CS105 Endsem: DIC on Discrete Structures

85 marks, 3 hrs

20 Nov 2023

Part A

1. (5*2=10 marks) True or False. Give a **short** justification for your answer.

   (a) For any $n \in \mathbb{N}$, if n^3 is odd, then so is n^2.

   (b) If a graph G has 5 vertices and 10 edges, then some vertex must have degree 4.

   (c) If a set A is infinite, then there is a bijection from $\mathbb{N}$ to $A \cup \mathbb{N}$.

   (d) Any two maximal matchings in a graph have the same size, i.e., number of edges.

   (e) Any theorem that can be shown by strong induction can also be proved using induction.

Solution. 1 mark for answer, 1 mark for explanation.

(a) True. Any Case analysis of n being odd or even will suffice. Just writing that n^3 is odd implies n is odd which implies n^2 is odd won't suffice. Can do a case analysis on the contrapositive too.

(b) True. Assume, to the contrary, that all vertices had degree ≤ 3.(the degrees can't exceed 4) Then sum of degrees$= 2 * 10 \leq 3 * 5 \implies 20 \leq 15$ which is absurd. Hence there exists a vertex of degree 4.

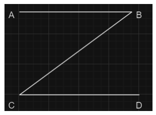
(c) False. Let A be the set of real numbers. Then $A \bigcup \mathbb{N} = A$ i.e. the set of real numbers itself. As done in class, there is no bijection between set of natural numbers and set of reals, hence given a counter example.

(d) False. Consider the graph on 4 vertices A, B, C, D and AB, BC, CD as the edges. Now,$\{BD\}$ and $\{AB, CD\}$ are both maximal matchings which however have different sizes.

(e) True, since weak induction implies strong induction.
Assume that weak induction holds, and you are given the strong induction hypotheses for some proposition $P(n)$. Define $Q(n) = \wedge_{i=0}^{n} P(i)$. $P(0) \implies Q(0)$, and since $\forall n \in \mathbb{N}$,
$P(0) \wedge P(1) \ldots \wedge P(n) \implies P(n+1)$, $Q(n) \implies Q(n) \wedge P(n+1) \implies Q(n+1)$. Now using weak induction we can claim $Q(n)$ ($\implies P(n)$) for all $n \in \mathbb{N}$.

2. (2*2.5=5 marks) Give examples for the following, if they exist. If not, explain why they don't exist with a short proof.

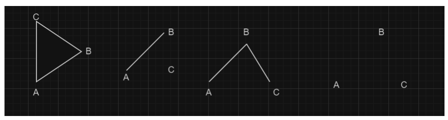
   (a) A graph G with at most 10 vertices and two matchings M, M' in G such that (i) there is an M-augmenting path in G and (ii) size of M' is strictly larger than size of M.

Solution. 

Consider the above graph, with $M = \{BC\}$ and $M' = \{AB, CD\}$. Then size of $M' = 2 >$ size of $M = 1$. Also A,B,C,D is an augmenting path for M. Also, the above graph is on 4 vertices which is indeed, ≤ 10

Grading Scheme 0.5 mark for drawing the graph , 1 for writing M, M' properly, 1 for giving the augmenting path correctly

(b) Draw all non-isomorphic graphs with 3 vertices.

Solution. 

Grading Scheme Number of components to marks mapping: 0:0, 1:1, 2:1.5, 3:2, 4:2.5

3. (5 marks) Prove or disprove: Every connected graph G with at least one edge has a closed walk in which every edge of G occurs exactly twice.

Solution. Double every edge: then every vertex certainly has even degree, so the graph is now Eulerian, and an Euler circuit on the new graph easily translates to a walk on G that traverses each edge exactly twice

Aliter:

We will induct on the number of edges (say, m).

Base Case: m=1, In this case the graph will have 2 vertices , say u, v and 1 edge uv, and thus the walk $u - v - u$ is closed with every edge occuring twice.

Induction Hypothesis: All connected graphs G with $m \leq k$ edges have a closed walk in which every edge occurs twice.

