MA 111

Calculus 2 : Double,Triple Integration and Vector Calculus

Notes

Double Integration

- A **closed bounded rectangle** $\mathcal{R}$ in $\mathbb{R}^2$ is a subset of the form

$$\mathcal{R} = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R} : a \leq x \leq b,\, c \leq y \leq d\}$$

- A partition $\mathcal{P}$ of a rectangle $\mathcal{R} = [a, b] \times [c, d]$ is the Cartesian product of partition $\mathcal{P}_1$ of $[a, b]$ and $\mathcal{P}_2$ of $[c, d]$.

$$\mathcal{P}_1 = \{x_0, x_1, ..., x_m\},\ a = x_0 < x_1 < x_2 < .... < x_m = b$$

$$\mathcal{P}_2 = \{y_0, y_1, ..., y_n\},\ c = y_0 < y_1 < y_2 < .... < y_n = b$$

$$\mathcal{P} = \{(x, y) |\, i \in \{0, 1, ..., m\},\ j \in \{0, 1, ..., n\}\}$$

The points in $\mathcal{P}$ break $\mathcal{R}$ into mn non-overlapping rectangles.

$$\mathcal{R} = \bigcup_{i=1,\, j=1}^{i=m,\, j=n} \mathcal{R}_{ij}, \quad \text{where} \quad \mathcal{R}_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$$

Area of each rectangle $\mathcal{R}_{ij}$ is $\Delta_{ij} = (x_{i+1} - x_i)(y_{j+1} - y_j)$

- Norm of partition $\mathcal{P}$ of $\mathcal{R}$ is

$$\|\mathcal{P}\| = \max\{(x_{i+1} - x_i),\ (y_{j+1} - y_j) \,|\, i = 0, 1, ..., m - 1 \ \ ,\ \ j = 0, 1, ..., n - 1\}$$

Darboux and Riemann Integrals

- Let $f : \mathcal{R} \to \mathbb{R}$ be a **bounded** function.

$$m(f) = \inf\{f(x, y) \,|\, (x, y) \in \mathcal{R}\} \quad \text{and} \quad M(f) = \sup\{f(x, y) \,|\, (x, y) \in \mathcal{R}\}$$

$$m_{ij}(f) = \inf\{f(x, y) \,|\, (x, y) \in \mathcal{R}_{ij}\} \quad \text{and} \quad M_{ij}(f) = \sup\{f(x, y) \,|\, (x, y) \in \mathcal{R}_{ij}\}$$

- Let $\mathcal{P}$ be a partition of $\mathcal{R}$.
Lower Darboux sum is defined as

$$L(f, \mathcal{P}) = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} m_{ij}(f) \Delta_{ij}$$

and the Upper Darboux sum is

$$U(f, \mathcal{P}) = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} M_{ij}(f) \Delta_{ij}$$

The upper and lower **Darboux integrals** are defined as

$$U(f) = \inf\{U(f, \mathcal{P}) \,|\, \mathcal{P} \text{ is a partition of } \mathcal{R}\}$$

$$L(f) = \sup\{L(f, \mathcal{P}) \,|\, \mathcal{P} \text{ is a partition of } \mathcal{R}\}$$

- A bounded function $f : \mathcal{R} \to \mathbb{R}$ is said to be **Darboux integrable** if $L(f) = U(f)$. The double integral of f is the common value $U(f) = L(f)$ and is denoted by $\iint_{\mathcal{R}} f$ or $\iint_{\mathcal{R}} f(x,y)\,dA$ or $\iint_{\mathcal{R}} f(x,y)\,dx\,dy$
- A bounded function $f : \mathcal{R} \to \mathbb{R}$ is integrable if and only if for every $\epsilon > 0$ $\exists$ partition $\mathcal{P}_\epsilon$ of $\mathcal{R}$ such that
$$|U(f,\mathcal{P}_\epsilon) - L(f,\mathcal{P}_\epsilon)| < \epsilon$$
- Let $t = \{t_{ij} | t_{ij} \in \mathcal{R}_{ij}\}$ be the set of tags of partition $\mathcal{P}$ of $\mathcal{R}$. **Riemann sum** of f associate to $(\mathcal{P}, t)$ is
$$S(f,\mathcal{P},t) = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} f(t_{ij})\Delta_{ij}$$
- A bounded function $f : \mathcal{R} \to \mathbb{R}$ is **Riemann integrable** if there exists $S \in \mathbb{R}$ such that for any $\epsilon > 0$ $\exists$ a $\delta > 0$ such that for every tagged partition $\mathcal{P}$
$$|S(f,\mathcal{P},t) - S| < \epsilon \implies \|\mathcal{P}\| < \delta$$
- The double integral geometrically gives signed volume.
A function is Darboux integrable if and only if it is Riemann integrable.
Every continuous function is intergrable.
- If $a < b$ and $c < d$
$$\iint_{[b,a]\times[c,d]} f(x,y)\,dx\,dy := -\iint_{[a,b]\times[c,d]} f(x,y)\,dx\,dy$$
$$\iint_{[a,b]\times[d,c]} f(x,y)\,dx\,dy := -\iint_{[a,b]\times[c,d]} f(x,y)\,dx\,dy$$
$$\iint_{[b,a]\times[d,c]} f(x,y)\,dx\,dy := \iint_{[a,b]\times[c,d]} f(x,y)\,dx\,dy$$
- **Domain additivity property :** f is integrable on $\mathcal{R}$ if and only f it is integrable over all its sub-rectangles $\mathcal{R}_{ij}$ and if exists, integral of f over $\mathcal{R}$ is sum of integrals of f over all $\mathcal{R}_{ij}$.
- **Cavalieri's principle :** Suppose two regions of space can be included between two parallel planes. If each parallel plane in between them intersects both regions in cross-sections of equal area, then the volumes of two regions are equal.

