Eigen values and Eigenvectors of a Matrix :-

Let A be a square matrix of order n, λ is any scalar and X be a Column vector of unknown, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

There exist a matrix eqn : $AX = \lambda X \Rightarrow [A - \lambda I] X = 0$ —(1)

where I is unit matrix of order n.

Clearly $X = 0$ is a solution of this eqn for any value of λ, but we are interested in finding non-trival Solution ie, $X \neq 0$. This system will have non-trivial Solution if and only if the determinant of Coefficient matrix is zero that is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ - & - & & \\ a_{m1} & a_{m2} & \cdots & a_{mn}-\lambda \end{vmatrix} = 0$$

This is called characteristic equation

Expanding this determinant we get equation of degree n in λ of the form $\lambda^n + k_1 \lambda^{n-1} + k_2 \lambda^{n-2} + \cdots + k_n = 0$ then the Roots $\lambda_1, \lambda_2 \cdots \lambda_n$ of this characteristic eqn. are called characteristic values or 'Eigenvalues' of matrix A. The Set of eigen values of matrix A is called Spectrum of A. Now, Corresponding to each eigenvalue the matrix eqn $[A - \lambda I] X = 0$ will have a non-zero Solution $X = [x_1, x_2 \cdots x_n]$ which is Known as characteristic vectors or 'Eigenvectors' of matrix A.

Properties of Eigenvalues and Eigenvectors :-

1. The eigen values of a Square matrix A are the roots of characteristic equation of A. Square matrix of order n has atleast one eigen value and at most n numerically different eigenvalues.

2. Corresponding to n different eigenvalues, we get n independent eigenvectors. But when two or more eigen values are equal, it may or may not be possible to get independent eigenvectors for repeated eigenvalues.

3. If X_i is solution for eigen value λ_i then from eqn(1) we have CX_i is also solution, where C is any Constant. Thus eigen vector Corresponding to a given eigenvalue is not unique.

4. The sum of eigen values of a matrix is the sum of elements of the principal diagonal. The sum of elements of principal diagonal of a matrix is called 'trace' of the matrix.

5. The determinant of a matrix A equals the product of eigen values of A.

6. The eigen values of a square matrix and its transpose are the same.

7. If λ is an eigenvalue of matrix A, then $\frac{1}{\lambda}$ is the eigenvalue of A^{-1}.

Q.1. Find the eigen values and eigen vectors of matrix $A = \begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$

Soln :- The characteristic eqn is

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & -2 \\ -5 & 4-\lambda \end{vmatrix} = 0$$

$$\left\{ \because [A - \lambda I] = \begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & -2 \\ -5 & 4-\lambda \end{bmatrix} \right.$$

$$\Rightarrow (1-\lambda)(4-\lambda) - 10 = 0 \Rightarrow \lambda^2 - 5\lambda - 6 = 0$$

$$\Rightarrow (\lambda - 6)(\lambda + 1) = 0$$

$$\Rightarrow \lambda = 6, -1$$

Thus the eigen values of A are 6, −1.

Corresponding to $\lambda = 6$, the eigen vectors are given by $[A - 6I]X = 0$

$$\Rightarrow \begin{bmatrix} 1-6 & -2 \\ -5 & 4-6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -5 & -2 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -5x_1 - 2x_2 = 0$ and $-5x_1 - 2x_2 = 0$, we get only one independent eqn: $-5x_1 - 2x_2 = 0 \Rightarrow \frac{x_1}{2} = \frac{x_2}{-5}$ giving eigenvector $(2, -5)$.

Corresponding to $\lambda = -1$, the eigen vectors are given by

$$[A + I]X = 0 \Rightarrow \begin{bmatrix} 2 & -2 \\ -5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We get only one independent equation: $2x_1 - 2x_2 = 0 \Rightarrow x_1 - x_2 = 0$

$\therefore \frac{x_1}{1} = \frac{x_2}{1}$ giving the eigen vector $(1, 1)$

Hence the two eigen vectors are $(2, -5)$ and $(1, 1)$.

Q. 2. Find eigen values and eigenvectors of $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

Soln :- The characteristic eqn of A is

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)[(3-\lambda)(2-\lambda)-2] - 2[(2-\lambda).1-1] + 1[2-(3-\lambda)] = 0$$

$$\Rightarrow \lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$$

Since $\lambda = 1$ satisfies it, we can write this eqn as

$$(\lambda-1)(\lambda^2 - 6\lambda + 5) = 0 \Rightarrow (\lambda-1)(\lambda-1)(\lambda-5) = 0 \Rightarrow \lambda = 1, 1, 5.$$

Therefore eigenvalues of A are $\lambda = 1, 1, 5$.

