MA-110 Linear Algebra and Differential Equations

Rekha Santhanam

Department of Mathematics
Indian Institute of Technology Bombay
Powai, Mumbai - 76

February 12, 2024
Lecture 16 D3

Question to think about

Show that to give a linear map from $T : \mathscr{M}_{2\times 2} \to \mathbb{R}^4$ it is sufficient to write down the image for $T(e_{11})$, $T(e_{12})$, $T(e_{21})$, $T(e_{22})$.

For instance create a linear transformation where $T(e_{11}) = (5,6,7,8)$, $T(e_{12}) = (1,2,3,4)$, $T(e_{21}) = (1,1,1,1)$ and $T(e_{22}) = (0,1,0,1)$

A general answer is given in the next slide.

- Consider $S : \mathscr{M}_{2\times 2} \to \mathbb{R}^4$ given by $S\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = (a, b, c, d)^T$.

Recall that $\{e_{11}, e_{12}, e_{21}, e_{22}\}$ is a basis of $\mathscr{M}_{2\times 2}$ such that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ae_{11} + be_{12} + ce_{21} + de_{22}$.
Observe that $S(e_{11}) = e_1, S(e_{12}) = e_2, S(e_{21}) = e_3,\ S(e_{22}) = e_4$.
Thus, $S(A) = aS(e_{11}) + bS(e_{12}) + cS(e_{21}) + dS(e_{22}) = ae_1 + be_2 + ce_3 + de_4 = (a, b, c, d)^T$.

General case:

If $\{v_1, \ldots, v_n\}$ is a basis of V, $T : V \to W$ is linear, $v \in V$,then $v = a_1v_1 + \cdots a_nv_n \Rightarrow T(v) = a_1T(v_1) + \cdots a_nT(v_n)$. Why? Thus,

$$\boxed{T \text{ is determined by its action on a basis,}}$$

i.e., for any n vectors $w_1, \ldots, w_n$ in W (not necessarily distinct), there is unique linear transformation $T : V \to W$ such that $T(v_1) = w_1, \ldots, T(v_n) = w_n$.

Important Observation: Let dim $(V) = n$, and $\mathscr{B} = \{v_1, \dots, v_n\}$ be a basis of V. Define $T : V \to \mathbb{R}^n$ by $T(v_i) = e_i$.

e.g., If $v = v_1 + v_n$, $T(v) =$? If $v = 3v_2 - 5v_3$, $T(v) =$?

If $v = a_1 v_1 + \cdots + a_n v_n$, $T(v) =$?

Thus $T(v) = [v]_{\mathscr{B}}$.

Is T a linear transformation? What is $N(T)$? What is $C(T)$?

Conclusion: $\boxed{\text{If dim (V) = n, then } V \simeq \mathbb{R}^n.}$

Question: Is $\mathscr{P}_3 \simeq \mathscr{M}_{2\times 2}$?

Key point: Composition of isomorphisms is an isomorphism, and inverse of an isomorphism is an isomorphism.

Exercise: Find 3 isomorphisms each from $\mathscr{P}_3$ to $\mathbb{R}^4$, and $\mathscr{M}_{2\times 2}$ to $\mathbb{R}^4$.

Example: $T(e_1) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $T(e_2) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, $T(e_3) = \begin{pmatrix} -5 \\ 0 \end{pmatrix}$

defines a linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$.

If $x = (x_1, x_2, x_3)^T$, then $T(x) = T(x_1 e_1 + x_2 e_2 + x_3 e_3) =$

$x_1 T(e_1) + x_2 T(e_2) + x_3 T(e_3) = x_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} + x_3 \begin{pmatrix} -5 \\ 0 \end{pmatrix}$, i.e., $T(x) = Ax$,

where $A = \begin{pmatrix} 3 & 2 & -5 \\ 1 & -1 & 0 \end{pmatrix}$. **Q:** $A_{*j} =?$

If $x = (x_1, x_2, x_3)^T$, then $T(x) = Ax$, where $A = \begin{pmatrix} 3 & 2 & -5 \\ 1 & -1 & 0 \end{pmatrix}$, i.e.,

$A_{*j} = T(e_j)$.

