MA 105 D3 Lecture 4

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Recap

Odds and ends about limits of functions of a real variable

Continuity

More about continuous functions

The rigourous definition of a limit of a function

Definition: A function $f : (a, b) \to \mathbb{R}$ is said to tend to (or converge to) a limit l at a point $x_0 \in [a, b]$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - l| < \epsilon$$

for all $x \in (a, b)$ such that $0 < |x - x_0| < \delta$. In this case, we write

$$\lim_{x \to x_0} f(x) = l,$$

or $f(x) \to l$ as $x \to x_0$ which we read as "$f(x)$" tends to l as x tends to x_0".

The the limit of a function may exist even if the function is not defined at that point because x_0 can be a or b.

Rules for limits

If $\lim_{x\to x_0} f(x) = l_1$ and $\lim_{x\to x_0} g(x) = l_2$, then

1. $\lim_{x\to x_0} f(x) \pm g(x) = l_1 \pm l_2$.
2. $\lim_{x\to x_0} f(x)g(x) = l_1 l_2$.
3. $\lim_{x\to x_0} f(x)/g(x) = l_1/l_2$. provided $l_2 \neq 0$

Theorem 5: As $x \to x_0$, if $f(x) \to l_1$, $g(x) \to l_2$ and $h(x) \to l_3$ for functions f, g, h on some interval (a, b) such that $f(x) \leq g(x) \leq h(x)$ for all $x \in (a, b)$, then

$$l_1 \leq l_2 \leq l_3.$$

As before, we have a second version.

Theorem 6: Suppose $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} h(x) = l$ and If $g(x)$ is a function satisfying $f(x) \leq g(x) \leq h(x)$ for all $x \in (a, b)$, then $g(x)$ converges to a limit as $x \to x_0$ and

$$\lim_{x\to x_0} g(x) = l$$

Limits at infinity

There is one further case of limits that we need to consider. This occurs when we consider functions defined on open intervals of the form $(-\infty, b)$, (a, ∞) or $(-\infty, \infty) = \mathbb{R}$ and we wish to define limits as the variable goes to plus or minus infinity. The definition here is very similar to the definition we gave for sequences. Let us consider the last case.

Definition: We say that $f : \mathbb{R} \to \mathbb{R}$ tends to a limit l as $x \to \infty$ (resp. $x \to -\infty$) if for all $\epsilon > 0$ there exists $X \in \mathbb{R}$ such that

$$|f(x) - l| < \epsilon,$$

whenever $x > X$ (resp. $x < X$), and we write

$$\lim_{x\to\infty} f(x) = l \quad \text{or} \quad \lim_{x\to-\infty} f(x) = l.$$

or, alternatively, $f(x) \to l$ as $x \to \infty$ or as $x \to -\infty$, depending on which case we are considering.

Limits from the left and right

If $f : (a, b) \to \mathbb{R}$ is a function and $c \in (a, b)$, then it is possible to approach c from either the left or the right on the real line.

We can define the limit of the function $f(x)$ as x approaches c from the left (if it exists) as a number l^- such that for all $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - l^-| < \epsilon$ whenever $|x - c| < \delta$ and $x \in (a, c)$.

Our notation for this is $\lim_{x \to c-} f(x) = l^-$, and it is also called the left hand (side) limit.

Exercise 2: Write down a definition for the limit of a function from the right. We usually denote the right hand (side) limit by $lim_{x \to c+} f(x)$. Show, using the definitions, that $\lim_{x \to c} f(x)$ exists if and only if the left hand and right hand limits both exist and are equal.

We can also think of the left hand limit as follows. We restrict our attention to the interval (a, c), that is we think of f as a function only on this interval. Call this restricted function f_a. Then, another way of defining the left hand limit is

$$\lim_{x \to c-} f(x) = \lim_{x \to c} f_a(x).$$

It should be easy to see that it is the same as the definition before One can make a similar definition for the right hand limit.

The notions of left and right hand limits are useful because sometimes a function is defined in different ways to the left and right of a particular point. For instance, $|x|$ has different definitions to the left and right of 0.

Calculating limits explicitly

As with sequences, using the rules for limits of functions together with the Sandwich theorem allows one to treat the limits of a large number of expressions once one knows a few basic ones:

(i) $\lim_{x\to 0} x^\alpha = 0$ if $\alpha > 0$, (ii) $\lim_{x\to\infty} x^\alpha = 0$ if $\alpha < 0$,
(iii) $\lim_{x\to 0} \sin x = 0$, (iv) $\lim_{x\to 0} \sin x/x = 1$
(v) $\lim_{x\to 0}(e^x - 1)/x = 1$, (vi) $\lim_{x\to 0} \ln(1+x)/x = 1$
We have not concentrated on trying to find limits of complicated expressions of functions using clever algebraic manipulations or other techniques. However, I can't resist mentioning the following problem.
Exercise 3: Find

$$\lim_{x\to 0} \frac{\sin(\tan x) - \tan(\sin x)}{\arcsin(\arctan x) - \arctan(\arcsin x)}.$$

I will give the solution next time, together with the history of the problem. Six years ago, one needed to know a couple of keywords in order to get the solution through google. But as of three years ago, just typing in the formula above into google will lead you to the solution.

