SSY285 - Home Assignment M1

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Introduction

In this assignment, dynamic model of a DC motor with the flywheel is to be done. As given by the assignment according to the figure 1, it consists of an electric motor which drives a flywheel and it is influenced by an external torque. Initially, state space model of the system is formulated and further analysis is done which will be explained in detail in the upcoming questions.

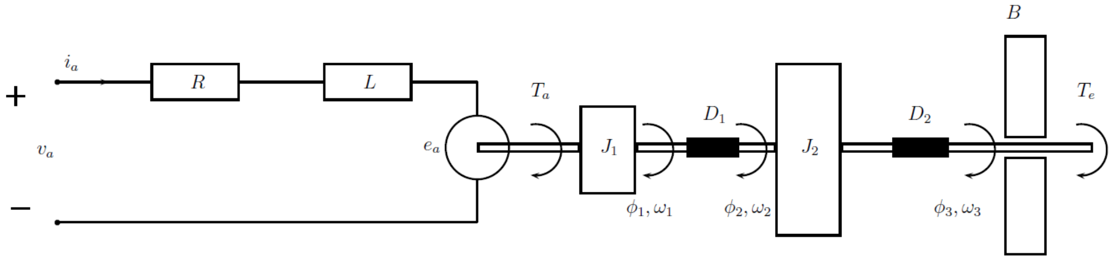

Figure 1: DC motor with flywheel

Various symbols used in this assignment and its respective description is given by the table 1. These symbols would be used in all the tasks of this assignment.

Table 1: Description of the symbols used in the assignment

Symbol	Description
v_a	External voltage applied to the rotor
i_a	Rotor current
e_a	Induced rotor voltage
L	Rotor Inductance
R	Rotor resistance
K_E	Coefficient related to induced voltage e_a
T_a	Rotor produced torque
K_T	Coefficient related to rotor torque constant driving rotor current i_a
ϕ_1, ϕ_2, ϕ_3	Angles
$\omega_1, \omega_2, \omega_3$	Angular speeds
J_1	Rotor inertia
J_2	Flywheel inertia
D_1, D_2	Torsional springs
B	Linear friction proportional to the angular speed
T_e	External torque applied to flywheel axis

Question A

According to the data given in this question, relations of induced rotor voltage and induced rotor torque is given by $e_a = K_E \cdot \omega_1$ and $T_a = K_T \cdot i_a$ respectively. The aim of this question is to formulate mathematical model using basic laws of electric and mechanics.

In order to formulate the electric equation, Kirchoff's law was used and electric equation at the input side as shown,

$$v_a = i_a \cdot R + L\frac{di_a}{dt} + e_a. \tag{1}$$

The equation (1) is rearranged to have it in state space form as shown,

$$\frac{di_a}{dt} = \frac{v_a}{L} - \frac{R}{L} \cdot i_a - \frac{K_E \cdot \omega_1}{L} \tag{2}$$

Torque balance equations are used to formulate the mechanics equation including rotor torque, rotor inertia and first torsional spring as shown,

$$T_a = K_T \cdot i_a = J_1\frac{d\omega_1}{dt} + D_1(\phi_1 - \phi_2). \tag{3}$$

It can be represented in state space form by re-arranging the above equation (3) as,

$$\frac{d\omega_1}{dt} = \frac{K_T}{J_1} \cdot i_a - \frac{D_1}{J_1} \cdot \phi_1 + \frac{D_1}{J_1} \cdot \phi_2. \tag{4}$$

Similar to equation (3), torque balance equation is used and it includes flywheel inertia and two torsional springs and it is rearranged in the state space form as shown,

$$D_1(\phi_1 - \phi_2) = J_2 \frac{d\omega_2}{dt} + D_2(\phi_2 - \phi_3) \tag{5}$$

$$\frac{d\omega_2}{dt} = \frac{D_1}{J_2} \cdot \phi_1 - (\frac{D_1}{J_2} + \frac{D_2}{J_2}) \cdot \phi_2 + \frac{D_2}{J_2} \cdot \phi_3. \tag{6}$$

