

Overview of Probabilistic Graphical Models

Representation: Part I

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This presentation is based **heavily** on the material by Daphne Koller ([1]), Nir Friedman ([1], [3]), and David Sontag ([2]). I have **directly copied** or otherwise incorporated parts of their work into many of these slides, citing the sources where appropriate. If you find this presentation useful, I highly recommend that you take some time to read their work.

Outline

Probability Review [1]

Bayesian Networks [1]

- Introduction

- I-Map to Factorization

- Factorization to I-Map

- Applications

Markov Networks [1]

- Introduction

- Independence

- Distributions to Graphs

- Log-Linear Models

Event Spaces

- We can formalize the notion of an *event* by defining a space of possible outcomes Ω .
- Further, we define an *event space* S so that we can attach probabilities to specific outcomes.
- The event space S must satisfy the following:
 - $\emptyset, \Omega \in S$, where \emptyset is the *empty event* and Ω is the *trivial event*.
 - Closure under union: $\alpha, \beta \in S \rightarrow \alpha \cup \beta \in S$.
 - Closure under complementation: $\alpha \in S \rightarrow \Omega - \alpha \in S$.

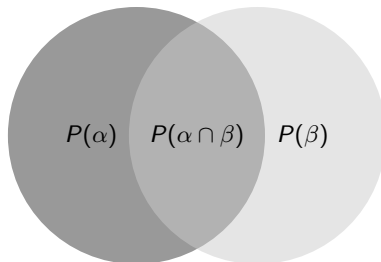
Probability Distributions

- A probability distribution over (Ω, S) is a mapping $P : S \rightarrow \mathbb{R}_0^+$. It must satisfy the following:
 - $P(\Omega) = 1$ — that is, the trivial event is given maximal probability.
 - If $\alpha, \beta \in S$ and $\alpha \cap \beta = \emptyset$, then $P(\alpha \cup \beta) = P(\alpha) + P(\beta)$.
If two outcomes are mutually disjoint, the probability of their union is the sum of their probabilities.
 - In this context, we view probabilities as *subjective degrees of belief* rather than as frequencies.

Conditional Probabilities

- How are our beliefs about an outcome affected in light of new evidence?
- This notion can be formalized using the following definition, which relates the areas of the shaded regions in the diagram.

$$P(\beta \mid \alpha) = \frac{P(\alpha \cap \beta)}{P(\alpha)}$$



Chain Rule

- From the definition of the conditional distribution, we have

$$P(\alpha \cap \beta) = P(\alpha)P(\beta \mid \alpha).$$

- More generally,

$$P(\alpha_1 \cap \dots \cap \alpha_k) = P(\alpha_1)P(\alpha_2 \mid \alpha_1) \cdots P(\alpha_k \mid \alpha_1 \cap \dots \cap \alpha_{k-1}),$$

but the order in which we choose to pull out variables does not matter.

- The fact that the chain rule allows us to decompose a joint distribution into *factors over smaller subsets of variables* becomes crucial later on.

Bayes Rule

- Decomposing the joint distribution in the definition of the conditional probability using the chain rule gives us Bayes rule:

$$P(\alpha \mid \beta) = \frac{P(\beta \mid \alpha)P(\alpha)}{P(\beta)}.$$

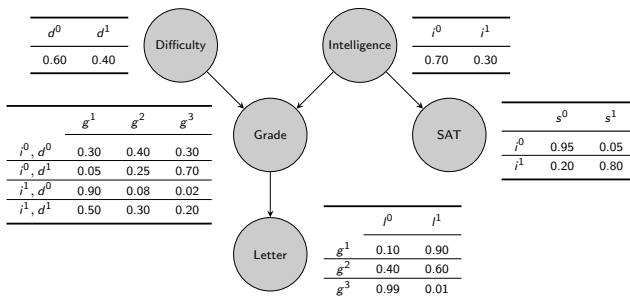
- Swapping an event on the left side of the pipe symbol with an event on the right works similarly when we are conditioning on several events:

$$P(\alpha \mid \beta \cap \gamma) = \frac{P(\beta \mid \alpha \cap \gamma)P(\alpha \mid \gamma)}{P(\beta \mid \gamma)}.$$

Independence and Conditional Independence

- A distribution P satisfies $(\alpha \perp \beta)$ if and only if $P(\alpha \cap \beta) = P(\alpha)P(\beta)$.
- A distribution P satisfies $(\alpha \perp \beta \mid \gamma)$ if and only if $P(\alpha \cap \beta \mid \gamma) = P(\alpha \mid \gamma)P(\beta \mid \gamma)$.
- Note the relationship between independence of variables and factorization of the joint distribution — it will play a very prominent role later on.

