MA110 - Linear Algebra & Differential Equations: Midsem

Code $\boxed{\text{A}}$: Solutions

Name: **Roll No.** $\boxed{}$

Tutorial Batch: D$\boxed{}$ **T**$\boxed{}$ **Max. marks: 60 (Weightage 40%)**

February 24th, 2024 **8:30 - 10:30 a.m.**

(1) The LU decomposition of $A = \begin{pmatrix} 2 & 1 & -1 \\ 4 & 0 & 1 \\ -4 & 2 & -3 \end{pmatrix}$ is given by [3]

$L = \begin{pmatrix} 1 & 0 & 0 \\ \boxed{2} & 1 & 0 \\ \boxed{-2} & \boxed{-2} & 1 \end{pmatrix}$ and $U = \begin{pmatrix} \boxed{2} & \boxed{1} & \boxed{-1} \\ 0 & \boxed{-2} & \boxed{3} \\ 0 & 0 & \boxed{1} \end{pmatrix}$.

(2) Let $W = \{x = (x_1, x_2, x_3, \dots) \in \mathbb{R}^\infty \mid x_n + 4\, x_{n-1} = 0 \text{ for all } n \geq 2\}$. The dimension of W is $\boxed{1}$ and a basis of W is given by $\boxed{(1, -4, 16, -64, \dots, (-1)^{n-1}4^{n-1}, \dots)}$. [2]

(3) A linear transformation $T : \mathcal{M}_{2\times 2} \to \mathcal{P}_3$, satisfying $T(e_{11}) = x + x^3$, $T(e_{12}) = 0$, $T(e_{21}) = 1 - x^2$ and $T(e_{22}) = 2x^2 + 3x$ is $T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \boxed{c} + \boxed{a+3d}\,x + \boxed{-c+2d}\,x^2 + \boxed{a}\,x^3$. [2]

(4) Let $v = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$, $w = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}^T$, and $W = \text{Span}\{v, w\}$. [4]

The length of v is $\boxed{\sqrt{3} \text{ units}}$ and $\text{proj}_v(w) = \boxed{\frac{2}{3}(1,1,1)^T}$.

An orthogonal basis of W constructed using Gram Schmidt method applied to $\{v, w\}$ is given by

$\boxed{\left\{\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T, \frac{1}{3}\begin{pmatrix} 1 & -2 & 1 \end{pmatrix}^T\right\}}$.

(5) Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}^T$ and $b = \begin{pmatrix} 1 & -1 & -1 \end{pmatrix}^T$. Then $A^T b = \boxed{\begin{pmatrix} -1 & -2 \end{pmatrix}^T}$. [3]

The least squares solution to the system $Ax = b$ is $\widehat{x} = \boxed{\begin{pmatrix} 1 & -1 \end{pmatrix}^T}$.

(6) Let $A = \begin{pmatrix} 3 & 1 & -1 & 2 \\ 3 & 0 & 1 & 3 \\ 0 & 1 & -2 & -1 \end{pmatrix}$. The dimension of $N(A)$ is $\boxed{2}$, [6]

a basis for $C(A^T)$ is $\boxed{\left\{\begin{pmatrix} 3 & 1 & -1 & 2 \end{pmatrix}^T, \begin{pmatrix} 3 & 0 & 1 & 3 \end{pmatrix}^T\right\}}$, a basis of $C(A)$ is $\boxed{\left\{\begin{pmatrix} 3 & 3 & 0 \end{pmatrix}^T, \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}^T\right\}}$,

and an x such that $Ax = \begin{pmatrix} 1 & 6 & -5 \end{pmatrix}^T$ is $\boxed{\begin{pmatrix} 2 & -5 & 0 & 0 \end{pmatrix}^T}$.

(7) Let $V = \mathcal{M}_{2\times 2}$, $T : V \to V$ be the linear transformation defined by $T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a+c & b+a \\ 0 & d+a \end{pmatrix}$,

and $\mathcal{B} = \left\{\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}$ be a basis of V. [3]

If $v = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $[v]_{\mathcal{B}} = \begin{pmatrix} \boxed{a} \\ \boxed{c-d} \\ \boxed{d} \\ \boxed{b-a} \end{pmatrix}$, and $[T]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} \boxed{1} & \boxed{1} & \boxed{1} & \boxed{0} \\ \boxed{-1} & \boxed{0} & \boxed{-1} & \boxed{0} \\ \boxed{1} & \boxed{0} & \boxed{1} & \boxed{0} \\ \boxed{1} & \boxed{-1} & \boxed{-1} & \boxed{1} \end{pmatrix}$.

(8) Let $v_1 = (1, 1, -1, -1)^T$, $v_2 = (-1, 1, -1, 1)^T$, $w_1 = (-1, 1, 1, -1)^T$, $w_2 = (1, 1, 1, 1)^T$, A be a 4×4 matrix. Suppose the special solutions of $(A - I)x = 0$ are v_1 and v_2, and that of $(A + I)x = 0$ are w_1 and w_2. Then $A = P\Lambda P^{-1}$, where

$$P = \begin{pmatrix} \boxed{1} & \boxed{-1} & \boxed{-1} & \boxed{1} \\ \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} \\ \boxed{-1} & \boxed{-1} & \boxed{1} & \boxed{1} \\ \boxed{-1} & \boxed{1} & \boxed{-1} & \boxed{1} \end{pmatrix}, \text{ and } \Lambda = \begin{pmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{-1} & 0 \\ 0 & 0 & 0 & \boxed{-1} \end{pmatrix}.$$ [2]

Furthermore, [4]

$$P^{-1} = \frac{1}{4} \begin{pmatrix} \boxed{1} & \boxed{1} & \boxed{-1} & \boxed{-1} \\ \boxed{-1} & \boxed{1} & \boxed{-1} & \boxed{1} \\ \boxed{-1} & \boxed{1} & \boxed{1} & \boxed{-1} \\ \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} \end{pmatrix}, \text{ and } A = \begin{pmatrix} \boxed{0} & \boxed{0} & \boxed{0} & \boxed{-1} \\ \boxed{0} & \boxed{0} & \boxed{-1} & \boxed{0} \\ \boxed{0} & \boxed{-1} & \boxed{0} & \boxed{0} \\ \boxed{-1} & \boxed{0} & \boxed{0} & \boxed{0} \end{pmatrix}$$

(9) Match each option (i) - (iv) on the left with the correct option (a) - (e) on the right: [4]

(i) $\left\{\frac{1}{\sqrt{2}}(1, 0, 1)^T, \frac{1}{\sqrt{2}}(1, 0, -1)^T, \frac{1}{\sqrt{2}}(1, -1, 0)^T\right\}$ — (a) is an orthonormal basis of $\mathbb{R}^3$

(ii) $\left\{\frac{1}{\sqrt{2}}(1, 0, 1)^T, \frac{1}{\sqrt{2}}(1, 0, -1)^T, \frac{1}{\sqrt{2}}(0, -1, 0)^T\right\}$ — (b) is a linearly dependent subset of $\mathbb{R}^3$

(iii) $\left\{\frac{1}{\sqrt{2}}(1, 0, 1)^T, (0, -1, 0)^T, \frac{1}{\sqrt{2}}(1, 0, -1)^T\right\}$ — (c) is a basis of $\mathbb{R}^3$ which is not orthogonal.

(iv) $\left\{(0, 0, 0)^T, (0, 1, 0)^T, \frac{1}{\sqrt{2}}(1, 0, 1)^T\right\}$ — (d) is an orthogonal set which is not a basis of $\mathbb{R}^3$

(e) is an orthogonal basis of $\mathbb{R}^3$ which is not orthonormal.

(i) - $\boxed{\text{(c)}}$ (ii) - $\boxed{\text{(e)}}$ (iii) - $\boxed{\text{(a)}}$ (iv) - $\boxed{\text{(b}}$

(10) Let V be a vector space with Span $\{v_1, \ldots, v_5\} = V$, where v_1, v_2, v_3 are linearly independent. Let $v_6 \in V$. Which of the following must be true? $\boxed{\text{(b),(e)}}$ [2]

(a) Span $\{v_1, v_2, v_3, v_4\} \neq V$
(b) v_1, v_3 are linearly independent.
(c) v_1, v_2, v_4 are linearly independent.
(d) v_1, v_2, v_3, v_4 are linearly dependent.
(e) $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ is a spanning set for V.

