Linear Classifiers

CS771: Introduction to Machine Learning

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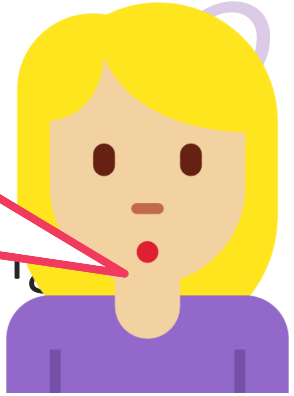
Decision Boundaries

Indeed, since we would have to not only predict for that data point, but also for other data points around it!



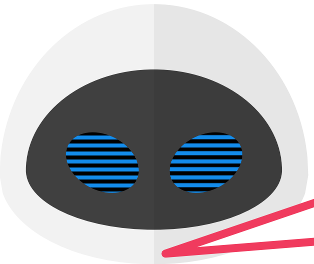
Regions of feature space where classifie
from one class to another class

These are also the regions where classifier may get confused and make a prediction with low confidence

All classifiers have such a decision boundary

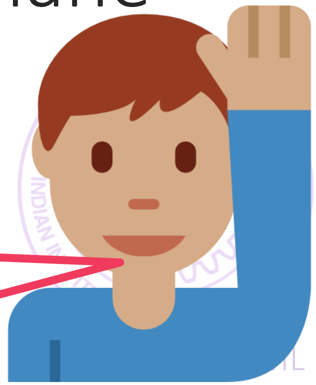
Easy to detect whether a test point is at decision boundary for linear classifiers – difficult to do so for most other classifiers, e.g. deep nets

Linear classifiers are those whose decision boundary is a line/plane



It might still get confused if there are 4 points equally close 2 of them red and 2 green ☺

For example, kNN will never get confused if k = 3 (or some odd number) and 2 classes



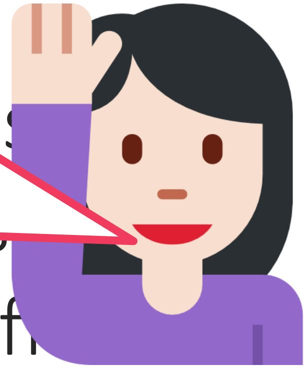
Linear Classifiers

Keep appearing again and ag

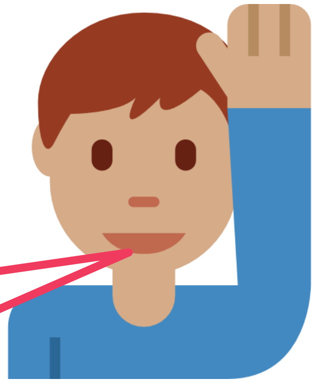
- LwP with 2 classes, Euclidea ... clas
- Even if Mahalanobis metric ... ss
- Decision stumps with a single feature also give a linear classifi

Extremely popular in ML

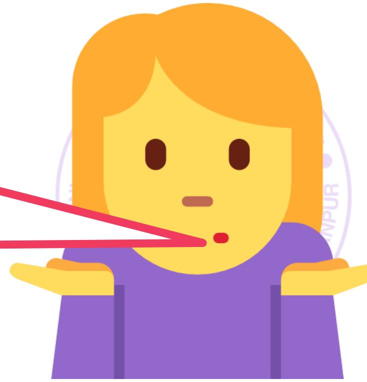
- Very small model size – just one vector (and one bias value)
- Very fast $\mathcal{O}(d)$ prediction time
- Used to build DTs, deep nets etc



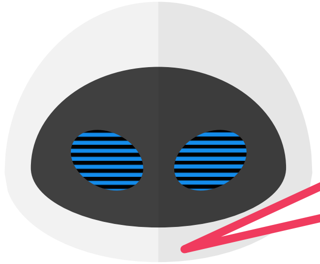
Before going forward, recall that linear classifiers are those that have a line or a plane as the decision boundary. A linear classifier is given by a model that looks like $(\mathbf{w}, b)$ and it makes predictions by looking at whether $\mathbf{w}^\top \mathbf{x} + b > 0$ or not



Learning classifiers directly will allow us to control many useful properties about them!

