Matrices

- A matrix is a rectangular arrangement of numbers or things in rows and columns.

A matrix having m rows and n columns is written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$ is called $m \times n$ matrix

where $a_{11}, a_{12}, a_{13} \cdots a_{mn}$ are elements of matrix A which may be any number. a_{11} reprents element in 1st row, 1st column, a_{12} element in 1st row, 2nd column. etc.
Thus a_{ij} is an element of ith row and jth column of matrix A.
Unlike determinants, a matrix does not have any value. If $m=n$ the matrix is said to be square matrix.

<u>Types of Matrices</u> :- (1) <u>Square matrix</u> :- If the no. of rows and columns of a matrix are equal, it is called square matrix. The diagonal of this matrix is called principal diagonal.
For example: $\begin{bmatrix} 1 & 4 \\ 5 & 8 \end{bmatrix}$ and $\begin{bmatrix} 1 & 4 & 7 \\ 6 & 8 & 9 \\ 2 & 7 & 5 \end{bmatrix}$ are square matrices because no. of rows and columns in each matrix is equal.
The first matrix has two rows and two columns while second matrix has three rows and three columns.
The determinant of a square matrix is denoted by $|A|$. For exp: If $A = \begin{bmatrix} 2 & 3 \\ 4 & 8 \end{bmatrix}$ then $|A| = \begin{vmatrix} 2 & 3 \\ 4 & 8 \end{vmatrix} = 16-12 = 4$.

2) <u>Diagonal matrix</u> :- A square matrix is called diagonal matrix if all its non-diagonal elements are zero. For example:
$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is a diagonal matrix having principal diagonal elements 2, 4, 3.

(3) Row matrix :- A matrix is called row matrix if it contains only one row. For exp. $\begin{bmatrix} 1 & 4 & 8 \end{bmatrix}$ is a row matrix.

(4) Column matrix :- A matrix is called column matrix if it contains only one column. For exp: $\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$

(5) Null matrix :- A matrix whose elements are all zero is called null or zero matrix.

For exp : $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(6) Equal matrices :- Two matrices A and B are called equal matrices if both are of same order and the corresponding elements in A and B are equal.

Exp : If $A = \begin{bmatrix} 4 & 7 \\ 8 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 7 \\ 8 & 9 \end{bmatrix}$ then $A = B$.

(7) Unit matrix :- A square matrix is called unit matrix if all its non-diagonal elements are zero and diagonal elements are unity. For exp.

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is unit matrix of order 2 and is denoted by I_2.

Similarly $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is unit matrix of order 3.

(8) Scalar matrix :- A square matrix is called scalar matrix if all its non-diagonal elements are zero and diagonal elements are equal. For exp:

$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is a scalar matrix.

(9) Triangular matrix :- There are two types of triangular matrices. (i) upper triangular (ii) lower triangular

A square matrix is called upper triangular matrix if all its elements below the principal diagonal are zero.

For exp : $\begin{bmatrix} 3 & 7 & 4 \\ 0 & 5 & -3 \\ 0 & 0 & 8 \end{bmatrix}$ is an upper triangular matrix.

A square matrix is called lower triangular matrix if all elements above principal diagonal are zero.

(10) Singular & non-Singular matrix :—
If determinant of a square matrix is zero. i.e. $|A| = 0$. Matrix A is called singular matrix otherwise it is non-singular matrix.

(11) Symmetric & Skew Symmetric matrix :—
A square matrix is said to be symmetric when $a_{ij} = a_{ji}$, where i and j stands for row and column resp.
If $a_{ij} = -a_{ji}$ the matrix is said to be skew symmetric.
In a skew symm. matrix all elements in principal diagonal are zero.

For exp : $A = \begin{bmatrix} 1 & 3 & 7 \\ 3 & 6 & 5 \\ 7 & 5 & 8 \end{bmatrix}$ is Symmetric matrix
Since $A = A'$ (transpose of A)

whereas $B = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}$ is skew symmetric matrix.

Transpose of a Matrix :— A matrix obtained by inter-changing rows to columns and columns to rows is called transpose of the matrix. For example

if $A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 6 & 8 \\ 3 & 7 & 9 \end{bmatrix}$ then transpose of A $(A') = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 6 & 7 \\ 5 & 8 & 9 \end{bmatrix}$

Operation on matrices

1. Addition of matrices :— If A and B are two matrices of same order, then their sum A+B is obtained by adding corresponding elements of A and B.

