Linear Algebra & Differential Equations
MA110

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Appendix I - Determinants

Reading Slide - Determinants: Key Properties

Let A and B $n \times n$, and c a scalar.

- True/False: $\det(A + B) = \det(A) + \det(B)$.
- True/False: $\det(cA) = c \det(A)$.
- $\det(AB) = \det(A)\det(B)$.
- $\det(A) = \det(A^T)$.
- If A is orthogonal, i.e., $AA^T = I$, then $\det(A) =$
- If $A = [a_{ij}]$ is triangular, then $\det(A) =$ ___
- A is invertible $\Leftrightarrow \det(A) \neq 0$.
  If this happens, then $\det(A^{-1}) =$ ___
- If $B = P^{-1}AP$ for an invertible matrix P,
  i.e., A and B are similar, then $\det(B) =$ ___
- If A is invertible, and $d_1, \ldots, d_n$ are the pivots of A,
  then $\det(A) =$ ___.

Reading Slide - Determinants: Defining Properties

Defn. The determinant function $\det : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ can be defined (uniquely) by its three basic properties.

• $\det(I) = 1$.

• The sign of determinant is reversed by a row exchange. Thus, if $B = P_{ij}A$, i.e., B is obtained from A by exchanging two rows, then $\det(B) = -\det(A)$. In particular, $\det(I) = 1 \Rightarrow \det(P_{ij}) = -1$.

• det is linear in each row separately, i.e. , we fix $n-1$ row vectors, say $v_2, \cdots, v_n$, then $\det \begin{pmatrix} - & v_2 & \cdots & v_n \end{pmatrix}^T : \mathbb{R}^n \to \mathbb{R}$ is a linear function.

I.e., for c, d in $\mathbb{R}$, and vectors u and v, if $A_{1*} = cu + dv$, we have $\det \begin{pmatrix} cu+dv & A_{2*} & \cdots & A_{n*} \end{pmatrix}^T$
$= c \det \begin{pmatrix} u & A_{2*} & \cdots & A_{n*} \end{pmatrix}^T + d \det \begin{pmatrix} v & A_{2*} & \cdots & A_{n*} \end{pmatrix}^T$.

There are n such equations (for n choices of rows).

Reading Slide - Determinants: Induced Properties

1. If two rows of A are equal, then $\det(A) = 0$.
   Proof. Suppose i-th and j-th rows of A are equal, i.e., $A_{i*} = A_{j*}$, then $A = P_{ij}A$.

   Hence $\det(A) = \det(P_{ij}A) = -\det(A) \Rightarrow \boxed{\det(A) = 0}$.

2. If B is obtained from A by $\boxed{R_i \mapsto R_i + aR_j,}$ then $\det(B) = \det(A)$.

3. If A is $n \times n$, and its row echelon form U is obtained without row exchanges, then $\det(U) = \det(A)$.

   Q: What happens if there are row exchanges? Exercise!

4. If A has a zero row, then $\det(A) = 0$.

   Proof: Let the ith row of A be zero, i.e., $A_{i*} = 0$.

   Let B be obtained from A by $\boxed{R_i = R_i + R_j}$, i.e., $B = E_{ij}(1)A$. Then $B_{i*} = B_{j*}$.

   Exercise: Complete the proof.

Reading Slide - Determinants: Special Matrices

5. If $A = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$ is diagonal, then det$(A) = a_1 \cdots a_n$.
   (Use linearity).

6. If $A = (a_{ij})$ is triangular, then det$(A) = a_{11} \ldots a_{nn}$.

   Proof. If all a_{ii} are non-zero, then by elementary row operations, A reduces to the diagonal matrix
   $\begin{pmatrix} a_{11} & & \\ & \ddots & \\ & & a_{nn} \end{pmatrix}$ whose determinant is $a_{11} \cdots a_{nn}$.

   If at least one diagonal entry is zero, then elimination will produce a zero row $\Rightarrow$ det$(A) = 0$.