Continuity - the definition

Definition: If $f : [a, b] \to \mathbb{R}$ is a function and $c \in [a, b]$, then f is said to be continuous at the point c if and only if

$$\lim_{x \to c} f(x) = f(c).$$

Thus, if c is one of the end points we require only the left or right hand limit to exist.

A function f on (a, b) (resp. $[a, b]$) is said to be continuous if and only if it is continuous at every point c in (a, b) (resp. $[a, b]$).

If f is not continuous at a point c we say that it is dicontinuous at c, or that c is a point of discontinuity for f.

Intuitively, continuous functions are functions whose graphs can be drawn on a sheet of paper without lifting the pencil of the sheet of paper. That is, there should be no "jumps" in the graph of the function.

Continuity of familiar functions: polynomials

What are the functions we really know or understand? What does "knowing" or understanding a function $f(x)$ even mean? Presumably, if we understand a function f, we should be able to calculate the value of the function $f(x)$ at any given point x. But if you think about it, for what functions $f(x)$ can you really do this?

One class of functions is the polynomial functions. More generally we can understand rational functions, that is functions of the form $R(x) = P(x)/Q(x)$ where $P(x)$ and $Q(x)$ are polynomials, since we can certainly compute the values of $R(x)$ by plugging in the value of x. How do we show that polynomials or rational functions are continuous (on $\mathbb{R}$)?

It is trivial to show from the definition that the constant functions and the function $f(x) = x$ are continuous. Because of the rules for limits of functions, the sum, difference, product and quotient (with non-zero denominator) of continuous functions are continuous. Applying this fact we see easily that $R(x)$ is continuous whenever the denominator is non-zero.

Continuity of other familiar functions

What are the other (continuous) functions we know? How about the trignometric functions? Well, here it is less clear how to proceed. After all we can only calculate $\sin x$ for a few special values of x ($x = 0, \pi/6, \pi/4, \ldots$ etc.). How can we show continuity when we don't even know how to compute the function?

Of course, if we define $\sin x$ as the y-coordinate of a point on the unit circle it seems intuitively clear that the y-coordinate varies continuously as the point varies on the unit circle, but knowing the precise definition of continuity this argument should not satisfy you.

We will not prove the continuity of $\sin x$ in this course, though we will given an idea of how this is done next week. So let us assume from now on that $\sin x$ is continuous. How can we show that $\cos x$ is continuous?

The composition of continuous functions

Theorem 8: Let $f : (a, b) \to (c, d)$ and $g : (c, d) \to (e, f)$ be functions such that f is continuous at x_0 in (a, b) and g is continuous at $f(x_0) = y_0$ in (c, d). Then the function $g(f(x))$ (also written as $g \circ f(x)$ sometimes) is continuous at x_0. So the composition of continuous functions is continuous.

Exercise 4: Prove the theorem above starting from the definition of continuity.

Using the theorem above we can show that $\cos x$ is continuous if we show that $\sqrt{x}$ is continuous, since $\cos x = \sqrt{1 - \sin^2 x}$ and we know that $1 - \sin^2 x$ is continuous since it is the product of the sums of two continuous functions ($(1 + \sin x)$ and $(1 - \sin x)$!).

Once we have the continuity of $\cos x$ we get the continuity of all the rational trignometric functions, that is functions of the form $P(x)/Q(x)$, where P and Q are polynomials in $\sin x$ and $\cos x$, provided $Q(x)$ is not zero.

The continuity of the square root function

Thus in order to prove the continuity of $\cos x$ (assuming the continuity of $\sin x$) we need only prove the continuity of the square root function.

The main observation is that continuity is a local property, that is, only the behaviour of the function near the point being investigated is important.

Let $x_0 \in [0, \infty)$. To show that the square root function is continuous at x_0 we need to show that $\lim_{y \to x_0} \sqrt{y} = \sqrt{x_0}$, that is we need to show that $|\sqrt{y} - \sqrt{x_0}| < \epsilon$ whenever $0 < |y - x_0| < \delta$ for some δ. First assume that $x_0 \neq 0$. Then

$$|\sqrt{y} - \sqrt{x_0}| = \left| \frac{y - x_0}{\sqrt{y} + \sqrt{x_0}} \right| < \frac{|y - x_0|}{\sqrt{x_0}}.$$

If we choose $\delta = \epsilon\sqrt{x_0}$, we see that

$$|\sqrt{y} - \sqrt{x_0}| < \epsilon,$$

which is what we needed to prove ($x_0 = 0$, exercise!). $\square$

The intermediate value theorem

One of the most important properties of continuous functions is the Intermediate Value Property (IVP). We will use this property repeatedly to prove other results.

