Divergence of a vector point function

If we want to consider the rate of change of a vector point function $\vec{f}$, there are two ways of operating the vector operator ∇ to the vector $\vec{f}$, namely $\nabla \cdot \vec{f}$ and $\nabla \times \vec{f}$. These two cases are called Divergence of a vector function and Curl of a vector function respectively.

If we consider a vector field as a fluid flow, then at every point in the flow, we need to measure the rate of flow of fluid from that point which is denoted by div $\vec{f}$ and the amount of spin possessed by the particles of fluid at that point is denoted by Curl $\vec{f}$.

Let $\vec{f} = f_1 \hat{i} + f_2 \hat{j} + f_3 \hat{k}$ be a vector function, where f_1, f_2, f_3 are scalar point functions which is differentiable at each point of the given space. Then divergence of $\vec{f}$ is denoted by $\nabla \cdot \vec{f}$ or div $\vec{f}$.

$$\nabla \cdot \vec{f} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (f_1 \hat{i} + f_2 \hat{j} + f_3 \hat{k})$$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}.$$

Note that Divergence of vector function is a scalar quantity since it is scalar product of ∇ and vector function $\vec{f}$.

If $\vec{f}$ is a constant vector then div $\vec{f} = 0$

<u>Physical interpretation of Divergence</u> :- div $\vec{v}$ gives the rate of flow of fluid per unit volume at a point. It is also known as 'Fluid flux'. Similarly if $\vec{v}$ represents the electric flow then div $\vec{v}$ is the amount of 'electric flux' which passes a unit volume in unit time.

If div $\vec{v} = 0$, the fluid is said to be incompressible i.e. there is no gain or loss in the volume of fluid, then $\vec{v}$ is called 'solenoidal'.

Q.1. Find div $\vec{V}$ where $\vec{V} = 3x^2y\,\hat{i} + z\hat{j} + x^2\hat{k}$

Soln :- we know that div $\vec{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$

Here, $V_1 = 3x^2y$, $V_2 = z$ and $V_3 = x^2$

Therefore, div $\vec{V} = 6xy$ Ans

Q.2. Find div $(3x^2\hat{i} + 5xy^2\hat{j} + xyz^3\hat{k})$ at the point $(1,2,3)$.

Soln :- Let $\vec{f} = 3x^2\hat{i} + 5xy^2\hat{j} + xyz^3\hat{k}$

$= f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ (say)

Then div $\vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = 6x + 10xy + 3xyz^2$

At $(1,2,3)$ div $\vec{f} = 6 + 20 + 54 = 80$ Ans

Q.3. Find div $\vec{f}$, where $\vec{f} = \text{grad}(x^3+y^3+z^3-3xyz)$

Soln :- Given $\vec{f} = \nabla(x^3+y^3+z^3-3xyz)$

$= (\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z})(x^3+y^3+z^3-3xyz)$

$= (3x^2-3yz)\hat{i} + (3y^2-3xz)\hat{j} + (3z^2-3xy)\hat{k}$

$= f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ (say)

Then div $\vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

$= 6x + 6y + 6z = 6(x+y+z)$ Ans

Q.4. Show that $\vec{V} = (y^2-z^2+3yz-2x)\hat{i} + (3xz+2xy)\hat{j}$
$+ (3xy-2xz+2z)\hat{k}$ is solenoidal.

Soln :- we know that if div $\vec{V} = 0$, $\vec{V}$ is called solenoidal

Since $\nabla\cdot\vec{V} = (\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z})[(y^2-z^2+3yz-2x)\hat{i} + (3xz+2xy)\hat{j}$
$+ (3xy-2xz+2z)\hat{k}]$

$= -2 + 2x - 2x + 2 = 0$

Hence $\vec{V}$ is solenoidal.

Curl of a vector point function :-

Let $\vec{f} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ be a vector point function, where f_1, f_2 and f_3 are scalar point functions. If f is differentiable at each point (x, y, z) in the given space then the Curl (or Rotation) of $\vec{f}$ is denoted by

Curl $\vec{f}$ or $\nabla \times \vec{f} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times (f_1\hat{i} + f_2\hat{j} + f_3\hat{k})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\hat{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\hat{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\hat{k}$$

Note that Curl $\vec{f}$ is a vector point function. It is vector product of ∇ and vector $\vec{f}$. If $\vec{f}$ is constant vector, then Curl $\vec{f} = 0$

Physical interpretation of curl :- Consider a rigid body rotating about a fixed axis with angular velocity $\vec{w} = w_1\hat{i} + w_2\hat{j} + w_3\hat{k}$. Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be position vector of any point $P(x, y, z)$ on the body. The linear velocity $\vec{v}$ at a point P is given by $\vec{v} = \vec{w} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ w_1 & w_2 & w_3 \\ x & y & z \end{vmatrix} = (w_2 z - w_3 y)\hat{i} + (w_3 x - w_1 z)\hat{j} + (w_1 y - w_2 x)\hat{k}$

thus Curl $\vec{v} = \nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ w_2 z - w_3 y & w_3 x - w_1 z & w_1 y - w_2 x \end{vmatrix} = 2(w_1\hat{i} + w_2\hat{j} + w_3\hat{k}) = 2\vec{w}$

So, $\vec{w} = \frac{1}{2}$ Curl $\vec{v}$. It follows that angular velocity at any point is equal to half the curl of linear velocity at that point of the body. Thus curl is a measure of rotation. The name 'curl' is used for rotation. It measures the rotation of fluid particles at any given point.