Fubini's Theorem

- Iterated integrals of f on rectangle $\mathcal{R}$ are

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

- **Fubini's theorem :** Let $\mathcal{R} := [a,b] \times [c,d]$ and $f : \mathcal{R} \to \mathbb{R}$ be an **integrable function**. Let I be the integral of f on $\mathcal{R}$. Then
  - If for each $x \in [a,b]$, the Riemann integral $\displaystyle\int_c^d f(x,y)\,dy$ exists, then the iterated integral $\displaystyle\int_a^b \int_c^d f(x,y)\,dy\,dx$ exists and is equal to I.
  - If for each $y \in [c,d]$, the Riemann integral $\displaystyle\int_a^b f(x,y)\,dx$ exists, then the iterated integral $\displaystyle\int_c^d \int_a^b f(x,y)\,dx\,dy$ exists and is equal to I.
- Using this theorem, we can also conclude that if both the iterated integrals exists but are unequal, then the integral (double integral) I doesn't exist.
- If $f : \mathcal{R} \to \mathbb{R}$ is a continuous function, then f is integrable over $\mathcal{R}$. Also both the iterated integrals exist and are equal to the double integral.
- Let $A : [a,b] \to \mathbb{R}$ and $B : [c,d] \to \mathbb{R}$ be Riemann integrable and $f(x,y) := A(x)B(y)$ $\forall$ $(x,y) \in \mathcal{R} = [a,b] \times [c,d]$. Then f is integrable on $\mathcal{R}$ and

$$\iint_{\mathcal{R}} f(x,y)\,dx\,dy = \left(\int_a^b A(x)\,dx \right) \left(\int_c^d B(y)\,dy \right)$$

- If f is **bounded and monotonic** in each of two variables, then f is integrable on $\mathcal{R}$.
- **Sets of measure zero :** Let $A \in \mathbb{R}^n$, we say A has measure zero in $\mathbb{R}^n$ if for every $\epsilon > 0$, there is a covering Q_1, Q_2,.... of A by countably many rectangles such that

$$\sum_{i=1}^{\infty} \text{volume}(Q_i) < \epsilon$$

  If A is closed, bounded and has measure zero, then the collection $\{Q_i\}_n$ can be chosen to be finite and we say that A has **content zero**.
  Let $f : [a,b] \to \mathbb{R}$ be a bounded function and $y := f(x)$. $S = \{(x,y) \,|\, y = f(x)\}$ has measure zero in $\mathbb{R}^2$.

- Let $\mathcal{R}$ be a closed rectangle in $\mathbb{R}$, let $f : \mathcal{R} \to \mathbb{R}$ be a bounded function. Let D be the set of all points in $\mathcal{R}$ where f is discontinuous. f is integrable over $\mathcal{R}$ if and only if D has measure zero.

Integrals over non rectangular sets

- Any bounded set $\mathcal{D}$ can be enclosed by a some rectangle $\mathcal{R}$. That is we can always find rectangle $\mathcal{R}$ such that $\mathcal{D} \subseteq \mathcal{R}$.
- Let $f : \mathcal{D} \to \mathbb{R}$ be a function. Let $\mathcal{R}$ be some rectangle that encloses $\mathcal{D}$. Then we define
$$f^*(x,y) := \begin{cases} f(x,y) & (x,y) \in \mathcal{D} \\ 0 & (x,y) \notin \mathcal{D} \end{cases}$$
- The function $f : \mathcal{D} \to \mathbb{R}$ is said to be integrable on $\mathcal{D}$ if f^* is integrable on $\mathcal{R}$ and the integral is defined as
$$\iint_{\mathcal{D}} f(x,y)\,dx\,dy := \iint_{\mathcal{R}} f^*(x,y)\,dx\,dy$$
- Let $\mathcal{D}_1, \mathcal{D}_2 \subseteq \mathcal{D}$ such that $\mathcal{D}_1 \cup \mathcal{D}_2 = \mathcal{D}$ and $\mathcal{D}_1 \cap \mathcal{D}_2$ has content zero. If f is integrable on $\mathcal{D}_1$ and $\mathcal{D}_2$ then f is integrable on $\mathcal{D}$ and
$$\iint_{\mathcal{D}} f = \iint_{\mathcal{D}_1} f + \iint_{\mathcal{D}_2} f$$
- **Boundary of a set :** Let $D \subseteq \mathbb{R}^n$ be a bounded set. A point $x \in \mathbb{R}^n$ is said to be a boundary point of D if there is a sequence $\{x_n\}_n$ in D and $\{y_n\}_n$ in $\mathbb{R}^n - D$, such that $\{x_n\}_n \to x$ and $\{y_n\}_n \to x$. The set of all such boundary points is called boundary and is denoted by ∂D.
- **Path :** A path γ in $\mathbb{R}^2$ (or $\mathbb{R}^3$) means a continuous function $\gamma : [a,b] \to \mathbb{R}^2$ (or $\gamma : [a,b] \to \mathbb{R}^3$).
  Path is said to be closed if $\gamma(a) = \gamma(b)$.
  By **curve** we mean image of γ in $\mathbb{R}^2$ (or $\mathbb{R}^3$). A *good* curve is always of measure zero.
- Let D be a bounded set whose boundary ∂D is given by finitely many closed, continuous curves. Then any bounded and continuous function $f : D \to \mathbb{R}$ is integrable over D.
- **Elementary Regions in $\mathbb{R}^2$**
  - **Type 1 :** Let $\mathcal{D} \subset \mathbb{R}^2$ be a bounded region. If $\forall a \in \mathbb{R}$, lines $x = a$ intersects $\mathcal{D}$ in an interval then $\mathcal{D}$ is of type 1.
  $$\mathcal{D} = \{(x,y) \,|\, \alpha \le x \le \beta,\, h_1(x) \le y \le h_2(x)\}$$
  Let $\mathcal{R} = [a,b] \times [c,d]$ such that $c < h_1(x)$ and $h_2(x) < d$ $\forall x \in [\alpha, \beta]$ and $a < \alpha$, $\beta < b$.
  $$\iint_{\mathcal{D}} f = \iint_{\mathcal{R}} f^* = \int_a^b \int_c^d f^*(x,y)\,dy\,dx = \int_\alpha^\beta \int_{h_1(x)}^{h_2(x)} f(x,y)\,dy\,dx$$
  - **Type 2 :** Let $\mathcal{D} \subset \mathbb{R}^2$ be a bounded region. If $\forall a \in \mathbb{R}$, lines $y = a$ intersects $\mathcal{D}$ in an interval then $\mathcal{D}$ is of type 2.
  $$\mathcal{D} = \{(x,y) \,|\, h_1(y) \le x \le h_2(y),\, \gamma \le y \le \delta\}$$
  Let $\mathcal{R} = [a,b] \times [c,d]$ such that $a < h_1(y)$ and $h_2(y) < b$ $\forall y \in [\gamma, \delta]$ and $c < \gamma$, $\delta < d$.
  $$\iint_{\mathcal{D}} f = \iint_{\mathcal{R}} f^* = \int_c^d \int_a^b f^*(x,y)\,dx\,dy = \int_\gamma^\delta \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$
  .
  - **Type 3 :** If the region is neither of type 1 nor of type 2, then it is of type 3.

