MA108/MA110 - Linear Algebra & Differential Equations: Endsem - $\boxed{\text{D}}$

Name: **Roll No.** $\boxed{}$

Tutorial Batch: D $\boxed{}$ **T** $\boxed{}$ **Max. marks: 40**

April 23, 2024 **14:00 - 16:00**

(1) For $x > 0$, $y_1(x) = \dfrac{\cos 4x}{x}$ is a solution of $xy'' + 2y' + 16xy = 0$. If y_2 is a solution of the DE satisfying $y_2(\frac{\pi}{8}) = 1 = y_2(\frac{\pi}{4})$, then for $x > 0$, $y_2(x)$ equals $\boxed{-\dfrac{\pi\cos 4x}{4x} + \dfrac{\pi \sin 4x}{8x}}$ [1]

(2) For $x \in \mathbb{R}$, $y_1(x) = \sin^3 x$ is a solution of the DE $y'' + ay' + by = \alpha\cos 3x + \beta \sin 3x$, $a, b, \alpha, \beta \in \mathbb{R}$. Let $y(x)$ denote the general solution of the DE. Then [1+1+1]

$\boxed{(a, b) = (0,\ 1)}$ $\boxed{(\alpha, \beta) = (0,\ 2)}$ $\boxed{y(x) = c_1\cos x + c_2 \sin x - \dfrac{1}{4}\sin 3x}$

(3) For $x > 0$, $x^{\frac{1}{3}}\sin(\ln x^{\frac{1}{3}})$ is a solution of $x^2y'' + axy' + by = 0, x > 0,\ a, b \in \mathbb{R}$. Then

$\boxed{a = \dfrac{1}{3}}$ $\boxed{b = \dfrac{2}{9}}$ [1+1]

(4) Let $Ly = y'' - y' + \dfrac{1}{4}y$.

(i) Let y_1, y_2 be two solutions of $Ly = 0$ satisfying $y_1(0) = 1, y_1'(0) = 0$; $y_2(0) = 0, y_2'(0) = 1$. Then [1+1]

$\boxed{y_1(x) = (1 - \dfrac{x}{2})e^{\frac{x}{2}}}$ $\boxed{y_2(x) = xe^{\frac{x}{2}}}$

(ii) If y_p is a particular solution of $Ly = \cos x$, then $\boxed{y_p(x) = -\dfrac{12}{25}\cos x - \dfrac{16}{25}\sin x}$ [2]

(5) Let $Ly = (1 - x)xy'' + (3x - 1)y' - 4y$. Let y_1, y_2 be linearly independent solutions of $Ly = 0$, $0 < x < 1$ with $y_1(x) = x^2, 0 < x < 1$; $y_2(\frac{1}{2}) = \frac{1}{2} - \frac{1}{4}\ln 2$, $y_2'(\frac{1}{2}) = \frac{5}{2} - \ln 2$. If $W(x)$ denotes the wronskian of (y_1, y_2) in that order, and $y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x)$ is a particular solution of $Ly = 2x^2(1 - x)^3$, then [1+1+1+1]

$\boxed{y_2(x) = x^2\ln x + 2x - \dfrac{1}{2}}$ $\boxed{W(0.5) = \dfrac{1}{8}}$

$\boxed{v_1(x) = -\dfrac{2}{3}x^3\ln x + \dfrac{2}{9}x^3 - 2x^2 + x}$ $\boxed{v_2(x) = \dfrac{2}{3}x^3}$

(6) Let $Ly = x^2y'' - 5xy' + 8y$. Let y_h denote the general solution of $Ly = 0, x > 0$, y_p denote a particular solution of $Ly = x^4 + \ln x, x > 0$ and $x^2D^2 + axD + b$ annihilates $\ln x$, $a, b \in \mathbb{R}$. Then [1+1+2]

$\boxed{y_h(x) = c_1x^2 + c_2x^4}$ $\boxed{(a, b) = (1,\ 0)}$ $\boxed{y_p(x) = \dfrac{3}{32} + \dfrac{1}{8}\ln x + \dfrac{x^4}{2}\ln x}$

(7) Let $Ly = y''' - 7y'' + 16y' - 12y$. Let y_h denote the general solution of $Ly = 0$. Then for $x \in \mathbb{R}$, [1]

$$\boxed{y_h(x) = c_1 e^{2x} + c_2 x e^{2x} + c_3 e^{3x}}$$

Particular solution of $Ly = e^{2x}(1 + e^x)$ is given by $\boxed{-\frac{1}{2}x^2 e^{2x} + x e^{3x}}$ [2]

(8) Let $L = x^3 D^3 + \alpha x^2 D^2 + \beta x D + \gamma$, $x > 0$ annihilates $x^2(\ln x)^2$, $\alpha, \beta, \gamma \in \mathbb{R}$. Also let y denote the general solution of $Ly = 0$. Then [2+1]

$\boxed{(\alpha, \beta, \gamma) = (-3,\ 7,\ -8)}$ $\boxed{y(x) = c_1 x^2 + c_2 x^2 \ln x + c_3 x^2 (\ln x)^2}$

(9) Let $a > 0$ be such that $y_1(x) = x^2$, $y_2(x) = x^4 \ln x, x > 0$ are solutions of $y'' + p(x)y' + q(x)y = 0, 0 < x < a$, where $p(x), q(x)$ are continuous in $(0,\ a)$. Let α denote the maximum value of a. Then [1+1+1+1]

$\boxed{\alpha = \frac{1}{\sqrt{e}}}$ $\boxed{p(x) = -\frac{10 \ln x + 7}{x(1 + 2\ln x)}}$

$\boxed{q(x) = \frac{4(3 + 4\ln x)}{x^2(1 + 2\ln x)}}$ $\boxed{\lim_{x \to 0+} x p(x) = -5}$

(10) Let $f(t) = \int_0^t \frac{4 \sin \tau}{\tau} d\tau$ and $\frac{d}{ds}(sF(s)) = \frac{\alpha}{s^2 + \beta s + 1}$, where $F(s)$ denotes the Laplace transform of f. Then [1+1]

$\boxed{(\alpha, \beta) = (-4,\ 0)}$ $\boxed{F(s) = \frac{4}{s} \cot^{-1} s}$

(11) Let $g : [0,\ \infty) \to \mathbb{R}, h : [0,\ \infty) \to \mathbb{R}$ be such that $g * h$ is the inverse Laplace transform of $F(s) = \frac{s}{(s-2)(s^2+4)}$, $s > 2$. [1+1]

(i) If $g(t) = e^{2t}$, then $h(t) =$ $\boxed{\cos 2t}$

(ii) If $g * h(t) = a \sin 2t + b \cos 2t + c e^{2t}, a, b, c \in \mathbb{R}$, then $(a, b, c) =$ $\boxed{(\frac{1}{4},\ -\frac{1}{4},\ \frac{1}{4})}$

12. Show that the IVP

$$\begin{aligned} y' &= 5(y-x)^{\frac{4}{5}} + 1,\ x \in (-\infty,\ a), \\ y(0) &= -2 \end{aligned}$$

has a unique solution for $a = 1$. Also find the largest value of a such that the IVP has a unique solution in $(-\infty,\ a)$. [4]

Solution and marking scheme same as in CODE A

13. Using Laplace transform technique, solve the DE

$$xy'' + (2x+3)y' + (x+3)y = \ 3e^{-x}, \ y(0) = 0.$$

[4]

Solution and marking scheme same as in CODE A