fORged by Machines Contest

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1 Introduction

Given historical demand data, we study the problem of optimal inventory control. The goal is to minimize the sum of holding cost and backorder cost. Because there is lead time in delivering inventory orders, any control policy needs to take into account the uncertainty of future demand. To this end, we propose a two-step data driven inventory control approach. Motivated by the insights from demand data, we propose a step-adjusted triple exponential smoothing (STES) algorithm in predicting future demand. Utilizing the forecasts from the algorithm, we find the optimal control policy in the second step.

Data analysis The provided monthly demand data from Jan. 1996 to Dec. 2005 demonstrates several important features. All figures are in the Appendix.

1. *Fixed ratio of monthly demand to annual demand.* For any given month, the ratio of demand observed during that month to that observed in the entire year is consistent over ten years (figure 4). For example, January on average accounts for 7.8% of the total annual demand with a standard deviation of just 0.3%.

2. *Increasing annual demand.* There is an upward trend in the annual demand. In particular, such increasing trend exists only in annual demand, but not in monthly demand.

3. *Start-of-year/quarter effect.* The demand observation in January contains more information regarding the annual demand than other months (figure 5). There is some weak evidence that this may also be true for each quarter, i.e., January, April, July and October.

Modeling approach. We tackle the inventory control problem in two steps. In the first step, we propose STES, which predicts future demand given historical demand. STES is a modification of the triple exponential smoothing algorithm (TES) [Winters, P. R. (1960)] . TES can capture seasonality and underlying trends in a time series. However, it cannot be used out of box in our context for two reasons: (1) Classical TES would assume a seasonal cycle of one year and a linear trend over all months whereas in the present context, a trend exists only over years, but not months. STES addresses this issue by assuming the monthly trend is a step function, which only takes positive values at year transitions. (2) TES assumes constant weights in smoothing the current and previous estimations. In STES, we put more weight on estimations implied by the starting month of each quarter.

In the second step, given the prediction of the next month's demand (both mean and variance), we solve a classical Newsvendor problem to obtain optimal order quantities, which will be delivered at the beginning of the next month.

2 Two-step data-driven inventory control

2.1 Forecasting algorithm: Step-adjusted Triple Exponential Smoothing (STES)

Demand model. We first introduce the demand model motivated by insights from data. Note the time t is the month index starting from the beginning of year 1996. For example, $t = 1$ refers to Jan. 1996, and $t = 121$ refers to Jan. 2006. Below are notations for demand model.

- $L = 12$: the length of the seasonal cycle is 12 months;
- $m(t) \in \{1, 2, \dots, 12\}$: the month of time t. For example, "$m(13) = 1$" means that time 13 is January, "$m(24) = 12$" means that time 24 is December;
- D_t: demand for time t. Denote its realization as d_t;
- S_t: annual demand for the year of time t. Denote its realization as s_t. For example, s_{121} is the realized annual demand for year 2006;
- B_t: trend in annual demand. Note it can be time-varying, i.e., different in different years;
- $\{C_t^1, C_t^2, \dots, C_t^{12}\}$: C_t^m is the ratio of demand in month m to annual demand during time t. For example, $C_{13}^1 = 7.8\%$ means that January accounts for 7.8% of the total annual demand in year 1997. Note this ratio vector can also be different in different years.

The demand D_t evolves over time according to a multiplicative trend (Equation 1) and an additive trend (Equation 2).

$$D_t = S_t \cdot C_t^{m(t)} + \epsilon_t \tag{1}$$
$$S_t = S_{t-1} + \mathbf{1}_{\{m(t)=1\}} \cdot B_t + e_t \tag{2}$$

where $\epsilon_t \sim^{i.i.d} \mathcal{N}(0, \sigma_1^2)$ and $e_t \sim^{i.i.d} \mathcal{N}(0, \sigma_2^2)$.

Predicting monthly demand. Given a realized time series of demand $\{d_t\}_{t=1}^T$, the challenge in predicting D_{t+1} is that S_{t+1}, $C_t^{m(t+1)}$ and B_{t+1} can be hidden. For example, suppose we are at the end of year 1996, when predicting demand for Jan. 1997 (D_{13}), we don't really know annual demand (S_{13}), the ratio vector (C_{13}^1) or year trend (B_{13}) for 1997. Therefore, we use smoothing algorithm to approximate these three hidden quantities. In particular, the STES algorithm is as follows.