Polar Coordinates

For any $(x, y) \in \mathbb{R}$, the polar coordinate (r, θ) is such that

$$x = r\cos\theta, \quad y = r\sin\theta, \ \ r \in [0, \infty), \ \ \theta \in [0, 2\pi)$$

- $$\mathcal{D}^* = \{(r, \theta) \in [0, \infty) \times [0, 2\pi) \,|\, (r\cos\theta, r\sin\theta) \in \mathcal{D}\}$$
  $$g(r, \theta) := f(r\cos\theta, r\sin\theta), \quad (r, \theta) \in \mathcal{D}^*$$

  The sun-rectangles in polar coordinates will have an area of $r\Delta r\Delta\theta$.

  $$\iint_{\mathcal{D}} f(x, y)\, dx\, dy = \iint_{\mathcal{D}^*} g(r, \theta) r\, dr\, d\theta$$

- $$\int_{-\infty}^{\infty} e^{-\alpha x^2}\, dx = \sqrt{\frac{\pi}{\alpha}}$$

  This can be easily evaluated by squaring and changing into polar coordinates.

Triple Integration

- Let $f : \mathcal{B} = [a,b] \times [c,d] \times [e,f] \to \mathbb{R}$. $\mathcal{B}_{ijk}$ is the cuboid defined by a partition $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2 \times \mathcal{P}_3$ with tags t_{ijk}.
  The Riemann sum will be
  $$S(f,\mathcal{P},t) = \sum_{i=0}^{l-1}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1} f(t_{ijk})\Delta_{ijk}$$
  Where Δ_{ijk} is volume of $\mathcal{B}_{ijk}$.
- The definitions of Riemann and Darboux integrals are same. Darboux and Riemann conditions are equivalent just like in single and double integration.
  The triple integral is denoted by $\iiint_{\mathcal{B}} f$ or $\iiint_{\mathcal{B}} f(x,y,z)\,dV$ or $\iiint_{\mathcal{B}} f(x,y,z)\,dx\,dy\,dz$
  Many other theorems also have the same behaviour here.
- **Fubini's Theorem :** If $f : \mathcal{B} \to \mathbb{R}$ is integrable on the cuboid $\mathcal{B}$, then if any of the iterated integrals exists, it is equal to the triple integral. For example,
  $$\iiint_{\mathcal{B}} f(x,y,z)\,dx\,dy\,dz = \int_a^b \int_c^d \int_e^f f(x,y,z)\,dz\,dy\,dx$$
  if exists.
- For a continuous function $f : \mathcal{B} \to \mathbb{R}$, all the iterated integrals exist and are equal to the triple integral.
- If $\mathcal{B}$ is not a cuboid, then we define a cuboid $\mathcal{R}$ which encloses $\mathcal{B}$ and extend the definition of f to f^*, just like in double integration.
- **Elementary Regions in $\mathbb{R}^3$**
  If the domain of the function is of the form
  $$\mathcal{B} = \{(x,y,z) \in \mathbb{R}^3 \mid \gamma_1(x,y) \le z \le \gamma_2(x,y),\ (x,y) \in \mathcal{D}\}$$
  where $\mathcal{D}$ is elementary region in $\mathbb{R}^2$.
  For example, if $\mathcal{D}$ is of type 1,
  $$\iiint_{\mathcal{B}} f(x,y,z)\,dx\,dy\,dz = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

The Jacobian

- **Change of variables in $\mathbb{R}^2$**
  Suppose we change coordinates (u, v) to the coordinates (x, y) with linear functions with translations (affine linear functions)
  $$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$
  A unit square in (u, v) coordinates will have an area of $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ in (x, y) coordinates.

- **Area element for a change of coordinates**
  Now the coordinates (x, y) and (u, v) are related by a general mapping (may or may not be linear) given by
  $$x = h_1(u, v), \quad y = h_2(u, v)$$
  Let us assume h_1 and h_2 are one-one, continuously differentiable functions.
  $$\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} \dfrac{\partial h_1}{\partial u} & \dfrac{\partial h_1}{\partial v} \\ \dfrac{\partial h_2}{\partial u} & \dfrac{\partial h_2}{\partial v} \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix}$$
  Area of rectangle with sides $(\Delta x, \Delta y)$ near $(h_1(u, v), h_2(u, v))$ will be $\quad \Delta u \Delta v \begin{vmatrix} \dfrac{\partial h_1}{\partial u} & \dfrac{\partial h_1}{\partial v} \\ \dfrac{\partial h_2}{\partial u} & \dfrac{\partial h_2}{\partial v} \end{vmatrix}_{(u,v)}$
  in (u, v) coordinates.

- **The Jacobian**
  The matrix
  $$J(h) = [J_{ij}]_{n \times n} = \left[\frac{\partial h_i}{\partial u_j} \right]_{n \times n}$$
  is the Jacobian matrix for the function $h = (h_1, ...h_n) : \mathbb{R}^n \to \mathbb{R}^n$.

- **Change of variables theorem**
  Let $h : \mathcal{A} \to \mathcal{B}$ be a $\mathcal{C}^1$ diffeomorphism of open bounded sets in $\mathbb{R}^n$. Let $f : \mathcal{B} \to \mathbb{R}$ be a continuous function.
  f is integrable over $\mathcal{B}$ if and only if the function $(f \circ h)|\det J(h)|$ is integrable over $\mathcal{A}$ and in this case,
  $$\int \cdots \int_{\mathcal{B}} f = \int \cdots \int_{\mathcal{A}} (f \circ g)|\det J(h)|$$
  h is a $\mathcal{C}^1$ diffeomorphism if it is differentiable, one-one, onto and $J(h)$ is continuous, invertible on $\mathcal{A}$, $h^{-1} : \mathcal{B} \to \mathcal{A}$ is also a $\mathcal{C}^1$ diffeomorphism.
  Let $\mathcal{A}^*$ and $\mathcal{B}^*$ be closed bounded sets with interiors $\mathcal{A}$ and $\mathcal{B}$ respectively. If $\partial\mathcal{A}^*$ and $\partial\mathcal{B}^*$ are of measure zero, then we can use $\mathcal{A}^*$ and $\mathcal{B}^*$ for integration.

