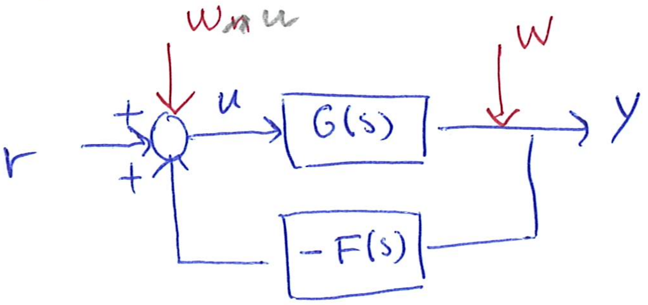
6.5 chapter 6. Closed loop system p. 147 →



$$G(s) = \frac{s-1}{s+1}$$

$$F(s) = \frac{s+2}{s-1}$$

Determine G_c, S, T. stable? Internally stable?

G_c - closed loop t.f. $= \dfrac{G}{1+FG}$

S - sensitivity function $= \dfrac{1}{1+FG}$ eq. 6.9, 10

T - complementary sensitivity function $\dfrac{G \cdot F}{1+GF}$

$$G_{ry} = G_c = \frac{Y}{R} = \frac{s-1}{2s+3}$$

$$S = \frac{1}{1+\frac{(\cancel{s-1})(s+2)}{(s+1)(\cancel{s-1})}} = \frac{s+1}{2s+3}$$

$$T = \frac{\cancel{s+1}}{2s+3} \cdot \frac{(\cancel{s-1})(s+2)}{(\cancel{s+1})(\cancel{s-1})}$$

$$G_{wu} = \frac{F}{1+FG} = \frac{(s+2)}{(s-1)} \cdot \frac{(s+1)}{(2s+3)}$$

↑ pole in +1 ⇒ unstable ∴ system is NOT internally stable!

Internally stable? def 6.1 page 151.
check all 4 t.f. from W_u & W to u & y

let $r=0$: $Y = W + G(W_n + (-F)\cdot Y)$

$$Y(1+GF) = W + G \cdot W_n$$

$$Y = \overset{S}{\boxed{\frac{1}{1+GF}}} \cdot W + \overset{G_c}{\boxed{\frac{G}{1+GF}}} \cdot W_n$$ ok.

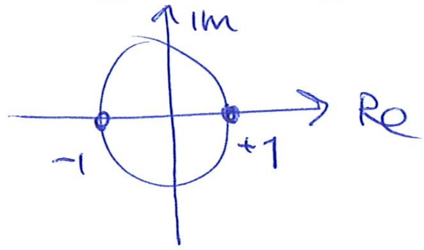
$$u = W_n + (-F)(W + G \cdot u) \Rightarrow u = \overset{\text{ok}}{\boxed{\frac{1}{1+FG}}} \cdot W_n - \boxed{\frac{F}{1+FG}} \cdot W$$

6.6 a) investigate stability, controllability, observability

$G = C(zI - A)^{-1}B$ $\qquad (zI - A)^{-1} = \begin{bmatrix} s & -1 \\ -1 & s \end{bmatrix}^{-1} = \frac{1}{s^2-1}\begin{bmatrix} s & 1 \\ 1 & s \end{bmatrix}$ ≪ Z not S!

$$G = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T \frac{1}{s^2-1} \underbrace{\begin{bmatrix} s & 1 \\ 1 & s \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\begin{bmatrix} s \\ 1 \end{bmatrix}} = \frac{s}{s^2-1}$$

pole in ± 1
discrete time
marginally stable



controllability: $S = [B \;\; AB]$

$AB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ⇒ full rank ⇒ controllable ok.

observability

$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix}$ $\qquad CA = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}$ $\qquad \mathcal{O} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ⇒ rank 2 controllable ok.

