MA 105 D3 Lecture 10

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Recap: The Darboux integral

Riemann integration

Partitions

Definition: Given a closed interval $[a, b]$, a partition P of $[a, b]$ is simply a collections of points

$$P = \{a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b\}.$$

We can think of the points of the partition as dividing the original interval $[a, b]$ into sub-intervals $I_j = [x_{j-1}, x_j]$, $1 \leq j \leq n$. Indeed $I = \cup_j I_j$ and if two sub-interval intersect, they have at most one point in common. Hence, the notation "partition".

Definition: A partition $P' = \{a = x'_0 < x'_1 < \ldots < x'_m = b\}$ is said to be a refinement of the partition P if for each $x_i \in P$, there exists an $x'_j \in P'$ such that $x_i = x'_j$ (more compactly, $P \subseteq P'$).

Intuitively, a refinement P' of a partition P will break some of the sub-intervals in P into smaller sub-intervals. Any two partitions P_1, P_2 have a common refinement - $P_1 \cup P_2$.

Lower and Upper sums

Given a partition $P = \{a = x_0 < x_1 < \ldots < x_{b-1} < x_n = b\}$ and a function $f : [a, b] \to \mathbb{R}$, we define two associated quantities. First we set:

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x) \quad \text{and} \quad m_i = \inf_{x \in [x_{i-1}, x_i]} f(x), \quad 1 \leq i \leq n$$

Defintion: We define the Lower sum as

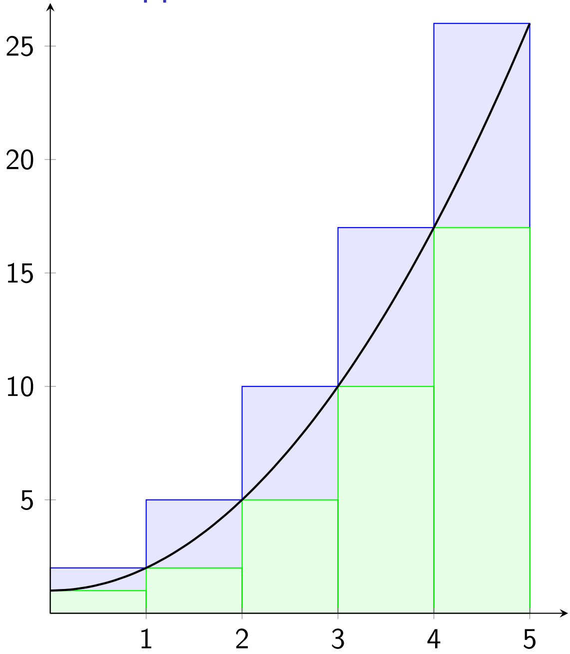
$$L(f, P) = \sum_{j=1}^{n} m_j (x_j - x_{j-1}).$$

Similarly, we can define the Upper sum as

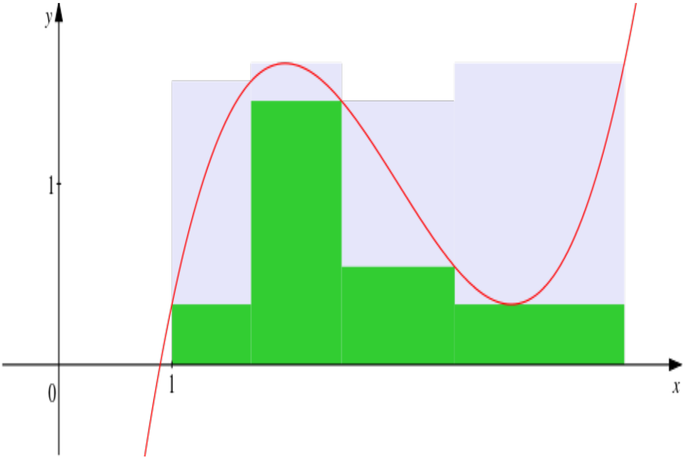
$$U(f, P) = \sum_{j=1}^{n} M_j (x_j - x_{j-1}).$$

In case the words "infimum" and "supremum" bother you, you can think "minimum" and "maximum most of time since we will usually be dealing with continuous functions on $[a, b]$.

