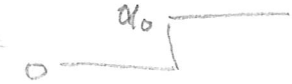
8.3 Step response experiments

Two inputs, two outputs

$u(t) =$ 

$U(s) = \frac{u_0}{s}$ here $u_0 = 1$

Determine Transfer function matrix $G(s)$

$$Y = G \cdot U \quad , \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \Rightarrow \begin{aligned} y_1 &= G_1 u_1 + G_2 u_2 \\ y_2 &= G_3 \cdot u_1 + G_4 \cdot u_2 \end{aligned}$$

$\Rightarrow G_1$ corresponds to from u_1 to Y_1 etc.

First order t.f. $G = \frac{K}{1+Ts}$

if $Y(s) = \frac{K}{1+Ts} \cdot U(s)$ $\quad u$-step $= \frac{u_0}{s}$

invers Laplace $\Rightarrow y(t) = K(1 - e^{-t/T})\, u_0$

$\therefore \; K = \lim_{t \to \infty} y(t)/u_0$

T is the time $y(T) = 0{,}63 K \cdot u_0$

From the figures: $\quad G_1$: $K = 2 \quad T \approx 1$

G_2: $K = 1{,}5 \quad T \approx 0{,}5$

$G_3 = G_4$: $K = 1 \quad T = 1$

(b) Find $W(s)$ such that WG is diagonal

$$\begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix} \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix} = \begin{bmatrix} \tilde{G}_{11} & 0 \\ 0 & \tilde{G}_{22} \end{bmatrix} \Rightarrow$$

OBS many functions $W(s)$ can satisfy this. Here is one example

$$W_1 \cdot G_2 + W_2 \cdot G_4 = 0 = W_1 \cdot \frac{1{,}5}{1 + 0{,}5s} + W_2 \cdot \frac{1}{1+s}$$

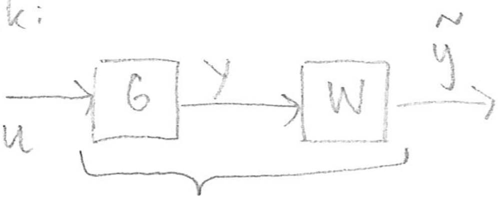
let $W_1 = G_4 = \frac{1}{1+s}$

$$W_2 = -G_2 = \frac{-1{,}5}{1 + 0{,}5s} = \frac{-3}{2+s}$$

$$W_3 G_1 + W_4 G_3 = W_3 \frac{2}{1+s} + W_4 \frac{1}{1+s} = 0 = (2W_3 + W_4)\frac{1}{1+s} = 0$$

let $W_3 = 1 \quad W_4 = -2$

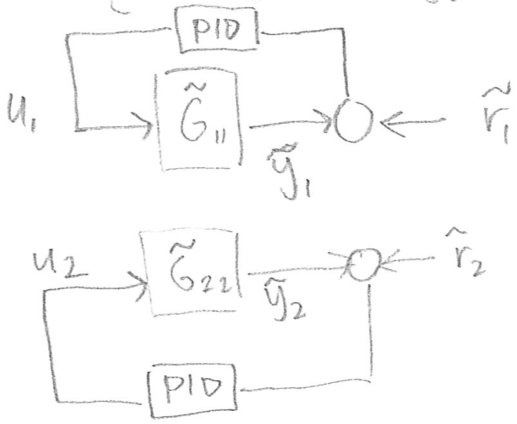
MIMO block:



