MA 105: D3 Lecture 17

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Partial Derivatives

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$(PV = mrT)$.

Differentiability

Exercise 4. Show that the function $f(x,y) = x^2 + y^2$ is differentiable at every point in $\mathbb{R}^2$ from first principles, that is from the definition of differentiability (how will you do it if we omit "from first principles" from the question)?

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Solution: The partial derivatives at (x_0, y_0) are $2x_0$ and $2y_0$. We have

$$(x_0 + h)^2 + (y_0 + k)^2 - 2x_0 h - 2y_0 h = h^2 + k^2 = \|(h, k)\|^2.$$

Obviously,

$$\lim_{(h,k)\to 0} \frac{(x_0 + h)^2 + (y_0 + k)^2 - 2x_0 h - 2y_0 h}{\|(h, k)\|} = \lim_{(h,k)\to 0} \|(h, k)\| = 0.$$

If we are not required to use the definition, we can observe that

the two partial derivatives ($2x$ and $2y$ respectively) are continous functions everywhere, and hence, also in any disc around any point (x_0, y_0). Hence $f(x, y)$ is differentiable at every point in $\mathbb{R}^2$.

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(a) $f(x, t) = e^{-2x}\cos(2\pi t)$, (b) $L(x, y, z) = xze^{-y^2-z^2}$

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Exercise 9. Show that the sum of the x-, y-, and z- intercepts of any tangent plane to the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{c}$ is a constant.

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Exercise 10. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function satisfying $f(tx, ty) = t^n f(x, y)$ for all t, x and y, and some positive integer n. Show that

$$x_0 \frac{\partial f}{\partial x}(x_0, y_0) + y_0 \frac{\partial f}{\partial y}(x_0, y_0) = nf(x_0, y_0).$$