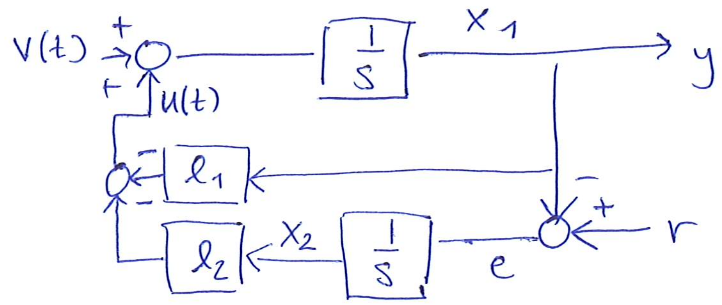
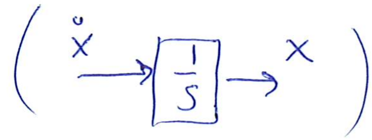
9.4 Determine the discrete time coefficient matrices required for the solution of the discrete time riccati eq. ← page 264-265 : A, B, C, N, M, Q_1, Q_2





process: $\dot{x}_1 = u + v$ controller: $u = -l_1 x_1 - l_2 x_2$

$\dot{x}_2 = e = r - x_1$

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} u + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{N} v + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$

continious time

let $r = 0$

$z = x \Rightarrow M = I$

$Q_1 = eye(2)$ $Q_2 = 1$

Discretization. Sampling with $T=1$

$$A_d = e^{AT} = I + AT + \frac{1}{2!}(AT)^2 + \frac{1}{3!}(AT)^3 + \dots$$

$$(AT)^2 = \begin{bmatrix} 0 & 0 \\ -T & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ -T & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A_d = I + AT = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -T & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -T & 1 \end{bmatrix}$$

$$B_d = \int_0^T e^{At} dt \cdot B = \int_0^T \begin{bmatrix} 1 & 0 \\ -t & 0 \end{bmatrix} dt \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} T & 0 \\ -T^2/2 & T \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} T \\ -T^2/2 \end{bmatrix}$$

$$\left[t\right]_0^T = T - 0$$

$$\left[\frac{-t^2}{2}\right]_0^T = -T^2/2 - 0$$

$B = N \Rightarrow B_d = N_d$

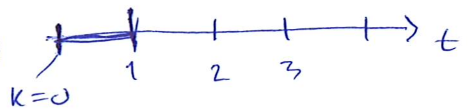
$y = [1 \ 0] x$

Discretization of the cost function:

$$V_c = \int_0^\infty (Mx)^T Q_{1c} M x + u^T Q_{2c} u \, dt = \int_0^T \dots + \int_T^{2T} \dots + \int \dots =$$

$$= \sum_{k=0}^{\infty} \left[\int_{kT}^{(k+1)T} \underbrace{x^T M^T Q_{1c} M x}_{2} + \underbrace{u^2}_{1} \, dt \right] = \sum_{k=0}^{\infty} \left(x^T Q_{1d} x + 2 x^T Q_{12}^T u + u^T Q_{2d} u \right)$$

1 $\quad \int_{kT}^{(k+1)T} u^2 \, dt = [1]_{kT}^{(k+1)T} \cdot u^2 = \left((k+1)T - kT\right) u^2 = T \cdot u^2 \qquad (*)$

2 

$$x(t) = e^{A(t-t_0)} x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)} B u(\tau) \, d\tau$$

$$\int_{kT}^{(k+1)T} \left[e^{A(t-t_0)} x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)} B u(\tau) \, d\tau \right]^T M^T Q_{1c} M \left[\dots \right] dt \qquad (**)$$

Terms that includes x but not u: (from $(**)$)

$$\int_{kT}^{(k+1)T} x^T(t_0) \underbrace{\left(e^{A(t-t_0)}\right)^T M^T Q_{1c} M \, e^{A(t-t_0)}}_{Q_{1d}} x(t_0) \, dt$$

for $t_0 = kT$ (or use $t_0 = 0$ & $\int_0^T$)

$$Q_{1d} = \int_0^T \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix} dt = \int_0^T \begin{bmatrix} 1+t^2 & -t \\ -t & 1 \end{bmatrix} dt =$$

$$= \begin{bmatrix} t + t^3/3 & -t^2/2 \\ -t^2/2 & t \end{bmatrix}_0^T = \begin{bmatrix} T + T^3/3 & -T^2/2 \\ -T^2/2 & T \end{bmatrix}_{\text{if } T=1} = \begin{bmatrix} 4/3 & 1/2 \\ -1/2 & 1 \end{bmatrix}$$

