Rank of a Matrix :- The rank of matrix A is said to be r if (i) It has at least one non-zero minor of order r
(ii) Every minor of A of order higher than r is zero.
In other words rank of a matrix A is equal to the highest ordered non-zero minor of A. It is denoted by $\rho(A)$

Since elementry transformations do not change the order and rank of a matrix, so a matrix can be reduced to echelon form in which maximum no. of zero rows are obtained by elementry row transformations. Then

Rank = Number of non-zero rows in echelon form of matrix.

Here, non-zero row is that row which does not contain all elements zero.

Rank of a matrix has following properties :-

(i) Every matrix will have a rank.
(ii) Rank of a matrix is unique.
(iii) Rank of null matrix $= 0$ and rank of not null matrix ≥ 1
(iv) If A is $m \times n$ matrix then $\rho(A) \leq$ minimum of m and n
(v) If A is non-singular $n \times n$ square matrix then $\rho(A) = n$
(vi) Rank of unit matrix of order $n = n$

Q.1. Find the rank of a matrix $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

Soln :- The minor of order $3 = 1(20-12) - 2(5-4) + 3(6-8) = 8-2-6 = 0$
therefore, Rank $\neq 3$. It must be less than 3.
Consider a minor of order $2 = \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = 4-2 = 2 \neq 0$. Hence there is at least one minor of order 2 which is not zero.
So Rank $= 2$.

Q. 2. Find the rank of matrix : $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

Soln :- Let $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

operating $R_2 - 2R_1$ and $R_3 - R_1$, $A \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix}$

operating $R_3 + R_2$ $\sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

This is the echelon form. Since number of non-zero rows is 2, so rank of A $= 2$ or $\rho(A) = 2$.

Q.3. Find the rank of the matrix $\begin{bmatrix} -1 & 2 & 3 & -2 \\ 2 & -5 & 1 & 2 \\ 3 & -8 & 5 & 2 \\ 5 & -12 & -1 & 6 \end{bmatrix}$

Soln:- We reduce the given matrix to echelon form by elementry row-transformations.

operating R_2+2R_1, R_3+3R_1 and R_4+5R_1 to given matrix

$$\sim \begin{bmatrix} -1 & 2 & 3 & -2 \\ 0 & -1 & 7 & -2 \\ 0 & -2 & 14 & -4 \\ 0 & -2 & 14 & -4 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 2 & 3 & -2 \\ 0 & -1 & 7 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{operating } R_3-2R_2 \text{ \& } R_4-2R_2$$

This matrix is in echelon form. Since number of non-zero rows is 2, So rank = 2.

<u>Normal form of a Matrix</u> :- Every non-zero matrix A of rank r can be reduced by a sequence of elementary transformation to the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$, where I_r is unit matrix of order r, is called normal form of matrix A.

<u>Property of Normal form</u>:- Corresponding to every matrix A of rank r, there exist non-singular matrices P and Q such that $PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

If A be $m \times n$ matrix, then P and Q are square matrix of order m and n respectively.

Q.4. Find the non-singular matrices P and Q such that PAQ is in the normal form where

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$$ Also find the rank of A.

Soln:- Let $A = I_3 A I_3$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Apply elementry operations on the matrix A until it is reduced to the normal form. Every elementry row operation will also be applied to pre-factor I_3 of the product on R.H.S. and every elementry column operation to the post factor I_3 on R.H.S.

operate $R_2 - R_1 \sim \begin{bmatrix}1&1&2\\0&1&1\\0&-1&-1\end{bmatrix} = \begin{bmatrix}1&0&0\\-1&1&0\\0&0&1\end{bmatrix} A \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

operate $C_2 - C_1$ & $C_3 - 2C_1 \sim \begin{bmatrix}1&0&0\\0&1&1\\0&-1&-1\end{bmatrix} = \begin{bmatrix}1&0&0\\-1&1&0\\0&0&1\end{bmatrix} A \begin{bmatrix}1&-1&-2\\0&1&0\\0&0&1\end{bmatrix}$

operate $C_3 - C_2 \sim \begin{bmatrix}1&0&0\\0&1&0\\0&-1&0\end{bmatrix} = \begin{bmatrix}1&0&0\\-1&1&0\\0&0&1\end{bmatrix} A \begin{bmatrix}1&-1&-1\\0&1&-1\\0&0&1\end{bmatrix}$

operate $R_3 + R_2 \sim \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix} = \begin{bmatrix}1&0&0\\-1&1&0\\-1&1&1\end{bmatrix} A \begin{bmatrix}1&-1&-1\\0&1&-1\\0&0&1\end{bmatrix}$