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b) state feedback with integral action,
*) include one more state, that is an integrator

$$x_3(k+1) = \sum_{n=0}^{k} e(n) = \underbrace{\left(\sum_{n=0}^{k-1} e(n)\right)}_{x_3(k)} + \underbrace{e(k)}_{r(k) - y(k)}$$

$r(k) = 0$, $\quad y(k) = x_1(k)$

$$x(k+1) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(k) \qquad y = [1 \;\; 0 \;\; 0]\, x$$

state feedback: $u(k) = -[l_1 \;\; l_2 \;\; l_3]\, x(k)$

closed loop poles in $\quad z = 0 \quad z = \rho e^{\pm j\phi}$

$x(k+1) = Ax + B(-Lx) = (A - BL)x$

closed loop poles z are found: $\det(zI - (A - BL)) = 0$

$$BL = \begin{bmatrix}1\\0\\0\end{bmatrix}[l_1\ l_2\ l_3] = \begin{bmatrix} l_1 & l_2 & l_3\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

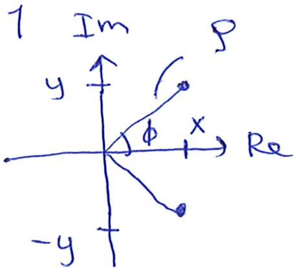
$$\begin{vmatrix} z+l_1 & l_2-1 & l_3 \\ -1 & z & 0 \\ 1 & 0 & z-1 \end{vmatrix} = (z+l_1)\left(z(z-1)-0\right) - (l_2-1)\left((-1)(z-1)-0\right) + l_3(0-z) = \dots$$

$$= z^3 + z^2(l_1-1) + z(\underbrace{l_2-1}-l_1-l_3) + \underbrace{1-l_2} = 0$$

pole in $z=0 \Rightarrow l_2 = 1$

poles in $z = \varrho e^{\pm j\phi}$

$\Rightarrow z = \varrho\cos\phi \pm i\varrho\sin\phi$



$\sin(\phi) = \dfrac{y}{\varrho}$

$\cos\phi = x/\varrho$

$$0 = (z-\varrho(\cos\phi + i\sin\phi))(z-\varrho(\cos\phi - i\sin\phi))$$

$$= z^2 + z\left(-\varrho(\cos\phi - \cancel{i\sin\phi}) - \varrho(\cos\phi + \cancel{i\sin\phi})\right) +$$

$$+\varrho^2 \underbrace{(\cos\phi + i\sin\phi)(\cos\phi - i\sin\phi)}_{\cos^2\phi - (i)^2\sin^2\phi \,=\, 1}$$

$(a+b)(a-b) = a^2 - b^2$

$$= z^2 + z(-2\varrho\cos\phi) + \varrho^2 = 0$$

$l_1 - 1 = -2\varrho\cos\phi \Rightarrow l_1 = 1 - 2\varrho\cos\phi$

$-l_1 - l_3 = \varrho^2 \Rightarrow l_3 = -l_1 - \varrho^2 = 2\varrho\cos\phi - 1 - \varrho^2$

Determine an observer. Both poles in $z = \nu$ $(0 < \nu < 1)$

original dynamics:

$$x(k+1) = Ax(k) + Bu(k) + NV_1(k)$$
$$y(k) = Cx(k) + V_2$$

we estimate the state $\hat{x}$

quality of estimate: $K(y - C\hat{x})$

our observer: $\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})$

Introduce the estimation error:

$\tilde{x} = x - \hat{x}$ with a good estimate $\tilde{x}$ is small

$$\dot{\tilde{x}} = Ax + \cancel{Bu} + NV_1 - (A\hat{x} + \cancel{Bu} + K\underbrace{y}_{(Cx - V_2)} - KC\hat{x})$$

$$= \underbrace{Ax - A\hat{x} - KCx + KC\hat{x}}_{(A-KC)\underbrace{(x-\hat{x})}_{\tilde{x}}} + NV_1 + KV_2$$

