CS105 Midsem: DIC on Discrete Structures

40 marks, 120min

23 Sep 2023

Instructions:

- Attempt *all* questions. Write all answers and proofs carefully. If you are *making any assumptions or using results proved in class, state them clearly.*
- All sets considered below are general sets that can be infinite. Hence, you must not assume that they are finite or countable, unless clearly specified otherwise.
- Recall that $\mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{N}$ denote, respectively, the set of real numbers, rational numbers, integers and natural numbers.
- In this course, one of our aims is to learn how to write good proofs, hence considerable weightage will be given to *clarity and completeness* of proofs.
- Do *not* copy or use any other unfair means. Offenders will be reported to the Disciplinary Action Committee.

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1. [6=2+2+2 marks] True or false. You MUST give a short justification or counterexample for each.
   (a) For any sets $A, B, C,\quad A \cup (B \cap C) = C \cup (B \cap A)$.
   (b) Any subset of an uncountable set is uncountable.
   (c) The set $(\mathbb{R} \cap (\mathbb{Z} \times \mathbb{Q}))$ is countable.

Solution.
[1 Mark for correctly stating True/False and 1 Mark for Justification/Counter-example.]

(a) **False**.
Counter-example : Take $A = \{1\}$, $B = \{2\}$ and $C = \{3\}$. Here $A \cup (B \cap C) = \{1\}$ and $C \cup (B \cap A) = \{3\}$. Thus $A \cup (B \cap C) \neq C \cup (B \cap A)$. This is just one example. We have given marks if any correct counter example was provided. [If this part was solved with Venn Diagram without explaining via counter-example, we have given 0.5 marks for the justification part.]

(b) **False**. *Counter-example*: Set of natural numbers $\mathbb{N}$ is a countable subset of the uncountable set of real numbers $\mathbb{R}$. There can be many counter-examples, we have given marks as long as proper explanation was given for the counter-example.

(c) **True**.
$(\mathbb{R} \cap (\mathbb{Z} \times \mathbb{Q})) = \phi$, which is a countable set. [If you have assumed, without realizing it, $\mathbb{R} \times \mathbb{R}$ instead of $\mathbb{R}$ and then given proper justification, partial marks have been given. If you have made it explicit, full marks are given.]

[**Common Mistakes** :

(a) Element belonging to a set is not the same as it being the subset of that set. For example, many have written set of Natural numbers $\mathbb{N}$ is a subset of Power set of $\mathbb{N}$, which is incorrect, the accurate thing to say would be $\mathbb{N}$ is an element in Power set of $\mathbb{N}$. As of subset, we can say $\{\mathbb{N}\}$ is a subset of Power set of $\mathbb{N}$.

(b) Many students have confused the definition of countable sets with countably infinite sets. Point to remember, finite sets are also countable sets. So, ϕ is also countable set.

(c) $\mathbb{N}$ is countably infinite, NOT uncountable.

(d) Not mentioning why $\mathbb{Z} \times \mathbb{Q}$ is countable. You need to mention that since $\mathbb{Q}$ and $\mathbb{Z}$ are countable sets, cartesian product $\mathbb{Z} \times \mathbb{Q}$ will also be countable.

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2. [4 marks] Let $f(0) = 1, f(1) = 2$ and for all $n \geq 1$, let $f(n+1) = f(n-1) + 2f(n)$. Prove by induction that for all $n \in \mathbb{N}$, $f(n) \leq 3^n$. Mention clearly if you are using weak or strong induction.

Solution.

(a) Base Case: [1 mark - Half mark deducted for not including one base case]

i. For $n = 0$: $f(0) = 1$, and $3^0 = 1$. So $f(0) \leq 3^0$

ii. For $n = 1$: $f(1) = 2$, and $3^1 = 3$. So $f(1) \leq 3^1$

(b) Inductive Hypothesis: [1 mark - Highlight strong induction is used - Marks deducted for not specifying strong or weak]
Assume that for some $k \geq 1$, the statement holds for all natural numbers less than or equal to k, i.e., for all $i \leq k$, we have $f(i) \leq 3^i$.