- **Notation**
  $$J = \frac{\partial(x_1, ..., x_n)}{\partial(u_1, ..., u_n)}$$

Vector Calculus

- Let $\mathcal{D}$ be a subset of $\mathbb{R}^n$.
  A **scalar field** on $\mathcal{D}$ is a map $f : \mathcal{D} \to \mathbb{R}$.
  A **vector field** on $\mathcal{D}$ is a map $\mathbf{F} : \mathcal{D} \to \mathbb{R}$.

- **Del operator (∇)**
  The del operator or the **gradient** operator is defined as

$$\nabla = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \widehat{e}_i$$

  in n-dimensions, where $\widehat{e}_i$ is a unit vector along x_i axis.

$$\nabla = \frac{\partial}{\partial x}\widehat{x} + \frac{\partial}{\partial y}\widehat{y} + \frac{\partial}{\partial z}\widehat{z}$$

  in 3-dimensions.

- **Gradient vector field**
  Let $f : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$ be a differentiable function. Then the vector field associated to ∇f is called gradient vector field. In 3-dimensions,

$$\nabla f = \frac{\partial f}{\partial x}\widehat{x} + \frac{\partial f}{\partial y}\widehat{y} + \frac{\partial f}{\partial z}\widehat{z}$$

- **Conservative vector field**
  A vector field $\mathbf{F}$ is called a conservative vector field if there exists a scalar function U such that $\mathbf{F} = -\nabla U$. In this case, U is called a potential function for $\mathbf{F}$.

Line Integrals

- **Flow lines of a Vector field**
  Let $\mathbf{F} : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}^n$ be a vector field. A **flow line** or an **integral curve** is a **path**, that is, a map $\mathbf{c} : [a, b] \to \mathcal{D}$ such that

$$\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t)), \qquad \forall\, t \in [a, b]$$

  $\mathbf{F}$ is called as the **velocity field** of the path $\mathbf{c}$.

- Let $\mathbf{F} = (P, Q, R)$ where $P, Q, R : \mathcal{D} \to \mathbb{R}$, then the flow lines of the vector field $\mathbf{F}$, $\mathbf{c}(t) = (x(t), y(t), z(t))$ are the solutions of the system of ordinary differential equations.

$$x'(t) = P(x(t), y(t), z(t))$$
$$y'(t) = Q(x(t), y(t), z(t))$$
$$z'(t) = R(x(t), y(t), z(t))$$

- **Path and Curve**
  A curve in $\mathbb{R}^n$ is the image of a path $\mathbf{c} : [a, b] \to \mathbb{R}^n$.
  Path $\mathbf{c}$ is called as **closed** if $\mathbf{c}(a) = \mathbf{c}(b)$.
  Path $\mathbf{c}$ is called as **simple** if $\mathbf{c}(t_1) \neq \mathbf{c}(t_2)$ for any $t_1 \neq t_2$ other than $t_1 = a$ and $t_2 = b$.
  If a $\mathcal{C}^1$ curve $\mathbf{c}$ is such that $\mathbf{c}'(t) \neq 0\ \forall\, t \in [a, b]$, the curve is called a **regular** or **non-singular parametrised** curve.

- **Line Integrals of Vector Fields**
  Let $\mathbf{F} : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field and $\mathbf{c} : [a, b] \to \mathcal{D}$ be a $\mathcal{C}^1$ path.
  The **line integral** of $\mathbf{F}$ over $\mathbf{c}$ is defined as

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} := \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt$$

  In 3-dimensions, $\mathbf{F} = (f_1, f_2, f_3)$ and $\mathbf{c}'(t) = (x(t), y(t), x(t))$,

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_a^b \left(f_1(\mathbf{c}(t)) \frac{dx(t)}{dt} + f_2(\mathbf{c}(t)) \frac{dy(t)}{dt} + f_3(\mathbf{c}(t)) \frac{dz(t)}{dt} \right) dt$$

- Let $\mathbf{c}_i : [a_i, a_{i+1}]$ be $\mathcal{C}^1$ paths with $\mathbf{c}_i(a_{i+1}) = \mathbf{c}_{i+1}(a_{i+1})\ \forall\, i \in \{1,\ 2,\ ...,\ n\}$. The union of paths $\mathbf{c}_i$ is written as $\mathbf{c} = \mathbf{c}_1 + \mathbf{c}_2 + ... + \mathbf{c}_n$. Then

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} := \int_{\mathbf{c}_1} \mathbf{F} \cdot d\mathbf{s} + ... + \int_{\mathbf{c}_n} \mathbf{F} \cdot d\mathbf{s}$$

  For a path $\mathbf{c}$ on $[a, b]$, the path traversed in the reverse direction, $\widetilde{\mathbf{c}}$, is also denoted by $-\mathbf{c}$. And

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = -\int_{-\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$$

- **Reparameterization**
  Let $\mathbf{c} : [a, b] \to \mathbb{R}^n$ be a non-singular ($\mathbf{c}'(t) \neq 0$) path. Let $h : [\alpha, \beta] \to [a, b]$ be a $\mathcal{C}^1$ diffeomorphism and $t = h(u)$. $\gamma(u)$ is called as a reparameterization of $\mathbf{c}(t)$, $\gamma(u) = \mathbf{c}(h(u))$. If $h(\alpha) = a$ and $h(\beta) = b$ (we will assume this), the orientation is not changed.

$$\int_{\gamma} \mathbf{F} \cdot d\mathbf{s} = \int_{\alpha}^{\beta} \mathbf{F}(\gamma(u)) \cdot \gamma'(u)\, du = \int_{\alpha}^{\beta} \mathbf{F}(\mathbf{c}(h(u))) \cdot \mathbf{c}'(h(u)) h'(u)\, du$$

$$= \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt = \int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$$

  If the reparameterization changes the orientation, then

$$\int_{\gamma} \mathbf{F} \cdot d\mathbf{s} = -\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$$

- Line integral along a geometric curve C is calculated by choosing a convenient parameterized path $\mathbf{c}$ traversing C in the given direction

$$\int_C \mathbf{F} \cdot d\mathbf{s} := \int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$$

  $\oint_C$ means line integral over a closed curve C.

