Wave-Particle Duality : complementarity

Quantum systems are neither pure particle nor a pure wave.

Particle Aspect of wave :-

In CM, Particles and waves are different

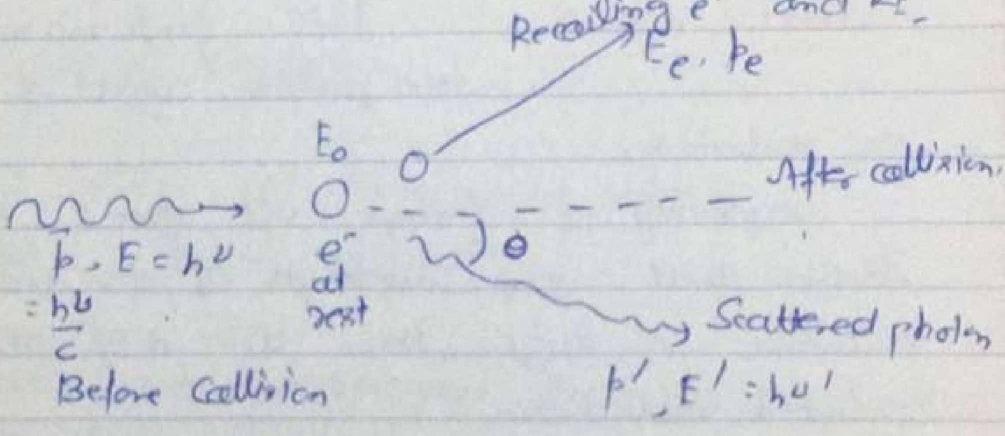
Compton Effect ⇒ 1923. Confirmation of the particle aspect of waves or radiation.

Scattering of X-rays off free e^-, he found that the wavelength of the scattered radiation is larger than the λ of the incident radiation.

It can be explained only by assuming that X-ray photon behave like particles. According to CM, incident and scattered radiation should have the same wavelength. But experimentally it is observed that the ~~intensity~~ $\Delta\lambda$ depends not on the intensity of the incident radiation, but only on the scattering angle.

Compton succeded in explaining his experimental results only after treating the radiation as stream of particles photons colliding elastically.

Elastic collisions → conservation of E & p.



Conservation of linear moment

$$\vec{p} = p_e + p'$$

$$p_e^2 = (\vec{p} - p')^2$$

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* Since, $E^2 = p^2c^2 + m_0^2c^4$

for photon, $E^2 = p^2c^2$, $p = \frac{E}{c} = \frac{h\nu}{c}$, also $E = h\nu = hc/\lambda$

$$p_e^2 = p^2 + p'^2 - 2pp'\cos\theta$$ (~~since vertical components are zero~~). cancel out

(taking vector dot product

$$\vec{p}_e \cdot \vec{p}_e = (\vec{p} - \vec{p}') \cdot (\vec{p} - \vec{p}')$$

$$= \vec{p}\cdot\vec{p} + \vec{p}'\cdot\vec{p}' - \vec{p}\cdot\vec{p}' - \vec{p}'\cdot\vec{p}$$

$$\vec{p}_e^{\,2} = \vec{p}^{\,2} + \vec{p}'^{\,2} - 2\vec{p}\vec{p}'\cos\theta.$$)

$$p_e^2 = \frac{h^2\nu^2}{c^2} + \frac{h^2\nu'^2}{c^2} - 2\,\frac{h\nu}{c}\times\frac{h\nu'}{c}\cos\theta$$

$$p_e^2 = \frac{h^2}{c^2}\left(\nu^2 + \nu'^2 - 2\nu\nu'\cos\theta\right) \quad \text{—— (1)}$$

Now, energy conservation,

$$E_0 = m_e c^2 \longrightarrow \text{before collisions.}$$

$$E_e = \sqrt{p_e^2 c^2 + m_e^2 c^4} \quad \text{—— (2)}$$

Using eqn (1) & (2), we have

$$E_e = h\sqrt{\nu^2 + \nu'^2 - 2\nu\nu'\cos\theta + \frac{m_e^2 c^4}{h^2}}$$

Now energies of incident and scattered photons are.

