# CS 105: DIC on Discrete Structures

Instructor : S. Akshay

Sept 26, 2023

Lecture 19 – Counting and Combinatorics

Solving Recurrence relations via generating functions

# Last few weeks

**Basic counting techniques and applications**

1. Sum and product, bijection, double counting principles
2. Binomial coefficients and binomial theorem, Pascal's triangle
3. Permutations and combinations with/without repetitions
4. Counting subsets, relations, Handshake lemma
5. Stirling's approximation: Estimating $n!$
6. Recurrence relations and one method to solve them.

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**Today**

Solving recurrence relations via generating functions.

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- Recall the recurrence for Catalan Numbers:
  $C(n) = \sum_{i=1}^{n-1} C(i)C(n-i)$ for $n > 1$, $C(0) = C(1) = 1$.
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## Reading assignment

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By solving, we mean give a closed-form expression for $n^{th}$ term.

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1. $\alpha^n = \alpha^{n-1} + \alpha^{n-2}$ implies $\alpha^{n-2}(\alpha^2 - \alpha - 1) = 0$.
2. So if $\alpha^2 - \alpha - 1 = 0$, the recurrence holds for all $n$.
3. Solving, $\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}$
4. Thus, general solution is $F_n = a(\frac{1+\sqrt{5}}{2})^n + b(\frac{1-\sqrt{5}}{2})^n$.
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(t+t^2)\phi(t) &= \sum_{n=0}^{\infty} F(n) t^n - 1 \\
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**thus, as before**

$$F(n) = \frac{\sqrt{5}+1}{2\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{\sqrt{5}-1}{2\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$$

# Properties of generating functions

**Definition**

The (ordinary) generating function for a sequence $a_0, a_1, \ldots \in \mathbb{R}$ is the infinite series $\phi(x) = \sum_{k=0}^{\infty} a_k x^k$.

- Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$, $g(x) = \sum_{k=0}^{\infty} b_k x^k$. Then
  1. If $f(x) = g(x)$, then $a_k = b_k$ for all $k$.
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## The extended binomial theorem

Let $u \in \mathbb{R}$, $(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k$.
If you don't like this, take $x \in \mathbb{R}, |x| < 1$.

# Simple examples using generating functions

**Standard identities:**

- $\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k$
- $\frac{1}{1-x^r} = \sum_{k=0}^{\infty} x^{rk}$
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2. Solve the recurrence $a_k = 8a_{k-1} + 10^{k-1}$ with $a_0 = 1, a_1 = 9$.

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  - Observe that $\phi(x) = (1 + x + x^2 + \ldots)^n = (1 - x)^{-n}$
  - Expand this by the extended binomial theorem and compare coefficients of $x^k$.
  - $a_k = \binom{-n}{k}(-1)^k = (-1)^k \binom{n+k-1}{k}(-1)^k = \binom{n+k-1}{k}$.
  - (H.W) What if there must be $\geq 1$ element of each type?
- Proving binomial identities: Show that $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.
  - Compare coefficients of $x^n$ in $(1 + x)^{2n} = ((1 + x)^n)^2$.
- (H.W) Write a recurrence for the number of derrangements. That is, no. of ways to arrange $n$ letters into $n$ addressed envelopes such that no letter goes to the correct envelope.

# Other applications of generating functions

- What is the number of ways $a_k$ of selecting $k$ elements from an $n$ element set if repetitions are allowed?
  - Let $\phi(x) = \sum_{k=0}^{\infty} a_k x^k$.
  - Observe that $\phi(x) = (1 + x + x^2 + \dots)^n = (1 - x)^{-n}$
  - Expand this by the extended binomial theorem and compare coefficients of $x^k$.
  - $a_k = \binom{-n}{k}(-1)^k = (-1)^k\binom{n+k-1}{k}(-1)^k = \binom{n+k-1}{k}$.
  - (H.W) What if there must be $\geq 1$ element of each type?
- Proving binomial identities: Show that $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.
  - Compare coefficients of $x^n$ in $(1 + x)^{2n} = ((1 + x)^n)^2$.
- (H.W) Write a recurrence for the number of derrangements. That is, no. of ways to arrange $n$ letters into $n$ addressed envelopes such that no letter goes to the correct envelope.
- (H.W) How many ways can a convex $n$-sided polygon be cut into triangles by adding non-intersecting diagonals (i.e., connecting vertices with non-crossing lines)? Write a recurrence and solve it!