LINEAR CONTROL SYSTEM DESIGN

Exercises with Solutions

Department of Electrical Engineering
Division of Systems and Control
CHALMERS

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Preface

This exercise compendium is a collection of material to be used in in the course Linear Control System Design (SSY285). The ambition has been to sort the exercises in this compendium such that the chapters agree with the corresponding chapters in the course textbook *Control Theory, Multivariable and Nonlinear Methods*, by Torkel Glad and Lennart Ljung (Taylor & Frances). Many of the exercises in this material have also been developed in their group at Department of Electrical Engineering, Linköping University, and they are marked (LiTH). A couple of exercises originates from *Feedback systems: Supplemental exercises and solutions* (Princton University Press) by Karl Johan Åström and Richard M. Murray, and those are marked (ÅM).

The majority, though, originates from previous course material and exam problems at the Automatic control group, Department of Electrical Engineering, Chalmers, formulated by myself, Claes Breitholtz and Balazs Kulcar. The answers and solutions mainly originates from my time as lecturer and teaching assistant in the course Multivariable Control - so if they are troublesome I am to blame.

Göteborg, October 2019

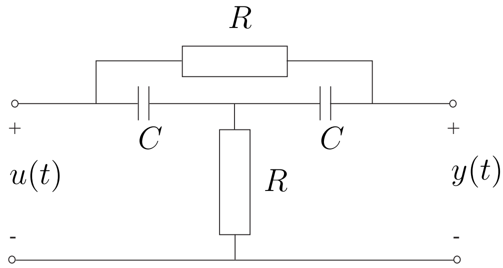
Torsten Wik

2 Representation of linear systems

2.1

(Chalmers)

Consider the T-net in the figure.



(a) Derive a state space model for the system! The input signal is the voltage u and the output is the voltage y.

(b) Determine the transfer function from u to y! Is this a minimum phase system?

2.2

(LiTH)

A simplified description of an AC-generator is as follows:

The coil current is I_m and the mechanical torque applied to the rotor is M. In the stator winding an alternating voltage with amplitude e is generated. This circuit is subjected to a load with resistance R. The angular rate is ω. This gives us

$$
\begin{aligned}
e &= R \cdot I_f && (I_f = \text{current amplitude in the secondary circuit}) \\
J\dot{\omega} &= M - M_e && (\text{applied torque minus induced voltage}) \\
M_e &= K_e \cdot \omega \cdot I_f \\
e &= C_e \cdot I_m \cdot \omega
\end{aligned}
$$

Consider e and ω as output signals, M and I_m as input signals, and R as a disturbance. Express the system on a state space form.

Linearize at the operating point

$$\omega_0 = R_0 = I_{m0} = M = 1$$

and set

$$K_e = C_e = 1, \;\; J = 1.$$

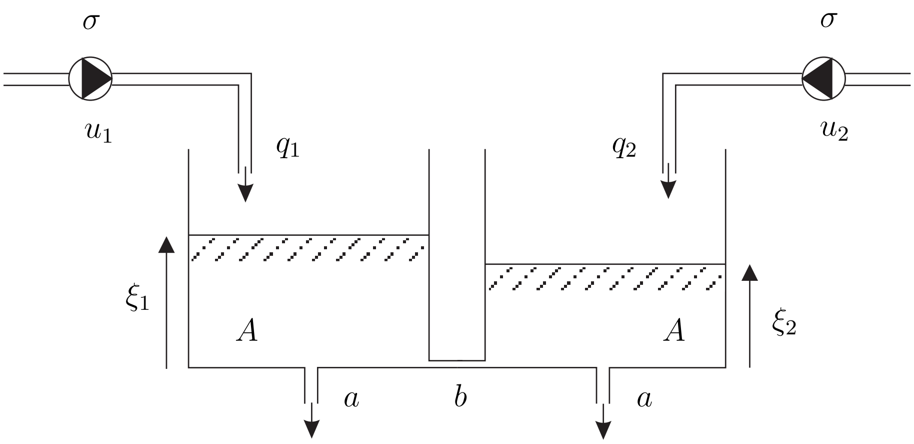
Derive the transfer function matrix from

$$u = \begin{bmatrix} \Delta M \\ \Delta I_m \\ \Delta R \end{bmatrix} \text{ to } \; y = \begin{bmatrix} \Delta \omega \\ \Delta e \end{bmatrix}.$$

2.3

(Chalmers)

Consider the two identical vessels in the figure below. The liquid can flow between the vessels through a tube with a cross-sectional area b. Each vessel has a cross-sectional area A and the outlet pipes has a cross-sectional area a.



The system input variables are the influent flows q_1 and q_2 to the two vessels. These are linearly dependent on the actual control signals u_i to the pumps according to

$$q_i = \sigma u_i, \qquad i = 1, 2.$$

Let the measured levels ξ_1 and ξ_2 be the states of the system.

(a) Derive a state space model for this system if the flows follows Bernoullis law.

(b) What stationary flows are required to maintain the levels $\bar{\xi}_1 = 9$ dm and $\bar{\xi}_2 =$ 8 dm? Use the following parameter values:

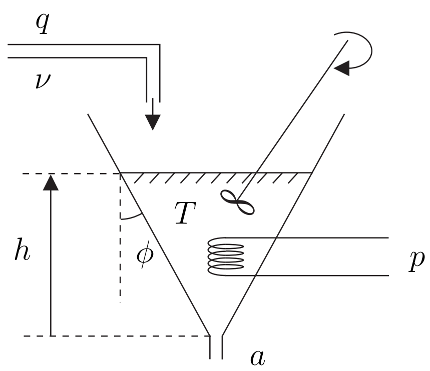
$$A = 1 \text{ m}^2,\; a = b = 1 \text{ dm}^2,\; g = 9.8 \text{ ms}^{-2} \text{ och } \sigma = 0.1 \text{ dm}^3\text{V}^{-1}\text{s}^{-1}.$$

(c) Determine the linearized state space model valid around this operating point.

2.4

(Chalmers)

Consider the funnel shaped vessel in the figure. A liquid having a density ρ and specific heat capacity c is poured into the well mixed vessel. This flow, q, having a temperature ν, and the applied heating power p are the inputs to the system. The outlet, having a cross-sectional area a is located at the very bottom of the vessel. The angle between the funnel walls and the vertical plane is ϕ.



(a) Let the temperature T in the vessel and the liquid level h be the states of the system. Determine the corresponding state-space model.

(b) Determine the corresponding linear state-space model for the operating point given by $h = \bar{h}$ and $T = \bar{T}$, when the influent temperature varies around $\bar{\nu}$. It can be assumed that $\bar{T} > \bar{\nu}$. (Begin by determining the stationary expressions for $\bar{q}$ och $\bar{p}$).

(c) What are the time constants of the system?

Use the parameters values

$$\begin{array}{llllllll} \phi = & 44.34^\circ & a = & 0.226 \text{ dm}^2 & g = & 9.8 \text{ ms}^{-2} & \rho = & 1.0 \text{ kg/dm}^3 \\ c = & 4\text{kJ/(kgK)} & \bar{h} = & 1.0 \text{ m} & \bar{T} = & 20^\circ C & \bar{\nu} = & 10^\circ C \end{array}$$

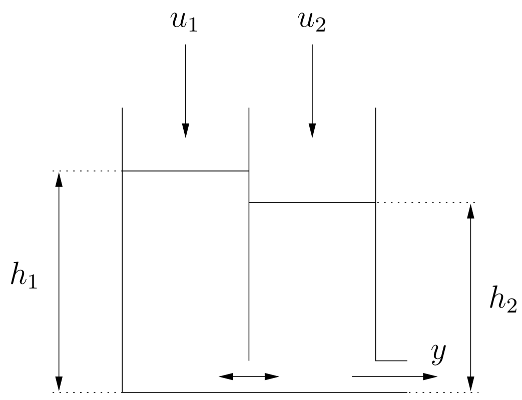
Introduce the parameter k, defined as

$$k = \frac{\pi \tan^2 \phi}{3} = 1,$$

to simplify the calculations.

2.5 (LiTH)

Consider a tank with two equal volumes separated by a vertical wall with a hole.



The influent flow to the left half is denoted u_1 and to the right half the influent is denoted u_2, both regarded as the systems input signals. The effluent flow y is the output, and is related to the liquid level in the right half:

$$y(t) = \alpha h_2(t)$$

The flow between the two volumes is proportional to the level difference:

$$f(t) = \beta(h_1(t) - h_2(t))$$

(flow from left to right).

Note that h_i, u_i are y to be considered as deviations from operating point values. As a consequence they do not necessarily have to be positive all the time.

(a) Assume $A_1 = A_2 = 1$ and determine the transfer function from u_1 and u_2 to y.

(b) Determine the stationary gain, i.e. the gain at frequency zero. (Note that this is a scalar).

(c) It turns out that this number is larger than 1. How is this possible - can the system produce more water than is added? Explain!

2.6 (Chalmers)

In an ideally stirred tank reactor a reversibel reaction takes place, where A becomes B and vice versa. The influent molecular concentrations of the substances are c_{Ai}

and c_{Bi} and inside the reactor they are c_A and c_B. The reactor volume V and the flow Q through the tank can be assumed to be constant.g

The rate at which A reacts into B is

$$r = f(c_A, c_B) = k \ln(c_A/c_B) \qquad (\text{mole m}^{-3}\text{s}^{-1}).$$

(a) Determine a state space model for the reactor, and also the inlet concentrations required to maintain the stationary operating point concentrations

$$\bar{c}_A = 0.2 \text{ mol m}^{-3} \text{ and } \bar{c}_B = 0.1 \text{ mol m}^{-3}.$$

(b) Determine the linearized state-space model for this operating point. Use

$$V = 1 \text{ m}^3, \quad Q = 1 \text{ dm}^3\text{s}^{-1} \text{ och } k = 10^{-4} \text{ mol m}^{-3}\text{s}^{-1}.$$

2.7

(LiTH)

Give a state-space realization of the following system:

$$G(s) = \begin{bmatrix} \frac{1}{(s+1)(s+2)} & \frac{s+3}{(s+1)(s^2+s+1)} \end{bmatrix}$$

2.8

(LiTH)

Give a state-space realization of the following system:

$$y(t) = \frac{p}{p^2+4p+4}u_1(t) + \frac{p-1}{p^2+5p+6}u_2(t)$$

2.9

(LiTH)

Give a state-space realization of the following system:

$$\ddot{y} + a_1\dot{y} + a_2 y = b_{11}\dot{u}_1 + b_{12}u_1 + b_{21}\dot{u}_2 + b_{22}u_2.$$

2.10 (LiTH)

Give a state-space realization of the following system:

$$\begin{cases} \dot{y}_1 + y_2 & = & \dot{u} + 2u \\ \dot{y}_2 + y_2 + y_1 & = & u \end{cases}$$

2.11 (ÅM)

(Keynesian economics) Keynes simple model for an economy is given by

$$Y(k) = C(k) + I(k) + G(k),$$

where Y, C, I and G are gross national product (GNP), consumption, investment and government expenditure for year k.

Consumption and investment are modeled by difference equations of the form

$$\begin{aligned} C(k+1) &= aY(k) \\ I[k+1] &= b(C[k+1] - C(k)), \end{aligned}$$

where a and b are parameters. The first equation implies that consumption increases with GNP but that the effect is delayed. The second equation implies that investment is proportional to the rate of change of consumption.

Show that the equilibrium value of the GNP is given by

$$Y_e = \frac{1}{1-a}(I_e + G_e),$$

where the parameter 1/(1 - a) is the Keynes multiplier (the gain from I or G to Y). With $a = 0.25$ an increase of government expenditure will result in a fourfold increase of GNP.

Also, show that the model can be written as the following discrete-time state model:

$$\begin{bmatrix} C(k+1) \\ I(k+1) \end{bmatrix} = \begin{bmatrix} a & a \\ ab-b & ab \end{bmatrix} \begin{bmatrix} C(k) \\ I(k) \end{bmatrix} + \begin{bmatrix} a \\ ab \end{bmatrix} G(k),$$
$$Y(k) = C(k) + I(k) + G(k).$$

2.12 (ÅM)

Consider the system

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x,$$

which is stable but not asymptotically stable. Show that if the system is driven by the bounded input $u = \cos t$ then the output is unbounded.

2.13

(Chalmers)

Consider the following linear model:

$$\frac{d}{dt}x(t) = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} u(t) + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \frac{d}{dt}u(t)$$

A linear model on the standard form

$$\begin{aligned} \frac{d}{dt}x &= Ax + Bu \\ y &= Cx \end{aligned}$$

can be achieved using the transformation

$$x(t) = Tz(t) + Su(t).$$

Determine suitable transformation matrices T and S, and the resulting state space model! (Note that $u(t)$ and $y(t)$ must be the same as in the original model though the original states x and the new states z are different!)

2.14

(ÅM)

Consider the simple queue model

$$\frac{dx}{dt} = \lambda - \mu\frac{x}{x+1} \tag{1}$$

based on continuous approximation, where λ is the arrival rate and μ is the service rate.

(a) Linearize the system around the equilibrium obtained with $\lambda = \lambda_e$ and $\mu = \mu_e$.

(b) The queue can be controlled by influencing the admission rate $\lambda = u\lambda_e$ or the service rate $\mu = u\mu_e$. Compute the transfer functions for these two control inputs and give the gains and the time constants of the system. Discuss the particular case when the ratio $r = \lambda_e/\mu_e$ goes to 1.

3 Properties of linear systems

3.1

(Chalmers)

A second order system is given by the state equations

$$\begin{aligned} \frac{d}{dt}x &= Ax + Bu \\ y &= Cx \end{aligned}$$

where the matrices B and C, and the *transition matrix* are

$$\begin{aligned} B &= \begin{bmatrix} -0.5 \\ 1 \end{bmatrix} \qquad\qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \\ e^{At} &= \begin{bmatrix} e^{-t}(\cos t - \sin t) & e^{-t}\sin t \\ -2e^{-t}\sin t & e^{-t}(\cos t \sin t) \end{bmatrix} \end{aligned}$$

(a) Determine the continuous time impulse response and the corresponding transfer function.

(b) Is the system stable?

(c) Assume that the system is sampled with a sampling time of 1.57 s and that the input is piecewise constant over a sampling interval. Determine the poles of the discrete time system. Is it stable?

3.2

(ÅM)

Consider the Keynesian macroeconomic model derived in Exercise 2.11,

$$\begin{aligned} \begin{bmatrix} C(k+1) \\ I(k+1) \end{bmatrix} &= \begin{bmatrix} a & a \\ ab-b & ab \end{bmatrix} \begin{bmatrix} C(k) \\ I(k) \end{bmatrix} + \begin{bmatrix} a \\ ab \end{bmatrix} G(k), \\ Y(k) &= C(k) + I(k) + G(k), \end{aligned}$$

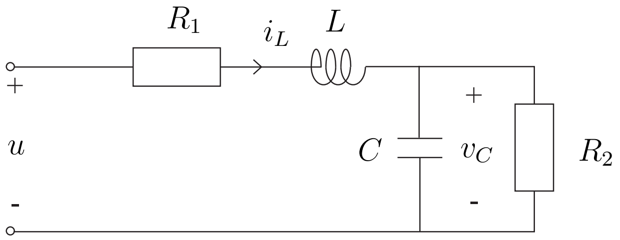
where Y, C, I and G are gross national product (GNP), consumption, investment and government expenditure for year k.

(a) For what values of the parameters a and b is the system stable?

(b) Assume that the system is in equilibrium with constant capital spending C, investment I and government expenditure G. Explore what happens when government expenditure increases by 10%. Use the values a = 0.25 and b = 0.5 then.

3.3 (Chalmers)

Consider the electrical circuit with input voltage u in the figure.



(a) Choose the current through the inductor and the voltage over the capacitor as states and determine a state-space model of the system on the form

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t).$$

(b) Determine the transition matrix $\Phi(t) = e^{At}$ for the case when

$$R_1 = R_2 = 1\,\Omega,\ L = 1\text{ H, and } C = 1\text{ F.}$$

(c) The circuit is connected to an accumulator, which has a very low inner resistance and an induced voltage of 2 V. The system is in a stationary state when Philip enters the laboratory. The first thing Philip does is to put his apartment key over the accumulators poles (at $t = 0$) with the consequence that the accumulator starts to boil.

Plot the values of the states for $0 < t < 3$ s.

3.4 (Chalmers)

A dynamic system is given by the following scalar differential equation together with an algebraic expression:

$$\begin{aligned} \frac{d}{dt}\xi &= -\xi + u\eta^3 \\ 0 &= -\eta + u^2 e^{\eta}. \end{aligned}$$

(a) A control system aiming at keeping the systems at a given stationary operating point ξ_o is to be designed. Determine the complete operating point (ξ_o, u_o, η_o) when $\xi_o = 1$.

(b) The systems input is u and its output is y. Determine a linear state-space model valid in a neighbourhood of the operating point determined in (a).

(c) How does the system's stability depend on the operating point?

3.5 (Chalmers)

A controllable and observable time invariant MIMO system is given by

$$\begin{bmatrix} \frac{d}{dt}x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$$

Show that for any linear transformation $x = Tz$, where T is non-singular, the system remains observable and controllable.

3.6 (Chalmers)

Consider the MIMO system

$$\frac{d}{dt}x(t) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & \alpha \\ 0 & 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & \beta \\ 0 & 0 \\ 1 & 0 \end{bmatrix} u(t)$$

Investigate the controllability of this system for different values of α and β.

3.7 (ÅM)

(Unreachable) Consider the dynamics of the two systems as

$$\frac{dz}{dt} = Az + Bu, \qquad \frac{dy}{dt} = Ay + Bu.$$

(a) If z and y have the same initial condition, they will always have the same state regardless of the input that is applied. Argue how this violates the definition of reachability for the systam state $x = \begin{bmatrix} z^T & y^T \end{bmatrix}^T$.

(b) Show that the corresponding controllability matrix $\mathcal{S}$ does not have full rank. What is the rank of the controllability matrix if the individual systems (A, B) is reachable?

3.8 (Chalmers)

There are two different organisms (A and B) in an autonomous biological process. The two organisms interact such that their growth rates are proportional to the product of their concentrations (c_A and c_B). At the same time the organisms die at a rate proportional to their individual concentration. The system is described by the following bilinear state equations:

$$\begin{aligned}\frac{d}{dt}c_A &= -c_A + \alpha c_A c_B \\ \frac{d}{dt}c_B &= -c_B + \beta c_A c_B\end{aligned}$$

The arithmetical average $c_M = 0.5(c_A + c_B)$ of their concetrations is measured.

(a) Determine the two possible steady states of the system and their corresponding linearized state equations for these operating points.

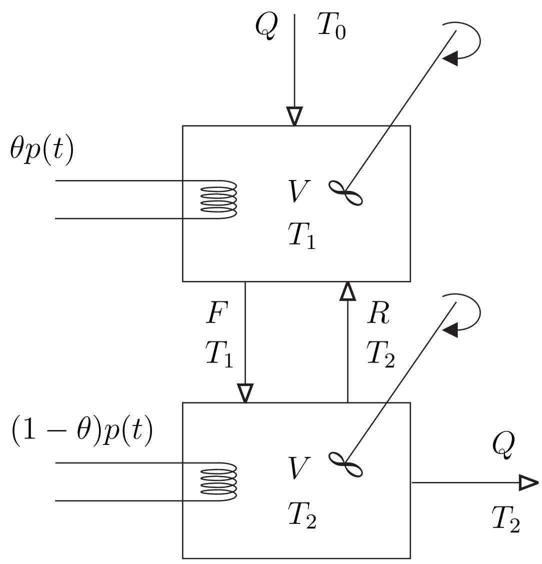
(b) Are the two models stable for all combinations of the process parameters α and β?

(c) Are the two models observable for all combinations of α and β?

3.9

(Chalmers)

Consider the system of two ideally stirred tanks in the figure. The supplied power $p(t)$ in the electrical heating is the systems only control signal, and the temperature variations in the incoming water is the main disturbance. The heating power is split by construction into two parts such that the upper tank is heated by $\theta p(t)$ and the lower tank is heated by $(1-\theta)p(t)$, where *theta* is a constant (between 0 and 1).



(a) Choose the temperatures in the two tanks as state variables and determine the corresponding linear state space model

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t).$$

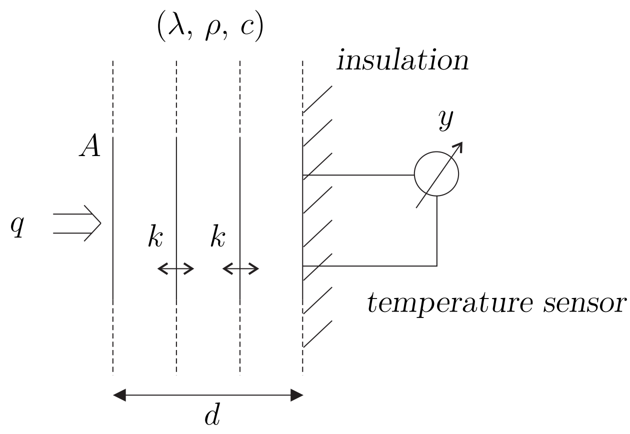
(b) Assume that the volumes are $V = 1$ m^3 and that the flows are $R = Q = 1$ m^3. Determine the condition number of the controllability matrix $\mathcal{S}(A, B)$ for the two cases $\theta = 0.6$ och $\theta = 0.4$.

(c) Is there a value of θ when the system no longer is controllable?

3.10

(Chalmers)

The figure below illustrates a homogeneous metal sheet with thickness d, surface area A, density ρ and specific heat capacity c. The heat conductivity of the material is λ.



To simplify the mathematical modeling the sheet is divided into N equally thick layers. On each of these layers we assume that the temperature is constant. The heat conduction can then be approximated with $N-1$ space discrete heat conductivities,

$$k = \frac{(N-1)\lambda}{d}.$$

Onto one of the sides of the sheet the control power q (W/m^2) is applied. On the other side, which is thermally insulated, an analogue temperature sensor is placed.

(a) Assume that three layers is a sufficient level of approximation. Determine the corresponding state space model..

(b) Investigate the observability and controllability of the model.

(c) Is the system stable?

3.11

(LiTH)

Consider the transfer function matrix

$$G(s) = \begin{bmatrix} \frac{1}{s+2} & -\frac{1}{s+2} & \frac{1}{s+2} \\ \frac{1}{s+2} & \frac{s+1}{s+2} & \frac{1}{s+2} \end{bmatrix}.$$

Determine the pole and zero polynomials of this system. What is the lowest order required for a state space realization having the same input-output behavior?

3.12

(LiTH)

Consider the MIMO-system

$$\dot{x}(t) = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(t)$$

Determine a minimum realization of the system.

3.13

(Chalmers)

The transfer function matrix G(s) is given:

$$G(s) = \begin{bmatrix} \frac{1}{s(s+1)} & \frac{1}{s} \\ \frac{1}{s+1} & \frac{2}{s+1} \end{bmatrix}$$

(a) Compute the pole polynomial $P(s)$ and the zero polynomial $N(s)$. Is the transfer function $G(s)$ a minimum phase system?

(b) Formulate a minimum order state-space model of the system $G(s)$.

3.14 (Chalmers)

Consider the system

$$\dot{x}(t) = \begin{bmatrix} -1.5 & -\beta \\ -1 & \gamma \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + \sqrt{\beta \cdot \gamma} u(t)$$

where β and γ are real valued finite scalars.

(a) Find values of β and γ for which the system becomes uncontrollable and unobservable.

(b) With $\beta = \gamma = 1$ find the transformation matrix T that diagonalizes the above representation and give the corresponding diagonalized state-space representation $(\tilde{A},\ \tilde{B},\ \tilde{C},\ \tilde{D})$.

3.15 (ÅM)

(Kalman decomposition) Consider a linear system characterized by the matrices

$$A = \begin{bmatrix} -2 & 1 & -1 & 2 \\ 1 & -3 & 0 & 2 \\ 1 & 1 & -4 & 2 \\ 0 & 1 & -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}, \quad D = 0.$$

Construct a Kalman decomposition for the system. (Hint: Try to diagonalize.)

4 Sampled Data System

4.1 (LiTH)

Consider the discrete time system

$$y(t+1) = 0.9y(t) + u(t),$$

which is controlled by a PI controller

$$u(t) = K_1 e(t) + \frac{K_2}{1 - q^{-1}} e(t),$$

where

$$e(t) = r(t) - y(t).$$

Set $K_1 = 1$ and determine for what values of K_2 the feedback system is stable!

