\frac{5}{4}\right)^{-\frac{1}{2}}$

$=\left(\frac{4}{5}\right)^{1/2}$

$=\sqrt{\frac{4}{5}}=\frac{2}{\sqrt{5}}$

Ques. If $x=\frac{1}{3}+\frac{1.3}{3.6}+\frac{1.3.5}{3.6.9}+.........\infty$, then find the value of x^2+2x

Ans. Add 1 both sides

$(1+x)=1+\frac{1}{3}+\frac{1.3}{3.6}+\frac{1.3.5}{3.6.9}+......\infty$

$(1+y)^n=1+ny+\frac{n(n-1)}{2}y^2+......\infty$

$\frac{1}{3}=ny\Rightarrow y=\frac{1}{3n}$ ...(i)

$\frac{1}{6}=\frac{n(n-1)}{2}y^2$

$\Rightarrow \frac{1}{6}=\frac{n(n-1)}{2}\cdot\frac{1}{9n^2}\Rightarrow 1=\frac{(n-1)}{3n}$

$\Rightarrow 3n=n-1\Rightarrow n=-\frac{1}{2}$

By eq^n (i)

$\frac{1}{3}=\left(\frac{-1}{2}\right)y\Rightarrow y=-\frac{2}{3}$

$\because 1+x=(1+y)^n$

$\Rightarrow 1+x=\left(1-\frac{2}{3}\right)^{-\frac{1}{2}}=\left(\frac{1}{3}\right)^{-\frac{1}{2}}=\sqrt{3}$

$\Rightarrow (1+x)^2=3 \Rightarrow x^2+2x+1=3$

$\Rightarrow x^2+2x=3-1=2$

FINDING NUMBER OF TERMS IN EXPANSION

$(x+y)^n \rightarrow$ no. of terms $=(n+1)$

$(x+y)^n={}^nC_0x^n+{}^nC_1x^{n-1}y^1+{}^nC_2x^{n-2}y^2+\ldots{}^nC_nx^0y^n$

$(x+y+z)^n \rightarrow$

$(x+y+z)^n=()\,x^0y^0z^n+()\,x^1y^0z^{n-1}+()\,x^0y^1z^{n-1}+()\,x^1y^1z^{n-2}+\ldots..$

no. of terms in $(x+y+z)^n$ is equal to no of ways in which n identical things can be distributed to 3 beggars.

No. of term in $(x+y+z)^n={}^{n+2}C_2$

$(a+b+c+d)^n \Rightarrow$ no of terms $={}^{n+3}C_3$

Important Note

- Number of terms in the expansion of $(x+y+z)^n={}^{n+2}C_2$
- The number of distinct terms in the expansion of $(x_1+x_2+\ldots+x_r)^n$ are ${}^{n+r-1}C_{r-1}$
- Sum of all the coefficients is obtained by substituting all the variables terms as unity i.e. 1.

TRINOMIAL THEOREM (FOR NATURAL INDEX)

Trinomial Expansion

$$(x+y+z)^n=\sum\frac{n!}{\alpha_1!\alpha_2!\alpha_3!}(x)^{\alpha_1}(y)^{\alpha_2}(z)^{\alpha_3}$$

where $\alpha_1, \alpha_2, \alpha_3$ are non negative integers such that $\alpha_1+\alpha_2+\alpha_3=n$

Examples:

$$(x+y+z)^{10}=\sum\frac{10!}{\alpha_1!\alpha_2!\alpha_3!}(x)^{\alpha_1}(y)^{\alpha_2}(z)^{\alpha_3}$$

where, $\alpha_1+\alpha_2+\alpha_3=10$

Ques. Find the coefficient of $a^3b^4c^5$ in the expansion of $(bc+ca+ab)^6$

Ans. $(bc+ca+ab)^6=\sum\frac{6!}{\alpha_1!\alpha_2!\alpha_3!}(bc)^{\alpha_1}.(ca)^{\alpha_2}.(ab)^{\alpha_3}$

$$=\sum\underbrace{\frac{6!}{\alpha_1!\alpha_2!\alpha_3!}}_{\text{Coefficient}}(b)^{\alpha_1+\alpha_3}(c)^{\alpha_1+\alpha_2}(a)^{\alpha_2+\alpha_3}$$