$$E = h\nu \quad , \quad E' = h\nu'$$

$$E + E_0 = E' + E_e \rightarrow \text{conservation of energy.}$$

$$h\nu + m_e c^2 = h\nu' + h\sqrt{\nu^2 + \nu'^2 - 2\nu\nu'\cos\theta + \frac{m_e^2 c^4}{h^2}}$$

$$\nu - \nu' + \frac{m_e c^2}{h} = \sqrt{\nu^2 + \nu'^2 - 2\nu\nu'\cos\theta + \frac{m_e^2 c^4}{h^2}}$$

Squaring both sides, we get.

$$\left(\nu - \nu' + \frac{m_e c^2}{h}\right)^2 = \nu^2 + \nu'^2 - 2\nu\nu'\cos\theta + \frac{m_e^2 c^4}{h^2}$$

$$(\nu - \nu')^2 + \left(\frac{m_e^2 c^2}{h}\right)^2 + 2(\nu - \nu')\left(\frac{m_e c^2}{h}\right)$$

$$= \nu^2 + \nu'^2 - 2\nu\nu'\cos\theta + \frac{m_e^2 c^4}{h^2}$$

$$\cancel{\nu^2} + \cancel{\nu'^2} - 2\nu\nu' + \frac{\cancel{m_e^2 c^4}}{h^2} + 2\frac{m_e c^2 \nu}{h} - 2\frac{m_e c^2 \nu'}{h}$$

$$= \cancel{\nu^2} + \cancel{\nu'^2} - 2\nu\nu'\cos\theta + \frac{\cancel{m_e^2 c^4}}{h^2}$$

$$\frac{m_e c^2 \nu}{h} - \frac{m_e c^2 \nu'}{h} = \nu\nu' - \nu\nu'\cos\theta$$

$$= \nu\nu'(1-\cos\theta)$$

$$\frac{m_e c^2}{h}(\nu - \nu') = \nu\nu'(1-\cos\theta)$$

$$\frac{\nu}{\nu\nu'} - \frac{\nu'}{\nu\nu'} = \frac{h}{m_e c^2}(1-\cos\theta)$$

$$\frac{1}{\nu'} - \frac{1}{\nu} = \frac{h}{m_e c^2}(1-\cos\theta)$$

$$\frac{c}{\nu'} - \frac{c}{\nu} = \frac{h}{m_e c}(1-\cos\theta)$$

$$\lambda' - \lambda = \frac{h}{m_e c}(1-\cos\theta)$$

$$\boxed{\Delta\lambda = \lambda' - \lambda = \frac{h}{m_e c}(1-\cos\theta)}$$

$$\lambda_c = \frac{h}{m_e c}$$

$\lambda_c \rightarrow$ compton wavelength.

$\Delta\lambda$ does not depend on the frequency but only depends on the scattering angle.

Ques $\Rightarrow$ γ-rays are scattered from e^- initially at rest. Assume photons are backscattered and their energies are much larger than the e^- rest mass energy, $E >> m_e c^2$

(a) Calculate the wavelength shift

(b) Show that the energy of the scattered photons is half the rest mass energy of the e^-, regardless of the energy of the incident photons.

(c) Calculate the e^- recoils kinetic energy if the incident photon is 150 MeV.

Sol^n. (a) When photon backscatter, $\theta = \pi$.

$$\Delta\lambda = \lambda' - \lambda = 2\lambda_c \sin^2\frac{\pi}{2} = 2\lambda_c$$

$$= 4.86 \times 10^{-12} \text{ m}.$$

(b) Since, the energy of the scattered photon,

$$E' = \frac{hc}{\lambda'} = \frac{hc}{\lambda + \frac{2h}{m_e c}} = \frac{m_e c^2}{\left(\frac{m_e c^2 \lambda}{hc} + 2\right)}$$

$$= \frac{m_e c^2}{\left(\frac{m_e c^2}{E} + 2\right)} = \frac{m_e c^2}{2}\left[\frac{m_e c^2}{2E} + 1\right]^{-1}$$

$$E' = \frac{m_e c^2}{2}\left[1 + \frac{m_e c^2}{2E}\right]^{-1}$$

If $E >> m_e c^2$,

then, $E' = \frac{m_e c^2}{2}\left[1 - \underbrace{\frac{m_e c^2}{2E}}_{\text{neglected}}\right]$

$\left(\frac{m_e c^2}{2E} << 1\right)$

$E' \simeq \frac{m_e c^2}{2} = 0.25 \text{ MeV}$

(c) If $E = 150 \text{ MeV}$.