(c) Inductive Step: [2 marks]
We need to prove that the statement holds for $k + 1$. i.e., we need to prove that

$$f(k+1) \leq 3^{k+1}$$

Consider the expression for $f(k+1)$:

$$f(k+1) = f(k-1) + 2f(k)$$

From our hypothesis:

$$f(k-1) + 2f(k) \leq 3^{k-1} + 2 \cdot 3^k$$

$$f(k+1) = f(k-1) + 2f(k) \leq 3^{k-1} + 2 \cdot 3^k = 3^{k-1}(1 + 2 * 3)$$

$$f(k+1) \leq 3^{k-1}(1 + 2 * 3) = 3^{k-1}(7) \leq 3^{k-1}(9) = 3^{k+1}$$

So, by the principle of strong induction, we have shown that for all $n \in \mathbb{N}$, $f(n) \leq 3^n$.

Aliter:
Another solution using Weak induction can be done by proving $f(k) \geq f(k-1)$ and using it for $f(k+1) \leq 3f(k)$.
$f(k) \geq f(k-1)$ because $2f(k) \geq 0, f(k-1) \geq 0$ making f(k+1) being greater than either of the terms on RHS.

[Common Mistakes: 2 Base cases have to be written. If using the fact that $f(k) \geq f(k-1)$, should formally include proof for it too.]

3. [10=3+3+4 marks] Counting and combinatorics

(a) A class has 20 students, 12 boys and 8 girls. How many ways are there to form a team of 5 students that has at least 1 boy and 1 girl?

(b) How many functions from $\{1, \ldots, n\}$ to $\{1, \ldots, n\}$ are there, that are *not* injective?

(c) Use double counting (i.e., counting the size of a suitably designed set in two different ways) to show the following identity. Using any other method will fetch partial marks.

$$\sum_{k=0}^{r} \binom{k}{m} \binom{r-k}{n} = \binom{r+1}{m+n+1}$$

Solution.

(a) Number of ways to choose 5 students out of 20 is $\binom{20}{5}$. If all of them are boys then we have $\binom{12}{5}$ ways to choose and if all of them are girls then we have $\binom{8}{5}$. But we want at least one boy and at least one girl in the team, hence the good way of selection is $\binom{20}{5}$ - $\binom{12}{5}$ - $\binom{8}{5}$

[1 Mark for first part (5 students). 1 Mark for correctly computing for all boys/girls, 1 Mark for the final answer.]

Alternate Solution *Case-based analyis*
Case 1: Choosing 1 boy and 4 girls in $\binom{12}{1}\binom{8}{4}$ ways.
Case 2: Choosing 2 boys and 3 girls in $\binom{12}{2}\binom{8}{3}$ ways.
Case 3: Choosing 3 boys and 2 girls in $\binom{12}{3}\binom{8}{2}$ ways.
Case 4: Choosing 4 boys and 1 girl in $\binom{12}{4}\binom{8}{1}$ ways.
Now, Sum rule produces the final answer as $\binom{12}{1}\binom{8}{4} + \binom{12}{2}\binom{8}{3} + \binom{12}{3}\binom{8}{2} + \binom{12}{4}\binom{8}{1}$.

[1/2 mark per case. 1 Mark using sum rule to get the final answer]

[**Common mistake** Arguments like choosing 1 boy from 12 in $\binom{12}{1}$ ways *and* choosing 1 girl from 8 in $\binom{8}{1}$ ways *and* choosing remaining 3 from 18 students in $\binom{18}{3}$ ways *simultaneously* is **incorrect**. This leads to overcounting. *(why?)*]

(b) Number of all functions from $\{1, 2, \ldots n\}$ to $\{1, 2 \ldots n\}$ are n^n, and the number of injective functions are $n!$. So number of functions which are not injective is $n^n - n!$

[1 Mark for correctly stating number of functions from [n] to [n], 1 Mark for no. of injective functions and finally 1 Mark for the final result]

(c) Suppose we have a set of $r+1$ items numbered through $\{0, 1, \ldots, r\}$ and we want to choose a subset of size $m+n+1$ from it. Hence total possible ways of doing this is $\binom{r+1}{m+n+1}$(RHS).