$\alpha_1+\alpha_2+\alpha_3=6$

$\alpha_1+\alpha_3=4 \Rightarrow 6-\alpha_2=4 \Rightarrow \alpha_2=2$

$\alpha_1+\alpha_2=5 \Rightarrow 6-\alpha_3=5 \Rightarrow \alpha_3=1$

$\alpha_2+\alpha_3=3 \Rightarrow 6-\alpha_1=3 \Rightarrow \alpha_1=3$

Coefficient $=\frac{6!}{2!.3!.1!}=\frac{6\times5\times\cancel{4}^{2}}{\cancel{2}}=60$

Ques. Find coefficient of $x^8y^8z^4$ in $(xy+yz+zx)^{10}$.

Ans. $(xy+yz+zx)=\frac{10!}{\alpha_1!\alpha_2!\alpha_3!}(xy)^{\alpha_1}(yz)^{\alpha_2}(zx)^{\alpha_3}$

$\alpha_1+\alpha_2+\alpha_3=10$

$$=\frac{10!}{\alpha_1!\alpha_2!\alpha_3!}x^{\alpha_1+\alpha_3}y^{\alpha_1+\alpha_2}z^{\alpha_1+\alpha_3}$$

$\alpha_1+\alpha_3=8 \Rightarrow \alpha_2=2$

$\alpha_1+\alpha_2=8 \Rightarrow \alpha_3=2$

$\alpha_2+\alpha_3=4 \Rightarrow \alpha_1=6$

coefficient $=\frac{10!}{6!2!2!}=1260$

Ques. Find the coefficient of x^3 in the expansion of $(2-x+3x^2)^5$

M-1 by T.T.

Ans. $\sum\frac{5!}{\alpha_1!\alpha_2!\alpha_3!}(2)^{\alpha_1}(-x)^{\alpha_2}(3x^2)^{\alpha_3}$

$$=\frac{5!}{\alpha_1!\alpha_2!\alpha_3!}(2)^{\alpha_1}(-1)^{\alpha_2}(3)^{\alpha_3}(x)^{\alpha_2+2\alpha_3}$$

α_1	α_2	α_3
2	3	0
3	1	1
	-1	2
	?	

$\boxed{\alpha_1+\alpha_2+\alpha_3=5}$

$\boxed{\alpha_2+2\alpha_3=3}$

Coeff of $x^3 = \frac{5!}{2!3!0!}(2)^2(-1)^3(3)^0 + \frac{5!}{3!1!1!}$
$(2)^2(-1)^1(3)^1$
$= -10 \times 4 - 20 \times 8 \times 3$
$= -40 - 480 = -520$

M-2 by B.T.

$T_{r+1} = {}^5C_r(2)^{5-r}(-x+3x^2)^r$

$r = 1 \quad {}^5C_1(2)^4(-x+3x^2)^1$

$r = 2 \quad {}^5C_2(2)^3(-x+3x^2)^2$

$r = 3 \quad {}^5C_3(2)^2(-x+3x^2)^3$

$r = 4 \quad {}^5C_2(2)^1(-x+3x^2)^4$

Coefficient of x^3 in

${}^5C_2(2)^3(-x+3x^2)^2 + {}^5C_3(2)^2(-x+3x^2)^3$

$= {}^5C_2(2)^3x^2(-1+3x)^2 + {}^5C_3 2^2 x^3(-1+3x)^3$

$= 80$ coeff of x^1 in $\underbrace{(-1+3x)^2}_{-6} + 40$ coeff of x^0 in $\underbrace{(-1+3x)^3}_{-1}$

$= 80(-6) + 40(-1)$

$= -480 - 40 = -520$

MULTINOMIAL THEOREM (FOR NATURAL TNDEX)

The formula by which any power of an algebraic expression consisting of more than two terms separated by an operator {+, -} is known as a Multinomial.

$$(x_1 + x_2 + x_3 + \ldots\ldots x_r)^n = \sum \frac{n!}{(\alpha_1!)(\alpha_2!)(\alpha_3!)\ldots\ldots(\alpha_r!)} x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}\ldots\ldots x_r^{\alpha_r}$$

where $\alpha_1, \alpha_2, \alpha_3$ are all non-negative integers and subject to a condition that $\alpha_1 + \alpha_2 + \alpha_3 + \ldots\ldots \alpha_r = n$

Important Note

The coefficient of x^r in the expansion of $(1-x)^{-n}$, where $n \in N$ is ${}^{n+r-1}C_r$

Ques. (i) Find coefficient of x^{17} in $(1-x)^{-2}$

$r = 17 \quad n = 2$

Ans. Coefficient: ${}^{n+r-1}C_r$

$= {}^{2+17-1}C_{17} = {}^{18}C_{17} = {}^{18}C_1 = 18$

(ii) Find coefficient of x^5 in $(1-x)^{-6}$.