Poles of the observer: $\det(zI - (A - \underbrace{KC}))) = 0$

$$\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} k_1 & 0 \\ k_2 & 0 \end{bmatrix}$$

$$\begin{vmatrix} z + k_1 & -1 \\ k_2 - 1 & z \end{vmatrix} = (z + k_1) z - (-1)(k_2 - 1) = 0$$

$$= z^2 + z \cdot k_1 + k_2 - 1 = (z - \nu)^2 = z^2 - 2\nu z + \nu^2$$

$\Rightarrow k_1 = -2\nu$

$k_2 = 1 + \nu^2$

6.8 $$x(k+1) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k)$$

$$y(k) = [1 \quad 0]\, x(k)$$

a) $u(k) = -Lx(k)$ can the poles of the closed loop be placed arbitrarily?

Yes. Iff it is controllable. Check rank(S).

$S = [B \ AB .. A^{n-1}B]$ $n = \#$ state here $n=2$ $n-1=1$

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\cancel{AAB = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}}$$

$$S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$ ← 2 independent rows ⇒ rank = 2 = full

Yes. controllable.

b) Dead beat controller. = put closed loop poles in origin.

$$u(k) = -LX = -[l_1 \ l_2]x = -l_1 x_1 - l_2 x_2$$

$$x(k+1) = Ax + B(-Lx) = \underbrace{(A - BL)}_{\text{"closed loop A-matrix"}} x(k)$$

$$A - BL = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix} [l_1 \ l_2]}_{\begin{bmatrix} l_1 & l_2 \\ 0 & 0 \end{bmatrix}} = \begin{bmatrix} 1-l_1 & 1-l_2 \\ 1 & 0 \end{bmatrix}$$

poles of closed loop syst
$\det(Is - (A - BL)) = 0$

$$\begin{vmatrix} s - 1 + l_1 & l_2 - 1 \\ -1 & s \end{vmatrix} = s^2 + s(l_1 - 1) + (l_2 - 1) = 0 = s^2$$

← double pole in 0

$\Rightarrow l_1 = 1 \quad l_2 = 1$

9.1 Given: process $\ddot{y} = u$

determine controller that minimizes:

$$\int_0^\infty (y^2(t) + \eta \cdot u^2(t))\, dt, \quad \eta > 0$$

Find closed loop poles.

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page 241-242 General system: $\dot{x} = Ax + Bu + Nv_1$

$z = Mx$

$y = Cx + v_2$

$\min V = \min \int z^T Q_1 z + u^T Q_2 u \, dt$

[if u & z scalars] $= \int Q_1 z^2 + Q_2 u^2 \, dt$

∴ our system let $x = \begin{bmatrix} \cancel{y} \\ \dot{y} \end{bmatrix}$ $\quad \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$

with $z = y \Rightarrow M = C$

$Q_1 = 1 \quad Q_2 = \eta$

$v_1 = v_2 = 0$

optimal controller: $u = -L\hat{x} = -\begin{bmatrix} l_1 & l_2 \end{bmatrix} \hat{x}$

eq 9.6, 9.7 $\quad L = Q_2^{-1} B^T S$

size: $[1 \times 2]$ scalar $\begin{bmatrix} 0 & 1 \end{bmatrix}$ 1×2 $\quad$ ~~2×2~~ or scalar ?!

size $n \times n$!

$$A^T S + SA + \underbrace{\underbrace{M^T Q_1 M}_{\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot 1 \cdot \begin{bmatrix} 1 & 0 \end{bmatrix}}}_{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} - SBQ_2^{-1} B^T S = 0 \qquad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{1}{\eta} \begin{bmatrix} 1 & 0 \end{bmatrix}$$

size 2×2

$\Rightarrow S = \begin{bmatrix} S_1 & S_2 \\ S_2 & S_3 \end{bmatrix}$

$$0 = \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}}_{\begin{bmatrix} 0 & 0 \\ s_1 & s_2 \end{bmatrix}} + \underbrace{\begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\begin{bmatrix} 0 & s_1 \\ 0 & s_2 \end{bmatrix}} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \underbrace{\underbrace{\begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\begin{bmatrix} s_2 \\ s_3 \end{bmatrix}} \frac{1}{\eta} \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}}_{\begin{bmatrix} s_2 & s_3 \end{bmatrix}}}_{\frac{1}{\eta}\begin{bmatrix} s_2^2 & s_2 s_3 \\ s_2 s_3 & s_3^2 \end{bmatrix}}$$

