Model questions on Rolle's theorem, Mean value theorem, Taylor's and Maclaurin's theorems:

Q.1. Verify Rolle's theorem for the following functions and find points in the interval where derivative vanishes:

(i) $f(x) = \sin x + \cos x - 1$ in $[0, \frac{\pi}{2}]$

(ii) $f(x) = e^{1-x^2}$ in $[-1, 1]$

(iii) $f(x) = (x^2-1)(x-2)$ in $[-1, 2]$

(iv) $f(x) = \log\left(\frac{x^2+ab}{(a+b)x}\right)$ in $[a, b]$, $a>0$

Q.2. Discuss the applicability of Rolle's theorem for the following functions:

(i) $f(x) = |x|$ in $[-1, 1]$

(ii) $f(x) = \begin{cases} x^2-4, & x \le 1 \\ 5x-8, & x > 1 \end{cases}$

Q.3. Verify Lagrange's mean value theorem for the following functions in the given interval and find 'c' of this theorem:

(i) $f(x) = \tan^{-1} x$ in $[0, 1]$

(ii) $f(x) = \sqrt{x^2-1}$ in $[1, 3]$

(iii) $f(x) = x(x-1)(x-2)$ in $[0, \frac{1}{2}]$

Q.4. Find a point on the curve $y = x^3$ where the tangent is parallel to the chord joining $(1, 1)$ and $(3, 27)$

Q.5. Expand $2x^3 + 7x^2 + x - 6$ in powers of $(x-2)$ using Taylor's theorem.

Q.6. Expand the following functions:

(i) $f(x) = e^x$ in power of $(x-1)$ upto four terms

(ii) $f(x) = \log \sin x$ in power of $(x-2)$.

(iii) $f(x) = \tan\left[x + \frac{\pi}{4}\right]$ in ascending powers of x upto terms in x^4.

Q.7. Evaluate $\sqrt{25.15}$ using Taylor's theorem

Q.8. (i) Expand $\log(1+e^x)$ by Maclaurin's theorem upto term x^4.

(ii) Expand $\tan^{-1}x$ in powers of x.

(iii) Expand $\log(1+x)$ in powers of x.

Q.9. (i) Expand $e^{ax}\sin by$ in powers of x and y upto third degree terms

(ii) Expand $e^x \sin y$ in powers of x and y upto third degree terms.

(iii) Expand $e^x \log(1+y)$ in powers of x and y upto third degree terms.

Q.10. (i) Find the Taylor's series expansion of x^y near the point $(1,1)$ upto second degree terms

(ii) Expand $e^x \sin y$ at $(-1, \frac{\pi}{4})$ in Taylor's series.

(iii) Expand $e^x \cos y$ about the point $(1, \frac{\pi}{4})$

(iv) Expand $x^2y + 3y - 2$ in powers of $(x-1)$ and $(y+2)$ using Taylor's theorem.

Answers to model questions :

Q.1. (i) $\frac{\pi}{4}$ (ii) 0 (iii) $\frac{2 \pm \sqrt{7}}{3}$ (iv) $\sqrt{ab}$

Q.2. (i) Not applicable (ii) Not applicable

Q.3. (i) $c = \sqrt{\frac{4-\pi}{\pi}}$ (ii) $c = \sqrt{2}$ (iii) $c = 0.2362$

Q.4. $\left(\frac{\sqrt{39}}{3}, \frac{13\sqrt{39}}{9}\right)$

Q.5. $f(x) = 40 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3$

Q.6. (i) $e^x = e\left[1 + (x-1) + \frac{(x-1)^2}{\lfloor 2} + \frac{(x-1)^3}{\lfloor 3} + \cdots\right]$

(ii) $\log \sin x = \log \sin 2 + (x-2)\cot 2 - \frac{(x-2)^2}{2}\operatorname{cosec}^2 2 + \frac{(x-2)^3}{3}\operatorname{cosec}^2 2 \cot 2 + \frac{2}{-} - - -$

(iii) $\tan\left[x + \frac{\pi}{4}\right] = 1 + 2x + 2x^2 + \frac{8}{3}x^3 + \frac{10}{3}x^4 + - - - -$

Q.7. $\sqrt{25.15} = 5 + 0.015 - 0.000225 + 0.0000000675 + - - -$
$= 5.01478$ (approx)

Q.8. (i) $\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + - - -$

(ii) $\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + - - - -$

(iii) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + - - - -$

Q.9. (i) $e^{ax}\sin by = by + abxy + \frac{1}{2}a^2bx^2y - \frac{b^3}{6}y^3 + - - -$

(ii) $e^x \sin y = y + xy + \frac{x^2y}{2} - \frac{y^3}{6} + - - -$

(iii) $e^x \log(1+y) = y + xy - \frac{y^2}{2} + \frac{x^2y}{2} - \frac{xy^2}{2} + \frac{y^3}{3} + - - -$

Q.10. (i) $x^y = 1 + (x-1) + (x-1)(y-1) + - - - -$

(ii) $e^x \sin y = \frac{1}{e\sqrt{2}}\left[1 + (x+1) + \left(y - \frac{\pi}{4}\right)\right] + - - - -$

(iii) $e^x \cos y = \frac{e}{\sqrt{2}}\left[1 + (x-1) - \left(y - \frac{\pi}{4}\right) + \frac{1}{2}(x-1)^2 - (x-1)\left(y - \frac{\pi}{4}\right) - \frac{1}{2}\left(y - \frac{\pi}{4}\right)^2 + - - - - -\right.$

(iv) $x^2y + 3y - 2 = -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + (x-1)^2(y+2)$.