Quiz 1: DIC on Discrete Structures

Solutions + Grading Scheme

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1. [2 marks] Prove that, if a, b are both prime numbers (> 1), and $a \neq b$, then $\log_b a$ is irrational.

Solution. Proof by contradiction.

- [1 mark] Let a, b be distinct primes numbers > 1. Suppose that $\log_b a$ was rational, i.e., $\log_b a = \frac{p}{q}$ for some $p, q \in \mathbb{N}$, $q \neq 0$.
- [0.5 marks] Then, $a = b^{\frac{p}{q}}$, which gives $a^q = b^p$.
- [0.5 marks] By fundamental theorem of arithmetic, this is possible only if $a = b$, since a and b are primes. [(Or any equivalent valid justification)]
  This is a contradiction.

Alternate way: Proof by contra-positive. Contrapositive of the statement is **If** $\log_b a$ **is rational, a and b are not BOTH prime or a = b. OR If** $\log_b a$ **is rational, either a or b is not prime or a = b.** Proof same as above.

2. [2 marks] Answer True or False. Also give a 1-line justification for each.

(a) Every non-empty subset of integers has a smallest element.

(b) If a proposition is true, then its converse must also be true.

Solution

(a) [0.5 marks] False.
[0.5 marks Valid justification needed. Stating this is not WOP is not enough, as that is not a justification!] E.g., consider the set of all integers divisible by 2, $S = \{\ldots, -4, -2, 0, 2, 4, 6, \ldots\}$. There is no smallest element in S.

(b) [0.5 marks] False.
[0.5 marks. Valid justification needed. Either counterexample or from truth table]
Eg., the proposition is true and converse is false in case of p = False and q = True. $p \implies q$ is True but $q \implies p$ is False.
For E.g., Counter example: If $x \geq 0, x^2 \geq 0$. Converse is not true. If $x^2 \geq 0, x \geq 0$. $x^2 = 25$, x can be -5 also

3. [3 marks] Prove by induction that for all $n \in \mathbb{N}$, 3^n is odd.

Solution.

- [1 mark] **Base case**: $n = 0$, $3^0 = 1$ is odd. [(-0.5 if base case starts with n=1)]
- [2 marks for remaining]
  Induction Hypothesis: Assume for some $k \geq 0$, 3^k is odd, i.e, $3^k = 2l + 1$, for some $l \geq 0$.
- **Induction step**: To show that 3^{k+1} is odd, i.e., to show that $3^{k+1} = 2l' + 1$, for some $l' \geq 0$.
  $3^{k+1} = 3 \cdot 3^k = 3 \cdot (2l + 1) = 6l + 3 = 6l + 2 + 1 = 2(3l + 1) + 1$
  Thus, $3^{k+1} = 2l' + 1$ where $l' = 3l + 1$.
- Hence by induction the statement is true for all $n \in \mathbb{N}$.

4. [4 marks] Prove or disprove: For any two sets A, B, there exists a surjection from A to B iff there exists an injection from B to A.

Solution **Note:** We need to assume B to be non-empty, since otherwise if A is non-empty then there exists an injection from B to A but no surjection from A to B (vacuously, since there exists no function from A to B). But, even if you haven't pointed out this corner case, and just proved the statement assuming $B \neq \emptyset$, you will get full marks.

Proof of the if part

Let f be an injection from B to A. Since $B \neq \emptyset$, pick an element b_0 from B. Then we would explicitly construct a surjection g from A to B as follows:

$$g(a) = \left\{ \begin{array}{ll} x \text{ s.t. } f(x) = a & \text{if } \exists x \in B \text{ s.t. } f(x) = a \\ b_0 & \text{otherwise} \end{array} \right\} \tag{1}$$

g is well defined since if a belongs to the image of f then its preimage will be a singleton because of the injectivity of f and thus $g(a) = f^{-1}(a)$ is well defined. And if a doesn't belong to the image of f then, $g(a) = b_0$ for a fixed b_0 is also well defined. Finally to prove the surjectivity of g, Let b_1 be an arbitrary element of B, then by the definition of g, we have $g(f(b_1)) = b_1$ and hence b_1 belongs to the image of g. Since we showed this for an arbitrary $b_1 \in B$, g is indeed surjective and we are done.

Proof of the only if part

Let s be a surjection from A to B. We will now explicitly construct an injection h from B to A

$$h(b) = \text{ any one (fixed) element of } s^{-1}(b)$$

h is well defined since $\forall b \in B, s^{-1}(b)$ is non-empty because of the surjectivity of s. Now for showing injectivity of h, assume to the contrary that $\exists a_1 \neq a_2 \in A$ such that $h(a_1) = h(a_2) = b'$(say),which means $b' \in s^{-1}(a_1)$ and $b' \in s^{-1}(a_2)$ and hence $s(b') = a_1 \neq a_2 = s(b')$ which conntradicts the well-definedness of s. Hence our assumption that $a_1 \neq a_2$ must be wrong and hence we have shown that $\forall a_1, a_2 \in A$ $h(a_1) = h(a_2) \implies a_1 = a_2$, which is precisely what h being an injection means, and hence h is injective and we are done.

