# MA 110 - Ordinary Differential Equations

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# Outline of the lecture

- Existence & uniqueness
- Picard's iteration

# Existence - Uniqueness Theorem

Let $R$ be a rectangle containing $(x_0, y_0)$ in the domain $D$,

- $f(x, y)$ be continuous at all points $(x, y)$ in $R : |x - x_0| < a, \ |y - y_0| < b$ and
- bounded in $R$, that is, $|f(x, y)| \leq K \ \ \forall (x, y) \in R$.

Then, the IVP $\boxed{y' = f(x, y), \ y(x_0) = y_0}$ has at least one solution $y(x)$ defined for all $x$ in the interval $|x - x_0| < \alpha$, where

$$\alpha = \min\left\{a, \frac{b}{K}\right\}.$$

In addition to the above conditions, if $f$ satisfies the Lipschitz condition with respect to $y$ in $R$, that is,

$$|f(x, y_1) - f(x, y_2)| \leq M|y_1 - y_2| \ \ \forall (x, y_1), (x, y_2) \text{ in } R,$$

then, the solution $y(x)$ defined at least for all $x$ in the interval $|x - x_0| < \alpha$, with $\alpha$ defined above is unique [1].

[1] Existence - Peano, Existence & uniqueness -Picard

# A quick check!

1. Is $f(x) = \sin x$ Lipschitz continuous over $\mathbb{R}$? Yes.
2. Is $f(x) = x^2$ globally Lipschitz continuous over $\mathbb{R}$ ? No.
   (Hint: $\left|\dfrac{f(x_1) - f(x_2)}{x_1 - x_2}\right| = |x_1 + x_2|$ )
   However, it is Lipschitz continuous over any closed interval of $\mathbb{R}$. We say that it is locally Lipschitz continuous over $\mathbb{R}$.
3. Is $f(x) = \dfrac{1}{x^2}$ globally Lipschitz continuous on $[\alpha, \infty)$ for any $\alpha > 0$? Yes.

# Example 1

Consider

$$y' = y^{1/3} \qquad y(0) = 0 \text{ in } R : |x| \leq a, |y| \leq b.$$

$f(x, y)$ is continuous in $R$ and hence existence is guaranteed.

But $\phi_1(x) = 0$ and $\phi_2(x) = \begin{cases} (\frac{2}{3}x)^{3/2} & \text{if } x \geq 0 \\ 0 & \text{if } x \leq 0 \end{cases}$ are solutions in $-\infty < x < \infty$.

Does this imply Lipschitz condition won't be satisfied?

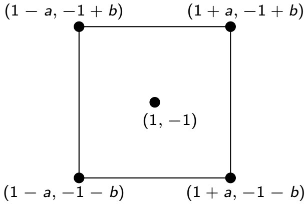
$$\frac{|f(x, y_1) - f(x, y_2)|}{|y_1 - y_2|} = \frac{|y_1^{1/3} - y_2^{1/3}|}{|y_1 - y_2|}.$$

Choosing $y_1 = \delta,\ y_2 = -\delta$, we see that the quotient is unbounded for small values of $\delta$ and hence Lipschitz condition is not satisfied.

Solution exists, but not unique.

# Example 2

Consider $y' = y^2,\ y(1) = -1$. Find $\alpha$ in the existence & uniqueness theorem.




$f(x, y) = y^2,\ f_y = 2y$ are continuous in the closed rectangle $R : |x - 1| \leq a, |y + 1| \leq b$.

$$|f(x, y)| = |y|^2 \leq |(-b - 1)|^2 \leq (b + 1)^2 \tag{1}$$

Now, $\alpha = \min \left\{ a, \dfrac{b}{(b+1)^2} \right\}$.

# Example 2 (contd..)

Consider

$$F(b) = \frac{b}{(b+1)^2}.$$

$F'(b) = \dfrac{1-b}{(b+1)^3} \implies$ the maximum value of $F(b)$ for $b > 0$ occurs at $b = 1$ (Why?); and we find $F(1) = \dfrac{1}{4}$.

Hence, if $\boxed{a \geq 1/4}$, $F(b) = \dfrac{b}{(b+1)^2} \leq a$ for all $b > 0$ and $\alpha = \min\{a, F(b)\} = F(b) = \dfrac{b}{(b+1)^2} \leq 1/4$, whatever be $a$.

If $\boxed{a < 1/4}$, then certainly $\alpha < 1/4$. Thus in any case, $\boxed{\alpha \leq 1/4}$.

For $b = 1, a \geq 1/4, \alpha = \min\{a, 1/4\} = 1/4$.

Thus the best possible $\alpha$ from the theorem gives that the IVP has a unique solution in $|x-1| \leq 1/4 \implies \boxed{3/4 \leq x \leq 5/4}$.

# Example 2 - Remarks

1. The theorem guarantees existence and uniqueness only in a very small interval!
2. The theorem DOES NOT give the largest interval where the solution is unique.
3. What is the solution in this case by separation of variables and where is it valid? Can you think of extending the solution to a larger interval?
   (Ans. $xy = -1$. Largest interval where solution exist is $(0, \infty)$)

# Picard's iteration method

[2] AIM : To solve

$$y' = f(x, y), \; y(x_0) = y_0 \tag{2}$$

METHOD

1. Integrate both sides of (2) to obtain

$$y(x) - y(x_0) = \int_{x_0}^{x} f(t, y(t))\, dt$$

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt \tag{3}$$

Note that any solution of (2) is a solution of (3) and vice-versa.

[2] Picard used this in his existence-uniqueness proof

2. Solve (3) by iteration:

$$\begin{aligned} y_1(x) &= y_0 + \int_{x_0}^{x} f(t, y_0)\, dt \\ y_2(x) &= y_0 + \int_{x_0}^{x} f(t, y_1(t))\, dt \\ &\vdots \\ y_n(x) &= y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\, dt \end{aligned}$$

3. Under the assumptions of existence-uniqueness theorem, the sequence of approximations converges to the solution $y(x)$ of (2). That is,

$$y(x) = \lim_{n\to\infty} y_n(x).$$

Solve : $y' = xy,\ y(0) = 1$ using Picard's iteration method.

1. The integral equation is

$$y(x) = 1 + \int_{x_0}^{x} ty\, dt.$$

2. The successive approximations are :

$$\begin{aligned}
y_1(x) &= 1 + \int_0^x t \cdot 1\, dt = 1 + \frac{x^2}{2}. \\
y_2(x) &= 1 + \int_0^x t(1 + \frac{t^2}{2})\, dt = 1 + \frac{x^2}{2} + \frac{x^4}{2 \cdot 4}. \\
&\vdots \\
y_n(x) &= 1 + (\frac{x^2}{2}) + \frac{1}{2!}(\frac{x^2}{2})^2 + \cdots + \frac{1}{n!}(\frac{x^2}{2})^n. \text{ (By induction )}
\end{aligned}$$

3. $\boxed{y(x) = \lim_{n\to\infty} y_n(x) = e^{x^2/2}.}$

1. Does uniform continuity $\Longrightarrow$ Lipschitz continuity ?
(No, consider $f(x) = \sqrt{x}, \;\; x \in [0, 2].$)
2. The value of $n$ such that the curves $x^n + y^n = C$ are the orthogonal trajectories of the family

$$y = \frac{x}{1 - Kx}$$

is . . . . . . . . .?
(Ans. DE for the given family of curves is $\dfrac{dy}{dx} = (\dfrac{y}{x})^2$. Finally, we get n=3).