4.2

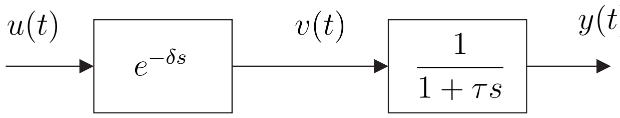
(Chalmers)

A first order system with time delay is given by $Y(s) = G(s)U(s)$, with the transfer-function

$$G(s) = \frac{e^{-\delta s}}{1 + \tau s},$$

where δ and τ are constants. The system is sampled with a sample time $h > \delta$ and the input signal $u(t)$ is constant on each sample interval.

The discrete time transfer function $H(z)$ corresponding to $G(s)$ is to be determined. A method where $G(s)$ is split into two parts, one describing the time delay and one describing the first order transfer function, is suggested (see figure):



(a) Derive a state space model for the process having $v(t)$ as input signal and $y(t)$ as output signal!

(b) Determine the convolution describing how $y(t)$ depends on $v(t)$!

(c) Determine $v(t)$ as a function of $u(t)$ for the, from a control perspective interesting, interval

$$kh - h < t < kh, \qquad k = 1, 2, \ldots$$

Inserting this expression into the convolution gives an expression from which a time discrete state space model can easily be derived. Then derive the transfer function $H(z)$ from this state space model!

4.3

(Chalmers)

Consider the following continuous time system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + u \end{aligned}$$

(a) Show that the system is controllable!

(b) Investigate if the controllability for the corresponding time discrete system with piecewise constant control depends on the sampling interval T. Is there a critical value of T?

5 Disturbance modelling

5.1

(LiTH)

A continuous stochastic process $u(t)$ has the power spectrum $\Phi_u(\omega)$. Show how the process can be represented by white noise passing through a linear filter, and in particular, determine the linear filter when

(a) $$\Phi_u(\omega) = \frac{a^2}{\omega^2 + a^2}.$$

(b) $$\Phi_u(\omega) = \frac{a^2 b^2}{(\omega^2 + a^2)(\omega^2 + b^2)}.$$

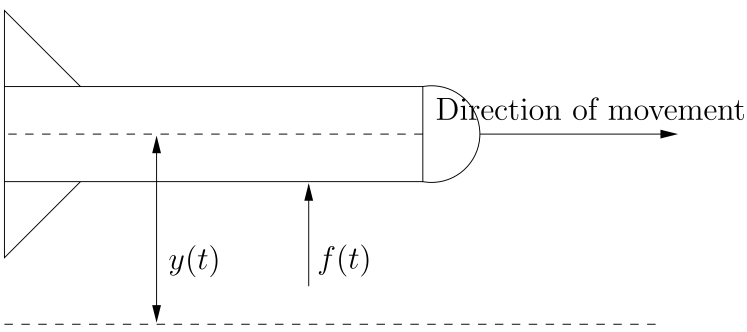
5.2

(LiTH)

A simple model for sideway control of a robot is

$$m\ddot{y}(t) = f(t),$$

where $y(t)$ is the position of the robot perpendicular to the direction of the robot, m is the mass of the robot and $f(t)$ is the force perpendicular to the direction of the robot.



The force $f(t)$ originates from the control, i.e. the rudder position $u(t)$, and a disturbance $z(t)$ from the wind.

$$f(t) = Ku(t) + z(t)$$

The robot is controlled by a computer sampling the signals at 10Hz. Derive a model for the robot dynamics suitable for a

(a) study of the robot behaviour after a sudden wind throw.

(b) study of the robot behaviour during normal wind conditions. The wind force $z(t)$ may then be described by a stochastic process, for which the value in each sampling is quite strongly correlated to the previous values. An auto regressive (AR) model of first order can be used:

$$z(t) + az(t-1) = w(t),$$

where $w(t)$ is white noice, and $a = -0.9$.

5.3

(LiTH)

Consider a missile with a position $z(t)$ driven by the drag force u. The air drag force is approximately

$$f = k_1\dot{z} + v,$$

where v is a more or less stochastic variations caused by the wind.

(a) Derive both a state space and an input/ouput model describing how z is affected by u and v.

(b) The disturbance v has been measured and the spectral density has been estimated to be

$$\Phi_v(\omega) = k_0 \cdot \frac{1}{\omega^2 + a^2}.$$

Extend the state space model in (a) such that the disturbance can be described by white noise.

5.4

(LiTH)

Assume in Exercise 5.3, that the position z is measured with a measurement noise $n(t)$, i.e.

$$y(t) = z(t) + n(t).$$

Derive a standard state space model where the noise is white with intensity 1 for the cases when

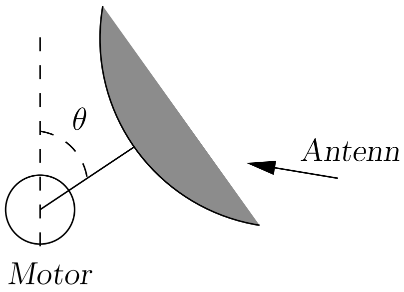
(a) $n(t)$ is white noise with intensity 0.1 ($\Phi_n(\omega) \equiv 0.1$).

(b) $\Phi_n(\omega) = 0.1\dfrac{\omega^2}{\omega^2 + b^2}$

(c) $\Phi_n(\omega) = 0.1\dfrac{1}{\omega^2 + b^2}$

5.5 (LiTH)

Consider the radar antenna in the figure



We want to achieve an improved estimate of the antenna position θ from noisy measurements θ_m. A better solution than simply averaging the measurement is to also account for the systems dynamics. To do so we need a model of the system.

The antenna dynamics can be described by

$$J\ddot{\theta}(t) + B\dot{\theta}(t) = \tau(t) + \tau_d(t),$$

where J is moment of inertia for the rotating parts of the antenna, B is a viscous friction coefficient, $\tau(t)$ is torque delivered by the engine and $\tau_d(t)$ is the torque generated by the wind. Assume that $\tau_d(t)$ can be modelled as white noise and that the torque from the engine is proportional to the voltage $\mu(t)$ over the motor:

$$\tau(t) = k\mu(t).$$

For simplicity, assume that the measurement error can be modelled as an additive white noise $e_m(t)$ uncorrelated with the wind:

$$\theta_m(t) = \theta(t) + e_m(t).$$

Discuss how a Kalman filter can be used to estimate $\theta(t)$ from $\theta_m(t)$.

Technical data:

$$\begin{aligned} B/J &= 4.6 \text{ s}^{-1} \\ k/J &= 0.787 \text{ rad/Vs}^2 \\ J &= 10 \text{ kg m}^2 \end{aligned}$$

$$E\ \tau_d(t)\tau_d(s) = v_d \cdot \delta(t-s) = 10 \cdot \delta(t-s) \text{ N}^2\text{m}^2$$

$$E\ e_m(t)e_m(s) = v_m\delta(t-s) = 10^{-7} \cdot \delta(t-s) \text{ rad}^2$$

5.6

(Chalmers)

A discrete time process is described by

$$x(k+1) = 0.8x(k) + v(k),$$

where $v(k)$ is white noise having zero mean and unit variance. Two different measurements of $x(k)$ are available:

$$y_1(k) = x(k) + w_1(k) \qquad \text{och} \qquad y_2(k) = x(k) + w_2(k).$$

Here $w_1(k)$ and $w_2(k)$ are independent white noises with zero mean and variance 1 and 2, respectively. Both $w_1(k)$ and $w_2(k)$ are independent of $v(k)$.

(a) The state $x(k)$ is estimated from both measurement channels using a stationary Kalman filter. Determine the stationary variance of the estimation error, i.e.

$$\tilde{P} = E\{(x(k|k) - \hat{x}(k|k))^2\}$$

(b) The Kalman filter can be described with transfer functions:

$$\hat{X}(z) = G_1(z)Y_1(z) + G_2(z)Y_2(z)$$

Determine the two transfer functions $G_1(z)$ and $G_2(z)$.

(c) Assume that the second measurement is delayed two samples, i.e.

$$y_2(k) = x(k-2) + w_2(k)$$

while $y_1(k)$ is the same as before. The state space model must then be modified before a stationary Kalman filter can be synthesized using the standard methods. Determine the necessary matrices A, N and C required for the solution of the Ricatti equations in this case.

(d) Investigate if the system in (c) is observable.

5.7

(Chalmers)

The discrete time stochastic signal $s(k)$ is defined by

$$s(k) = v(k-1) + a\,v(k-2), \qquad |a| < 1,$$

where $v(k)$ is a white noise.

The signal is measured under influence of white measurement noise according to

$$y(k) = s(k) + w(k).$$

The white noises $v(k)$ and $w(k)$ are independent and have the following properties:

$$E\{v(k)\} = E\{w(k)\} = 0 \qquad\qquad E\{v^2(k)\} = E\{w^2(k)\} = 1.$$

(a) Describe the system on a state space form

$$\begin{aligned} x(k+1) &= Ax(k) + Nv(k) \\ y(k) &= Cx(k) \end{aligned}$$

by choosing the states $x_1(k) = s(k)$ and $x_2(k) = v(k-1)$.

(b) Determine the matrix $\tilde{K}$ in a stationary Kalman filter for estimation of the state $x(k)$ from the the signal $y(k)$.

5.8

(Chalmers)

Among many other processes, growth in some biological systems can be described by so-called *random walk* processes, where the change at each time instant is completely random. Such a process can be described by

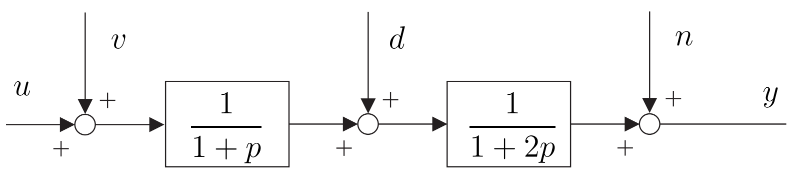
$$\begin{aligned} x(t+1) &= x(t) + v_1(t) \\ y(t) &= Cx(t) + v_2(t) \end{aligned}$$

where $v_1(t)$ and $v_2(t)$ are uncorrelated white noises with variances R_1 and R_2. Determine how to calculate an optimal estimate $\hat{x}(t|t)$ from the measurement $y(t)$. What conditions on the matrix C are required for the method to work satisfactorily?

5.9

(Chalmers)

Consider the block diagram in the figure.



A spectrum for the disturbance d has been determined:

$$\Phi_d(\omega) = \frac{1}{\omega^2 + 1}$$

(a) Write the system on a state space form

$$\begin{aligned} \frac{d}{dt}x(t) &= Ax(t) + Bu(t) + Ne(t) \\ y(t) &= Cx(t) + n(t), \end{aligned}$$

where $e(t)$ is a white noise having intensity 1.

(b) Suppose we have determined the spectrum for the measurement noise n:

$$\Phi_n(\omega) = \frac{\omega^2 + 4}{\omega^2 + 9}.$$

Write this system on a state space form

$$\begin{aligned}
\frac{d}{dt}x(t) &= Ax(t) + Bu(t) + Nv_1(t) \\
y(t) &= Cx(t) + v_2(t),
\end{aligned}$$

where $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is white noise having the intensity $R = \begin{bmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{bmatrix}$.

(c) What will the intensity matrix R be if n and d are independent?

5.10

(Chalmers)

For the time discrete double integrator system

$$\begin{aligned}
x(t+1) &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)
\end{aligned}$$

only the position state is measured ($y = x_1$).

(a) Design an observer such that the observer poles are in $z = 0$ and $z = 0.8$.

(b) Show that the estimated position in this observer is the actual measurement, and that the observer dynamics only affects the estimation of the speed $\hat{x}_2$. (This is called a reduced order observer.)

5.11

(ÅM)

(Discrete-time random walk) Suppose that we wish to estimate the position of a particle that is undergoing a random walk in one dimension (i.e., along a line). We model the position of the particle as

$$x(k+1) = x(k) + u(k),$$

where x is the position of the particle and u is a white noise processes with $E\{u(i)\} = 0$ and $E\{u(i)u(j)\} = R_u\delta(i-j)$. We assume that we can measure x subject to additive, zero-mean, Gaussian white noise with covariance 1.

a) Compute the expected value and covariance of the particle as a function of k.

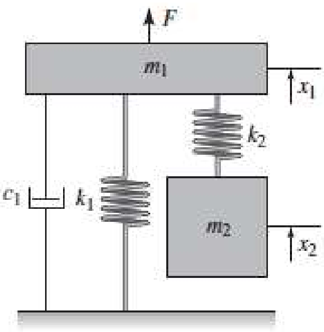
b) Construct a Kalman filter to estimate the position of the particle given the noisy measurements of its position. Compute the steady-state expected value and covariance of the error of your estimate.

c) Suppose that $E\{u(0)\} = \mu \neq 0$ but is otherwise unchanged. How would your answers to parts (a) and (b) change?

5.12

(ÅM)

(Vibration absorber) Damping vibrations is a common engineering problem. A schematic diagram of a damper is shown below: The disturbing vibration is a sinusoidal



Figur 1: A vibration absorber [AM]

force acting on mass m_1, and the damper consists of the mass m_2 and the spring k_2.

(a) Show that the transfer function from disturbance force to height x_1 of the mass m_1 is

$$G_{x_1F} = \frac{m_2s^2 + k_2}{m_1m_2s^4 + m_2c_1s^3 + (m_1k_2 + m_2(k_1 + k_2))s_2 + k_2c_1s + k_1k_2}$$

(b) How should the mass m_2 and the stiffness k_2 be chosen to eliminate a sinusoidal oscillation with frequency ω_0.

5.13

(Chalmers)

A scalar dynamic system can be described by an autoregressive (AR) process

$$x(k+1) = 0.8x(k) + v(k)$$

where v is white noise of variance 1. The state of the system is measured by two sensors having independent white Guassian noise being indenpendent of the process disturbance v. The variance of the noise of the first sensor is 1. The other sensor is less accurate and has variance 2.

(a) Determine the optimal stationary observer (minimizing the estimation error variance) $\hat{x}(k|k-1)$ based on measurements up to time $k-1$. How large is the estimation error variance.

(b) To improve the capacity one may calculate the estimate $\hat{x}(k)$ directly from the measurement at time k. How much can the variance be reduced then?

5.14

(Chalmers)

Consider

$$\begin{aligned} x(k+1) &= 0.2x(k) + u(k) + v_1(k) \\ y(k) &= 0.6x(k) + v_2(k) \end{aligned}$$

where v and e are independent white noises having zero mean and unit variance.

(a) Determine the Kalman filter for the estimate $\hat{x}_p(k) = \hat{x}(k|k-1)$.

(b) Determine the Kalman filter for the estimate $\hat{x}_f(k) = \hat{x}(k|k)$. How much is the variance of the estimation error improved by also using the 'current' measurement $y(k)$?

6 The closed system

6.1

(ÅM)

(State feedack for Keynes economic model) As studied in Exercise 2.11 and 3.2 Keynes model for a national economy is a simple discrete time model described by

$$\begin{aligned} \begin{bmatrix} C(k+1) \\ I(k+1) \end{bmatrix} &= \begin{bmatrix} a & a \\ ab-b & ab \end{bmatrix} \begin{bmatrix} C(k) \\ I(k) \end{bmatrix} + \begin{bmatrix} a \\ ab \end{bmatrix} G(k), \\ Y(k) &= C(k) + I(k) + G(k), \end{aligned}$$

where C denotes consumption, I investment, G government expenditure and Y the GNP. Let the time increment be a quarter year and let the parameters be $a = 0.8$ and $b = 1.25$. Show that the system is marginally stable and then design a state feedback such that the closed loop system has two eigenvalues at $\lambda = 0.5$.

6.2

(Chalmers)

A process is described by the following state-space model:

$$\begin{aligned}\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\end{aligned}$$

The system is to be controlled with a state feedback $u = Kr - Lx$, where r is a reference and $L = \begin{bmatrix} l_1 & l_2 \end{bmatrix}$ is the feedback gain matrix.

(a) Determine L such that the feedback system has the poles $p = -1.6 \pm j1.2$.

(b) Determine the transfer function from $R(s)$ to $Y(s)$ for the feedback system.

(c) Determine the reference gain K such that there will be no static errors ($e = r - y$) at stepwise changes in the reference.

6.3

(Chalmers)

For the system in Exercise 6.2 assume that step disturbances are added to the control signal. To eliminate the effect of those the controller is extended with integral action.

(a) Set up a state-space model for the extended system.

(b) Determine the feedback gain L_e for the extended system resulting in closed loop poles in $p = -1 + j$, $p = -1 - j$ and $p = -2$.

(c) Determine the closed system's transfer function from $R(s)$ to $Y(s)$.

6.4

(LiTH)

For a system G in feedback control with a controller F the following four transfer functions need to be stable for the closed system to fullfill the requirements on internal stability:

$$\begin{aligned} G_{w_u u} &= (I + FG)^{-1} & G_{wu} &= -(I + FG)^{-1}F \\ G_{w_u y} &= (I + GF)^{-1}G & G_{wy} &= (I + GF)^{-1} \end{aligned}$$

Show that

$$\begin{bmatrix} G_{w_u u} & G_{wu} \\ G_{w_u y} & G_{wy} \end{bmatrix} = \begin{bmatrix} I & F \\ -G & I \end{bmatrix}^{-1}$$

6.5

(LiTH)

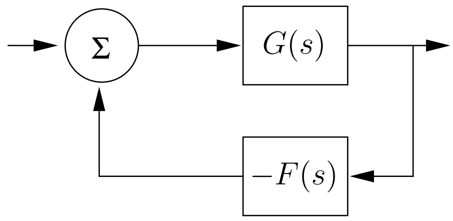
The system

$$G(s) = \frac{s-1}{s+1}$$

is in feedback control with

$$F(s) = \frac{s+2}{s-1}$$

according to the figure.



Determine G_c, T and S. Are they stable? Is the system internally stable?

6.6

(Chalmers)

Consider the discrete time system

$$\begin{aligned} x(t) &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(t-1) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t-1) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{aligned}$$

(a) Investigate the stability, controllability and observability of the system.

(b) Determine a state feedback with integral action such that the closed loop poles are $z = 0$, $z = \rho e^{j\phi}$ och $z = \rho e^{-j\phi}$, dr $0 < \rho < 1$.

(c) Determine an observer for the process states such that both poles of the observer are in $z = \nu$, dr $0 < \nu < 1$. (Note that the integral state should not be estimated by the observer.)

6.7

(ÅM)

(Inverted pendulum with rate sensor) Consider a normalized inverted pendulum with a rate sensor described by

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Design a controller based on state feedback, and an observer, such that the matrices $A - BL$ and $A - KC$ have the characteristic polynomials $s^2 + a_1 s + a_2$ and $s^2 + b_1 s + b_2$ with all coefficients positive. Show that the controller transfer function always has a pole in the right half plane.

6.8

(Chalmers)

Consider

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) \end{aligned}$$

(a) By the use of state feedback $u(k) = -Lx(k)$ can the poles of the closed loop system be placed arbitrarily?

(b) A *deadbeat* controller takes a system of order n to the origin in at most n samples (if there is no disturbance). It is in fact the fastest controller of all, but it comes of course with the cost of high control signal activity. Determine a deadbeat controller for this system.

7 Control Design Limitations

7.1

(LiTH)

For the following system

$$G(s) = \frac{s-3}{s+1}$$

we want a complementary sensitivity function

$$T(s) = \frac{5}{s+5}.$$

(a) Determine a feedback $F_r = F_y = F$ which results in this T. Will it work??

(b) Suggest an alternative T, still having the bandwidth 5 rad/s, but results in a stable systems with $F_r = F_y = F$!

(c) What is the corresponding sensitivity function?

(d) Can a solution with different F_r and F_y be a good alternative?

7.2

(LiTH)

The loop gain of a stable continuous time system has a zero in $s = 3$ and a time-delay of 1.0 seconds. What is the highest realistic cut-off frequency if the loop gain is monotonically decreasing with increasing frequency?

7.3

(LiTH)

Give an example of a system for which one cannot find a controller giving a stable system with good sensitivity reduction at low frequencies and a good damping of sensor noise at high frequencies.

7.4

(LiTH)

A multivariable system should dampen all disturbances (v) by a factor at least 10 below 0.1 rad/s. It should also dampen measurement noise (n) by a factor at least 10 over 2 rad/s. Constant disturbances should be dampened by a factor at least 100 in a steady state.

(a) Formulate conditions on S and T (their singular values) to ensuring this.

(b) Translate the specifications to conditions on the loop transfer GF.

(c) Formulate the conditions using $\|\cdot\|_\infty$ and weight functions W_S and W_T according to text book.

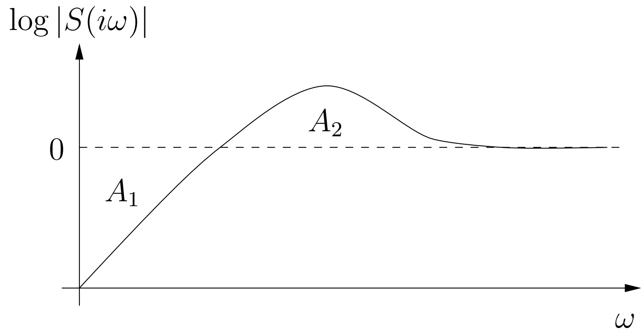
(d) What cut-off frequncy and phase margin can be expected in (b). What becomes the lower limit of $\|T\|_\infty$?

(e) How does this lower limit on $\|T\|_\infty$ agree with the requirements in (c)?

7.5

(LiTH)

A control system has the sensitivity S according to the figure.



What can be said about the loop transfer if the surface A_2 is larger than the surface of A_1?

7.6 (LiTH)

The following specifications are to hold for a certain feedback system:

(i) Disturbance on the output below 2 rad/s should be dampened by at least a factor 1000.

(ii) The system should remain stable in spite of a model uncertainty

$$|\Delta G| \leq 100|G|$$

for frequencies above 20 rad/s, where G is the nominal transfer function of the open loop system and ΔG is its absolute error.

Can this be achieved with a linear time invariant controller?

7.7 (LiTH)

The following specification are to hold for a SISO system

$$|S(i\omega)| \leq 10^{-3}, \qquad \omega \leq 1$$
$$|T(i\omega)| \leq 10^{-3}, \qquad \omega \geq 100$$

(a) Give two non-constant weight functions W_S and W_T that will guarantee this.

(b) When trying to find a controller satidfying these design criteria, e.g. using the methods in Chapter 10 in the text book, one will fail. Could this have been concluded already from the beginning?

8 Controller structures and design

8.1

(LiTH)

Let

$$G(s) = \begin{bmatrix} \dfrac{1}{s+2} & \dfrac{10}{s+1} \\ \dfrac{1}{s+5} & \dfrac{5}{s+3} \end{bmatrix}$$

(a) Determine RGA($G(0)$).

(b) What input-output pairing should be avoided?

8.2

(LiTH)

Consider the following multivariable system

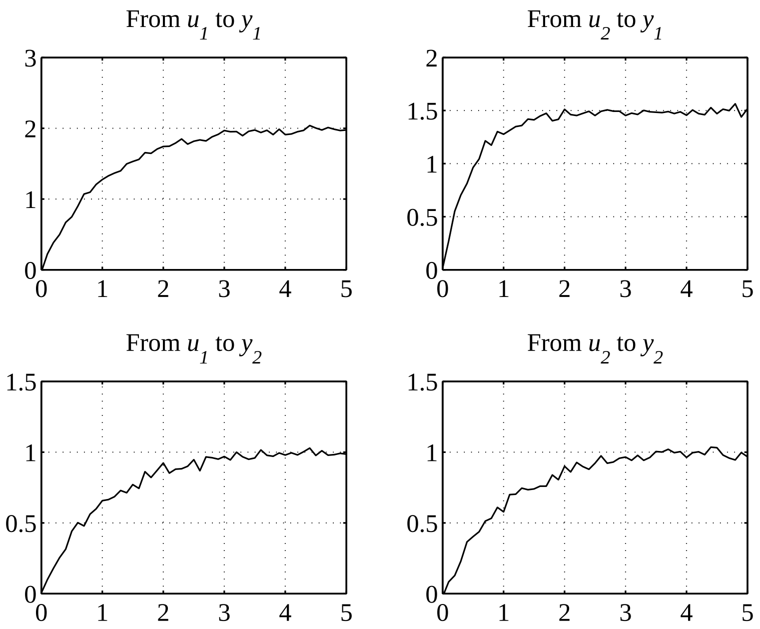
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{0.1s+1} \begin{bmatrix} \dfrac{0.6}{s+1} & -0.4 \\ 0.3 & 0.6 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Assume we eant to control the system with a diagonal (decentralized) controller and use *relative gain array* (RGA) to decide which control signal should control which output. Further assume that we aim for a crossover frequency $\omega_c = 10$ rad/s. How should the signals be paired?

