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Roll No : 12345 (e.g., 190040001) **Dept.**: CSE (e.g., CSE)

IIT Bombay
CS 6002: SAMD
Quiz 2, 2024-25-II
Date: March 21, 2025

CS 6002: Selected Areas of Mechanism Design

Total: **20 marks,** *Duration:* **45 minutes, ATTEMPT ALL QUESTIONS**

Instructions:

1. This question-and-answersheet booklet contains a total of 5 sheets of paper (10 pages, pages 2 and 10 are blank). Please verify.
2. Write your roll number and department on **every side of every sheet** (except the blank sheet) of this booklet. Use only **black/blue ball-point pen**. The first 5 minutes of additional time is given exclusively for this activity.
3. Write final answers neatly with a pen **only in the given boxes**.
4. Use the rough sheets for scratch works / attempts to solution. **Write only the final solution (which may be a sequence of logical arguments) in a precise and succinct manner in the boxes provided**. Do not provide unnecessarily elaborate steps. The space within the boxes are sufficient for the correct and precise answers.
5. Submit your answerscripts to the teaching staff when you leave the exam hall or the time runs out (whichever is earlier). **Your exam will not be graded if you fail to return the paper**.
6. **This is a closed book, notes, internet exam. No communication device, e.g., cellphones, iPad, etc., is allowed**. Keep it switched off in your bag and keep the bag away from you. If anyone is found in possession of such devices during the exam, that answerscript may be disqualified for evaluation and DADAC may be invoked.
7. Two A4 assistance sheets (**text on both sides**) are allowed for the exam.
8. **After you are done with your exam or the exam duration is over, please DO NOT rush to the desk for submitting your paper**. Please remain seated until we collect all the papers, count them, and give a clearance to leave your seat.

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Problem 1 (8 points). Shapley Value. Answer the following questions.

(a) Consider a *transferrable utility* (TU) coalitional game $\langle N, v\rangle$, where $n = |N|$. Let $a \in R^n$ be a vector. Compute the Shapley value of the game defined by the characteristic function as follows.

$$v(S) = \left(\sum_{i \in S} a_i\right)^2, \forall S \subseteq N.$$

Find the expression of the Shapley value in its most concise form, i.e., express $\mathrm{Sh}_i(N, v)$ only in terms of a_i and $v(N)$. Show the essential steps and arguments of your derivation. **1 + 3 points.**

$$Sh_i(N,v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} v(P_i(\pi) \cup \{i\}) - v(P_i(\pi)) \quad \text{--- } (1)$$

Now note: $v(P_i(\pi) \cup \{i\}) = \left(\sum_{j \in P_i(\pi)} a_j + a_i\right)^2$

and $v(P_i(\pi)) = \left(\sum_{j \in P_i(\pi)} a_j\right)^2$

$$\Rightarrow v(P_i(\pi) \cup \{i\}) - v(P_i(\pi)) = a_i\left(a_i + 2\sum_{j \in P_i(\pi)} a_j\right)$$

Substituting this in (1), we get

$$Sh_i(N,v) = \frac{1}{n!} a_i^2 \cdot n! + \frac{1}{n!} 2a_i \underbrace{\sum_{\pi \in \Pi(N)} \sum_{j \in P_i(\pi)} a_j}_{(2)}$$

write (2) as

$$\underbrace{\sum_{\pi \in \Pi(N)} \sum_{j \in N}}_{\text{this trick we have used in class multiple times}} I\{j \in P_i(\pi)\} a_j = \sum_{j \in N} \sum_{\pi \in \Pi(N)} I\{j \in P_i(\pi)\} a_j$$

order of summation switch

$$= \sum_{j \in N} a_j \underbrace{\sum_{\pi \in \Pi(N)} I\{j \in P_i(\pi)\}}_{= \frac{n!}{2} \ \forall j \neq i} = n! \frac{1}{2} \sum_{j \in N \setminus \{i\}} a_j = n! \frac{1}{2}\left(\sqrt{v(N)} - a_i\right)$$

since over all permutations π, j will be before i exactly half the time

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Substituting for (2) we get

$$Sh_i(N,v) = a_i^2 + 2a_i \cdot \frac{1}{2}\left(\sqrt{v(N)} - a_i\right)$$

$$\boxed{Sh_i(N,v) = a_i\left(\cancel{a_i +} \sqrt{v(N)}\right)}$$

(b) In the class we defined Shapley value as

$$\mathrm{Sh}_i(N, v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} \left(v(P_i(\pi) \cup \{i\}) - v(P_i(\pi))\right),$$

where $\Pi(N)$ is the set of all possible permutations of the n players and $P_i(\pi)$ is the set of predecessors of i in π. We will call this representation as the **permutation** representation. Show that this expression can also be written in the **set-based** representation defined as follows.

$$\mathrm{Sh}_i(N, v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n - |S| - 1)!}{n!} \left(v(S \cup \{i\}) - v(S)\right)$$

To prove this, start from the permutation representation and arrive at the set-based representation. Show the essential steps and arguments of your derivation. **1 + 3 points.**

