MA-110 Linear Algebra and Differential Equations

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Lecture 3 D3

Recap Choosing pivots: Two examples

Example 2: 3 equations in 3 unknowns (u, v, w)
$0u + v + 2w = 1, \quad 0u + 6v + 4w = -2, \quad 0u + 7v - 2w = -9.$

$$[A|b] = \begin{pmatrix} \mathbf{0} & \mathbf{1} & 2 & | & 1 \\ \mathbf{0} & 6 & 4 & | & -2 \\ \mathbf{0} & 7 & -2 & | & -9 \end{pmatrix} \to \begin{pmatrix} 0 & \mathbf{1} & 2 & | & 1 \\ 0 & 0 & \mathbf{-8} & | & -8 \\ 0 & 0 & -16 & | & -16 \end{pmatrix}$$

Coefficient of u is 0 in every equation. The first pivot is 1 and we eliminate v from the second and third equations. Solve for w and v to get $w = 1$, and $v = -1$.

Note: $(0, -1, 1)$ is a solution of the system. So is $(1, -1, 1)$. In general, $(*, -1, 1)$ is a solution, for any real number $*$.

Observe: Unique solution is not an option. Why? This system has infinitely many solutions.

Q: Does such a system always have infinitely many solutions?
A: Depends on the constant vector b.

Exercise: Find 3 vectors b for which the above system has (i) no solutions (ii) infinitely many solutions.

Summary: Pivots

- Can a pivot be zero? No (since we need to divide by it).
- If the first pivot (coefficient of 1st variable in 1st equation) is zero, then interchange it with next equation so that you get a non-zero first pivot. Do the same for other pivots.
- If the coefficient of the 1st variable is zero in every equation, consider the 2nd variable as 1st and repeat the previous step.
- Consider system of n equations in n variables.

  The non-singular case, i.e. the system has **exactly** n pivots: The system has a unique solution.

  The singular case, i.e., the system has **atmost** $n-1$ pivots: The system has no solutions, i.e., it is inconsistent, or it will have infinitely many solutions, provided it is consistent.

What is a matrix?

A *matrix* is a collection of numbers arranged into a fixed number of rows and columns.
If a matrix A has m rows and n columns, the size of A is $m \times n$.

The rows of A are denoted $A_{1*}, A_{2*}, \ldots, A_{m*}$, i.e., $A = \begin{pmatrix} A_{1*} \\ A_{2*} \\ \vdots \\ A_{m*} \end{pmatrix}$,

the columns are denoted $A_{*1}, A_{*2} \ldots, A_{*n}$, i.e.,
$A = \begin{pmatrix} A_{*1} & A_{*2} & \cdots & A_{*n} \end{pmatrix}$, and the (i,j)th entry is A_{ij} (or a_{ij}).

Operations on Matrices: Matrix Addition

Example 1. We know how to add two row or column vectors.

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} + \begin{pmatrix} -3 & -2 & -1 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 2 \end{pmatrix} \quad \text{(component-wise)}$$

$\boxed{\text{We can add matrices if and only if they have the same size}}$,

and the addition is component-wise.
Example 2.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \end{pmatrix} + \begin{pmatrix} -1 & -4 & -2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -2 & 1 \\ 2 & 2 & 5 \end{pmatrix}$$

Thus

$$(A+B)_{i*} = A_{i*} + B_{i*} \text{ and } (A+B)_{*j} = A_{*j} + B_{*j}$$

Linear Systems: Multiplying a Matrix and a Vector

One row at a time (dot product): The system

$2u+v+w=5, \quad 4u-6v=-2, \quad -2u+7v+2w=9$

can be rewritten using dot product as follows:

$$\begin{pmatrix}2 & 1 & 1\end{pmatrix}\begin{pmatrix}u\\ v\\ w\end{pmatrix}=5, \quad \begin{pmatrix}4 & -6 & 0\end{pmatrix}\begin{pmatrix}u\\ v\\ w\end{pmatrix}=-2 \quad \text{and}$$

$$\begin{pmatrix}-2 & 7 & 2\end{pmatrix}\begin{pmatrix}u\\ v\\ w\end{pmatrix}=9.$$

Write the system in the $Ax=b$ form:

$$\begin{pmatrix}2 & 1 & 1\\ 4 & -6 & 0\\ -2 & 7 & 2\end{pmatrix}\begin{pmatrix}u\\ v\\ w\end{pmatrix}=\begin{pmatrix}2u+v+w\\ 4u-6v\\ -2u+7v+2w\end{pmatrix}=\begin{pmatrix}5\\ -2\\ 9\end{pmatrix}$$

Note: No. of columns of A = length of the vector x.