- **Arc-length parameterization**
  One of the convenient parameterizations is to take the parameter to be length of curve from the starting point.
  Length $l(\mathbf{c})$ of curve $\mathbf{c}$ of path $\mathbf{c} : [a,b] \to \mathbb{R}^3$ is

$$l(\mathbf{c}) = \int_a^b \|\mathbf{c}'(t)\|\, dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt$$

  Let $s : [a,b] \to [0, l(\mathbf{c})]$ be defined as ($\mathbf{c}$ is non-singular)

$$s(t) = \int_a^t \|\mathbf{c}'(u)\|\, du$$

  Let $h : [0, l(\mathbf{c})] \to [a,b]$ be s^{-1}. Let $\widetilde{\mathbf{c}}(u) = \mathbf{c}(h(u))$ be a reparameterization. $\widetilde{\mathbf{c}}$ is called as the arc-length parameterzation.

$$\int_{\widetilde{\mathbf{c}}} \cdot d\mathbf{s} = \int_0^{l(\mathbf{c})} \mathbf{F}(\widetilde{\mathbf{c}}(u)) \cdot \widetilde{\mathbf{c}}'(u)\, du = \int_0^{l(\mathbf{c})} \mathbf{F}(\mathbf{c}(h(u))) \cdot \frac{\mathbf{c}(h(u))}{\|\mathbf{c}'(h(u))\|}\, du$$

$$= \int_0^{l(\mathbf{c})} \mathbf{F}(\mathbf{c}(h(u))) \cdot \mathbf{T}(h(u))\, du$$

  where $\mathbf{T}(t) = \dfrac{\mathbf{c}'(t)}{\|\mathbf{c}'(t)\|}$, is the unit tangent vector along the curve.

- **For reparameterization, the path must be non-singular.**

- **Line Integral of a Scalar Function**
  Let $f : \mathcal{D} \to \mathbb{R}$ be a continuous scalar function. and $\mathbf{c} : [a,b] \to \mathcal{D}$ be a non-singular path. The line integral of f along $\mathbf{c}$ is

$$\int_{\mathbf{c}} f\, ds := \int_a^b f(\mathbf{c}(t)) \|\mathbf{c}'(t)\|\, dt$$

Conservative Vector Fields

- **Fundamental Theorem of Calculus**
  Let $\mathbf{c} : [a,b] \to \mathcal{D} \subset \mathbb{R}^n$ be a smooth path and $f : \mathcal{D} \to \mathbb{R}$ be a differentiable function with continuous gradient field on $\mathbf{c}$.
  Then,

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = f(\mathbf{c}(b)) - f(\mathbf{c}(a))$$

- Let $\mathbf{F}$ be any continuous conservative field, with $\mathbf{F} = \nabla f$ for some $\mathcal{C}^1$ scalar function f. Then for any smooth path $\mathbf{c}$,

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = f(\mathbf{c}(b)) - f(\mathbf{c}(a))$$

  Line integral of a conservative vector field between two given points is independent of the path traversed.

- **Connected**
  A set $\mathcal{D} \subset \mathbb{R}^n$ is called connected if it cannot be written as a disjoint union of two non-empty subsets $\mathcal{D}_1 \cup \mathcal{D}_2$ with $\mathcal{D}_1 = \mathcal{D} \cap U_1$ and $\mathcal{D}_2 = \mathcal{D} \cap U_2$, where U_1 and U_2 are open sets.

- **Path Connected**
  A set $\mathcal{D} \subset \mathbb{R}^n$ is called path connected if any two points in the set can be connected by a curve (image of a **continuous** path) inside $\mathcal{D}$.
  An open set is connected if and only if it is path connected.
  In general path connected sets are connected.

- **Converse**
  Let $\mathbf{F} : \mathcal{D} \to \mathbb{R}^3$ be a continuous vector field on a connected open region (implies path connected) $\mathcal{D}$ in $\mathbb{R}^3$. If the line integral of $\mathbf{F}$ is independent of path in $\mathcal{D}$, then $\mathbf{F}$ is a conservative vector field in $\mathcal{D}$.

- **Necessary condition for conservative fields**
  Let $\mathbf{F}(x, y, z) = f_1(x, y, z)\widehat{x} + f_2(x, y, z)\widehat{y} + f_3(x, y, z)\widehat{z}$ be a conservative vector field and f_1, f_2, f_3 have continuous first order partial derivatives on an open region $\mathcal{D}$, then

  $$\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x}, \quad \frac{\partial f_1}{\partial z} = \frac{\partial f_3}{\partial x}, \quad \frac{\partial f_2}{\partial z} = \frac{\partial f_3}{\partial y} \quad \text{on } \mathcal{D}$$

  Satisfying above conditions may not mean that the field is conservative (the field may need to satisfy other requirements). However, not satisfying definitely means non-conservative.

- **Simply connected domain**
  A subset $\mathcal{D}$ of $\mathbb{R}^n$ for $n = 2, 3$ is simply connected if $\mathcal{D}$ is connected and any simple closed curve lying in $\mathcal{D}$ encloses a region that is in $\mathcal{D}$. That is, simply connected region shouldn't have a hole.

- If $\mathcal{D}$ is a simply connected open set, then the necessary condition (partial derivative test) becomes sufficient.

Green's Theorem

- **Jordan curve theorem**
  If $\mathbf{c} : [a,b] \to \mathbb{R}^2$ is a simple closed path, then $\mathbb{R}^2 - \mathbf{c}([a,b])$ is divided into two connected regions - 'interior' and 'exterior' such that any path from one to other will have to intersect $\mathbf{c}([a,b])$. The bounded region is the 'interior' and the unbounded region is the 'exterior'.

- **Positive and negative orientations in $\mathbb{R}^2$**

  - By convention, the positive orientation of a simple closed curve corresponds to the anti-clockwise direction.
  - The boundary curve C of a bounded region $\mathcal{D} \subset \mathbb{R}^2$ is positively oriented if the region $\mathcal{D}$ always lies to the left of an observer walking along the curve .
  - Positive orientation of a curve C in $\mathbb{R}^2$ is given by the vector field $\widehat{z} \times \vec{n}_{\text{out}}$, where $\vec{n}_{\text{out}}$ is the unit normal vector field pointing outward along the curve.

