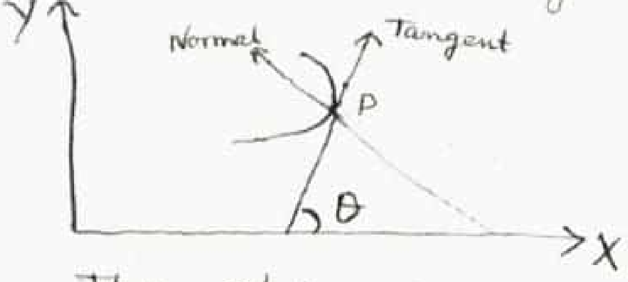
Tangents and Normals

A line which touches the curve at a single point is called tangent at a point and if a line is perpendicular to the tangent, it is called normal.



The slope of a tangent to the curve $y = f(x)$ at the point P is given by $\frac{dy}{dx} = \tan\theta$

The slope of normal to the curve $y = f(x)$ at point P is given by $\frac{-1}{\text{slope of tangent at P}} = \frac{-1}{dy/dx} = -\frac{dx}{dy}$

<u>Equation of tangent at a point</u> :- Let $P(x_1, y_1)$ be a point in a curve. Then equation of a line passing through $P(x_1, y_1)$ with slope m is given by $y - y_1 = m(x - x_1)$.

If the line is tangent to the curve at P, then $m = \tan\theta = \frac{dy}{dx}$

Hence eqn. of the tangent at $P(x_1, y_1)$ is

$$y - y_1 = \frac{dy_1}{dx_1}(x - x_1), \text{ where } \frac{dy_1}{dx_1} = \frac{dy}{dx} \text{ at } (x_1, y_1)$$

<u>Equation of normal at $P(x_1, y_1)$</u> is

$$y - y_1 = -\frac{dx_1}{dy_1}(x - x_1)$$

Q. Find equation of tangent and normal to the curve $y^2 = 3x^2 + 1$ at the point $(1, 2)$.

Soln :- Given $y^2 = 3x^2 + 1 \Rightarrow 2y\frac{dy}{dx} = 6x \Rightarrow \frac{dy}{dx} = \frac{3x}{y}$

$\frac{dy}{dx}$ at the point $(1, 2) = \frac{3 \times 1}{2} = \frac{3}{2}$.

$\therefore$ slope of the tangent $= \frac{3}{2}$

So the eqn. of tangent is $y - 2 = \frac{3}{2}(x - 1)$

or, $3x - 2y + 1 = 0$

The slope of the normal $= -\frac{2}{3}$

So the eqn. of normal is $y - 2 = -\frac{2}{3}(x - 1)$

or, $2x + 3y - 8 = 0$

Tangent plane and Normal to a Surface :-

The equation of tangent plane to the Surface $F(x,y,z)=0$ at the point $P(x_1,y_1,z_1)$ is given by

$$\frac{\partial F}{\partial x}(x-x_1)+\frac{\partial F}{\partial y}(y-y_1)+\frac{\partial F}{\partial z}(z-z_1)=0$$

The equation of normal to the Surface at $P(x_1,y_1,z_1)$ is

$$\frac{x-x_1}{\partial F/\partial x}=\frac{y-y_1}{\partial F/\partial y}=\frac{z-z_1}{\partial F/\partial z}.$$

Q.1. Find the equation of tangent plane and normal to the surface $x^2+2y^2+3z^2=12$ at the point $(1,2,-1)$

Soln :- Given $F(x,y,z)=x^2+2y^2+3z^2-12$

$$\frac{\partial F}{\partial x}=2x,\ \frac{\partial F}{\partial y}=4y,\ \frac{\partial F}{\partial z}=6z$$

At the point $(1,2,-1)$ $\frac{\partial F}{\partial x}=2,\ \frac{\partial F}{\partial y}=8,\ \frac{\partial F}{\partial z}=-6$

Hence eqn. of tangent plane at $(1,2,-1)$ is

$$2(x-1)+8(y-2)-6(z+1)=0 \Rightarrow 2x+8y-6z=24$$

or $x+4y-3z=12$.

The eqn. of normal at $(1,2,-1)$ is given by

$$\frac{x-1}{2}=\frac{y-2}{8}=\frac{z+1}{-6} \text{ or, } \frac{x-1}{1}=\frac{y-2}{4}=\frac{z+1}{-3}$$

Q.2. Find the equations of tangent plane and the normal line to the Surface $2x^2+y^2+2z-3=0$ at the point $(2,1,-3)$

Soln :- Given surface is $F(x,y,z)=2x^2+y^2+2z-3=0$

$$\frac{\partial F}{\partial x}=4x,\ \frac{\partial F}{\partial y}=2y,\ \frac{\partial F}{\partial z}=2$$

At the point $(2,1,-3)$ $\frac{\partial F}{\partial x}=8,\ \frac{\partial F}{\partial y}=2,\ \frac{\partial F}{\partial z}=2$