$$0 = \begin{bmatrix} 1 - s_2^2/\eta & s_1 - s_2 s_3/\eta \\ s_1 - s_2 s_3/\eta & 2s_2 - s_3^2/\eta \end{bmatrix}$$

$$\Rightarrow s_2 = \sqrt{\eta}$$

$$s_3^2 = 2 s_2 \cdot \eta \Rightarrow s_3 = \sqrt{2}\, \eta^{3/4}$$

$$s_1 = \frac{s_2 s_3}{\eta} = \sqrt{2}\, \eta^{1/4}$$

$$L = \frac{1}{\eta} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix} = \begin{bmatrix} s_2 & s_3 \end{bmatrix} \frac{1}{\eta} = \begin{bmatrix} \eta^{-0,5} & \sqrt{2}\, \eta^{-1/4} \end{bmatrix}$$

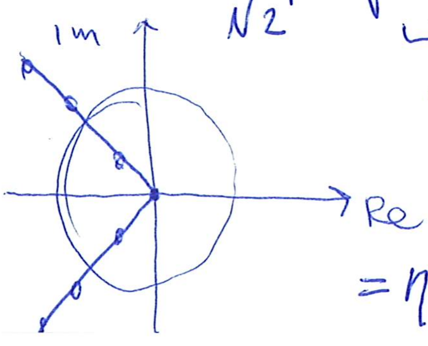
$$\left\{ \text{if } \mu = \eta^{-1/4} \right\} = [\mu^2, \sqrt{2}\mu]$$

Closed loop poles:

$$\dot{X} = AX + B(-LX) \Rightarrow \underbrace{(sI - A + BL)}_{\det(..) = 0} X = 0$$

$$0 = \begin{vmatrix} s & -1 \\ \mu^2 & s + \sqrt{2}\mu \end{vmatrix} = s^2 + s\sqrt{2}\mu + \mu^2 = \Big(s + \underbrace{\frac{\sqrt{2}\mu}{2}}_{\mu/\sqrt{2}}\Big)^2 - \left(\frac{\sqrt{2}\mu}{2}\right)^2 + \mu^2$$

$$\Rightarrow s = -\frac{\mu}{\sqrt{2}} \pm \sqrt{\underbrace{\frac{\mu^2}{2} - \mu^2}_{-\frac{\mu^2}{2}}} = -\frac{\mu}{\sqrt{2}} \pm i\frac{\mu}{\sqrt{2}}$$



If μ increases $\Rightarrow$ poles more far from origin $\Rightarrow$ faster syst

$= \eta$ decreases

9.3 Process: $(s+1)\Delta H(s) = \Delta Q_1(s) + \Delta Q_2(s)$

deviations in level, controlled flow & disturbance flow

Loss function: $V = \sum_{t=0}^{\infty} \{ x_1^2(t) + x_2^2(t) + u^2(t) \}$

where $x_1(t) = \Delta h(t)$ $x_2(t)$ is an integral state

$x_2(k+1) = \sum_{h=0}^{k} e(h) = \underbrace{\sum_{h=0}^{k-1} e(h)}_{x(k)} + e(k)$

$x_2(k+1) - x_2(k) = e(k)$

a) Determine matrices needed to solve riccati eq.

page 264-265 Discrete Time optimal controller

general setup: $x(k+1) = A x(k) + B u(k) + N v_1(k)$

$$z(k) = M x(k)$$
$$y(k) = C x(k) + v_2(k)$$
$$\min \sum z^T Q_1 z + u^T Q_2 u$$

from process $(s+1) x_1 = u + v_1 \Rightarrow \dot{x}_1 = -x_1 + u + v_1$

$y = x_1$

convert to discrete time:

