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Lyapunov Iterations for Solving Coupled Algebraic Riccati Equations of Nash Differential Games and Algebraic Riccati Equations of Zero-Sum Games

T-Y. Li
Department of Mathematics, Michigan State University,
E. Lansing, MI 48824, USA

Z. Gajic
Department of Electrical and Computer Engineering, Rutgers University,
Piscataway, NJ 08855-0909, USA

Abstract

In this paper we study the symmetric coupled algebraic Riccati equations corresponding to the steady state Nash strategies. Under control-oriented assumptions, imposed on the problem matrices, the Lyapunov iterations are constructed such that the proposed algorithm converges to the nonnegative (positive) definite stabilizing solution of the coupled algebraic Riccati equations. In addition, the problem order reduction is achieved since the obtained Lyapunov equations are of the reduced-order and can be solved independently. As a matter of fact a parallel synchronous algorithm is obtained. A high-order numerical example is included in order to demonstrate the efficiency of the proposed algorithm. In the second part of this paper we have proposed an algorithm, in terms of the Lyapunov iterations, for finding the positive *semidefinite* stabilizing solution of the algebraic Riccati equation of the zero-sum differential games. The similar algebraic Riccati type equations appear in the H_∞ optimal control and related problems.

1 Coupled Algebraic Riccati Equations of Nash Differential Games

The solutions of the coupled algebraic Riccati equations produce the answers to some important problems of modern control theory, for example, the differential games with conflict of interest and simultaneous decision making (Nash strategies), (Starr and Ho, 1969; Basar, 1991), the H_∞ optimal control problems (Bernstein and Haddad, 1989; Basar and Bernhard, 1991), the optimal control of jump linear systems (Mariton, 1990).

In this paper we solve the symmetric coupled algebraic Riccati equations corresponding to the steady state Nash strategies of the linear-quadratic differential game problem. At the present time *there is no global efficient method for solving these coupled algebraic Riccati equations*. The obtained solution is stabilizing one, nonnegative (positive) definite and valid under the stabilizability-detectability assumptions imposed on the problem matrices. The proposed algorithm is of the reduced-order and can be implemented as a synchronous parallel algorithm (Bertsekas and Tsitsiklis, 1991). As a matter of fact, it will be clear from the convergence proof that this algorithm is based on the successive approximations technique of dynamic programming (Bellman, 1954, 1957, 1961; Larson, 1967; Bertsekas, 1987). The method of successive approximations is the main tool in solving the functional equation of dynamic programming. It has been used in several control theory papers, for example (Vaisbord, 1963; Mil'shtein, 1964; Leake and Liu, 1967; Kleinman, 1968; Levine and Vilas, 1973; Mageriou, 1977). This method can be used as a very powerful decomposition technique which simplifies computations. In the work of (Mil'shtein, 1964), an approximate convergent method for synthesis of the optimal control system is investigated. The approach is based on a combination of the ideas of Lyapunov's second method and Bellman's method of successive approximations. Convergent suboptimal control sequences were also obtained in (Bellman, 1954, 1961) and (Vaisbord, 1963; Kleinman, 1968; Mageriou, 1977).

A controlled linear dynamic system corresponding to the Nash differential game strategies is given by

$$\dot{x} = Ax + B_1 u_1 + B_2 u_2, \quad x(t_0) = x_0 \tag{1}$$

where $x \in \Re^n$ is a state vector, $u_1 \in \Re^{m_1}$ and $u_2 \in \Re^{m_2}$ are control inputs (for the reason of simplicity we limit our attention to two control agents), A, B_1, and B_2 are constant matrices of appropriate dimensions. With each control agent a quadratic type function is associated

$$J_1(u_1, u_2, x_0) = \frac{1}{2}\int_{t_0}^{\infty}\left(x^T Q_1 x + u_1^T R_{11} u_1 + u_2^T R_{12} u_2\right) dt \tag{2}$$

$$J_2(u_1, u_2, x_0) = \frac{1}{2}\int_{t_0}^{\infty}\left(x^T Q_2 x + u_1^T R_{21} u_1 + u_2^T R_{22} u_2\right) dt \tag{3}$$

Weighting matrices are symmetric and

$$\begin{aligned} &Q_i \geq 0, \quad i = 1,2; \quad \text{(positive semidefinite)} \\ &R_{ii} > 0, \quad i = 1,2; \quad \text{(positive definite)} \\ &R_{ij} \geq 0, \quad i = 1,2; \quad j = 1,2; \quad i \neq j \end{aligned} \tag{4}$$

The optimal solution to the given problem leads to the so-called Nash optimal strategies u_1^* and u_2^* satisfying

$$J_1(u_1^*, u_2^*) \leq J_1(u_1, u_2^*), \quad J_2(u_1^*, u_2^*) \leq J_1(u_1^*, u_2) \tag{5}$$

It was shown in (Starr and Ho, 1969) that the *closed-loop* Nash optimal strategy is given by

$$u_i^* = -R_{ii}^{-1} B_i^T K_i x, \quad i = 1, 2 \tag{6}$$

where $K_i, i = 1, 2$, satisfy the coupled algebraic Riccati equations

$$\begin{aligned} &K_1 A + A^T K_1 + Q_1 - K_1 S_1 K_1 - K_2 S_2 K_1 \\ &-K_1 S_2 K_2 + K_2 Z_2 K_2 = \mathcal{N}_1(K_1, K_2) = 0 \end{aligned} \tag{7}$$

$$\begin{aligned} &K_2 A + A^T K_2 + Q_2 - K_2 S_2 K_2 - K_2 S_1 K_1 \\ &-K_1 S_1 K_2 + K_1 Z_1 K_1 = \mathcal{N}_2(K_1, K_2) = 0 \end{aligned} \tag{8}$$

with

$$S_i = B_i R_{ii}^{-1} B_i^T, \; i = 1, 2; \; Z_i = B_i R_{ii}^{-1} R_{ji} R_{ii}^{-1} B_i^T, \; i, j = 1, 2, \; i \neq j$$

The existence of the nonlinear optimal Nash strategies was established in (Basar, 1974), so that (6), in fact, are the best linear optimal feedback strategies. Since a linear control law is highly desirable from the practical point of view, the linear feedback strategies (6) attract the attention of many researchers.