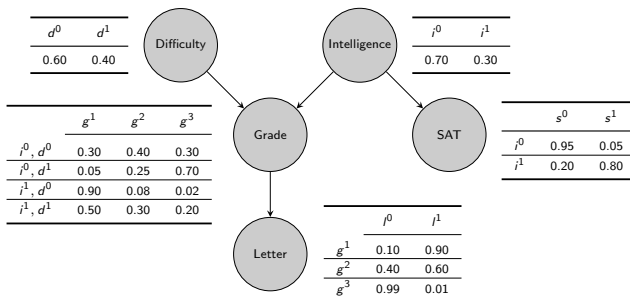
The Student Model



- In the student model, $Val(D) = \{easy, hard\}$, $Val(I) = \{low, high\}$, $Val(G) = \{A, B, C\}$, $Val(S) = \{low, high\}$, and $Val(L) = \{weak, strong\}$.
- The joint distribution is

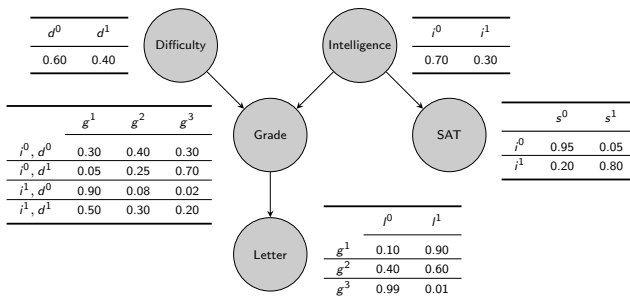
$$P(i, d, g, s, l) = P(i)P(d)P(g \mid i, d)P(s \mid i)P(l \mid g).$$

The Student Model



- Suppose that initially, we only have a student's recommendation letter (which is weak) and transcript (indicating that he received a 'C' in the course). How does finding that he received a high SAT score affect our beliefs about his intelligence?

The Student Model



- What is the largest set \mathbf{W} such that
 - $(L \perp \mathbf{W} \mid G)$?
 - $(S \perp \mathbf{W} \mid I)$?
 - $(G \perp \mathbf{W} \mid \pi(G))$, where $\pi(G)$ returns the set of parents of G ?
 - $(D \perp \mathbf{W})$?
 - $(D \perp \mathbf{W} \mid L)$?

Conclusions from the Student Model

- We view a Bayesian network as a generative model in which we sample the variables in topological order [2].
- The parents can be thought of as “generating” their children — that is, once we know about the values of the parents, the child is “shielded” from probabilistic influence by non-descendants.
- Consequently, we can think of a Bayesian network as representing a set of conditional independence assertions.

I-Maps

- Let K be any graph object associated with a set of independencies $I(K)$. We say that K is an I-map for a set of independencies I if $I(K) \subseteq I$.
- That is, all the conditional independence assertions that hold in $I(K)$ also hold in I .

I-Maps and Factorization

- Given that G is an I-map for P , can we simplify the representation of P [3]?
- Applying $I_\ell(G)$ to each factor in the naive decomposition proves that if G is an I-map for P , then P factorizes according to G . That is,

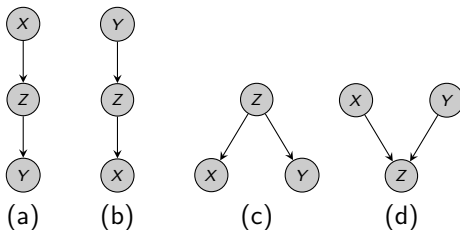
$$P(X_1, \dots, X_n) = \prod_{i=1}^n P(X_i \mid \pi(X_i)).$$

- The converse is also true.
- The compact factorization can result in an exponential reduction in the number of parameters that need to be specified!

I-Maps and Factorization

- Can we go the other way around and recover the graph G given the factorization of the distribution P ?
- We will first take a detour and explore more deeply how probabilistic influence flows across a graph.

Path Blockage [3]



The four possible edge trail combinations from X to Y via Z : (a) An indirect causal effect; (b) An indirect evidential effect; (c) A common cause; (d) A common effect.