Torque balance equation is again used and it includes linear friction, applied external torque and second torsional spring. it is also rearranged in the state space form as shown,

$$B \cdot \frac{d\phi_3}{dt} - T_e = D_2(\phi_2 - \phi_3) \tag{7}$$

$$\frac{d\phi_3}{dt} = \frac{D_2}{B} \cdot \phi_2 - \frac{D_2}{B} \cdot \phi_3 + \frac{T_e}{B}. \tag{8}$$

Relation between rotor angles and their respective angular speeds are used in the state space equations as shown

$$\frac{d\phi_1}{dt} = \omega_1 \tag{9}$$

$$\frac{d\phi_2}{dt} = \omega_2. \tag{10}$$

Hence, the equations(2,4,6, 8, 9 and 10) form the linear differential equations and they are 6 in total. The system inputs in this case is v_a **and** T_e.

Question B

In this question, it is assumed that $L \approx 0$. State variables $x(t)$ and inputs for vector $u(t)$ of the system needs to be chosen and continuous time state space equation of the form

$$\dot{x(t)} = A \cdot x(t) + B \cdot u(t) \tag{11}$$

needs to be formulated. Since $L = 0$, the equation (1) is modified as

$$\underbrace{L \frac{di_a}{dt}}_{\approx 0} = v_a - i_a \cdot R - e_a \tag{12}$$

$$v_a = i_a \cdot R + e_a \tag{13}$$

and it is written in terms of i_a as shown

$$i_a = (\frac{v_a}{R} - \frac{e_a}{R}) = (\frac{v_a}{R} - \frac{K_E \cdot \omega_1}{R}). \tag{14}$$

Equation(14) in equation(4)

$$\frac{d\omega_1}{dt} = \frac{K_T}{J_1} \cdot (\frac{v_a}{R} - \frac{K_E \cdot \omega_1}{R}) - \frac{D_1}{J_1} \cdot \phi_1 + \frac{D_1}{J_1} \cdot \phi_2 \tag{15}$$

$$\frac{d\omega_1}{dt} = (\frac{K_T}{J_1 \cdot R}) \cdot v_a - (\frac{K_T \cdot K_E}{J_1 \cdot R}) \cdot \omega_1 - \frac{D_1}{J_1} \cdot \phi_1 + \frac{D_1}{J_1} \cdot \phi_2 \tag{16}$$

By using the equations (16, 6, 8, 9 and 10), state vector $x(t)$, inputs for the vector $u(t)$, Matrix A and B can be formulated as shown,

$$\underbrace{\begin{bmatrix} \frac{d\omega_1}{dt} \\ \frac{d\omega_2}{dt} \\ \frac{d\phi_1}{dt} \\ \frac{d\phi_2}{dt} \\ \frac{d\phi_3}{dt} \end{bmatrix}}_{\dot{x(t)}} = \underbrace{\begin{bmatrix} -\frac{K_T \cdot K_E}{J_1 \cdot R} & 0 & -\frac{D_1}{J_1} & \frac{D_1}{J_1} & 0 \\ 0 & 0 & \frac{D_1}{J_2} & -(\frac{D_1}{J_2} + \frac{D_2}{J_2}) & \frac{D_2}{J_2} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{D_2}{B} & -\frac{D_2}{B} \end{bmatrix}}_{Matrix-A} \underbrace{\begin{bmatrix} \omega_1 \\ \omega_2 \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} \frac{K_T}{J_1 \cdot R} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{B} \end{bmatrix}}_{Matrix-B} \underbrace{\begin{bmatrix} v_a \\ T_e \end{bmatrix}}_{u(t)} \tag{17}$$

Question C

In this question output $y(t)$ of the system in a state space model which is related to state and input, as $y(t) = Cx(t) + Du(t)$ needs to be formulated. Here Matrices C and D needs to be found for the following two cases.

