MA 105 D3 Lecture 14

Ravi Raghunathan

Department of Mathematics

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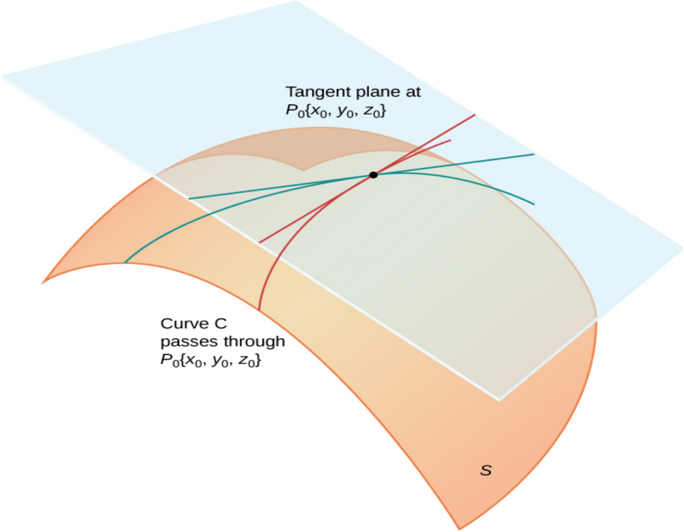
The tangent plane in a picture



https://openstax.org/books/calculus-volume-3/pages/4-4-tangent-planes-and-linear-approximations

The tangent plane

Let $f(x, y)$ be a function which has both partial derivatives. In the two variable case we need to look at the distance between the surface $z = f(x, y)$ and its tangent plane.

Let us first recall how to find the equation of a plane passing through the point $P = (x_0, y_0, z_0)$. It is the graph of the function

$$z = g(x, y) = z_0 + a(x - x_0) + b(y - y_0).$$

Let us determine the tangent plane to $z = f(x, y)$ passing through a point $P = (x_0, y_0, z_0)$ *on the surface*. In other words, we have to determine the constants a and b.

If we fix the y variable and treat $f(x,y)$ only as a function of x, we get a curve. Similarly, if we treat $g(x,y)$ as function only of x, we obtain a line. The tangent to the curve must be the same as the line passing through (x_0, y_0, z_0), and, in any event, their slopes must be the same. Thus, we must have

$$\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial g}{\partial x}(x_0, y_0) = a.$$

Arguing in exactly the same way, but fixing the x variable and varying the y variable we obtain

$$\frac{\partial f}{\partial y}(x_0, y_0) = \frac{\partial g}{\partial y}(x_0, y_0) = b.$$

Hence, the equation of the tangent plane to $z = f(x,y)$ at the point (x_0, y_0) is

$$z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

The tangent plane to the sphere

Exercise: Find the equation of the tangent plane to the hemisphere $z = f(x, y) = \sqrt{1 - x^2 - y^2}$ at a point (x_0, y_0).

Solution: The partial derivatives are

Differentiability for functions of two variables

We now define differentiability for functions of two variables by imitating the one variable definition, but using the "$o(h)$" version.
We let $(x, y) = (x_0, y_0) + (h, k) = (x_0 + h, y_0 + k)$
Definition A function $f : U \to \mathbb{R}$ is said to be differentiable at a point (x_0, y_0) if $\frac{\partial f}{\partial x}(x_0, y_0)$, and $\frac{\partial f}{\partial y}(x_0, y_0)$ exist and

$$\lim_{(h,k)\to 0} \frac{\left| f(x_0 + h, y_0 + k) - f(x_0, y_0) - \frac{\partial f}{\partial x}(x_0, y_0)h - \frac{\partial f}{\partial y}(x_0, y_0)k \right|}{\|(h, k)\|} = 0,$$

This is saying that the distance between the tangent plane and the surface is going to zero even after dividing by $\|(h, k)\|$. We could rewrite this as

$$\left| f((x_0, y_0) + (h, k)) - f(x_0, y_0) - \frac{\partial f}{\partial x}(x_0, y_0)h - \frac{\partial f}{\partial y}(x_0, y_0)k \right| = \varepsilon(h, k)\|(h, k)\|$$

where $\varepsilon(h, k)$ is a function that goes to 0 as $\|(h, k)\| \to 0$.This form of differentiability now looks exactly like the one variable version case (put $o(h, k) = \varepsilon(h, k)\|(h, k)\|$).