(11) Let A be an $m \times n$ matrix, with $m \neq n$, and $S : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation defined by $S(u) = Au$. Consider the following statements:

(a) The rows of A are linearly dependent.
(b) $N(A^T) = 0$.
(c) The columns of A span $\mathbb{R}^m$.
(d) $N(A) = 0$.
(e) rank$(A) = n$.
(f) rank$(A) = m$.

(i) The statements from (a) - (f) which imply that S is one-one are: $\boxed{\text{(d), (e)}}$ [2]

(ii) The statements from (a) - (f) which imply that S is onto are: $\boxed{\text{(b), (c), (f)}}$ [2]

(12) The statements (a)-(g) are needed to prove the statement: [3]
"Given $T : V \to W$ linear, $N(T)$ is a subspace of V."
The correct order in which these statements are to be written to give a proof is
$\boxed{\text{(d)}}$ $\boxed{\text{(f)}}$ $\boxed{\text{(b)}}$ $\boxed{\text{(a)}}$ $\boxed{\text{(g)}}$ $\boxed{\text{(e)}}$ $\boxed{\text{(c)}}$

(a) $\Rightarrow T(v_1) = 0$, and $T(v_2) = 0$.
(b) Let $v_1, v_2 \in N(T)$.
(c) $\Rightarrow a_1v_1 + a_2v_2 \in N(T)$, i.e., $N(T)$ is closed under linear combinations.
(d) $T(0) = 0$
(e) $\Rightarrow T(a_1v_1 + a_2v_2) = 0$, since T is linear.
(f) $\Rightarrow 0 \in N(T)$.
(g) $\Rightarrow a_1T(v_1) + a_2T(v_2) = 0$ for every $a_1, a_2 \in \mathbb{R}$.

(13) Consider the statement: [2]
"Let V be a vector space, W_1, W_2 be subspaces, and $W = W_1 \cup W_2$. Then W is a subspace of V".
Give a counter-example (with justification) to show that the statement is false.

Let $W_1 = \{(x, 0) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$ and $W_2 = \{(0, y) \in \mathbb{R}^2 \mid y \in \mathbb{R}\}$.
These are lines through the origin in $\mathbb{R}^2$ and hence subspaces of $\mathbb{R}^2$.
Consider $(1, 0) \in W_1 \subseteq W_1 \cup W_2$ and $(0, 1) \in W_2 \subseteq W_1 \cup W_2$. Then $(1, 0) + (0, 1) = (1, 1)$.
Note $(1, 1) \notin W_1$ since its second coordinate is non-zero and $(1, 1) \notin W_2$ since the first is non-zero.
Therefore, $(1, 1) \notin W_1 \cup W_2$.
This implies $W_1 \cup W_2$ is not closed under vector addition and is not a subspace of $\mathbb{R}^2$.

(14) Let A be a $m \times n$ matrix and B be a $n \times p$ matrix. Show that $\text{rank}(AB) \leq \text{rank}(A)$. [2]

First note that any column in AB is the product of A with a column of B, that is $(AB)_{*j} = AB_{*j}$.
Now $AB_{*j} = b_{1j}A_{*1} + \ldots + b_{nj}A_{*n}$ and therefore, each column of AB is a linear combination of columns of A.
Thus $(AB)_{*j} \in C(A)$. Since, $C(A)$ is closed under linear combinations, $C(AB) \subseteq C(A)$.
Thus, $\text{rank}(AB) = \dim(C(AB)) \leq \dim(C(A)) = \text{rank}(A)$.

(15) Let A be an 5×3 matrix.
(i) Prove that AA^T is not invertible. [2]

Since A is a 5×3 matrix, $\text{rank}(A) \leq 3$.
By Q14, we see that $\text{rank}(AA^T) \leq \text{rank}(A)$, i.e., $\text{rank}(AA^T) \leq 3$.
In particular, AA^T is a 5×5 matrix such that $\text{rank}(AA^T) \neq 5$, and hence AA^T is not invertible

(ii) Prove that $N(A) = N(A^TA)$. [2]

If $x \in N(A)$, then $Ax = 0$, and hence $(A^TA)(x) = A^T(Ax) = 0$. This proves $N(A) \subseteq N(A^TA)$.
For the other inclusion, if $x \in N(A^TA)$, then $(A^TA)(x) = 0$.
In particular, $x^T(A^TA)x = 0$, i.e., $(x^TA^T)(Ax) = (Ax)^TAx = 0$.
Thus $||Ax|| = 0$, which implies $Ax = 0$, i.e., $x \in N(A)$, proving $N(A^TA) \subseteq N(A)$.

(iii) Show that $\text{rank}(A^TA) = \text{rank}(A)$. [1]

By (ii), we see that $N(A) = N(A^TA)$, and in particular, $\dim(N(A)) = \dim(N(A^TA))$.
Thus, since both A^TA and A have 3 columns each, by the rank-nullity theorem, we get
$\text{rank}(A) = 3 - \dim(N(A) = 3 - \dim(N(A^TA) = \text{rank}(A^TA)$.

(iv) If the columns of A are linearly independent, then show that A^TA is invertible. [1]

Since the columns of A are linearly independent, we get $\text{rank}(A^TA) = \text{rank}(A) = 3$.
This forces A^TA to be invertible, since A^TA is a 3×3 matrix.

(16) Let $f_1 = 1$, $f_2 = x + x^2$, $f_3 = 1 + x + x^2 + x^3$, $f_4 = 2x + 2x^2 + x^3$ be polynomials in $\mathcal{P}_3$.
(i) Show that f_1, f_2, f_3, f_4 are linearly dependent. [2]

Note that $f_3 = f_4 - f_2 + f_1$, i.e., $f_1 - f_2 - f_3 + f_4 = 0$.
This defines a non-trivial linear relation between f_1, f_2, f_3 and f_4 making them linearly dependent.

(ii) Find $f \in \mathcal{P}_3$ which is not in $\text{Span}\{f_1, f_2, f_3, f_4\}$. Justify your answer. [2]

Consider any element $f \in \text{Span}\{f_1, f_2, f_3, f_4\}$. Then $f = af_1 + bf_2 + cf_3 + df_4$.
This implies $f = a + bx + bx^2 + c + cx + cx^2 + cx^3 + 2dx + 2dx^2 + dx^3$,
i.e., $f = (a + c) + (b + c + 2d)x + (b + c + 2d)x^2 + (c + d)x^3$.
This implies for any $f \in \text{Span}\{f_1, f_2, f_3, f_4\}$, the coefficient of x and x^2 are the same.
In particular, $x \in \mathcal{P}_3$ is not in the $\text{Span}\{f_1, f_2, f_3, f_4\}$.

(17) Show that if T is a bijective linear transformation, then T^{-1} is a linear transformation. [4]

Let the domain and codomain of T be V and W respectively. Since T is a bijection, its inverse T^{-1} exists. Moreover, for $v \in V$, $w \in W$, we have $T^{-1}(w) = v$ if and only if $T(v) = w$.
For any scalars a, b, and $w_1, w_2 \in W$, we need to show that $T^{-1}(aw_1+bw_2) = aT^{-1}(w_1)+bT^{-1}(w_2)$.
Since T is a bijection, we can find $v_1, v_2 \in V$, such that $T(v_i) = w_i$ and hence $T^{-1}(w_i) = v_i$ for $i = 1, 2$.
Since T is linear, we have $T(av_1+bv_2) = aT(v_1)+bT(v_2)$, and hence $av_1+bv_2 = T^{-1}(aT(v_1)+bT(v_2))$.
Since $v_i = T^{-1}(w_i)$, and $w_i = T(v_i)$, for $i = 1, 2$, we get $aT^{-1}(w_1) + bT^{-1}(w_2) = T^{-1}(aw_1 + bw_2)$, proving T^{-1} is a linear transformation.

