SSY285 - Home Assignment M2

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December 6, 2019

Introduction

In this assignment, analysis of linear state space model of the DC motor with flywheel as given by in the previous assignment according to the figure 1 is done, it consists of an electric motor which drives a flywheel and it is influenced by an external torque. In this assignment we deal with if the system is controllable, observable, stabilizable or detectable for non zero, check the condition number and if the resulting discrete state space model is of minimal order, etc.

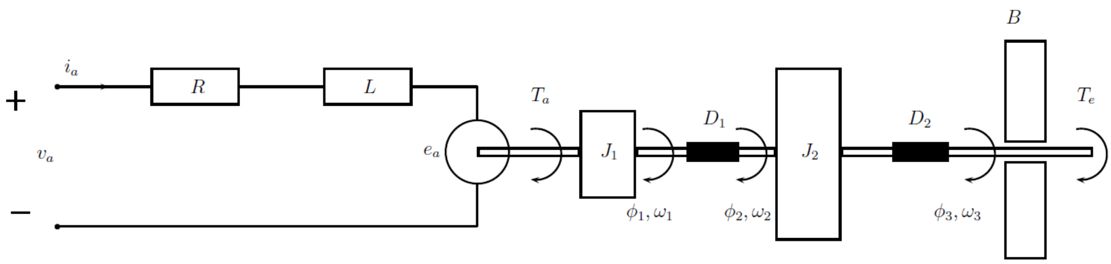

Figure 1: DC motor with flywheel

Various symbols used in this assignment and its respective description is given by the table 1. These symbols would be used in all the tasks of this assignment.
According to the assignment matrices A,B,C and D is to be taken from the previous assignment. Hence, the respective matrices are as follows:

$$A = \begin{bmatrix} -\frac{K_T \cdot K_E}{J_1 \cdot R} & 0 & -\frac{D_1}{J_1} & \frac{D_1}{J_1} & 0 \\ 0 & 0 & \frac{D_1}{J_2} & -(\frac{D_1}{J_2} + \frac{D_2}{J_2}) & \frac{D_2}{J_2} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{D_2}{B} & -\frac{D_2}{B} \end{bmatrix} \tag{1}$$

Table 1: Description of the symbols used in the assignment

Symbol	Description
v_a	External voltage applied to the rotor
i_a	Rotor current
e_a	Induced rotor voltage
L	Rotor Inductance
R	Rotor resistance
K_E	Coefficient related to induced voltage e_a
T_a	Rotor produced torque
K_T	Coefficient related to rotor torque constant driving rotor current i_a
ϕ_1, ϕ_2, ϕ_3	Angles
$\omega_1, \omega_2, \omega_3$	Angular speeds
J_1	Rotor inertia
J_2	Flywheel inertia
D_1, D_2	Torsional springs
B	Linear friction proportional to the angular speed
T_e	External torque applied to flywheel axis

$$B = \begin{bmatrix} \frac{K_T}{J_1 \cdot R} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{B} \end{bmatrix} \tag{2}$$

$$C_{case_a} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{3}$$

$$C_{case_b} = \begin{bmatrix} -\frac{K_E}{R} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{D_2}{B} & -\frac{D_2}{B} \end{bmatrix} \tag{4}$$

$$D_{case_a} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{5}$$

$$D_{case_b} = \begin{bmatrix} \frac{1}{R} & 0 \\ 0 & \frac{1}{B} \end{bmatrix} \tag{6}$$

The value for the parameters are given by the table 2.

Table 2: Parameters and its values

Parameter	Value
R	$1\,\Omega$
K_E	$10^{-1}\,\mathrm{V\,s\,rad^{-1}}$
K_T	$10^{-1}\,\mathrm{N\,m\,A^{-1}}$
J_1	$10^{-5}\,\mathrm{kgm^2}$
J_2	$4\times10^{-5}\,\mathrm{kgm^2}$
D_1	$20\,\mathrm{N\,m\,rad^{-1}}$
D_2	$2\,\mathrm{N\,m\,rad^{-1}}$
B	$2\times10^{-3}\,\mathrm{N\,m\,s}$

Question A

In this task it will be checked if the system is controllable or observable for two different cases of C and D matrices.
To check if the system is controllable, then controllabiltiy matrix needs to be a full rank matrix equal to the number of states and in our case the matrix would be as shown,

$$S_c = \begin{bmatrix} B & AB & A^2B & A^3B & A^4B \end{bmatrix}. \tag{7}$$

Symbolic variables were used in the matrices, the rank of the controllability matrix is found to be 5. **Since the rank of the matrices is equal to the number of states, it is a full rank matrix. System is controllable**.

