Ordinary Differentiation

The process of finding derivative of a function is called differentiation. The Concept of derivative is a basic tool of calculus. It helps to measure the instantaneous rate of change of one variable with respect to another variable. The maximum and minimum value of a function, tangents and normals to a curve are evaluated by using derivatives. It also helps determine partial derivative and total derivative of a function of two or more variables.

Derivative of a function at a point :- Let $y = f(x)$ be a function of x, then $y + \delta y = f(x+\delta x)$ where δx is increment in independent variable x and δy is corresponding increment in dependent variable y.

Then $\delta y = f(x+\delta x) - f(x)$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x}$$

Now $\frac{\delta y}{\delta x}$ represents average rate of change and this becomes instantaneous rate of change as $\delta x \to 0$.

Thus $$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

If the limit on right exists finitely then this limit is called derivative of $f(x)$ at the point x. and is denoted by $f'(x)$ or $$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

It is also known as 'first principle' of differentiation.

A function $f(x)$ is said to be differentiable at a point $x = a$ if L.H. derivative at $(x=a)$ = R.H. derivative at $(x=a)$

ie, $$\lim_{h \to 0} \frac{f(a-h) - f(a)}{-h} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

A function may not be derivable at all points in an interval. Thus $\sqrt{x}$ is not derivable at $x = 0$ Since $\lim_{h \to 0} \frac{\sqrt{0+h} - \sqrt{h}}{h}$ does not exist. However if a function is derivable at all points of interval $[a, b]$ then it is said to be derivable in the interval [a, b].

Imp. Results :- 1. Every differentiable function is Continuous but every continuous function is not differentiable.

2. Every polynomials, exponential and const. functions are differentiable.

3. logarithmic, trigonometric and inverse trigonometric functions are differentiable in their domain.

Rules of Derivative :- 1. Sum and difference rule

Let $y = f(x) \pm g(x)$, then $\frac{dy}{dx} = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$

2. Product rule :- Let $y = f(x) \cdot g(x)$

then $\frac{dy}{dx} = g(x) \cdot \frac{d}{dx}f(x) + f(x) \cdot \frac{d}{dx}g(x)$

3. Quotient rule :- Let $y = \frac{f(x)}{g(x)}$; $g(x) \neq 0$

then $\frac{dy}{dx} = \frac{g(x) \cdot \frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{[g(x)]^2}$

4. chain rule :- Let $y = f(u)$ and $u = f(x)$

then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

Some Std. derivatives :-

$\frac{d}{dx}(\text{Constant}) = 0$

$\frac{d}{dx}(x^n) = nx^{n-1}$

$\frac{d}{dx}(e^x) = e^x$

$\frac{d}{dx}\log_e x = \frac{1}{x}$; $x > 0$

$\frac{d}{dx}(a^x) = a^x \log_e a$

$\frac{d}{dx}\sin x = \cos x$

$\frac{d}{dx}\cos x = -\sin x$

$\frac{d}{dx}\tan x = \sec^2 x$

$\frac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x . \cot x$

$\frac{d}{dx}\sec x = \sec x . \tan x$

$\frac{d}{dx}\cot x = -\operatorname{cosec}^2 x$

$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$

$\frac{d}{dx}\cot^{-1} x = \frac{-1}{1+x^2}$

$\frac{d}{dx}\operatorname{cosec}^{-1} x = \frac{-1}{x\sqrt{x^2-1}}$

$\frac{d}{dx}\sec^{-1} x = \frac{1}{x\sqrt{x^2-1}}$

Derivative of Implicit functions :- Suppose $f(x,y)=0$ be a function in x and y which can not be expressed in the form $y=f(x)$, then such function is called implicit function. For example: Find $\frac{dy}{dx}$ for

$$x^2+2y = 2axy.$$

The given fun. is implicit fun. Differentiate both sides w.r.t. x, we get $2x+2\frac{dy}{dx} = 2a\left(x\frac{dy}{dx}+y\cdot 1\right)$

$$\Rightarrow \frac{dy}{dx}(2-2ax) = 2ay-2x$$

$$\Rightarrow \frac{dy}{dx} = \frac{ay-x}{1-ax}.$$

Derivative of parametric function :- A relation expressed between two variables x and y in the form $x=f(t)$, $y=g(t)$ is said to be parametric function with t as parameter. Then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \quad \left(\frac{dx}{dt} \neq 0\right)$$