Theorem 9: Suppose $f : [a, b] \to \mathbb{R}$ is a continuous function. For every u between $f(a)$ and $f(b)$ there exists $c \in [a, b]$ there such that $f(c) = u$.

Functions which have this property are said to have the Intermediate Value Property. Theorem 9 can thus be restated as saying that continuous functions have the IVP.

We will not be proving this property - it is a consequence of the completeness of the real numbers. Intuitively, this is clear. Since one can draw the graph of the function without lifting one's pencil off the sheet of paper, the pencil must cut every line $y = e$ with e between $f(a)$ and $f(b)$.

The IVT in a picture

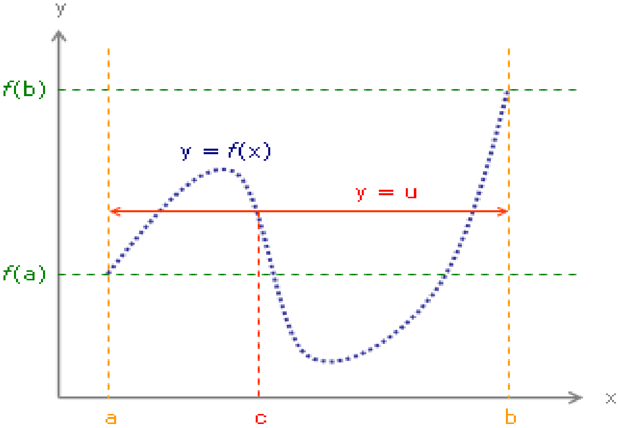

Image created by Enoch Lau see
http://en.wikipedia.org/wiki/File:Intermediatevaluetheorem.png

Zeros of functions

One of the most useful applications of the intermediate value property is to find roots of polynomials, or, more generally, to find zeros of continuous functions, that is to find points $x \in \mathbb{R}$ such that $f(x) = 0$.

Theorem 10: Every polynomial of odd degree has at least one real root.

Proof: Let $P(x) = a_n x^n + \ldots + a_0$ be a polynomial of odd degree. We can assume without loss of generality that $a_n > 0$. It is easy to see that if we take $x = b > 0$ large enough, $P(b)$ will be positive. On the other hand, by taking $x = a < 0$ small enough, we can ensure that $P(a) < 0$. Since $P(x)$ is continuous, it has the IVP, so there must be a point $x_0 \in (a, b)$ such that $f(x_0) = 0$. $\square$

The IVP can often be used to get more specific information. For instance, it is not hard to see that the polynomial $x^4 - 2x^3 + x^2 + x - 3$ has a root that lies between 1 and 2.

Continuous functions on closed, bounded intervals

The other major result on continuous functions that we need is the following. A closed bounded interval is one of the form $[a, b]$, where $-\infty < a$ and $b < \infty$.

Theorem 11: A continuous function on a closed bounded interval $[a, b]$ is bounded and attains its infimum and supremum, that is, there are points x_1 and x_2 in $[a, b]$ such that $f(x_1) = m$ and $f(x_2) = M$, where m and M denote the infimum and supremum respectively.

We defined infimum and supremum for sequences previously. The definition for functions of a real variable is the same: Let $X \subset \mathbb{R}$ and let $f : X \to \mathbb{R}$ be a function. A real number M is called the supremum of $f(x)$ (on X) if
1. If $f(x) \leq M$ for all $x \in X$.
2. If for some real number M_1 $f(x) \leq M_1$ for all $x \in X$, then $M \leq M_1$.

Relaxing the conditions

Again, we will not prove Theorem 11, but will use it quite often. Note the contrast with open intervals. The function $1/x$ on $(0, 1)$ does not attain a maximum - in fact it is unbounded. Similarly the function $1/x$ on $(1, \infty)$ does not attain a minimum, although, it is bounded below and the infimum is 0.

Exercise 5: In light of the above theorem, can you find a continuous function $g : (a, b) \to \mathbb{R}$ for part (i) of Exercise 1.11, with $c \in (a, b)$?

The function $\sin \frac{1}{x}$

Let us look at Exercise 1.13 part (i).

Consider the function defined as $f(x) = \sin \frac{1}{x}$ when $x \neq 0$, and $f(0) = 0$. Is this function is continuous at $x = 0$.

How about $x \neq 0$? Why is $f(x)$ continuous? Because it is a compostion continuous functions (sine and $1/x$).

Let us look at the sequence of points $x_n = 2/(2n+1)\pi$.

Clearly $x_n \to 0$ as $n \to \infty$.

For these points $f(x_n) = \pm 1$. This means that no matter how small I take my δ, there will be a point $x_n \in (0, \delta)$, such that $|f(x_n)| = 1$. But this means that $|f(x) - f(0)| = |f(x)|$ cannot be made smaller than 1 no matter how small δ may be. Hence, f is not continuous at 0. The same kind of argument will show that there is no value that we can assign $f(0)$ to make the function $f(x)$ continuous at 0.

You can easily check that $f(x)$ has the IVP. However, we have proved that it is not continuous. So IVP $\not\Rightarrow$ continuity.