MA 105 D3 Lecture 8

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The proof of Taylor's theorem

A very explicit calculation

Analytic functions

The Taylor polynomials

Given a function $f(x)$ which is n times differentiable at some point x_0 in an interval I, we can associate to it a family of polynomials $P_0(x), P_1(x), \ldots P_n(x)$ called the Taylor polynomials of degrees $0, 1, \ldots n$ at x_0 as follows.

We let $P_0(x) = f(x_0)$,

$$P_1(x) = f(x_0) + f^{(1)}(x_0)(x - x_0),$$

$$P_2(x) = f(x_0) + f^{(1)}(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2$$

We can continue in this way to define

$$P_n(x) = f(x_0) + f^{(1)}(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

Taylor's Theorem

The Taylor polynomials are rigged exactly so that the degree n Talyor polynomial has the same first n derivatives at the point x_0 as the function $f(x)$ has, that is, $P^{(k)}(x_0) = f^{(k)}(x_0)$ for all $0 \leq k \leq n$, where $f^{(0)} = f(x)$ by convention.

Taylor's Theorem says that we can recover a lot of information about the function from the Taylor polynomials.

Theorem 19: Let I be an open interval and suppose that $[a, b] \subset I$. Suppose that $f \in \mathcal{C}^n(I)$ $(n \geq 0)$ and suppose that $f^{(n)}$ is differentiable on I. Then there exists $c \in (a, b)$ such that

$$f(b) = P_n(b) + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1},$$

where $P_n(x)$ denotes the Taylor polynomial of degree n at a.

The proof of Taylor's theorem

Proof: From the definition, we see that

$$P_n(b) = f(a) + f^{(1)}(a)(b-a) + \cdots \frac{f^{(n}(a)}{n!}(b-a)^n$$

Consider the function

$$F(x) = f(b) - f(x) - f^{(1)}(x)(b-x) - \frac{f^{(2)}(x)}{2!}(b-x)^2 - \ldots - \frac{f^{(n)}(x)}{n!}(b-x)^n.$$

Clearly $F(b) = 0$, and

$$F^{(1)}(x) = -\frac{f^{(n+1)}(x)(b-x)^n}{n!}. \qquad (1)$$

We would like to apply Rolle's Theorem here, but $F(a) \neq 0$. So consider

$$g(x) = F(x) - \left(\frac{b-x}{b-a}\right)^{n+1} F(a)$$

(this is similar to the method by which we reduced the MVT to Rolle's Theorem), and we see that $g(a) = 0$. Applying Rolle's Theorem we see that there is a $c \in (a, b)$ such that $g'(c) = 0$.

This yields

$$F^{(1)}(c) = -(n+1)\left[\frac{(b-c)^n}{(b-a)^{n+1}}\right]F(a). \tag{2}$$

We can eliminate $F^{(1)}(c)$ using (1). This gives

$$-(n+1)\left[\frac{(b-c)^n}{(b-a)^{n+1}}\right]F(a) = -\frac{f^{(n+1)}(c)(b-c)^n}{n!},$$

from which we obtain

$$F(a) = \frac{(b-a)^{n+1}}{(n+1)!}f^{(n+1)}(c).$$

This proves what we want. $\square$

Power series

As we have explained in the previous slide the "correct" (both from the point of view of proofs and of computation) way to define a function like e^x is via convergent series involving non-negative integer powers of x. Such series are called power series and such functions should be viewed as the natural generalizations of polynomials.

The nice thing about power series is that once we know that they converge in some interval $(a-r, a+r)$ around a, it is not hard to show that the functions that they define are continuous functions. In fact, it is not too hard to show that they are smooth functions (that is, that all their derivatives exist), and in fact, the derivative is nothing but the power series obtained by differentiating term by term which also converges in $(a-r, a+r)$!

Thus when functions are given by convergent power series, we can automatically conclude that they are smooth.

Calculating the values of functions

As we have also mentioned several times, calculators and computers calculate the values of various common functions like trigonometric polynomials and expressions in $\log x$ and e^x by using Taylor series.

The great advantage of Taylor series is that one can estimate the error since we have a simple formula for the error which can be easily estimated. For instance, for the function $\sin x$, the n-th derivative is either $\pm \sin x$ or $\pm \cos x$, so in either case $|f^{(n}(x)| \leq 1$. Hence,

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$$

If we take $x = 1$, and we want to compute $\sin 1$ to an error of less than 10^{-16}, we need only make sure that $(n+1)! > 10^{16}$, which is achieved when $n \geq 21$. (Can you find a value of n which works for any value of x?)

Computing the vaues of $\sin x$

First, remember that $\sin x$ is periodic, so we only have to look at the values of x between $-\pi$ and π.

But we can do better, because $\sin(-x) = -\sin x$. So we only have to bother about the interval $[0, \pi]$.

We can do still better! Once we know $\sin x$ in $[0, \pi/2]$, we can easily figure out what it is in $[\pi/2, \pi]$.

So finally, it is enough to find the desired value of n for $x \in [0, \pi/2]$.