The existence of Nash strategies (6) and a solution of the coupled algebraic Riccati equations (7)-(8) were studied in Papavassilopoulos et al., 1979, using the Brower fixed point theorem by imposing norm conditions on the given matrices. These conditions are, in general, difficult to test and they are not very useful from a practical point of view. We are interested in imposing the control oriented assumptions Wonham, 1968; Kucera, 1972, which accompanied with a convenient algorithm will lead to the required solution of (7)-(8). We will show that the obtained solution is a nonnegative (positive) definite and stabilizing one. In addition, the proposed algorithm operates only on two decoupled standard algebraic Riccati equations (initialization) and performs iterations on two algebraic Lyapunov equations; thus, it operates on the reduced-order problems, and from a computational point of view, the algorithm is extremely efficient.

So far the existing algorithms for solving (7)-(8) Krikelis and Rekasius, (1971); Tabak (1975) are of the local type, that is, they are faced with the problems of finding very good initial guesses. Furthermore, the algorithm proposed in (Tabak, 1975) does not necessarily converge even when the initial guesses are close to the optimal ones, as was pointed out in Olsder (1975).

On the other hand, it is not known how to generate the initial guesses for the Newton-type algorithm used in Krikelis and Rekasius (1971), such that the algorithm converges to the stabilizing solutions of (7)-(8). Note that the differential coupled Riccati equations of Nash differential games corresponding to the finite time optimization problems were studied in Jodar and Abou-Kandil, 1989; Abou-Kandil et al., 1993. In Abou-Kandil et al., 1993, the nonsymmetric coupled algebraic Riccati equations corresponding to the *open-loop* Nash strategies were also studied. Equations (7)-(8) have been studied for special classes of systems in (Khalil and Kokotovic, 1979; Khalil, 1980) - singularly perturbed systems, and (Ozguner and Perkins, 1977; Petrovic and Gajic, 1988) weakly coupled systems.

2 The Lyapunov Iterations for the Linear-Quadratic Nash Games

The considered algorithm is originally proposed by the authors in Gajic and Li, 1988; see also Gajic and Shen, 1993, pp. 359, where only the algorithm and simulation results were presented. In this paper we give the convergence proof. It is shown that the algorithm converges to the nonnegative (positive) definite stabilizing solution of (7)-(8) under the following control-oriented assumption.

Assumption 2.1 *Either the triple* $(A, B_1, \sqrt{Q_1})$ *or* $(A, B_2, \sqrt{Q_2})$ *is stabilizable-detectable.* (weaker conditions than controllable-observable)

These conditions are quite natural since at least one control agent has to be able to control and observe unstable modes. Because the game is a noncooperative one, the assumption that their joint effect will take care of unstable modes seems to be very idealistic.

Let us suppose that $(A, B_1, \sqrt{Q_1})$ is stabilizable-detectable. Then a unique positive definite solution of an auxiliary algebraic Riccati equation

$$K_1^{(0)} A + A^T K_1^{(0)} + Q_1 - K_1^{(0)} S_1 K_1^{(0)} = 0 \tag{9}$$

exists such that $\left(A - S_1 K_1^{(0)}\right)$ is a stable matrix. By plugging $K_1 = K_1^{(0)}$ in (8) we get the second auxiliary Riccati equation as

$$\begin{aligned} &K_2^{(0)} \left(A - S_1 K_1^{(0)}\right) + \left(A - S_1 K_1^{(0)}\right)^T K_2^{(0)} \\ &+ \left(Q_2 + K_1^{(0)} Z_1 K_1^{(0)}\right) - K_2^{(0)} S_2 K_2^{(0)} = 0 \end{aligned} \tag{10}$$

Since $\left(A - S_1 K_1^{(0)}\right)$ is a stable matrix and $Q_2 + K_1^{(0)} Z_1 K_1^{(0)}$ is a positive semidefinite matrix, the corresponding closed-loop matrix

$\left(A - S_1 K_1^{(0)} - S_2 K_2^{(0)}\right)$ is stable.

In fact, the triple $\left(A - S_1 K_1^{(0)}, B_2, \sqrt{Q_2 + K_1^{(0)} S_1 K_1^{(0)}}\right)$ is stabilizable-detectable and the stabilizing $K_2^{(0)}$ is uniquely determined. In the following we will use the solutions of (9)-(10), that is, $K_1^{(0)}$ and $K_2^{(0)}$ to initialize our algorithm.