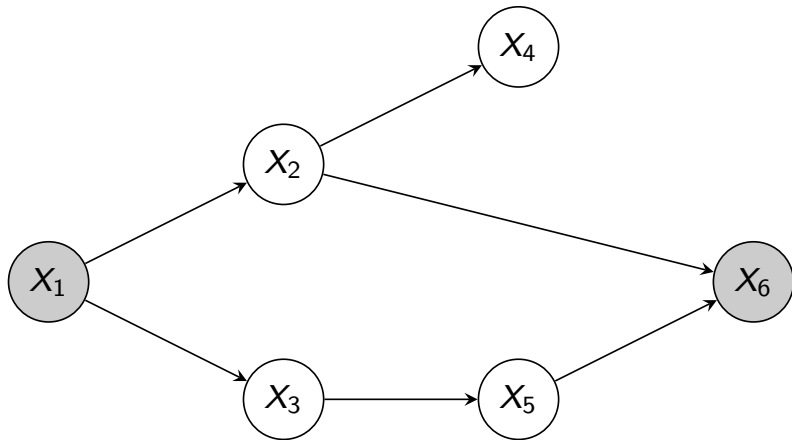
- Edge trails (a)–(c) are active if and only if Z is not observed.
- Edge trail (d) is active if and only if either Z or one of Z 's descendants is observed. Intuitively, this can be understood as a consequence of intercausal reasoning.

D-Separation

- We say that \mathbf{X} and \mathbf{Y} are d-separated (“directed-separated”) given \mathbf{Z} , denoted $\text{d-sep}_G(\mathbf{X}; \mathbf{Y} \mid \mathbf{Z})$, if there is no active trail between any node $X \in \mathbf{X}$ and $Y \in \mathbf{Y}$ given \mathbf{Z} .
- We use $I(G)$ to denote the set of independencies (aka *global Markov independencies*) that correspond to d-separation:

$$I(G) = \{(\mathbf{X} \perp \mathbf{Y} \mid \mathbf{Z}) : \text{d-sep}_G(\mathbf{X}; \mathbf{Y} \mid \mathbf{Z})\}.$$

D-Separation [2]



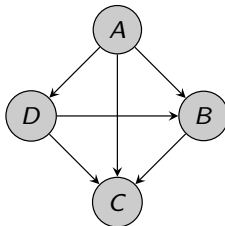
Independence and D-Separation

- We now return to make the connection between the factorization of P and the construction of G .
- For any distribution G that factorizes over G , we have that we have that if X and Y are not d-separated given \mathbf{Z} in G , then X and Y are dependent in all distributions P that factorize over G .
- Otherwise, G would contain some independence assertion that is not reflected in P , a contradiction.

Independence and D-Separation

- For almost all distributions P that factorize over G , that is, for all distributions except for a set of measure zero in the space of CPD parameterizations, we have that $I(P) = I(G)$.
- D-separation reduces statistical independencies in graphs (hard) to connectivity in graphs (easy) [2].

Minimal I-Maps [3]



- By itself, the concept of an I-map is not sufficient for us to get very far.
- The graph depicted above (a *complete* DAG) is an I-map for *any* distribution P .

Minimal I-Maps [3]

- A graph K is a minimal I-map for a set of independence assertions I if K is an I-map for I , and the removal of even a single edge from K renders it not an I-map.
- Taking $I = I(P)$ or $I = I(K')$, we can talk about K as being a minimal I-map for a distribution or another graph.

Algorithm for Constructing a Minimal I-Map

Require: An ordering X_1, \dots, X_n of random variables in \mathcal{X}

Require: A set of independencies I

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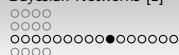
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for $i = 1, \dots, n$ **do**

\mathbf{U} is the current candidate for parents of X_i

$\mathbf{U} \leftarrow \{X_1, \dots, X_{i-1}\}$



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Find the minimal set \mathbf{U} satisfying

$(X_i \perp \{X_1, \dots, X_{i-1}\} - \mathbf{U} \mid \mathbf{U})$

for $\mathbf{U}' \subseteq \{X_1, \dots, X_{i-1}\}$ **do**

if $\mathbf{U}' \subset \mathbf{U}$ and $(X_i \perp \{X_1, \dots, X_{i-1}\} - \mathbf{U}' \mid \mathbf{U}') \in I$ **then**

$\mathbf{U} \leftarrow \mathbf{U}'$

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$\mathbf{U} \leftarrow \mathbf{U}'$

