MA 109 Quiz-1 Time: 55 minutes Code: B

Division: D3 Tutorial batch: T6

Roll number: 22B1053 Name: KAVYA GUPTA

Total marks: 20

Instructions:

(1) Write your name, roll number, Division, tutorial batch clearly. Failing to which will attract a penalty of 2 marks.
(2) You have to write your answers on this booklet only. No extra paper can be added to this booklet. However you can use extra papers for your rough work.
(3) After the examination is over at 9:10 AM, you have to upload scanned copy of this answer paper (except rough work) to respective google classroom.
(4) This answer booklet and Rough work (hard copy) should be returned to the invigilators separately.

[PTO]

Code: B

(1) Fill the blanks:

(a) If $\lim_{x\to 3} \frac{f(x)}{x^3} = 3$, then the value of

$$\lim_{x\to 3} \frac{f(x)}{x} = \underline{\quad 27 \quad}.$$

[2]

(b) Let $\alpha, \beta \in \mathbb{R}$ with $\beta > 0$. Let

$$f(x) = \begin{cases} x^\alpha \sin \frac{1}{x^{2\beta}} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

The necessary and sufficient condition on α and β for which f is differentiable on $\mathbb{R}$ is

$\underline{\alpha - 1 > 0}$.

The necessary and sufficient condition on α and β for which the derivative f' of f is continuous on $\mathbb{R}$ is

$\underline{\alpha - 2\beta > 1}$. ~~and $\alpha > 1$~~

[1 +1]

(c) Let $f : [0,1] \to \mathbb{R}$ be a differentiable function on $[0,1]$ such that $f(\frac{1}{n}) = 1 + \frac{1}{n}$ for all $n \in \mathbb{N}$. Then the value of

$$f'(0) = \underline{\quad 1 \quad}.$$

[2]

(d) The (exact) number of real roots of the equation $x\left(x^{111} - e^{-5x} - 2022\right) = 0$ is

$\underline{\quad 2 \quad}$.

[2]

(e) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function on $\mathbb{R}$ with $f(0) = 2$ and $f(x) = f(x^2)$ for all $x \in \mathbb{R}$. The value of

$$f\left(\frac{1}{\sqrt{5}}\right) = \underline{\quad 2 \quad}.$$

~~[2]~~ → 3

(2) Tick the correct option/options. (There may be more than one correct answers, for each wrong answer there will be a **penalty** of $\frac{1}{2}$ marks).

✓(a) If the sequence $\{x_n + \frac{x_n}{n}\}$ converges then $\{x_n\}$ converges.
(b) If the sequence $\{x_n^2 + \frac{x_n}{n}\}$ converges then $\{x_n\}$ converges.
(c) If the sequence $\{x_n^2\}$ converges then $\{x_n\}$ converges.
✓(d) If the sequence $\{x_n^2\}$ converges to 0 then $\{x_n\}$ converges to 0.

[2]

[PTO]

Q③ [Contd.] { Beginning of solution from below of ques. } $\Longrightarrow$ $S = \text{Sup}\{a_n\}$ exists as a_n is bounded!!

and monotonic increasing and bounded above sequences are convergent

Code: B

$\Longrightarrow$ $\{x_n\}$ is convergent!!

(3) Let $\{a_n\}$ be a bounded sequence in $\mathbb{R}$. For each $n \in \mathbb{N}$, we define $x_n = \inf\{a_k \mid k > n\}$. Prove that the sequence $\{x_n\}$ is convergent.