Thus $\begin{bmatrix}I_2&0\\0&0\end{bmatrix} = PAQ$. where $P = \begin{bmatrix}1&0&0\\-1&1&0\\-1&1&1\end{bmatrix}$ and $Q = \begin{bmatrix}1&-1&-1\\0&1&-1\\0&0&1\end{bmatrix}$

Since A reduced to normal form $\begin{bmatrix}I_2&0\\0&0\end{bmatrix}$

Hence rank of $A = 2$.

Q.5. Reduce the matrix $\begin{bmatrix}3&2&-1\\4&2&6\\7&4&5\end{bmatrix}$ to normal form and hence find its rank.

Soln :- Using elementry row/column operations in the given matrix A.

$$A = \begin{bmatrix}3&2&-1\\4&2&6\\7&4&5\end{bmatrix} \overset{R_1\to R_1-R_2}{\sim} \begin{bmatrix}-1&0&-7\\4&2&6\\7&4&5\end{bmatrix} \overset{R_1\to -R_1}{\sim} \begin{bmatrix}1&0&7\\4&2&6\\7&4&5\end{bmatrix}$$

$R_2 \to R_2 - 4R_1$, $R_3 \to R_3 - 7R_1$ $\qquad R_2 \to \frac{1}{2}R_2 \qquad R_3 \to R_3 - 4R_2$

$$\sim \begin{bmatrix}1&0&7\\0&2&-22\\0&4&-44\end{bmatrix} \sim \begin{bmatrix}1&0&7\\0&1&-11\\0&4&-44\end{bmatrix} \sim \begin{bmatrix}1&0&7\\0&1&-11\\0&0&0\end{bmatrix}$$

$C_3 \to C_3 - 7C_1 \qquad C_3 \to C_3 + 11C_2$

$$\sim \begin{bmatrix}1&0&0\\0&1&-11\\0&0&0\end{bmatrix} \sim \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix} \sim \begin{bmatrix}I_2&0\\0&0\end{bmatrix}$$

which is in normal form. Hence Rank of $A = 2$.

Q.6. Find the values of a and b such that Rank of matrix

$A = \begin{bmatrix}1&-2&3&1\\2&1&-1&2\\6&-2&a&b\end{bmatrix}$ is 2.

Soln :- we reduce the given matrix A to echelon form by Row-operations.

$$A \sim \begin{bmatrix}1&-2&3&1\\0&5&-7&0\\0&-5&a+3&b-6\end{bmatrix}$$ on opeating $R_3 \to R_3 - 3R_2$ and $R_2 \to R_2 - 2R_1$

$$A \sim \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 5 & -7 & 0 \\ 0 & 0 & a-4 & b-6 \end{bmatrix}$$ on operating $R_3 \to R_3 + R_2$

Since rank of A is given to be 2, $a-4=0 \Rightarrow a=4$ and $b-6=0 \Rightarrow b=6$.

<u>Solution of System of linear equations</u>:- The most important use of matrices occurs in the Solution of System of linear equations or linear system. Such systems are used in engineering, Computer Science, physics, economics, statistics etc For instance in electrical and Computer networks, traffic flow, optimization process and many other field.

A linear system of m equations in n unknowns $x_1, x_2 \cdots x_n$ is given by

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots \quad \cdots \quad \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_n \end{aligned}\right\} \quad \text{--- (1)}$$

The system is called linear because each variable x_i appears in the first degree only. Here $a_{11}, a_{21} \cdots a_{mn}$ are given numbers called Coefficients and $b_1, b_2 \cdots b_n$ are also given numbers. The above linear eqns can be written as $AX = B$ — (2)

where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ - & - & - & - \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ is called Coefficient matrix

$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is Column matrix of unknowns and $B = \begin{bmatrix} b_1 \\ b_2 \\ - \\ b_n \end{bmatrix}$ is Column matrix of Constants.