Case A

$y_1(t) = \phi_2$
$y_2(t) = \omega_2$

It can be written as,

$$\underbrace{\begin{bmatrix} \phi_2 \\ \omega_2 \end{bmatrix}}_{y(t)} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}}_{Matrix-C} \underbrace{\begin{bmatrix} \omega_1 \\ \omega_2 \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{Matrix-D} \underbrace{\begin{bmatrix} v_a \\ T_e \end{bmatrix}}_{u(t)} \tag{18}$$

Case B

$y_1(t) = i_a$
$y_2(t) = \omega_3$
In this case, output equations for the state space model is possible and is shown as

$$i_a = (\frac{v_a}{R} - \frac{e_a}{R}) = (\frac{v_a}{R} - \frac{K_E \cdot \omega_1}{R}) \tag{19}$$

$$\omega_3 = \frac{d\phi_3}{dt} = \frac{D_2}{B} \cdot \phi_2 - \frac{D_2}{B} \cdot \phi_3 + \frac{1}{B} \cdot T_e \tag{20}$$

$$y(t) = C \cdot x(t) + D \cdot u(t) \tag{21}$$

Matrix C and D is written by using the equations (19 and 20) as shown

$$\underbrace{\begin{bmatrix} i_a \\ \omega_3 \end{bmatrix}}_{y(t)} = \underbrace{\begin{bmatrix} -\frac{K_E}{R} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{D_2}{B} & -\frac{D_2}{B} \end{bmatrix}}_{Matrix-C} \underbrace{\begin{bmatrix} \omega_1 \\ \omega_2 \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} \frac{1}{R} & 0 \\ 0 & \frac{1}{B} \end{bmatrix}}_{Matrix-D} \underbrace{\begin{bmatrix} v_a \\ T_e \end{bmatrix}}_{u(t)} \tag{22}$$

Question D

In this question, values for the parameters is given as shown in table 2. The aim is to find the matrix A in matlab (code is shown in appendix) and to find whether the input-output of the system is stable for both the cases as mentioned in question C. By

Table 2: Parameters and its values

Parameter	Value
R	$1\,\Omega$
K_E	$10^{-1}\,\mathrm{V\,s\,rad^{-1}}$
K_T	$10^{-1}\,\mathrm{N\,m\,A^{-1}}$
J_1	$10^{-5}\,\mathrm{kgm^2}$
J_2	$4 \times 10^{-5}\,\mathrm{kgm^2}$
D_1	$20\,\mathrm{N\,m\,rad^{-1}}$
D_2	$2\,\mathrm{N\,m\,rad^{-1}}$
B	$2 \times 10^{-3}\,\mathrm{N\,m\,s}$

substituting parameters in table 2 in matrix A as mentioned in equation (17) we get

$$A = \begin{bmatrix} -1000 & 0 & -2\times10^6 & 2\times10^6 & 0 \\ 0 & 0 & 5\times10^5 & 5.5\times10^5 & 5\times10^4 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1000 & -1000 \end{bmatrix} \tag{23}$$

and its eigen values is

$$A_{eig} = \begin{bmatrix} -391.38838 + 1479.0829i \\ -391.38838 - 1479.0829i \\ 5.571\,982\,6 \times 10^{-13} \approx 0 \\ -270.83468 \\ -946.38856 \end{bmatrix} \tag{24}$$

Stability

In order to calculate stability for case A, A,B,C and D matrices is chosen from equations (17 and 19). Transfer function of the system is found and poles zeros are calculated(calculation is shown in the matlab code as shown in Appendix). Poles are same as the eigen values as shown in equation (24). zeros for the four transfer functions is as shown,

$$Zeros_A(1,1) = -1000, \quad Zeros_A(1,2) = 10^3 \begin{bmatrix} -0.5000 + 1.3228i \\ -0.5000 - 1.3228i \end{bmatrix} \tag{25}$$

$$Zeros_A(2,1) = \begin{bmatrix} 0 \\ -1000 \end{bmatrix}, \quad Zeros_A(2,2) = 10^3 \begin{bmatrix} 0 \\ -0.5000 + 1.3228i \\ -0.5000 + 1.3228i \end{bmatrix} \tag{26}$$

Since, all the zero's and pole's real part either lies on the imaginary axis or is at the negative real axis, system is marginally stable.