Terms including only u (not x) from (*)(*)

for $t_0 = 0$ ($\Rightarrow k = 0$)

$$\underbrace{u(0)^T \int_0^T \left(\left(\int_0^t e^{A(t-\tau)} d\tau\, B \right)^T M^T Q_{1c} M \left(\int_0^t e^{A(t-\tau)} d\tau\, B \right) \right) dt\, u(0)}$$

$Q_{2d} = \uparrow + T$

$\nwarrow$ from (*)

$$\Gamma(t) = \int_0^t e^{A(t-\tau)} d\tau \cdot B = \begin{bmatrix} t \\ -t^2/2 \end{bmatrix}$$

$$Q_{2d} = \int_0^T [t \;\; -t^2/2] \cdot I \cdot \begin{bmatrix} t \\ -t^2/2 \end{bmatrix} dt + T =$$

$$= \int_0^T (t^2 + t^4/4)\, dt + T = \left[\frac{t^3}{3} + \frac{t^5}{20} \right]_0^T + T =$$

$$= \frac{T^3}{3} + \frac{T^5}{20} + T \underset{\{\text{for } T=1\}}{=} \frac{20+3+60}{60} = \frac{83}{60}$$

Terms including x & u (from (*)(*))

$$\int_{kT}^{(k+1)T} \Big[e^{A(t-t_0)} x(t_0) \Big]^T \underbrace{M^T Q_{1c} M}_{I} \big(\Gamma(t)\, u(t_0) \big) + \underbrace{[\Gamma(t)\, u(t_0)]^T M^T Q_{1c} M \big(e^{A(t-t_0)} x(t_0) \big)}_{\text{This is a scalar, } S} dt$$

$\Rightarrow S = S^T$

$$= 2 \times x(t_0)^T \underbrace{\int_0^T [e^{At}]^T \Gamma(t)\, dt}_{= Q_{12}^T} \cdot u(t_0)$$

$$Q_{12}^T = \int_0^T \underbrace{\begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ -t^2/2 \end{bmatrix}}_{\begin{bmatrix} t + t^3/2 \\ -t^2/2 \end{bmatrix}} dt = \begin{bmatrix} t^2/2 + t^4/8 \\ -t^3/6 \end{bmatrix}_0^T = \begin{bmatrix} T^2/2 + T^4/8 \\ -T^3/6 \end{bmatrix} \underset{T=1 \Rightarrow}{=} \begin{bmatrix} 5/8 \\ -1/6 \end{bmatrix}$$

9.7 system: $\begin{cases} z = \frac{1}{p+1} u + \frac{1}{p+1} v \\ y = z + e \end{cases}$

v & e disturbances with spectra $\phi_v(\omega) = r_1$

$\phi_e(\omega) = 1$

The aim is to minimize $E\{ q_1 z^2(t) + u^2(t) \}$

a) Determine the loop transfer of the resulting closed loop system

1. write your system in a standard form
2. estimate the states using a Kalman Filter (p.128)
3. use linear quadratic optimization (p. 242) $u = -L\hat{x}$
4. calculate loop transfer

(1) $\begin{cases} \dot{x} = Ax + Bu + Nv_1 \\ y = Cx + v_2 \end{cases}$

$\dot{z} + z = u + v \quad x = z \Rightarrow A = -1 \quad B = 1$

$\underset{\substack{\uparrow \\ \text{white noise}}}{Nv_1} = v \quad N = 1 \quad R_1 = r_1$ (or $v = G(p)\, e$ ← white

$\phi_v = G(i\omega)\, \underbrace{\phi_e}_{1}\, G^*(i\omega) = r_1$

$\Rightarrow G(i\omega) = \sqrt{r_1}$

$\Rightarrow N = \sqrt{r_1}$ & $R_1 = 1$)

$y = z + e$

$\Rightarrow C = 1 \quad R_2 = 1$

no correlation $v_1\ v_2 \Rightarrow R_{12} = 0$

$\min\{ q_1 z^2 + u^2 \} \Rightarrow Q_1 = q_1 \quad Q_2 = 1$

(2) Kalman filter (continuous time)

$$K = (P\underset{1}{C^T} + N\underset{0}{\cancel{R_{12}}})\underset{1}{R_2^{-1}} = p$$

$$\underbrace{AP}_{-p} + \underbrace{PA^T}_{-p} - (\underbrace{PC^T}_{p} + N\underset{0}{\cancel{R_{12}}})\underbrace{R_2^{-1}}_{1}(\underbrace{PC^T}_{p} + N\underset{0}{\cancel{R_{12}}})^T + \underbrace{NR_1N^T}_{r_1} = 0$$

$$-2p - p^2 + r_1 = 0 \qquad p^2 + 2p - r_1 = 0 = (p+1)^2 - 1 - r_1 = 0$$

$$p = -1 \pm \sqrt{1+r_1} \qquad r_1 > 0,\ p > 0 \Rightarrow p = -1 + \sqrt{1+r_1} = K$$