$r = 5 \quad n = 6$

Ans. Coefficient: ${}^{n+r-1}C_r = {}^{6+5-1}C_5 = {}^{10}C_5$

Ques. If $|x| < 1$ then the coefficient of x^n in the expansion of $(1 + x + x^2 + \ldots\infty)^2$ will be [Pb.CET 2023]

(a) 1
(b) n
(c) n + 1 ✓
(d) None of these

Ans. $\left(\frac{1}{1-x}\right)^2 = \frac{1}{(1-x)^2} = (1-x)^{-2}$

$n = 2, r \to n$

Coefficient ${}^{n+n-1}C_r$

$= {}^{2+n-1}C_n = {}^{n+1}C_n$

$= {}^{n+1}C_1 = n + 1$

P&C QUESTIONS USING MULTINOMIAL THEOREM

Ques. There are infinite number of identical red, white, green & blue balls available.

(a) select 10 balls;

Ans. **M-1**

$x_1 + x_2 + x_3 + x_4 = 10 \Rightarrow {}^{13}C_3$

M-2 by B.T.

Coeff of x^{10} in $(x^0 + x^1 + x^2 + x^3 \ldots..)(x^0 + x^1 + x^2 + \ldots..)(x^0 + x^1 + x^2 + x^3 \ldots..)(x^0 + x^1 + x^2 + \ldots..)$

Power of x in I[st] GP represent no. of Red balls

Power of x in 2[nd] GP represent no. of white balls

Power of x in 3[rd] GP represent no. of Green balls

Power of x in 4[th] GP represent no. of Blue balls

coeff of x^{10} in $\left(\frac{1}{1-x}\right)^4 = (1-x)^{-4}$

$r = 10, n = 4$

${}^{n+r-1}C_r = {}^{4+10-1}C_{10} = {}^{13}C_{10} = {}^{13}C_3$

(b) select 10 balls such that at least 1 ball of each colour is there.

Ans. M-1

$x_1 + x_2 + x_3 + x_4 = 10 \Rightarrow {}^9C_3$

M-2 by B.T

Coeff of x^{10} in $(x^1 + x^2 + x^3 \ldots\ldots)(x^1 + x^2 + \ldots\ldots)$
$(x^1 + x^2 \ldots\ldots)(x^1 + x^2 + \ldots\ldots)$

Power of x in I^{st} GP represent no. of Red balls

Power of x in 2^{nd} GP represent no. of white balls

Power of x in 3^{rd} GP represent no. of Green balls

Power of x in 4^{th} GP represent no. of Blue balls

Coeff of x^{10} in $\left(\frac{x}{1-x}\right)^4 = x^4(1-x)^{-4}$

Coeff of x^6 in $(1-x)^{-4}$, $n = 4, r = 6$

$${}^{n+r-1}C_r = {}^9C_6 = {}^9C_3$$

Ques. Find the number of integral solutions for the system $x_1 + x_2 + x_3 = 10$, $0 \le x_1 \le 6$, $0 \le x_2 \le 7$, $0 \le x_3 \le 8$.

Ans. Coeff of x^{10} in

$(x^0 + x^1 + \ldots\ldots + x^6)(x^0 + x^1 + \ldots\ldots + x^7)(x^0 + x^1 + \ldots\ldots + x^8)$

$= \left(\frac{1-x^7}{1-x}\right)\left(\frac{1-x^8}{1-x}\right)\left(\frac{1-x^9}{1-x}\right)$

