MA 110 Endsem TSC

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Average freshie 1 night before Endsem: click here

Hi, and welcome to the MA110 Endsem TSC. We plan to give you a quick formula revision and then focus on solving questions relevant to the exam.

Please raise your hand at any time for clarifications or to answer, interactive classes help all of us remember it better.

Major disclaimer:

These are **not** official course slides by any means. This is just a small recap to go over every concept broadly and give you an idea to understand things intuitively. The only resource which actually has *all* the information you need to do well, are the prof's slides. So, be sure to go through them as well.

ODEs

We know what an ODE is. The *order* of an ODE is the order of the highest derivative in the equation. The equation

$$\sin\left(\frac{d^2y}{dx^2}\right) = \left(\frac{dy}{dx}\right)^3$$

has order 2.
The ODE is said to be *linear* if it is of the form

$$a_n(x)y^{(n)}(x) + \cdots + a_0(x)y = b(x)$$

for some $n > 0$ and functions $a_0, \ldots, a_n, b$ of x.

ODE Solutions and Orthogonal Trajectories

ODE Solutions: Consider the general ODE: $y^{(n)} = f(x, y, y', \ldots, y^{(n-1)})$.

- **Explicit Solution:** Function ϕ defined on interval I such that $\phi^{(n)}(x) = f(x, \phi(x), \ldots, \phi^{(n-1)}(x))$ for all $x \in I$.
- **Implicit Solution:** Relation $g(x, y) = 0$ defining at least one explicit solution.

Orthogonal Trajectories: Given a family of curves $F(x, y, \lambda) = 0$:

- Differentiate and eliminate λ, substitute y' with $-\frac{1}{y'}$ to find orthogonal trajectories.
- Solution gives family of trajectories, e.g., $y = cx$.

Separable and Homogeneous ODEs

Separable ODEs: An ODE in the form $M(x) + N(y)y' = 0$ is separable, often written as $M(x)\,dx + N(y)\,dy = 0$.

- **Solution:** If $H_1'(x) = M(x)$ and $H_2'(y) = N(y)$, then $H_1(x) + H_2(y) = c$ where $c \in \mathbb{R}$.

Homogeneous Functions: A function f of n-variables is homogeneous of degree d if $f(tx_1, \ldots, tx_n) = t^d f(x_1, \ldots, x_n)$ for $t \neq 0$.
Homogeneous ODEs: An ODE $M(x, y) + N(x, y)y' = 0$ is homogeneous if M and N are homogeneous functions of equal degree.

- **Solution Method:** Substituting $y = xv$, the ODE often becomes separable in v, simplifying the solution process.

Exact ODEs and Integrating Factors

Exact ODEs: An ODE $M(x,y) + N(x,y)y' = 0$ is exact if $\exists u(x,y)$ such that:

$$\frac{\partial u}{\partial x} = M, \quad \frac{\partial u}{\partial y} = N$$

Solution: $u(x,y) = c$, where $c \in \mathbb{R}$ and $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.
Finding $u(x,y)$: Integrate M w.r.t. x:

$$u(x,y) = \int M(x,y)\,dx + k(y)$$

Differentiate and equate to N, solving for $k'(y)$ and integrating to find $k(y)$.
Integrating Factors: If non-exact, seek $\mu(x,y)$ making $\mu M\,dx + \mu N\,dy$ exact. If $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$ is only a function of x:

$$\mu = \exp\left(\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\,dx\right)$$

Initial Value Problems (IVP) and Lipschitz Continuity

Initial Value Problem (IVP): An IVP is defined by an ODE of the form $y' = f(x, y)$, with initial conditions $y(x_0) = y_0$. **Existence:** If f is continuous and bounded within a rectangle $R = (x_0 - a, x_0 + a) \times (y_0 - b, y_0 + b)$, and $|f(x, y)| \leq K$, then there exists a solution defined on $(x_0 - \delta, x_0 + \delta)$ where $\delta = \min\{a, b/K\}$.

Lipschitz Continuity: A function f is *Lipschitz continuous* if there exists a constant L such that:

$$|f(x_1) - f(x_2)| \leq L|x_1 - x_2|$$

for all x_1, x_2 in some interval I. For functions of two variables, f is Lipschitz with respect to y if:

$$|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2|$$

This property ensures continuous solutions in y but not necessarily in x.

Uniqueness and Picard's Iteration Method

Uniqueness Theorem: For an IVP $y' = f(x,y), y(x_0) = y_0$ where f is continuous and bounded in R and satisfies Lipschitz condition with respect to y, the solution is unique on the interval $(x_0 - \delta, x_0 + \delta)$.

Picard's Iteration Method: Solves the IVP by converting the differential equation into an integral equation:

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt$$

Starting from $y_0(x) = y_0$, iteratively define:

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))\, dt$$

Under the conditions of the existence-uniqueness theorem, this sequence converges to the solution $y(x)$.