$A_d = e^{AT}$ $B_d = \int_0^T e^{At} B\,dt$

for x_1: $A_d = e^{-T}$ $B_d = \int_0^T e^{-t} dt = \left[\frac{e^{-t}}{-1}\right]_0^T = \frac{e^{-T}}{-1} - \frac{e^0}{-1} = 1 - e^{-T}$

x_2: $x_2(k+1) = x_2(k) + \underbrace{e(k)}_{r - y = x_1}$

$$x(k+1) = \begin{bmatrix} e^{-T} & 0 \\ -1 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1-e^{-T} \\ 0 \end{bmatrix} u(k) + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix} r(k)}_{r(k) \hat{=} 0} + \begin{bmatrix} 1-e^{-T} \\ 0 \end{bmatrix} v_1(k)$$

$y(k) = [1 \ \ 0]\, x(k)$

There we have A, B, C, N

we want to keep y at stationary point

from the loss function:

u is a scalar $\Rightarrow u^T Q_2 u = Q_2 u^2 \Rightarrow Q_2 = 1$

with $z = x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Mx \Rightarrow M = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$z^T Q_1 z = \underbrace{[x_1 \;\; x_2]}_{(1\times 2)} Q_1 \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{(2\times 1)} = q_1 x_1^2 + 2q_{12} \cdot x_1 x_2 + q_2 x_2^2$$

Size $Q_1 = 2\times 2$ $\begin{bmatrix} q_1 & q_{12} \\ q_{12} & q_2 \end{bmatrix}$

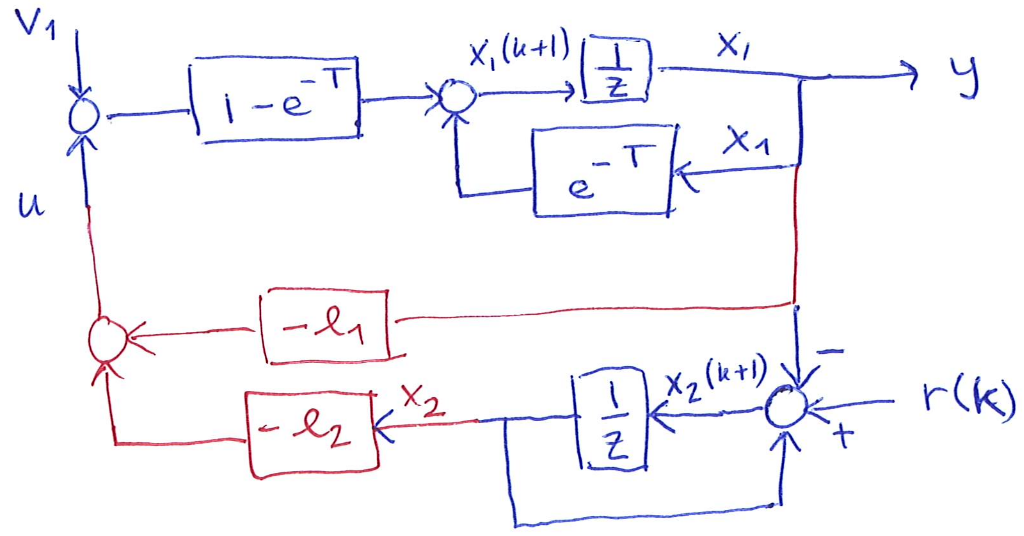
$\Rightarrow Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore$ All needed ~~equate~~ matrices given.

(b) Draw a block diagram with SISO-block.

(1) $x_1(k+1) = e^{-T} x(k) + (1-e^{-T})(u(k) + v_1(k))$

(2) $u(k) = -Lx = -[\ell_1 \;\; \ell_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\ell_1 x_1 - \ell_2 x_2$



(3) $x_2(k+1) = x_2(k) + r(k) - x_1(k)$