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Roll No : 12345678 **Dept.**:
e.g., 190040001 e.g., CSE

IIT Bombay
CS 405/6001: GT&AMD
Endsem, 2024-25-I
Date: November 21, 2024

CS 405/6001: Game Theory and Algorithmic Mechanism Design

Total: **50 marks,** *Duration:* $2\frac{1}{2}$ **hours, ATTEMPT ALL QUESTIONS**

Instructions:

1. This question-and-answersheet booklet contains a total of 6 sheets of paper (12 pages, page 2 is blank). Please verify.
2. Write your roll number and department on **every side of every sheet** (except the blank sheet) of this booklet. Use only **black/blue ball-point pen**. The first 5 minutes of additional time is given exclusively for this activity.
3. Write final answers neatly with a pen **only in the given boxes**.
4. Use the rough sheets for scratch works / attempts to solution. **Write only the final solution (which may be a sequence of logical arguments) in a precise and succinct manner in the boxes provided**. Do not provide unnecessarily elaborate steps. The space within the boxes are sufficient for the correct and precise answers.
5. Submit your answerscripts to the teaching staff when you leave the exam hall or the time runs out (whichever is earlier). **Your exam will not be graded if you fail to return the paper**.
6. **This is a closed book, notes, internet exam. No communication device, e.g., cellphones, iPad, etc., is allowed**. Keep it switched off in your bag and keep the bag away from you. If anyone is found in possession of such devices during the exam, that answerscript may be disqualified for evaluation and DADAC may be invoked.
7. Two A4 assistance sheets (text **text on both sides**) are allowed for the exam.
8. **After you are done with your exam or the exam duration is over, please DO NOT rush to the desk for submitting your paper**. Please remain seated until we collect all the papers, count them, and give a clearance to leave your seat.

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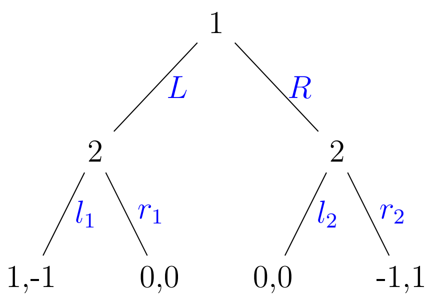
Problem 1 (10 points). The following theorem may be useful in this problem.

Theorem 1.1 (von Neumann (1948))**.** In any two-player zero sum *perfect information extensive form game*, where reaching an utility of 1, −1, and 0 are considered 'win', 'loss', and 'draw' respectively, one and only one of the following must be true.

- Player 1 has a winning strategy.
- Player 2 has a winning strategy.
- Each of the two players has a strategy guaranteeing at least a draw.

Now, answer the following questions.

(a) Consider the following *perfect information extensive form game*. The number on the vertices of the game tree represents the player at that vertex, and the actions on the edges represent the actions available to the player at the vertex above that edge. Find all the **subgame-perfect Nash equilibria** of this game. **2 points.**



(L, r_1, r_2)

(b) Represent the above game as a *normal form game*. **2 points.**

	l_1, l_2	l_1, r_2	r_1, l_2	r_1, r_2
L	1,−1	1,−1	0,0	0,0
R	0,0	−1,1	0,0	−1,1

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(c) Find all the **pure strategy Nash equilibria** of the above normal-form game. **2 points.**

(L, r_1, l_2) and (L, r_1, r_2) are the PSNEs

(d) Consider the normal form representation of a **two player zero-sum perfect information extensive form** game, where each box represents the outcomes: I (Player I wins), II (Player II wins), and D (a draw). Can such a perfect information extensive form game exist? (Yes/No) **2 points.**

Player II

Player I	s_{II}^1	s_{II}^2	s_{II}^3	s_{II}^4
s_{I}^1	D	I	II	I
s_{I}^2	I	II	I	D
s_{I}^3	I	I	II	II

No.

(e) Explain your answer to the previous part of this question. If the answer is yes, describe the game. If not, explain why not. The part of this question will not get any credit if the answer to the previous part is incorrect. **2 points.**

This follows from Theorem 1.1 above.
This game does not have a winning or draw guaranteeing strategy for any of the players, which violates the theorem.