Lower and Upper Darboux sums



A picture for a non-monotonic function



https://upload.wikimedia.org/wikipedia/commons/thumb/5/59/Darboux.svg/700px-Darboux.svg.png

The Darboux integrals

We now define the lower Darboux integral of f by

$$L(f) = \sup\{L(f, P)\colon P \text{ is a partition of } [a, b]\},$$

where the supremum is taken over all partitions of $[a, b]$.

and similarly the upper Darboux integral of f by

$$U(f) = \inf\{U(f, P)\colon P \text{ is a partition of } [a, b]\},$$

and again the infimum is over all partitions of $[a, b]$. (This time there is no escaping inf and sup!)

If $L(f) = U(f)$, then we say that f is Darboux-integrable and define

$$\int_a^b f(t)\,dt := U(f) = L(f).$$

This common value of the two integrals is called the Darboux integral.

Useful properties of the Darboux sums

Since, for any partition P, $L(f, P) \leq U(f, P)$, we have

$$L(f) \leq U(f).$$

In fact, for any two partitions P_1 and P_2, we have

$$L(f, P_1) \leq U(f, P_2).$$

One of the most useful properties of the Darboux sums is the following. If P' is a refinement of P then obviously

$$L(f, P) \leq L(f, P') \leq U(f, P') \leq U(f, P).$$

Riemann Sums

There is another way of getting at the integral due to Riemann which may be a little more intuitive and is better for calculation. This is done via Riemann sums.

To define the notion of a Riemann sum we need one more piece of data. Suppose that for each of the intervals I_j we are given a point $t_j \in I_j$. We will denote the collection of points t_j by t. The pair (P, t) is sometimes called a tagged partition.

Definition: We define the Riemann sum associated to the function f, and the tagged partition (P, t) by

$$R(f, P, t) = \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1}).$$

The norm of a partition

As must be clear, the Lower sum, Upper sum and Riemann sum all give approximations to the area between the lines $x = a$ and $x = b$ and between the curve $y = f(x)$ and the x-axis and

$$L(f, P) \leq R(f, P, t) \leq U(f, P).$$

The point is to make this statement quantitatively precise.

We define the norm of a partition P (denoted $\|P\|$) by

$$\|P\| = \max_j \{|x_j - x_{j-1}|\}, \quad 1 \leq j \leq n.$$

The norm gives some measure of the "size" of a partition, in particular, it allows us to say whether a partition is big or small.

When the size of the partition is small, it means that every interval in the partition is small.

The Riemann integral

Definition 1: A function $f : [a, b] \to \mathbb{R}$ is said to be Riemann integrable if for some $R \in \mathbb{R}$ and every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|R(f, P, t) - R| < \epsilon,$$

whenever $\|P\| < \delta$. In this case R is called the Riemann integral of the function f on the interval $[a, b]$.

In other words, for all sufficiently "small" or "fine" partitions, the Riemann sums must be within ϵ of R.

Notice, that as long as $\|P\|$ is small, it doesn't matter exactly where the x_j's or the t_j's are in the interval $[a, b]$.

Also notice that if P' is a refinement of P, then $\|P'\| \leq \|P\|$.

The Riemann integral continued

Intuitively, we can see that the smaller or finer the partition, the better the area under the curve is represented by the Riemann sum.

The reason that the Riemann integral is useful is because the definition we have given is actually equivalent to the following apparently weaker definition.

Definition 2: A function $f : [a, b] \to \mathbb{R}$ is said to be Riemann integrable if for some $R \in \mathbb{R}$ and every $\epsilon > 0$ there exists a partition P such that for every tagged refinement of (P', t') of P with $\|P'\| \leq \delta$,

$$|R(f, P', t') - R| < \epsilon.$$

The nice thing about the above definition is that one only has to check that $|R(f, P', t') - R|$ is small for for refinements of a fixed partition, and not for all partitions.

