CHAPTER-2
POLYNOMIALS

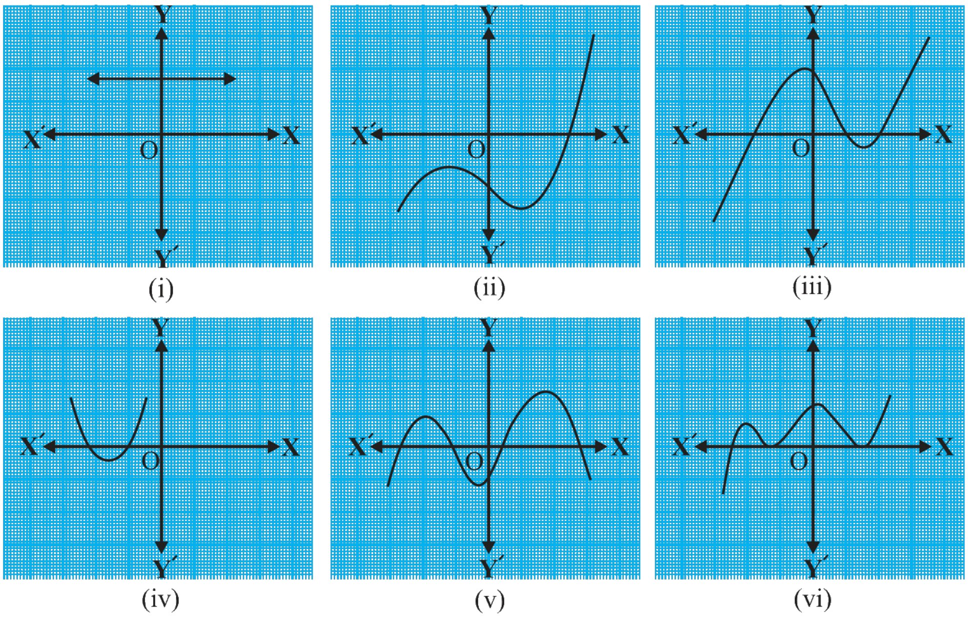

Fig 2.10

Example 4:

Find a quadratic polynomial, the sum and product of whose zeroes are -3 and 2, respectively.

Solution: Let the quadratic polynomial be $ax^2 + bx + c$, and its zeroes be α and β. We have:

$$\alpha + \beta = -3, \quad \alpha\beta = 2$$

Assuming $a = 1$, then:

$$b = -(\alpha + \beta) = 3, \quad c = \alpha\beta = 2$$

So, one quadratic polynomial which fits the given conditions is:

$$x^2 + 3x + 2$$

You can check that any other quadratic polynomial that fits these conditions will be of the form:

$$k(x^2 + 3x + 2), \quad \text{where } k \in \mathbb{R}$$

Cubic Polynomial Example

Let us now look at cubic polynomials. Do you think a similar relation holds between the zeroes of a cubic polynomial and its coefficients?

Let us consider:

$$p(x) = 2x^3 - 5x^2 - 14x + 8$$

You can check that $p(x) = 0$ for $x = 4, -2, \frac{1}{2}$. Since $p(x)$ can have at most three zeroes, these are the zeroes of $2x^3 - 5x^2 - 14x + 8$. Now,

Sum of zeroes:

$$4 + (-2) + \frac{1}{2} = \frac{1}{2} = -\frac{-5}{2} = -\frac{\text{Coefficient of } x^2}{\text{Coefficient of } x^3}$$

Product of zeroes:

$$4 \cdot (-2) \cdot \frac{1}{2} = -4 = \frac{-8}{2} = \frac{-\text{Constant term}}{\text{Coefficient of } x^3}$$

Sum of products of zeroes taken two at a time:

$$4 \cdot (-2) + (-2) \cdot \frac{1}{2} + 4 \cdot \frac{1}{2} = -8 - 1 + 2 = -7 = \frac{-14}{2} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^3}$$

In general, it can be proved that if α, β, γ are the zeroes of the cubic polynomial:

$$ax^3 + bx^2 + cx + d$$

Then:

$$\alpha + \beta + \gamma = -\frac{b}{a}, \quad \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}, \quad \alpha\beta\gamma = -\frac{d}{a}$$

Example 5:

Verify that 3, -1, and $\frac{1}{3}$ are the zeroes of the cubic polynomial $p(x) = 3x^3 - 5x^2 - 11x - 3$, and then verify the relationship between the zeroes and the coefficients.

Solution: Comparing the given polynomial with $ax^3 + bx^2 + cx + d$, we get:

$$a = 3, \quad b = -5, \quad c = -11, \quad d = -3$$

Now we check each zero:

$$p(3) = 3(3)^3 - 5(3)^2 - 11(3) - 3 = 81 - 45 - 33 - 3 = 0$$

$$p(-1) = 3(-1)^3 - 5(-1)^2 - 11(-1) - 3 = -3 - 5 + 11 - 3 = 0$$

$$p\left(\frac{1}{3}\right) = 3\left(\frac{1}{3}\right)^3 - 5\left(\frac{1}{3}\right)^2 - 11\left(\frac{1}{3}\right) - 3 = \frac{1}{9} - \frac{5}{9} - \frac{11}{3} - 3 = 0$$

Therefore, 3, -1, and $\frac{1}{3}$ are the zeroes of $3x^3 - 5x^2 - 11x - 3$. So we take:

$$\alpha = 3, \quad \beta = -1, \quad \gamma = \frac{1}{3}$$

Now we verify the relations:

$$\alpha + \beta + \gamma = 3 + (-1) + \frac{1}{3} = \frac{7}{3} = -\frac{b}{a} = -\frac{-5}{3}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = 3 \cdot (-1) + (-1) \cdot \frac{1}{3} + \frac{1}{3} \cdot 3 = -3 - \frac{1}{3} + 1 = -\frac{7}{3} = \frac{c}{a} = \frac{-11}{3}$$

$$\alpha\beta\gamma = 3 \cdot (-1) \cdot \frac{1}{3} = -1 = -\frac{d}{a} = -\frac{-3}{3}$$

Hence, the relationships between the zeroes and coefficients are verified.