General case: If $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then

for $x = (x_1, \ldots, x_n)^T$ in $\mathbb{R}^n$,

$$T(x) = x_1 T(e_1) + \cdots x_n T(e_n) = Ax,$$

where $A = \begin{pmatrix} T(e_1) & \cdots & T(e_n) \end{pmatrix} \in \mathscr{M}_{m \times n}$, i.e., $A_{*j} = T(e_j)$.

Defn. A is called the *standard matrix* of T. Thus

$$\boxed{\text{Linear transformations from } \mathbb{R}^n \text{ to } \mathbb{R}^m}$$

$$\boxed{\text{are in one-one correspondence with } m \times n \text{ matrices.}}$$

Question : Can you imitate this if V and W are not $\mathbb{R}^n$ and $\mathbb{R}^m$?

Matrix Associated to a Linear Map: Example

$S : \mathscr{P}_2 \to \mathscr{P}_1$ given by $S(a_0 + a_1x + a_2x^2) = a_1 + 4a_2x$ is linear.

Question: Is there a matrix associated to S?
Expected size: 2×3. Why?

Idea: Construct an associated linear map $\mathbb{R}^3 \to \mathbb{R}^2$.
Use coordinate vectors! Fix bases $\mathscr{B} = \{1, x, x^2\}$ of $\mathscr{P}_2$, and $\mathscr{C} = \{1, x\}$ of $\mathscr{P}_1$ to do this.

Identify $f = a_0 + a_1x + a_2x^2 \in \mathscr{P}_2$ with $[f]_{\mathscr{B}} = (a_0, a_1, a_2)^T \in \mathbb{R}^3$,

and $S(f) \in \mathscr{P}_1$ with $[S(f)]_{\mathscr{C}} = (a_1, 4a_2)^T \in \mathbb{R}^2$.

The associated linear map $S' : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by
$S'(a_0, a_1, a_2)^T = (a_1, 4a_2)^T$, i.e., $S'([f]_{\mathscr{B}}) = [S(f)]_{\mathscr{C}}$, i.e.,
S' is defined by $S'(e_1) = (0, 0)^T$, $S'(e_2) = (1, 0)^T$, $S'(e_3) = (0, 4)^T \Rightarrow$ the standard matrix of S' is $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

Q: How is A related to S?

Observe: $A_{*1} = [S(1)]_{\mathscr{C}}$, $A_{*2} = [S(x)]_{\mathscr{C}}$, $A_{*3} = [S(x^2)]_{\mathscr{C}}$.

Example: The matrix of $S(a_0 + a_1x + a_2x^2) = a_1 + 4a_2x$, w.r.t. the bases $\mathscr{B} = \{1, x, x^2\}$ of $\mathscr{P}_2$ and $\mathscr{C} = \{1, x\}$ of $\mathscr{P}_1$ is $A =$ $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ and $\boxed{A_{*1} = [S(1)]_{\mathscr{C}}, A_{*2} = [S(x)]_{\mathscr{C}}, A_{*3} = [S(x^2)]_{\mathscr{C}}.}$

General Case: If $T : V \to W$ is linear, then the matrix of T w.r.t. the ordered bases $\mathscr{B} = \{v_1, \dots, v_n\}$ of V, and $\mathscr{C} = \{w_1, \dots, w_m\}$ of W, denoted $[T]_{\mathscr{C}}^{\mathscr{B}}$, is

$A = \begin{pmatrix} [T(v_1)]_{\mathscr{C}} & \cdots & [T(v_n)]_{\mathscr{C}} \end{pmatrix} \in \mathscr{M}_{m \times n}.$

Example: Projection onto the line $x_1 = x_2$

$P\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{x_1+x_2}{2} \\ \frac{x_1+x_2}{2} \end{pmatrix}$ has standard matrix $\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$.

This is the matrix of P w.r.t. the standard basis.

Question: What is $[P]_{\mathscr{B}}^{\mathscr{B}}$ where $\mathscr{B} = \{(1, 1)^T, (-1, 1)^T\}$?

Conclusion: The matrix of a transformation depends on the chosen basis. Some are better than others!