MA 110 - Ordinary Differential Equations

Santanu Dey

Department of Mathematics,
Indian Institute of Technology Bombay,
Powai, Mumbai 76
santanudey@iitb.ac.in

March 28, 2024

Outline of the lecture

- Method of undetermined coefficients for 2^{nd} order ODE
- n^{th} order

Example 6

Find a particular solution of

$$y'' + 4y = 3\cos 2t.$$

Since $r(t) = 3\cos 2t$, you would look for solutions of the form

$$y(t) = a\cos 2t + b\sin 2t.$$

Thus,

$$y'(t) = -2a\sin 2t + 2b\cos 2t,$$
$$y''(t) = -4a\cos 2t - 4b\sin 2t.$$

Substituting in the given DE, we get:

$$(-4a\cos 2t - 4b\sin 2t) + 4(a\cos 2t + b\sin 2t) = 3\cos 2t.$$

But the lhs is 0! So can't solve for a and b.

Method of Undetermined Coefficients

Why this ...? The problem was that $\sin 2t$ and $\cos 2t$ are also solutions of the associated homogeneous ODE: $y'' + 4y = 0$. When we search for solutions of a particular form, we need to make sure that it's not a solution of the associated homogeneous equation.

We now modify the proposed solution as:

$$y(t) = at\cos 2t + bt\sin 2t.$$

Then,

$$y'(t) = (b - 2at)\sin 2t + (a + 2bt)\cos 2t,$$

$$y''(t) = -4at\cos 2t - 4bt\sin 2t - 4a\sin 2t + 4b\cos 2t.$$

Substituting, we get:

$$-4a\sin 2t + 4b\cos 2t = 3\cos 2t.$$

Thus, $a = 0$, $b = \dfrac{3}{4}$, and a particular solution is $y(t) = \dfrac{3}{4}t\sin 2t$.

If

$$r(x) = r_1(x) + r_2(x) + \ldots + r_n(x),$$

where $r_i(x)$ are e^{ax} or $\sin ax$ or $\cos ax$ or polynomials in x, consider the n subproblems

$$y'' + py' + qy = r_i(x).$$

If $y_i(x)$ is a particular solution of this problem, then,

$$y_p(x) = y_1(x) + y_2(x) + \ldots + y_n(x)$$

is a particular solution of

$$y'' + py' + qy = r(x).$$

Example 7

Find a particular solution of

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t + 4t^2 - 1 - 8e^t\cos 2t.$$

Here,

$$r(t) = r_1(t) + r_2(t) + r_3(t) + r_4(t).$$

We need to solve

$$y'' - 3y' - 4y = r_i(t),$$

get a particular solution $y_i(t)$, and then

$$y(t) = y_1(t) + y_2(t) + y_3(t) + y_4(t)$$

is a particular solution of the given problem. Thus, a particular solution is

$$y(t) = -\frac{1}{2}e^{2t} - \frac{5}{17}\sin t + \frac{3}{17}\cos t - t^2 + \frac{3}{2}t - \frac{11}{8} + \frac{10}{13}e^t\cos 2t + \frac{2}{13}e^t\sin 2t.$$

n^{th} ORDER DE

Consider an n-th order linear ODE :

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_n(x)y = g(x).$$

Assume that the functions $a_0(x), a_1(x), \ldots, a_n(x), g(x)$ are continuous on an interval I. Also assume that $a_0(x) \neq 0$ for every $x \in I$.

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = r(x).$$

is called a n-th order linear ODE in standard form.
If $r(x) \equiv 0$ that is,

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = 0$$

then the ODE is said to be homogeneous. Otherwise it is called non-homogeneous.

Initial Value Problem- Existence/Uniqueness

An IVP for n^{th} order will be of the form

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = 0$$
$$y(x_0) = k_0, y^{(1)}(x_0) = k_1, \ldots, y^{(n-1)}(x_0) = k_{n-1}$$

with $x_0 \in I$.

Existence - Uniqueness theorem : If $p_i(x)$ are continuous throughout an interval I containing x_0, then the IVP has a unique solution on I.

Note that both existence and uniqueness are guaranteed on the same I where continuity of the coefficients is given.

Wronskian

The Wronskian of n differentiable functions $y_1(x), y_2(x), \cdots, y_n(x)$ is defined by

$$W(y_1, \cdots, y_n) := \begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{vmatrix}.$$

Suppose that

$$\boxed{y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = 0}$$

has continuous coefficients on an open interval I. Then n solutions $y_1, y_2, \cdots, y_n$ of the DE on I are linearly dependent iff their Wronskian is 0 at some $x_0 \in I$.