In a different counting setting, consider any $m+1$ element subset of the same $r+1$ element set where k is largest. For a fixed k we have to choose m more elements from 0 to $k-1$ numbered items (k elements), in $\binom{k}{m}$ ways to get the $m+1$ sized subset. Now, to extend that $m+1$ sized subset to $m+n+1$, one can choose rest of the n elements among the items numbered greater than k, in $\binom{r-k}{n}$ ways. (With this method of selection, one can be guaranteed that each of the $m+n+1$ items will always be unique.)

Hence summing over all possible choices for the item labelled k we get the total number of $m+n+1$ elements from $r+1$ elements to be $\sum_{k=0}^{r} \binom{k}{m}\binom{r-k}{n}$(LHS).

[1 mark for RHS computation, 3 marks for computing LHS correctly. Using any techniques (like Induction or Binomial coefficent calculation) other than Double counting will fetch at most 2 marks (according to correct arguments).

Extremely sloppy arguments with *somewhat* correct idea (for LHS) will fetch 1 mark only]

4. [7=3+3+1 marks] Consider the relation $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x - y \in \mathbb{Q}\}$ on real numbers $\mathbb{R}$.

(a) Show that R is an equivalence relation.

(b) Under R, what are the equivalence classes of [1], $[\frac{1}{2}]$ and $[\pi]$? Recall that the equivalence class $[x]$ under R, for any $x \in \mathbb{R}$, is defined as $[x] = \{y \in \mathbb{R} \mid (x, y) \in R\}$.

(c) Is the set of equivalence classes of R finite? Why or why not?

Solution.

(a) **Reflexive**:

We have $\forall x \in \mathbb{R}$, $x - x = 0 \in \mathbb{Q} \implies (x, x) \in R$. Hence the given relation is reflexive.

Symmetric:

We have $\forall x, y \in \mathbb{R}$ such that $(x, y) \in R$, $x - y \in \mathbb{Q} \implies x - y = p/q$ for some integers p and $q \neq 0$. Hence, $y - x = -p/q \in \mathbb{Q} \implies (y, x) \in R$. Hence the given relation is symmetric.

Transitive:

We have $\forall x, y, z \in \mathbb{R}$ such that $(x, y) \in R$ and $(y, z) \in R$, $x - y = p_1/q_1$ and $y - z = p_2/q_2$. Hence, $x - z = x - y + y - z = p_1/q_1 + p_2/q_2 = (p_1q_2 + p_2q_1)/(q_1q_2) \in \mathbb{Q} \implies (x, z) \in R$. Hence the given relation is transitive.

Therefore, the given relation is an equivalence relation.

(b)
- $[1] = \mathbb{Q}$ (that is, the partition is the set of all rational numbers). For any $x \in \mathbb{R}$, $1-x \in \mathbb{Q} \iff 1 - x = p/q$ (for some integers p and $q \neq 0$) $\iff x = 1 - p/q \iff x \in \mathbb{Q}$. The last step comes from the closure property of rationals under subtraction.
- $[1/2] = \mathbb{Q}$ (by the same argument as above).
- $[\pi] = \{\pi + x | x \in \mathbb{Q}\}$.

(c) The set of equivalence classes of R is infinite. Observe that $[\sqrt{p_1}] \neq [\sqrt{p_2}]$ for primes $p_1 \neq p_2$. This can be shown using contradiction as follows: Assume that $\sqrt{p_1} R \sqrt{p_2}$ (that is, they belong in the same equivalence class). Then, $\sqrt{p_1} - \sqrt{p_2} = p/q \implies p_1 + p_2 - 2\sqrt{p_1p_2} = p^2/q^2$ (squaring both the sides) $\implies \sqrt{p_1p_2} \in \mathbb{Q}$ (using closure properties of rationals). But this is a contradiction since $\sqrt{p_1p_2}$ is an irrational number (the proof for this is similar to showing $\sqrt{2}$ is irrational).

Since there are infinitely many primes, and the square root of each belongs to a different equivalence class, we can conclude that the the equivalence classes for the given relation are infinite.

