## Symmetric and Skew Symmetric matrix :-

1. For Symmetric matrix, $A^T = A$ and for Skew-Symmetric matrix, $A^T = -A$

2. Every Square matrix can be expressed as sum of Symmetric and Skew-Symmetric matrix.

Let $A$ be the given square matrix, then $A = \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T)$

Let $B = \frac{1}{2}(A+A^T)$ and $C = \frac{1}{2}(A-A^T)$

$\therefore B^T = [\frac{1}{2}(A+A^T)]^T = \frac{1}{2}[A^T + (A^T)^T] = \frac{1}{2}(A^T+A) = B$

$\Rightarrow B = \frac{1}{2}(A+A^T)$ is a symmetric matrix

Again $C^T = [\frac{1}{2}(A-A^T)]^T = \frac{1}{2}[A^T - (A^T)^T] = \frac{1}{2}(A^T - A) = -C$

$\Rightarrow C = \frac{1}{2}(A-A^T)$ is Skew symm. matrix

Hence $A$ can be expressed as sum of Symmetric and Skew Symmetric matrix.

Q1. Express the matrix $A$ as the sum of symmetric and Skew-Symmetric matrix, where $A = \begin{bmatrix} 4 & 2 & -3 \\ 1 & 3 & -6 \\ -5 & 0 & -7 \end{bmatrix}$

Soln :- We have $A^T = \begin{bmatrix} 4 & 1 & -5 \\ 2 & 3 & 0 \\ -3 & -6 & -7 \end{bmatrix}$

then $A + A^T = \begin{bmatrix} 8 & 3 & -8 \\ 3 & 6 & -6 \\ -8 & -6 & -14 \end{bmatrix}$ and $A - A^T = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -6 \\ -2 & 6 & 0 \end{bmatrix}$

$\therefore A = \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T) = \begin{bmatrix} 4 & 1.5 & -4 \\ 1.5 & 3 & -3 \\ -4 & -3 & -7 \end{bmatrix} + \begin{bmatrix} 0 & 0.5 & 1 \\ -0.5 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$

**Orthogonal matrix** :- A square matrix $A$ is called an orthogonal matrix if $A \cdot A^T = A^T \cdot A = I$, where $I$ is unit matrix.

Q2. Show that $\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$ is orthogonal

Soln :- Let $A$ bee the given matrix. Then

$$A \cdot A^T = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \times \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}^2$$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$ Hence $A$ is orthogonal.

Q3. Determine $a$, $b$ and $c$ so that $A$ is orthogonal matrix where $A = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$

Soln :- For orthogonal matrix, we have $A.A^T = I$

$$\therefore A.A^T = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix} \times \begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4b^2+c^2 & 2b^2-c^2 & -2b^2+c^2 \\ 2b^2-c^2 & a^2+b^2+c^2 & a^2-b^2-c^2 \\ -2b^2+c^2 & a^2-b^2-c^2 & a^2+b^2+c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equating the Corresponding elements, we have

$\left.\begin{matrix} 4b^2+c^2 = 1 \\ 2b^2 - c^2 = 0 \end{matrix}\right\}$ solving we get, $6b^2 = 1$ or $b = \pm\frac{1}{\sqrt{6}}$ and $c = \pm\frac{1}{\sqrt{3}}$

Also $a^2+b^2+c^2 = 1 \Rightarrow a^2+\frac{1}{6}+\frac{1}{3} = 1 \Rightarrow a^2 = \frac{1}{2}$ or $a = \pm\frac{1}{\sqrt{2}}$

Hence $a = \pm\frac{1}{\sqrt{2}}$, $b = \pm\frac{1}{\sqrt{6}}$ and $c = \pm\frac{1}{\sqrt{3}}$

**Adjoint of a square matrix :-** Let $A$ be a square matrix then adjoint of $A$ is transpose of Cofactors of matrix $A$.

For example: If $A = \begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

then Cofactors of $|A|$ is given by $A_{11} = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3$, $A_{12} = -\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = -1$

$A_{13} = \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = 5$, $A_{21} = -\begin{vmatrix} 5 & 3 \\ 2 & 1 \end{vmatrix} = 1$, $A_{22} = \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -1$, $A_{23} = -\begin{vmatrix} 2 & 5 \\ 1 & 2 \end{vmatrix} = 1$

$A_{31} = \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = 7$, $A_{32} = -\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = 5$, $A_{33} = \begin{vmatrix} 2 & 5 \\ 3 & 1 \end{vmatrix} = -13$

$$\therefore \text{Adj } A = \begin{bmatrix} -3 & -1 & 5 \\ 1 & -1 & 1 \\ 7 & 5 & -13 \end{bmatrix}^T = \begin{bmatrix} -3 & 1 & 7 \\ -1 & -1 & 5 \\ 5 & 1 & -13 \end{bmatrix}$$

**Properties of adjoint matrix :-** (1) If $A$ and $B$ are two non-Singular square matrices of same order, then $\text{adj}(AB) = (\text{adj } B)(\text{adj } A)$

2. The product of a matrix and its adjoint is equal to unit matrix multiplied by the determinant $A$.

Thus $A.(\text{adj } A) = |A|.I$, where $I$ is unit matrix of same order that of $A$.

