MA 110 Midsem TSC

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Linear Equations

A set of linear equations can be written in the form of $Ax = b$, where A is a matrix of appropriate order. It is also called the coefficient matrix sometimes. For instance if we have m equations in n variables, then A has size $m \times n$, which is to say A has m rows and n columns.

Gaussian Elimination

We illustrate it by an example. Consider the system

$$\begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix} (x_1, x_2, x_3)^t = (5, 5, -4)^t$$

Working out the example

We write it in the following form, which is called the augmented matrix $(= [A \mid b])$.

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & 5 \\ -2 & 7 & 2 & -4 \end{array}\right) \xrightarrow{R_2-2R_1, R_3+R_1} \left(\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -5 \\ 0 & 8 & 3 & 1 \end{array}\right)$$

$$\xrightarrow{R_3+R_2} \left(\begin{array}{ccc|c} \mathbf{2} & 1 & 1 & 5 \\ 0 & \mathbf{-8} & -2 & -5 \\ 0 & 0 & \mathbf{1} & -4 \end{array}\right)$$

The numbers in bold are called pivots, Observe they are three in number, can you comment if the system has a unique solution?
The matrix which we get at the end is called **Echelon form** of A.

Pivots

We make the following remarks about Pivots.

1. A pivot is always non-zero.
2. Pivot is the first non-zero element in the Echelon form.
3. If a $n \times n$ matrix A has n pivots then for any $b \in \mathbb{R}^n$, the system $Ax = b$ has a unique solution.

Matrices

Addition and scalar multiplication is defined entrywise, The rows of A are denoted $A_{1*}, A_{2*}, \ldots, A_{m*}$, i.e., $A = \begin{pmatrix} A_{1*} \\ A_{2*} \\ \vdots \\ A_{m*} \end{pmatrix}$,
the columns are denoted $A_{*1}, A_{*2} \ldots, A_{*n}$, i.e.,
$A = \begin{pmatrix} A_{*1} & A_{*2} & \cdots & A_{*n} \end{pmatrix}$, and the (i,j) th entry is A_{ij} (or a_{ij}). Let A be $m \times n$ and B be $n \times r$, Then define

$$AB = [AB_{*1} AB_{*2} \cdots AB_{*r}]$$

Properties of Matrix multiplication

- (associativity) $(AB)C = A(BC)$
- (distributivity) $A(B + C) = AB + AC$

$$(B + C)D = BD + CD$$

- (non-commutativity) $AB \neq BA$, in general. Find examples.
- (Identity) Let $I = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$ be $n \times n$. If A is $n \times n$, then $AI = A = IA$.

- A is symmetric if $A = A^T$
- A is skew symmetric if $A = -A^T$
- $A + A^T, AA^T$ are always symmetric and $A - A^T$ is always skew symmetric.

Inverse of a matrix

Given A of size $n \times n$, we say B is an inverse of A if $AB = I = BA$.
If this happens, we say A is invertible.

- An inverse may not exist. Find an example. Hint: $n = 1$.
- An inverse of A, if it exists, has size $n \times n$.
- If the inverse of A exists, it is unique, and is denoted A^{-1}.

Remark: $(BA)^{-1} = A^{-1}B^{-1}$

Some comments

1. If A is invertible then $Ax = b$ has unique solution for any b, which is?
2. Gaussian elimination will produce n pivots for an invertible matrix.
3. If A is invertible then $Ax = 0$ cannot have a non-zero solution.
4. Upper triangular matrix is invertible, then all of its diagonal elements are non-zero.
5. Finding A^{-1} is equivalent to finding solutions of $Ax_i = e_i$.

Elementary Matrix

Define $E_{ij}(\lambda)$ to be the matrix obtained by adding λ times the *ith* row to the *jth* row in the identity matrix.
How is the matrix $E_{ij}(\lambda)A$ related to A,
What is the inverse of $E_{ij}(\lambda)$?

Permutation matrix

Similarly P_{ij} is the matrix which interchanges the *ith* and the *jth* row.
How is the matrix $P_{ij}A$ related to A,
What is the inverse of P_{ij}?

