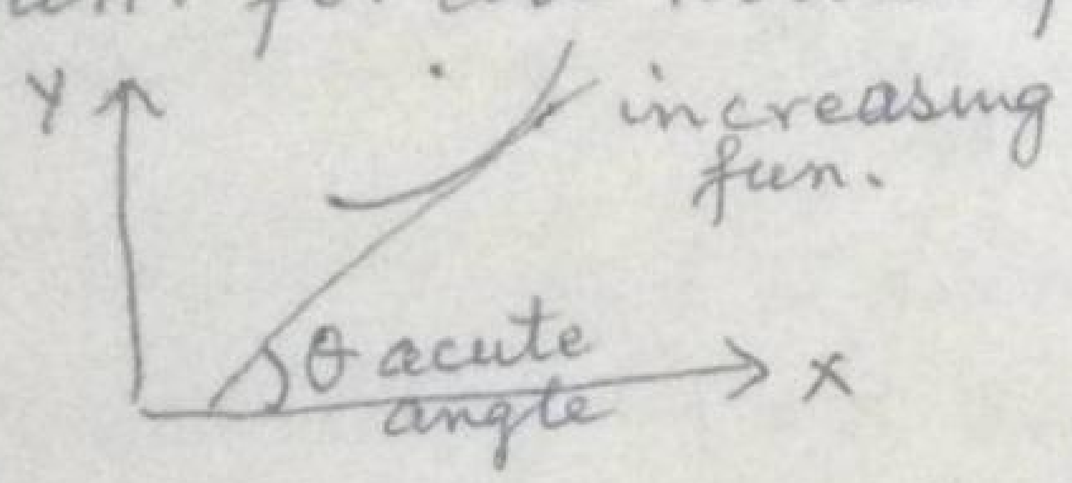
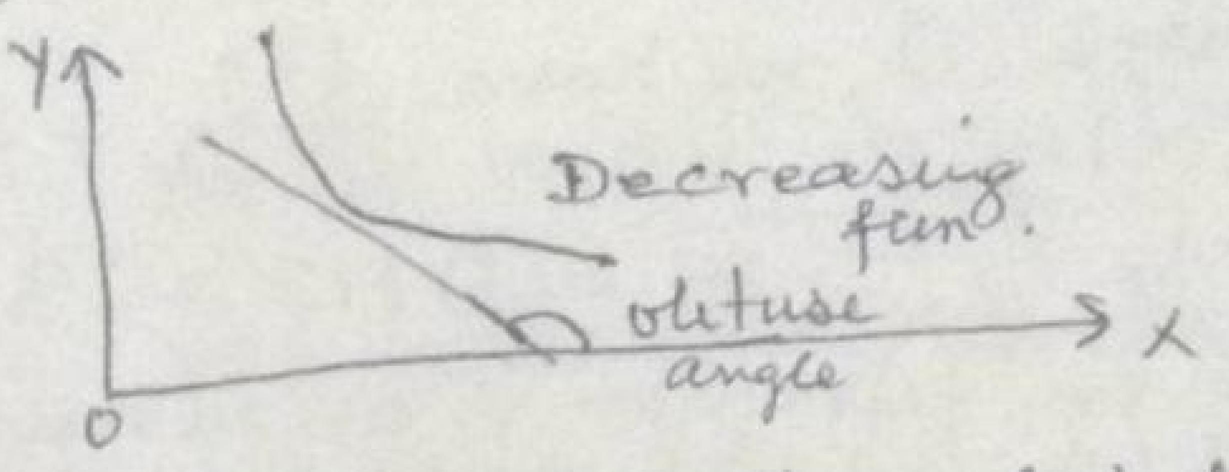
Maxima and Minima

Increasing & Decreasing functions :- If the value of a function $y = f(x)$ increases as x increases then $f(x)$ is said to be increasing function of x. A function $\phi(x)$ is said to be a decreasing function if value of $\phi(x)$ decreases as x increases. For example: $f(x) = x^2 + 3$ is an increasing fun. for all values of x and $\phi(x) = \frac{1}{x}$ is a decreasing fun. for all values of x