The following algorithm is proposed in Gajic and Li, 1988; see also Gajic and Shen, 1993, pp. 359 for solving the coupled algebraic Riccati equations (7)-(8)

Algorithm 1:

$$\begin{aligned}\left(A - S_1 K_1^{(i)} - S_2 K_2^{(i)}\right)^T K_1^{(i+1)} + K_1^{(i+1)} \left(A - S_1 K_1^{(i)} - S_2 K_2^{(i)}\right) = \\ = \overline{Q_1^{(i)}} = -\left(Q_1 + K_1^{(i)} S_1 K_1^{(i)} + K_2^{(i)} Z_2 K_2^{(i)}\right), \quad i = 0, 1, 2, \ldots\end{aligned} \tag{11}$$

$$\begin{aligned}\left(A - S_1 K_1^{(i)} - S_2 K_2^{(i)}\right)^T K_2^{(i+1)} + K_2^{(i+1)} \left(A - S_1 K_1^{(i)} - S_2 K_2^{(i)}\right) = \\ = \overline{Q_2^{(i)}} = -\left(Q_2 + K_1^{(i)} Z_1 K_1^{(i)} + K_2^{(i)} S_2 K_2^{(i)}\right), \quad i = 0, 1, 2, \ldots\end{aligned} \tag{12}$$

with initial conditions $K_1^{(0)}$ and $K_2^{(0)}$ obtained from (9)-(10).

This algorithm is based on the Lyapunov iterations. Even though it looks like this algorithm has the form of Kleinman (1968), it is quite easy to show that this is not the case. Note that Kleinman's algorithm is used to solve the regular algebraic Riccati equation. In our paper we are concerned with the problem of solving coupled algebraic Riccati equations, where the coupling comes through the nonlinear quadratic terms, so that our problem is much more complex. Also, it can be shown that the proposed algorithm (11)-(12) is not of the Newton type. As a matter of fact, the Kleinman algorithm is equivalent to the Newton method. Interestingly enough, it can be shown that Kleinman's algorithm can be obtained by using the successive approximations of dynamic programming, and for the regular algebraic Riccati equation this is equivalent to using the Newton method to solve it.

The algorithm (11)-(12) has the feature given in the following theorem.

Theorem 2.1 *Under Assumption 2.1 the unique nonnegative definite stabilizing solution of the coupled algebraic Riccati equations (7)-(8) exists. It is obtained by performing Lyapunov iterations (11)-(12).*

The proof of this theorem is based on the successive approximations technique of dynamic programming.

Proof: Consider the linear-quadratic optimal control problem of minimizing the following performance criteria

$$J_1(u_1,u_2,x(t)) = \frac{1}{2}\int_t^{\infty}\left(x^TQ_1x+u_1^TR_{11}u_1+u_2^TR_{12}u_2\right)d\tau \tag{13}$$

$$J_2(u_1,u_2,x(t)) = \frac{1}{2}\int_t^{\infty}\left(x^TQ_2x+u_1^TR_{21}u_1+u_2^TR_{22}u_2\right)d\tau \tag{14}$$

along the trajectories of dynamic system (1). Note that the optimization problem defined by (1) and (13)-(14) is more general than the one with fixed initial time (Kirk, 1970). Corresponding Hamiltonians for the Nash differential game for each control agent are given by Starr and Ho, 1969:

$$\begin{aligned} H_1\left(x,u_1,u_2^*,\tfrac{\partial J_1^*}{\partial x}\right) &= \tfrac{1}{2}\left\{x^TQ_1x+u_1^TR_{11}u_1+u_2^{*^T}R_{12}u_2^*\right\} \\ &+\left(\tfrac{\partial J_1^*}{\partial x}\right)^T\{Ax+B_1u_1+B_2u_2^*\} \end{aligned} \tag{15}$$

$$\begin{aligned} H_2\left(x,u_1^*,u_2,\tfrac{\partial J_2^*}{\partial x}\right) &= \tfrac{1}{2}\left\{x^TQ_2x+u_1^{*^T}R_{21}u_1^*+u_2^TR_{22}u_2\right\} \\ &+\left(\tfrac{\partial J_2^*}{\partial x}\right)^T\{Ax+B_1u_1^*+B_2u_2\} \end{aligned} \tag{16}$$

The necessary conditions for the Nash optimal strategies are (Starr and Ho, 1969)

$$\min_{u_1} H_1(x,u_1,u_2^*) = 0 \quad\Rightarrow\quad u_1^* = -R_{11}^{-1}B_1^T\left(\frac{\partial J_1^*}{\partial x}\right)^T \tag{17}$$

$$\min_{u_2} H_2(x,u_1^*,u_2) = 0 \quad\Rightarrow\quad u_2^* = -R_{22}^{-1}B_2^T\left(\frac{\partial J_2^*}{\partial x}\right)^T \tag{18}$$

The successive approximations technique of dynamic programming applied to (1) and (13)-(16) is composed of the following two steps.

Step 1. Take any stabilizable linear control law $u_1^{(0)}(x(t))$ and $u_2^{(0)}(x(t))$, for example $u_1^{(0)}(x(t)) = -R_{11}^{-1}B_1^TK_1^{(0)}x(t)$ and $u_2^{(0)}(x(t)) = -R_{22}^{-1}B_2^TK_2^{(0)}x(t)$ with $K_1^{(0)}$ and $K_2^{(0)}$ being symmetric, and evaluate the expression for the performance criterion

$$\begin{aligned} J_1^{(0)} &= \frac{1}{2}\int_t^{\infty}[x^T(\tau)Q_1x(\tau)+u_1^{(0)^T}(x(\tau))R_{11}u_1^{(0)}(x(\tau)) \\ &\quad +u_2^{(0)^T}(x(\tau))R_{12}u_2^{(0)}(x(\tau))]d\tau \\ &= \frac{1}{2}\int_t^{\infty}x^T(\tau)\left[Q_1+K_1^{(0)}S_1K_1^{(0)}+K_2^{(0)}Z_2K_2^{(0)}\right]x(\tau)\,d\tau \end{aligned} \tag{19}$$

along the trajectories of the system

$$\dot{x}(t) = Ax(t) + B_1 u_1^{(0)}(x(t)) + B_2 u_2^{(0)}(x(t)) = \left(A - S_1 K_1^{(0)} - S_2 K_2^{(0)}\right) x(t) \tag{20}$$

