MA 105 D3 Lecture 6

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Recap

Maxima and minima

Properties of differentiable functions

The second derivative

Maxima and minima

Let $X \subset \mathbb{R}$ and let $f : X \to \mathbb{R}$ be a function (you can think of X as an open, closed or half-open interval, for instance).

Definition: The function f is said to attain a maximum (resp. minimum) at a point $x_0 \in X$ if $f(x) \leq f(x_0)$ (resp. $f(x) \geq f(x_0)$) for all $x \in X$.

If X is a closed bounded interval and f is a continuous function, the Extreme Value Theorem tells us that the maximum and minimum are actually attained.

Definition: Let $f : X \to \mathbb{R}$ be a function and x_0 be in X. Suppose there is an sub-interval $x_0 \in (c, d) \subset X$ such that $f(x_0) \geq f(x)$ (resp. $f(x_0) \leq f(x)$) for all $x \in (c, d)$, then f is said to have a local maximum (resp. local minimum) at x_0.

Theorem 13: (Fermat's Theorem) If $f : X \to \mathbb{R}$ is differentiable and has a local minimum or maximum at a point $x_0 \in X$, $f'(x_0) = 0$.

Rolle's Theorem

Theorem 13 is known as Fermat's theorem. It can be used to prove Rolle's Theorem.

Theorem 14: Suppose $f : [a, b] \to \mathbb{R}$ is a continuous function which is differentiable in (a, b) and $f(a) = f(b)$. Then there is a point x_0 in (a, b) such that $f'(x_0) = 0$.

Proof: Since f is a continuous function on a closed bounded interval Theorem 11 tells us that f must attain its minimum and maximum somewhere in $[a, b]$. If both the minimum and maximum are attained at the end points, f must be the constant function, in which case we know that $f'(x) = 0$ for all $x \in (a, b)$. Hence, we can assume that at least one of the minimum or maximum is attained at an interior point x_0 and Theorem 13 shows that $f'(x_0) = 0$ in this case. $\square$

One easy consequence: If $P(x)$ is a polynomial of degree n with n real roots, then all the roots of $P'(x)$ are also real. (How do we know that polynomials are differentiable?)

Problems centered around Rolle's Theorem

Exercise 2.3: Let $f : [a, b] \to \mathbb{R}$ be continuous and suppose f is differentiable on (a, b). If $f(a)$ and $f(b)$ are of opposite signs and $f'(x) \neq 0$ for all $x \in (a, b)$, then there is a unique point x_0 in (a, b) such that $f(x_0) = 0$. Solution: Since the

Intermediate Value Theorem guarantees the existence of a point x_0 such that $f(x_0) = 0$, the real point of this exercise is the uniqueness. Suppose there were two points $x_1, x_2 \in (a, b)$

such that $f(x_1) = f(x_2) = 0$. Applying Rolle's Theorem, we see that there would exist $c \in (x_1, x_2)$ such that $f'(c) = 0$ contradicting our hypothesis, This proves the exercise. Let us

look at Exercise 2.8(i): Find a function f which satisfies all the given conditions, or else show that no such function exists: $f''(x) > 0$ for all $x \in \mathbb{R}$ and $f'(0) = 1, f'(1) = 1$. Solution:

Apply Rolle's Theorem to $f'(x)$ to conclude that such a function cannot exist.

The Mean Value Theorem

Rolle's theorem is a special case of the Mean Value Theorem (MVT).

Theorem 15: Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function and that f is differentiable in (a, b). Then there is a point x_0 in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(x_0).$$

$\square$

Proof: Apply Rolle's Theorem to the function

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a} \cdot (x - a)$$

(Why does one think of the function $g(x)$?)

Applications of the MVT

Here is an application of the MVT which you have probably always taken for granted:

Theorem 16: If f satisfies the hypotheses of the MVT, and further $f'(x) = 0$ for every $x \in (a, b)$, f is a constant function.

Indeed, if $f(c) \neq f(d)$ for some two points $c < d$ in $[a, b]$,

$$0 \neq \frac{f(d) - f(c)}{d - c} = f'(x_0),$$

for some $x_0 \in (c, d)$, by the MVT. This contradicts the hypothesis. $\square$

Applications of the MVT continued

Consider Exercise 2.6.:

Let f be continuous on $[a, b]$ and differentiable on (a,b). If $f(a) = a$ and $f(b) = b$, show that there exist distinct $c_1, c_2 \in (a, b)$ such that $f'(c_1) + f'(c_2) = 2$.

Solution: The idea is that the function clearly has an average rate of growth equal to 1 on the interval $[a, b]$. If the derivative at some point is less than 1, there must be another point where it is greater than 1 so that the sum adds up to 2. How to use this idea?

Split the interval into two pieces - $[a, \frac{a+b}{2}]$ and $[\frac{a+b}{2}, b]$ - and apply the MVT to each interval.

Darboux's Theorem

Another interesting property of differentiable functions is that their derivatives have the IVP (intermediate value property). This fact is sometimes called Darboux's Theorem.

Theorem 17: Let $f : (a, b) \to \mathbb{R}$ be a differentiable function. If c, d, $c < d$ are points in (a, b), then for every u between $f'(c)$ and $f'(d)$, there exists an x in $[c, d]$ such that $f'(x) = u$.

Proof: We can assume, without loss of generality, that $f'(c) < u < f'(d)$, otherwise we can take $x = c$ or $x = d$. Define $g(t) = ut - f(t)$. This is a continuous function on $[c, d]$, and hence, by Theorem 11 must attain its extreme values. These extreme values cannot occur at c or d since $g'(c) = u - f'(c) > 0$ and $g'(d) = u - f'(d) < 0$ (contradicts Fermat's Theorem). It follows that there exists $x \in (c, d)$ where g takes an extreme value. By Fermat's Theorem $g'(x) = 0$ which yields $f'(x) = u$. $\square$

Continuity of the first derivative

We have just seen that the derivative satisfies the IVP. Can we find a function which is differentiable but for which the derivative is not continuous?