$\tilde{Y} = W \cdot Y$

$$\tilde{Y} = \begin{bmatrix} \tilde{G}_{11} & 0 \\ 0 & \tilde{G}_{22} \end{bmatrix} u \Rightarrow \begin{matrix} \tilde{y}_1 = \tilde{G}_{11} u_1 \\ \tilde{y}_2 = \tilde{G}_{22} u_2 \end{matrix}$$

they are decoupled

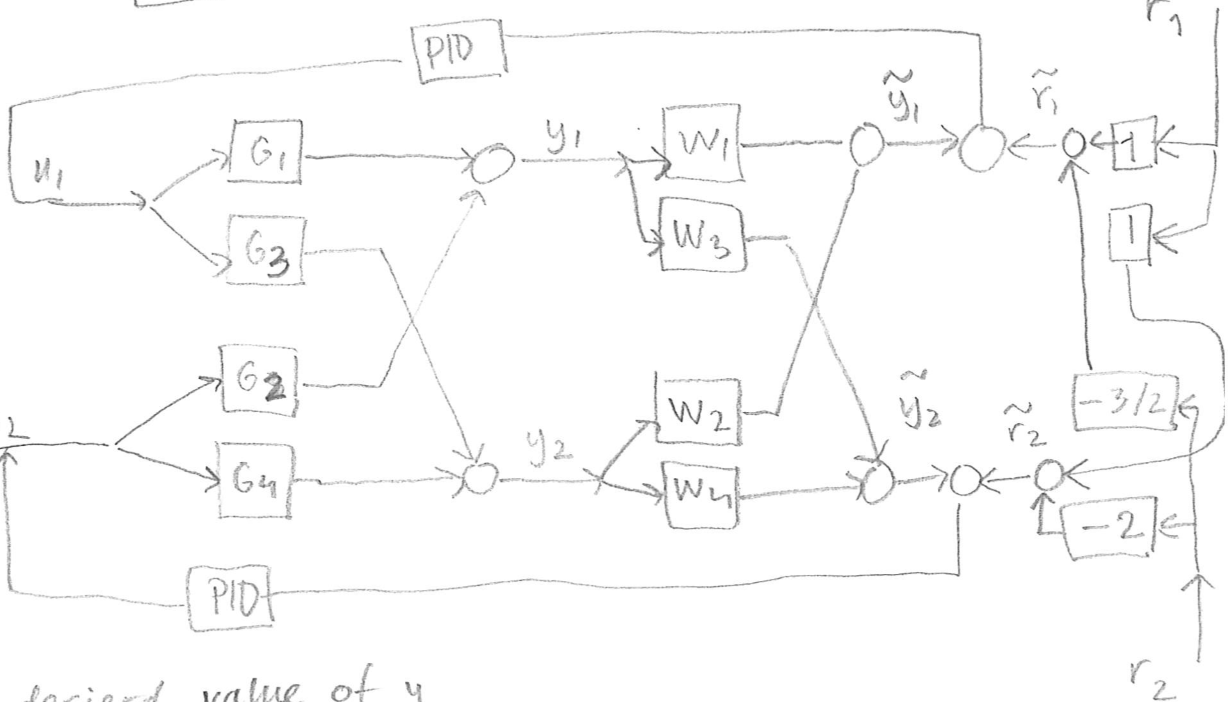


$\tilde{Y} = W Y$

$\tilde{y}_1 = W_1 \cdot y_1 + W_2 \cdot y_2$

$\tilde{y}_2 = W_3 \cdot y_1 + W_4 \cdot y_2$

with all blocks:



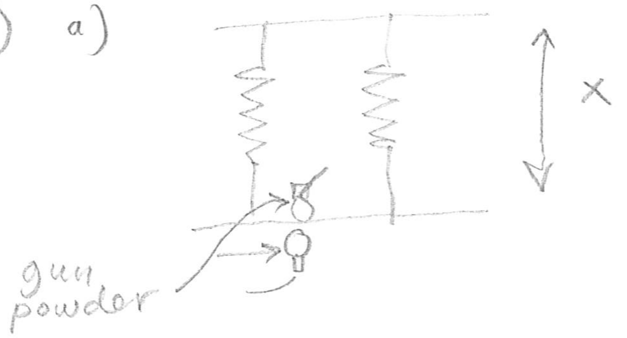
r is the desired value of y

$\Rightarrow \tilde{r}$ & r relates in the same way as $\tilde{y}$ & y $\Rightarrow \tilde{r} = W \cdot r$