(3) LQ-optim. p. 242

$$L = \underbrace{Q_2^{-1}}_{1}\underbrace{B^T}_{1}S = S$$

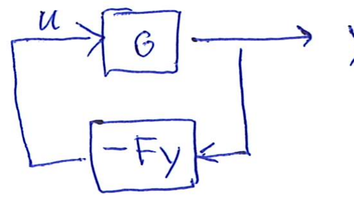
$$\underbrace{A^TS}_{-s} + \underbrace{SA}_{-s} + \underbrace{M^TQ_1M}_{q_1} - \underbrace{SBQ_2^{-1}BS}_{1} = 0$$

$$-2s + q_1 - s^2 = 0 \Rightarrow (s+1)^2 - 1 - q_1 = 0$$

$$s = -1 + \sqrt{1+q_1}$$

(4) Loop transfer $= G \cdot F_y$

General syst:



Assume all inputs $= 0$

i.e. $v_1 = 0$ $e = 0$ $r = 0$

our syst:

$$\begin{cases} \dot{x} = Ax + Bu + Nv_1 \\ y = Cx + e \end{cases}$$

inputs $= 0$

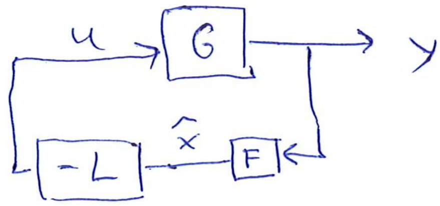
$$Y = \underbrace{C(sI-A)^{-1}B}_{G = \frac{1}{s+1}} \cdot u + C(sI-A)^{-1}N \cdot \underset{0}{v_1} + \underset{0}{e}$$

Observer:

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x}) \\ u = -L\hat{x} \end{cases}$$

$$\hat{x}(sI - A + BL + KC) = K \cdot Y$$

$$\hat{x} = \underbrace{(sI - \underset{-1}{A} + \underset{1}{B}L + KC)^{-1} K \cdot Y}_{\dfrac{K}{s+1+L+K} = F}$$



$$F_y = L \cdot F$$

Loop Transfer: $GF_y = \dfrac{1}{s+1} \cdot \dfrac{LK}{s+1+L+K}$

$$= \frac{(-1+\sqrt{1+q_1})(-1+\sqrt{1+r_1})}{(s+1)(s+\cancel{1}+\cancel{-1}+\sqrt{1+q_1}-1+\sqrt{1+r_1})}$$

(b) Different role of r_1 & q_1 ? symmetry $\Rightarrow$ r_1 & q_1 effect the loop transfer in the same way

9.13

system: $\begin{cases} \dot{x} = x + u + v_1 \\ z = x \\ y = x + v_2 \end{cases}$

Disturbances v_i are white with variances R_i

a) Determine the controller that minimizes

$$V = \int_0^\infty Q_1 x^2 + Q_2 u^2 dt$$

1. write your system in standard form
2. estimate the states using Kalman Filter
3. linear quadratic optimization $u = -L\hat{x}$

Theorem 9.1 page 242

1. $A = 1 \quad B = 1 \quad N = 1 \quad M = 1 \quad C = 1 \quad R_1 = R_1 \quad R_2 = R_2 \quad R_{12} = 0$

