Relations

1. A relation from a set A to itself is called a relation on A. Let R_1, R_2 be relations on a set A. Then $R_1.R_2$ is a relation on A defined by $(a, c) \in R_1.R_2$ if and only if there exists $b \in A$ such that $(a, b) \in R_1$ and $(b, c) \in R_2$, for all $a, c \in A$. The operation . is called composition. Prove that . is an associative operation. Does there exist an identity for this operation? Does there always exist an inverse, and if not, for what relations does there exist one? Prove that if R_1, R_2 are functions/ injections/ surjections/ bijections, then $R_1.R_2$ satisfies the same property. Does the converse of this statement hold? Note that A may not be a finite set.

2. Let R be a relation from A to B and let R^{-1} be the relation from B to A defined by $(b, a) \in R^{-1}$ iff $(a, b) \in R$. Suppose that R contains a 1-1 function f and R^{-1} contains a 1-1 function g. Prove that R contains a bijection from A to B. This is a stronger form of the Schroder-Bernstein theorem, which is a particular case, when R is $A \times B$. Note that your argument should hold for any sets A and B and not only finite sets. What properties of sets are used in the proof? Using this, show that there exists a bijection from the real-numbers to the power-set of the natural numbers.

3. This is another (non-constructive) proof of Hall's theorem by induction on $|R|$. Let R be a relation from A to B that satisfies Hall's condition $|R(X)| \geq |X|$ for all subsets $X \subseteq A$. Suppose there exists a pair $(a, b) \in R$ such that $R - (a, b)$ satisfies Hall's condition. Then we can find a 1-1 function in $R - (a, b)$. Suppose that for every pair $(a, b) \in R$, $R - (a, b)$ does not satisfy Hall's condition. Then prove that R itself must be a 1-1 function from A to B.

4. Let R be a relation from A to B for finite sets A and B, such that there exists a number $k \geq 1$ such that $|R(a)| \geq k$ for all $a \in A$ and $|R^{-1}(b)| \leq k$ for all $b \in B$. Prove that R contains a 1-1 function from A to B. Let $\mathcal{A}$ be the set of all subsets of size k of $\{1, 2, \ldots, n\}$ where $k < n/2$. Let $\mathcal{B}$ be the set of all subsets of size $k+1$ of $\{1, 2, \ldots, n\}$. The relation R from $\mathcal{A}$ to $\mathcal{B}$ is defined by $(X, Y) \in R$ iff $X \subseteq Y$ for a subset $X \in \mathcal{A}$ and $Y \in \mathcal{B}$. Prove that R contains a 1-1 function from $\mathcal{A}$ to $\mathcal{B}$. Describe one such function f explicitly by showing how to compute $f(X)$ given $X \in \mathcal{A}$. Try to extend this to the set of all multisubsets of size k of a multiset of size n.

5. Suppose there are n students and m companies. A relation R from students to companies is defined by $(s, c) \in R$ iff the student s is to be interviewed by company c. The interviews are conducted in slots of 15 minutes. A student can be interviewed by at most one company in a slot and a company can interview at most one student in a slot. Suppose that every student is to be interviewed by at most D companies and every company has to interview at most D students. Prove that it is always possible to schedule the interviews so that at most D slots are required. Is it always possible to find such a schedule such that the slots for every company are consecutive, though they may start at different times? If so prove it, else find a counterexample.