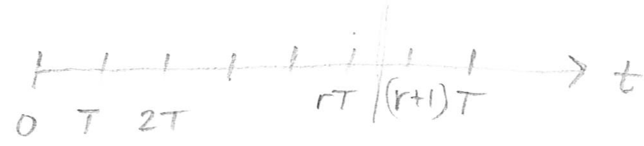
However, one alternative is to not include the dynamics $\Rightarrow \tilde{r} = W(0) \cdot r$

$$W(0) = \begin{bmatrix} 1 & -3/2 \\ 1 & -2 \end{bmatrix}$$

9.5 a)



Firing gun powder in time intervall T



m_k - amount of powder exploding at time kT

$u_k = m_k$ or $u_k = -m_k$

scaled system:

$$\ddot{x} + x = \sum_{k=-\infty}^{\infty} u_k \delta(t - kT)$$

let $y_k = x(kT)$ Give state space model.

integrate from 0 to t where $rT \leq t < (r+1)T$

$$\dot{x} + \int_0^t x(s)\,ds = \int_0^t \sum_{k=-\infty}^{\infty} u_k \delta(\tau - kT)\,d\tau = \sum_{k=0}^{r} u_k \underbrace{\int_0^t \delta(\tau - kT)\,d\tau}_{=1}$$

introduce $V(t) = V_r = \sum_{k=0}^{r} u_k$ for $rT \leq t < (r+1)T$

State; $z_1 = \int_0^t x(s)\,ds \Rightarrow \dot{z}_1 = \underbrace{x(t)} = z_2$

$z_2 = x(t)$ $\quad \dot{z}_2 = -\underbrace{\int x(s)}_{z_1} + V(t)$

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ 1 \end{bmatrix} V(t)$$

$$y(t) = x(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} z$$

Sampling: (chapter 4)

$$\begin{cases} A_d = e^{AT} \\ B_d = \int_0^T e^{As}\,ds\, B \\ C_d = C \end{cases}$$

$e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$

$$\begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix}^{-1} = \frac{1}{s^2+1}\begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix}$$

$$\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2+\omega^2}$$

$$A_d = \begin{bmatrix} \cos(T) & \sin(T) \\ -\sin(T) & \cos(T) \end{bmatrix} \quad B_d = \int_0^T \begin{bmatrix} \cos(\tau) & \sin(\tau) \\ -\sin(\tau) & \cos(\tau) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\tau =$$

$$= \int_0^T \begin{bmatrix} \sin\tau \\ \cos\tau \end{bmatrix} d\tau = \begin{bmatrix} -\cos\tau \\ \sin(\tau) \end{bmatrix}_0^T = \begin{bmatrix} -\cos T - (-1) \\ \sin T - 0 \end{bmatrix} = \begin{bmatrix} 1-\cos(T) \\ \sin(T) \end{bmatrix}$$

now we have the sampling with input $v(t)$

but we want input u_k

for $rT \leq t < (r+1)T$

$$v(t) = \sum_{k=0}^{r} u_k = \underbrace{\sum_{k=0}^{r-1} u_k}_{v(t) \text{ for } (r-1)T \leq t < rT} + u_k \Big|_{k=r}$$

$$v(k+1) = v(k) + u_k$$

$$\begin{bmatrix} z_1(k+1) \\ z_2(k+1) \\ v(k+1) \end{bmatrix} = \begin{bmatrix} \cos(T) & \sin(T) & 1-\cos T \\ -\sin(T) & \cos(T) & \sin(T) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_1(k) \\ z_2(k) \\ v(k) \end{bmatrix} + \begin{bmatrix} 1-\cos(T) \\ \sin(T) \\ 1 \end{bmatrix} u_k$$

$$y(k) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ v_k \end{bmatrix}$$

9.5 b

$$x(t+1) = F x(t) + G u(t)$$
$$y(t) = H x(t)$$

n # states

u - scalar

* All states are measurable
* at time $k=0$, apply control low

so that $X(N) = 0$ & $\sum_{k=0}^{N} u(k) = 0$

How can such a control law be determined?