- **Green's Theorem**

  - Let $\mathcal{D}$ be a bounded region in $\mathbb{R}^2$ with a positively oriented boundary $\partial\mathcal{D}$ consisting of finite number of non-intersecting simple, closed and piecewise continuously differentiable curves.
  - Let Ω be an open set in $\mathbb{R}^2$ such that $(\mathcal{D} \cup \partial\mathcal{D}) \subset \Omega$ and let $\mathbf{F} : \Omega \to \mathbb{R}^2$ be a vector field with $\mathbf{F} = f_1\widehat{x} + f_2\widehat{y}$ where $f_1, f_2 : \Omega \to \mathbb{R}^2$ are $\mathcal{C}^1$ functions.
    Then,
    $$\int_{\partial\mathcal{D}} \mathbf{F} \cdot d\mathbf{r} = \int_{\partial\mathcal{D}} f_1\, dx + f_2\, dy = \iint_{\mathcal{D}} \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dx\, dy$$

Curl and Divergence

- **Curl**
  Curl of a vector field $\mathbf{F} = (f_1, f_2, f_3)$, $\nabla \times \mathbf{F}$ is defined as
  $$\nabla \times \mathbf{F} := \begin{vmatrix} \widehat{x} & \widehat{y} & \widehat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$
  Curl of a vector field indicates the 'rotation'.

- Curl of velocity field is twice the angular velocity at the point.
  If curl of field is zero at a point, then that point is free from rotations.
  A curl free field is called as **irrotational** field.

- **Curl of a Gradient Field**
  Let $\mathbf{F} = \nabla f$ for some $\mathcal{C}^2$ scalar function f. Then

$$\nabla \times \nabla f = \begin{vmatrix} \widehat{x} & \widehat{y} & \widehat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = 0$$

  As for any $\mathcal{C}^2$ function, $\dfrac{\partial^2 f}{\partial u \partial v} = \dfrac{\partial^2 f}{\partial v \partial u}$.
  So $\nabla \times \mathbf{F} = 0$ is a necessary condition for smooth vector field $\mathbf{F}$ to be a gradient field.

- **Green's theorem using curl**
  Let $\mathbf{F} = f_1\widehat{x} + f_2\widehat{y}$ be a $\mathcal{C}^1$ vector field on an open connected region $\mathcal{D}$ with $\partial\mathcal{D}$ positively oriented. Then,

$$\int_{\partial\mathcal{D}} \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{D}} [(\nabla \times \mathbf{F}) \cdot \widehat{z}]\, dx\, dy$$

- **Conservative field and curl in $\mathbb{R}^2$**
  Let Ω be an open, simply connected region in $\mathbb{R}^2$ and vector field $\mathbf{F} = f_1\widehat{x} + f_2\widehat{y}$ is such that f_1 and f_2 have continuous first order partial derivatives on Ω. Then, $\mathbf{F}$ is a conservative field if and only if $\nabla \times \mathbf{F} = 0$ in Ω.

- **Divergence**
  Lef $\mathbf{F} = (f_1, f_2, f_3)$ be a vector field. The divergence $\nabla \cdot \mathbf{F}$ of $\mathbf{F}$ is a scalar function defined by

$$\nabla \cdot \mathbf{F} := \left(\frac{\partial}{\partial x}\widehat{x} + \frac{\partial}{\partial y}\widehat{y} + \frac{\partial}{\partial z}\widehat{z}\right) \cdot (f_1\widehat{x} + f_2\widehat{y} + f_3\widehat{z}) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

- Let vector field $\mathbf{v}$ represent the velocity field of a fluid in $\mathbb{R}^2$. Cosider a fluid element at a point P (x, y) at $t = 0$. Let the point move under the velocity field with its coordinates at a time t given by $(X, Y) = (X(x, y, t), Y(x, y, t))$. Now consider the time evolution of small area Δ of fluid element initially at point P.

$$\Delta(t) = J(x, y, t)\Delta_0$$

  Where, $J(x, y, t)$ is the Jacobian

$$J(x, y, t) = \begin{vmatrix} \frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} \\ \frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial y} \end{vmatrix}$$

  The rate of change of $\Delta(t)$ is $\frac{\partial \Delta}{\partial t} = \frac{\partial J}{\partial t}\Delta_0$

$$\frac{\partial J}{\partial t} = (\nabla \cdot \mathbf{v})J$$

- Divergence of a velocity field of fluid measures the expansion/compression of fluid.
  A divergence free field is called as **incompressible** field.
  Divergence free vector field is area preserving.

- **Divergence of Curl**
  Divergence of any curl is zero.

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0$$

- **Green's theorem using divergence**
  Let $\partial\mathcal{D}$ be a non-singular, positively oriented curve in $\mathbb{R}^2$, parameterized by $\mathbf{c} : [a,b] \to \mathbb{R}^3$ such that $\mathbf{c}(t) = (x(t), y(t), 0)$. The unit tangent $\mathbf{T}$ to the curve and the outward normal $\mathbf{n}$ are
  $$\mathbf{T}(\mathbf{t}) = \frac{\mathbf{c}'(t)}{\|\mathbf{c}'(t)\|}, \qquad \mathbf{n}(t) = \mathbf{T}(t) \times \widehat{z} \quad \forall\, t \in [a,b]$$
  Divergence form or normal form of Green's theorem :
  $$\int_{\partial\mathcal{D}} \mathbf{F} \cdot \mathbf{n}\, ds = \iint_{\mathcal{D}} \nabla \cdot \mathbf{F}\, dx\, dy$$
  where $ds = \|\mathbf{c}'(t)\|\, dt$.

Surface Integrals

- A curve is a 1-D object as one parameter is sufficient to describe it. A surface is a 2-D object, so, two parameters are required.

- **Definition of a Surface**
  Let $\mathcal{D} \subset \mathbb{R}^2$ be a path connected subset. A **parameterized surface** is a continuous function $\phi : \mathcal{D} \to \mathbb{R}^3$. The image $S = \phi(\mathcal{D})$ of parameterized surface ϕ is called as **geometric surface**.
  For $(u,v) \in \mathcal{D}$, $\phi(u,v)$ is a vector in $\mathbb{R}^3$ given by
  $$\phi(u,v) = (x(u,v), y(u,v), z(u,v))$$
  ϕ is said to be **smooth** if the fuctions x, y and z have continuous partial derivatives in an open subset of $\mathbb{R}^2$ containing $\mathcal{D}$.