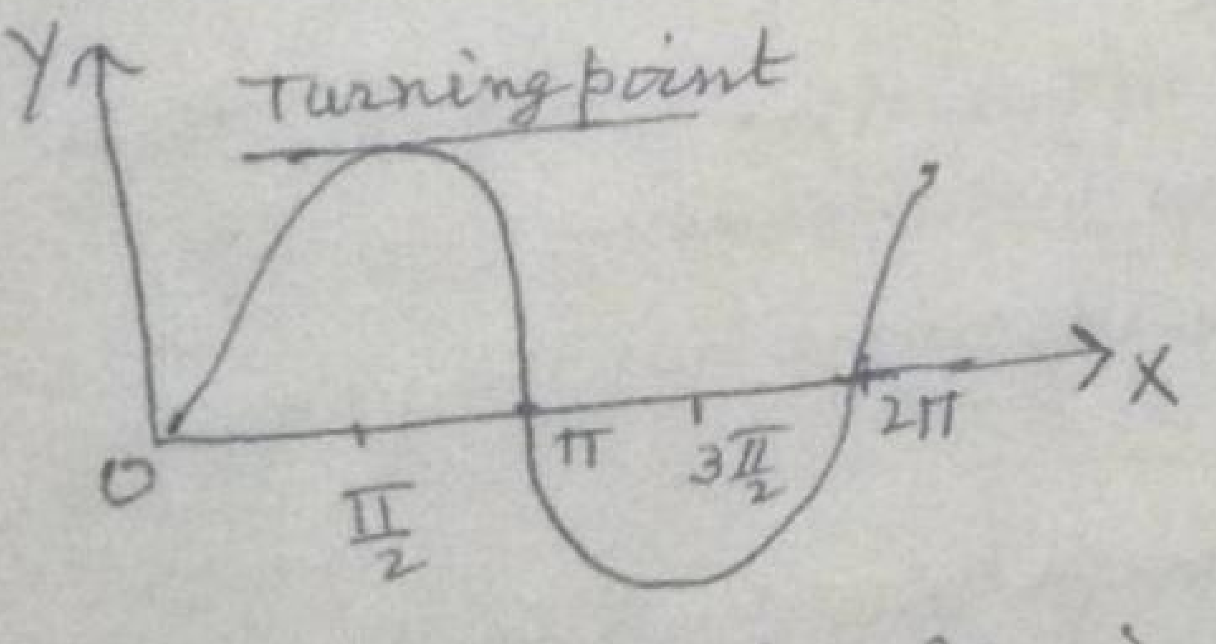
In case of increasing fun. the tangent at any point of the curve makes an acute angle θ with +ve direction of x-axis. So,

$$\tan\theta = +ve \text{ or } \frac{dy}{dx} > 0$$

Similarly in case of decreasing fun. the tangent at any point of the curve makes obtuse angle with +ve direction of x-axis, So

$$\tan\theta = -ve \text{ or } \frac{dy}{dx} < 0$$

Turning point or Stationary point :- The point at which the function changes its nature is called turning point. The increasing fun. may change to decreasing one and vice-versa. For example, the $\sin x$ is an increasing fun. between 0 to $\frac{\pi}{2}$ and decreasing between $\frac{\pi}{2}$ to $\frac{3\pi}{2}$. At $\frac{\pi}{2}$ the function is neither increasing nor decreasing but stationary. The tangent at turning point is parallel to x-axis.



$$\therefore \frac{dy}{dx} = 0$$

Conditions for Maxima & Minima :- Let $f(x)$ be maxima at $x = a$. Here $\frac{dy}{dx}$ changes from +ve to −ve as it passes through $x = a$. So derivative of $\frac{dy}{dx}$ should be −ve at $x = a$

or, $\frac{d^2y}{dx^2} < 0$ at $x = a$

Hence there are two conditions for maxima

(I) $\frac{dy}{dx} = 0$ (II) $\frac{d^2y}{dx^2} < 0$ (or −ve)

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Similarly two conditions for minima are

(i) $\frac{dy}{dx} = 0$ (ii) $\frac{d^2y}{dx^2} > 0$ (or +ve)

Example :- Find the points at which the function $y = x^3 + 6x^2 - 15x + 5$ has maximum and minimum value.