For exp :- if $A = \begin{bmatrix} 2 & 4 & 7 \\ 3 & 0 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 6 & -3 \\ 4 & 7 & 8 \end{bmatrix}$

then $A+B = \begin{bmatrix} 2+0 & 4+6 & 7-3 \\ 3+4 & 0+7 & 5+8 \end{bmatrix} = \begin{bmatrix} 2 & 10 & 4 \\ 7 & 7 & 13 \end{bmatrix}$

Remark :- If two matrices are not of the same order, their sum is not defined.

For exp : if $A = \begin{bmatrix} 2 & 5 \\ 7 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ then A+B is not defined because A and B are of different order.

2. Difference of matrices :– If A and B are two matrices of the same order, then their difference A–B is obtained by subtracting elements of B from corresponding elements of A.

For exp: if $A = \begin{bmatrix} 4 & 3 \\ 6 & 9 \\ 9 & 12 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 8 \\ 11 & 7 \\ 6 & 9 \end{bmatrix}$

then $A - B = \begin{bmatrix} 4-3 & 3-8 \\ 6-11 & 9-7 \\ 9-6 & 12-9 \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ -5 & 2 \\ 3 & 3 \end{bmatrix}$

Properties of matrix addition :–

(i) Commutative law of addition :- If A and B are matrices of same order then $A + B = B + A$

(ii) Associative law of addition :- If A, B and C are matrices of same order then $A + (B + C) = (A + B) + C$

(iii) Existence of additive identity :- If A is any matrix then $A + O = A = O + A$, where O is a zero matrix of same order as that of A.

(iv) Existence of additive inverse :- If A is any matrix then $A + (-A) = O$ where $-A$ is additive inverse of matrix A.

Scalar multiplication of a matrix :- If A is a matrix and K be a scalar (any number) then multiplying every element of A by K gives scalar multiplication KA.

For exp: if $A = \begin{bmatrix} 2 & 5 & 6 \\ -4 & 3 & 7 \end{bmatrix}$ and $K = 8$, then

$KA = 8\begin{bmatrix} 2 & 5 & 6 \\ -4 & 3 & 7 \end{bmatrix} = \begin{bmatrix} 8\times2 & 8\times5 & 8\times6 \\ 8\times(-4) & 8\times3 & 8\times7 \end{bmatrix} = \begin{bmatrix} 16 & 40 & 48 \\ -32 & 24 & 56 \end{bmatrix}$

Properties of Scalar multiplication :- If A and B are two matrices of the same order and K, L are any number then

(i) $K(A + B) = KA + KB$

(ii) $(K + L)A = KA + LA$

(iii) $1 . A = A$

For exp: If $K = 3$, $A = \begin{bmatrix} 4 & 7 \\ 2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 1 \\ 0 & 6 \end{bmatrix}$

then $K(A + B) = 3 . \begin{bmatrix} 1 & 8 \\ 2 & 11 \end{bmatrix} = \begin{bmatrix} 3 & 24 \\ 6 & 33 \end{bmatrix}$

and $KA + KB = 3\begin{bmatrix} 4 & 7 \\ 2 & 5 \end{bmatrix} + 3\begin{bmatrix} -3 & 1 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 12 & 21 \\ 6 & 15 \end{bmatrix} + \begin{bmatrix} -9 & 3 \\ 0 & 18 \end{bmatrix} = \begin{bmatrix} 3 & 24 \\ 6 & 33 \end{bmatrix}$

$\therefore K(A + B) = KA + KB$.

Matrix Multiplication :- Let A and B be two matrices then their product A.B is defined only if number of Columns of A is equal to number of Rows of B. For example

① If $A = \begin{bmatrix} 4 & 7 \\ 6 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 8 \\ 3 & 7 \end{bmatrix}$ find A.B

Here the product AB is defined because no. of Columns of A is equal to no. of rows of B (each = 2)

$$\therefore AB = \begin{bmatrix} 4 & 7 \\ 6 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 8 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 4\times1+7\times3 & 4\times8+7\times7 \\ 6\times1+2\times3 & 6\times8+2\times7 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & 81 \\ 12 & 62 \end{bmatrix}$$

② $A = \begin{bmatrix} 1 & 5 & 7 \\ 0 & 6 & 8 \end{bmatrix}$, $B = \begin{bmatrix} 7 & 8 \\ 9 & 8 \\ 6 & 5 \end{bmatrix}$ find AB

The matrix A is of order 2x3 and B is of order 3x2. Since no. of Columns of A = no. of rows of B = 3. Hence product AB is defined as

$$AB = \begin{bmatrix} 1\times7+5\times9+7\times6 & 1\times8+5\times8+7\times5 \\ 0\times7+6\times9+8\times6 & 0\times8+6\times8+8\times5 \end{bmatrix}$$

$$= \begin{bmatrix} 94 & 83 \\ 102 & 88 \end{bmatrix}$$

Properties of matrix multiplication :- (1) matrix multiplication is not necessarily Commutative.