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Problem 2 (10 points). A (finite) square matrix $A = [a_{i,j}, i, j = 1, \ldots, n]$, is called **anti-symmetric** if $a_{i,j} = -a_{j,i}$ for all $i, j = 1, \ldots, n$.

(a) Consider a **two-player zero-sum** game (also called *matrix game*, since the payoffs of the players can be represented by a matrix) that has an **anti-symmetric** payoff matrix. Recall that the value of a matrix game is the expected utility of the row player in a *mixed strategy Nash equilibrium* (MSNE). It is known that for a matrix game given by the matrix A, its MSNE $(\mathbf{x}^*, \mathbf{y}^*)$ and value v satisfy

$$\max_{\mathbf{x}} \min_{\mathbf{y}} \ \mathbf{x}^\top A \ \mathbf{y} = \min_{\mathbf{y}} \max_{\mathbf{x}} \ \mathbf{x}^\top A \ \mathbf{y} = \mathbf{x}^{* \top} A \ \mathbf{y}^* = v.$$

Find the value of the game in mixed strategies. Show the minimal precise steps to obtain your answer.

6 points.

Value, $v = \max_{x} \min_{y} \ x^T A y$

$= \max_{x} \min_{y} \ y^T A^T x$ since $a^T b = b^T a$

$= \max_{x} \min_{y} \ y^T(-A) x$ since $A^T = -A$

$= \max_{y} \min_{x} \ -x^T A y$ interchanging the notation $x \leftrightarrow y$

$= -\left(\min_{y} \max_{x} \ x^T A y\right)$ negative sign changes the maximization into minimization and vice versa

$= -v$

$\Rightarrow \boxed{v = 0}$

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(b) Let $G = (V, E)$ be a directed graph, where V is the set of vertices, and E is the set of edges. A directed edge from vertex x to vertex y is denoted by (x, y). Suppose that the graph is complete, i.e., for every pair of edges $x, y \in V$, either $(x, y) \in E$ or $(y, x) \in E$, but not both. In particular, $(x, x) \in E$ for all $x \in V$.

Define a matrix game in which the set of pure strategies of the two players is V, and the payoff function of the row player is defined as follows:

$$u(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y, (x, y) \in E, \\ -1, & \text{if } x \neq y, (x, y) \notin E. \end{cases}$$

Find the value of this matrix game. Hint: you may use the result from the previous part of this question, however, the arguments supporting why such a result can be applied should be explained in detail. **4 points.**

Since x and y are the vertices of the directed graph described here, the utility matrix of the row player $A = [a_{ij}]$, $i, j = 1, 2, \ldots, n$ is given by

$$a_{ij} = \begin{cases} 0 & \text{if } i = j \text{ , hence } a_{ji} = 0 \\ 1 & \text{if } i \neq j \text{ and } (i,j) \in E \Rightarrow (j,i) \notin E \text{ , hence } a_{ji} = -1 \\ -1 & \text{if } i \neq j \text{ and } (i,j) \notin E \\ & \Rightarrow (j,i) \in E \text{, hence } a_{ji} = 1 \end{cases}$$

Therefore we find that $a_{ij} = -a_{ji} \ \forall i, j = 1, \ldots, n$.

$\Rightarrow$ A is an antisymmetric matrix.

Hence the value of this matrix game is zero.

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Problem 3 (10 points). Assignment Game. Consider the following setup (called the assignment game) with two types of agents. The first type of agents is **seller**. There are n sellers having houses to sell at a cost $c_1, c_2, \ldots, c_n$ respectively. The set of sellers is denoted by S. The second type of agents is **buyer**. There are n buyers, denoted by the set B, and v_{ij} denotes the valuation of buyer i for house j, where $i \in B, j \in S$. An independent coordinator runs the mechanism of house-selling, where the following two decisions are taken.

