Consistency of linear system of equations :- If a system of linear equations has one or more solution, it is said to be Consistent, otherwise it is called inconsistent.

Consider the system of m linear equations Containing n unknowns $x_1, x_2 \ldots x_n$. To determine whether the linear equations are consistent or not, we Consider the Ranks of matrices.

Let $AX = B$ be matrix form of given system of eqn (1), where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ - & - & - & - \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ - \\ x_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ - \\ b_n \end{bmatrix}$$

then $$[A : B] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ - & - & - & - & - \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_n \end{bmatrix}$$ is called 'augmented matrix'

The matrix equation $AX = B$ need not always have a solution. It may have no solution or unique solution or infinite number of solutions.

Consistency theorem :- For system of non-homogeneous linear eqn $AX = B$

(I) If rank of augmented matrix $[A:B]$ = Rank of Coefficient matrix A ie, $\rho[A:B] = \rho(A)$, the system is Consistent.

(II) If $\rho[A:B] \neq \rho(A)$, the system is inconsistent and have no solution.

(III) If $\rho[A:B] = \rho(A) = n$ (number of variables), the system has unique solution.

(IV) If $\rho[A:B] = \rho(A) < n$, the system has infinite number of solutions.

For system of homogeneous linear eqn. $AX = 0$:-

(I) $X = 0$ is always a solution, that is $x_1 = x_2 = \cdots x_n = 0$ which is called trivial solution. Thus system of homogeneous linear equations is always consistent.

(II) If $\rho(A) = n$ (number of variables), the system has only trivial solution : $x_1 = x_2 = \cdots x_n = 0$.

(III) If $\rho(A) < n$, the system has infinite number of solutions.

So, homogeneous linear equations has either trivial solution or infinite solutions.

To test the consistency of system of equations, reduce the augmented matrix $[A:B]$ and coefficient matrix A into echelon form by elementry row operations and find their ranks. If $\rho[A:B] = \rho(A)$, the system is said to be consistent otherwise inconsistent.

Q.5. Test for Consistency and Solve : $x+2y-z=3$, $3x-y+2z=1$, $2x-2y+3z=2$, $x-y+z=-1$

Soln :- The given system of equations can be written in matrix form as $AX = B$

where $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ 1 \\ 2 \\ -1 \end{bmatrix}$

Augmented matrix is $[A:B] = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{bmatrix}$

operating $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$, $R_4 \to R_4 - R_1$; operating $R_2 \to R_2 - R_3$; operating $R_3 \to R_3 - 6R_2$, $R_4 \to R_4 - 3R_2$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & 5 & 20 \\ 0 & 0 & 2 & 8 \end{bmatrix}$$

$R_3 \to \frac{1}{5}R_3$, $R_4 \to \frac{1}{2}R_4$; $R_4 \to R_4 - R_3$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The no. of non-zero rows in the echelon form is 3. Hence $\rho[A:B] = 3$. Also $A \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ so $\rho(A) = 3$

Thus $\rho[A:B] = \rho(A)$, So the given system of eqn. is consistent.
Further we have $\rho[A:B] = \rho(A) = 3 =$ number of variables.
Therefore the system of eqn. has a unique solution.
Rewriting the equation from the augmented matrix, we have

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 4 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{aligned} x + 2y - z &= 3 \\ -y &= -4 \\ z &= 4 \end{aligned}\right\}$$

solving we get $x = -1$, $y = 4$, $z = 4$ is required solution.

Q.6. Test for Consistency and solve:

$$x + y + z = -3, \quad 3x + y - 2z = -2, \quad 2x + 4y + 7z = 7.$$

Soln:- The matrix form of given eqn is $AX = B$

where $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -2 \\ 2 & 4 & 7 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} -3 \\ -2 \\ 7 \end{bmatrix}$

and $[A:B] = \begin{bmatrix} 1 & 1 & 1 & -3 \\ 3 & 1 & -2 & -2 \\ 2 & 4 & 7 & 7 \end{bmatrix}$

operating $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$; operating $R_3 \to R_3 + R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & -3 \\ 0 & -2 & -5 & 7 \\ 0 & 2 & 5 & 13 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & -3 \\ 0 & -2 & -5 & 7 \\ 0 & 0 & 0 & 20 \end{bmatrix}$$

The no. of non-zero rows in Echelon form is 3. Hence $\rho[A:B] = 3$

But $A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -5 \\ 0 & 0 & 0 \end{bmatrix}$ So, $\rho(A) = 2$

Since $\rho[A:B] \neq \rho(A)$. Hence given system of equations is inconsistent. The system has no solution.

