MA-110 Linear Algebra

Rekha Santhanam

Department of Mathematics
Indian Institute of Technology Bombay
Powai, Mumbai - 76

January 4, 2024
Lecture 1 D3

Some Class Policies

Moodle:

- We will use Moodle to communicate with you.
  Please check the course page frequently.
- Lecture slides and tutorial problems will be posted here.
- A file with more detailed information will be posted soon.

Evaluation: 100 marks are waiting to be earned:

Quizzes (2)	10 marks each
Midsem	40 marks
Final Exam	40 marks
Total	100 marks

Academic Honesty: Be honest.
Do not to violate the academic integrity of the Institute.
Any form of academic dishonesty will invite severe penalties.

What is Linear Algebra?

`It is the theory of solving simultaneous linear equations in a finite number of unknowns.`

$\boxed{\text{Example :}}$ Let M be a new delivery app we brought into the market.

If we charge a flat rate of Rs 30 per delivery we get 200 customers. When priced at Rs 20 per delivery we get 450.

Assuming d, the number of deliveries and c, cost per delivery relate linearly, we have,

$$d = -25c + 950.$$

Clearly $(d, c) = (950, 0)$ is a solution. Is $(d, c) = (950, 0)$ the only solution of

$$d = -25c + 950?$$

What is Linear Algebra?

Is $(d, c) = (950, 0)$ the only solution of

$$d = -25c + 950?$$

This equation has several solutions; $(d, c) = (50, -300)$, $(10, 700)$, $(0.2, 945)$, $(-100, 13450)$, etc.

Are all these solutions **permissible?**

Definitely not $(50, -300)$, $(0.2, 945)$ or $(-100, 13450)$. Further assume delivery costs force the following linear relation on the number of deliveries

$$\text{Then, } d = 10c + 250.$$

Solve $d = 10c + 250$, $d = -25c + 950$ simultaneously to get $(20, 450)$.

Key note: In general, we want all possible solutions to the given system, i.e., without any constraints, unlike the introductory example.

Solving equations, Example

Solve the system: (1) $2x + y = 5$, (2) $x + 2y = 4$.

Elimination of variables: Eliminate x by $(2) - 1/2 \times (1)$ to get $y = 1$, or

Cramer's Rule (determinant): $y = \dfrac{\begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}} = \dfrac{8-5}{4-1} = 1$

In either case, back substitution gives $x = 2$

We could also solve for x first and use back substitution for y.
Why ?

Key Note: For a large system, say 100 equations in 100 variables, elimination method is preferred, since computing 101 determinants of size 100×100 is time-consuming.

Row method:

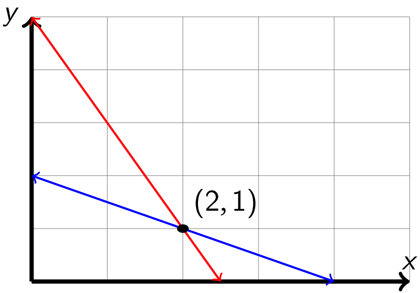
$\boxed{2x+y=5}$

and

$\boxed{x+2y=4}$

represent lines in $\mathbb{R}^2$ passing through $\boxed{(0,5)}$ and $\boxed{(5/2,0)}$ and through $\boxed{(0,2)}$ and $\boxed{(4,0)}$ respectively.

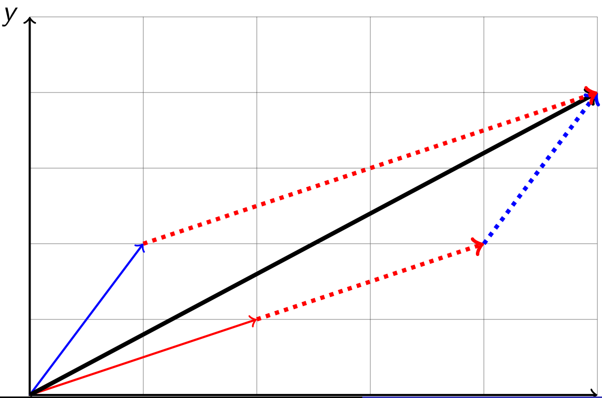
The intersection of the two lines is the unique point $(2,1)$. Hence $x=2$ and $y=1$ is the solution of above system of linear equations.



Column method: The system is $$x\begin{pmatrix}2\\1\end{pmatrix}+y\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}5\\4\end{pmatrix}.$$

We need to find a *linear combination* of the column vectors on LHS to produce the column vector on RHS.

Geometrically this is same as completing the parallelogram with given directions and diagonal.