Proof for n th order - $\Longrightarrow$

Let $y_1, \cdots, y_n$, be linearly dependent in I. That is, $\exists$ non-trivial $k_1, \cdots, k_n$ such that

$$\begin{aligned}
&k_1 y_1(x) + \cdots + k_n y_n(x) = 0\\
&k_1 y_1'(x) + \cdots + k_n y_n'(x) = 0\\
&\vdots\\
&k_1 y_1^{(n-1)}(x) + \cdots + k_n y_n^{(n-1)}(x) = 0
\end{aligned}$$

For $x_0 \in I$, in particular,

$$\begin{pmatrix} y_1(x_0) & y_2(x_0) & \cdots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \cdots & y_n'(x_0) \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

has a non-trivial solution $\Longrightarrow W(y_1, \cdots, y_n)(x_0) = 0$.

Conversely, let $W(y_1, \cdots, y_n)(x_0) = 0$ for some $x_0 \in I$.
Consider the linear system of equations :

$$\begin{aligned}
&k_1 y_1(x_0) + \cdots + k_n y_n(x_0) = 0\\
&k_1 y_1'(x_0) + \cdots + k_n y_n'(x_0) = 0\\
&\vdots\\
&k_1 y_1^{(n-1)}(x_0) + \cdots + k_n y_n^{(n-1)}(x_0) = 0
\end{aligned}$$

$W(y_1, \cdots, y_n)(x_0) = 0 \Longrightarrow \exists$ non-trivial $k_1, \cdots, k_n$ solving the above linear system.

Let

$$y(x) = k_1 y_1(x) + k_2 y_2(x) + \cdots + k_n y_n(x).$$

Now, $y(x_0) = y'(x_0) = \cdots = y^{(n-1)}(x_0) = 0$.
By existence-uniqueness theorem, $y(x) \equiv 0$ is the unique solution of the IVP

$$y^{(n)} + p_1(x) y^{(n-1)} + \ldots + p_n(x) y = 0, y(x_0) = \cdots = y^{n-1}(x_0) = 0$$

$$\implies k_1 y_1(x) + k_2 y_2(x) + \cdots + k_n y_n(x) = 0$$

with $k_1, k_2, \cdots, k_n$ not all identically zero.
Hence, $y_1, y_2, \cdots, y_n$ are l.d.

Abel's formula

Theorem

If $y_1, y_2, \ldots, y_n$ are solutions of

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = 0,$$

then,

$$W(y_1, \ldots, y_n)(x) = W(y_1, \ldots, y_n)(x_0)e^{-\int_{x_0}^{x} p_1(t)dt}.$$

Proof

Proceeding as in the proof for second order case, we need to show

$$W' = -p_1(x)W.$$

Notice that the derivative of

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

is

$$\begin{vmatrix} y_1' & y_2' \\ y_1' & y_2' \end{vmatrix} + \begin{vmatrix} y_1 & y_2 \\ y_1'' & y_2'' \end{vmatrix} = \begin{vmatrix} y_1 & y_2 \\ y_1'' & y_2'' \end{vmatrix}.$$

In this, substituting $y_i'' = -p_1(x)y_i' - p_2(x)y_i$, we get

$$\begin{vmatrix} y_1 & y_2 \\ -p_1(x)y_1' & -p_1(x)y_2' \end{vmatrix} + \begin{vmatrix} y_1 & y_2 \\ p_2(x)y_1 & p_2(x)y_2 \end{vmatrix}.$$

The second determinant is zero and the claim is thus proved.

Proof

The derivative of $W(y_1, y_2, \ldots, y_n)$ is the sum of n determinants with derivative being taken in the first row in the first one, in the second row in the second one, etc. Except the last one, all vanish because two rows are identical. In the last one, make the substitution

$$y_i^{(n)} = -p_1(x)y_i^{(n-1)} - p_2(x)y_i^{(n-2)} - \ldots - p_n(x)y_i.$$

Expand this into sum of n determinants. Once again, all but one vanish. The non-vanishing one gives

$$-p_1(x) \cdot W(y_1, y_2, \ldots, y_n).$$

Thus, for solutions of linear homogeneous DE's, the Wronskian is either never zero or identically zero on I!

Basis of solutions & General solution (n^{th} order)

If $p_1(x), \ldots, p_n(x)$ are continuous on an open interval I, then

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = 0$$

has a basis of solutions $y_1, \ldots, y_n$ on I.

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x)$$

is the general solution of the DE.
Every solution $y = Y(x)$ of the DE has the form

$$Y(x) = C_1 y_1(x) + \cdots + C_n y_n(x),$$

where $C_1, \cdots, C_n$ are arbitrary constants. (Prove this !)