Taylor's theorem for Single variable

Let $f(x)$ be a function of x. If the function $f(x+h)$ can be expanded in Convergent Series of positive integral powers of h, then the expansion is given by

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{\lfloor 2}f''(x) + \frac{h^3}{\lfloor 3}f'''(x) + \cdots + \frac{h^n}{\lfloor n}f^n(x) + \cdots$$

Proof :- Let $f(x+h) = A_0 + A_1h + A_2h^2 + A_3h^3 + \cdots + A_nh^n + \cdots$ —(1)

where $A_0, A_1, A_2 \cdots$ are Constants to be determined.

Differentiating eqn.(1) successively w.r.t. h, we get

$$f'(x+h) = A_1 + 2A_2h + 3A_3h^2 + \cdots + nA_nh^{n-1} + \cdots$$
$$f''(x+h) = 2A_2 + 3\times 2A_3h + \cdots + n(n-1)A_nh^{n-2} + \cdots$$
$$f'''(x+h) = 3\times 2A_3 + \cdots + n(n-1)(n-2)A_nh^{n-3} + \cdots$$

Putting $h=0$ in each of the above eqns. we get

$$A_0 = f(x),\ A_1 = f'(x),\ A_2 = \frac{f''(x)}{\lfloor 2} \cdots A_n = \frac{f^n(x)}{\lfloor n}$$

Substituting these values in eqn (1), we have

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{\lfloor 2}f''(x) + \frac{h^3}{\lfloor 3}f'''(x) + \cdots + \frac{h^n}{\lfloor n}f^n(x) + \cdots \quad (2)$$

which proves the theorem.

Deductions :- (1) Interchanging h and x in equation (2) we get expansion in powers of x as

$$f(x+h) = f(h) + xf'(h) + \frac{x^2}{\lfloor 2}f''(h) + \frac{x^3}{\lfloor 3}f'''(h) + \cdots + \frac{x^n}{\lfloor n}f^n(h) + \cdots$$

(2) If we replace x by a in eqn.(2), we get

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{\lfloor 2}f''(a) + \frac{h^3}{\lfloor 3}f'''(a) + \cdots + \frac{h^n}{\lfloor n}f^n(a) + \cdots \quad (3)$$

Called Taylor's Series, Converges to $f(a+h)$

(3) If we put $a+h = x$ in above eqn. we have

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{\lfloor 2}f''(a) + \cdots + \frac{(x-a)^n}{\lfloor n}f^n(a) + \cdots$$

Which is expansion of $f(x)$ in powers of $(x-a)$.

(4) If we put $a=0, h=x$ in eqn (3), we get Maclaurin's series. Hence

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \cdots + \frac{x^n}{\lfloor n} f^n(0) + \cdots$$

<u>Examples</u>:- Q.(1). If $f(x) = x^3 + 8x^2 + 15x - 24$. Calculate the value of $f\left(\frac{11}{10}\right)$ by using Taylor's series.

Soln :- we have $f(x) = x^3 + 8x^2 + 15x - 24$

By Taylor's series, we have

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{\lfloor 2} f''(x) + \frac{h^3}{\lfloor 3} f'''(x) + \cdots$$

We take $x = 1$ and $h = \frac{1}{10} = 0.1$. Also

$f'(x) = 3x^2 + 16x + 15 \Rightarrow f'(1) = 3 + 16 + 15 = 34$

$f''(x) = 6x + 16 \qquad \Rightarrow f''(1) = 6 + 16 = 22$

$f'''(x) = 6$ and $f^{iv}(x) = 0$

Therefore, $f\left(\frac{11}{10}\right) = f\left(1 + \frac{1}{10}\right) = f(1) + 0.1 f'(1) + \frac{0.01}{2} f''(1) + \frac{0.001}{6} f'''(1)$

$= 0 + 3.4 + 0.11 + 0.001 = 3.511$ <u>Ans</u>.

Q.(2). Expand $\log \sin x$ in powers of $(x-3)$.