Aliter
Here is a proof by contradiction. We will prove something stronger i.e. the set of equivivalence classes here is uncountable. Assume, to the contrary, that this set is actually countable. Then let an enumeration of the set of equivalence classes be $\{C_0, C_1, \ldots, C_n, \ldots \ldots\}$(if this set is finite of size m, then this enumeration is only till C_m). For every class C_i, pick a (fixed) real x_i in this class.(note that since each C_i is non-empty we can indeed do this). Now, following is an injection f,from R to $\mathbb{N} \times \mathbb{Q}$:

$$\forall x \in \mathbb{R}, \text{ if x is in the class } C_j, \text{ then } f(x) = \{j, x - x_j\}$$

This function is well defined since every real number belongs to exactly one of the equivalence classes, and since x_j's are fixed, every x is mapped to exacly one element. Also, for every real x, the first coordinate of $f(x)$ is indeed in N because of the enumeration, its second coordinate will be $x - x_j$, but since x, x_j belong to the same equivalence class we must have $x - x_j \in \mathbb{Q}$ and hence the second coordinate of $f(x)$ belongs to $\mathbb{Q}$, so f indeed maps every real to exactly one element in $\mathbb{N} \times \mathbb{Q}$ and so f is indeed well defined.
Now to argue for the injectivity, lets say we had $f(x) = f(y)$ for some reals x and y. Then since the first coordinates of $f(x), f(y)$ are the same, they belong to the same equivalence class (say C_k), thus equating the second coordinates gives $x - x_k = y - x_k \implies x = y$. Hence we have shown that $\forall x, y \in \mathbb{R}, f(x) = f(y) \implies x = y$, hence f must be injective.
Thus since there is an injection between $\mathbb{R}$ and $\mathbb{N} \times \mathbb{Q}$ we must have that:

$$|\mathbb{R}| \leq |\mathbb{N} \times \mathbb{Q}| = |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$$

The first equality comes from the fact that $\mathbb{Q}$ is countably infinte and second from the fact that $\mathbb{N} \times \mathbb{N}$ is countably infinite. Hence we have $|\mathbb{R}| \leq \mathbb{N}$ which means $\mathbb{R}$ is countable, which is clearly not true(proved in class) and thus we arrive at a contradiction.Hence our assumption that the set of equivalence classes is countable should be incorrect. Therefore, the set of equivalence classes here is uncountable, and hence, infinite too!

Aliter 2

Consider the set $\{n\pi | n \in \mathbb{N}\}$. We claim that the equivalence classes corresponding to any two elements in S are distinct. For that, assume, to the contrary that $[m\pi] = [n\pi]$ for some $m \neq n \in \mathbb{N}$. Then this would mean that $(m-n)\pi \in \mathbb{Q}$. Now since $m-n$ is non zero rational, we can conlcude from this that π is rational. (as division of a rational by another rational is a rational). This however is false since π is irrational and hence our assumption that $[m\pi] = [n\pi]$ for some $m \neq n \in \mathbb{N}$ must be wrong. Thus to every element of S, a unique(upto S) equivalence class can be assigned, Now since S is infinite, we have found infinitely many equivalence classes. Hence there must indeed be an infinte number of equivalence classes wrt R. Note that in this proof, any irrational would work, nothing special about π.

[**Grading Scheme**]:

(a) [1 + 1 + 1 (One mark each for correctly showing the relation is reflexive, symmetric and transitive).]

(b) [1 + 1 + 1 (1 mark each for the correct answers, justification not needed,since wasn't asked in the problem statement).]

(c) [0.5 (for mentioning infinite) + 0.5 (for correct justification)]

5. [12=2+3+3+4 marks] Let $(P, \preceq)$ be a (non-empty) poset with $|P| = n$ and let $Q \subseteq P$. Q is called a *down-set* if whenever $x \in Q, y \in P$ and $y \preceq x$, then $y \in Q$. Let $\mathcal{D}(P)$ denote the set of all down-sets of P and $|\mathcal{D}(P)|$ its cardinality.

   (a) Prove that $(\mathcal{D}(P), \subseteq)$ is a poset.

   (b) What is $|\mathcal{D}(P)|$ if: (i) P is a chain; (ii) P is an anti-chain? Justify.

   (c) We say that R is an *up-set* if whenever $x \in R, y \in P$ and $x \preceq y$, then $y \in R$. Prove or disprove: Q is a down-set of P iff $P \setminus Q$ is an up-set of P.

   (d) Suppose the number of anti-chains in P is k. Then what is $|\mathcal{D}(P)|$ (as a function of k and n)? Justify.