Soln :- Given $y = x^3 + 6x^2 - 15x + 5$

then $\frac{dy}{dx} = 3x^2 + 12x - 15$ — (1)

For maxima and minima, $\frac{dy}{dx} = 0 \Rightarrow 3x^2 + 12x - 15 = 0$

$\Rightarrow x = 1, -5.$

Again diff. eqn (1), we have $\frac{d^2y}{dx^2} = 6x + 12$

(i) when $x = 1$, $\frac{d^2y}{dx^2} = 6(1) + 12 = 18$, which is +ve.

$\therefore$ The given function is minimum at $x = 1$

The minimum value of the function at $x=1$ $= (1)^3 + 6(1)^2 - 15(1) + 5$

$= 1 + 6 - 15 + 5 = -3$ Ans

(ii) when $x = -5$, $\frac{d^2y}{dx^2} = 6(-5) + 12 = -18$, which is −ve

$\therefore$ The given function is maximum at $x = -5$

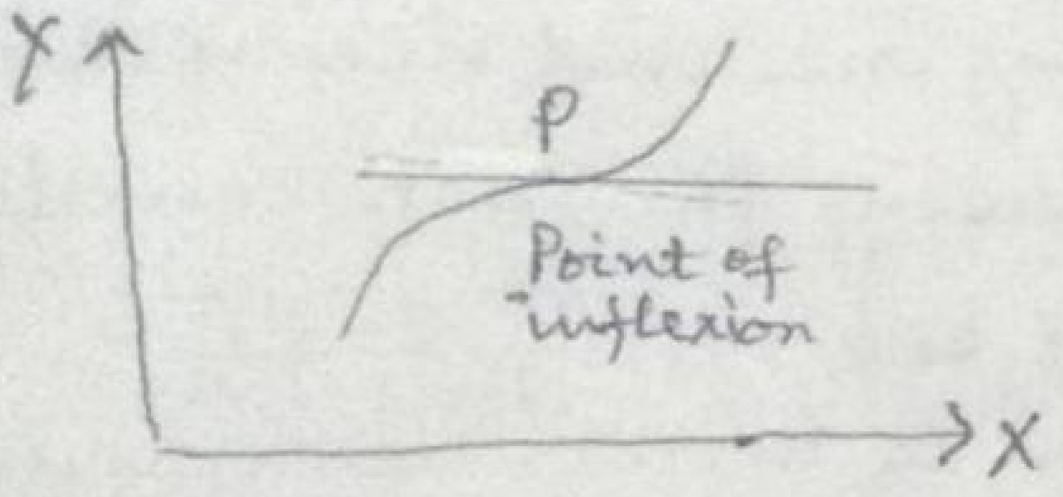
and the maximum value of function at $x = -5$ $= (-5)^3 + 6(-5)^2 - 15(-5) + 5$

$= -125 + 150 + 75 + 5$

$= 105$

Ans

Point of inflexion :- Let us consider a function $y = f(x)$ at P, where tangent is parallel to x-axis. So, $\frac{dy}{dx} = 0$



But before and after the point P, the function $y = f(x)$ is increasing, so $\frac{dy}{dx}$ does not change its sign while passing through P.

Hence the Condition for a point of inflexion are

(i) $\frac{dy}{dx} = 0$ (ii) $\frac{d^2y}{dx^2} = 0$ (iii) $\frac{d^3y}{dx^3} \neq 0$

Example :- Find the point of inflexion for the function $y = 11 - 12x + 6x^2 - x^3$

Soln :- we have $\frac{dy}{dx} = -12 + 12x - 3x^2$

For maxima and minima $\frac{dy}{dx} = 0 \Rightarrow 3x^2 - 12x + 12 = 0 \therefore x = 2$

Now, $\frac{d^2y}{dx^2} = 12 - 6x$, At $x = 2$, $\frac{d^2y}{dx^2} = 0$ but $\frac{d^3y}{dx^3} = -6 \neq 0$

Hence $x = 2$ is a point of inflexion.

