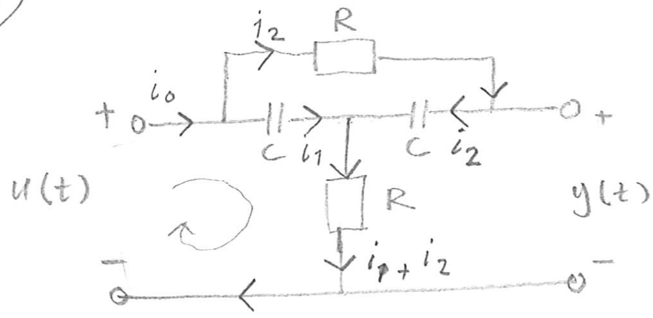
2o1

Consider the T-net



$$i_0 = i_1 + i_2$$

I) $u = V_{c_1} + R(i_1+i_2)$

II) $u = V_{c_1} - V_{c_2} + Y$

III) $R i_2 + V_{c_2} = V_{c1}$

Time derivative of voltage over the capacitor:

$$\ast \begin{cases} \dot{V}_{c_1} = \frac{1}{C} \cdot i_1 \\ \dot{V}_{c_2} = \frac{1}{C} i_2 \end{cases}$$

~~equation for~~

choose states: $X_1 = V_{c_1}$
$X_2 = -V_{c_2}$

II) $Y = -X_1 - X_2 + u \leftarrow$

III) $\cancel{R} i_2 = (X_1 + X_2)\frac{1}{R}$

I) $R i_1 + \underbrace{R i_2}_{x_1+x_2} = -X_1 + u$

$$i_1 = [-2x_1 - x_2 + u]\frac{1}{R}$$

$$\ast \Rightarrow \quad \dot{X}_1 = \frac{1}{CR}(-2x_1 - x_2 + u)$$

$$\dot{X}_2 = \frac{-1}{CR}(x_1 + x_2)$$

a) Derive a state space model

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

Input: voltage u

output: voltage Y

normal assumption:

no current through y

Known

Kirchhoffs laws:

voltage $\sum V_k = 0$ in a closed loo

current $\sum I_k = 0$ from/into a nod

Resistor $V_R = R \cdot i_R$

Capacitor $V_C = \frac{1}{C} \int_0^t i_c \, d\tau$

* Write known equations

* Determine states X where we can express $\dot{x} = ..$

State - Space Model

$$\begin{cases} \dot{X} = \frac{1}{RC}\begin{bmatrix} -2 & -1 \\ -1 & -1 \end{bmatrix} X + \begin{bmatrix} \frac{1}{CR} \\ 0 \end{bmatrix} u \\ Y = \begin{bmatrix} -1 & -1 \end{bmatrix} X + u \end{cases}$$

2016

Determine transfer function from $u$ to $y$

i.e. $G$: $Y(s) = G(s) \cdot U(s)$

Minimum phase system? Yes, if no zeros in RHP.

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

Laplace transform: $\mathcal{L}\{x(t)\} = X(s)$

$$\mathcal{L}\{\dot{x}(t)\} = s \cdot X(s)$$

$$sX = AX + BU$$

$$(sI - A)X = BU$$

$$X = (sI - A)^{-1} BU$$

$$Y = C \cdot (sI - A)^{-1} B \cdot U + DU$$

$$Y = \underbrace{\left[C(sI - A)^{-1}B + D\right]}_{G(s)} U$$

$$\boxed{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{2}} \cdot \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}}$$

$$(sI - A)^{-1} = \begin{bmatrix} s + \frac{2}{RC} & \frac{1}{RC} \\ \frac{1}{RC} & s + \frac{1}{RC} \end{bmatrix}^{-1} = \frac{1}{\underbrace{\left(s + \frac{2}{RC}\right)\left(s + \frac{1}{RC}\right) - \frac{1}{(RC)^2}}_{s^2 + s \cdot \frac{3}{RC} + \frac{1}{(RC)^2}}} \begin{bmatrix} s + \frac{1}{RC} & \frac{-1}{RC} \\ \frac{-1}{RC} & s + \frac{2}{RC} \end{bmatrix} =$$

$$C(sI - A)^{-1}B + D = \begin{bmatrix} -1 & -1 \end{bmatrix} \frac{1}{k} \underbrace{\begin{bmatrix} s + \frac{1}{RC} & -1/RC \\ -1/RC & s + 2/RC \end{bmatrix} \begin{bmatrix} \frac{1}{RC} \\ 0 \end{bmatrix}}_{\begin{bmatrix} \frac{s}{RC} + \frac{1}{(RC)^2} \\ -1/(RC)^2 \end{bmatrix}} + 1 =$$

