MA 110 - Ordinary Differential Equations

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Outline of the lecture

- Separable ODE
- Equations reducible to separable form

Separable ODE - Example 2

Find the solution to the initial value problem:

$$\frac{dy}{dx} = \frac{y\cos x}{1+2y^2}; \;\; y(0) = 1.$$

Assume $y \neq 0$. Then,

$$\frac{1+2y^2}{y}dy = \cos x \; dx.$$

Integrating,

$$\ln|y| + y^2 = \sin x + c.$$

As $y(0) = 1$, we get $c = 1$. Hence a particular solution to the IVP is

$$\ln|y| + y^2 = \sin x + 1.$$

Note: $y \equiv 0$ is a solution to the DE but it is not a solution to the given IVP.

Separable ODE - Example 3

Escape velocity.
A projectile of mass m moves in a direction perpendicular to the surface of the earth. Suppose v_0 is its initial velocity. We want to calculate the height the projectile reaches.
Its weight at height x (from the surface of the earth) is given by

$$w(x) = \frac{GmM_e}{(R+x)^2}, \quad \text{and} \quad g = \frac{GM_e}{R^2}.$$

$$\text{Hence} \qquad w(x) = \frac{mgR^2}{(R+x)^2},$$

where M_e and R are mass and radius, resp., of the earth.
Neglect force due to air resistance and other celestial bodies.

Therefore, the equation of motion is

$$m\frac{dv}{dt} = -\frac{mgR^2}{(R+x)^2}; \; v(0) = v_0.$$

By chain rule,

$$\frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = v \cdot \frac{dv}{dx}.$$

Thus,

$$v \cdot \frac{dv}{dx} = -\frac{gR^2}{(R+x)^2}.$$

This ODE is separable. Linear or non-linear? (NL)
Separating the variables and integrating, we get:

$$\frac{v^2}{2} = \frac{gR^2}{R+x} + c.$$

For $x = 0$, we get $\frac{v_0^2}{2} = gR + c$, hence, $c = \frac{v_0^2}{2} - gR$, and,

$$v = \pm\sqrt{v_0^2 - 2gR + \frac{2gR^2}{R + x}}.$$

Suppose the body reaches the maximum height H. Then $v = 0$ at this height.

$$v_0^2 - 2gR + \frac{2gR^2}{(R + H)} = 0.$$

Thus,

$$v_0^2 = 2gR - \frac{2gR^2}{R + H} = 2gR\left(\frac{H}{R + H}\right).$$

The escape velocity is found by taking limit as $H \to \infty$. Thus,

$$v_e = \sqrt{2gR} \sim 11 \text{ km/sec}.$$

Method of separation of variables doesn't yield all solutions!

Solve $y' = 3y^{2/3}, \; y(0) = 0$.

$y \equiv 0$ is a solution.

If $y \neq 0$, $\dfrac{dy}{y^{2/3}} = 3dx \Longrightarrow 3y^{1/3} = 3(x+c) \Longrightarrow y = (x+c)^3$.

Initial condition yields $c = 0$.

Hence $\boxed{y = x^3}$ and $\boxed{y = 0}$ are solutions which satisfy the initial conditions.

Consider

$$\phi_k(x) = \begin{cases} 0 & -\infty < x \leq k \\ (x-k)^3 & k < x < \infty \end{cases}$$

Are these functions solutions of the DE? YES.

There are infinitely many functions which are solutions of the DE.

Homogeneous functions

Definition

A function $f(x_1, \ldots, x_n)$ is called homogeneous if

$$f(tx_1, \ldots, tx_n) = t^d f(x_1, \ldots, x_n)$$

for some $d \in \mathbb{Z}$ and for all $t \neq 0$.

The number d is called the degree of $f(x_1, \ldots, x_n)$.

Examples :

$f(x, y) = x^2 + xy + y^2$ is homogeneous of degree 2.

$f(x, y) = y + x\cos^2\left(\frac{y}{x}\right)$ is homogeneous of degree 1.

Definition

The first order ODE

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

is called homogeneous if M and N are homogeneous of equal degree.

Example :

$$(y^2 - x^2)\frac{dy}{dx} + 2xy = 0.$$

Homogeneous ODE's - Reduction to variable separable form

Let

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

where M and N are homogeneous of degree d. Put

$$\frac{y}{x} = v.$$

Then,

$$\frac{dy}{dx} = x\frac{dv}{dx} + v.$$

Substituting this in the given ODE, we get:

$$M(x,xv) + N(x,xv)\left(x\frac{dv}{dx} + v\right) = 0.$$

Thus,

$$x^d M(1,v) + x^d N(1,v)\left(x\frac{dv}{dx} + v\right) = 0.$$

Homogeneous ODE's

Let $x \neq 0$. Then,

$$M(1,v) + N(1,v) \cdot v + N(1,v) \cdot x\frac{dv}{dx} = 0.$$

Thus,

$$\frac{dx}{x} + \frac{N(1,v)}{M(1,v) + N(1,v) \cdot v}dv = 0.$$

This is a separable equation.

Remark : What is important for the above method to work is that the ODE can be put into the form

$$y' = f(\frac{y}{x}).$$

Solve the ODE:

$$(y^2 - x^2)\frac{dy}{dx} + 2xy = 0.$$

Put $y = vx$. Thus, $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$.
Substituting this in the given ODE, we get:

$$(v^2x^2 - x^2)\left(v + x\frac{dv}{dx}\right) + 2x^2v = 0.$$

Thus, for $x \neq 0$,

$$(v^2 - 1)v + x(v^2 - 1)\frac{dv}{dx} + 2v = 0;$$

i.e.,

$$(v^3 + v) + x(v^2 - 1)\frac{dv}{dx} = 0.$$

Homogeneous ODE's

Thus, we have the separable ODE:

$$\frac{v^2 - 1}{v(v^2+1)} dv + \frac{dx}{x} = 0.$$

Integrating, we get:

$$\ln|x| + \int \left(\frac{2v}{v^2+1} - \frac{1}{v} \right) dv = 0.$$

Thus,

$$\ln|x| + \ln(v^2+1) - \ln|v| = c_1.$$

Hence,

$$\frac{x(v^2+1)}{v} = 2c,$$

or

$$y^2 + x^2 = 2cy,$$

which is

$$x^2 + (y-c)^2 = c^2.$$