Gaussian Elimination again

Let

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix}$$

Find the Echelon form for A

Null space of a matrix

Let A be $m \times n$, then

1. The null space of $A, N(A)$ contains vectors from $\mathbb{R}^n$.
2. $Ax = 0 \Leftrightarrow x$ is in $N(A)$.
3. Observe that the null space is closed under linear combinations.

Column Space

The column space of a matrix A, $C(A)$ is the linear combination of all columns of A

1. A vector b is in the column space of A iff $Ax = b$ has a solution.
2. Column space is closed under linear combinations.

Questions

1. A 3×4 matrix can have atmost   pivots.
2. Find a polynomial $p(t)$ of degree 2 such that $p(1) = 6, p(2) = 15 \& p(3) = 28$
3. Decompose $\begin{pmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{pmatrix}$ into a product of lower and upper triangular matrix.

Questions

1. Let $u = \begin{pmatrix} 7 \\ 2 \\ 5 \end{pmatrix}$, $v = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$ and $w = \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix}$. Use the fact that $2u - 3v - w = 0$ to solve the system.
$$\begin{pmatrix} 7 & 3 \\ 2 & 1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix}.$$
2. Construct a matrix whose column space contains $(1, 1, 1)$ and whose nullspace is the line of multiples of $(1, 1, 1, 1)$.
3. Reduce A and B to their echelon forms, find their ranks, the free and the dependent variables.
$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$
Find the special solutions to $Ax = 0$ and $Bx = 0$, and their nullspaces.

Vector Spaces

Definition: A **vector space** (or **linear space**) **V** over a field **F** consists of a set on which two operations (called **addition** and **scalar multiplication**, respectively) are defined so that for each pair of elements x, y, in **V** there is a unique element x + y in V, and for each element a in **F** and each element x in **V** there is a unique element ax in V, such that the following conditions hold:

Vector Spaces

1. (VS 1) For all x, y in V, $x + y = y + x$ (commutativity of addition).
2. (VS 2) For all x, y, z in V, $(x + y) + z = x + (y + z)$ (associativity of addition).
3. (VS 3) There exists an element in V denoted by 0 such that $x + 0 = x$ for each x in V.
4. (VS 4) For each element x in V there exists an element y in V such that $x + y = 0$.
5. (VS 5) For each element x in V, $1x = x$.
6. (VS 6) For each pair of elements a, b in F and each element x in V, $(ab)x = a(bx)$.
7. (VS 7) For each element a in F and each pair of elements x, y in V, $a(x + y) = ax + ay$.
8. (VS 8) For each pair of elements a, b in F and each element x in V, $(a + b)x = ax + bx$.

Subspaces

Definition: A subset **W** of a vector space **V** over a field **F** is called a subspace of **V** if **W** is a vector space over **F** with the operations of addition and scalar multiplication defined on **V**.
Lemma: Let V be a vector space and W a subset of V. Then W is a subspace of V if and only if the following three conditions hold for the operations defined in V.

1. $0 \in W$.
2. $x + y \in W$ whenever $x \in W$ and $y \in W$.
3. $cx \in W$ whenever $c \in F$ and $x \in W$.

Basis

Definition: A set of vectors $\mathcal{B} \subset V$ is called a basis if and only if they are linearly independent and $\mathcal{S}\text{pan}\{\mathcal{B}\} = V$

Lemma: Every vector space V has a basis.

- A basis is not a unique set of vectors
- The dimension of a vector space is defined as the cardinality of the basis. Obviously, for this to make sense all sets of basis must have the same cardinality.

Question: We have seen that a basis is not unique. Let $\mathcal{C}$ be the set of all basis $\mathcal{B}$ for V. Is this set $\mathcal{C}$ also a vector space under usual definitions of addition and scalar multiplication?

