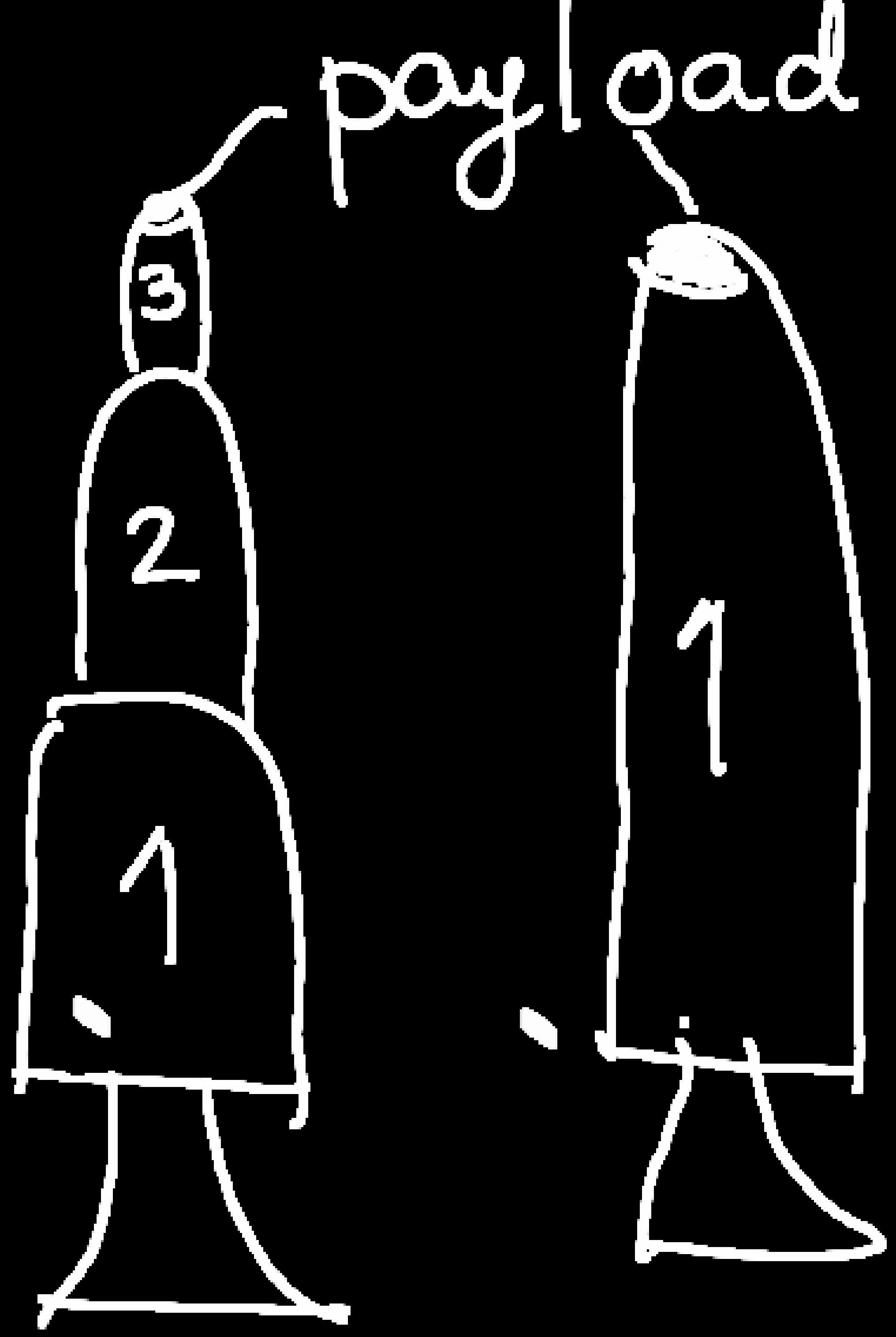
Multistaging



$$\Delta u_i = u_{eq}\big|_i \ln R_i$$

$$R_i = \frac{1+\lambda_i}{\epsilon_i+\lambda_i}$$

$$\underset{\text{Terminal velocity}}{u_n} = \sum_{i=1}^{n} \Delta u_i \;, \quad n\text{- no. of stages}$$

$$= \sum_i u_{eq}\big|_i \ln R_i$$

$$u_n = n\,u_{eq} \ln R = n u_{eq} \ln\left(\frac{1+\lambda}{\epsilon+\lambda}\right)$$