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Code $\boxed{B}$: Solutions

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Tutorial Batch: D$\boxed{}$ **T**$\boxed{}$ **Max. marks: 60 (Weightage 40%)**

February 24th, 2024 **8:30 - 10:30 a.m.**

(1) The LU decomposition of $B = \begin{pmatrix} 2 & 1 & -1 \\ -4 & 2 & 0 \\ 4 & -2 & 1 \end{pmatrix}$ is given by [3]

$L = \begin{pmatrix} 1 & 0 & 0 \\ \boxed{-2} & 1 & 0 \\ \boxed{2} & \boxed{-1} & 1 \end{pmatrix}$ and $U = \begin{pmatrix} \boxed{2} & \boxed{1} & \boxed{-1} \\ 0 & \boxed{4} & \boxed{-2} \\ 0 & 0 & \boxed{1} \end{pmatrix}$.

(2) Let $W = \{x = (x_1, x_2, x_3, \ldots) \in \mathbb{R}^\infty \mid x_n + 5\, x_{n-1} = 0 \text{ for all } n \geq 2\}$. The dimension of W is $\boxed{1}$ and a basis of W is given by $\boxed{\{1, -5, 25, -125, \ldots, (-1)^{n-1}5^{n-1}, \ldots)\}}$. [2]

(3) A linear transformation $T : \mathcal{M}_{2\times 2} \to \mathcal{P}_3$, satisfying $T(e_{11}) = 2x^2 + 3x$, $T(e_{12}) = x + x^3$, $T(e_{21}) = 0$ and $T(e_{22}) = 1 - x^2$ is $T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \boxed{\text{d}} + \boxed{\text{3a+b}}\, x + \boxed{\text{2a - d}}\, x^2 + \boxed{\text{b}}\, x^3$. [2]

(4) Let $v = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$, $w = \begin{pmatrix} -1 & 0 & -1 \end{pmatrix}^T$, and $W = \text{Span}\{v, w\}$. [4]
The length of v is $\boxed{\sqrt{3} \text{ units}}$ and $\text{proj}_v(w) = \boxed{-\frac{2}{3}(1,1,1)^T}$.
An orthogonal basis of W constructed using Gram Schmidt method applied to $\{v, w\}$ is given by
$\boxed{\left\{\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T, \frac{1}{3}\begin{pmatrix} -1 & 2 & -1 \end{pmatrix}^T\right\}}$.

(5) Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}^T$ and $b = \begin{pmatrix} 1 & -1 & -1 \end{pmatrix}^T$. Then $A^T b = \boxed{\begin{pmatrix} -2 & -1 \end{pmatrix}^T}$. [3]
The least squares solution to the system $Ax = b$ is $\widehat{x} = \boxed{\begin{pmatrix} -1 & 1 \end{pmatrix}^T}$.

(6) Let $B = \begin{pmatrix} 3 & 1 & -1 & 2 \\ 0 & 1 & -2 & -1 \\ 3 & 0 & 1 & 3 \end{pmatrix}$. The dimension of $N(B)$ is $\boxed{2}$, [6]
a basis for $C(B^T)$ is $\boxed{\left\{\begin{pmatrix} 3 & 1 & -1 & 2 \end{pmatrix}^T, \begin{pmatrix} 0 & 1 & -2 & -1 \end{pmatrix}^T\right\}}$, a basis of $C(B)$ is $\boxed{\left\{\begin{pmatrix} 3 & 0 & 3 \end{pmatrix}^T, \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}^T\right\}}$,
and an x such that $Bx = \begin{pmatrix} 1 & -5 & 6 \end{pmatrix}^T$ is $\boxed{\begin{pmatrix} 2 & -5 & 0 & 0 \end{pmatrix}^T}$.

(7) Let $V = \mathcal{M}_{2\times 2}$, $T : V \to V$ be the linear transformation defined by $T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a + c & b + a \\ 0 & d + a \end{pmatrix}$,
and $\mathcal{B} = \left\{\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}$ be a basis of V. [3]

If $v = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $[v]_\mathcal{B} = \begin{pmatrix} \boxed{\text{a}} \\ \boxed{\text{c-d}} \\ \boxed{\text{d}} \\ \boxed{\text{b-a}} \end{pmatrix}$, and $[T]_\mathcal{B}^\mathcal{B} = \begin{pmatrix} \boxed{1} & \boxed{1} & \boxed{1} & \boxed{0} \\ \boxed{-1} & \boxed{0} & \boxed{-1} & \boxed{0} \\ \boxed{1} & \boxed{0} & \boxed{1} & \boxed{0} \\ \boxed{1} & \boxed{-1} & \boxed{-1} & \boxed{1} \end{pmatrix}$.

(8) Let $v_1 = (-1, 1, 1, -1)^T$, $v_2 = (1, 1, 1, 1)^T$, $w_1 = (1, 1, -1, -1)^T$, $w_2 = (-1, 1, -1, 1)^T$, B be a 4×4 matrix. Suppose the special solutions of $(B + I)x = 0$ are v_1 and v_2, and that of $(B - I)x = 0$ are

w_1 and w_2. Then $B = P\Lambda P^{-1}$, where

$$P = \begin{pmatrix} \boxed{-1} & \boxed{1} & \boxed{1} & \boxed{-1} \\ \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} \\ \boxed{1} & \boxed{1} & \boxed{-1} & \boxed{-1} \\ \boxed{-1} & \boxed{1} & \boxed{-1} & \boxed{1} \end{pmatrix}, \text{ and } \Lambda = \begin{pmatrix} \boxed{-1} & 0 & 0 & 0 \\ 0 & \boxed{-1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{pmatrix}.$$ [2]

Furthermore, [4]

$$P^{-1} = \frac{1}{4}\begin{pmatrix} \boxed{-1} & \boxed{1} & \boxed{1} & \boxed{-1} \\ \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} \\ \boxed{1} & \boxed{1} & \boxed{-1} & \boxed{-1} \\ \boxed{-1} & \boxed{1} & \boxed{-1} & \boxed{1} \end{pmatrix}, \text{ and } B = \begin{pmatrix} \boxed{0} & \boxed{0} & \boxed{0} & \boxed{-1} \\ \boxed{0} & \boxed{0} & \boxed{-1} & \boxed{0} \\ \boxed{0} & \boxed{-1} & \boxed{0} & \boxed{0} \\ \boxed{-1} & \boxed{0} & \boxed{0} & \boxed{0} \end{pmatrix}$$

(9) Match each option (i) - (iv) on the left with the correct option (a) - (e) on the right: [4]

(i) $\left\{\frac{1}{\sqrt{2}}(1,0,1)^T, \frac{1}{\sqrt{2}}(1,0,-1)^T, \frac{1}{\sqrt{2}}(1,-1,0)^T\right\}$

(ii) $\left\{\frac{1}{\sqrt{2}}(1,0,1)^T, \frac{1}{\sqrt{2}}(1,0,-1)^T, \frac{1}{\sqrt{2}}(0,-1,0)^T\right\}$

(iii) $\left\{\frac{1}{\sqrt{2}}(1,0,1)^T, (0,-1,0)^T, \frac{1}{\sqrt{2}}(1,0,-1)^T\right\}$

(iv) $\left\{(0,0,0)^T, (0,1,0)^T, \frac{1}{\sqrt{2}}(1,0,1)^T\right\}$

(a) is an orthonormal basis of $\mathbb{R}^3$

(b) is a linearly dependent subset of $\mathbb{R}^3$

(c) is a basis of $\mathbb{R}^3$ which is not orthogonal.

(d) is an orthogonal set which is not a basis of $\mathbb{R}^3$

(e) is an orthogonal basis of $\mathbb{R}^3$ which is not orthonormal.

(i) - $\boxed{\text{(c)}}$ (ii) - $\boxed{\text{(e)}}$ (iii) - $\boxed{\text{(a)}}$ (iv) - $\boxed{\text{(b)}}$

(10) Let V be a vector space with Span $\{v_1, \ldots, v_5\} = V$, where v_1, v_2, v_3 are linearly independent. Let $v_6 \in V$. Which of the following must be true? $\boxed{\text{(a), (c)}}$ [2]

(a) $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ is a spanning set for V.

