Total Differentiation

Important deductions:- Let $z = f(x, y)$ then

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} \quad \text{——— (1)}$$

If $t = x$, then $\frac{dx}{dt} = 1$

putting in (1) we get $\frac{dz}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx}$ — (2)

Differentiating of Implicit Function :-

Let implicit fun. $z = f(x, y) = 0$

or, $\frac{dz}{dx} = 0$

putting in eqn (2) we have

$$0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx}$$

or, $\frac{\partial f}{\partial y}\cdot\frac{dy}{dx} = -\frac{\partial f}{\partial x} \Rightarrow \frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$ ——— (3)

We can find $\frac{d^2y}{dx^2}$ by diff. eqn (3)

Let $\frac{\partial f}{\partial x} = p$, $\frac{\partial f}{\partial y} = q$, $\frac{\partial^2 f}{\partial x^2} = r$, $\frac{\partial^2 f}{\partial x \partial y} = s$, $\frac{\partial^2 f}{\partial y^2} = t$

From (3) $\frac{dy}{dx} = \frac{-p}{q}$

$$\therefore \frac{d^2y}{dx^2} = -\frac{\left[q\frac{dp}{dx} - p\frac{dq}{dx}\right]}{q^2}. \quad \text{——— (4)}$$

But $\frac{dp}{dx} = \frac{\partial p}{\partial x} + \frac{\partial p}{\partial y}\cdot\frac{dy}{dx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\cdot\frac{dy}{dx}$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y \partial x}\cdot\frac{dy}{dx} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y \partial x}\left[\frac{-\partial f/\partial x}{\partial f/\partial y}\right]$$

$$= r - s\cdot\frac{p}{q} = \frac{qr - ps}{q}$$

Similarly $\frac{dq}{dx} = \frac{\partial q}{\partial x} + \frac{\partial q}{\partial y}\cdot\frac{dy}{dx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\left(\frac{-p}{q}\right)$

$$= \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y^2}\cdot\frac{p}{q} = s - t\frac{p}{q} = \frac{qs - tp}{q}$$

Substituting in eqn (4), we get

$$\frac{d^2y}{dx^2} = -\frac{\left[q \cdot \frac{qr-ps}{q} - p \cdot \frac{qs-tp}{q}\right]}{q^2}$$

$$= -\frac{(q^2r - 2pqs + p^2t)}{q^3}$$

Q. 1. Find $\frac{dy}{dx}$ if $x^3+3x^2y+6xy^2+y^3=1$

Soln :- Let $f(x,y) = x^3+3x^2y+6xy^2+y^3-1 = 0$

$$\frac{\partial f}{\partial x} = 3x^2+6xy+6y^2$$

$$\frac{\partial f}{\partial y} = 3x^2+12xy+3y^2$$

$$\therefore \frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{3x^2+6xy+6y^2}{3x^2+12xy+3y^2} = -\frac{x^2+2xy+2y^2}{x^2+4xy+y^2}$$

Q. 2. Find $\frac{dy}{dx}$ when $(\cos x)^y = (\sin y)^x$

Soln :- Let $f(x,y) = (\cos x)^y - (\sin y)^x = 0$

$$\therefore \frac{\partial f}{\partial x} = y(\cos x)^{y-1}(-\sin x) - (\sin y)^x \log \sin y$$

$$= -[y \sin x (\cos x)^{y-1} + (\sin y)^x \log \sin y]$$

and $\frac{\partial f}{\partial y} = (\cos x)^y \log \cos x - x(\sin y)^{x-1}\cos y$

then $$\frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y} = \frac{y\sin x(\cos x)^{y-1} + (\sin y)^x \log \sin y}{(\cos x)^y \log \cos x - x(\sin y)^{x-1}\cos y} \quad \text{—(1)}$$

putting $(\cos x)^y$ for $(\sin y)^x$ in eqn. (1)

$$\frac{dy}{dx} = \frac{y\sin x(\cos x)^{y-1} + (\cos x)^y \log \sin y}{(\cos x)^y \log \cos x - \frac{x(\cos x)^y}{\sin y} \cdot \cos y}$$

$$= \frac{(\cos x)^y[y\tan x + \log \sin y]}{(\cos x)^y[\log \cos x - x\cot y]} = \frac{y\tan x + \log \sin y}{\log \cos x - x\cot y}$$

Home Assignment

Q. 1. Evaluate the following limits

(i) $\lim_{x \to 0} \frac{e^{3x}-1}{\tan x}$ (ii) $\lim_{x \to 0} \frac{\log \tan x}{\log x}$

(iii) $\lim_{x \to \frac{\pi}{4}} \frac{\sec^2 x - 2\tan x}{1+\cos 4x}$ (iv) $\lim_{x \to 1} \frac{x - x^x}{1+\log x - x}$

(v) $\lim_{x \to 0} \left(\frac{1}{x} - \cot x\right)$ (vi) $\lim_{x \to \frac{\pi}{2}} (\cos x)^{\cos x}$

(vii) $\lim_{x \to 0} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right)$ (viii) $\lim_{x \to \infty} x\left(e^{\frac{1}{x}}-1\right)$

Q. 2. Verify Euler's theorem for the function:

$$u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$$

Q. 3. If $u = e^{xyz}$ find the value of $\frac{\partial^3 u}{\partial x \partial y \partial z}$.

Q. 4. If $u = \log(x^3+y^3+z^3-3xyz)$ show that

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$$

Q. 5. If z be Homogeneous function of degree n, show that

(i) $x.\frac{\partial^2 z}{\partial x^2} + y.\frac{\partial^2 z}{\partial x \partial y} = (n-1)\frac{\partial z}{\partial x}$

(ii) $x^2.\frac{\partial^2 z}{\partial x^2} + 2xy.\frac{\partial^2 z}{\partial x \partial y} + y^2.\frac{\partial^2 z}{\partial y^2} = n(n-1)z$

Q. 6. If $u = \tan^{-1}\frac{x^3+y^3}{x-y}$, prove that $x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = \sin 2u$.

Q. 7. If $u = f(r)$ and $x = r\cos\theta$, $y = r\sin\theta$ prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r}f'(r)$$

Q. 8. If $u = u(y-z, z-x, x-y)$ prove that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

Q.9. If $x^3+y^3-3axy=0$; find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

Q.10. If $u = x\log xy$ where $x^3+y^3+3xy=1$. Find $\frac{du}{dx}$

Answers :-

Q.1. (I) 3 (II) 1 (III) $\frac{1}{2}$ (IV) 2 (V) 0 (VI) 1 (VII) $-\frac{1}{3}$ (VIII) 1.

Q.3. $\frac{\partial^3 u}{\partial x \partial y \partial z} = e^{xyz}(1+3xyz+x^2y^2z^2)$

Q.9. $\frac{dy}{dx} = \frac{ay-x^2}{y^2-ax}$ and $\frac{d^2y}{dx^2} = \frac{2a^3xy}{(ax-y^2)^3}$

Q.10. $\frac{du}{dx} = 1+\log xy - \frac{x}{y}\cdot\frac{x^2+y}{x+y^2}$