$u_{eq}\big|_i$ & R_i are same

Payload ratio, $\lambda_i = \dfrac{M_{o\,i+1}}{M_{oi} - M_{o\,i+1}}$

$$1/\lambda_i = \frac{M_{oi} - M_{o\,i+1}}{M_{o\,i+1}} = \frac{M_{oi}}{M_{o\,i+1}} - 1$$

$$\boxed{\frac{M_{oi}}{M_{o\,i+1}} = \frac{1+\lambda_i}{\lambda_i}}$$

① $$\frac{M_{01}}{M_{02}} = \frac{1+\lambda_1}{\lambda_1}$$

② $$\frac{M_{02}}{M_{03}} = \frac{1+\lambda_2}{\lambda_2}$$

$\vdots$

ⓝ $$\frac{M_{0n}}{M_\ell} = \frac{1+\lambda_n}{\lambda_n}$$

$$\frac{M_{01}}{M_\ell} = \prod_{i=1}^{n} \frac{1+\lambda_i}{\lambda_i} = \left(\frac{1+\lambda}{\lambda}\right)^n$$

$$\boxed{\left(\frac{M_{01}}{M_\ell}\right)^{1/n} - 1 = 1/\lambda}$$

$$u_n = n\, u_{eq} \ln\left(\frac{1+\lambda}{\epsilon+\lambda}\right) = n\, u_{eq} \ln\left(\frac{1+{}^{1}\!/\!\lambda}{1+\epsilon/\lambda}\right).$$

$$u_n = n\, u_{eq} \ln\left[\frac{\left(\frac{M_{01}}{M_l}\right)^{1/n}}{1+\epsilon\left(\left(\frac{M_{01}}{M_l}\right)^{1/n}-1\right)}\right]$$

Escape velocity $V_{esc} = 11.2\ km/s = u_n$

$u_n/u_{eq} \simeq 3.2$ $\quad u_{eq} \simeq 3500\ m/s$

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Multi-staging

$$\frac{u_n}{u_{eq}} = n \ln \left[\frac{\left(\frac{M_{01}}{M_L}\right)^{1/n}}{\in\left\{\left(\frac{M_{01}}{M_L}\right)^{1/n} - 1\right\} + 1} \right]$$