Reading Slide - Formula for Determinant: 2×2 case

Write $(a, b) = (a, 0) + (0, b)$, the sum of vectors in coordinate directions. Similarly write $(c, d) = (c, 0) + (0, d)$. By linearity,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}.$$

For an $n \times n$ matrix, each row splits into n coordinate directions, so the expansion of det(A) has n^n terms.
However, when two rows are in same coordinate direction, that term will be zero, e.g.,

$$\begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} = 0, \quad \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix} = 0, \quad \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} = -\begin{vmatrix} c & 0 \\ 0 & b \end{vmatrix} = -bc \Rightarrow \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The non-zero terms have to come in different columns. So, there will be $n!$ such terms in the $n \times n$ case.

Reading Slide - Formula for Determinant: $n \times n$ case

For $n \times n$ matrix $A = (a_{ij})$,

$$\det(A) = \sum_{\text{all permutations } P} (a_{1\alpha_1} \ \ldots \ a_{n\alpha_n}) \det(P).$$

The sum is over $n!$ permutations of numbers $(1, \ldots, n)$. Here a permutation $(i_1, i_2, \ldots, i_n)$ of $(1, 2, \ldots, n)$ corresponds to the product of permutation matrices $P = \begin{bmatrix} e_{i_1}^T \\ \vdots \\ e_{i_n}^T \end{bmatrix}$. Then $\det(P) = +1$ if the number of row exchanges in P needed to get I is even, and -1 if it is odd.

Reading Slide - Cofactors: 3×3 Case

$$
\begin{aligned}
\text{det}(A) &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\
&= a_{11}\,a_{22}\,a_{33}\,(1) + a_{11}\,a_{23}\,a_{32}\,(-1) + a_{12}\,a_{21}\,a_{33}\,(-1) \\
&\quad + a_{12}\,a_{23}\,a_{31}\,(1) + a_{13}\,a_{21}\,a_{32}\,(1) + a_{13}\,a_{22}\,a_{31}\,(-1) \\
&= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) \\
&\quad + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\
&= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\
&= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \text{ where,}
\end{aligned}
$$

$$
C_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix},\; C_{12} = (-1)\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix},\; C_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.
$$

Reading Slide - Cofactors: $n \times n$ Case

Let C_{1j} be the coefficient of a_{1j} in the expansion

$$\text{det}(A) = \sum_{\text{all permutations } P} (a_{1\alpha_1} \dots a_{n\alpha_n}) \, \text{det}(P)$$

Then $\boxed{\text{det}(A) = a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{1n}}$ where,

$$\begin{aligned} C_{1j} &= \Sigma a_{2\beta_2} \dots a_{n\beta_n} \text{det}(P) \\ &= (-1)^{1+j} \text{det} \begin{bmatrix} a_{21} & . & a_{2(j-1)} & a_{2(j+1)} & . & a_{2n} \\ \vdots & . & \vdots & . & \vdots & \\ a_{n1} & . & a_{n(j-1)} & a_{j+1} & . & a_{nn} \end{bmatrix} \\ &= (-1)^{1+j} \text{det}(M_{1j}), \end{aligned}$$

where M_{1j} is obtained from A by deleting the 1^{st} row and j^{th} column.

Extra Reading Slides - Determinants

The following set of slides contain some extra reading material on determinants for interested students of MA110 (Spring 2024).

$\det(AB) = \det(A)\det(B)$ (Proof)

7.

$$\boxed{\det(AB) = \det(A)\det(B)}$$

Proof. We may assume that B are invertible. Else, $\mathrm{rank}(AB) \le \mathrm{rank}B \neq n \Rightarrow \mathrm{rank}(AB) \neq n \Rightarrow AB$ is not invertible.

$\boxed{\text{Hint:}}$ For fixed B, show that the function d defined by

$$d(A) = \det(AB)/\det(B)$$

satisfies the following properties

0.1 $d(I) = 1$.

0.2 If we interchange two rows of A, then d changes its sign.

0.3 d is a linear function in each row of A.

Then d is the unique determinant function det and $\det(AB) = \det(A)\det(B)$. □

Determinants of Transposes (Proof)

8.

$$\boxed{\det(A) = \det(A^T)}$$

Proof. With U, L, and P, as usual write $PA = LU$
$\Rightarrow A^T P^T = U^T L^T$ Since U and L are triangular, we get

$$\det(U) = \det(U^T) \quad \text{and} \quad \det(L) = \det(L^T).$$

Since $PP^T = I$ and $\det(P) = \pm 1$, we get
$\det(P) = \det(P^T)$.