Similarly, evaluate for the second control agent

$$\begin{aligned} J_2^{(0)} &= \frac{1}{2}\int_t^{\infty} [x^T(\tau) Q_2 x(\tau) + u_1^{(0)^T}(x(\tau)) R_{21} u_1^{(0)}(x(\tau)) \\ &\quad + u_2^{(0)^T}(x(\tau)) R_{22} u_2^{(0)}(x(\tau))] d\tau \\ &= \frac{1}{2}\int_t^{\infty} x^T(\tau)\left[Q_2 + K_1^{(0)} Z_1 K_1^{(0)} + K_2^{(0)} S_2 K_2^{(0)}\right] x(\tau)\, d\tau \end{aligned} \tag{21}$$

From (19) and (21) we are also able to find the expressions for $\frac{\partial J_1^{(0)}}{\partial x}(t)$ and $\frac{\partial J_2^{(0)}}{\partial x}(t)$ along (20). These expressions will be determined later.

Step 2. For the known value of $\frac{\partial J_1^{(0)}}{\partial x}(t)$ find a new approximation for the control law by minimizing with respect to u_1 the "partially frozen" Hamiltonian

$$\begin{aligned} &H_1\left(x, u_1, u_2^*, \left(\tfrac{\partial J_1}{\partial x}\right)^{(0)}\right) = \tfrac{1}{2}\{x^T(t) Q_1 x(t) + u_1^T(t) R_{11} u_1(t) \\ &+ u_2^{*^T}(t) R_{21} u_2^*(t)\} + \left(\tfrac{\partial J_1}{\partial x}(t)\right)^{(0)^T} (Ax(t) + B_1 u_1(t) + B_2 u_2^*(t)) \end{aligned} \tag{22}$$

The minimization produces a stabilizing control given by

$$u_1^{(1)}(t) = -R_{11}^{-1} B_1^T \left(\frac{\partial J_1}{\partial x}(t)\right)^{(0)} \tag{23}$$

Similarly, the minimization of

$$\begin{aligned} &H_2\left(x, u_1^*, u_2, \left(\tfrac{\partial J_2}{\partial x}\right)^{(0)}\right) = \tfrac{1}{2}\{x^T(t) Q_2 x(t) + u_1^{*^T}(t) R_{21} u_1^*(t) \\ &+ u_2^T(t) R_{22} u_2(t)\} + \left(\tfrac{\partial J_2}{\partial x}(t)\right)^{(0)^T} (Ax(t) + B_1 u_1^*(t) + B_2 u_2(t)) \end{aligned} \tag{24}$$

with respect to u_2 produces the stabilizing control for the second agent as

$$u_2^{(1)}(t) = -R_{22}^{-1} B_2^T \left(\frac{\partial J_2}{\partial x}(t)\right)^{(0)} \tag{25}$$

Note that $\frac{\partial J_1}{\partial x}$ along the system trajectory can be calculated from (1), (14), namely, by using the indentity

$$\frac{dJ_1}{dt} = \frac{\partial J_1}{\partial x}\frac{dx}{dt} = -\frac{1}{2}\left(x^T(t) Q_1 x(t) + u_1^T(t) R_{11} u_1(t) + u_2^T(t) R_{21} u_2(t)\right) \tag{26}$$

Under the stabilizing control laws $u_1^{(0)}(x(t))$ and $u_2^{(0)}(x(t))$ the last equality produces

$$\begin{aligned} &\tfrac{\partial J_1^{(0)}}{\partial x}\left(A - S_1K_1^{(0)} - S_2K_2^{(0)}\right)x(t) \\ &= -\tfrac{1}{2}x^T(t)\left[Q_1 + K_1^{(0)}S_1K_1^{(0)} + K_2^{(0)}S_2K_2^{(0)}\right]x(t) \end{aligned} \tag{27}$$

This simple partial differential equation has a solution of the form

$$J_1^{(0)} = \frac{1}{2}x^T(t)K_1^{(1)}x(t) \tag{28}$$

By using the fact that

$$\frac{\partial J_1^{(0)}}{\partial x} = K_1^{(1)}x(t) \tag{29}$$

we get

$$\begin{aligned} &x^T(t)K_1^{(1)}\left(A - S_1K_1^{(0)} - S_2K_2^{(0)}\right)x(t) \\ &= -\tfrac{1}{2}x^T(t)\left[Q_1 + K_1^{(0)}S_1K_1^{(0)} + K_2^{(0)}S_2K_2^{(0)}\right]x(t) \end{aligned} \tag{30}$$

Using the standard symmetrization technique known from the derivations of the Riccati equation, that is

$$x^TMx = \frac{1}{2}x^T\left(M + M^T\right)x, \quad \text{for any square matrix} \quad M \tag{31}$$

we get

$$\begin{aligned} &\left(A - S_1K_1^{(0)} - S_2K_2^{(0)}\right)^T K_1^{(1)} + K_1^{(1)}\left(A - S_1K_1^{(0)} - S_2K_2^{(0)}\right) \\ &= -\left(Q_1 + K_1^{(0)}S_1K_1^{(0)} + K_2^{(0)}S_2K_2^{(0)}\right) \end{aligned} \tag{32}$$