Here is the standard example:

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \\ = 0 & \text{if } x = 0. \end{cases}$$

This function will be differentiable at 0 but its derivative will not be continuous at that point. In order to see this you will need to study the function in Exercise 1.13(ii). This will show that $f'(0) = 0$.

On the other hand, if we use the product rule when $x \neq 0$ we get

$$f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x},$$

which does not go to 0 as $x \to 0$.

Back to maxima and minima

We will assume that $f : [a, b] \to \mathbb{R}$ is a continuous function and that f is differentiable on (a, b). A point x_0 in (a, b) such that $f'(x_0) = 0$ often called a stationary point. We will assume further that $f'(x)$ is differentiable at x_0, that is, that the second derivative $f''(x_0)$ exists. We formulate the Second Derivative Test below.

Theorem 18: With the assumptions above:

1. If $f''(x_0) > 0$, the function has a local minimum at x_0.
2. If $f''(x_0) < 0$, the function has a local maximum at x_0.
3. If $f''(x_0) = 0$, no conclusion can be drawn.

The proof of the Second Derivative Test

Proof: The proofs are straightforward. For instance, to prove the first part we observe that

$$0 < f''(x_0) = \lim_{h \to 0} \frac{f'(x_0 + h)}{h}.$$

It follows that for $|h|$ small enough, $f'(x_0 + h) < 0$, if $h < 0$ and $f'(x_0 + h) > 0$ if $h > 0$. It follows that $f(x_0)$ is decreasing to the left of x_0 and increasing to the right of x_0. Hence, x_0 must be a local minimum. A similar argument yields the second case. $\square$

If the third case of the theorem above occurs, the function may be changing from concave to convex. In this case x_0 is called a point of inflection. An example of this phenomenon is given by $f(x) = x^3$ at $x = 0$.

Concavity and convexity

Let I denote an interval (open or closed or half-open).
Definition: A function $f : I \to \mathbb{R}$ is said to be concave (or sometimes concave downwards) if

$$f(tx_1 + (1-t)x_2) \geq tf(x_1) + (1-t)f(x_2)$$

for all x_1 and x_2 in I and $t \in [0,1]$.
Similarly, a function is said to be convex (or concave upwards) if

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2).$$

By replacing the $\geq$ and $\leq$ signs above by strict inequalities we can define strictly concave and strictly convex functions.

Note that if $f(x)$ is a concave function, $-f(x)$ is a convex function, so it is really enough to study one class or the other. Convex functions occur in many areas of mathematics.

Examples of concave and convex functions

Here are some examples of convex functions.

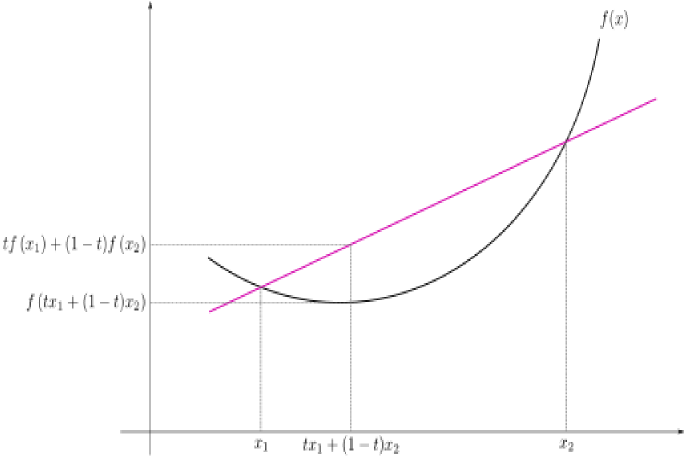
1. $f(x) = x^2$ on $\mathbb{R}$.
2. $f(x) = x^3$ on $[0, \infty)$.
3. $f(x) = e^x$ on $\mathbb{R}$.

Examples of concave functions include

1. $f(x) = -x^2$
2. $f(x) = x^3$ on $(-\infty, 0]$
3. $f(x) = \log x$ on $(0, \infty)$.

For a convex function f and point $c \in (x_1, x_2)$, the point $(c, f(c))$ always lies below the line joining $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

Convexity illustrated graphically



Propeties of Convex functions

Convex functions have many nice properties. For instance, it is easy to show that convex functions are continuous (do this!). More is true.

Exercise 1. Every convex function is Lipschitz continuous (a function is Lipschitz continuous if it satisfies the inequality given in Exercise 1.16 but with $\alpha = 1$). In fact, much more is true. A convex function is actually differentiable at all but at most countably many points.

A differentiable function is convex if and only if its derivative is monotonically increasing. Moreover, if a function is both differentiable and convex, it is continuously differentiable, that is, its derivative is continuous (feel free to try proving these facts).

Convexity and the second derivative (not yet covered in class)

A twice differentiable function on an interval will be convex if its second derivative is everywhere non-negative. If the second derivative is positive, the function will be strictly convex.

However, the converse of the second statement above is not true. Can you give a counter-example to the converse of the second statement?

How about $f(x) = x^4$?

Definition: A point of inflection x_0 for a function f is a point where the function changes its behavior from concave to convex (or vice-versa). At such a point $f''(x_0) = 0$, but this is only a necessary, not a sufficient condition. (Why?)
If further, we also assume that the lowest order (≥ 2) non-zero derivative is odd, then we get a sufficient condition.