MA-110 Linear Algebra and Differential Equations

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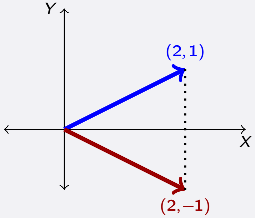
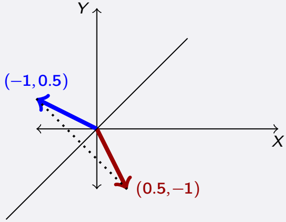
February 6, 2024
Lecture 10 D3

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}$. Let $\mathbf{x} = (2,1)^T$. What is $\mathbf{A}x$? How does A transform x?
A reflects vectors across the X-axis.

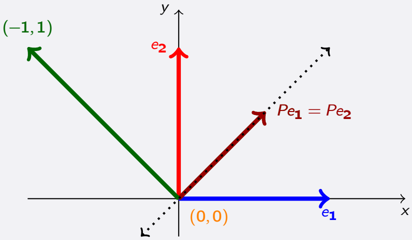


Let $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $B\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$. If $\mathbf{x} = (-1, 0.5)^T$, then $\mathbf{B}x = (0.5, -1)^T$. How does B transform x?
B reflects vectors across the line $x_1 = x_2$.



Q: Do reflections preserve scalar multiples? Sums of vectors?

- $P = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$ transforms $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ to $Px = \begin{pmatrix} \frac{x_1+x_2}{2} \\ \frac{x_1+x_2}{2} \end{pmatrix}$.



$Pe_1 = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = Pe_2$.

P transforms the vector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ to the origin.

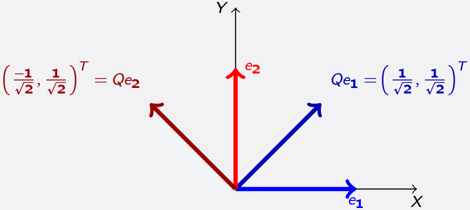
Question: Geometrically, how is P transforming the vectors?
Answer: Projects onto the line $x_1 = x_2$.

Question: What happens to sums of vectors when you project them? What about scalar multiples?

Question: Understand the effect of $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ on e_1 and e_2 and interpret what P represents geometrically!

Let $Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \cos(45^\circ) & -\sin(45^\circ) \\ \sin(45^\circ) & \cos(45^\circ) \end{pmatrix}$.

How does Q transform the standard basis vectors $\mathbf{e_1}$ and $\mathbf{e_2}$?



Q: What does the transformation $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto Qx$ represent geometrically?

Rotations also map sum of vectors to sum of their images and a scalar multiple of a vector to the scalar multiple of its image.

• An $m \times n$ matrix A transforms a vector x in $\mathbb{R}^n$ into the vector Ax in $\mathbb{R}^m$. Thus $T(x) = Ax$ defines a function $T : \mathbb{R}^n \to \mathbb{R}^m$.

• The domain of T is ___. The codomain of T is ___.

• Let $b \in \mathbb{R}^m$. Then b is in $C(A) \Leftrightarrow Ax = b$ is consistent $\Leftrightarrow T(x) = b$, i.e., b is in the image (or range) of T.
Hence, the range of T is ___.

Example: Let $A = \begin{pmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{pmatrix}$. Then $T(x) = Ax$ is a function with domain $\mathbb{R}^4$, codomain $\mathbb{R}^3$, and range equal to
$C(A) = \{(a, b, c)^T \mid 2a - b - c = 0\} \subseteq \mathbb{R}^3$.

Question: How does T transform sums and scalar multiples of vectors?
Ans. Nicely! For scalars a and b, and vectors x and y,
$T(ax + by) = A(ax + by) = aAx + bAy = aT(x) + bT(y)$. Thus

$$\boxed{T \text{ takes linear combinations to linear combinations.}}$$

Defn. Let V and W be vector spaces.

- A *linear transformation* from V to W is a function $T : V \to W$ such that for $x, y \in V$, scalars a and b,

$$\boxed{T(ax + by) = aT(x) + bT(y)}$$

i.e., T takes linear combinations of vectors in V to the linear combinations of their images in W.

- If T is also a bijection, we say T is a *linear isomorphism*.
- The *image* (or *range*) of T is defined to be
$$C(T) = \{y \in W \mid T(x) = y \text{ for some } x \in V\}.$$
- The *kernel* (or *null space*) of T is defined as
$$N(T) = \{x \in V \mid T(x) = 0\}.$$

Main Example: Let A be an $m \times n$ matrix. Define $T(x) = Ax$.

- This defines a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$.
- The image of T is the column space of A, i.e., $C(T) = C(A)$.
- The kernel of T is the null space of A, i.e., $N(T) = N(A)$.

Linear Transformations: Examples

Which of the following functions are linear transformations?

- $g : \mathbb{R}^3 \to \mathbb{R}^3$ defined as $g(x_1, x_2, x_3)^T = (x_1, x_2, 0)^T$

$$ag(x) + bg(y) \; = ag\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + bg\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} ax_1 \\ ax_2 \\ 0 \end{pmatrix} + \begin{pmatrix} by_1 \\ by_2 \\ 0 \end{pmatrix} \; = \begin{pmatrix} ax_1 + by_1 \\ ax_2 + by_2 \\ 0 \end{pmatrix}$$

$= g(ax + by)$ is a linear transformation.

Exercise: Find $N(g)$ and $C(g)$.

- $h : \mathbb{R}^3 \to \mathbb{R}^3$ defined as $h(x_1, x_2, x_3)^T = (x_1, x_2, 5)^T$.

Note: $h(0 + 0) \neq h(0) + h(0)$.

Observe: A linear transformation must map $0 \in V$ to $0 \in W$.

- $f : \mathbb{R}^2 \to \mathbb{R}^4$ defined by $f(x_1, x_2)^T = (x_1, 0, x_2, x_2^2)^T$.

Note: f transforms the Y-axis in $\mathbb{R}^2$ to $\{(0, 0, y, y^2)^T \mid y \in \mathbb{R}\}$.

Observe: A linear transformation must transform a subspace of V into a subspace of W.

- $S : \mathscr{M}_{2\times 2} \to \mathbb{R}^4$ defined by $S\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = (a, b, c, d)^T$is a linear

transformation.

Observe: S is also a bijection, and hence an isomorphism!

S is onto $\Rightarrow C(S) = \mathbb{R}^4$, and $S(A) = S(B) \Rightarrow A = B$,

i.e., S is one-one. In particular, $N(S) = \{0\}$.

Show that the following functions are linear transformations.

$T : \mathbb{R}^\infty \to \mathbb{R}^\infty$ defined by $T(x_1, x_2, \ldots) = (x_1 + x_2, x_2 + x_3, \ldots,)$.
Exercise: What is $N(T)$? What is $C(T)$?