$K_e = E - E' \simeq 150 - 0.25 = 149.75 \text{ MeV}$

Photoelectric Effect ⇒

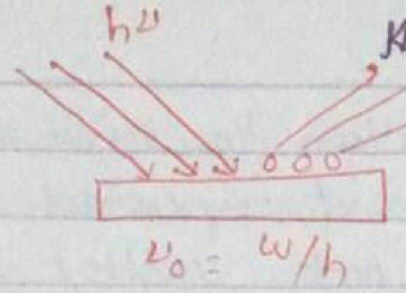
It directly gives a direct confirmation of the energy quantization of light.
In 1887, Hertz discovered the photo-electric effect.
Following experimental laws are discovered:

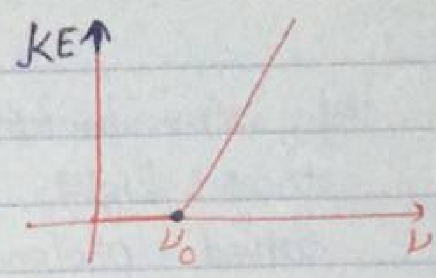
* If frequency < threshold frequency of metal (ν_0) then no electron will eject.

* No matter how low the intensity, e^- will be ejected instantly if $\nu > \nu_0$

* The no. of e^- will be increased if intensity increase.

* The KE of e^- depends on the frequency but not on the intensity of the beam. KE varies linearly with the incident frequency.





These findings cannot explained within the context of a purely classical picture of radiation.

According to classical physics, since $I \propto a^2$, any frequency with sufficient intensity can supply the necessary energy to free the e^- from metal.

Also, e^- would keep on absorbing energy at a continuous rate until it gained a sufficient amount then it would leave the metal.

The above conclusions however diagrees utterly with experimental observation. Thus these concepts are indeed erroneous

Einstein succeeded in 1905 in giving a theoretical explanation for the dependence of of photoelectric emission on the frequency of the incident radiation. He assumed that light is made of corpuscles called photons. When photon incident it transmits all its energy to an e^- near surface.

$$\boxed{h\nu = W + K}$$

where K is the KE of e^- leaving the surface of metal.

$$K = h\nu - W = h\nu - h\nu_0$$
$$K = h(\nu - \nu_0)$$

$\nu_0 = W/h$ is called the threshold or cutoff frequency of the metal.

Since K cannot be negative hence, PEE cannot occur for $\nu < \nu_0$

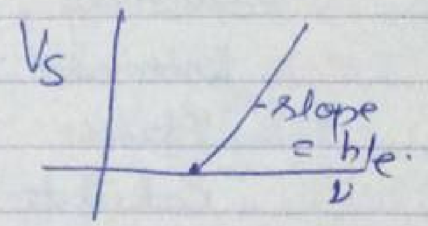
- Lenard measured the KE of e^- experimentally (Cathode + Anode) → evacuated glass → Photocurrent.

V_s → Stopping potential at which all of the electrons, even the most energetic ones will be turned back before reaching the collector. hence the photo-electric current ceases completely. Now,

$$eV_s = \frac{1}{2} m_e v^2 = K.$$

Thus, $eV_s = h\nu - \omega$

$$V_s = \frac{h}{e}\nu - \frac{\omega}{e}$$



$$\boxed{V_s = \frac{hc}{e\lambda} - \frac{\omega}{e}}$$

V_s versus ν is a straight line with the slope now given by h/e.

Millikan in 1916 gave a systematic experimental confirmation of Einstein's photoelectric theory. and found value of h.

In summary, the photoelectric effect does provide compelling evidence for the corpuscular nature of the electromagnetic radiation.

Indeterministic Nature of the Microphysical world ⇒

Waves are not localized in space. It is impossible to trace the motion of individual electrons. These findings inspired Heissberg to postulate the indeterministic nature of the microphysical world and Born to introduce the probabilistic interpretation of quantum mechanics.