 Now set \mathbf{U} to be the parents of X_i

for $X_j \in \mathbf{U}$ **do**

 Add $X_j - X_i$ to G

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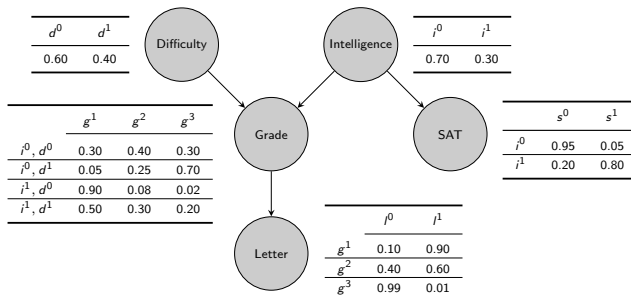
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for $X_j \in \mathbf{U}$ **do**

 Add $X_j - X_i$ to G

return G

Constructing a Minimal I-Map



- We now apply the algorithm to the variables in the student model, listed in topological order: D, I, S, G, L .

Constructing a Minimal I-Map



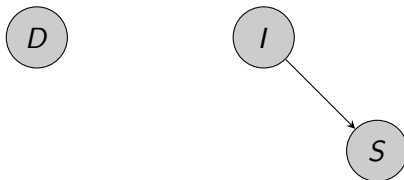
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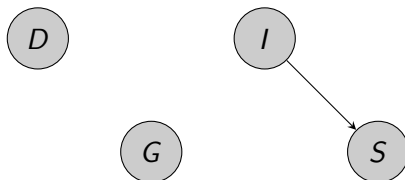
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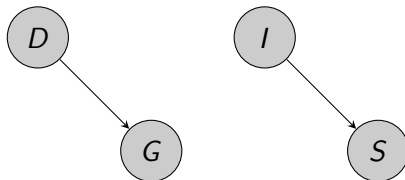
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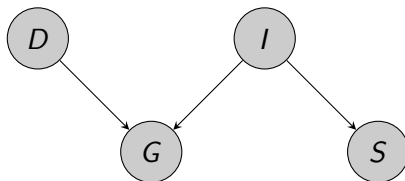
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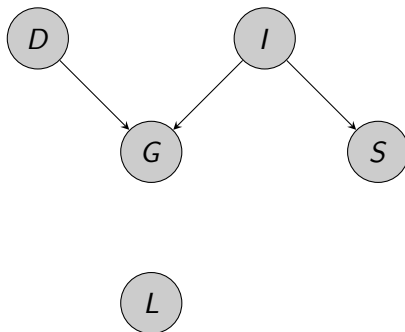
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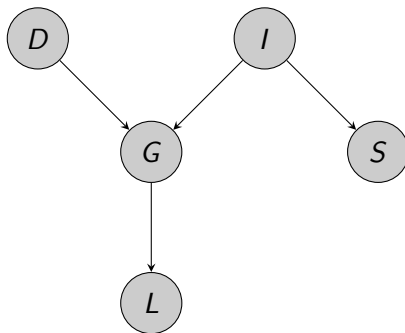
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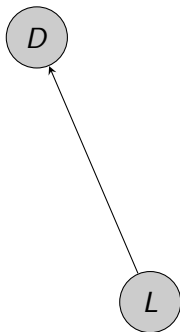
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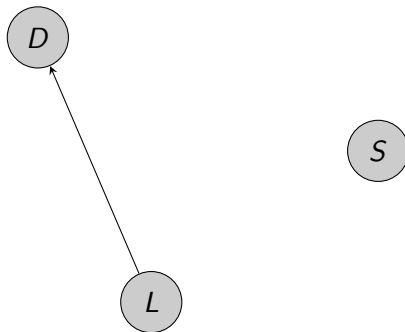
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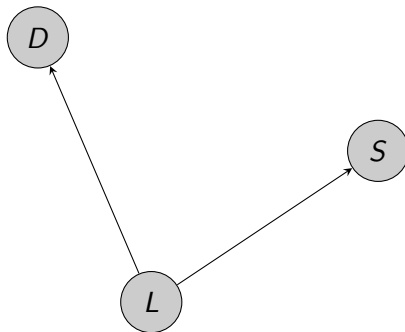