PSLV – 4
GSLV – 3

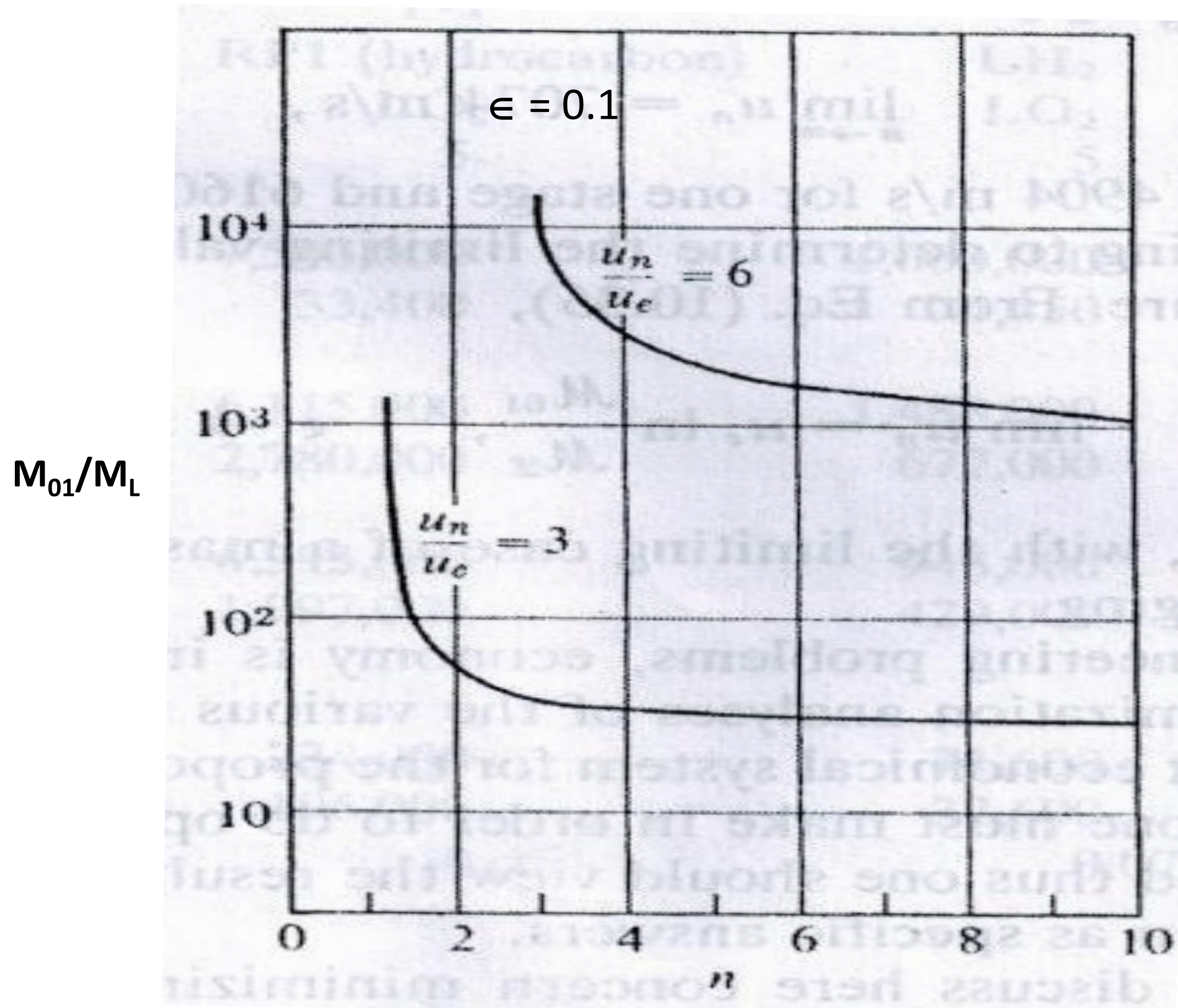

Fig. 10.8 Variation of overall mass ratio with number of stages for fixed terminal velocity ratios; similar stages and structural coefficient ∈ = 0.1.
(Mechanics and Thermodynamics of Propulsion by Philip Hill and Carl Peterson, Second Edition, Dorling Kindersley India Pvt. Ltd., Noida, 2010)

