MA-110 Linear Algebra and Differential Equations

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Recap

• If A is invertible then the system $Ax = b$ has a unique solution for every b.

• Since $x = 0$ is always a solution of $Ax = 0$, if $Ax = 0$ has a non-zero solution, then A is not invertible by the last remark.

• If A is invertible, then the Gaussian elimination of A produces n pivots.

• A diagonal matrix A is invertible if and only if A is _____? .

• Since $AB = \begin{pmatrix} AB_{*1} & AB_{*2} \cdots AB_{*n} \end{pmatrix}$and $I = \begin{pmatrix} e_1 & e_2 \cdots e_n \end{pmatrix}$,
if $B = A^{-1}$, then B_{*j} is a solution of $Ax = e_j$ for all j.

• Strategy to find A^{-1}: Let A be an $n \times n$ invertible matrix.
Solve $Ax = e_1$, $Ax = e_2$, ..., $Ax = e_n$.

Solutions to Multiple Systems

Q: Let $A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$, $b_1 = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$ $b_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$. Solve for $Ax = b_1$ and $Ax = b_2$.

Do we apply Gaussian Elimination on two augmented matrices?

Rephrased question: Let $B = \begin{pmatrix} b_1 & b_2 \end{pmatrix}$. Is there a matrix C such that $AC = B$, i.e., such that $AC_{*1} = b_1$, $AC_{*2} = b_2$?

$$[A|B] = \left(\begin{array}{ccc|cc} 0 & 1 & -1 & -1 & 1 \\ 1 & 0 & 2 & 2 & 0 \\ 1 & 2 & 0 & 0 & 2 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|cc} \mathbf{1} & 0 & 2 & 2 & 0 \\ 0 & 1 & -1 & -1 & 1 \\ 1 & 2 & 0 & 0 & 2 \end{array}\right)$$

$$\xrightarrow{R_3 - R_1} \left(\begin{array}{ccc|cc} 1 & 0 & 2 & 2 & 0 \\ 0 & \mathbf{1} & -1 & -1 & 1 \\ 0 & 2 & -2 & -2 & 2 \end{array}\right) \xrightarrow{R_3 - 2R_2} \left(\begin{array}{ccc|cc} 1 & 0 & 2 & 2 & 0 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Q: Are $Ax = b_1$ and $Ax = b_2$ both consistent?

Solutions to Multiple Systems (Contd.)

Q: Given matrices A, $B = \begin{pmatrix} b_1 & b_2 \end{pmatrix}$, is there a matrix C such that $AC = B$?

$$[A|B] = \begin{pmatrix} 0 & 1 & -1 & | & -1 & 1 \\ 1 & 0 & 2 & | & 2 & 0 \\ 1 & 2 & 0 & | & 0 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 2 & | & 2 & 0 \\ 0 & 1 & -1 & | & -1 & 1 \\ 0 & 0 & 0 & | & 0 & 0 \end{pmatrix}$$

A solution to $Ax = b_1$ is $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, and to $Ax = b_2$ is $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

(Verify)! So $C = \begin{pmatrix} e_3 & e_2 \end{pmatrix}$ works! Is it unique?

Revisit the question about matrix inverses. Can you find inverse of a matrix this way?

Finding inverse of matrix

Strategy : . Let A be an $n \times n$ matrix. If $v_1, v_2, \ldots, v_n$ are solutions of $Ax = e_1$, $Ax = e_2$, $\ldots$, $Ax = e_n$ respectively, then if it exists, $\boxed{A^{-1} = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}.}$

If $Ax = e_j$ is not solvable for some j, then A is not invertible.

Thus, finding A^{-1} reduces to solving multiple systems of linear equations with the same coefficient matrix.

Consider the previous example, A. Is it invertible?

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

Observe: In the above process, we used a *row exchange*: $R_1 \longleftrightarrow R_2$ and *elimination using pivots*: $R_3 = R_3 - R_1$, $R_3 = R_3 - 2R_2$. Row operations can be achieved by left multiplication by special matrices.

Example: $E\mathbf{x} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} u \\ v-2u \\ w \end{pmatrix}.$

If $A = \begin{pmatrix} A_{*1} & A_{*2} & A_{*3} \end{pmatrix}$, then $EA = \begin{pmatrix} EA_{*1} & EA_{*2} & EA_{*3} \end{pmatrix}$.

Thus, EA has the same effect on A as the row operation $R_2 \mapsto R_2 + (-2)R_1$ on the matrix A.

Note: E is obtained from the identity matrix I by the row operation $R_2 \mapsto R_2 + (-2)R_1$.

Such a matrix (diagonal entries 1 and atmost one off-diagonal entry non-zero) is called an *elementary* matrix.

Notation: $\boxed{E := E_{21}(-2).}$ Similarly define $E_{ij}(\lambda)$.