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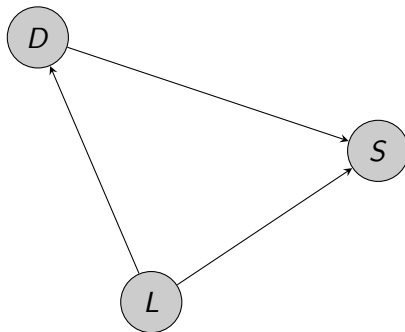
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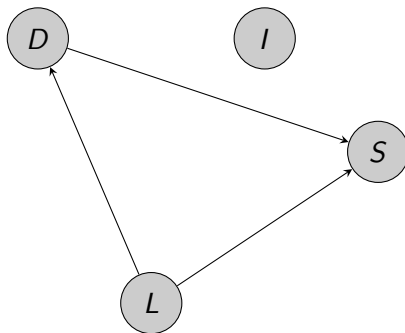
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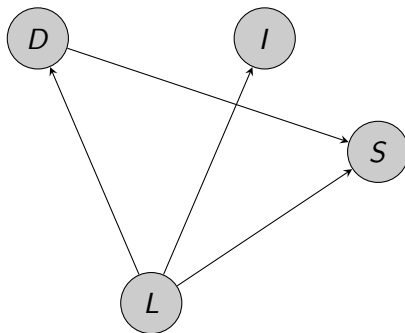
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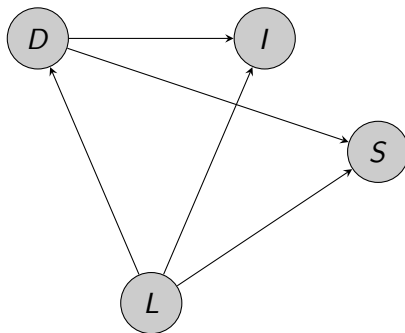
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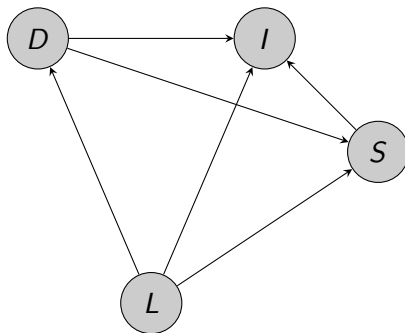
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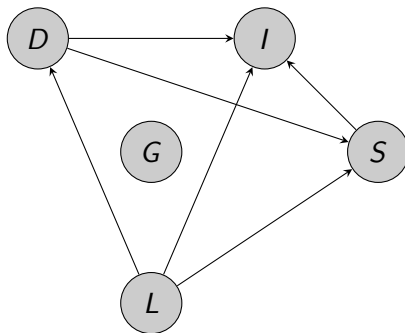
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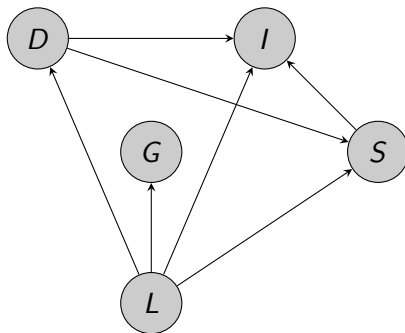
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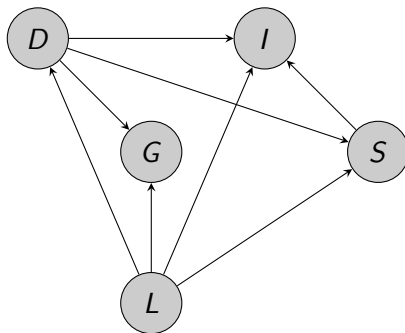
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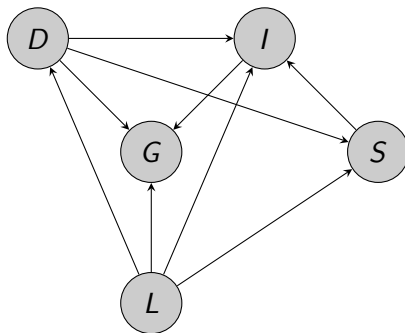
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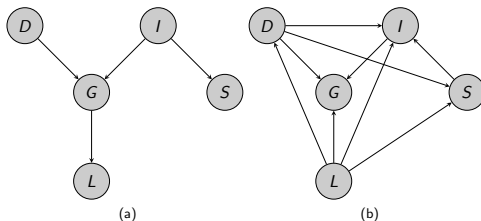
Constructing a Minimal I-Map



Constructing a Minimal I-Map



Construction of Minimal I-Maps

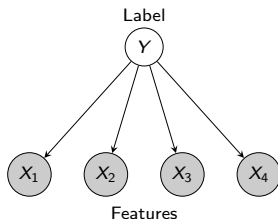


- Both graphs (a) and (b) are valid I-maps for G , but have completely different edge configurations.
- Ironically, we cannot “read off” the independence assertions from a minimal I-map. Even minimal I-maps fail to capture some or all of the independencies that hold in the distribution.