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Multi-staging

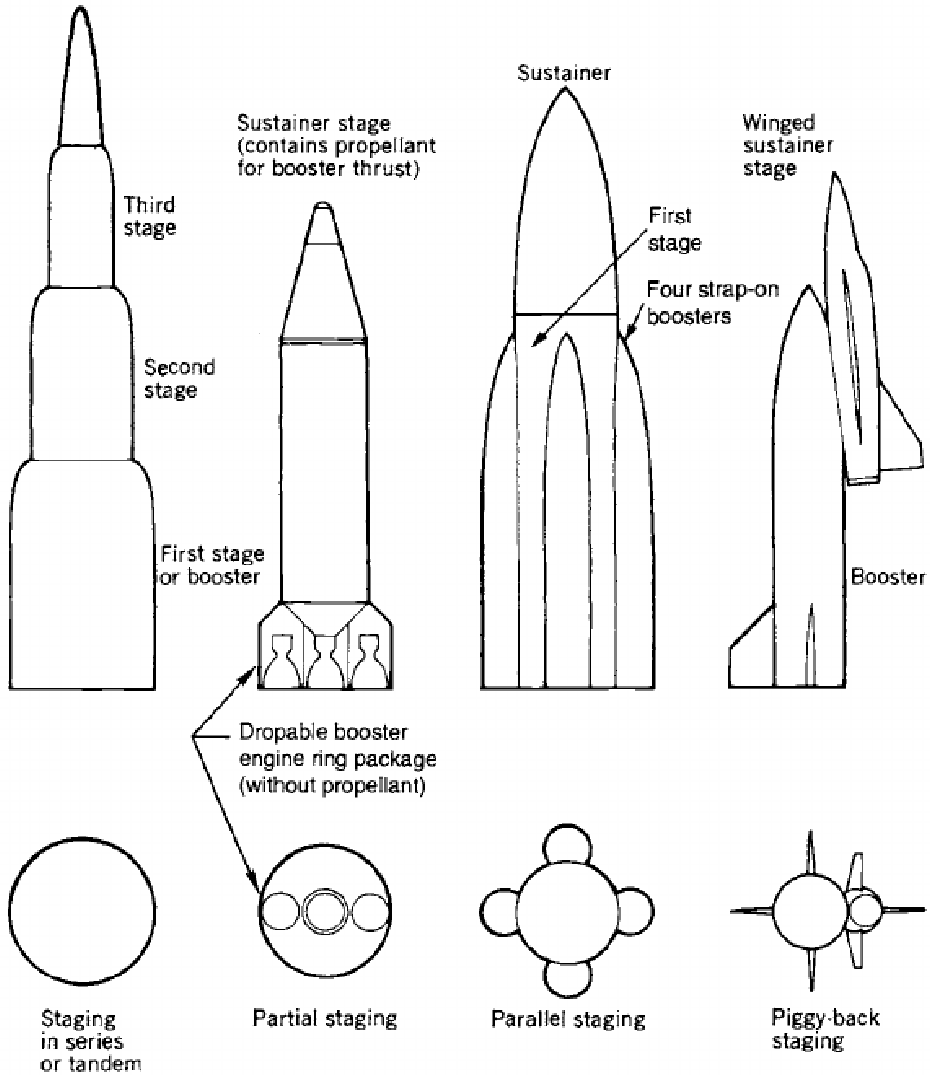

Fig. 4.14 Simplified schematic sketches of four geometric configurations for assembling individual stages into a launch vehicle.
(Rocket propulsion elements by G P Sutton and O Biblarz)

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Multistage optimization Reading assignment 1

- **Structural coefficients ($\in_i$) are known**
- For all the stages the **equivalent exhaust velocities are same ($u_{eq\text{-}i} = u_{eq}$) and known**
- To **find payload ratios (λ_i)** which will result in **maximum terminal velocity (u_n)** subject to constraint that the vehicle **payload mass (M_L) and initial vehicle mass (M_{01}) is fixed**

Maximize: $$\frac{u_n}{u_{eq}} = \sum_{i=1}^{n} \ln(R_i) = \sum_{i=1}^{n} \ln\left(\frac{1+\lambda_i}{\in_i+\lambda_i}\right) = \sum_{i=1}^{n} \mathrm{F}(\lambda_i)$$

Constraint: $$\frac{M_{01}}{M_L} = \prod_{i=1}^{n}\left(\frac{1+\lambda_i}{\lambda_i}\right)$$ **or** $$\frac{M_L}{M_{01}} = \prod_{i=1}^{n}\left(\frac{\lambda_i}{1+\lambda_i}\right)$$

$$\ln\left(\frac{M_L}{M_{01}}\right) = \ln\left[\prod_{i=1}^{n}\left(\frac{\lambda_i}{1+\lambda_i}\right)\right] = \sum_{i=1}^{n} \ln\left(\frac{\lambda_i}{1+\lambda_i}\right) = \sum_{i=1}^{n} \mathrm{G}(\lambda_i)$$

