Linear Algebra (MA106 & MA110 First Half)
Tutorial Problems

Most of these problems are from reference texts for this course. We will add the new tutorial problems to this same file each week. For the latest problems, see the last few pages.

Tutorial 1: Wednesday, 10th Jan 2024

1. Sketch the three lines, and decide if the system is solvable. If yes, find the solution set.
$$x + 2y = 2, \quad x - y = 2, \quad y = 1$$

2. For the equations $x + y = 4,\ 2x - 2y = 4$, draw the row picture (two intersecting lines) and the column picture (combination of two columns equal to the column vector $(4, 4)$ on the right side).

3. Describe the intersection of the three hyperplanes in a four dimensional space
$$u + v + w + z = 6, \quad u + w + z = 4, \quad u + w = 2$$
Is it a line or a point or an empty set? What is the intersection if the fourth hyperplane $u = -1$ is included? Find a different fourth equation that leaves us with no solution.

4. Under what conditions on y_1, y_2, y_3, do the points $(0, y_1), (1, y_2), (2, y_3)$ lie on a line?

5. Starting with $x + 4y = 7$, find the equation for the parallel line through $x = 0, y = 0$. Find the equation of another line that meets the first at $x = 3, y = 1$.

6. Starting with a first plane $u + 2v - w = 6$, find the equation for
   (a) the parallel plane through the origin.
   (b) a second plane through origin that also contains the points $(6, 0, 0)$ and $(2, 2, 0)$.
   (c) a third plane that meets the first and second in the point $(4, 1, 0)$.

7. It is impossible for a system of linear equations to have exactly two solutions. Explain why.
   (a) If (x, y, z) and (X, Y, Z) are two solutions of system of linear equations in 3 unknowns, what is another one?
   (b) If 25 planes in $\mathbb{R}^3$ meet at two points, where else do they meet?

8. Show that the set $\left\{ c_1 \begin{pmatrix} -2 \\ 5 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ -15/2 \end{pmatrix} \mid c_1, c_2 \in \mathbb{R} \right\}$ describes a line. Does it describe a line through the origin?

9. Fill in the blanks.
   (a) For four linear equations in two unknowns x and y, the row picture shows four ________. The column picture is in ________ dimensional space. The equations have no solutions unless the vector on the right-hand side is a linear combination of ________.

(b) If a linear system is consistent, then the solution is unique if and only if the following is true about the columns containing pivots: ---------.

(c) A 3×4 matrix can have at most ---- pivots.

(d) A 4×3 matrix can have at most ---- pivots.

10. Choose a coefficient a that makes this system singular. Then choose a right-hand side b that makes it solvable. Find two solutions in that singular case.

$$2x + ay = 16, \quad 4x + 8y = b.$$

11. What test on b_1, and b_2 decides whether these two equations allow a solution? How many solutions will they have? Draw the column picture.

$$3x - 2y = b_1, \quad 6x - 4y = b_2$$

12. If the following system is consistent for all values of c and d, what can you say about the coefficients a and b?

$$2x_1 + 4x_2 = d, \quad ax_1 + bx_2 = c$$

13. Find h and k, if they exist, such that the following system $x_1 + hx_2 = 2, \quad 4x_1 + 8x_2 = k$ has (a) no solution, (b) a unique solution, and (c) many solutions.

14. Which number b leads later to a row exchange? Which b leads to a missing pivot? In that singular case find a non-zero solution x, y, z.

$$x + by = 0, \quad x - 2y - z = 0, \quad y + z = 0$$

15. Apply elimination (circle the pivots) and back-substitution to solve

$$2x - 3y = 3, \quad 4x - 5y + z = 7, \quad 2x - y - 3z = 5$$

16. (a) Verify that $(1, 1)$ is a solution to $3x + y = 4$. Find the solution set of this system.

    (b) Find two systems of equations such that the solution set is $\{(1, 1)\}$.

17. Use elimination to solve

    (a)

$$u + v + w = 6, \quad u + 2v + 2w = 11, \quad 2u + 3v - 4w = 3$$

    (b)

$$u + v + w = 7, \quad u + 2v + 2w = 10, \quad 2u + 3v - 4w = 3$$

18. Find a polynomial $p(t) = a_0 + a_1t + a_2t^2$ such that $p(1) = 6$, p(2)=15, p(3)=28.

19. Consider a 3×3 system in variables u, v and w, with three (nonzero) pivots. State true or false with explanation:

    (a) If the third equation starts with a zero coefficient (it begins with $0u$) then no multiple of equation 1 will be subtracted from equation 3.

