MA 105 Lecture 9

Ravi Raghunathan

Department of Mathematics

August 28

The Darboux integral

A Feynman story

Riemann integration

Partitions

Definition: Given a closed interval $[a, b]$, a partition P of $[a, b]$ is simply a collections of points

$$P = \{a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b\}.$$

We can think of the points of the partition as dividing the original interval $[a, b]$ into sub-intervals $I_j = [x_{j-1}, x_j]$, $1 \leq j \leq n$. Indeed $I = \cup_j I_j$ and if two sub-interval intersect, they have at most one point in common. Hence, the notation "partition".

Definition: A partition $P' = \{a = x'_0 < x'_1 < \ldots < x'_m = b\}$ is said to be a refinement of the partition P if for each $x_i \in P$, there exists an $x'_j \in P'$ such that $x_i = x'_j$.

Intuitively, a refinement P' of a partition P will break some of the sub-intervals in P into smaller sub-intervals. Any two partitions have a common refinement - why?.

Lower and Upper sums

Given a partition $P = \{a = x_0 < x_1 < \ldots < x_{b-1} < x_n = b\}$ and a function $f : [a, b] \to \mathbb{R}$, we define two associated quantities. First we set:

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x) \quad \text{and} \quad m_i = \inf_{x \in [x_{i-1}, x_i]} f(x), \quad 1 \leq i \leq n$$

Defintion: We define the Lower sum as

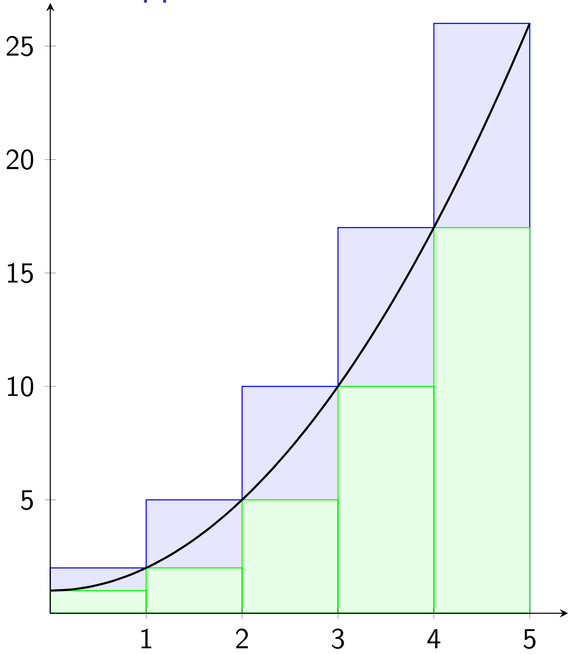
$$L(f, P) = \sum_{j=1}^{n} m_j (x_j - x_{j-1}).$$

Similarly, we can define the Upper sum as

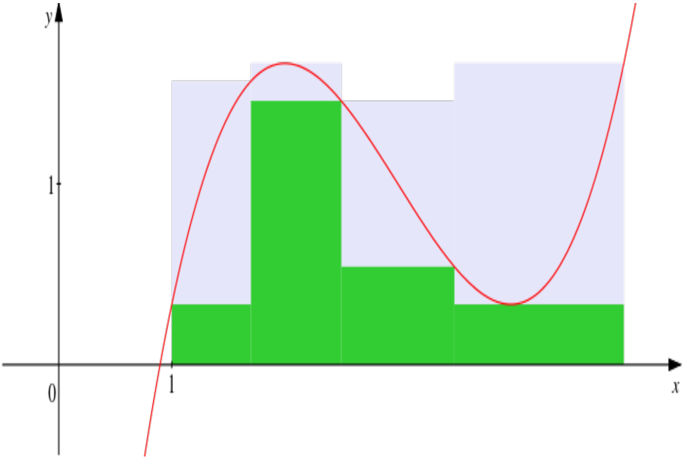
$$U(f, P) = \sum_{j=1}^{n} M_j (x_j - x_{j-1}).$$

In case the words "infimum" and "supremum" bother you, you can think "minimum" and "maximum most of time since we will usually be dealing with continuous functions on $[a, b]$.

Lower and Upper Darboux sums



A picture for a non-monotonic function



https://upload.wikimedia.org/wikipedia/commons/thumb/5/59/Darboux.svg/700px-Darboux.png

One basic example

In order to illustrate what we are saying we will take the following basic example. Let $[a, b] = [0, 1]$ and let $f(x) = x$.

One of the most natural partitions on an interval is a partition that divides the interval into sub-intervals of equal length. For $[0, 1]$, this is

$$P_n = \{0 < 1/n < 2/n < \ldots < (n-1)/n < 1\}.$$

On the interval $I_j = [\frac{j-1}{n}, \frac{j}{n}]$, where does the function $f(x) = x$

take its minimum? its maximum?

Clearly, the minimum $m_j = \frac{j-1}{n}$ is attained at $\frac{j-1}{n}$ and the maximum $M_j = \frac{j}{n}$ at $\frac{j}{n}$. And finally, $\frac{j}{n} - \frac{j-1}{n} = \frac{1}{n}$, for all $1 \leq j \leq n$.

An example of a refinement of P_n is P_{2n}, or, more generally, P_{kn} for any natural number k.

The Darboux integrals

We now define the lower Darboux integral of f by

$$L(f) = \sup\{L(f, P) \colon P \text{ is a partition of } [a, b]\},$$

where the supremum is taken over all partitions of $[a, b]$.

and similarly the upper Darboux integral of f by

$$U(f) = \inf\{U(f, P) \colon P \text{ is a partition of } [a, b]\},$$

and again the infimum is over all partitions of $[a, b]$. (This time there is no escaping inf and sup!)

