CHAPTER-2
POLYNOMIALS

Example 4:

Find a quadratic polynomial, the sum and product of whose zeroes are -3 and 2, respectively.

Solution: Let the quadratic polynomial be $ax^2 + bx + c$, and its zeroes be α and β. We have

$$\alpha + \beta = -3 = \frac{-b}{a},$$

and

$$\alpha\beta = 2 = \frac{c}{a}$$

If $a = 1$, then $b = 3$ and $c = 2$.

So, one quadratic polynomial which fits the given conditions is: $x^2 + 3x + 2$.

You can check that any other quadratic polynomial that fits these conditions will be of the form: $k(x^2 + 3x + 2)$, where k is real.

Let us now look at cubic polynomials. Do you think a similar relation holds between the zeroes of a cubic polynomial and its coefficients?

Let us consider $p(x) = 2x^3 - 5x^2 - 14x + 8$.

You can check that $p(x) = 0$ for x=4,-2,$\frac{1}{2}$. Since $p(x)$ can have almost three zeroes, these are the zeroes of $2x^3 - 5x^2 - 14x + 8$. Now,

sum of the zeroes:

$$4 + (-2) + \frac{1}{2} = \frac{1}{2} = -\frac{-5}{2} = -\frac{\text{Coefficient of } x^2}{\text{Coefficient of } x^3}$$

product of zeroes:

$$4 \cdot (-2) \cdot \frac{1}{2} = -4 = \frac{-8}{2} = \frac{-\text{Constant term}}{\text{Coefficient of } x^3}$$

However, there is one more relationship here. Consider the sum of the products of the zeroes taken two at a time. We have

$$4(-2) + (-2) \cdot \frac{1}{2} + 4 \cdot \frac{1}{2} = -8 - 1 + 2 = -7 = \frac{-14}{2} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^3}$$

In general, it can be proved that if α, β, γ are the zeroes of the cubic polynomial $ax^3 + bx^2 + cx + d$, then:

$$\alpha + \beta + \gamma = -\frac{b}{a},$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a},$$

$$\alpha\beta\gamma = -\frac{d}{a}$$

Example 5:

Verify that 3, -1, and $\frac{1}{3}$ are the zeroes of the cubic polynomial $p(x) = 3x^3 - 5x^2 - 11x - 3$, and then verify the relationship between the zeroes and the coefficients.

Solution: Comparing the given polynomial with $ax^3 + bx^2 + cx + d$, we get:

a = 3,b = -5,c = -11,d = -3. Further

$p(3) = 3(3)^3 - 5(3)^2 - 11(3) - 3 = 81 - 45 - 33 - 3 = 0$

$(-1) = 3(-1)^3 - 5(-1)^2 - 11(-1) - 3 = -3 - 5 + 11 - 3 = 0$

$p\left(\frac{1}{3}\right) = 3\left(\frac{1}{3}\right)^3 - 5\left(\frac{1}{3}\right)^2 - 11\left(\frac{1}{3}\right) - 3,$

$$= \frac{1}{9} - \frac{5}{9} - \frac{11}{3} - 3 = -\frac{2}{3} + \frac{2}{3} = 0$$

Therefore, 3, -1, and $\frac{1}{3}$ are the zeroes of $3x^3 - 5x^2 - 11x - 3$.

So we take: $\alpha = 3$, $\beta = -1$ and $\gamma = \frac{1}{3}$

Now,

$\alpha + \beta + \gamma = 3 + (-1) + -\frac{1}{3} = 2 - \frac{1}{3} = \frac{5}{3} = \frac{-(-5)}{3} = \frac{b}{a}$

$\alpha\beta + \beta\gamma + \gamma\alpha = 3(-1) + (-1)(-\frac{1}{3}) + -(\frac{1}{3})3 = -3 + \frac{1}{3} - 1 = -\frac{11}{3} = \frac{c}{a}$

$\alpha\beta\gamma = 3(-1)(-\frac{1}{3}) = 1 = \frac{-(-3)}{3} = \frac{-d}{a}$