To check if the system is observable, then observability matrix needs to be a full matrix equal to the number of states and in our case the matrix would be as shown,

$$S_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \\ CA^4 \end{bmatrix} \tag{8}$$

Again, symbolic variables were used, there by finding the observibility matrices for two cases. It is found that rank of matrices for case A is 5 and for case b it is 4. **Since the rank of the matrices in case A is full rank and it is equal to the number of states, it is observable. Where as the rank of the matrices in case B is not full rank, it is not observable**.

Question B

In this question, it is interesting to know whether the system is stabilizable or detectable for non zero, positive and finite values of R.

To find this, we do PBH test.
According to this test, to find if the system is stabilizable

$$\begin{bmatrix} A - \lambda I & B \end{bmatrix} \tag{9}$$

$\forall\lambda$ must have full rank. To find if the system is detectable

$$\begin{bmatrix} A - \lambda I \\ C \end{bmatrix} \tag{10}$$

$\forall\lambda$ must have full rank.
We find that the rank for all the eigen values is 5 and is full rank when checked for stabilizability. Hence it is **stabilizable** .
When checked for observability for case A, for all the eigen values the rank is 5 and is full rank. Hence for **case A the system is detectable** .
For case B it is found that rank for all eigen values is

$$Rank_{detect_b} = \begin{bmatrix} 4 \\ 5 \\ 5 \\ 5 \\ 5 \end{bmatrix} \tag{11}$$

Since, one of the ranks are not full, **for case B system is not detectable.**

Question C

The numerical parameter values from assignment M-1 is tabulated as shown in the table 2, these values are implemented to check if the system is controllable in equation (7) and observable in equation (8).
The controllability matrix for the given system with numerical parameter values is as shown in equation (12). The rank of the controllability matrix is found to be 4 from

equation (13). Since the rank of the matrices is not equal to the number of states, therefore the system said to be uncontrollable. The possible reason for this could be precision of numerical parameter values in the matrix, the cancellation of poles and zeroes.

Similarly, the observability matrix for two different cases with numerical parameter values is as shown in equation (14) and equation (16). The rank of the observability matrix for case - A is found to be 4 from equation (15), and rank for observability matrix for case - B is found to be 4 from equation (17). Since the rank of the matrices is not equal to the number of states, therefore the system said to be not observable for both cases.

$$C = \begin{bmatrix} 1 \times 10^4 & 0 & -1 \times 10^7 & 0 & -1 \times 10^{10} & 0 & 3 \times 10^{13} & 5 \times 10^{13} & -14.6484 & -1 \times 10^{17} \\ 0 & 0 & 0 & 2.5 \times 10^7 & 5 \times 10^9 & -2.5 \times 10^{10} & -5 \times 10^{12} & 1.125 \times 10^{13} & -7.75 \times 10^{15} & -1 \times 10^{16} \\ 0 & 0 & 1 \times 10^4 & 0 & -1 \times 10^7 & 0 & -1 \times 10^{10} & 0 & 3 \times 10^{13} & 5 \times 10^{13} \\ 0 & 0 & 0 & 0 & 0 & 2.5 \times 10^7 & 5 \times 10^9 & -2.5 \times 10^{10} & -5 \times 10^{12} & 1.125 \times 10^{13} \\ 0 & 500 & 0 & -5 \times 10^5 & 0 & 5 \times 10^8 & 0 & -4.75 \times 10^{11} & 5 \times 10^{12} & 4.5 \times 10^{14} \end{bmatrix} \tag{12}$$

