Q.4. Show that $\text{Curl}(\text{Curl}\,\vec{f}) = \text{grad}(\text{div}\,\vec{f}) - \nabla^2\vec{f}$

Soln :- Let $\vec{f} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$

$$\therefore \text{curl}\,\vec{f} = \nabla\times\vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \hat{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) + \hat{j}\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) + \hat{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

Now, $\text{Cur}(\text{curl}\,\vec{f}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} & \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} & \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \end{vmatrix}$

$$= \sum \hat{i}\left[\frac{\partial}{\partial y}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\right]$$

$$= \sum \hat{i}\left[\left(\frac{\partial^2 f_2}{\partial y\partial x} + \frac{\partial^2 f_3}{\partial z\partial x}\right) - \left(\frac{\partial^2 f_1}{\partial y^2} + \frac{\partial^2 f_1}{\partial z^2}\right)\right]$$

$$= \sum \hat{i}\left[\left(\frac{\partial^2 f_2}{\partial y\partial x} + \frac{\partial^2 f_3}{\partial z\partial x}\right) - \left(\frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_1}{\partial y^2} + \frac{\partial^2 f_1}{\partial z^2} - \frac{\partial^2 f_1}{\partial x^2}\right)\right]$$

$$= \sum \hat{i}\left[\frac{\partial}{\partial x}\left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) - \left(\frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_1}{\partial y^2} + \frac{\partial^2 f_1}{\partial z^2}\right)\right]$$

$$= \sum \hat{i}\left[\frac{\partial}{\partial x}(\text{div}\,\vec{f}) - \nabla^2 f_1\right]$$

$$= \sum \hat{i}\,\frac{\partial}{\partial x}(\nabla\cdot\vec{f}) - \nabla^2\sum \hat{i} f_1$$

$$= \text{grad}\,(\nabla\cdot\vec{f}) - \nabla^2\vec{f} \qquad \text{Proved}$$

Alternative method :- we have $\text{curl}(\text{curl}\,\vec{f}) = \nabla\times(\nabla\times\vec{f})$

Treating ∇ as vector ∇_1 and ∇_2 and using formula for vector triple product :- $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$

Thus $\nabla\times(\nabla\times\vec{f}) = \nabla_1\times(\nabla_2\times\vec{f}) = \nabla_2(\nabla_1\cdot\vec{f}) - (\nabla_1\cdot\nabla_2)\vec{f}$

now dropping suffix of ∇, we get, $\nabla\times(\nabla\times\vec{f}) = \nabla(\nabla\cdot\vec{f}) - (\nabla\cdot\nabla)\vec{f}$

$$= \nabla(\nabla\cdot\vec{f}) - \nabla^2\vec{f}$$

$$= \text{grad}(\nabla\cdot\vec{f}) - \nabla^2\vec{f}.$$

Q.5. Prove that $\nabla^2 u = 0$ if $u = x^2 - y^2$

Soln :- $\nabla u = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 - y^2) = 2x\hat{i} - 2y\hat{j}$

$\nabla\cdot(\nabla u) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(2x\hat{i} - 2y\hat{j}) = 2 - 2 = 0$

$\Rightarrow \nabla^2 u = 0$ Hence proved.

Q.6. Show that $\vec{f} = (y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}$ is (a) Solenoidal and (b) irrotational.

Soln :(a) $\nabla \cdot \vec{f} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left[(y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}\right]$

$= -2 + 2x - 2x + 2$

$= 0$

So $\nabla \cdot \vec{f} = 0$ which implies that $\vec{f}$ is Solenoidal.

(b) We have $\nabla \times \vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 - z^2 + 3yz - 2x & 3xz + 2xy & 3xy - 2xz + 2z \end{vmatrix}$

$= \hat{i}(3x - 3x) - \hat{j}(3y - 2z + 2z - 3y) + \hat{k}(3z + 2y - 2y - 3z)$

So $\nabla \times \vec{f} = 0$. Therefore $\vec{f}$ is irrotational.

Q.7. Prove that $\nabla^2(r^n) = n(n+1)r^{n-2}$

Soln :- Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, So $|\vec{r}| = r = \sqrt{x^2 + y^2 + z^2}$

Then $r^2 = x^2 + y^2 + z^2$

Differentiating w.r.t. x partially, we get

$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

From definition of laplacian operator ∇^2 we have

$\nabla^2(r^n) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(r^n)$ —— I

But, $\frac{\partial^2}{\partial x^2}(r^n) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} r^n\right) = \frac{\partial}{\partial x}\left(n r^{n-1}\frac{\partial r}{\partial x}\right) = \frac{\partial}{\partial x}\left(n r^{n-1} \cdot \frac{x}{r}\right)$, Since $\frac{\partial r}{\partial x} = \frac{x}{r}$

$= \frac{\partial}{\partial x}(n r^{n-2} x) = n\left[(n-2) r^{n-3}\frac{\partial r}{\partial x} \cdot x + r^{n-2}\right]$

$= n(n-2) r^{n-3} \cdot \frac{x}{r} \cdot x + n r^{n-2} = n(n-2) r^{n-4} x^2 + n r^{n-2}$

Similarly $\frac{\partial^2}{\partial y^2}(r^n) = n(n-2) r^{n-4} y^2 + n r^{n-2}$

and $\frac{\partial^2}{\partial z^2}(r^n) = n(n-2) r^{n-4} z^2 + n r^{n-2}$

Putting these in equ. I, we have

$\nabla^2(r^n) = n\left[(n-2) r^{n-4}(x^2 + y^2 + z^2) + 3r^{n-2}\right]$

$= n\left[(n-2) r^{n-4} \cdot r^2 + 3r^{n-2}\right] = n\left[(n-2) r^{n-2} + 3r^{n-2}\right]$

$= n\left[(n-2+3) r^{n-2}\right]$

$= n(n+1) r^{n-2}$ Hence Proved