Constructing Minimal I-Maps

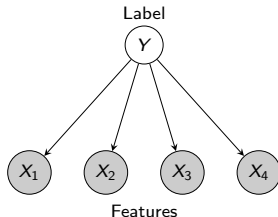
- A more restrictive definition called the perfect map (P-map) captures the independencies in a given distribution P .
- We say that a graph K is a P-map for P if $I(K) = I(P)$.
- Through an involved process, it is possible to construct a PDAG (partially directed acyclic graph) from a distribution P that encodes all P-maps in the I-equivalence class of P .
- Unfortunately, not all distributions have P-maps.

Email Classification [2]



- We now shift our focus to a few applications of Bayesian networks.
- To generate an email, recall that we can sample the variables in the Bayesian network in topological order.
 - First, we sample $y \sim P(Y)$ to decide whether or not the email is spam.
 - Then, $\forall i \in [1, n]$ sample $x_i \sim P(X_i \mid Y = y)$.

Email Classification [2]

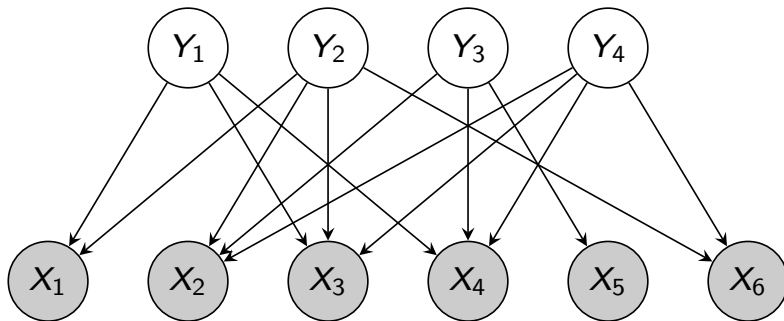


- To determine whether an email is spam given the features \mathbf{X}_i , we need to compute

$$\begin{aligned} P(Y | \mathbf{X}_i) &= \frac{P(Y, \mathbf{X}_i)}{P(\mathbf{X}_i)} \\ &= \frac{P(Y) \prod_{i=1}^n P(Y | X_i)}{\sum_{y \in Y} P(Y, \mathbf{X}_i)} \\ &= \frac{P(Y) \prod_{i=1}^n P(Y | X_i)}{\sum_{y \in Y} P(Y) \prod_{i=1}^n P(Y | X_i)}. \end{aligned}$$

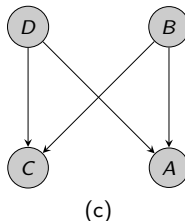
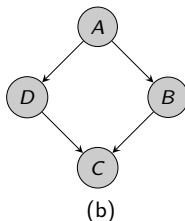
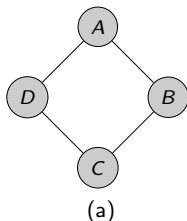
Naive Bayes [2]

Diseases (e.g. “pneumonia”, “flu”, “common cold”)



Findings (e.g. “cough”, “fever”, “fast breathing”)

Motivation



A professor misspeaks in class, causing four students to have a misconception. This model predicts whether or not each of them continues to carry the misconception after the students meet in pairs (Alice and Bob, Bob and Charles, Charles and Debbie, and Debbie and Alice).

- We want to encode $(A \perp C \mid D, B)$ and $(B \perp D \mid A, C)$. Why do (b) and (c) fail to capture these independencies?
- The fact that the a CPD involving a variable in a Bayesian network can only be over itself and its parents poses limitations.

What is a Markov Network?

- Similarly to Bayesian networks, a Markov network, or Markov random field (MRF) is an undirected graphical model with one node per variable.
- Unlike Bayesian networks, the non-negative potential functions (or *factors*) are associated with cliques of variables.

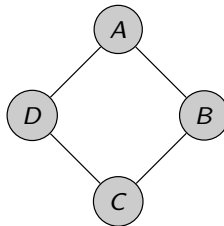
$$P(X_1, \dots, X_n) = \frac{1}{Z} \tilde{P}(X_1, \dots, X_n),$$

where the normalizing constant (or *partition function*) Z is defined as

$$Z = \sum_{X_1, \dots, X_n} \tilde{P}(X_1, \dots, X_n).$$

- A distribution of this kind is called a *Gibbs distribution*.