For exp: Find $\frac{dy}{dx}$ for $x=t^3$, $y=t^2$

Here, $\frac{dx}{dt} = 3t^2$ and $\frac{dy}{dt} = 2t$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{3t^2} = \frac{2}{3t}$$

Logarithmic differentiation :- Let $y=u^v$. Taking natural logarithm (with base e) on both sides and differentiating w.r.t. x we get $\log y = v\log u$

or, $\frac{d}{dx}(\log y) = \frac{d}{dx}(v\log u) \Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = v\cdot\frac{1}{u}\frac{du}{dx} + \log u\cdot\frac{dv}{dx}$

For exp: Let $y=x^x$. $\Rightarrow \log y = \log x^x = x\log x$

Differentiating w.r.t x, $\frac{1}{y}\cdot\frac{dy}{dx} = x\cdot\frac{1}{x} + \log x\cdot 1$

$$\Rightarrow \frac{dy}{dx} = y(1+\log x) = x^x(1+\log x)$$

Differentiation of a function w.r.t. another function :- Let $y=f(x)$ and $z=g(x)$ are two functions, then

$$\frac{dy}{dz} = \frac{dy/dx}{dz/dx}.$$

For exp :- Differentiate $\sin^3 x$ w.r.t. $\cos^3 x$

Let $y=\sin^3 x$ and $z=\cos^3 x$

$$\therefore \frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{3\sin^2 x \cos x}{-3\cos^2 x . \sin x} = -\frac{\sin x}{\cos x} = -\tan x.$$

<u>Higher order derivatives</u> :- Let $y = f(x)$

then 2nd order derivative is $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$

3rd order derivative is $\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$

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nth order derivative is $\frac{d}{dx}\left(\frac{d^{n-1}}{dx^{n-1}}\right) = \frac{d^ny}{dx^n}$

(i) The nth derivative of $(ax+b)^m = m(m-1)(m-2)\text{----}(m-n+1) \times a^n(ax+b)^{m-n}$

(ii) " " $e^{ax} = a^n e^{ax}$

(iii) " " $\sin(ax+b) = a^n \sin(ax+b+\frac{n\pi}{2})$

(iv) " " $\cos(ax+b) = a^n \cos(ax+b+\frac{n\pi}{2})$

<u>Leibnitz's theorem</u> for nth Derivative of product of <u>two functions</u> :- Let u and v are function of x, then

$$\frac{d^n}{dx^n}(u.v) = {}^nC_0\, u_n . v + {}^nC_1\, u_{n-1} . v_1 + {}^nC_2\, u_{n-2} . v_2 + \text{---} \text{----} + {}^nC_r\, u_{n-r}\, v_r + \text{---} + {}^nC_n\, u . v_n$$

(Q) If $x = \tan(\log y)$, prove that $(1+x^2)y_{n+1} + (2nx-1)y_n + n(n-1)y_{n-1} = 0$

Given that $x = \tan(\log y)$

$\Rightarrow \tan^{-1}x = \log y \Rightarrow y = e^{\tan^{-1}x}$

Differentiating w.r.t. x, we get

$y_1 = e^{\tan^{-1}x}\left(\frac{1}{1+x^2}\right)$

or, $(1+x^2)y_1 = y$ — I

Differentiating eqn. I n times by Leibnitz theorem

$(1+x^2)y_{n+1} + ny_n(2x) + \frac{n(n-1)}{2}y_{n-1}(2) = y_n$

or, $(1+x^2)y_{n+1} + (2nx-1)y_n + n(n-1)y_{n-1} = 0$ proved.

