Example

Given:

equivalent exhaust velocity (u_{eq}) = 3,048 m/s

initial rocket mass (M_0) = 15,000 kg

propellant mass = 12,000 kg

Burnout time (t_b) = 100 s

If the rocket is fired vertically, find the burnout height (h_b) and maximum height (h_{max}), neglect drag, assume constant exhaust mass flow rate, and constant acceleration due to gravity [= at earth's surface (g_0) = 9.81 m/s^2]

Solution:

burnout mass (M_b) = 15,000 – 12,000 = 3,000 kg

R = M_0/M_b = 15,000/3,000 = 5

Exhaust mass flow ($\dot{m}$): $\dot{m} = -\frac{dM}{dt} = -\frac{M_b - M_0}{t_b} = -\frac{3{,}000-15{,}000}{100} = 120\,\frac{kg}{s}$

Thrust (T_h): $T_h = \dot{m}u_{eq} = 120 \times 3048 = 365{,}760\ N = 366\ kN$

Initial weight (M_0g_0): $M_0 g_0 = 15{,}000 \times 9.81 = 147{,}150\ N = 147\ kN$

note: the thrust is higher than the initial weight, hence the vehicle can accelerate

specific impulse: $I_{sp} = \frac{u_{eq}}{g_0} = \frac{3{,}048}{9.81} = 311\ s$

Adapted from slides of Prof. V. Arghode, IITK

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Example

burnout height (h_b):

$$h_b = u_{eq} t_b \left[1 - \frac{\ln(R)}{R-1}\right] - g_0 \frac{t_b^2}{2} = 3048 \times 100 \left[1 - \frac{\ln(5)}{5-1}\right] - 9.81 \frac{100^2}{2}$$
$$= 182{,}161 - 49{,}050 = 133{,}111\, m = 133\, km$$

maximum height (h_{max}):

$$h_{max} = \frac{u_{eq}^2 [\ln(R)]^2}{2 g_0} - u_{eq} t_b \left[\frac{R}{R-1} \ln(R) - 1\right]$$
$$= \frac{3048^2 [\ln(5)]^2}{2 \times 9.81} - 3048 \times 100 \left[\frac{5}{5-1} \ln(5) - 1\right] = 1{,}226{,}533 - 308{,}396 = 918{,}137 m = 918\, km$$

rocket speed at burnout (u_b):

$$u_b = u_{eq} \ln(R) - g_0 t_b = 3048 \times \ln(5) - 9.81 \times 100 = 4906 - 981 = 3{,}925\, m/s$$

Assignment

time to reach maximum height (t_{max}):

$$\boxed{t_{max} = \frac{u_{eq} \ln(R)}{g_0}} = \frac{3048 \times \ln(5)}{9.81} = 500\, s$$

rocket acceleration (du/dt):

at take off (t = 0):

$$\frac{du}{dt} = u_{eq} \left(1 - \frac{1}{R}\right) \frac{1}{t_b} - g_0 = 3048 \left(1 - \frac{1}{5}\right) \frac{1}{100} - 9.81 = 14.57 \frac{m}{s^2}$$

$\mathbf{1.5g_0}$

at burnout ($t = t_b$):

$$\frac{du}{dt} = u_{eq} R \left(1 - \frac{1}{R}\right) \frac{1}{t_b} - g_0 = 3048 \times 5 \left(1 - \frac{1}{5}\right) \frac{1}{100} - 9.81 = 112.11 \frac{m}{s^2}$$

$\mathbf{11.4g_0}$

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Example

If burnout time (t_b) is increased to 200 s:

$$h_b = 3048 \times 200\left[1 - \frac{\ln(5)}{5-1}\right] - 9.81\,\frac{200^2}{2}$$
$$= 364{,}322 - 196{,}200 = 168{,}122\ m = 168\ km$$

$$h_{max} = \frac{3048^2[\ln(5)]^2}{2\times 9.81} - 3048 \times 200\left[\frac{5}{5-1}\ln(5) - 1\right] = 1{,}226{,}533 -$$
$$616{,}792 = 609{,}741 m = 610\ km$$

$$u_b = 3048 \times \ln(5) - 9.81 \times 200 = 4906 - 981 = 2{,}944\ m/s$$

↓ than that for $t_b = 100s$

Thus for longer burnout time, the burnout speed as well as maximum height reduces

$$t_{max} = \frac{3048\times\ln(5)}{9.81} = 500\ s$$

$$\frac{du}{dt}(t=0) = 3048\left(1-\frac{1}{5}\right)\frac{1}{200} - 9.81 = 2.38\,\frac{m}{s^2} \quad \mathbf{0.24g_0}$$

$$\frac{du}{dt}(t=t_b) = 3048\times 5\left(1-\frac{1}{5}\right)\frac{1}{200} - 9.81 = 51.15\,\frac{m}{s^2} \quad \mathbf{5.21g_0}$$

