MA-110 Linear Algebra and Differential Equations

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Lecture 21 D3

By induction,

$$w_r := v_r - \text{proj}_{\text{Span}\{w_1,\dots w_{r-1}\}}(v_r) =$$
$$v_r - \text{proj}_{w_1}(v_r) - \text{proj}_{w_2}(v_r) - \cdots - \text{proj}_{w_{r-1}}(v_r)$$
$$= v_r - \frac{w_1^T v_r}{\|w_1\|^2} w_1 - \frac{w_2^T v_r}{\|w_2\|^2} w_2 - \cdots - \frac{w_{r-1}^T v_r}{\|w_{r-1}\|^2} w_{r-1}$$

Now take $q_1 = \frac{w_1}{\|w_1\|}$, $q_2 = \frac{w_2}{\|w_2\|}$, $\dots$, $q_r = \frac{w_r}{\|w_r\|}$. Then $\{q_1, \dots, q_r\}$ is an orthonormal set and
$W = \text{Span}\{v_1, \dots, v_r\} = \text{Span}\{w_1, \dots, w_r\} = \text{Span}\{q_1, \dots, q_r\}$.

In particular, $\{q_1, q_2, \dots, q_r\}$ is an *orthonormal basis* for W.

Exercise: Show that if $\{w_1, \dots, w_r\}$ is an orthogonal set, then

$$\boxed{\text{proj}_{\text{Span}\{w_1,\dots w_{i-1}\}}(v_i) = \text{proj}_{w_1}(v_i) + \text{proj}_{w_2}(v_i) + \cdots + \text{proj}_{w_{i-1}}(v_i).}$$

Gram-Schmidt Method: Example

Q: Let $S = \left\{ v_1 = \begin{pmatrix} 3 \\ 1 \\ -1 \\ 3 \end{pmatrix}, v_2 = \begin{pmatrix} -5 \\ 1 \\ 5 \\ -7 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 1 \\ -2 \\ 8 \end{pmatrix} \right\}$ and $W =$ Span(S). Find an orthonormal basis for W.

Exercise: First verify that $\{v_1, v_2, v_3\}$ are linearly independent. (Check that rank of $\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$ is 3). Hence S is a basis of W.

Use Gram-Schmidt method: $w_1 = v_1,\ w_2 = v_2 - \left(\frac{w_1^T v_2}{\|w_1\|^2} \right) w_1$

$\Rightarrow$

$$w_2 = v_2 - \left(\frac{-15+1-5-21}{9+1+1+9} \right) w_1 = v_2 - \left(\frac{-40}{20} \right) w_1 = v_2 + 2w_1$$
$$= \begin{pmatrix} 1 & 3 & 3 & -1 \end{pmatrix}^T.$$

Observe: $v_1,\ v_2 \in \text{Span}\{w_1, w_2\},\ w_1,\ w_2 \in \text{Span}\{v_1, v_2\} \Rightarrow$ $\text{Span}\{v_1, v_2\} = \text{Span}\{w_1, w_2\}$.

Gram-Schmidt Method: Example (Contd.)

Recall $w_1 = \begin{pmatrix} 3 & 1 & -1 & 3 \end{pmatrix}^T$, $w_2 = \begin{pmatrix} 1 & 3 & 3 & -1 \end{pmatrix}^T$, and $v_3 = \begin{pmatrix} 1 & 1 & -2 & 8 \end{pmatrix}^T$. (Check $w_1^T w_2 = 0$).

Now $w_3 = v_3 - \left(\dfrac{w_1^T v_3}{\|w_1\|^2} \right) w_1 - \left(\dfrac{w_2^T v_3}{\|w_2\|^2} \right) w_2 =$

$v_3 - \left(\dfrac{3+1+2+24}{20} \right) w_1 - \left(\dfrac{1+3-6-8}{20} \right) w_2$

$$\Rightarrow \quad w_3 = \begin{pmatrix} 1 \\ 1 \\ -2 \\ 8 \end{pmatrix} - \frac{3}{2} \begin{pmatrix} 3 \\ 1 \\ -1 \\ 3 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ 3 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 1 \\ 3 \end{pmatrix}.$$

Check $w_1^T w_3 = 0 = w_2^T w_3$ and $\text{Span}\{v_1, v_2, v_3\} = \text{Span}\{w_1, w_2, w_3\}$. Hence $\{w_1, w_2, w_3\}$ is an orthogonal basis of W. An orthonormal basis for W is $\left\{ \dfrac{1}{\sqrt{20}} w_1, \dfrac{1}{\sqrt{20}} w_2, \dfrac{1}{\sqrt{20}} w_3 \right\}$.

Diagonalizing Symmetric Matrices: Example

Example: Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. Then

$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{bmatrix}$ and

$\det(A - \lambda I) =$

$$(1-\lambda)[(1-\lambda)^2 - 1] - 1[1-\lambda-1] + 1[1-(1-\lambda)]$$

$= (3-\lambda)\lambda^2$ Eigenvalues: $\lambda_1 = 3, \lambda_2 = 0, \lambda_3 = 0$.

