MA-110 Linear Algebra and Differential Equations

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Lecture 9 D3

Reading Slide: Vector Spaces definition continued

Let x, y and z be vectors, a and b be scalars The vector addition and scalar multiplication are required to satisfy the following axioms:

- $x + y = y + x$ $\boxed{\text{Commutativity of addition}}$
- $(x + y) + z = x + (y + z)$ $\boxed{\text{Associativity of addition}}$
- There is a unique vector 0, such that $x + 0 = x$
  $\boxed{\text{Existence of additive identity}}$
- For each x, there is a unique $-x$ such that $x + (-x) = 0$
  $\boxed{\text{Existence of additive inverse}}$
- $1 * x = x$ $\boxed{\text{Unit property}}$
- $(a + b) * x = a * x + b * x, \quad a * (x + y) = a * x + a * y$
  $(ab) * x = a * (b * x)$ $\boxed{\text{Compatibility}}$

Notation: For a scalar a, and a vector x, we denote $a * x$ by ax.

Subspaces: Definition and Examples

If V is a vector space, and W is a non-empty subset, then W is a *subspace* of V if:

$$\boxed{x,\ y \text{ in } W, \quad a,\ b \text{ in } \mathbb{R} \Rightarrow a * x + b * y \text{ are in } W.}$$

i.e., linear combinations stay in the subspace.

Examples:

1. $\{0\}$: The zero subspace and $\mathbb{R}^n$ itself.
2. $\{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0\}$ is not a subspace of $\mathbb{R}^2$. Why?
3. The line $x - y = 1$ is not a subspace of $\mathbb{R}^2$. Why?
   Exercise: A line not passing through the origin is not a subspace of $\mathbb{R}^2$.
4. The line $x - y = 0$ is a subspace of $\mathbb{R}^2$. Why?
   Exercise: Any line passing through the origin is a subspace of $\mathbb{R}^2$.

Vector Spaces: Examples

1. $V = \{0\}$, the space consisting of only the zero vector.
2. $V = \mathbb{R}^n$, the n-dimensional space.
3. $V = \mathbb{R}^\infty =$ sequences of real numbers, e.g., $x = (0,1,0,2,0,3,0,4,\dots)$, with component-wise addition and scalar multiplication.
4. $V = \mathscr{M}_{m\times n}$, the set of $m \times n$ matrices, with entry-wise $+$ and $*$.
5. $V = \mathscr{P}$, the set of polynomials, e.g. $1+2x+3x^2+\cdots+2023x^{2022}$, with term-wise $+$ and $*$.
6. $V = \mathscr{C}[0,1]$, the set of continuous real-valued functions on the closed interval $[0,1]$. e.g., x^2, e^x are vectors in V. How about $1/x$ and $1/(x-5)$?Are they vectors in V?
   Vector addition and scalar multiplication are pointwise: $(f+g)(x) = f(x)+g(x)$ and $(a*f)(x) = af(x)$.

Subspaces: More Examples

5. Let A be an $m \times n$ matrix.
   The null space of A, $N(A)$, is a subspace of $\mathbb{R}^n$.
   The column space of A, $C(A)$, is a subspace of $\mathbb{R}^m$.
   Recall: They are both closed under linear combinations.
6. The set of 2×2 symmetric matrices is a subspace of $\mathscr{M}$.
   The set of 2×2 lower triangular matrices is also a subspace of $\mathscr{M}$.

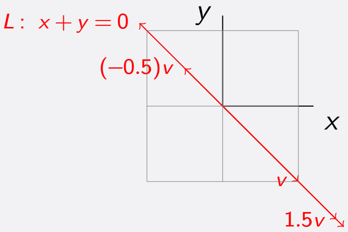
   Q. Is the set of invertible 2×2 matrices a subspace of $\mathscr{M}$?
7. The set of convergent sequences is a subspace of $\mathbb{R}^\infty$.
   What about the set of sequences convergent to 1?
8. The set of differentiable functions is a subspace of $\mathscr{C}[0,1]$.
   Is the same true for the set of functions integrable on $[0,1]$?Create your own examples.
9. See the tutorial sheet for many more examples!

Exercise: **(i)** A subspace must contain the 0 vector!
(ii) Show that a subspace of a vector space is a vector space.

What are all the subspaces of $\mathbb{R}^2$?

- $V = \{\begin{pmatrix}0 & 0\end{pmatrix}^T\}$.
- $V = \mathbb{R}^2$.
- What if V is neither of the above?

Example: Suppose V contains a non-zero vector, say $v = \begin{pmatrix}-1 & 1\end{pmatrix}^T$.



V must contain the entire line $L: \ x + y = 0$, i.e., all multiples of v.

Examples: Subspaces of $\mathbb{R}^2$

Let V be a subspace of $\mathbb{R}^2$ containing $v_1 = \begin{pmatrix} -1 & 1 \end{pmatrix}^T$. Then V must contain the entire line $L: \; x + y = 0$.
If $V \neq L$, it contains a vector v_2, which is not a multiple of v_1, say $v_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$.

Observe: $A = \begin{pmatrix} v_1 & v_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$ has two pivots,

$\Leftrightarrow$ A is invertible.
$\Leftrightarrow$ for any v in $\mathbb{R}^2$, $Ax = v$ is solvable,
$\Leftrightarrow$ v is in $C(A)$,
$\Leftrightarrow$ v can be written as a linear combination of v_1 and v_2.
$\Rightarrow$ v is in V, i.e., $V = \mathbb{R}^2$

To summarise: A subspace of $\mathbb{R}^2$, which is non-zero, and not $\mathbb{R}^2$, is a line passing through the origin.

Linear Span: Definition

Given a collection $S = \{v_1, v_2, \dots, v_n\}$ in a vector space V, the *linear span* of S, denoted $\text{Span}(S)$ or $\text{Span}\{v_1, \dots, v_n\}$, is the set of all linear combinations of v_1, $v_2, \dots, v_n$, i.e.,

$$\boxed{\text{Span}(S) = \{v = a_1 v_1 + \cdots + a_n v_n, \text{ for scalars } a_1, \dots, a_n\}.}$$

Let $\{v_1, \dots, v_n\}$ be n vectors in $\mathbb{R}^n$, $A = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$.

Note:

1. If $v_1, \dots, v_n$ are in $\mathbb{R}^m$, $\text{Span}\{v_1, \dots, v_n\} = C(A)$. Thus v is in $\text{Span}\{v_1, \dots, v_n\} \iff Ax = v$ is consistent.
2. $\text{Span}\{v_1, \dots, v_n\} = \mathbb{R}^m \iff Ax = v$ is consistent for all $v \in \mathbb{R}^m$ $\iff A$ has m pivots. This implies, $m \leq n$.
3. Let $m = n$. Then A is invertible $\iff A$ has n pivots $\iff$ $Ax = v$ is consistent for every v in $\mathbb{R}^n \iff \text{Span}\{v_1, \dots, v_n\} =$ $\mathbb{R}^n$.

Example: $\text{Span}\{e_1, \dots, e_n\} = \mathbb{R}^n$.