How large N?

introduce an extra state:

$$V(t) = \sum_{k=0}^{t-1} u(k) = \underbrace{\sum_{k=0}^{t-2} u(k)}_{V(t-1)} + u(t-1)$$

$$\Rightarrow v(t+1) = v(t) + u(t)$$

$$\Rightarrow \begin{bmatrix} x(t+1) \\ V(t+1) \end{bmatrix} = \underbrace{\begin{bmatrix} F & 0 \\ 0 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x(t) \\ V(t) \end{bmatrix} + \underbrace{\begin{bmatrix} G \\ 1 \end{bmatrix}}_{B} u(t)$$

- Discrete state space
- You want to put all your states to 0

↳ This is fastest done with a dead beat controller

$u = -L \cdot X_{new}$ determine L such that

↑ vector $n+1$

the closed loops poles are in origin ⇒ eig$(A - BL)$ in origin

(9.8) $\begin{cases} z = \dfrac{1}{p+1} \cdot u + \dfrac{1}{p+1} \cdot v \\ y = z + e \end{cases}$

v - disturbance $= \dfrac{1}{p+\varepsilon} \cdot v_1$

v_1 & e - white noise

$\phi_{v_1}(\omega) = \phi_e = 1$

a) Determine LQG controller, minimizing $E z^2 + E u^2$ when $\varepsilon \to 0$

$z(p+1) = u + v$

$\dot{v} = -\varepsilon v + v_1$ ← white

$\dot{z} = -z + u + v$

let $x_1 = z$ $\quad x_2 = v$

$$\dot{x} = \underbrace{\begin{bmatrix} -1 & 1 \\ 0 & -\varepsilon \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} u + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{N} v_1$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} x + v_2$$

$R_1 = \phi_{v_1} = 1$

$R_2 = \phi_e = 1$

$R_{12} = 0$ ← no correlati

Working order: (i) Kalman Filter to estimate $\hat{x}$

$\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})$

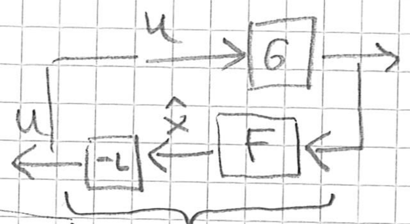
(ii) Feedback: $u = -L\hat{x}$

L determind from LQ-solution

(i)+(ii) ⇒ controller

$\dot{\hat{x}} = A\hat{x} + B(-L\hat{x}) + K(y - C\hat{x})$

$\Rightarrow \hat{x} = \underbrace{(sI - A + BL + KC)^{-1} K}_{F} y$



F_s

$u = -L\hat{x} = -\underbrace{\boxed{L(sI - A + BL + KC)^{-1} K}}_{F_s} \cdot y$

(iii) static gain of $\overbrace{\text{sensitivity function}}^{S(s)}$:

$S(s) = \dfrac{1}{1 + F_y G}$ $\quad$ static gain: $S(0) = \dfrac{1}{1 + F_y(0)\, G(0)}$

(i) page 128: $K = (PC^T + \cancel{NR_{12}}) R_2^{-1}$

$$0 = AP + PA^T - (PC^T + \cancel{NR_{12}}) \underbrace{R_2^{-1}}_{1} (PC^T + \cancel{NR_{12}})^T + NR_1N^T$$

n = # states

$n = 2 \quad A\ [n \times n], \quad P\ [n \times n] = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} 1 [0\ 1]$

$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

AP: $\begin{bmatrix} -1 & 1 \\ 0 & -\varepsilon \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} = \begin{bmatrix} -P_{11} + P_{12} & P_{22} - P_{12} \\ -\varepsilon P_{12} & -\varepsilon P_{22} \end{bmatrix}$

PA^T: $\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & -\varepsilon \end{bmatrix} = \begin{bmatrix} P_{12} - P_{11} & -\varepsilon P_{12} \\ P_{22} - P_{12} & -\varepsilon P_{22} \end{bmatrix}$

