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Roll No : 12345   **Dept.**: CSE
e.g., 190040001   e.g., CSE

IIT Bombay
CS 6002: SAMD
Midsem, 2024-25-II
Date: March 1, 2025

CS 6002: Selected Areas of Mechanism Design

Total: **40 marks,** *Duration:* **2 hours, ATTEMPT ALL QUESTIONS**

Instructions:

1. This question-and-answersheet booklet contains a total of 7 sheets of paper (14 pages, pages 2 and 14 are blank). Please verify.
2. Write your roll number and department on **every side of every sheet** (except the blank sheet) of this booklet. Use only **black/blue ball-point pen**. The first 5 minutes of additional time is given exclusively for this activity.
3. Write final answers neatly with a pen **only in the given boxes**.
4. Use the rough sheets for scratch works / attempts to solution. **Write only the final solution (which may be a sequence of logical arguments) in a precise and succinct manner in the boxes provided**. Do not provide unnecessarily elaborate steps. The space within the boxes are sufficient for the correct and precise answers.
5. Submit your answerscripts to the teaching staff when you leave the exam hall or the time runs out (whichever is earlier). **Your exam will not be graded if you fail to return the paper**.
6. **This is a closed book, notes, internet exam. No communication device, e.g., cellphones, iPad, etc., is allowed**. Keep it switched off in your bag and keep the bag away from you. If anyone is found in possession of such devices during the exam, that answerscript may be disqualified for evaluation and DADAC may be invoked.
7. Two A4 assistance sheets (**text on both sides**) are allowed for the exam.
8. **After you are done with your exam or the exam duration is over, please DO NOT rush to the desk for submitting your paper**. Please remain seated until we collect all the papers, count them, and give a clearance to leave your seat.

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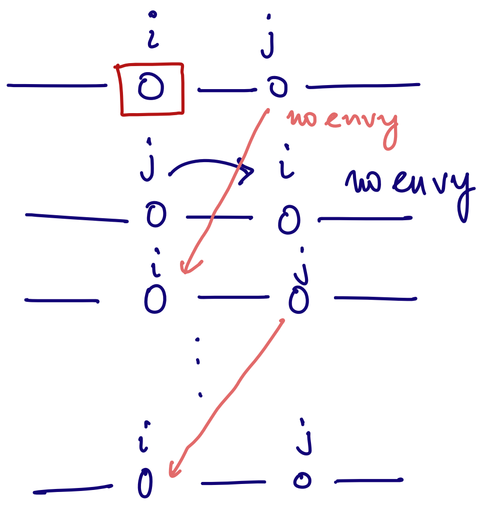
Problem 1 (10 points). Fair division of indivisible goods. Consider a fair division setting where items are goods (represented by the set M) and every agent has additive valuations, i.e., the value of agent i for a bundle $S \subseteq M$ is given by $v_i(S) = \sum_{a \in S} v_i(a)$. Consider the following statements and assert whether they are *true* or *false* along with their justifications.

(a) The **round-robin** (RR) algorithm requires the same order of the agents in every round of the algorithm in order to ensure **envy-freeness upto one good** (EF1)? (Yes/No) **1 point.**

The statement implies that having a permutation σ_1 of the players in one round of RR and a different permutation σ_2 of the players in another round can violate EF1.

No

(b) Explain your answer above. If your answer is 'yes', then provide a small counterexample (preferably with as few agents and items as possible) where changing the RR order in different rounds violate EF1. Otherwise, prove it formally and succinctly from first principles, i.e., not using the results derived in class (however, you are free to use the arguments that are similar). [Note: this part of the question will get no credit if the previous part is incorrect] **4 points.**



Consider an arbitrary pair of agents i and j. We know that if j appears before i in any round of RR, then j doesn't envy i since it gets to pick first. So, the envy can exist only in those rounds where j comes after i. But there too, we know that j prefers his item of a round than the item picked by i in the next round where i came before j. Hence, except the very first item in the first round where i came before j, j does not envy i's remaining bundle. This ensures EF-1 since i and j were arbitrary.