$$C_{Rank} = \begin{bmatrix} 4 \end{bmatrix} \tag{13}$$

$$O_A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 5 \times 10^5 & -5.5 \times 10^5 & 5 \times 10^4 \\ 0 & 0 & 5 \times 10^5 & -5.5 \times 10^5 & 5 \times 10^4 \\ 5 \times 10^5 & -5.5 \times 10^5 & 0 & 5 \times 10^7 & -5 \times 10^7 \\ 5 \times 10^5 & -5.5 \times 10^5 & 0 & 5 \times 10^7 & -5 \times 10^7 \\ -5 \times 10^8 & 5 \times 10^7 & -1.275 \times 10^{12} & 1.2525 \times 10^{12} & 2.25 \times 10^{10} \\ -5 \times 10^8 & 5 \times 10^7 & -1.275 \times 10^{12} & 1.2525 \times 10^{12} & 2.25 \times 10^{10} \\ -7.75 \times 10^{11} & 1.2525 \times 10^{12} & 1.025 \times 10^{15} & -1.005 \times 10^{15} & -2 \times 10^{13} \end{bmatrix} \tag{14}$$

$$O_{A-Rank} = \begin{bmatrix} 4 \end{bmatrix} \tag{15}$$

$$
O_B = \begin{bmatrix} -0.1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1\times10^{3} & -1\times10^{3} \\ 100 & 0 & 2\times10^{5} & -2\times10^{5} & 0 \\ 0 & 1\times10^{3} & 0 & -1\times10^{6} & 1\times10^{6} \\ 1\times10^{5} & -2\times10^{5} & -2\times10^{8} & 2\times10^{8} & 0 \\ 0 & -1\times10^{6} & 5\times10^{8} & 4.5\times10^{8} & -9.5\times10^{8} \\ -3\times10^{8} & 2\times10^{8} & -3\times10^{11} & -3.1\times10^{11} & -1\times10^{10} \\ 5\times10^{8} & 4.5\times10^{8} & -5\times10^{11} & -4\times10^{11} & 9\times10^{11} \\ -1.4648\times10^{-4} & 3.1\times10^{11} & 7\times10^{14} & -7.2\times10^{14} & 2\times10^{13} \\ -1\times10^{12} & -4\times10^{11} & -7.75\times10^{14} & 1.6525\times10^{15} & -8.7750\times10^{14} \end{bmatrix} \tag{16}
$$

$$
O_{B-Rank} = \begin{bmatrix} 4 \end{bmatrix} \tag{17}
$$

$$
C_N = \sqrt{\frac{\lambda_{max}}{\lambda_{min}}} \tag{18}
$$

From the above equations, the system is said to be uncontrollable and unobservable for numerical parameter values. The condition number is evaluated to know the stability status of the matrix, as shown in equation (18). Eignen values for the condition number for $C, O_A and O_B$ is chosen from the matrices $CC^T, O_A O_A^T$ and $O_B O_B^T$ respectively. With the large condition number, it is close to instability, whereas condition number close to 1 is far from being unstable.

The condition number is evaluated for a controllable matrix and observable matrix for two different cases. From equation (19, 20 and 21), the values for above cases are large and it is close to instability.

$$
Cond_{N_c} = 1.189 \times 10^{9} \tag{19}
$$

$$
Cond_{N_{Oa}} = 2.066 \times 10^{12} \tag{20}
$$

$$
Cond_{N_{Ob}} = 2.626 \times 10^{13} \tag{21}
$$

The ranks of the controllability matrix and observability matrix are not identical for the matrix with symbolic variables and the numerical parameter values. From the condition numbers, the system is predicted to be unstable. Further, to know whether the system is stabilizable or detectable, we use the PBH test for stabilizability and detectability, as shown in equation (9 and 10). The stabilizability for the given system with numerical parameter values is as shown in equation (22). The rank of the stabilizability matrix is found to be 5 from equation (23), which is similar to the matrix with symbolic variables. Hence it is said to be stabilizable.

$$S = \begin{bmatrix} -53.6114 & 0 & -2\times10^6 & 2\times10^6 & 0 & 1\times10^4 & 0 \\ 0 & 946.3885 & 5\times10^5 & -5.5\times10^5 & 5\times10^4 & 0 & 0 \\ 1 & 0 & 946.3885 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 946.3885 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1\times10^3 & -53.6114 & 0 & 500 \end{bmatrix} \tag{22}$$

$$S_{Rank} = \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \\ 5 \end{bmatrix} \tag{23}$$

Similarly, the detectability matrix for the given system with numerical parameter values for two different cases is as shown in equation (24 and 26). The rank of the detectability matrix for case A is found to be 5 and full rank from equation (25), which is similar to the matrix with symbolic variables. Hence for case A, the system is said to be detectable.

