# MA 110 - Ordinary Differential Equations

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# Outline of the lecture

- Laplace Transforms
- Examples
- Existence
- Properties

# Existence of Laplace transforms

- For a given $f$, $L(f)$ may or may not exist.
- Sufficient conditions under which convergence is guaranteed for the integral in the definition of the Laplace transform is that $f$ is piecewise continuous and is of exponential order.
- Piecewise continuity - The function is continuous except possibly for finitely many jump discontinuities.
- A function $f$ is said to be of exponential order if there exists $a \in \mathbb{R}$ and positive constants $t_0$ and $K$ such that

$$|f(t)| \leq Ke^{at},$$

for all $t \geq t_0 > 0$.

# Exponential Order

- In other words, we say $f$ is of exponential order if there exists a constant $a$ such that $e^{-at}|f(t)|$ is bounded for all sufficiently large values of $t$.
- That is, if $f$ is of exponential order and the values $f(t)$ of $f$ become infinite as $t \to \infty$, these values cannot become infinite more rapidly than a multiple of $K$ of the corresponding $e^{at}$ values of some constant $a$.

# Examples

1. Every bounded function is of exponential order with the constant $a = 0$. Further, $\sin bt$ and $\cos bt$ are of exponential order. Also, if $|f(t)| \leq K$ for $t \geq t_0 > 0$, then $f$ is of exponential order.
2. $e^{\alpha t} \sin bt$ is of exponential order, with constant $a = \alpha$.
3. $t^n$ for $n > 0$ is of exponential order, since for $a > 0$, $\lim_{t \to \infty} e^{-at} t^n = 0$ and thus, there exists $K > 0$ and $t_0 > 0$ such that
$$e^{-at}|f(t)| = e^{-at} t^n < K, \ \text{ for } t > t_0.$$
4. $e^{t^2}$ is not of exponential order, for in this case,
$$e^{-at}|f(t)| = e^{t^2 - at}$$
and this becomes unbounded as $t \to \infty$, no matter what is value of $a$.
5. Sum of functions of exponential order is also of exponential order.

# Existence theorem

**Theorem**

*Suppose $f(t)$ is piecewise continuous on $[0, \alpha]$ for all $\alpha > 0$. Further suppose*

$$|f(t)| \leq Ke^{at},$$

*for $t \geq t_0 > 0$, where $K > 0$, $a$, $t_0 \in \mathbb{R}$. Then*

$$L(f)(s) = F(s) = \int_0^\infty e^{-st} f(t) dt$$

*exists for $s > a$.*

Proof: We have:

$$L(f)(s) = \int_0^{t_0} e^{-st} f(t) dt + \int_{t_0}^\infty e^{-st} f(t) dt.$$

As $f$ is piecewise continuous on $[0, t_0]$, $\displaystyle\int_0^{t_0} e^{-st} f(t) dt$ exists.

# Proof contd...

We need to show that $\int_{t_0}^{\infty} e^{-st} f(t)dt$ converges.

For $t \geq t_0$, we have:

$$|e^{-st} f(t)| \leq e^{-st} K e^{at} = K e^{-(s-a)t}.$$

For $s > a$, $\int_{t_0}^{\infty} e^{-(s-a)t} dt$ converges. Hence,

$$\int_{t_0}^{\infty} |e^{-st} f(t)| dt,$$

and thus

$$\int_{t_0}^{\infty} e^{-st} f(t) dt$$

converges.

# Remark

The conditions in the theorem are sufficient but not necessary for the existence of $L(f)$.
Example:

$$\frac{1}{\sqrt{t}} \to \infty$$

as $t \to 0$. Hence it is not piecewise continuous on $[0, b]$ for any $b > 0$. But we will prove later that $\boxed{L(\frac{1}{\sqrt{t}})(s) = \sqrt{\frac{\pi}{s}}.}$

Let $f, g : [0, \infty) \to \mathbb{R}$ be functions such that $L(f)$ and $L(g)$ exist. Let $a, b \in \mathbb{R}$. Then,

$$L(af(t) + bg(t)) = aL(f(t)) + bL(g(t)).$$

Proof:

$$\begin{aligned} L(af + bg) &= \int_0^\infty e^{-st}(af(t) + bg(t))dt \\ &= \int_0^\infty e^{-st} af(t)dt + \int_0^\infty e^{-st} bg(t)dt \\ &= aL(f) + bL(g). \end{aligned}$$

# Example 1

$$
\begin{aligned}
L(e^{i\omega t}) &= \int_0^\infty e^{-st} e^{i\omega t}\, dt \\
&= \int_0^\infty e^{-(s-i\omega)t}\, dt \\
&= \left.\frac{e^{-(s-i\omega)t}}{-(s-i\omega)}\right|_{t=0}^{\infty} = \frac{1}{s-i\omega} \\
&= \frac{s+i\omega}{s^2+\omega^2}.
\end{aligned}
$$

Hence,

$$L(\cos\omega t + i\sin\omega t) = \frac{s+i\omega}{s^2+\omega^2}.$$

Using linearity,

$$\boxed{L(\cos\omega t) = \frac{s}{s^2+\omega^2}, \qquad L(\sin\omega t) = \frac{\omega}{s^2+\omega^2}.}$$

