Quiz 2: DIC on Discrete Structures

25 marks, 55min

27 Oct 2023

Instructions:

- Attempt *all* questions. Write all answers and proofs carefully. If you are *making any assumptions or using results proved in class, state them clearly.*
- In this course, one of our aims is to learn how to write good proofs, hence considerable weightage will be given to *clarity and completeness* of proofs.
- Do *not* copy or use any other unfair means. Offenders will be reported to the Disciplinary Action Committee.

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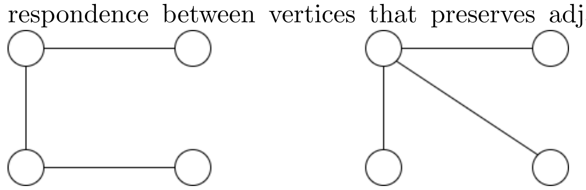
1. [2+2=4 marks] True or False. If True write a 2-3 line justification. If False, give a counter-example or a justification as applicable. Recall that a simple graph has at most one edge between two vertices and does not have self-loops.

   (a) If two simple graphs have the same number of vertices and edges, then they must be isomorphic.

   (b) If a simple graph G has 50 vertices, each having degree 5, then the number of edges of G is 100.

[1 Mark for True/False + 1 Mark for Justification/Counter-example.]

(a) **False.** The number of edges alone does not determine the structure of the graph, as the arrangement of edges and vertices matters. Isomorphism requires a one-to-one correspondence between vertices that preserves adjacency. See the counter-example below



[1 mark is given only if a valid counter-example is provided. We gave 0.5 marks if only the definition of the isomorphic graph was mentioned without any counter-example or a valid justification. A valid justification for this must not only mention the definition but also state that just having same number of edges and vertices cannot imply a bijection between edges and vertices.]
In the above examples, graphs G and H have same number of vertices (4) and edges (3), but they are not isomorphic. This can be seen for instance, by observing that H has a vertex of degree 3, while G does not.

(b) **False.** Handshaking lemma - Let's denote the number of edges in the graph by E. Since each vertex has a degree of 5, the total degree of all vertices is $50 \times 5 = 250$, which counts each edge twice. Thus, we have $2E = 250$, which leads to $E = 125$. Therefore, the number of edges in the graph is 125, not 100. Hence, the given statement is false.
[Full Marks for the justification are given if the formula is correctly used (even if the name "Handshaking lemma" is not mentioned). In the rare instances where students have provided valid counter-example for this part, we have given full marks.]

2. [3+3 = 6 marks] A piece of wire is 120 cm long.

   (a) Can one bend it to form the edges of a cube, each of whose edges is 10cm? Why or why not?

   (b) What is the smallest number of **cuts** one must make in the wire, so as to be able to form the required cube? Justify. (Note that the question asks for number of cuts, not number of pieces.)

(a) No, it can't be bent to form the edges of the cube. [1mark]

- The cube has 12 edges each of length 10cm. The minimum length required to form the cube = 12*10 cm = 120 cm. And we have only 120 cm wire. So, each edge has to be traversed exactly once.
- Since, each edge has to be traversed only once, this can be modelled as finding an Eulerian Trail. [1 mark]
- The wire can be bent to form the cube iff there exists an Eulerian Trail. Eulerian trail can be closed (if all vertices have even degree) or open (if start and end vertices have odd degree and all other vertices have even degree.)
- **Thus, Eulerian trail is possible only if the number of odd vertices is 0 or 2.**
- All vertices in this graph have degree 3 (all odd) and there are 8 of them. Since, the number of odd degree vertices in this graph is greater than 2, we can't form an Eulerian trail. [1 mark]

[1 mark for Eulerian trail + 1 mark for reason]

(b) Number of cuts = 3. [1mark]

- Since there are 8 odd degree vertices, each Eulerian trail can reduce the number of odd degree vertices by either 0 or 2.
- Eulerian trail can reduce the number of odd degree vertices by 0 if the start and end vertex are same. Else, it will reduce the number of odd degree vertices by 2.
- So, there must be atleast 4 Eulerian trails. So the number of cuts required is 3.