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Maxima and minima of function of two variables :-

Defⁿ :- A function $f(x, y)$ is said to have a maximum value at $x = a, y = b$ if $f(a,b) > f(a+h, b+k)$ for all small values of h and k. Similarly $f(x,y)$ is said to have minimum value at $x = a, y = b$ if $f(a,b) < f(a+h, b+k)$ for small values of h and k.

A max. or min. value of a function is called its 'extreme value'.

Conditions for $f(x, y)$ to be max. or min.

Necessary Condition :- The necessary condition for $f(x,y)$ to have max. or min. values at (a, b) are that $f_x(a, b) = 0$ and $f_y(a, b) = 0$; where $f_x(a,b) = \frac{\partial f}{\partial x}$ at (a,b) & $f_y(a,b) = \frac{\partial f}{\partial y}$ at (a,b)

Sufficient Conditions :- If $f_x(a,b) = 0$, $f_y(a,b) = 0$ $f_{xx}(a,b) = r$, $f_{xy}(a,b) = s$, $f_{yy}(a,b) = t$ then

(i) $f(a,b)$ is maximum value if $rt - s^2 > 0$ and $r < 0$ (or $t < 0$) at (a, b).

(ii) $f(a, b)$ is minimum value if $rt - s^2 > 0$ and $r > 0$ (or $t > 0$) at (a, b).

(iii) $f(a, b)$ is not an extreme value if $rt - s^2 < 0$ at (a,b), then (a, b) is a saddle point.

(iv) If $rt - s^2 = 0$ the test is inconclusive.

Stationary point & stationary value : — A point (a, b) at which $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ is called stationary or turning point. The value of $f(x, y)$ at stationary point (a, b) is called stationary value.

Thus every extreme value is a stationary value but the converse may not be true.

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Q.1. Find the maximum and minimum values of
$x^3+y^3-3y-12x+20$.

Soln :- Let $f(x,y) = x^3+y^3-12x-3y+20$.

$\therefore f_x = \frac{\partial f}{\partial x} = 3x^2-12$; $f_y = \frac{\partial f}{\partial y} = 3y^2-3$

$f_{xx} = \frac{\partial^2 f}{\partial x^2} = 6x$; $f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 0$ and $f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = 6y$

when $f_x = 0$, we have $3x^2-12=0$
" $f_y = 0$, we have $3y^2-3=0$
Solving above eqns, we get $x=\pm 2$ and $y=\pm 1$
So the stationary points are $(-2,-1)$ $(-2,1)$ $(2,-1)$ $(2,1)$
From table we have

Points	$r=f_{xx}$	$s=f_{xy}$	$t=f_{yy}$	$rt-s^2$	Extreme value
$(-2,-1)$	-12	0	-6	$72>0$	Max. at $(-2,-1)$
$(-2,1)$	-12	0	6	$-72<0$	No extreme value (saddle point)
$(2,-1)$	12	0	-6	$-72<0$	No extreme value (saddle point)
$(2,1)$	12	0	6	$72>0$	Minimum at $(2,1)$

Therefore maximum value at $(-2,-1) = (-2)^3+(-1)^3-12(-2)$
$-3(-1)+20 = 38$

and minimum value at $(2,1) = 2^3+1^3-12(2)-3(1)+20 = 2$

Q.2. Determine the points where the function
$f(x,y) = x^3+y^3-3axy$ has a maximum or minimum.

Soln :- we have $f_x = \frac{\partial f}{\partial x} = 3x^2-3ay$, $f_y = \frac{\partial f}{\partial y} = 3y^2-3ax$

$r = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 6x$, $s = f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -3a$, $t = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$
$= 6y$.