$= (1 - x^7)(1 - x^8)(1 - x^9)(1 - x)^{-3}$

Coeff of x^{10} in $(1 - x^7 - x^8 + x^{15})(1 - x^9)(1 - x)^{-3}$

$= (1 - x^7 - x^8)(1 - x^9)(1 - x)^{-3}$

$\Rightarrow$ Coeff of x^{10} in $(1 - x^7 - x^8 - x^9)(1 - x)^{-3}$

$$= \underbrace{(1-x)^{-3}}_{\boxed{r = 10}\ \boxed{n = 3}} - x^7 \underbrace{(1-x)^{-3}}_{\boxed{r = 3}\ \boxed{n = 3}} - x^8 \underbrace{(1-x)^{-3}}_{\boxed{r = 2}\ \boxed{n = 3}} - x^9 \underbrace{(1-x)^{-3}}_{\boxed{r = 1}\ \boxed{n = 3}}$$

$= {}^{3+10-1}C_{10} - {}^{3+3-1}C_3 - {}^{3+2-1}C_2 - {}^{3+1-1}C_1$

$= {}^{12}C_{10} - {}^5C_3 - {}^4C_2 - {}^3C_1$

$= \frac{12 \times 11}{2} - 10 - 6 - 3$

$= 66 - 16 - 3 = 50 - 3 = 47$

Short Notes

BINOMIAL THEOREM FOR NATURAL INDEX

- $(a+x)^n = {}^nC_0a^n + {}^nC_1a^{n-1}x + {}^nC_2a^{n-2}x^2 + \ldots + {}^nC_r\,a^{n-r}x^r + \ldots + {}^nC_nx^n$
- $(a-x)^n = {}^nC_0a^n - {}^nC_1a^{n-1}x + {}^nC_2a^{n-2}x^2 - \ldots + (-1)^r\,{}^nC_r\,a^{n-r}x^r + \ldots + (-1)^n\,{}^nC_nx^n$
- $(a+x)^n + (a-x)^n = \sum\limits_{r=0}^{n}{}^nC_rx^r\left[1+(-1)^r\right] = 2({}^nC_0\,a^n + {}^nC_2\,a^{n-2}x^2 \ldots)$
- $(a+x)^n - (a-x)^n = \sum\limits_{r=0}^{n}{}^nC_rx^r\left[1-(-1)^r\right] = 2({}^nC_1\,a^{n-1}x + {}^nC_3\,a^{n-3}x^3 \ \ldots\ldots)$
- $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_rx^r + \ldots + {}^nC_nx^n = \sum\limits_{r=0}^{n}{}^nC_rx^r$
- $(1-x)^n = {}^nC_0 - {}^nC_1x + {}^nC_2x^2 + \ldots + (-1)^r\,{}^nC_rx^r + \ldots + (-1)^n\,{}^nC_nx^n = \sum\limits_{r=0}^{n}(-1)^r\,{}^nC_rx^r$
- Number of terms in the expansion of $(x+y)^n$ is $(n+1)$ i.e. one more than the index n.
- Binomial Expansion is a Homogeneous expansion i.e. the sum of indices of x and y for each and every term is n.
- The binomial coefficients of the terms $({}^nC_0, {}^nC_1 \ldots)$ equidistant from the beginning and the end are equal. i.e. ${}^nC_r = {}^nC_{r-1}$
- The r^{th} term from end is $(T_{n-r+2})^{th}$ term from beginning.
- r^{th} term from end in $(x+y)^n$ is same as r^{th} term from beginning in $(y+x)^n$

GENERAL TERM

- The general term or the $(r+1)^{th}$ term in the expansion of $(x+y)^n$ is given by $T_{r+1} = {}^nC_rx^{n-r}y^r$
- General term for the expansion of $(1+x)^n$ is given by $T_{r+1} = {}^nC_rx^r$

MIDDLE TERM

- If n is even, there is only one middle term which is given by $T_{[(n)/2]+1} = {}^nC_{n/2} \cdot x^{n/2} \cdot y^{n/2}$
- If n is odd, there are two middle terms which are $T_{(n+1)/2}$ and $T_{[(n+1)/2]+1}$

In Greatest Integer & Fractional Part Questions

- If we add I + f & f' to cancel out irrational terms then always I + f + f ' = even & f + f ' = 1
- If we subtract I + f & f' to cancel out the irrational terms then always I + f - f ' = even & f - f ' = 0

BASIC PROPERTIES OF BINOMIAL EXPANSION OF $(a+x)^n$

For convenience, the coefficients ${}^nC_0, {}^nC_1, \ldots, {}^nC_r, \ldots, {}^nC_n$ are usually denoted by $C_0, C_1, \ldots, C_r, \ldots, C_n$ respectively