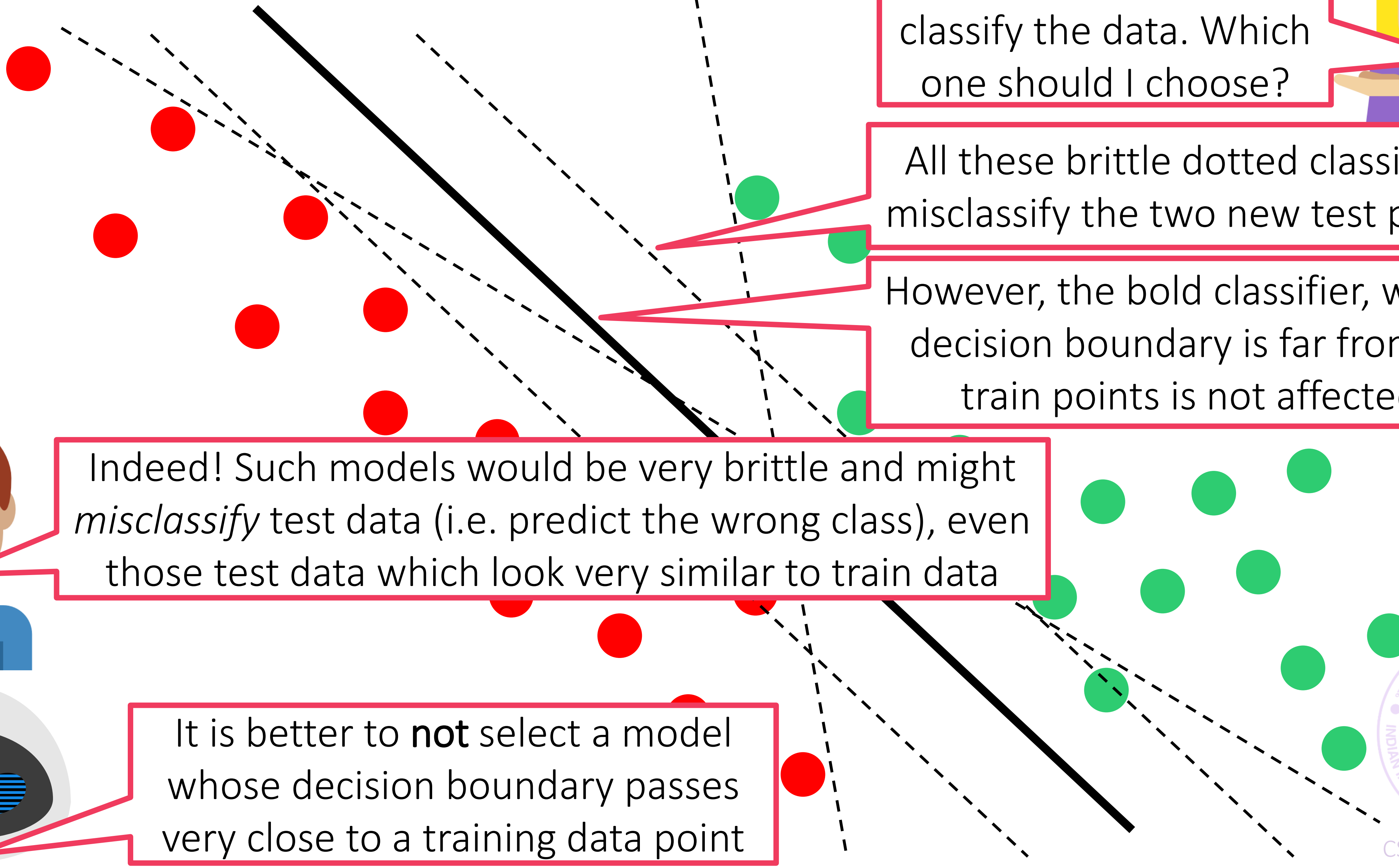


Instead of indirectly getting a linear classifier via LwP + Mahalanobis etc etc, can't we learn one directly?