The Misconception Model [2]



- We introduce single-node potentials $\phi_A, \phi_B, \phi_C, \phi_D$ to represent probabilities that individuals correctly work out the misconceptions themselves.
- We introduce pairwise potentials $\phi_{AB}, \phi_{BC}, \phi_{CD}, \phi_{DA}$ to model whether partners agree after their meeting.

The Misconception Model [2]

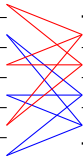
- The joint distribution is

$$P(A, B, C, D) = \frac{1}{Z} \phi(a) \phi(b) \phi(c) \phi(d) \\ \phi(a, b) \phi(b, c) \phi(c, d) \phi(d, a).$$

- Entries in the potentials need not be normalized; we can only judge relative importance by normalizing (scaling by Z) before comparison.

Factor Product

a^1	b^1	0.5
a^1	b^2	0.8
a^2	b^1	0.1
a^2	b^2	0
a^3	b^1	0.3
a^3	b^2	0.9



b^1	c^1	0.5
b^1	c^2	0.7
b^2	c^1	0.1
b^2	c^2	0.2

a^1	b^1	c^1	$0.50 \cdot 0.50 = 0.25$
a^1	b^1	c^2	$0.50 \cdot 0.70 = 0.35$
a^1	b^2	c^1	$0.80 \cdot 0.10 = 0.08$
a^1	b^2	c^2	$0.80 \cdot 0.20 = 0.16$
a^2	b^1	c^1	$0.10 \cdot 0.50 = 0.05$
a^2	b^1	c^2	$0.10 \cdot 0.70 = 0.07$
a^2	b^2	c^1	$0.00 \cdot 0.10 = 0.00$
a^2	b^2	c^2	$0.00 \cdot 0.20 = 0.10$
a^3	b^1	c^1	$0.30 \cdot 0.50 = 0.15$
a^3	b^1	c^2	$0.30 \cdot 0.70 = 0.21$
a^3	b^2	c^1	$0.90 \cdot 0.10 = 0.09$
a^3	b^2	c^2	$0.90 \cdot 0.20 = 0.18$

$$\phi(A, B) \times \phi(B, C) = \phi(A, B, C)$$

Independence [2]

- Consider a Markov network $A-B-C$ with the joint distribution

$$P(a, b, c) = \frac{1}{Z} \phi_{AB}(a, b) \phi_{BC}(b, c).$$

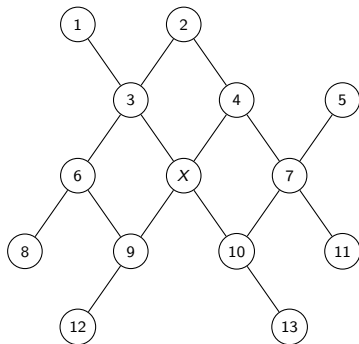
- First, we show that $P(a \mid b)$ can be computed using only $\phi_{AB}(a, b)$:

$$\begin{aligned} P(a \mid b) &= \frac{P(a, b)}{P(b)} \\ &= \frac{\frac{1}{Z} \sum_{c'} \phi_{AB}(a, b) \phi_{BC}(b, c')}{\frac{1}{Z} \sum_{a', c'} \phi_{AB}(a', b) \phi_{BC}(b, c')} \\ &= \frac{\phi_{AB}(a, b) \sum_{c'} \phi_{BC}(b, c')}{\sum_{a'} \phi_{AB}(a', b) \sum_{c'} \phi_{BC}(b, c')} \\ &= \frac{\phi_{AB}(a, b)}{\sum_{a'} \phi_{AB}(a', b)}. \end{aligned}$$

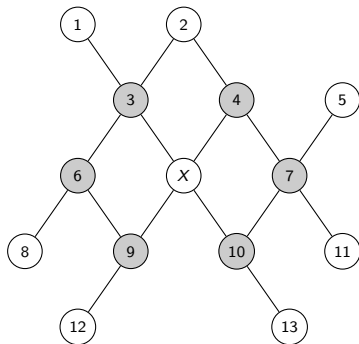
Independence [2]

- The probability of a variable conditioned on its neighbors depends *only* on the potentials involving that node.
- Observing even a single variable in a path between two variables impedes the flow of probabilistic influence, causing that path to become inactive.
- This means that a path $X_1 \cdots X_n$ is active given a subset of observed variables \mathbf{Z} if and only if none of the X_i 's is in \mathbf{Z} .