**Inverse of a Matrix** :- If A and B are two square matrices of same order, such that $AB = BA = I$ then B is called inverse of A and is denoted by $B = A^{-1}$.
Thus $A \cdot A^{-1} = A^{-1} \cdot A = I$

**Properties of inverse Matrix** :- (1). The inverse of the inverse of the matrix is itself. So $(A^{-1})^{-1} = A$

(2) Inverse of a matrix is unique.

Proof :- Let B and C are two inverse matrices of a given matrix A.

Then $AB = BA = I \quad \therefore B = A^{-1}$
and $AC = CA = I \quad \therefore C = A^{-1}$

But $C(AB) = (CA)B \Rightarrow C.I = I.B \Rightarrow C = B$

Hence inverse of matrix A is unique.

(3) If A and B are non-singular matrices of same order, then $(AB)^{-1} = B^{-1}A^{-1}$.

Proof :- Since $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = A.A^{-1} = I$

Similarly $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$

$\therefore (AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1})AB = I$

By definition of inverse of a matrix, $B^{-1}A^{-1}$ is inverse of AB.

Hence $B^{-1}A^{-1} = (AB)^{-1}$.

**To find inverse of a matrix using adjoint matrix** :-

We know that $A.(\text{adj } A) = |A| I$

Since $|A| \neq 0$, dividing both sides by $|A|$ we get

$\frac{A.(\text{adj } A)}{|A|} = I$. Now multiplying both sides by $A^{-1}$, we have

$\frac{I.(\text{adj } A)}{|A|} = I.A^{-1} \Rightarrow \frac{\text{adj } A}{|A|} = A^{-1}$.

Q4. Find the adjoint and inverse of $A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$

Soln :- The Cofactors of $|A|$ are $A_{11} = \begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix} = 10$, $A_{12} = -\begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} = -15$

$A_{13} = \begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix} = 5$, $A_{21} = -\begin{vmatrix} 3 & 4 \\ 2 & 4 \end{vmatrix} = -4$, $A_{22} = \begin{vmatrix} 2 & 4 \\ 1 & 4 \end{vmatrix} = 4$, $A_{23} = -\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = -1$

$A_{31} = \begin{vmatrix} 3 & 4 \\ 3 & 1 \end{vmatrix} = -9$, $A_{32} = -\begin{vmatrix} 2 & 4 \\ 4 & 1 \end{vmatrix} = 14$, $A_{33} = \begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} = -6$

So, $\text{adj}(A) = \begin{bmatrix} 10 & -15 & 5 \\ -4 & 4 & -1 \\ -9 & 14 & -6 \end{bmatrix}^T = \begin{bmatrix} 10 & -4 & -9 \\ -15 & 4 & 14 \\ 5 & -1 & -6 \end{bmatrix}$

Now $|A| = 2(12-2) - 3(16-1) + 4(8-3) = 20 - 45 + 20 = -5 \neq 0$

So inverse exists. Therefore

$$A^{-1} = \frac{\text{adj } A}{|A|} = -\frac{1}{5}\begin{bmatrix} 10 & -4 & -9 \\ -15 & 4 & 14 \\ 5 & -1 & -6 \end{bmatrix}$$

## Elementry Transformation of a Matrix :-

The following operations on a matrix are known as elementry transformations :

(1) Interchange of any two rows or columns.

(2) Multiplication of any row or column by a non-zero number.

(3) Addition of a constant multiple of one row/column to another row/column.

Above elementry transformations donot change either the order or rank of a matrix. The value of minors may get changed by transformation but their zero or non-zero character remains unaffected.

**Equivalent matrix** :- Two matrices A and B are said to be equivalent if one can be obtained from other by sequence of elementry transformations. Two equivalent matrices have same order and the same rank. It is denoted by $A \sim B$

**Elementry matrix** :- A matrix obtained from a unit matrix by a single elementry transformation is called elementary matrix. For example:

Let $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then $R_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $R_1 + 3R_2 = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

are elementry matrices.

## Inverse of a Matrix by elementry transformation (Gauss-Jordan method

The elementry row-transformations which reduce a given square matrix A to the unit matrix, when applied to unit matrix I give inverse of A is known as 'Gauss-Jordan method'. This method is used to calculate the inverse of matrix A when A is of large order. It is one of the best computational method for finding inverse of a matrix.

**Working rule** :- write $A = I.A$. Perform elementry row transformations on A of the left side and on I of the right side so that A is reduced to I and I of R.H.S. is reduced to $A^{-1}$.

Q. use Gauss-Jordan method to find the inverse of the matrix $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

Soln :- Let A denote the given matrix. Then

$$A = I.A \Rightarrow \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

we reduce L.H.S to unit matrix by elementry Row-operations.

operating $R_2 - R_1$ and $R_3 + 2R_1$

$$\sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & -6 \\ 0 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} A$$

operating $R_3 + R_2$

$$\sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & -6 \\ 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} A$$

operating $\frac{1}{2}R_2$ and $(-\frac{1}{4})R_3$

$$\sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1/2 & 0 \\ -1/4 & -1/4 & -1/4 \end{bmatrix} A$$

operating $R_1 - 3R_3$ and $R_2 + 3R_3$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7/4 & 3/4 & 3/4 \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{bmatrix} A$$

operating $R_1 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 3/2 \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{bmatrix} A$$

Equating this with $I = A^{-1}A$, we get

$$A^{-1} = \begin{bmatrix} 3 & 1 & 3/2 \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{bmatrix}$$