Initialization. Let $\hat{S}_1 = d_1 \cdot 12$, $\hat{B}_1 = \frac{s_N - s_1}{N-1}$, $\hat{C}_1^m = \frac{1}{N} \sum_{i=1}^N \frac{d_{m+12(i-1)}}{s_{m+12(i-1)}}$ for $m = 1, \ldots, 12$, where $N = \left\lfloor \frac{T}{12} \right\rfloor$ is the number of full years in data.

Iteration. For each $t \geq 1$, after observing the realization of d_t,

$$\hat{B}_t = \mathbf{1}_{\{m(t)=1\}} \cdot [\beta \cdot (\hat{S}_t - \hat{S}_{t-1}) + (1-\beta) \cdot \hat{B}_{t-1}] \tag{3}$$
$$\hat{C}_t^{m(t)} = \gamma \cdot \frac{d_t}{\hat{S}_t} + (1-\gamma) \cdot \hat{C}_t^{m(t)} \tag{4}$$
$$\hat{S}_{t+1} = \begin{cases} \alpha_q \cdot \frac{d_t}{\hat{C}_t^{m(t)}} + (1-\alpha_q) \cdot \hat{S}_t, & \text{if } m(t) \in \{1,4,7,10\} \\ \alpha_m \cdot \frac{d_t}{\hat{C}_t^{m(t)}} + (1-\alpha_m) \cdot (\hat{S}_t + \mathbf{1}_{\{m(t)=12\}} \cdot \hat{B}_t), & \text{if } m(t) \notin \{1,4,7,10\} \end{cases} \tag{5}$$
$$\hat{D}_{t+1} = \hat{S}_{t+1} \cdot \hat{C}_t^{m(t+1)} \tag{6}$$

where α_m and α_q are monthly and quarterly smoothing factors for annual demand, respectively. β is the smoothing factor for the annual trend. γ is the smoothing factor for seasonality trend.

Equation 3 and 4 update year trend and seasonality trend respectively. Equation 5 and 6 predict annual and monthly demand respectively. We estimate $\hat{Var}(D_{t+1})$ by the following, from

$$\hat{Var}(D_{t+1}) = \frac{1}{N} \sum_{n:m(n)=m(t+1)} (\hat{D}_n - d_n)^2. \tag{7}$$

The R code for STES can be found on Github.[1]

[1] `https://github.com/adityas16/f0Rged/blob/master/STES.R`Github

2.2 Inventory model

Suppose the inventory system is at the end of month $t-1$, we are now making a decision for the number of orders which will be delivered at the beginning of month t. The timeline of the system is shown in Figure 1. The notation is introduced below.

- For convenience, denote $\mu_t = \hat{D}_t$, $\sigma_t^2 = \hat{Var}(D_t)$;
- x_{t-1}^{end}: the ending inventory level at the end of month $t-1$. A positive value refers to leftover inventory, which incurs holding cost. A negative value refers to backorder quantities, which incurs backorder cost;
- x_t: the starting inventory level at month t. Naturally, $x_t = x_{t-1}^{end}$;
- a_t: replenishment delivered at month t;
- y_t: inventory position at the beginning of month t, which is $x_t + a_t$.

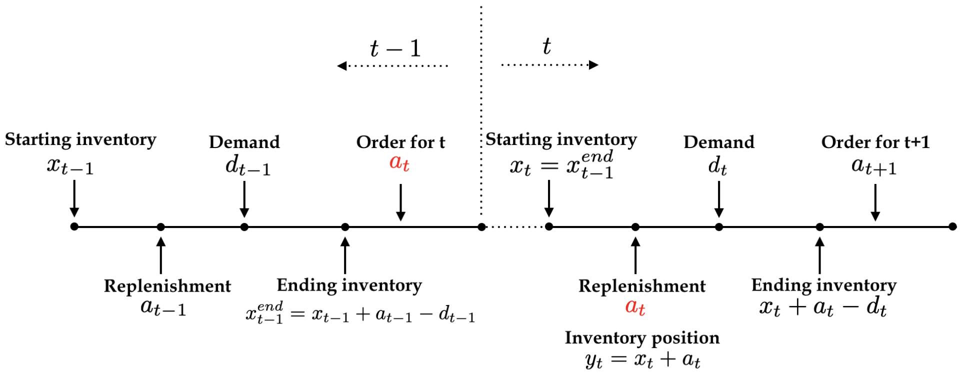

Figure 1: Timeline for inventory flow: month $t-1$ (left) and month t (right).

