MA 109 D3 Lecture 12

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September 4, 2023

Recap

The Mean Value Theorem for integration

Functions of severable variables

Limits and continuity

A remark on Definition 2 of Riemann integrability

Recall that the second definition of Riemann integrability was the following:

Definition 2: A function $f : [a, b] \to \mathbb{R}$ is said to be Riemann integrable if for some $R \in \mathbb{R}$ and every $\epsilon > 0$ there exists $\delta > 0$ and a partition P, such that for every tagged refinement (P', t') of P with $\|P''\| < \delta$,

$$|R(f, P', t') - R| < \epsilon. \qquad (1)$$

Suppose f is Riemann integrable in the sense above and let P be the chosen partition. Let P' be a refinement of P such that $\|P'\| < \delta$. If P'' is any refinement of P', it is a refinement of P, $\|P''\| < \delta$ and (1) holds for P''. Thus, by replacing P by P' we can reformulate Definition 2 as follows:

A function $f : [a, b] \to \mathbb{R}$ is said to be Riemann integrable if for some $R \in \mathbb{R}$ and every $\epsilon > 0$, a partition P, such that for every tagged refinement (P', t') of P

$$|R(f, P', t') - R| < \epsilon.$$

The Fundamental Theorem - Part I

Theorem 24 (Part I): Let $f : [a, b] \to \mathbb{R}$ be a continuous function, and let

$$F(x) = \int_a^x f(t)dt$$

for any $x \in [a, b]$. Then $F(x)$ is continuous on $[a, b]$, differentiable on (a, b) and

$$F'(x) = f(x),$$

for all $x \in (a, b)$.

Theorem 24 (Part II): Let $f : [a, b] \to \mathbb{R}$ be given and suppose there exists a continuous function $g : [a, b] \to \mathbb{R}$ which is differentiable on (a, b) and which satisfies $g'(x) = f(x)$. Then, if f is Riemann integrable on $[a, b]$,

$$\int_a^b f(t)dt = g(b) - g(a).$$

Tutorial Problem 4.4

Exercise 4.4 Compute

(a) $\frac{d^2y}{dx^2}$, if

$$x = \int_0^y \frac{dt}{\sqrt{1+t^2}}$$

(b) $\frac{dF}{dx}$, if for $x \in \mathbb{R}$

$$(i) F(x) = \int_1^{2x} \cos(t^2) dt$$

and

$$(ii) F(x) = \int_0^{x^2} \cos(t) dt.$$

Problem 4.5

Let p be a real number and let f be a continuous function on $\mathbb{R}$ that satisfies the equation $f(x+p) = f(x)$ for all $x \in \mathbb{R}$. Show that the integral

$$\int_a^{a+p} f(t)dt$$

has the same value for every real number a.

(Hint: Consider $F(a) = \int_a^{a+p} f(t)dt$.)

The logarithm

For $x \in (0, \infty)$ we define

$$f(x) = \int_1^x \frac{1}{t} dt.$$

Then, for any y, define $g(x) = f(xy)$

Differentiating with respect to x we see that $g'(x) = f'(x)$ Hence,

$$f(x) = g(x) + C,$$

for some constant C. Set $x = 1$ to obtain $C = -f(y)$. Thus,

$$f(xy) = f(x) + f(y).$$

The logarithm and exponential functions

The function $f(x)$ is usually denoted $\ln x$. Since $f'(x) = \frac{1}{x} > 0$, whenever $x > 0$, we see that f is (strictly) monotonic increasing and concave.

By computing the Darboux lower sums associated to $\ln x$, we can easily check that $\ln x > 1$ if $x \geq 3$. By the intermediate value theorem, it follows that there exists a real number e, such that $\ln e = 1$.

It is not hard to see that f must have an inverse function. This is the exponential function sometimes denoted $\exp(x)$. Clearly $\exp(x + y) = \exp(x) \cdot \exp(y)$. Again, it requires some work to see that

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

When $x = 1$ we will obtain a formula for e!

The Mean Value Theorem for Integration

Let $f : [a, b] \to \mathbb{R}$ be a continuous function. The slide that was projected in the last class unnecessarily had the condition that f was differentiable. We apply the Mean Value Theorem to the function

$$F(x) = \int_a^x f(t)dt.$$

This says that there exists $c \in (a, b)$ such that

$$\frac{F(b) - F(a)}{b - a} = F'(c).$$

But this is the same as saying

$$\int_a^b f(t)dt = f(c)(b - a).$$

This is the Mean Value Theorem for integration.

Functions with range contained in $\mathbb{R}$

We will be interested in studying functions $f : \mathbb{R}^m \to \mathbb{R}$, when $m = 2, 3$. We have already mentioned how limits of such functions can be studied in the first few lectures. Before doing this in detail, however, we will study certain other features of functions in two and three variables.

The most basic thing one needs to understand about a function is the domain on which it is defined. Very often a function is given by a formula which makes sense only on some subset of $\mathbb{R}^m$ and not on the whole of $\mathbb{R}^m$. When studying functions of two or more variables given by formulæ it makes sense to first identify this subset, which is sometimes call the natural domain of the function, and to describe it geometrically if possible.