Equation (1) is said to be non-homogeneous linear equations. Any set of values of $x_1, x_2 \cdots x_n$ which satisfy eqn (1) is called Solution of non-homogeneous linear eqn.

<u>Solution of Non-homogeneous linear eqn</u>:-

(i) <u>Matrix inversion method</u>:- Let eqn (1) be equivalent to matrix eqn $AX = B$. where A is non-Singular Coefficient matrix. Since A is non-Singular, A^{-1} exists. Multiply both sides of above eqn by A^{-1}. we get $A^{-1}.AX = A^{-1}B \Rightarrow I.X = A^{-1}B$ $\Rightarrow X = A^{-1}.B$. Thus given system of eqn. can be solved using inverse of A. If $|A| \neq 0$, then $X = A^{-1}B$ has unique solution (since A^{-1} is unique)

Q. 1. Solve the equations by matrix inversion method:

$3x+y+2z=3,\ 2x-3y-z=-3,\ x+2y+z=4.$

Soln :- The matrix equation is given by $AX=B$ or $X=A^{-1}.B$

where

$$A=\begin{bmatrix}3&1&2\\2&-3&-1\\1&2&1\end{bmatrix},\quad X=\begin{bmatrix}x\\y\\z\end{bmatrix} \text{ and } B=\begin{bmatrix}3\\-3\\4\end{bmatrix}$$

Now $|A| = 3(-3+2)-1(2+1)+2(4+3) = -3-3+14 = 8 \neq 0$

and Cofactors are given by $A_{11}=-1,\ A_{12}=-3,\ A_{13}=7,\ A_{21}=3,\ A_{22}=1$ ---- $A_{33}=-11$

then Cofactor matrix of A is $[A_{ij}]=\begin{bmatrix}-1&-3&7\\3&1&-5\\5&7&-11\end{bmatrix}$

So, $Adj(A)=[A_{ij}]^T=\begin{bmatrix}-1&3&5\\-3&1&7\\7&-5&-11\end{bmatrix}$

$\therefore A^{-1}=\frac{1}{|A|}adj(A) = \frac{1}{8}\begin{bmatrix}-1&3&5\\-3&1&7\\7&-5&-11\end{bmatrix}$

Substituting A^{-1} and B in the eqn $X=A^{-1}.B$ we get

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\frac{1}{8}\begin{bmatrix}-1&3&5\\-3&1&7\\7&-5&-11\end{bmatrix}\times\begin{bmatrix}3\\-3\\4\end{bmatrix}=\frac{1}{8}\begin{bmatrix}-3-9+20\\-9-3+28\\21+15-44\end{bmatrix}=\begin{bmatrix}1\\2\\-1\end{bmatrix}$$

Hence $x=1,\ y=2,\ z=-1$ is the req. Solution.

(ii) Method of determinant (Cramer's rule) :- Let A be non-singular matrix. we define A_i = matrix obtained by replacing the ith Column of A by B of eqn (2). Then by Cramer's rule, Solution of linear equations is given as $x_i = \frac{|A_i|}{|A|}$ where $i=1,2,\ldots n$. So $x_1=\frac{|A_1|}{|A|}$, $x_2=\frac{|A_2|}{|A|}$ ... So on.

