MA-110 Linear Algebra and Differential Equations

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Lecture 13 D3

Associated to an $m \times n$ matrix A, we have four subspaces:

- The **column space** of A: $C(A) = \text{Span}\{A_{*1}, \ldots A_{*n}\}$
$= \{v : Ax = v \text{ is consistent}\} \subseteq \mathbb{R}^m$.
- The **null space** of A: $N(A) = \{x : \; Ax = 0\} \subseteq \mathbb{R}^n$.
- The **row space** of $A = \text{Span}\{A_{1*}, \ldots, A_{m*}\} = C(A^T) \subseteq \mathbb{R}^n$.
- The **left null space** of $A = \{x : x^T A = 0\} = N(A^T) \subseteq \mathbb{R}^m$.

Question: Why are the row space and the left null space subspaces?

Recall: Let U be the echelon form of A, and R its reduced form.

$$\text{Then } \boxed{N(A) = N(U) = N(R).}$$

Observe: The rows of U(and R) are linear combinations of the rows of A, and vice versa $\Rightarrow$ their row spaces are same, i.e.,

$$\boxed{C(A^T) = C(U^T) = C(R^T).}$$

We compute bases and dimensions of these special subspaces.

We illustrate how to find a basis and the dimension of the Null Space $N(A)$, the Column Space $C(A)$, and the Row Space $C(A^T)$ by using the following example.

Let $A = \begin{pmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{pmatrix}$.

Recall:

- The reduced form of A is $R = \begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.
- The 1st and 2nd are pivot columns $\Rightarrow$ rank(A) = 2.
- $v = \begin{pmatrix} a & b & c \end{pmatrix}^T$ is in $C(A) \Leftrightarrow Ax = v$ is solvable $\Leftrightarrow 2a - b - c = 0$.
- We can compute special solutions to $Ax = 0$. The number of special solutions to $Ax = 0$ is the number of free variables.

For $A = \begin{pmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{pmatrix}$, reduced form $R = \begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

$$N(A) = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} -c+2d \\ -c-2d \\ c \\ d \end{pmatrix} = c \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} 2 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

$= \text{Span}\left\{ w_1 = \begin{pmatrix} -1 & -1 & 1 & 0 \end{pmatrix}^T, w_2 = \begin{pmatrix} 2 & -2 & 0 & 1 \end{pmatrix}^T \right\}.$

w_1, w_2 are linearly independent (Why?)

$\Rightarrow \mathscr{B} = \{w_1, w_2\}$ forms a basis for $N(A) \Rightarrow \dim(N(A)) = 2$.

$\boxed{\text{A basis for } N(A) \text{ is the set of special solutions.}}$

$\boxed{\dim(N(A)) = \text{ no. of free variables } = \text{ no. of variables - rank}(A)}$

$\dim(N(A))$ is called nullity(A).

Show: $w = (-3, -7, 5, 1)^T$ is in $N(A)$. Find $[w]_{\mathscr{B}}$.

For $A = \begin{pmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{pmatrix}$, reduced form $R = \begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Write $A = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 \end{pmatrix}$ and $R = \begin{pmatrix} w_1 & w_2 & w_3 & w_4 \end{pmatrix}$.

Recall: Relations between the column vectors of A are the same as the relations between column vectors of R.

$\Rightarrow$ $Ax = v_3$ has a solution has the same solution as $Rx = w_3$, and $Ax = v_4$ has a same solution as $Rx = w_4$.

Particular solutions are $(1,1,0,0)^T$ and $(-2,2,0,0)^T$ respectively $\Rightarrow$ $v_3 = v_1 + v_2$, $v_4 = -2v_1 + 2v_2$.

Observe:

- v_1 and v_2 correspond to the pivot columns of A.
- $\{v_1, v_2\}$ are linearly independent. Why?
- $C(A) = \mathrm{Span}\{v_1, \ldots, v_4\} = \mathrm{Span}\{v_1, v_2\}$.

Thus $\mathscr{B} = \{v_1, v_2\}$ is a basis of $C(A)$. **Ques:** What is $[v_i]_{\mathscr{B}}$?

The Rank-Nullity Theorem

More generally, for an $m \times n$ matrix A,

• Let $\operatorname{rank}(A) = r$. The r pivot columns are linearly independent since their reduced form contains an $r \times r$ identity matrix.

• Each non-pivot column A_{*j} of A can be written as a linear combination of the pivots columns, by solving $Ax = A_{*j}$. Thus

$$\boxed{\text{A basis for } C(A) \text{ is given by the pivot columns of } A.}$$

$$\boxed{\dim(C(A)) = \text{no. of pivot variables} = \operatorname{rank}(A).}$$

• A basis for $N(A)$ is given by the special solutions of A. Thus

$$\boxed{\dim(N(A)) = \text{no. of free variables} = \operatorname{nullity}(A).}$$

Rank-Nullity Theorem: Let A be an $m \times n$ matrix. Then

$$\dim(C(A)) + \dim(N(A)) = \text{no. of variables} = n$$

Recall: If $A = \begin{pmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{pmatrix}$, then $R = \begin{pmatrix} {\color{red}1} & {\color{red}0} & 1 & -2 \\ {\color{red}0} & {\color{red}1} & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Recall: R is obtained from A by taking non-zero scalar multiples of rows and their sums $\Rightarrow C(R^T) = C(A^T)$.

