MA 105 Endsem TSC

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https://github.com/K1ngPat/FreshieCourses

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Average freshie 1 night before Endsem: click here

Hi, and welcome to the MA105 Endsem TSC. We plan to give you a quick formula revision and then focus on solving questions relevant to the exam.

Please raise your hand at any time for clarifications or to answer, interactive classes help all of us remember it better.

Major disclaimer:

These are **not** official course slides by any means. This is just a small recap to go over every concept broadly and give you an idea to understand things intuitively. The only resource which actually has *all* the information you need to do well, are the prof's slides. So, be sure to go through them as well.

Credits for helping with the slides to Agnipratim Nag, who took the TSC last year :)

2D Nostalgia

A multiple integral of a function f is considered to be integrable if and only if, for arbitrarily small ϵ, there exists a partition P_ϵ such that:

$$U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$$

Fubini's Theorem:
Consider an integrable function f, on a rectangular domain R $[a, b] \times [c, d]$.

$$\int_a^b \int_c^d f(x, y)\, dy\, dx = \int_a^b \left(\int_c^d f(x, y) dy \right) dx = \int_c^d \left(\int_a^b f(x, y) dx \right) dy$$

If every term in the above expression is well-defined.

IMAX 3D

Again, Fubini says that if f is integrable, any iterated integral that exists **has to be equal** to the triple integral.

Also, keep in mind that if (for example) the order of integration is dxdydz, the limits of z will be constants, of y will be functions of z, and of x will be functions of x and y.

Jacob? Jacob!

The Jacobian is given by the determinant of the derivative matrix of $x = h_1(u, v)$ and $y = h_2(u, v)$:

$$\begin{pmatrix} \frac{\partial h_1}{\partial u} & \frac{\partial h_1}{\partial v} \\ \frac{\partial h_2}{\partial u} & \frac{\partial h_2}{\partial v} \end{pmatrix}$$

$|\mathbf{J}|$ gives us $\frac{dA_{XY}}{dA_{UV}}$. There used as such:

Theorem (Change of variables)

Let D be a closed and bounded set, with the continuous function f defined on it. Consider a change of variable by a one-one differentiable function h, which has non-zero $|J|$, which maps D in XY to D^* in UV. Then:

$$\iint_D f(x, y)\, dx\, dy = \iint_{D^*} f(x(u, v), y(u, v)) * |J(u, v)|\, du\, dv$$

They are not pajamas!

These are three concepts which we will use very often:

1. Divergence: Acts on a **vector** and produces a **scalar**.
2. Curl: Acts on a **vector** and produces a **vector**.
3. Gradient: Acts on a **scalar** and produces a **vector**.

They are all denoted using the "del" operator which you can think of as a vector:

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

- Divergence is $\vec{\nabla}.\vec{F}$
- Curl is $\vec{\nabla} \times \vec{F}$
- Gradient is $\vec{\nabla} f$

Get in line

The line integral of a vector field $\vec{F}$ (x, y, z) over a curve C is denoted by $\int_C \vec{F}.\vec{ds}$. To evaluate this, we:

- First, write the curve C as $\vec{C}(t) = (x(t), y(t), z(t))$ for t $\in$ [a, b] (may be piecewise differentiable)
- The $\vec{ds}$ vector earlier referred to, is:

$$\vec{ds} = < x'(t), y'(t), z'(t) > dt$$

- Now, take the dot product of $\vec{F}(t)$ and $\vec{ds}$ and integrate on the domain [a, b].
- Any parametrization of the curve will lead to same answer, but traversing it the other way will lead to negative of the answer.

Conservative ~~Economic Policies~~ Fields

A vector field is said to be **conservative** if it is the gradient of a scalar function f, which is $\vec{F} = \nabla f$.

Using the Fundamental Theorem of Calculus, for a differentiable function f on a continuous smooth path, we have:

$$\int_{start}^{end} \nabla f \, ds = f(end) - f(start)$$

- Conservative $\implies$ Path Independent
- Path Independent $\not\Longrightarrow$ Conservative
- Path Independent + Simply Connected Domain $\implies$ Conservative

If the area does not have any holes and does not consist of two or more pieces, then it is simply connected.