The myopic policy. At the end of month $t-1$, we observe x_t and make decisions for a_t by taking into account the uncertainty in future demand. Specifically, we calculate the optimal order for the next month, i.e., a_t, by solving a single period Newsvendor problem (Porteus, 2002). That is, when making decision for a_t, we only consider the distribution of demand D_t and minimize the expected cost for month t. Let y_t^M be the optimal inventory position in the myopic Newsvendor policy. We have,

$$y_t^M = \Phi_t^{-1}(\frac{c_p}{c_p + c_h}) = \mu_t + k\sigma_t, \tag{8}$$

where $k = \frac{2}{3}$ (when holding cost c_h=1, penalty costs $c_p = 3$), μ_t and σ_t are the mean and standard deviation of D_t respectively.

Ideally, the derivation of optimal order a_t shall take into account the distribution of future demand starting from month t to month T. Next in Section 2.2.1, we discuss the probability that this myopic policy is sub-optimal.

2.2.1 Empirical probability that the myopic policy is sub-optimal

Denote y_t^* the optimal inventory position for time t.

First, at the end of the planning horizon, $y_T^* = y_T^M$. Secondly, the action a_{T-1} derived from Newsvendor problem (Equation 8) is sub-optimal only if $x_T > y_T^M$. If $x_T \leq y_T^M$, the gap quantity $(y_T^M - x_T)$ can be ordered. However, if $x_T > y_T^M$, the leftover quantity $(x_T - y_T^M)$ can not be discarded.

In the following analysis, we estimate the probability that $x_T > y_T^M$ conditional on $y_{T-1} = y_{T-1}^M$. First, notice that

$$x_T > y_T^M \iff D_{T-1} < y_{T-1}^M - y_T^M = \mu_{T-1} + k\sigma_{T-1} - (\mu_T + k\sigma_T)$$

Based on our current estimations from data, we have $k = \frac{2}{3}$, and $\forall t$, $\mu_{t-1} - \mu_t \leq 26.25$, $\sigma_{t-1} - \sigma_t < 1.6$, $\mu_t \in [85, 105]$, $\sigma_t < 4$. Thus,

$$\mathbb{P}(a_{T-1} \text{ is sub-optimal}) \leq \mathbb{P}(D_{T-1} < 26.5) \approx O(10^{-50}).$$

There are three main reasons for the small probability of sub-optimality: (1) The mean demand in successive periods is steady without outstanding jumps; (2) The variance in demand forecasts is low, resulting in a low chance of large amount of leftover; (3) The standard deviation of predicted demand (≈ 3) is extremely small compared with its mean (≈ 90), resulting in high chance of positive orders.

We have shown in a two-month problem, the myopic Newsvendor solution is almost optimal. Recursively, we can see that overall, this solution concept is almost optimal for a twenty-four-month planning problem. As a matter of fact, in our numerical experiment (Section 4), there is never a time when $x_t > y_t^M$, indicating that the myopic policy we derive is optimal.

3 Code

3.1 Requirements

R version 3.4.4 or later. All the code has been written in R and only uses in-built R libraries.

3.2 Execution steps

1. main.R is the script that needs to be run (which will in turn source 4 other scripts in the folder)

2. From the terminal or command line, change directory to the folder containing the main.R script

3. Place the file containing the demand for 2006 and 2007 in the folder and name it as 'Test-Demand.csv'. The format of this file must be the same as was provided in 'Ten-Year-Demand.csv'.

4. Run the command : *(Path to R installation)/bin/Rscript main.R*. This must be done from the same folder that the script is in so that main.R script can load the other scripts.