Row method

A linear equation in 3 variables represents a plane in a 3 dimensional space $\mathbb{R}^3$.

Example: (1)

$$\boxed{x+2y+3z=6}$$

represents a plane passing through: $(0,0,2)$, $(0,3,0)$, $(6,0,0)$.

Example: (2)

$$\boxed{x+2y+3z=0}$$

represents a plane passing through: $(-2,1,0)$, $(-1,-1,1)$, $(2,-1,0)$.

In Example (2) we are looking for (x,y,z) such that $(x,y,z)\cdot(1,2,3)=0$, i.e., plane (2) is the set of all vectors perpendicular to the vector (1,2,3).

Example 1: (1) $x + 2y + 3z = 6$ (2) $x + 2y + 3z = 0$.

The two equations represent planes with normal vector (1,2,3) and are parallel to each other. **Exercise :** Prove this.

How many solutions can we find? There are *no solutions*.

Example 2: (1) $x + 2y + 3z = 0$ (2) $-x + 2y + z = 0$

The two equations represent planes passing through (0,0,0).

The intersection is non-empty, i.e., the system has at least one solution.

In fact, the *solution set* is a line passing through the origin.

Exercise: Find all the solutions in the second example.

3 equations in 3 variables

- Solving 3 by 3 system by the **row method** means finding an intersection of three planes, say P_1, P_2, P_3.

  This is same as the intersection of a line L (intersection of P_1 and P_2, if they are non-parallel) with the plane P_3.

- If the line L does not intersect the plane P_3, then the linear system has no solution, i.e., the system is *inconsistent*. Same is true if P_1 and P_2 were parallel.

- If the line L is contained in the plane P_3, then the system has infinitely many solutions.

  In this case, every point of L is a solution.

- **Exercise:** Workout some examples.

Linear Combinations

Column method:

Consider the 3×3 system:
x+2y+3z=2, -2x+3y=-5, -x+5y+2z=-4. Equivalently,

$$x\begin{pmatrix}1\\-2\\-1\end{pmatrix}+y\begin{pmatrix}2\\3\\5\end{pmatrix}+z\begin{pmatrix}3\\0\\2\end{pmatrix}=\begin{pmatrix}2\\-5\\-4\end{pmatrix}$$

We want a *linear combination* of the column vectors on LHS which is equal to RHS.

Observe: • $x = 1, y = -1, z = 1$ is a solution. **Q:** Is it unique?

• Since each column represents a vector in $\mathbb{R}^3$ from origin, we can find the solution geometrically, as in the 2×2 case.

Q: Can we do the same when number of variables are > 3?

Use other solving techniques to answer such questions.

Gaussian Elimination

Example: $\mathbf{2}u+v+w=5, \quad 4u-6v=-2, \quad -2u+7v+2w=9.$

Algorithm: Eliminate u from last 2 equations by $(2)-\frac{4}{\mathbf{2}}\times(1)$, and $(3)-\frac{-2}{\mathbf{2}}\times(1)$ to get the *equivalent system:*

$$\mathbf{2}u+v+w=5, \qquad \text{-}\mathbf{8}v-2w=-12, \qquad \mathbf{8}v+3w=14$$

The coefficient used for eliminating a variable is called a *pivot*. The first pivot is 2. The second pivot is -8. The third pivot is 1. Eliminate v from the last equation to get an equivalent *triangular system*:

$$\mathbf{2}u+v+w=5, \qquad \text{-}\mathbf{8}v-2w=-12, \qquad \mathbf{1}\cdot w=2$$

Solve this triangular system by *back substitution*, to get the *unique solution*
$w=2, \quad v=1, \quad u=1.$

Matrix notation ($A\vec{x} = \vec{b}$) for linear systems

Consider the system

$2u + v + w = 5, \quad 4u - 6v = -2, \quad -2u + 7v + 2w = 9.$

Let $\vec{x} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$ be the unknown vector, and $\vec{b} = \begin{pmatrix} 5 \\ -2 \\ 9 \end{pmatrix}$.

The coefficient matrix is $A = \begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix}$.

If we have m equations in n variables, then A has m rows and n columns, the column vector $\vec{b}$ has size m, and the unknown vector $\vec{x}$ has size n.

Notation: From now on, we will write $\vec{x}$ as x and $\vec{b}$ as b.

Some things to think about

- What are all the ways two different lines can intersect?
  What are all possible ways three different lines can intersect?
- What are all the ways two different planes can intersect?
  What are all possible ways three different plane can intersect?
- What is (if any) the geometric significance of the equation $x + y + z + w = 0$?
- Does the elimination method change the system of equations?
- Why does the solution set remain same all through the elimination method?