    (b) If the third equation has zero as its second coefficient (it contains $0v$ then no multiple of equation 2 will be subtracted from equation 3.

    (c) If the third equation contains $0u$ and $0v$ then no multiple of equation 1 or no multiple of equation 2 will be subtracted from equation 3.

20. Suppose a 3×5 coefficient matrix for a system has three pivot columns. Is the system consistent? Why or why not?

21. Suppose A is a 3×3 matrix and b is a 3×1 column vector such that $Ax = b$ does not have a solution. Does there exist a 3×1 column vector y such that $Ax = y$ has a unique solution?

22. Suppose A is a 3×4 matrix and b is a 3×1 column vector such that $Ax = b$ does not have a solution. Does there exist a 3×1 column vector y such that $Ax = y$ has a unique solution?

23. Let $A = \begin{pmatrix} 1 & -5 & 4 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 7 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$.

   (a) Find all possible solutions to $Ax = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

   State true or false with explanation: $Ax = d$ is consistent for any 4×1 matrix d.

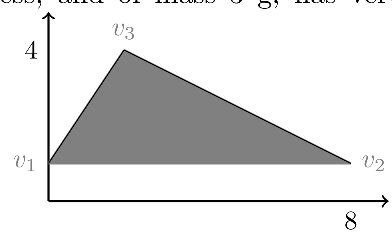
   (b) Find all possible solutions to $Bx = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

   State true or false with explanation: $Bx = b$ is consistent for any 3×1 matrix b.

24. Let $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$, B be 2×2 matrices such that $AB = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Show that $BA = I_2$.

25. Write your CPI as a linear combination (or weighted sum) of your grades.

26. A thin triangular plate of uniform density and thickness, and of mass 3 g, has vertices at



$v_1 = (0, 1)$, $v_2 = (8, 1)$, and $v_3 = (2, 4)$, as in the figure.

   (a) Find the (x, y)-coordinates of the centre of mass of the plate. (Hint: Find the centroid).

   (b) The balance point of the plate coincides with the centre of mass of a system consisting of three 1 gram point masses located at the vertices. Determine how to distribute an additional mass of $6g$ at the three vertices of the plate to move the balance point to $(2, 2)$. $\left(\text{Center of mass, of point masses } m_j \text{ located at } v_j, j = 1, \ldots, n, \text{ is given by } \frac{\sum_{j=1}^n m_j v_j}{\sum_{j=1}^n m_j}\right)$.

27. Consider an economy with three sectors: Fuels and Power, Manufacturing, and Services. Fuels and Power sells 80% of its output to Manufacturing, 10% to services and retains the rest. Manufacturing sells 10% of its out put to Fuels and Power, 80% to Services, and retains the rest. Services sells 20% to Fuels and Power, 40% to Manufacturing, and retains the rest.

Develop a system of equations that leads to prices at which each sector's income matches its expenses. Then write the augmented matrix that can be row reduced to find these prices.

28. Limestone, $CaCO_3$, neutralizes the acid H_3O, in acid rain by the following unbalanced equation.

$$H_3O + CaCO_3 \to H_2O + Ca + CO_2.$$

Balance this equation.

Tutorial 2: Wednesday, 17th Jan 2024

1. Let A and B be $n \times n$ matrices. State true or false with explanation:
   (a) $(AB)^T = B^T A^T$.
   (b) If $AB = 0$ then $A = 0$ or $B = 0$.
   (c) The zero matrix is diagonal.
   (d) If A is upper triangular, then so is A^T.
   (e) The identity matrix I is upper triangular.
   (f) Every lower triangular matrix is symmetric.
   (g) If A is symmetric and skew-symmetric, then $A = 0$
   (h) If A and B are triangular, then so is $A + B$.

