Limit :- The limit of a function is a fundamental concept in the study of Differential Calculus. Here we shall study the meaning of limit and methods of evaluating limits of various kinds.

Let $f(x)$ be a function of x such that $f(x) = \frac{x^2-4}{x-2}$. At $x=2$, $f(2) = \frac{0}{0}$ which is meaningless.

We observe that when x approaches 2 either from left or right, the value of $f(x)$ goes closer and closer to 4.

x	$f(x) = \frac{x^2-4}{x-2}$
1.9	3.9
1.99	3.99
1.999	3.999
	4
2.001	4.001
2.01	4.01
2.1	4.1

We denote as

$$\lim_{x \to 2} \frac{x^2-4}{x-2} = 4$$

We can define limit of a function at a point as follows:-

A function $f(x)$ is said to tend to limit l as x tends to a if $|f(x)-l| \to 0$ as $|x-a| \to 0$.

Which is denoted as $\lim_{x \to a} f(x) = l$

Properties of limits :-

(I) $\lim_{x \to a} c.f(x) = c.\lim_{x \to a} f(x)$; where c is a Constant.

(II) $\lim_{x \to a} (f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$

(III) $\lim_{x \to a} (f(x) \cdot g(x)) = \lim_{x \to a} f(x) . \lim_{x \to a} g(x)$

(IV) $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$; provided $\lim_{x \to a} g(x) \neq 0$

(V) $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$, provided $\sqrt[n]{\lim_{x \to a} f(x)}$ is real number.

Some Important Limits :-

(i) $\lim_{x\to 0} \frac{\sin x}{x} = 1$ (ii) $\lim_{x\to 0} \frac{x}{\sin x} = 1$

(iii) $\lim_{x\to 0} \frac{\tan x}{x} = 1$ (iv) $\lim_{x\to a} \frac{x^n - a^n}{x-a} = na^{n-1}$

(v) $\lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a$ (vi) $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$

Methods of evaluating limits :-

1. Evaluation of $\lim_{x\to a} \frac{f(x)}{g(x)}$; where $\lim_{x\to a} f(x) = 0$ & $\lim_{x\to a} g(x) = 0$

(a) Factorise $f(x)$ and $g(x)$ and cancell the Common factor in $\frac{f(x)}{g(x)}$ and then apply limits.

For exp : Evaluate $\lim_{x\to 2} \frac{x^2-4}{x-2}$

At $x = 2$, $\frac{x^2-4}{x-2} = \frac{4-4}{2-2} = \frac{0}{0}$ which is meaningless.

Factorising Numerator $(x^2-4) = (x+2)(x-2)$

we get $\lim_{x\to 2} \frac{(x+2)(x-2)}{(x-2)} = \lim_{x\to 2} (x+2) = 4$ Ans

(b) method of Substitution :- Put $x = a+h$ in $\frac{f(x)}{g(x)}$ and Simplify $\frac{f(a+h)}{g(a+h)}$. Apply limits then.

For Exp : Evaluate $\lim_{x\to 2} \frac{x^2-3x+2}{x^2-x-2}$.

At $x=2$, $\frac{x^2-3x+2}{x^2-x-2} = \frac{0}{0}$. Let $x = 2+h$ $\therefore h\to 0$ as $x\to 2$

$$\therefore \lim_{x\to 2} \frac{x^2-3x+2}{x^2-x-2} = \lim_{h\to 0} \frac{(2+h)^2 - 3(2+h)+2}{(2+h)^2-(2+h)-2} = \lim_{h\to 0} \frac{h+1}{h+3}$$

$$= \frac{\lim_{h\to 0}(h+1)}{\lim_{h\to 0}(h+3)} = \frac{0+1}{0+3} = \frac{1}{3} \quad \text{Ans}$$

(c) method of rationalisation :— Rationalise $\frac{f(x)}{g(x)}$ which involve square roots. Simplify Num. and Deno. and then apply limits. For exp :- Evaluate $\lim_{x\to 0} \frac{\sqrt{2-x}-\sqrt{2+x}}{x}$

At $x=0$, given limit is $\frac{0}{0}$ which is meaningless or indeterminate.