Lagrangian: $$\mathrm{L}(\lambda_i, \alpha)\big|_{i=1,\ldots,n} = \sum_{i=1}^{n} \mathrm{F}(\lambda_i) + \alpha\left[\sum_{i=1}^{n} \mathrm{G}(\lambda_i) - \ln\left(\frac{M_L}{M_{01}}\right)\right]$$

where, α is the Lagrange multiplier

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Multistage optimization

maximizing the Lagrangian

n equations (for partial derivative with k = 1,…, n)

$$\frac{\partial \mathrm{L}(\lambda_i,\alpha)|_{i=1,\ldots,n}}{\partial \lambda_k} = \frac{\partial \sum_{i=1}^{n} \mathrm{F}(\lambda_i)}{\partial \lambda_k} + \alpha \frac{\partial \sum_{i=1}^{n} \mathrm{G}(\lambda_i)}{\partial \lambda_k} = 0$$

1 equation (for partial derivative with α)

$$\frac{\partial \mathrm{L}(\lambda_i,\alpha)|_{i=1,\ldots,n}}{\partial \alpha} = \sum_{i=1}^{n} \mathrm{G}(\lambda_i) - \ln\left(\frac{M_L}{M_{01}}\right) = 0$$

which specifies the constraint

note: we have n+1 equations and n (λ_i) + 1 (α) unknowns and hence all λ_i can be found

$$\frac{\partial \sum_{i=1}^{n} \mathrm{F}(\lambda_i)}{\partial \lambda_k} = \frac{\partial \ln\left(\frac{1+\lambda_k}{\epsilon_k+\lambda_k}\right)}{\partial \lambda_k} = \frac{1}{\left(\frac{1+\lambda_k}{\epsilon_k+\lambda_k}\right)}\left[\frac{1}{\epsilon_k+\lambda_k} - \frac{1+\lambda_k}{(\epsilon_k+\lambda_k)^2}\right] = \frac{1}{1+\lambda_k} - \frac{1}{\epsilon_k+\lambda_k}$$

$$\frac{\partial \sum_{i=1}^{n} \mathrm{G}(\lambda_i)}{\partial \lambda_k} = \frac{\partial \ln\left(\frac{\lambda_k}{1+\lambda_k}\right)}{\partial \lambda_k} = \frac{1}{\left(\frac{\lambda_k}{1+\lambda_k}\right)}\left[\frac{1}{1+\lambda_k} - \frac{\lambda_k}{(1+\lambda_k)^2}\right] = \frac{1}{\lambda_k} - \frac{1}{1+\lambda_k}$$

for each k we have:

$$\frac{1}{1+\lambda_k} - \frac{1}{\epsilon_k+\lambda_k} + \alpha\left(\frac{1}{\lambda_k} - \frac{1}{1+\lambda_k}\right) = 0$$

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Multistage optimization

$$\frac{1}{1+\lambda_k} - \frac{1}{\epsilon_k+\lambda_k} + \frac{\alpha}{\lambda_k(1+\lambda_k)} = 0$$

$$\frac{1}{1+\lambda_k} + \frac{\alpha}{\lambda_k(1+\lambda_k)} = \frac{1}{\epsilon_k+\lambda_k}$$

$$\frac{\lambda_k+\alpha}{\lambda_k(1+\lambda_k)} = \frac{1}{\epsilon_k+\lambda_k}$$

$$\lambda_k \in_k + \alpha \in_k + \cancel{\lambda_k^2} + \alpha\lambda_k = \lambda_k + \cancel{\lambda_k^2}$$

$$\lambda_k \in_k + \alpha \in_k + \alpha\lambda_k = \lambda_k$$

$$\alpha \in_k = \lambda_k - \alpha\lambda_k - \lambda_k \in_k$$

$$\lambda_k = \frac{\alpha\in_k}{1-\alpha-\in_k}$$

note: if all the structural coefficients ($\in_k$) are same for all the stages then all the payload ratios (λ_k) are also same as α is a constant

$$\frac{M_{01}}{M_L} = \left(\frac{1+\lambda}{\lambda}\right)^n \qquad \lambda = \frac{1}{\left(\frac{M_{01}}{M_L}\right)^{\frac{1}{n}} - 1}$$