- **Tangent planes of a surface**
  The normal vector corresponding to the tangential plane of the surface at a point corresponding to (u_0, v_0) is obtained by considering the cross product of tangent vectors to curves $\mathbf{c}_1(u) = \phi(u, v_0)$ and $\mathbf{c}_2(v) = \phi(u_0, v)$.
  $$\mathbf{T}_1 = \mathbf{c}_1'(u_0) = \frac{\partial\phi}{\partial u}(u_0, v_0) = \phi_u(u_0, v_0)$$
  $$\mathbf{T}_2 = \mathbf{c}_2'(v_0) = \frac{\partial\phi}{\partial v}(u_0, v_0) = \phi_v(u_0, v_0)$$
  The normal vector of the tangent plane is parallel to $\mathbf{n}$
  $$\mathbf{n} = \mathbf{T}_1 \times \mathbf{T}_2 = \phi_u \times \phi_v$$
  The equation of tangent plane is given by
  $$\mathbf{n}(u_0, v_0) \cdot (\vec{r} - \phi(u_0, v_0)) = 0$$
  where $\vec{r} = x\widehat{x} + y\widehat{y} + z\widehat{z}$.

- **Non-singular**
  A surface ϕ is called **regular or non-singular** parameterized surface if ϕ is $\mathcal{C}^1$ and $\phi_u \times \phi_v \neq 0$ at all points.

- **Unit normal**
  For any regular (non-singular) surface parameterized by $\phi : \mathcal{D} \to \mathbb{R}^3$, the unit normal $\widehat{n}$ to surface at a point P $= \phi(u_0, v_0)$ is defined by

$$\widehat{n} := \frac{\phi_u \times \phi_v}{\|\phi_u \times \phi_v\|}$$

- **Surface Area**
  Let $\phi : \mathcal{D} \to \mathbb{R}^3$ be a smooth parameterized surface, where $\mathcal{D}$ is a bounded, path-connected subset of $\mathbb{R}^2$ with boundary $\partial\mathcal{D}$ of content zero.
  Let $(u, v) \in \mathcal{D}$ and $h, k \in \mathbb{R}$ with $h \to 0, k \to 0$. Let $P = \phi(u, v), P_1 = \phi(u + h, v), P_2 = \phi(u, v + k)$ and $Q = \phi(u + h, v + h)$. The area of parallelogram PP_1P_2Q is
  $\|\phi_u(u, v) \times \phi_v(u, v)\||hk|$. So, area of ϕ will be

$$\text{Area}(\phi) = \iint_{\mathcal{D}} \|(\phi_u \times \phi_v)(u, v)\| \, du \, dv$$

  Similar to $ds = \|\gamma'(t)\| \, dt$ in case of a curve, we have

$$dS = \|\phi_u \times \phi_v\| \, du \, dv$$

  in case of a surface.

- **Area Vector**
  A rectangle of sides du and dv at a point (u, v) in $\mathcal{D}$ will be a parallelogram in $\phi(\mathcal{D})$ with sides $\phi_u \, du$ and $\phi_v \, dv$. The area vector of this parallelogram will be

$$d\mathbf{S} = (\phi_u \times \phi_v) \, du \, dv$$

  It has a magnitude $dS = \|d\mathbf{S}\|$.

$$d\mathbf{S} = \widehat{n} \, dS$$

  where $\widehat{n}$ is the unit normal vector at the point.
  Let $\phi(u, v) = (x, (u, v), y(u, v), z(u, v))$,

$$d\mathbf{S} = \begin{vmatrix} \widehat{x} & \widehat{y} & \widehat{z} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} du \, dv$$

$$dS = \sqrt{\left[\frac{\partial(y, z)}{\partial(u, v)}\right]^2 + \left[\frac{\partial(z, x)}{\partial(u, v)}\right]^2 + \left[\frac{\partial(x, y)}{\partial(u, v)}\right]^2} \, du \, dv$$

  in terms of three Jacobians.

- **Surface integral of a Scalar function**
  Any bounded scalar function $f : S \to \mathbb{R}$ can be integrated over a surface as

$$\iint_S f \, dS = \iint_{\mathcal{D}} f(x, y, z) \sqrt{\left[\frac{\partial(y, z)}{\partial(u, v)}\right]^2 + \left[\frac{\partial(z, x)}{\partial(u, v)}\right]^2 + \left[\frac{\partial(x, y)}{\partial(u, v)}\right]^2} \, du \, dv$$

provided the RHS integral exists.

Let $\Sigma = \bigcup_{i=1}^{n} S_i$, where surfaces S_i intersect only along their boundary curves. Then

$$\iint_{\Sigma} = \sum_{i=1}^{n} \iint_{S_i} f\, dS$$

- **Surface integral of a Vector field**
  Let $\mathbf{F}$ be a bounded vector field on $\mathbb{R}^3$ sucht that the domain of $\mathbf{F}$ contains the non-singular parameterized surface $\phi : \mathcal{D} \to \mathbb{R}^3$. The surface integral of $\mathbf{F}$ over S is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} := \iint_S \mathbf{F} \cdot \widehat{n}\, dS := \iint_{\mathcal{D}} \mathbf{F}(\phi(u,v)) \cdot (\phi_u \times \phi_v)\, du\, dv$$

- **Reparameterization of a Surface**
  Let $\phi : \mathcal{D} \to \mathbb{R}^3$ be a smooth parameterized surface and $\mathcal{D}$ be path-connected subset of $\mathbb{R}^2$ with an area. Let $\mathcal{E}$ be a path-connected subset of $\mathbb{R}^2$ with an area. Let $h : \mathcal{E} \to \mathcal{D}$ be a continuously differentiable one-one function such that $h(\mathcal{E}) = \mathcal{D}$ and its Jacobian $J(h)$ doesn't vanish on $\mathcal{E}$. The smooth surface $\psi = \phi \circ h$ is called as reparameterization of ϕ.

$$(\psi_p \times \psi_q)(p,q) = [(\phi_u \times \phi_v) \circ h(p,q)]J(h(p,q))$$

  The surface integral of a continuous scalar field over a smooth surface is invariant under reparameterization upto a sign (sign may change).