Partial Derivatives:- Let $z=f(x,y)$ be a function of two variables x and y. If we keep y const. and x varies then z becomes a function of x only. The derivative of z w.r.t. x keeping y const. is called partial derivative of z w.r.t x and is denoted by symbol

$$\frac{\partial z}{\partial x} \text{ or } \frac{\partial f}{\partial x}$$

Then $\frac{\partial z}{\partial x} = \lim_{\partial x \to 0} \frac{f(x+\partial x, y)-f(x,y)}{\partial x}$

Similarly the partial derivative of z w.r.t. y keeping x as const. is denoted by $\frac{\partial z}{\partial y}$ or $\frac{\partial f}{\partial y}$

$$\frac{\partial z}{\partial y} = \lim_{\partial y \to 0} \frac{f(x,y+\partial y)-f(x,y)}{\partial y}$$

Some notation of partial derivatives:- $\frac{\partial z}{\partial x} = p$, $\frac{\partial z}{\partial y} = q$

$$\frac{\partial^2 z}{\partial x^2} = r, \quad \frac{\partial^2 z}{\partial x \partial y} = s, \quad \frac{\partial^2 z}{\partial y^2} = t.$$

Partial derivative of higher order:- $\frac{\partial^2 z}{\partial x^2}$ or $f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$

$$\frac{\partial^2 z}{\partial y^2} \text{ or } f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right), \quad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) \text{ or } f_{xy}$$

Note that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$

Q① If $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$ then find the value of

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}.$$

Soln:- Given $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$

$$\therefore \frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot \frac{1}{y} + \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right)$$

$$= \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}$$

$$x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2} \quad \text{— I}$$

Also $\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\left(-\frac{x}{y^2}\right) + \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = \frac{-x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}$

$$y\frac{\partial u}{\partial y} = -\frac{x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2} \quad \text{— II}$$

Adding eqn I and II, we have $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$ Ans

Q② If $u = x^2\tan^{-1}\frac{y}{x} - y^2\tan^{-1}\frac{x}{y}$

show that $\frac{\partial^2 u}{\partial y \partial x} = \frac{x^2-y^2}{x^2+y^2}$

Soln:- $z = x^2\tan^{-1}\frac{y}{x} - y^2\tan^{-1}\frac{x}{y}$

$$\frac{\partial z}{\partial x} = 2x\tan^{-1}\frac{y}{x} + x^2 \cdot \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) - y^2 \cdot \frac{1}{1+\frac{x^2}{y^2}}\left(\frac{1}{y}\right)$$

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or, $\frac{\partial z}{\partial x} = 2x \tan^{-1}\frac{y}{x} - \frac{x^2 y}{x^2+y^2} - \frac{y^3}{x^2+y^2}$

$= 2x\tan^{-1}\frac{y}{x} - \frac{y(x^2+y^2)}{x^2+y^2} = 2x\tan^{-1}\frac{y}{x} - y$

$\therefore \frac{\partial^2 z}{\partial y \partial x} = 2x \cdot \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x} - 1 = 2 \cdot \frac{x^2}{x^2+y^2} - 1$

$= \frac{x^2-y^2}{x^2+y^2}$ Proved

Q) If $u = e^{xyz}$, find value of $\frac{\partial^3 u}{\partial x \partial y \partial z}$

Soln :- $u = e^{xyz}$ $\therefore \frac{\partial u}{\partial z} = e^{xyz}(xy)$

$\frac{\partial^2 u}{\partial y \partial z} = e^{xyz}(x) + e^{xyz}(xz)(xy)$

$= e^{xyz}(x + x^2yz)$

and $\frac{\partial^3 u}{\partial x \partial y \partial z} = e^{xyz}(1+2xyz) + e^{xyz}(yz)\cdot(x+x^2yz)$