(b) Span $\{v_1, v_2, v_3, v_4\} \neq V$

(c) v_1, v_3 are linearly independent.

(d) v_1, v_2, v_4 are linearly independent.

(e) v_1, v_2, v_3, v_4 are linearly dependent.

(11) Let A be an $m \times n$ matrix, with $m \neq n$, and $S : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation defined by $S(u) = Au$. Consider the following statements:

(a) The rows of A are linearly dependent.

(b) $N(A^T) = 0$.

(c) The columns of A span $\mathbb{R}^m$.

(d) $N(A) = 0$.

(e) $\text{rank}(A) = n$.

(f) $\text{rank}(A) = m$.

(i) The statements from (a) - (f) which imply that S is onto are: $\boxed{\text{(b), (c), (f)}}$ [2]

(ii) The statements from (a) - (f) which imply that S is one-one are: $\boxed{\text{(d), (e)}}$ [2]

(12) The statements (a)-(g) are needed to prove the statement: [3]
"Given $T : V \to W$ linear, $N(T)$ is a subspace of V."
The correct order in which these statements are to be written to give a proof is
$\boxed{\text{(c)}}$ $\boxed{\text{(f)}}$ $\boxed{\text{(b)}}$ $\boxed{\text{(g)}}$ $\boxed{\text{(d)}}$ $\boxed{\text{(e)}}$ $\boxed{\text{(a)}}$

(a) $\Rightarrow a_1v_1 + a_2v_2 \in N(T)$, i.e., $N(T)$ is closed under linear combinations.

(b) Let $v_1, v_2 \in N(T)$.

(c) $T(0) = 0$

(d) $\Rightarrow a_1T(v_1) + a_2T(v_2) = 0$ for every $a_1, a_2 \in \mathbb{R}$.

(e) $\Rightarrow T(a_1v_1 + a_2v_2) = 0$, since T is linear.

(f) $\Rightarrow 0 \in N(T)$.

(g) $\Rightarrow T(v_1) = 0$, and $T(v_2) = 0$.

(13) Consider the statement: [2]

"Let V be a vector space, W_1, W_2 be subspaces, and $W = W_1 \cup W_2$. Then W is a subspace of V".

Give a counter-example (with justification) to show that the statement is false.

Let $W_1 = \{(x, 0) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$ and $W_2 = \{(0, y) \in \mathbb{R}^2 \mid y \in \mathbb{R}\}$.
These are lines through the origin in $\mathbb{R}^2$ and hence subspaces of $\mathbb{R}^2$.
Consider $(1, 0) \in W_1 \subseteq W_1 \cup W_2$ and $(0, 1) \in W_2 \subseteq W_1 \cup W_2$. Then $(1, 0) + (0, 1) = (1, 1)$.
Note $(1, 1) \notin W_1$ since its second coordinate is non-zero and $(1, 1) \notin W_2$ since the first is non-zero.
Therefore, $(1, 1) \notin W_1 \cup W_2$.
This implies $W_1 \cup W_2$ is not closed under vector addition and is not a subspace of $\mathbb{R}^2$.

(14) Let A be a $m \times n$ matrix and B be a $n \times p$ matrix. Show that $\operatorname{rank}(AB) \leq \operatorname{rank}(A)$. [2]

First note that any column in AB is the product of A with a column of B, that is $(AB)_{*j} = AB_{*j}$.
Now $AB_{*j} = b_{1j}A_{*1} + \ldots + b_{nj}A_{*n}$ and therefore, each column of AB is a linear combination of columns of A.
Thus $(AB)_{*j} \in C(A)$. Since, $C(A)$ is closed under linear combinations, $C(AB) \subseteq C(A)$.
Thus, $\operatorname{rank}(AB) = \dim(C(AB)) \leq \dim(C(A)) = \operatorname{rank}(A)$.

(15) Let A be an 5×3 matrix.

(i) Prove that AA^T is not invertible. [2]

Since A is a 5×3 matrix, $\operatorname{rank}(A) \leq 3$.
By Q14, we see that $\operatorname{rank}(AA^T) \leq \operatorname{rank}(A)$, i.e., $\operatorname{rank}(AA^T) \leq 3$.
In particular, AA^T is a 5×5 matrix such that $\operatorname{rank}(AA^T) \neq 5$, and hence AA^T is not invertible

(ii) Prove that $N(A) = N(A^T A)$. [2]

If $x \in N(A)$, then $Ax = 0$, and hence $(A^T A)(x) = A^T(Ax) = 0$. This proves $N(A) \subseteq N(A^T A)$.
For the other inclusion, if $x \in N(A^T A)$, then $(A^T A)(x) = 0$.
In particular, $x^T(A^T A)x = 0$, i.e., $(x^T A^T)(Ax) = (Ax)^T Ax = 0$.
Thus $||Ax|| = 0$, which implies $Ax = 0$, i.e., $x \in N(A)$, proving $N(A^T A) \subseteq N(A)$.

(iii) Show that $\operatorname{rank}(A^T A) = \operatorname{rank}(A)$. [1]

By (ii), we see that $N(A) = N(A^T A)$, and in particular, $\dim(N(A)) = \dim(N(A^T A))$.
Thus, since both $A^T A$ and A have 3 columns each, by the rank-nullity theorem, we get
$\operatorname{rank}(A) = 3 - \dim(N(A) = 3 - \dim(N(A^T A) = \operatorname{rank}(A^T A)$.

(iv) If the columns of A are linearly independent, then show that $A^T A$ is invertible. [1]

Since the columns of A are linearly independent, we get $\operatorname{rank}(A^T A) = \operatorname{rank}(A) = 3$.
This forces $A^T A$ to be invertible, since $A^T A$ is a 3×3 matrix.

(16) Let $f_1 = 1$, $f_2 = x + x^2$, $f_3 = 1 + x + x^2 + x^3$, $f_4 = 2x + 2x^2 + x^3$ be polynomials in $\mathcal{P}_3$.

(i) Show that f_1, f_2, f_3, f_4 are linearly dependent. [2]

Note that $f_3 = f_4 - f_2 + f_1$, i.e., $f_1 - f_2 - f_3 + f_4 = 0$.
This defines a non-trivial linear relation between f_1, f_2, f_3 and f_4 making them linearly dependent.

(ii) Find $f \in \mathcal{P}_3$ which is not in $\operatorname{Span}\{f_1, f_2, f_3, f_4\}$. Justify your answer. [2]

Consider any element $f \in \operatorname{Span}\{f_1, f_2, f_3, f_4\}$. Then $f = af_1 + bf_2 + cf_3 + df_4$.
This implies $f = a + bx + bx^2 + c + cx + cx^2 + cx^3 + 2dx + 2dx^2 + dx^3$,
i.e., $f = (a + c) + (b + c + 2d)x + (b + c + 2d)x^2 + (c + d)x^3$.
This implies for any $f \in \operatorname{Span}\{f_1, f_2, f_3, f_4\}$, the coefficient of x and x^2 are the same.
In particular, $x \in \mathcal{P}_3$ is not in the $\operatorname{Span}\{f_1, f_2, f_3, f_4\}$.

(17) Show that if T is a bijective linear transformation, then T^{-1} is a linear transformation. [4]

Let the domain and codomain of T be V and W respectively. Since T is a bijection, its inverse T^{-1} exists. Moreover, for $v \in V$, $w \in W$, we have $T^{-1}(w) = v$ if and only if $T(v) = w$.

For any scalars a, b, and $w_1, w_2 \in W$, we need to show that $T^{-1}(aw_1+bw_2) = aT^{-1}(w_1)+bT^{-1}(w_2)$. Since T is a bijection, we can find $v_1, v_2 \in V$, such that $T(v_i) = w_i$ and hence $T^{-1}(w_i) = v_i$ for $i = 1, 2$.

Since T is linear, we have $T(av_1+bv_2) = aT(v_1)+bT(v_2)$, and hence $av_1+bv_2 = T^{-1}(aT(v_1)+bT(v_2))$. Since $v_i = T^{-1}(w_i)$, and $w_i = T(v_i)$, for $i = 1, 2$, we get $aT^{-1}(w_1) + bT^{-1}(w_2) = T^{-1}(aw_1 + bw_2)$, proving T^{-1} is a linear transformation.

