MA 105 D3 Lecture 7

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Recap

Towards Taylor's Theorem - higher derivatives

Arnold's problem

Concavity and convexity

Let I denote an interval (open or closed or half-open).
Definition: A function $f : I \to \mathbb{R}$ is said to be concave (or sometimes concave downwards) if

$$f(tx_1 + (1-t)x_2) \geq tf(x_1) + (1-t)f(x_2)$$

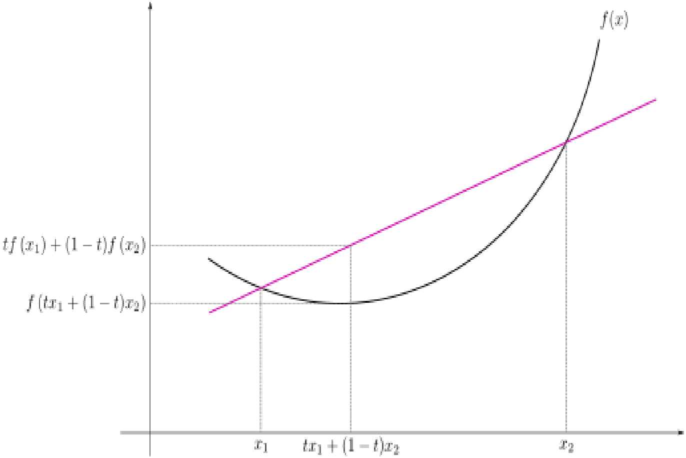
for all x_1 and x_2 in I and $t \in [0, 1]$.
Similarly, a function is said to be convex (or concave upwards) if

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2).$$

By replacing the $\geq$ and $\leq$ signs above by strict inequalities we can define strictly concave and strictly convex functions.

Note that if $f(x)$ is a concave function, $-f(x)$ is a convex function, so it is really enough to study one class or the other. Convex functions occur in many areas of mathematics.

Convexity illustrated graphically



Convexity and the second derivative

A twice differentiable function on an interval will be convex if its second derivative is everywhere non-negative. If the second derivative is positive, the function will be strictly convex.

However, the converse of the second statement above is not true. Can you give a counter-example to the converse of the second statement?

How about $f(x) = x^4$?

Definition: A point of inflection x_0 for a function f is a point where the function changes its behavior from concave to convex (or vice-versa). At such a point $f''(x_0) = 0$, but this is only a necessary, not a sufficient condition. (Why?)
If further, we also assume that the lowest order (≥ 2) non-zero derivative is odd, then we get a sufficient condition.

Smooth functions

We will now introduce some notation. The space $\mathcal{C}^k(I)$, will denote the space of k times continuously differentiable functions on an (open) interval I, for some fixed $k \in \mathbb{N}$, that is, the space of functions for which k derivatives exist and such that the k-th derivative is a continuous functions.

The space $\mathcal{C}^\infty(I)$ will consist of functions that lie in $\mathcal{C}^k(I)$ for every $k \in \mathbb{N}$. Such functions are called smooth or infinitely differentiable functions.

From now on we will denote the k-th derivative of a function $f(x)$ by $f^{(k)}(x)$.

Our aim will be to enlarge the class of functions we understand using the polynomials as stepping stones.

The Taylor polynomials

Given a function $f(x)$ which is n times differentiable at some point x_0 in an interval I, we can associate to it a family of polynomials $P_0(x), P_1(x), \dots P_n(x)$ called the Taylor polynomials of degrees $0, 1, \dots n$ at x_0 as follows.

We let $P_0(x) = f(x_0)$,

$$P_1(x) = f(x_0) + f^{(1)}(x_0)(x - x_0),$$

$$P_2(x) = f(x_0) + f^{(1)}(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2$$

We can continue in this way to define

$$P_n(x) = f(x_0) + f^{(1)}(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

Taylor's Theorem

The Taylor polynomials are rigged exactly so that the degree n Talyor polynomial has the same first n derivatives at the point x_0 as the function $f(x)$ has, that is, $P^{(k)}(x_0) = f^{(k)}(x_0)$ for all $0 \leq k \leq n$, where $f^{(0)} = f(x)$ by convention.

Taylor's Theorem says that we can recover a lot of information about the function from the Taylor polynomials.

Theorem 19: Let I be an open interval and suppose that $[a, b] \subset I$. Suppose that $f \in \mathcal{C}^n(I)$ $(n \geq 0)$ and suppose that $f^{(n)}$ is differentiable on I. Then there exists $c \in (a, b)$ such that

$$f(b) = P_n(b) + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1},$$

where $P_n(x)$ denotes the Taylor polynomial of degree n at a.

Remarks on Taylor's Theorem and some examples

Remark 1: When $n = 0$ in Taylor's Theorem we get the MVT. When $n = 1$, Taylor's Theorem is called the Extended Mean Value Theorem.