Not your favourite shade of Green

Green's Theorem:

Theorem

1. *Let D be a bounded region in $\mathbb{R}^2$ with a positively oriented boundary ∂D consisting of a finite number of non-intersecting simple closed piecewise C^1 curves.*
2. *Let Ω be an open set in $\mathbb{R}^2$ such that $D \cup \partial D \subset \Omega$ and consider F_1 and F_2 which are C^1 functions from Ω to $\mathbb{R}^2$.*

Then:

$$\int\limits_{\partial D} F_1\,dx + F_2\,dy = \iint\limits_{D} \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\,dA$$

Green's Theorem v2.0, Beta release

Green's Theorem - Divergence Edition:

$$\int_{\partial D} \vec{F}.\vec{n}\, dr = \iint_{D} \vec{\nabla}.\vec{F}\, dA$$

Green's Theorem - Curl Edition:

$$\int_{\partial D} \vec{F}.\vec{dr} = \iint_{D} (\vec{\nabla} \times \vec{F}).\vec{k}\, dA$$

Surfer? Surface? Surf Excel?

Evaluating the surface integral of a **scalar field** goes like:

$$\iint_D f(x, y, z)\, dS = \iint_E f(x(u, v), y(u, v), z(u, v))||\vec{n}(u, v)||\, du\, dv$$

$$\text{where } \vec{n}(u, v) = \vec{\phi}_u(u, v) \times \vec{\phi}_v(u, v)$$

where f(u, v) is obtained by substituting x, y and z as functions of (u, v) into f(x, y, z). E is the domain in UV space, whose image is D.

To evaluate area of a surface, simply put f = 1.

For a **vector field**, a surface integral looks like:

$$\iint_D \vec{F}.\vec{dS} = \iint_E \vec{F}(u, v).\vec{n}(u, v)\, du\, dv$$

$\vec{n}$ is the usual normal vector that we have defined earlier. After taking the dot product, the surface integral is business as usual.

Can't carry England alone

Stoke's Theorem is Green's but *different*.

Theorem

- *Let S be a piecewise C^2, bounded, oriented surface in $\mathbb{R}^2$ whose piecewise smooth intrinsic boundary ∂S consists of a finite number of non-intersecting simple closed curves along with their induced orientation.*
- *Let $\vec{F}$ be a C^1 vector field.*

Then:

$$\int\limits_{\partial D} \vec{F}.\vec{dr} = \iint\limits_{D} \overrightarrow{curl F}.\vec{dA}$$

- Gradient field $\implies$ Zero curl
- Zero curl $\not\Longrightarrow$ Gradient field (Example?)
- Zero curl + Simply Connected $\implies$ Gradient field

Wait, ye physics mai nahi tha?

Gauss' Theorem:

Theorem

- *Let D be a closed and bounded subset of $\mathbb{R}^3$ whose boundary ∂D consists of a finite number of non intersecting piecewise smooth surfaces without any edges and is positively oriented*
- *Let $\vec{F}$ be a C^1 vector field in D.*
  Then:

$$\iint\limits_{\partial D} \vec{F}.\, d\vec{S} = \iiint\limits_{D} \vec{\nabla}.\vec{F}\, dV$$

A few things:

- F is a curl field $\implies \vec{\nabla}.\vec{F} = 0$
- $\vec{\nabla}.\vec{F} = 0 \not\Longrightarrow$ F is a curl field
- $\vec{\nabla}.\vec{F} = 0 +$ Simply Connected $\implies$ F is a curl field

Question 1: A simple change of variables

Question: If D is the region $|x| + |y| \leq 1$ then evaluate $\int\int_D e^{x+y} dxdy$

Question 2:Using Fubini

Question: Calculate $\int_2^4 \int_{4/x}^{(20-4x)/(8-x)} (y-x)dydx$ by reversing the order of integration.

Question 3: Regular partitions

Verify that $\int\int_D f(x, y)dxdy$ where f is $(x + y)^2$ and D is the square with vertices $(0, 0), (0, 1), (1, 0)$ and $(1, 1)$ is equal to $lim_{n\to\infty}\Sigma_{i=0}^{n-1}\Sigma_{j=0}^{n-1}f(t_{ij})\frac{1}{n^2}$ for some tagging of the partitions.

Question 4 : Let's warm up

The sphere $x^2 + y^2 + z^2 = 25$ is intersected by the plane $z = 3$. Consider the solid R bounded by the closed surface $S_0 = S_1 \cup S_2$, where S_1 is the part of the sphere corresponding to $z \geq 3$, and S_2 is the part of the plane $z = 3$ lying on or inside the sphere. If the outward unit normal to R is $\cos\alpha\mathbf{i} + \cos\beta\mathbf{j} + \cos\gamma\mathbf{k}$, then compute the value of the surface integral

$$\iint_S (xz\cos\alpha + yz\cos\beta + \cos\gamma)dS$$

when (a) S is the spherical cap S_1, and (b) S is the planar base S_2. Furthermore, compute the value of this surface integral when S is the complete boundary S_0 by using (a) and (b), and also by using the Gauss' divergence theorem.