Due to the fact that $A - S_1K_1^{(0)} - S_2K_2^{(0)}$ is a stable matrix and the right-hand side of equation (32) is negative semidefinite, it follows that a unique nonnegative definite solution $K_1^{(1)}$ exists. *Note that if we assume that the penalty matrix Q_1 is positive definite then the corresponding solution of (32) will be also positive definite.* From (23) and (29) we have

$$\frac{\partial J_1^{(0)}}{\partial x}(t) = K_1^{(1)}x(t) \Rightarrow u_1^{(1)}(t) = -R_{11}^{-1}B_1^TK_1^{(1)}x(t) \tag{33}$$

Similarly, performing operations (26)-(33) for the second control agent we get

$$\frac{\partial J_2^{(0)}}{\partial x}(t) = K_2^{(1)}x(t) \Rightarrow u_2^{(1)}(t) = -R_{22}^{-1}B_2^TK_2^{(1)}x(t) \tag{34}$$

By repeating steps 1 and 2 now with $u_1^{(1)}(x(t))$ and $u_2^{(1)}(x(t))$ we get $u_1^{(2)}(x(t))$ and $u_2^{(2)}(x(t))$ as well as $K_1^{(2)}$ and $K_2^{(2)}$. Continuing the same procedure, we get the sequences of the solution matrices. It is easy to show (minimization technique in the negative gradient direction) that these sequences are convergent since $K_1^{(m)}, K_2^{(m)}, m = 1, 2, \ldots$ cause the corresponding Hamiltonians to tend to zero. In addition, let $K_1^{(\infty)}$ and $K_2^{(\infty)}$ be the limit points of the corresponding sequences; then from (11)-(12) we have

$$\begin{aligned} &\left(A - S_1K_1^{(\infty)} - S_2K_2^{(\infty)}\right)^T K_1^{(\infty)} + K_1^{(\infty)}\left(A - S_1K_1^{(\infty)} - S_2K_2^{(\infty)}\right) \\ &\quad + \left(Q_1 + K_1^{(\infty)}S_1K_1^{(\infty)} + K_2^{(i)}Z_2K_2^{(\infty)}\right) = 0 \end{aligned} \tag{35}$$

$$\begin{aligned} &\left(A - S_1K_1^{(\infty)} - S_2K_2^{(\infty)}\right)^T K_2^{(\infty)} + K_2^{(\infty)}\left(A - S_1K_1^{(\infty)} - S_2K_2^{(\infty)}\right) \\ &\quad + \left(Q_2 + K_1^{(\infty)}Z_1K_1^{(\infty)} + K_2^{(\infty)}S_2K_2^{(\infty)}\right) = 0 \end{aligned} \tag{36}$$

that is, $K_1^{(\infty)}$ and $K_2^{(\infty)}$ satisfy (7)-(8) so that they represent the sought solutions of these equations.

Note that the stronger result than the one stated in Theorem 2.1 can be similarly obtained by assuming that the penalty matrices Q_1 and Q_2 are positive definite (see (32) and the corresponding comment below it).

Assumption 2.2 *The state penalty matrices satisfy* $Q_1 > 0,\ Q_2 > 0$.

In that case we have the following theorem.

Theorem 2.2 *Under Assumptions 2.1 and 2.2 the unique positive definite stabilizing solution of the coupled algebraic Riccati equations (7)-(8) exists. It is obtained by performing Lyapunov iterations (11)-(12).*

Numerical Example 1: In order to demonstrate the efficiency of the proposed algorithm we have run a tenth-order example, which is in fact a system of 110 nonlinear algebraic equations. Matrices A, B_1, and B_2 have been chosen randomly, whereas the choice of matrices Q_1, Q_2, R_{11}, R_{12}, and R_{22} assures that Assumption 2.1 is satisfied. These matrices are given by

$$B_1^T = \begin{bmatrix} -2.036 & 1.560 & -0.907 & -1.214 & 0.813 & 0.044 & -0.750 & 0.901 & 0.913 & 0.084 \\ 0.637 & 0.447 & 1.154 & -1.091 & -0.575 & 0.729 & 0.690 & -1.826 & 0.635 & -0.209 \end{bmatrix}$$

$$B_2^T = \begin{bmatrix} -1.648 & 0.171 & -0.380 & -1.465 & 1.854 & 0.015 & 0.458 & 0.255 & 0.274 & -0.502 \\ -0.759 & 1.256 & -1.076 & -0.101 & 0.745 & 1.717 & -0.091 & -1.304 & -0.763 & 1.345 \end{bmatrix}$$

