MA 110 - Ordinary Differential Equations

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Outline of the lecture

- Orthogonal Trajectories
- Lipschitz continuity
- Existence & uniqueness

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Orthogonal Trajectories

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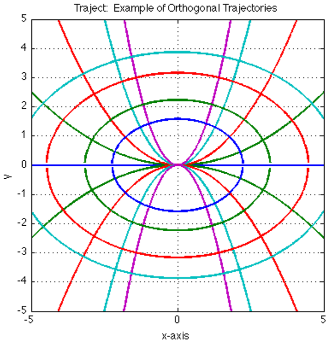

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(Leaving a part certain trajectories that are vertical lines!)

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Hence the orthogonal trajectories are given by $\boxed{y = kx}$.

Definitions

1. Let f be a real function defined on D, where D is either a domain or a closed domain of the xy plane. The function f is said to be bounded in D if there exists a positive number M such that
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Consider

$$|g(x_2) - g(x_1)| \leq M|x_2 - x_1| \ \ \forall x_1,\, x_2 \text{ in the domain of } g.$$

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At each point $(a, g(a))$, the entire graph of g lies between the lines

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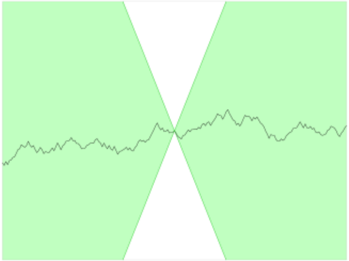

Example : x^2 is Lipschitz in $[1, 2]$.

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$$P_1(x, y_1, f(x, y_1)) \;\&\; P_2(x, y_2, f(x, y_2))$$

on the surface $z = f(x, y)$, and let α $(0 \le \alpha \le \pi/2)$ denote the angle that the chord joining P_1 and P_2 makes with the xy- plane.

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  holds in D, then $\tan\alpha$ is bounded in absolute value.
- That is, the chord joining P_1 and P_2 is bounded away from being perpendicular to the xy- plane.
- Further, this bound is independent of the points (x, y_1) and (x, y_2) belonging to D.

Lipschitz condition $\implies$ Continuity ?

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Note that the condition of Lipschitz continuity implies nothing concerning the continuity of f with respect to x.

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Sufficiency

Result : If f is such that $\dfrac{\partial f}{\partial y}$ exists and is bounded for all $(x, y) \in D$, then f satisfies Lipschitz condition w.r.t. y in D, where the Lipschitz constant

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That is, f satisfies Lipschitz condition.

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$$f(x, y) = y^2 \text{ defined in } D : |x| \leq a,\ |y| \leq b.$$

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(Verify Lipschitz condition directly!)

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Existence of bounded derivative f_y is a sufficient condition for Lipschitz condition to hold true (not necessary).

Existence - Uniqueness Theorem

Let R be a rectangle containing (x_0, y_0) in the domain D,

- $f(x, y)$ be continuous at all points (x, y) in $R : |x - x_0| < a,\ |y - y_0| < b$ and

[1] Existence - Peano, Existence & uniqueness -Picard

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Then, the IVP $\boxed{y' = f(x, y), \ y(x_0) = y_0}$ has at least one solution $y(x)$ defined for all x in the interval $|x - x_0| < \alpha$, where

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