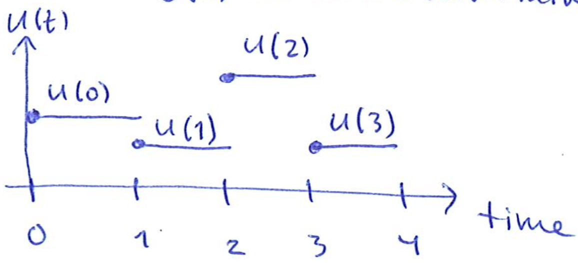
4.2) $Y(s) = G(s)U(s)$ $\qquad G(s) = \dfrac{e^{-\delta s}}{1+\tau s}$

The discrete time t-f $H(z)$ corresponding to $G(s)$ is to be determined

Sample time $h > \delta$

$u(t)$ is constant on each sample interval



a) $u(t) \rightarrow \boxed{e^{-\delta s}} \xrightarrow{v(t)} \boxed{\dfrac{1}{1+\tau s}} \rightarrow y(t)$

Derive a state space model $\quad v(t)$ input $\quad y(t)$ output

$$Y(s) = \frac{1}{1+\tau s} \cdot V(s) \quad \overset{\mathcal{L}^{-1}}{\Rightarrow} \quad y(t) + \tau \dot{y}(t) = v(t)$$

$$\dot{y} = -\frac{1}{\tau} y + \frac{1}{\tau} v \qquad \text{let } x = y$$

$$\therefore \quad A = -\frac{1}{\tau} \quad B = 1/\tau \quad C = 1$$

b) Convolution: $\mathcal{L}\left\{\int_0^t y_1(\sigma)\, y_2(t-\sigma)\, d\sigma\right\} = Y_1(s)\, Y_2(s)$

Describe how $y(t)$ depends on $v(t)$

page 43: Solving system equation

$$x(t) = e^{A(t-t_0)} x(t_0) + \int_{t_0}^{t} e^{A(t-\sigma)} B u(\sigma)\, d\sigma \qquad \text{general}$$

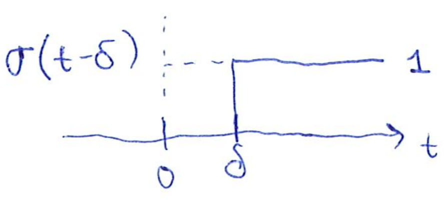
for us: $y(t) = x(t)$ $\quad$ input is not u but v

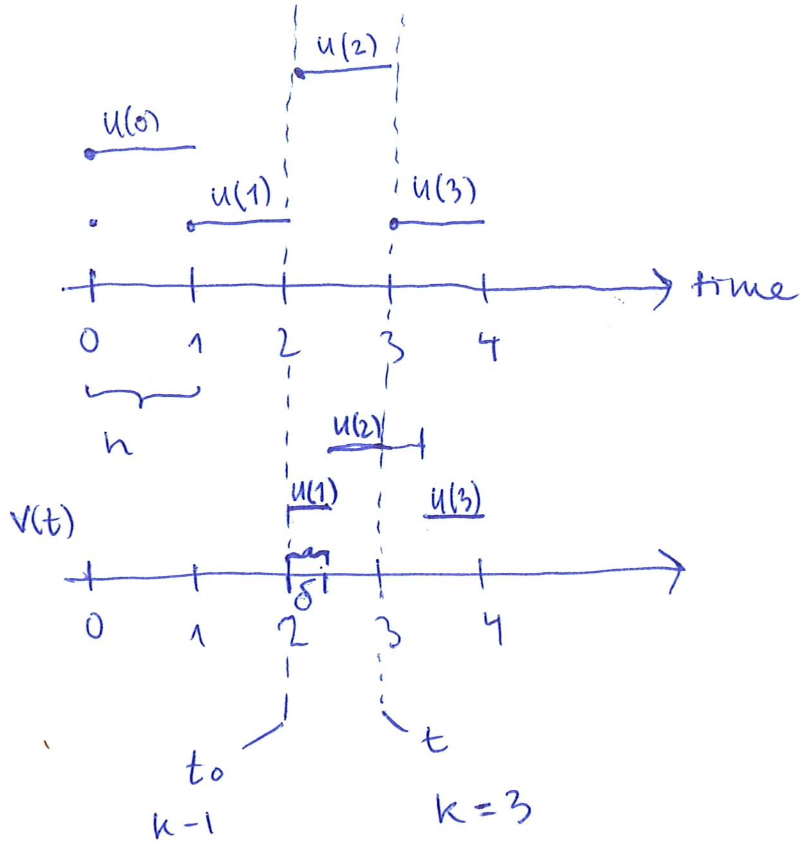
$$\Rightarrow \quad y(t) = e^{A(t-t_0)} x(t_0) + \int_{t_0}^{t} e^{A(t-\sigma)} B v(\sigma)\, d\sigma$$