↓ than 100 s

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Rocket trajectory

in y'-direction:

$$M\frac{du}{dt} = T_h - D - Mg\cos(\theta)$$

$$T_h = \dot{m}u_{eq} \qquad u_{eq} = u_e + \frac{(P_e - P_a)A_e}{\dot{m}}$$

$$M\frac{du}{dt} = \dot{m}u_{eq} - D - Mg\cos(\theta)$$

$$du = \frac{\dot{m}u_{eq}}{M}dt - \frac{D}{M}dt - g\cos(\theta)\,dt$$

$$\Delta u = \frac{\dot{m}u_{eq}}{M}\Delta t - \frac{D}{M}\Delta t - g\cos(\theta)\,\Delta t$$

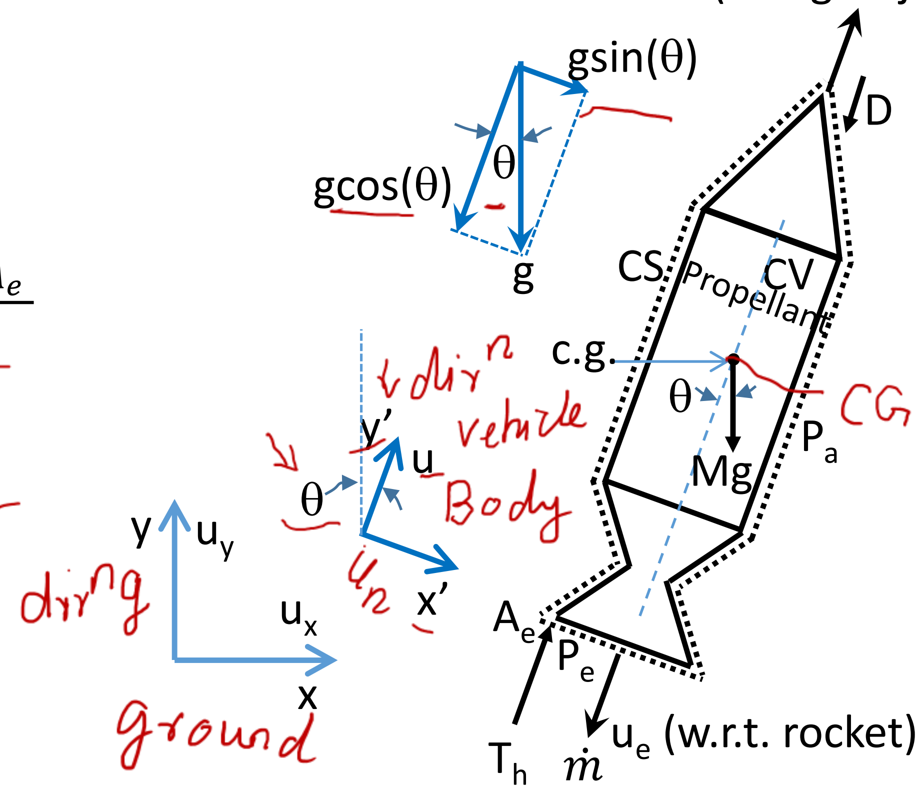
$$\boxed{\Delta u = \left[\frac{\dot{m}u_{eq}}{M} - \frac{D}{M} - g\cos(\theta)\right]\Delta t}$$

in x'-direction:

$$M\frac{du_n}{dt} = Mg\sin(\theta)$$

$$du_n = g\sin(\theta)\,dt$$

$$\boxed{\Delta u_n = g\sin(\theta)\,\Delta t}$$



$g = f(h) \rightarrow f(t)$

$\theta = f(t)$

$M = f(t)$

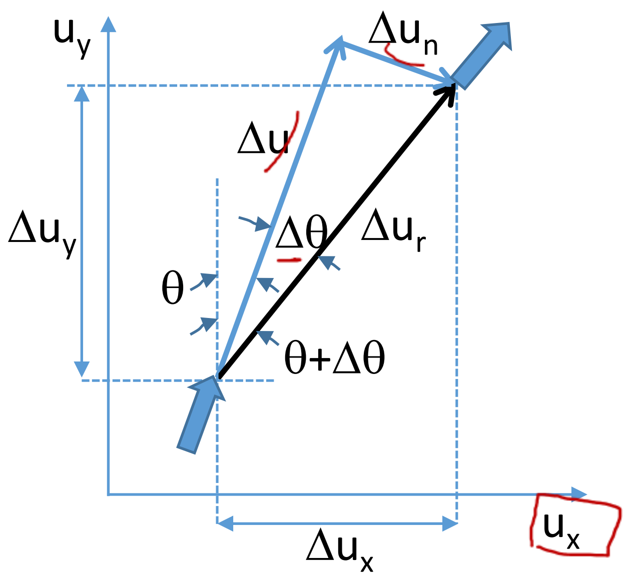
$D = f(\rho, u) \rightarrow f(t)$

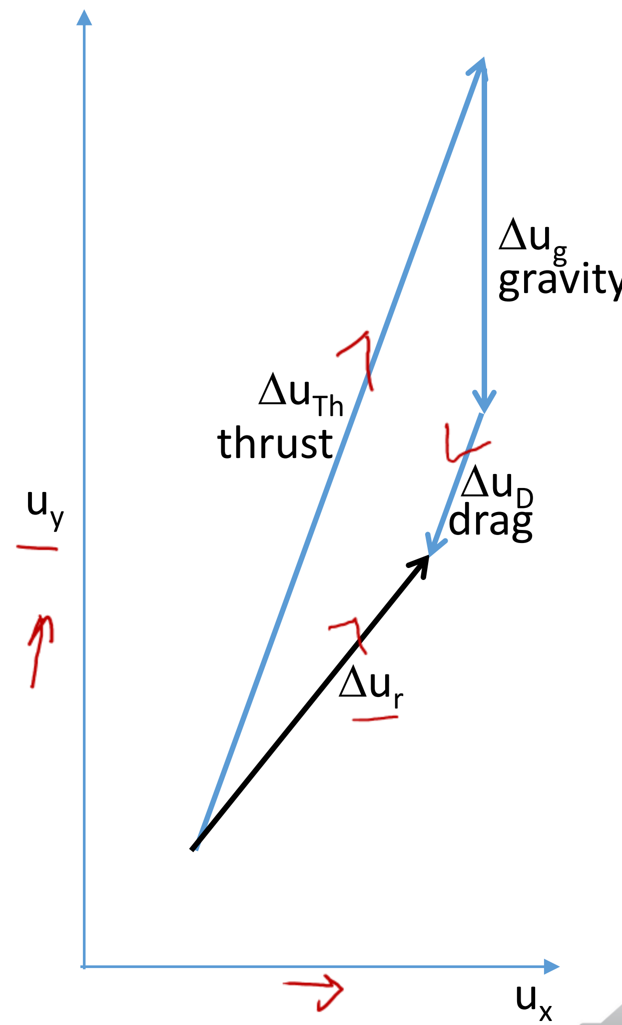
$\dot{m} = f(t)$

$u_{eq} = f(t)$

in each time interval (Δt): g, θ, M, D, $\dot{m}$ and u_{eq} are assumed constant

Rocket trajectory





change in rocket speed in time increment (Δt):

$$\Delta u_r = \sqrt{(\Delta u)^2 + (\Delta u_n)^2}$$

change in rocket direction in time increment (Δt):

$$\Delta\theta = \tan^{-1}\left(\frac{\Delta u_n}{\Delta u}\right)$$

x-component of change in rocket speed in Δt:

$$\Delta u_x = \Delta u_r \sin(\theta + \Delta\theta)$$

y-component of change in rocket speed in Δt:

$$\Delta u_y = \Delta u_r \cos(\theta + \Delta\theta)$$

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Rocket trajectory

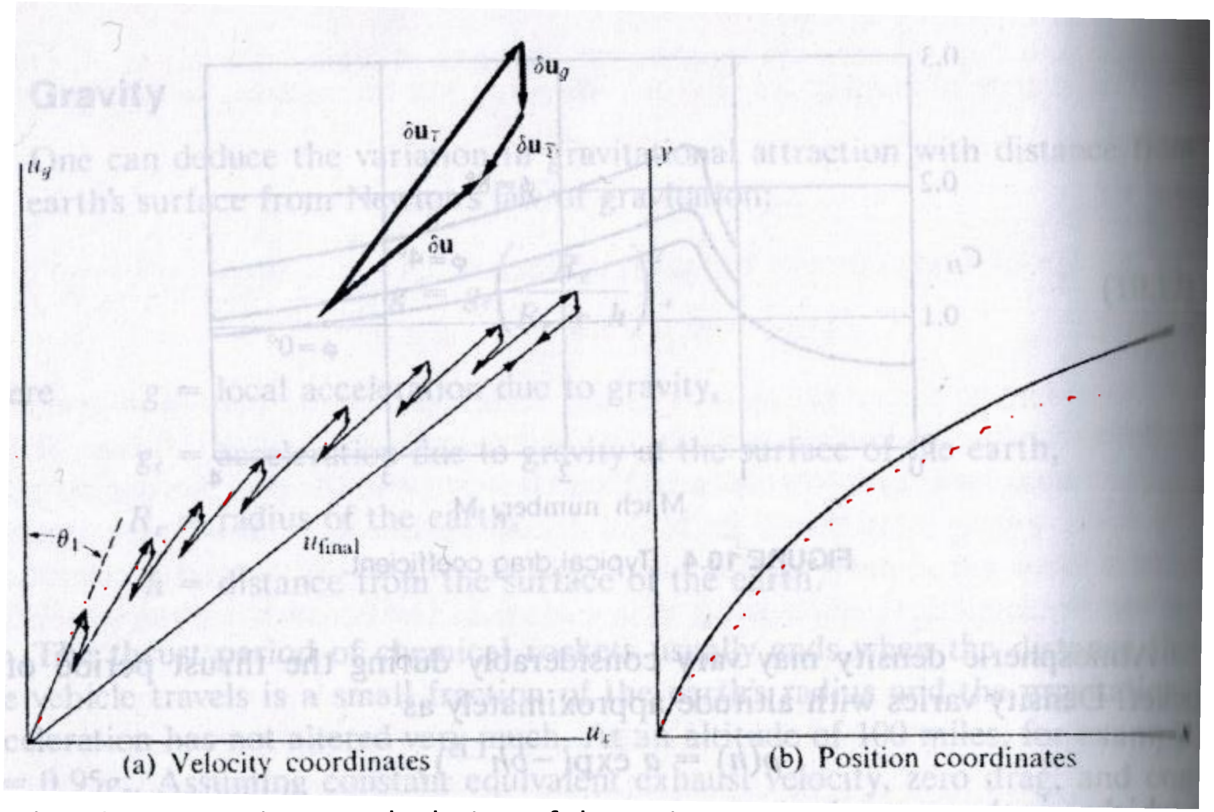

Fig. 10.5 Approximate calculation of the trajectory.
(Mechanics and Thermodynamics of Propulsion by Philip Hill and Carl Peterson, Second Edition, Dorling Kindersley India Pvt. Ltd., Noida, 2010)

distance travelled along x-direction in Δt:

$$\Delta x = u_x \Delta t$$

distance travelled along y-direction in Δt:

$$\Delta y = u_y \Delta t$$

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Rocket trajectory

- Rocket fired from rest initialize the parameters → at t = 0:
  - $u_x, u_y, u, u_n, u_r, D, x, y = 0$
  - $\theta = \theta_0, M = M_0, g = g_0, \dot{m} = \dot{m}_0, u_{eq} = u_{eq\text{-}0}$
- At small time increment, $t_{new} = t_{old} + \Delta t$: ← choosen

$$\Delta u(t_{new}) = \left[\frac{\dot{m}(t_{old})u_{eq}(t_{old})}{M(t_{old})} - \frac{D(t_{old})}{M(t_{old})} - g(t_{old})\cos[\theta(t_{old})]\right]\Delta t$$

$$\Delta u_n(t_{new}) = g(t_{old})\sin[\theta(t_{old})]\,\Delta t$$

$$\Delta u_r(t_{new}) = \sqrt{[\Delta u(t_{new})]^2 + [\Delta u_n(t_{new})]^2}$$

$$\Delta\theta(t_{new}) = \tan^{-1}\left[\frac{\Delta u_n(t_{new})}{\Delta u(t_{new})}\right]$$

$$\theta(t_{new}) = \theta(t_{old}) + \Delta\theta(t_{new})$$

$$\Delta u_x(t_{new}) = \Delta u_r \sin[\theta(t_{new})]$$

$$u_x(t_{new}) = u_x(t_{old}) + \Delta u_x(t_{new})$$

$$\Delta u_y(t_{new}) = \Delta u_r \cos[\theta(t_{new})]$$

$$u_y(t_{new}) = u_y(t_{old}) + \Delta u_y(t_{new})$$

note: x and y components of velocity (u_x, u_y) as a function of time can be obtained

Rocket trajectory

$$\Delta x(t_{new}) = u_x(t_{new})\Delta t$$

$$x(t_{new}) = x(t_{old}) + \Delta x(t_{new})$$

$$\Delta y(t_{new}) = u_y(t_{new})\Delta t$$

$$y(t_{new}) = y(t_{old}) + \Delta y(t_{new})$$

$$\Delta M(t_{new}) = -\dot{m}(t_{old})\Delta t$$

$$M(t_{new}) = M(t_{old}) + \Delta M(t_{new})$$

note: x and y coordinates as a function of time can be obtained

- Update the parameters → at $t = t_{new}$:
  - g, θ, M, D, $\dot{m}$ and u_{eq}
- Repeat the process

Combustion