8.3

(Chalmers)

Step response experiments were carried out on a system with two inputs and two outputs, i.e. first u_1 was increased one unit and the two plots to the left were registered. Then u_2 was increased one unit and the two steps to the right were registered.



(a) Give an approximate transfer function matrix $G(s)$ with 4 first order transfer functions that agree with the step responses.

(b) Try to find a compensator $W(s)$, of as low order as possible, such that WG is diagonal (decoupling) and draw a block diagram with only SISO blocks showing how the system can be controlled with two SISO PID controllers.

8.4

(Chalmers)

System identification performed on process input/output data has been used to derive a time discrete transfer function matrix from u to y:

$$H(q) = \begin{bmatrix} \frac{2}{q-0.5} & \frac{1}{q-1} \\ \frac{1}{q-0.5} & \frac{3}{q-0.5} \end{bmatrix}$$

(a) Decouple this MIMO system completely with a compensator $W(q)$ such that $W(q)H(q)$ is diagonal.

(b) The next step is to design two SISO feedback controllers, $\tilde{F}_1$ and $\tilde{F}_2$, for the decoupled system WH (do not carry out the design!). Express your resulting MIMO controller from y_1 and y_2 to u_1 and u_2 in terms of the two SISO controllers and your elements in W (you may assume zero setpoint.)

9 Linear Quadratic Optimization

9.1

(LiTH)

Consider the double integrator process

$$\ddot{y}(t) = u(t).$$

Determine the controller that minimizes

$$\int_0^\infty (y^2(t) + \eta \cdot u^2(t))dt \quad ; \quad \eta > 0.$$

What will the closed loop poles be and how will the system response change when η is reduced?

9.2

(Chalmers)

The time discrete process

$$\begin{aligned} x_1(t+1) &= x_2(t) \\ x_2(t+1) &= 0.2x_2(t) + u(t) \end{aligned}$$

is to be controlled from the initial value $x(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ such that the following loss function is minimized:

$$V = \sum_{t=0}^{\infty} \{x_1^2(t) + 0.5u^2(t)\}.$$

(a) Determine the optimal control law $u(t) = -Lx(t)$.

(b) Investigate for which values of L the feedback system is stable. Illustrate the result in the parameter plane (l_1, l_2).

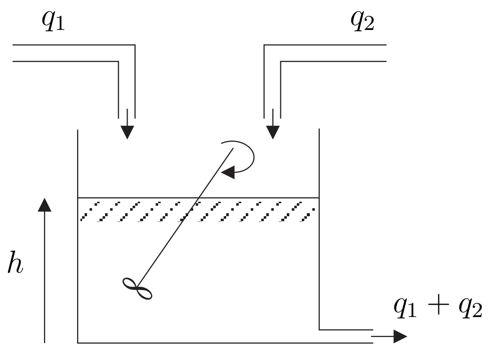
9.3

(Chalmers)

A linearized model of a tank with one outlet and two inlets has been derived (see figure):

$$(s+1)\Delta H(s) = \Delta Q_1(s) + \Delta Q_2(s),$$

where ΔH, ΔQ_1 and ΔQ_2 are the deviations in level, controlled flow and disturbance flow from the operating point $(\bar{h}, \bar{q}_1, \bar{q}_2)$.



A digital control system for regulating the level h around a set point is to be constructed. There should be no stationary offset from the set point level after step load disturbances in q_2. The sampling interval is so short that a loss function for linear quadratic optimization can be formulated directly:

$$V = \sum_{t=0}^{\infty} \{x_1^2(t) + x_2^2(t) + u^2(t)\}$$

where $x_1(t) = \Delta h(t)$, and $x_2(t)$ is an integral state.

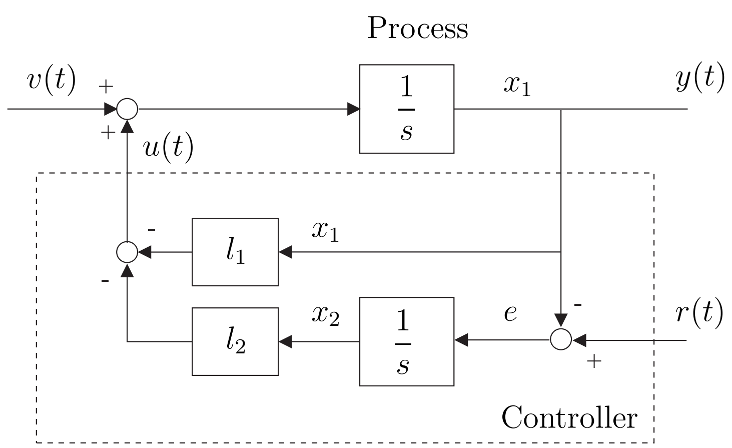
(a) Give the matrices needed to solve the Riccati equation for this case.

(b) Draw a block diagram with SISO-blocks showing in detail how the control system is structured.

9.4

(Chalmers)

A system that can be described by a scalar integration is to be PI-controlled. In this case we would like to implement the PI-controller as an optimal discrete time controller using LQ-methods. The corresponding block diagram is shown in the figure.



The criterion that is to be minimized is formulated in *continuous time* as an integral

$$V = \int_0^\infty \{x_1^2(t) + x_2^2(t) + u^2(t)\}dt$$

The sampling time is $T = 1$ and the control signal is constant on each sampling interval.

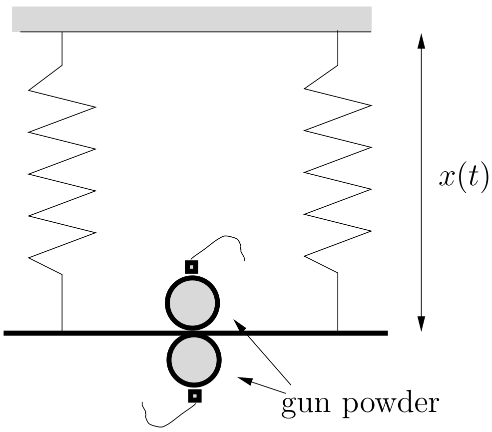
Determine the discrete time coefficient matrices required for the solution of the discrete time Riccati equation yielding the control law $L = \begin{bmatrix} l_1 & l_2 \end{bmatrix}$.

9.5

(LiTH)

This problem consists of two parts, though they can both be solved independently of each other.

(a) Consider a spring-suspended fundament plate without dampers.



The plate is controlled by firing gun powder in time intervals T. These can be of varying size and directed both upwards and downwards. Let m_k be the amount of powder exploding at time $t = kT$ and let $u_k = m_k$ or $u_k = -m_k$ if the explosion is in the upper container or in the container underneath the plate.

The scaled system can be described by

$$\ddot{x}(t) + x(t) = \sum_{k=-\infty}^{+\infty} u_k \delta(t - kT). \tag{2}$$

Let $y_k = x(kT)$ and give a standard state space model for the relation between the sequence $\{u_k\}$ and the sequence $\{y_k\}$.

(b) Consider the system

$$\begin{aligned} x(t+1) &= Fx(t) + Gu(t) \\ y(t) &= Hx(t) \end{aligned}$$

The dimension of x is n and u is a scalar.

All states are measurable and at time $k = 0$ a control law is to be applied such that

$$x(N) = 0$$

and

$$\sum_{k=0}^{N} u(k) = 0.$$

How can such a control law be determined? How large must N be to succeed?

9.6

(LiTH)

Consider the antenna in Exercise 5.5. Now, an automatic control of the antenna is to be implemented. A suitable measure of the performance of the closed loop (controlled) system is

$$J = E\{\theta^2(t) + \rho\mu^2(t)\},$$

where ρ is a constant to be used for tuning. Determine an optimal control law for the system. How should it be combined with the previously derived Kalman filter?

9.7

(LiTH)

Consider the system

$$\begin{aligned} z &= \frac{1}{p+1}u + \frac{1}{p+1}v \\ y &= z + e \end{aligned}$$

where v and e are disturbances with spectra

$$\Phi_v(\omega) \equiv r_1 \quad \text{respektive} \quad \Phi_e(\omega) \equiv 1.$$

The aim is to minimize

$$E\{q_1 z^2(t) + u^2(t)\}.$$

(a) Determine the loop transfer of the resulting closed loop system.

(b) What is the difference if the role of r_1 and q_1 here?

(c) Sketch the loop transfer. What happens if $r_1 \to \infty$ and when $q_1 \to \infty$?

9.8

(LiTH)

Consider the system

$$
\begin{aligned}
z &= \frac{1}{p+1}u + \frac{1}{p+1}\nu \\
y &= z + e
\end{aligned}
$$

where ν is a very low frequent disturbance,

$$\nu = \frac{1}{p+\varepsilon}v$$

where v and e are white noise with unit spectra, i.e. $\Phi_v(\omega) \equiv \Phi_e(\omega) \equiv 1$.

(a) Determine the LQG controller minimizing

$$Ez^2 + Eu^2$$

when $\varepsilon \to 0$.

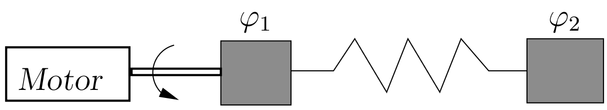
What will the static gain of the sensitivity function become?

(b) Use output LTR (LTR(y)) to determine L. What will the static gain of the sensitivite function become now?

9.9

(LiTH)

The below figure illustrates an electric motor driving two rotating masses connected by a weak shaft



φ_1 and φ_2 are the angular positions of the masses, and ω_1 and ω_2 are the angular velocities. Their moments of inertia are both 10. The spring constant of the shaft is k and their dampening is 0.1. The input signal is the voltage to the electric motor. With the states

$$
\begin{aligned}
x_1 &= \varphi_1 - \varphi_2 \\
x_2 &= \omega_1 \\
x_3 &= \omega_2
\end{aligned}
$$

we get the state-space model

$$\dot{x} = \begin{bmatrix} 0 & 1 & -1 \\ -\frac{1}{2}\omega_0^2 & -0.01 & 0.01 \\ \frac{1}{2}\omega_0^2 & 0.01 & -0.01 \end{bmatrix} x + \begin{bmatrix} 0 \\ \omega_0 \\ 0 \end{bmatrix} u$$

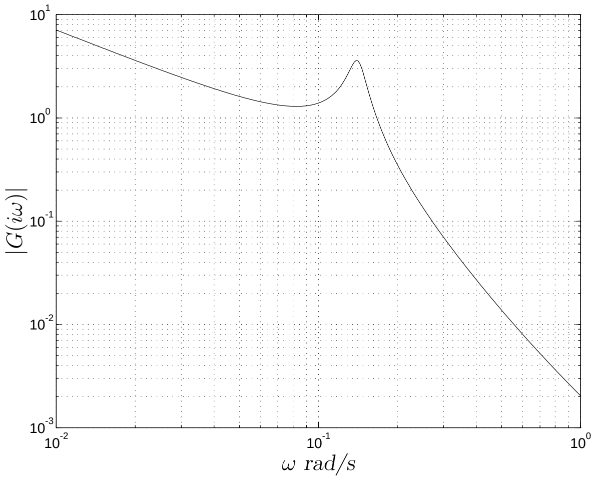
$$z = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x$$

where

$$\omega_0^2 = \frac{k}{50}.$$

The Bode diagram for this system is shown in the below figure for $k = 1$. The resonance peak occurs for ω_0. The spring constant is not exactly known, but has a value around 1. In any way, we want to designa controller that gives a stable system in spite of variations in k (around 1).

How can the model above be extended with a noise model such that we can get robustness against variations in k when we use LQG design? Give a concrete example of such an extended system.



9.10

(LiTH)

Consider the system

$$\dot{x} = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} x + \begin{bmatrix} -4 \\ 8 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} x$$

We whish to minimize the criteria

$$V(T) = \int_0^T (x^T(t)x(t) + u^2(t))dt.$$

Is it possible to find a controller $u = -Lx$ such that $V(T) < \infty$ when $T \to \infty$?

9.11

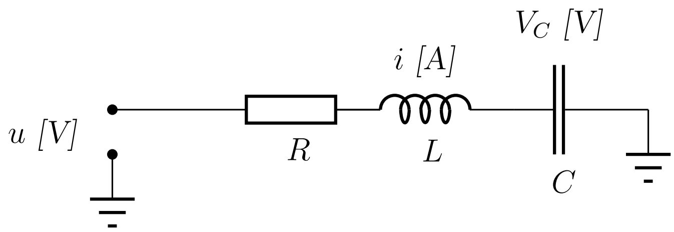
(LiTH)

Assume we have a model that very precisely describe the system, except in a narrow frequency region where the model uncertainties are considerable. Explain how this can be integrated in the LQG-design to achieve a closed loop system robust to these uncertainties. (No calculations are neccessary.)

9.12

(LiTH)

An electrical circuit is illustrated in the figure below.



Introduce the state variables $x_1 = V_C$ and $x_2 = i$. The parameter values

$$R = 5\ \Omega, \qquad L = 0.1\ H, \qquad C = 1000\ \mu F$$

then gives the following state space model

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} 0 & 1000 \\ -10 & -50 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 10 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)\end{aligned}$$

Determine a state space feedback that minimizes the loss function

$$J = \int_0^\infty \left(x_2^2(t) + 0.01 u^2(t) \right) dt.$$

(This loss function is formulated as a mean to reduce the power losses without to large control signals.)

9.13

(LiTH)

Consider the system

$$\frac{1}{s-1}$$

realized on state-space form with disturbances as

$$\begin{aligned}\dot{x} &= x + u + v_1 \\ z &= x \\ y &= x + v_2\end{aligned}$$

The disturbances v_i are white with variances R_i.

(a) Determine the controller that minimizes

$$V = \int_0^\infty Q_1 x^2 + Q_2 u^2 dt.$$

(b) Show that the controller only depends on the ratios $\alpha = Q_1/Q_2$ and $\beta = R_1/R_2$.

(c) Determine the closed system's poles, the loop transfer and the cut-off frequncy as a function of α and β. Comment on the relations.

9.14 (ÅM)

Consider the double integrator described as:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Find a state feedback that minimizes the quadratic cost function

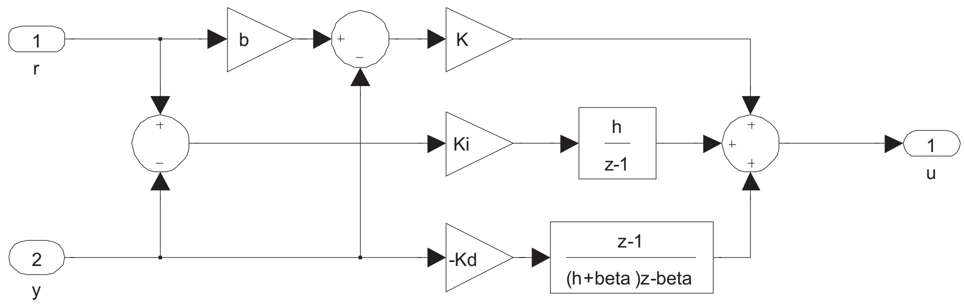
$$J = \frac{1}{2}\int_0^\infty (q_1 x_1^2 + q_2 x_2^2 + q_u u^2) dt$$

where $q_1 \geq 0$ is the penalty on position, $q_2 \geq 0$ is the penalty on velocity, and $q_u > 0$ is the penalty on control actions. Analyze the coefficients of the closed loop characteristic polynomial and explore how they depend on the penalties.

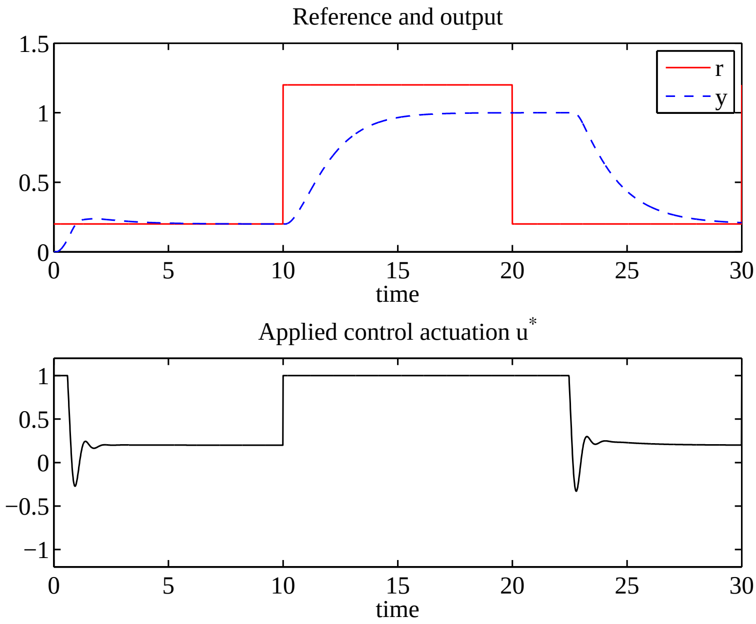
10 Miscellaneous

10.1 (Chalmers)

Below is a block diagram of a time discrete controller.



The following behavior was observed:



(a) What is this phenomena called?

(b) Modify the block diagram to improve the performance in situations like this one!

10.2

(Chalmers)

A DC motor with an attached load has the following transfer function from supplied voltage to angular position

$$G(s) = \frac{1}{s(1+s)}$$

(a) Design a continuous time PI controller (with two parameters) that gives a phase margin $\varphi_m = 45°$ and a crossover frequency $\omega_c = 0.5$.

(b) The controller is implemented digitally with a zero order hold circuit and a sampling interval $h = 0.02$ s. Based on the phase shift of the hold function, estimate how much extra phase margin that is needed in the continuous time design to achieve the desired $45°$ phase margin in the implemented system?

10.3

(Chalmers)

Consider a system

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 0 & 0 \\ a & 1 \end{bmatrix} x(k) + \begin{bmatrix} b \\ 0 \end{bmatrix} u(k) + v(k) \\ y(k) &= x_2(k) \end{aligned}$$

(a) For which values of a and b is the system reachable and observable?

(b) Assuming both states are available for feedback, determine the control law that minimizes

$$J = E\{y^2 + u^2\}$$

when $a = 1$ and $b = 1$.

(c) Use the deterministic relation between u and x_1 to determine the minimizing feedback controller $H_c(q)$ from y to u.

10.4

(Chalmers)

The process

$$G(s) = \frac{Y(s)}{U(s)} = \frac{e^{-Ts}}{s+1}$$

is to be controlled with piecewise constant control signal with a sampling interval $h = T$ s.

(a) Determine a time discrete state space model of the process.

(b) By coincidence(!) the time delay is $T = \ln 2$ s. Determine the state feedback $u_{FB}(k) = -Lx(k)$ that gives a double pole in 0.4 and reformulate it as $u_{FB}(k) = H(q)y(k)$.

(c) Determine the feed forward gain K_r such that $u(k) = K_r r(k) - u_{FB}(k)$ gives the correct stationary gain from r to y.

SOLUTIONS

2.1 (a) For example

$$\begin{aligned}
\frac{d}{dt}x(t) &= \frac{1}{RC}\begin{bmatrix} -2 & -1 \\ -1 & -1 \end{bmatrix} x(t) + \frac{1}{RC}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
y(t) &= \begin{bmatrix} -1 & -1 \end{bmatrix} x(t) + u(t)
\end{aligned}$$

(b)

$$G(s) = \frac{(1+RCs)^2}{1+3RCs+R^2C^2s^2}$$

which is a minimum phase system.

2.2 The given equations gives

$$\begin{aligned}
J\dot{\omega} &= M - K_e\omega I_f = \{I_f R = e\} \\
&= M - K_e\omega\frac{e}{R} = \{e = C_e I_m \omega\} \\
&= M - K_e\frac{C_e}{R}\omega^2 I_m
\end{aligned}$$

With $x = \omega$ as state, and the given parameters and input and output vector we have

$$\begin{aligned}
\dot{x} &= u_1 - \frac{x^2}{u_3}u_2 = f(x,u) \\
y &= \begin{bmatrix} x \\ xu_2 \end{bmatrix} = h(x,u)
\end{aligned}$$

Linearizing around the operating point $\bar{x} = \bar{u}_1 = \bar{u}_2 = \bar{u}_3 = 1$ gives linear state space model

$$\begin{aligned}
\Delta\dot{x} &= -2\Delta x + \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}\Delta u \\
\Delta y &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}\Delta x + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}
\end{aligned}$$

$$\begin{bmatrix} \Delta\omega \\ \Delta e \end{bmatrix} = \frac{1}{s+2}\begin{bmatrix} 1 & -1 & 1 \\ 1 & s+1 & 1 \end{bmatrix}\begin{bmatrix} \Delta M \\ \Delta I_m \\ \Delta R \end{bmatrix}$$

2.3 (a) If we assume $h_1 > h_2$ the mass balances over the vessels are

$$\begin{aligned}\frac{d}{dt}(Ah_1) &= q_1 - a\sqrt{2gh_1} - b\sqrt{2g(h_1-h_2)}\\ \frac{d}{dt}(Ah_2) &= q_2 - a\sqrt{2gh_2} + b\sqrt{2g(h_1-h_2)}\end{aligned}$$

(b) The stationary flows are

$$\begin{aligned}\bar{q}_1 &= a\sqrt{2g\bar{h}_1} + b\sqrt{2g(\bar{h}_1-\bar{h}_2)}\\ \bar{q}_2 &= a\sqrt{2g\bar{h}_2} - b\sqrt{2g(\bar{h}_1-\bar{h}_2)}\end{aligned}$$

With parameter values $\bar{q}_1 = 56$ l/s $\bar{q}_2 = 25.6$ l/s.

(c) Using the notation $\Delta h_i = h_i - \bar{h}_i$, $\Delta q_i = q_i - \bar{q}_i$ and $\Delta u_i = \sigma \Delta q_i$ the linearized model becomes

$$\begin{aligned}\frac{d}{dt}\begin{bmatrix}\Delta h_1\\ \Delta h_2\end{bmatrix} &= \begin{bmatrix}-\dfrac{7}{75} & \dfrac{7}{100}\\ \dfrac{7}{100} & -\dfrac{7(4+\sqrt{2})}{400}\end{bmatrix}\begin{bmatrix}\Delta h_1\\ \Delta h_2\end{bmatrix} + 10^{-3}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\Delta u_1\\ \Delta u_2\end{bmatrix}\\ \begin{bmatrix}\Delta h_1\\ \Delta h_2\end{bmatrix} &= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\Delta h_1\\ \Delta h_2\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\Delta u_1\\ \Delta u_2\end{bmatrix}\end{aligned}$$

2.4 (a) The nonlinear state-space model follows from a heat balance and a mass balance, where the effluent flow is assumed to obey Bernoullis law:

$$\begin{aligned}\frac{d}{dt}T &= \frac{1}{\rho ckh^3}(\rho cq(\nu - T) + P)\\ \frac{d}{dt}h &= \frac{1}{3kh^2}(q - a\sqrt{2gh})\end{aligned}$$

(b) Equilibrium in the operating point, i.e. all time derivatives equal to zero, gives

$$\begin{aligned}\bar{q} &= a\sqrt{2g\bar{h}}\\ \bar{p} &= \rho c\bar{q}(\bar{T} - \bar{\nu})\end{aligned}$$

The linearized model becomes

$$\frac{d}{dt}\begin{bmatrix}\Delta T\\ \Delta h\end{bmatrix} = \begin{bmatrix}-\frac{\bar{q}}{k\bar{h}^3} & 0\\ 0 & -\frac{-a\sqrt{2g}}{6k\bar{h}^2\sqrt{\bar{h}}}\end{bmatrix}\begin{bmatrix}\Delta T\\ \Delta h\end{bmatrix} + \begin{bmatrix}\frac{\bar{\nu}-\bar{T}}{k\bar{h}^3} & \frac{1}{\rho c_p k\bar{h}^3} & \frac{\bar{q}}{k\bar{h}^3}\\ \frac{1}{3k\bar{h}^2} & 0 & 0\end{bmatrix}\begin{bmatrix}\Delta q\\ \Delta p\\ \Delta \nu\end{bmatrix}$$

(c) With parameter values inserted $\bar{q} = 0.01$ m^{3}s^{-1} and $\bar{p} = 400$ kW. This gives the linearized model

$$\frac{d}{dt}\begin{bmatrix}\Delta T\\ \Delta h\end{bmatrix} = 10^{-3}\begin{bmatrix}-10 & 0\\ 0 & -1.7\end{bmatrix}\begin{bmatrix}\Delta T\\ \Delta h\end{bmatrix} + \begin{bmatrix}-10 & 0.00025 & 0.01\\ 0.33 & 0 & 0\end{bmatrix}\begin{bmatrix}\Delta q\\ \Delta p\\ \Delta \nu\end{bmatrix}$$

(d) The linearized systems two poles p_i equal the eigenvalues of the system matrix A, i.e. $\det(p_i I - A) = 0$. The time constants are $T_i = -1/p_i$, which equals 100 s and 600 s, respectively.

