MA-110 Linear Algebra and Differential Equations

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Summary: Eigenvalues and Characteristic Polynomial

Let A be $n \times n$.

1. The *characteristic polynomial* of A is $\det(A-\lambda I)$ (of degree n) and its roots are the *eigenvalues* of A.
2. For each eigenvalue λ, the associated *eigenspace* is $N(A-\lambda I)$. To find it, solve $(A-\lambda I)v=0$. Any non-zero vector in $N(A-\lambda I)$ is an *eigenvector* associated to λ.
3. If A is a diagonal matrix with diagonal entries $\lambda_1, \cdots, \lambda_n$, then its eigenvalues are $\lambda_1, \cdots, \lambda_n$ with associated eigenvectors $e_1, \cdots, e_n$ respectively.
4. Write $\det(A-\lambda I) = (\lambda_1 - \lambda)\cdots(\lambda_n - \lambda)$ and expand.

$$\begin{aligned} \text{Trace of } A &= a_{11} + \cdots + a_{nn} \quad \text{(sum of diagonal entries)} \\ &= \lambda_1 + \ldots + \lambda_n \\ \det(A) &= \lambda_1 \cdots \lambda_n \end{aligned}$$

Thus: If λ_1, …, λ_n are real numbers, then $\mathrm{Tr}(A) =$ sum of eigenvalues, and $\det(A) =$ product of eigenvalues.

Similarity and Eigenvalues

Defn. The $n \times n$ matrices A and B are *similar*, if there exists an invertible matrix P such that $P^{-1}AP = B$.

Observe: If $B = P^{-1}AP$, then (i) $\det(A) = \det(B)$, and (ii) $B^n = P^{-1}A^nP$ for each n.

Theorem: If A and B are similar, then they have the same characteristic polynomial. In particular, they have the same eigenvalues, $\det(A) = \det(B)$ and $\mathrm{Trace}(A) = \mathrm{Trace}(B)$.

Proof. Given: $B = P^{-1}AP$. prove: $\det(A - \lambda I) = \det(B - \lambda I)$.
Note: It is enough to prove that $A - \lambda I$ and $B - \lambda I$ are similar!
Indeed, $B - \lambda I = P^{-1}AP - \lambda P^{-1}P$

$$= P^{-1}(A - \lambda I)P. \qquad \square$$

Write $\det(A - \lambda I) = (\lambda_1 - \lambda) \cdots (\lambda_n - \lambda)$. Compare constant coeff.: $\det(A) = \lambda_1 \cdots \lambda_n = \det(B)$; Compare coeff. of λ^{n-1}:
Sum of diagonal entries $= a_{11} + \cdots + a_{nn} =$ Trace of $A =$ $\lambda_1 + \ldots + \lambda_n =$ Trace of B.

Ques: How are eigenvalues of A and B related?

Diagonizability: Introduction

Note: Finding roots of characteristic polynomials (and hence eigenvalues) is difficult in general.
For $n \geq 5$, no formula exists for roots. (Abel, Galois)
For $n = 3, 4$, formulae for root exist, but not easy to use.

Defn. An $n \times n$ matrix A is diagonalizable if A is similar to a diagonal matrix Λ, i.e., there is an invertible matrix P and a diagonal matrix Λ such that $P^{-1}AP = \Lambda$.

Importance of Diagonalizability:

Let the $n \times n$ matrix A be diagonalizable, i.e., $P^{-1}AP = \Lambda$, where P is invertible and Λ is diagonal. If this happens,

- The eigenvalues of A are the diagonal entries of Λ,
- $\det(A)$ is the product of the diagonal entries of Λ, and
- $\text{Trace}(A) =$ sum of the diagonal entries of Λ.
- **Other Information:** e.g., what is $\text{Trace}(A^n)$?

Example: $A = \begin{pmatrix} 1 & 5 & 6 \\ 0 & 2 & -4 \\ 0 & 0 & 3 \end{pmatrix}$ is triangular.

$\det(A - \lambda I) = (1-\lambda)(2-\lambda)(3-\lambda)$.
Eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$.

Note: If A is triangular, its eigenvalues are on the diagonal

Eigenvectors: $v_1 = e_1$, $v_2 = \begin{pmatrix} 5 & 1 & 0 \end{pmatrix}^T$, $v_3 = \begin{pmatrix} -7 & -4 & 1 \end{pmatrix}^T$.
(**How?**)Further, $\{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$. Hence
$P = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$ is invertible , and
$AP = \begin{pmatrix} Av_1 & Av_2 & Av_3 \end{pmatrix} = \begin{pmatrix} v_1 & 2v_2 & 3v_3 \end{pmatrix} = P\Lambda$, where
$\Lambda = \begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix}$. Thus $P^{-1}AP = \Lambda$, i.e., A is diagonalizable.