(i) If x_1, x_2, x_3 be components of eigen vector corresponding to eigenvalue $\lambda = 1$, then

$$[A - \lambda I]X = 0 \Rightarrow \begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

putting $\lambda = 1$, we have $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_1 + 2x_2 + x_3 = 0$. Since all three equations are the same, we get only one independent eqn. So taking x_1 and x_2 as free variables, let $x_1 = 1$ and $x_2 = 0$, we get $x_3 = -1$ and taking $x_1 = 0$ and $x_2 = 1$, we get $x_3 = -2$. Hence the eigen vectors are $(1, 0, -1)$ and $(0, 1, -2)$

(ii) Corresponding to $\lambda = 5$, the eigen vector is given by

$$[A - 5I]X = 0 \Rightarrow \begin{bmatrix} 2-5 & 2 & 1 \\ 1 & 3-5 & 1 \\ 1 & 2 & 2-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow -3x_1 + 2x_2 + x_3 = 0$, $x_1 - 2x_2 + x_3 = 0$ and $x_1 + 2x_2 - 3x_3 = 0$

Solving the first two equations, by cross multiplication rule we have

$$\frac{x_1}{2+2} = \frac{x_2}{1+3} = \frac{x_3}{6-2} \Rightarrow \frac{x_1}{4} = \frac{x_2}{4} = \frac{x_3}{4} \Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

So eigenvectors corresponding to $\lambda = 5$ is $(1, 1, 1)$.

Hence the three eigen vectors are $(1, 0, -1)$, $(0, 1, -2)$ and $(1, 1, 1)$

Also every non-zero multiple of these vectors is also eigen vectors.

Q.3. Find the eigen values and eigenvectors of the matrix.

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

Soln :- The characteristic eqn. of A is

$$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

Since $\lambda = 1$, satisfies the eqn, we can write this eqn as

$$(\lambda-1)(\lambda^2 - 5\lambda + 6) = 0 \Rightarrow (\lambda-1)(\lambda-2)(\lambda-3) = 0$$

$$\Rightarrow \lambda = 1, 2, 3$$

So eigenvalues of A are $\lambda = 1, 2, 3$.

To find eigenvectors for corresponding eigenvalues we consider the matrix equation :

$$[A-\lambda I]X = 0 \Rightarrow \begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{--- (1)}$$

(i) Eigen vector corresponding eigen value $\lambda = 1$ is given by putting $\lambda = 1$ in the above eqn.

$$\therefore \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} -x_3 = 0 \Rightarrow x_3 = 0 \\ x_1 + x_2 + x_3 = 0 \\ 2x_1 + 2x_2 + 2x_3 = 0 \end{array}$$

The last two eqns are the same. Let $x_1 = K \therefore y = -K$

So eigenvector $X_1 = \begin{bmatrix} K \\ -K \\ 0 \end{bmatrix} = K\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$; K is any non-zero number.

(ii) Eigen vector corresponding to $\lambda = 2$ is given by putting $\lambda = 2$ in eqn (1)

$$\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} -x_1 + 0x_2 - x_3 = 0 \Rightarrow x_1 + 0x_2 + x_3 = 0 \\ x_1 + 0x_2 + x_3 = 0 \\ 2x_1 + 2x_2 + x_3 = 0 \end{array}$$

The first two equs. are the same, Taking last two equations, we have $\frac{x_1}{0-2} = \frac{x_2}{2-1} = \frac{x_3}{2-0} \Rightarrow \frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{-2}$. Eigenvectors $X_2 = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$

(iii) Similarly Eigen vector corresponding to $\lambda = 3$ is given by putting $\lambda = 3$ in (1)

we have $\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} -2x_1 + 0.x_2 - x_3 = 0 \\ x_1 - x_2 + x_3 = 0 \\ 2x_1 + 2x_2 + 0x_3 = 0 \end{array}$

Taking first two equations, $\frac{x_1}{0-1} = \frac{x_2}{-1+2} = \frac{x_3}{2-0} \Rightarrow \frac{x}{1} = \frac{x_2}{-1} = \frac{x_3}{-2}$

$\therefore$ eigenvector corresponding to $\lambda = 3$, $X_3 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$

Hence the three eigenvectors are $(1,-1,0)(2,-1,-2),(1,-1,-2)$

Also every non-zero multiple of these vectors is also eigen vectors.

Cayley-Hamilton Theorem :- Every square matrix satisfies its own characteristic equation.