That is exactly what we will do today!

The “best” Linear Classifier



It seems infinitely many classifiers perfectly classify the data. Which one should I choose?

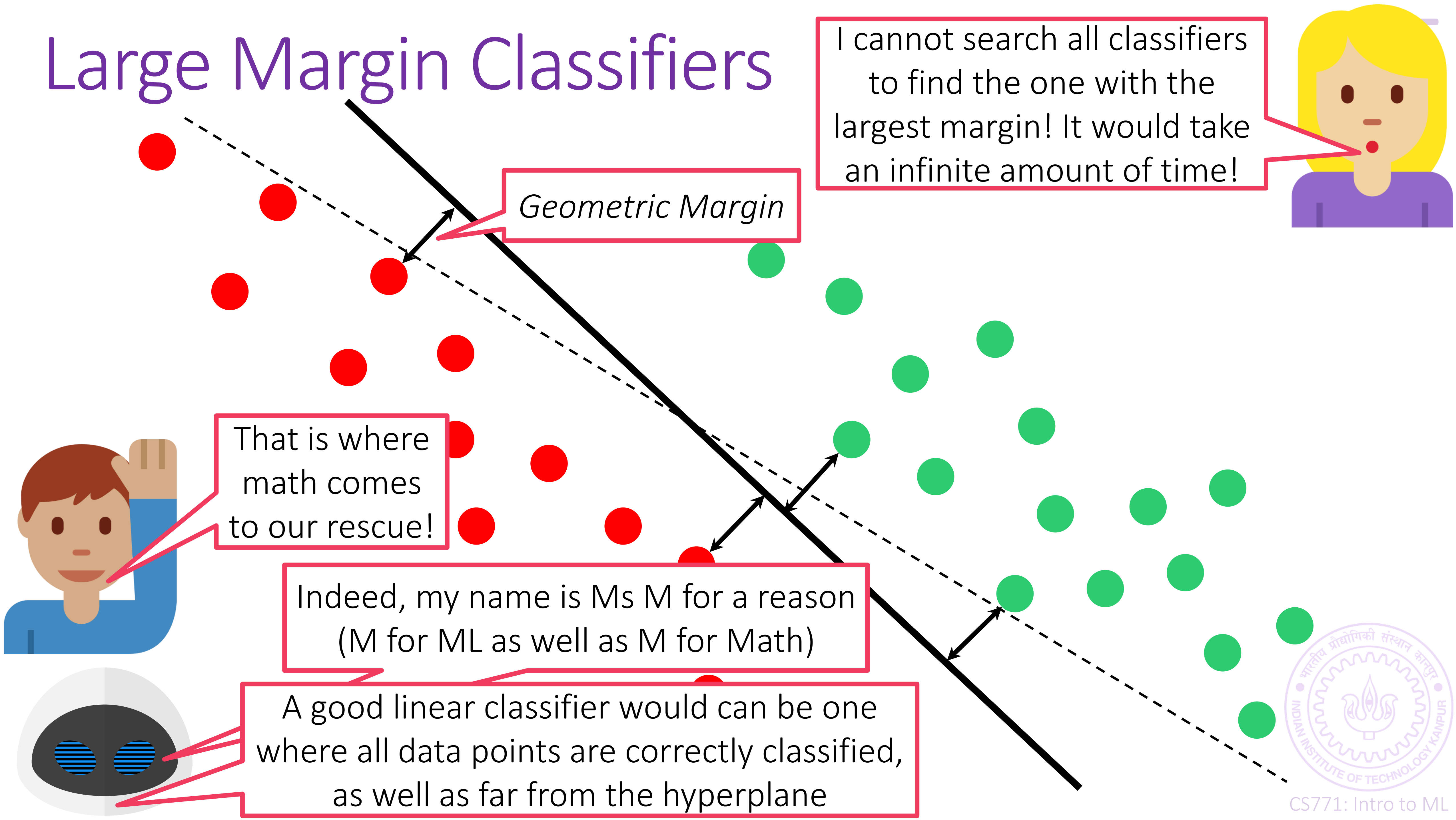
All these brittle dotted classifiers misclassify the two new test points

