MA105 TSC

Ekansh Ravi Shankar, Om Joglekar, Tejas Shende, Pratik Sahoo

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Definition of a limit of a sequence

A sequence a_n tends to a limit L, if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n - L| < \epsilon$$

whenever $n > N$. This is equivalent to writing

$$\lim_{n \to \infty} a_n = L$$

A sequence that does not converge is said to diverge, or to be divergent.

Limits Algebra for Sequences

If a_n and b_n are two convergent sequences, then

a) $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$

b) $\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$

c) $\lim_{n\to\infty}(a_n/b_n) = \lim_{n\to\infty} a_n / \lim_{n\to\infty} b_n$, provided $\lim_{n\to\infty} b_n \neq 0$.

Sandwich Theorem

Theorem 1: If a_n, b_n and c_n are convergent sequences such that $a_n \leq b_n \leq c_n$ for all n, then

$$\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n \leq \lim_{n\to\infty} c_n$$

Theorem 2: Suppose $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n$. If b_n is a sequence satisfying $a_n \leq b_n \leq c_n$ for all n, then b_n converges and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n$$

Here, we do not assume that b_n converges, we get it for free.

Bounded Sequences

A sequence a_n is said to be bounded if there is a real number $M > 0$ such that $|a_n| \leq M$ for every $n \in \mathbb{N}$.

Theorem: Every convergent sequence is bounded.

A sequence a_n is said to be bounded above (resp. bounded below) if $a_n \leq M$ (resp. $a_n \geq m$) for some $M \in \mathbb{R}$ (resp. $m \in \mathbb{R}$).

Theorem: A monotonically increasing (resp. decreasing) sequence which is bounded above (resp. below) converges.

Cauchy Sequences

A sequence a_n in $\mathbb{R}$ is said to be a Cauchy sequence if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n - a_m| < \epsilon$$

for all $m, n > N$.

Theorem: Every Cauchy sequence in $\mathbb{R}$ converges.

Theorem: Every convergent sequence is Cauchy.

Definition of a limit of a function

A function $f : (a, b) \to \mathbb{R}$ is said to tend to (or converge to) a limit L at a point $x_0 \in [a, b]$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - L| < \epsilon$$

for all $x \in (a, b)$ satisfying $0 < |x - x_0| < \delta$. This is equivalent to writing

$$\lim_{x \to x_0} f(x) = L$$

Note: The limits algebra for functions is similar to that of sequences, and so are the Sandwich theorems.

Limits from the left and right

The limit of the function $f(x)$ as x approaches c from the left is a number L such that for all $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $|x - c| < \delta$ and $x \in (a, c)$.
The notation for this is $\lim_{x \to c-} f(x) = L$ and is also called the Left Hand Limit.

A similar definition can be written for the Right Hand Limit.

Continuity

If $f : [a, b] \to \mathbb{R}$ is a function and $c \in [a, b]$, then f is said to be continuous at the point c if and only if

$$\lim_{x \to c} f(x) = f(c)$$

A function f on (a, b) (resp. $[a, b]$) is said to be continuous if and only if it is continuous at every point c in (a, b) (resp. $[a, b]$).

A continuous function on a closed and bounded interval $[a, b]$ is bounded and attains its infimum and supremum.

Sequential Continuity

A function $f(x)$ is continuous at a point a if and only if for every sequence x_n converging to a, the sequence $f(x_n)$ converges to $f(a)$.

Intermediate Value Theorem

Suppose $f : [a, b] \to \mathbb{R}$ is a continuous function. For every u between $f(a)$ and $f(b)$ there exists $c \in [a, b]$ such that $f(c) = u$.

Extreme Value Theorem

Every continuous function, on a closed bounded interval, attains its supremum and infimum on that interval.
(Hence the words supremum and infimum can be replaced with the words maximum and minimum)

Differentiability

$f : (a, b) \to \mathbb{R}$ is said to be differentiable at a point $c \in (a, b)$ if

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

exists.
The value of the limit is denoted by $f'(c)$ and is called the derivative of f at c.

Chain rule for $F(x) = f(g(x))$ states that $F'(x) = f'(g(x))g'(x)$

Maxima and minima

A function f is said to attain a maximum (resp. minimum) at a point $x_0 \in X$ if $f(x) \leq f(x_0)$ (resp. $f(x) \geq f(x_0)$) for all $x \in X$.