Row Operations: Permutation Matrices

Example: $P\,x = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} v \\ u \\ w \end{pmatrix}$

If $A = \begin{pmatrix} A_{*1} & A_{*2} & A_{*3} \end{pmatrix}$, then $PA = \begin{pmatrix} PA_{*1} & PA_{*2} & PA_{*3} \end{pmatrix}$.

Thus PA has the same effect on A as the row interchange $R_1 \longleftrightarrow R_2$.

Note: We get P from the I by interchanging first and second rows. A matrix is called a *permutation* matrix if it is obtained from identity by row exchanges (possibly more than one).

Notation: $\boxed{P = P_{12}.}$ Similarly define P_{ij}.

Remark: Row operations correspond to multiplication by elementary matrices $E_{ij}(\lambda)$ or permutation matrices P_{ij} on the left.

Elementary Matrices: Inverses

For any $n \times n$ matrix A, observe that the row operations $R_2 \mapsto R_2 - 2R_1, R_2 \mapsto R_2 + 2R_1$ leave the matrix unchanged. In matrix terms, $E_{21}(2)E_{21}(-2)A = IA = A$ since

$$E_{21}(-2)\ E_{21}(2) = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

- If $E_{21}(\lambda) = \begin{pmatrix} 1 & 0 & 0 \\ \lambda & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, what is your guess for $E_{21}(\lambda)^{-1}$?

Verify.

- Let $P_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e_2^T \\ e_1^T \\ e_3^T \end{pmatrix}$. What is P_{12}^T? $P_{12}^T P_{12}$? P_{12}^{-1}?

Permutation Matrices: Inverses

Notice that the row interchange $R_1 \longleftrightarrow R_2$ followed by $R_1 \longleftrightarrow R_2$ leaves a matrix unchanged.
In matrix terms, $P_{12}P_{12}A = IA = A$, since

$$P_{12}P_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

• Let P_{ij} be obtained by interchanging the ith and jth rows of I. Show that $P_{ij}^T = P_{ij} = P_{ij}^{-1}$.

• Let $P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} e_3^T \\ e_1^T \\ e_2^T \end{pmatrix}$. Show that $P = P_{12}P_{23}$.
Hence, $P^{-1} = (P_{12}P_{23})^{-1} = P_{23}^{-1}P_{12}^{-1} = P_{23}^TP_{12}^T = P^T$.

$$\text{Consider} \quad \begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 9 \end{pmatrix} \quad (Ax = b)$$

$\boxed{\text{Step 1}}$ Eliminate u by $R_2 \mapsto R_2 + (-2)R_1, \quad R_3 \mapsto R_3 + R_1$.

This corresponds to multiplying both sides on the left first by $E_{21}(-2)$ and then by $E_{31}(1)$. The equivalent system is:

$$E_{31}(1)E_{21}(-2)Ax = E_{31}(1)E_{21}(-2)b, \text{ i.e.,}$$

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 5 \\ -12 \\ 14 \end{pmatrix}.$$

$\boxed{\text{Step 2}}$ Eliminate v by $R_3 \mapsto R_3 + R_2$,
i.e., multiply both sides by $E_{32}(1)$ to get $Ux = c$,
where $U = E_{32}(1)E_{31}(1)E_{21}(-2)A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{pmatrix}$ and

$$c = E_{32}(1)E_{31}(1)E_{21}(-2)b = \begin{pmatrix} 5 \\ -12 \\ 2 \end{pmatrix}.$$

Elimination changed A to an upper triangular matrix and reduced the problem to solving $Ux = c$.

Observe: The pivots of the system $Ax = b$ are *the diagonal entries of* U.

Triangular Factorization

Thus $\boxed{Ax = b}$ is equivalent to $\boxed{Ux = c}$.

where

$$E_{32}(1)\ E_{31}(1)\ E_{21}(-2)\ A = U$$

Multiply both sides by $E_{32}(-1)$ on the left:

$$E_{31}(1)\ E_{21}(-2)\ A = E_{32}(-1)U$$

Multiply first by $E_{31}(-1)$ and then $E_{21}(2)$ on the left:

$$A = E_{21}(2)\ E_{31}(-1)\ E_{32}(-1)\ U = LU$$

where U is upper triangular, which is obtained by forward elimination, with diagonal entries as pivots and

$$L = E_{21}(2)\ E_{31}(-1)\ E_{32}(-1).$$

Triangular Factorization

Note that each $E_{ij}(a)$ is a *lower triangular*. Product of lower triangular matrices is lower triangular. In particular L is lower triangular, where

$$L = E_{21}(2)\ E_{31}(-1)\ E_{32}(-1) =$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \mathbf{2} & 1 & 0 \\ \mathbf{-1} & \mathbf{-1} & 1 \end{pmatrix}$$

Observe: L is lower triangular with diagonal entries 1 and *below the diagonals are* the multipliers.
$(2, -1, -1$ in the earlier example).