5. The output will be written to two files: (1)*inventory_state.csv* contains the state of the inventory at the beginning and end of the month, quantity ordered and the costs for each of the 24 months from 121-144. (2)*aggregate_stats.csv* contains the mean and total backorder and holding costs over all periods.

6. Samples of all the input file and the 2 output files have been included in the 'samples' folder

4 Test results from known data

The following charts summarize the results from our test of using data from the 1995-2003 to predict and run inventory control on the years 2004 and 2005.

References

[Porteus 2002] Porteus, Evan L.: *Foundations of stochastic inventory theory.* Stanford University Press, 2002

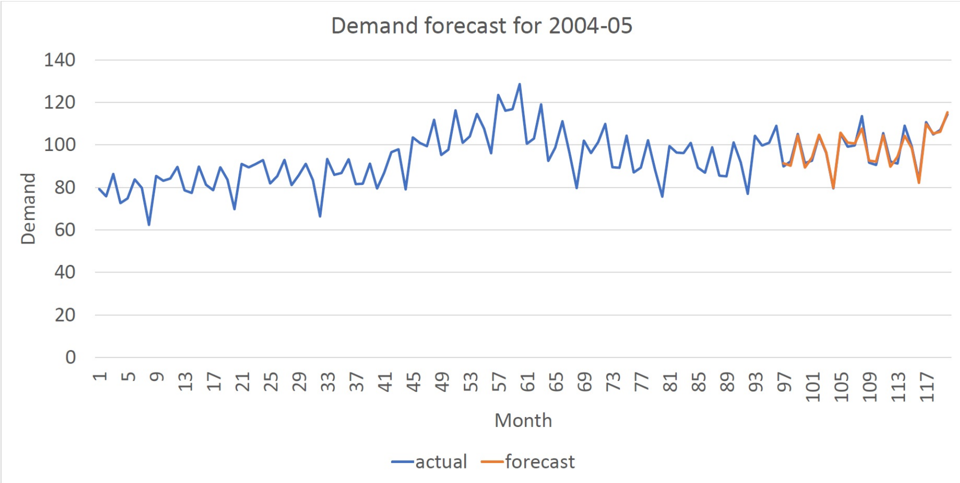

Figure 2: Results from forecasting demand for 2004 and 2005 using data till 2003. The RMSE of the prediction was 2.05 which was a 16% improvement over the RMSE from using the standard ets code (RMSE=2.48)

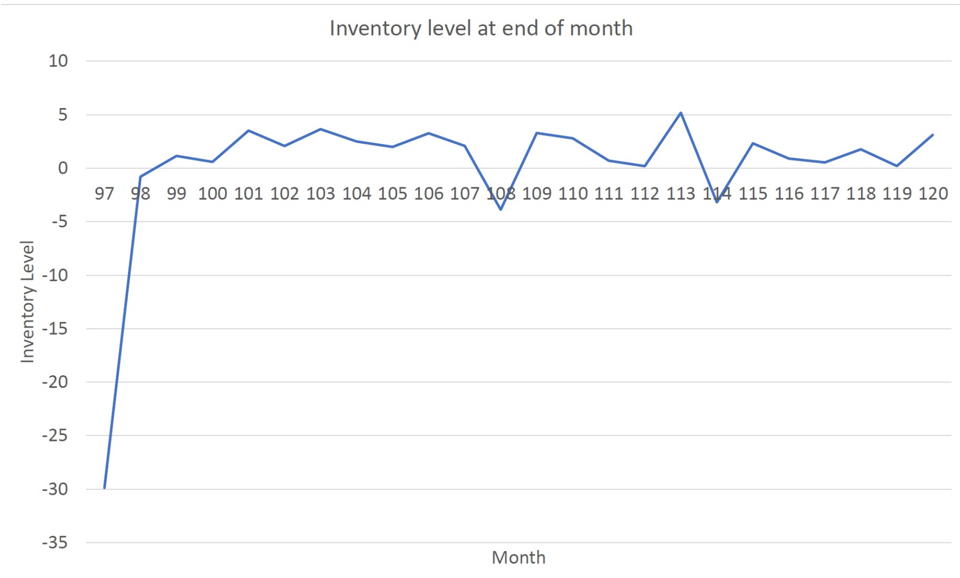

Figure 3: The average cost incurred is 2.84. Note we exclude cost of month 97 because it is not controlled by the policy.