MA110 - Linear Algebra & Differential Equations: Midsem

Code $\boxed{\text{C}}$: Solutions

Name: **Roll No.** $\boxed{}$

Tutorial Batch: D $\boxed{}$ **T** $\boxed{}$ **Max. marks: 60 (Weightage 40%)**

February 24th, 2024 **8:30 - 10:30 a.m.**

(1) The LU decomposition of $C = \begin{pmatrix} 2 & 1 & -1 \\ 4 & 0 & 1 \\ -4 & 2 & -3 \end{pmatrix}$ is given by [3]

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \boxed{2} & 1 & 0 \\ \boxed{-2} & \boxed{-2} & 1 \end{pmatrix} \text{ and } U = \begin{pmatrix} \boxed{2} & \boxed{1} & \boxed{-1} \\ 0 & \boxed{-2} & \boxed{3} \\ 0 & 0 & \boxed{1} \end{pmatrix}.$$

(2) Let $W = \{x = (x_1, x_2, x_3, \ldots) \in \mathbb{R}^\infty \mid x_n + 4\, x_{n-1} = 0 \text{ for all } n \geq 2\}$. The dimension of W is $\boxed{1}$ and a basis of W is given by $\boxed{(1, -4, 16, -64, \ldots, (-1)^{n-1}4^{n-1}, \ldots)}$. [2]

(3) A linear transformation $T : \mathcal{M}_{2\times 2} \to \mathcal{P}_3$, satisfying $T(e_{11}) = x + x^3$, $T(e_{12}) = 0$, $T(e_{21}) = 1 - x^2$ and $T(e_{22}) = 2x^2 + 3x$ is $T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \boxed{c} + \boxed{a+3d}\,x + \boxed{-c+2d}\,x^2 + \boxed{a}\,x^3$. [2]

(4) Let $v = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$, $w = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}^T$, and $W = \text{Span}\{v, w\}$. [4]

The length of v is $\boxed{\sqrt{3} \text{ units}}$ and $\text{proj}_v(w) = \boxed{\frac{2}{3}(1,1,1)^T}$.

An orthogonal basis of W constructed using Gram Schmidt method applied to $\{v, w\}$ is given by $\boxed{\left\{\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T, \frac{1}{3}\begin{pmatrix} 1 & -2 & 1 \end{pmatrix}^T\right\}}$.

(5) Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}^T$ and $b = \begin{pmatrix} 1 & -1 & -1 \end{pmatrix}^T$. Then $A^T b = \boxed{\begin{pmatrix} -1 & -2 \end{pmatrix}^T}$. [3]

The least squares solution to the system $Ax = b$ is $\widehat{x} = \boxed{\begin{pmatrix} 1 & -1 \end{pmatrix}^T}$.

(6) Let $A = \begin{pmatrix} 3 & 1 & -1 & 2 \\ 3 & 0 & 1 & 3 \\ 0 & 1 & -2 & -1 \end{pmatrix}$. The dimension of $N(A)$ is $\boxed{2}$, [6]

a basis for $C(A^T)$ is $\boxed{\left\{\begin{pmatrix} 3 & 1 & -1 & 2 \end{pmatrix}^T, \begin{pmatrix} 3 & 0 & 1 & 3 \end{pmatrix}^T\right\}}$, a basis of $C(A)$ is $\boxed{\left\{\begin{pmatrix} 3 & 3 & 0 \end{pmatrix}^T, \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}^T\right\}}$,

and an x such that $Ax = \begin{pmatrix} 1 & 6 & -5 \end{pmatrix}^T$ is $\boxed{\begin{pmatrix} 2 & -5 & 0 & 0 \end{pmatrix}^T}$.

(7) Let $V = \mathcal{M}_{2\times 2}$, $T : V \to V$ be the linear transformation defined by $T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a+c & b+a \\ 0 & d+a \end{pmatrix}$, and $\mathcal{B} = \left\{\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}$ be a basis of V. [3]

If $v = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $[v]_{\mathcal{B}} = \begin{pmatrix} \boxed{a} \\ \boxed{c-d} \\ \boxed{d} \\ \boxed{b-a} \end{pmatrix}$, and $[T]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} \boxed{1} & \boxed{1} & \boxed{1} & \boxed{0} \\ \boxed{-1} & \boxed{0} & \boxed{-1} & \boxed{0} \\ \boxed{1} & \boxed{0} & \boxed{1} & \boxed{0} \\ \boxed{1} & \boxed{-1} & \boxed{-1} & \boxed{1} \end{pmatrix}$.

(8) Let $v_1 = (1, 1, -1, -1)^T$, $v_2 = (-1, 1, -1, 1)^T$, $w_1 = (-1, 1, 1, -1)^T$, $w_2 = (1, 1, 1, 1)^T$, A be a 4×4 matrix. Suppose the special solutions of $(A - I)x = 0$ are v_1 and v_2, and that of $(A + I)x = 0$ are w_1 and w_2. Then $A = P\Lambda P^{-1}$, where

$$P = \begin{pmatrix} \boxed{1} & \boxed{-1} & \boxed{-1} & \boxed{1} \\ \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} \\ \boxed{-1} & \boxed{-1} & \boxed{1} & \boxed{1} \\ \boxed{-1} & \boxed{1} & \boxed{-1} & \boxed{1} \end{pmatrix}, \text{ and } \Lambda = \begin{pmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{-1} & 0 \\ 0 & 0 & 0 & \boxed{-1} \end{pmatrix}. \qquad [2]$$

Furthermore, [4]

$$P^{-1} = \frac{1}{4}\begin{pmatrix} \boxed{1} & \boxed{1} & \boxed{-1} & \boxed{-1} \\ \boxed{-1} & \boxed{1} & \boxed{-1} & \boxed{1} \\ \boxed{-1} & \boxed{1} & \boxed{1} & \boxed{-1} \\ \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} \end{pmatrix}, \text{ and } A = \begin{pmatrix} \boxed{0} & \boxed{0} & \boxed{0} & \boxed{-1} \\ \boxed{0} & \boxed{0} & \boxed{-1} & \boxed{0} \\ \boxed{0} & \boxed{-1} & \boxed{0} & \boxed{0} \\ \boxed{-1} & \boxed{0} & \boxed{0} & \boxed{0} \end{pmatrix}$$

(9) Match each option (i) - (iv) on the left with the correct option (a) - (e) on the right: [4]

(i) $\left\{\frac{1}{\sqrt{2}}(1,0,1)^T, \frac{1}{\sqrt{2}}(1,0,-1)^T, \frac{1}{\sqrt{2}}(1,-1,0)^T\right\}$ (a) is an orthonormal basis of $\mathbb{R}^3$

(ii) $\left\{\frac{1}{\sqrt{2}}(1,0,1)^T, \frac{1}{\sqrt{2}}(1,0,-1)^T, \frac{1}{\sqrt{2}}(0,-1,0)^T\right\}$ (b) is a linearly dependent subset of $\mathbb{R}^3$

(iii) $\left\{\frac{1}{\sqrt{2}}(1,0,1)^T, (0,-1,0)^T, \frac{1}{\sqrt{2}}(1,0,-1)^T\right\}$ (c) is a basis of $\mathbb{R}^3$ which is not orthogonal.

(iv) $\left\{(0,0,0)^T, (0,1,0)^T, \frac{1}{\sqrt{2}}(1,0,1)^T\right\}$ (d) is an orthogonal set which is not a basis of $\mathbb{R}^3$

(e) is an orthogonal basis of $\mathbb{R}^3$ which is not orthonormal.

(i) - $\boxed{\text{(c)}}$ (ii) - $\boxed{\text{(e)}}$ (iii) - $\boxed{\text{(a)}}$ (iv) - $\boxed{\text{(b}}$

(10) Let V be a vector space with Span $\{v_1, \dots, v_5\} = V$, where v_1, v_2, v_3 are linearly independent. Let $v_6 \in V$. Which of the following must be true? $\boxed{\text{(b),(e)}}$ [2]

(a) Span $\{v_1, v_2, v_3, v_4\} \neq V$ (b) v_1, v_3 are linearly independent.
(c) v_1, v_2, v_4 are linearly independent. (d) v_1, v_2, v_3, v_4 are linearly dependent.
(e) $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ is a spanning set for V.