However, the bold classifier, whose decision boundary is far from all train points is not affected

Indeed! Such models would be very brittle and might *misclassify* test data (i.e. predict the wrong class), even those test data which look very similar to train data

It is better to **not** select a model whose decision boundary passes very close to a training data point

Large Margin Classifiers



Large Margin Classifiers

Fact: distance of origin from hyperplane $\mathbf{w}^\top\mathbf{x} + b = 0$ is $|b|/\|\mathbf{w}\|_2$

Fact: distance of a point $\mathbf{p}$ from this hyperplane is $|\mathbf{w}^\top\mathbf{p} + b|/\|\mathbf{w}\|_2$

Given train data $\left\{\left(\mathbf{x}^i, y^i\right)\right\}_{i=1}^n$ for a binary classfn problem where $\mathbf{x}^i \in \mathbb{R}^d$ and $y^i \in \{-1,1\}$, we want two things from a classifier

Demand 1: classify every point correctly – how to ask this politely?

One way: demand that for all $i = 1 \ldots n$, $\operatorname{sign}\left(\mathbf{w}^\top\mathbf{x}^i + b\right) = y^i$

Easier way: demand that for all $i = 1 \ldots n$, $y^i \cdot (\mathbf{w}^\top\mathbf{x}^i + \mathrm{b}) \geq 0$

Demand 2: not let any data point come close to the boundary

Demand that $\min\limits_{i=1\ldots n} \left|\mathbf{w}^\top\mathbf{x}^i + b\right|/\|\mathbf{w}\|_2$ *be as large as possible*

Geometric Margin

Support Vector Machines

Just a fancy way of saying

> *Please find me a linear classifier that perfectly classifies the train data while keeping data points as far away from the hyperplane as possible*

Let us simplify this optimization problem

The mathematical way of writing this request is the following

Constraints

$$\max_{\mathbf{w},b} \left\{ \min_{i=1\ldots n} \left|\mathbf{w}^\top \mathbf{x}^i + b\right| / \|\mathbf{w}\|_2 \right\}$$

Objective

such that $y^i \cdot (\mathbf{w}^\top \mathbf{x}^i + \mathrm{b}) \geq 0$ for all $i = 1 \ldots n$

This is known as an *optimization problem* with an *objective* and lots of *constraints*

This looks so complicated, how will I ever find a solution to this optimization problem?

Constrained Optimi

Constraints are usually specified using math equations. The set of points that satisfy *all* the constraints is called the *feasible set* of the optimization problem

HOW WE MUST SPEAK TO MS M

Objective

Constraints

$$\min_x f(x)$$

such that $p(x) < 0$

and $q(x) > 0$ etc. etc.

You optimization problem has no solution since no point satisfies all your constraints ☹