If Curl $\vec{v} = 0$, then vector $\vec{v}$ is called 'irrotational vector'

Laplacian operator (∇^2) :- The Laplacian operator is denoted as $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

If ϕ is scalar point function, then $\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$

Laplace eq. is given by $\nabla^2\phi = 0$

Q.1. Find Curl $\vec{f}$, where $\vec{f} = \text{grad}(x^3+y^3+z^3-3xyz)$

Soln :- Given $\vec{f} = \nabla(x^3+y^3+z^3-3xyz)$

$$= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x^3+y^3+z^3-3xyz)$$

$$= (3x^2-3yz)\hat{i}+(3y^2-3xz)\hat{j}+(3z^2-3xy)\hat{k}$$

$\therefore$ Curl $\vec{f} = \nabla\times\vec{f} = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times\vec{f}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix} = \hat{i}(-3x+3x)+\hat{j}(-3y+3y)+\hat{k}(-3z+3z)$$

$$= \hat{i}(0)+\hat{j}(0)+\hat{k}(0)$$

$$= \vec{0} \quad \text{Ans}$$

Q.2. Show that the vector $\vec{v} = (yz)\hat{i}+(zx)\hat{j}+(xy)\hat{k}$ is irrotational

Soln :- Since Curl $\vec{v} = \nabla\times\vec{v} = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times\left[yz\hat{i}+zx\hat{j}+xy\hat{k}\right]$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & zx & xy \end{vmatrix} = \hat{i}(x-x)+\hat{j}(y-y)+\hat{k}(z-z)$$

$$= \vec{0}$$

Hence $\vec{v}$ is irrotational.

Q.3. Show that (i) $\text{div}(\text{grad} f) = \nabla^2 f$
(ii) Curl $(\text{grad}\,\phi) = \vec{0}$ (iii) $\text{div}(\text{curl}\,\vec{v}) = 0$

Soln : (i) $\text{div}(\text{grad} f) = \nabla\cdot\nabla f = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}\frac{\partial f}{\partial x}+\hat{j}\frac{\partial f}{\partial y}+\hat{k}\frac{\partial f}{\partial z}\right)$

$$= \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2} = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)f$$

$$= \nabla^2 f \quad \text{Hence Proved}$$

(ii) Curl $(\text{grad}\,\phi) = \nabla\times\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times\left(\hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}\right)$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial\phi/\partial x & \partial\phi/\partial y & \partial\phi/\partial z \end{vmatrix} = \hat{i}\left(\frac{\partial^2\phi}{\partial y\partial z}-\frac{\partial^2\phi}{\partial z\partial y}\right)+\hat{j}\left(\frac{\partial^2\phi}{\partial z\partial x}-\frac{\partial^2\phi}{\partial x\partial z}\right)+\hat{k}\left(\frac{\partial^2\phi}{\partial x\partial y}-\frac{\partial^2\phi}{\partial y\partial x}\right)$$

$$= \hat{i}(0)+\hat{j}(0)+\hat{k}(0) = \vec{0} \quad \text{Hence Proved}$$

(iii) To prove $\text{div}(\text{curl}\,\vec{v}) = 0$. Let $\vec{v} = v_1\hat{i}+v_2\hat{j}+v_3\hat{k}$, then

Curl $\vec{v} = \nabla\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_1 & v_2 & v_3 \end{vmatrix} = \hat{i}\left(\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z}\right)+\hat{j}\left(\frac{\partial v_1}{\partial z}-\frac{\partial v_3}{\partial x}\right)+\hat{k}\left(\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y}\right)$

$\therefore \text{div}(\text{curl}\,\vec{v}) = \nabla\cdot(\nabla\times\vec{v})$

$$= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\left[\hat{i}\left(\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z}\right)+\hat{j}\left(\frac{\partial v_1}{\partial z}-\frac{\partial v_3}{\partial x}\right)+\hat{k}\left(\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y}\right)\right]$$

$$= \frac{\partial^2 v_3}{\partial x\partial y}-\frac{\partial^2 v_2}{\partial x\partial z}+\frac{\partial^2 v_1}{\partial y\partial z}-\frac{\partial^2 v_3}{\partial y\partial x}+\frac{\partial^2 v_2}{\partial z\partial x}-\frac{\partial^2 v_1}{\partial z\partial y}$$

$$= 0 \quad \text{hence proved.}$$