Remark 2: The Taylor polynomials are nothing but the partial sums of the Taylor Series associated to a $\mathcal{C}^\infty$ function about (or at) the point a:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(b-a)^k.$$

We can show that this series converges provided we know that the difference $f(x) - P_n(x) = R_n(x)$ can be made less than any $\epsilon > 0$ when n is sufficiently large. We will see how to do this for certain simple functions like e^x or $\sin x$.

The Taylor series for e^x

Let us show that the Taylor series for the function e^x about the point 0 is a convergent series for any value of $x = b \geq 0$ and that it converges to the value e^b (a similar proof works for $b < 0$).

In this case, at any point a, $f^{(n)}(a) = e^a$, so at $a = 0$ we obtain $f^{(n)}(0) = 1$. Hence the series about 0 is

$$\sum_{k=0}^{\infty} \frac{b^k}{k!}.$$

If we look at $R_n(b) = e^b - s_n(b)$ we obtain

$$|R_n(b)| = \frac{e^c b^{n+1}}{(n+1)!} \leq \frac{e^b b^{n+1}}{(n+1)!},$$

since $c \leq b$. As $n \to \infty$ this clearly goes to 0. This shows that the Taylor series of e^b converges to the value of the function at each real number b.

Defining functions using Taylor series

Instead of finding the Taylor series of a given function we can reverse the process and define functions using convergent series. Thus, one can define the function e^x as

$$e^x := \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

In this case, we have to first show that the series on the right hand side converges for a given value of x, in which case the definition above makes sense.

We show the convergence of such series by showing that they are Cauchy series. This means that we do not have to guess at a value of the limit.

Power series

As we have explained in the previous slide the "correct" (both from the point of view of proofs and of computation) way to define a function like e^x is via convergent series involving non-negative integer powers of x. Such series are called power series and such functions should be viewed as the natural generalizations of polynomials.

The nice thing about power series is that once we know that they converge in some interval $(a-r, a+r)$ around a, it is not hard to show that the functions that they define are continuous functions. In fact, it is not too hard to show that they are smooth functions (that is, that all their derivatives exist). Thus when functions are given by convergent power series, we can automatically conclude they are smooth. This is the advantage of defining functions in this way.

Calculating the values of functions

As we have also mentioned several times, calculators and computers calculate the values of various common functions like trigonometric polynomials and expressions in $\log x$ and e^x by using Taylor series.

The great advantage of Taylor series is that one can estimate the error since we have a simple formula for the error which can be easily estimated. For instance, for the function $\sin x$, the n-th derivative is either $\pm \sin x$ or $\pm \cos x$, so in either case $|f^{(n}(x)| \leq 1$. Hence,

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$$

If we take $x = 1$, and we want to compute $\sin 1$ to an error of less than 10^{-16}, we need only make sure that $(n+1)! > 10^{16}$, which is achieved when $n \geq 21$. (Can you find a value of n which works for any value of x?)

Arnold's problem

Recall that you were asked to find the following limit (Exercise 3 of the previous lecture).

$$\lim_{x\to 0} \frac{\sin(\tan x) - \tan(\sin x)}{\arcsin(\arctan x) - \arctan(\arcsin x)}.$$

The problem above was posed by the Russian mathematician Vladimir I. Arnold (see his book "Huygens and Barrow, Newton and Hooke") as an example of a problem that seventeenth century mathematicians could solve very easily, but that modern mathematicians, even with all their extra machinery and knowledge can't. In fact, it was his habit to put up this problem while lecturing to eminent mathematicians in leading universities and challenge them to solve it within ten minutes.

Notice that the limit that we have to calculate has the form

$$\frac{f(x) - g(x)}{g^{-1}(x) - f^{-1}(x)}.$$

The solution to Arnold's problem

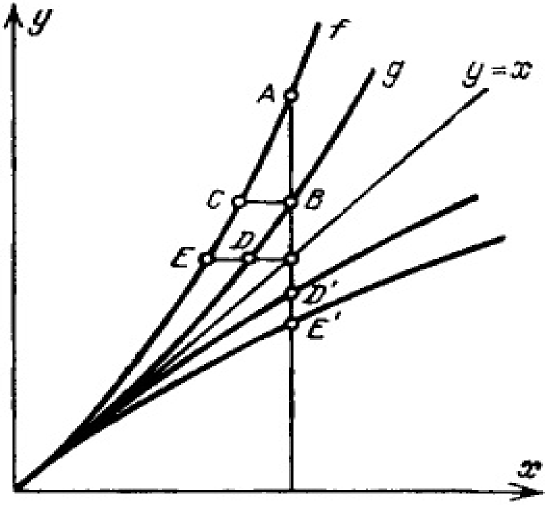


Fig. 37.
Calculation of the limit |AB|/|D′E′|

V. I. Arnold

The preceeding picture was taken from V. I. Arnold's book "Huygens and Barrow, Newton and Hooke (Birkhauser 1990)

V. I. Arnold (1936-2008) worked in geometry, differential equations and mathematical physics. He was one of the most important mathematicians of the twentieth century.