Hence eqn. of the tangent plane at $(2,1,-3)$ is

$$8(x-2)+2(y-1)+2(z+3)=0$$

$$\Rightarrow 8x+2y+2z-12=0 \Rightarrow 4x+y+z-6=0$$

Equation of normal is $\frac{x-2}{8}=\frac{y-1}{2}=\frac{z+3}{2}$

or, $\frac{x-2}{4}=\frac{y-1}{1}=\frac{z+3}{1}$

Maxima and minima of function of two variables :-

<u>Def^n</u> :- A function $f(x,y)$ is said to have a maximum value at $x=a$, $y=b$ if $f(a,b) > f(a+h, b+k)$ for all small values of h and k. Similarly $f(x,y)$ is said to have minimum value at $x=a$, $y=b$ if $f(a,b) < f(a+h, b+k)$ for small values of h and k.
A max. or min. value of a function is called its 'extreme value'.

Conditions for $f(x,y)$ to be max. or min.

Necessary Condition :- The necessary condition for $f(x,y)$ to have max. or min. values at (a,b) are that
$f_x(a,b)=0$ and $f_y(a,b)=0$; where $f_x(a,b) = \frac{\partial f}{\partial x}$ at (a,b)
& $f_y(a,b) = \frac{\partial f}{\partial y}$ at (a,b)

<u>Sufficient Conditions</u> :- If $f_x(a,b)=0$, $f_y(a,b)=0$
$f_{xx}(a,b)=r$, $f_{xy}(a,b)=s$, $f_{yy}(a,b)=t$ then

(i) $f(a,b)$ is maximum value if $rt-s^2>0$ and $r<0$ (or $t<0$) at (a,b).

(ii) $f(a,b)$ is minimum value if $rt-s^2>0$ and $r>0$ (or $t>0$) at (a,b).

(iii) $f(a,b)$ is not an extreme value if $rt-s^2<0$ at (a,b). then (a,b) is a saddle point.

(iv) If $rt-s^2=0$ the test is inconclusive.

<u>Stationary point & Stationary value</u> :— A point (a,b) at which $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$ is called stationary or turning point. The value of $f(x,y)$ at stationary point (a,b) is called stationary value.

Q.1. Find the maximum and minimum values of $x^3+y^3-3y-12x+20$.

Soln :- Let $f(x,y) = x^3+y^3-12x-3y+20$.

$\therefore f_x = \frac{\partial f}{\partial x} = 3x^2-12$; $f_y = \frac{\partial f}{\partial y} = 3y^2-3$

$f_{xx} = \frac{\partial^2 f}{\partial x^2} = 6x$; $f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 0$ and $f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = 6y$

when $f_x = 0$, we have $3x^2-12=0$

" $f_y = 0$, we have $3y^2-3=0$

Solving above eqns, we get $x = \pm 2$ and $y = \pm 1$

So the stationary points are $(-2,-1)$ $(-2,1)$ $(2,-1)$ $(2,1)$

From table we have

Points	$r = f_{xx}$	$s = f_{xy}$	$t = f_{yy}$	$rt-s^2$	Extreme value
$(-2,-1)$	-12	0	-6	$72>0$	Max, at $(-2,-1)$
$(-2,1)$	-12	0	6	$-72<0$	No extreme value (saddle point)
$(2,-1)$	12	0	-6	$-72<0$	No extreme value (saddle point)
$(2,1)$	12	0	6	$72>0$	Minimum at $(2,1)$

Therefore maximum value at $(-2,-1) = (-2)^3+(-1)^3-12(-2)-3(-1)+20 = 38$

and minimum value at $(2,1) = 2^3+1^3-12(2)-3(1)+20 = 2$

Q.2. Determine the points where the function $f(x,y) = x^3+y^3-3axy$ has a maximum or minimum.

Soln :- we have $f_x = \frac{\partial f}{\partial x} = 3x^2-3ay$, $f_y = \frac{\partial f}{\partial y} = 3y^2-3ax$

$r = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 6x$, $s = f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -3a$, $t = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = 6y$.

For extreme points, $f_x = f_y = 0$

$\therefore 3x^2-3ay = 0$ and $3y^2-3ax = 0$

Solving these eqns, we get two stationary points as $(0,0)$ and (a,a)

Thus $rt - s^2 = 36xy - 9a^2$

At $(0,0)$, $rt - s^2 = -9a^2$ (negative). So there is no extreme point at origin $(0,0)$.

At (a,a), we have $rt - s^2 = 36a^2 - 9a^2 = 27a^2 > 0$

Also r at (a,a) is equal to $6a$.

If a is +ve, then r is +ve and $f(x,y)$ will have a minimum at (a,a).

If a is −ve, then r is −ve, so $f(x,y)$ will have a maximum at (a,a) for $a<0$.