Let A be a square matrix of order n, λ be any scalar and I is unit matrix of nth order. The characteristic equation is given as $|A-\lambda I|=0 \Rightarrow (-1)^n[\lambda^n + a_1\lambda^{n-1} + \cdots + a_n] = 0$

where $a_1, a_2 \cdots a_n$ are constants.

Replacing λ by A, gives $A^n + a_1 A^{n-1} + \cdots + a_n I = 0$ — (1)

This theorem can be used to find A^{-1} also as given below:

Premultiplying eqn (1) by A^{-1}, gives

$$A^{n-1} + a_1 A^{n-2} + \cdots + a_n A^{-1} = 0$$

$$\therefore A^{-1} = -\frac{1}{a_n}[A^{n-1} + a_1 A^{n-2} + \cdots + a_{n-1} I]$$

Q.1. Verify Cayley-Hamilton theorem for matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ and hence find A^{-1}.

Soln :- Cayley-Hamilton theorem states that every square matrix satisfies its characteristic equation.

The characteristic equation for given matrix A is

$$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 5 = 0$$

Hence A must satisfy its characteristic eqn. We have to show that $A^2 - 5I = 0$ — (1)

Since $A^2 = A \times A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$

$$\therefore A^2 - 5I = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence Cayley-Hamilton theorem is verified.

To find A^{-1} multiply both sides of eqn (1) with A^{-1}, we get

$$A^{-1}.A^2 - 5I.A^{-1} = 0 \Rightarrow A - 5A^{-1} = 0 \Rightarrow A^{-1} = \frac{1}{5}A = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & -1/5 \end{bmatrix}$$

Q.2. Verify Cayley Hamilton theorem for the matrix

$A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix}$ and hence find A^{-1}.

Soln :- The characteristic eqn for the matrix A is given by

$$|A-\lambda I|=0 \Rightarrow \begin{vmatrix} 1-\lambda & 0 & -4 \\ 0 & 5-\lambda & 4 \\ -4 & 4 & 3 \end{vmatrix}=0 \Rightarrow (1-\lambda)[(5-\lambda)(3-\lambda)-16] -4[4(5-\lambda)]=0$$

$$\Rightarrow \lambda^3-9\lambda^2-9\lambda+81=0$$

To verify Cayley - Hamilton theorem, we have to show that $A^3-9A^2-9A+81I=0$ —(1)

For the given matrix A we have

$$A^2 = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 17 & -16 & -16 \\ -16 & 41 & 32 \\ -16 & 32 & 41 \end{bmatrix}$$

and $A^3 = A^2 . A = \begin{bmatrix} 17 & -16 & -16 \\ -16 & 41 & 32 \\ -16 & 32 & 41 \end{bmatrix} . \begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 81 & -144 & -180 \\ -144 & 333 & 324 \\ -180 & 324 & 315 \end{bmatrix}$

Putting these in eqn (1), we get

$$A^3-9A^2-9A+81I = \begin{bmatrix} 81 & -144 & -180 \\ -144 & 333 & 324 \\ -180 & 324 & 315 \end{bmatrix} - 9\begin{bmatrix} 17 & -16 & -16 \\ -16 & 41 & 32 \\ -16 & 32 & 41 \end{bmatrix}$$

$$-9\begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix} + 81\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since A satisfies its characteristic eqn, so Cayley - Hamilton theorem is verified.

To find A^{-1} we multiply both sides of eqn (1) by A^{-1}. Thus

$$A^{-1}. A^3 - 9A^{-1}. A^2 - 9A^{-1}A + 81 I . A^{-1} = 0$$

$$\Rightarrow A^2-9A-9I+81A^{-1}=0 \Rightarrow A^{-1}=\frac{1}{81}[-A^2+9A+9I]$$

$$\Rightarrow A^{-1} = \frac{1}{81}\left\{\begin{bmatrix} -17 & 16 & 16 \\ 16 & -41 & -32 \\ 16 & -32 & -41 \end{bmatrix} + \begin{bmatrix} 9 & 0 & -36 \\ 0 & 45 & 36 \\ -36 & 36 & 27 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}\right\}$$

$$= \frac{1}{81}\begin{bmatrix} 1 & 16 & -20 \\ 16 & 13 & 4 \\ -20 & 4 & -5 \end{bmatrix} = \begin{bmatrix} 1/81 & 16/81 & -20/81 \\ 16/81 & 13/81 & 4/81 \\ -20/81 & 4/81 & -5/81 \end{bmatrix}$$