2. Prove or disprove.
   (a) If a 2×2 matrix A is such that $AB = BA$ for all 2×2 matrices B, then A is a constant multiple of the identity matrix.
   (b) Let A be a matrix. There does not exist a matrix B such that $BA = 2A$.
   (c) Product of triangular matrices is triangular.
   (d) Inverse of an invertible triangular matrix is triangular.
   (e) Inverse of an invertible symmetric matrix is symmetric.
   (f) If u and v are solutions to $Ax = b$ then so is $(u + v)$.
   (g) Given a square matrix A, if $Ax = b$ has a solution for all b, then the solutions are all unique.
   (h) If $A^2 = A$, then $A = I$ or $A = 0$.

3. By trial and error find examples of 2 by 2 matrices such that
   (a) $A^2 = -I$, A having only real entries.
   (b) $B^2 = 0$, although $B \neq 0$.
   (c) $CD = -DC$, not allowing the case $CD = 0$.
   (d) $EF = 0$, although no entries of E or F are zero.

4. Let $C = \begin{pmatrix} 2 & 1 \\ -1 & 2 \\ 2 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 6 & -4 & 0 \\ 4 & -2 & 0 \\ -1 & 0 & 3 \end{pmatrix}$.

(a) Find a 2×2 matrix X, if it exists such that $CX = \begin{pmatrix} 1 & 3 \\ 3 & 1 \\ 1 & 3 \end{pmatrix}$.

(b) Find all column vectors X such that $DX = 3X$.

5. What three elementary matrices $E_{21},\ E_{31},\ E_{32}$ put $A = \begin{pmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{pmatrix}$ into triangular form U? Multiply the E's to get one matrix M that does the elimination to give $MA = U$.

6. Fill in the blanks.

   (a) Let A be a 3×3 matrix, with no row exchanges are needed in elimination to get U. Suppose $a_{33} = 7$ and the third pivot is 5.
   (i) If you change a_{33} to 11, what is the third pivot?
   (ii) What should you change a_{33} to, so that there is a zero in the third pivot position?

   (b) To obtain the entry in row 3, column 4 of AB we need to multiply the ____ row of ____ with the ____ column of ____ .

   (c) If a 5×5 matrix has __ number of pivots, then it is invertible.

7. Find A such that

$$A \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ A \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\ \text{and } A \begin{pmatrix} 2 \\ 3 \\ 8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

   How is A related to the matrix $B = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{pmatrix}$?

8. Let A be $m \times n$, and b be an $m \times 1$ vector. If $Ax = 0$ has a unique solution, what can you say about the number of solutions for $Ax = b$ for some b?

9. Factor A into LU and write down the upper triangular system $Ux = c$ which appears after elimination, for

$$Ax = \begin{pmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix}$$

10. How could you factor A into a product UL, upper triangular times lower triangular? Would they be the same factors as in $A = LU$?

11. Solve as two triangular system, without multiplying LU to find A:

$$LUx = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$$

12. For which numbers c, will A have LU decomposition ?

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & c & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

13. Find the inverses of

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 3 & 0 & 0 \\ 4 & 5 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 4 & 5 \end{pmatrix}, \ A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

14. If A, B and C are $n \times n$ matrices such that $AB = I_n$, and $CA = I_n$, then show that $B = C$.

15. (a) If P_1 and P_2 are permutation matrices, so is P_1P_2. This still has the rows of I in some order. Give examples with $P_1P_2 \neq P_2P_1$ and $P_3P_4 = P_4P_3$.

    (b) Find the inverses of the permutation matrices

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

    (c) Explain for permutations why P^{-1} is always the same as P^T. Show that the 1's are in the right place to give $PP^T = I$.

16. Suppose A is invertible and you exchange its first two rows to reach B. Is the new matrix B invertible? How would you find B^{-1} from A^{-1}?

17. Let A and B be $n \times n$. Show that $I - AB$ is invertible if $I - BA$ is invertible. Start from $B(I - AB) = (I - BA)B$.

18. This matrix has a remarkable inverse. Find A^{-1} by elimination on $[A \mid I]$. Extend it to 5×5 "alternating matrix in 1, -1" and guess its inverse.

$$A = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

19. (a) There are sixteen 2 by 2 matrices whose entries are 1's and 0's. How many are invertible?

    (b) If you put 1's and 0's at random into the entries of a 10 by 10 matrix, is it more likely to be invertible or singular?

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