$= e^{xyz}[1+2xyz+xyz+x^2y^2z^2]$

$= e^{xyz}[1+3xyz+x^2y^2z^2]$ Ans

Q) If $z = f(x+ct) + \phi(x-ct)$ prove that $\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$

Soln :- we have $\frac{\partial z}{\partial x} = f'(x+ct) \cdot \frac{\partial}{\partial x}(x+ct) + \phi'(x-ct)\frac{\partial}{\partial x}(x-ct)$

$= f'(x+ct)\cdot 1 + \phi'(x-ct)\cdot 1$

$= f'(x+ct) + \phi'(x-ct)$

and $\frac{\partial^2 z}{\partial x^2} = f''(x+ct) + \phi''(x-ct)$ — I

Again $\frac{\partial z}{\partial t} = f'(x+ct)\frac{\partial}{\partial t}(x+ct) + \phi'(x-ct)\frac{\partial}{\partial t}(x-ct)$

$= c f'(x+ct) - c\phi'(x-ct)$

and $\frac{\partial^2 z}{\partial t^2} = c^2 f''(x+ct) + c^2\phi''(x-ct) = c^2[f''(x+ct) + \phi''(x-ct)]$ — II

From eqn I and II, we get $\frac{\partial^2 z}{\partial x^2} = c^2 \frac{\partial^2 z}{\partial t^2}$ Proved

Q) If $u = f(r)$ and $x = r\cos\theta$, $y = r\sin\theta$, prove that

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r}f'(r)$

Soln :- we have $\frac{\partial u}{\partial x} = f'(r)\cdot\frac{\partial r}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2} = f''(r)\cdot\left(\frac{\partial r}{\partial x}\right)^2 + f'(r)\cdot\frac{\partial^2 r}{\partial x^2}$

Similarly $\frac{\partial^2 u}{\partial y^2} = f''(r)\left(\frac{\partial r}{\partial y}\right)^2 + f'(r)\frac{\partial^2 r}{\partial y^2}$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r)\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right] + f'(r)\left[\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2}\right]$ — I

Now to find $\frac{\partial r}{\partial x}$ and $\frac{\partial r}{\partial y}$, we write $r = (x^2+y^2)^{\frac{1}{2}}$

$\therefore \frac{\partial r}{\partial x} = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2x = \frac{x}{r}$ and $\frac{\partial^2 r}{\partial x^2} = \frac{r(1) - x\cdot\frac{\partial r}{\partial x}}{r^2} = \frac{r - \frac{x^2}{r}}{r^2}$

$= \frac{y^2}{r^3}$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial^2 r}{\partial y^2} = \frac{x^2}{r^3}$

putting these values of $\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}$ etc in eqn I, we get

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = f''(r)\left[\frac{x^2}{r^2} + \frac{y^2}{r^2}\right] + f'(r)\left[\frac{y^2}{r^3} + \frac{x^2}{r^3}\right] = f''(r) + \frac{1}{r}f'(r)$

Proved

Homogeneous function :– An expression of the form

$$a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \cdots + a_n y^n$$

in which every term is of n^{th} order is called homogeneous function of degree n. This can be rewritten as

$$x^n \left[a_0 + a_1\left(\frac{y}{x}\right) + a_2\left(\frac{y}{x}\right)^2 + \cdots + a_n\left(\frac{y}{x}\right)^n\right]$$

Thus any function $f(x,y)$ which can be expressed in the form $x^n \phi\left(\frac{y}{x}\right)$ is called homogeneous function of degree n in x & y.

For exp: $x^3 \cos\left(\frac{y}{x}\right)$ is homogenous function of degree 3 in x & y.

Euler's theorem on homogeneous function :– If u be a homogeneous function of degree n in x and y, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

(Q) If $u = \sin^{-1}\frac{x^2+y^2}{x+y}$ prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$.