For extreme points, $f_x = f_y = 0$

$\therefore 3x^2-3ay=0$ and $3y^2-3ax=0$

Solving these eqns, we get two stationary points as $(0,0)$ and (a,a)

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Thus $rt - s^2 = 36xy - 9a^2$

At $(0,0)$, $rt - s^2 = -9a^2$ (negative). So there is no extreme point at origin $(0,0)$.

At (a,a), we have $rt - s^2 = 36a^2 - 9a^2 = 27a^2 > 0$

Also r at (a,a) is equal to $6a$.

If a is +ve, then r is +ve and $f(x,y)$ will have a minimum at (a,a).

If a is −ve, then r is −ve, so $f(x,y)$ will have a maximum at (a,a) for $a<0$.

Q.3. A rectangular box, open at the top is to have volume of 32 c.c. Find the dimensions of the box requiring least material for its construction.

Soln :- Let x, y, z be length, breadth and height of the rectangular box. In order to find the dimension of the box requiring least material for its construction, it is sufficient to find the least surface area.

Let S be the surface area.

Given volume $= xyz = 32 \Rightarrow z = \frac{32}{xy}$ — (1)

Surface Area $S = xy + 2yz + 2zx$ —— (2)

Eliminating z from (2) with the help of (1), we get

$$S = xy + 2(y+x)\frac{32}{xy} = xy + 64\left(\frac{1}{x} + \frac{1}{y}\right)$$

$\therefore \frac{\partial S}{\partial x} = y - \frac{64}{x^2} = 0$ and $\frac{\partial S}{\partial y} = x - \frac{64}{y^2} = 0$

Solving these equations, we get $x = 4, y = 4$

Now $r = \frac{\partial^2 S}{\partial x^2} = \frac{128}{x^3}$, $s = \frac{\partial^2 S}{\partial x \partial y} = 1$, $t = \frac{\partial^2 S}{\partial y^2} = \frac{128}{y^3}$

At $x=4, y=4$; $rt - s^2 = \frac{128}{64} \times \frac{128}{64} - 1 = 2\times 2 - 1 = 3$

and r is also +ve

Hence S is minimum for $x=4, y=4, z=2$ <u>Ans</u>

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Lagrange's method of undetermined multipliers :-

In many situations it is required to find the max. or min. value of a function whose variables are connected by some given relation. Lagrange's method is very helpful in those condition.

Let $f(x,y,z)$ be a function of three variables x,y,z and the variables be connected by the relation

$$\phi(x,y,z) = 0 \quad \text{--- (1)}$$

For $f(x,y,z)$ to have stationary values, it is necessary that $\frac{\partial f}{\partial x}=0$, $\frac{\partial f}{\partial y}=0$, $\frac{\partial f}{\partial z}=0$

$$\Rightarrow \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0 \quad \text{--- (2)}$$

Also differentiating eq. (1), $\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = 0$ --- (3)

Multiply (3) by λ and add to (2), we get

$$\left(\frac{\partial f}{\partial x}+\lambda\frac{\partial \phi}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y}+\lambda\frac{\partial \phi}{\partial y}\right)dy + \left(\frac{\partial f}{\partial z}+\lambda\frac{\partial \phi}{\partial z}\right)dz = 0$$

$$\Rightarrow \frac{\partial f}{\partial x}+\lambda\frac{\partial \phi}{\partial x}=0,\quad \frac{\partial f}{\partial y}+\lambda\frac{\partial \phi}{\partial y}=0,\quad \frac{\partial f}{\partial z}+\lambda\frac{\partial \phi}{\partial z}=0$$

on solving these three equns. together with (1) we can find the values of x, y, z and λ for which $f(x,y,z)$ has stationary value.