2.5 (a)

$$
\begin{aligned}
\dot{h} &= \begin{bmatrix} -\beta/A_1 & \beta/A_1 \\ \beta/A_2 & -(\beta+\alpha)/A_2 \end{bmatrix} h + \begin{bmatrix} 1/A_1 & 0 \\ 0 & 1/A_2 \end{bmatrix} u \\
y &= \begin{bmatrix} 0 & \alpha \end{bmatrix} h
\end{aligned}
$$

$$
\begin{aligned}
G(s) &= \frac{\alpha}{s^2 A_1 A_2 + s(\alpha A_1 + \beta(A_1 + A_2)) + \alpha\beta} \begin{bmatrix} \beta & sA_1 + \beta \end{bmatrix} \\
&= \frac{1}{s^2 + (2\beta + \alpha)s + \alpha\beta} \begin{bmatrix} \alpha\beta & \alpha(s+\beta) \end{bmatrix}
\end{aligned}
$$

(b)

$$
\begin{aligned}
G(0) &= \begin{bmatrix} 1 & 1 \end{bmatrix} \\
\bar{\sigma}(G(0)) &= \max_i \sqrt{\lambda_i\left(G^*(0)G(0)\right)} = \sqrt{2}
\end{aligned}
$$

2.6 (a) Mass balances for the two substances over the tank reactor give

$$
\begin{aligned}
\frac{d}{dt} c_A &= \frac{Q}{V}(c_{Ai} - c_A) - k\ln(c_A/c_B) \\
\frac{d}{dt} c_B &= \frac{Q}{V}(c_{Bi} - c_B) + k\ln(c_A/c_B)
\end{aligned}
$$

The stationary concentrations in the influent are given by

$$
\begin{aligned}
\bar{c}_{Ai} &= \frac{V}{Q} k\ln(\bar{c}_A/\bar{c}_B) + \bar{c}_A \\
\bar{c}_{Bi} &= \frac{V}{Q} k\ln(\bar{c}_B/\bar{c}_A) + \bar{c}_B
\end{aligned}
$$

With parameter values inserted $\bar{c}_{Ai} = 0.269\,\mathrm{mol\,m^{-3}}$ and $\bar{c}_{Bi} = 0.0307\,\mathrm{mol\,m^{-3}}$.

(b)

$$
\frac{d}{dt}\begin{bmatrix} \Delta c_A \\ \Delta c_B \end{bmatrix} = 10^{-3}\begin{bmatrix} -1.5 & 1 \\ 0.5 & -2 \end{bmatrix}\begin{bmatrix} \Delta c_A \\ \Delta c_B \end{bmatrix} + 10^{-3}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \Delta c_{Ai} \\ \Delta c_{Bi} \end{bmatrix}
$$

2.7 Write the system on lowest common denominator

$$\begin{aligned} Y(s) &= \frac{(s^2+s+1)}{(s+1)(s+2)(s^2+s+1)}U_1(s) + \frac{(s+2)(s+3)}{(s+1)(s+2)(s^2+s+1)}U_2(s) \\ &= \frac{(s^2+s+1)}{(s^4+4s^3+6s^2+5s+2)}U_1(s) + \frac{(s^2+5s+6)}{(s^4+4s^3+6s^2+5s+2)}U_2(s) \end{aligned}$$

The system can now be realized on observable canonical form

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -4 & 1 & 0 & 0 \\ -6 & 0 & 1 & 0 \\ -5 & 0 & 0 & 1 \\ -2 & 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 5 \\ 1 & 6 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x(t) \end{aligned}$$

2.8 On common denominator:

$$y(t) = \frac{1}{p^3+7p^2+16p+12} \begin{bmatrix} p^2+3p & p^2+p-2 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

Observable canonical form:

$$\dot{x}(t) = \begin{bmatrix} -7 & 1 & 0 \\ -16 & 0 & 1 \\ -12 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 1 \\ 3 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(t)$$

2.9 Laplace transformation gives

$$Y(s) = \frac{(b_{11}s+b_{12})}{(s^2+a_1s+a_2)}U_1(s) + \frac{(b_{21}s+b_{22})}{(s^2+a_1s+a_2)}U_2(s)$$

On observable canonical form:

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -a_1 & 1 \\ -a_2 & 0 \end{bmatrix} x(t) + \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{aligned}$$

2.10 Laplace transformation gives

$$A(s)Y(s) = B(s)U(s)$$

where

$$A(s) = \begin{bmatrix} s & 1 \\ 1 & (s+1) \end{bmatrix} \qquad B(s) = \begin{bmatrix} s+2 \\ 1 \end{bmatrix}$$

Multiplication with $A^{-1}(s)$ gives

$$Y(s) = A^{-1}(s)B(s)U(s)$$

i.e.

$$Y(s) = \begin{bmatrix} \dfrac{s^2+3s+1}{s^2+s-1} \\ \dfrac{-2}{s^2+s-1} \end{bmatrix} U(s) = \begin{bmatrix} \dfrac{2s+2}{s^2+s-1} + 1 \\ \dfrac{-2}{s^2+s-1} \end{bmatrix} U(s)$$

On controllable canonical form:

$$\dot{x}(t) = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 2 & 2 \\ 0 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

2.11 Using $C[k+1] = aY[k]$, we have

$$Y[k+1] = C[k+1] + I[k+1] + G[k+1] = aY[k] + I[k+1] + G[k+1].$$

At equilibrium, $Y[k+1] = Y[k] = Y_e$, $I[k+1] = I_e$, and $G[k+1] = G_e$, so that

$$\begin{aligned} Y_e &= Y[k+1] = aY[k] + I[k+1] + G[k+1] = aY_e + I_e + G_e, \\ \Rightarrow Y_e &= \frac{1}{1-a}(I_e + G_e). \end{aligned}$$

To obtain the state-space representation of this system, we have:

$$\begin{aligned} C[k+1] &= aY[k] = aC[k] + aI[k] + aG[k], \\ I[k+1] &= b(C[k+1] - C[k]) = b((aC[k] + aI[k] + aG[k]).C[k]) \\ &= (ab-b)C[k] + abI[k] + abG[k], \end{aligned}$$

which is equivalent to

$$\begin{aligned} \begin{bmatrix} C[k+1] \\ I[k+1] \end{bmatrix} &= \begin{bmatrix} a & a \\ ab-b & ab \end{bmatrix} \begin{bmatrix} C[k] \\ I[k] \end{bmatrix} + \begin{bmatrix} a \\ ab \end{bmatrix} G[k], \\ Y[k] &= C[k] + I[k] + G[k]. \end{aligned}$$

2.12 For the given system with zero initial conditions we have

$$
\begin{aligned}
y(t) &= C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau = C\int_0^t e^{A(t-\tau)}B\cos\tau d\tau = \int_0^t \sin(t-\tau)\cos\tau d\tau \\
&= \frac{1}{2}\int_0^t (\sin t + \sin(t-2\tau))\,d\tau = \frac{t}{2}\sin t
\end{aligned}
$$

Notice that the dynamics matrix A has eigenvalues $s = \pm i$ and that the input is $\cos t = (e^{it} + e^{-it})/2$.

2.13 The system is on the form

$$
\begin{aligned}
\dot{x} &= \tilde{A}x + \tilde{B}u + \tilde{N}\dot{u} \\
y &= \tilde{C}x
\end{aligned}
$$

If $z = Tx + Su$ and T is an invertible 2×2 matrix (i.e. has rank 2) then

$$
\begin{aligned}
\dot{z} &= T^{-1}\tilde{A}Tz + T^{-1}(\tilde{A}S + \tilde{N})u + T^{-1}(\tilde{N} - S)\dot{u} \\
y &= \tilde{C}Tz + \tilde{C}Su
\end{aligned}
$$

By choosing $S = \tilde{N}$ we can eliminate the input time derivative in the model. Then, by choosing

$$
T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}
$$

for example, the new model becomes

$$
\begin{aligned}
\frac{d}{dt}z &= \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} z + \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} u \\
y &= \begin{bmatrix} 1 & 0 \end{bmatrix} z + \begin{bmatrix} 0 & 1 \end{bmatrix} u
\end{aligned}
$$

2.14 (a) The equilibrium queue length x_e is given by

$$
u_e\lambda_e = \mu_e\frac{x_e}{1+x_e}
$$

Hence,

$$
x_e = \frac{u_e\lambda_e}{\mu_e - u_e\lambda_e} = \frac{1}{1-r}
$$

On the other hand, we can write

$$
f_x(x,u) = \frac{\mu_e}{(1-x)^2}, \quad , f_u(x,u) = \lambda_e
$$

The linearized model becomes

$$
\frac{d(x-x_e)}{dt} = f_x(x_e,u_e)(x-x_e) + f_u(x_e,u_e)(u-u_e)
$$

(b) Laplace transformation of the linearized model gives

$$P(s) = \frac{f_u(x_e, u_e)}{s + f_x(x_e, u_e)} = \frac{\lambda_e}{s + \mu_e/(1 - x_e)^2}$$

The queue length goes to infinity as μ_e approaches λ_e and the transfer function approaches an integrator ($G(s) = \lambda_e/s$).

3.1 (a) The output at $t > 0$ for an arbitrary input signal is

$$y(t) = C\left(e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau\right)$$

With $x(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$ and $u(t) = \delta(t)$ we get the impulse response

$$g(t) = y(t) = Ce^{At}B = 0.5e^{-t}(3\sin t - \cos t)$$

The corresponding transfer functions is

$$G(s) = \mathcal{L}^{-1}\{g(t)\} = \frac{1 - 0.5s}{s^2 + 2s + 2}$$

(b) The system is stable since $\lim_{t\to\infty} g(t) = 0$.

(c)

$$A_d = e^{Ah} = \begin{bmatrix} -0.21 & 0.21 \\ -0.420 & \end{bmatrix}$$

$\lambda_{1,2}(A) = -0.104 \pm 0.2750i \quad \Rightarrow \quad |\lambda| = 0.29 < 1 \quad \Rightarrow \quad$ stable.

3.2 The eigenvalues of the dynamics matrix can be found by solving for the roots of

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - a & -a \\ b - ab & \lambda - ab \end{vmatrix} = \lambda^2 - a(1+b)\lambda + ab$$

which gives

$$\lambda_{1,2} = \frac{(a + ab) \pm \sqrt{(a + ab)^2 - 4ab}}{2}$$

To guarantee $\lambda_{1,2} < 1$, we need to have

(a) if $(a + ab)^2 - 4ab < 0$, the eigenvalues are complex and $|\lambda_1| = |\lambda_2|$, so $|\lambda_{1,2}| < 1 \quad \Rightarrow \quad |\lambda_{1,2}| = |ab| < 1$

(b) if $(a+ab)^2 - 4ab \geq 0$, the eigenvalues are real. In this case we need $|\lambda_1| < 1$ and $|\lambda_2| < 1$. Assuming $a, b > 0$, the first eigenvalue is larger and the condition becomes

$$\lambda_1 = \frac{(a+ab) + \sqrt{(a+ab)^2 - 4ab}}{2} < 1$$

When the system is at equilibrium, C, I, and G are all constants, so $C = aC + aI + aG$, $I = b(a-1)C + abI + abG$. We can express C and I using G:

$$C = \frac{a}{1-a}G, \qquad I = 0$$

When there is a 10% increase in G, using a = 0.25 and b = 0.5, we can calculate that C will increase by 3.33% while I will not change.

3.3 (a)

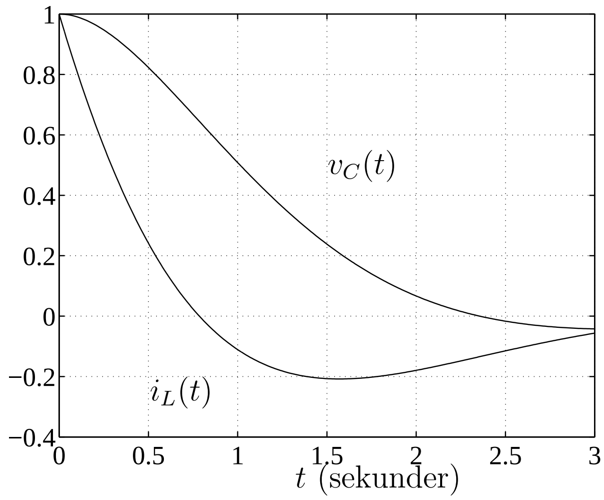
$$\frac{d}{dt}\begin{bmatrix} i_L(t) \\ v_C(t) \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L} & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{R_2 C} \end{bmatrix} \begin{bmatrix} i_L(t) \\ v_C(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} u(t)$$

(b)

$$\Phi(t) = \begin{bmatrix} e^{-t}\cos t & -e^{-t}\sin t \\ e^{-t}\sin t & e^{-t}\cos t \end{bmatrix}$$

(c) In a steady state $u = 2$ V implies $i_L = 1$ A and the voltage $v_C = 1$ V. Short circuit of the accumulator implies a step change in the input signal of 2 V. The response becomes

$$\begin{aligned} i_L(t) &= e^{-t}(\cos t - \sin t) \\ v_C(t) &= \sqrt{2}e^{-t}\sin(t + \frac{\pi}{4}) \end{aligned}$$



3.4 (a) (ξ_o, u_o, η_o) =(1, 0.6017, 1.1845)

(b)

$$\begin{aligned}
\frac{d}{dt}\Delta\xi &= -\Delta\xi - 52.6\Delta u \\
\Delta y &= 1.18\Delta\xi - 21.3\Delta u
\end{aligned}$$

(c) The system has a stable pole in -1 independently of ξ_o.

3.5 The transformation gives $\mathcal{O}_z = \mathcal{O}_x T$ and $\mathcal{S}_z = T^{-1}\mathcal{S}_x$.

T non-singular $\Rightarrow T$ and T^{-1} have full rank $(= n) \Rightarrow \mathcal{O}_z$ and $\mathcal{S}_z$ have full rank $(= n)$ if and only if $\mathcal{O}_x$ and $\mathcal{S}_x$ have full rank.

3.6

$$\mathcal{S} = \begin{bmatrix} 0 & \beta & 0 & -\beta & 0 & \beta \\ 0 & 0 & \alpha & 0 & -\alpha & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \end{bmatrix}$$

The system is controllable if both $\alpha \neq 0$ and $\beta \neq 0$ ($\Rightarrow$ three linearly independent columns).

3.7 (a) For a reachable system it should be possible to drive the system (using u) to any state $x^* \in \mathcal{R}^n$. However, since z and y obey the same equation they cannot be taken to independent states zst and y^*.

(b) The above can naturally also be seen from the controllability matrix $\mathcal{S}$:

$$\mathcal{S} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{2n-1}B \\ B & AB & A^2B & \cdots & A^{2n-1}B \end{bmatrix}$$

The number of independent columns are always the same as the number of independent rows, and clearly $\mathcal{S}$ can have at most n independent rows while the system is of order $2n$. Therefore, the system is not reachable (controllable). If (A, B) is reachable the first n rows are independent and the remaning rows are simply duplicates. Therefore, the controllability matrix $\mathcal{S}$ will have rank n.

3.8 (a)

$$SS1: \begin{cases} \bar{c}_A = 0 \\ \bar{c}_B = 0 \end{cases} \qquad \text{and} \qquad SS2: \begin{cases} \bar{c}_A = 1/\beta \\ \bar{c}_B = 1/\alpha \end{cases}$$

Model 1:

$$\begin{aligned}\Delta\dot{c} &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \Delta c \\ \Delta c_M &= \begin{bmatrix} 0.5 & 0.5 \end{bmatrix} \Delta c\end{aligned}$$

Model 2:

$$\begin{aligned}\Delta\dot{c} &= \begin{bmatrix} 0 & \alpha/\beta \\ \beta/\alpha & 0 \end{bmatrix} \Delta c \\ \Delta c_M &= \begin{bmatrix} 0.5 & 0.5 \end{bmatrix} \Delta c\end{aligned}$$

(b) Model 1 is stable and Model 2 is unstable.

(c) Model 1 is not observable (but detectable since it is stable) and Model 2 is observable if $\alpha/\beta \neq \beta/\alpha$.

3.9 (a)

$$\frac{d}{dt}\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} -\frac{Q+R}{V} & \frac{R}{V} \\ \frac{Q+R}{V} & -\frac{Q+R}{V} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} + \frac{1}{\rho c V}\begin{bmatrix} \theta \\ (1-\theta) \end{bmatrix} u + \begin{bmatrix} \frac{Q}{V} \\ 0 \end{bmatrix} T_0$$

(b) The condition number $\kappa(M)$ for a non-quadratic matrix M is the ratio between the largest and the smallest singular value, which is the quotient of the eigenvalues of MM^T or M^TM (the same value).

$$\mathcal{S} = \frac{1}{\rho c}\begin{bmatrix} \theta & 1-3\theta \\ 1-\theta & 4\theta-2 \end{bmatrix}$$

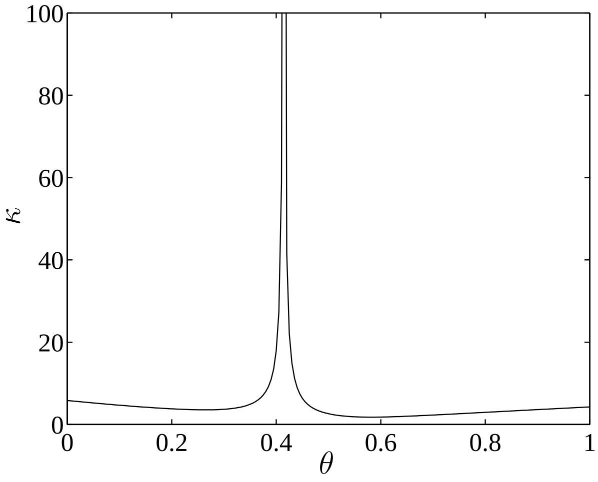
$$\mathcal{S}\mathcal{S}^T = \frac{1}{\rho^2 c^2}\begin{bmatrix} 10\theta^2-6\theta+1 & -13\theta^2+11\theta-2 \\ -13\theta^2+11\theta-2 & 17\theta^2-18\theta+5 \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix}$$

$$\det(\lambda I - \mathcal{S}\mathcal{S}^T) = 0 \quad \Rightarrow$$

$$\lambda = \frac{1}{2}\left(s_{11} + s_{22} \pm \sqrt{4(s_{12}^2 - s_{11}s_{22}) + (s_{11}+s_{22})^2}\right)$$

$$\kappa(\mathcal{S}) = \frac{\sigma_{max}}{\sigma_{min}} = \sqrt{\frac{\lambda_{max}}{\lambda_{min}}} = \begin{cases} \frac{0.847}{0.047} = 18 & \text{when } \theta = 0.4 \\ \frac{1}{0.57} = 1.8 & \text{when } \theta = 0.6 \end{cases}$$

From a control point of view, the condition number should be as small as possible, i.e. $\theta = 0.6$ is to prefer. From the figure it can be seen that the lowest condition number is when $\theta \approx 0.6$. When a matrix is close to singular, the singular value approaches infinity because one eigenvalue is close to zero.



(c) $\det \mathcal{S} = 0 \quad \Rightarrow \quad \theta = -1 \pm \sqrt{2}$, i.e. $\mathcal{S}$ looses its full rank when $\theta = \sqrt{2} - 1 \approx 0.414$.

3.10 (a)

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -\omega & \omega & 0 \\ \omega & -2\omega & \omega \\ 0 & \omega & -\omega \end{bmatrix} x(t) + \begin{bmatrix} \omega/k \\ 0 \\ 0 \end{bmatrix} q(t) \\ y(t) &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x(t) \end{aligned}$$

where $\omega = 3k/(\rho c d)$.

(b) The system is both controllable and observable.

(c) The system is marginally stable (one pole in the origin).

3.11 The transfer function matrix has the minors

$$-\frac{1}{(s+2)^2} - \frac{(s+1)}{(s+2)^2} = -\frac{1}{s+2}$$

when the first column is removed,

$$\frac{1}{(s+2)^2} - \frac{1}{(s+2)^2} = 0$$

when the second column is removed, and

$$\frac{(s+1)}{(s+2)^2} + \frac{1}{(s+2)^2} = \frac{1}{(s+2)}$$

when the third column is removed. Furthermore, the matrix elements themselves are always minors as well. The least common denominator to all the minors is the pole polynamoial, i.e.

$$p(s) = (s+2)$$

Thus the system has one pole in $s = -2$ and the system can therefore be realized by a state space model having only one state.

The maximal minors are

$$-\frac{1}{s+2}, \qquad 0, \qquad \frac{1}{(s+2)}$$

and hence the zero polynomial is a constant, i.e. there are no zeros.

3.12 **Alternative 1**. The output signal y depends only on x_1 and x_2. From an analysis of the system matrix A (study row 1 and row 2) we conclude that neither do the states x_1 and x_2 depend on x_3. Hence we may eliminate the third state:

$$\begin{aligned} \dot{\tilde{x}}(t) &= \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \tilde{x}(t) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tilde{x}(t), \end{aligned}$$

where $\tilde{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$. The controllability and observability matrices for this system have full ranks and consequently this is a minimum realization.

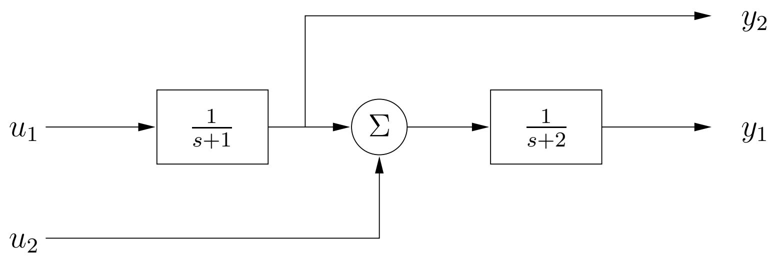
Alternative 2. The transfer function matrix $G(s) = C(sI - A)^{-1}B$ is

$$G(s) = \begin{bmatrix} \frac{1}{(s+2)(s+1)} & \frac{1}{s+2} \\ \frac{1}{s+1} & 0 \end{bmatrix}$$

i.e.

$$\begin{cases} y_1 = \dfrac{1}{(s+2)(s+1)} u_1 + \dfrac{1}{s+2} u_2 \\ y_2 = \dfrac{1}{s+1} u_1 \end{cases}$$

This corresponds to the block diagram in the figure



Introduce one state for each block, for example $x_1 = y_1$ and $x_2 = y_2$. This gives the same minimum realization as in Alt. 1.

3.13 (a) Since $\det(G(s)) = \frac{1-s}{s(s+1)^2}$, the pole polynomial is $P(s) = s(s+1)^2$ and the zero polynomial is $N(s) = 1 - s$. The system poles are consequently located at -1 (2 poles) and at 0. Also, the system has only one zero, located at 1 (right half plane). Therefore, the system zero is unstable and the system is a non minimum phase system.