Heisenberg's Uncertainty Principle ⇒

According to classical Physics, the future behaviour of the physical system can be determined exactly if the initial conditions are determined (known). Thus, Classical Physics is completely deterministic.
But microphysical particle is represented by a wave and cannot be localized. The classical concepts of exact position, momentum and unique path of a particle therefore

make no sense at the microscopic scale. This is the essence of Heisenberg's uncertainty principle. It states that:

"If the x-component of the momentum of a particle is measured with an uncertainty Δp_x, then its x-position cannot, at the same time be measured more accurately than $\Delta x = \hbar/(2\Delta p_x)$. Thus,

$$\Delta x \Delta p_x \geq \frac{\hbar}{2}, \quad \Delta y \Delta p_y \geq \frac{\hbar}{2}, \quad \Delta z \Delta p_z \geq \frac{\hbar}{2}"$$

Although, it is possible to measure x & p_x of a particle accurately, it is not possible to measure these two simultaneously to an arbitrary accuracy.

According to de Broglie's relation, $p = h/\lambda$

if λ is short, p will be high. That is if $\Delta x \to 0$, $\Delta p_x \to \infty$.

Heisenberg's uncertainty principle can be generalized to any pair of complementary or canonically conjugate dynamical variables.

It is impossible to devise an experiment that can measure simultaneously two complementary variables to arbitrary accuracy (if this were ever achieved, the theory of Quantum Mechanics would collapse.

Also,

$$\boxed{\Delta E \, \Delta t \geq \frac{\hbar}{2}}$$

If the two measurements are separated by a time interval Δt, the measured energies will differ by an amount ΔE which can no way be smaller than $\hbar / \Delta t$.

This can be attributed to the fact that when the first measurement is carried out, the system becomes perturbed and it takes it to a long time to return its initial, unperturbed state.

Expectation Values :

From wave f^n → get information about the probability density for the particle.
In this section, we see how to extract the wide variety of additional information regarding particle. We can know about the momentum, energy etc. of the particle using wave f^n.

The probability, $P(x,t)\,dx = \psi^*(x,t)\psi(x,t)\,dx$
The average or expectation value of x of the particle at instant t is

$$\bar{x} = \int_{-\infty}^{\infty} x\, P(x,t)\, dx$$

or, $$\boxed{\bar{x} = \int_{-\infty}^{\infty} \psi^* x \psi \, dx}$$

or, $$\bar{x} = \frac{\int_{-\infty}^{\infty} \psi^* x \psi \, dx}{\int_{-\infty}^{\infty} P\, dx}$$

since $\int_{-\infty}^{\infty} P\, dx = 1$

Similarly,

$$\overline{x^2} = \int_{-\infty}^{\infty} \psi^* x^2 \psi \, dx.$$

$$\overline{f(x)} = \int_{-\infty}^{\infty} \psi^* f(x) \psi \, dx.$$

$$\bar{p} = \int_{-\infty}^{\infty} \psi^*(x,t) \, p \, \psi(x,t) \, dx.$$

p cannot be written in terms of x because of uncertainty principle.
Considering the free particle wave f^n,

$$\psi(x,t) = \cos(kx - \omega t) + i\sin(kx - \omega t)$$

$$\frac{\partial \psi}{\partial x} = -k\sin(kx-\omega t) + ik\cos(kx-\omega t)$$

$$= ik[\cos(kx-\omega t) + i\sin(kx-\omega t)]$$

Since,

$$= ik[\psi(x,t)]$$

Since, $k = \dfrac{p}{\hbar}$

$$\frac{\partial \psi}{\partial x} = \frac{ip}{\hbar} \psi(x,t)$$

$$p[\psi(x,t)] = -i\hbar \frac{\partial}{\partial x}[\psi(x,t)].$$

or, $\boxed{\hat{p} \rightarrow -i\hbar \frac{\partial}{\partial x}.}$

Similarly,

$$\frac{\partial \psi}{\partial t} = -i\omega[\psi(x)]$$

Since, $\omega = \frac{E}{\hbar}$

$$E[\psi(x,t)] = i\hbar \frac{\partial}{\partial t}[\psi(x,t)]$$

$$\boxed{\hat{E} \longrightarrow i\hbar \frac{\partial}{\partial t}}$$

So,

$$\frac{p^2}{2m} + V(x,t) = E \text{ gives}$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V\psi = i\hbar \frac{\partial \psi}{\partial t}$$ – which is SE

Thus, $\hat{p}$ and $\hat{E}$ are correct.