Multiplication of a Matrix and a Vector

Dot Product (row method): Ax is obtained by taking dot product of each row of A with x.

If $A = \begin{pmatrix} A_{1*} \\ A_{2*} \\ A_{3*} \end{pmatrix}$, then $\boxed{Ax = \begin{pmatrix} A_{1*} \cdot x \\ A_{2*} \cdot x \\ A_{3*} \cdot x \end{pmatrix}}$

Linear Combinations (column method):
The column form of the system
$2u + v + w = 5, \quad 4u - 6v = -2, \quad -2u + 7v + 2w = 9$ is:

$$u \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} + v \begin{pmatrix} 1 \\ -6 \\ 7 \end{pmatrix} + w \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

Thus Ax is a linear combination of columns of A, with the coordinates of x as weights, i.e., $\boxed{Ax = uA_{*1} + vA_{*2} + wA_{*3}}$.

An Example

Let $A = \begin{pmatrix} 1 & 3 & -3 & -1 \\ 1 & 2 & 0 & -2 \\ 1 & 0 & -2 & 0 \end{pmatrix}$, $x = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, and $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

$A_{1*} = \begin{pmatrix} 1 & 3 & -3 & -1 \end{pmatrix}$, $A_{2*} = \begin{pmatrix} 1 & 2 & 0 & -2 \end{pmatrix}$ $A_{3*} = ?$.

Then $A_{1*} \cdot x = ?$, $A_{2*} \cdot x = 0$, $A_{3*} \cdot x = 0$, hence $Ax = \begin{pmatrix} ? \\ ? \\ ? \end{pmatrix}$.

Q: What is Ae_1? **A:** The first column A_{*1} of A.

Exercise:

What should x be so that $Ax = A_{*j}$, the jth column of A?

Observe: No. of rows of Ax = No. of rows of A,
and No. of columns of Ax = No. of columns of x.

Question: What can you say about the solutions of $Ax = 0$?

$\boxed{\text{Two matrices } A \text{ and } B \text{ can be multiplied if and only if}}$

$\boxed{\text{no. of columns of } A = \text{no. of rows of } B}$.

If A is $m \times \underline{n}$ and B is $\underline{n} \times r$, then AB is $m \times r$.

Key Idea: We know how to multiply a matrix and a vector.

Column wise: Write B column-wise, i.e., let
$B = \begin{pmatrix} B_{*1} & B_{*2} & \cdots & B_{*r} \end{pmatrix}$. Then

$$AB = \begin{pmatrix} AB_{*1} & AB_{*2} & \cdots & AB_{*r} \end{pmatrix}$$

Note: Each B_{*j} is a column vector of length n. Hence, AB_{*j} is a column vector of length m. So, the size of AB is $m \times r$.

Row wise: Write A row-wise, i.e., let $A_{1*}, \ldots, A_{m*}$ be the rows of A. Then

$$AB = \begin{pmatrix} A_{1*} \\ \vdots \\ A_{m*} \end{pmatrix} B = \begin{pmatrix} A_{1*}B \\ \vdots \\ A_{m*}B \end{pmatrix}$$

Note: Each A_{i*} is a row vector of size $1 \times n$. Hence, $A_{i*}B$ is a row vector of size $1 \times r$. So, the size of AB is $m \times r$.

Working Rule:

The entry in the ith row and jth column of AB is the dot product of the ith row of A with the jth column of B, i.e., $(AB)_{ij} = A_{i*} \cdot B_{*j}$.

If A is $m \times n$ and B is $n \times r$.

a) $(AB)_{ij} = A_{i*} \cdot B_{*j}$ = (ith row of A) $\cdot$ (jth column of B)

b) jth column of AB $=A \cdot$ (jth column of B) , i.e., $(AB)_{*j} = AB_{*j}$.

c) ith row of AB =(ith row of A) $\cdot$ B , i.e., $(AB)_{i*} = A_{i*}B$.

Properties of Matrix Multiplication:

- (associativity) $(AB)C = A(BC)$. Why?
- (distributivity) $A(B + C) = AB + AC$. How to verify?

  $(B + C)D = BD + CD$. Why?
- (non-commutativity) $AB \neq BA$, in general. Why? Find examples.