(b)

$$\begin{aligned}
y_1(s) &= \frac{1}{s(s+1)}u_1 + \frac{1}{s}u_2 = \frac{1}{s}u_1 - \frac{1}{s+1}u_1 + \frac{1}{s}u_2 = \frac{1}{s}(u_1+u_2) - \frac{1}{s+1}u_1 \\
y_2(s) &= \frac{1}{s+1}u_1 + \frac{2}{s+1}u_2 = \frac{2}{s+1}(u_1+u_2) - \frac{1}{s+1}u_1
\end{aligned}$$

Set

$$x_1 = \frac{1}{s}(u_1+u_2), \quad x_2 = \frac{1}{s+1}u_1, \quad x_3 = \frac{1}{s+1}(u_1+u_2).$$

The system can therefore be written as

$$\begin{aligned}
\dot{x}_1 &= u_1 + u_2 \\
\dot{x}_2 &= -x_2 + u_1 \\
\dot{x}_3 &= -x_3 + 2u_1 + 2u_2
\end{aligned}$$

$$\begin{aligned}
\dot{x} &= \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 2 & 2 \end{bmatrix}}_{B} u \\
y &= \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{C} x
\end{aligned}$$

3.14 (a) The system is unobservable when

$$\det \mathcal{O} = \det\left(\begin{bmatrix} C \\ CA \end{bmatrix}\right) = -\beta = 0 \;\Rightarrow\; \beta = 0.$$

The system is uncontrollable when

$$\det \mathcal{S} = \det\left(\begin{bmatrix} B & AB \end{bmatrix}\right) = 2\gamma + 4\beta + 2 = 0 \;\Rightarrow \gamma = -1.$$

(b) Based on the eigenvalue decomposition and $\beta = \gamma = 1$, the eigenvalues of A (poles of the system) are $p_1 = -1.85,\; p_2 = 1.35$.

The normalized eigenvectors are

$$e_1 = \begin{bmatrix} -0.94 \\ -0.33 \end{bmatrix} \quad \text{and} \quad e_2 = \begin{bmatrix} 0.33 \\ -0.94 \end{bmatrix}.$$

A diagonalizing transformation matrix is therefore

$$T = [e_1 \ e_2] = \begin{bmatrix} -0.94 & 0.33 \\ -0.33 & -0.94 \end{bmatrix}.$$

$$\tilde{A} = T^{-1}AT = \begin{bmatrix} -1.85 & 0 \\ 0 & 0.35 \end{bmatrix}; \qquad \tilde{B} = T^{-1}B = \begin{bmatrix} -1.6 \\ -1.55 \end{bmatrix};$$
$$\tilde{C} = CT = \begin{bmatrix} -0.94 & 0.33 \end{bmatrix}; \qquad \tilde{D} = D = 1$$

3.15 By diagonalizing A:

$$\tilde{A} = TAT^{-1} = \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

where

$$T = \begin{bmatrix} 0 & -1/6 & 1/6 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & -1/2 & 1/2 & -2 \\ 0 & 1/3 & -1/3 & 1 \end{bmatrix}$$

The new B and C matrices become

$$\tilde{B} = TB = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}, \quad \tilde{C} = CT^{-1} = \begin{bmatrix} 6 & 0 & 0 & 0 \end{bmatrix}.$$

The Kalman decomposition can then be constructed by reordering the modes:

$$\begin{aligned} \dot{z} &= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} z + \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} u, \\ y &= \begin{bmatrix} 0 & 6 & 0 & 0 \end{bmatrix} z. \end{aligned}$$

4.1

Process transfer function: $\quad G_T(q) = \dfrac{1}{q - 0.9}$

PI controller: $\quad G_{PI}(q) = \dfrac{(1 + K_2)q}{q - 1}$

Characteristic equation: $q^2 + (K_2 - 0.9)q - 0.1 = 0$

Apply the Möbius transformation

$$q = \frac{1+s}{1-s}$$

which gives a one-to-one mapping between the unit disc $|q| < 1$ and the left half plane $\mathrm{Re}\{s\} < 0$.

$$\begin{aligned} 0 &= (1+s)^2 + (K_2 - 0.9)(1+s)(1-s) - 0.1(1-s)^2 \\ &= s^2(1.8 - K_2) + 2.2s + K_2 \end{aligned}$$

The Routh Hurwitz Criterion, for example, can be used to show that the roots in s are in the LHP if all coefficients in a second order polynomial have the same sign. Hence, the system is stable for $0 < K_2 < 1.8$.

4.2 (a)

$$G_{vy}(s) = \frac{1}{1+\tau s} \quad \Longleftrightarrow \quad \begin{cases} \dot{x} = -(1/\tau)x + (1/\tau)v \\ y = x \end{cases}$$

(b) The analytical solution is

$$y(t) = x(t) = e^{At_0}x(t_0) + \int_{t_0}^{t} e^{A(t-\sigma)}Bv(\sigma)d\sigma$$

where $A = -1/\tau$ and $B = 1/\tau$.

(c) Let $t_0 = kh - h$ and $t = kh$. Since we have a time delay $\delta < h$, $v(t) = u(t-\delta)$. Inserting this into the integral above and split the integral into the two intervals where the input signal is constant gives:(argument k corresponds to $t = kh$)

$$\begin{aligned} y(k) &= e^{-h/\tau}y(k-1) + \\ &\quad + \frac{1}{\tau}\int_{kh-h}^{kh-h+\delta} e^{-(kh-\sigma)/\tau}d\sigma \cdot u(k-2) + \\ &\quad + \frac{1}{\tau}\int_{kh-h+\delta}^{kh} e^{-(kh-\sigma)/\tau}d\sigma \cdot u(k-1) \\ &= ay(k-1) + b_2u(k-2) + b_1u(k-1) \end{aligned}$$

where

$$\begin{aligned} b_2 &= e^{-(h-\delta)/\tau} - e^{-h/\tau} \\ b_1 &= 1 - e^{-(h-\delta)/\tau} \end{aligned}$$

Z-transformation then gives

$$H(z) = z^{-1}\frac{b_1 + b_2z^{-1}}{1 - az^{-1}}$$

4.3 (a) The controllability matrix is

$$\mathcal{S}(A,B) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

which has full rank. Hence, the time continuous system is controllable.

(b) Discretization gives

$$x(t+1) = \begin{bmatrix} \cos T & \sin T \\ -\sin T & \cos T \end{bmatrix} x(t) + \begin{bmatrix} 1-\cos T \\ \sin T \end{bmatrix} u(t)$$

The controllability matrix is then

$$\begin{aligned} \mathcal{S} &= \begin{bmatrix} 1-\cos T & \cos T - \cos^2 T + \sin^2 T \\ \sin T & -\sin T + 2\sin T\cos T \end{bmatrix} \\ \det \mathcal{S} &= 2\sin T(\cos T - 1) \end{aligned}$$

which looses rank for $T = n\pi, \quad n = 1, 2, 3, \ldots$. Hence, the discrete system is not reachable. Since A is invertible, the system is not controllable either. Thus, controllability of a continuous time system can be lost by sampling!

5.1 $\Phi_u(\omega)$ is an even function. Make the spectral factorization

$$\Phi_u(\omega) = G(i\omega)G(-i\omega)\Phi_e(\omega)$$

where $G(s)$ has poles and zeros in the LHP and $\Phi_e = 1$ (white noise).

(a)

$$\Phi_u(\omega) = \frac{a^2}{\omega^2 + a^2}\Phi_e(\omega) = \frac{a}{i\omega + |a|} \cdot \frac{a}{-i\omega + |a|}$$

Thus, the linear filter becomes

$$G(s) = \frac{a}{s + |a|}, \qquad a \neq 0.$$

(b) In the same way

$$\begin{aligned} \Phi_u(\omega) &= \frac{a^2 b^2}{(\omega^2 + a^2)(\omega^2 + b^2)}\Phi_e(\omega) \\ &= \frac{ab}{(i\omega + |a|)(i\omega + |b|)} \frac{ab}{(-i\omega + |a|)(-i\omega + |b|)} \\ &\Rightarrow \quad G(s) = \frac{ab}{(s + |a|)(s + |b|)} \end{aligned}$$

5.2 Introduce the states $x_1 = y$, $x_2 = \dot{y} \quad \Rightarrow$

$$\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} f(t) \\
&= Ax(t) + Bf(t) \\
y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \\
&= Cx(t)
\end{aligned}$$

Assume $f(t)$ piecewise constant and set the sampling time to $T = 0.1 \quad \Rightarrow$

$$\begin{aligned}
x(t+T) &= \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0.005/m \\ 0.1/m \end{bmatrix} f(t) = Fx(t) + Gf(t) \\
y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) = Cx(t)
\end{aligned}$$

$$\begin{aligned}
y(t) &= C(qI - F)^{-1} G f(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} q-1 & -0.1 \\ 0 & q-1 \end{bmatrix}^{-1} \begin{bmatrix} 0.005/m \\ 0.1/m \end{bmatrix} f(t) \\
&= \frac{0.005}{m} \frac{q+1}{(q-1)^2} f(t) = \frac{0.005}{m} \cdot \frac{q^{-1}(1+q^{-1})}{(1-q^{-1})^2} f(t)
\end{aligned}$$

or formulated as a difference equation

$$y(t) - 2y(t-T) + y(t-2T) = \frac{0.005}{m}(f(t-T) + f(t-2T))$$

(a) A wind throw at time $\tau \Rightarrow$ Let $f(t) = f \cdot \delta_\tau(t)$ in the equations above.

(b) Let $(1 - 0.9q^{-1})z(t) = w(t)$ where $w(t)$ is white noise.

$$f(t) = Ku(t) + z(t) = \frac{1}{1 - 0.9q^{-1}} w(t)$$

$$\begin{aligned}
y(t) &= \frac{0.005}{m} \cdot \frac{q^{-1}(1+q^{-1})}{(1-q^{-1})^2} \cdot \frac{1}{1-0.9q^{-1}} (w(t) + K(1-0.9q^{-1})u(k)) \\
&= \frac{0.005}{m} \frac{q^{-1}(1+q^{-1})}{1 - 2.9q^{-1} + 2.8q^{-2} - 0.9q^{-3}} (w(t) + K(1-0.9q^{-1})u(k))
\end{aligned}$$

The difference equation becomes

$$\begin{aligned}
& y(t) - 2.9y(t-T) + 2.8y(t-2T) - 0.9y(t-3T) \\
&= \frac{0.005}{m} \{K(u(k-1) + 0.1u(k-2) - 0.9u(k-3)) + w(t-T) + w(t-2T)\}
\end{aligned}$$

or formulated as a state space model (a new state must be added)

$$\begin{bmatrix} x_1(t+T) \\ x_2(t+T) \\ z(t+T) \end{bmatrix} = \begin{bmatrix} 1 & 0.1 & \frac{0.005}{m} \\ 0 & 1 & \frac{0.1}{m} \\ 0 & 0 & 0.9 \end{bmatrix} x(t) + \begin{bmatrix} \frac{0.005K}{m} \\ \frac{0.1K}{m} \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w(t+T)$$

5.3 (a) Newtons law of motion gives $m\ddot{z} = u - f$, where m is the mass of the missile and u is the drag force.

Input/output model:

$$\ddot{z} + \frac{k_1}{m}\dot{z} = \frac{1}{m}(u - v)$$

State space model: Let $x_1 = z$, $x_2 = \dot{z}$ $\Rightarrow$ $\dot{x}_1 = x_2$,

$$\dot{x}_2 = \frac{1}{m}(u - f) = \frac{1}{m}(u - k_1 x_2 - v)$$

which implies

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -\dfrac{k_1}{m} \end{bmatrix} x + \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix} u + \begin{bmatrix} 0 \\ -\dfrac{1}{m} \end{bmatrix} v$$

$$z = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

(b) Description of v:

$$\Phi_v(\omega) = |H(i\omega)|^2 \Phi_e(\omega)$$

Thus, $H(s) = \frac{\sqrt{k_0}}{s+|a|}$, i.e., $\dot{v} + |a|v = \sqrt{k_0}\, e$. Intruduce an additional state: $\dot{x}_3 = -|a|x_3 + e$:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\dfrac{k_1}{m} & -\dfrac{1}{m} \\ 0 & 0 & -|a| \end{bmatrix} x + \begin{bmatrix} 0 \\ \dfrac{1}{m} \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ \sqrt{k_0} \end{bmatrix} e$$

$$z = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x$$

The input/output form becomes

$$(p^2 + \frac{k_1 p}{m})\, z = \frac{1}{m}\left(u - \frac{\sqrt{k_0}}{p + |a|}\, e\right)$$

5.4 (a) With $\{A, B, C, N\}$ as in 5.3(a) we get the standard form

$$\begin{cases} \dot{x} = Ax + Bu + Ne \\ y = Cx + n \end{cases}$$

where n has spectral density, $\Phi_n = 0.1$.

(b) A noise with spectral density Φ_n can be generated as an output from a system having a transfer function

$$G_n(s) = \frac{s}{s + |b|}$$

where the input is white noise with spectral density $\Phi_{w_n} = 0.1$. A corresponding state space model is

$$\begin{cases} \dot{x}_4 &= -|b|x_4 + |b|w_n \\ n &= -x_4 + w_n \end{cases}$$

We get the following extended state space model

$$\dot{x} = \begin{bmatrix} A & 0 \\ 0 & -|b| \end{bmatrix} x + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} N & 0 \\ 0 & |b| \end{bmatrix} \begin{bmatrix} e \\ w_n \end{bmatrix}$$
$$y = \begin{bmatrix} C & -1 \end{bmatrix} x + w_n$$

(c) In the same way as in (b) we model the noise by a state space model corresponding to a transfer function $G_n(s) = 1/(s + |b|)$ driven by white noise having a spectral density $\Phi_{w_n} = 0.1$:

$$\dot{x}_4 + |b|x_4 = w_n.$$

We get the following extended state space model

$$\dot{x} = \begin{bmatrix} A & 0 \\ 0 & -|b| \end{bmatrix} x + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} N & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e \\ w_n \end{bmatrix}$$
$$y = \begin{bmatrix} C & 1 \end{bmatrix} x$$

5.5 Introduce the states, input, and noise according to

$$x(t) = \begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \end{bmatrix}, \quad u(t) = \mu(t), \quad v_1(t) = \frac{1}{J}\tau_d(t), \quad v_2(t) = e_m(t)$$

and set $\alpha = B/J$; $H = k/J$; $\gamma = 1/J$. The systems state space model then becomes

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -\alpha \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ H \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v_1(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + v_2(t)$$

The Riccati equation for the Kalman filter is

$$\dot{P}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -\alpha \end{bmatrix} P(t) + P(t) \begin{bmatrix} 0 & 0 \\ 1 & -\alpha \end{bmatrix} + R_1 - P(t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} R_2^{-1} \begin{bmatrix} 1 & 0 \end{bmatrix} P(t)$$

where the intensities of the white noises v_1 and v_2 are $R_{12} = \begin{bmatrix} 0 & 0 \end{bmatrix}$ and

$$R_1 = \begin{bmatrix} 0 & 0 \\ 0 & \gamma^2 v_d \end{bmatrix}, \qquad R_2 = v_m$$

In a stationary state, i.e. ($\dot{P} \equiv 0$), the equations for the matrix components are

$$
\begin{aligned}
2p_{12}(t) - \frac{p_{11}^2(t)}{v_m} &= 0 \\
p_{22}(t) - \alpha p_{12}(t) - \frac{p_{11}(t)p_{12}(t)}{v_m} &= 0 \\
-2\alpha p_{22}(t) + \gamma^2 v_d - \frac{p_{12}^2(t)}{v_m} &= 0
\end{aligned}
$$

If we eliminate p_{12} and p_{22} we get

$$\frac{p_{11}^4}{4v_m^3} + \frac{\alpha p_{11}^3}{v_m^2} + \frac{\alpha^2 p_{11}^2}{v_m} - \gamma^2 v_d = 0$$

Introduce

$$p_{11} = v_m \cdot p'_{11}$$

which gives

$$p'^4_{11} + 4\alpha p'^3_{11} + 4\alpha^2 p'^2_{11} - 4\gamma^2 \frac{v_d}{v_m} = 0$$

$$(p'^2_{11} + 2\alpha p'_{11})^2 - 4\gamma^2 \frac{v_d}{v_m} = 0$$

$$p'^2_{11} + 2\alpha p'_{11} - 2\gamma \sqrt{\frac{v_d}{v_m}} = 0$$

Now, we set $\beta = \gamma\sqrt{v_d/v_m}$, which gives

$$p'_{11} = -\alpha + \sqrt{\alpha^2 + 2\beta}$$

The entire solution is

$$P = v_m \begin{bmatrix} -\alpha + \sqrt{\alpha^2 + 2\beta} & \alpha^2 + \beta - \alpha\sqrt{\alpha^2 + 2\beta} \\ \alpha^2 + \beta - \alpha\sqrt{\alpha^2 + 2\beta} & -\alpha^3 - 2\alpha\beta + (\alpha^2 + \beta)\sqrt{\alpha^2 + 2\beta} \end{bmatrix}$$

The stationary Kalman gain becomes

$$K = \begin{bmatrix} -\alpha + \sqrt{\alpha^2 + 2\beta} \\ \alpha^2 + \beta - \alpha\sqrt{\alpha^2 + 2\beta} \end{bmatrix}$$

and if we use the numerical values we get

$$K = \begin{bmatrix} 40.36 \\ 814.3 \end{bmatrix}$$

with an error covariance

$$P = 10^{-5} \begin{bmatrix} 0.4036 & 8.143 \\ 8.143 & 366.1 \end{bmatrix}$$

The filter for the estimation of θ becomes

$$\dot{\hat{x}} = \begin{bmatrix} 0 & 1 \\ 0 & -\alpha \end{bmatrix} \hat{x} + \begin{bmatrix} 0 \\ H \end{bmatrix} \mu(t) + K(y - \begin{bmatrix} 1 & 0 \end{bmatrix} \hat{x})$$

with K as above.

5.6 (a)

$$A = 0.8 \qquad B = 0 \qquad N = 1$$

$$C = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$R_1 = 1 \qquad R_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad R_{12} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

The variance of the prediction error is

$$\begin{aligned} P &= E\{(x(k) - \hat{x}(k|k-1))^2\} \\ &= APA^T + NR_1N^T - (APC^T)(CPC^T + R_2)^{-1}(APC^T)^T \end{aligned}$$

$$\Rightarrow \quad P = 1.281$$

The variance of the estimation error is

$$\begin{aligned} \tilde{P} &= E\{(x(k) - \hat{x}(k|k))^2\} \\ &= (I - \tilde{K}C)P(I - \tilde{K}C)^T + \tilde{K}R_2\tilde{K}^T \\ \tilde{K} &= PC^T(CPC^T + R_2)^{-1} = \begin{bmatrix} 0.438 & 0.2192 \end{bmatrix} \end{aligned}$$

$$\Rightarrow \quad \tilde{P} = 0.4384$$

(b)

$$\begin{aligned} \hat{x}(k|k) &= \hat{x}(k|k-1) + \tilde{K}\nu(k) \\ &= (I - \tilde{K}C)\hat{x}(k|k-1) + \tilde{K}y(k) \\ \hat{x}(k|k-1) &= A\hat{x}(k-1|k-1) + Bu(t) + \hat{\nu}_1(k-1|k-1) \\ \hat{\nu}_1(k-1|k-1) &= NR_{12}(\ldots) = 0 \end{aligned}$$

$$\Rightarrow \quad \hat{x}(k|k) = \underbrace{(I - \tilde{K}C)A}_{\mu}\,\hat{x}(k-1|k-1) + \tilde{K}y(k)$$

Z-transformation ($\hat{X}(z) = \mathcal{Z}\{\hat{x}(k|k)\}$) gives

$$(1 - \mu z^{-1})\hat{X}(z) = \tilde{K}Y(z)$$

$$\Rightarrow \quad \hat{X}(z) = \frac{1}{1 - \mu z^{-1}}\tilde{K}Y(z) = \underbrace{\frac{0.438}{1 - 0.274z^{-1}}}_{G_1(z)} Y_1(z) + \underbrace{\frac{0.219}{1 - 0.274z^{-1}}}_{G_2(z)} Y_2(z)$$

(c) Let

$$\begin{cases} x_1(k) = x(k) \\ x_2(k) = x(k-1) \\ x_3(k) = x(k-2) \end{cases} \quad \Rightarrow$$

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 0.8 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} v(k) \\ y(k) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} w_1(k) \\ w_2(k) \end{bmatrix} \end{aligned}$$

(d)

$$\mathcal{O}(A,C) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0.8 & 0 & 0 \\ 0 & 1 & 0 \\ 0.64 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

For example row 1,2 and 4 i $\mathcal{O}(A,C)$ are linearly independent, which implies full rank (min(no. of columns, no. of rows)=3). Hence, the system is observable.

5.7 (a) Since $v(k)$ is a white noise, the value at every sample is completely independent of the value at any other sample. We may therefore equally well use the following system description:

$$s(k) = v(k) + a\,v(k-1) \quad \Rightarrow$$

$$\begin{aligned} \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} &= \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} v(k) \\ y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + w(k) \end{aligned}$$

(b) $R_1 = 1$, $R_2 = 1$ och $R_{12} = 0$.

$$P = APA^T + NR_1N^T - (APC^T + NR_{12})(CPC^T + R_2)^{-1}(APC^T + NR_{12})^T$$

$$\Rightarrow \quad \begin{cases} p_{11} = a^2 p_{22} + 1 - \dfrac{a^2 p_{12}^2}{1 + p_{11}} \\ p_{12} = 1 \\ p_{22} = 1 \end{cases}$$

$$\mu = \frac{a^2}{2} \quad \Rightarrow \quad p_{11} = \mu + \sqrt{1 + \mu^2}$$

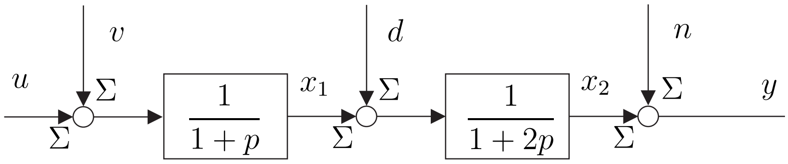
$$\tilde{K} = PC^T(CPC^T + R_2)^{-1} = \begin{bmatrix} \dfrac{\mu + \sqrt{1+\mu^2}}{1 + \mu + \sqrt{1+\mu^2}} \\ \dfrac{1}{1 + \mu + \sqrt{1+\mu^2}} \end{bmatrix}$$

5.8

$$\hat{x}(t|t) = R_1 C^T (C R_1 C^T + R_2)^{-1} y(t)$$

The matrix C must have full rank.

5.9 Introduce the states x_1 and x_2 according to the figure below.



(a)

$$x_1 = \frac{1}{1+p} u \quad \Leftrightarrow \quad \dot{x}_1 = -x_1 + u$$

Spectral factorization gives

$$\Phi_d = \frac{1}{1+\omega^2} \quad \Leftrightarrow \quad d(t) = \frac{1}{1+p} e(t), \qquad e \text{ white noise with } \Phi_e = 1.$$

$$d = x_3 \quad \Rightarrow \quad \dot{x}_3 = -x_3 + e$$

$$x_2 = \frac{1}{1+2p}(x_1 + d) \quad \Rightarrow \quad x_2 + 2\dot{x}_2 = x_1 + x_3$$

We get

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0.5 & -0.5 & 0.5 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e$$

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + n$$

(b) Spectral factorization gives

$$n(t) = \frac{p+2}{p+3} v_2(t), \quad v_2 \text{ vitt brus med } \Phi_{v_2} = 1.$$

$$n = \frac{p+3}{p+3} v_2 - \underbrace{\frac{1}{p+3} v_2}_{x_4} = v_2 - x_4$$

We get $\dot{x}_4 = -3x_4 + v_2$ and

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0.5 & -0.5 & 0.5 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e \\ v_2 \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & 1 & 0 & -1 \end{bmatrix} x + v_2$$

(c) n and d independent $\Rightarrow$ e and v_2 independent

$$\begin{aligned} \Rightarrow \quad R_1 &= \begin{bmatrix} \Phi_e & 0 \\ 0 & \Phi_{v_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ R_{12} &= \begin{bmatrix} \Phi_{ev_2} \\ \Phi_{v_2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ R_2 &= \Phi_{v_2} = 1 \end{aligned}$$

5.10 (a) $K^T = \begin{bmatrix} 1 & 0.2 \end{bmatrix}$

(b) The proof is based on the fact that the first row in the matrix $I - KC$ are zeros.