$PC^T \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} P_{11} \\ P_{12} \end{bmatrix} \qquad \begin{bmatrix} P_{11} \\ P_{12} \end{bmatrix} [P_{11}\ P_{12}] = \begin{bmatrix} P_{11}^2 & P_{11}P_{12} \\ P_{11}P_{12} & P_{12}^2 \end{bmatrix}$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} P_{12} - P_{11} & P_{22} - P_{12} \\ -\varepsilon P_{12} & -\varepsilon P_{22} \end{bmatrix} + \begin{bmatrix} P_{12} - P_{11} & -\varepsilon P_{12} \\ P_{22} - P_{12} & -\varepsilon P_{22} \end{bmatrix} - \begin{bmatrix} P_{11}^2 & P_{11}P_{12} \\ P_{11}P_{12} & P_{12}^2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

[1,1]: $0 = P_{12} - P_{11} + P_{12} - P_{11} - P_{11}^2 \Rightarrow P_{11}^2 + 2P_{11} - 2P_{12} = 0$

$(P_{11} + 1)^2 = 2P_{12} + 1$

1,2: $0 = P_{22} - P_{12} - \varepsilon P_{12} - P_{11}P_{12}$

2,1: $0 = -\varepsilon P_{12} + P_{22} - P_{12} - P_{11}P_{12}$

2,2: $0 = -\varepsilon P_{22} - \varepsilon P_{22} - P_{12}^2 + 1$

for $\varepsilon \to 0 \quad P_{12}^2 = 1 \Rightarrow P_{12} = 1$

$[1,1] \Rightarrow P_{11} = -1 \pm \sqrt{2 \underset{1}{\underset{\shortparallel}{P_{12}}} + 1} = -1 + \sqrt{3}$

$[1,2] \Rightarrow P_{22} = P_{12} + P_{11}P_{12} = 1 + (-1 + \sqrt{3}) = \sqrt{3}$

$(2,1) \Rightarrow P_{22} = P_{12} + P_{11}P_{12}$ ok.

$P = \begin{bmatrix} \sqrt{3} - 1 & 1 \\ 1 & \sqrt{3} \end{bmatrix} \qquad K = \begin{bmatrix} P_{11} \\ P_{12} \end{bmatrix} = \begin{bmatrix} \sqrt{3} - 1 \\ 1 \end{bmatrix}$

(ii) page 242. $L = Q_2^{-1} B^T S = [1\ 0]\begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix} = [s_1\ \ s_2]$

$Q_1 = 1$

$\min \int_0^\infty x_1^2 + u^2 = \int (Mx)^T Q_1 (Mx) + u^T Q_2 u \quad \Rightarrow \quad Q_2 = 1$

$\begin{bmatrix}1\\0\end{bmatrix}[1\ 0] = \begin{bmatrix}1&0\\0&0\end{bmatrix}$

$Mx = x_1 = [1\ 0]\begin{bmatrix}x_1\\x_2\end{bmatrix}$

$A^T S + SA + \overbrace{M^T Q_1 M}^{} - SBQ_2^{-1}B^T S = 0 \qquad M = C$ if $Mx = y$

$$\begin{bmatrix}-1 & 0 \\ 1 & -\varepsilon\end{bmatrix}\begin{bmatrix}s_1 & s_2 \\ s_2 & s_3\end{bmatrix} = \begin{bmatrix}-s_1 & -s_2 \\ s_1 - \varepsilon s_2 & s_2 - \varepsilon s_3\end{bmatrix} \quad \begin{bmatrix}s_1 & s_2 \\ s_2 & s_3\end{bmatrix}\begin{bmatrix}-1 & 1 \\ 0 & -\varepsilon\end{bmatrix} = \begin{bmatrix}-s_1 & s_1 - \varepsilon s_2 \\ -s_2 & s_2 - \varepsilon s_3\end{bmatrix}$$