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(c) For any given goods allocation instance with additive valuations, which is given by the set of players N, the set of items M, and the valuations of the agents for the items, every possible EF1 allocation is **achievable** by the RR algorithm with a suitable choice of RR order. (Yes/No) **1 point.**

Here 'achievability' implies that an EF1 allocation on that instance will be obtained in the RR algorithm if it is run with some choice of RR order.

No

(d) Explain your answer above. If your answer is 'yes', then prove it formally and succinctly from first principles, i.e., not using the results derived in class (however, you are free to use the arguments that are similar). Otherwise, provide a small counterexample (preferably with as few agents and items as possible) where a final EF1 allocation cannot be achived via RR and why. [Note: this part of the question will get no credit if the previous part is incorrect] **4 points.**

Consider the following example:

	g_1	g_2	g_3	g_4
1	0.7	0.7	1	1
2	1	1	0.7	0.7

The allocation $1 \to \{g_1, g_2\}$
$2 \to \{g_3, g_4\}$
is EF1.

But if any of its suballocation where one item has not yet been allocated, that partial allocation is not EF1. Since RR always maintains the invariant of EF1 at every step of the algorithm, this allocation is not achievable via RR.

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Problem 2 (10 points). Stronger notion of envy-freeness with indivisible goods. In the class, we discussed two algorithms, namely, round-robin (RR) and envy-cycle elimination (ECE), for computing an allocation satisfying EF1. A fairness notion stronger than EF1 is **envy-freeness up to any good** (EFX), which states that any pairwise envy can be eliminated by removing any good from the envied bundle. Formally, an allocation $A = (A_1, \ldots, A_n)$ satisfies EFX if for every pair of agents i, j and **every** good $g \in A_j$, we have $v_i(A_i) \geqslant v_i(A_j \setminus \{g\})$. Observe that an EFX allocation satisfies EF1.

(a) Tick the correct statement in the additive valuation setting. **2 points.**

- RR satisfies EFX but ECE does not.
- Both RR and ECE satisfy EFX.
- Neither RR nor ECE satisfy EFX. ✓
- ECE satisfies EFX but RR does not.

(b) Explain your answer. If you claim an algorithm satisfies EFX, then prove it formally. Otherwise, provide a small counterexample (preferably with 2 agents and at most 4 goods). [Note: this part of the question will get no credit if the previous part is incorrect] **4 points.**

Consider the following example:

	g_1	g_2	g_3
1	1	2	3
2	1	2	3

For RR consider the picking order to be (1,2), then the allocation is
$\{g_1, g_3\}, \{g_2\}$
which is not EFX (agent 2 envies agent 1 even without g_1)

For ECE, consider the item order is $g_1 \to g_2 \to g_3$
WLOG, suppose agent 1 is given g_1, then the final allocation is again $\{g_1, g_3\}, \{g_2\}$, which is not EFX due to the reason explained earlier.

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(c) Consider the indivisible goods problem under additive valuations. Show that when agents have identical rankings of the goods (but not necessarily identical numerical values), an EFX allocation can be computed in polynomial time. For example, in the instance given below, both agents rank the goods as $g_1 \succ g_2 \succ g_3$ but have different numerical valuations for them. Also note that the **ranking of bundles** need not be identical; indeed, agent 2 prefers the bundle $\{g_2, g_3\}$ over $\{g_1\}$, while agent 1 prefers the bundle $\{g_1\}$ over $\{g_2, g_3\}$ in the example below. You need to mention the algorithm in bulleted list (pseudocode not needed) and explain its correctness. [Hint: an amalgamation of the algorithms we discussed in class may be useful] **4 points.**

	g_1	g_2	g_3
1	11	7	2
2	8	7	5

Algorithm:

- order the items in decreasing order of the preference (which is common for all agents)
- allocate these items via envy cycle elimination algorithm.