5.11 (a) The solution for the process at time k is given by

$$x(k) = x(0) + \sum_{i=0}^{k-1} u(i).$$

The expected value is

$$E\{x(k)\} = E\{x(0)\} + \sum_{i=0}^{k-1} E\{u(i)\} = E\{x(0)\} = \mu_x$$

The (co-)variance is given by $E\{(x(k)\mu_x)^2\}$, where we have subtracted off the mean since it is nonzero. Since the input is a white noise random process, it follows that

$$E\{(x(k) - \mu_x)^2\} = \sum_{i=0}^{k-1} E\{u^2(i)\} = kR_u$$

(all of the cross terms between the us disappear).

(b) The Kalman filter for the process is given by

$$\begin{aligned} \hat{x}(k+1) &= \hat{x}(k+1) + K(k)(x(k) - \hat{x}(k)) \\ L(k) &= P(k)(R_w + P(k))^{-1} \\ P(k+1) &= (1 - K(k))^2 P(k) + R_u + K^2(k) R_w \\ P(0) &= Ex^2(0). \end{aligned}$$

where $R_w = 1$ is the covariance of the measurement noise The steady-state properties of the error, e, are given by $E\{e\} = 0$ and $E\{e^2\} = P$, where P satisfies:

$$P = APA^T + R_u - APC^T(R_w + CPC^T)^{-1}CPA^T.$$

Substitution of the values of $A = 1$ and $C = 1$ and solving for positive root of P yields:

$$P = \frac{1}{2}R_u + \frac{1}{2}\sqrt{R_u^2 + 4R_uR_w}$$

(c) If the mean of u is nonzero, then you get a biased random walk. In this case, the mean and variance of the process are given by

$$\begin{aligned} E\{x(k)\} &= E\{x(0)\} + \mu \\ E\{(x(k) - \mu_x)^2\} &= k + k^2 \end{aligned}$$

The Kalman filter can be updated to take into account the (known) mean of u and hence this term can be subtracted out. The filter equations become

$$\hat{x}(k+1) = \hat{x}(k) + \mu + K(k)(x(k) - \hat{x}(k)).$$

The statistics for the error are unchanged because we subtract the mean.

5.12 (a) The system's dynamics can be described by the following equations

$$\begin{aligned} m_1\ddot{x}_1 + c_1\dot{x}_1 + k_1x_1 &= F - F_2 \\ m_2\ddot{x}_2 &= F_2 \\ F_2 &= k_2(x_1 - x_2) \end{aligned}$$

Taking the Laplace transform of the above equations gives the transfer function G_{x_1F}.

(b) The transfer function has a zero at $s = i\sqrt{k_2/m_2}$, which means that transmission of sinusoidal signals with frequency $\sqrt{k_2/m_2}$ is zero. Therefore, we can set $k_2/m_2 = \omega_0^2$ to eliminate disturbances at frequency ω_0.

5.13 The full system with a measurement equation for the case of having two sensors measuring the same state is

$$\begin{aligned} x(k+1) &= 0.8x(k) + v_1(k), \qquad v_1 \sim WGN(0, R_1) \\ \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} x(k) + v_2(k), \qquad v_2 \sim WGN(0, R_2) \end{aligned}$$

With the information given we thus have

$$A = 0.8, \quad B = 0, \quad N = 0, \quad C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$R_1 = 1, \quad R_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad R_{12} = \begin{bmatrix} 0 & 0 \end{bmatrix} \quad (v_1 \ \&\ v_2 \text{ independent.})$$

(a) The minimum variance estimate is given by the stationary Kalman filter (predictor case), which gives the estimate $\hat{x}(k+1) = \hat{x}(k+1|k)$ based on the measurement $y(k)$.

$$\begin{aligned}
\hat{x}(k+1) &= 0.8\hat{x}(k) + K(y(k) - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \hat{x}(k)) \\
K &= APC^T(CPC^T + R_2)^{-1} \\
&= 0.8P \begin{bmatrix} 11 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} P \begin{bmatrix} 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right)^{-1} \\
&= \frac{0.8P}{3P+2} \begin{bmatrix} 2 & 1 \end{bmatrix}
\end{aligned}$$

where the estimation error variance is given by the solution to

$$\begin{aligned}
P &= APA^T + R_1 - \underbrace{APC^T(CPC^T + R_2)^{-1}}_{K} CPA^T \\
&= 0.64P + 1 - \frac{0.8P}{3P+2} \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} 0.8P \\
0 &= (1 - 0.36P)(3P+2) - 3 \cdot 0.64P^2 \\
&= 2 + 2.28P + P^2 \\
\Rightarrow P &= 1.28 \quad \text{(-0.52 incorrect since } P \text{ must be larger than zero)} \\
\Rightarrow K &= 0.175 \begin{bmatrix} 2 & 1 \end{bmatrix}
\end{aligned}$$

(b) The case when we use the measurement $y(k)$ to estimate $x(k)$, or equivalently $y(k+1)$ to estimate $x(k+1)$, is called the filter case, and is given by

$$\begin{aligned}
\hat{x}(k|k) &= \hat{x}(k) + \tilde{K}(y(k) - C\hat{x}(k)) \\
\tilde{K} &= PC^T(CPC^T + R_2)^{-1} \\
&= \frac{P}{3P+2} \begin{bmatrix} 2 & 1 \end{bmatrix} = 0.22 \begin{bmatrix} 2 & 1 \end{bmatrix}
\end{aligned}$$

The variance of this estimate is

$$\begin{aligned}
P(k|k) &= P - \underbrace{PC^T(CPC^T + R_2)^{-1}}_{\tilde{K}} CP \\
&= P(1 - 3 \cdot 0.22)
\end{aligned}$$

Thus, a 66% reduction of the estimation error variance.

6.1 By calculating the reachability matrix, it can be observed that the system is reachable if both a and b are different from zero. The characteristic polynomial of the closed loop system is

$$\det(\lambda I - A) = \lambda^2 - (a + ab)\lambda + ab$$

With the given numerical values, the eigenvalues are

$$\lambda = 0.9 \pm 0.4354i,$$

we have $|\lambda| = 1$ which implies that the eigenvalues are on the unit circle. Since the sampling time is a quarter year the corresponding continuous system has the eigenvalues

$$s = -\frac{1}{h} log\lambda = \pm 1.8041i,$$

To find a control law that stabilizes the system we introduce the state feedback $u = -Kx + k_r r = -k_1 C - k_2 I + k_r r$. Which gives the following dynamics matrix of the closed loop system

$$A - BK = \begin{bmatrix} a - ak_1 & a - ak_2 \\ ab - b - abk_1 & ab - abk_2 \end{bmatrix}$$

The matrix has the characteristic polynomial

$$\det(\lambda I - A) = \lambda^2 - (a + ab - ak_1 - abk_2)\lambda - ab(k_2 - 1)$$

Requiring that this polynomial is equal to the desired closed loop polynomial and solving it for $k_1$and k_2 gives

$$k_1 = 1 - \frac{2\alpha - \alpha^2}{a}$$
$$k_2 = 1 - \frac{\alpha^2}{ab}$$

where α is the eigenvalue of the closed-loop system. Moreover, The steady state solution of the closed loop system for constant reference input $r = r_0$ is given by

$$x_0 = (I - A + BK)^{-1} B k_r r_0$$

the steady-state output is then

$$y_0 = Cx_0 + Dk_r r_0 = \big(C(I - A + BK)^{-1}B + D\big)k_r r_0.$$

to satisfy $y_0 = r_0$ we need to have:

$$k_r = \frac{(1-\alpha)^2}{\alpha + (1-\alpha)^2}.$$

Having $\alpha = 0.5$, the control law becomes

$$u = -0.0625C - 0.75I + 0.238r$$

Using the simple model the government should thus consider the economic situation each quarter and make corresponding adjustments in public spending. Notice that in this particular case the governments actions are most sensitive to changes in investment.

6.2 (a) $L = \begin{bmatrix} 0.6 & 1.2 \end{bmatrix}$

(b) $$\frac{Y(s)}{R(s)} = \frac{K(s+1)}{s^2 + 3.2s + 4}$$

(c) $K = 4$

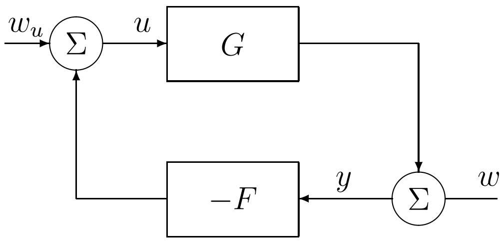
6.3 (a)

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -2 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r \\ y &= \begin{bmatrix} -1 & 1 & 0 \end{bmatrix} x \end{aligned}$$

(b) $L_e = \begin{bmatrix} -3 & 2 & -4 \end{bmatrix}$

(c) $$G(s) = \frac{Y(s)}{R(s)} = \frac{4(s+1)}{(s+2)(s^2+2s+2)}$$

6.4 Consider the blockscheme



from which we conclude

$$y = (I + GF)^{-1}(w + Gw_u) = G_{wy}w + G_{w_uy}w_u$$

and

$$u = (I + FG)^{-1}(w_u - Fw) = G_{w_uu}w_u + G_{wu}w,$$

which gives the I/O model

$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} G_{w_uu} & G_{wu} \\ G_{w_uy} & G_{wy} \end{bmatrix} \begin{bmatrix} w_u \\ w \end{bmatrix}.$$

Trivially, we also have

$$w_u = u + Fy$$

and

$$w = y - Gu$$

which on matrix form becomes

$$\begin{bmatrix} w_u \\ w \end{bmatrix} = \begin{bmatrix} I & F \\ -G & I \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix}.$$

Thus, we have proven

$$\begin{bmatrix} G_{w_u u} & G_{wu} \\ G_{w_u y} & G_{wy} \end{bmatrix}^{-1} = \begin{bmatrix} I & F \\ -G & I \end{bmatrix}$$

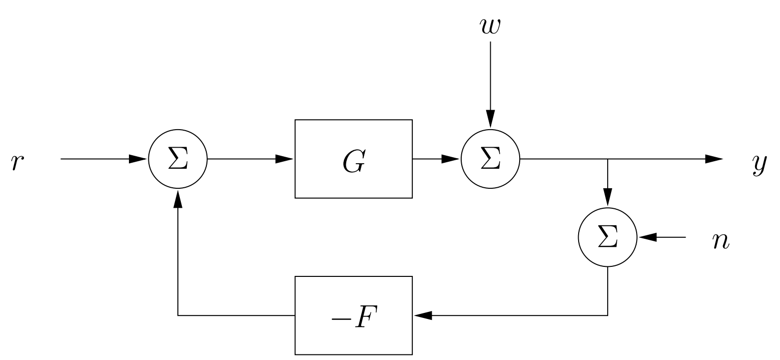
Alternatively, show that the product of the two matrices equals the identity matrix.

6.5 With the transfer functions(c.f. figure)

$$G = \frac{s-1}{s+2}, \quad F = \frac{s+2}{s-1},$$

$$Y = G(R - F(Y+N)) + W \;\Rightarrow\; (1+GF)Y = GR - GFN + W$$
$$\Rightarrow\; Y = (1+GF)^{-1}GR - (1+GF)^{-1}GFN + (1+GF)^{-1}W$$

.



Now

$$G_c = G_{ry} = (1+GF)^{-1}G = \frac{s-1}{2s+3}$$
$$S = G_{wy} = (1+GF)^{-1} = \frac{s+1}{2s+3}$$
$$T = 1 - S = \frac{s+2}{2s+3}$$

which are all stable.

Checking the following four transfer functions

$$
\begin{aligned}
G_{w_u u} &= (1+FG)^{-1} = \frac{s+1}{2s+3} \\
G_{wu} &= -(1+FG)^{-1}F = -\frac{(s+2)(s+1)}{(s-1)(2s+3)} \\
G_{w_u y} &= (1+GF)^{-1}G = \frac{s-1}{2s+3} \\
G_{wy} &= (1+GF)^{-1} = \frac{s+1}{2s+3}
\end{aligned}
$$

we conclude that the system is not internally stable because G_{wu} is not stable.

6.6 (a)

$$
\begin{aligned}
\det(zI-A) = 0 \quad &\Rightarrow \quad z = \pm 1 \quad \Rightarrow \quad \text{marginellt stabilt} \\
\mathcal{S}(A,B) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad &\Rightarrow \quad \text{rang=2} \quad \Rightarrow \quad \text{styrbart} \\
\mathcal{O}(A,C) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad &\Rightarrow \quad \text{rang=2} \quad \Rightarrow \quad \text{observerbart}
\end{aligned}
$$

(b) $L = \begin{bmatrix} 1-2\rho\cos\phi & 1 & 2\rho\cos\phi - 1 - \rho^2 \end{bmatrix}$

(c) $K^T = \begin{bmatrix} 1+\nu^2 & -2\nu \end{bmatrix}$

6.7 First by checking the reachability and the observability matrix, we make sure that the system is reachable and observable and there exist a solution to the problem.

The characteristic polynomials of the closed-loop are given by

$$
\det(sI-(A-BL)) = det \begin{bmatrix} s & -1 \\ l_1-1 & s+l_2 \end{bmatrix} = s^2 + l_2 s + l_1 - 1.
$$

Comparing with the desired characteristic polynomials we obtain

$$
l_1 = 1 + a_2, \quad l_2 = a_1.
$$

Furthermore,

$$
\det(sI-(A-KC)) = det \begin{bmatrix} s & k_1-1 \\ -1 & s+k_2 \end{bmatrix} = s^2 + k_2 s + k_1 - 1.
$$

Comparing with the given characteristic polynomials and solve it for l_1 and l_2, we obtain

$$
k_1 = 1 + b_2, \quad k_2 = b_1 \tag{3}
$$

The controller transfer function is then given by

$$G(s) = L(sI - A + BL + CK)^{-1}K = \begin{bmatrix} l_1 & l_2 \end{bmatrix} \begin{bmatrix} s & k_1 - 1 \\ l_1 - 1 & sl_2 + k_2 \end{bmatrix}^{-1} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

$$\begin{aligned} &= \frac{s(l_1k_1 + l_2k_2) + l_1k_2 + l_2k_1}{s_2 + s(l_2 + k_2) - (l_1 - 1)(k_1 - 1)} \\ &= \frac{s(1 + a_2 + b_2 + a_1b_1 + a_2b_2) + a_1 + b_1 + a_1b_2 + a_2b_1}{s^2 + s(a_1 + b_1) - a_2b_2} \end{aligned}$$

The controller poles are the zeros of the polynomial $s^2 + s(a_1 + b_1) - a_2b_2$. Since $a_2b_2 > 0$, one controller pole is always in the right half plane.

6.8 (a) The controllability matrix

$$\mathcal{S}(A, B) = \begin{bmatrix} A & AB \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

has full rank (2). Hence the system is controllable, which means that the poles can be placed arbitrarily.

(b) A dead beat controller has all its poles in the origin. Since the system is controllable we can also place the closed system's poles there.
With the feedback $u(k) = -Lx(k)$ we have

$$\left.\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ u(k) &= -Lx(k) \end{aligned}\right\} \Rightarrow x(k+1) = (A - BL)x(k)$$

The closed loop poles are therefore given by $P(\lambda) = \det(\lambda I - (A - BL)) = 0$. All poles in the origin means that the pole polynomial must also be given by $P(\lambda) = (\lambda - 0)^n = \lambda^n$. With $L = \begin{bmatrix} l_1 & l_2 \end{bmatrix}$ we have

$$\begin{aligned} \det(\lambda I - (A - BL)) &= \det \begin{bmatrix} \lambda - 1 + l_1 & -1 + l_2 \\ -1 & \lambda \end{bmatrix} \\ &= \lambda^2 + (l_1 - 1)\lambda + l_2 - 1 \\ &= \lambda^2 \end{aligned}$$

Polynomial identification

$$\left.\begin{aligned} l_1 - 1 &= 0 \\ l_2 - 1 &= 0 \end{aligned}\right\} \Rightarrow l_1 = l_2 = 1$$

Thus, the feedback of the deadbeat controller is $u(k) = -\begin{bmatrix} 1 & 1 \end{bmatrix} x(k)$.

7.1 (a) The complementary sensitivity function (and closed loop system $G_c(s)$) is given by

$$T(s) = \frac{F(s)G(s)}{1 + F(s)G(s)}$$

To determine the controller $F(s)$ that yields a desired $T(s)$ we can express $F(s)$ as a function of $T(s)$ and $G(s)$. Rewriting the above expression gives

$$T(s) + T(s)F(s)G(s) = F(s)G(s)$$

and

$$T(s) = G(s)(1 - T(s))F(s),$$

which gives

$$F(s) = G^{-1}(s)\frac{T(s)}{1 - T(s)}.$$

With the given $T(s)$ and $G(s)$ this gives

$$F(s) = \frac{5(s+1)}{s(s-3)}.$$

This means that the zero in $s = 3$ will be cancelled when we regard the system from reference to outpu signal. If we look att the relation between reference and control signal, we have

$$U(s) = \frac{5(s+1)}{(s-3)(s+5)}R(s),$$

which implies that the feedback system has a pole in $s = 3$, and consequently is unstable.

(b) The bandwidth 5 rad/s can also be achieved if we keep the zero and instead place a pole in $s = -3$, i.e.

$$T(s) = \frac{5(s-3)}{(s+3)(s+5)}.$$

The relation

$$F(s) = G^{-1}(s)\frac{T(s)}{1 - T(s)}$$

then gives

$$F(s) = \frac{5(s+1)}{(s^2 + 3s + 30)}.$$

In this case there is no cancellation and all transfer functions of the closed loop system are stable.

(c) The sensitivity function becomes

$$S(s) = 1 - T(s) = \frac{s^2 + 3s + 30}{(s+3)(s+5)}.$$

Note that $S(0) = 2$, which implies that constant process disturbances are amplified by a factor 2. In other words this design is not particularly efficient.

(d) A controller with different F_r och F_y (also called 2DoF control) gives addition degrees of freedom. One may, for example, choose a feedback F_y that gives desired S och T, and adjust the reference compensator F_r to get the desired closed loop bandwidth.

7.2 The loop gain for a system with a zero in $s = 3$ and a time delay of 1 s can be expressed as

$$L(s) = e^{-s}(3 - s)\bar{L}(s)$$

or

$$L(s) = e^{-s}\frac{(3 - s)}{(3 + s)}(3 + s)\bar{L}(s)$$

The argument of the frequency function is now

$$\arg L(i\omega) = -\omega - 2\arctan\frac{\omega}{3} + \arg((3 + i\omega)\bar{L}(i\omega))$$

According to the prerequisites the amplitude curve is decreasing monotonously. According to Bode's relation this implies

$$\arg((3 + i\omega)\bar{L}(i\omega)) \leq 0.$$

This means that

$$\arg L(i\omega) \leq -\omega - 2\arctan\frac{\omega}{3}$$

and that the phase margin is

$$\varphi_m = \pi + \arg L(i\omega_c) \leq \pi - \omega_c - 2\arctan\frac{\omega_c}{3}$$

Let us study the special case when $\varphi_m = 0$ and assume that we have equality above. Then

$$0 = \pi - \omega_c - 2\arctan\frac{\omega_c}{3}$$

i.e.

$$\omega_c \approx 2.$$

The highest possible crossover frequency is consequently ca 2 rad/s.

7.3

7.4 (a) The requirement $|S(i\omega)| = \bar{\sigma}(S(i\omega))$ and $|T(i\omega)| = \bar{\sigma}(T(i\omega))$ can be formulated as

$$|S(i\omega)| \leq \tfrac{1}{10}, \quad \omega \leq 0.1, \qquad\qquad |T(i\omega)| \leq \tfrac{1}{10}, \quad \omega \geq 2$$
$$|S(0)| \leq \tfrac{1}{100}$$

(b) The corresponding requirement on GF_y becomes

$$|G(0)F_y(0)| > 100$$
$$|G(i\omega)F_y(i\omega)| > 10, \quad \omega \leq 0.1$$
$$|G(i\omega)F_y(i\omega)| < \frac{1}{10}, \quad \omega \geq 2$$

(c) The requirements in (a) can be formulated using weight functions W_S and W_T such that

$$|S(i\omega)| \leq |W_S^{-1}(i\omega)|, \quad \forall\omega$$
$$|T(i\omega)| \leq |W_T^{-1}(i\omega)|, \quad \forall\omega.$$

If W_S^{-1} and W_T^{-1} is chosen to be of first order,

$$W_S^{-1}(s) = a_1\left(1+\frac{s}{b_1}\right), \qquad W_T^{-1}(s) = \frac{a_2}{s}\left(1+\frac{s}{b_2}\right)$$

we get

$$W_S^{-1}(s) = \frac{1}{100}(1+100s), \qquad W_T^{-1}(s) = \frac{0.14}{s}\left(1+\frac{s}{2}\right)$$

(d) In this qustion we assume we have a SISO system! From the requirements in (b) we conclude that $|GF| = 1$ must occur between $\omega = 0.1$ and $\omega = 2$. If we assume that $20\log_{10}|GF|$ decreases linearly with $\log_{10}\omega$ the curve crosses 0 dB approximately at $\omega_c = 0.4$.
Equation (7.20) gives

$$\arg G(j\omega_c) \approx 90^\circ \frac{d(\log|GF|)}{d(\log\omega)} = -138^\circ$$

which gives the phase margin $\varphi_m = 42^\circ$.
Under "normala" stability margins and in absence of time delays the closed loop system have a resonance peak approximately give by

$$M_p = \| T \|_\infty \geq \frac{1}{2\sin\frac{\varphi_m}{2}} = 1.4$$

(e)

$$|T(i\omega_c)| = 1.4$$
$$|W_T^{-1}(i\omega_c)| = \frac{0.14}{0.45}\sqrt{1+\frac{0.45^2}{2^2}} = 0.32$$

Thus, there is no solution to this choice of weight functions.

7.5 Bode's sensitivity integral for stable systems with stable controllers states that the area A_1 and the area A_2 should be the same. If $A_2 > A_1$ the loop transfer GF is unstable, and either G is unstable, F unstable, or both unstable.

7.6 The first requirement means that

$$|S(i\omega)| < 10^{-3} \qquad \omega \leq 2,$$

where

$$S(s) = \frac{1}{1 + F(s)G(s)}.$$

When $|F(i\omega)G(i\omega)|$ is "large" we have

$$|S(i\omega)| \approx \frac{1}{|F(i\omega)G(i\omega)|},$$

which gives

$$|F(i\omega)G(i\omega)| > 10^3 \qquad \omega \leq 2.$$

Furthermore, for the control system to be stable in spite of a model uncertainty

$$|\Delta G(i\omega)| \leq 100|G(i\omega)| \qquad \omega \geq 20,$$

where $\Delta G(s)$ denotes the absolute model error in $G(s)$. Thus, the relative model error should satisfy

$$\left|\frac{\Delta G(i\omega)}{G(i\omega)}\right| \leq 100.$$

To maintain stability

$$|T(i\omega)| < 10^{-2} \qquad \omega \geq 20,$$

where

$$T(s) = \frac{F(s)G(s)}{1 + F(s)G(s)}.$$

When $|F(i\omega)G(i\omega)|$ is "small" we have

$$|T(i\omega)| \approx |F(i\omega)G(i\omega)|$$

which gives the requirement

$$|F(i\omega)G(i\omega)| < 10^{-2} \qquad \omega \geq 20.$$

To satisfy both requirements, the loop gain must decrease from 10^3 to 10^{-2} on the interval $\omega = 2$ to $\omega = 20$, i.e. 100 dB per decade (slope -5). Then, according to Bode's relation, $\arg G(i\omega) \approx -5 \cdot 90°$ within this interval. This means that the feedback system is unstable. Thus, the requirements cannot be fulfilled.

7.7 (a) We can, for example, choose $W_S(s) = 1000/s$ and $W_T(s) = 10\sqrt{2}s/(1 + 0.01s)$, which gives $|W_S(j\omega)| \geq 10^3\ \forall\omega \leq 1$ and $|W_T(j\omega)| \geq 10^3\ \forall\omega \geq 100$.

(b) The relations (7.12) and (7.13) gives $|GF| > 1000\ \forall\omega \leq 1$ and $|GF| < 10^{-3}\ \forall\omega \geq 100$. From this follows that ω_c is between $\omega = 1$ and $\omega = 100$. The slope of the loop gain becomes approximately

$$\frac{20\log_{10} 10^{-3} - 20\log_{10} 1000}{\log_{10} 100 - \log_{10} 1} = -60\text{dB/dekad}.$$

Bode's relation then gives that $\arg GF(j\omega_c) \approx -270°$, i.e. the system is unstable.

