

MA-110 Linear Algebra and Differential Equations

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Lecture 4 D3

- We discussed how number of pivots and solution set is related.
- Last class we discussed various matrix operations.
- We can add any two matrices of same size.
- We can multiply two matrices only if the number of columns in first matrix is same as the number of rows in the second matrix.
- Matrix multiplication is associative. It is distributive with matrix addition.
- Matrix multiplication is not commutative.

Matrix Multiplication: Examples

Examples:

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \end{pmatrix}, I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ (Identity)}$$

- $AB =$??

- size of BA is $___ \times ___$

- $BA = \begin{pmatrix} 4 & 10 & 7 \\ 4 & 18 & 10 \end{pmatrix},$

- and $IA = A = AI.$

Matrix Multiplication: Examples

Examples:

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(\text{Permutation}) \quad P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (e_2 \ e_1 \ e_3)$$

$$\text{Then } AP = (Ae_2 \ Ae_1 \ Ae_3) = (A_{*2} \ A_{*1} \ A_{*3})$$

Exercise: Find EA and PA .

Question: Can you obtain EA and PA directly from A ? How?

Transpose A^T of a Matrix A

Defn. The i -th row of A is the i -th column of A^T , the transpose of A and vice-versa.

Hence if $A_{ij} = a$, then $(A^T)_{ji} = a$.

Example: If $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & 1 \end{pmatrix}$, then $A^T = \begin{pmatrix} 1 & 0 \\ 2 & -2 \\ 3 & 1 \end{pmatrix}$.

- If A is $m \times n$, then A^T is $n \times m$.
- If A is upper triangular, then A^T is lower triangular.
- $(A^T)^T = A$, $(A + B)^T = A^T + B^T$.
- $(AB)^T = B^T A^T$. *Proof.* Exercise.

Symmetric Matrix

Defn. If $A^T = A$, then A is called a **symmetric** matrix.

Note: A symmetric matrix is always $n \times n$.

Examples: $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are **symmetric**.

- If A, B are symmetric, then AB may **NOT be symmetric**.

In the above case, $AB = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$.

- If A and B are **symmetric**, then $A + B$ is symmetric. **Why?**
- If A is a $n \times n$ matrix, $A + A^T$ is symmetric. **Why?**
- For any $m \times n$ matrix B , BB^T and $B^T B$ are symmetric. **Why?**

Exercise: If $A^T = -A$, we say that A is *skew-symmetric*.

Verify if similar observations are true for skew-symmetric matrices.

Inverse of a Matrix

Defn. Given A of size $n \times n$, we say B is an inverse of A if $AB = I = BA$. If this happens, we say A is *invertible*.

- What would be the *inverse* of $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$?
- An *inverse may not exist*. Find an example. *Hint: $n = 1$.*
- An inverse of A , if it exists, *has size $n \times n$* .
- If the inverse of A exists, it is *unique*, and is denoted A^{-1} . *Why unique?*

Proof. Let B and C be inverses of A .

$$\begin{aligned} \Rightarrow BA &= I && \text{by definition of inverse.} \\ \Rightarrow (BA)C &= IC && \text{multiply both sides on the right by } C. \\ \Rightarrow B(AC) &= IC && \text{by associativity.} \\ \Rightarrow BI &= IC && \text{since } C \text{ is an inverse of } A. \\ \Rightarrow B &= C && \text{by property of the identity matrix } I. \end{aligned}$$

Inverse of a Matrix

- If A and B are invertible, what about AB ? AB is invertible, with inverse $(AB)^{-1} = B^{-1}A^{-1}$.

Proof. Exercise.

- If A, B are invertible, what about $A + B$? $A + B$ may not be invertible.

Example: $I + (-I) = (0)$.

- If A is invertible, what about A^T ? A^T is invertible with inverse $(A^T)^{-1} = (A^{-1})^T$.

Proof. Use $AA^{-1} = I$. Take transpose.

- If A is symmetric and invertible then, is A^{-1} symmetric?

Yes. *Proof.* Exercise!

- (Identity) $I^{-1} = I$.

Inverses and Linear Systems

- If A is invertible then the system $Ax = b$ has a solution, for every constant vector b , namely $x = A^{-1}b$. Is this **unique**?
- Since $x = 0$ is always a solution of $Ax = 0$, if $Ax = 0$ has a non-zero solution, then A is **not invertible** by the last remark.
- If A is invertible, then the Gaussian elimination of A produces n pivots.

Exercise:

1. A diagonal matrix A is invertible if and only if _____.
(Hint: When are the diagonal entries pivots?)
2. When is an upper triangular matrix invertible?
 - Since $AB = (AB_{*1} \ AB_{*2} \ \cdots \ AB_{*n})$ and $I = (e_1 \ e_2 \ \cdots \ e_n)$, if $B = A^{-1}$, then B_{*j} is a solution of $Ax = e_j$ for all j .
 - Strategy to find A^{-1} : Let A be an $n \times n$ invertible matrix. Solve $Ax = e_1$, $Ax = e_2$, \dots , $Ax = e_n$.

Things to think about

- Complete the proofs left as exercise.
- Currently we are unable to show that if $AB = I$ then $BA = I$ for square matrices A and B . Why so?
- Can you rephrase what we proved about transposes as a property of the transpose function from the set of $m \times n$ matrices to $n \times m$ matrices?