For case B, the detectability matrix is found, as shown in equation (27). Since one of the ranks is not full, the system is not detectable for case B. Thus, from the PBH test, the system is said to be stabilizable, detectable for case A and not detectable for case B. The reason for this is the precision of numerical parameter values in the controllability matrix and observability matrix, which deviates the respective matrix rank.

$$D_A = \begin{bmatrix} -53.6114 & 0 & -2\times10^6 & 2\times10^6 & 0 \\ 0 & 946.3885 & 5\times10^5 & -5.5\times10^5 & 5\times10^4 \\ 1 & 0 & 946.3885 & 0 & 0 \\ 0 & 1 & 0 & 946.3885 & 0 \\ 0 & 0 & 0 & 1\times10^3 & -53.6114 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{24}$$

$$D_{A-Rank} = \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \\ 5 \end{bmatrix} \tag{25}$$

$$D_B = \begin{bmatrix} -53.6114 & 0 & -2\times10^6 & 2\times10^6 & 0 \\ 0 & 946.3885 & 5\times10^5 & -5.5\times10^5 & 5\times10^4 \\ 1 & 0 & 946.3885 & 0 & 0 \\ 0 & 1 & 0 & 946.3885 & 0 \\ 0 & 0 & 0 & 1\times10^3 & -53.6114 \\ -0.1000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1\times10^3 & -1\times10^3 \end{bmatrix} \tag{26}$$

$$D_{B-Rank} = \begin{bmatrix} 5 \\ 5 \\ 4 \\ 5 \\ 5 \end{bmatrix} \tag{27}$$

Question D

In this question, state transition matrix is computed for discrete sampling time interval of T_s =1 ms, where A_C denotes the corresponding continuous time parameter matrix.

$$A_d = e^{A_c T_s} = \mathscr{L}^{-1}((SI - A)^{-1}) \tag{28}$$

From equation (28), where $(SI - A)^{-1}$ is called resolvent of A. It maps the initial state to the state at time T_s =1 ms, thus A_d is evaluated as shown in the equation (29).

$$A_d = e^{A_c T_s} = \begin{bmatrix} -0.0489 & 0.5946 & -782.8328 & 773.7475 & 9.0853 \\ 0.1491 & 0.7749 & 342.5299 & -370.9589 & 28.4290 \\ 4.49993\times10^{-4} & 2.3406\times10^{-4} & 0.4010 & 0.5964 & 0.0026 \\ 5.8514\times10^{-5} & 9.1912\times10^{-4} & 0.2076 & 0.7747 & 0.0175 \\ 1.3088\times10^{-5} & 3.5054\times10^{-4} & 0.0585 & 0.5686 & 0.3729 \end{bmatrix} \tag{29}$$

Question E

In this question, the task is to calculate the discrete time input matrix B_d which is mentioned in (2) for sampling time T_s = 1 ms with ZOH principle is given by the expression,

$$B_d = \int_0^{T_s} e^{A_c \cdot t} B_c dt \tag{30}$$

$$B_d = A_c^{-1} \cdot [e^{A_c \cdot t} - 1] \cdot B_c \tag{31}$$

Either by using equation (31) or by c2d command in MatLab which discretizes the continuous-time dynamic system model using zero-order hold on the inputs and a sample time of given Ts, the discrete time input matrix is given by,

$$B_d = \begin{bmatrix} 4.499 & 1.309 \\ 0.5851 & 8.763 \\ 0.003154 & 0.0002857 \\ 0.0001595 & 0.003211 \\ 2.857 \times 10^{-5} & 0.3168 \end{bmatrix} \tag{32}$$

Question F

As found in question C, in case of continious time system, the rank of controllability matrix was 4 equation (13), when the actual values were used, unlike the symbolic variables. After discretizing the system, controlability test was performed again and the result is as shown.