To know whether the system described by case B is stable, similar steps is followed as explained for case A. Poles remain the same for this case as matrix A is not changed and matrix B is same as used in the previous case. Zeros for the system is as shown,

$$Zeros_B(1,1) = 10^3 \begin{bmatrix} -0.0015 + 1.5834i \\ -0.0015 - 1.5834i \\ -0.9553 \\ -0.0417 \\ 0 \end{bmatrix}, \quad Zeros_B(1,2) = 0 \tag{27}$$

$$Zeros_B(2,1) = 0, \quad Zeros_B(2,2) = 10^3 \begin{bmatrix} 0 \\ -0.3893 + 1.4775i \\ -0.3893 - 1.4775i \\ -0.1106 + 0.1748i \\ -0.1106 - 0.1748i \end{bmatrix} \tag{28}$$

Similar to the previous case, all the zero's and pole's real part is either lies on the imaginary axis or is at the negative real axis, system is marginally stable.

Question E

As asked,the initial states, applied rotor voltage and external torque are all zero. Then the applied rotor voltage is increased step wise and is changed from 0 to 10 V. As the system is energised, the angular velocities in the system start increasing and reach steady state at 0.02 s as indicated by the simulation figure 2. More oscillations are observed in ω_1, since the pole ϕ_1 is at origin, making the system marginally stable. since all the speeds are interlinked as seen in the equations (4,6 and8) ω_2 and ω_3 do not have a smooth rise for the step response Once all the angular velocities reach steady state, an external

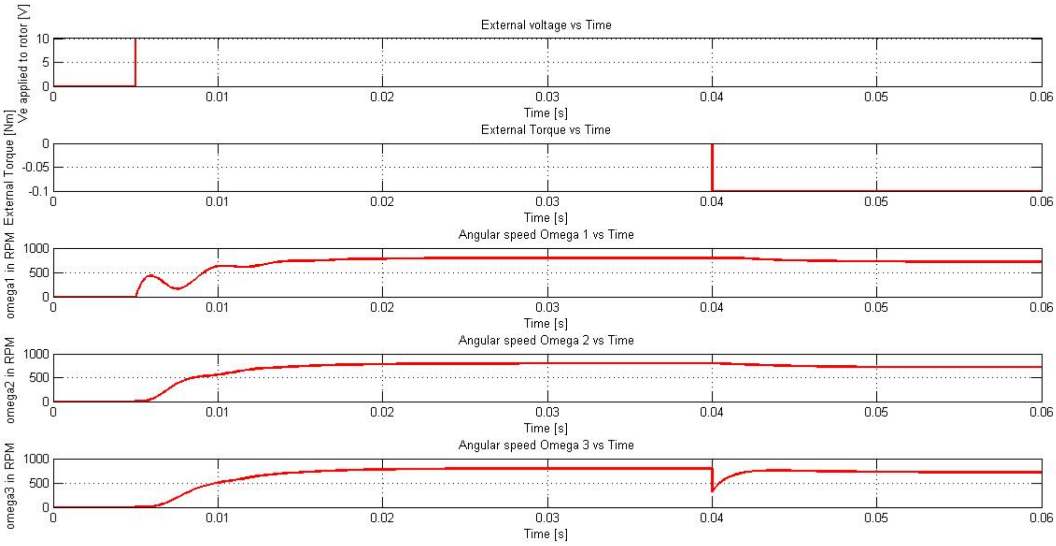

Figure 2: Simulation results of the system is shown, first two subplots shows the step inputs from V_a and T_e respectively. Rest of the three subplots shows the response of angular speeds ω_1, ω_2 and ω_3 for the step inputs respectively

reducing torque of $0.1\,\mathrm{N\,m}$ is applied step wise at $0.04\,\mathrm{s}$ as seen in simulation. As the applied torque is directly affecting ω_3 as per the equation (8), sudden drop is observed in ω_3. All angular velocities settle to a new steady state value (the value is reduced when compared to before of the application of reducing torque) after the application of reducing external torque.