$(\varepsilon \to 0$

$$\underbrace{SBQ_2^{-1}B^T}_{\begin{bmatrix}1\\0\end{bmatrix}[1\ 0] = \begin{bmatrix}1&0\\0&0\end{bmatrix}} S = \begin{bmatrix}s_1 & s_2 \\ s_2 & s_3\end{bmatrix}\begin{bmatrix}1&0\\0&0\end{bmatrix} S = \begin{bmatrix}s_1 & 0 \\ s_2 & 0\end{bmatrix}\begin{bmatrix}s_1 & s_2 \\ s_2 & s_3\end{bmatrix} = \begin{bmatrix}s_1^2 & s_1 s_2 \\ s_1 s_2 & s_2^2\end{bmatrix}$$

$$\begin{bmatrix}0&0\\0&0\end{bmatrix} = \begin{bmatrix}-s_1 & -s_2 \\ s_1 - \varepsilon s_2 & s_2 - \varepsilon s_3\end{bmatrix} + \begin{bmatrix}-s_1 & s_1 - \varepsilon s_2 \\ -s_2 & s_2 - \varepsilon s_3\end{bmatrix} + \begin{bmatrix}1&0\\0&0\end{bmatrix} - \begin{bmatrix}s_1^2 & s_1 s_2 \\ s_1 s_2 & s_2^2\end{bmatrix}$$

$[1,1]$: $0 = -s_1 - s_1 + 1 - s_1^2 \Rightarrow s_1^2 + 2s_1 - 1 = 0$

$(s_1 + 1)^2 - 1 - 1 = 0$

$s_1 = -1 + \sqrt{1+1}$

$[1,2]$: $0 = -s_2 + s_1 - s_1 s_2 - \varepsilon s_2$

same ok.

$(2,1)$: $0 = s_1 - s_2 - s_1 s_2 - \varepsilon s_2$

$[2,2)$: $0 = s_2 + s_2 - s_2^2 - 2\varepsilon s_3 \quad | \quad s_2^2 - 2s_2 = (s_2 - 1)^2 - 1 = 0$

$0 = s_2(2 - s_2) - 2\varepsilon s_3 \quad | \quad s_2 = 1 \pm \sqrt{1}$

$s_2 = 0$ or $s_2 = 2$

$s_2 + s_1 s_2 = s_1$

$$s_2 = \frac{s_1}{1 + s_1} = \frac{-1 + \sqrt{2}}{\cancel{1} - \cancel{1} + \sqrt{2}} = \frac{-1}{\sqrt{2}} + \frac{\sqrt{2}}{\sqrt{2}} = 1 - \frac{1}{\sqrt{2}}$$

$(a-b)(a+b)$
$a^2 - b^2$

$$s_3 = \frac{s_2(2 - s_2)}{2\varepsilon} = \frac{\left(1 - \frac{1}{\sqrt{2}}\right)\left(\overbrace{2 - 1}^{1} + \frac{1}{\sqrt{2}}\right)}{2\varepsilon} \quad \frac{\left(1 - \frac{1}{\sqrt{2}}\right)\left(1 + \frac{1}{\sqrt{2}}\right)}{2\varepsilon} = \frac{1 - \frac{1}{2}}{2\varepsilon} = \frac{1}{4\varepsilon}$$

(iii) static gain $F_y(0) = L(\underset{0}{sI} - A + BL + KC)^{-1}K =$

$$= [l_1\ l_2]\Big(\underbrace{\begin{bmatrix}1 & -1\\ 0 & \varepsilon\end{bmatrix} + \underbrace{\begin{bmatrix}1\\0\end{bmatrix}[l_1\ l_2]}_{\begin{bmatrix}l_1 & l_2\\ 0 & 0\end{bmatrix}} + \underbrace{\begin{bmatrix}k_1\\k_2\end{bmatrix}[1\ 0]}_{\begin{bmatrix}k_1 & 0\\ k_2 & 0\end{bmatrix}}}_{\begin{bmatrix}1+l_1+k_1 & -1+l_2\\ k_2 & \varepsilon\end{bmatrix}}\Big)^{-1}\begin{bmatrix}k_1\\k_2\end{bmatrix}$$