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Multistage optimization

$$\lambda_k = \frac{\alpha \in_k}{1-\alpha-\in_k}$$

note: for different structural coefficients ($\in_k$), the Lagrange multiplier (α) needs to be determined to calculate payload ratios (λ_k) for each stage, this is done by using the last equation (partial derivative w.r.t. α) or the constraint equation

$$\frac{M_L}{M_{01}} = \prod_{i=1}^{n}\left(\frac{\lambda_i}{1+\lambda_i}\right) = \prod_{i=1}^{n}\left(\frac{\frac{\alpha\in_i}{1-\alpha-\in_i}}{1+\frac{\alpha\in_i}{1-\alpha-\in_i}}\right) = \prod_{i=1}^{n}\left(\frac{\alpha}{1-\alpha}\times\frac{\in_i}{1-\in_i}\right) = \left(\frac{\alpha}{1-\alpha}\right)^n \prod_{i=1}^{n}\left(\frac{\in_i}{1-\in_i}\right)$$

$$\frac{M_{01}}{M_L} = \left(\frac{1}{\alpha}-1\right)^n \prod_{i=1}^{n}\left(\frac{1}{\in_i}-1\right)$$

$$\frac{\frac{M_{01}}{M_L}}{\prod_{i=1}^{n}\left(\frac{1}{\in_i}-1\right)} = \left(\frac{1}{\alpha}-1\right)^n \qquad \left[\frac{\frac{M_{01}}{M_L}}{\prod_{i=1}^{n}\left(\frac{1}{\in_i}-1\right)}\right]^{\frac{1}{n}} + 1 = \frac{1}{\alpha}$$

$$\alpha = \frac{1}{\left[\frac{\left(\frac{M_{01}}{M_L}\right)}{\prod_{i=1}^{n}\left(\frac{1}{\in_i}-1\right)}\right]^{\frac{1}{n}}+1}$$

note: once the Lagrange multiplier (α) is determined the payload ratios (λ_k) for each stage can be obtained

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Example

Given:

equivalent exhaust velocity (u_{eq}) = 3,048 m/s

initial rocket mass (M_0) = 15,000 kg

payload mass (M_L) = 1,000 kg

neglect drag and gravity and consider two stages (n = 2)

Find the terminal velocity (u_n) if (a) $\in_1 = \in_2 = 0.143$ (b) $\in_1 = 0.1$, $\in_2 = 0.2$

Solution:

(a) for same structural coefficients for both the stages ($\in_1 = \in_2$):

$$\in_1 = \frac{M_{S1}}{M_{01}-M_{02}} = \in_2 = \frac{M_{S2}}{M_{02}-M_L} = 0.143$$

For maximum terminal velocity, the payload ratios are also same ($\lambda_1 = \lambda_2 = \lambda$)

$$\lambda_1 = \lambda_2 = \lambda = \frac{1}{\left(\frac{M_{01}}{M_L}\right)^{\frac{1}{n}}-1} = \frac{1}{\left(\frac{15000}{1000}\right)^{\frac{1}{2}}-1} = 0.348$$

$$\frac{u_n}{u_{eq}} = n\ln\left(\frac{1+\lambda}{\in+\lambda}\right) = 2\ln\left(\frac{1+0.348}{0.143+0.348}\right) = 2.02$$

$$u_n = 2.02 \times 3048 = 6154\, m/s$$

Example

$$\lambda_2 = \frac{M_L}{M_{02} - M_L} = 0.348 = \frac{1000}{M_{02} - 1000} \qquad M_{02} = 3873\ kg$$

$$\lambda_1 = \frac{M_{02}}{M_{01} - M_{02}} = 0.348 = \frac{M_{02}}{15000 - M_{02}}$$