Induction step We will now use this to prove it for $m = k + 1$. Let uv be any one of the edges. Consider the graph $G' = G \setminus \{uv\}$ (that is the graph obtained by removing the edge uv from G).This leads to two cases:

(a) **Case 1-** G' is connected

In this case, as G' has k edges and is connected, we can use induction hypothesis to get a closed walk W starting at u,(and such that every edge occurs twice in G') , Then appending W with uvu gives a closed walk in G in which every edge occurs twice.

(b) **Case 2-** G' is not connected

Then G' will have two connected components C_1, C_2. WLOG let $u \in C_1, v \in C_2$. Then as C_1 has $\leq$ k edges, we can use induction hypothesis to get a closed walk W_1 (where each edge occurs twice) in C_1 starting at u. we can similarly get a closed walk W_2 in C_2 starting at v where each edge occurs twice. Then consider the closed walk W in G that starts at u and follows W_1,then goes to v via the edge uv and then follows W_2 starting at v, and the gets back to u via the edge uv. Clearly, W is closed (starts and ends at u) and covers every edge twice. (W_1, W_2 anyway do and the only other edge covered is uv, which is twice). Thus $W = uW_1u - vW_2v - u$ is a desired closed walk in this case.

This completes the induction step.

Grading Scheme full or nothing for the first solution. For the second proof, 1 mark for base case, 2 marks per case in the induction step.

For other solutions, full or partial may be given accordingly

Common Mistakes: Adding the (n+1)th edge/ vertex during Induction step. Wordily giving a description of going through a path and coming back without proving each edge is covered exactly twice.

4. (14+6=20 marks) Recurrences and counting

   (a) (5+6+3=14 marks) Consider a row of n numbered chairs, where a child is sitting on each chair. Each child may move by at most one seat (to left or right) or stay in place. Let a_n be the number of ways in which these children can be rearranged in the chairs (i.e., each chair has exactly one child, no chair is empty or has two children in it).

      i. Write a recurrence for a_n, along with its initial conditions.

      ii. Now, suppose the n chairs are placed in a circle (with each child in a chair as before). Let b_n be the number of ways they can be arranged now? Write a recurrence for b_n and solve it.

      iii. Using the recurrences, or otherwise, compute values of b_7 and a_8. You may assume any facts about Fibonacci, Catalan numbers etc that were taught in class.

Solution.

i. Let's look at the child on the last seat. If they don't switch with anyone, total number of rearrangements is a_{n-1}. If the last child and the second last child switch places, total number of rearrangements a_{n-2}. Note: these are the only valid movements for the last child. Thus, the recursion becomes $a_n = a_{n-1} + a_{n-2}$. Initial Conditions $a_1 = 1$, $a_2 = 2$. 4 marks for getting the correct equation, 1 mark for getting in terms of only b, 1 mark for solving.

ii. Choose the child on the last seat i.e. the seat numbered n (valid for $n \geq 3$).
1/2 mark each for the 3 below conditions, and 0.5 more for circular movement.
2 marks for getting the final recursion (in terms of b_n), with 1 mark cut if recursion is in terms of a_n.
2 marks for correctly solving it. Partial marks if b_1 and b_2 not explicitly mentioned, as they won't follow the same recursion(why?)

A. Remains at the same place (The problem reduced to row with n-1 chairs).
B. Swaps with the left child i.e. child on seat 1 (The problem reduced to row with n-2 chairs).
C. Swap with the right child i.e. child on seat n-1 (The problem reduced to row with n-2 chairs).

Apart from that everyone can move 1 seat left. Similarly, everyone can move 1 seat right. This is possible here due to the circular arrangement. As the chairs are numbers, these are also considered as different arrangements. So the recurrence for $b_n = a_{n-1} + 2$*$a_{n-2} + 2$. Let $c_n = b_n - 2 = a_{n-1} + 2 * a_{n-2}$

$$c_n = a_{n-2} + a_{n-3} + 2 * a_{n-3} + 2 * a_{n-4} \tag{1}$$
$$c_n = (a_{n-2} + 2 * a_{n-3}) + (a_{n-3} + 2 * a_{n-4}) \tag{2}$$
$$c_n = c_{n-1} + c_{n-2} \tag{3}$$
$$b_n - 2 = b_{n-1} - 2 + b_{n-2} - 2 \tag{4}$$

