MA108/MA110 - Linear Algebra & Differential Equations: Endsem - A

Name: **Roll No.**

Tutorial Batch: D T **Max. marks: 40**

April 23, 2024 **14:00 - 16:00**

(1) For $x > 0$, $y_1(x) = \dfrac{\cos x}{x}$ is a solution of $xy'' + 2y' + xy = 0$. If y_2 is a solution of the DE satisfying $y_2(\frac{\pi}{2}) = 1 = y_2(\pi)$, then for $x > 0$, $y_2(x)$ equals $\boxed{-\dfrac{\pi\cos x}{x} + \dfrac{\pi \sin x}{2x}}$ [1]

(2) For $x \in \mathbb{R}$, $y_1(x) = \cos^3 x$ is a solution of the DE $y'' + ay' + by = \alpha\cos x + \beta \sin x$, $a, b, \alpha, \beta \in \mathbb{R}$. Let $y(x)$ denote the general solution of the DE. Then [1+1+1]

$\boxed{(a,b) = (0,\ 9)}$ $\boxed{(\alpha, \beta) = (6,\ 0)}$ $\boxed{y(x) = c_1 \cos 3x + c_2 \sin 3x + \frac{3}{4}\cos x}$

(3) For $x > 0$, $\sqrt{x}\sin(\ln\sqrt{x})$ is a solution of $x^2y'' + axy' + by = 0, x > 0,\ a, b \in \mathbb{R}$. Then

$\boxed{a = 0}$ $\boxed{b = \frac{1}{2}}$ [1+1]

(4) Let $Ly = y'' + y' + \frac{1}{4}y$.

(i) Let y_1, y_2 be two solutions of $Ly = 0$ satisfying $y_1(0) = 1, y_1'(0) = 0$; $y_2(0) = 0, y_2'(0) = 1$. Then [1+1]

$\boxed{y_1(x) = (1 + \frac{x}{2})e^{-\frac{x}{2}}}$ $\boxed{y_2(x) = xe^{-\frac{x}{2}}}$

(ii) If y_p is a particular solution of $Ly = \cos x$, then $\boxed{y_p(x) = -\frac{12}{25}\cos x + \frac{16}{25}\sin x}$ [2]

(5) Let $Ly = (1 - x)xy'' + 2xy' - 2y$. Let y_1, y_2 be linearly independent solutions of $Ly = 0$, $0 < x < 1$ with $y_1(x) = x, 0 < x < 1; y_2(\frac{1}{2}) = -\frac{3}{4} + \ln 2$, $y_2'(\frac{1}{2}) = 2\ln 2 - 1$. If $W(x)$ denotes the wronskian of (y_1, y_2) in that order, and $y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x)$ is a particular solution of $Ly = x^2(1-x)^3$, then [1+1+1+1]

$\boxed{y_2(x) = x^2 - 2x\ln x - 1}$ $\boxed{W(0.5) = \frac{1}{4}}$

$\boxed{v_1(x) = -\frac{x^2}{4} + \frac{2}{3}x^3 \ln x - \frac{2}{9}x^3 + \frac{x^2}{2}}$ $\boxed{v_2(x) = \frac{x^3}{3}}$

(6) Let $Ly = x^2y'' - 4xy' + 6y$. Let y_h denote the general solution of $Ly = 0, x > 0$, y_p denote a particular solution of $Ly = x^2 + \ln x, x > 0$ and $x^2D^2 + axD + b$ annihilates $\ln x$, $a, b \in \mathbb{R}$. Then [1+1+2]

$\boxed{y_h(x) = c_1x^2 + c_2x^3}$ $\boxed{(a,b) = (1,\ 0)}$ $\boxed{y_p(x) = \frac{5}{36} + \frac{1}{6}\ln x - x^2 \ln x}$

(7) Let $Ly = y''' - 4y'' + 5y' - 2y$. Let y_h denote the general solution of $Ly = 0$. Then for $x \in \mathbb{R}$, [1]

$$\boxed{y_h(x) = c_1 e^x + c_2 x e^x + c_3 e^{2x}}$$

Particular solution of $Ly = e^x(1 + e^x)$ is given by $\boxed{-\frac{x^2}{2}e^x + xe^{2x}}$ [2]

(8) Let $L = x^3D^3 + \alpha x^2 D^2 + \beta x D + \gamma$, $x > 0$ annihilates $\sqrt{x}\,(\ln x)^2$, $\alpha, \beta, \gamma \in \mathbb{R}$. Also let y denote the general solution of $Ly = 0$. Then [2+1]

$\boxed{(\alpha, \beta, \gamma) = (\frac{3}{2}, \frac{1}{4}, -\frac{1}{8})}$ $\boxed{y(x) = c_1\sqrt{x} + c_2\sqrt{x}\ln x + c_3\sqrt{x}(\ln x)^2}$

(9) Let $a > 0$ be such that $y_1(x) = x$, $y_2(x) = x^2 \ln x, x > 0$ are solutions of $y'' + p(x)y' + q(x)y = 0, 0 < x < a$, where $p(x), q(x)$ are continuous in $(0,\ a)$. Let α denote the maximum value of a. Then [1+1+1+1]

$\boxed{\alpha = \frac{1}{e}}$ $\boxed{p(x) = -\frac{3 + 2\ln x}{x(1 + \ln x)}}$

$\boxed{q(x) = \frac{3 + 2\ln x}{x^2(1 + \ln x)}}$ $\boxed{\lim_{x \to 0+} xp(x) = -2}$