$$A = \begin{bmatrix} -1.944 & 0.572 & 1.446 & -0.576 & 0.736 & -0.601 & -0.722 & -0.088 & 0.977 & 0.380 \\ 1.440 & 0.393 & 1.023 & -0.711 & 1.282 & -0.679 & 0.010 & 0.588 & 1.281 & -1.414 \\ -0.881 & 1.058 & -1.492 & 1.113 & -1.728 & 0.498 & 0.313 & 1.509 & -1.536 & -0.264 \\ -1.170 & -1.055 & -0.058 & -0.723 & -0.939 & 1.453 & -1.087 & -0.486 & 1.066 & 0.235 \\ 0.736 & -0.569 & 1.449 & -1.383 & 0.116 & -0.052 & 1.387 & 0.659 & -1.658 & -1.437 \\ 0.014 & 0.658 & 0.586 & -0.850 & -0.074 & -1.335 & -0.261 & -1.021 & -0.449 & 1.444 \\ -0.734 & 0.621 & 0.422 & -0.369 & -0.395 & -0.453 & 1.228 & 0.213 & -1.380 & 1.307 \\ 0.820 & -1.746 & 0.178 & -0.860 & -1.235 & -0.902 & 0.390 & -0.656 & -1.658 & 1.329 \\ 0.831 & 0.569 & 1.408 & 1.500 & 1.396 & -0.605 & 0.387 & -0.729 & 1.717 & 1.309 \\ 0.051 & -0.224 & 1.394 & 0.104 & -1.742 & -0.386 & -0.047 & -0.505 & -1.135 & 1.392 \end{bmatrix}$$

$$R_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad R_{12} = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}, \quad R_{21} = \begin{bmatrix} 5 & 0 \\ 0 & 6 \end{bmatrix}, \quad R_{22} = \begin{bmatrix} 7 & 0 \\ 0 & 8 \end{bmatrix}, \quad Q_1 = I_{10}, \; Q_2 = I_{10}$$

The obtained results are really remarkable since only after 8 iterations we got very good convergence. These results are presented in Table 1. The errors are defined as the absolute values of the largest elements in matrices $\mathcal{N}_1\left(K_1^{(i)}, K_2^{(i)}\right)$ and $\mathcal{N}_2\left(K_1^{(i)}, K_2^{(i)}\right)$ where i stands for the number of iterations.

Simulation results presented in Table 1 are obtained by using the software package L-A-S (Bingulac and Vanlandingham, 1993).

Iteration	*error 1*	*error 2*
1	$1.5283 \times 10^{+2}$	$1.4193 \times 10^{+2}$
2	$1.6726 \times 10^{+1}$	$4.0585 \times 10^{+1}$
3	$3.1057 \times 10^{+0}$	$1.2188 \times 10^{+1}$
4	2.3207×10^{-1}	$2.4337 \times 10^{+0}$
5	1.2386×10^{-1}	7.6489×10^{-2}
6	4.1600×10^{-3}	6.9948×10^{-5}
7	7.0661×10^{-4}	3.0096×10^{-7}
8	2.4374×10^{-5}	9.2183×10^{-8}

Table 1: Simulation results for a system of 110 nonlinear scalar equations

3 Algorithm for Solving the Generalized Algebraic Riccati Equation

Consider *the generalized algebraic Riccati equation* appearing in the H_∞ optimal control problems and zero-sum differential games (Basar, 1991; Hewer, 1993; Zhou and Khargonekar, 1987; Bernstein and Haddad, 1989; Mageirou, 1976; Peterson, 1988; Basar and Bernhard, 1991)

$$A^T P + PA + Q - P\left(\frac{1}{\gamma^2}S - Z\right)P = 0 \tag{37}$$

where A, Q, S, Z, and P are real constant matrices of dimensions $n \times n$ and γ is a real positive parameter $0 < \gamma \leq \infty$. In addition, matrices S, Z, *and* Q *are positive semidefinite.* An important feature of equation (37) which distinguishes this equation from the standard linear-quadratic optimal control algebraic Riccati equation is that the matrix $\frac{1}{\gamma^2}S - Z$ is in general indefinite. Equation (37) or its forms also appear in the stabilization of uncertain systems (Peterson and Hollot, 1986; Peterson, 1988), disturbance attenuation problems (Peterson, 1987), and decentralized stabilization (Mageirou and Ho, 1977).

In this paper we develop an elegant and simple algorithm which converges globally to the *positive semidefinite stabilizing* solution of (37) under stabilizability-detectability assumptions. The algorithm is given in terms of the standard algebraic Riccati equations, which have to be solved iteratively. We have also presented the Lyapunov iterations version of the proposed algorithm. Note that the Lyapunov iterations for finding the *positive definite* solution of (37), based on the "Bellman approximation in policy space," were obtained in Mageirou and Ho, 1977; Mageirou, 1977, see also Basar, 1991. However, that algorithm is different from the corresponding one presented in this paper and *it cannot be used* for finding the positive semidefinite stabilizing solution of (37), which is of interest for H_∞ optimal control problems.

The generalized algebraic Riccati equation (37) will be solved under the following assumption.

Assumption 3.1 *The triple* $\left(A, \sqrt{S}, \sqrt{Q}\right)$ *is stabilizable-detectable.*

We propose the following algorithm for solving (37).

Algorithm 2:

$$A^T P^{(i+1)} + P^{(i+1)}A - \tfrac{1}{\gamma^2}P^{(i+1)}SP^{(i+1)} + Q + P^{(i)}ZP^{(i)} = 0$$

$$\textit{with the initial condition obtained from} \tag{38}$$

$$A^T P^{(0)} + P^{(0)}A + Q - \tfrac{1}{\gamma^2}P^{(0)}SP^{(0)} = 0$$

The properties of this algorithm are stated in the next theorem.

Theorem 3.1 *The proposed algorithm converges globally under Assumption 3.1 to the positive semidefinite stabilizing solution of the generalized algebraic Riccati equation (37), assuming that such a solution exists.*

Note that Assumption 3.1 may be tightened so that the algorithm converges to the positive definite stabilizing solution of (37), see Remark 1.