Definition: A linear transformation $T : V \rightarrow W$ is a function which satisfies the linearity operations i.e.,

- $T(u + v) = T(u) + T(v) \ \forall u, w \in V$
- $T(\alpha v) = \alpha T(v) \quad \forall \alpha \in F$ and $v \in V$

Corollary

$T(0) = 0$

Isomorphism: A linear transformation $T : V \rightarrow W$ is called an isomorphism iff it is both one-one and onto.
Question: Let $T : V \rightarrow W$ be an isomorphism and $\mathcal{B}$ a basis for V. Prove that $T(\mathcal{B})$ is a basis for W. Use this to prove $\exists\ T : V \rightarrow W$ which is an isomorphism iff the dimension of V and W are the same

Matrix Representation of Linear Transformations

A linear transformation $T : V \to W$ between finite dimensional vector spaces can be represented using matrices.
Note: This matrix is not unique and depends on the choice of basis in both V and W.

Let $\mathcal{B}$ be a basis for a vector space V of dimension n. Then let $[v]_{\mathcal{B}}$ represent the n x 1 vector with the corresponding coefficients for each of the basis vectors.

Question: Prove that $T : V \to \mathbb{R}^n$ given by $T(v) = [v]_{\mathcal{B}}$ is a linear isomorphism

Let $\mathcal{B}$ be a basis for V and $\mathcal{C}$ be a basis for W, then the matrix of transformations $M_{\mathcal{B}}^{\mathcal{C}}(T)$ for $T : V \to W$ is defined as

$$M_{\mathcal{B}}^{\mathcal{C}}(T) = \begin{bmatrix} [T(v_1)]_{\mathcal{C}} & [T(v_2)]_{\mathcal{C}} & \dots & [T(v_n)]_{\mathcal{C}} \end{bmatrix}$$

where $\{v_1, v_2, \dots v_n\} = \mathcal{B}$

Note: This matrix completely defines $T : V \to W$ because a linear transformation is completely defined by its action on a basis.

Question: Find the matrix of transformations for the linear transformation $T : V \to W$ given by $T(v) = v^T$ and basis $\mathcal{B}$ and $\mathcal{B}^T$

Rank- Nullity Theorem

Given any linear transformation $T : V \to W$, we can study some interesting properties related to them.
Definition: The image space of T denoted by $C(T)$ is the set of vectors $w \in W$ such that $\exists v \in V,\ T(v) = w$.
Definition: The null space or the kernel of T denoted by $N(T)$ is the set of all vectors $v \in V$, $T(v) = 0$ verify!! both of them are vector spaces

Theorem

Rank Nullity Theorem states that the sum of the dimension of $C(T)$ and $N(T)$ is equal to the dimension of V.

Question: Given $N(T)$ and $C(T)$ and V, W, is it always possible to define a unique linear transformation using these??

Determinants

The study of determinants is an interesting topic because they help provide a heuristic to reason about linear transformations.

Definition: Let $V = \mathbb{R}^n$, then the determinant is a function $f : V^n \to \mathbb{R}$ which has the following properties:

- Multilinearity: $f(v_1, v_2, \ldots, \alpha a + \beta b, \ldots, v_n) = \alpha f(v_1, v_2, \ldots, a, \ldots, v_n) + \beta f(v_1, v_2, \ldots, b, \ldots, v_n)$
- Alternating: $f(v_1, v_2, \ldots, v_i, \ldots, v_j, \ldots, v_n) = -f(v_1, v_2, \ldots, v_j, \ldots, v_i, \ldots, v_n)$
- $f(I) = 1$ where I is the identity matrix

Note: The normal way we use to calculate determinants is in fact a function which satisfies these properties.

Determinants

Some useful properties of determinants which help us reason about linear transformations:

- A matrix is invertible iff its determinant is non zero
- $det(CD) = det(C)det(D)$
- $det(CD) = det(DC)$
- $det(\lambda A) = \lambda^n det(A)$ where n is the dimension of the matrix.
- $det(C + D) \neq det(C) + det(D)$
- If any matrix of transformation in your favourite basis has non zero determinant, then the linear transformation is an isomorphism.
- A basis transformation is also a linear transformation. Using properties of determinants, it is easy to show that this a linear isomorphism

Dot Products (Inner Product)