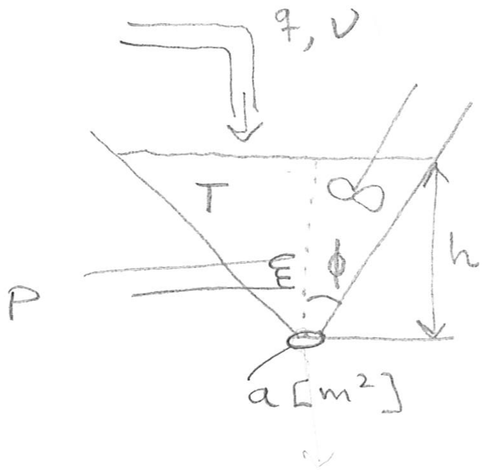
$$= \frac{1}{k}\left(\frac{-s}{RC} - \cancel{\frac{1}{(RC)^2}} + \cancel{\frac{1}{(RC)^2}}\right) + 1 \cdot \frac{k}{k} = \frac{s^2 + \overset{2}{\cancel{3}}s/RC + \frac{1}{(RC)^2} \cancel{- s/RC}}{k} =$$

extend by $(RC)^2$

$$= \frac{\overbrace{(RCs)^2 + 2sRC + 1}^{(RCs + 1)^2}}{(RCs)^2 + 3sRC + 1} = \frac{(RC \cdot s + 1)^2}{(RCs)^2 + 3sRC + 1}$$

$\therefore$ double zeros in $\frac{-1}{RC}$ $\Rightarrow$ Left half plane $\Rightarrow$ Yes. Minimum phase.

2.4



a) states: T, h

Determine state space model, i.e.

$\frac{dT}{dt}$ & $\frac{dh}{dt}$

Known:

volume;

$V = \frac{1}{3} \cdot \pi r^2 \cdot h$

* Use volumebalance (or massbal.) & energybalance

$\overset{\text{accumulated}}{acc} =$ flow in − flow out

$\tan\phi = \frac{r}{h}$

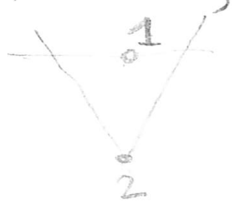
② $\frac{dV}{dt} = q_{in} - q_{out}$ — depends on the pressur i.e. the height h

$[m^3/s]$

$V = \underbrace{\boxed{\frac{\pi}{3} \tan^2\phi}}_{k} h^3 = k \cdot h^3$ ①

Bernoulli's principle

~~Epressure~~, potential energy & kinetic energy − constant



1. potential energy $mgh$ } conserved
2. kinetic energy $\frac{mv^2}{2}$

$mgh = \frac{mv^2}{2} \Rightarrow \boxed{v = \sqrt{2gh}}$ flow at point 2:

$\left[\frac{m^3}{s}\right] = a \cdot v = a\sqrt{2gh}$

$\frac{d}{dt}$ of ① into ② ⇒

③ $\dot{V} = 3kh^2 \cdot \dot{h} = q_{in} - a\sqrt{2gh} \Rightarrow \boxed{\dot{h} = \frac{1}{3kh^2}(q_{in} - a\sqrt{2gh})}$

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# Energy balance

$$\text{Acc} = \text{flow in} - \text{flow out} + \underset{P}{\text{generated}} - \underset{0}{\text{consumptio}}$$

$$\left[\frac{J}{s}\right] \frac{d}{dt}\left(V c \rho \cdot T\right) = q_{in} \rho c \cdot \upsilon - \underbrace{a\sqrt{2gh}}_{\frac{m^3}{s}} \rho c \cdot T + P$$

$$[m^3]\left[\frac{J}{kgK}\right]\left[\frac{kg}{m^3}\right] K \qquad \left[\frac{m^3}{s}\right] \qquad \frac{kg}{m^3}\,\frac{J}{kgK}\; K$$

$$c\rho\left(\underbrace{\frac{dV}{dt}}_{q - a\sqrt{2gh}} \cdot T + \underbrace{V}_{kh^3} \cdot \boxed{\frac{dT}{dt}}\right)$$

solve for this

$$\boxed{\frac{dT}{dt} = \frac{1}{\rho c k h^3}\left(\rho c q_{in}(\upsilon - T) + P\right)}$$

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(2.4b) Linearization.