$$\therefore \lim_{x\to 0} \frac{\sqrt{2-x}-\sqrt{2+x}}{x} = \lim_{x\to 0}\left(\frac{\sqrt{2-x}-\sqrt{2+x}}{x} \times \frac{\sqrt{2-x}+\sqrt{2+x}}{\sqrt{2-x}+\sqrt{2+x}}\right)$$

$$= \lim_{x\to 0} \frac{-2}{\sqrt{2-x}+\sqrt{2+x}} = \frac{-2}{2\sqrt{2}} = -\frac{1}{\sqrt{2}} \quad \text{Ans}$$

2. Evaluation of $\lim_{x\to a} \frac{x^n-a^n}{x-a} = na^{n-1}; a>0$

For Exp :- Evaluate $\lim_{x\to 2} \frac{x^{10}-1024}{x^4-16}$.

The given limit can be written as $\lim_{x\to 2} \frac{\frac{x^{10}-2^{10}}{x-2}}{\frac{x^4-2^4}{x-2}}$

$$= \frac{\lim_{x\to 2} \frac{x^{10}-2^{10}}{x-2}}{\lim_{x\to 2} \frac{x^4-2^4}{x-2}} = \frac{10.2^{10-1}}{4.2^{4-1}} = 160 \quad \underline{\text{Ans}}$$

3. Evaluation of limit as $\lim_{x\to 0} \frac{e^x-1}{x} = 1$

For Exp :- Evaluate $\lim_{x\to 0} \frac{e^{2x}-1}{x}$

Soln :- $\lim_{x\to 0} \frac{e^{2x}-1}{x} = \lim_{x\to 0}\left(\frac{e^{2x}-1}{2x}\times 2\right) = 2.\lim_{2x\to 0}\left(\frac{e^{2x}-1}{2x}\right) = 2\times 1 = 2$

4. Evaluation of Trigonometrical limits.

Exp :- Evaluate (i) $\lim_{x\to 0} \frac{\sin ax}{\sin bx}$ (ii) $\lim_{x\to 0} \frac{\tan 8x}{\sin 2x}$

Soln :- (i) $\lim_{x\to 0} \frac{\sin ax}{\sin bx} = \lim_{x\to 0} \frac{\frac{\sin ax}{ax}.ax}{\frac{\sin bx}{bx}.bx} = \frac{a}{b}.\frac{\lim_{ax\to 0}\frac{\sin ax}{ax}}{\lim_{bx\to 0}\frac{\sin bx}{bx}}$

$$= \frac{a}{b}.\frac{1}{1} = \frac{a}{b} \quad \underline{\text{Ans}}$$

(ii) $\lim_{x\to 0} \frac{\tan 8x}{\sin 2x} = \lim_{x\to 0}\left[\frac{\tan 8x}{8x}\times\frac{2x}{\sin 2x}\times\frac{8}{2}\right]$

$$= 4.\lim_{8x\to 0}\frac{\tan 8x}{8x}.\lim_{2x\to 0}\frac{2x}{\sin 2x}$$

$$= 4\times 1\times 1 = 4 \qquad \left(\because \lim_{x\to 0}\frac{x}{\sin x} = 1 \ \& \lim_{x\to 0}\frac{\tan x}{x} = 1\right)$$

(iii) Evaluate $\lim_{\theta\to 0} \frac{1-\cos n\theta}{1-\cos m\theta}$

Soln :- $\lim_{\theta\to 0} \frac{1-\cos n\theta}{1-\cos m\theta} = \lim_{\theta\to 0} \frac{1-(1-2\sin^2\frac{n\theta}{2})}{1-(1-2\sin^2\frac{m\theta}{2})}$

$$= \lim_{\theta\to 0} \frac{\sin^2\frac{n\theta}{2}}{\sin^2\frac{m\theta}{2}} = \lim_{\theta\to 0}\left(\frac{\sin\frac{n\theta}{2}}{\frac{n\theta}{2}}\right)^2\left(\frac{m\theta/2}{\sin\frac{m\theta}{2}}\right)^2\left(\frac{\frac{n\theta}{2}}{\frac{m\theta}{2}}\right)^2$$

$$= \left(\lim_{\theta\to 0}\frac{\sin\frac{n\theta}{2}}{\frac{n\theta}{2}}\right)^2.\left(\lim_{\theta\to 0}\frac{\frac{m\theta}{2}}{\sin\frac{m\theta}{2}}\right)^2.\left(\lim_{\theta\to 0}\frac{n\theta/2}{m\theta/2}\right)^2$$

$$= 1^2\times 1^2\times\frac{n^2}{m^2} = \frac{n^2}{m^2} \quad \underline{\text{Ans}}$$