The derivative as a linear map

We can rewrite the differentiability criterion once more as follows. We define the 1×2 matrix

$$Df(x_0, y_0) = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \end{pmatrix}.$$

A 1×2 matrix can be multiplied by a column vector (which is 2×1 matrix) to give a real number. In particular:

$$\begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} = \frac{\partial f}{\partial x}(x_0, y_0)h + \frac{\partial f}{\partial y}(x_0, y_0)k$$

that is,

$$Df(x_0, y_0) \begin{pmatrix} h \\ k \end{pmatrix} = \frac{\partial f}{\partial x}(x_0, y_0)h + \frac{\partial f}{\partial y}(x_0, y_0)k$$

The definition of differentiability can thus be reformulated using matrix notation.

Definition: The function $f(x, y)$ is said be differentiable at a point (x_0, y_0) if there exists a matrix denoted $Df((x_0, y_0))$ with the property that

$$f((x_0, y_0) + (h, k)) - f(x_0, y_0) - Df(x_0, y_0) \begin{pmatrix} h \\ k \end{pmatrix} = p(h, k)\|(h, k)\|,$$

for some function $p(h, k)$ which goes to zero as (h, k) goes to zero. Viewing the derivative as a matrix allows us to view it as a linear map from $\mathbb{R}^2 \to \mathbb{R}$. Given a 1×2 matrix A and two column vectors v and w, we see that

$$A \cdot (v + w) = A \cdot v + A \cdot w \quad \text{and} \quad A \cdot (\lambda v) = \lambda(A \cdot v),$$

for any real number λ. As we have seen before, functions satisfying the above two properties are called linear functions or linear maps. Thus, the map $v \to A \cdot v$ gives a linear map from $\mathbb{R}^2$ to $\mathbb{R}$.

The matrix $Df(x_0, y_0)$ is called the Derivative matrix of the function $f(x, y)$ at the point (x_0, y_0).

The Gradient

When viewed as a row vector rather than as a matrix, the Derivative matrix is called the gradient and is denoted $\nabla f(x_0, y_0)$. Thus

$$\nabla f(x_0, y_0) = \left(\frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0) \right).$$

In terms of the coordinate vectors $\mathbf{i}$ and $\mathbf{j}$ the gradient can be written as

$$\nabla f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)\mathbf{i} + \frac{\partial f}{\partial y}(x_0, y_0)\mathbf{j}.$$

A criterion for differentiability

Before we state the criterion, we note that with our definition of differentiability, every differentiable function is continuous.

Theorem 26: Let $f : U \to \mathbb{R}$. If the partial derivatives $\frac{\partial f}{\partial x}(x, y)$ and $\frac{\partial f}{\partial y}(x, y)$ exist and are continuous in a neighbourhood of a point (x_0, y_0) (that is in a region of the plane of the form $\{(x, y) \mid \|(x, y) - (x_0, y_0)\| < r\}$ for some $r > 0$. Then f is differentiable at (x_0, y_0).

We omit the proof of this theorem. However, we note that a function whose partial derivatives exist and are continuous is said to be continuously differentiable or of class $\mathcal{C}^1$. The theorem says that every function that is $\mathcal{C}^1$ in a small disc around a point is differentiable at that point.

Three variables

For the next few slides, we will assume that $f : U \to \mathbb{R}$ is a function of three variables, that is, U is a subset of $\mathbb{R}^3$. In this case, if we denote the variables by x, y and z, we get three partial derivatives as follows: we hold two of the variables constant and vary the third. For instance if y and z are kept fixed while x is varied, we get the partial derivative with respect to x at the point (a, b, c):

$$\frac{\partial f}{\partial x}(a, b, c) = \lim_{x \to a} \frac{f(x, b, c) - f(a, b, c)}{x - a}.$$

In a similar way we can define the partial derivatives

$$\frac{\partial f}{\partial y}(a, b, c) \quad \text{and} \quad \frac{\partial f}{\partial z}(a, b, c).$$

Once we have the three partial derivatives we can once again define the gradient of f:

$$\nabla f(a, b, c) = \left(\frac{\partial f}{\partial x}(a, b, c), \frac{\partial f}{\partial y}(a, b, c), \frac{\partial f}{\partial z}(a, b, c) \right).$$

Differentiability in three variables

Exercise 1: Formulate a definition of differentiability for a function of three variables.

Exercise 2: Formulate the analogue of Theorem 26 for a function of three variables.

We can also define differentiability for functions from $\mathbb{R}^m$ to $\mathbb{R}^n$ where m and n are any positive integers. We will do this in detail in this course when m and n have the values 1 and 2 and 3.