Back to our example

Using Definition 2 of the Riemann integral it is easy to see that the function $f(x) = x$ is Riemann integrable.

Let $\epsilon > 0$ be arbitrary. For our fixed partition we take $P = P_n$ where $n > \frac{1}{\epsilon}$ is some fixed number. Moreover, if (P', t') is any refinement of P_n we have

$$L(f, P_n) \leq L(f, P') \leq R(f, P', t') \leq U(f, P') \leq U(f, P_n),$$

whence it follows that (remember $U(f, P_n) - 1/2 + 1/2n$ and $L(f, P_n) = 1/2 - 1/2n$)

$$|R(f, P', t') - \frac{1}{2}| < \epsilon.$$

The example continued

As the preceding example shows, Definition 2 of the Riemann integral is really easy to work with. Why do we then care about Definition 1 or the Darboux integral?

The reason is that while Definition 2 is good for showing that a given function is Riemann integrable, the other definitions are often better for proving the *abstract properties* of integrals.

In fact, this will be clear in the tutorial exercises. You will see that sometimes the Darboux integral is better than the Riemann integral.

Before going any further we will formally state what we have already been referring to for several slides.

Comparison with the Darboux integral

Theorem 20: The Riemann integral (using either definition) exists if and only if the Darboux integral exists and in this case the two integrals are equal.

With this theorem in hand, we see that the function $f(x) = x$ is also Darboux integrable.

How does one prove Theorem 20? It is not too hard but it takes some work and is roundabout.

The easiest way is to proceed as follows. It is clear that if f is Riemann integrable in the sense of Definition 1, it is Riemann integrable in the sense of Definition 2. Next, one shows that if f is Riemann integrable in the sense of Definition 2, then it is Darboux integrable. And finally, one can show that if the Darboux integral exists, then the Riemann integral exists in the sense of Definition 1. An interested student can try this as an exercise.

The main theorem for Riemann integration

From now on we will use any of the three definitions - the Darboux definition, Definition 1 and Definition 2 for the integral interchangeably and we will use only the words Riemann integral.

The main theorem of Riemann integration is the following:

Theorem 21: Let $f : [a, b] \to \mathbb{R}$ be a function that is bounded, and continuous at all but finitely many points of $[a, b]$. Then f is Riemann integrable on $[a, b]$.

In fact, one can allow even countably many discontinuities and the Theorem will remain true.

Exercise 1: Those of you who have an extra interest in the course should think about trying to prove both Theorem 21 and the extension to countably many discontinuities (Warning: there is one crucial fact about continuous functions that we have not covered that you will have to discover for yourself).

An example of a function that is not Darboux integrable

Here is a function that is not Darboux integrable of $[0, 1]$. Let

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$$

It should be clear that no matter what partition one takes the infimum on any sub-interval in the partition will be 0 and the supremum will be 1.

From this one can see immediately that

$$L(f, P) = 0 \neq 1 = U(f, P),$$

for every P, and hence that $L(f) = 0 \neq 1 = U(f)$.

Another property of the Riemann Integral

Theorem 23: Suppose f is Riemann integrable on $[a, b]$ and $c \in [a, b]$. Then

$$\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt.$$

Proof: First we note that if $c = a$ or $c = b$, there is nothing to prove.

Next, if $c \in (a, b)$ we proceed as follows. If P_1 is a partition of $[a, c]$ and P_2 is a partition of $[c, b]$, then $P_1 \cup P_2 = P'$ is obviously a partition of $[a, b]$. Thus, partitions of the form $P_1 \cup P_2$ constitute a subset of the set of all partitions of $[a, b]$. For such partitions P' we have

$$L(f, P') = L(f, P_1) + L(f, P_2).$$

Let us denote by $L(f)_{[a,c]}$ (resp. $L(f)_{[c,b]}$) the Darboux lower integral of f on the interval $[a, c]$ (resp. $[c, b]$).