Second Order Homogeneous Linear ODEs

Second Order Homogeneous Linear ODE: Given the equation:

$$y'' + p(x)y' + q(x)y = 0$$

where $p(x)$ and $q(x)$ are continuous functions on an open interval I.
Existence and Uniqueness Theorem: For any $x_0 \in I$ and constants a, b (initial conditions):

- There is a unique solution y defined on I that satisfies:

$$y(x_0) = a, \quad y'(x_0) = b$$

Dimension of Solution Space:

- The solution space of the equation is a two-dimensional real vector space, indicating the general solution can be expressed as a linear combination of two linearly independent solutions.

Wronskian and Abel's Formula

Wronskian: For differentiable functions y_1 and y_2:

$$W(y_1, y_2)(x) := \det \begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix}$$

Linear Dependence and Wronskian: If y_1 and y_2 are solutions to $y'' + p(x)y' + q(x)y = 0$, then: 1. They are linearly dependent on I if and only if their Wronskian is zero everywhere on I. 2. If their Wronskian is zero at any point in I, it is zero everywhere on I.

Abel's Formula (Wronskian's Differential Equation): If y_1 and y_2 solve $y'' + p(x)y' + q(x)y = 0$:

$$W'(x) = -p(x)W(x)$$

$$W(x) = W(x_0)\exp\left(-\int_{x_0}^{x} p(t)\,dt\right)$$

ODEs with Constant Coefficients and Cauchy-Euler Equations

ODEs with Constant Coefficients: The general form of the ODE is:

$$y'' + py' + qy = 0$$

where p and q are real numbers. Solve by finding the roots $m^2 + pm + q = 0$:

- **Case 1:** Real and distinct roots m_1, m_2. Basis for solution: $\{\ e^{m_1 x},\ e^{m_2 x}\ \}$.
- **Case 2:** Real repeated root m_1. Basis for solution: $\{\ e^{m_1 x},\ xe^{m_1 x}\ \}$.
- **Case 3:** Complex roots $a \pm ib$. Basis for solution: $\{\ e^{ax}\cos(bx),\ e^{ax}\sin(bx)\ \}$.

Cauchy-Euler ODE: Given by:

$$x^2 y'' + pxy' + qy = 0$$

Solve by dividing through by x^2 and finding roots of $m(m-1) + pm + q = 0$:

- **Case 1:** Real and distinct roots m_1, m_2. Basis for solution: $\{\ x^{m_1},\ x^{m_2}\ \}$.
- **Case 2:** Real repeated root m_1. Basis for solution: $\{\ x^{m_1},\ x^{m_1}\log(x)\ \}$.
- **Case 3:** Complex roots $a \pm ib$. Basis for solution: $\{\ x^a\cos(b\log(x)),\ x^a\sin(b\log(x))\ \}$.

Higher-Order Linear Homogeneous ODEs

General Form: An n-th order linear homogeneous ODE is given by:

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_0(x)y = 0$$

where coefficients $p_0, \ldots, p_{n-1}$ are continuous on an interval I.
Existence-Uniqueness Theorem: Given $x_0 \in I$ and initial conditions $k_0, \ldots, k_{n-1}$,

$$y(x_0) = k_0,\ y'(x_0) = k_1, \ldots,\ y^{(n-1)}(x_0) = k_{n-1}$$

there exists a unique solution y, defined on I.
Dimension of Solution Space: The solution space for this ODE is an n-dimensional vector space.

Wronskian and Applications

Wronskian for n Functions: The Wronskian of n functions $y_1, \ldots, y_n$ is defined as:

$$W(y_1, \ldots, y_n)(x) = \det \begin{bmatrix} y_1(x) & \cdots & y_n(x) \\ y_1'(x) & \cdots & y_n'(x) \\ \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{bmatrix}$$

Linear Dependence: Functions are linearly dependent iff their Wronskian vanishes at any point in I.
Abel's Theorem: For solutions of $y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_0(x)y = 0$,

$$W'(x) = -p_{n-1}(x)W(x)$$

$$W(x) = W(x_0)e^{-\int_{x_0}^{x} p_{n-1}(t)\,dt}$$

Solving Non-Homogeneous ODEs

General Non-Homogeneous ODE:

$$y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_0 y = r(x)$$

Given LI solutions $y_1, \ldots, y_n$ of the homogeneous part, a particular solution y_p can be determined by:

$$y_p = v_1 y_1 + \cdots + v_n y_n$$

where v_i's are functions solved from the Wronskian system.
Method of Variation of Parameters: Solve the system where the matrix of derivatives equals the non-homogeneity:

$$\begin{bmatrix} y_1 & \cdots & y_n \\ y_1' & \cdots & y_n' \\ \vdots & \ddots & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \\ \vdots \\ v_n' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ r(x) \end{bmatrix}$$

Solving Differential Equations

Constant Coefficients ODE: Solve the homogeneous ODE:

$$y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_0 y = 0$$

Characteristic Equation:

$$m^n + p_{n-1}m^{n-1} + \cdots + p_0 = 0$$

Solutions based on roots:

- Real roots: $e^{m_0 x}, xe^{m_0 x}, \ldots, x^k e^{m_0 x}$
- Complex roots: $x^k e^{ax}\cos(bx), x^k e^{ax}\sin(bx)$

Cauchy-Euler ODE:

$$x^n y^{(n)} + \cdots + p_0 y = 0$$

Characteristic equation simplified:

$$m(m-1)\cdots(m-(n-1)) + \cdots + p_0 = 0$$

Typical Solutions: $x^{m_0}, \ldots, x^{m_0}(\log(x))^k$

Method of Undetermined Coefficients: To solve $y^{(n)} + \cdots + p_0 y = g(x)$:

- Assume a particular form $y_p = x^{\mu}(a_0 + \cdots + a_k x^k)e^{mx}$, adjusting for the type of $g(x)$.
- Differentiate y_p, substitute into the ODE, and match coefficients to solve for $a_0, \ldots, a_k$.

Introduction to Laplace Transform

Definition: The Laplace transform of a function $f : (0, \infty) \to \mathbb{R}$, denoted by $L(f)$, is defined as:

$$L(f)(s) = \int_0^\infty e^{-st} f(t)\, dt$$

This transform is typically defined for $s > a$ where a is determined based on the growth of $f(t)$.

Existence Conditions: For $L(f)(s)$ to exist, f must be piecewise continuous and of exponential order on $(0, \infty)$. Specifically, if there exist constants a, t_0, K such that $|f(t)| \leq Ke^{at}$ for all $t > t_0$, then $L(f)(s)$ exists for $s > a$.

Properties and Functions of Laplace Transform

Heaviside and Convolution:

- Heaviside function $u_c(t)$:

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$

- Convolution of f and g:

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

  Convolution is commutative, associative, and distributive over addition.

Key Properties:

- Linearity: $L(af + bg) = aL(f) + bL(g)$
- Shifting: $L(e^{at}f(t)) = F(s-a)$
- Scaling: $L(f(ct)) = \frac{1}{c}F\left(\frac{s}{c}\right)$
- Derivatives: $L(f')(s) = sF(s) - f(0)$
- Convolution: $L(f * g) = L(f) \cdot L(g)$

Laplace Transform of Common Functions and Inverse Transform

Laplace Transform of Common Functions:

- $L(t^n) = \frac{n!}{s^{n+1}}$ for n integer.
- $L(e^{at}) = \frac{1}{s-a}$
- $L(\sin(\omega t)) = \frac{\omega}{s^2+\omega^2}$
- $L(\cos(\omega t)) = \frac{s}{s^2+\omega^2}$

Inverse Laplace Transforms:

- If $L(f) = L(g)$, then $f(t) = g(t)$ at all continuity points.
- For rational expressions, partial fraction decomposition is used.
- Example for inverse transform:

$$L^{-1}\left(\frac{1}{(s-a)^n}\right) = \frac{t^{n-1}e^{at}}{(n-1)!}$$

Application of Inverse Laplace Transform: Using properties and tables of inverse transforms, complex time-domain solutions can be computed from simpler s-domain representations.

Question 1: A simple substitution

Question: Find constraints on α, β s.t. every solution of

$$y''(x) + \alpha y'(x) + \beta y(x) = 0$$

tends to 0 as $x \to \infty$

Question 2: Understanding Annihilators

Find an annihilator of $x^3 + x^2 e^x$

Question 3: What was it

Find inverse Laplace transform of:

$$\frac{s^2 - 3s}{s^2 - 6s + 178}$$

Question 4: Laplace to the rescue

Solve the IVP:

$$y'' + 4y' + 4y = x^3 e^{-2x}$$

Given $y(0) = 1, y'(0) = 3$

Question 5 : Am I Dreaming

Do there exist continuous functions $p(x), q(x)$ on $\mathbb{R}$ such that $e^{x^2}\sin\left(x^2\right) + x^4\cos(x)$ is a solution of:

$$y'' + p(x)y' + q(x)y = 0$$

Question 6: In a galaxy far, far away

Given that the Laplace transform of a function $y(x)$ and its derivative $y'(x)$ exist, and $L(y) = \frac{e^{\frac{-1}{s^2}}}{s}$. Find $y(0)$.

Question 7: Took me embarrassingly long

Let $g(t) = \int_0^t (x-t)^2 \cos x dx$ and $f(t) = \begin{cases} 0 & t < 1 \\ g(t-1) & t \geq 1 \end{cases}$

Find Laplace transform of $f(t)$.

Question 8:Pretty Standard no?

Find a particular solution of the following inhomogeneous Cauchy-Euler equation:

$$x^2y'' - 6y = \ln(x)$$

Find Laplace transform of:

$$f(x) = n \iff n - 1 \leq x < n \forall n \in \mathbb{N}$$

Question 10: Closing off

Find a second solution of

$$(x^2 - x)y'' + (x+1)y' - y = 0$$

Given that $(1 + x)$ is a solution.