c) Determine $v(t)$ as a function of $u(t)$

From block diagram: $\quad V(s) = e^{-\delta s} U(s)$

$$\overset{\mathcal{L}^{-1}}{\Rightarrow} \quad v(t) = u(t-\delta) \underbrace{\sigma(t-\delta)}_{=1} \Leftarrow \text{let } u(t) = 0 \text{ if } t < 0$$





at time intervall

$[kh-h, \; kh]$ or short

$[k-1, \; k]$

from t_0 to t:

$t = kh \qquad t - t_0 = h$

$t_0 = kh - h$

$v(t) = u(t-\delta)$

$$v(t) = \begin{cases} u(k-2) & t \in [k-1, k-1+\delta] \\ u(k-1) & t \in [k-1+\delta, k] \end{cases}$$

Insert this into the convolution:

$y(kh) = y(k)$

$y(kh-h) = y(k-1)$

$$y(k) = e^{A\cdot h} y(k-1) + \underbrace{\int_{hk-h}^{hk-h+\delta} e^{A(k-\sigma)} B\cdot u(k-2)\, d\sigma}_{b_2} + \underbrace{\int_{hk-h+\delta}^{kh} e^{A(k-\sigma)} B u(k-1)\, d\sigma}_{b_1}$$

$$b_2 = B \cdot \int_{hk-h}^{hk-h+\delta} e^{Akh - A\sigma} d\sigma = B \left[\frac{1}{-A} e^{A(k^h - \sigma)} \right]_{hk-h}^{h\cdot k - h + \delta} = \frac{B}{-A} \left(e^{A(\cancel{kh} - (\cancel{kh} - h + \delta))} - e^{A(\cancel{kh} - (\cancel{kh} - h))} \right)$$

$$= \frac{B}{-A} \left(e^{A(h-\delta)} - e^{Ah} \right)$$

$$" \frac{1/\tau}{-(-1/\tau)} = 1$$

$$y(k) = a \cdot y(k-1) + b_2 u(k-2) + b_1 u(k-1)$$

Z-transform $\mathcal{Z}\{y(k-\beta)\} = Y(z) \cdot z^{-\beta}$

$$Y(z) = a Y(z) z^{-1} + b_2 U(z) z^{-2} + b_1 U(z) z^{-1}$$

$$Y(z) = \underbrace{\boxed{\frac{b_2 z^{-2} + b_1 z^{-1}}{1 - a z^{-1}}}}_{H(z)} \cdot U(z)$$

4.3) $\dot{x}_1 = x_2$
$\dot{x}_2 = -x_1 + u$

a) show it is controllable.

page 45.
$S = [B \; AB \dots A^{n-1}B]$
controllable iff S has full rank.

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

2 states $\Rightarrow n = 2$

$$AB = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

2 independent rows
rank = 2 = full rank.

b) Corresponding discrete system. u is piecewise constant
Sampling intervall T. Controllable?

page 69. Discrete Time. controllable iff S full rank.