Thus $\det(A) = \det(A^T)$. $\square$

Determinants and Invertibility (Proof)

9. A is invertible if and only if $\det(A) \neq 0$.
   By elimination, we get an upper triangular matrix U, a lower triangular matrix L with diagonal entries 1, and a permutation matrix P, such that $PA = LU$.

   Observation 1: If A is singular, then $\det(A) = 0$.
   This is because elimination produces a zero row in U and hence $\det(A) = \pm \det(U) = 0$.

   Observation 2: If A is invertible, then $\det(A) \neq 0$.
   This is because elimination produces n pivots, say $d_1, \ldots, d_n$, which are non-zero. Then U is upper triangular, with diagonal entries $d_1, \ldots, d_n \Rightarrow$
   $\det(A) = \pm \det(U) = \pm d_1 \cdots d_n \neq 0$.
   Thus we have: A invertible $\Rightarrow \det(A) =$ $\pm$(product of pivots).

   Exercise: If AB is invertible, then so are A and B.
   Exercise: A is invertible if and only if A^T is invertible.

Determinant: Geometric Interpretation (2×2)

INVERTIBILITY: Very often we are interested in knowing when a matrix is invertible. Consider $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then A is invertible if and only if A has full rank.

If a, c both are zero then clearly rank$(A) < 2 \Rightarrow A$ is not invertible. Assume $a \neq 0$, else, interchange rows. The row operations $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{R_2 - c/aR_1} \begin{bmatrix} a & b \\ 0 & d - cb/a \end{bmatrix}$ show that A is invertible if and only if $d - cb/a \neq 0$, i.e., $ad - bc \neq 0$.

AREA: The area of the parallelogram with sides as vectors $v = (a, b)$ and $w = (c, d)$ is equal to $ad - bc$. Thus,

A 2×2 matrix A is singular $\Leftrightarrow$

its columns are on the same line

$\Leftrightarrow$ the area is zero.

Determinant: Geometric Interpretation

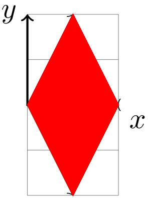
- Test for invertibility: An $n \times n$ matrix A is invertible $\Leftrightarrow$ $\det(A) \neq 0$.

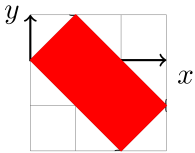
- n-dimensional volume: If A is $n \times n$, then $|\det(A)| =$ the volume of the box (in n-dimensional space $\mathbb{R}^n$) with edges as rows of A.

Examples: (1) The volume (area) of a line in $\mathbb{R}^2 = 0$.

(2) The determinant of the matrix $A = \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}$ is $\boxed{-4}$.

(3) Let's compute the volume of the box (parallelogram) with edges as rows of A or columns of A.





$\boxed{=4}$

Expansion along the i-th row (Proof)

If C_{ij} is the coefficient of a_{ij} in the formula of $\det(A)$, then $\boxed{\det(A) \ = \ a_{i1}\,C_{i1} + \ldots + a_{in}\,C_{in}}$, where C_{ij} is determined as follows:

By $i-1$ row exchanges on A, get the matrix
$B = \begin{pmatrix} A_{i*} & A_{1*} & .. & A_{(i-1)*} & A_{(i+1)*} & .. & A_{n*} \end{pmatrix}^T$
Since $\det(A) = (-1)^{i-1}\det(B)$, we get

$$C_{ij}(A) = (-1)^{i-1}C_{1j}(B) = (-1)^{i-1}(-1)^{j-1}\det(M)$$

where M is obtained from B by deleting 1^{st} row and j^{th} column. Here M is obtained from B by deleting its first row, and j-th column, and hence from A by deleting i-th row and j-th column. Write M as M_{ij}. Then $\boxed{C_{ij} = (-1)^{i+j}\det(M_{ij})}$

Expansion along the j-th column (Proof)

Note that $$C_{ij}(A^T) = C_{ji}(A).$$

Hence, if we write $A^T = (b_{ij})$, then

$$\begin{aligned}
\text{det}(A) &= \text{det}(A^T) \\
&= b_{j1}\, C_{j1}(A^T) + \ldots + b_{jn}\, C_{jn}(A^T) \\
&= a_{1j}\, C_{1j}(A) + \ldots + a_{nj}\, C_{nj}(A)
\end{aligned}$$

This is the expansion of det(A) along j-th column of A.