[2 marks for reason]

Alternate method: (using theorem in class) For a connected graph with exactly 2k odd vertices and at least one edge, the minimum number of trails that decompose it is max{k, 1}. [Full marks for correct reasoning using theorem]
Common mistakes:

(a) 0.5 marks cut for not mentioning that Eulerian Trail is possible even if there is a pair of odd degree vertices.

3. [2 + 4 + 4 = 10 marks] A flag consists of n horizontal stripes, where each stripe can be any one of three colors, red, blue and green, such that no two adjacent stripes have same color.

   (a) How many such flags are possible (as a function of n)? Why?

Now suppose that, in order to avoid flying the flag upside down, the top and bottom stripes are required to be of different colors. Let a_n denote the number of such flags with n stripes.

   (b) Write a recurrence relation for a_n, along with initial conditions.

   (c) Solve the recurrence to obtain an expression for a_n in terms of n. Simplify it as much as possible.

(a) [1 mark for answer and 1 mark for correct justification]

Starting from the top, the first stripe has 3 possible colors to choose from and all the subsequent stripes will have 2 possible colors to choose from. Thus total is $3 * 2^{n-1}$ different possible colorings.
Note: the flag is unique from top to bottom, rotating will give us a different flag.

(b) [1 mark for correct recurrence relation, 1 mark for correct base cases ($a_1 = 0, a_2 = 6$), and 2 marks for correct justification. Partial marks awarded based on students' answers.]
Let a_n be the number of flags such that their first and last color don't match while b_n be the number of flags such that their first and last colors do match.
Therefore $a_n = 2 * b_{n-1} + a_{n-1}$. This is if the first and second last color match, then the last color has 2 valid possibilities while if the first and second last color don't match, then the last color has 1 valid possibility.
And $b_n = a_{n-1}$. This is if the first and second last color match, then the first and last can't match while if the first and second last don't match, then the last color has 1 possibility for it to match with the first color.
By substituting the second recurrence into the first, we get $a_n = a_{n-1} + 2 * a_{n-2}$

(c) [1 mark for correct answer, 3 marks for correct procedure.]
For solving linear recurrences, we can assume the solution to be a^n. Thus, the polynomial will become $x^2 - x - 2 = 0$. Solving this yields $x = 2$ and $x = -1$. Thus, the general solution to our recurrence relation will be of the form $a * 2^n + b * (-1)^n$ where a and b are constants. Base case will be a_1 and a_2. For $n = 1$, the single stripe flag will always violate the condition. Hence, $a_1 = 0$. For $n = 2$, the first stripe has 3 possibilities while the second stripe has 2, thus 6 total flags i.e. $a_2 = 6$.
Now substituting for $n = 1$ and $n = 2$, we get 2 equations in a and b which are $2 * a - b = 0$ and $4 * a + b = 6$. Solving these will yield $a = 1$ and $b = 2$.
Thus, the final answer is $a_n = 2^n + 2 * (-1)^n$

Note: Alternate solutions include forming different linear recurrences. As long as the relation is correct, full marks have been awarded.

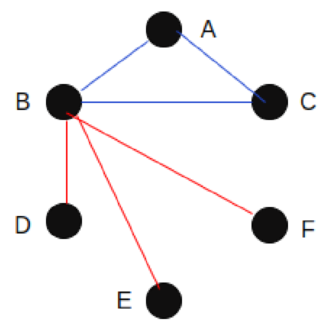
A common mistake that a lot of people did were taking $a_1 = 3$. Vacuously the first and last stripes are same if only 1 stripe exists on the flag. Thus, a_1 must be 0.
Not a mistake but a suggestion is to always try to make a homogenous linear recurrence relation, if possible, as that will be the easiest to solve amongst all. Minimal guessing and calculation will be involved in it.

4. [5 marks] Suppose we take a complete (simple) graph on 6 vertices where all edges are colored red or blue. We saw in class that it must have one monochromatic triangle (i.e., triangle whose all sides are red or all are blue). Show that it must have 2 monochromatic triangles! That is, it must be that there must be two triangles, such that their edges are all red or all blue. (It may be that one of them has all red and other has all blue or both triangles are all blue or both are all red).