I want to find an unknown x

that gives me the best value

on f

ld do! x

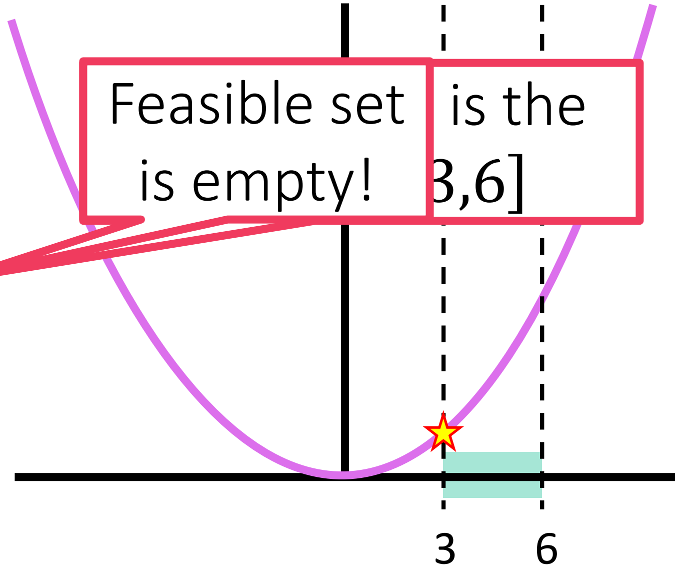
must satisfy these conditions

All I am saying is, of the values of x that satisfy my conditions, find me the one that gives the best value according to f

$$\min_x x^2$$

s.t. $x \geq 6$

and $x \leq 3$



Back to SVMs

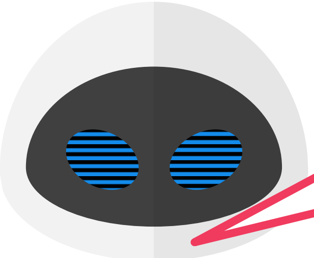
Assume there do exist models that perfectly classify all train data

Consider one such model $(\mathbf{w}, b)$ which classifies train data perfectly

Now, $\left|\mathbf{w}^\top \mathbf{x}^i + b\right| / \|\mathbf{w}\|_2 = \left|y^i \cdot \left(\mathbf{w}^\top \mathbf{x}^i + b\right)\right| / \|\mathbf{w}\|_2$ as $y^i \in \{-1, 1\}$

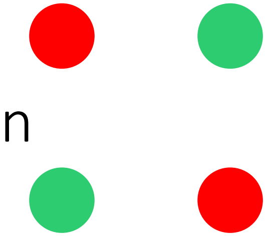
Thus, geometric margin is same as $\min_{i=1\ldots n} \left|y^i \cdot \left(\mathbf{w}^\top \mathbf{x}^i + b\right)\right| / \|\mathbf{w}\|_2 = \min_{i=1\ldots n} y^i \cdot \left(\mathbf{w}^\top \mathbf{x}^i + b\right) / \|\mathbf{w}\|_2$ since model has perfect classification!

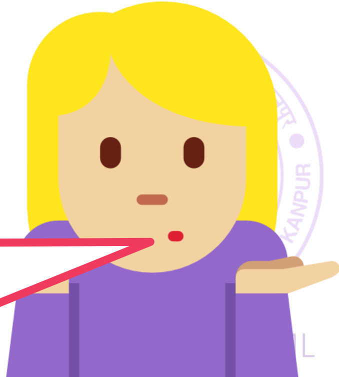
We will use this useful fact to greatly simplify the optimization problem



We will remove this assumption later

What if train data is *non-linearly separable* i.e no linear classifier can perfectly classify it? For example





Support Vector Mach

Called the *functional margin*. Note that geometric margin = functional margin/$\|\mathbf{w}\|_2$

Let i_0 be the data point that comes closest to the hyperplane i.e.

$$\min_{i=1\ldots n} y^i \cdot \left(\mathbf{w}^\top \mathbf{x}^i + b\right) = y^{i_0} \cdot \left(\mathbf{w}^\top \mathbf{x}^{i_0} + b\right)$$

Recall that all this discussion holds only for a perfect classifier $(\mathbf{w}, b)$

Let $\epsilon = y^{i_0} \cdot \left(\mathbf{w}^\top \mathbf{x}^{i_0} + b\right)$ and consider $\tilde{\mathbf{w}} = \mathbf{w}/\epsilon, \tilde{b} = b/\epsilon$