To find $N(A-3I)$, solve $A - 3I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$N(A)$ is the plane $x + y + z = 0$. Hence, the associated eigenvectors are $v_1 = (1,1,1)^T$, $v_2 = (-1,0,1)^T$ and $v_3 = (0,-1,1)^T$.

Example: $A = Q\Lambda Q^T$

A has eigenvalues $\lambda_1 = 3, \lambda_2 = 0, \lambda_3 = 0$ with associated eigenvectors $v_1 = (1,1,1)^T$, $v_2 = (-1,0,1)^T$ and $v_3 = (0,-1,1)^T$. Note that v_2 and v_3 are linearly independent in $N(A)$. Observe $v_1^T v_2 = 0 = v_1^T v_3$.

How do we get an orthogonal Q such that $A = Q\Lambda Q^T$, where Λ is diagonal with entries 3, 0, 0 on the diagonal?

Steps: 1. Let $u_1 = v_1/\|v_1\|$.

2. Start with the basis $\{v_2, v_3\}$ of $N(A)$, and apply the Gram-Schimdt process to get an orthonormal basis $\{u_2, u_3\}$ for $N(A)$. Note that u_2 and u_3 are eigenvectors of A associated to $\lambda = 0$, and are linearly independent since they are non-zero orthogonal vectors.

3. Then $Q = [u_1 \quad u_2 \quad u_3]$ is orthogonal, and $Q^{-1}AQ = \Lambda$.

4. Since $Q^{-1} = Q^T$, $A = Q\Lambda Q^T$.

Diagonalizing Symmetric Matrices

Let A be a symmetric matrix, which is diagonalizable. Then there is an orthogonal matrix Q, and a diagonal matrix Λ such that $A = Q\Lambda Q^T$.

Observe: Eignevectors corresponding to distinct eigenvalues are orthogonal.

Proof. Let λ and μ be distinct eigenvalues of A with associated eigenvectors v and w repectively. Now,
$\lambda(v^T w) = (\lambda v)^T w = (Av)^T w = v^T(A^T w) = v^T(Aw) = \mu(v^T w)$.
Since $\lambda \neq \mu$, this imples $v^T w = 0$, proving the result.

$\boxed{\text{Step 1:}}$ Find the eigenvalues and the respective eigenvectors.

$\boxed{\text{Step 2:}}$ Use Gram-Schmidt process to get an orthogonal basis for each eignespace.

Theorem: (Real Spectral Theorem)
Every symmetric matrix (with real entries) is diagonalizable, and hence decomposes as above.

QR Factorization

Let $A = \begin{pmatrix} v_1 & \cdots & v_r \end{pmatrix}$ be an $n \times r$ matrix of rank r. Then $v_1, \ldots, v_r$ are linearly independent vectors in $\mathbb{R}^n$. By the Gram-Schmidt method, we get an orthonormal basis $\{q_1, \ldots, q_r\}$ of $C(A)$, where $q_i = \dfrac{w_i}{\|w_i\|}$ and $w_1 = v_1$, and for $k > 1$,

$$w_k = v_k - \left(\frac{w_1^T v_k}{\|w_1\|^2}\right) w_1 - \cdots - \left(\frac{w_{k-1}^T v_k}{\|w_{k-1}\|^2}\right) w_{k-1}.$$

Let $Q = \begin{pmatrix} q_1 & \ldots & q_r \end{pmatrix}$. $\boxed{\text{How are } A \text{ and } Q \text{ related?}}$

Note that $\text{Span}\{v_1, \ldots, v_k\} = \text{Span}\{w_1, \ldots, w_k\} = \text{Span}\{q_1, \ldots, q_k\}$ for all k. If $v_k = c_1 q_1 + \ldots + c_k q_k$, then $c_1 = q_1^T v_k, \quad c_2 = q_2^T v_k, \quad \ldots, \quad c_k = q_k^T v_k$. Thus

Hence $v_k = (q_1^T v_k) q_1 + \ldots + (q_k^T v_k) q_k$.

QR factorization (Contd.)

$$v_k = (q_1^T v_k)q_1 + \ldots + (q_k^T v_k)q_k \qquad \text{for each } k.$$

Therefore

$$\begin{pmatrix} v_1 & v_2 & \ldots & v_r \end{pmatrix} = \begin{pmatrix} q_1 & q_2 & \ldots & q_r \end{pmatrix} \begin{pmatrix} q_1^T v_1 & q_1^T v_2 & & q_1^T v_r \\ 0 & q_2^T v_2 & & q_2^T v_r \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & q_r^T v_r \end{pmatrix}$$

i.e. $\boxed{A = QR}$, where the columns of Q form an orthonormal set and R is an invertible $r \times r$ matrix. **Q:** Why is R invertible? This is called QR-factorization of A.

- If A is invertible $n \times n$, then $\boxed{A = QR}$, where Q is an orthogonal matrix and R is an invertible upper triangular matrix, both are $n \times n$ matrices.