5. Evaluation of limits at infinity :-

Exp :- Evaluate (i) $\lim_{x\to\infty} \frac{1}{x}$ (ii) $\lim_{x\to\infty} \frac{4x^2+5x+6}{3x^2+4x+5}$

(iii) $\lim_{x\to\infty} \left(\sqrt{x+2}-\sqrt{x}\right)$ (iv) $\lim_{x\to\infty} \left(1+\frac{1}{x^2}\right)^x$

Solution :- (i) $\lim_{x\to\infty} \frac{1}{x}$. The variable x can be made as large as we like.

$\therefore \frac{1}{x}$ grows smaller and smaller as x becomes larger and larger.

$\therefore \frac{1}{x} \to 0$, so $\lim_{x\to\infty} \frac{1}{x} = 0$ Ans

(ii) $\lim_{x\to\infty} \frac{4x^2+5x+6}{3x^2+4x+5}$. Here degree of num = degree of den. = 2

thus we divide num. and deno. by x^2.

$$= \lim_{x\to\infty} \frac{\frac{4x^2}{x^2}+\frac{5x}{x^2}+\frac{6}{x^2}}{\frac{3x^2}{x^2}+\frac{4x}{x^2}+\frac{5}{x^2}} = \lim_{x\to\infty} \frac{4+\frac{5}{x}+\frac{6}{x^2}}{3+\frac{4}{x}+\frac{5}{x^2}}$$

$$= \frac{4+0+0}{3+0+0} = \frac{4}{3} \text{ Ans}$$

(iii) $\lim_{x\to\infty} \left(\sqrt{x+2}-\sqrt{x}\right) = \lim_{x\to\infty} \left(\sqrt{x+2}-\sqrt{x} \times \frac{\sqrt{x+2}+\sqrt{x}}{\sqrt{x+2}+\sqrt{x}}\right)$

$$= \lim_{x\to\infty} \frac{(x+2)-x}{\sqrt{x+2}+\sqrt{x}} = \lim_{x\to\infty} \frac{2}{\sqrt{x+2}+\sqrt{x}}$$

$$= \lim_{x\to\infty} \frac{\frac{2}{\sqrt{x}}}{\sqrt{\frac{x+2}{x}}+\sqrt{\frac{x}{x}}} = \lim_{x\to\infty} \frac{\frac{2}{\sqrt{x}}}{\sqrt{1+\frac{2}{x}}+1} = \frac{0}{\sqrt{1+0}+1} = \frac{0}{2} = 0$$

(iv) $\lim_{x\to\infty} \left(1+\frac{1}{x^2}\right)^x = \lim_{x\to\infty} \left[\left(1+\frac{1}{x^2}\right)^{x^2}\right]^{\frac{1}{x}} = \lim_{x\to\infty} e^{\frac{1}{x}} = 1$ Ans

$\left\{\because \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e\right.$

Indeterminate Forms

The following are called indeterminate forms :-

$\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, \infty-\infty, \infty^0, 1^\infty$

L'Hospital rule for Evaluating indeterminate forms :-

Let $f(x)$ and $\phi(x)$ be two functions, such that $f(a)=0, \phi(a)=0$ then according to L'Hospital rule,

$$\lim_{x\to a}\frac{f(x)}{\phi(x)} = \frac{f'(a)}{\phi'(a)}$$

Again if $f(a)=0, \phi(a)=0, f'(a)=0$ and $\phi'(a)=0$

then $\lim_{x\to a}\frac{f(x)}{\phi(x)} = \frac{f''(a)}{\phi''(a)}$

Similarly $\lim_{x\to a}\frac{f(x)}{\phi(x)} = \frac{f'''(a)}{\phi'''(a)}$, if $f''(a)=0$ & $\phi''(a)=0$

Procedure: (i) Let $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\infty}{\infty}$

we write it as $\frac{\frac{1}{g(x)}}{\frac{1}{f(x)}}$ which becomes $\frac{0}{0}$ as $x\to\infty$

then apply L'Hospital rule.