Note this gives us $y^i \cdot \left(\tilde{\mathbf{w}}^\top \mathbf{x}^i + \tilde{b}\right) \geq 1$ for all $i = 1 \ldots n$ as well as $\min_{i=1\ldots n} y^i \cdot \left(\tilde{\mathbf{w}}^\top \mathbf{x}^i + \tilde{b}\right)/\|\tilde{\mathbf{w}}\|_2 = 1/\|\tilde{\mathbf{w}}\|_2$ (as $y^{i_0} \cdot \left(\tilde{\mathbf{w}}^\top \mathbf{x}^{i_0} + \tilde{b}\right) = 1$)

Thus, instead of searching for $(\mathbf{w}, b)$, easier to search for $\left(\tilde{\mathbf{w}}, \tilde{b}\right)$

$$\min_{\tilde{\mathbf{w}}, \tilde{b}} \left\{\|\tilde{\mathbf{w}}\|_2^2\right\}$$

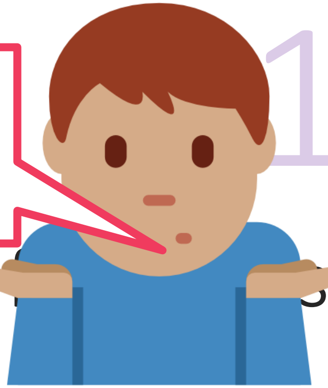
such that $y^i \cdot (\tilde{\mathbf{w}}^\top \mathbf{x}^i + \tilde{b}) \geq 1$ for all $i = 1 \ldots n$

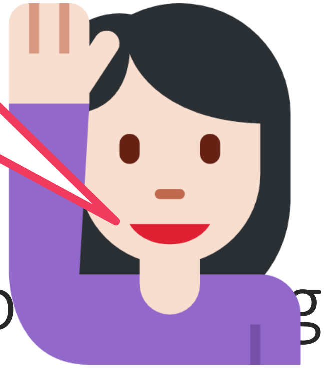
The C-SVM Te

What prevents me from misusing the slack variables to learn a model that misclassifies every data point?

For *linearly separable* cases v

The C term prevents you from doing so. If we set C to be a large value (it is a hyper-parameter), then it will penalize solutions that misuse slack too much





$$\text{s.t. } y^i \cdot (\tilde{\mathbf{w}}^\top \mathbf{x}$$

If a linear classifier cannot perfectly classify data, then find mo

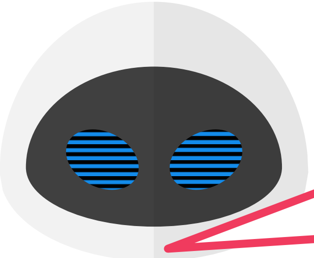
$$\min_{\tilde{\mathbf{w}},\tilde{b},\{\xi_i\}} \frac{1}{2}\|\tilde{\mathbf{w}}\|_2^2 + C\sum_{i=}^{n}$$

Having the constraint $\xi_i \geq 0$ prevents us from misusing slack to artificially inflate the margin

$$\text{s.t. } y^i \cdot \left(\tilde{\mathbf{w}}^\top \mathbf{x}^i + \tilde{b}\right) \geq 1 - \xi_i \text{ fo}$$

as well as $\xi_i \geq 0$ for all $i \in [n]$

Recall English phrase *"cut me some slack"*



The ξ_i terms are called *slack variables*. They allow some data points to come close to the hyperplane or be misclassified altogether

From C-SVM to Loss Functions

We can further simplify the previous optimization problem

Note ξ_i basically allows us to have $y^i \cdot \left(\tilde{\mathbf{w}}^\top \mathbf{x}^i + \tilde{b}\right) < 1$ (even < 0)

Thus, the amount of slack we want is just $\xi_i = 1 - y^i \cdot \left(\tilde{\mathbf{w}}^\top \mathbf{x}^i + \tilde{b}\right)$

However, recall that we must also satisfy $\xi_i \geq 0$

Another way of saying that if you already have $y^i \cdot \left(\tilde{\mathbf{w}}\right.$
then you don't need any slack i.e. you should have $\xi_i = 0$ in this case

$$\boxed{[x]_+ = \max\{x, 0\}}$$

Thus, we need only set $\xi_i = \left[1 - y^i \cdot \left(\tilde{\mathbf{w}}^\top \mathbf{x}^i + \tilde{b}\right)\right]_+$

The above is nothing but the popular hinge loss function!