# Example 2

$$
\begin{aligned}
L(\cosh at) &= L\Big(\frac{e^{at}+e^{-at}}{2}\Big) \\
&= \frac{1}{2}L(e^{at}) + \frac{1}{2}L(e^{-at}) \\
&= \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) = \frac{s}{s^2-a^2}.
\end{aligned}
$$

Hence, $\boxed{L(\cosh at) = \frac{s}{s^2-a^2}.}$

Similarly, $\boxed{L(\sinh at) = \frac{a}{s^2-a^2}.}$

# Example 3

$$
\begin{aligned}
L(te^{i\omega t}) &= \int_0^\infty e^{-st} te^{i\omega t}\, dt = \int_0^\infty te^{-(s-i\omega)t}\, dt \\
&= \left. \frac{te^{-(s-i\omega)t}}{-(s-i\omega)} \right|_{t=0}^{\infty} + \int_0^\infty \frac{e^{-(s-i\omega)t}}{s-i\omega}\, dt \\
&= \left. \frac{1}{(s-i\omega)} \frac{e^{-(s-i\omega)t}}{(-(s-i\omega))} \right|_{t=0}^{\infty} \\
&= \frac{1}{(s-i\omega)^2} = \frac{1}{s^2-\omega^2-2is\omega} \frac{s^2-\omega^2+2is\omega}{s^2-\omega^2+2is\omega} \\
&= \frac{s^2-\omega^2}{(s^2-\omega^2)^2+4s^2\omega^2} + i\frac{2s\omega}{(s^2-\omega^2)^2+4s^2\omega^2} \\
&= \frac{s^2-\omega^2}{(s^2+\omega^2)^2} + i\frac{2s\omega}{(s^2+\omega^2)^2}.
\end{aligned}
$$

Hence,

$$
\boxed{L(t\cos\omega t) = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}, \qquad L(t\sin\omega t) = \frac{2s\omega}{(s^2+\omega^2)^2}.}
$$

# Property 2 : I Shifting theorem ($s$ shifting)

If $L(f(t)) = F(s)$, then $L(e^{at} f(t)) = F(s-a)$.

Proof :

$$\begin{aligned} L(e^{at} f(t)) &= \int_0^\infty e^{-st} e^{at} f(t)\, dt \\ &= \int_0^\infty e^{-(s-a)t} f(t)\, dt \\ &= F(s-a). \end{aligned}$$

Examples :

1. $L(t^2) = \dfrac{2}{s^3} \Longrightarrow L(e^{-t} t^2) = \dfrac{2}{(s+1)^3}$.
2. $L(\cos \omega t) = \dfrac{s}{s^2+\omega^2} \Longrightarrow L(e^{at} \cos \omega t) = \dfrac{s-a}{(s-a)^2+\omega^2.}$

# Exercises

Find the Laplace transforms of

1. $t^n e^{at}$
2. $\cosh at \cos at$
3. $e^{-t} \sin^2 t$

# Property 3 : Scaling

$$\boxed{\text{If } L(f) = F(s), \text{ then } L(f(ct)) = \frac{1}{c} F\left(\frac{s}{c}\right), \ c > 0.}$$

Proof: Let $\xi = ct$. Then, $d\xi = c\,dt$.

$$\begin{aligned} L(f(ct)) &= \int_0^\infty e^{-st} f(ct)\,dt \\ &= \int_0^\infty e^{-\left(\frac{s\xi}{c}\right)} \frac{1}{c} f(\xi)\,d\xi \\ &= \frac{1}{c} \int_0^\infty e^{-\left(\frac{s\xi}{c}\right)} f(\xi)\,d\xi \\ &= \frac{1}{c} F\left(\frac{s}{c}\right). \end{aligned}$$

Example :

$$L(e^t) = \frac{1}{s-1} \Longrightarrow L(e^{at}) = \frac{1}{a} \frac{1}{\left(\frac{s}{a} - 1\right)} = \frac{1}{s-a}.$$

# Property 4 : Differentiation

I.

- Suppose $f$ is continuous,
- $f'$ is piecewise continuous on $[0, a]$ for all $a > 0$,
- $|f(t)| \leq Ke^{\alpha t}$, for $t \geq t_0 > 0$, where $K > 0$, $t_0$, $\alpha \in \mathbb{R}$.

Then, $L(f')(s)$ exists for $s > \alpha$ and

$$L(f') = sL(f) - f(0).$$

II.

- Suppose $f, f', \ldots, f^{(n-1)}$ are continuous
- $f^{(n)}$ is piecewise continuous on $[0, a]$, for all $a > 0$,
- For all $t \geq t_0 > 0$, $|f^{(i)}(t)| \leq Ke^{\alpha t}$, $0 \leq i \leq n-1$, where $K > 0$, $t_0$, $\alpha \in \mathbb{R}$.

Then, $L(f^{(n)})(s)$ exists for all $s > \alpha$ and

$$L(f^{(n)}) = s^n L(f) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - s f^{(n-2)}(0) - f^{(n-1)}(0).$$