Soln :- Let $f(x) = \log \sin x$

then $f(x) = f(3 + (x-3)) = f(3+h)$, where $h = x-3$

By Taylor's series expansion, we have

$$f(3+h) = f(3) + h f'(3) + \frac{h^2}{\lfloor 2} f''(3) + \frac{h^3}{\lfloor 3} f''(3) + \cdots$$

Since $f(x) = \log \sin x$, we have

$f'(x) = \frac{\cos x}{\sin x} = \cot x$; $f''(x) = -\mathrm{cosec}^2 x$

$f'''(x) = -2\,\mathrm{cosec}\, x(-\mathrm{cosec}\, x \cot x) = 2\,\mathrm{cosec}^2 x \cot x$

$+ \cdots$

Hence $\log \sin x = f(3+h) = \log \sin 3 + (x-3)\cot 3 - \frac{(x-3)^2}{2}\mathrm{cosec}^2 3$

$+ \frac{(x-3)^3}{3}\mathrm{cosec}^2 3 \cot 3 + \cdots$

Q.(3). Prove that $\log(x+h) = \log h + \frac{x}{h} - \frac{x^2}{2h^2} + \frac{x^3}{3h^3} - \cdots$

Soln :- Here we have to expand $\log(x+h)$ in powers of x. using Taylor's theorem :

$$f(x+h) = f(h) + x f'(h) + \frac{x^2}{\lfloor 2} f''(h) + \frac{x^3}{\lfloor 3} f'''(h) + \cdots \quad \text{—— (1)}$$

Let $f(x+h) = \log(x+h)$

we have $f(x) = \log x \Rightarrow f(h) = \log h$

$f'(x) = \frac{1}{x} \Rightarrow f'(h) = \frac{1}{h}$

$f''(x) = -\frac{1}{x^2} \Rightarrow f''(h) = -\frac{1}{h^2}$

$f'''(x) = \frac{2}{x^3} \Rightarrow f'''(h) = \frac{2}{h^3}$

Putting these values of $f(h), f'(h), f''(h)$ etc in eqn. (1)

we get $\log(x+h) = \log h + x\left(\frac{1}{h}\right) + \frac{x^2}{\lfloor 2}\left(-\frac{1}{h^2}\right) + \frac{x^3}{\lfloor 3}\left(\frac{2}{h^3}\right) + \cdots$

$= \log h + \frac{x}{h} - \frac{x^2}{2h^2} + \frac{x^3}{3h^3} - \cdots$ Proved

Q.(4) Expand : (i) e^x in power of $(x-1)$ upto four terms

(ii) $\tan^{-1} x$ in power of $(x - \frac{\pi}{4})$

Soln : (i) Let $f(x) = e^x$. Expanding it in Taylor's series

we have $e^x = f(x) = f[1 + (x-1)] = f(1) + (x-1) f'(1) + \frac{(x-1)^2}{\lfloor 2} f''(1) + \frac{(x-1)^3}{\lfloor 3} f'''(1) + \cdots$ —— (1)

Here $f(x) = e^x \Rightarrow f(1) = e$

$f'(x) = e^x \Rightarrow f'(1) = e$

$f''(x) = e^x \Rightarrow f''(1) = e$ and $f'''(x) = e^x \Rightarrow f'''(1) = e$ so on.

Putting these values in eqn (1).

$e^x = e + (x-1)e + \frac{(x-1)^2}{\lfloor 2} e + \frac{(x-1)^3}{\lfloor 3} e + \cdots$

$= e\left[1 + (x-1) + \frac{(x-1)^2}{\lfloor 2} + \frac{(x-1)^3}{\lfloor 3} + \cdots\right]$

(ii) Let $f(x) = \tan^{-1} x$. Expanding by Taylor's series,

we have $\tan^{-1} x = f(x) = f\left[\frac{\pi}{4} + (x - \frac{\pi}{4})\right]$

$= f(\frac{\pi}{4}) + (x - \frac{\pi}{4}) f'(\frac{\pi}{4}) + \frac{(x-\frac{\pi}{4})^2}{\lfloor 2} f''(\frac{\pi}{4}) + \cdots$ —— (1)

Now $f(x) = \tan^{-1} x \Rightarrow f(\frac{\pi}{4}) = \tan^{-1} \frac{\pi}{4}$

$f'(x) = \frac{1}{1+x^2} \Rightarrow f'(\frac{\pi}{4}) = \frac{1}{1 + \frac{\pi^2}{16}}$

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$$f''(x) = \frac{-2x}{(1+x^2)^2} \Rightarrow f''\left(\frac{\pi}{4}\right) = \frac{-\pi}{2\left(1+\frac{\pi^2}{16}\right)^2} \quad \text{and so on.}$$

Substituting these values in eqn (1), we get

$$\tan^{-1}x = \tan^{-1}\frac{\pi}{4} + \left(x-\frac{\pi}{4}\right)\cdot\frac{1}{1+\frac{\pi^2}{16}} - \pi\frac{\left(x-\frac{\pi}{4}\right)^2}{4\left(1+\frac{\pi^2}{16}\right)^2} + \cdots$$

Maclaurin's series in single variable:-

If the function $f(x)$ can be expanded in a convergent series of positive integral power of x, then

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2}f''(0) + \frac{x^3}{\lfloor 3}f'''(0) + \cdots + \frac{x^n}{\lfloor n}f^n(0) + \cdots$$

is called Maclaurin's infinite series.

