MA 110 - Ordinary Differential Equations

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Outline of the lecture

- Integrating factors
- Bernoulli equation

Warm up!

1. The value of b that makes $(xy^2 + bx^2y)dx + (x + y)x^2dy = 0$ exact is ....
2. The value of r for which the DE $y'' + y' - 6y = 0$ has solutions of the form $y = e^{rt}$ are ....
   (Ans. $(r^2 + r - 6)e^{rt} = 0$. So, $r = 2, -3$)
3. The solution of $-ydx + (x + \sqrt{xy})dy = 0$ is ....
4. The solution of $y' = e^{3x} - y$ is ....
   (Ans. $ye^x = \int Q(x)e^x\, dx + C$ where $Q(x) = e^{3x}$)
5. Let $L := x^2\dfrac{d^2}{dx^2} + 2x\dfrac{d}{dx} + 2$ be a differential operator. $L(x^3)$ is ....
   (Ans. $L(x^3) = 6x^3 + 6x^3 + 2x^3 = 14x^3$)

Integrating Factors

Suppose the first order ODE

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

is not exact; that is, $M_y \neq N_x$. In this situation, we try to find a function $\mu(x,y)$ such that

$$\mu \cdot M + \mu \cdot N\frac{dy}{dx} = 0$$

is exact; i.e.,

$$(\mu \cdot M)_y = (\mu \cdot N)_x.$$

Thus,

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x.$$

That is, $\mu(x,y)$ satisfies the DE

$$\boxed{\mu_y M - \mu_x N + (M_y - N_x)\mu = 0}.$$

Such a function $\mu(x,y)$ is called an integrating factor of the given ODE.

Integrating Factor - function of x alone

In practice, we start by looking for an IF which depends only on one variable x or y, because it may be difficult to solve the PDE $\mu_y M - \mu_x N + (M_y - N_x)\mu = 0$.

Case 1 :
Suppose $\boxed{\mu \text{ is a function of } x \text{ alone}}$. That is, $\mu = \mu(x), \mu_y = 0$.
Then, the PDE above reduces to

$$\mu_x N = (M_y - N_x)\,\mu.$$

Thus,

$$\frac{d\mu}{dx} = \left(\frac{M_y - N_x}{N}\right)\mu.$$

If further, $\boxed{\dfrac{M_y - N_x}{N} \text{ is a function of } x}$ then the above DE is separable & we try to solve it to find $\mu(x)$.

$$\boxed{\mu(x) = e^{\int \frac{M_y - N_x}{N}\,dx}.}$$

Integrating Factor - function of y alone

Case 2 :

If we assume $\boxed{\mu \text{ to be a function of } y \text{ alone}}$ in the PDE

$$\mu_y M - \mu_x N + (M_y - N_x)\mu = 0,$$

then we get an analogous equation:

$$\frac{d\mu}{dy} = \left(\frac{N_x - M_y}{M}\right)\mu.$$

If further, $\boxed{\dfrac{N_x - M_y}{M} \text{ is a function of } y}$ then the above DE is separable & we try to solve it to find $\mu(y)$.

$$\boxed{\mu(y) = e^{\int \frac{N_x - M_y}{M}\, dy}}.$$

Example 1

Solve the ODE:

$$(8xy - 9y^2) + (2x^2 - 6xy)\frac{dy}{dx} = 0.$$

Let $M = 8xy - 9y^2$ and $N = 2x^2 - 6xy$.
Thus, $M_y = 8x - 18y$ and $N_x = 4x - 6y$. As $M_y \neq N_x$, the given ODE is not exact.
We first try to find an IF depending only upon one variable.
Note that

$$\frac{M_y - N_x}{N} = \frac{4x - 12y}{2x(x - 3y)} = \frac{2}{x}, \text{ a function of } x \text{ alone.}$$

Hence by the earlier discussion, we have:

$$\frac{d\mu}{dx} = \frac{2}{x}\mu.$$

Solving this separable ODE, we get $\ln|\mu| = \ln x^2$. Hence, $\boxed{\mu(x) = x^2}$ can be chosen as an IF for the given ODE.

Multiplying the given ODE by $\mu(x) = x^2$, we get:

$$(8x^3y - 9x^2y^2) + (2x^4 - 6x^3y)\frac{dy}{dx} = 0.$$

Check that this is an exact ODE. (How?)
To solve this exact ODE, we need to find $u(x,y)$ such that

$$8x^3y - 9x^2y^2 = u_x \ \& \ 2x^4 - 6x^3y = u_y.$$

To find $u(x,y)$:
Step I: $u(x,y) = 2x^4y - 3x^3y^2 + k(y)$.
Step II: $2x^4 - 6x^3y = u_y = 2x^4 - 6x^3y + k'(y)$.
Thus, $k'(y) = 0$. Hence,

$$u(x,y) = 2x^4y - 3x^3y^2 = c$$

is a solution of the given ODE.