Definition: Let V be a vector space, then the dot product is a function $f : V \times V \to \mathbb{R}$ with the following properties:

- $f(u+v,w) = f(u,w) + f(v,w)$
- $f(\alpha u, v) = \alpha f(u,v)$
- $f(u,v) = f(v,u)$
- $f(u,u) \geq 0$ and $f(u,u) = 0$ iff $u = 0$

Note: Two vectors u, v are said to be orthogonal iff $f(u,v) = 0$

Note: It is convenient to represent this instead by use of a dot such that $f(u,v) = u.v$

Corollary:

$u.0 = 0.u = 0$

Orthogonal Spaces

Definition: Let V be a vector space equipped with a dot product . and W be a subspace of V. The set of all vectors $u \in V$ such that $u.v = 0$ $\forall v \in W$ also form a subspace (verify!!) in V. Let us denote this by U. U is called the orthogonal subspace of W.

The orthogonal subspace has some very interesting properties:

- $U \cap W = \{0\}$. Prove this!!
- $dim(U) + dim(W) = dim(V)$. Does this remind you of something :)

Question: Let $v \in V$ be a fixed vector. Prove that $T(u) = u.v$ is a linear transformation.

Question: Let $\mathcal{B}$ be a basis of U and $\mathcal{C}$ be a basis of W, then prove that $\mathcal{B} \cup \mathcal{C}$ is a basis of V.

Orthogonal Basis

Definition: Let V be a vector space equipped with a dot product ., then an orthonormal basis set $\mathcal{B}$ is a set which is first of all a basis and all pairs of basis vectors are mutually orthogonal.
Note: It is easy to prove existence of such a basis for finite dimensional vector spaces using Gram- Schmidt Orthogonalization

The great thing about an orthogonal basis is that it is extremely easy to calculate the coefficients of the basis vectors.

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n \text{ then } c_i = v.v_i / v_i.v_i$$

Projection Matrices

Let us now consider the following problem of projecting a vector onto a subspace. We tackle this problem by constructing an orthonormal basis for the subspace and then projecting a vector onto it.

Note: Let $\mathcal{B}$ be an orthonormal basis for the subspace W and P represent the matrix $P = [v_1 v_2 \dots v_n]$, then the projection matrix $\pi = PP^T$, this is simply utilizing the above fact.

Note: The norm of a vector u is defined as $\sqrt{u.u}$

Eigenvalues and Eigenvectors

Definition: Let A be a matrix and v be a vector such that $Av = \lambda v$, then λ is called an eigenvalue of A and v is an eigenvector corresponding to eigenvalue λ.

Some properties which follow are:

- The set of all eigenvectors corresponding to an eigenvalue λ also forms a vector space and is known as the eigenspace.
- Eigenvectors corresponding to distinct eigenvalues are linearly independent.
- If all the eigenvalues corresponding to a matrix are distinct, then its eigenspace spans $\mathbb{R}^n$
- $det(A - \lambda I)$ is known as the characteristic polynomial of A. Roots of this polynomial are known as the eigenvalues of A

Diagonalization

Definition: A matrix A is said to be diagonalizable if $\exists$ an invertible matrix P such that $P^{-1}AP = \Lambda$ where Λ is diagonal.

Some properties are:

- A matrix A is diagonalizable iff eigenspace of A has n linearly independent vectors.
- n distinct eigenvalues provides a sufficient condition for n linearly independent vectors.
- $P = [v_1 v_2 \dots v_n]$ where v_i are n linearly independent vectors
- $A = P\Lambda P^{-1}$ is known as the eigenvalue decomposition of A

Question: The Cayley Hamilton theorem states that the matrix A satisfies its characteristic equation. Prove the Cayley Hamilton theorem for diagonalizable matrices.

Questions:

Let $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$. Define addition of elements of V coordinatewise, and for $(a_1, a_2) \in V$ and $c \in \mathbb{R}$, define

$$c \cdot (a_1, a_2) = \begin{cases} (0,0) & \text{if } c = 0, \\ (ca_1, a_2/c) & \text{if } c \neq 0. \end{cases}$$

Is V a vector space over $\mathbb{R}$ with these operations?

