CS 228 : Logic in Computer Science

Krishna. S

Satisfiability of FOL

Given a formula in FOL over some signature τ, is it satisfiable?

Herbrand Theory

- Named after the French mathematician Jacques Herbrand
- Famous for Herbrand's Theorem, which allows a certain reduction from FOL to propositional logic
- Herbrand's theorem allows reducing a FOL formula φ in Skolem Normal Form to an infinite set $E(\varphi)$ of propositional formulae s.t. φ is satisfiable iff $E(\varphi)$ is satisfiable
- If $E(\varphi)$ is not satisfiable, then $\emptyset \in Res^*(E(\varphi))$, and we can derive this in finite number of steps
- As $E(\varphi)$ may be infinite, there is no way to say $\emptyset \notin Res^*(E(\varphi))$.

Herbrand Universe

Herbrand Universe

Let τ be a signature. The Herbrand universe for τ is the set of all ground terms.

Herbrand Universe

Herbrand Universe

Let τ be a signature. The Herbrand universe for τ is the set of all ground terms.

- If τ contains a constant c and unary function f, then the Herbrand universe contains $c, f(c), f(f(c)), f(f(f(c))), \ldots$

Herbrand Universe

Herbrand Universe

Let τ be a signature. The Herbrand universe for τ is the set of all ground terms.

- If τ contains a constant c and unary function f, then the Herbrand universe contains $c, f(c), f(f(c)), f(f(f(c))), \ldots$
- If τ contains a constant c and unary function f and binary function g, then the Herbrand universe contains distinct ground terms $c, g(c, c), f(c), g(c, f(c)), g(g(f(f(c)), c), f(c)), \ldots$

Herbrand Universe

- If τ has no constants, then the Herbrand universe is empty
- If τ has no functions, then the Herbrand universe consists of the constants of τ and is finite
- If τ has constants and functions, then the Herbrand universe is infinite

Herbrand Structures

Herbrand Structures

A structure $\mathcal{A}$ over a signature τ is a Herbrand structure if its underlying universe is a Herbrand universe.

- A Herbrand structure gives the natural interpretation to the constants and functions in τ : a constant *c* is interpreted as the element *c* in the universe,

Herbrand Structures

Herbrand Structures

A structure $\mathcal{A}$ over a signature τ is a Herbrand structure if its underlying universe is a Herbrand universe.

- A Herbrand structure gives the natural interpretation to the constants and functions in τ : a constant c is interpreted as the element c in the universe,
- If the signature τ has no relations or no constants, there is a unique Herbrand structure for τ

Herbrand Structures

- If the signature τ has relations and constants, then there are many Herbrand structures over τ depending on how you interpret them.

Herbrand Structures

- If the signature τ has relations and constants, then there are many Herbrand structures over τ depending on how you interpret them.
  - If τ contains a constant c and a binary relation R, then
  - $\mathcal{A} = (U^{\mathcal{A}} = \{c\}, R^{\mathcal{A}} = \{(c, c)\})$ is a Herbrand structure for τ.
  - $\mathcal{A} = (U^{\mathcal{A}} = \{c\}, R^{\mathcal{A}} = \{\})$ is a Herbrand structure for τ.
- If τ contains a constant c, function f and a unary relation R, then
  - $\mathcal{A} = (U^{\mathcal{A}} = \{c, f(c), f(f(c)), \ldots, \}, R^{\mathcal{A}} = \{c, f(c)\})$ is a Herbrand structure for τ.
  - $\mathcal{A} = (U^{\mathcal{A}} = \{c, f(c), f(f(c)), \ldots, \}, R^{\mathcal{A}} = \{c, f(c), f(f(f(f(c))))\})$ is a Herbrand structure, and so on.

Herbrand Signature

Let Γ be a set of sentences over a signature τ.

- The Herbrand signature for Γ is denoted τ_H.
- $\tau_H = \tau \cup \{c\}$ if τ contains no constants, else it is τ.
- The Herbrand universe for Γ denoted $H(\Gamma)$ is the Herbrand universe for τ_H.

Herbrand Model

A Herbrand model for Γ is a Herbrand structure M over τ_H such that $M \models \varphi$ for all $\varphi \in \Gamma$.