$$\in_1 = \frac{M_{S1}}{M_{01} - M_{02}} = 0.143 = \frac{M_{S1}}{15000 - 3873} \qquad M_{S1} = 1589\ kg$$

$$\in_2 = \frac{M_{S2}}{M_{02} - M_L} = 0.143 = \frac{M_{S2}}{3873 - 1000} \qquad M_{S2} = 411\ kg$$

$$M_S = M_{S1} + M_{S2} = 1589 + 411 = 2000\ kg$$

the total propellant mass (M_P): $M_P = M_{01} – M_L – M_S = 15000 – 1000 – 2000 = 12000$ kg

Stage 1:

M_{01} = 15,000 kg

M_{S1} = 1,589 kg

$M_{L1} = M_{02}$ = 3,873 kg

M_{P1} = 9,538 kg

M_{b1} = 5,462 kg

$\lambda_1 = 0.348$

$\in_1 = 0.143$

$R_1 = 2.745$

Stage 2:

M_{02} = 3,873 kg

M_{S2} = 411 kg

$M_{L2} = M_L$ = 1,000 kg

M_{P2} = 2,462 kg

M_{b1} = 1,411 kg

$\lambda_2 = 0.348$

$\in_2 = 0.143$

$R_2 = 2.745$

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Example

(b) For different structural coefficients for both the stages ($\in_1 \neq \in_2$):

Suppose: $\in_1 = \frac{M_{S1}}{M_{01}-M_{02}} = 0.1$ and $\in_2 = \frac{M_{S2}}{M_{02}-M_L} = 0.2$

$$\alpha = \frac{1}{\left[\frac{\left(\frac{M_{01}}{M_L}\right)}{\prod_{i=1}^{n}\left(\frac{1}{\in_i}-1\right)}\right]^{\frac{1}{n}}+1} = \frac{1}{\left[\frac{\left(\frac{15000}{1000}\right)}{\left(\frac{1}{0.1}-1\right)\left(\frac{1}{0.2}-1\right)}\right]^{\frac{1}{2}}+1}$$

$$\alpha = 0.6077$$

$$\lambda_k = \frac{\alpha \in_k}{1-\alpha-\in_k} \qquad \lambda_1 = \frac{0.6077 \times 0.1}{1-0.6077-0.1} = 0.208 \qquad \lambda_2 = \frac{0.6077 \times 0.2}{1-0.6077-0.2} = 0.632$$

Example

$$\frac{u_n}{u_{eq}} = \sum_{i=1}^{n} \ln\left(\frac{1+\lambda_i}{\epsilon_i+\lambda_i}\right) = \ln\left(\frac{1+0.208}{0.1+0.208}\right) + \ln\left(\frac{1+0.632}{0.2+0.632}\right) = 1.367 + 0.674 = 2.04$$

$$u_n = 2.04 \times 3048 = 6220\ m/s$$

$$\lambda_2 = \frac{M_L}{M_{02}-M_L} = 0.632 = \frac{1000}{M_{02}-1000}$$

$$M_{02} = 2582\ kg$$

$$\lambda_1 = \frac{M_{02}}{M_{01}-M_{02}} = 0.208 = \frac{M_{02}}{15000-M_{02}}$$

$$\epsilon_1 = \frac{M_{S1}}{M_{01}-M_{02}} = 0.1 = \frac{M_{S1}}{15000-2582} \qquad M_{S1} = 1242\ kg$$

$$\epsilon_2 = \frac{M_{S2}}{M_{02}-M_L} = 0.2 = \frac{M_{S2}}{2582-1000} \qquad M_{S2} = 316\ kg$$

$$M_S = M_{S1} + M_{S2} = 1242 + 316 = 1558\ kg$$

note: M_S for the two cases are not same as this constraint is not imposed while optimization

Example

the constraint is: fixed M_{01} (15000 kg) and M_L (1000 kg)

hence, the total propellant mass (M_P) will be different for the two cases:

$M_P = M_{01} – M_L – M_S = 15000 – 1000 – 1558 = 12442$ kg

Stage 1:

$M_{01} = 15{,}000$ kg

$M_{S1} = 1{,}242$ kg

$M_{L1} = M_{02} = 2{,}582$ kg

$M_{P1} = 11{,}176$ kg

$M_{b1} = 3{,}824$ kg

$\lambda_1 = 0.208$

$\in_1 = 0.1$

$R_1 = 3.923$

Stage 2:

$M_{02} = 2{,}582$ kg

$M_{S2} = 316$ kg

$M_{L2} = M_L = 1{,}000$ kg

$M_{P2} = 1{,}266$ kg

$M_{b1} = 1{,}316$ kg

$\lambda_2 = 0.632$

$\in_2 = 0.2$

$R_2 = 1.962$

Example

Now, if M_S is 1558 kg, M_P = 12442 kg: and fixed M_{01} (15000 kg), M_L (1000 kg)

and both the stages have same structural coefficients ($\in_1 = \in_2$)

$$\in_1 = \frac{M_{S1}}{M_{01}-M_{02}} = \in_2 = \frac{M_{S2}}{M_{02}-M_{0L}}$$

$$M_{S1} = M_{S2}\left(\frac{M_{01}-M_{02}}{M_{02}-M_{0L}}\right)$$

$$M_{S1} + M_{S2} = M_{S2}\left(\frac{M_{01}-M_{02}}{M_{02}-M_{0L}} + 1\right) = M_S$$

$$M_{S2}\left(\frac{M_{01}-M_L}{M_{02}-M_{0L}}\right) = M_S$$

$$\frac{M_{S2}}{M_{02}-M_{0L}} = \in_2 = \frac{M_S}{M_{01}-M_L} = \in$$

$$\in_1 = \in_2 = \in = \frac{1558}{15000-1000} = 0.111$$

$$\lambda_1 = \lambda_2 = \lambda = \frac{1}{\left(\frac{M_{01}}{M_L}\right)^{\frac{1}{n}}-1} = \frac{1}{\left(\frac{15000}{1000}\right)^{\frac{1}{2}}-1} = 0.348$$

$$\frac{u_n}{u_{eq}} = n\ln\left(\frac{1+\lambda}{\in+\lambda}\right) = 2\ln\left(\frac{1+0.348}{0.111+0.348}\right) = 2.155$$

$$u_n = 2.155 \times 3048 = 6567\ m/s$$

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Example

$$\lambda_2 = \frac{M_L}{M_{02}-M_L} = 0.348 = \frac{1000}{M_{02}-1000} \qquad M_{02} = 3873\ kg$$

$$\lambda_1 = \frac{M_{02}}{M_{01}-M_{02}} = 0.348 = \frac{M_{02}}{15000-M_{02}}$$

$$\in_1 = \frac{M_{S1}}{M_{01}-M_{02}} = 0.111 = \frac{M_{S1}}{15000-3873} \qquad M_{S1} = 1238\ kg$$

$$\in_2 = \frac{M_{S2}}{M_{02}-M_L} = 0.111 = \frac{M_{S2}}{3873-1000} \qquad M_{S2} = 320\ kg$$

$$M_S = M_{S1} + M_{S2} = 1238 + 320 = 1558\ kg$$

$M_P = M_{01} – M_L – M_S = 15000 – 1000 – 1558 = 12442$ kg

Stage 1:

M_{01} = 15,000 kg

M_{S1} = 1,238 kg

$M_{L1} = M_{02}$ = 3,873 kg

M_{P1} = 9,889 kg

M_{b1} = 5,111 kg

$\lambda_1 = 0.348$

$\in_1 = 0.111$

$R_1 = 2.935$

Stage 2:

M_{02} = 3,873 kg

M_{S2} = 320 kg

$M_{L2} = M_L$ = 1,000 kg

M_{P2} = 2,553 kg

M_{b1} = 1,320 kg

$\lambda_2 = 0.348$

$\in_2 = 0.111$

$R_2 = 2.935$