* Operating point - stationarity : time derivatives $=0$

* Given: $\underbrace{\bar{h}, \bar{T}}_{x_0}, \bar{v}$

states: $x = \begin{bmatrix} T \\ h \end{bmatrix}$ what is our inputs? $u = \begin{bmatrix} q \\ P \\ v \end{bmatrix}$

$u_0$: $\dot{h} = 0 \Rightarrow \bar{q}_{in} = a\sqrt{2g\bar{h}}$

$\dot{T} = 0 \Rightarrow \bar{P} = -\rho c \bar{q}_{in}(\bar{v} - \bar{T})$

now we have $x_0$ & $u_0$

Linearization: fr. Taylor expansion if $\dot{x} = f(x)$

$$\frac{d}{dt}(\Delta x) = \left.\frac{\partial f}{\partial x}\right|_{x_0 u_0} \cdot \Delta x + \left.\frac{\partial f}{\partial u}\right|_{x_0 u_0} \Delta u$$

$$\Delta x = \begin{bmatrix} T - \bar{T} \\ h - \bar{h} \end{bmatrix}$$

$$\Delta u = \begin{bmatrix} q - \bar{q} \\ P - \bar{P} \\ v - \bar{v} \end{bmatrix}$$

Here: $\dot{x} = \begin{bmatrix} \frac{dT}{dt} \\ \frac{dh}{dt} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$

$$\frac{\partial f_1}{\partial \underset{T}{x_1}} = \frac{1}{\cancel{\rho c} k h^3} \cancel{\rho c} q(-1)\Big|_{x_0 u_0} = \frac{-1}{k\bar{h}^3} a\sqrt{2gh}$$

$$\frac{\partial f_1}{\partial x_2} = \frac{-3}{\rho c k h^4} \underbrace{(\rho c q(v-T) + P)}_{0}\Big|_{u_0 x_0} = 0$$

$$\frac{\partial f_1}{\partial q} = \frac{\cancel{\rho c}(v-T)}{\cancel{\rho c} k h^3}\Big|_{x_0 u_0}$$

$$\frac{\partial f_1}{\partial P} = \frac{1}{\rho c k h^3}\Big|_{x_0 u_0}$$

$$\frac{\partial f_1}{\partial v} = \frac{\cancel{\rho c} q}{\cancel{\rho c} k h^3}\Big|_{x_0 u_0}$$

$$\frac{\partial f_2}{\partial x_1} = 0$$

$$f_2 = \frac{q}{3kh^2} - \frac{a\sqrt{2g}}{3k} \underbrace{h^{-2} \cdot h^{0,5}}_{h^{-1,5}}$$

$$\frac{\partial f_2}{\partial \underset{h}{x_2}} = \frac{-2q}{3kh^3} + 1{,}5 \underset{h^{-3} \cdot h^{0,5}}{\boxed{h^{-2,5}}} \frac{a\sqrt{2g}}{3k}$$

$$= \frac{-2a\sqrt{2gh}}{3kh^3} + \frac{1{,}5\, a\sqrt{2gh}}{3kh^3} \left[\frac{\times 2}{\times 2}\right]$$

$$= \frac{-a\sqrt{2gh}}{6kh^3}$$

$$\frac{\partial f_2}{\partial q} = \frac{1}{3kh^2} \qquad \frac{\partial f_2}{\partial P} = 0 \qquad \frac{\partial f_2}{\partial v} = 0$$

$$\frac{d}{dt}\begin{bmatrix}\Delta T\\ \Delta h\end{bmatrix} = \begin{bmatrix}\frac{-a\sqrt{2g\bar{h}}}{k\bar{h}^3} & 0\\ 0 & \frac{-a\sqrt{2g\bar{h}}}{6k\bar{h}^3}\end{bmatrix}\begin{bmatrix}\Delta T\\ \Delta h\end{bmatrix} + \begin{bmatrix}\frac{(\bar{V}-\bar{T})}{k\bar{h}^3} & \frac{1}{\rho c k\bar{h}^3} & \frac{\bar{q}}{k\bar{h}^3}\\ \frac{1}{3k\bar{h}^2} & 0 & 0\end{bmatrix}\begin{bmatrix}\Delta q\\ \Delta p\\ \Delta V\end{bmatrix}$$

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**2,4c** Time constant of the system?

* Eigenvalues $\overset{\lambda_i}{}$ of the system matrix $\overset{\Delta}{A}$ ← (for reachable & observable systems)
are the poles of the system.