(2) Associative law of multiplication. If A, B and C are matrices of order $m\times n$, $n\times p$ and $p\times q$ respectively then $(AB)C = A(BC)$

(3) Matrix multiplication is distributive with respect to addition :- $A(B+C) = AB + AC$

If A, B and C are of order $m\times n$, $n\times p$, $p\times q$ respectively.

Q1. If $A = \begin{bmatrix} 1 & 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 3 & 7 \\ 6 & 8 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 4 \\ 5 & 6 \\ 1 & 0 \end{bmatrix}$

Show that $A(B+C) = AB + AC$.

Soln :- $B+C = \begin{bmatrix} 2 & 1 \\ 3 & 7 \\ 6 & 8 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ 5 & 6 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 8 & 13 \\ 7 & 8 \end{bmatrix}$

L.H.S. $A(B+C) = \begin{bmatrix} 1 & 3 & 4 \end{bmatrix} \times \begin{bmatrix} 5 & 5 \\ 8 & 13 \\ 7 & 8 \end{bmatrix}$

$= \begin{bmatrix} 1\times5+3\times8+4\times7 & 1\times5+3\times13+4\times8 \end{bmatrix}$

$= \begin{bmatrix} 5+24+28 & 5+39+32 \end{bmatrix} = \begin{bmatrix} 57 & 76 \end{bmatrix}$

Now, $AB = \begin{bmatrix} 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 7 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 1\times2+3\times3+4\times6 & 1\times1+3\times7+4\times8 \end{bmatrix}$

$= \begin{bmatrix} 2+9+24 & 1+21+32 \end{bmatrix}$

$= \begin{bmatrix} 35 & 54 \end{bmatrix}$

$AC = \begin{bmatrix} 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 5 & 6 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1\times3+3\times5+4\times1 & 1\times4+3\times6+4\times0 \end{bmatrix}$

$= \begin{bmatrix} 3+15+4 & 4+18+0 \end{bmatrix} = \begin{bmatrix} 22 & 22 \end{bmatrix}$

R.H.S. $= AB+AC = \begin{bmatrix} 35 & 54 \end{bmatrix} + \begin{bmatrix} 22 & 22 \end{bmatrix} = \begin{bmatrix} 57 & 76 \end{bmatrix}$

$\therefore A(B+C) = AB+AC$ Proved.

Q.2. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = 0$

Soln :- Here $A^2 = A\times A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \times \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix}$

$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$

$5A = 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 5\times3 & 5\times1 \\ 5\times-1 & 5\times2 \end{bmatrix} = \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix}$

$7I = 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$ ($\because$ I is unit matrix of order 2)

$\therefore A^2 - 5A + 7I = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

$= \begin{bmatrix} 8-15 & 5-5 \\ -5+5 & 3-10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

$= \begin{bmatrix} -7+7 & 0+0 \\ 0+0 & -7+7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

$\therefore A^2 - 5A + 7I = 0$ Proved

Q.3. If $f(x) = x^3 + 4x^2 - x$, find $f(A)$, when $A = \begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix}$

Soln :- Given $f(x) = x^3 + 4x^2 - x$

then $f(A) = A^3 + 4A^2 - A$

Now, $A^2 = A\times A = \begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 0\times0+1\times2 & 0\times1+1\times(-3) \\ 2\times0-3\times2 & 2\times1+(-3)\times(-3) \end{bmatrix}$

$= \begin{bmatrix} 0+2 & 0-3 \\ 0-6 & 2+9 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -6 & 11 \end{bmatrix}$

$$\therefore A^3 = A^2 \times A = \begin{bmatrix} 2 & -3 \\ -6 & 11 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 2\times 0 + (-3)2 & 2\times 1 + 3\times 3 \\ -6\times 0 + 11\times 2 & -6\times 1 + 11\times(-3) \end{bmatrix}$$

$$= \begin{bmatrix} 0-6 & 2+9 \\ 0+22 & -6-33 \end{bmatrix} = \begin{bmatrix} -6 & 11 \\ 22 & -39 \end{bmatrix}$$

$$4A^2 = 4\begin{bmatrix} 2 & -3 \\ -6 & 11 \end{bmatrix} = \begin{bmatrix} 8 & -12 \\ -24 & 44 \end{bmatrix}$$

$$\therefore f(A) = A^3 + 4A^2 - A = \begin{bmatrix} -6 & 11 \\ 22 & -39 \end{bmatrix} + \begin{bmatrix} 8 & -12 \\ -24 & 44 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} -6+8-0 & 11-12-1 \\ 22-24-2 & -39+44+3 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -4 & 8 \end{bmatrix}$$