This is true for $n \geq 5$ as the recusion for b_n was true for $n \geq 3$. Thus c_n follows the fibonacci recursion. We have seen the general solution for this is $c_n = a * (\frac{1+\sqrt{5}}{2})^{n-3} + b * (\frac{1-\sqrt{5}}{2})^{n-3}$. We can take n-3 instead of n as the 3 power will be absolved into the constants. Taking n-3 is beneficial in finding the solution for a and b as the base cases will be n = 3 and n = 4. Taking the base cases $b_1 = 1$, $b_2 = 2$, $b_3 = 6$, $b_4 =$ i.e. $c_3 = 4$, $c_4 = 7$. Plugging these in the general solution, we get $c_n = (2+\sqrt{5}) * (\frac{1+\sqrt{5}}{2})^{n-3} + (2-\sqrt{5}) * (\frac{1-\sqrt{5}}{2})^{n-3}$ and $b_n = c_n + 2$. Therefore, final solution is $b_1 = 1$, $b_2 = 2$ and
$b_n = (2+\sqrt{5}) * (\frac{1+\sqrt{5}}{2})^{n-3} + (2-\sqrt{5}) * (\frac{1-\sqrt{5}}{2})^{n-3} + 2$ for $n \geq 3$.

iii. $b_7 = 31$ and $a_8 = 34$.
1 mark for correct answer and 0.5 for some calculation for both.

(b) (6 marks) Over the alphabet $S = \{a, b, c, d\}$, how many n-length strings are there, which have an even number of a's? Express your answer as a function of n, in as simple a form as possible.

Solution.

Recurrence Solution:

i. Let x_n be the number of strings of length n with even occurrence of a. $x_n = 3{*}x_{n-1} + y_{n-1}$.

ii. Let y_n be the number of strings of length n with odd occurrence of a. $y_n = 3{*}y_{n-1} + x_{n-1}$.

iii. So the recurrence relation is $x_n = 6{*}x_{n-1}$ - $8{*}x_{n-2}$

iv. Solving them with base case $x_1 = 3$, $x_2 = 10$, we get $(4^n + 2^n)/2$.

2 marks for Recurrence relation. 2 marks for Explanation. 2 marks for the Solution

Counting Solution:

i. The number of strings without a and b are 2^n. (Contains even number of a's)

ii. Out of remaining 4^n strings, the number of strings with even number of a's and odd number of a's are same. (Swapping the first occurrence of a or b gives the bijection function mapping from odd number of a's to even number of a's.

iii. So the number of strings with even number of a's is $(4^n - 2^n)/2 + 2^n = (4^n + 2^n)/2$.

6 marks if fully correct. Partial marks based on the approach

Binomial equation method:

i. Let the number of a's in the string be $x = \{0, 2, 4, 6, \ldots\}$. The number of strings with x a's is $\binom{n}{x} \cdot 3^{(n-x)}$.

ii. The total number of strings can be expressed as the sum

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \cdot 3^{(n-2k)}.$$

iii. Since we only need even values in x, it can be obtained by

$$\frac{(3+1)^n + (3-1)^n}{2},$$

which gives $\frac{4^n+2^n}{2}$.

6 marks if fully correct. -1 for not considering 0 a's in the string. Partial marks based on the approach if the odd length string case is wrongly done

Part B

5. (24 marks) A poset $(S, \preceq)$ is called a *well-partial order* if every infinite sequence of elements $x_0, x_1 \ldots$ from S contains at least one pair of elements x_i, x_j for $0 \leq i < j$ such that $x_i \preceq x_j$.