[Rubrics: 5 marks for correct proof. 2 marks if tried making cases on number of red and blue edges from a vertex and correctly proving only one case. 1 mark if cases identified but no case solved fully. No marks for showing existence of only one monochromatic triangle/incorrect proof for two monochromatic triangles. Partial marks deducted for small errors (justification for deducting marks provided in the answer sheet)]

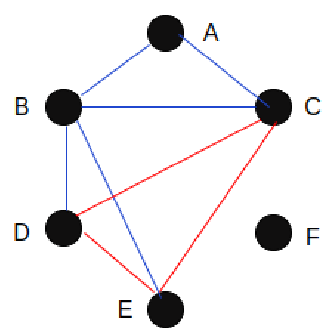
Let the vertices be A, B, C, D, E, F. We start by assuming the fact that we already know there must be at least *one* such triangle. Without loss of generality, let us say that ABC is the triangle, and again WLOG, assume it is blue.

Now, we claim for each vertex of the triangle must have at least one blue edge. If, by any chance, a vertex (say B) within this primary triangle were to exhibit three remaining edges (BD, BE and BF), all colored as red, it either forges a connection between all three vertices at the base (D,E and F) through blue edges, thereby forming a secondary triangle, or results in two of them connected by a red edge, subsequently forming a red triangle.



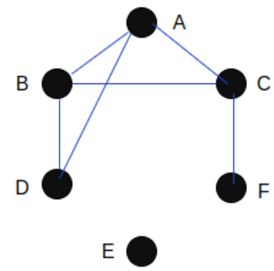
If all the edges from any vertex from ΔABC are red

However, if any vertex in the primary triangle possess more than one blue edge (for instance, BD and BE are blue edges),we either have a red triangle (Δ DCE), as shown, or as an alternative, one of the edges shown as red must instead be blue, leading in the creation of a blue triangle.



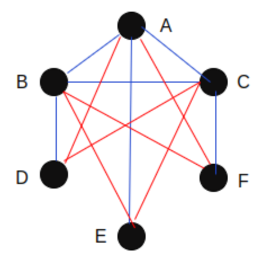
If any two edges the edges are blue

Therefore, for each of the vertices A, B and C within the primary triangle, the three remaining edges comprise *exactly* one blue edge and two red edges. Furthermore, each of the three blue edges, originating from each vertices of the primary triangle, must connect to distinct vertices, D, E and F in order to prevent the occurrence of an additional blue triangle.



If any two blue edges of ΔABC goes into same vertices.

Now, to avoid the creation of another blue triangle, specific edges must be red (as depicted in the diagram below). But then, to prevent the formation of a red triangle in this arrangement,all the edges connecting to any two of the lower-tier vertices D, E and F, must be blue, results in the emergence of a second monochromatic blue triangle.



To avoid blue triangle certain edges must be red

Remark: This construction also shows that it is possible to 2-color a K_6 so as to have exactly two monochromatic triangles. However, it is not known, for general n, what is the minimum number of monochromatic triangles is in a 2-colored K_n.

Alternate Counting Solution

The total number of traingles in $K_6 = \binom{6}{3} = 20$. We shall now show that the total number of dichromatic triangles (triangles having edges of both the colors) cannot exceed 18.

Observe that any dichromatic triangle will have two dichromatic angles (angles formed by edges of different color). Also, each angle is unique to a triangle. Hence the number of dichromatic triangles will be half of the number of dichromatic angles.

Now, for any vertex, say there are r red edges and b blue edges adjacent to it, with $r + b = 5$. Then, the number of dichromatic angles around the vertex will be $rb \leq 6$ (enumerate all the possibilities and verify that this is indeed the case). Hence the total dichromatic angles in the graph will be less than or equal to $6 \times 6 = 36$.

As shown earlier, the number of dichromatic traingles $= 1/2$ number of dichromatic angles $\leq 1/2 \times 36 = 18$.

Hence the number of monochromatic triangles $= 20-$ number of dichromatic triangles $\geq 20-18 = 2$