- **Allocation:** a function $f(v, c)$, where $v = [v_{ij}, i \in B, j \in S]$ and $c = (c_j, j \in S)$ that outputs a ~~perfect~~ matching ~~(where all the sellers are matched to the buyers)~~ that matches each seller to ~~exactly~~ at most one buyer. Now, if a buyer i is matched to seller j, i.e., $f_{ij}(v, c) = 1$.
  and each buyer to at most one seller
- **Payment:** the payment of agent $k \in B \cup S$ is given by $p_k(v, c)$. Buyer i's utility is $\sum_{j \in S} v_{ij} f_{ij}(v, c) - p_i(v, c)$ and seller j's utility value is $-c_j - p_j(v, c)$. Payment can be either positive or non-positive, its sign determine the direction of the payment. Observe that these utilities are quasi-linear.
  $-c_j \sum_{i \in B} f_{ij}(v,c)$

(a) Describe the above setup as a problem of mechanism design with transfers as defined in the class: specify the (i) set of players, (ii) set of feasible allocations, (iii) the valuations for each feasible allocation, and finally (iv) the payment functions. **4 points.**

(i) Player set $= N = B \cup S$ $\quad B =$ set of buyers
$S =$ set of sellers

(ii) Feasible allocations : $f(v,c)$ s.t.

$$f_{ij}(v,c) = \begin{cases} 1 & \text{if } i \text{ is allocated house } j \quad i \in B, j \in S \\ 0 & \text{ow} \end{cases}$$

$$\text{s.t.} \quad \sum_{i \in B} f_{ij}(v,c) \leq 1 \quad \forall j \in S, \quad \sum_{j \in S} f_{ij}(v,c) \leq 1 \quad \forall i \in B.$$

(iii) Valuation v_i for a feasible allocation f.

$$v_i(f) = \sum_{j \in S} f_{ij}(v,c)\, v_{ij} \quad \forall i \in B$$

$$v_j(f) = -c_j \sum_{i \in B} f_{ij}(v,c) \quad \forall j \in S$$

(iv) payment p_i is a function of the players' reported types $(\hat{v}, \hat{c})$ and the allocation $f(\hat{v}, \hat{c})$.

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(b) Express mathematically the meaning of f being allocatively efficient in this setting? You may write an optimization problem. **2 points.**

$$\max \sum_{i \in B} \sum_{j \in S} f_{ij}(v_{ij} - c_j)$$

$$\text{s.t. } \sum_{i \in B} f_{ij} \leq 1 \quad \forall j \in S, \quad \sum_{j \in S} f_{ij} \leq 1 \quad \forall i \in B$$

$$f_{ij} \in \{0,1\}.$$

The solution f^* is allocatively efficient.

(c) Consider *two* buyers and *two* sellers with valuations $\begin{pmatrix} 7 & 6 \\ 10 & 5 \end{pmatrix}$ (rows correspond to the buyers and columns to the sellers) and costs $(3, 5)$ for the two houses respectively. What are the Vickrey-Clarke-Groves (VCG) payments made by each agent? Write only the answers for each player.

4 points.

Payment of buyer 1: $7 + 0 - (-5 + 7) = 5$

Payment of buyer 2: $4 + 0 - (-3 + 1) = 6$

Payment of seller 1: $1 - (10 + 1) = -10$

Payment of seller 2: $7 - (6 + 7) = -6$

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Problem 4 (10 points). Revenue Optimal Auction. Consider a mechanism to sell a single indivisible object to two buyers, $N = \{1, 2\}$. The buyers valuations, $t_i, i = 1, 2$, are independent and uniformly distributed over the interval $[a_i, b_i]$, where $0 \leqslant a_i < b_i$, respectively for $i = 1, 2$. Assume that $b_1 > b_2$ and $a_1 > a_2$. Let the mechanism be given by $(\mathbf{f}, \mathbf{p})$, where $f_i(t_1, t_2)$ denote the probability of allocating the item to buyer i and $p_i(t_1, t_2)$ denote the payment paid by buyer i to the mechanism designer, $i = 1, 2$.