If $L(f) = U(f)$, then we say that f is Darboux-integrable and define

$$\int_a^b f(t)\,dt := U(f) = L(f).$$

This common value of the two integrals is called the Darboux integral.

Back to the example

Let us calculate $L(f, P_n)$ and $U(f, P_n)$ in the example we gave.

$$L(f, P_n) = \sum_{j=1}^{n} \frac{(j-1)}{n} \cdot \frac{1}{n} = \sum_{j=0}^{n-1} \frac{j}{n^2}.$$

This can be evaluated explicitly:

$$L(f, P_n) = \frac{n(n-1)}{2} \cdot \frac{1}{n^2} = \frac{1}{2} - \frac{1}{2n}.$$

Similary, we can check that

$$U(f, P_n) = \frac{n(n+1)}{2} \cdot \frac{1}{n^2} = \frac{1}{2} + \frac{1}{2n}.$$

Can we conclude that the Darboux integral is $1/2$ by letting $n \to \infty$? Unfortunately, no.

A diversion: How to calculate powers of e in your head?

From Richard Feynman's "Surely you're joking Mr. Feynman!" (pages 173-174):

One day at Princeton I was sitting in the lounge and overheard some mathematicians talking about the series for e to the x power which is $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$. Each term you get by multiplying the preceding term by x and dividing by the next number. For example, to get the next term after $x^3/3!$ you multiply that term by x and divide by 4. It's very simple.

When I was a kid I was excited by series, and had played with this thing. I had computed e using that series, and had seen how quickly the new terms became very small.

I mumbled something about how it was easy to calculate e to any power using that series (you just substitute the power for x).
"Oh yeah?" they said. "Well, the what's e to the 3.3?" said some joker - I think it was Tukey.
I say, "That's easy. It's 27.11."

Feynman's anecdote continued

Tukey knows it isn't so easy to compute all that in your head.
"Hey! How'd you do that?"
Another guy says, "You know Feynman, he's just faking it. It's not really right."
They go to get a table, and while they're doing that, I put on a few more figures: "27.1126," I say.
They find it in the table. "It's right! But how'd you do it!"
"I just summed the series."
"Nobody can sum the series that fast. You must just happen to know that one. How about e^3?"
"Look," I say. "It's hard work! Only one a day!"
"Hah! It's a fake!" they say, happily.
"All right," I say, "It's 20.085."
They look in the book as I put a few more figures on. They're all excited now, because I got another one right.

Here are these great mathematicians of the day, puzzled at how I can compute e to any power! One of them says, “He just can't be substituting and summing - it's too hard. There's some trick. You couldn't do just any old number like e to the 1.4.”
I say, “It's hard work, but for you, OK. It's 4.05.”
As they're looking it up, I put on a few more digits and say, “And that's the last one for the day!” and walk out.

What happened was this: I happened to know three numbers - the logarithm of 10 to the base e (needed to convert numbers from base 10 to base e), which is 2.3026 (so I knew that e to the 2.3 is very close to 10), and because of radioactivity (mean-life and half-life), I knew the log of 2 to the base e, which is .69315 (so I also knew that e to the .7 is nearly equal to 2). I also knew e (to the 1), which is 2.71828.

The first number they gave me was e to the 3.3, which is e to the 2.3 (10) times e, or 27.18. While they were sweating about how I was doing it, I was correcting for the extra .0026 - 2.3026 is a little high.

I knew I couldn't do another one; that was sheer luck. But then the guy said e to the 3: that's e to the 2.3 times e to the .7, or ten times two. So I knew it was 20.something, and while they were worrying how I did it, I adjusted for the .693.

Now I was sure I couldn't do another one, because the last one was again by sheer luck. But the guy said e to the 1.4 which is e to the .7 times itself. So all I had to do is fix up 4 a little bit!

They never did figure out how I did it.

https://en.wikipedia.org/wiki/Richard_Feynman Richard Feynman (1918-1988)

Useful properties of the Darboux sums

Since, for any partition P, $L(f, P) \leq U(f, P)$, we have

$$L(f) \leq U(f).$$

In fact, for any two partitions P_1 and P_2, we have

$$L(f, P_1) \leq U(f, P_2).$$

This is easy to see - the lower sum computes the sum of the areas of rectangles that lie entirely below the curve while the upper sum computes the sum of the areas of rectangles whose "tops" lie above the curve.

One of the most useful properties of the Darboux sums is the following. If P' is a refinement of P then obviously

$$L(f, P) \leq L(f, P') \leq U(f, P') \leq U(f, P).$$

Riemann Sums

There is another way of getting at the integral due to Riemann which may be a little more intuitive and is better for calculation. This is done via Riemann sums.

To define the notion of a Riemann sum we need one more piece of data. Suppose that for each of the intervals I_j we are given a point $t_j \in I_j$. We will denote the collection of points t_j by t. The pair (P, t) is sometimes called a tagged partition.

Definition: We define the Riemann sum associated to the function f, and the tagged partition (P, t) by

$$R(f, P, t) = \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1}).$$

The norm of a partition

As must be clear, the Lower sum, Upper sum and Riemann sum all give approximations to the area between the lines $x = a$ and $x = b$ and between the curve $y = f(x)$ and the x-axis and

$$L(f, P) \leq R(f, P, t) \leq U(f, P).$$

The point is to make this statement quantitatively precise.

We define the norm of a partition P (denoted $\|P\|$) by

$$\|P\| = \max_j \{|x_j - x_{j-1}|\}, \quad 1 \leq j \leq n.$$

The norm gives some measure of the "size" of a partition, in particular, it allows us to say whether a partition is big or small.

When the size of the partition is small, it means that every interval in the partition is small.