- $C_0 + C_1 + C_2 + \ldots.. + C_n = 2^n$
- $C_0 - C_1 + C_2 - \ldots.. + C_n = 0$
- $C_0 + C_2 + C_4 + \ldots.. = C_1 + C_3 + C_5 + \ldots.. = 2^{n-1}$
- ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$
- $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n-r+1}{r}$
- $r\,{}^nC_r = n\,{}^{n-1}C_{r-1}$
- $\dfrac{{}^nC_r}{r+1} = \dfrac{{}^{n+1}C_{r+1}}{n+1}$.
- There are (n+1) terms in expansion.
- Binomial coefficients of terms equidistant from beginning and end are equal.
- The general term of the expansion is ${}^nC_r\,a^{n-r}x^r$, this is in fact the $(r+1)^{th}$ term from the beginning.

- If n is even, there is only one middle term namely $\left(\frac{n}{2}+1\right)^{th}$ and is equal to ${}^{n}C_{\frac{n}{2}}a^{\frac{n}{2}}.x^{\frac{n}{2}}$.
- If n is odd, there are two middle terms namely $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$ terms respectively.
- Binomial coefficient of middle term is the greatest binomial coefficient occurring in the expansion.
- Ratio of two consecutive terms in the expansion of $(x+a)^n$ is $=\left(\frac{n-r+1}{r}\right)\left|\frac{a}{x}\right|=\left(\frac{n+1}{r}-1\right)\left|\frac{a}{x}\right|$

BINOMIAL THEOREM FOR ANY NEGATIVE OR RATIONAL INDEX

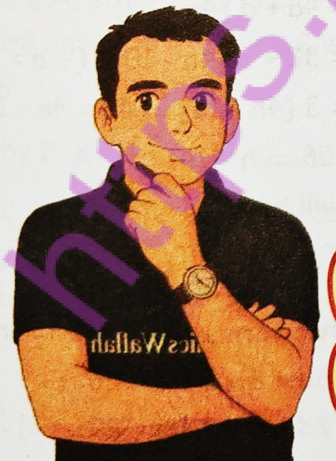
- $$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\ldots+\frac{n(n-1)(n-2)\ldots(n-(r-1))}{r!}x^r+\ldots\infty$$

  The above expansion is valid if and only if $|x| < 1$.
- If n is a negative integer or a fraction, then ${}^{n}C_r$ will be written as $\frac{(n)(n-1)\ldots(n-(r-1))}{r!}$
- If n is a negative integer or a fraction, then the number of terms in the series are infinite. Hence, the series never terminates.
- If first term is not 1 (unity), then make first term unity in the following way $(x+y)^n=x^n\left(1+\frac{y}{x}\right)^n$ in such a way that $\left|\frac{y}{x}\right|<1$.
- If x is so small that square and higher powers of x can be neglected, then $(1+x)^n=1+nx$
- $(1+x)^{-1}=1-x+x^2-x^3+x^4-\ldots.\infty$
- $(1-x)^{-1}=1+x+x^2+x^3+x^4+\ldots.\infty$
- $(1+x)^{-2}=1-2x+3x^2-4x^3+\ldots.\infty$
- $(1-x)^{-2}=1+2x+3x^2+4x^3+\ldots.\infty$
- $(1+x)^{-3}=1-3x+6x^2-10x^3+\ldots..\infty$
- $(1-x)^{-3}=1+3x+6x^2+10x^3+\ldots..\infty$

MULTINOMIAL THEOREM FOR NATURAL INDEX

- $$(x_1+x_2+x_3+\ldots+x_r)^n=\sum\frac{n!}{(\alpha_1!)(\alpha_2!)(\alpha_3!)\ldots(\alpha_r!)}x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}\ldots x_r^{\alpha_r}$$

  where, α_1, α_2, α_3, ..., α_r are all non-negative integers and subject to a condition that $\alpha_1+\alpha_2+\alpha_3+\ldots+\alpha_r=n$
- The number of distinct terms in the expansion of $(x_1+x_2+x_3+\ldots+x_r)^n$ are ${}^{n+(r-1)}C_{(r-1)}$.
- Sum of all the coefficients is obtained by substituting all the variables terms equal to unity i.e. 1.



"Small terms add up to powerful results — just like your consistent efforts."