Proof :- Let $f(x) = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \cdots + A_n x^n + \cdots \quad (1)$

where $A_0, A_1, A_2 \cdots$ are constants (independent of x) to be determined. Differentiating eqn (1) w.r.t x successively we get

$$f'(x) = A_1 + 2A_2 x + 3A_3 x^2 + \cdots + nA_n x^{n-1} + \cdots$$

$$f''(x) = 2\times 1 A_2 + 3\times 2 A_3 x + \cdots + n(n-1)A_n x^{n-2} + \cdots$$

$$f'''(x) = 3\times 2 A_3 + 4\times 3\times 2 A_4 x + \cdots n(n-1)(n-2)A_n x^{n-3} + \cdots$$

Putting $x=0$ in each of these equations, we get $A_0 = f(0)$, $A_1 = f'(0)$, $A_2 = \frac{f''(0)}{\lfloor 2}$, $A_3 = \frac{f'''(0)}{\lfloor 3}$ ---

Substituting these values in eqn (1)

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2}f''(0) + \frac{x^3}{\lfloor 3}f'''(0) + \cdots$$

Example :- Q.(1) Expand $\log(1+e^x)$ by Maclaurin's theorem upto term of x^4.

Soln :- Let $f(x) = \log(1+e^x)$. Then $f(0) = \log 2$.

Differentiating w.r.t. x, we get

$$f'(x) = \frac{e^x}{1+e^x} \Rightarrow f'(0) = \frac{1}{2}$$

$$f''(x) = \frac{(1+e^x)e^x - e^x(e^x)}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2} \Rightarrow f''(0) = \frac{1}{4}$$

$$f'''(x) = \frac{(1+e^x)^2 e^x - 2(1+e^x)e^x e^x}{(1+e^x)^4} = \frac{e^x - e^{2x}}{(1+e^x)^3} \Rightarrow f'''(0) = 0$$

$$f^{iv}(x) = \frac{(1+e^x)^3(e^x - 2e^{2x}) - 3(1+e^x)^2 e^x(e^x - e^{2x})}{(1+e^x)^6}$$

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or, $f^{IV}(x) = \dfrac{(1+e^x)(e^x - 2e^{2x}) - 3e^x(e^x - e^{2x})}{(1+e^x)^4} \Rightarrow f^{IV}(0) = -\frac{1}{8}$

Putting the values of $f(0), f'(0), f''(0), f'''(0)$ in the Maclaurin's series:

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \frac{x^4}{\lfloor 4} f^{IV}(0) + \cdots$$

we get $f(x) = \log 2 + \frac{x}{2} + \frac{x^2}{2}\left(\frac{1}{4}\right) + \frac{x^3}{6}(0) + \frac{x^4}{24}\left(-\frac{1}{8}\right) + \cdots$

$$\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \cdots$$

Q.(2) Expand the following function in power series

(I) $\tan x$ (II) $\log(1+x)$

Soln :- Let $f(x) = \tan x \Rightarrow f(0) = 0$

$f'(x) = \sec^2 x = 1 + \tan^2 x \Rightarrow f'(0) = 1 + 0 = 1$

$f''(x) = 2\tan x \cdot \sec^2 x = 2\tan x(1+\tan^2 x) \Rightarrow f''(0) = 0$

$f'''(x) = 2\sec^2 x + 6\tan^2 x \sec^2 x$

$= 2(1+\tan^2 x) + 6\tan^2 x(1+\tan^2 x) = 2 + 8\tan^2 x + 6\tan^4 x$

$\Rightarrow f'''(0) = 2 + 0 + 0 = 2$

$f^{IV}(x) = 16\tan x \sec^2 x + 24\tan^3 x \sec^2 x$

$= 16\tan x + 40\tan^3 x + 24\tan^5 x$

$\Rightarrow f^{IV}(0) = 0 + 0 + 0 = 0$

$f^{V}(x) = 16\sec^2 x + 120\tan^2 x \cdot \sec^2 x + 120\tan^4 x \cdot \sec^2 x$

$\Rightarrow f^{V}(0) = 16(1) + 0 + 0 = 16$

Putting values of $f(0), f'(0), f''(0)$ etc in Maclaurin's series:

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \frac{x^4}{\lfloor 4} f^{IV}(0) + \frac{x^5}{\lfloor 5} f^{V}(0) + \cdots$$

we have $\tan x = 0 + x + 0 + \frac{x^3}{6}(2) + 0 + \frac{x^5}{120}(16) + \cdots$

or $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots$ Ans

(II) Let $f(x) = \log(1+x) \Rightarrow f(0) = \log 1 = 0$

$f'(x) = \frac{1}{1+x} \Rightarrow f'(0) = 1$

$f''(x) = -\frac{1}{(1+x)^2} \Rightarrow f''(0) = -1$

$f'''(x) = \frac{2}{(1+x)^3} \Rightarrow f'''(0) = 2$

$f^{IV}(x) = \frac{-6}{(1+x)^4} \Rightarrow f^{IV}(0) = -6$ and so on.

By Maclaurin's theorem, we have

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \frac{x^4}{\lfloor 4} f^{IV}(0) + \cdots$$

$$\therefore \log(1+x) = 0 + x(1) + \frac{x^2}{\lfloor 2}(-1) + \frac{x^3}{\lfloor 3}(2) + \frac{x^4}{\lfloor 4}(-6) + \cdots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad \text{Ans.}$$

Taylor's series for a function of two variables :-

If $f(x, y)$ and its partial derivatives are finite and Continuous for all points (x, y) and h and k are small increments in x and y, then

$$f(x+h, y+k) = f(x, y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x, y) + \frac{1}{\lfloor 2}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x, y) + \cdots$$

Deductions :- (1) put $x = a$, $y = b$ in above eqn. we get

$$f(a+h, y+b) = f(a, b) + [h f_x(a, b) + k f_y(a, b)] + \frac{1}{\lfloor 2}[h^2 f_{xx}(a, b) + 2hk f_{xy}(a, b) + k^2 f_{yy}(a, b)] + \cdots \quad (1)$$

(2) If we put $a+h = x$, $b+k = y$ in eqn (1) we get

$$f(x, y) = f(a, b) + [(x-a) f_x(a, b) + (y-b) f_y(a, b)] + \frac{1}{\lfloor 2}[(x-a)^2 f_{xx}(a, b) + 2(x-a)(y-b) f_{xy}(a, b) + (y-b)^2 f_{yy}(a, b)] + \cdots \quad (2)$$

Maclaurin's Series for a function of two variables :-

This is special case of Taylor's Series. Put $a = 0$, $b = 0$ in eqn (2) above, we get

$$f(x, y) = f(0, 0) + [x f_x(0, 0) + y f_y(0, 0)] + \frac{1}{\lfloor 2}[x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)] + \cdots$$

where $f_x = \frac{\partial f}{\partial x}$, $f_y = \frac{\partial f}{\partial y}$, $f_{xx} = \frac{\partial^2 f}{\partial x^2}$, $f_{yy} = \frac{\partial^2 f}{\partial y^2}$ and $f_{xy} = \frac{\partial f}{\partial x \partial y}$.

Q.(1) Expand $f(x,y) = e^{xy}$ in Taylor Series at $(1,1)$ up to Second degree terms.

Soln :- Let $f(x,y) = e^{xy}$. By Taylor Series expansion

$$f(x,y) = f(a,b) + [(x-a)f_x(a,b) + (y-b)f_y(a,b)] + \frac{1}{\underline{|2}}[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)] + \cdots \quad (1)$$

Here $a=1$, $b=1$, therefore

$$f(x,y) = e^{xy} \Rightarrow f(1,1) = e$$
$$f_x(x,y) = y e^{xy} \Rightarrow f_x(1,1) = e$$
$$f_y(x,y) = x e^{xy} \Rightarrow f_y(1,1) = e$$
$$f_{xx}(x,y) = y^2 e^{xy} \Rightarrow f_{xx}(1,1) = e$$
$$f_{xy}(x,y) = xy\, e^{xy} \Rightarrow f_{xy}(1,1) = e$$
$$f_{yy}(x,y) = x^2 e^{xy} \Rightarrow f_{yy}(1,1) = e$$

Putting these values in eqn (1) gives

$$f(x,y) = e + [(x-1)e + (y-1)e] + \frac{1}{\underline{|2}}[(x-1)^2 e + 2(x-1)(y-1)e + (y-1)^2 e] + \cdots$$

$$\therefore e^{xy} = e[1 + (x-1) + (y-1) + (x-1)^2 + 2(x-1)(y-1) + (y-1)^2 + \cdots]$$

Q.(2). Expand $x^2y + \sin y + e^x$ in Taylor Series about $(1,\pi)$ upto Second degree.