Exercise 5.1: Find the natural domains of the following functions:
(i) $\frac{xy}{x^2-y^2}$

Clearly this function is defined whenever the denominator is not zero, in other words when $x^2 - y^2 \neq 0$.
The natural domain is thus

$$\mathbb{R}^2 \setminus \{(x, y) \,|\, x^2 - y^2 = 0\},$$

that is, $\mathbb{R}^2$ minus the pair of straight lines with slopes ± 1.
(ii) $f(x, y) = \log(x^2 + y^2)$
This function is defined whenever $x^2 + y^2 \neq 0$, in other words, in $\mathbb{R}^2 \setminus \{(0, 0)\}$.

Level curves and contour lines

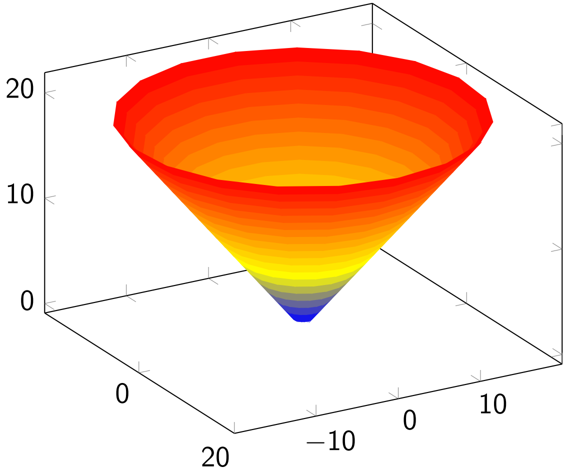
The second thing one should do with a function from $\mathbb{R}^2 \to \mathbb{R}$ is to study its range. This is done in different ways.

One way is to study the level sets of the functions. These are the sets of the form $\{(x, y) \in \mathbb{R}^2 \mid f(x, y) = c\}$, where c is a constant. The level set "lives" in the xy-plane.

One can also plot (in three dimensions) the surface $z = f(x, y)$. By varying the value of c in the level curves one can get a good idea of what the surface looks like.

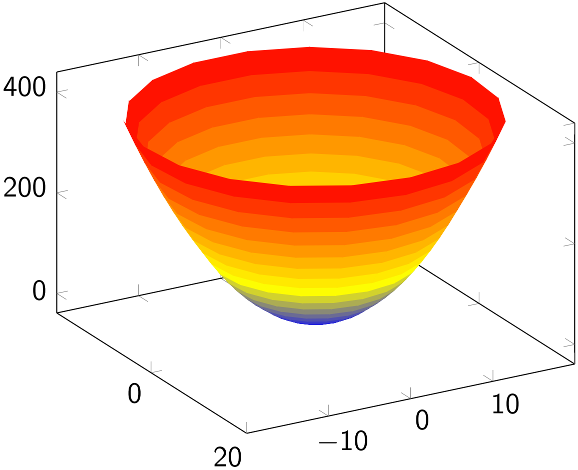
When one plots the $f(x, y) = c$ for some constant c one gets a curve. Such a curve is usually called a contour line (the contour "lives" in the $z = c$ plane).

I have a couple of pictures in the next two slides to illustrate the point.



This is the graph of the function $z = \sqrt{x^2 + y^2}$ lying above the xy-plane. It is a right circular cone.

The contour lines $z = c$ give circles lying on planes parallel to the xy-plane.The curves given by $z = f(x, 0)$ and $z = f(0, y)$ give pairs of straight lines in the planes $y = 0$ and $x = 0$.



This is the graph of the function $z = x^2 + y^2$ lying above the xy-plane. It is a paraboloid of revolution.
The contour lines $z = c$ give circles lying on planes parallel to the xy-plane.The curves $z = f(x, 0)$ or $z = f(y, 0)$ give parabolæ lying in the planes $y = 0$ and $x = 0$. Exercise 5.2.(ii).

Limits

We have already said what it means for a function of two or more variables to approach a limit. We simply have to replace the absolute value function on $\mathbb{R}$ by the distance function on $\mathbb{R}^m$. We will do this in two variables. The three variable definition is entirely analogous. We will denote by U a set in $\mathbb{R}^2$.

Definition: A function $f : U \to \mathbb{R}$ is said to tend to a limit l as $x = (x_1, x_2)$ approaches $c = (c_1, c_2)$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - l| < \epsilon,$$

whenever $0 < \|x - c\| < \delta$ with $x \in U$.

We recall that

$$\|x\| = \sqrt{x_1^2 + x_2^2}.$$

Continuity

Before talking about continuity we remark the following. In the plane $\mathbb{R}^2$ it is possible to approach the point c from infinitely many different directions - not just from the right and from the left. In fact, one may not even be approaching the point c along a straight line! Hence, to say that a function from $\mathbb{R}^2$ to $\mathbb{R}$ possesses a limit is actually imposing a strong condition - for instance, the limits along all possible curves leading to the point must exist and all these (infinitely many) limits must be equal.

Once we have the notion of a limit, the definition of continuity is just the same as for functions of one variable.

Definition: The function $f : U \to \mathbb{R}$ is said to be continuous at a point c if

$$\lim_{x \to c} f(x) = f(c).$$

The rules for limits and continuity

The rules for addition, subtraction, multiplication and division of limits remain valid for functions of two variables (or three variables for that matter). Nothing really changes in the statements or the proofs.

Using these rules, we can conclude, as before, that the sum, difference, product and quotient of continuous functions are continuous (as usual we must assume that the denominator of the quotient is non zero).