Computing the values of $\sin x$

We know that the remainder term satisfies

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}.$$

Hence, we need

$$\frac{|x|^{n+1}}{(n+1)!} < 10^{-16}.$$

We know that it is enough to look at $x \leq \pi/2$. Let us be a little careless and allow $x \leq 2$ (so we won't get the best possible n, maybe).

We already know that $1/(n+1)! < 10^{-16}$ if $n \geq 21$. Now $|x|^{22} \leq 2^{22}$. If we take $n = 31$, we see that $|x|^{32} \leq 2^{22} \cdot 2^{10}$,

$$1/(n+1)! = 1/32! < 10^{-16} \cdot 10^{-10} \cdot 2^{-10}.$$

Smooth functions and Taylor series

Given a smooth function $f(x)$ on $a \in I \subset \mathbb{R}$ we can write down its associated Taylor polynomials $P_n(x)$ around any point a in $\mathbb{R}$. Here are some natural questions that arise. Let us take $a = 0$ in what follows.

Question 1. When $x = 0$, obviously $P_n(0) = f(0)$ for all n. Do the Taylor polynomials $P_n(x)$ (around 0, say) always converge as $n \to \infty$ for $x \neq 0, x \in I$? at least for all x in some sub-interval $(c, d) \ni 0$?

Question 2. If $P_n(x)$ converges as $n \to \infty$ does it necessarily converge to $f(x)$?

We will answer the second question.

Smooth but not analytic

The standard example is the function

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ e^{-1/x} & \text{if } x > 0 \end{cases}$$

Notice that $f^{(k)}(0) = 0$ for all $k \geq 0$. Hence $P_n(x) = 0$ for all $n \geq 0$! Hence, $\lim_{n\to\infty} P_n(x) = 0$. Thus the Taylor polynomials $P_n(x)$ around 0 converge to 0 for any $x \in \mathbb{R}$.

But obviously, they do not converge to the value of the function, since $f(x) > 0$ if $x > 0$!

In this case the Taylor series does a very poor job of approximating the function. Indeed, the remainder term $R_n(x) = f(x)$ for all $x > 0$.

Thus, when we use Taylor series to approximate a function in an interval I, we must make sure that $R_n(x) \to 0$ as $n \to \infty$, for all $x \in I$.

Analytic functions

We say that a function $f(x)$ is analytic in an (open(interval I, if for each point $a \in I$, the Taylor polynomials of the function $f(x)$ around a, converge to $f(x)$ in some (possibly smaller) interval containing a. This means that $R_n(x) \to 0$ for all x in some interval $a \in (c, d) \subseteq I$

The functions $\sin x$, $\cos x$ and e^x are analytic on all of $\mathbb{R}$. The function $\tan x$ is analytic in $(-\pi/2, \pi/2)$. The function $\log x$ is analytic in $(0, \infty)$.

The purpose of this discussion is to alert you to the fact that Taylor series may not always do a good job of approximating a given function. You have to prove something before using Taylor series.

Question 1 even more subtle: See
`https://math.stackexchange.com/questions/620290/`
`is-it-possible-for-a-function-to-be-smooth-everywhere-anal`

L'Hôpital's rule

Suppose f and g are $\mathcal{C}^1$ functions in an interval I containing 0. By the MVT, for $x \in I$,

$$f(x) = f(0) + f^{(1)}(c_1)x \quad \text{and} \quad g(x) = g(0) + g^{(1)}(c_2)x$$

for $0 < c_1, c_2 < x$.
If $f(0) = g(0) = 0$,

$$\lim_{x \to 0} f(x)/g(x) = \lim_{x \to 0} f^{(1)}(c_1)x/g^{(1)}(c_2)x = \lim_{x \to 0} f^{(1)}(c_1)/g^{(1)}(c_2).$$

But $f^{(1)}$ and $g^{(1)}$ are continuous functions and as $x \to 0$, $c_1, c_2 \to 0$. Hence,

$$\lim_{x \to 0} f^{(1)}(c_1)/g^{(1)}(c_2) = f^{(1)}(0)/g^{(1)}(0)$$

If the functions are in $\mathcal{C}^n$, and $f^{(k)}(0) = g^{(k)}(0) = 0$ for all $k < n$, we can apply the MVT repeatedly (or we can apply Taylor's theorem directly) to get $f^{(n)}(0)/g^{(n)}(0)$ as the limit.

About the Quiz:

You will need to enter two numbers (A and B) at the top of your paper below your roll number.

1. Fill in the numbers "A" and "B" above as follows:
   If the last digit a of your roll number satisfies $0 \leq a \leq 4$, let $A = a + 5$. If $5 \leq a \leq 9$, let $A = a$.
   If the second-last digit b of your roll number satisfies $0 \leq b \leq 4$, let $B = b + 5$. If $5 \leq b \leq 9$, let $B = b$. Thus $5 \leq A, B \leq 9$.
   Example: Your Roll number is 23B0092. Then $A = 7$ and $B = 9$.

2. **You must use these values of A and B below. Using the wrong value of A or B even in one question may lead to the loss of all marks in this quiz.**