Applications: 1. Computing A^{-1}

If $C = (C_{ij})$: cofactor matrix of A, then $\boxed{A\,C^T = \det(A)\,I}$

i.e.,

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & \dots & C_{n1} \\ \vdots & \ddots & \vdots \\ C_{1n} & \dots & C_{nn} \end{bmatrix} = \begin{bmatrix} \det(A) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \det(A) \end{bmatrix}$$

Proof. We have seen that $a_{i1}C_{i1} + \ldots + a_{in}C_{in} = \det(A)$. Now

$$a_{11}C_{21} + a_{12}C_{22} + \ldots + a_{1n}C_{2n} = \det \begin{bmatrix} a_{11} & \dots & a_{1n} \\ a_{11} & \dots & a_{1n} \\ a_{31} & \dots & a_{3n} \\ . & & . \\ a_{n1} & \dots & a_{nn} \end{bmatrix} = 0.$$

Similarly, if $i \neq j$, then $a_{i1}C_{j1} + a_{i2}C_{j2} + \ldots + a_{in}C_{jn} = 0$. $\square$

Remark. If A is invertible, then $A^{-1} = \dfrac{1}{\det(A)} C^T$.

For $n \geq 4$, this is *not* a good formula to find A^{-1}.
Use elimination to find A^{-1} for $n \geq 4$.
This formula is of theoretical importance.

Applications: 2. Solving $Ax = b$

Cramer's rule: If A is invertible, the $Ax = b$ has a unique solution.

$$x = A^{-1}b = \frac{1}{\det(A)}\, C^T b = \frac{1}{\det(A)} \begin{bmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & & \vdots \\ C_{1n} & \cdots & C_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

Hence

$$x_j = \frac{1}{\det(A)}(b_1 C_{1j} + b_2 C_{2j} + \cdots + b_n C_{nj}) = \frac{1}{\det(A)} \det(B_j),$$

where B_j is obtained by replacing j^{th} column of A by b, and $\det(B_j)$ is computed along the j^{th} column.

Remark: For $n \geq 4$, use elimination to solve $Ax = b$. Cramer's rule is of theoretical importance.

Applications: 3. Volume of a box

$\boxed{\text{Assume the rows of } A \text{ are mutually orthogonal}}$. Then

$$AA^T = \begin{pmatrix} A_{1*} \\ \vdots \\ A_{n*} \end{pmatrix} \begin{pmatrix} (A_{1*})^T & \dots & (A_{n*})^T \end{pmatrix} = \begin{pmatrix} l_1^2 & & 0 \\ & \ddots & \\ 0 & & l_n^2 \end{pmatrix}$$

where $l_i = \sqrt{(A^i)^T \cdot A^i}$ is the length of A^i. Since
$\det(A) = \det(A^T)$,

we get $\boxed{|\det(A)| = l_1 \cdots l_n}$.

Since the edges of the box spanned by rows of A are at right angles, the volume of the box

$$= \text{ the product of lengths of edges}$$

$$= |\det(A)|.$$

Applications: 4. A Formula for Pivots

Observation: If row exchanges are not required, then the first k pivots are determined by the top-left $k \times k$ submatrices $\widetilde{A}_k$ of A.

Example. If $A = [a_{ij}]_{3\times 3}$, then $\widetilde{A}_1 = (a_{11})$, $\widetilde{A}_2 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $\widetilde{A}_3 = A$.

Assume the pivots are $d_1, \ldots, d_n$, obtained without row exchange. Then

- $\det(\widetilde{A}_1) = a_{11} = d_1$
- $\det(\widetilde{A}_2) = d_1\, d_2 = \det(A_1)\, d_2$
- $\det(\widetilde{A}_3) = d_1\, d_2\, d_3 = \det(A_2)\, d_3 \;\; etc.,$
- If $\det(\widetilde{A}_k) = 0$, then we need a row exchange in elimination.
- Otherwise the k-th pivot is $\boxed{d_k = \det(\widetilde{A}_k)/\det(\widetilde{A}_{k-1})}$