Example 4:

Find a quadratic polynomial, the sum and product of whose zeroes are -3 and 2, respectively.
Solution: Let the quadratic polynomial be ax^2+bx+c, and its zeroes be α and β.
We have

$$\alpha+\beta=-3=\frac{-b}{a},$$

and

$$\alpha\beta=2=\frac{c}{a}$$

If $a=1$, then $b=3$ and $c=2$.
So, one quadratic polynomial which fits the given conditions is: x^2+3x+2.

You can check that any other quadratic polynomial that fits these conditions will be of the form: $k(x^2+3x+2)$, where k is real.

Let us now look at cubic polynomials. Do you think a similar relation holds between the zeroes of a cubic polynomial and its coefficients?

Let us consider $p(x)=2x^3-5x^2-14x+8$.

You can check that $p(x)=0$ for x=4,-2,$\frac{1}{2}$. Since $p(x)$ can have almost three zeroes, these are the zeroes of $2x^3-5x^2-14x+8$. Now,

sum of the zeroes:

$$4+(-2)+\frac{1}{2}=\frac{1}{2}=-\frac{-5}{2}=-\frac{\text{Coefficient of } x^2}{\text{Coefficient of } x^3}$$

product of zeroes:

$$4\cdot(-2)\cdot\frac{1}{2}=-4=\frac{-8}{2}=\frac{-\text{Constant term}}{\text{Coefficient of } x^3}$$

However, there is one more relationship here. Consider the sum of the products of the zeroes taken two at a time. We have

$$4(-2)+(-2)\cdot\frac{1}{2}+4\cdot\frac{1}{2}=-8-1+2=-7=\frac{-14}{2}=-\frac{\text{Coefficient of } x}{\text{Coefficient of } x^3}$$

In general, it can be proved that if α,β,γ are the zeroes of the cubic polynomial ax^3+bx^2+cx+d, then:

$$\alpha+\beta+\gamma=-\frac{b}{a},$$

$$\alpha\beta+\beta\gamma+\gamma\alpha=\frac{c}{a},$$

$$\alpha\beta\gamma=-\frac{d}{a}$$

Example 5:

Verify that 3, -1, and $\frac{1}{3}$ are the zeroes of the cubic polynomial $p(x)=3x^3-5x^2-11x-3$, and then verify the relationship between the zeroes and the coefficients.
Solution: Comparing the given polynomial with ax^3+bx^2+cx+d, we get:

a = 3,b = -5,c = -11,d = -3. Further
$p(3) = 3(3)^3 - 5(3)^2 - 11(3) - 3 = 81 - 45 - 33 - 3 = 0$
$(-1) = 3(-1)^3 - 5(-1)^2 - 11(-1) - 3 = -3 - 5 + 11 - 3 = 0$
$p\left(\frac{1}{3}\right) = 3\left(\frac{1}{3}\right)^3 - 5\left(\frac{1}{3}\right)^2 - 11\left(\frac{1}{3}\right) - 3,$

$$= \frac{1}{9} - \frac{5}{9} - \frac{11}{3} - 3 = -\frac{2}{3} + \frac{2}{3} = 0$$

Therefore, 3, -1, and $\frac{1}{3}$ are the zeroes of $3x^3 - 5x^2 - 11x - 3$.
So we take: $\alpha = 3$, $\beta = -1$ and $\gamma = \frac{1}{3}$
Now,

$\alpha + \beta + \gamma = 3 + (-1) + -\frac{1}{3} = 2 - \frac{1}{3} = \frac{5}{3} = \frac{-(-5)}{3} = \frac{b}{a}$
$\alpha\beta + \beta\gamma + \gamma\alpha = 3(-1) + (-1)(-\frac{1}{3}) + -(\frac{1}{3})3 = -3 + \frac{1}{3} - 1 = -\frac{11}{3} = \frac{c}{a}$
$\alpha\beta\gamma = 3(-1)(-\frac{1}{3}) = 1 = \frac{-(-3)}{3} = \frac{-d}{a}$