(ii) Let $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=\infty$

then $\lim_{x\to a} f(x)\times g(x) = \lim_{x\to a}\frac{f(x)}{\frac{1}{g(x)}} = \frac{0}{0}$ then apply L'Hospital rule.

Q Evaluate $\lim_{x\to 1}\frac{x^5-2x^3-4x^2+9x-4}{x^4-2x^3+2x-1}$ $\left[\frac{0}{0}\text{form}\right]$

Apply L'Hospital rule.

$= \lim_{x\to 1}\frac{5x^4-6x^2-8x+9}{4x^3-6x^2+2}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim_{x\to 1}\frac{20x^3-12x-8}{12x^2-12x}$ $= \left[\frac{0}{0}\right]$

$= \lim_{x\to 1}\frac{60x^2-12}{24x-12} = \frac{48}{12} = 4$ Ans

Q Evaluate $\lim_{x\to 0}\frac{\log\tan x}{\log x}$ $\left[\frac{0}{0}\text{ form}\right]$

Apply L'Hospital rule, $= \lim_{x\to 0}\frac{\frac{1}{\tan x}\cdot\sec^2 x}{\frac{1}{x}} = \lim_{x\to 0}\frac{x}{\sin x.\cos x}$

$= \lim_{x\to 0}\frac{2x}{\sin 2x} = 1$ Ans

Q Evaluate $\lim_{x\to 0}\left(\frac{1}{x}-\cot x\right)$

$= \lim_{x\to 0}\left(\frac{1}{x}-\frac{\cos x}{\sin x}\right) = \lim_{x\to 0}\left(\frac{\sin x - x\cos x}{x\sin x}\right)$ $\left[\frac{0}{0}\text{form}\right]$

Applying L Hospital rule, $= \lim_{x\to 0}\frac{\cos x-\cos x-x(-\sin x)}{x\cos x+\sin x}$ $\left[\frac{0}{0}\text{ "}\right]$

$= \lim_{x\to 0}\frac{x\sin x}{x\cos x+\sin x}$ $\left[\frac{0}{0}\text{form}\right]$

$= \lim_{x\to 0}\frac{x\cos x+\sin x}{-x\sin x+\cos x+\cos x} = \frac{0+0}{0+1+1} = \frac{0}{2} = 0$ Ans

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(5) Evaluate $\lim_{x\to 1} (2-x)^{\tan \frac{\pi x}{2}}$

Let $y = \lim_{x\to 1} (2-x)^{\tan \frac{\pi x}{2}}$

Taking log both sides,

$$\log y = \lim_{x\to 1} \tan \frac{\pi x}{2} \cdot \log(2-x)$$

$$= \lim_{x\to 1} \frac{\log(2-x)}{\cot \frac{\pi x}{2}} \quad \left[\frac{0}{0} \text{ form}\right]$$

Apply L'Hospital rule

$$\log y = \lim_{x\to 1} \frac{-\frac{1}{2-x}}{-\frac{\pi}{2} \operatorname{cosec}^2 \frac{\pi x}{2}} = \frac{-1}{-\frac{\pi}{2}} = \frac{2}{\pi}$$

$$\therefore y = e^{\frac{2}{\pi}} \Rightarrow \lim_{x\to 1} (2-x)^{\tan \frac{\pi x}{2}} = e^{2/\pi} \quad \underline{\text{Ans}}$$

(6) Evaluate $\lim_{x\to 0} \frac{\log x}{\cot x}$

We have $\lim_{x\to 0} \frac{\log x}{\cot x}$ $\left[\frac{\infty}{\infty} \text{ form}\right]$ since $\log 0 = \infty$

Applying L'Hospital rule, $= \lim_{x\to 0} \frac{\frac{1}{x}}{-\operatorname{cosec}^2 x}$ $\left(\frac{\infty}{\infty} \text{ form}\right)$

$$= \lim_{x\to 0} -\frac{\sin^2 x}{x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} -\frac{2\sin x \cos x}{1} = 0 \quad \underline{\text{Ans}}$$