(10) Let $f(t) = \int_0^t \frac{\sin\tau}{\tau}d\tau$ and $\frac{d}{ds}(sF(s)) = \frac{\alpha}{s^2 + \beta s + 1}$, where $F(s)$ denotes the Laplace transform of f. Then [1+1]

$\boxed{(\alpha, \beta) = (-1,\ 0)}$ $\boxed{F(s) = \frac{1}{s}\cot^{-1} s}$

(11) Let $g : [0,\ \infty) \to \mathbb{R}$, $h : [0,\ \infty) \to \mathbb{R}$ be such that $g * h$ is the inverse Laplace transform of $F(s) = \frac{s}{(s+1)(s^2+4)}, s > 0$. [1+1]

(i) If $g(t) = e^{-t}$, then $h(t) = \boxed{\cos 2t}$

(ii) If $g * h(t) = a\sin 2t + b\cos 2t + ce^{-t}, a, b, c \in \mathbb{R}$, then $(a, b, c) = \boxed{(\frac{2}{5}, \frac{1}{5}, -\frac{1}{5})}$

12. Show that the IVP

$$\begin{aligned} y' &= 5(y-x)^{\frac{4}{5}} + 1,\ x \in (-\infty,\ a), \\ y(0) &= -2 \end{aligned}$$

has a unique solution for $a = 1$. Also find the largest value of a such that the IVP has a unique solution in $(-\infty,\ a)$. [4]

Solution: Denote the IVP by (1). Set $u = y - x$. Then (1) becomes $u' = 5u^{\frac{4}{5}}$, $u(0) = -2$.
Using separation variables method, we get $u = (x - 2^{\frac{1}{5}})^5 \Rightarrow\ y(x) = x + (x - 2^{\frac{1}{5}})^5, x \in \mathbb{R}$ as a solution of the IVP (1) [1]

The solution $y(x) = x + (x - 2^{\frac{1}{5}})^5, x \in \mathbb{R}$ is increasing and meets the line $y = x$ at $x = 2^{\frac{1}{5}}$ and $y = y(x), x < 2^{\frac{1}{5}}$ is below the line $y = x$.
$f(x, y) = (y - x)^{\frac{4}{5}} + 1$ is bounded and continuous on any finite rectangle containing $(x_0, y(x_0))$ for each fixed $x_0 \in \mathbb{R}$. Also since $f_y(x, y) = \frac{4}{5(y-x)^{\frac{1}{5}}}$, $y \neq x$ is bounded on any finite rectangle which (strictly) lies above or below the line $y = x$, f is Lipschitz continuous in y on any finite rectangle which (strictly) lies above or below the line $y = x$.

Hence using the existence uniqueness theorem, the IVP $y' = f(x, y), y(a) = a + (a - 2^{\frac{1}{5}})^5 := y_0$ has a unique solution for each $a < 2^{\frac{1}{5}}$ in $(a - \alpha,\ a + \alpha)$ for some $\alpha > 0$. Since $y = x + (x - 2^{\frac{1}{5}})^5$ is a solution satisfying $y(a) = y_0$, it is the unique solution for the IVP (1) for $x < a$ for any $a < 2^{\frac{1}{5}}$. In particular IVP (1) has a unique solution in the interval $(-\infty,\ 1)$. [1]

Consider

$$y_1(x) = \begin{cases} x + (x - 2^{\frac{1}{5}})^5 & \text{for} \quad x \leq 2^{\frac{1}{5}} \\ x & \text{for} \quad x > 2^{\frac{1}{5}}. \end{cases}$$

$y_1'(2^{\frac{1}{5}})$ exists and hence y_1 is differentiable at each $x \in \mathbb{R}$.
Also $y = x$ is a solution of the IVP $y' = f(x, y), y(2^{\frac{1}{5}}) = 2^{\frac{1}{5}}$.
Hence one can see that $y = y_1(x), x \in \mathbb{R}$ is a solution of the IVP (1). Therefore, $y = y_1(x), x < a$ is a second solution of the IVP for $a > 2^{\frac{1}{5}}$. [1]

Hence maximum value of a is $2^{\frac{1}{5}}$ [1]

13. Using Laplace transform technique, solve the DE

$$xy'' + (2x+3)y' + (x+3)y = \ 3e^{-x}, \ y(0) = 0.$$

[4]

Solution Set $Y = L(y)$, the Laplace transform of y. Take Laplace transform on the DE and using linearity property of Laplace transform, we get

$$\begin{aligned} L(xy'') + 2L(xy') + 3L(y') + L(xy) + 3L(y) &= \frac{3}{s+1}. \\ \Rightarrow \ -\frac{d}{ds}(s^2Y - sy(0) - y'(0)) - 2\frac{d}{ds}(sY - y(0)) + 3(sY - y(0)) - Y' + 3Y &= \frac{3}{s+1}. \end{aligned}$$

[1]

$$\Rightarrow \ (-s^2 - 2s - 1)Y' + (1+s)Y = \frac{3}{s+1}$$

$$\Rightarrow \ Y' - \frac{1}{(1+s)}Y = -\frac{3}{(s+1)^3}$$

Solving, we get

$$Y(s) = \frac{1}{(1+s)^2} + c(1+s)$$

[1]

Since using the result, $L(f)(s) \to 0$ as $s \to \infty$ for any piecewise continuous function f with exponential order, we have $Y(s) \to 0$ as $s \to \infty$, hence $c = 0$. [1]

Therefore

$$Y = \frac{1}{(1+s)^2} = -\frac{d}{ds}\Big(\frac{1}{1+s}\Big).$$

$$\Rightarrow \ y = xe^{-x}, x \geq 0$$

is a solution. [1]