(11) Let A be an $m \times n$ matrix, with $m \neq n$, and $S : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation defined by $S(u) = Au$. Consider the following statements:

(a) The rows of A are linearly dependent. (b) $N(A^T) = 0$.
(c) The columns of A span $\mathbb{R}^m$. (d) $N(A) = 0$.
(e) rank$(A) = n$. (f) rank$(A) = m$.

(i) The statements from (a) - (f) which imply that S is one-one are: $\boxed{\text{(d), (e)}}$ [2]

(ii) The statements from (a) - (f) which imply that S is onto are: $\boxed{\text{(b), (c), (f)}}$ [2]

(12) The statements (a)-(g) are needed to prove the statement: [3]
"Given $T : V \to W$ linear, $N(T)$ is a subspace of V."
The correct order in which these statements are to be written to give a proof is
$\boxed{\text{(d)}}$ $\boxed{\text{(f)}}$ $\boxed{\text{(b)}}$ $\boxed{\text{(a)}}$ $\boxed{\text{(g)}}$ $\boxed{\text{(e)}}$ $\boxed{\text{(c)}}$

(a) $\Rightarrow T(v_1) = 0$, and $T(v_2) = 0$.
(b) Let $v_1, v_2 \in N(T)$.
(c) $\Rightarrow a_1v_1 + a_2v_2 \in N(T)$, i.e., $N(T)$ is closed under linear combinations.
(d) $T(0) = 0$
(e) $\Rightarrow T(a_1v_1 + a_2v_2) = 0$, since T is linear.
(f) $\Rightarrow 0 \in N(T)$.
(g) $\Rightarrow a_1T(v_1) + a_2T(v_2) = 0$ for every $a_1, a_2 \in \mathbb{R}$.

(13) Consider the statement: [2]
"Let V be a vector space, W_1, W_2 be subspaces, and $W = W_1 \cup W_2$. Then W is a subspace of V".
Give a counter-example (with justification) to show that the statement is false.

Let $W_1 = \{(x, 0) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$ and $W_2 = \{(0, y) \in \mathbb{R}^2 \mid y \in \mathbb{R}\}$.
These are lines through the origin in $\mathbb{R}^2$ and hence subspaces of $\mathbb{R}^2$.
Consider $(1, 0) \in W_1 \subseteq W_1 \cup W_2$ and $(0, 1) \in W_2 \subseteq W_1 \cup W_2$. Then $(1, 0) + (0, 1) = (1, 1)$.
Note $(1, 1) \notin W_1$ since its second coordinate is non-zero and $(1, 1) \notin W_2$ since the first is non-zero.
Therefore, $(1, 1) \notin W_1 \cup W_2$.
This implies $W_1 \cup W_2$ is not closed under vector addition and is not a subspace of $\mathbb{R}^2$.

(14) Let A be a $m \times n$ matrix and B be a $n \times p$ matrix. Show that $\text{rank}(AB) \leq \text{rank}(A)$. [2]

First note that any column in AB is the product of A with a column of B, that is $(AB)_{*j} = AB_{*j}$.
Now $AB_{*j} = b_{1j}A_{*1} + \ldots + b_{nj}A_{*n}$ and therefore, each column of AB is a linear combination of columns of A.
Thus $(AB)_{*j} \in C(A)$. Since, $C(A)$ is closed under linear combinations, $C(AB) \subseteq C(A)$.
Thus, $\text{rank}(AB) = \dim(C(AB)) \leq \dim(C(A)) = \text{rank}(A)$.

(15) Let A be an 5×3 matrix.
(i) Prove that AA^T is not invertible. [2]

Since A is a 5×3 matrix, $\text{rank}(A) \leq 3$.
By Q14, we see that $\text{rank}(AA^T) \leq \text{rank}(A)$, i.e., $\text{rank}(AA^T) \leq 3$.
In particular, AA^T is a 5×5 matrix such that $\text{rank}(AA^T) \neq 5$, and hence AA^T is not invertible

(ii) Prove that $N(A) = N(A^T A)$. [2]

If $x \in N(A)$, then $Ax = 0$, and hence $(A^T A)(x) = A^T(Ax) = 0$. This proves $N(A) \subseteq N(A^T A)$.
For the other inclusion, if $x \in N(A^T A)$, then $(A^T A)(x) = 0$.
In particular, $x^T(A^T A)x = 0$, i.e., $(x^T A^T)(Ax) = (Ax)^T Ax = 0$.
Thus $||Ax|| = 0$, which implies $Ax = 0$, i.e., $x \in N(A)$, proving $N(A^T A) \subseteq N(A)$.

(iii) Show that $\text{rank}(A^T A) = \text{rank}(A)$. [1]

By (ii), we see that $N(A) = N(A^T A)$, and in particular, $\dim(N(A)) = \dim(N(A^T A))$.
Thus, since both $A^T A$ and A have 3 columns each, by the rank-nullity theorem, we get
$\text{rank}(A) = 3 - \dim(N(A) = 3 - \dim(N(A^T A) = \text{rank}(A^T A)$.

(iv) If the columns of A are linearly independent, then show that $A^T A$ is invertible. [1]

Since the columns of A are linearly independent, we get $\text{rank}(A^T A) = \text{rank}(A) = 3$.
This forces $A^T A$ to be invertible, since $A^T A$ is a 3×3 matrix.

(16) Let $f_1 = 1$, $f_2 = x + x^2$, $f_3 = 1 + x + x^2 + x^3$, $f_4 = 2x + 2x^2 + x^3$ be polynomials in $\mathcal{P}_3$.
(i) Show that f_1, f_2, f_3, f_4 are linearly dependent. [2]

Note that $f_3 = f_4 - f_2 + f_1$, i.e., $f_1 - f_2 - f_3 + f_4 = 0$.
This defines a non-trivial linear relation between f_1, f_2, f_3 and f_4 making them linearly dependent.

(ii) Find $f \in \mathcal{P}_3$ which is not in $\text{Span}\{f_1, f_2, f_3, f_4\}$. Justify your answer. [2]

Consider any element $f \in \text{Span}\{f_1, f_2, f_3, f_4\}$. Then $f = af_1 + bf_2 + cf_3 + df_4$.
This implies $f = a + bx + bx^2 + c + cx + cx^2 + cx^3 + 2dx + 2dx^2 + dx^3$,
i.e., $f = (a + c) + (b + c + 2d)x + (b + c + 2d)x^2 + (c + d)x^3$.
This implies for any $f \in \text{Span}\{f_1, f_2, f_3, f_4\}$, the coefficient of x and x^2 are the same.
In particular, $x \in \mathcal{P}_3$ is not in the $\text{Span}\{f_1, f_2, f_3, f_4\}$.

(17) Show that if T is a bijective linear transformation, then T^{-1} is a linear transformation. [4]

Let the domain and codomain of T be V and W respectively. Since T is a bijection, its inverse T^{-1} exists. Moreover, for $v \in V$, $w \in W$, we have $T^{-1}(w) = v$ if and only if $T(v) = w$.
For any scalars a, b, and $w_1, w_2 \in W$, we need to show that $T^{-1}(aw_1 + bw_2) = aT^{-1}(w_1) + bT^{-1}(w_2)$. Since T is a bijection, we can find $v_1, v_2 \in V$, such that $T(v_i) = w_i$ and hence $T^{-1}(w_i) = v_i$ for $i = 1, 2$.
Since T is linear, we have $T(av_1 + bv_2) = aT(v_1) + bT(v_2)$, and hence $av_1 + bv_2 = T^{-1}(aT(v_1) + bT(v_2))$. Since $v_i = T^{-1}(w_i)$, and $w_i = T(v_i)$, for $i = 1, 2$, we get $aT^{-1}(w_1) + bT^{-1}(w_2) = T^{-1}(aw_1 + bw_2)$, proving T^{-1} is a linear transformation.

