MA-110 Linear Algebra and Differential Equations

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Lecture 15 D3

Defn. Let V and W be vector spaces.

- A *linear transformation* from V to W is a function $T : V \to W$ such that for $x, y \in V$, scalars a and b,

$$\boxed{T(ax + by) = aT(x) + bT(y)}$$

i.e., T takes linear combinations of vectors in V to the linear combinations of their images in W.

- If T is also a bijection, we say T is a *linear isomorphism*.
- The *image* (or *range*) of T is defined to be
$$C(T) = \{y \in W \mid T(x) = y \text{ for some } x \in V\}.$$
- The *kernel* (or *null space*) of T is defined as
$$N(T) = \{x \in V \mid T(x) = 0\}.$$

Main Example: Let A be an $m \times n$ matrix. Define $T(x) = Ax$.

- This defines a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$.
- The image of T is the column space of A, i.e., $C(T) = C(A)$.
- The kernel of T is the null space of A, i.e., $N(T) = N(A)$.

Show that the following functions are linear transformations.

$T:\mathbb{R}^\infty \to \mathbb{R}^\infty$ defined by $T(x_1,x_2,\ldots)=(x_1+x_2,x_2+x_3,\ldots,)$.
Exercise: What is $N(T)$?

$S:\mathbb{R}^\infty \to \mathbb{R}^\infty$ defined by $S(x_1,x_2,\ldots)=(x_2,x_3,\ldots)$.
Exercise: Find $C(S)$, and a basis of $N(S)$.

Let $T:\mathscr{P}_2 \to \mathscr{P}_1$ be $S(a_0+a_1x+a_2x^2)=a_1+4a_2x$.

Exercise: Show that dim $(N(T))=1$, and find $C(T)$.

Let $D:\mathscr{C}^\infty([0,1]) \to \mathscr{C}^\infty([0,1])$ defined as $Df=\frac{df}{dx}$.

Exercise: Is $D^2=D\circ D$ linear? What about D^3?

Exercise: What is $N(D)$? $N(D^2)$? $N(D^k)$?

Question: Is integration linear?

Observe: Images and null spaces are subspaces!
Of which vector space?

Properties of Linear transformations

Let $\mathscr{B} = \{v_1, \ldots, v_n\} \subseteq V$, $T : V \to W$ be linear, and $T(\mathscr{B})$ $= \{T(v_1), \ldots, T(v_n)\}$. Then:

- $T(au + bv) = aT(u) + bT(v)$. In particular, $T(0) = 0$.
- $N(T)$ is a subspace of V. Why? $C(T)$ is a subspace of W. Why?
- If $\text{Span}(\mathscr{B}) = V$, is $\text{Span}\{T(\mathscr{B})\} = W$?

Note: It is $C(T)$.

Conclusion: (i) If dim $(V) = n$, then dim $(C(T)) \leq n$.
(ii) T is onto $\Leftrightarrow \text{Span}\{T(\mathscr{B}\}) = C(T) = W$.

- $T(u) = T(v) \Leftrightarrow u - v \in N(T)$.

Conclusion: T is one-one $\Leftrightarrow N(T) = 0$.• If $\mathscr{B} \subseteq V$ is linearly independent, is $\{T(\mathscr{B})\} \subseteq W$ linearly independent?

Hint: $a_1 T(v_1) + \cdots a_n T(v_n) = 0 \Rightarrow a_1 v_1 + \cdots a_n v_n \in N(T)$.

- $S : U \to V$, $T : V \to W$ are linear $\Rightarrow T \circ S : U \to W$ is linear. **Exercise:**

Show that $N(S) \subseteq N(T \circ S)$. How are $C(T \circ S)$ and $C(T)$ related?

Recall: A linear map $T : V \to W$ is an *isomorphism* if T is also a bijection.
Notation: $V \simeq W$.

Ques: If $T : V \to W$ is an isomorphism, is $T^{-1} : W \to V$ linear?

Recall: T is one-one $\Leftrightarrow N(T) = 0$ & T is onto $\Leftrightarrow C(T) = W$.

Thus $\boxed{T \text{ is an isomorphism} \Leftrightarrow N(T) = 0 \text{ and } C(T) = W.}$

Example: If V is the subspace of convergent sequences in $\mathbb{R}^\infty$, then $L : V \to \mathbb{R}$ given by $L(x_1, x_2, \ldots) = \lim_{n\to\infty}(x_n)$ is linear.
What is $N(L)$? $C(L)$? Is L one-one or onto?

Exercise: Given $A \in \mathscr{M}_{m\times n}$, let $T(x) = Ax$ for $x \in \mathbb{R}^n$.
Then T is an isomorphism $\Leftrightarrow m = n$ and A is invertible.

Exercise: In the previous examples, identify linear maps which are one-one, and those which are onto.

Example: $S\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = (a, b, c, d)^T$ is an isomorphism since $N(S) = 0$ and $C(S) = \mathbb{R}^4$. Thus $\mathscr{M}_{2\times 2} \simeq \mathbb{R}^4$. What is S^{-1}?

Question to think about

Show that to give a linear map from $T : \mathscr{M}_{2\times 2} \to \mathbb{R}^4$ it is sufficient to write down the image for $T(e_{11})$, $T(e_{12})$, $T(e_{21})$, $T(e_{22})$.

For instance create a linear transformation where $T(e_{11}) = (5,6,7,8)$, $T(e_{12}) = (1,2,3,4)$, $T(e_{21}) = (1,1,1,1)$ and $T(e_{22}) = (0,1,0,1)$

A general answer is given in the next slide.

- Consider $S : \mathscr{M}_{2\times 2} \to \mathbb{R}^4$ given by $S\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = (a, b, c, d)^T$.

Recall that $\{e_{11}, e_{12}, e_{21}, e_{22}\}$ is a basis of $\mathscr{M}_{2\times 2}$ such that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ae_{11} + be_{12} + ce_{21} + de_{22}$.
Observe that $S(e_{11}) = e_1, S(e_{12}) = e_2, S(e_{21}) = e_3,\ S(e_{22}) = e_4$.
Thus, $S(A) = aS(e_{11}) + bS(e_{12}) + cS(e_{21}) + dS(e_{22}) = ae_1 + be_2 + ce_3 + de_4 = (a, b, c, d)^T$.

General case:

If $\{v_1, \ldots, v_n\}$ is a basis of V, $T : V \to W$ is linear, $v \in V$,then $v = a_1 v_1 + \cdots a_n v_n \Rightarrow T(v) = a_1 T(v_1) + \cdots a_n T(v_n)$. Why? Thus,

$$\boxed{T \text{ is determined by its action on a basis,}}$$

i.e., for any n vectors $w_1, \ldots, w_n$ in W (not necessarily distinct), there is unique linear transformation $T : V \to W$ such that $T(v_1) = w_1, \ldots, T(v_n) = w_n$.