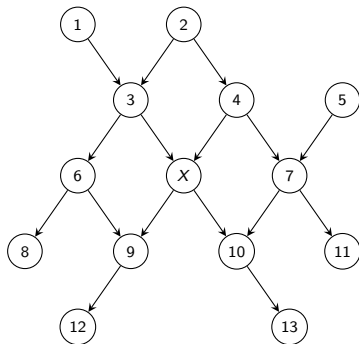
Markov Blanket: Undirected Graph



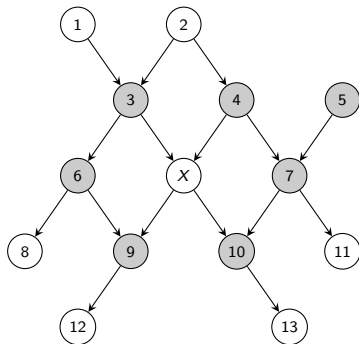
Markov Blanket: Undirected Graph



Markov Blanket: Directed Graph



Markov Blanket: Directed Graph



Equivalence of Local and Global Independencies

- We can now formulate a definition for $I_\ell(H)$ in terms of the Markov blankets of each $X \in H$:

$$I_\ell(H) = \{(X \perp \mathcal{X} - \{X\} - \text{MB}_H(X) \mid \text{MB}_H(X)) : X - Y \notin H\}.$$

- The following three statements are equivalent for a positive distribution P :
 1. $P \models I_\ell(H).$
 2. $P \models I_p(H).$
 3. $P \models I(H).$

Hammersley-Clifford Theorem

- Let P be a positive distribution over \mathcal{X} , and H a Markov network graph over \mathcal{X} . If H is an I-map for P , then P is a Gibbs distribution over H .
- For positive distributions, *all four conditions* — factorization and the three types of Markov assumptions — are all equivalent.

Log-Linear Models

- A distribution P is a log-linear model over a Markov network H if it is associated with
 - a set of features $F = \{f_1(\mathbf{D}_1), \dots, f_k(\mathbf{D}_k)\}$, where each \mathbf{D}_i is a complete subgraph in H , and
 - a set of weights w_1, \dots, w_k ,

such that

$$P(X_1, \dots, X_n) = \frac{1}{Z} \exp \left[- \sum_{i=1}^k w_i f_i(\mathbf{D}_i) \right].$$

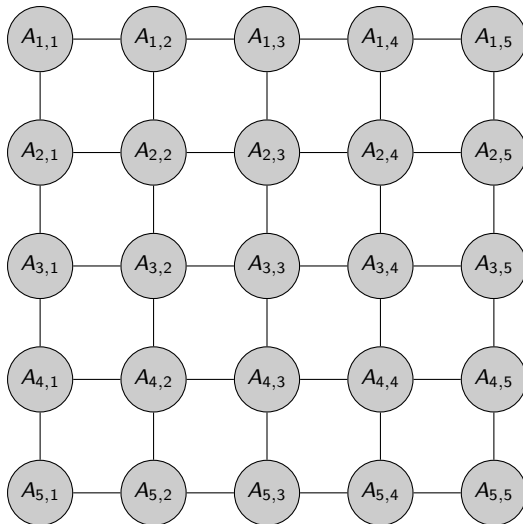
Log-Linear Models: The Ising Model




- The Ising model is a model for the energy of a particular system involving a system of interacting atoms.
- Each atom is associated with a binary variable $X_i = \{+1, -1\}$, whose value defines the direction of the atom's spin.
- The energy function associated with the edges is defined by

$$\epsilon_{ij}(x_i, x_j) = w_{i,j}x_i x_j.$$

- When two atoms X_i, X_j have the same spin, they make a contribution w_{ij} ; otherwise, they make a contribution $-w_{ij}$.
- We also include single node potentials u_i that bias a particular atom to have one spin or another.

Log-Linear Models: The Ising Model



-  Daphne Koller and Nir Friedman, *Probabilistic Graphical Models: Principles and Techniques*, MIT Press, 1st Edition, 2009.
-  David Sontag, lecture slides on Probabilistic Graphical Models. Accessible at <http://cs.nyu.edu/~dsontag/courses/pgm12/>.
-  Nir Friedman, lecture slides on the theory of Bayesian networks. Accessible at classes.soe.ucsc.edu/cmsps290c/Spring04/paps/nir2.pdf.