(a) Are the following posets well-partial orders? Why or why not? (2 + 3 marks)

(i) Natural numbers with the divisibility relation $(\mathbb{N}, |\ )$

(ii) Natural numbers with the usual ordering relation $(\mathbb{N}, \leq)$

(b) Prove or disprove: every infinite sequence of $(\mathbb{N}, \leq)$ has an *infinite* non-decreasing subsequence. Recall that a sequence $x_0, x_1 \dots$ is non-decreasing if $x_i \leq x_{i+1}$ for all $i \geq 0$. (5 marks)

(c) For any $k \in \mathbb{N}, k \geq 1$, consider the set $\mathbb{N}^k$ of vectors of k natural numbers and $\leq^k$ the component-wise ordering, defined by, $(a_1, \dots, a_k) \leq^k (b_1, \dots, b_k)$ if $a_i \leq b_i$ for all $1 \leq i \leq k$. (4+5 marks)

   i. Show that $(\mathbb{N}^k, \leq^k)$ is a poset. Is it a total order? Why or why not?

   ii. Show that $(\mathbb{N}^k, \leq^k)$ is a well-partial order.

(d) *Show that a poset is a well-partial order iff it has no infinite strictly decreasing sequence and no infinite anti-chain. Recall that an anti-chain is a set of elements in which any pair are incomparable. (5 marks)

Solution.

(a) i. **False**: Consider an infinite sequence of distinct primes $\{p_0, p_1, \dots\}$ (where $p_i \neq p_j$ when $i \neq j$). For any two indices $0 \leq i < j$, $p_i \nmid p_j$. So, there exists an infinite sequence for the given poset, for which the given property does not hold, implying the given poset is not a well-partial order.
1 mark for answer, 1 mark for proof

ii. **True**: Given any infinite sequence $\{x_0, x_1, \dots\}$, consider first $x_0 + 1$ numbers in the sequence after x_0: $x_1, x_2, \dots, x_{x_0+1}$. Either this finite sub sequence has an element $x \geq x_0$, in which case, we found the required pair and the sequence would satisfy the given property, OR all the elements in the sub sequence are less than x_0.By pigeon hole principle, some number must repeat in the sub sequence. Say it repeats at indices $i < j$. Then $x_i \leq x_j$, giving us the required indices, and hence the sequence satisfies the property. Therefore we can conclude that all the sequences satisfy the given property, and hence the given poset is well-partial order.
1 mark for answer, 2 mark for proof

(b) i. Consider any infinite sequence $s = (x_1, x_2, \dots)$. If for some x_i, $x_i > x_j$ for all $j > i$, then some $z \in \{0, \dots, x_i - 1\}$ occurs infinitely often in $(x_{i+1}, x_{i+2}, \dots)$, and we have our non-decreasing sub-sequence as $(z, z, \dots)$. Otherwise for all i, we have some $j > i$ such that $x_i \leq x_j$, which again allows us to construct the required sub-sequence.

ii. **Alternate Solution** Claim: In any infinite sequence of $(\mathbb{N}, \leq)$, the longest non-decreasing sequence is infinite. Let's assume the claim is false i.e. there exists an infinite sequence where the longest non-decreasing sequence is finite. Let's look at the last term of that sequence, say x_k. Now there are 2 possibilities, either there exists a number greater than or equal to x_k after it or all the numbers after it are less than it. Case 1) we can include said number in the given sequence to get a longer sequence, thus a contradiction. Case 2) as infinite numbers occur after x_k and all of them are less than it, by pigeon hole principle, at least one number will repeat infinitely. We'll choose the sequence of this number repeating infinitely to arrive at a longer sequence, thus a contradiction. Thus our assumption was false which implies our claim was true. Our claim proves the required statement.

5 marks for correct proof, partials given based on progress made

Solution.

(c) i. **Reflexive**: For any $x \in \mathbb{N}^k$, $x_i \leq x_i \forall 1 \leq i \leq k$. Therefore $x \preceq x$.

Anti Symmetric: Given any two $x \neq y \in \mathbb{N}^k$, such that $x \prec y$, we have $\forall i \leq k, x_i \leq y_i$ and $\exists i, x_i \neq y_i$. Hence $\exists i, x_i < y_i \implies \exists i, y_i \not\leq x_i \implies y \not\preceq x$.