Correctness:

- The addition of a good is always to an agent whose bundle is not envied by anyone (due to ECE algorithm). By the choice of the algorithm every item added in any round is the least preferred

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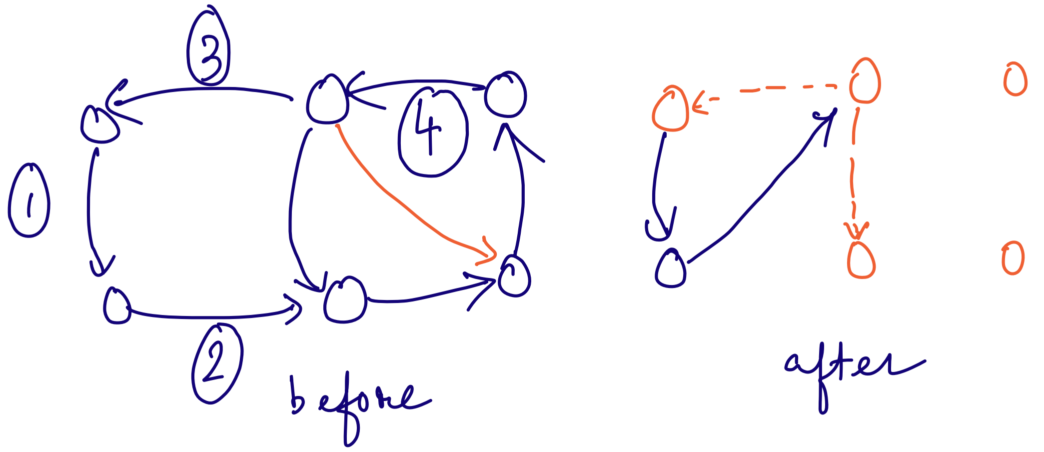
item. Hence it maintains EFX for adding items.

- For resolving EC, note that the current allocation with envy cycle is EFX by the previous argument. When the envy cycle is resolved, the agents in the cycle becomes better off, hence EFX is maintained for them. For the agents outside the cycle, since the bundles remain the same, they still remain EFX.



in the figure above, the same argument used for ECE for EF1 can be extended for EFX proof as well.

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Problem 3 (10 points). Coalitional and Bargaining games. Recall that, in the class we have seen two definitions of **convex** transferrable utility coaltional games.

Definition 1. For every pair of coalitions S and T the following holds:

$$v(S) + v(T) \leqslant v(S \cup T) + v(S \cap T).$$

Definition 2. For every coalitions X and Y such that $X \subseteq Y \subseteq N$ and for every $i \in N \setminus Y$ the following holds:

$$v(X \cup \{i\}) - v(X) \leqslant v(Y \cup \{i\}) - v(Y).$$

(a) Prove that Definition 1 implies Definition 2. **2 points.**

Put $S = X \cup \{i\}$, $T = Y$

by def 1: $v(X \cup \{i\}) + v(Y) \leq v(Y \cup \{i\}) + v(X)$

which is what we wanted to show.

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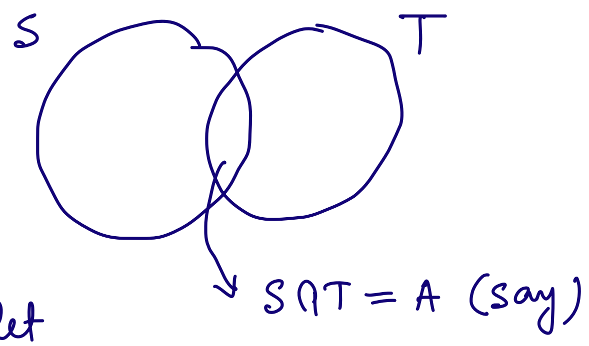
(b) Prove that Definition 2 implies Definition 1. **3 points.**

Consider the most general case where S and T are not contained in one another define $A := T \cap S$.