[**Grading Scheme-** 2 marks for each of the parts involved. For each part, 1 mark for the construction and 1/2 for injection/surjection justification and 1/2 for well-definedness justification]

5. Which of the following sets are countable? Justify with formal proof. You may assume that countable union of countable sets is countable.

   (a) [4 marks] Set of all functions from $\mathbb{N}$ to $\mathbb{N}$.

   (b) [5 marks] Set of all non-increasing functions from $\mathbb{N}$ to $\mathbb{N}$. A function $f : \mathbb{N} \to \mathbb{N}$ is said to be non-increasing if for all $x, y \in \mathbb{N}$, if $x \leq y$ then $f(x) \geq f(y)$.

Solution.

(a) [**(0.5 marks)**] The set of all functions from $\mathbb{N}$ to $\mathbb{N}$ is **uncountable**.

[**(3.5 marks for correct and formal justification)**]

Method 1: Cantor's Diagonalization

[**(Deducting 0.5 marks for each error in the proof)**]

[**(1 mark, if Cantor diagonalisation mentioned but nothing else shown)**]

[**(2.5 marks for constructing the new function to arrive at contradiction correctly)**]

Let us assume on the contrary that the set of all functions from $\mathbb{N}$ to $\mathbb{N}$ be countable. Let this set be called $S = \{f | f : \mathbb{N} \to \mathbb{N}\}$. Then there exists a bijection g from $\mathbb{N}$ to S. Let $f_i = g(i) \quad \forall i \in \mathbb{N}$ (just for ease of notation). Construct a new function $f' : \mathbb{N} \to \mathbb{N}$:

$$f'(k) = f_k(k) + 1 \quad \forall k \in \mathbb{N}$$

Claim: $f' \notin S$

Proof: Assume on the contrary that $f' \in S$. Then $\exists i \in \mathbb{N}$ such that $f_i = f'$ (since g is surjective and hence any function $f \in S$ has a pre image $i \in \mathbb{N}$). But the two functions disagree at the input i ($f'(i) = f_i(i) + 1 \neq f_i(i)$). Hence $f_i \neq f'$. Hence by contradiction, $f' \notin S$.

Hence we constructed a new function $f' : \mathbb{N} \to \mathbb{N}$ which does not belong to S. But this is a contradiction since S is the set of all functions from $\mathbb{N} \to \mathbb{N}$.

Hence by contradiction, we can say S is uncountable.

Method 2: Constructing a surjection from S to $\mathcal{P}(\mathbb{N})$

[**(Deducting 0.5 marks for each error in the proof)**]

[**(0 mark if only shown as a subset of $\mathcal{P}(\mathbb{N} \times \mathbb{N})$ or any similar argument)**]

We shall construct a function $g : S \to \mathcal{P}(\mathbb{N})$:

$$g(f) = \{i \in \mathbb{N} \quad | \quad f(i) = 1\}$$

This is essentially a subset of $\mathbb{N}$ consisting of all numbers where f takes value 1.

Claim: The function g is surjective.

Proof: Consider any subset $X \subseteq \mathbb{N}$. Then one of its pre image is the function f:

$$f(i) = \begin{cases} 1 & \text{if } i \in X \\ 0 & \text{Otherwise} \end{cases}$$

It is easy to verify that f is indeed a pre image of X. Hence for all subsets $X \subseteq \mathbb{N}$, we can find a pre image. This implies g is surjective.

Claim: There exists no bijection from S to $\mathbb{N}$

Proof: Assume on the contrary that there exists a bijection $h : \mathbb{N} \to S$. Then the composition $f' = g \circ h$ is a surjection from $\mathbb{N}$ to $\mathcal{P}(\mathbb{N})$. Consider the injective function $f'' : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ as $f(k) = \{k\} \quad \forall k \in \mathbb{N}$. But this implies that there is a bijection from natural numbers to its power set! Hence by contradiction we can say there is no bijection from $\mathbb{N}$ to S.

Hence the set S is uncountable.

(b) [(0.5)] The set of all non-increasing functions from N to N is countable.

Method 1: Using Schroeder Bernstein Theorem from set of all non-increasing functions to the set of all natural numbers

[(1)] Surjection: It's an infinite set so there exists a surjection from the set of all non-increasing functions to the set of all natural numbers.