Now,

$$\bar{p} = \int_{-\infty}^{\infty} \psi^* \hat{p} \psi \, dx.$$

$$\bar{p} = \int_{-\infty}^{\infty} \psi^* \left(-i\hbar \frac{\partial}{\partial x}\right) \psi \, dx$$

$$\boxed{\bar{p} = -i\hbar \int_{-\infty}^{\infty} \psi^* \frac{\partial \psi}{\partial x} \, dx.}$$

Similarly,

$$\bar{E} = i\hbar \int_{-\infty}^{\infty} \psi^* \frac{\partial \psi}{\partial t} \, dx$$

$$\bar{E} = \int_{-\infty}^{\infty} \psi^* \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V\right) \psi(x) \, dx.$$

Another derivation of Schroedinger time dependent wave eqⁿ

We know that,

$$E = \frac{p^2}{2m} + V \Rightarrow E\psi = \frac{p^2}{2m}\psi + V\psi$$

Writing the above eqⁿ in form of operator as:

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{-\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V\psi$$

Operating the above operator on a wave fⁿ ψ, we get,

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V\psi$$

$$\Rightarrow \boxed{-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V\psi = i\hbar \frac{\partial \psi}{\partial t}}$$

Time Independent Schroedinger Wave eqⁿ :-

In many situations, the potential energy of a particle does not depend directly upon time. The wave f^n can be written as,

$$\psi = A \exp[i(kx - \omega t)]$$

$$= A \exp(ikx)\, e^{-i\omega t}$$

$$\psi = A \exp(ikx) \exp(-i\omega t).$$

$$\psi = A\, \phi(x) \exp(-i\omega t) \quad \text{——— (1)}$$

Differentiating eq^n (1) w.r.t. x, we get

$$\frac{\partial \psi}{\partial x} = A \exp(-i\omega t) \frac{\partial \phi}{\partial x}$$

or, $$\frac{\partial^2 \psi}{\partial x^2} = A \exp(-i\omega t) \frac{\partial^2 \phi}{\partial x^2} \quad \text{——— (2)}$$

Also, differentiating eq^n (1), w.r.t. t, we get,

$$\frac{\partial \psi}{\partial t} = \phi(x) \cdot \exp(-iEwt) \times (-iw)$$

$$\frac{\partial \psi}{\partial t} = -iw\phi \exp(-iwt) \quad \text{——} \; (3)$$

Putting eqn (2) & (3), in time dependent SE, we get,

$$i\hbar(-iw\phi \exp(-iwt))$$

$$= \frac{-\hbar^2}{2m} \times \exp(-iwt) \frac{\partial^2 \phi}{\partial x^2}$$

$$+ V\phi \exp(-iwt)$$

$$\hbar w \phi \exp(-iwt) = \frac{-\hbar^2}{2m} \exp(-iwt) \frac{\partial^2 \phi}{\partial x^2}$$

$$+ V\phi \exp(-iwt)$$

$$\cancel{\hbar} \times \frac{E}{\cancel{\hbar}} \phi = \frac{-\hbar^2}{2m} \frac{\partial^2 \phi}{\partial x^2} + V\phi$$

$$E\phi = \frac{-\hbar^2}{2m}\frac{\partial^2\phi}{\partial x^2} + V\phi$$

$$\boxed{\frac{-\hbar^2}{2m}\frac{\partial^2\phi}{\partial x^2} + V\phi = E\phi}$$

↓

Time-Independent SE.

Question: Prove that the e^- can not exist outside the nucleus using Heisenberg's uncertainty principle.

Proof: Size of nucleus is 10^{-14} m

$$\Delta x = 10^{-14} m$$

Now,

$$\Delta p = \frac{\hbar}{2\Delta x} = \frac{h}{2\pi \times \Delta x} = \frac{6.625 \times 10^{-34}}{2 \times 3.14 \times 10^{-14}}$$

$$= 10^{-20} \text{ kg m/s}$$

Thus, $|\vec{p}| = 10^{-20}$ kg m/s

The relativistic energy of e^- in the nucleus

$$E = \sqrt{m_0^2c^4 + p^2c^2} = \sqrt{(9.1\times10^{-31})^2(3\times10^8)^4 + (10^{-20})^2\times(3\times10^8)^2}$$

$$= 1.6\times10^{-12}\,J = 10\,MeV$$

Thus, if e^- are present in the nucleus, then on being emitted they should have kinetic energy of the order of 10 MeV. However, the particles emitted from the nucleus have energy of 2 to 3 MeV. Thus e^-, cannot exist in the nucleus.

which is a time-dependent SE.