2. $$K = (PC^T + N\underbrace{\cancel{R_{12}}}_{0})R_2^{-1} = \frac{P}{R_2}$$

$$AP + PA^T + NR_1N^T - (PC^T + N\underbrace{\cancel{R_{12}}}_{0})R_2^{-1}(PC^T + N\underbrace{\cancel{R_{12}}}_{0})^T = 0$$

$$2P + R_1 - \frac{P^2}{R_2} = 0 \qquad P^2 - 2R_2P - R_1R_2 = 0$$

$$(P - R_2)^2 - R_2^2 - R_1R_2 = 0 \Rightarrow P = R_2 + \sqrt{R_2^2 + R_1R_2} \quad (P > 0)$$

$$\Rightarrow \boxed{K = P/R_2 = 1 + \sqrt{1 + R_1/R_2}}$$

3. $$L = Q_2^{-1}B^TS = S/Q_2$$

$$\underbrace{A^TS + SA}_{2S} + \underbrace{M^TQ_1M}_{Q_1} - \underbrace{SBQ_2^{-1}B^TS}_{S^2/Q_2} = 0$$

$$S^2 - 2Q_2S - Q_1Q_2 = 0 \Rightarrow S = Q_2 + \sqrt{Q_2^2 + Q_1Q_2}$$

$$\boxed{L = 1 + \sqrt{1 + Q_1/Q_2}}$$

The optimal controller that minimizes V is:

(b) $u = -L\hat{x}$

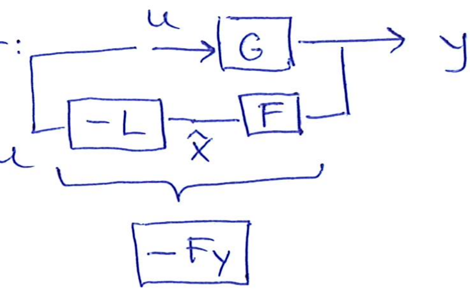
where $\hat{x}$: $\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})$

K & L is given by: $K = 1 + \sqrt{1 + R_1/R_2}$

$L = 1 + \sqrt{1 + Q_1/Q_2}$

Hence the controller only depends on the ratios R_1/R_2 and Q_1/Q_2.

(c) Loop transfer:



$G \cdot F_y$ = loop transfer

$$\begin{cases} \dot{x} = Ax + Bu + Nv_1 \\ y = Cx + v_2 \end{cases}$$

$$Y = \underbrace{C(sI-A)^{-1}B}_{G}\,u + \overbrace{C(sI-A)^{-1}Nv_1 + v_2}^{\text{assume all inputs} = 0}$$

$$G = \frac{1}{s-1}$$

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x}) \\ u = -Lx \end{cases}$$

$$\hat{x} = \underbrace{(sI - A + BL + KC)^{-1}K}_{F} \cdot Y$$

$$F = \frac{K}{s - 1 + L + K} \qquad F_y = L \cdot F = \frac{KL}{s-1+L+K}$$

Loop Transfer: $G \cdot F_y = \dfrac{1}{(s-1)} \cdot \dfrac{KL}{(s-1+L+K)}$

Closed systems poles: Given by $1 + GF_y = 0$

$$(s-1)(s-1+\underbrace{L+K}_{2+\sqrt{1+\alpha}+\sqrt{1+\beta}}) + \underbrace{KL}_{(1+\sqrt{1+\alpha})(1+\sqrt{1+\beta}) = 1+\gamma+K+\gamma K} = 0$$

$$2 + \underbrace{\sqrt{1+\alpha}}_{\gamma} + \underbrace{\sqrt{1+\beta}}_{K} \qquad (1+\underbrace{\sqrt{1+\alpha}}_{\gamma})(1+\underbrace{\sqrt{1+\beta}}_{K}) = 1+\gamma+K+\gamma K$$

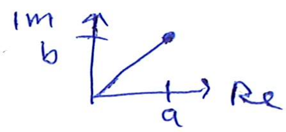
$$s^2 + s(-\cancel{2} + \cancel{2} + \gamma + K) + \underbrace{1 - 2 - \gamma - K + \overbrace{1+\gamma+K+\gamma K}}\, = 0$$

$$s^2 + s(\gamma + K) + \gamma K = (s+\gamma)(s+K) = 0$$

$\therefore$ poles in $s = -\gamma \quad s = -K \Rightarrow s = \begin{cases} -\sqrt{1+\alpha} \\ -\sqrt{1+\beta} \end{cases}$

Cross over frequency ω_c: $|G(j\omega_c)F_y(j\omega_c)| = 1$

$$1 = \left| \frac{KL}{(j\omega_c - 1)(j\omega_c - 1 + L + K)} \right| \qquad |a+bi| = \sqrt{a^2+b^2}$$



Let $\omega_c^2 = \xi \Rightarrow \ldots$

$$\xi = -1 + (K+L) + 1{,}5(K+L)^2 - 2(K+L)^3 + 0{,}5(K+L)^4 + 2K^2L^2$$

if $K+L$ increases $\Rightarrow$ ω_c will increase $\Rightarrow$ faster system