Transitive:

Therefore, $(\mathbb{N}^k, \leq^k)$ is a poset. Note however that $(1, 2, 2, \ldots, 2)$ and $(2, 1, 1, \ldots, 1)$ are incomparable, and hence the given poset is not total for $k > 1$ (It is for $k = 1$).

0.5,1,2 based on how many of 3 properties from reflexive, transitive and antisymmetric proved. 1 mark for it is not a total order for all k, and 1 mark for counter-example. If proved for all k not a total order, 1 mark is cut as it is a total order for k = 0

ii. We shall prove by induction on k.

Base Case $k = 1$. In this case, the poset reduces to $(\mathbb{N}, \leq)$, which we have already show is a well-partial order.

Induction Step: Let us assume that the poset $(\mathbb{N}^k, \leq^k)$ is a well-partial order. We will then show that $(\mathbb{N}^{k+1}, \leq^{k+1})$ is a well-partial order.

Consider any infinite sequence be $x_0, x_1, \ldots$. Let each $x_i = (a_i^1, \ldots, a_i^{k+1})$.

Consider the infinite sequence of natural numbers $a_0^1, a_1^1, a_2^1, \ldots$ (essentially considering the first element from each of the x_i's). By (b), we know there exists an infinite non-decreasing sub sequence $a_{i_1}^1, a_{i_2}^1, \ldots$

Now consider the infinite sub sequence $x_{i_1}^1, x_{i_2}^1, \ldots$. Consider then the infinite sub sequence ${x'_{i_1}}^1, {x'_{i_2}}^1, \ldots$, where $x'_i = (a_i^2, \ldots, a_i^{k+1})$. This is a sub sequence over $\mathbb{N}^k$, and by induction hypothesis, $(\mathbb{N}^k, \leq^k)$ is a well-partial order. Hence, in this sequence, there exists indices $i < j$ such that $x'_i \preceq x'_j \implies \forall 2 \leq l \leq k+1, a_i^l \leq a_j^l$. Since the sub sequence was non decreasing for the first element of x_i's, we also have $a_i^1 \leq a_j^1$. Hence $x_i \preceq x_j$. Hence the given sequence satisfies the property. Hence $(\mathbb{N}^{k+1}, \leq^{k+1})$ is a well-partial order.

Therefore, by induction, $(\mathbb{N}^k, \leq^k)$ is a well-partial order for all $k \geq 1$.

5 marks for correct proof, partials given based on progress made. Constructive proof is also given marks.

(d) If an infinite strictly decreasing sequence or an infinite anti-chain existed in the poset, it would be a contradiction to the definition of a well partial order.

Now assume that the partial order is not *well*. Hence, there exists an $s = (x_1, x_2, \ldots)$ such that for all i, $\forall j > i, (x_i > x_j$ or x_i, x_j are incomparable).

Define $S = \{x_i \,|\, x_j, x_i \text{ are incomparable } \forall j > i\}$. First observe that S is an antichain, for if some $x_i, x_j \in S$ were comparable with $i < j$, then it would lead to the contradiction $x_i \notin S$. If S is finite, then we have some k (maximum index in S) such that $\forall i > k$, $\exists j > i, x_i > x_j$, and hence we can construct a strictly decreasing sub-sequence.

Otherwise S is an infinite anti-chain. Hence, we have proven that if some partial order is not *well* $\iff$ there is either an infinite strictly decreasing subsequence or an infinite anti-chain in the poset.

2 marks for forward and 3 marks for reverse.

6. (9 marks) A table with m rows and n columns is filled with non-negative integers such that each row and each column contains at least one positive integer. Moreover, if a row and a column intersect in a positive integer, then the sums of their integers are the same. Using Hall's theorem or otherwise, prove that $m = n$.

Solution.

- **Bipartite Graph Construction** : We first form a bipartite graph $\mathcal{G}$ where the **rows** correspond to the vertices on the left-hand-side (say $\mathbb{R}$) and the **columns** correspond to the vertices on the right hand side (say $\mathbb{C}$). Now, we consider an edge existing between a node in $\mathbb{R}$ and that in $\mathbb{C}$ if in the original table their corresponding entry is a positive value.

  + 2 marks for **correct** bipartite graph construction based on the problem statement. + 1 mark if student has constructed the graph partially.