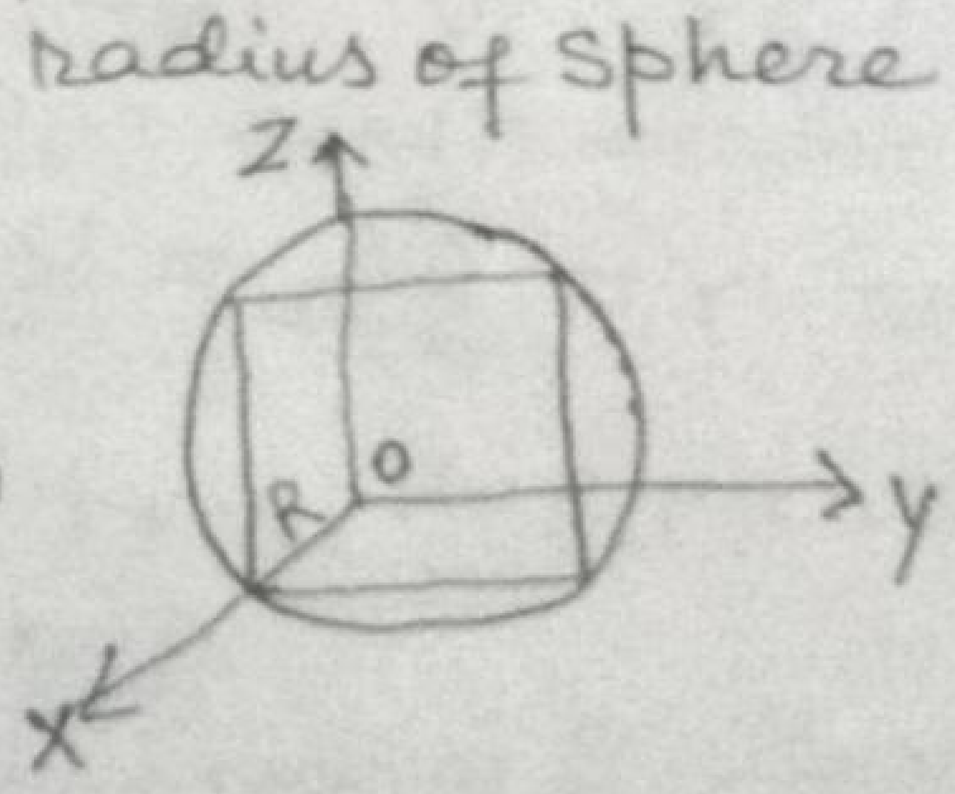
Q.(1). Show that the rectangular solid of maximum volume that can be inscribed in a sphere is a cube.

Soln:- Let $2x, 2y, 2z$ be the length, breadth and height of a rectangular solid. Let R be radius of sphere.

volume of solid $V = 8xyz$

and $x^2+y^2+z^2 = R^2$

or $\phi(x,y,z) = x^2+y^2+z^2-R^2 = 0$ --- (1)



By Lagrange's equations, we have

$$\frac{\partial V}{\partial x}+\lambda\frac{\partial \phi}{\partial x}=0 \Rightarrow 8yz+\lambda(2x)=0 \quad \text{--- (2)}$$

$$\frac{\partial V}{\partial y}+\lambda\frac{\partial \phi}{\partial y}=0 \Rightarrow 8xz+\lambda(2y)=0 \quad \text{--- (3)}$$

$$\frac{\partial V}{\partial z}+\lambda\frac{\partial \phi}{\partial z}=0 \Rightarrow 8xy+\lambda(2z)=0 \quad \text{--- (4)}$$

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From eqn (2), we have $2\lambda x = -8yz \Rightarrow 2\lambda x^2 = -8xyz$

From eqn (3) " $2\lambda y = -8xz \Rightarrow 2\lambda y^2 = -8xyz$

From eqn (4) " $2\lambda z = -8xy \Rightarrow 2\lambda z^2 = -8xyz$

Therefore, $2\lambda x^2 = 2\lambda y^2 = 2\lambda z^2$

or $x^2 = y^2 = z^2 \Rightarrow x = y = z.$

Hence rectangular solid is a Cube Proved

Q(2). Find the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Soln :- Given $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

or $\phi(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$

Let $2x, 2y, 2z$ be length, breadth and height of the rectangular parallelopiped inscribed in the ellipsoid

volume $V = 2x \cdot 2y \cdot 2z = 8xyz$

The problem is to maximize $8xyz$ subject to $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Now, $\frac{\partial V}{\partial x} = 8yz$, $\frac{\partial V}{\partial y} = 8xz$, $\frac{\partial V}{\partial z} = 8xy$

