MA 110 - Ordinary Differential Equations

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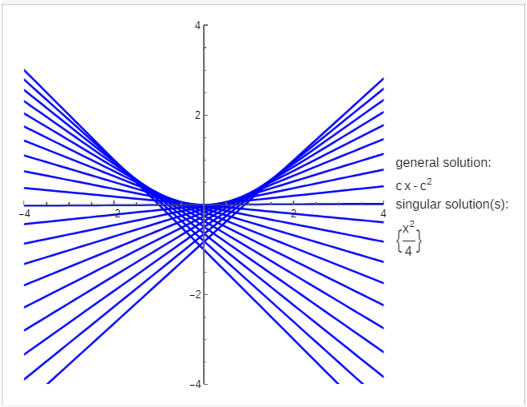
Outline of the lecture

- Equations reducible to separable form
- Exact equations

1. Solve $(4x+2y+5)y' + (2x+y-1) = 0$.
   Hint :
   Substitute $v = 2x + y$. Reduces to separable form.
2. Solve $y' = \dfrac{x+y-3}{x-y-1}$.
   Hint :
   - Substitute $x = x_1 + h,\ y = y_1 + k$ for some $h,\ k$ which will be determined.
   - $\dfrac{dy_1}{dx_1} = \dfrac{x_1 + y_1 + h + k - 3}{x_1 - y_1 + h - k - 1}$.
   - Choose $h,\ k$ such that $h + k - 3 = 0,\ h - k - 1 = 0$. This choice makes the equation homogeneous.
   - Formal Solution : $e^{\tan^{-1}(\frac{y-1}{x-2})} = C\sqrt{(x-2)^2 + (y-1)^2}$.

1. The DE $e^x y' + 3y = x^2 y$ is linear & separable. TRUE OR FALSE?
2. The DE $yy' + 3x = 0$ is linear & separable. TRUE OR FALSE?
3. Is the DE $\dfrac{dx}{dt} = \dfrac{x + 2xt + \cos t}{1 + t^2}$ linear/non-linear & separable/ not separable?
4. For the linear differential equation $\dfrac{dy}{dx} + \dfrac{x}{1+x} y = 1 + x$, the integrating factor is ——?
   (Integrating factor $= e^{\int P(x)dx}$ for $y' + P(x)y = Q(x)$.)
5. $y = cx - c^2$ is a general solution of $y'^2 - xy' + y = 0$. But $y = x^2/4$ is a singular solution of the ODE because it cannot be obtained from the general solution.

Solutions of $y'^2 - xy' + y = 0$



Exact ODE's

Definition

A first order ODE

$$M(x, y) + N(x, y)y' = 0$$

is called exact, if there is a function $u(x, y)$ such that

$$\frac{\partial u}{\partial x} = M \ \& \ \frac{\partial u}{\partial y} = N.$$

Example : Is

$$(2x + y^2) + 2xy\frac{dy}{dx} = 0$$

exact? Consider the function $u(x, y) = x^2 + xy^2$.

Exact ODE's

$\boxed{\text{Recall from calculus}}$ Given a function $u(x, y)$ with continuous first partial derivatives, its differential is

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy.$$

If the ODE $M(x, y) + N(x, y)y' = 0$ is exact, then there exist such $u(x, y)$ with $\frac{\partial u}{\partial x} = M$ & $\frac{\partial u}{\partial y} = N$, and hence

$$0 = M(x, y)dx + N(x, y)dy = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = du.$$

Integrating $du = 0$, we get $\boxed{u(x, y) = c}$ as an implicit/formal solution to the given ODE.

Example : by inspection

Solve the DE:

$$(2x + y^2) + 2xy\frac{dy}{dx} = 0.$$

Consider the function $u(x, y) = x^2 + xy^2$. Note that

$$\frac{\partial u}{\partial x} = 2x + y^2, \; \frac{\partial u}{\partial y} = 2xy.$$

Hence $x^2 + xy^2 = c$ is the solution of the given ODE.

Working Rule

Given an exact ODE $M(x,y) + N(x,y)y' = 0$, the function $u(x,y)$ can be found either by inspection or by the following method:

1. Integrate $\dfrac{\partial u}{\partial x} = M(x,y)$ with respect to x to obtain
$$u(x,y) = \int M(x,y)dx + k(y),$$
where $k(y)$ is a constant of integration. (y is treated as a constant during integration).
2. To determine $k(y)$, differentiate the above equation with respect to y, to obtain
$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\int M(x,y)dx\right) + k'(y).$$
3. As the given ODE is exact, we get
$$N(x,y) = k'(y) + \frac{\partial}{\partial y}\left(\int M(x,y)dx\right).$$
We use this to determine $k(y)$ and hence u.

Test for exactness

Theorem

Let M, N and their first order partial derivatives exist and be continuous in a region $D \subseteq \mathbb{R}^2$. We have:

1. *If $M(x,y)dx + N(x,y)dy = 0$ is an exact differential equation, then $M_y = N_x$.*
2. *If D is convex, then*
   $M_y = N_x \implies M(x,y)dx + N(x,y)dy = 0$ *is exact.*

Proof: Let the ODE be exact. So there is a u such that $M = \dfrac{\partial u}{\partial x}$ and $N = \dfrac{\partial u}{\partial y}$. Then,

$$M_y = \frac{\partial^2 u}{\partial y \partial x} \;\&\; N_x = \frac{\partial^2 u}{\partial x \partial y}.$$

By the theorem on mixed partials, $M_y = N_x$.

Conversely, let D be convex, and $M_y = N_x$. Consider the vector field

$$H(x, y) = (M(x, y), N(x, y)).$$

By our assumptions, H is continuously differentiable throughout D. The curl of H is given by

$$\nabla \times H = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix} = (N_x - M_y)\mathrm{k} = 0.$$

As D is convex, "curl free is grad"; i.e., there is a function $\phi(x, y)$ such that

$$H = \nabla\phi = (\phi_x, \phi_y).$$

Hence $\phi_x = M$, $\phi_y = N$ and thus $Mdx + Ndy = 0$ is exact.

Example

Solve the DE:

$$(y\cos x + 2xe^y) + (\sin x + x^2 e^y - 1)y' = 0.$$

Let $M = y\cos x + 2xe^y$ and $N = \sin x + x^2 e^y - 1$.

Do we have an exact DE?

How to find $u(x, y)$ such that $u_x = M$ and $u_y = N$?

Example contd...

1. $$u(x, y) = \int (y\cos x + 2xe^y)dx + k(y) = y\sin x + x^2e^y + k(y).$$
2. $$u_y = \sin x + x^2e^y + k'(y) = \sin x + x^2e^y - 1.$$
3. Thus, $k'(y) = -1$.
4. Choosing $k(y) = -y$, we obain :

$$u(x, y) = y\sin x + x^2e^y - y = c$$

as an implicit solution (Why implicit?) to the given DE.

Remarks

1. Given $u(x, y) = c$, this will define a unique differentiable function ϕ in a neighbourhood of and passing through (x_0, y_0), if

$$u(x_0, y_0) = c, \ \frac{\partial u}{\partial y}(x_0, y_0) \neq 0.$$

2. The method fails if attempt to solve non-exact equations. Consider $(3x + y^2) + (x^2 + xy)y' = 0$. Is the equation exact ? Does the method work?

Can we use integrating factors!?