Soln :- Let $f(x,y) = x^2y + \sin y + e^x$. By Taylor's expansion

$$f(x,y) = f(a,b) + [(x-a)f_x(a,b) + (y-b)f_y(a,b)] + \frac{1}{\underline{|2}}[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)] + \cdots \quad (1)$$

Here $a=1$, $b=\pi$. So

$$f(x,y) = x^2y + \sin y + e^x \Rightarrow f(1,\pi) = \pi + \sin\pi + e = \pi + e$$
$$f_x(x,y) = 2xy + e^x \Rightarrow f_x(1,\pi) = 2\pi + e$$
$$f_y(x,y) = x^2 + \cos y \Rightarrow f_y(1,\pi) = 0$$
$$f_{xx}(x,y) = 2y + e^x \Rightarrow f_{xx}(1,\pi) = 2\pi + e$$
$$f_{xy}(x,y) = 2x \Rightarrow f_{xy}(1,\pi) = 2$$
$$f_{yy}(x,y) = -\sin y \Rightarrow f_{yy}(1,\pi) = 0$$

Substituting these values in eqn. (1) we get

$$f(x,y) = (\pi+e) + [(x-1)(2\pi+e) + (y-\pi)(0)] + \frac{1}{\underline{|2}}[(x-1)^2(2\pi+e) + 2(x-1)(y-\pi)(2) + (y-\pi)^2(0)] + \cdots$$

$$\therefore x^2y+\sin y+e^x=(\pi+e)+(2\pi+e)(x-1)+\tfrac{1}{2}(x-1)^2(2\pi+e)$$
$$+2(x-1)(y-\pi)+\cdots$$

Q.(3) Expand $e^{ax}\sin by$ in powers of x and y upto third degree terms.

Soln:- Here the points are not given, so we expand $e^{ax}\sin by$ as Maclaurin's Series about the point $(0,0)$

Let
$$f(x,y)=e^{ax}\sin by \Rightarrow f(0,0)=0$$
$$f_x(x,y)=ae^{ax}\sin by \Rightarrow f_x(0,0)=0$$
$$f_{xx}(x,y)=a^2e^{ax}\sin by \Rightarrow f_{xx}(0,0)=0$$
$$f_{xxx}(x,y)=a^3e^{ax}\sin by \Rightarrow f_{xxx}(0,0)=0$$
$$f_y(x,y)=be^{ax}\cos by \Rightarrow f_y(0,0)=b$$
$$f_{yy}(x,y)=-b^2e^{ax}\sin by \Rightarrow f_{yy}(0,0)=0$$
$$f_{yyy}(x,y)=-b^3e^{ax}\cos by \Rightarrow f_{yyy}(0,0)=-b^3$$
$$f_{xy}(x,y)=ab\,e^{ax}\cos by \Rightarrow f_{xy}(0,0)=ab$$
$$f_{xxy}(x,y)=a^2b\,e^{ax}\cos by \Rightarrow f_{xxy}(0,0)=a^2b$$
$$f_{xyy}(x,y)=-ab^2e^{ax}\sin by \Rightarrow f_{xyy}(0,0)=0$$

By Maclaurin's Series, we have
$$f(x,y)=f(0,0)+[xf_x(0,0)+yf_y(0,0)]+\frac{1}{\lfloor 2}[x^2f_{xx}(0,0)+2xyf_{xy}(0,0)+y^2f_{yy}(0)]+\frac{1}{\lfloor 3}[x^3f_{xxx}(0,0)+3x^2yf_{xxy}(0,0)+3xy^2f_{xyy}(0,0)+y^3f_{yyy}(0,0)]+\cdots$$
$$\therefore e^{ax}\sin by=0+[x(0)+y(b)]+\frac{1}{\lfloor 2}[x^2(0)+2xy(ab)+y^2(0)]+\frac{1}{\lfloor 3}[x^3(0)+3x^2y(a^2b)+3xy^2(0)+y^3(-b^3)]+\cdots$$
$$=by+\frac{1}{\lfloor 2}ab(2xy)+\frac{1}{\lfloor 3}[3x^2y(a^2b)-b^3y^3]+\cdots$$
$$=by+abxy+\frac{1}{2}a^2bx^2y-\frac{b^3}{6}y^3+\cdots$$

Q.4. Expand $e^x\log(1+y)$ in powers of x and y upto third degree terms.

Soln:- Here points is not given, so we expand $e^x\log(1+y)$ by Maclaurin's Series about $(0,0)$.

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