In the first representation, since the sum is over all permutations π, $P_i(\pi)$ (the predecessor set of i in π) will take all possible subsets $S \subseteq N \setminus \{i\}$. Hence we can write the expression as follows

$$Sh_i(N,v) = \frac{1}{n!} \sum_{S \subseteq N \setminus \{i\}} \sum_{\pi \in \Pi(N):\, P_i(\pi) = S} \left[v(S \cup \{i\}) - v(S)\right]$$

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$$Sh_i(N,v) = \frac{1}{n!} \sum_{S \subseteq N\setminus\{i\}} \sum_{\pi \in \Pi(N):\, P_i(\pi) = S} \underbrace{\left[v(S\cup\{i\}) - v(S)\right]}$$

note: this term is a constant for the inner summation

$$= \frac{1}{n!} \sum_{S \subseteq N\setminus\{i\}} v(S\cup\{i\}) - v(S) \underbrace{\left(\sum_{\pi \in \Pi(N):\, P_i(\pi) = S} 1\right)}$$

Note:

given S

the number of permutations π that has this structure is $|S|!\,(n-|S|-1)!$

$$\underbrace{\qquad\qquad}_{\text{permutation contains the same set of agents } S}\; i \;\underbrace{\qquad\qquad}_{N\setminus\{i\}\cup S}$$

Substituting the value above in the expression, we get

$$Sh_i(N,v) = \sum_{S \subseteq N\setminus\{i\}} \frac{|S|!\,(n-|S|-1)!}{n!} \left(v(S\cup\{i\}) - v(S)\right)$$

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Problem 2 (7 points). Dual Games. For every TU coalitional game $\langle N, v\rangle$, where $n = |N|$, define its **dual** game $\langle N, v^*\rangle$ as follows:

$$v^*(S) = v(N) - v(N \setminus S).$$

(a) Find the relationship between the Shapley value of a TU game $\text{Sh}_i(N, v)$ and its dual game $\text{Sh}_i(N, v^*)$. Show the essential steps and arguments of your derivation. **2 points.**

Using the alternative definition of Shapley value as obtained in Problem 1(b).

$$\text{Sh}_i(N, v^*) = \sum_{S \subseteq N\setminus\{i\}} \frac{|S|!\,(n-|S|-1)!}{n!} \Big(\underbrace{v^*(S\cup\{i\}) - v^*(S)}_{v(N\setminus S) - v(N\setminus(S\cup\{i\}))}\Big)$$

Set $T = N\setminus(S\cup\{i\}) \Rightarrow |T| = n-|S|-1 \Rightarrow n-|T|-1 = |S|$

also we can always define a bijection s.t. $S \subseteq N\setminus\{i\}$ corresponds to a unique $T = N\setminus(S\cup\{i\}) \subseteq N\setminus\{i\}$

Hence $\text{Sh}_i(N, v^*) = \sum_{T\subseteq N\setminus\{i\}} \frac{|T|!\,(n-|T|-1)!}{n!}\big(v(T\cup\{i\}) - v(T)\big)$

$= \text{Sh}_i(N, v)$, Hence equal.

(b) Prove or disprove: the core of a TU game $\langle N, v\rangle$ is nonempty if and only if the core of its dual $\langle N, v^*\rangle$ is nonempty. To prove, start from the definitions of TU game. To disprove, provide a concise counterexample. State first which of the two you are doing. **1 + 4 points.**

The statement is false. Even if $C(N, v) \neq \phi$, $C(N, v^*)$ can be empty.

Here is an example.

Consider $N = \{1, 2\}$, $v(1) = v(2) = 0$ and $v(1,2) = 1$

Note $(0,1) \in C(N, v) \neq \phi$.

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But by applying Bondareva-Shapley thm on (N, v^*) we get

$$v^*(N) = 0 < v^*(1) + v^*(2) = 2$$

Hence, $C(N, v^*) = \phi$.

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Problem 3 (5 points). **Convex Games and Shapley Value.** Recall that, a definition of **convex** TU game is as follows.

Definition 1. For every coalitions X and Y such that $X \subseteq Y \subseteq N$ and for every $i \in N \setminus Y$ the following holds:

$$v(X \cup \{i\}) - v(X) \leqslant v(Y \cup \{i\}) - v(Y).$$

Prove or disprove: the Shapley value of a convex game is always in its core. To prove, start from the definitions of TU game and Shapley value. To disprove, provide a concise counterexample. State first which of the two you are doing. **1 + 4 points.**

In the class, we have shown that for every permutation $\pi \in \Pi(N)$ define the imputation x^π given by

$$x_i^\pi = v(P_i(\pi) \cup \{i\}) - v(P_i(\pi)), \quad \forall i \in N,$$

is in the core of a convex game (N, v). By definition of Shapley value, it is the average of each of these imputations over all permutations π, i.e.,

$$Sh_i(N,v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} x_i^\pi$$

Since each $x^\pi \in C(N,v)$ and core is a convex set, we conclude that $Sh(N,v) \in C(N,v)$ for any arbitrary convex game.

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END OF QUESTION PAPER. GOOD LUCK!