page 85-86. System sampling

$$x((k+1)T) = A_d\, x(kT) + B_d\, u(kT)$$

$$y(kT) = Cx(kT)$$

where: $A_d = e^{AT}$ $\quad B_d = \int_0^T e^{At} B \, dt$

eq. 4.11: $e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$ $\qquad \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos(t)$

$$(sI - A)^{-1} = \begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix}^{-1} = \frac{1}{s^2+1}\begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix} \qquad \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin(t)$$

here $t = T \Rightarrow$

$$A_d = e^{AT} = \begin{bmatrix} \cos(T) & \sin(T) \\ -\sin(T) & \cos(T) \end{bmatrix}$$

$$B_d = \int_0^T e^{At}\begin{bmatrix} 0 \\ 1 \end{bmatrix} dt = \int_0^T \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix} dt = \begin{bmatrix} -\cos(t) \\ \sin(t) \end{bmatrix}_0^T = \begin{bmatrix} 1 - \cos(T) \\ \sin(T) \end{bmatrix}$$

$$A_d B_d = \begin{bmatrix} \cos T & \sin T \\ -\sin T & \cos T \end{bmatrix}\begin{bmatrix} 1 - \cos T \\ \sin T \end{bmatrix} = \begin{bmatrix} \cos(T) - \cos^2(T) + \sin^2(T) \\ \cos(T)\sin(T) - \sin(T) + \cos(T)\sin(T) \end{bmatrix}$$

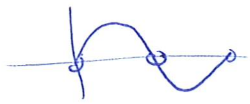
$$S = \begin{bmatrix} B_d & A_d B_d \end{bmatrix} = \begin{bmatrix} 1-\cos(T) & \cos(T)-\cos^2(T)+\sin^2(T) \\ \sin(T) & 2\cos(T)\sin(T) - \sin(T) \end{bmatrix}$$

if $\det(S) = 0 \Rightarrow S$ has <u>not</u> full rank

(for example if $T=0 \Rightarrow S = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ not full rank!)

$$\det(S) = (1-\cos(T))(2\cos(T)\sin(T)-\sin(T)) - (\sin(T))\cdot(\cos(T)-\cos^2(T)+\sin^2(T)) =$$

$$= \dots = 2\cdot\sin(T)(\cos(T)-1) = 0 \quad \text{if } T = n\cdot\pi$$



5.6

Process: $X(k+1) = 0{,}8X(k) + V(k)$

white noise $V(k) \sim N(0,1)$

2 measurements are avaliable:

$y_1 = X(k) + w_1(k) \qquad w_1 \sim N(0,1)$

$y_2 = X(k) + w_2(k) \qquad w_2 \sim N(0,2)$

$w_1\ w_2\ V$ - independent.

page 137-138 Kalman Filter Discrete Time
(continious time - starts at page 124)

General form: $X(k+1) = AX(k) + Bu(t) + NV_1(t)$

$y(k) = CX(k) + Du(t) + V_2(t)$

we have two measurments $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} X(k) + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

$\therefore \quad A = 0{,}8 \quad B = 0 \quad N = 1$

$C = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad V_2 = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

"V_1 & V_2 white noisees with intensities R_1 & R_2, cross spectrum R_{12}"

eq 5.70 $\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$ intensities $\begin{bmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{bmatrix}$ dimensions: $\begin{bmatrix} 1 & [0 \quad 0] \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \end{bmatrix}$

in our case: $V_1 = V, \quad R_1 = 1$

$V_2 = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad R_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ zeros as w_1 & w_2 independent

V_1 & V_2 independent $\Rightarrow R_{12} = [0 \quad 0]$

now we have described our system in the "standard Kalman way"

5.6a The variance of the estimation error $\tilde{P}$

$$\tilde{P} = E\left\{ \left(x(k|k) - \hat{x}(k|k)\right)^2 \right\}$$

estimate of
x given measurments at k
~~[illegible]~~

(if $\tilde{x} = x - \hat{x}$ $\quad E\{\tilde{x}\tilde{x}^T\} = E\left\{ \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_n \end{bmatrix} \begin{bmatrix} \tilde{x}_1 & \tilde{x}_2 & \dots & \tilde{x}_n \end{bmatrix} \right\} =$

$$= E\{\hat{x}_1^2 + \tilde{x}_2^2 + \dots \hat{x}_n^2\} = \hat{P} \text{)}$$

Between eq 5.100 & 5.101 "The error is":

$$E\{\tilde{x}(k|k)\tilde{x}^T(k|k)\} = (I - \tilde{K}C)P(I - \tilde{K}C)^T + \tilde{K}R_2\tilde{K}^T$$