Q. 2. Solve the system of linear eqns. by Cramer's rule:

$3x+y+2z=3,\ 2x-3y-z=-3,\ x+2y+z=4$

Soln :- Let A be the Coefficient matrix. Then

$$|A|=\begin{vmatrix}3&1&2\\2&-3&-1\\1&2&1\end{vmatrix}=3(-3+2)-1(2+1)+2(4+3)=-3-3+14=8\neq 0$$

So A is non-singular matrix. Applying Cramer's rule we get

$$x=\frac{1}{|A|}\begin{vmatrix}3&1&2\\-3&-3&-1\\4&2&1\end{vmatrix}=\frac{8}{8}=1,\quad y=\frac{1}{|A|}\begin{vmatrix}3&3&2\\2&-3&-1\\1&4&1\end{vmatrix}=\frac{16}{8}=2$$

and $$z=\frac{1}{|A|}\begin{vmatrix}3&1&3\\2&-3&-3\\1&2&4\end{vmatrix}=\frac{-8}{8}=-1$$

Hence the req. solution is $x=1,\ y=2$ and $z=-1$.

Homogeneous linear equations :- Consider the system of m homogenous equations in n unknowns $x_1, x_2 \cdots x_n$ given as

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0$$
$$\text{-- -- -- -- -- -- --}$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0$$

The matrix form of this equation is $AX = 0$

where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ - & - & - & - \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ - \\ x_n \end{bmatrix}$, $0 = \begin{bmatrix} 0 \\ 0 \\ - \\ 0 \end{bmatrix}$

It is clear that $x_1 = 0, x_2 = 0 \cdots x_n = 0$ ie $X = 0$ is always a solution of homogeneous linear eqn $AX = 0$. This solution is called trivial solution or zero solution. So homogeneous system is always consistent.

Theorem :- (1) A system of homogeneous linear eqn. $AX=0$ has only zero or trivial solution if coefficient matrix A is non-singular ie, $|A| \neq 0$.

(2) The system of eqn. $AX=0$ has non-zero or non-trivial solution if and only if A is a singular matrix, ie $|A| = 0$

Method for finding solution of eqn. $AX=0$:- Find the rank r of the coefficient matrix A by reducing it to echelon form by elementry row operations.

(1) If $r = n$ (number of variables) then system of equations have only trivial solution (ie zero solution).

(2) If $r < n$ then system of equations have an infinite number of non-trivial solutions.

Q.3. Solve : $x_1 - x_2 + x_3 = 0$, $x_1 + 2x_2 - x_3 = 0$, $2x_1 + x_2 + 3x_3 = 0$

Soln :- This is homogeneous linear eqn. We write it in matrix form $AX = 0$, where

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We note that $|A| = 1(6+1) + 1(3+2) + 1(1-4) = 7+5-3 = 9 \neq 0$

Thus A is non-singular. Also rank of A = no. of variables $= 3$. Hence the given system of homogeneous equation has only trivial solution : $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$.

Q.4. Solve : $x+3y-2z=0,\ 2x-y+4z=0,\ x-11y+14z=0$

Soln :- The matrix form of given system of homogeneous equations is $AX=0$, where

$$A=\begin{bmatrix}1 & 3 & -2\\ 2 & -1 & 4\\ 1 & -11 & 14\end{bmatrix},\quad X=\begin{bmatrix}x\\ y\\ z\end{bmatrix},\quad 0=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

we note that $|A|=\begin{vmatrix}1 & 3 & -2\\ 2 & -1 & 4\\ 1 & -11 & 14\end{vmatrix}=30-72+42=0$

Therefore A is Singular, that is Rank of $A < n$ (no. of variables).
Thus the given system has a non-trivial solution and have infinite number of solutions.
Now we reduce matrix A to echelon form by row operations for the eqn $AX=0$. we have

$$\begin{bmatrix}1 & 3 & -2\\ 2 & -1 & 4\\ 1 & -11 & 14\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 3 & -2\\ 0 & -7 & 8\\ 0 & -14 & 16\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}\quad \text{operating } R_2\to R_2-2R_1,\ R_3\to R_3-R_1$$

$$\sim\begin{bmatrix}1 & 3 & -2\\ 0 & -7 & 8\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}\quad \text{" } R_3\to R_3-2R_2$$

So we have, $\left.\begin{aligned}x+3y-2z&=0\\ -7y+8z&=0\end{aligned}\right\}$ solving, we get $y=\frac{8}{7}z,\ x=-\frac{10}{7}z$.

Giving different values to $z=1,2,3\ldots$ we get infinite number of solutions.