Practice Problems

September 20, 2023

1 Problems on Induction:

1. The function $f : \mathbb{N} \rightarrow \mathbb{N}$ is defined as follows.
   $f(n) = n/2$ if n is even
   f(n) = 3n + 1 if n is 1 mod 4(i.e,4 divides n-1)
   f(n) = 3n - 1 if n is 3 mod 4

Prove that for any n, there exists a k such that $f^k(n) = 1$, where f^k is the function f composed with itself k times.

2. Prove Binomial Theorem with induction. Binomial theorem states that for all $n > 1$,we have $(x + y)^n = \Sigma_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

3. Show that n straight lines divide a 2-D plane into $(n^2 + n + 2)/2$ regions, provided that no two such lines are parallel and no three meet at a single point.

4. Prove that we can cover a $2^n \times 2^n$ chessboard, $n \geq 2$ with L-shaped tiles such that no two tiles overlap and all but one of the 4 centre squares are covered.

5. In a cricket tournament, every two teams played against each other exactly once. After all the games were over, an ordered list of teams was created such that any two teams T_i and T_j appear consecutive in the list if and only if T_i won against T_j.Prove that an ordered list can be created that covers all the teams.

2 Problems on sets, bijections, equivalence relations and partial orders:

1. For any two sets A,B, is it the case that $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$? If yes, prove your statement, else give a counter-example. (note: A and B are arbitrary sets, so you cannot assume them to be finite or even countable!)

2.Show that the set of real numbers that are solutions of quadratic equations $ax^2 + bx + c = 0$,where a,b,c are integers is countable.

3. Consider the poset $(\mathcal{P}(S), \subseteq)$ for a finite set S. A chain $A_1 \subseteq A_2 \subseteq ... \subseteq A_m$ is defined to be a symmetric chain if $|A_{i+1}| = A_i + 1$ and $|A_i| + |A_m| = |S|$. prove that the set $\mathcal{P}(S)$ can be partitioned into symmetric chains.

4. Let R denote the relation on integers defined by: aRb iff a-b is divisible by 11. is R an equivalence relation on $\mathbb{Z}$? If yes prove it and count the number of equivalence classes. If no, provide a counterexample and specify which defining property is violated.

5. let $A_1, A_2, ...$ be an infinite sequence of sets such that the intersection of any finite number of sets in the sequence is not empty. Is it true that there exists an element x such that $x \in a_i$ for all i? If so, prove it, else give an example for which it is false. Suppose $a_1, a_2, ...$is an infinite sequence of numbers such that the GCD of any finite set of numbers in the sequence is greater than 1. Prove that there exists a prime p such that p divides a_i for all i.

6.let f be a function from set A to itself. Suppose, There exists a number k such that f^k is the identity function . Prove that f is a bijection. Prove that the converse is true if A is finite but is not true if A is infinite. For a finite set A with n elements, what is the smallest number k such that for every bijection f from A to A , $f^k = I$.prove your answer.

7.For a finite set S, let $P(S)$ be the set of partitions of S. Define the relation $\preceq$ on $P(S)$ as $A \preceq B \iff \forall x, y \in S, x \sim_A y \Rightarrow x \sim_B y$ where $\sim_A, \sim_B$ are the equivalence relations defined by the partitions. This is equivalent to saying that all the equivalence classes of A are contained in/are a refinement of those of B.
Show that $\preceq$ is a partial order. Show that $(P(S), \preceq)$ has a minimum, maximum element, and that it is a lattice.

let $\vee, \wedge$ stand for the lub,glb operations.Show that these need not follow the distributive laws i.e $a \vee (b \wedge c) \neq (a \vee b) \wedge (a \vee c), a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$using a partition lattice.

3 Problems on counting:

1. Using counting/combinatorial arguments, prove that for all a, b ,and prime p , p^2 divides $\binom{pa}{pb} - \binom{a}{b}$.

(You may use the fact, that p divides $\binom{p}{k}$ when $0 < k < p$)

(Hint: Use Double Counting on pairs (j,k) for d(j)=k.)