MA110 - Linear Algebra & Differential Equations: Midsem

Code $\boxed{D}$: Solutions

Name: **Roll No.** $\boxed{}$

Tutorial Batch: D$\boxed{}$ **T**$\boxed{}$ **Max. marks: 60 (Weightage 40%)**

February 24th, 2024 **8:30 - 10:30 a.m.**

(1) The LU decomposition of $D = \begin{pmatrix} 2 & 1 & -1 \\ -4 & 2 & 0 \\ 4 & -2 & 1 \end{pmatrix}$ is given by [3]

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \boxed{-2} & 1 & 0 \\ \boxed{2} & \boxed{-1} & 1 \end{pmatrix} \text{ and } U = \begin{pmatrix} \boxed{2} & \boxed{1} & \boxed{-1} \\ 0 & \boxed{4} & \boxed{-2} \\ 0 & 0 & \boxed{1} \end{pmatrix}.$$

(2) Let $W = \{x = (x_1, x_2, x_3, \ldots) \in \mathbb{R}^\infty \mid x_n + 5\, x_{n-1} = 0 \text{ for all } n \geq 2\}$. The dimension of W is $\boxed{1}$ and a basis of W is given by $\boxed{\{1, -5, 25, -125, \ldots, (-1)^{n-1}5^{n-1}, \ldots)\}}$. [2]

(3) A linear transformation $T : \mathcal{M}_{2\times 2} \to \mathcal{P}_3$, satisfying $T(e_{11}) = 2x^2 + 3x$, $T(e_{12}) = x + x^3$, $T(e_{21}) = 0$ and $T(e_{22}) = 1 - x^2$ is $T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \boxed{\text{d}} + \boxed{\text{3a+b}}\, x + \boxed{\text{2a - d}}\, x^2 + \boxed{\text{b}}\, x^3$. [2]

(4) Let $v = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$, $w = \begin{pmatrix} -1 & 0 & -1 \end{pmatrix}^T$, and $W = \text{Span}\{v, w\}$. [4]
The length of v is $\boxed{\sqrt{3} \text{ units}}$ and $\text{proj}_v(w) = \boxed{-\frac{2}{3}(1, 1, 1)^T}$.
An orthogonal basis of W constructed using Gram Schmidt method applied to $\{v, w\}$ is given by
$\boxed{\left\{\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T, \frac{1}{3}\begin{pmatrix} -1 & 2 & -1 \end{pmatrix}^T\right\}}$.

(5) Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}^T$ and $b = \begin{pmatrix} 1 & -1 & -1 \end{pmatrix}^T$. Then $A^T b = \boxed{\begin{pmatrix} -2 & -1 \end{pmatrix}^T}$. [3]
The least squares solution to the system $Ax = b$ is $\widehat{x} = \boxed{\begin{pmatrix} -1 & 1 \end{pmatrix}^T}$.

(6) Let $B = \begin{pmatrix} 3 & 1 & -1 & 2 \\ 0 & 1 & -2 & -1 \\ 3 & 0 & 1 & 3 \end{pmatrix}$. The dimension of $N(B)$ is $\boxed{2}$, [6]
a basis for $C(B^T)$ is $\boxed{\left\{\begin{pmatrix} 3 & 1 & -1 & 2 \end{pmatrix}^T, \begin{pmatrix} 0 & 1 & -2 & -1 \end{pmatrix}^T\right\}}$, a basis of $C(B)$ is $\boxed{\left\{\begin{pmatrix} 3 & 0 & 3 \end{pmatrix}^T, \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}^T\right\}}$,
and an x such that $Bx = \begin{pmatrix} 1 & -5 & 6 \end{pmatrix}^T$ is $\boxed{\begin{pmatrix} 2 & -5 & 0 & 0 \end{pmatrix}^T}$.

(7) Let $V = \mathcal{M}_{2\times 2}$, $T : V \to V$ be the linear transformation defined by $T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a + c & b + a \\ 0 & d + a \end{pmatrix}$,
and $\mathcal{B} = \left\{\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}$ be a basis of V. [3]

If $v = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $[v]_{\mathcal{B}} = \begin{pmatrix} \boxed{\text{a}} \\ \boxed{\text{c-d}} \\ \boxed{\text{d}} \\ \boxed{\text{b-a}} \end{pmatrix}$, and $[T]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} \boxed{1} & \boxed{1} & \boxed{1} & \boxed{0} \\ \boxed{-1} & \boxed{0} & \boxed{-1} & \boxed{0} \\ \boxed{1} & \boxed{0} & \boxed{1} & \boxed{0} \\ \boxed{1} & \boxed{-1} & \boxed{-1} & \boxed{1} \end{pmatrix}$.

(8) Let $v_1 = (-1, 1, 1, -1)^T$, $v_2 = (1, 1, 1, 1)^T$, $w_1 = (1, 1, -1, -1)^T$, $w_2 = (-1, 1, -1, 1)^T$, B be a 4×4 matrix. Suppose the special solutions of $(B + I)x = 0$ are v_1 and v_2, and that of $(B - I)x = 0$ are

w_1 and w_2. Then $B = P\Lambda P^{-1}$, where

$$P = \begin{pmatrix} \boxed{-1} & \boxed{1} & \boxed{1} & \boxed{-1} \\ \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} \\ \boxed{1} & \boxed{1} & \boxed{-1} & \boxed{-1} \\ \boxed{-1} & \boxed{1} & \boxed{-1} & \boxed{1} \end{pmatrix}, \text{ and } \Lambda = \begin{pmatrix} \boxed{-1} & 0 & 0 & 0 \\ 0 & \boxed{-1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{pmatrix}.$$ [2]

Furthermore, [4]

$$P^{-1} = \frac{1}{4}\begin{pmatrix} \boxed{-1} & \boxed{1} & \boxed{1} & \boxed{-1} \\ \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} \\ \boxed{1} & \boxed{1} & \boxed{-1} & \boxed{-1} \\ \boxed{-1} & \boxed{1} & \boxed{-1} & \boxed{1} \end{pmatrix}, \text{ and } B = \begin{pmatrix} \boxed{0} & \boxed{0} & \boxed{0} & \boxed{-1} \\ \boxed{0} & \boxed{0} & \boxed{-1} & \boxed{0} \\ \boxed{0} & \boxed{-1} & \boxed{0} & \boxed{0} \\ \boxed{-1} & \boxed{0} & \boxed{0} & \boxed{0} \end{pmatrix}$$

(9) Match each option (i) - (iv) on the left with the correct option (a) - (e) on the right: [4]

(i) $\left\{\frac{1}{\sqrt{2}}(1,0,1)^T, \frac{1}{\sqrt{2}}(1,0,-1)^T, \frac{1}{\sqrt{2}}(1,-1,0)^T\right\}$

(ii) $\left\{\frac{1}{\sqrt{2}}(1,0,1)^T, \frac{1}{\sqrt{2}}(1,0,-1)^T, \frac{1}{\sqrt{2}}(0,-1,0)^T\right\}$

(iii) $\left\{\frac{1}{\sqrt{2}}(1,0,1)^T, (0,-1,0)^T, \frac{1}{\sqrt{2}}(1,0,-1)^T\right\}$

(iv) $\left\{(0,0,0)^T, (0,1,0)^T, \frac{1}{\sqrt{2}}(1,0,1)^T\right\}$

(a) is an orthonormal basis of $\mathbb{R}^3$

(b) is a linearly dependent subset of $\mathbb{R}^3$

(c) is a basis of $\mathbb{R}^3$ which is not orthogonal.

(d) is an orthogonal set which is not a basis of $\mathbb{R}^3$

(e) is an orthogonal basis of $\mathbb{R}^3$ which is not orthonormal.

(i) - $\boxed{\text{(c)}}$ (ii) - $\boxed{\text{(e)}}$ (iii) - $\boxed{\text{(a)}}$ (iv) - $\boxed{\text{(b)}}$

(10) Let V be a vector space with Span $\{v_1, \dots, v_5\} = V$, where v_1, v_2, v_3 are linearly independent. Let $v_6 \in V$. Which of the following must be true? $\boxed{\text{(a), (c)}}$ [2]

(a) $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ is a spanning set for V.