Question F

In this task, the transfer function from the input-applied rotor voltage to the output-rotor current and ω_3 is derived. Transfer function is

$$G(s) = \begin{bmatrix} \frac{s^5+1000s^4+2.55\times10^6s^3+2.5\times10^9s^2+1\times10^{11}s-0.03109}{s^5+2000s^4+3.55\times10^6s^3+3.05\times10^9s^2+6\times10^{11}s+0.2381} \\ \frac{5e12s}{s^5+2000s^4+3.55\times10^6s^3+3.05\times10^9s^2+6\times10^{11}s+0.2381} \end{bmatrix} \tag{29}$$

$$Poles = 10^3 \cdot \begin{bmatrix} -0.391388379995381 + 1.479082902413484i \\ -0.391388379995381 - 1.479082902413484i \\ -0.946388564469211 + 0.000000000000000i \\ -0.270834675540029 + 0.000000000000000i \\ -0.000000000000000 + 0.000000000000000i \end{bmatrix} \tag{30}$$

$$Zeros(1) = 10^3 \cdot \begin{bmatrix} -0.001459838945321 + 1.583414552397049i \\ -0.001459838945321 - 1.583414552397049i \\ -0.955330289481925 + 0.000000000000000i \\ -0.041750032627434 + 0.000000000000000i \\ 0.000000000000000 + 0.000000000000000i \end{bmatrix} \tag{31}$$

$$Zeros(2) = 0 \tag{32}$$

Observing the poles and zeroes, it can be concluded that, four poles are on the left half of S plane and one pole is at origin, which makes the system marginally stable. Four zeros are on left half of S plane and one zero is at origin. Since, by definition, a system is said to be minimal phase if there is no zero on right half of S plane, **given system is minimal phase.**

Appendix

Code to calculate matrix A, its eigen values, transfer functions, poles and zeros for question C,D and F.

```
R = 1; %Ohms
K_E = 0.1; %Vsrad^-1
K_T = 0.1; %NmA^-1
J1 = 10^(-5) ;%kgm^2
J2 = 4*10^(-5) ;%kgm^2
D1 = 20 ; % Nmrad^-1
D2 = 2 ; % Nmrad^-1
B = 2*10^-3 ; %Nms

A = [(-K_T*K_E/(J1*R)) 0 (-D1/J1) D1/J1 0;
    0 0 D1/J2 -(D1+D2)/J2 D2/J2;
    1 0 0 0 0;
    0 1 0 0 0;
    0 0 0 D2/B -D2/B];

A_eig = eig(A);

digitsOld = digits(8);
A_vpa = vpa(A_eig) % To have better readability

%%
%Question D

%Case A
B_d = [(K_T/(J1*R)) 0 ;
    0 0;
    0 0;
    0 0;
    0 1/B];
C_a = [0 0 0 1 0;
       0 1 0 0 0];
D_a = [0 0 ;
       0 0];
C_b = [-K_E/R 0 0 0 0;
    0 0 0 D2/B -D2/B];
D_b = [1/R 0 ;
    0 1/B ];
sys_a = ss(A,B_d,C_a,D_a);
Gs_a = tf(sys_a)
%To find zeros and poles
for n = 1:1:2
    for m = 1:1:2
```

```
[zz_a,pp_a,kk_a] = tf2zp(Gs_a.numerator{n,m},Gs_a.denominator{n,m})
    end
end

sys_b = ss(A,B_d,C_b,D_b);
Gs_b = tf(sys_b)
%To find zeros and poles
for n = 1:1:2
   for m = 1:1:2
[zz_a,pp_b,kk_b] = tf2zp(Gs_b.numerator{n,m},Gs_b.denominator{n,m})
   end
end

%%
%Question F
B_mat = [(K_T/(J1*R)) ;
    0 ;
    0 ;
    0 ;
    0 ]
C = [-K_E/R 0 0 0 0;
    0 0 0 D2/B -D2/B]
D = [1/R ;
    0 ]
I = eye(5);


sys = ss(A,B_mat,C,D);
Gs = tf(sys)
%To find zeros and poles
for n = 1:1:2
[zz,pp,kk] = tf2zp(Gs.numerator{n,1},Gs.denominator{n,1})
end
```