Before we proceed with the proof of the algorithm's global convergence we need the following lemma which follows from Hewer, 1993; Peterson, 1988.

Lemma 3.1 *If it exists the positive semidefinite stabilizing solution of (37) is unique.*

In addition, it is easy to observe that if P is stabilizing, that is, $A - \frac{1}{\gamma^2}SP + ZP$ is stable, then the feedback matrix $A - \frac{1}{\gamma^2}SP$ is stable too.

Proof: Under Assumption 3.1 the unique positive semidefinite stabilizing solution of the standard algebraic Riccati equation given by

$$A^T P^{(0)} + P^{(0)} A - \frac{1}{\gamma^2} P^{(0)} S P^{(0)} + Q = 0 \tag{39}$$

exists. Subtracting (39) from (37) and assuming that the positive semidefinite stabilizing solution of (37) exists, we get

$$\begin{aligned} \left(A - \tfrac{1}{\gamma^2}SP\right)^T \left(P - P^{(0)}\right) + \left(P - P^{(0)}\right)\left(A - \tfrac{1}{\gamma^2}SP\right) \\ = -\tfrac{1}{\gamma^2}\left(P - P^{(0)}\right) S \left(P - P^{(0)}\right) - PZP \end{aligned} \tag{40}$$

Since $\left(A - \frac{1}{\gamma^2}SP\right)$ is stable and the right side of (40) is negative semidefinite, it follows that

$$P - P^{(0)} \geq 0 \;\Rightarrow\; P \geq P^{(0)} \tag{41}$$

Consider now the next iteration of (38), that is

$$A^T P^{(1)} + P^{(1)} A - \frac{1}{\gamma^2} P^{(1)} S P^{(1)} + \left(Q + P^{(0)} Z P^{(0)}\right) = 0 \tag{42}$$

Since

$$Q + \frac{1}{\gamma^2} P^{(0)} S P^{(0)} \geq Q \tag{43}$$

we have

$$P^{(1)} \geq P^{(0)} \tag{44}$$

Continuing the same procedure for $i = 2, 3, \ldots$, it can be shown by using the properties of the algebraic Riccati equation that

$$Q + P^{(i)} Z P^{(i)} \geq Q + P^{(i-1)} Z P^{(i-1)} \Rightarrow P^{(i+1)} \geq P^{(i)} \tag{45}$$

hence, we have from (38)

$$P^{(i+1)} \geq P^{(i)} \geq ...P^{(0)} \geq 0, \quad i = 0, 1, 2, ... \tag{46}$$

This monotonically nondecreasing sequence $\{P^{(i)}\}$ has the upper bound. To show this we subtract (38) from (37) which produces

$$\begin{aligned} &\left(A - \tfrac{1}{\gamma^2} SP\right)^T \left(P - P^{(i+1)}\right) + \left(P - P^{(i+1)}\right)\left(A - \tfrac{1}{\gamma^2} SP\right) \\ &= -\tfrac{1}{\gamma^2}\left(P - P^{(i+1)}\right) S \left(P - P^{(i+1)}\right) - \left(PZP - P^{(i)} Z P^{(i)}\right) \end{aligned} \tag{47}$$

Established relation (41) and the induction arguments imply

$$P \geq P^{(i+1)}, \quad i = 0, 1, 2, ... \tag{48}$$

Thus, the required positive semidefinite stabilizing solution of (37) represents the upper bound for the sequence (46). The bounded sequence defined by (46) and (48) is convergent by the monotonic convergence of positive operators (Wonham, 1968; Kantorovich and Akilov, 1964). Assuming that $P^{(\infty)}$ is the limit point of the sequence $\{P^{(i)}\}$ we have

$$A^T P^{(\infty)} + P^{(\infty)} A - \frac{1}{\gamma^2} P^{(\infty)} S P^{(\infty)} + Q + P^{(\infty)} Z P^{(\infty)} = 0 \tag{49}$$

Since equations (37) and (49) are identical, it follows that the proposed algorithm converges to the required solution of (37). Since by Lemma 3.1 the stabilizing solution of (37) is unique, the proposed algorithm is globally convergent. This completes the convergence proof of the proposed algorithm and proves stated Theorem 3.1.

Remark 1. By tightening Assumption 3.1 to the triple $\left(A, \sqrt{S}, \sqrt{Q}\right)$ is stabilizable-observable, the proposed algorithm produces *the positive definite* stabilizing solution of (37). This is important for zero-sum differential games where the required solution is positive definite.

Note that an algorithm for solving (37) of zero-sum differential games, in terms of the Lyapunov iterations, has been proposed in Mageirou and Ho, 1977; Mageirou, 1977 in the form

$$\left(A + QM^{(i)}\right)^T M^{(i+1)} + M^{(i+1)}\left(A + QM^{(i)}\right) = (S - Z) + M^{(i)} Q M^{(i)} \tag{50}$$

with $M^{(0)}$ being any antistabilizing initial guess and

$$P^{(i+1)} = M^{(i+1)^{-1}}, \quad P^{(\infty)} = M^{(\infty)^{-1}} \to P \tag{51}$$

Remark 2. The first obvious drawback of algorithm (50)-(51) is that it produces only the positive definite solution (P must be invertible) so that it cannot be used for the H^∞ optimal control problems. Secondly, finding the stabilizing initial guess for high order problems is computationally involved.