(b) Span $\{v_1, v_2, v_3, v_4\} \neq V$

(c) v_1, v_3 are linearly independent.

(d) v_1, v_2, v_4 are linearly independent.

(e) v_1, v_2, v_3, v_4 are linearly dependent.

(11) Let A be an $m \times n$ matrix, with $m \neq n$, and $S : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation defined by $S(u) = Au$. Consider the following statements:

(a) The rows of A are linearly dependent.

(b) $N(A^T) = 0$.

(c) The columns of A span $\mathbb{R}^m$.

(d) $N(A) = 0$.

(e) rank$(A) = n$.

(f) rank$(A) = m$.

(i) The statements from (a) - (f) which imply that S is onto are: $\boxed{\text{(b), (c), (f)}}$ [2]

(ii) The statements from (a) - (f) which imply that S is one-one are: $\boxed{\text{(d), (e)}}$ [2]

(12) The statements (a)-(g) are needed to prove the statement: [3]
"Given $T : V \to W$ linear, $N(T)$ is a subspace of V."
The correct order in which these statements are to be written to give a proof is
$\boxed{\text{(c)}}$ $\boxed{\text{(f)}}$ $\boxed{\text{(b)}}$ $\boxed{\text{(g)}}$ $\boxed{\text{(d)}}$ $\boxed{\text{(e)}}$ $\boxed{\text{(a)}}$

(a) $\Rightarrow a_1v_1 + a_2v_2 \in N(T)$, i.e., $N(T)$ is closed under linear combinations.

(b) Let $v_1, v_2 \in N(T)$.

(c) $T(0) = 0$

(d) $\Rightarrow a_1T(v_1) + a_2T(v_2) = 0$ for every $a_1, a_2 \in \mathbb{R}$.

(e) $\Rightarrow T(a_1v_1 + a_2v_2) = 0$, since T is linear.

(f) $\Rightarrow 0 \in N(T)$.

(g) $\Rightarrow T(v_1) = 0$, and $T(v_2) = 0$.

(13) Consider the statement: [2]
"Let V be a vector space, W_1, W_2 be subspaces, and $W = W_1 \cup W_2$. Then W is a subspace of V".
Give a counter-example (with justification) to show that the statement is false.

Let $W_1 = \{(x,0) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$ and $W_2 = \{(0,y) \in \mathbb{R}^2 \mid y \in \mathbb{R}\}$.
These are lines through the origin in $\mathbb{R}^2$ and hence subspaces of $\mathbb{R}^2$.
Consider $(1,0) \in W_1 \subseteq W_1 \cup W_2$ and $(0,1) \in W_2 \subseteq W_1 \cup W_2$. Then $(1,0)+(0,1) = (1,1)$.
Note $(1,1) \notin W_1$ since its second coordinate is non-zero and $(1,1) \notin W_2$ since the first is non-zero.
Therefore, $(1,1) \notin W_1 \cup W_2$.
This implies $W_1 \cup W_2$ is not closed under vector addition and is not a subspace of $\mathbb{R}^2$.

(14) Let A be a $m \times n$ matrix and B be a $n \times p$ matrix. Show that $\operatorname{rank}(AB) \leq \operatorname{rank}(A)$. [2]

First note that any column in AB is the product of A with a column of B, that is $(AB)_{*j} = AB_{*j}$.
Now $AB_{*j} = b_{1j}A_{*1} + \ldots + b_{nj}A_{*n}$ and therefore, each column of AB is a linear combination of columns of A.
Thus $(AB)_{*j} \in C(A)$. Since, $C(A)$ is closed under linear combinations, $C(AB) \subseteq C(A)$.
Thus, $\operatorname{rank}(AB) = \dim(C(AB)) \leq \dim(C(A)) = \operatorname{rank}(A)$.

(15) Let A be an 5×3 matrix.
(i) Prove that AA^T is not invertible. [2]

Since A is a 5×3 matrix, $\operatorname{rank}(A) \leq 3$.
By Q14, we see that $\operatorname{rank}(AA^T) \leq \operatorname{rank}(A)$, i.e., $\operatorname{rank}(AA^T) \leq 3$.
In particular, AA^T is a 5×5 matrix such that $\operatorname{rank}(AA^T) \neq 5$, and hence AA^T is not invertible

(ii) Prove that $N(A) = N(A^TA)$. [2]

If $x \in N(A)$, then $Ax = 0$, and hence $(A^TA)(x) = A^T(Ax) = 0$. This proves $N(A) \subseteq N(A^TA)$.
For the other inclusion, if $x \in N(A^TA)$, then $(A^TA)(x) = 0$.
In particular, $x^T(A^TA)x = 0$, i.e., $(x^TA^T)(Ax) = (Ax)^TAx = 0$.
Thus $||Ax|| = 0$, which implies $Ax = 0$, i.e., $x \in N(A)$, proving $N(A^TA) \subseteq N(A)$.

(iii) Show that $\operatorname{rank}(A^TA) = \operatorname{rank}(A)$. [1]

By (ii), we see that $N(A) = N(A^TA)$, and in particular, $\dim(N(A)) = \dim(N(A^TA))$.
Thus, since both A^TA and A have 3 columns each, by the rank-nullity theorem, we get
$\operatorname{rank}(A) = 3 - \dim(N(A) = 3 - \dim(N(A^TA) = \operatorname{rank}(A^TA)$.

(iv) If the columns of A are linearly independent, then show that A^TA is invertible. [1]

Since the columns of A are linearly independent, we get $\operatorname{rank}(A^TA) = \operatorname{rank}(A) = 3$.
This forces A^TA to be invertible, since A^TA is a 3×3 matrix.

(16) Let $f_1 = 1$, $f_2 = x + x^2$, $f_3 = 1 + x + x^2 + x^3$, $f_4 = 2x + 2x^2 + x^3$ be polynomials in $\mathcal{P}_3$.
(i) Show that f_1, f_2, f_3, f_4 are linearly dependent. [2]

Note that $f_3 = f_4 - f_2 + f_1$, i.e., $f_1 - f_2 - f_3 + f_4 = 0$.
This defines a non-trivial linear relation between f_1, f_2, f_3 and f_4 making them linearly dependent.

(ii) Find $f \in \mathcal{P}_3$ which is not in $\operatorname{Span}\{f_1, f_2, f_3, f_4\}$. Justify your answer. [2]

Consider any element $f \in \operatorname{Span}\{f_1, f_2, f_3, f_4\}$. Then $f = af_1 + bf_2 + cf_3 + df_4$.
This implies $f = a + bx + bx^2 + c + cx + cx^2 + cx^3 + 2dx + 2dx^2 + dx^3$,
i.e., $f = (a+c) + (b+c+2d)x + (b+c+2d)x^2 + (c+d)x^3$.
This implies for any $f \in \operatorname{Span}\{f_1, f_2, f_3, f_4\}$, the coefficient of x and x^2 are the same.
In particular, $x \in \mathcal{P}_3$ is not in the $\operatorname{Span}\{f_1, f_2, f_3, f_4\}$.

(17) Show that if T is a bijective linear transformation, then T^{-1} is a linear transformation. [4]

Let the domain and codomain of T be V and W respectively. Since T is a bijection, its inverse T^{-1} exists. Moreover, for $v \in V$, $w \in W$, we have $T^{-1}(w) = v$ if and only if $T(v) = w$.
For any scalars a, b, and $w_1, w_2 \in W$, we need to show that $T^{-1}(aw_1+bw_2) = aT^{-1}(w_1)+bT^{-1}(w_2)$.
Since T is a bijection, we can find $v_1, v_2 \in V$, such that $T(v_i) = w_i$ and hence $T^{-1}(w_i) = v_i$ for $i = 1, 2$.
Since T is linear, we have $T(av_1+bv_2) = aT(v_1)+bT(v_2)$, and hence $av_1+bv_2 = T^{-1}(aT(v_1)+bT(v_2))$.
Since $v_i = T^{-1}(w_i)$, and $w_i = T(v_i)$, for $i = 1, 2$, we get $aT^{-1}(w_1) + bT^{-1}(w_2) = T^{-1}(aw_1 + bw_2)$, proving T^{-1} is a linear transformation.