Let $f : X \to \mathbb{R}$ be a function and x_0 be in X. Suppose there is a sub-interval $x_0 \in (c, d) \subset X$ such that $f(x_0) \geq f(x)$ (resp. $f(x_0) \leq f(x)$) for all $x \in (c, d)$, then f is said to have a local maximum (resp. local minimum) at x_0.

Theorem: If $f : X \to \mathbb{R}$ is differentiable and has a local minimum or maximum at a point $x_0 \in X$, $f'(x_0) = 0$.

Rolle's Theorem

Suppose $f : [a, b] \to \mathbb{R}$ is a continuous function which is differentiable in (a, b) and $f(a) = f(b)$. Then there is a point x_0 in (a, b) such that $f'(x_0) = 0$.

Mean Value Theorem

Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function and that f is differentiable in (a, b). Then there is a point x_0 in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(x_0)$$

Darboux's Theorem

Let $f : (a, b) \to \mathbb{R}$ be a differentiable function. If $c, d, c < d$ are points in (a, b), then for every u between $f'(c)$ and $f'(d)$, there exists an x in $[c, d]$ such that $f'(x) = u$.

Second Derivative Test

$f : [a, b] \to \mathbb{R}$ is a continuous function and that f is differentiable on (a, b). A point x_0 in (a, b) such that $f'(x_0) = 0$, is called a stationary point. We will assume that $f'(x)$ is differentiable at x_0, that is, the second derivative $f''(x_0)$ exists. With the assumptions above:

a) If $f''(x_0) > 0$, the function has a local minimum at x_0.

b) If $f''(x_0) < 0$, the function has a local maximum at x_0.

c) If $f''(x_0) = 0$, no conclusion can be drawn.

Concavity and convexity

A function $f : I \to \mathbb{R}$ is said to be concave (concave downwards) if

$$f(tx_1 + (1-t)x_2) \geq tf(x_1) + (1-t)f(x_2)$$

for all x_1 and x_2 in I and $t \in [0, 1]$. Similarly, a function is said to be convex (or concave upwards) if

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$$

Remarks on inflection point

A function is said to have an inflection point at $x = c$ if the concavity of the function changes at c
If a function f has an inflection point at $x = c$ then $f''(c) = 0$ **provided** the second derivative exists at $x = c$

Taylor Polynomials

We can associate a function $f(x)$ which is n time differentiable at some point x_0 in an interval I to a family of polynomials $P_0(x), P_1(x), \dots P_n(x)$, called the Taylor polynomials of order $0, 1, \dots, n$ at x_0 as follows.

$$P_n(x) = f(x_0) + f^{(1)}(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

Taylor's Theorem

Let $f \in \mathcal{C}^n(I)$ for some open interval I containing a, and suppose that $f^{(n+1)}$ exists on this interval. Then for each $b \neq a \in I$, there exists c between a and b such that

$$f(b) = P_n(b) + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$$

where P_n denotes the Taylor polynomial of order n at a.
We denote the function $f(b) - P_n(b)$ by $R_n(b)$.

Ratio test for convergence of a series

Let $\sum_{k=1}^{\infty} a_k$ be an infinite series and let

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = L$$

Then,

a) If $L < 1$, then the above series is convergent.
b) If $L > 1$, then the above series is divergent.
c) If $L = 1$, then the test is inconclusive.

Partitions

Given a closed interval $[a, b]$, a partition P of $[a, b]$ is simply a collection of points

$$P = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}.$$

A partition $P' = \{a = x'_0 < x'_1 < \cdots < x'_m = b\}$ is said to be a refinement of the partition P if for each $x_i \in P$, there exists an $x'_j \in P'$ such that $x_i = x'_j$

Lower and Upper sums

Given a partition $P = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$ and a bounded function $f : [a, b] \to \mathbb{R}$, we define the lower and upper sum. First, we set:

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x) \text{ and } m_i = \inf_{x \in [x_{i-1}, x_i]} f(x), \ 1 \leq i \leq n.$$

