Ans-2) : The main objective of image sharpening is to highlight the transitions in pixel intensity values.

→ Image sharpening is obtained by applying spatial differentiation across the pixel values. As discussed in lectures, we can approximate derivatives by expanding their Taylor series expansions.

* For a 1-D func^n $f(x)$:

$$\boxed{\frac{\partial f}{\partial x} = f(x+1) - f(x)}$$

* For a 2-D func^n $f(x, y)$:

$$\frac{\partial^2 f}{\partial x^2} = f(x+1, y) + f(x-1, y) - 2f(x, y)$$

⇒ Computing Laplacian filtered image

→ The Laplacian for an image $f(x, y)$ is defined as :

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

Putting the Taylor series approximations of these second order derivatives into the above eqⁿ

$$\therefore \nabla^2 f = f(x+1, y) + f(x-1, y) - 2f(x, y) + f(x, y+1) + f(x, y-1) - 2f(x, y)$$

$$\therefore \boxed{\nabla^2 f = f(x+1, y) + f(x-1, y) + f(x, y+1) + f(x, y-1) - 4f(x, y)} \quad - (1)$$

→ We know for a given func^n $f(x, y)$

$$\boxed{f(x,y) \ast \delta(x-x_0, y-y_0) = f(x-x_0, y-y_0)}$$

where $\delta(x-x_0, y-y_0) =$ 2-D impulse func^n $= \begin{cases} 1; & x=x_0, y=y_0 \\ 0; & \text{otherwise} \end{cases}$

and $\ast$ denotes the convolution operator

→ Using the above result in ①,

$$\therefore \nabla^2 f = f(x,y) \ast \delta(x+1, y) + f(x,y) \ast \delta(x-1, y)$$
$$+ f(x,y) \ast \delta(x, y+1) + f(x,y) \ast \delta(x, y-1)$$
$$-4 f(x,y) \ast \delta(x, y)$$

As convolution operation is distributive,

$$\boxed{\therefore \nabla^2 f = f(x,y) \ast \underbrace{\left[\delta(x+1,y) + \delta(x-1,y) + \delta(x,y+1) + \delta(x,y-1) - 4\delta(x,y) \right]}_{w(x,y)}}$$

→ We can create a new kernel / mask / filter which will help us use the prev. eq^n practically. A mask of the type

$$w(x,y) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

will do the job. The diagonal constants are also incorporated in this while keeping (x,y) as its centre, with -4 as origin / center. This is also called a Laplacian kernel.

→ We can obtain a sharpened image g(x,y) from input f(x,y) by using the Laplacian in the foll. way

$$g(x,y) = f(x,y) + c\left[\nabla^2 f(x,y)\right]$$

→ We can again use $f(x,y) \star \delta(x,y) = f(x,y)$ in our eq^n to obtain:

$$g(x,y) = f(x,y) \star \delta(x,y) + c\nabla^2[f(x,y)]$$

$$= f(x,y) \star \delta(x,y) + c\, f(x,y) \star w'(x,y)$$

where $w'(x,y) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$$\therefore g(x,y) = f(x,y) \star [\delta(x,y) + c \cdot w'(x,y)]$$

→ As our ~~cente~~ centre is defined at a negative value $= -4$ $\therefore$ we'll take $c = -1$ to have a net "subtractive" effect.

$$\therefore\; g(x,y) = f(x,y) \star \left[\delta(x,y) - w'(x,y)\right]$$

$$= f(x,y) \star \left[\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right]$$

$$= f(x,y) \star \begin{bmatrix} 0-0 & 0-1 & 0-0 \\ 0-1 & 1-(-4) & 0-1 \\ 0-0 & 0-1 & 0-0 \end{bmatrix}$$

$$= f(x,y) \star \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 0 \end{bmatrix}}_{w(x,y)}$$

$$\therefore\; \boxed{\begin{gathered} g(x,y) = f(x,y) \star w(x,y) \\ \text{where } w(x,y) = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 0 \end{bmatrix} \end{gathered}}$$

(Ans)