This is equal to $\tilde{P}$

To do:

1) calculate P from eq 5.100
2) calculate $\tilde{K}$ from eq. 5.101
3) calculate $\tilde{P}$ (found between 5.100 & 5.101)

1) $P = APA^T + NR_1N^T - (APC^T + NR_{12})(CPC^T + R_2)^{-1}(APC^T + NR_{12})^T$

size of P?

having n states - A: [n × n]

in our case n = 1

⇒ P is a scalar

$$\underbrace{A \; P \; A^T}_{[n\times n]}, \quad A, P, A^T: [n\times n]$$

(if n = 2 ⇒ $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}$ & you will get 4 coupled equations from 1))

... ⇒ $P = \alpha \pm \beta = \begin{cases} 1{,}2806 \\ -0{,}52 \end{cases}$ but by definition $P \geq 0$!

2) $\tilde{K} = P C^T (C P C^T + R_2)^{-1} = P [a+c \;,\; b+d] = [0{,}438 \;\; 0{,}22] = [k_1 \;\; k_2]$

C^T: $[1 \; 1]$; size 2×2: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

3) $\tilde{P} = (I - \tilde{K} C) P (I - \tilde{K} C)^T + \tilde{K} R_2 \tilde{K}^T =$

scalare case $I = 1$

$\tilde{K}C$: $[k_1 \; k_2] \begin{bmatrix} 1 \\ 1 \end{bmatrix} = k_1 + k_2$

$\tilde{K} R_2 \tilde{K}^T$: $[k_1 \; k_2] \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = [k_1 \;\; 2k_2] \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = k_1^2 + 2k_2^2$

$= \ldots = (1 - k_1 - k_2) P (1 - k_1 - k_2) + k_1^2 + 2k_2^2 = 0{,}4384$

5.6b The Kalman Filter can be described by t.f.
$\hat{X}(z) = G_1(z) Y_1(z) + G_2 Y_2(z)$. Determine G_1 & G_2

eq 5.102 $\Rightarrow$ $\hat{x}(t|t) = \hat{x}(t|t-1) + \tilde{K}(y(t) - C\hat{x}(t|t-1) - \underbrace{Du(t)}_{=0 \text{ for us.}})$

$\hat{x}$ - prediction of x

$= (I - \tilde{K}C)\underbrace{\hat{x}(t|t-1)}_{} + \tilde{K}\, y(t)$

$\hat{x}(t|t-1) = A\hat{x}(t-1|t-1) + \underbrace{Bu(t-1)}_{B=0} + \underbrace{\hat{v}_1(t-1|t-1)}_{=0 \text{ if } R_{12}=0}$

i.e. no correlation between system noise & measurement noise

$\underbrace{\hat{x}(t|t)}_{\hat{x}(t)} = (I - \tilde{K}C) A \underbrace{\hat{x}(t-1|t-1)}_{\hat{x}(t-1)} + \tilde{K} y(t)$ $\qquad \mathcal{Z}(\hat{x}(t-1)) = z^{-1}\hat{X}(z)$

$\hat{x}(z) = \underbrace{(I - \tilde{K}C)A}_{M} z^{-1}\hat{X}(z) + \tilde{K} Y(z)$ $\qquad \hat{X}(z) = \frac{1}{1 - Mz^{-1}} \underbrace{\tilde{K}}_{[k_1 \;\; k_2]} \underbrace{Y(z)}_{\begin{bmatrix} Y_1(z) \\ Y_2(z) \end{bmatrix}} =$