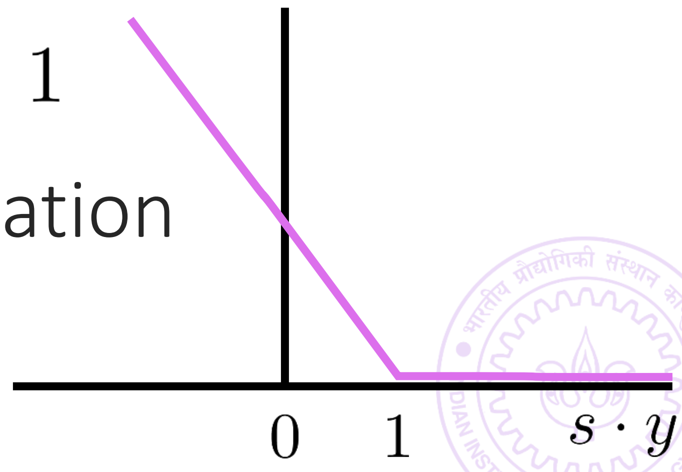
Hinge Loss

Captures how well as a classifier classified a data point

Suppose on a data point $(\mathbf{x}, y), y \in \{-1,1\}$, a model gives prediction score of s (for a linear model $(\mathbf{w}, b)$, we have $s = \mathbf{w}^\top \mathbf{x} + b$)

We obviously want $s \cdot y \geq 0$ for correct classification but we also want $s \cdot y \gg 0$ for large margin – hinge loss function captures both

$$\ell_{\text{hinge}}(s, y) = [1 - s \cdot y]_+ = \begin{cases} 0 & \text{if } s \cdot y \geq 1 \\ 1 - s \cdot y & \text{if } s \cdot y < 1 \end{cases}$$



Note that hinge loss not only penalizes misclassification but also correct classification if the data point gets too close to the hyperplane!

Final Form of C-SVM

Recall that the C-SVM optimization finds a model by solving

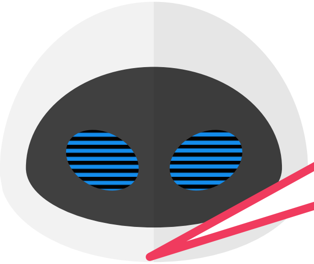
$$\min_{\tilde{\mathbf{w}}, \tilde{b}, \{\xi_i\}} \frac{1}{2}\|\tilde{\mathbf{w}}\|_2^2 + C\sum_{i=1}^{n} \xi_i$$

$$\text{s.t. } y^i \cdot \left(\tilde{\mathbf{w}}^\top \mathbf{x}^i + \tilde{b}\right) \geq 1 - \xi_i \text{ for all } i \in [n]$$

$$\text{as well as } \xi_i \geq 0 \text{ for all } i \in [n]$$

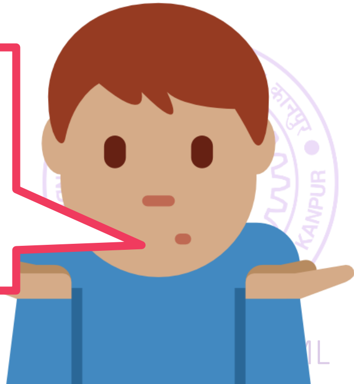
Using the previous discussion, we can rewrite the above very simply