We define the Lower sum as

$$L(f, P) = \sum_{j=1}^{n} m_j (x_j - x_{j-1})$$

Similarly, we define the Upper sum as

$$U(f, P) = \sum_{j=1}^{n} M_j (x_j - x_{j-1})$$

Darboux Integral

The lower Darboux integral of f is defined by

$$L(f) = \sup\{L(f, P) : P \text{ is a partition of } [a, b]\}.$$

Similarly, the upper Darboux integral of f is defined by

$$U(f) = \inf\{U(f, P) : P \text{ is a partition of } [a, b]\}.$$

If $U(f) = L(f)$, then we say that f is Darboux-integrable and we define

$$\int_a^b f(t)dt := U(f) = L(f)$$

Riemann Sum and Riemann Integral

We define the Riemann sum associated to the function f and the tagged partition (P, t) by

$$R(f, P, t) = \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1})$$

A function $f : [a, b] \to R$ is said to be Riemann integrable if for some $R \in \mathbb{R}$ and every $\epsilon > 0$, there exists $\delta > 0$ such that

$$|R(f, P, t) - R| < \epsilon,$$

for any tagged partition (P, t) of $[a, b]$ having $||P|| < \delta$.

Properties of Riemann integration

Theorem: Let $f : [a, b] \to \mathbb{R}$ be a function that is bounded, and continuous at all but countably many points of $[a, b]$. Then f is Riemann integrable on $[a, b]$.

Assuming f and g are Riemann integrable, the following are true.

$$\int_a^b [f(t) + g(t)]dt = \int_a^b f(t)dt + \int_a^b g(t)dt$$

$$\int_a^b cf(t)dt = c\int_a^b f(t)dt \qquad \text{for any } c \in \mathbb{R}$$

$$\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt \qquad \text{if } c \in [a, b]$$

Fundamental Theorem of Calculus Part I and II

Let $f : [a, b] \to \mathbb{R}$ be a continuous function, and let

$$F(x) = \int_a^x f(t)dt$$

for any $x \in [a, b]$. Then $F(x)$ is continuous on $[a, b]$, differentiable on (a, b) and

$$F'(x) = f(x)$$

for all $x \in (a, b)$.

Let $f : [a, b] \to \mathbb{R}$ be given and suppose there exists a continuous function $g : [a, b] \to \mathbb{R}$ which is differentiable on (a, b) and which satisfies $g'(t) = f(t)$. Then, if f is Riemann integrable on $[a, b]$,

$$\int_a^b f(t)dt = g(b) - g(a)$$

Arc length

Suppose x and y are continuously differentiable functions mapping from $[a, b] \to \mathbb{R}$ and we have a parameterised curve given by $(x(t), y(t))$, we then define the arc length of the curve C as

$$l(C) = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} dt$$

Limit and Continuity of a function with domain $\mathbb{R}^2$

A function $f : U \to \mathbb{R}$ is said to tend to a limit L as $x = (x_1, x_2)$ approaches $c = (c_1, c_2)$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - L| < \epsilon,$$

whenever $0 < ||x - c|| < \delta$.

The function $f : U \to \mathbb{R}$ is said to be continuous at a point c if

$$\lim_{x \to c} f(x) = f(c).$$

Partial and Directional derivatives

The partial derivative of $f : U \to \mathbb{R}$ with respect to x_1 at the point (a, b) is defined by

$$\frac{\partial f}{\partial x_1}(a, b) := \lim_{x_1 \to a} \frac{f((x_1, b)) - f((a, b))}{x_1 - a} = \lim_{t \to 0} \frac{f((a + t, b)) - f(a, b)}{t}$$

Similarly, we can define the partial derivative with respect to the variable x_2.

The directional derivative of f in the direction of a unit vector $v = (v_1, v_2)$ at a point $x = (x_1, x_2)$ is defined as

$$\nabla_v f(x) = \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t} = \lim_{t \to 0} \frac{f((x_1 + tv_1, x_2 + tv_2)) - f((x_1, x_2))}{t}$$

Differentiability for functions of two variables

A function $f : U \to \mathbb{R}$ is said to be differentiable at a point (x_0, y_0) if $\frac{\partial f}{\partial x}(x_0, y_0)$ and $\frac{\partial f}{\partial y}(x_0, y_0)$ exist and

$$\lim_{(h,k)\to(0,0)} \frac{\left|f(x_0+h, y_0+k) - f(x_0, y_0) - \frac{\partial f}{\partial x}(x_0, y_0)h - \frac{\partial f}{\partial y}(x_0, y_0)k\right|}{||(h,k)||} = 0$$

If $f : U \to \mathbb{R}$ is differentiable at (x_0, y_0), then f is continuous at (x_0, y_0).