- **Proof by Contradiction**: ***Assumption***: Now let us assume that a set $\mathbb{S}$ of rows exist where $\mathbb{S} \subseteq \mathbb{R}$ for which there only exist the total set $n(\mathbb{S})$ (where $n(\mathbb{S})$ indicates the neighbors of the vertices in $\mathbb{S}$) of columns such that $|n(\mathbb{S})| < |\mathbb{S}|$.

  Let us denote each row of $\mathbb{R}$ as $r_i \ \forall \ i \in [|\mathbb{R}|]$. Now let the sum of $|\mathbb{S}|$ rows be denoted as $s_1, s_2, \ldots, s_{|\mathbb{S}|}$. That is $\sum_{j \in r_i} r_{ij} = s_i$ (where i denotes the ith row r_i and j denotes the jth element in r_i).

  Using the property given, each of the $|n(\mathbb{S})|$ columns has sum equal to one of the s_i, since they intersect at a positive integer atleast. Therefore, the total sum of integers in the $\mathbb{S}$ rows, when calculated from column side is at most a sum of a subset of the s_i since every other entries outside are all nonnegative. But since all $s_i > 0$, this leads to a contradiction.

  Hence the total sum of integers in S rows when calculated from column's is at most a sum of a subset of s_i.

  +2 = + 1 marks for correct proof by contradiction proof, +1 mark marks for correctly using the property of equal sums across a particular row and col for positive interesection

- **Applying Hall's Theorem**: Using Hall's Theorem we can therefore conclude that there is a matching from the rows to columns which further implies that the number of columns is at least the number of rows. Again using symmetry we can conclude that the number of rows is atleast the number of columns, thereby we have the equality $m = n$.

  +2 marks for correctly applying Hall's Theorem

+3 marks for overall proof completion.
+1 or +2 marks if some point missing or incompletely stated. Whether +1 or +2 will depend on the validity of the proof.

7. (6+6=12 marks) Let $T(n)$ denote the maximum number of edges that a graph with n vertices can have, if it does not contain a triangle (i.e., a subgraph isomorphic to K_3).

   (a) Prove that for all $n \in \mathbb{N}$, $T(n) \leq \lfloor \frac{n^2}{4} \rfloor$.

   (b) If G is a graph with n vertices and at least $\lfloor \frac{n^2}{4} \rfloor + 1$ edges, show that it contains at least $\lfloor \frac{n}{2} \rfloor$ triangles.

Solution.

(a) **Grading Scheme** 2 marks for showing the maximality of x and partitioning the graph
2 marks for showing the triangle-free graph is bounded by the complete bipartite graph
2 marks for proving the maximality of bipartite graph
Showing the result by assuming a special case, such as the graph being bipartite, earns at most 2 marks

Number of different proofs are available for this simple, yet elegant theorem, first proposed by Willem Mantel.
Let G be an n-vertex triangle-free simple graph. Let x be a vertex of maximum degree, with $k = deg(x)$. Since G has no triangles, there are no edges among neighbours of x *(Why?)*. Hence summing the degrees of x and its non-neighbours counts at least one endpoint of every edge, i.e., $\sum_{v \notin N(x)} d(v) >= e(G)$. We sum over $n-k$ vertices (all the non-neighbours of x), each having degree at most k (as k is the maximum degree in the graph, according to our assumption). So, $e(G) <= (n-k)k$.
Since $(n-k)k$ counts the edges in complete bipartite graph $K_{n-k,k}$, we have now proved that $e(G)$ is bounded by the size of some complete bipartite subgraph (biclique) with n vertices. Moving a vertex of $K_{n-k,k}$ from the set of size $n-k$ gains $k-1$ edges and loses $n-k$ edges. The net gain is $2k-1-n$ which is positive for $2k > n+1$ and negative for $2k < n+1$. Thus $e(K_{n-k,k})$ is maximised when k is $\lfloor n/2 \rfloor$ or $\lceil n/2 \rceil$. The product is then $n^2/4$ for even n and $(n^2-1)/4$ for odd n. Thus $e(G) <= \lfloor n^2/4 \rfloor$.
(*Note:* You can proof the maximality of $K_{n-k,k}$ using simple calculus also. Although discrete approach is preferable as it restricts k to be an integer.)