$= \left(\frac{k_1}{1 - Mz^{-1}}\right) Y_1(z) + \frac{k_2}{1 - Mz^{-1}} Y_2(z)$

5.6c

$$x(k+1) = 0{,}8\,x(k) + v(k)$$
$$y_1(k) = x(k) + w_1(k)$$
$$y_2(k) = x(k-2) + w_2(k)$$

y must be a function of the state.
extend the state vector. Previous: one state x(k

let $x_{new}(k) = \begin{bmatrix} x(k) \\ x(k-1) \\ x(k-2) \end{bmatrix}$ $\quad x_{new}(k+1) = \begin{bmatrix} x(k+1) \\ x(k) \\ x(k-1) \end{bmatrix} = \begin{matrix} A\ldots \\ x_1 \\ x_2 \end{matrix}$

new model:

$$x_{new}(k+1) = \underbrace{\begin{bmatrix} 0{,}8 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{A} x_{new}(k) + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{N} v(k)$$

$$y(k) = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{C} x_{new} + \underbrace{\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}}_{V_2}$$

5.6d observable? $\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$ $\quad n = 3 \Rightarrow CA^2$

$$CA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0{,}8 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0{,}8 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad (CA)A = \begin{bmatrix} 0{,}8^2 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

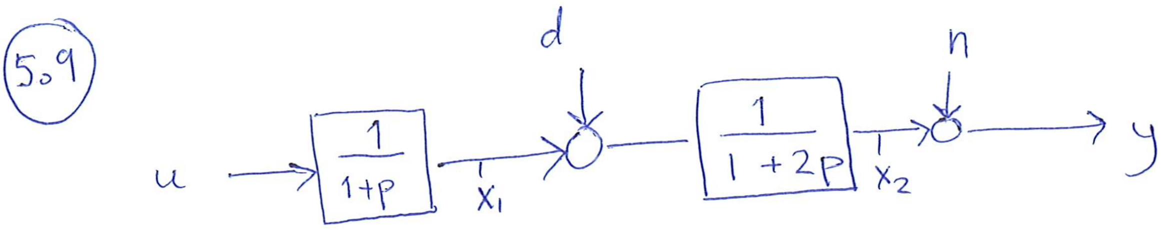
$$\mathcal{O} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0{,}8 & 0 & 0 \\ 0 & 1 & 0 \\ 0{,}8^2 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

size $[6 \times 3] \Rightarrow$ max rank is 3

we find 3 independent rows $\rightarrow$ rank = 3

$\therefore$ system is observable!

5.9



spectrum for the disturbance d: $\Phi_d(w) = \frac{1}{w^2+1}$

a) write system on state space form

$$\begin{cases} \dot{x} = Ax + Bu + Ne \\ y = Cx + n \end{cases}$$

e - white noise

$x_1 = \frac{1}{1+p} \cdot u \Rightarrow x_1 + \dot{x}_1 = u \Rightarrow \dot{x}_1 = -x_1 + u$ ok.

$x_2 = \frac{1}{1+2p}(d + x_1) \Rightarrow x_2 + 2\dot{x}_2 = d + x_1 \Rightarrow \dot{x}_2 = \frac{1}{2}(x_1 - x_2 + d)$

$y = x_2 + n$

we need to express d in terms of e

$e \rightarrow \boxed{G(p)} \rightarrow d$

page 106 chap. 5.3 Spectral Description

" To describe the properties of a disturbance"

" how its energy is distributed over different frequencies "

$d(t) = G(p) \cdot e(t)$ $\quad \Phi_d = G(iw) \cdot \Phi_e \cdot G^*(iw)$

Adjoint

$A = (a_{ij})$

$A^* = (\overline{a_{ji}})$

$\Phi_e = R$ if e is white noise, here $R = 1$

$\Phi_d = G(iw) \cdot 1\, G(-iw) = \frac{1}{w^2+1} = \frac{1}{(wi+1)} \frac{1}{(-wi+1)}$

$\Rightarrow G(p) = \frac{1}{p+1}$

$d = \frac{1}{p+1} \cdot e \Rightarrow \dot{d} = -d + e$ we got a 3d state $x_3 = d$

page 1/2

$$\dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 1/2 & -1/2 & \frac{1}{2} \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e$$

$$y = [\,0 \quad 1 \quad 0\,]\, x + n$$

(b) spectrum for measurement noise $\phi_n = \dfrac{w^2+4}{w^2+9}$

write on a state space form

where $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is white noise with intensity $R = \begin{bmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{bmatrix}$

$$n(t) = G(p) \cdot v_2$$

$$\phi_n = G(iw) \underbrace{\phi_{v_2}}_{R_2 = 1} G(-iw) = \frac{w^2+4}{w^2+9} \Rightarrow G(p) = \frac{p+2}{p+3}$$

$$n = \frac{p+2\,(+1-1)}{p+3} \cdot v_2 = \frac{p+3-1}{p+3} v_2 = v_2 + \frac{-1}{p+3} \cdot v_2$$

$$= v_2 - x_4$$

express n as a function of v_2 & states.