Solution.

(a) Any collection of sets is a poset under the $\subseteq$ relation, since it is a partial order. $A \subseteq A$ is true by definition of a subset, thus the relation is reflexive. $A \subseteq B$ and $B \subseteq A \implies A = B$ because of definition of subset. Thus, the relation is antisymmetric. $A \subseteq B$ and $B \subseteq C \implies A \subseteq C$ because of the definition of subset relation. Thus, the relation is also transitive. Hence, the relation is a partial order.
[2 marks for showing all 3 properties.]
[1 mark for showing 2 properties.]
[1/2 marks for showing only 1 property. 0 otherwise]

(b) (i) Let the poset be $A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n$. If for a down-set Q, $A_k \in Q$, then for all $i \leq k$ $A_i \subseteq A_k \implies A_i \in Q$. Hence, a down-set can only be of the form $\{A_i, i \leq k\}$ where $k = 1, 2, \ldots, n$ or ϕ, therefore, $|\mathcal{D}(P)| = n + 1 = |P| + 1$.
(ii) Let the poset be $\{A_1, A_2, \ldots, A_n\}$ with an empty $\subseteq$. Any subset Q of P is a down-set, because $\forall\, y \in P\, \{(\exists\, x \in Q\, y \preceq x) \implies y \in Q\}$ is vacuously true since $\exists\, x \in Q\, y \preceq x \iff y \in Q$ for any $y \in P$. Hence, $|\mathcal{D}(P)| = 2^{|P|}$.

[0.5 marks for correct answer and 1 mark for correct justification for both parts. -1/2 for b(i) if just constructed the down-sets without explaining why other subsets of P are not downsets.]

(c) **Forward Direction**: We prove the claim by contradiction. Let Q be a down-set such that $P \setminus Q = R$ is not an up-set. R is not an up-set $\implies \exists\, y \in P\, \{y \notin R \wedge (\exists\, x \in R\, x \preceq y)\}$ which is the same as $\exists\, x \notin Q\, \exists\, y \in Q\, (x \preceq y)$ which contradicts the claim that Q is a down-set. Hence, Q is down-set $\implies P \setminus Q$ is an up-set.
Reverse Direction: We prove the claim by contradiction. Let Q be a subset of P such that $P \setminus Q = R$ is an up-set and Q is not a down-set. Q is not a down-set $\implies \exists, x \in Q, y \in P$ and $y \preceq x$ but $y \notin Q \implies y \in R$ which is the same as $\exists, x \in P, y \in R, (y \preceq x)$ and $x \in Q x \notin R$ which contradicts the claim that R is an up-set. Hence, $p \setminus Q$ is an up-set $\implies$ Q is a down-set.

[1.5 marks for forward and reverse direction. -1/2 if reverse direction proof written as similarly above. Partials based on headway in problem.]

(d) **Method 1**:
We will show a bijection from down-sets of P to anti-chains of P. Let f be a map from any down-set D to the set of maximal elements in D.

Claim: set of maximal elements is an anti-chain. Hence f is a map from down-sets to anti-chains.

Let's assume it is not, then $\exists m, m' \in M$ (set of maximal elements) such that $m \preceq m'$ and $m \neq m'$. However, m is supposed to be a maximal element of the corresponding downset(D) $\implies \nexists m'' \in D$ such that $m \preceq m''$ and $m \neq m''$. Thus a contradiction, hence the set of maximal elements is an anti-chain.

Claim: f is a well defined function. Let $f(D) = A_1$ and $f(D) = A_2$. Wlog, assume $\exists m \in A_1$ such that $m \notin A_2$. $m \in A_1 \implies \nexists m' \in D$ such that $m \preceq m'$ and $m \neq m'$. $m \notin A_2 \implies \exists m' \in D$ such that $m \preceq m'$ and $m \neq m'$. Thus, this is a contradiction. Hence, no such m exists $\implies A_1 \subseteq A_2$. Similarly, $\nexists m$ such that $m \in A_2$ and $m \notin A_1 \implies A_2 \subseteq A_1 \implies A_1 = A_2$. Thus, f is a well defined function.

