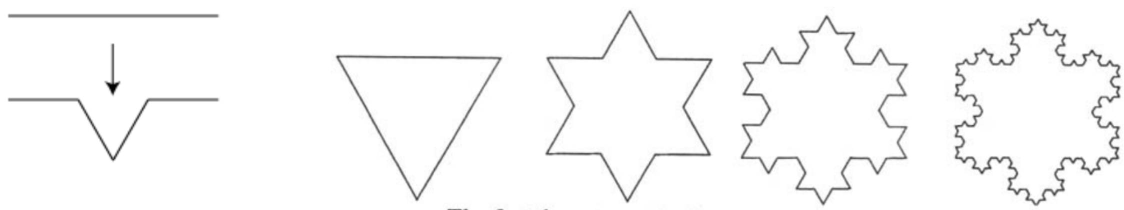
CS105 2023 Discrete Structures
Pop Quiz solutions

Grade A if Q1 and Q2.1 are fully correct and Q2.2 partial. Grade B if Q1 is correct, but Q2 is only partially correct. Grade C if Q1 is incorrect or only partially correct.

1 **(0.5 marks)** Base case: $K(0)$ has $4^0 \cdot 3 = 3$ line segments. Hence the base case holds.
(0.5 marks) Induction Hypothesis: Assume that the claim is true for $n = k$, i.e., $K(k)$ has $4^k \cdot 3$ line segments.



The first four stages in the construction of the Koch snowflake

(1 mark) Induction step: For $n = k + 1$: In constructing $K(k+1)$ from $K(k)$, each segment develops a protrusion, leading to 4 shorter segments, hence

$$\#\text{ line segments in } K(k+1) = 4 \cdot \#\text{ line segments in } K(k) = 4 \cdot (4^k \cdot 3) \text{ (by indn hyp) } = 4^{k+1} \cdot 3$$

Conclusion: Hence, $\forall n \in \mathbb{N}, K(n)$ has $4^n \cdot 3$ line segments.

2.1 Let $P(n)$ denote the perimeter of $K(n)$. We can claim $P(n) = P(0)\left(\frac{4}{3}\right)^n = 3 \cdot \left(\frac{4}{3}\right)^n$.

Base case: For $n = 0, P(0) = 3 = 3 \cdot \left(\frac{4}{3}\right)^0$

Induction Hypothesis: Assume the claim is true for $n = k$, that is $P(k) = 3 \cdot \left(\frac{4}{3}\right)^k$

Induction step: For $n = k + 1$: Each step duplicates the middle third of each segment, leading to an increase by a factor of $\frac{4}{3}$, hence $P(k+1) = \frac{4}{3}P(k) = 3 \cdot \left(\frac{4}{3}\right)^{k+1}$. (by ind. hypothesis)

Conclusion: Hence, this holds for all $n \in \mathbb{N}$.

Grading Scheme: 1 mark for correct result and 1 mark for correct proof. No partial marks.

2.1 Alternate Solution:
First we prove that all the line segments of $K(n)$ have length 3^{-n}.
Base case: each side of $K(0)$ is of length $1 = 3^{-0}$.
Induction hypothesis: Assume claim is true for $n = k$, that is sides of $K(k)$ are 3^{-k} long.

Induction step: For $n = k + 1$: The construction of $K(k+1)$ replaces each side of length l with 4 segments of length $l/3$. Hence each side of $K(k+1)$ is of length $3^{-k}/3 = 3^{-(k+1)}$
Conclusion: Our claim holds for all $n \in \mathbb{N}$.
Now, as we know the number of sides from 1st question and the length of each side, the perimeter is nothing but product of both. Therefore

$$P(n) = 3^{-n} \times (4^n \cdot 3)$$
$$P(n) = 3 \cdot \left(\frac{4}{3}\right)^n$$

2.2 As shown above, the length of all line segments in $K(n)$ is 3^{-n}
Now, letting $A(n)$ denote the area of $K(n)$, we can express $A(n+1)$ as:

$$\begin{aligned} A(n+1) &= A(n) + \# \text{ of protrusions} \cdot \text{area of each protrusion} \\ &= A(n) + 4^n \cdot 3 \times \frac{\sqrt{3}}{4}\left(3^{-(n+1)}\right)^2 \\ &= A(n) + \frac{1}{4\sqrt{3}}\left(\frac{4}{9}\right)^n \end{aligned}$$

Now we claim*:
$$A(n) = \frac{2\sqrt{3}}{5} - \frac{3\sqrt{3}}{20}\left(\frac{4}{9}\right)^n$$

Base case: $A(0) = \dfrac{\sqrt{3}}{4} = \dfrac{2\sqrt{3}}{5} - \dfrac{3\sqrt{3}}{20}$
Induction Hypothesis: Assume the claim holds true for $n = k$, that is $A(k) = \frac{2\sqrt{3}}{5} - \frac{3\sqrt{3}}{20}\left(\frac{4}{9}\right)^k$
Induction step: Assuming the expression for $A(k)$,

$$\begin{aligned} A(k+1) &= A(k) + \frac{1}{4\sqrt{3}}\left(\frac{4}{9}\right)^k \\ A(k+1) &= \frac{2\sqrt{3}}{5} - \frac{3\sqrt{3}}{20}\left(\frac{4}{9}\right)^k + \frac{1}{4\sqrt{3}}\left(\frac{4}{9}\right)^k \\ &= \frac{2\sqrt{3}}{5} - \frac{1}{5\sqrt{3}}\left(\frac{4}{9}\right)^k \\ &= \frac{2\sqrt{3}}{5} - \frac{3\sqrt{3}}{20}\left(\frac{4}{9}\right)^{k+1} \end{aligned}$$

$\implies$ this holds for all $n \in \mathbb{N}$.
Grading Scheme: 1 mark for correct result and 1 mark for correct proof.
Note: There can be other solutions for this question, this is not the only way.
Observe that as $n \to \infty$, the area converges to $\dfrac{2\sqrt{3}}{5}$ but the perimeter blows up to ∞.

**Think about how to arrive at such a claim. Hint: Sum of geometric progression*