Soln :– Here u is not a homogeneous function. Let z be funct. of u

$$z = \sin u = \frac{x^2+y^2}{x+y} = x\frac{\left(1+\frac{y}{x}\right)^2}{\left(1+\frac{y}{x}\right)}$$

Thus z is homogeneous function of degree 1 in x and y. Hence by Euler's theorem

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 1.z = z \quad \text{— I}$$

But $\frac{\partial z}{\partial x} = \cos u.\frac{\partial u}{\partial x}$ and $\frac{\partial z}{\partial y} = \cos u\frac{\partial u}{\partial y}$

So eqn I becomes, $x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \sin u$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{\sin u}{\cos u} = \tan u \quad \text{Proved}$$

(Q) If $u = \log_e\left(\frac{x^4+y^4}{x+y}\right)$ show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3$

Soln :– Here u is not a homogeneous function,

$$\text{Let } z = e^u = \frac{x^4+y^4}{x+y} = \frac{x^4\left[1+\left(\frac{y}{x}\right)^4\right]}{x\left[1+\left(\frac{y}{x}\right)\right]} = x^3\phi\left(\frac{y}{x}\right)$$

Here z is homogeneous function of degree 3.

By Euler's theorem, $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3.z$ — I

where $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial x} = e^u.\frac{\partial u}{\partial x}$ and $\frac{\partial z}{\partial y} = e^u.\frac{\partial u}{\partial y}$

hence eqn I becomes $e^u\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] = 3.e^u \Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3$ Proved

(Q) If $u = \tan^{-1}\frac{x^3+y^3}{x-y}$ prove that $x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = \sin 2u$

Soln :– Here u is not homogeneous function.

$$\text{Let } z = \tan u = \frac{x^3+y^3}{x-y} = \frac{x^3\left[1+\left(\frac{y}{x}\right)^3\right]}{x\left[1-\frac{y}{x}\right]} = x^2.\frac{1+\left(\frac{y}{x}\right)^3}{1-\left(\frac{y}{x}\right)}$$

$$= x^2.\phi\left(\frac{y}{x}\right)$$

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So that z is homogeneous function of x, y of order 2.

By Euler's theorem, $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y} = 2.z$ — I

when $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} = \sec^2 u.\frac{\partial u}{\partial x}$

and $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} = \sec^2 u.\frac{\partial u}{\partial y}$

putting values of $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ in equ I, we get

$$\sec^2 u\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right] = 2.\tan u$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \frac{2\sin u.\cos^2 u}{\cos u} = 2\sin u\cos u = \sin 2u$$ Proved

(Q) If z is a homogeneous function of degree n in x and y show that $x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)z$.

Soln:- By Euler's theorem, $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y} = nz$ — I

Differentiating I partially w.r.t. x, we get

$$x\frac{\partial^2 z}{\partial x^2}+\frac{\partial z}{\partial x}+y\frac{\partial^2 z}{\partial x\partial y} = n\frac{\partial z}{\partial x}$$

$$\Rightarrow x\frac{\partial^2 z}{\partial x^2}+y\frac{\partial^2 z}{\partial x\partial y} = (n-1)\frac{\partial z}{\partial x} \quad \text{— II}$$

Again differentiating I partially w.r.t. y, we get

$$x\frac{\partial^2 z}{\partial y\partial x}+\frac{\partial z}{\partial x}+y\frac{\partial^2 z}{\partial y^2} = n\frac{\partial z}{\partial y}$$

$$\Rightarrow x\frac{\partial^2 z}{\partial x\partial y}+y\frac{\partial^2 z}{\partial y^2} = (n-1)\frac{\partial z}{\partial y} \quad \text{— III}$$

multiplying II by x and III by y and adding, we get

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2} = (n-1)\left(x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right)$$

$$= (n-1)nz = n(n-1)z$$

Proved

Total Differentiation :- In partial diff. of a function of two or more variables, only one variable varies. But in total differentiation, increments are given in all the variables.