- **Oriented Surface**
  A surface S is said to be orientable if there exists a continuous vector field $\mathbf{F} : S \to \mathbb{R}^3$ such that for each point P in S, $\mathbf{F}(P)$ is a unit vector normal to the surface S at P. Some surfaces like Mobius strip cannot have a continuous vector field representing the normal vectors. They are non-orientable.

- **Preserving and Reversing of orientation**
  Consider an oriented geometric surface S that is described as $\mathcal{C}^1$ non-singular parameterized surface $\phi(u,v)$. Let

$$\widehat{n}_\phi = \frac{\phi_u \times \phi_v}{\|\phi_u \times \phi_v\|}$$

  If the unit normal vector $\widehat{n}_\phi$ agrees with the given orientation $\widehat{n}$ of the surface S, we say that ϕ is orientation preserving. Otherwise we say that ϕ is orientation reversing.
  Let ϕ_1 and ϕ_2 be parameterizations of oriented surface S.
  If both are orientation preserving or both are orientation reversing,

$$\iint_{\phi_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{\phi_2} \mathbf{F} \cdot d\mathbf{S}$$

  If one of them is preserving and the other reversing,

$$\iint_{\phi_1} \mathbf{F} \cdot d\mathbf{S} = -\iint_{\phi_2} \mathbf{F} \cdot d\mathbf{S}$$

Stokes Theorem

- **Homeomorphism**
  Let ψ be a function from $\mathcal{D}_1 \subset \mathbb{R}^n$ to $\mathcal{D}_2 \subset \mathbb{R}^m$. The mapping $\psi : \mathcal{D}_1 \to \mathcal{D}_2$ is called homeomorphism if ψ is continuous, bijective map such that $\psi^{-1} : \mathcal{D}_2 \to \mathcal{D}_1$ is also continuous.
  The spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ are homeomorphic only if $m = n$.
  An **open** ball in $\mathbb{R}^2$ of radius r is homeomorphic to $\mathbb{R}^2$.

- **Boundary**
  A surface $S \subset \mathbb{R}^3$ is called as a surface without a boundary if for every point $P \in S$, there is an open subset $U \subseteq \mathbb{R}^3$ containing P such that $U \cap S$ is homeomorphic to $\mathbb{R}^2$.
  A surface $S \subset \mathbb{R}^3$ is called as a surface with a boundary if for every point $P \in S$, there is an open subset $U \subseteq \mathbb{R}^3$ containing P such that $U \cap S$ is homeomorphic to either $\mathbb{R}^2$ or the upper half plane $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 \,|\, y \geq 0\}$. A point $P \in S$ lies in the boundary ∂S if there is an open subset $U \subseteq \mathbb{R}^3$ and a homeomorphism $\psi : U \cap S \to \mathbb{H}$ such that $\psi(P) \in \mathbb{H} = \{(x, y) \in \mathbb{R}^2 \,|\, y = 0\}$. The boundary ∂S is a curve.

- **Induced orientation**
  Let S be an oriented surface with boundary that is a disjoint union simple, closed, piecewise non-singular parameterized curves. Let $\widehat{n}(P)$ denote the prescribed unit normal at all interior points $P \in S$. The direction of $\widehat{n}$ induces orientation of ∂S :
  If you walk in the positive direction around ∂S with head pointing in the direction of $\widehat{n}$, then the surface will always be on the left.
  Alternatively we can use the right hand thumb rule. If the right hand thumb points in the direction of $\widehat{n}$, the direction in which the fingers curl is the positive orientation of the boundary.
  If $\mathcal{D}$ is a path-connected subset on $\mathbb{R}^2$ and $\phi : \mathcal{D} \to \mathbb{R}^3$ is a smooth orientation preserving parameterization of the surface S, then $\phi(\partial\mathcal{D}) = \partial S$ and the induced orientation of ∂S corresponds to the positive orientation of $\partial\mathcal{D}$ with respect to $\mathcal{D}$.

- **Stokes Theorem**
  Let S be a bounded piecewise smooth oriented surface (at least $\mathcal{C}^2$) with non-empty boundary ∂S. Suppose S is a closed subset of $\mathbb{R}^3$.
  Let the boundary ∂S of S be a disjoint union of simple closed curves each of which is a piecewise non-singular parameterized curve with the induced orientation.
  Let $\mathbf{F} = f_1\widehat{x} + f_2\widehat{y} + f_3\widehat{z}$ be a $\mathcal{C}^1$ vector field defined on an open subset containing S.
  Then,
  $$\int_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

- If two different oriented surfaces S_1 and S_2 have the **same boundary** C, then from Stokes theorem,
  $$\iint_{S_1} (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_{S_2} (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

- Let S be a closed oriented smooth surface in $\mathbb{R}^3$ with $\partial S = \phi$. Suppose $\mathbf{F}$ be a vector field on an open subset containing S. Then (using domain additivity)

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = 0$$

- Let $\mathbf{F}$ be a smooth vector field on an open subset $\mathcal{D}$ of $\mathbb{R}^3$ such that $\nabla \times \mathbf{F} = 0$ on $\mathcal{D}$. Suppose S is a bounded oriented, piecewise $\mathcal{C}^2$ surface in $\mathcal{D}$ and let ∂S denote its boundary with the induced orientation. Then

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{s} = 0$$

  If $\mathbf{F}$ is defined on $\mathbb{R}^3$, then $\mathbf{F}$ is a gradient field on $\mathcal{D}$.

Gauss's Divergence Theorem

- **Closed Surfaces**
  A surface S in $\mathbb{R}^3$ is said to be closed if it is bounded, complement is open and the boundary of S is empty.

- **Simple soild region**
  A region $W \subset \mathbb{R}^3$, which is simultaneously Type 1, Type 2 and Type 3 and the boundary of the region is a closed surface, is called a simple soild region.

- **Gauss's Divergence Theorem**
  If W is a closed, bounded and simple soild region in $\mathbb{R}^3$.
  Let the boundary ∂W of W be a closed surface and positively oriented.
  Let $\mathbf{F}$ be a smooth vector field on an open subset on $\mathbb{R}^3$ containing W.
  Then,

$$\iint_{\partial W} \mathbf{F} \cdot d\mathbf{S} = \iiint_W (\nabla \cdot \mathbf{F})\, dx\, dy\, dz$$

- **Flux**
  Flux of a vector field $\mathbf{F}$ across an oriented surface S is

$$\text{flux}(\mathbf{F}) = \iint_S \mathbf{F} \cdot d\mathbf{S}$$