Alternate Proof:
1 marks for showing base case
1 marks for the induction hypothesis
2 marks for giving the correct argument for picking the right vertices and
2 marks for completion of the induction step correctly
We proceed by induction on n, number of vertices in a graph.
Base cases: For $n = 1$ and $n = 2$ the result is trivial. (*Note* we need 2 base cases to start. Why?)
Induction hypothesis: Let the statement holds for all graphs of size $<= n-2$.
Induction Step: We will prove the statement for all triangle-free graphs of size $<= n$. Let G be any triangle-free graph with n vertices. Pick a pair of connected vertices x and y. Then $deg(x) + deg(y) <= n$. If the sum of the degrees of x and y is more than n, then they have a common vertex z, and so xyz is a triangle. Now, delete x and y from G to get G'. Then all the edges adjacent to them from G will decrease the number of edges in G' by at most $(n-1)$ *(Why?)*. By induction hypothesis, G' has at most $\lfloor (n-2)^2/4 \rfloor$ edges. Then number of edges in G is $<= \lfloor (n-2)^2/4 \rfloor + (n-1) = \lfloor n^2/4 \rfloor$.

(b) 1 mark for illustrating base cases.
1 mark for stating induction hypothesis correctly.
2 marks for picking right vertices and stating proper arguments about the neighbourhoods of the vertices and, 2 marks for stitching everything correctly and complete the induction step
We will prove the result by induction on n. For $n = 3$, a subgraph with three edges contains one triangle, as expected. Similarly, for $n = 4$ also the result is trivial to check.

Assume now that a graph with $n-2$ vertices and at least $\lfloor (n-2)^2/4 \rfloor + 1$ edges contains at least $\lfloor (n-2)/2 \rfloor$ triangles. We will prove the required result also holds for n. Suppose that we have a graph G on n vertices with $\lfloor n^2/4 \rfloor + 1 = \lfloor (n-2)^2/4 \rfloor + (n-1) + 1$ edges but with fewer than $\lfloor n/2 \rfloor$ triangles. Let x and y be two vertices which are joined by an edge but are not contained in a triangle. This is certainly possible, since $3(\lfloor n/2 \rfloor - 1) <= \lfloor n^2/4 \rfloor + 1$. Therefore, as usual $deg(x) + deg(y) <= n$ (proved in previous part). Moreover, the neighbourhoods $N(x)$ and $N(y)$ of x and y must be disjoint. We now know that the graph $G' = G - x, y$ contains at least $\lfloor (n-2)^2/4 \rfloor + 1$ edges. It must therefore contain at least $\lfloor (n-2)/2 \rfloor$ triangles. But the number of edges between $N(x)$ and $N(y)$ is at most $\lfloor (n-2)^2/4 \rfloor$. Therefore, one of $N(x)$ and $N(y)$ must contain an edge. This yields one further triangle and proves the result.

Common Mistakes:

1. for proving Mantel's Theorem, part(a), many students starts something like "Suppose the claim is true when $n = k$ and add a new vertex to form a triangle-free graph with $k + 1$ vertices," and then completes the prove somehow.
   This argument is wrong, because by doing so you are not considering **all** triangle-free graphs with $k + 1$ vertices, but only the extremal $k-$vertex graph. This graph does appears in the extremal graph with $k + 1$ vertices, but we cannot use that fact before proving it, It *may* possible that the largest example with $k + 1$ vertices arises by adding a new vertex of high degree to a non-maximal example with k vertices.
   This is commonly known as **Induction Trap**. For further reading about this common mistake refer DB West Introduction to Graph Theory (2nd Edition) chapter 1 section 1.3.24 , 1.3.25 and 1.3.26.

2. for part(b) Assuming the graph structure as a complete bipartite graph, specifically $K_{n/2,n/2}$, along with the addition of one more edge in one partition, without a substantial proof using induction or alternative methods, merits only 1 mark.