(a) Find an **incentive compatible** (where reporting true valuation is the best reponse of every buyer) and **individually rational** (where every buyer gets non-negative utility from participation) selling mechanism under this setting that maximizes the expected revenue of the seller. You may refer to any result discussed in the class but mention that explicitly. Your answer should provide the details of the allocation and payment rules in a mathematical manner.

Allocation: **2 points.**

The mechanism will be Myerson's optimal mechanism, for which we first obtain the virtual valuations of the agents, given by

$$\phi_i(t_i, t_{-i}) = 2t_i - b_i \qquad i = 1,2$$

The allocation is

Case 1: if $\phi_i(t_i, t_{-i}) < 0$ for $i = 1, 2$, $f_i(t_i, t_{-i}) = 0$ for $i = 1, 2$.

Case 2: $2t_1 - b_1 > 2t_2 - b_2 \geqslant 0 \quad f_1(t_1, t_2) = 1 = 1 - f_2(t_1, t_2)$

$2t_1 - b_1 = 2t_2 - b_2 \geqslant 0 \quad f_1(t_1, t_2) = \alpha = 1 - f_2(t_1, t_2) \quad \alpha \in [0,1]$

$0 \leqslant 2t_1 - b_1 < 2t_2 - b_2 \quad f_1(t_1, t_2) = 0 = 1 - f_2(t_1, t_2)$

Payment: **2 points.**

Case 1: if $\phi_i(t_i, t_{-i}) < 0 \ \forall i = 1,2 \Rightarrow p_i(t_i, t_{-i}) = 0 \ \forall i = 1,2$

Case 2:

	p_1	p_2
$2t_1 - b_1 > 2t_2 - b_2$	$\max\left\{\frac{b_1}{2}, t_2 + \frac{b_1 - b_2}{2}\right\}$	0
$2t_1 - b_1 = 2t_2 - b_2$	$\alpha\left(t_2 + \frac{b_1 - b_2}{2}\right)$	$(1-\alpha)\left(t_1 + \frac{b_2 - b_1}{2}\right)$
$2t_1 - b_1 < 2t_2 - b_2$	0	$\max\left\{\frac{b_2}{2}, t_1 + \frac{b_2 - b_1}{2}\right\}$

Since the question asks for one allocation and payment $\alpha = 0$ and simplified and will be given full credit.

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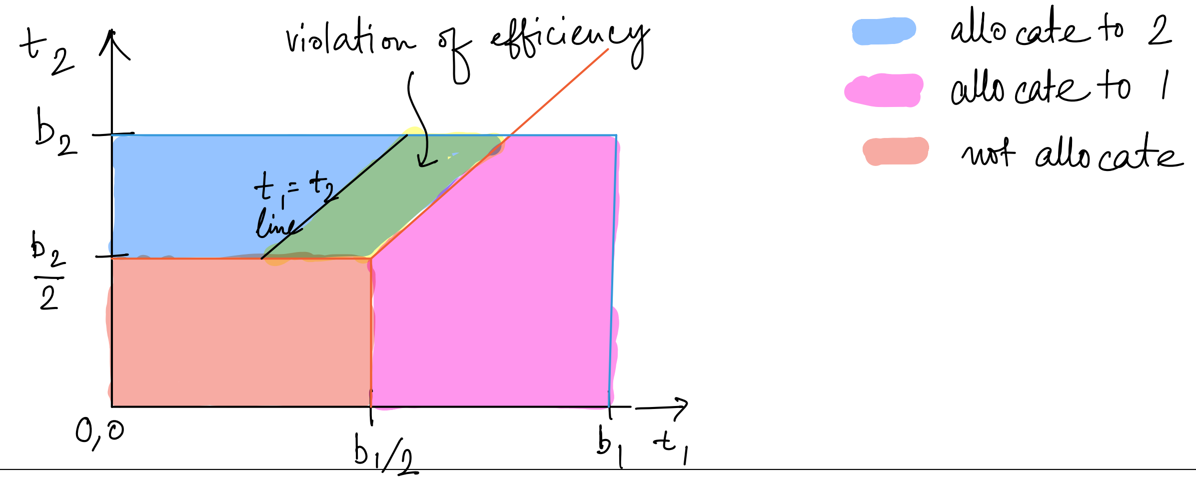
(b) Draw a figure on a 2-D plane with the x and y axes showing the values of t_1 and t_2 respectively. Mark the regions with the allocation decisions according to the revenue maximizing auction as obtained in the previous part of this question. **2 points.**