Free particle Wave f^n and Wave packets ⇒

Wave packets ⇒

A localized wave f^n is called a wave packet. It consists of a group of waves of slightly different wavelengths so chosen that they interfere constructively over a small region of space and destructively elsewhere. Mathematically, we can carry out this type of interference or superposition by means of Fourier transforms. We can construct the wave packet $\psi(x,t)$ by superposing the plane waves (propagating along the x-axis) of different frequencies.

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{i(kx-\omega t)} dk$$

$\phi(k)$ is the amplitude of the wave packet.

At $t=0$,

$$\psi_0(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{ikx} \, dk \, .$$

where $\phi(k)$ is the fourier transform of $\psi_0(x)$.

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi_0(x) e^{-ikx} \, dx$$

Free particle wave f^n :

This is the simplest one-dimensional problem because it corresponds to $V(x)=0$. In this case, SWE is

$$-\frac{\hbar^2}{2m} \frac{d^2\psi(x)}{dx^2} + 0 = E\psi(x)$$

$$\frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2}\psi(x) = 0$$

$* \quad k' = k = \frac{p}{\hbar}$

$$\left(\frac{d^2}{dx^2} + k'^2\right) \psi(x) = 0$$

where, $k'^2 = \frac{2mE}{\hbar^2}$

The general soln of above eqn is.

$$\psi(x) = A_+ e^{ik'x} + A_- e^{-ik'x}$$

eigenfn

where A_+ and A_- are arbitrary constants

The complete wave fn.

$$\psi(x,t) = A_+ e^{i(k'x - \omega t)} + A_- e^{-i(k'x + \omega t)}$$

wave fn. — wave travelling to the right — wave travelling to the left.

Since, there are no restrictions, hence p and E can take any values. It is a simple problem but presents a no. of physical subtleties. Let us discuss three of them:

(i) $$P_{\pm}(x,t) = |\psi_{\pm}(x,t)|^2 = |A_{\pm}|^2$$

are constant and does not depend on x ~~&~~ t.

(ii) The speed of plane wave is,

$$v_{wave} = \frac{\omega}{k'} = \frac{E}{\hbar k'} = \frac{\hbar^2 k'^2/2m}{\hbar k'}$$

$$= \frac{\hbar k}{2m}$$

$$v_{classical} = \frac{p}{m} = \frac{\hbar k}{m} = 2 v_{wave}.$$

This means that the particle travels twice as fast as the wave that represents it.

(iii) The wave f^n is not normalizable:

$$\int_{-\infty}^{\infty} \psi_{\pm}^{*} \psi_{\pm}(x,t)\, dx = |A_{\pm}|^2 \int_{-\infty}^{\infty} dx = \infty$$

So, A must be zero because $\int_{-\infty}^{\infty} dx = \infty$
This is not physical.

Thus, the solⁿ $\psi_{\pm}(x,t)$ is unphysical. A free particle cannot have sharply defined momenta and energy. Thus, the solⁿ in this cannot be plane waves but it should be wave packets:

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k)\, e^{i(kx-\omega t)}\, dk.$$

The wave packet solⁿ cures and avoids all subtleties raised above.

In summary, a free particle cannot be represented by a single plane wave; but it has to be represented by a wave packet.

Probability Current =>

We know that, the probability density,

$$P = \psi^* \psi$$

$$\frac{\partial P}{\partial t} = \left[\psi^* \frac{\partial \psi}{\partial t} + \frac{\partial \psi^*}{\partial t} \psi \right] \quad \text{--- (1)}$$

Now, the time-dependent Schrodinger wave eqn is :

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x)\psi \quad \text{--- (2)}$$

and its complex conjugate ie,

$$-i\hbar \frac{\partial \psi^*}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi^*}{\partial x^2} + V(x)\psi^*$$

($\because V^* = V$ potential is always real)

$$\frac{\partial \psi^*}{\partial t} = \frac{\hbar^2}{2mi\hbar} \frac{\partial^2 \psi^*}{\partial x^2} - \frac{V(x)\psi^*}{i\hbar} \quad \text{--- (3)}$$

$$\frac{\partial \psi^*}{\partial t} = \frac{-i\hbar}{2m} \frac{\partial^2 \psi^*}{\partial x^2} + \frac{iV(x)\psi^*}{\hbar}$$