Q.7. Determine for what values of λ and μ the following equations have (i) no solution (ii) a unique solution (iii) infinite number of solutions :

$$x+y+z=6, \quad x+2y+3z=10, \quad x+2y+\lambda z=\mu$$

Soln :- The matrix form of the given eqn. is $AX=B$, where

$$A=\begin{bmatrix}1&1&1\\1&2&3\\1&2&\lambda\end{bmatrix}, \quad X=\begin{bmatrix}x\\y\\z\end{bmatrix}, \quad B=\begin{bmatrix}6\\10\\\mu\end{bmatrix}, \quad [A:B]=\begin{bmatrix}1&1&1&6\\1&2&3&10\\1&2&\lambda&\mu\end{bmatrix}$$

operating $R_2 \to R_2-R_1$, $R_3 \to R_3-R_1$; operating $R_3 \to R_3-R_2$

$$[A:B] \sim \begin{bmatrix}1&1&1&6\\0&1&2&4\\0&1&\lambda-1&\mu-6\end{bmatrix} \sim \begin{bmatrix}1&1&1&6\\0&1&2&4\\0&0&\lambda-3&\mu-10\end{bmatrix}$$

(i) There is no solution if $\rho(A) \neq \rho(A:B)$

$\Rightarrow \lambda-3=0$ or $\lambda=3$ and $\mu-10\neq 0$ or $\mu \neq 10$.

(ii) There is unique solution if $\rho(A)=\rho(A:B)=3=$ no. of variable

$\Rightarrow \lambda-3\neq 0$ or $\lambda\neq 3$ and μ may have any value.

(iii) There are infinite solutions if $\rho(A)=\rho(A:B)=2$

$\Rightarrow \lambda-3=0$ or $\lambda=3$ and $\mu-10=0$ or $\mu=10$

Q.8. For what values of 'K' the system of eqn. $x+y+z=2$, $x+2y+z=-2$, $x+y+(K-5)z=K$ has no solution.

Soln :- The matrix form of given eqn. is $AX=B$, where

$$A=\begin{bmatrix}1&1&1\\1&2&1\\1&1&K-5\end{bmatrix}, \quad X=\begin{bmatrix}x\\y\\z\end{bmatrix}, \quad B=\begin{bmatrix}2\\-2\\K\end{bmatrix} \text{ and}$$

$$[A:B]=\begin{bmatrix}1&1&1&2\\1&2&1&-2\\1&1&K-5&K\end{bmatrix} \sim \begin{bmatrix}1&1&1&2\\0&1&0&-4\\0&0&K-6&K-2\end{bmatrix} \quad \begin{matrix}R_2\to R_2-R_1\\R_3\to R_3-R_1\end{matrix}$$

The above system of eqn. has no solution if $\rho(A:B)\neq\rho(A)$

$\Rightarrow K-6=0$ or $K=6$ and $K-2\neq 0$ or $K\neq 2$.

Q.9. Determine 'b' such that the system of homogeneous equation

$$x+y+3z=0, \quad 2x+y+2z=0, \quad 4x+3y+bz=0$$

has non-trivial solution. Find the non-trivial solution.

Solu :- The matrix form of system of homogeneous eqn. is $AX=0$

where $A=\begin{bmatrix}1&1&3\\2&1&2\\4&3&b\end{bmatrix}$, $X=\begin{bmatrix}x\\y\\z\end{bmatrix}$ and $O=\begin{bmatrix}0\\0\\0\end{bmatrix}$

This homogeneous system will have non-trivial solution only if $|A|=0$. Thus for non-trivial solution, we have

$$\begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 2 \\ 4 & 3 & b \end{bmatrix} = 0 \Rightarrow 1(b-6) - 1(2b-8) + 3(6-4) = 0 \Rightarrow -b+8 = 0$$
$$\Rightarrow b = 8$$

Thus for non-trivial solution $b = 8$.

The coefficient matrix for non-trivial solution is

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 2 \\ 4 & 3 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -4 \\ 0 & -1 & -4 \end{bmatrix} \quad \text{operating } R_2 \to R_2 - 2R_1, \; R_3 \to R_3 - 4R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -4 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - R_4$$

The no. of non-zero rows in the echlon form is 2. So $\rho(A) = 2$.

Since $\rho(A) <$ number of variables (3). So the system has infinite number of solutions. Rewriting the equation in matrix form $AX = 0$ we have

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} x + y + 3z &= 0 \\ -y - 4z &= 0 \end{aligned}$$

Solving, $y = -4z$ and $x = z$. Taking $z = 1, 2, 3 \ldots$ we get infinite number of non-trivial solutions.