Time constants $T_i = -1/\lambda_i$

$\det(\lambda_i I - A) = 0 \Rightarrow \lambda_i$

why? $G(s) = C\underbrace{(sI-A)^{-1}}B$

poles of G ⟶ $\frac{1}{\det(sI-A)}\cdot[\;]$

Time constant: $G(s) = \frac{H(s)}{(1+Ts)} = \frac{H_2(s)}{(s-\lambda)} = \frac{H_2(s)/-\lambda}{(\frac{s}{-\lambda}+1)}$

$\Rightarrow T = \frac{-1}{\lambda}$

Given parameter values: $A = 10^{-3}\begin{bmatrix}-10 & 0\\ 0 & -1{,}7\end{bmatrix}$

$\det(\lambda_i I - A) = \begin{bmatrix}\lambda + 10^{10^{-3}} & 0\\ 0 & \lambda + 1{,}7^{10^{-3}}\end{bmatrix} = (\lambda + 10^{\cdot 10^{-3}})(\lambda + 1{,}7^{\cdot 10^{-3}}) = 0$

$\lambda_1^{\cancel{2}} = -10^{\cdot 10^{-3}}$ $\lambda_2 = -1{,}7^{\cdot 10^{-3}}$

$T = \frac{-1}{\lambda} = \begin{pmatrix}\frac{1}{104}\\ \frac{1}{1{,}7\cdot 10^3}\end{pmatrix} = \begin{bmatrix}100\\ 588\end{bmatrix}[s]$ use SI-units!

## (2.7) Transfer function to state space

page 35-37
example
2.5

$$G(s) = \begin{bmatrix} \frac{1}{(s+1)(s+2)} & \frac{s+3}{(s+1)(s^2+s+1)} \end{bmatrix} = [G_1 \; G_2]$$

Dimensions: $Y(s) = G(s)\,U(s)$ (assum $n=1$) $\Rightarrow$

$[1\times n] \; [1\times 2] \; [2\times n]$

$$Y(s) = [G_1 \; G_2]\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = G_1U_1 + G_2U_2$$

General state space form

$\dot{x} = Ax + Bu$ $\quad\mathcal{L}:\quad sX = AX + Bu \rightarrow X = (sI-A)^{-1}Bu$

$y = Cx$ $\quad\quad Y = CX$

$$Y = \underbrace{C(sI-A)^{-1}B}_{G(s) = [G_1 \; G_2]}\, u$$

$B \cdot U$

$[\beta \; 2] \; [2\times n]$ $\quad B = [B_1 \; B_2]$

$$G(s) = C(sI-A)^{-1}[B_1 \; B_2] = \begin{bmatrix} \underbrace{C(sI-A)^{-1}B_1}_{G_1} & \underbrace{C(sI-A)^{-1}B_2}_{G_2} \end{bmatrix}$$

- Describe system using a common denominator
- Use the observable canonical form $\Rightarrow$ $G_1$ & $G_2$ will have the same A & C matrices.
- combinde $B_1$ & $B_2$ $\rightarrow$ $B = [B_1 \; B_2]$ $\Rightarrow$ mission compleated

2.7 fortsättning

$$Y = \frac{u_1}{(s+1)(s+2)} + \frac{u_2(s+3)}{(s+1)(s^2+s+1)} =$$

$$= \frac{(s^2+s+1)u_1 + (s+2)(s+3)u_2}{(s^2+s+1)(s+1)(s+2)} =$$

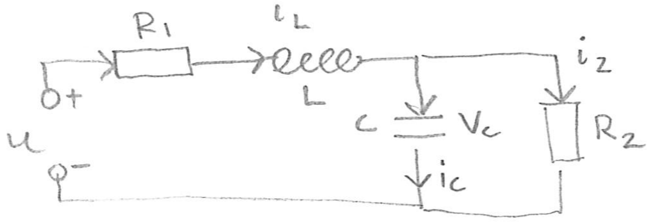
$$= \frac{[0\cdot s^3 + 1\cdot s^2 + 1\cdot s + 1]u_1 + [0\cdot s^3 + s^2 + 5s + 6]u_2}{s^4 + 4s^3 + 6s^2 + 5s + 2}$$

Observable canonical form (page 36)

$$\dot{x} = \begin{bmatrix} -4 & 1 & 0 & 0 \\ -6 & 0 & 1 & 0 \\ -5 & 0 & 0 & 1 \\ -2 & 0 & 0 & 0 \end{bmatrix} x + \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 5 \\ 1 & 6 \end{bmatrix}}_{B = [B_1\ B_2]} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$y = [1\ 0\ 0\ 0]\,x$$

(3.3)