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~~Solution of Wave eqn:~~

From eqn (2),

$$\frac{\partial \psi}{\partial t} = \frac{-\hbar^2}{2m\, i\hbar}\frac{\partial^2 \psi}{\partial x^2} + \frac{V(x)\psi}{i\hbar}$$

$$\frac{\partial \psi}{\partial t} = \frac{-\hbar}{2mi}\frac{\partial^2 \psi}{\partial x^2} + \frac{V(x)\psi}{i\hbar}$$

$$\frac{\partial \psi}{\partial t} = \frac{-i\hbar}{2mi^2}\frac{\partial^2 \psi}{\partial x^2} + \frac{V(x)\psi}{i\hbar}$$

$$\frac{\partial \psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \psi}{\partial x^2} - \frac{i\,V(x)\psi}{\hbar} \quad (4)$$

($V = V^*$ ∵)

Similarly, from eqn (3),

$$\frac{\partial \psi^*}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2 \psi^*}{\partial x^2} + \frac{i\,V(x)\psi^*}{\hbar} \quad (5)$$

Putting eqⁿ (4) and (5) in eqⁿ (1) we get,

$$\frac{\partial P}{\partial t} = \psi^* \left[\frac{i\hbar}{2m} \frac{\partial^2 \psi}{\partial x^2} - \frac{iV\psi}{\hbar} \right]$$

$$+ \left[-\frac{i\hbar}{2m} \frac{\partial^2 \psi^*}{\partial x^2} + \frac{iV\psi^*}{\hbar} \right] \psi$$

$$\frac{\partial P}{\partial t} = \frac{i\hbar}{2m} \psi^* \frac{\partial^2 \psi}{\partial x^2} - \frac{iV}{\hbar} \cancel{\psi^* \psi}$$

$$- \frac{i\hbar}{2m} \frac{\partial^2 \psi^*}{\partial x^2} \cdot \psi + \frac{iV}{\hbar} \cancel{\psi^* \psi}$$

$$\frac{\partial P}{\partial t} = \frac{i\hbar}{2m} \left[\psi^* \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi^*}{\partial x^2} \psi \right]$$

$$\underbrace{\frac{\partial P}{\partial t} = \frac{i\hbar}{2m} \left[\psi^* \cdot \frac{\partial^2 \psi}{\partial x^2} - \psi \frac{\partial^2 \psi^*}{\partial x^2} \right]}$$

Now, we define,

$$\frac{\partial}{\partial x}\left[\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}\right]$$

$$= \psi^* \frac{\partial^2 \psi}{\partial x^2} + \cancel{\frac{\partial \psi^*}{\partial x} \frac{\partial \psi}{\partial x}} - \psi \frac{\partial^2 \psi^*}{\partial x^2} - \cancel{\frac{\partial \psi}{\partial x} \frac{\partial \psi^*}{\partial x}}$$

$$= \left[\psi^* \frac{\partial^2 \psi}{\partial x^2} - \psi \frac{\partial^2 \psi^*}{\partial x^2}\right]$$

Hence,

$$\frac{\partial P}{\partial t} = \frac{i\hbar}{2m} \frac{\partial}{\partial x}\left[\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}\right]$$

we define,

$$\boxed{j(r,t) = \frac{\hbar}{2im}\left[\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}\right]}$$

$$\frac{\partial P}{\partial t} = -\frac{\partial j}{\partial x}$$

probability current

$$\boxed{\frac{\partial P}{\partial t} + \frac{\partial j}{\partial x} = 0} \rightarrow \text{conservation of Probability.}$$

This is continuity eqⁿ.

This means the probability does not depends upon time.

It can be proved as

$$\frac{d}{dt}\int_{-\infty}^{\infty} dx \; |\psi(x,t)|^2 .$$

$$= \int_{-\infty}^{\infty} dx \; \frac{\partial}{\partial t} |\psi(x,t)|^2$$

$$= \int_{-\infty}^{\infty} dx \; \left(-\frac{\partial j}{\partial x}\right)$$

$$= -[\, j(\infty, t) - j(-\infty, t)]$$

$$= 0 .$$

we assume that the wave f^n vanishes at infinity, i.e.

$$\boxed{\frac{d}{dt}(P) = 0 .}$$