Vector Calculus

Scalar and vector point function :-

A variable quantity whose value at any point in space depends upon the position of the point, is called a point function.

There are two types of point functions.

1. Scalar point function :- A function $\phi(x, y, z)$ is called scalar point function if it associates a scalar with every point in space. For example: temperature distribution at any instant, density of a body, distribution of atmospheric pressure.
2. Vector point function :- If a function $\vec{F}(x, y, z)$ defines a vector at every point in space, then $\vec{F}(x, y, z)$ is called vector point function and the region of space R is called vector field.
For example: velocity of moving fluid at any instant, the gravitational force at any point in space are examples of vector point function.
3. Level surface :- The surface drawn in space containing all those points where $\phi(x, y, z)$ has same value is known as level surface.
4. Vector differential operator :- It is denoted by the symbol '∇' called Del or nabla.
$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}.$$
5. Derivative of vector function :- A vector function $\vec{V}(t)$ is said to be differentiable at a point t, if the following limit exists.
$$\frac{d}{dt}\vec{V}(t) \text{ or } \vec{V}'(t) = \lim_{\Delta t \to 0} \frac{\vec{V}(t+\Delta t) - \vec{V}(t)}{\Delta t}$$
$\vec{V}'(t)$ is called derivative of $\vec{V}(t)$.

The rules of diff. calculus holds for differentiation of vector function also. Thus

$(c\vec{V})' = c\vec{V}'$ where c is a constant.

$(\vec{u} \pm \vec{V})' = \vec{u}' \pm \vec{V}'$.

$(\vec{u} \cdot \vec{V})' = \vec{u}' \cdot \vec{V} + \vec{u} \cdot \vec{V}'$ and $(\vec{u} \times \vec{V})' = \vec{u}' \times \vec{V} + \vec{u} \times \vec{V}'$

<u>Gradient of a Scalar point function</u> :- The gradient of scalar point function ϕ is defined as

$$\nabla\phi \text{ or grad } \phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}.$$

The gradient at any point of scalar point fun. $\phi(x,y,z)$ is a vector that is normal to level surface $\phi(x,y,z) = c$. The magnitude of gradient is the rate of change of $\phi(x,y,z)$ in the direction of normal to surface at the point $P(x,y,z)$.

For example: In a room, the temp. is different at different points, gradient determines this difference. The roof of a house is built with a gradient to enable rain water to run down the roof.

<u>Properties of gradient</u> :- (1) If f and g are two scalar point functions, then

$$\text{grad}(f \pm g) \text{ or } \nabla(f \pm g) = \nabla f \pm \nabla g$$

(2) If $f(x,y,z)$ is constant, then $\nabla f = 0$

(3) $\nabla(f \cdot g) = f\nabla g + g\nabla f$.

(4) $\nabla(c \cdot f) = c\nabla f$, where c is a constant.

<u>Directional derivative</u> :- The directional derivative in the direction of vector $\vec{a}$ is $\nabla\phi \cdot \hat{a}$, where $\hat{a}$ is unit vector. The directional derivative of ϕ is maximum along the normal to the surface $\phi(x,y,z) = c$. and the magnitude of this maximum is given by $|\nabla\phi|$

Q.1: Find a unit vector normal to the surface $x^3+y^3+z^3+3xyz=3$ at the point $(1,2,-1)$.

Soln :- Let the given surface be $\phi(x,y,z)=x^3+y^3+z^3+3xyz-3=0$

Since $\nabla\phi$ is a vector normal to the surface ϕ

$\therefore \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z} = \hat{i}(3x^2+3yz)+\hat{j}(3y^2+3xz)+\hat{k}(3z^2+3xy)$

At $(1,2,-1)$ $\nabla\phi = -3\hat{i}+9\hat{j}+9\hat{k}$ which is a vector normal to the given surface at $(1,2,-1)$

Hence unit vector normal to given surface at $(1,2,-1)$

$$= \frac{-3\hat{i}+9\hat{j}+9\hat{k}}{\sqrt{9+81+81}} = \frac{1}{\sqrt{19}}(-\hat{i}+3\hat{j}+3\hat{k})$$ Ans

Q.2. Find directional derivative of x^2+y^2+4xz at $(1,-2,2)$ in the direction of vector $2\hat{i}-2\hat{j}-\hat{k}$