(7) Evaluate $\lim_{x\to \frac{\pi}{2}} \frac{\log\left(x-\frac{\pi}{2}\right)}{\tan x}$

The given limit is $\left(\frac{\infty}{\infty} \text{ form}\right)$ using L'Hospital rule, we have

$$= \lim_{x\to \frac{\pi}{2}} \frac{\frac{1}{x-\frac{\pi}{2}}}{\sec^2 x} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x\to \frac{\pi}{2}} \frac{\cos^2 x}{x-\frac{\pi}{2}} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to \frac{\pi}{2}} -\frac{2\cos x \sin x}{1} = 0 \quad \underline{\text{Ans}}$$

(8) Evaluate $\lim_{x\to\infty} x \tan \frac{1}{x}$

For given limit, we have $\lim_{x\to\infty} x \tan \frac{1}{x}$ $(0 \times \infty \text{ form})$

$$= \lim_{x\to\infty} \frac{\tan \frac{1}{x}}{\frac{1}{x}} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{h\to 0} \frac{\tan h}{h}, \text{ Taking } h = \frac{1}{x}$$

$$= 1 \quad \underline{\text{Ans}}$$

Continuity of a function

A function $f(x)$ is said to be Continuous at $x=a$ if

(i) $f(a)$ exist

(ii) Left hand limit $\displaystyle\lim_{x\to a-0} f(x) =$ Right hand limit $\displaystyle\lim_{x\to a+0} f(x) = f(a)$

Q. Is the function $f(x) = x\sin\frac{1}{x}$, $x \neq 0$ and $f(0)=0$ Continuous at the origin.

Soln :- Given $f(x) = x\sin\frac{1}{x}$

$$\text{L.H.}\lim_{x\to 0-0} f(x) = \lim_{x\to 0-h} x\sin\frac{1}{x} = \lim_{h\to 0}(0-h)\sin\frac{1}{0-h}$$
$$= \lim_{h\to 0} h\sin\frac{1}{h} = 0$$

$$\text{R.H.}\lim_{x\to 0+0} f(x) = \lim_{x\to 0+h} x\sin\frac{1}{x} = \lim_{h\to 0}(0+h)\sin\frac{1}{0+h}$$
$$= \lim_{h\to 0} h\sin\frac{1}{h} = 0$$

~~Now~~, $\text{L.H.}\displaystyle\lim_{x\to 0-0} f(x) = \text{R.H.}\lim_{x\to 0+0} f(x) = f(0) = 0$

Hence the given function is Continuous at $x=0$ Ans

Q. A function $f(x)$ is defined as

$$f(x) = -x, \text{ when } x \le 0$$
$$= x, \text{ when } 0 < x < 1$$
$$= 2-x, \text{ when } x \ge 1$$

Show that $f(x)$ is Continuous at $x=0$ and also at $x=1$

Soln :- (i) For Continuity at $x=0$

$$\text{L.H.}\lim_{x\to 0-0} f(x) = \lim_{x\to 0-h}(-x) = \lim_{h\to 0} h = 0$$
$$\text{R.H.}\lim_{x\to 0+0} f(x) = \lim_{x\to 0+h}(x) = \lim_{h\to 0} h = 0$$

and $f(0) = 0$

Since $\text{L.H.}\displaystyle\lim_{x\to 0} f(x) = \text{R.H}\lim_{x\to 0} f(x) = f(0)$

Hence given function $f(x)$ is Continuous at $x=0$ Ans

(ii) For Continuity at $x=1$

$$\text{L.H.}\lim_{x\to 1-0} f(x) = \lim_{x\to 1-h} x = \lim_{h\to 0}(1-h) = 1$$
$$\text{R.H.}\lim_{x\to 1+0} f(x) = \lim_{x\to 1+h}(2-x) = \lim_{h\to 0}[2-(1+h)] = 1$$

and $f(1) = 2-1 = 1$

Since $\text{L.H.}\displaystyle\lim_{x\to 1} f(x) = \text{R.H}\lim_{x\to 1} f(x) = f(1) = 1$

$\therefore$ The given function $f(x)$ is Continuous at $x=1$ Ans