$$\begin{bmatrix}1+l_1+k_1 & -1+l_2\\ k_2 & \varepsilon\end{bmatrix}^{-1} = \frac{1}{\varepsilon-(k_2(-1+l_2))}\begin{bmatrix}\varepsilon & 1-l_2\\ -k_2 & 1+l_1+k_1\end{bmatrix}$$

$$F_y(0) = \underbrace{[l_1\ l_2]\frac{1}{k_2-k_2l_2}\begin{bmatrix}0 & 1-l_2\\ -k_2 & 1+l_1+k_1\end{bmatrix}}\begin{bmatrix}k_1\\k_2\end{bmatrix}$$

$$\frac{1}{k_2(1-l_2)}[-l_2k_2 \quad l_1 - \cancel{l_1l_2} + l_2 + \cancel{l_1l_2} + l_2k_1]\begin{bmatrix}k_1\\k_2\end{bmatrix} =$$

$$= (\cancel{-l_2k_2k_1} + l_1k_2 + l_2k_2 + \cancel{l_2k_1k_2})\frac{1}{k_2(1-l_2)} =$$

$$= \frac{l_1+l_2}{1-l_2} = \left\{\begin{matrix}l_1 = s_1 = -1+\sqrt{2}\\ l_2 = s_2 = 1-1/\sqrt{2}\end{matrix}\right\} = \frac{-\cancel{1}+\sqrt{2}+\cancel{1}-1/\sqrt{2}}{\cancel{1-1}+1/\sqrt{2}} =$$

$$= \frac{2-1}{1} = 1$$

$$u \longrightarrow \boxed{G} \longrightarrow y \quad \left\{\begin{matrix}\dot{x} = Ax+Bu\\ y = Cx\end{matrix}\right. \Rightarrow Y = \underbrace{C(sI-A)^{-1}B}_{G}\cdot u$$

$$G(0) = [1\ 0]\underbrace{\begin{bmatrix}1 & -1\\ 0 & \varepsilon\end{bmatrix}^{-1}}_{\begin{bmatrix}1 & 1/\varepsilon\\ 0 & 1/\varepsilon\end{bmatrix}}\begin{bmatrix}1\\0\end{bmatrix} = [1\ \ 1/\varepsilon]\begin{bmatrix}1\\0\end{bmatrix} = 1$$

$$S(0) = \frac{1}{1+F_y(0)G(0)} = \frac{1}{2}$$

Determine L using LTR   Loop Transfer Recovery   p.262

LQ design $\Rightarrow$ very good robustness & sensitivity properties

$u = -Lx$

requiers measurements of all states x

- if x is not measurable:
  - * use observer to estimate $\hat{x}$
    drawback: robustness not guaranteed
  - * use (input- or output-) LTR
    tune parameters, to ensure good robustness

<u>output</u> - LTR (page 263)

- calculate kalman gain K (as previous done)
- L is determined as LQ-feedback with

$$M = C, \quad Q_1 = \alpha Q_2 \quad \text{with} \quad \alpha \to \infty$$

$$L_{LTR} = Q_2^{-1} B^T S = [s_1 \;\; s_2] \quad \text{as previous}$$

$$S: \; A^T S + SA + \underbrace{C^T \alpha \cdot C}_{\begin{bmatrix}1\\0\end{bmatrix} \alpha [1 \;\; 0] = \begin{bmatrix}\alpha & 0\\ 0 & 0\end{bmatrix}} - SBQ_2^{-1}B^T S = 0$$

only index [1:1] will change: $0 = -s_1 - s_1 + \alpha - s_1^2 \Rightarrow s_1 = -1 \pm \sqrt{1+\alpha}$

$$s_2 + s_1 s_2 = s_1 \Rightarrow s_2 = \frac{s_1}{1+s_1} = \frac{-1+\sqrt{1+\alpha}}{1-1+\sqrt{1+\alpha}} = 1 - \frac{1}{\sqrt{1+\alpha}}$$

verify: $S(0) = \dfrac{1}{1+F_y(0)G(0)} \to 0$ as $\alpha \to \infty$