Diagonalisation of a Matrix :-

If a square matrix A of order n has n linear independent eigen vectors, then a matrix P can be found such that $P^{-1}AP$ is a diagonal matrix.

The matrix P which diagonalises A is called 'modal matrix' of A and the resulting diagonal matrix D is known as 'Spectral matrix' of A. The matrix P is found by grouping the eigen vectors of A into square matrix and the diagonal matrix has the eigen values of A as its elements.

The transformation of matrix A by non-singular matrix P is called 'Similarity transformation'.

Proof :- Let A be a square matrix of order 3. Let λ_1, λ_2 and λ_3 be its eigen values and

$$X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \text{ and } X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$$ be corresponding eigen vectors.

Representing $$[X_1\, X_2\, X_3] = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} = P \text{ (say)}$$

then $$AP = A[X_1\, X_2\, X_3] = [AX_1, AX_2, AX_3] = [\lambda_1 X_1, \lambda_2 X_2, \lambda_3 X_3]$$

$$= \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \lambda_3 x_3 \\ \lambda_1 y_1 & \lambda_2 y_2 & \lambda_3 y_3 \\ \lambda_1 z_1 & \lambda_2 z_2 & \lambda_3 z_3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \times \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$= P.D$, where D is the diagonal matrix.

$\therefore AP = PD \Rightarrow P^{-1}AP = P^{-1}P.D = D$ which proves the theorem.

A square matrix D of order n is called similar to a square matrix A of order n if $D = P^{-1}AP$, where P is non-singular matrix of order $n \times n$

Q1. Diagonalise the matrix $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ Also find the modal matrix P.

Soln :- The characteristic eqn is $|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^3 - 7\lambda^2 + 36 = 0$$

Since $\lambda = -2$ satisfies it, we can write this eqn as

$$(\lambda+2)(\lambda^2 - 9\lambda + 18) = 0 \Rightarrow (\lambda+2)(\lambda-3)(\lambda-6) = 0$$

Thus eigenvalues of A are $\lambda = -2, 3, 6$.

If x, y, z be the components of eigenvector corresponding to eigenvalue λ, we have

$$[A-\lambda I]X = 0 \Rightarrow \begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Putting $\lambda = -2$, we have $3x+y+3z=0$, $x+7y+z=0$, $3x+y+3z=0$

The first and third eqns are same. So solving first two equations, we get $\frac{x}{-20} = \frac{y}{0} = \frac{z}{20} \Rightarrow \frac{x}{-1} = \frac{y}{0} = \frac{z}{1}$

Hence eigenvector corresponding to $\lambda = -2$ is $[-1, 0, 1]$

Also every non-zero multiple of this vector is an eigenvector.

Similarly eigenvectors corresponding to $\lambda = 3$ and $\lambda = 6$ are non-zero multiples of vectors $[1, -1, 1]$ and $[1, 2, 1]$

Hence three eigenvectors are $(-1, 0, 1), (1, -1, 1), (1, 2, 1)$

Writing these eigenvectors as three columns, the modal matrix P is given by $P = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

So, $P^{-1} = \frac{1}{|P|}(\text{adj } P) = \frac{1}{6}\begin{bmatrix} -3 & 0 & 3 \\ 2 & -2 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

Thus diagonal matrix $D = P^{-1}AP = \frac{1}{6}\begin{bmatrix} -3 & 0 & 3 \\ 2 & -2 & 2 \\ 1 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$ Ans

Powers of a Matrix :- We can obtain powers of a matrix by using diagonalisation.

Let A be a square matrix and P is a non-Singular matrix Such that $D = P^{-1}AP$

$$\therefore \quad D^2 = (P^{-1}AP)(P^{-1}AP) = P^{-1}A^2P \quad (\because PP^{-1} = I)$$

In general, $D^n = P^{-1}A^nP$ —— (1)

To obtain A^n, premultiply (1) by P and post-multiply by P^{-1}

then $PD^nP^{-1} = PP^{-1}A^nPP^{-1} = I.A^n.I = A^n$

Q.2. Find a matrix P which transform the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ to diagonal form, hence find A^4.

Soln :- The characteristic eqn of matrix A is $|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\Rightarrow (\lambda-1)(\lambda-2)(\lambda-3) = 0$$

$$\therefore \lambda = 1, 2, 3.$$

For $\lambda = 1$, eigenvectors are given by

$$\begin{bmatrix} 1-1 & 0 & -1 \\ 1 & 2-1 & 1 \\ 2 & 2 & 3-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} 0x_1 + 0x_2 - x_3 &= 0 \\ x_1 + x_2 + x_3 &= 0 \end{aligned}\right\}$$

Solving by Cross-multiplication rule

$$\frac{x_1}{0+1} = \frac{x_2}{-1-0} = \frac{x_3}{0} \Rightarrow \frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{0}$$

$\therefore$ Eigenvector is $[1, -1, 0]$

Similarly for $\lambda = 2$, eigenvectors are given by

$$\begin{bmatrix} 1-2 & 0 & -1 \\ 1 & 2-2 & 1 \\ 2 & 2 & 3-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} x_1 + 0x_2 + x_3 &= 0 \\ 2x_1 + 2x_2 + x_3 &= 0 \end{aligned}\right]$$

Solving, we get

$$\frac{x_1}{0-2} = \frac{x_2}{2-1} = \frac{x_3}{2-0} = \frac{x_1}{-2} = \frac{x_2}{1} = \frac{x_3}{2}$$

eigen vector is $[-2, 1, 2]$

and for $\lambda = 3$, eigen vectors are given by

$$\begin{bmatrix} 1-3 & 0 & -1 \\ 1 & 2-3 & 1 \\ 2 & 2 & 3-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} -2x_1 + 0x_2 - x_3 &= 0 \\ x_1 - x_2 + x_3 &= 0 \end{aligned}\right]$$

solving, we get

$$\frac{x_1}{0-1} = \frac{x_2}{-1+2} = \frac{x_3}{2-0} \Rightarrow \frac{x}{-1} = \frac{x_2}{1} = \frac{x_3}{2}$$

$\therefore$ Eigenvectors are $[-1, 1, 2]$

The modal matrix $P = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$, $|P| = 1(2-2) + 1(-4+2) = -2$

$$\therefore P^{-1} = -\frac{1}{2}\begin{bmatrix} 0 & 2 & -1 \\ 2 & 2 & 0 \\ -2 & -2 & -1 \end{bmatrix}$$

Now, $P^{-1}AP = \begin{bmatrix} 0 & -1 & 1/2 \\ -1 & -1 & 0 \\ 1 & 1 & 1/2 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = D$$

$\therefore A^4 = PD^4P^{-1} = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix} \times \begin{bmatrix} 0 & -1 & 1/2 \\ -1 & -1 & 0 \\ 1 & 1 & 1/2 \end{bmatrix}$

$$= \begin{bmatrix} -49 & -50 & -40 \\ 65 & 66 & 40 \\ 130 & 130 & 81 \end{bmatrix}$$

Ans