Observe: The non-zero rows of R will span $C(A^T)$, and they contain an identity submatrix $\Rightarrow$ they are linearly independent.

Thus, the non-zero rows of R form a basis for $C(R^T) = C(A^T)$.

Exercise: Give two different basis for $C(A^T)$.

Since the number of non-zero rows of R = number of pivots of A, we have:

$$\boxed{\dim\ C(A^T) = \text{no. of pivots of } A = \operatorname{rank}(A).}$$

Recall: $\dim\ C(A^T) = \operatorname{rank}(A^T)$. Thus,

$$\boxed{\operatorname{rank}(A^T) = \dim\ (C(A^T)) = \operatorname{rank}(A)}$$

The no. of columns of A^T is m.
By Rank-Nullity Theorem, $\text{rank}(A^T) + \dim(N(A^T)) = m$.

Hence:

$$\boxed{\dim(N(A^T)) = m - \text{rank}(A).}$$

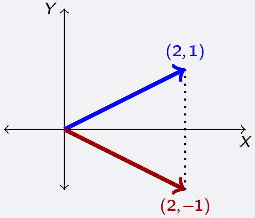
Exercise: Complete the example by finding a basis for $N(A^T)$.

$A = \begin{pmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{pmatrix}$, reduced form $R = \begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

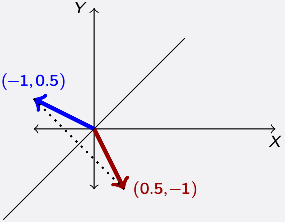
Question. Can you use R to compute the basis for $N(A^T)$? Why not?

A. Need the reduced form of A^T which is $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}$. Let $\mathbf{x} = (2, 1)^T$. What is $\mathbf{A}x$? How does A transform x?
A reflects vectors across the X-axis.

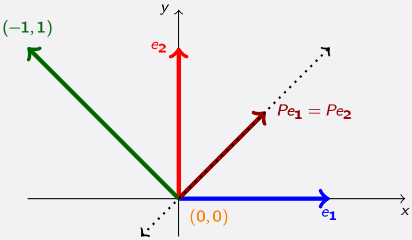


Let $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $B\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$. If $\mathbf{x} = (-1, 0.5)^T$, then $\mathbf{B}x = (0.5, -1)^T$. How does B transform x?
B reflects vectors across the line $x_1 = x_2$.



Q: Do reflections preserve scalar multiples? Sums of vectors?

- $P = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$ transforms $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ to $Px = \begin{pmatrix} \frac{x_1+x_2}{2} \\ \frac{x_1+x_2}{2} \end{pmatrix}$.



$Pe_1 = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = Pe_2$.

P transforms the vector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ to the origin.

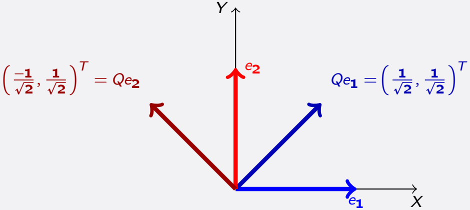
Question: Geometrically, how is P transforming the vectors?
Answer: Projects onto the line $x_1 = x_2$.

Question: What happens to sums of vectors when you project them? What about scalar multiples?

Question: Understand the effect of $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ on e_1 and e_2 and interpret what P represents geometrically!

Let $Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \cos(45^\circ) & -\sin(45^\circ) \\ \sin(45^\circ) & \cos(45^\circ) \end{pmatrix}$.

How does Q transform the standard basis vectors $\mathbf{e_1}$ and $\mathbf{e_2}$?



Q: What does the transformation $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto Qx$ represent geometrically?

Rotations also map sum of vectors to sum of their images and a scalar multiple of a vector to the scalar multiple of its image.

• An $m \times n$ matrix A transforms a vector x in $\mathbb{R}^n$ into the vector Ax in $\mathbb{R}^m$. Thus $T(x) = Ax$ defines a function $T : \mathbb{R}^n \to \mathbb{R}^m$.

• The domain of T is ___. The codomain of T is ___.

• Let $b \in \mathbb{R}^m$. Then b is in $C(A) \iff Ax = b$ is consistent $\iff T(x) = b$, i.e., b is in the image (or range) of T.
Hence, the range of T is ___.

Example: Let $A = \begin{pmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{pmatrix}$. Then $T(x) = Ax$ is a function with domain $\mathbb{R}^4$, codomain $\mathbb{R}^3$, and range equal to
$C(A) = \{(a, b, c)^T \mid 2a - b - c = 0\} \subseteq \mathbb{R}^3$.

Question: How does T transform sums and scalar multiples of vectors?
Ans. Nicely! For scalars a and b, and vectors x and y,
$T(ax + by) = A(ax + by) = aAx + bAy = aT(x) + bT(y)$. Thus

$$\boxed{T \text{ takes linear combinations to linear combinations.}}$$

Defn. Let V and W be vector spaces.

• A *linear transformation* from V to W is a function $T : V \to W$ such that for $x, y \in V$, scalars a and b,

$$\boxed{T(ax + by) = aT(x) + bT(y)}$$

i.e., T takes linear combinations of vectors in V to the linear combinations of their images in W.