$$C = \begin{bmatrix} 4.49 & 1.30 & -2.21 & 10.30 & -0.14 & 17.75 & 2.68 & 14.53 & 1.69 & 8.2448 \\ 0.5851 & 8.7630 & 2.1461 & 14.8988 & 1.9680 & 12.4413 & 0.786 & 9.7852 & 0.3682 & 8.5262 \\ 0.0032 & 0.0003 & 0.0035 & 0.0055 & 0.0019 & 0.0203 & 0.0034 & 0.0372 & 0.0059 & 0.0484 \\ 0.0002 & 0.0032 & 0.0016 & 0.016 & 0.0038 & 0.0302 & 0.0052 & 0.0411 & 0.0057 & 0.0502 \\ 0.0000 & 0.3168 & 0.0005 & 0.1231 & 0.0020 & 0.0608 & 0.0037 & 0.0456 & 0.0048 & 0.0462 \end{bmatrix} \tag{33}$$

$$C_{Rank} = \begin{bmatrix} 5 \end{bmatrix} \tag{34}$$

After discretization, The system controllability matrix has full rank of 5 and hence it is stabilizable.

Similarly, observability test was done for case a and case b. Results of observability test for case a and case b are:

$$O_A = \begin{bmatrix} 0 & 0 & 0 & 1.0000 & 0 \\ 0 & 1.0000 & 0 & 0 & 0 \\ 0.0001 & 0.0009 & 0.2076 & 0.7749 & 0.0175 \\ 0.1491 & 0.7749 & 342.5000 & -371.0000 & 28.4300 \\ 0.0003 & 0.0015 & 0.5141 & 0.4385 & 0.0473 \\ 0.2410 & 0.4386 & 210.6741 & -239.1848 & 28.3796 \\ 0.0005 & 0.0019 & 0.6047 & 0.3230 & 0.0722 \\ 0.1348 & 0.3230 & -1.9645 & -19.8030 & 21.5989 \\ 0.0005 & 0.0022 & 0.5885 & 0.3196 & 0.0918 \\ 0.0398 & 0.3196 & 1.5031 & -19.8057 & 18.1090 \end{bmatrix} \tag{35}$$

$$O_{ARank} = \begin{bmatrix} 5 \end{bmatrix} \tag{36}$$

$$O_B = \begin{bmatrix} -0.0001 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0000 & -1.0000 \\ 0.0000 & -0.0001 & 0.0783 & -0.0774 & -0.0009 \\ 0.0000 & 0.0006 & 0.1491 & 0.2063 & -0.3554 \\ 0.0000 & -0.0001 & -0.0090 & 0.0121 & -0.0031 \\ 0.0002 & 0.0006 & 0.2410 & -0.1291 & -0.1119 \\ -0.0000 & -0.0001 & -0.0512 & 0.0547 & -0.0035 \\ 0.0002 & 0.0004 & 0.1348 & -0.1090 & -0.0258 \\ -0.0000 & -0.0000 & -0.0131 & 0.0153 & -0.0022 \\ 0.0001 & 0.0004 & 0.0398 & -0.0426 & 0.0027 \end{bmatrix} \tag{37}$$

$$O_{BRank} = \begin{bmatrix} 5 \end{bmatrix} \tag{38}$$

When the system was in continious time domain, as mentioned in equation (15) and in equation (17), rank of observability matrix was 4 in both case a and case b, but after discretization, the rank of observability matrix is 5 in both case a and case b, which suggests that the system after discretization is observable and hence detectable.

Since the discrete time state-space model is both detectable and observable, it is of minimal order.

The discretized systems matrix A_d was obtained in question D as shown in equation (29). The eigen values of A_d are as follows.

$$\text{Eigen values of } A_d = \begin{bmatrix} 0.6761 \\ 0.6761 \\ 1.0000 \\ 0.7627 \\ 0.3881 \end{bmatrix} \tag{39}$$

Since four of the eigen values are inside the unit circle on Z plane and one eigen value is on unit circle, the discretised system is marginally stable.