(c) Is this allocation *efficient*? Write yes/no in the box below and write the justification if your answer is *yes* OR show the region in the previous figure itself where it is violated if your answer is *no*. The second part of this question will not get any credit if the first answer is incorrect. **1 + 1 points.**

No, the violation of efficiency is shown in the figure above.

(d) Find the optimal expected revenue of the seller in case of a single bidder, $n = 1$, and drop the subscripts of a and b. Write the answers only. (Hint: Treat the cases $a < b/2$ and $a \geqslant b/2$ separately.) **2 points.**

Case 1: $a < \frac{b}{2}$, expected revenue $= \frac{b}{2} P\left(\frac{b}{2} < t < b\right) = \frac{b^2}{4(b-a)}$

Case 2: $a \geqslant \frac{b}{2}$, expected revenue $= a \cdot 1 = a$

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Problem 5 (10 points). Divisible Resource Allocation. The GPU server time (a divisible resource) is being allocated among a set of n users, who are seeking to run their code on the server. Each user can be allocated different amounts of time on the server. If $x_i \in [0, 1]$ fraction of the resource is allocated to the agent i, then he pays x_i^2 amount of money for the allocated resource. Note that $\sum_{i=1}^{n} x_i = 1$ for any valid allocation vector $\mathbf{x}$. Moreover, the value that agent i derives for the x_i amount of resource is proportional to x_i given by $\theta_i x_i$, where θ_i is agent i's private information.

(a) Assuming the utility is quasi-linear in allocation and payment, write down the expression of utility of agent i for an allocation of x_i fraction of the resource. **1 ~~2~~ point~~s~~.**

$u_i(x) = \theta_i x_i - x_i^2$.

(b) What type of preference does each agent have *over their consumed resource*? **1 ~~2~~ point~~s~~.**

Every agent has <u>single-peaked</u> preferences over their consumed resource.

(c) The GPU owner wants to design a mechanism in this setting such that it reveals the private types of the agents truthfully, and is Pareto efficient (no agent can be made happier without making at least one agent unhappier) and anonymous. What mechanism would you recommend to solve this problem? Why? **2 + 2 points.**

The mechanism will be the uniform rule (due to Sprumont).
For task sharing domain, uniform rule satisfies strategyproofness, Pareto efficiency, and anonymity. Since the resource allocation domain is of similar nature, the same mechanism can satis

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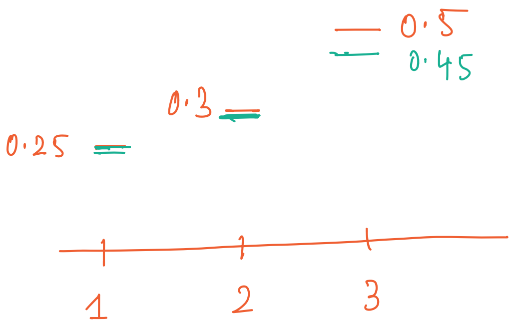
(d) If there are three agents in this mechanism and they have $\theta_i, i = 1, 2, 3$, to be 0.5, 0.6, and 1.0 respectively, what will be the final allocated resource fraction for each of them? Write the values of $x_i, i = 1, 2, 3$ and briefly explain how you arrived at them. **2 + 2 points.**

Since utilities are of the above form, the peak of each agent is at $\theta_i/2$, i.e., at 0.25, 0.3, and 0.5 for these three agents. Now the sum of peaks $\sum_{i=1}^{3} p_i = 1.05 > 1$



Therefore, the uniform rule should apply the water level filling from below.

Hence, the allocations will be

$\boxed{0.25, 0.3, 0.45 \text{ for the three agents}}$

END OF QUESTION PAPER. GOOD LUCK!