* Determine $\dot{x} = Ax + Bu$

* states: $x_1 = i_L$, $x_2 = V_c$

ⓐ

(i) $V_L = L \cdot \frac{d}{dt}(i_L)$

(ii) $V_c = \frac{1}{C}\int_0^t i_c \, d\tau$

$$(i) \Rightarrow \frac{d}{dt}(i_L) = \frac{V_L}{L} = \dot{x}_1$$

$$(ii) \Rightarrow \frac{d}{dt}(V_c) = \frac{1}{C} i_c = \dot{x}_2$$

$$I) \quad u = R_1 \cdot \underset{x_1}{\underset{\uparrow}{i_L}} + L(\underset{\dot{x}_1}{\underset{\uparrow}{\dot{i}_L}}) + \underset{x_2}{\underset{\uparrow}{V_c}} \Rightarrow$$

$$II) \quad \overset{x_2}{\overset{\downarrow}{V_c}} = R_2 \cdot i_2$$

$$III) \quad \underset{x_1}{\underset{\uparrow}{i_L}} = \underset{\dot{x}_2 \cdot C}{\underset{\uparrow}{i_c}} + \boxed{i_2}$$

$$\Rightarrow x_2 = R_2(i_L - i_c) = R_2(x_1 - \dot{x}_2 \cdot C)$$

$$R_2 \dot{x}_2 C = -x_2 + R_2 x_1$$

$$I \Rightarrow \dot{x}_1 = [u - R_1 x_1 - x_2]\frac{1}{L}$$

$$(II, III) \Rightarrow \dot{x}_2 = \frac{1}{CR_2}[R_2 x_1 - x_2]$$

$$\dot{x} = \begin{bmatrix} -\frac{R_1}{L} & -\frac{1}{L} \\ \frac{1}{C} & \frac{-1}{CR_2} \end{bmatrix} x + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} u$$

save on the board

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ⓑ Determine $\phi(t) = e^{At}$

Given $R_1 = R_2 = 1$, $L = 1$, $H = 1$

$$\Rightarrow A = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$$

useful relations of matrix exponentials:

$$\begin{cases} e^X = \sum_{k=0}^{\infty} \frac{1}{k!} X^k \\ \mathcal{L}\{e^{tX}\} = (sI - X)^{-1} \end{cases}$$

$$\Rightarrow e^{tA} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$$

$$(sI - A)^{-1} = \begin{bmatrix} s+1 & 1 \\ -1 & s+1 \end{bmatrix}^{-1} = \frac{1}{(s+1)^2+1}\begin{bmatrix} s+1 & -1 \\ 1 & s+1 \end{bmatrix}$$

$$\frac{s+1}{(s+1)^2+1} = F(s+1) \Rightarrow F(s) = \frac{s}{s^2+1}$$

$$\mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a}$$

$$\mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2}$$

$$\mathcal{L}\{e^{-at}f(t)\} = F(s+a)$$ shift theorem

$$F(s) = \frac{s}{s^2+1} \Rightarrow f(t) = \mathcal{L}^{-1}\{F(s)\} = \cos(t)$$

$$F(S+1) \Rightarrow e^{-t}\cos(t)$$

$$H(S+1) = \frac{1}{(s+1)^2+1} \qquad H(s) = \frac{1}{s^2+1} \qquad h(t) = \sin(t) \qquad h(t+1) = e^{-t}\sin(t)$$

$$\phi(t) = e^{At} = e^{-t}\begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}$$

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ⓒ Plot the states for $0 < t < 3s$

Given: $u = 2V$ at $t = 0$ (steady state)

short circuit: $u = 0$ for $t > 0$

page 43. "Solving the system Equations"

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau$$

i) use $t_0 = 0$, calculate $x(t_0)$

ii) plug in matrices from a), b).

i) state space model at steady state.

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} i_{L0} \\ V_{c0} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\overset{=2}{u_0}$$

2nd row $\Rightarrow i_{L0} = V_{c0}$ 1st row: $V_{c0} = -i_{L0} + 2$ $\therefore V_{c0} = i_{L0} = 1$

$$x(t) = e^{-t}\begin{bmatrix}\cos(t) & -\sin(t)\\ \sin(t) & \cos(t)\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} + \int_0^t e^{\cdots}\cdot B\cdot \underbrace{u(\tau)}_{=0\ \forall t>0}\, d\tau$$

$$\Rightarrow x_1(t) = i_L(t) = e^{-t}\left(\cos(t) - \sin(t)\right)$$

$$x_2(t) = V_c(t) = e^{-t}\left(\sin(t) + \cos(t)\right)$$