Question 5 : Stokes-y

Evaluate $\iint_S \mathbf{F} \cdot \mathbf{n} dS$, where $F = \left(1 - y^2, 1 - z^2, 1 - x^2\right)$, and S is that portion of the plane $2x + 2y + z = 6$ included in the first octant, and n is the outward unit normal to S using Stokes' theorem.

Question 6 : Why so Green ?

6. Let C be the astroid given by the parametric equations

$$x = 2\sin^3\theta, \quad y = 2\cos^3\theta, \quad 0 \le \theta \le 2\pi.$$

(i) Using Green's theorem, find the area of the planar region bounded by the closed curve C.
(ii) Evaluate the line integral

$$\oint_C \left(2x + y^2\right) dy - \left(2x^2 + y\right) dx$$

Question 7 : Another one I&g

Evaluate
$\iint_S \left(z^2x + e^{y^2}\right) dy \wedge dz + \left(x^2y + e^{x^2}\right) dz \wedge dx + \left(y^2z + e^{x^2+y^2}\right) dx \wedge dy$,
where S is the surface of the upper hemisphere $z^2 = 9 - x^2 - y^2$ that is above the XY plane.

Question 8 : A basic One

Find the work done by the force field $\mathbf{F} = \left(y - x^2\right)\mathbf{i} + \left(z - y^2\right)\mathbf{j} + \left(x - z^2\right)\mathbf{k}$ along the curve $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}, 0 \leq t \leq 1$, from $(0,0,0)$ to $(1,1,1)$?

Question 9 : Gauss-pel truth

Find the net outward flux of the field

$$\mathbf{F} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\rho^3}, \quad \rho = \sqrt{x^2 + y^2 + z^2}$$

across the boundary of the region $D : 0 < b^2 \leq x^2 + y^2 + z^2 \leq a^2$

Question 10 : Just for some physics-y vibes

The following vector fields represent the velocity of a gas flowing in space. Find the divergence of each vector field and interpret its physical meaning.

(a) Expansion: $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

(b) Compression: $\mathbf{F}(x, y, z) = -x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$

(c) Rotation about the z-axis: $\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j}$

(d) Shearing along parallel horizontal planes: $\mathbf{F}(x, y, z) = z\mathbf{j}$

Question 11 : Back to GDT

Let S be the part of the unit sphere in the first octant, i.e., $S = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1, x > 0, y > 0, z > 0\}$, oriented with $\mathbf{n} = (x, y, z)$. Let

$$F(x, y, z) = \left(x + \frac{1}{x} + \frac{y^2 + z^2}{2}, y + \frac{1}{y} + \frac{z^2 + x^2}{2}, z + \frac{1}{z} + \frac{x^2 + y^2}{2}\right).$$

Compute the surface integral $\iint_S F \cdot d\mathbf{S}$.

Question 12 : I'm out of captions

(i) Consider the vector field $\mathbf{F} = \left(2xy + z^3\right)\mathbf{i} + x^2\mathbf{j} + 3xz^2\mathbf{k}$ on $\mathbb{R}^3$. Determine a potential function ϕ on $\mathbb{R}^3$ such that $\mathbf{F} = \nabla\phi$. Also, find the work done by the vector field $\mathbf{F}$ in moving an object from $(1, -2, 1)$ to $(3, 1, 4)$.

(ii) A vector field $\mathbf{v}$ on $\mathbb{R}^3$ is the product of a differentiable scalar field ψ and the gradient of a scalar field ϕ, where ϕ has continuous first and second order partial derivatives. Compute the real number $\mathbf{v} \cdot \operatorname{curl} \mathbf{v}$.

Question 13 : Why should we listen to Gauss ?

Find the outward flux of the vector field $\mathbf{F} = \left(xz, 0, x^2z\right)$ through the surface S of the cylinder $x^2 + y^2 = 1$ whose bottom is the unit disk in the plane $z = 0$ and whose top is the unit disk in the plane $z = 1$. Also find this flux by applying Gauss' Divergence Theorem.

Question 14: Around the world in 2-D

Evaluate the line integral

$$I = \int_C z dx + (x + e^{y^2}) dx + (y + e^{z^2}) dx$$

where C is the curve which is the intersection of the plane $y + z = 3$ and the cylinder $x^2 + y^2 = 4$. Orient C counterclockwise as viewed from above.

Can we go home now?

Not so fast :)

Don't stress too much about grades. In the words of one of my professors, how many marks you score in the endsem will not affect the number of zeroes in your salary. So,

Are we there yet?

Yep, we're done. If there are any corrections that you feel need to be made to the slides, do let me know. Any other feedback is welcome, you can always reach out to us. Enjoy your holidays, and all the best for the endsem!