Also $\frac{\partial \phi}{\partial x} = \frac{2x}{a^2}$, $\frac{\partial \phi}{\partial y} = \frac{2y}{b^2}$, $\frac{\partial \phi}{\partial z} = \frac{2z}{c^2}$

Using Lagrange's method, we have

$$\frac{\partial V}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 \Rightarrow 8yz + \lambda \cdot \frac{2x}{a^2} = 0 \quad \text{--- (1)}$$

$$\frac{\partial V}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0 \Rightarrow 8xz + \lambda \cdot \frac{2y}{b^2} = 0 \quad \text{--- (2)}$$

$$\frac{\partial V}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0 \Rightarrow 8xy + \lambda \cdot \frac{2z}{c^2} = 0 \quad \text{--- (3)}$$

Multiply (1), (2) and (3) by x, y, z respectively and adding we get, $24xyz + 2\lambda\left[\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right] = 0$

$$\Rightarrow 24xyz + 2\lambda(1) = 0$$

$$\Rightarrow \lambda = -12xyz$$

Putting the value of λ in (1) we get

$$8yz + (-12xyz)\frac{2x}{a^2} = 0 \Rightarrow 8yz\left(1 - \frac{3x^2}{a^2}\right) = 0$$

$$\Rightarrow x = \frac{a}{\sqrt{3}}$$

Similarly from (2) and (3), we get

$$y = \frac{b}{\sqrt{3}} \text{ and } z = \frac{c}{\sqrt{3}}$$

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$\therefore$ volume of greatest rectangular parallelopiped $= 8xyz$

$$= 8\left(\frac{a}{\sqrt{3}}\right)\left(\frac{b}{\sqrt{3}}\right)\left(\frac{c}{\sqrt{3}}\right) = \frac{8abc}{3\sqrt{3}}$$

Q.(3) The temperature T at any point (x, y, z) in space is $T = 400xyz^2$. Find the highest temperature at the surface of a unit sphere $x^2+y^2+z^2=1$.

Soln :- Given $T = 400xyz^2$

and $x^2+y^2+z^2=1$ or, $\phi(x,y,z) = x^2+y^2+z^2-1$

The problem is to maximize T subject to $x^2+y^2+z^2=1$

Using Lagrange's method, we have

$$\frac{\partial T}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0 \Rightarrow 400yz^2 + \lambda(2x) = 0 \quad \text{— (1)}$$

$$\frac{\partial T}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0 \Rightarrow 400xz^2 + \lambda(2y) = 0 \quad \text{— (2)}$$

$$\frac{\partial T}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0 \Rightarrow 800xyz + \lambda(2z) = 0 \quad \text{— (3)}$$

Multiply (1) by x, (2) by y and (3) by z and adding together, we get

$$400xyz^2 + 2\lambda x^2 + 400xyz^2 + 2\lambda y^2 + 800xyz^2 + 2\lambda z^2 = 0$$

$$\Rightarrow 1600xyz^2 + 2\lambda(x^2+y^2+z^2) = 0$$

$$\Rightarrow 1600xyz^2 + 2\lambda(1) = 0 \Rightarrow \lambda = -800xyz^2$$

Putting the value of λ in (1) we get

$$400yz^2 + 2x(-800xyz^2) = 0$$

$$\Rightarrow 400yz^2 - 1600x^2yz^2 = 0 \Rightarrow 1-4x^2=0 \Rightarrow x = \pm\frac{1}{2}$$

Similarly putting value of λ in (2) and (3) we get

$$y = \pm\frac{1}{2} \text{ and } z = \pm\frac{1}{\sqrt{2}}$$

on putting values of x, y, z in $T = 400xyz^2$, we get

$$T = 400 \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = 50$$

$\therefore$ The highest temp. at the surface of unit sphere $= 50$ Ans

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