Appendix

Code to calculate controllability, observability, stabilizability, detectability, state transition matrix $e^{A_c t}$, B_d and to know if the discrete system is stable.

```
R = 1; %Ohms
K_E = 0.1; %Vsrad^-1
K_T = 0.1; %NmA^-1
J1 = 10^(-5) ;%kgm^2
J2 = 4*10^(-5) ;%kgm^2
D1 = 20 ; % Nmrad^-1
D2 = 2 ; % Nmrad^-1
B = 2*10^-3 ; %Nms

%syms R K_E K_T J1 J2 D1 D2 B one zero real



A = [(-K_T*K_E/(J1*R)) 0 (-D1/J1) D1/J1 0;
    0 0 D1/J2 -(D1+D2)/J2 D2/J2;
    1 0 0 0 0;
    0 1 0 0 0;
    0 0 0 D2/B -D2/B];

B_d = [(K_T/(J1*R)) 0 ;
    0 0;
    0 0;
    0 0;
    0 1/B];

%Case A
C_a = [0 0 0 1 0;
       0 1 0 0 0];
D_a = 0;
%Case B
C_b = [-K_E/R 0 0 0 0;
    0 0 0 D2/B -D2/B];
D_b = [1/R 0 ;
    0 1/B ];

%%
% Question A - Controllability and stability

%Controllability matrix S_c

S_c = [B_d A*B_d A^2*B_d A^3*B_d A^4*B_d]
S_c_rank = rank(S_c)
```

```
S_o_a = [C_a;
    C_a*A;
    C_a*A^2;
    C_a*A^3;
    C_a*A^4];
S_o_rank_a = rank(S_o_a)

S_o_b = [C_b;
    C_b*A;
    C_b*A^2;
    C_b*A^3;
    C_b*A^4];

S_o_rank_b = rank(S_o_b)

%% %% To find if the system is stabilisable or detectable - PBH test

A_eig = eig(A);
Rank_stable = zeros(5,1);
Rank_detect_a = zeros(5,1);
Rank_detect_b = zeros(5,1);
for n = 1:1:5
    Stabilisability = [A - A_eig(n,1)*eye(5) B_d];
    Rank_stable(n) = rank(Stabilisability);
end

for n = 1:1:5
    Detectablility_a = [A - A_eig(n,1)*eye(5) ; C_a];
    Rank_detect_a(n) = rank(Detectablility_a);
end

for n = 1:1:5
    Detectablility_b = [A - A_eig(n,1)*eye(5) ; C_b];
    Rank_detect_b(n) = rank(Detectablility_b);
end


%% To check the condition number of Controllability and observability
    matrices

Cond_C = S_c * S_c' ;
Cond_C_eig = vpa(eig(Cond_C));
Cond_Nc = sqrt(abs(max(Cond_C_eig)/min(Cond_C_eig)));


Cond_Oa = S_o_a * S_o_a' ;
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```
Cond_Oa_eig = vpa(eig(Cond_Oa));
Cond_N_oa = sqrt(abs(max(Cond_Oa_eig)/min(Cond_Oa_eig)));


Cond_Ob = S_o_b * S_o_b' ;
Cond_Ob_eig = vpa(eig(Cond_Ob));
Cond_N_ob = sqrt(abs(max(Cond_Ob_eig)/min(Cond_Ob_eig)));

%% Ts = 1ms, Find Ad

syms s real
% test = [0 -1; 2 -3]
t = 1e-3;
A_dis  = double (ilaplace([s*eye(5)- A]^-1,t))
A_dis_test = double(expm(A*t))
B_dis = double(inv(A)*expm(A*t)*B_d -  inv(A)*B_d)

%% Another method to find Ad and Bd
sys_a = ss(A,B_d,C_a,D_a);
sys_b = ss(A,B_d,C_b,D_b);

D_sys_a = c2d(sys_a,t);
D_sys_b = c2d(sys_b,t);
Eig_Ad = eig(D_sys_a.A);
G_da = tf(D_sys_a);

%% To check if the resulting discrete state space model is minimal order
% and if the eigen values are in region of stability

P = zeros(5,1)
for n = 1:1:5
    P(n,1) = abs(Eig_Ad(n,1))
end
```