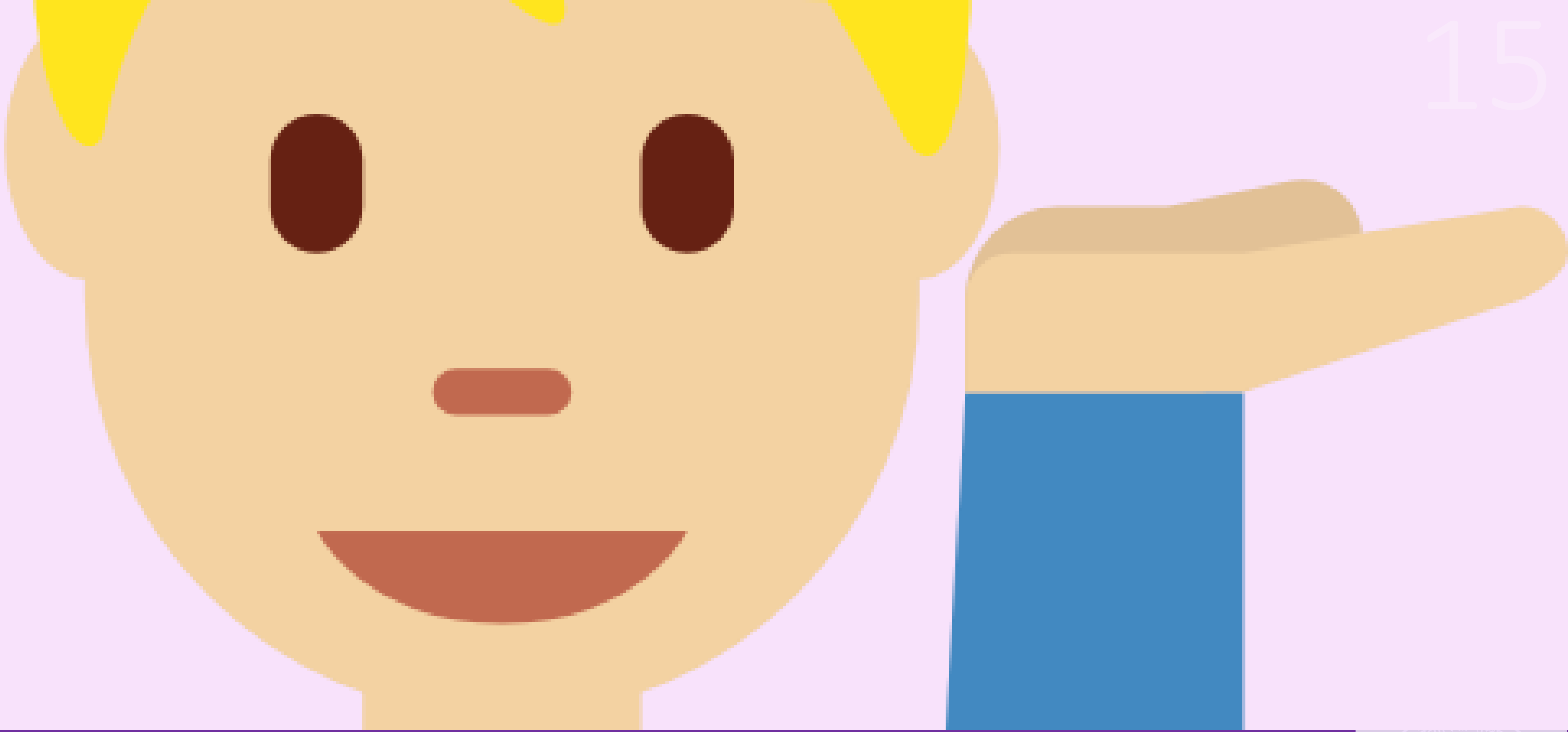
$$\min_{\tilde{\mathbf{w}}, \tilde{b}} \frac{1}{2}\|\tilde{\mathbf{w}}\|_2^2 + C\sum_{i=1}^{n} \ell_{\text{hinge}}\left(\tilde{\mathbf{w}}^\top \mathbf{x}^i + \tilde{b}, y^i\right)$$



This is where calculus and other math topics come to our rescue ☺

Agreed this is simpler than before but I still don’t know how to use this to find the model





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