8.1 (a)

$$\text{RGA}(G(0)) = G(0) .* G^{-T}(0) = \begin{bmatrix} -\frac{5}{7} & \frac{12}{7} \\ \frac{12}{7} & -\frac{5}{7} \end{bmatrix}$$

(b) Avoid the pairing $u_1 \leftrightarrow y_1$ och $u_2 \leftrightarrow y_2$.

8.2

$$\text{RGA}(G(s)) = \begin{bmatrix} \frac{3}{s+4} & \frac{s+1}{s+4} \\ \frac{s+1}{s+4} & \frac{3}{s+4} \end{bmatrix}$$

For the DC gain we have

$$\text{RGA}(G(0)) = \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix}$$

Since all elements of $\text{RGA}(G(0))$ are positive, all combinations are possible. At the crossover frequency we have

$$\text{RGA}(G(10i)) = \begin{bmatrix} \frac{12-30i}{116} & \frac{104+30i}{116} \\ \frac{104+30i}{116} & \frac{12-30i}{116} \end{bmatrix}.$$

Elements closest to unity is achieved if u_1 is used to control y_2 and u_2 control y_1.

8.3 (a) The responses are recognized as those of first order systems without time delay, i.e.

$$G_{ij} = \frac{K_{ij}}{1 + sT_{ij}}$$

Since there is no delay T_{ij} is the time when $y_{ij} = 0.63K_{ij}$ and $K_{ij} = \lim_{t\to\infty} y_{ij}$. From the figure we therefore deduce

$$G(s) = \begin{bmatrix} \dfrac{2}{1+s} & \dfrac{1.5}{1+0.5s} \\ \dfrac{1}{1+s} & \dfrac{1}{1+s} \end{bmatrix}$$

(b) Assign a compensator W such that

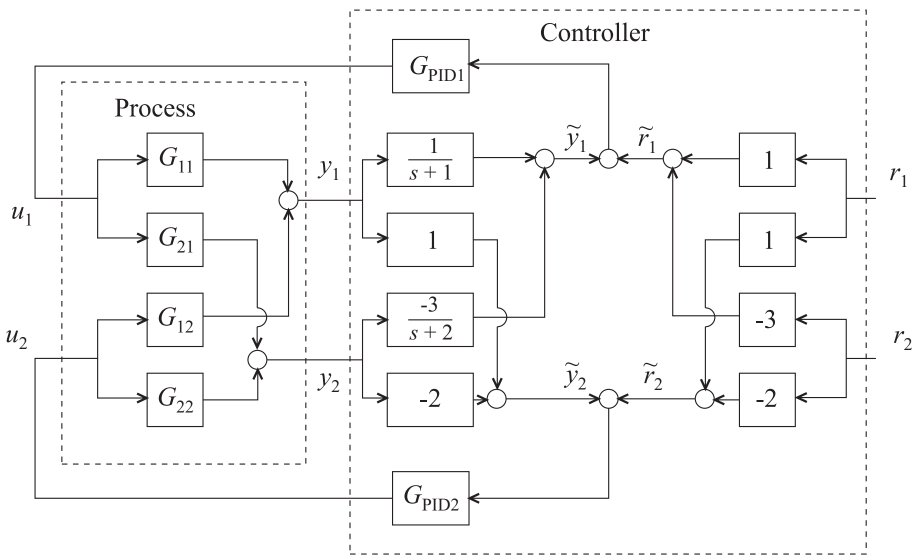
$$W(s)G(s) = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \begin{bmatrix} \tilde{G}_{11} & 0 \\ 0 & \tilde{G}_{22} \end{bmatrix}$$

This gives

$$\begin{aligned} W_{11}G_{12} + W_{12}G_{22} &= W_{11}\frac{3}{s+2} + W_{12}\frac{1}{1+s} = 0 \\ W_{21}G_{21} + W_{22}G_{21} &= (2W_{21} + W_{22})\frac{1}{1+s} = 0 \end{aligned}$$

Try $W_{21} = 1$, $W_{22} = -2$, $W_{11} = \dfrac{1}{s+1}$ and $W_{12} = -\dfrac{3}{s+2}$.

$$\begin{aligned} \tilde{G}_{11} &= \ldots = \frac{1-s}{(s+1)^2(s+2)} \\ \tilde{G}_{22} &= \ldots = \frac{s-1}{(1+s)(2+s)} \end{aligned}$$



8.4 (a)

$$\tilde{H}(q) = W(q)H(q) = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \begin{bmatrix} \dfrac{2}{q-0.5} & \dfrac{1}{q-1} \\ \dfrac{1}{q-0.5} & \dfrac{3}{q-0.5} \end{bmatrix} = \begin{bmatrix} \tilde{H}_{11} & 0 \\ 0 & \tilde{H}_{22} \end{bmatrix}$$

The antidiagonal elements gives

$$\begin{aligned} W_{11}\frac{1}{q-1} + W_{12}\frac{3}{q-0.5} &= 0 \\ W_{21}\frac{2}{q-0.5} + W_{22}\frac{1}{q-0.5} &= 0 \end{aligned}$$

The solution to this is not unique, but aiming for the simple we can try

$$\begin{aligned} W_{21} &= 1 \quad \Rightarrow \quad W_{22} = -2 \\ W_{11} &= \frac{3}{q-0.5} \quad \Rightarrow \quad W_{12} = -\frac{1}{q-1} \end{aligned}$$

All W_{ij} should be stable, which they are.
Now, the resulting diagonal elements that we will design our SISO controllers for should be non-zero and not become unnecessary difficult to control. With the above compensator W we get

$$\begin{aligned} \tilde{H}_{11} &= W_{11}\frac{2}{q-0.5} + W_{12}\frac{1}{q-0.5} \\ &= \frac{5q-5.5}{(q-1)(q-0.5)^2} \\ \tilde{H}_{22} &= W_{21}\frac{1}{q-1} + W_{22}\frac{3}{q-0.5} \\ &= -\frac{1.5}{q-0.5} \end{aligned}$$

which is OK.

9.1 The process can be written on state-space form as

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{aligned}$$

$$V = \int_0^\infty (y^2 + \eta \cdot u^2)\, dt \qquad \Rightarrow \qquad Q_1 = 1,\ Q_2 = \eta$$

Since $z = y = Cx \quad \Rightarrow \quad M = C$ and the stationary Riccati equation becomes

$$A^T\bar{S} + \bar{S}A + M^T Q_1 M - \bar{S}BQ_2^{-1}B^T\bar{S} = 0$$

Let

$$\bar{S} = \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix} \Rightarrow$$

The Riccati equation can then be written as

$$\begin{bmatrix} 0 & 0 \\ s_1 & s_2 \end{bmatrix} + \begin{bmatrix} 0 & s_1 \\ 0 & s_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \frac{1}{\eta} \cdot \begin{bmatrix} s_2^2 & s_2 s_3 \\ s_2 s_3 & s_3^2 \end{bmatrix} = 0$$

$$\begin{cases} 1 - \frac{s_2^2}{\eta} = 0 \\ s_1 - \frac{s_2 s_3}{\eta} = 0 \\ 2s_2 - \frac{s_3^2}{\eta} = 0 \end{cases} , \quad \bar{S} \text{ positive definite} \quad \Rightarrow \quad \begin{cases} s_1 = \sqrt{2} \cdot \eta^{1/4} \\ s_2 = \eta^{1/2} \\ s_3 = \sqrt{2} \cdot \eta^{3/4} \end{cases}$$

$$\begin{aligned} L = Q_2^{-1} B^T \bar{S} &= \frac{1}{\eta} \cdot \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2}\eta^{1/4} & \eta^{1/2} \\ \eta^{1/2} & \sqrt{2} \cdot \eta^{-3/4} \end{bmatrix} \\ &= \frac{1}{\eta} \cdot \begin{bmatrix} \eta^{1/2} & \sqrt{2}\eta^{3/4} \end{bmatrix} = \begin{bmatrix} \eta^{-1/2} & \sqrt{2} \cdot \eta^{-1/4} \end{bmatrix} \end{aligned}$$

Let $\mu = \eta^{-1/4} \quad \Rightarrow \quad L = \begin{bmatrix} \mu^2 & \sqrt{2} \cdot \mu \end{bmatrix}$

$$\begin{aligned} u = -Lx \quad \Rightarrow \quad \dot{x} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} \mu^2 & \sqrt{2} \cdot \mu \end{bmatrix} \cdot x \\ &= \begin{bmatrix} 0 & 1 \\ -\mu^2 & -\sqrt{2}\mu \end{bmatrix} x \end{aligned}$$

$$0 = \begin{vmatrix} s & -1 \\ \mu^2 & s + \sqrt{2} \cdot \mu \end{vmatrix} = s^2 + \sqrt{2}\mu s + \mu^2$$

$$\begin{aligned} s &= -\frac{\mu}{\sqrt{2}} \pm \sqrt{\frac{\mu^2}{2} - \mu^2} = -\frac{\mu}{\sqrt{2}} \pm i \cdot \frac{\mu}{\sqrt{2}} = \\ &= -\frac{\mu}{\sqrt{2}} \cdot (1 \pm i) = -\frac{1}{\sqrt{2} \cdot \eta^{1/4}} \cdot (1 \pm i) \end{aligned}$$

If η is reduced the distance between the poles and the origin will increase, which corresponds to a faster behaviour (to the cost of increased control signal activity, c.f. the cost function).

9.2 (a)

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0.2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad Q_2 = 0.5$$

$$\mathcal{S}(A, B) = \begin{bmatrix} 0 & 1 \\ 1 & 0.2 \end{bmatrix} \quad \Rightarrow \quad \text{controllable} \quad \Rightarrow \quad \text{stabilizable}$$

Thus, there exist an optimal feedback given by

$$\begin{aligned} L &= (B^T SB + Q_2)^{-1} B^T SA \\ S &= A^T SA + Q_1 - A^T SB(B^T SB + Q_2)^{-1} B^T SA \end{aligned}$$

where $S = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}$ is symmetric and positive definite.

$$\Rightarrow \quad \begin{cases} s_{11} = 1 \\ s_{12} = 0 \\ s_{22} = 1 + 0.04 s_{22} - \dfrac{0.04 s_{22}^2}{s_{22} + 0.5} \end{cases}$$

$$\Rightarrow \quad s_{22} = 1.013 \quad \Rightarrow \quad L = \begin{bmatrix} 0 & 0.134 \end{bmatrix}$$

The feedback becomes $u(t) = -0.134 x_2(t)$.

(b) With the feedback $u(t) = -Lx(t)$ we have

$$x(t+1) = (A - BL)x(t)$$

which is stable if the eigenvalues to $A - BL$ is inside the unit circle. Let $L = \begin{bmatrix} l_1 & l_2 \end{bmatrix}$.

$$\det(\lambda I - A + BL) = 0 \quad \Rightarrow \quad \lambda^2 + (l_2 - 0.2)\lambda + l_1 = 0$$

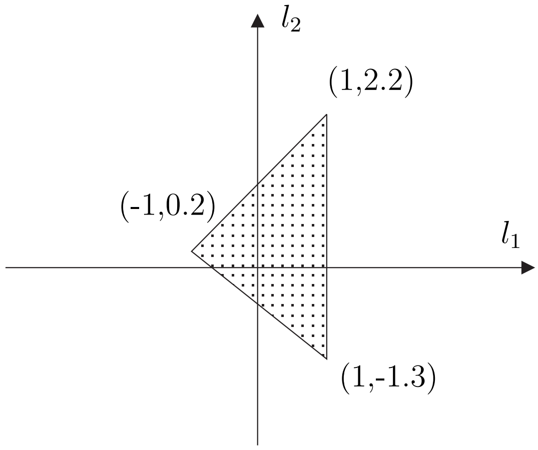
Introduce the Möbius transformation

$$s = \frac{\lambda + 1}{\lambda - 1} \quad \Rightarrow \quad \lambda = \frac{s+1}{s-1}$$

from the region inside the unit circle to the left half plane.

$$\Rightarrow \quad (0.8 + l_2 + l_1)s^2 + 2(1 - l_1)s + 1.2 - l_2 + l_1 = 0$$

Routh-Hurwitz analysis of second order systems gives that the system is stable if all coefficients have the same sign, in this particular case all coefficients smust be positive (see figure).



9.3 (a) The derived model can be written as a state space model

$$\begin{aligned}\frac{d}{dt}x_1(t) &= -x(t) + u(t) + v_1(t)\\ y(t) &= x(t)\end{aligned}$$

where $x_1(t) = \Delta h(t)$, $u(t) = \Delta q_1(t)$ and $v(t) = \Delta q_2(t)$.
Discretization with the sampling interval T gives

$$\begin{aligned}x_1(t+1) &= e^{-T}x_1(t) + (1-e^{-T})u(t) + (1-e^{-T})v_1(t)\\ y(t) &= x_1(t)\end{aligned}$$

Here, we have assumed that also the disturbance flow is constant on every sampling interval.
To avoid stationary errors after step disturbances we introduce the integral state

$$x_2(t) = \frac{1}{q-1}e(t) = \frac{1}{q-1}(r(t) - y(t))$$

$$\Rightarrow \quad x_2(t+1) = x_2(t) - x_1(t) + r(t)$$

The extended model, with integral action for the LQG-design becomes

$$\begin{aligned}x(t+1) &= \underbrace{\begin{bmatrix} e^{-T} & 0 \\ -1 & 1 \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 1-e^{-T} \\ 0 \end{bmatrix}}_{B} u(t) + \underbrace{\begin{bmatrix} 1-e^{-T} \\ 0 \end{bmatrix}}_{N} v_1(t)\\ &\quad + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(t)\\ z(t) &= \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{M} x(t)\\ y(t) &= \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} x(t)\end{aligned}$$

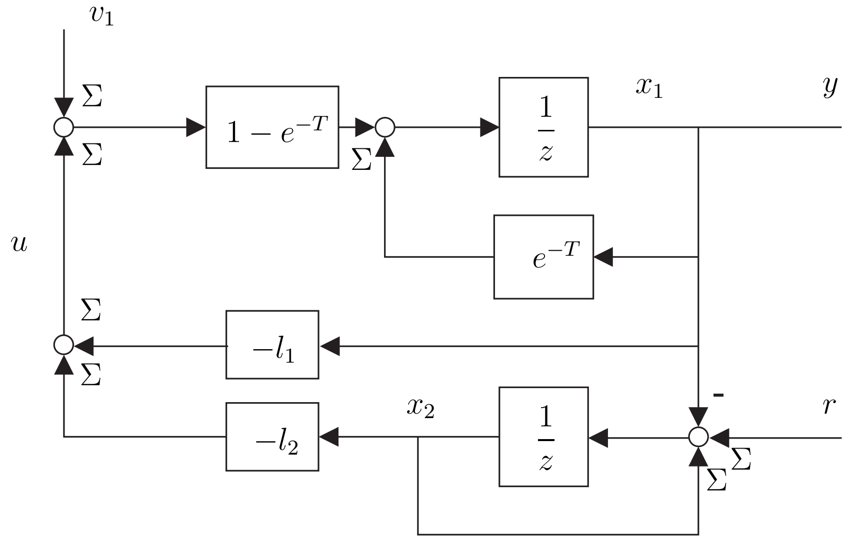
where $r(t) = 0$ when the level should be $\bar{h}$.
The loss function to be minimized now is

$$V = \| z \|_{Q_1}^2 + \| u \|_{Q_2}^2$$

where $Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $Q_2 = 1$.
The optimal control law $u(t) = -Lx(t)$ is given by

$$\begin{aligned}L &= (B^TSB + Q_2)^{-1}B^TSA\\ S &= A^TSA + M^TQ_1M - A^TSB(B^TSB + Q_2)^{-1}B^TSA\end{aligned}$$

(b)



9.4 The system can be written as a state space model,

$$\begin{aligned}
\frac{d}{dt}x(t) &= \underbrace{\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} u(t) + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{N_c} v(t) \\
z(t) &= \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{M} x(t) \\
y(t) &= \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} x(t)
\end{aligned}$$

Discretization gives

$$\begin{aligned}
F &= e^{AT} = I + AT + \frac{1}{2}(AT)^2 + \ldots = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \\
G &= \int_0^T e^{As} ds B = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix} \\
N &= \int_0^T e^{As} ds N_c = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}
\end{aligned}$$

The continuous time criterion to be minimized can be written

$$V_c = \| z \|^2_{Q_1^c} + \| u \|^2_{Q_2^c}$$

dr $Q_1^c = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ och $Q_2^c = 1$. In discrete time this corresponds to the minimization of

$$V_d = E\{x^T(t)Q_1x(t) + 2x^TQ_{12}^Tu(t) + u^T(t)Q_2u(t)\}$$

With

$$\Gamma(t) = \int_0^t e^{As} B ds = \begin{bmatrix} t \\ \frac{t^2}{2} \end{bmatrix}$$

$$\begin{aligned}
Q_1 &= \int_0^T (e^{At})^T M^T Q_1^c M e^{At} dt &= \begin{bmatrix} 4/3 & -1/2 \\ -1/2 & 1 \end{bmatrix} \\
Q_{12}^T &= \int_0^T \Gamma^T(t) M^T Q_1^c M e^{At} dt &= \begin{bmatrix} 5/8 & -1/6 \end{bmatrix} \\
Q_2 &= \int_0^T \Gamma^T(t) M^T Q_1^c M \Gamma(t) dt + T Q_2^c = \frac{83}{60}
\end{aligned}$$

The discrete time control law $u(t) = -Lx(t)$ then follows from the solution of the Ricattiequation

$$\begin{aligned}
S &= A^T S A + M^T Q_1 M - (A^T S B + Q_{12})(B^T S B + Q_2)^{-1}(A^T S B + Q_{12})^T \\
L &= (B^T S B + Q_2)^{-1}(A^T S B + Q_{12})^T
\end{aligned}$$

9.5 (a) Integrate the equation (2) from 0 to t:

$$\dot{x}(t) + \int_0^t x(s) ds = \sum_{k=0}^{r} u_k; \quad t_r \le t < t_{r+1}$$

Introduce

$$v_r = \sum_{k=0}^{r} u_k$$

and the function

$$v(t) = v_r \text{ om } r_r \le t < t_{r+1}$$

This is piecewise continuous and with

$$z(t) = \int_0^t x(s) ds$$

we get the state space model

$$\begin{aligned}
\frac{d}{dt} \begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v(t); \\
x(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix}
\end{aligned}$$

Sampling implies

$$\begin{aligned}
\begin{bmatrix} z(kT+T) \\ x(kT+T) \end{bmatrix} &= \begin{bmatrix} \cos T & \sin T \\ -\sin T & \cos T \end{bmatrix} \begin{bmatrix} z(kT) \\ x(kT) \end{bmatrix} + \begin{bmatrix} 1-\cos T \\ \sin T \end{bmatrix} v_k \\
y_k &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} z(kT) \\ x(kT) \end{bmatrix}
\end{aligned}$$

$$\begin{bmatrix} z(kT+T) \\ x(kT+T) \\ v_{k+1} \end{bmatrix} =$$

$$\begin{bmatrix} \cos T & \sin T & 1-\cos T \\ -\sin T & \cos T & \sin T \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z(kT) \\ x(kT) \\ v_k \end{bmatrix} + \begin{bmatrix} 1-\cos T \\ \sin T \\ 1 \end{bmatrix} u_k$$

$$y_k = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} z(kT) \\ x(kT) \\ v_k \end{bmatrix}$$

(b) Introduce

$$v(t) = \sum_{k=0}^{t-1} u(k)$$

as state variable:

$$\begin{bmatrix} x(t+1) \\ v(t+1) \end{bmatrix} = \underbrace{\begin{bmatrix} F & 0 \\ 0 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} G \\ 1 \end{bmatrix} u(t)$$

and use state space deadbeat, i.e., introduce $u(t) = l_1 x(t) + l_2 v(t)$ and determine l_1 and l_2 such that the closed loop system has both poles (the eigenvalues of $A - BL$) in the origin. For deadbeat control the system reaches the origin in the same number of samples as the degree of the numerator of the transfer function, which in this case is $\leq n = \deg F$ since we have a strictly proper system (because there is no D-matrix).

9.6 From Exercise 5.5 we have the system model

$$\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ 0 & -\alpha \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ H \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v_1(t) \\
y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + v_2(t)
\end{aligned}$$

Since v_1 and v_2 are white noises with intensities R_1 and R_2 we can assume that x and u are stationary stochastic processes. Minimizing the expected value of $J = E\{\theta^2(t) + \rho\mu^2(t)\}$ is then equal to

$$\min \int x^T(t) Q_1 x(t) + u(t) Q_2 u(t) dt = \min \|x\|^2_{Q_1} + \|u\|^2_{Q_2}$$

where

$$Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \text{och} \qquad Q_2 = \rho.$$

The theorem of separation implies that the optimal feedback is to combine the LQ state feedback L with the Kalman estimation, i.e. $u(t) = -L\hat{x}(t)$ where $\hat{x}(t)$ is the estimation of $\begin{bmatrix} \theta & \dot{\theta} \end{bmatrix}^T$ using the Kalman filter in exercise 5.5.

The feedback $L = Q_2^{-1}B^TS$ is given by the unique, positive semidefinite and symmetrical solution S of the Ricatti equation

$$0 = A^TS + SA + Q_1 - SBQ_2^{-1}B^TS$$

$$\begin{aligned} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 1 & -\alpha \end{bmatrix} \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix} + \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -\alpha \end{bmatrix} + \\ &\quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix} \begin{bmatrix} 0 \\ H \end{bmatrix} \frac{1}{\rho} \begin{bmatrix} 0 & H \end{bmatrix} \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix} \end{aligned}$$

Component by component

$$\begin{aligned} 0 &= 1 - \frac{H^2}{\rho} s_{12}^2 \\ 0 &= -\frac{H^2}{\rho} s_{12}s_{22} + s_{11} - \alpha s_{12} \\ 0 &= -\frac{H^2}{\rho} s_{22}^2 + 2s_{12} - 2\alpha s_{22} \end{aligned}$$

$S \geq 0 \quad \Rightarrow \quad s_{11} \geq 0$ and $s_{22} \geq 0$ give the solution

$$\begin{aligned} s_{11} &= \frac{\sqrt{\rho}}{H} \sqrt{\alpha^2 + \frac{2H}{\sqrt{\rho}}} \\ s_{12} &= \frac{\sqrt{\rho}}{H} \\ s_{22} &= \frac{\rho}{H^2} \left(-\alpha + \sqrt{\alpha^2 + \frac{2H}{\sqrt{\rho}}} \right) \end{aligned}$$

The feedback L is given by

$$\begin{aligned} L = \frac{1}{\rho} \begin{bmatrix} 0 & H \end{bmatrix} \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix} &= \frac{H}{\rho} \begin{bmatrix} s_{12} & s_{22} \end{bmatrix} \\ = \begin{bmatrix} \frac{1}{\sqrt{\rho}} & \frac{1}{H} \left(-\alpha + \sqrt{\alpha^2 + \frac{2H}{\sqrt{\rho}}} \right) \end{bmatrix} \end{aligned}$$

Thus, the optimal control law is

$$\mu(t) = \begin{bmatrix} -\frac{1}{\sqrt{\rho}} & \frac{1}{H} \left(\alpha - \sqrt{\alpha^2 + \frac{2H}{\sqrt{\rho}}} \right) \end{bmatrix} \hat{x}(t)$$

where $\hat{x}(t)$ is the estimation of $\begin{bmatrix} \theta & \dot{\theta} \end{bmatrix}^T$ with the Kalman filter in exercise 5.5.

9.7

$$\begin{aligned} z &= \frac{1}{p+1}u + \frac{1}{p+1}v \quad \Rightarrow \quad G(s) = \frac{1}{s+1} \\ y &= z + e, \qquad\qquad\qquad \Phi_v(\omega) \equiv r_1, \quad \Phi_e(\omega) \equiv 1 \end{aligned}$$

$$\text{minimera } J = E\{q_1 z^2(t) + u^2(t)\}$$

(a) If we define the state $x_1 = z$ we can write the system on the form

$$\begin{cases} \dot{x}_1 &= -x_1 + u + v \\ y &= x_1 + e \\ z &= x_1 \end{cases}$$

i.e. $A = -1, B = 1, M = 1, Q_1 = q_1$ och $Q_2 = 1$. $Q_{12} = 0$.
In this case we cannot measure the state, but only y. According to the separation theorem we will minimize J if we

1. Estimate the state using a Kalman filter.
2. Use feedback of the Kalman filter estimate, i.e. $u(t) = -L\hat{x}(t)$, where L is determined as the standard LQ-solution for the case when we measure the states without noise.