Injection: Let p0, p1, p2, ... be a fixed enumeration of the prime numbers (Possible as prime numbers are countable).

[(2)] Claim: There exists a_i, such that the function is a constant after that.

Construct a set $S = \{x | f(x) < f(x-1) \text{ and } x > 0\}$. This set will be finite as the function is non increasing and the first element, $f(0)$ is finite. If the set is empty, it means the function is a constant function and $a_i = 0$. Otherwise, there must exist a maximum element in the set S, say m, and $a_i = m$. *(Can prove $a_i = m$ via contradiction)*

[(1)] Just map the function to the product, $\prod_{j=0}^{a_i} pj^{f(j)+1}$. This will be an injection as prime factorization uniquely determines the function (Fundamental Theorem of Arithmetic). So 2 functions cannot map to the same number.

[(0.5)] Thus, using schroeder Bernstein theorem, the set S has the same cardinality as the set N. Thus S is countable.

[Maximum 3 Marks will be given if you have taken product of "infinite" primes.]

Infinite primes is an issue as fundamental thoerem of arithmetic states that any natural number can be written as a product of finite primes. Thus, product of infinite primes is not a natural number!!

Method 2: [(1)] Let S be the set of all non increasing functions from N to N and $G(K)$ is the set of all non-increasing functions from N to N such that

$$\forall_{f \in G(K)}, f(0) = k$$

Thus

$$S = \bigcup_{k \in N} G(k)$$

Claim: $\forall_{K \in N}, G(K)$ is countable.

[3] Base Case: For $k = 0, G(k)$ contains only one function which is the constant function i.e. $\forall_{x \in N} f(x) = 0$. Thus, $G(0)$ is countable.

Induction Hypothesis: For some k, $\forall_{b<k} G(b)$ is countable.

Induction Step: Let $C(k)$ denote the constant function i.e. $\forall_{x \in N} f(x) = k$.

For any other function in G(K), there will be a smallest element, say x, such that $G(x) = \alpha < k$ (as the function is non-increasing) *(Use well ordering for this)*.

Claim: Total number of functions such that $f(x) = \alpha$ is $G(\alpha)$.

Say f is a function such that

$$f(i) = \begin{cases} k & if\ i < x \\ g(i-x) & \text{Otherwise} \end{cases}$$

We can see $g(x)$ is a non increasing function which belongs to the set $G(\alpha)$ as $g(0) = \alpha$. An f can be uniquely defined by a g and similarly an f can uniquely define a g. Thus, this is a bijection from the set of functions with $f(0) = k$ and $f(x) = \alpha$ with x being the smallest l for which $f(l) \neq k$

Thus, $G(K)$ can be partitioned over the pair (x, α) where $x \in N$ and $\alpha \in \{0, 1, ..., k-1\}$. Therefore,

$$G(k) = C(k) \cup \bigcup_{x \in N} \bigcup_{\alpha=0}^{k-1} G(\alpha)$$

Our induction hypothesis states $G(\alpha)$ is countable in the above limits. Also, it's given that countable union of countable sets is also countable. Thus using the above statements, we can conclude $G(k)$ is countable.

Therefore using strong induction, we can claim $\forall_{k \in N}, G(K)$ is countable.

[0.5] Thus finally, again using the statement that the countable union of countable sets is countable, we can conclude our set S of all non-increasing functions from N to N is countable.

Method 3:

[2] Claim: There exists a_i, such that the function is a constant after that.

Construct a set $S = \{x | f(x) < f(x-1) \text{ and } x > 0\}$. This set will be finite as the function is non increasing and the first element, $f(0)$ is finite. If the set is empty, it means the function is a constant function and $a_i = 0$. Otherwise, there must exist a maximum element in the set S, say m, and $a_i = m$. *(Can prove $a_i = m$ via contradiction)*

[1.5] Now we can divide the total set of functions based on 3 parameters vis-a-vis $f(0)$, a_i and $f(a_i)$. Call this $S(n, m, c)$. n is $f(0)$, m is a_i and c is $f(a_i)$ Now we just need to decide the function values for the set $\{1, 2, ..., m-1\}$ to uniquely determine the function. Note, these can only map to values from the set $\{c+1, ..., n\}$. Total number of such functions is $(m-1)^{n-c}$. Non-increasing functions will be a subset of these. As the total number is finite, $S(n, m, c)$ will also be finite.

Now we can just take union over n, m and c to get S, our desired set.

$$S = \bigcup_{n \in N} \bigcup_{m \in N} \bigcup_{c \leq n} S(n, m, c)$$

[1] As given, countable union of countable sets is countable. Thus, the countable union of $S(n, m, c)$ (finite) is countable and hence the total number of non-increasing functions from N to N are finite.

Note: There can be many more solutions with little changes here and there in the above solutions.