A Data analysis charts

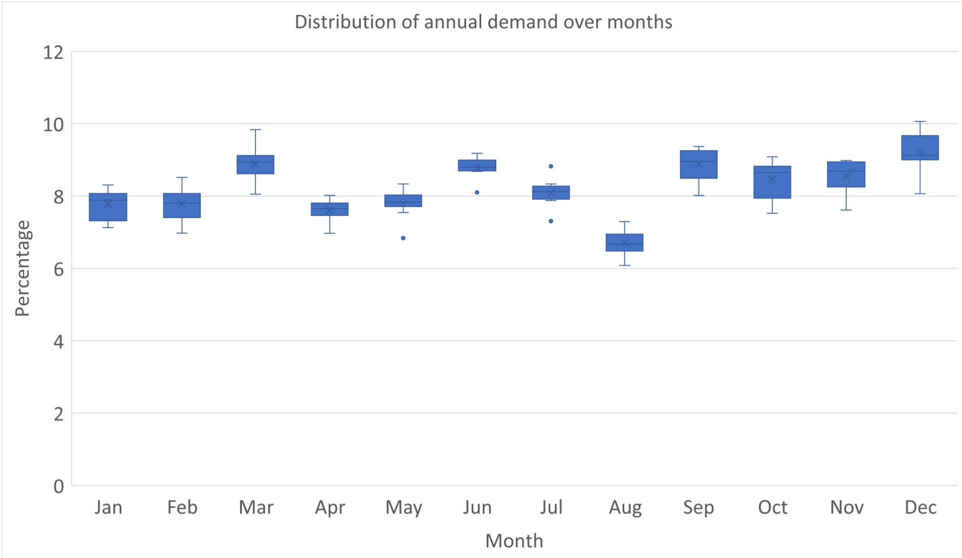

Figure 4: Monthly demand as a percentage of total annual demand. The variation in the percentage is very small.

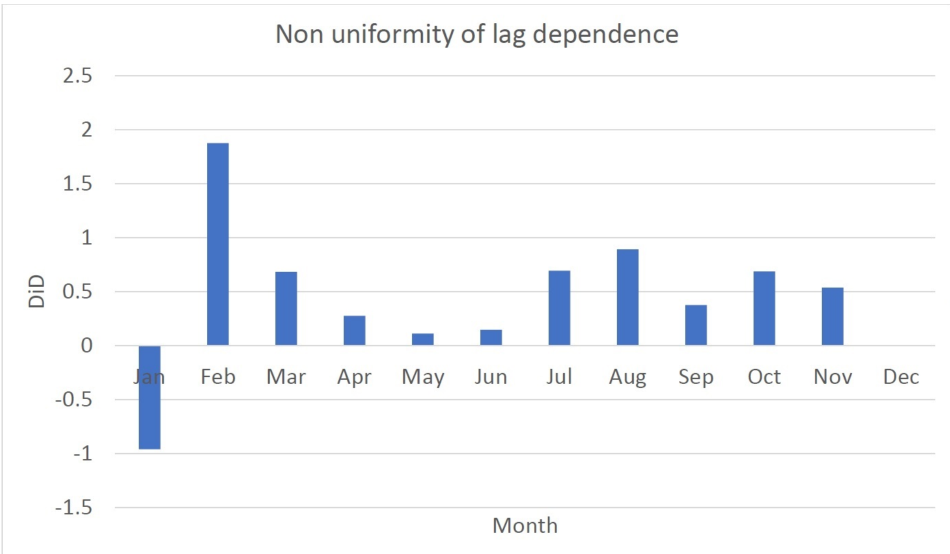

Figure 5: For each year, we calculate $|d_m - d_{m-2}| - |d_m - d_{m-1}|$ for each month m and this plot shows its average over all years. The large value in February implies that the demand in January has a high weight in explaining February's demand.