Example 2

Solve the DE: $-y + x\dfrac{dy}{dx} = 0.$

Check that this is not an exact DE.

Let $M(x,y) = -y$ and $N(x,y) = x$.

To find a possible IF μ: note that $\dfrac{N_x - M_y}{M} = -\dfrac{2}{y}$, a function of y alone.

By the earlier discussion, we obtain :

$$\frac{d\mu}{dy} = -\frac{2}{y}\mu.$$

Thus, $\ln|\mu| = -2\ln|y|$.

So we choose

$$\mu(y) = \frac{1}{y^2}$$

as an IF.

Then, $\dfrac{-y + x\dfrac{dy}{dx}}{y^2} = 0$ is exact. Thus, $d\left(-\dfrac{x}{y}\right) = 0.$

Therefore, solution is given by $\boxed{\dfrac{x}{y} = c.}$

Consider

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \qquad (n = 0,\ 1 \text{ yields linear equations!})$$

$\boxed{\text{Claim :}}$ Let $n \neq 0, 1$.

Then the transformation $\boxed{v = y^{1-n}}$ reduces the Bernoulli equation to a linear equation in v.

$\boxed{\text{Justification :}}$

Let $v = y^{1-n}$.

$$\frac{dv}{dx} = (1-n)\ y^{\cancel{1}-n-\cancel{1}}\ \frac{dy}{dx}$$

That is,

$$\frac{dy}{dx} = \frac{1}{1-n}\ y^n\ \frac{dv}{dx}.$$

Substituting in the DE,

$$\frac{1}{1-n}\,y^n\,\frac{dv}{dx} + P(x)y = Q(x)y^n$$
$$\frac{1}{1-n}\,\frac{dv}{dx} + P(x)v = Q(x) \ \ (\text{assuming } y \neq 0)$$

Hence,

$\dfrac{dv}{dx} + (1-n)P(x)v = Q(x)(1-n)$, which is a linear DE in v.

Solve : $\dfrac{dy}{dx} + y = xy^3$.

Let $\boxed{v = y^{-2}}$.

$\dfrac{dv}{dx} = -2y^{-3}\dfrac{dy}{dx} \implies -\dfrac{1}{2}\dfrac{dv}{dx} + v = x$

That is, $\dfrac{dv}{dx} - 2v = -2x$ (linear equation in v)

Integrating factor is e^{-2x}.

$$
\begin{aligned}
ve^{-2x} &= -\int 2xe^{-2x}\,dx + C \\
&= \frac{2xe^{-2x}}{-2} - \int \cancel{2}\frac{e^{-2x}}{\cancel{2}} + C \\
&= xe^{-2x} + \frac{e^{-2x}}{2} + C
\end{aligned}
$$

$$\implies \frac{1}{y^2} = x + \frac{1}{2} + Ce^{2x}.$$

Equations reducible to linear equations

Consider

$$\boxed{\frac{d}{dy}(f(y))\frac{dy}{dx} + P(x)f(y) = Q(x),}$$

where f is an unknown function of y.
Set $\boxed{v = f(y)}$.
Then,

$$\frac{dv}{dx} = \frac{dv}{dy} \cdot \frac{dy}{dx} = \frac{d}{dy}(f(y))\frac{dy}{dx}.$$

Hence the given equation is

$$\frac{dv}{dx} + P(x)v = Q(x), \text{ which is linear in } v.$$

Remark : Bernoulli DE is a special case when $f(y) = y^{1-n}$.

Example

Solve : $\cos y \dfrac{dy}{dx} + \dfrac{1}{x}\sin y = 1.$

Set $\boxed{v = \sin y}$.

Then,

$$\frac{dv}{dx} = \frac{dv}{dy} \cdot \frac{dy}{dx} = \cos y \frac{dy}{dx}.$$

Hence the given equation is

$$\frac{dv}{dx} + \frac{1}{x} v = 1, \quad \text{which is linear in } v.$$

That is,

$$\begin{aligned} e^{\int \frac{1}{x}\, dx} v(x) &= \int e^{\int \frac{1}{x}\, dx} dx + C \\ \Longrightarrow x\, v(x) &= \frac{x^2}{2} + C \\ \sin y &= \frac{x}{2} + \frac{C}{x}. \end{aligned}$$