Thus

$$\dot{\hat{x}} = A\hat{x} + Bu + K(Y - C\hat{x}),$$

where $K = PC^T R_2^{-1}$ and P is the positive semi-definite symmetric solution to the stationary Riccati equation

$$AP + PA^T + NR_1N^T - PC^TR_2^{-1}CP = 0$$

which in our case becomes the scalar equation

$$P^2 + 2P - r_1 = 0 \quad \Rightarrow \quad P = -1 + \sqrt{1 + r_1}$$

which gives

$$K = -1 + \sqrt{1 + r_1}.$$

Now, use the feed back $u = -L\hat{x}$, where $L = Q_2^{-1}B^TS$, S is the solution to

$$A^TS + SA + M^TQ_1M - SBQ_2^{-1}B^TS = 0$$

which with $M = 1, Q_1 = q_1$ and $Q_2 = 1$ gives

$$L = S = -1 + \sqrt{1 + q_1}.$$

The loop transfer becomes

$$
\begin{aligned}
G(s)F_y(s) &= \frac{1}{s+1}L\frac{1}{1+s+L+K}K = \\
&= \frac{(-1+\sqrt{1+r_1})(-1+\sqrt{1+q_1})}{(s+1)(s-1+\sqrt{1+r_1}+\sqrt{1+q_1}}
\end{aligned}
$$

(b) r_1 and q_1 have the same effect on the loop transfer becuase of symmetry.

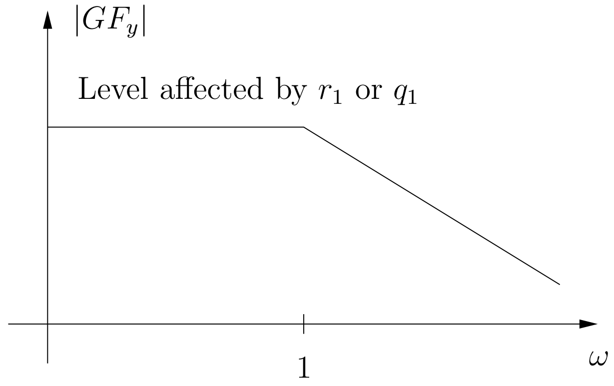
(c)

$$
G(s)F_y(s) = \frac{(-1+\sqrt{1+r_1})(-1+\sqrt{1+q_1})}{(s+1)(s-1+\sqrt{1+r_1}+\sqrt{1+q_1})}
$$

Let r_1 or $q_1 \to \infty$. What happens?
$r_1 \to \infty \Rightarrow G(s)F_y(s) = \frac{(-1+\sqrt{1+q_1})}{(s+1)\frac{(s-1+\sqrt{1+r_1}+\sqrt{1+q_1})}{-1+\sqrt{1+r_1}}} \to \frac{-1+\sqrt{1+q_1}}{s+1}$ In the same way

$$
\lim_{q_1 \to \infty} G(s)F_y(s) = \frac{-1+\sqrt{1+r_1}}{s+1}
$$

By varying q_1 and/or r_1 we can thus shape the loop transfer according to the sketch below.



9.8 (a) Working order:

(i) Determine a Kalmanfilter: $\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})$.

(ii) Make feed back: $u = -L\hat{x}$, with L being LQ-solution.

(i) + (ii) gives the controller $F_y = L(sI - A + BL + KC - KDL)^{-1}K$.

(i) Let $x_1 = z,\ x_2 = \nu,\ v_1 = v,\ v_2 = e$ and $x = (x_1, x_2)^T$. This gives

$$
\begin{cases}
\dot{x} = \underbrace{\begin{bmatrix} -1 & 1 \\ 0 & -\epsilon \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} u + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{N} v_1 \\
y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} x + v_2
\end{cases}
$$

Furthermore,

$$\begin{array}{rclclcl} R_1 & = & \Phi_{v_1}(\omega) & = & \Phi_v(\omega) & = & 1 \\ R_2 & = & \Phi_{v_2}(\omega) & = & \Phi_e(\omega) & = & 1 \\ R_{12} & = & \Phi_{v_1 v_2} & = & 0 & & \end{array}$$

The Kalmanfilter is given by $K = PC^T R_2^{-1}$ with P being the positive definite symmetric solution to

$$AP + PA^T + NR_1N^T - PC^TR_2^{-1}CP = 0.$$

Let $P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$.

This gives $\lim_{\epsilon\to 0} P = \begin{bmatrix} \sqrt{3}-1 & 1 \\ 1 & \sqrt{3} \end{bmatrix}$ and

$K = PC_TR_2^{-1} = \begin{bmatrix} \sqrt{3}-1 \\ 1 \end{bmatrix}$

(ii) Determine L by

$$\min_L \int_0^\infty x_1^2(t) + u^2(t)dt = \min_L \int_0^\infty y^TQ_1y + u^TQ_2udt$$

where $Q_1 = Q_2 = 1$.
Optimal L is given by $L = Q_2^{-1}B^TS$ where S is the positive definite symmetric solution to

$$A^TS + SA + C^TQ_1C - SBQ_2^{-1}B^TS = 0.$$

Let $S = \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}$.

This gives $\lim_{\epsilon\to 0} S = \begin{bmatrix} \sqrt{2}-1 & 1-\dfrac{1}{\sqrt{2}} \\ 1-\dfrac{1}{\sqrt{2}} & * \end{bmatrix}$ and

$L = Q_2^{-1}B^TS = \begin{bmatrix} \sqrt{2}-1 & 1-\dfrac{1}{\sqrt{2}} \end{bmatrix}$

The LQG-controller then becomes

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x}) \\ u = -L\hat{x} \end{cases}$$

with K and L according to above.

The static gain of the sensitivity function ($\epsilon = 0$):
(i) and (ii) gives

$$F_y = L(sI - A + BL + KC - KDL)^{-1}K = \left\{ \begin{array}{c} D = 0 \\ S = 0 \end{array} \right\} = 1$$

$$S(0) = \frac{1}{1 + F_y(0)G(0)} = \frac{1}{1+1} = \frac{1}{2}.$$

(a) Determine L using LTR(y):

$$\begin{aligned}&L_{ltr} = Q_2^{-1}B^TS\\ &S : A^TS + SA + C^T\rho C - SBQ_2^{-1}B^TS = 0\end{aligned}$$

$$\begin{aligned}\Rightarrow S &= \begin{bmatrix} \sqrt{1+\rho}-1 & \dfrac{\sqrt{1+\rho}-1}{\sqrt{1+\rho}+\epsilon} \\ \dfrac{\sqrt{1+\rho}-1}{\sqrt{1+\rho}+\epsilon} & * \end{bmatrix}\\ \Rightarrow L_{ltr} &= \begin{bmatrix} \sqrt{1+\rho}-1 & \dfrac{\sqrt{1+\rho}-1}{\sqrt{1+\rho}+\epsilon} \end{bmatrix}\end{aligned}$$

The static gain of the sensitivity function, $S(0) \to 0$ as $\rho \to \infty$.

9.9

$$\begin{aligned}\dot{x} &= \overbrace{\begin{bmatrix} 0 & 1 & -1 \\ -\frac{1}{2}w_0^2 & -0.01 & 0.01 \\ \frac{1}{2}w_0^2 & 0.01 & -0.01 \end{bmatrix}}^{A} x + \overbrace{\begin{bmatrix} 0 \\ w_0 \\ 0 \end{bmatrix}}^{B} u\\ z &= \underbrace{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}}_{M} x\end{aligned}$$

$$w_0^2 = \frac{k}{50}$$

From the Bode diagram for $k = 1$ we read the resonance peak frequency $w_0 \approx 0.14$.

Include measurement noise: Let $y = z + v_2$, where v_2 is coloured (not white). With the control policy $u = -L\hat{x} + \tilde{p}r$ we can write z as

$$z = G_c r - T v_2 + \tilde{s} v_1.$$

Here, we can see that by letting the spectrum of v_2 be large for $w = w_0$, T will be forced to be small there. Therefore we let v_2 be a coloured noise with a frequency peak at w_0. This can be ackomplished by choosing poles in $-0.01 \pm 0.14i$ and a zero in 0, i.e.

$$v_2 = \frac{k_2 p}{p^2 + 0.021p + 0.02} w,$$

where w is a white noise.

A realization of v_2 is

$$\dot{x}_v = \underbrace{\begin{bmatrix} -0.02 & -0.02 \\ 1 & 0 \end{bmatrix}}_{A_v} x_v + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B_v} w$$
$$v_2 = \underbrace{\begin{bmatrix} k_2 & 0 \end{bmatrix}}_{C_v} x_v$$

Now, extending the model with the noise model gives

$$\dot{x}_e = \begin{bmatrix} A & 0 \\ 0 & A_v \end{bmatrix} x_e + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ B_v \end{bmatrix} w, \qquad x_e = \begin{bmatrix} x \\ x_v \end{bmatrix}$$

$$y = \begin{bmatrix} M & C_v \end{bmatrix} x_e$$

$$z = \begin{bmatrix} M & 0 \end{bmatrix} x_e$$

$$v_2 = \begin{bmatrix} 0 & C_v \end{bmatrix} x_e$$

with A, B, M, A_v, B_v, C_v as above.

9.10 When $T \to \infty$ we get the standard stationary LQ-problem. The requirement for the existance of a state feedback $u(t) = -Lx(t)$ that minimizes the criteria is then that (A, B) is stabilizable.

Such a feedback gives a stable system where the trajectories x are given by

$$x(t) = e^{(A-BL)t}x(0).$$

Since the system is stable, the states will approach 0 exponentioally, i.e. there exist $\alpha > 0$ such that

$$||x(t)|| \leq e^{-\alpha t}||x(0)||.$$

Inserting $u(t) = -Lx(t)$ the criteria becomes

$$V(\infty) = \int_0^\infty x^T(t)x(t) + (-Lx(t))^2 \, dt = \int_0^\infty x^T(I + L^T L)x \, dt.$$

Since $||x^T Rx|| \leq ||x|| \, ||R|| \, ||x|| = ||R|| \, ||x||^2$ we get

$$V(\infty) \leq ||I + L^T L|| \, ||x(0)||^2 \int_0^\infty e^{-2\alpha} \, dt = ||I + L^T L|| \, ||x(0)||^2 \frac{1}{2\alpha} < \infty.$$

Hence, if (A, B) is stabilizable $V(\infty) < \infty$.

The controllability matrix is

$$\mathcal{S}(A,B) = \begin{bmatrix} -4 & -12 \\ 8 & 24 \end{bmatrix},$$

which only has rank 1. We must therefore investigate the stability. The pole polynomial for the feedback system with $u = -\begin{bmatrix} l_1 & l_2 \end{bmatrix}$ is

$$p(\lambda) = \det(\lambda I - A + BL) = \lambda^2 + \lambda(-5 + 8l_2 - 4l_1) + 2(1 - 4l_1)(1 - 4l_2)$$

The poles will be in LHP if all coefficients of this polynomial has the same sign (easily shown with Routh Hurwitz criteria for second order polynomials), which holds if $l_1 \ll 0$ and $l_2 < 0.25$, for example. Thus, the system is stabilizable and therefore $V(\infty) < \infty$.

9.11 Assume the process is described by

$$\begin{aligned} z &= Gu + w \\ y &= Mz + n \end{aligned}$$

where w is process disturbances and n is measurement noise.

LQG

1. estimates the states from the known control signal u, the measurement y and the (incorrect) model (A, B, C, D and "the size" of the disturbances and noise),
2. feed back the estimated states based on the process model such that the wrong control signal is used for the minimization.

To decrease the effects of the model errors we can

- consider the model uncertainties as a process disturbance and let for example w be white noise filtered through a band pass filter designed to let disturbances through mainly in the frequency region where there are large model uncertainties.
- Conversely, we may choose the measurement noise n to be filtered such that it is small for these frequencies.

As a result, the Kalman filter will trust measurements more than the model in frequencies where the uncertainties are large.

- By increasing the weight matrix for the control signal in the loss function, we also get a more careful control, which in general results in a more robust system (for stable systems).

Reformulate the system into a state space model and calculate K and L.

9.12 The controller to be determined is a standard LQ controller with A and B according to the state space model. The weight matrices in the standard loss function are

$$M = I, \quad Q_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{och} \quad Q_2 = 0.01$$

The corresponding solution is

$$\begin{aligned} S &= 10^{-4} \begin{bmatrix} 0.618 & 0 \\ 0 & 61.8 \end{bmatrix} \\ L &= \begin{bmatrix} 0 & 6.18 \end{bmatrix} \end{aligned}$$

9.13 Theorem 9.1 with $A = 1$, $B = 1$, $N = 1$, $C = 1$ and $M = 1$ gives the sought LQG controller. The solutions to the Riccati equations are

$$\begin{aligned} P &= R_2 + \sqrt{R_1 R_2 + R_2^2} \\ K &= 1 + \sqrt{\beta + 1} \\ S &= Q_1 + \sqrt{Q_1 Q_2 + Q_2^2} \\ L &= 1 + \sqrt{\alpha + 1} \end{aligned}$$

The feedback $F_y(s)$ follows from Laplace transformation of the feedback $u = -L\hat{x}$, the Kalmanfilter equation $\dot{\hat{x}} = \ldots$ and eliminating $\hat{X}(s)$ such that one gets $U(s) = -F_y(s)Y(s)$. The result of this becomes

$$F_y(s) = \frac{LK}{s - 1 + K + L} = \frac{(1 + \sqrt{\alpha + 1})(1 + \sqrt{\beta + 1})}{s + 1 + \sqrt{\alpha + 1} + \sqrt{\beta + 1}}$$

The closed loop system poles are given by $1 + GF_y = 0$. If we let $\gamma = \sqrt{\alpha + 1}$ and $\kappa = \sqrt{\beta + 1}$ we get the characteristic equation

$$s^2 + s(\gamma + \kappa) + \gamma\kappa = (s + \gamma)(s + \kappa) = 0.$$

Consequently, the poles are $s = -\sqrt{\beta + 1}$ och $s = -\sqrt{\alpha + 1}$.

The crossover frquency ω_c is defined by

$$|G(j\omega_c)F_y(j\omega_c)| = 1.$$

Taking the square of this gives, after considerable manipulations,

$$\omega_c^2 = -1 + (K + L) + 1.5(K + L)^2 - 2(K + L)^3 + 0.5(K + L)^4 + 2K^2L^2$$

If $\frac{R_1}{R_2}$ or $\frac{Q_1}{Q_2}$ increases $\Rightarrow K + L$ increases $\Rightarrow \omega_c$ increases $\Rightarrow$ faster system.

9.14 The solution is given by the algebraic Riccati equation as

$$SA + A^T S - SBQ_u^{-1} B^T S + Q_x = 0$$

where

$$\begin{aligned} Q_x &= \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix} \\ Q_u &= q_u \end{aligned}$$

Furtheremore S is chosen as

$$S = \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix}$$

Therefore, elements of the algebraic Riccati equation becomes

$$\begin{aligned} \text{element} \quad 11 &: \; 0 - \frac{s_{12}^2}{q_u} + q_1 = 0 \\ \text{element} \quad 12 &: \; s_{11} - \frac{s_{12}s_{22}}{q_u} = 0 \\ \text{element} \quad 22 &: \; 2s_{12} - \frac{s_{22}^2}{q_u} + q_2 = 0. \end{aligned}$$

The positive definite solution for S is

$$\begin{aligned} s_{11} &= \sqrt{q_1 q_2} + 2q_1\sqrt{q_1 q_u} \\ s_{12} &= \sqrt{q_1 q_u} \\ s_{22} &= \sqrt{q_2 q_u} + 2q_u\sqrt{q_1 q_u} \end{aligned}$$

and the controller gains are

$$K = Q_u^{-1} B^T S = \frac{1}{q_u}\begin{bmatrix} s_{12} & s_{22} \end{bmatrix} = \begin{bmatrix} \sqrt{q_1/q_u} & \sqrt{q_2/q_u + 2\sqrt{q_1/q_u}} \end{bmatrix}$$

The closed loop characteristic polynomial is

$$\det(sI - A + BK) = \det\begin{bmatrix} s & -1 \\ k_1 s + k_2 & \end{bmatrix} = s^2 + k_2 s + k_1$$

Comparing this with the standard second order polynomial $s2 + 2\zeta_0\omega_0 s + \omega_0^2$ we can conclude that

$$\begin{aligned} q_1 &= q_u\omega_0^4 \\ q_2 &= 2q_u(2\zeta_0^2 - 1)\omega_0^2 \end{aligned}$$

Notice that:

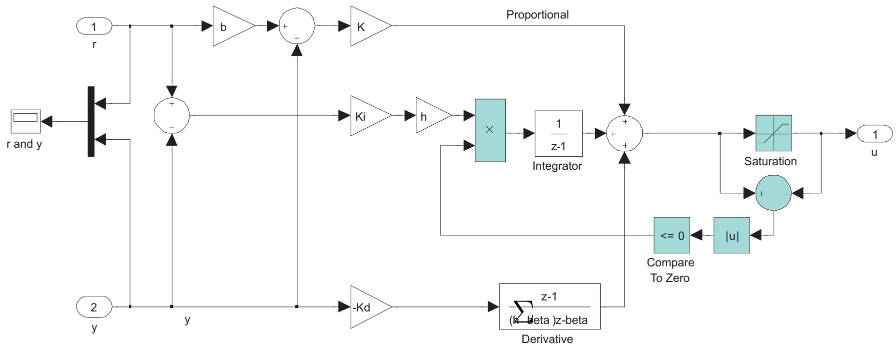
- The controller gains depend on the ratios q_1/qu and $q2/qu$.
- The weight q_1 is proportional to the fourth power of the frequency ω_0 which implies that large changes in q_1 are required to obtain significant changes in response speed.

- The weight q_2 is proportional to the second power of the frequency ω_0.
- Critical damping ($\zeta_0 = \sqrt{2}/2$) is obtained for $q_2 = 0$.

It is also true that changing weights on states associated with position influences response speed while those corresponding to velocity influences damping.

10.1 (a) Windup. Because of control signal saturation (not in the controller) the output y can never reach the desired reference r. The integrator (the summation $h/(z-1)$) then increases more and more (windup) and when the reference is changed it will take time before the control output decreases below saturation.

(b) The remedy is called anti-windup and can be implemented as in the figure.



10.2 (a) $\arg\{G(j\omega_c)G_{PI}(j\omega_c)\} \equiv -180° + \varphi_m$ and $|G(j\omega_c)G_{PI}(j\omega_c)| \equiv 1$ give

$$G_{PI} = 0.53\left(1 + \frac{1}{6s}\right)$$

(b) The sample and hold can be described by

$$\begin{aligned} G_h(s) &= \frac{1}{sh}(1 - e^{-sh}) \\ &= \frac{1}{sh}(1 - (1 - sh + \frac{(sh)^2}{2} + \ldots)) \\ &= 1 - \frac{sh}{2} + \ldots \\ &\approx e^{-sh/2} \end{aligned}$$

$$\Rightarrow \quad \arg\{G_h(j\omega_c)\} = \frac{-\omega_c h}{2}\frac{180}{\pi} = \frac{0.9}{\pi} = 0.29°,$$

which can be neglected.

10.3 (a) The controllability matrix is

$$W_c = \begin{bmatrix} b & 0 \\ 0 & ab \end{bmatrix}$$

which has full rank iff $ab \neq 0 \quad \Leftrightarrow \quad$ reachable (full controllability).
The observability matrix is

$$W_o = \begin{bmatrix} 0 & 1 \\ a & 1 \end{bmatrix}$$

which has full rank iff $a \neq 0 \quad \Leftrightarrow \quad$ observable.

(b) $u(k) = -Lx(k)$ where L is the solution to the stationary LQ problem minimizing

$$J = E\left\{ \sum_0^\infty x^T \underbrace{\begin{bmatrix} x0 & 0 \\ 0 & 1 \end{bmatrix}}_{Q_1} x + u \underbrace{\cdot 1 \cdot}_{Q_2} u \right\}$$

L is given by the stationary Ricatti equations

$$\begin{aligned} L &= (Q_2 + \Gamma^T S \Gamma)^{-1}(\Gamma^T S \Phi + Q_{12}^T) = \dots \\ &= \frac{s_{12}}{1 + s_{11}} \begin{bmatrix} 1 & 1 \end{bmatrix} \\ S &= \Phi^T S \Phi + Q_1 - (\Phi^T S \Gamma + Q_{12}) L \end{aligned}$$

where $Q_{12} = \begin{bmatrix} 0 & 0 \end{bmatrix}$ and S is symmetric and positive semidefinite, which has the solution

$$\begin{aligned} S &= \begin{bmatrix} 1.618 & 1.618 \\ 1.618 & 2.618 \end{bmatrix} \\ L &= \begin{bmatrix} 0.618 & 0.618 \end{bmatrix} \end{aligned}$$

(c) $x_1(k+1) = u(k) \quad \Rightarrow \quad x_1(k) = u(k-1)$ and hence, since $y(k) = x_2(k)$ the control law gives the controller

$$u(k) = -0.618u(k-1) - 0.618y(k) \qquad \Rightarrow \qquad u(k) = \underbrace{\frac{-0.618}{1 + 0.618q^{-1}}}_{H_c(q)} y(k)$$

10.4 (a) A state space model corresponding to G is

$$\begin{aligned} \dot{x}(t) &= -x(t) + u(t - T) \\ y(t) &= x(t) \end{aligned}$$

Sampling with sampling time $h = T$ gives

$$x(k+1) = \varphi x(k) + \gamma u(k-1)$$

where

$$\begin{aligned} \varphi &= e^{-h} \\ \gamma &= \int_0^h e^{-\tau} d\tau = 1 - e^{-h} \end{aligned}$$

Introducing $x_2(k) = u(k-1)$ gives

$$\begin{aligned} \underbrace{\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix}}_{\tilde{x}(k+1)} &= \underbrace{\begin{bmatrix} e^{-h} & 1-e^{-h} \\ 0 & 0 \end{bmatrix}}_{\Phi} \underbrace{\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}}_{\tilde{x}(k)} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\Gamma} u(k) \\ y(k) &= \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \end{aligned}$$

(b)

$$u_{FB}(k) = -\underbrace{\begin{bmatrix} l_1 & l_2 \end{bmatrix}}_{L} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \quad \Rightarrow \quad \tilde{x}(k+1) = (\Phi - \Gamma L)\tilde{x}(k)$$

With $h = \ln 2$ inserted the closed loop poles are given by

$$P(\lambda) = \det(\lambda I - (\Phi - \Gamma L)) = \ldots = \lambda^2 - \lambda(0.5 - l_2) + 0.5(l_1 - l_2) = 0$$

This should equal $P(\lambda = (\lambda - 0.4)^2 = \lambda^2 - 0.8\lambda + 0.16 = 0$
Polynomial identification gives $l_1 = 0.02$ and $l_2 = -0.3$. Then we have

$$\begin{aligned} u_{FB}(k) &= -0.02x_1(k) + 0.3x_2(k) \\ &= -0.02x_1(k) + 0.3u_{FB}(k-1) \\ &= -0.02y(k) + 0.3u_{FB}(k-1) \end{aligned}$$

Hence

$$u_{FB}(k) = \underbrace{\frac{-0.02}{1 - 0.3q^{-1}}}_{H(q)} y(k)$$

(c)

$$\begin{aligned} u(k) &= K_r r(k) - H(q)y(k) \\ x_1(k+1) &= 0.5x_1(k) + 0.5u(k-1) \end{aligned}$$

Since the system is stable (2 stable poles) we can assume steady state:

$$\begin{aligned} \bar{u} &= K_r \bar{r} - H(1)\bar{y} \\ \bar{y} &= 0.5\bar{x}_1 + 0.5\bar{u} \quad \Rightarrow \quad \bar{y} = \bar{u} \end{aligned}$$

Correct stationaary gain $\equiv \bar{r} = \bar{y}$, which imply

$$1 = K_r + \frac{0.02}{1 - 0.3} \quad \Rightarrow \quad K_r = 0.971$$