Gradient and Tangent Planes

The gradient of $f : U \to \mathbb{R}$, where $U \subseteq \mathbb{R}^m$, at a point $(a_1, a_2, \ldots, a_m) \in U$ is denoted as $\nabla f(a_1, a_2, \ldots, a_m)$.

$$\nabla f(a_1, a_2, \ldots, a_m) = \left(\frac{\partial f}{\partial x_1}(a_1, a_2, \ldots, a_m), \ldots, \frac{\partial f}{\partial x_m}(a_1, a_2, \ldots, a_m) \right).$$

Questions!

Hope you're following us till now. From this point on, we're gonna be discussing questions.

Question 1

Prove that $f(x) = x^2$ is continuous at all points in its domain.

Blank

Damn.

Question 2

Suppose that f is continuous on $[0, 1]$ and $f(0) = f(1)$. Let n be any natural number. Prove that there is some number x such that

$$f(x) = f(x + \frac{1}{n})$$

Question 2 with Hint

Suppose that f is continuous on $[0, 1]$ and $f(0) = f(1)$. Let n be any natural number. Prove that there is some number x such that

$$f(x) = f(x + \frac{1}{n})$$

Hint: Think of $g(x) = f(x) - f(x + \frac{1}{n})$ and what would happen if it was never 0.

Blank

Damn.

Question 3

Let f be a function such that $|f(x)| \leq x^2$ for all x. Prove that f is differentiable at 0.

Blank

Damn.

Question 4

Show that the tangent plane to the surface $z = x^2 - y^2$ at $(3, 3, 0)$ intersects the surface in 2 perpendicular lines.

Blank

Damn.

Question 5

For $x > 0$, prove that

$$\left| ln(1+x) - (x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}) \right| \leq \frac{x^5}{5}$$

Blank

Damn.

Question 6

Let $S_n = \frac{5}{n}(\sum_{i=1}^{n}\left(\frac{i}{n}\right) + \sum_{i=n+1}^{2n}\left(\frac{i}{n}\right)^{\frac{3}{2}})$

Blank

Damn.

Question 7

Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

is convergent iff $p > 1$.

Blank

Damn.

Question 8

For a function $f : \mathbb{R}^2 \to \mathbb{R}$, at the point (a, b), state true or false:

1. Partial derivatives exist $\implies$ f is continuous
2. All directional derivatives exist $\implies$ f is continuous
3. All directional derivatives exist $\implies$ f is differentiable
4. Partial derivatives exist and are continuous $\implies$ f is continuous
5. Partial derivatives exist and are bounded $\implies$ f is differentiable
6. Partial derivatives exist and are continuous $\implies$ f is differentiable
7. All directional derivatives exist and are continuous $\implies$ f is differentiable

Blank

Damn.

Question 9

Prove or disprove: Let $\{x_n\}$ be a sequence of positive real numbers such that the sequence $\{x_{n+1} - x_n\}$ converges to 0, then $\{x_n\}$ is convergent.

Blank

Damn.

Question 10 Part 1

Consider the function

$$f(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0) \\ \frac{3x^2y - y^3}{x^2+y^2} & \text{otherwise} \end{cases}$$

Check if f is continuous at (0,0).

Question 10 Part 2

Consider the function

$$f(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0) \\ \frac{3x^2y - y^3}{x^2+y^2} & \text{otherwise} \end{cases}$$

Find $f_y(x,0)$ for $x \neq 0$

Question 10 Part 3

Consider the function

$$f(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0) \\ \frac{3x^2y - y^3}{x^2+y^2} & \text{otherwise} \end{cases}$$

Verify if f_y is continuous at the origin.

Blank

Damn.

Question 11

Given a C^2 function f on [0,1] such that

$$f\left(\frac{1}{n}\right) = 0 \quad \forall n \in \mathbb{N}$$

Prove that,

$$f(0) = f'(0) = f''(0) = 0$$

Blank

Damn.

Question 12

Consider $f : [0, 2\pi] \to \mathbb{R}$ defined as:

$$f(x) = \begin{cases} cos(x) & \text{if } x \text{ is rational} \\ sin(x) & \text{otherwise} \end{cases}$$

Blank

Damn.

Question 13

Prove that there exists a unique $x \in [0, 10]$ such that

$$\int_0^x ln(3 + t^2 + e^t)dt = \int_x^{10} ln(3 + t^2 + e^t)dt$$