[3]

Given : $x_n = \inf\{a_k \mid k > n\}$

which also means $x_n = \inf\{a_{n+1}, a_{n+2}, \ldots a_\infty\}$

Similarly, $x_{n+1} = \inf\{a_{n+2}, a_{n+3}, \ldots\}$

Claim: If $m = \inf\{a, x_1, x_2, x_3, \ldots x_n\}$

then $m = \inf\{a, M\}$ where $M = \inf\{x_i \mid i \in \mathbb{N}\}$

Proof : $m + \varepsilon \geq x_i$ or $m + \varepsilon \geq a$ for some i

and $M + \varepsilon \geq x_k$ for some k, for some $\varepsilon > 0$

so clearly m functions as M and sometimes on 'a'

If $a > M$ then $m = M$ and a could come under M also $\{M + \varepsilon \geq a\} \Rightarrow$ True

If $a < M$ then $m = a$, $\{a + \varepsilon \geq M\} \Rightarrow$ True.

Hence our claim is correct.

so from the claim we deduce that

$$x_n = \inf\{a_{n+1}, x_{n+1}\}$$

because x_{n+1} covers all elements of x_n except a

so if $a_{n+1} = x_{n+1} \Rightarrow x_{n+1} = x_n$

if $a_{n+1} < x_{n+1} \Rightarrow x_n = a < x_{n+1}$

if $a_{n+1} > x_{n+1} \Rightarrow x_n = x_{n+1}$

[PTO]

Considering all cases $\boxed{x_n \leq x_{n+1}}$ $\forall n \in \mathbb{N}$

or $\{x_n\}$ is monotonically increasing

And $x_n \leq a_K$ for some K; & $a_K \leq S$, $S \in \mathbb{R}$

where $S = \text{Sup}\{a_n\} \Longrightarrow x_n \leq S \Longrightarrow x_n$ is bounded above

[Look at Top]

Q③ [Contd.] Question 3 $\longrightarrow$ $S = \sup\{a_n\}$ exists because a_n is a bounded sequence and so $x_n \leq S$ means x_n is also bounded above so montonic increasing & bounded above sequences are convergent

Code: B

$\Rightarrow \{x_n\}$ is convergent

(4) Let $f : [0,1] \to \mathbb{R}$ be a continuous on $[0,1]$ and differentiable on $(0,1)$. Suppose $f(0) = f(1) = 0$ and there exists $x_0 \in (0,1)$ with $x_0 \neq \frac{1}{2}$ such that $f(x_0) = 1$. Prove that $|f'(c)| > 2$ for some $c \in (0,1)$. [4]

Ans 4

Function f is continuous in $[0,1]$ and differentiable in $(0,1)$, it satisfies the cond^n (condition) of MVT (Mean Value Theorem) i.e. for $x_1, x_2 \in [0,1]$ $\exists\, x \in (x_1, x_2)$ $\{x_1 < x_2\}$ such that

$$f'(x) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

There will be 2 cases depending on which side x_0 lies of $1/2$ $\Rightarrow$

Case ①: $x_0 > 1/2 \Rightarrow x_0 > \frac{1}{2} \longrightarrow 1 - x_0 < \frac{1}{2}$

or

$$\frac{1}{1-x_0} > 2 \quad \text{— Ⓐ}$$

Apply MVT from x_0 to 1, for some $c \in (x_0, 1)$

$$f'(c) = \frac{f(1) - f(x_0)}{1 - x_0} = \frac{0-1}{1-x_0} = \frac{-1}{1-x_0} < -2$$

(from Ⓐ)

$\Rightarrow f'(c) < -2$

or $|f'(c)| > 2$.

Case ②: $x_0 < \frac{1}{2} \Rightarrow x_0 < \frac{1}{2} \Rightarrow \frac{1}{x_0}$ or $\frac{1}{x_0 - 0} > 2$ — Ⓑ

Apply MVT from 0 to x_0, for some $c \in (0, x_0)$

$$f'(c) = \frac{f(x_0) - f(0)}{x_0 - 0} = \frac{1-0}{x_0} = \frac{1}{x_0} > 2$$

(from Ⓑ)

$\Longrightarrow f'(c) > 2$

or

$|f'(c)| > 2$

Hence proved that for $x_0 \neq \frac{1}{2}$, some c exists in $(0,1)$ where $|f'(c)| > 2$