Example: If $\mathscr{B} = \{v_1, v_2, v_3\}$, and $T(v) = Av$, then
$[T]_{\mathscr{B}}^{\mathscr{B}} =$ ___.

Eigenvalue Decomposition (EVD)

Question: What is the advantage of a basis of $\mathbb{R}^n$ consisting of eigenvectors?

Let A be an $n \times n$ matrix with n eigenvectors $v_1, \ldots, v_n$, associated to eigenvalues $\lambda_1, \ldots, \lambda_n$. If $\mathscr{B} = \{v_1, \ldots, v_n\}$ is a basis of $\mathbb{R}^n$, then the matrix $P = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$ is invertible.

Moreover, $AP = A\begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix} = \begin{pmatrix} Av_1 & \cdots & Av_n \end{pmatrix}$

$$= \begin{pmatrix} \lambda_1 v_1 & \cdots & \lambda_n v_n \end{pmatrix} = P\Lambda, \text{ where } \Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}.$$

Therefore $P^{-1}AP = \Lambda$, i.e., A is similar to a diagonal matrix.

Thus: $\boxed{\text{Eigenvectors diagonalize a matrix}}$

Eigenvalue Decomposition (EVD): Let A be diagonalizable. With notation as above, we have $A = P\Lambda P^{-1}$.
This is called as the eigenvalue decompostion (EVD) of A.

Theorem A is diagonalizable $\Leftrightarrow$ A has n linearly independent eigenvectors. In particular, $\mathbb{R}^n$ has a basis consisting of eigenvectors of A.

Proof. ($\Leftarrow$): Done! To prove ($\Rightarrow$), assume $P = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$ is an invertible matrix such that $P^{-1}AP = \Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$.

Then $AP = P\Lambda$, i.e. $\begin{pmatrix} Av_1 & \dots & Av_n \end{pmatrix} = \begin{pmatrix} \lambda_1 v_1 & \dots & \lambda_n v_n \end{pmatrix}$.

Therefore $v_1, \dots, v_n$ are eigenvectors of A. They are linearly independent since P is invertible. $\square$

Question: Is every matrix is diagonalizable? **A:** No.

Examples: $Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ no eigenvalues (over $\mathbb{R}$)!

$P = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ not enough eigenvectors!

When is A Diagonalizable?

Ques: When does A have n linearly independent eigenvectors?

• If $v_1, \dots, v_r$ are eigenvectors of A associated to distinct eigenvalues $\lambda_1, \dots, \lambda_r$, then $v_1, \dots, v_r$ are linearly independent.

Proof. Suppose $v_1, \dots, v_r$ are linearly dependent. Choose a linear relation involving minimum number of v_i's, say

$$(1) \quad a_1 v_1 + \cdots + a_t v_t = 0. \quad (1 < t \leq r,\ t \text{ is minimal, } a_i \neq 0)$$

Apply A to get $\quad a_1 \lambda_1 v_1 + \cdots + a_t \lambda_t v_t = 0 \qquad (2)$

$\lambda_1(1) - (2)$ gives $\quad a_2(\lambda_1 - \lambda_2) v_2 + \cdots + a_t(\lambda_1 - \lambda_t) v_t = 0,$

which contradicts the minimality of t. $\square$

• If A has n distinct eigenvalues, then A is diagonalizable.

Proof. If $v_1, \dots, v_n$ are eigenvectors associated to distinct eigenvalues $\lambda_1, \dots, \lambda_n$, then $\{v_1, \dots, v_n\}$ is linearly independent. Then $P = \begin{pmatrix} v_1 & \dots & v_n \end{pmatrix}$ is invertible, and $P^{-1}AP = \Lambda$ as seen earlier. Hence A is diagonalizable. $\square$

• If λ is an eigenvalue of A, μ is an eigenvalue of B, is $\lambda\mu$ an eigenvalue of AB?

False Proof. $$ABx = A(\mu x) = \mu(Ax) = \lambda\mu x.$$

This is false since A and B may not have same eigenvector x.

• **Example:** $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

The eigenvalues of A and B are $0,0$ and that of AB are $1,0$.

• Eigenvalues of $A+B$ are NOT $\lambda+\mu$.

In above example, $A+B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ has eigenvalues $1,-1$.

• If A and B have same eigenvectors associated to λ and μ, then $\lambda\mu$ and $\lambda+\mu$ are eigenvalues of AB and $A+B$ respectively.

Question: When do A and B have the same eigenvectors?