we get a new state: $x_4 = \dfrac{1}{p+3} \cdot v_2$

$$\dot{x}_4 = -3x_4 + v_2$$

$$\dot{x} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1/2 & -1/2 & 1/2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} e + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} v_2}$$

e & v_2 are white noise

this is our $N \cdot v_1$

$$v_1 = \begin{bmatrix} e \\ v_2 \end{bmatrix} \qquad N = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$v_2 = v_2$$

note v_1 & v_2 are now correlated!

5.9c Determine R if n & d are independent.

$\Rightarrow$ e & V_2 are independent. $R_1 = \begin{bmatrix} \phi_e & 0 \\ 0 & \phi_{V2} \end{bmatrix}$

$R_2 = \phi_{V_2}$

$$R = \begin{bmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{bmatrix} = \begin{bmatrix} \phi_e & 0 & \begin{bmatrix} \cdot \\ \cdot \end{bmatrix} \\ 0 & \phi_{V_2} & \\ [\cdot & \cdot] & \phi_{V_2} \end{bmatrix}$$

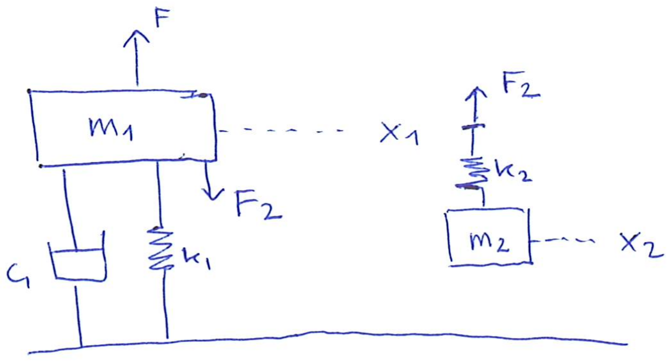
$$R_{12} = \begin{bmatrix} \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

correlation between $V_1 = \begin{bmatrix} e \\ V_2 \end{bmatrix}$ & V_2

e - white noise with intensity $1 \Rightarrow \phi_e = 1$

V_2 - white noise, just given intensity R_2

5.12 Vibration absorber



Newtons 2nd law $\Sigma F = ma$

$$F_2 = k_2(x_1 - x_2)$$

$$m_1 \ddot{x}_1 = F - F_2 - k_1 x_1 - c_1 \dot{x}_1$$

$$m_2 \ddot{x}_2 = F_2$$

States: $x = \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix}$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-(k_1+k_2)}{m_1} & \frac{-c_1}{m_1} & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ k_2/m_2 & 0 & \frac{-k_2}{m_2} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/m_1 \\ 0 \\ 0 \end{bmatrix} F$$

$$y = [1 \ 0 \ 0 \ 0]\, x$$

Find $G_{x_1 F}$

$\dot{x} = Ax + Bu \qquad u = F$

$Y = Cx$

$$Y = \underbrace{c\,(sI - A)^{-1} B}_{\text{what we search for}} \cdot u$$

$$(sI - A)^{-1} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix}$$

$$c \cdot (sI - A)^{-1} = [1 \ 0 \ 0 \ 0]\begin{bmatrix} & \\ & \end{bmatrix} = [a \ b \ c \ d]$$

$$[a \ b \ c \ d] \cdot \begin{bmatrix} 0 \\ 1/m_1 \\ 0 \\ 0 \end{bmatrix} = b \cdot \frac{1}{m_1}$$

You need to find one element of the inversed 4×4 matrix...

$$G_{x_1 F} = b \cdot \frac{1}{m_1} \qquad b = \ldots$$