Soln :- Let the scalar point function $\phi(x,y,z)=x^2+y^2+4xz$

then $\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z} = (2x+4z)\hat{i}+2y\hat{j}+4x\hat{k}$

At $(1,-2,2)$ $\nabla\phi = 10\hat{i}-4\hat{j}+4\hat{k}$

Thus directional derivative at $(1,-2,2)$ along the vector $2\hat{i}-2\hat{j}-\hat{k}$ is $(10\hat{i}-4\hat{j}+4\hat{k})\cdot\frac{2\hat{i}-2\hat{j}-\hat{k}}{\sqrt{9}}$

$$= \frac{1}{3}(20+8-4) = 8$$ Ans

Q.3. If $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$. Show that

(i) $\text{grad}\, r = \frac{\vec{r}}{r}$

(ii) $\text{grad}(\vec{a}\cdot\vec{r}) = \vec{a}$, where $\vec{a}$ is const. vector

Soln :- Given $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$

$\therefore |\vec{r}|$ or $r = \sqrt{x^2+y^2+z^2} \Rightarrow r^2 = x^2+y^2+z^2$

Diff. partially w.r.t. x, $2r\frac{\partial r}{\partial x} = 2x$ or $\frac{\partial r}{\partial x} = \frac{x}{r}$

Similarly diff. partially w.r.t. y and z, we get

$$\frac{\partial r}{\partial y} = \frac{y}{r} \text{ and } \frac{\partial r}{\partial z} = \frac{z}{r}$$

$\therefore$ grad $r = \nabla r = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)r$

$$= \hat{i}\frac{\partial r}{\partial x}+\hat{j}\frac{\partial r}{\partial y}+\hat{k}\frac{\partial r}{\partial z} = \hat{i}\left(\frac{x}{r}\right)+\hat{j}\left(\frac{y}{r}\right)+\hat{k}\left(\frac{z}{r}\right)$$

$$= \frac{x\hat{i}+y\hat{j}+z\hat{k}}{r} = \frac{\vec{r}}{r}$$ proved

(ii) To show that $\nabla(\vec{a}\cdot\vec{r}) = \vec{a}$, where $\vec{a}$ is Const. vector

Soln :- Let $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, where a_1, a_2, a_3 are Const.

$\therefore \vec{a}\cdot\vec{r} = (a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\cdot(x\hat{i}+y\hat{j}+z\hat{k})$

$= a_1x+a_2y+a_3z$

and $\nabla(\vec{a}\cdot\vec{r}) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(a_1x+a_2y+a_3z)$

$= \hat{i}a_1+\hat{j}a_2+\hat{k}a_3 = \vec{a}$ proved

Q.4. In what direction from $(3,1,-2)$ is the directional derivative of $\phi = x^2y^2z^4$ maximum? Also find the magnitude of this maximum.

Soln :- The directional derivative of ϕ is max. for $\nabla\phi$ and the magnitude of this max. is $|\nabla\phi|$

$\therefore \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)x^2y^2z^4 = 2xy^2z^4\hat{i}+2yx^2z^4\hat{j}$
$+4z^3x^2y^2\hat{k}$

At $(3,1,-2)$ $\nabla\phi = 96\hat{i}+288\hat{j}-288\hat{k} = 96(\hat{i}+3\hat{j}-3\hat{k})$ Ans

Also the magnitude of this maximum $= |\nabla\phi| = 96\sqrt{1+9+9} = 96\sqrt{19}$

Q.5. The temperature of points in space is given by $T(x,y,z) = x^2+y^2-z$. A mosquito located at $(1,1,2)$ desires to fly in such a direction that it will get warm as soon as possible. In what direction should it move?

Soln :- Since temperature T is a scalar point function. ∇T will give maximum rate of change of temp. along the normal to the surface $T = x^2+y^2-z$.

$\therefore \nabla T = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)x^2+y^2-z = 2x\hat{i}+2y\hat{j}-\hat{k}$

At $(1,1,2)$ $\nabla T = 2\hat{i}+2\hat{j}-\hat{k}$

Thus unit vector normal to the surface at $(1,1,2)$ is

$$\frac{2\hat{i}+2\hat{j}-\hat{k}}{\sqrt{4+4+1}} = \frac{1}{3}(2\hat{i}+2\hat{j}-\hat{k})$$ Which is the req. direction

Ans