Note that Algorithm 2 can be reformulated in terms of the Lyapunov iterations by having in mind that the solution of any standard algebraic Riccati equation can be obtained by performing iterations on the Lyapunov algebraic equations (Kleinman, 1968). The "linearized" version of Algorithm 2 is given by

Algorithm 3:

$$\begin{aligned}&\left(A-\tfrac{1}{\gamma^2}SL^{(i)}\right)^T L^{(i+1)}+L^{(i+1)}\left(A-\tfrac{1}{\gamma^2}SL^{(i)}\right)\\&\quad=-\left(Q+\tfrac{1}{\gamma^2}L^{(i)}SL^{(i)}+L^{(i)}ZL^{(i)}\right)\\&\text{with}\quad A^TL^{(0)}+L^{(0)}A+Q-\tfrac{1}{\gamma^2}L^{(0)}SL^{(0)}=0\end{aligned}\tag{52}$$

It should be pointed out that the sequence generated by Lyapunov iterations (52) is closer to the required positive semidefinite stabilizing solution than the corresponding sequence generated by the Riccati iterations (38). To see this, observe that both sequences start at the same initial point and that from (38) and (52) we have

$$\begin{aligned}&\left(A-\frac{1}{\gamma^2}SL^{(i)}\right)^T\left(L^{(i+1)}-P^{(i+1)}\right)+\left(L^{(i+1)}-P^{(i+1)}\right)\left(A-\frac{1}{\gamma^2}SL^{(i)}\right)=\\&-\frac{1}{\gamma^2}\left(P^{(i+1)}-L^{(i)}\right)S\left(P^{(i+1)}-L^{(i)}\right)-\left(L^{(i)}ZL^{(i)}-P^{(i)}ZP^{(i)}\right)\end{aligned}\tag{53}$$

Since $L^{(0)}=P^{(0)}$ it follows that $L^{(1)}\geq P^{(1)}$; then by induction, it is easy to establish that

$$L^{(i+1)}\geq P^{(i+1)},\qquad i=0,1,2,...\tag{54}$$

Other technical details establishing the complete convergence proof for Algorithm 3 can be obtained similarly to those of Algorithm 2.

Numerical Example 2: In order to demonstrate the efficiency of the proposed algorithm we solve the following example. The problem matrices for (37) are given by

$$A=\begin{bmatrix}1&0&3&0\\0&-2&3&0\\0&1&-3&0\\1&0&0&-4\end{bmatrix},\quad Q=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

$$S=B_1R_1^{-1}B_1^T,\quad Z=B_2R_2^{-1}B_2^T,\quad R_1=I_2,\quad R_2=I_1,\quad \gamma=1$$

$$B_1^T=\begin{bmatrix}0.0116&0.6020&0.9215&0.5565\\0.5450&0.0730&0.5565&0.3834\end{bmatrix}$$

$$B_2^T=\begin{bmatrix}0.4814&0.3909&0.4087&0.5591\end{bmatrix}$$

The open-loop eigenvalues of the matrix A are $\{-4.3028, -4, -0.6972, 1\}$, which indicates that this system is open-loop unstable. The rank of the observability matrix is 3 so that this system is not observable. However, the system is both stabilizable and detectable. Note that matrices B_1 and B_2 are obtained by using a random number generator. Simulation results are obtained using MATLAB. The proposed Algorithm 2 has produced the positive *semidefinite* stabilizing solution with accuracy of $O\left(10^{-5}\right)$ after 16 iterations. It is interesting to point out that the same accuracy is obtained after the same number of iterations by using Algorithm 3 based on the Lyapunov iterations. Results for the trace of the matrix P per iteration for both the Riccati iterations algorithm (38) and the Lyapunov iterations algorithm (52) are given in Table 2. Note that the Lyapunov iterations results (column

Iteration	Algorithm 2 $traceP^{(i)}$	Algorithm 3 $traceP^{(i)}$
1	4.2911	4.4778
2	4.7192	4.8320
3	4.9459	5.0093
4	5.0678	5.1026
5	5.1337	5.1526
6	5.1694	5.1796
7	5.1886	5.1941
8	5.1990	5.2020
9	5.2046	5.2062
10	5.2078	5.2085
11	5.2093	5.2097
12	5.2101	5.2104
13	5.2106	5.2107
14	5.2109	5.2109
15	5.2110	5.2110
16	5.2111	5.2111
Optimal =	5.2111	5.2111

Table 2: Solution of the generalized algebraic Riccati equation

3) are closer to the desired solution in each iteration than those from the Riccati iterations (column 2), as established in (54). The obtained stabilizing

positive semidefinite solution is given by

$$P^{(16)} = \begin{bmatrix} 3.0103 & 0.3834 & 2.1315 & 0 \\ 0.3834 & 0.3875 & 0.5205 & 0 \\ 2.1315 & 0.5205 & 1.8134 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \geq 0$$

The eigenvalues of the closed-loop matrix are obtained as

$$\lambda\left\{A - (S - Z)\, P^{(16)}\right\} = \left\{ \begin{array}{c} -4.1317 \\ -4.0000 \\ -1.3959 \pm j0.5865 \end{array} \right\}$$

4 Conclusions

An iterative algorithm leading to the nonnegative (positive) definite stabilizing solution of coupled algebraic Riccati equations is constructed. Computational requirements are reduced considerably since the problem decomposition is achieved and only reduced-order Lyapunov equations have to be solved. In addition, a simple and elegant algorithm is presented for solving the algebraic Riccati equation of H^∞ optimal control and zero-sum differential games. This algorithm, in fact, finds the desired positive semidefinite stabilizing solution, assuming that it exists. The algorithm's initial guess is easily obtained.

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