Let $z=f(x,y)$. If $\partial x, \partial y$ be increments in x and y respectively. Let ∂z be corresponding increment in z, then total diff. Coefficient, $dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$

Proof :- Let $z=f(x,y)$ — I and $z+\partial z = f(x+\partial x, y+\partial y)$ — II

II−I, $\partial z = f(x+\partial x, y+\partial y) - f(x,y)$ — III

Adding and Subtracting $f(x, y+\partial y)$ on R.H.S of III

$$\partial z = f(x+\partial x, y+\partial y) - f(x, y+\partial y) + f(x, y+\partial y) - f(x,y)$$

$$\Rightarrow \partial z = \frac{f(x+\partial x, y+\partial y) - f(x, y+\partial y)}{\partial x}\partial x + \frac{f(x,y+\partial y) - f(x,y)}{\partial y}\partial y$$

on taking limit $\partial x \to 0, \partial y \to 0$

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

Differentiation of Composite function :- Let $z=f(x,y)$ where $x=\phi(t)$ and $y=\psi(t)$. So z is a Composite function of t.

Dividing dz by dt in the total diff. formula, we get

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

Thus $\frac{dz}{dt}$ is Called total diff. Coefficient of z.

Corollary :- Let $z=f(x,y)$ where $x=\phi(u,v)$ and $y=\psi(u,v)$ then from above formula, we get

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u}$$

and $$\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v}$$

Q (1) If $u = x^3+y^3$ where $x = a\cos t$, $y = b\sin t$ find $\frac{du}{dt}$

Soln :- Given $u = x^3+y^3$, $x = a\cos t$ and $y = b\sin t$

$$\therefore \frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$$= (3x^2)(-a\sin t) + (3y^2)(b\cos t)$$

$$= -3a^3\cos^2 t\sin t + 3b^3\sin^2 t\cos t$$ Ans

Q.(2) Find $\frac{du}{dt}$ if $u = x^3y^2 + x^2y^3$, where $x = at^2$, $y = 2at$

Soln :- Given $u = x^3y^2 + x^2y^3$

$$\therefore \frac{\partial u}{\partial x} = 3x^2y^2 + 2xy^3$$

and $$\frac{\partial u}{\partial y} = 2x^3y + 3x^2y^2$$

Also given $x = at^2$, $y = 2at$

then $\frac{dx}{dt} = 2at$, $\frac{dy}{dt} = 2a$

we know that $\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$

$$\therefore \frac{du}{dt} = (3x^2y^2 + 2xy^3)2at + (2x^3y + 3x^2y^2)2a$$

$$= [3(at^2)^2(2at)^2 + 2(at^2)(2at)^3]2at$$

$$+ [2(at^2)^2(2at) + 3(at^2)^2(2at)^2]2a$$

$$= 8a^5t^6(4t+7)$$

Q.(3) If $u = e^x y z^2$, find du

Soln :- we have $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$

Therefore, $du = (e^x y z^2)dx + e^x z^2 dy + 2e^x y z dz$

Q.(4) If $u = xy + yz + zx$, where $x = \frac{1}{t}$, $y = e^t$ and $z = e^{-t}$ find du/dt.

Soln :- Given $u = xy + yz + zx$

then $\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$

$$= (y+z)\left(-\frac{1}{t^2}\right) + (x+z)e^t + (y+x)(-e^{-t})$$

$$= -\left(\frac{e^t + e^{-t}}{t^2}\right) + \left(\frac{1}{t} + e^{-t}\right)e^t - e^{-t}\left(e^t + \frac{1}{t}\right)$$

(Replacing values of x, y, z)

$$= -\frac{1}{t^2}(e^t + e^{-t}) + \frac{1}{t}(e^t - e^{-t})$$