Questions:

Let $S = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$. For (a_1, a_2), $(b_1, b_2) \in S$, and $c \in \mathbb{R}$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 - b_2)$$

and

$$c \cdot (a_1, a_2) = (ca_1, ca_2).$$

Is V a vector space over $\mathbb{R}$ with these operations?

Let S be a nonempty set and F a field. Let $\mathcal{C}(S, F)$ denote the set of all functions $f \in \mathcal{F}(S, F)$ such that $f(s) = 0$ for all but a finite number of elements of S. We want to prove that $\mathcal{C}(S, F)$ is a subspace of $\mathcal{F}(S, F)$.

Questions:

Let W be a subspace of a vector space V over a field F. For any $v \in V$, the set $v + W = \{v + w : w \in W\}$ is called the coset of W containing v. It is customary to denote this coset by $v + W$ rather than $\{v\} + W$.

(a) Prove that $v + W$ is a subspace of V if and only if $v \in W$.

(b) Prove that $v_1 + W = v_2 + W$ if and only if $v_1 - v_2 \in W$.

Addition and scalar multiplication by scalars of F can be defined in the collection $S = \{v + W : v \in V\}$ of all cosets of W as follows:

$$(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W \quad \text{for all } v_1, v_2 \in V,$$

and

$$a(v + W) = av + W \quad \text{for all } v \in V \text{ and } a \in F.$$

Prove that the set S is a vector space with the operations defined. This vector space is called the quotient space of V modulo W and is denoted by V/W. Comment on the dimension of V/W

Questions:

Let V and W be vector spaces and $dim(V) > dim(W)$. Prove that there is no one-one linear transformation $T : V \to W$.

Let V be a vector space, and let $T : V \to V$ be linear. A subspace W of V is said to be T-invariant if $T(x) \in W$ for every $x \in W$, that is, $T(W) \subseteq W$. Prove that the subspaces $\{0\}$, V, $C(T)$, and $N(T)$ are all T-invariant.

Questions:

Prove the Cauchy Schwartz inequality $|v||w| \geq |v.w|$ using properties of dot product

Questions:

For each of the following matrices $A \in M_{n\times n}(F)$,

1. Determine all the eigenvalues of A.
2. For each eigenvalue λ of A, find the set of eigenvectors corresponding to λ.
3. If possible, find a basis for F^n consisting of eigenvectors of A.
4. If successful in finding such a basis, determine an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

(a)

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \text{ for } F = \mathbb{R}$$

(b)

$$A = \begin{pmatrix} 0 & -2 & -3 \\ 1 & -1 & 2 \\ 2 & 2 & 5 \end{pmatrix} \text{ for } F = \mathbb{R}$$

Questions:

Label the following statements as true or false.

(a) Any linear operator on an n-dimensional vector space that has fewer than n distinct eigenvalues is not diagonalizable.

(b) Two distinct eigenvectors corresponding to the same eigenvalue are always linearly dependent.

(c) If λ is an eigenvalue of a linear operator T, then each vector in E_λ is an eigenvector of T.

(d) If λ_1 and λ_2 are distinct eigenvalues of a linear operator T, then $E_{\lambda_1} \cap E_{\lambda_2} = \{0\}$.

(e) Let $A \in M_{n\times n}(F)$ and $\beta = \{v_1, v_2, \dots, v_n\}$ be an ordered basis for F^n consisting of eigenvectors of A. If Q is the $n \times n$ matrix whose jth column is v_j $(1 \leq j \leq n)$, then $Q^{-1}AQ$ is a diagonal matrix.

(f) A linear operator T on a finite-dimensional vector space is diagonalizable if and only if the multiplicity of each eigenvalue λ equals the dimension of E_λ.

(g) Every diagonalizable linear operator on a nonzero vector space has at least one eigenvalue.

Questions:

Consider a vector space V and all possible linear transformations $T : V \to \mathbb{R}$. Then prove that this set also forms a vector space and find its dimension. Infact, this is known as the dual space of V.