Finally, the rules for the partial derivatives of sums, differences, products and quotients of functions $f, g : U \to \mathbb{R}$, ($U \subset \mathbb{R}^m$, $m = 2, 3$) are exactly analogous to those for the derivative of functions of one variable.

The Chain Rule

We now study the situation where we have composition of functions. We assume that $x, y : I \to \mathbb{R}$ are differentiable functions from some interval (open or closed) to $\mathbb{R}$. Thus the pair $(x(t), y(t))$ defines a function from I to $\mathbb{R}^2$. Suppose we have a function $f : \mathbb{R}^2 \to \mathbb{R}$ which is differentiable. We would like to study the derivative of the composite function $z(t) = f(x(t), y(t))$ from I to $\mathbb{R}$.

Theorem 27: With notation as above

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

For a function $w = f(x, y, z)$ in three variables the chain rule takes the form

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

Clarifications on the notation

The form in which I have written the chain rule is the standard one used in many books (both in engineering and mathematics). However, it is not very good notation. For instance, in the formula

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

the letter z is being used for two different functions: both as a function $z(t)$ from $\mathbb{R}$ to $\mathbb{R}$ on the left hand side, and as a function $z(x, y)$ from $\mathbb{R}^2$ to $\mathbb{R}$. If one wants to be precise one should write the chain rule as

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Similarly, for the function w we should write

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

Verifying the chain rule in a simple case

Example: Let us verify this rule in a simple case. Let $z = xy$, $x = t^3$ and $y = t^2$.

Then $z = t^5$ so $z'(t) = 5t^4$. On the other hand, using the chain rule we get

$$z'(t) = y \cdot 3t^2 + x \cdot 2t = 3t^4 + 2t^4 = 5t^4.$$

Example: A continuous mapping $c : I \to \mathbb{R}^n$ of an interval I to $\mathbb{R}$ is called a curve in $\mathbb{R}^n$, $(n = 2, 3)$.
In what follows, we will assume that all the curves we have are actually differentiable, not just continuous. We will say what this means below.

An application to tangents of curves

Let us consider a curve $c(t)$ in $\mathbb{R}^3$. Each point on the curve will be given by a triple of coordinates which will depend on t. That is, the curve can be described by a triple of functions $(g(t), h(t), k(t))$. Saying that $c(t)$ is a differentiable function of t, means that each of $g(t), h(t), k(t)$ are differentiable functions from $\mathbb{R} \to \mathbb{R}$. If we write

$$c(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}, \quad \text{then} \quad c'(t_0) = g'(t_0)\mathbf{i} + h'(t_0)\mathbf{j} + k'(t_0)\mathbf{k},$$

represents its tangent or velocity vector at the point $c(t_0)$.

Tangents to curves on surfaces

So far our example has nothing to do with the chain rule. Suppose $z = f(x, y)$ is a surface, and $c(t) = (g(t), h(t), f(g(t), h(t)))$ lies on the $z = f(x, y)$. (Here we are assuming that $f : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function!) Let us compute the tangent vector to the curve at $c(t_0)$. It is given by

$$c'(t_0) = g'(t_0)\mathbf{i} + h'(t_0)\mathbf{j} + k'(t_0)\mathbf{k},$$

where $k(t) = f(g(t), h(t))$. Using the chain rule we see that

$$k'(t_0) = \frac{\partial f}{\partial x}(g(t_0), h(t_0))g'(t_0) + \frac{\partial f}{\partial y}(g(t_0), h(t_0))h'(t_0).$$

We can further show that this tangent vector lies on the tangent plane to the surface $z = f(x, y)$. Indeed we have already seen that the tangent plane has the equation

$$z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0).$$

A normal vector to this plane is given by

$$\left(-\frac{\partial f}{\partial x}(x_0, y_0), -\frac{\partial f}{\partial y}(x_0, y_0), 1\right).$$

Thus, to verify that the tangent vector lies on the plane, we need only check that its dot product with normal vector is 0. But this is now clear.

Just to give a concrete example of what we are talking about, take a curve $(g(t), h(t))$ in the unit disc $x^2 + y^2 \leq 1$ in the xy plane. Then $\left(g(t), h(t), \sqrt{1 - g(t)^2 - h(t)^2}\right)$ lies on the upper hemisphere $z = \sqrt{1 - x^2 - y^2}$. For concreteness, we can take $I = \left[0, \frac{1}{\sqrt{2}}\right]$, $g(t) = t$ and $h(t) = t^2$.