Claim: f is injective. If down-sets D_1, D_2 have same set of maximal elements, i.e. $f(D_1) = f(D_2)$, then wlog any $x \in D_1$ is either a maximal element i.e. $x \in f(D_1) = f(D_2)$ hence in D_2, else $x \preceq y$ for some $y \in f(D_1) = f(D_2)$, but D_2 is also a down-set so $x \in D_2 \implies D_1 \subseteq D_2$. Similarly, $D_2 \subseteq D_1 \implies D_1 = D_2$. Thus, f is an injection.

Claim: f is surjective. To see this take any anti-chain A in P. Take its down-ward closure, i.e., $D(A) = \{y \in P \mid y \preceq x, x \in A\}$. Mini-claim : $D(A)$ is a down-set. Let's assume it is not. Thus, $\exists y \in D(A)$ and $x \in P$ such that $x \preceq y$ and $x \notin D(A)$. However, $x \preceq y$ and as $y \in D(A), y \preceq m$ for some $m \in A$. Therefore, $x \preceq y$ and $y \preceq m \implies x \preceq m \implies x \in D(A)$. Thus a contradiction. Hence our assumption was false and hence $D(A)$ is a downset. Thus, for every antichain A of P, there exists a downset $D(A)$ such that $f(D(A))$ is the antichain A itself. Therefore, f is a surjection.

Therefore, f is a bijection from the set of down-sets to the set of anti-chains. As we are only considering finite posets, both these sets are finite and thus their size is the same. Therefore, number of down-sets is k.

Method 2:
Given M an anti-chain, let $f(M)$ be $\{x \in P \,|\, x \preceq m, m \in M\}$ [downwards closure of M].
Claim: $f(M)$ is a down-set for any anti-chain M.
Proof: Consider any $x \in P$ such that $x \preceq y$ for some y in $f(M)$. $y \in f(M) \implies y \preceq m$ for some $m \in M$. Hence, $x \preceq y \preceq m \implies x \preceq m \implies x \in f(M)$.
Given $D \in \mathcal{D}(P)$, let $g(D)$ be the set of maximal elements of D.

Claim: $g(f(M)) = M$ for all $M \in$ {set of anti-chains of P}.
Proof: Assume that some $m \in M$ is not a maximal element in $f(M) \implies \exists x \in f(M)$ such that $x \neq m$ and $m \preceq x$. $x \in f(M) \implies x \preceq m'$ for some $m' \in M$, hence $m \preceq x \preceq m'$ with $x \neq m$. Given that M is an anti-chain, this implies $m = m'$ leading to a contradiction. Hence $m \in g(f(M))$ for all $m \in M \implies M \subseteq g(f(M))$.
Now let x be a maximal element of $f(M)$ such that $x \notin M$. But, $x \in f(M) \implies x \preceq m$ for some $m \in M (\implies m \neq x, m \in f(M))$, which is a contradiction. Hence $g(f(M)) \subseteq M$, completing the proof.

Claim: $f(g(D)) = D$ for all $D \in \mathcal{D}(P)$.
Proof: Consider any $x \in f(g(D)) \implies x \preceq m$ for some $m \in g(D) \implies m \in D$, hence $x \in D$ by the down-set property $\implies f(g(D)) \subseteq D$.
Now assume to the contrary that $D' = D \setminus f(g(D)) \neq \phi$. Let m' be a maximal element of D'. Since $m' \notin f(g(D)) \implies m' \notin g(D)$, it is not a maximal element of D. Hence, $\exists x \in D$ such that $x \neq m'$ and $m' \preceq x$. Since m' is maximal in D', $x \notin D' \implies x \in f(g(D)) \implies x \preceq m$ for some $m \in g(D) \implies m' \preceq x \preceq m \implies m' \preceq m \implies m' \in f(g(D))$, leading to a contradiction. Hence, $D = f(g(D))$.

Treating f as a function from the set of anti-chains to the set of down-sets, we have shown that g is a left-inverse $\implies f$ is injective, and also that g is a right-inverse $\implies f$ is surjective. Hence, f is a bijection $\implies |\mathcal{D}(P)| = |\{\text{set of anti-chains of } P\}| = k$.

[1 Mark for correct answer and 3 marks for correct proof. Partials based on headway in problem.]