Since both sets are finite, let us number the elements of $T \setminus A = \{t_1, t_2, \dots, t_k\}$

Note that $A \subseteq S$ and $t_1, t_2, \dots, t_k \notin S$. Hence defn. 2 can be applied to get

$$v(\cancel{S \cup \{t_1\}}) - v(S) \geq v(\cancel{A \cup \{t_1\}}) - v(A)$$

$$v(S \cup \{\cancel{t_1}, t_2\}) - v(\cancel{S \cup \{t_1\}}) \geq v(A \cup \{\cancel{t_1}, t_2\}) - v(\cancel{A \cup \{t_1\}})$$

$$\cancel{\vdots}$$

$$v(S \cup \{t_1, t_2, \dots, t_k\}) - v(\cancel{S \cup \{t_1, \dots, t_{k-1}\}}) \geq v(A \cup (T \setminus A)) - v(\cancel{A \cup \{t_1 \dots t_{k-1}\}})$$

adding all these inequalities, we get

$$v(S \cup T) - v(S) \geq v(T) - v(S \cap T)$$

which is what we wanted to prove.

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(c) Consider a solution concept ϕ for the space $\mathcal{F}$ of **bargaining games**. Recall the following properties of a solution concept (these are not the complete formal definitions and are used only to remind).

(i) ϕ is called **symmetric (SYM)** if for all symmetric bargaining games (S, d), $\phi_1(S, d) = \phi_2(S, d)$.

(ii) ϕ is **efficient (EFF)** if for all $(S, d) \in \mathcal{F}$, we have $\phi(S, d) \in \mathrm{PO}(S)$.

(iii) ϕ is **covariant under positive affine transformations (CPAT)** if for all $(S, d) \in \mathcal{F}$, and for every $g, h \in \mathbb{R}^2, g \gg (0, 0)$, $\phi(gS + h, gd + h) = g\phi(S, d) + h$.

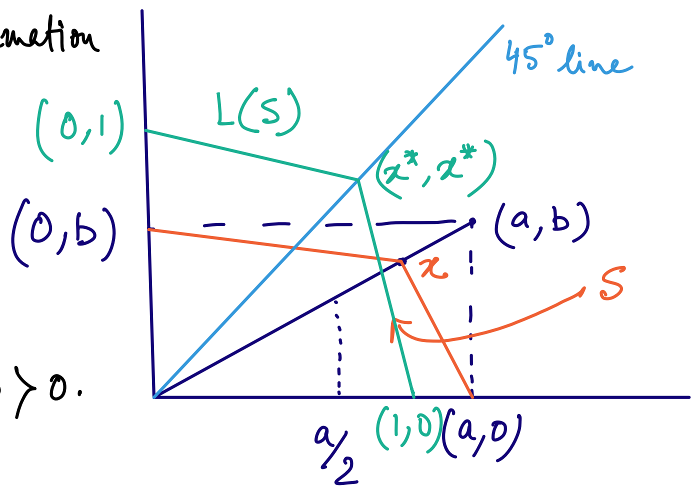
Let ϕ be a solution concept (for $\mathcal{F}$) satisfying SYM, EFF, and CPAT. Let $a, b > 0$ be positive real numbers, and let $x = (x_1, x_2)$ be a point on the ray emanating from $(0, 0)$ and passing through (a, b), satisfying $x_1 > a/2$ and $a \neq b$. Let S be the quadrangle whose vertices are $(0, 0)$, $(a, 0)$, $(0, b)$, and x. With proper arguments, find the value of $\phi(S, (0, 0))$. **2 + 3 points.**

Consider the affine transformation

$$L(z) = \left(\frac{1}{a}, \frac{1}{b}\right) z$$

$$= \left(\frac{z_1}{a}, \frac{z_2}{b}\right)$$

this is well defined since $a, b > 0$.



After this transformation $L(S)$ is again a quadrangle having points at $(0,0)$, $(0,1)$, $(1,0)$ and (x^*, x^*) where $x^* = \frac{x_1}{a} = \frac{x_2}{b}$ (since x lies on the line joining $(0,0)$ and (a,b))

Since $x_1 > \frac{a}{2} \Rightarrow x_2 > \frac{b}{2}$

$\Rightarrow x^* > \frac{1}{2}$. hence using SYM and EFF on $L(S)$

We get $\Phi(L(S), L(0,0)) = (x^*, x^*)$

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Also L is an invertible affine transformation.

Hence using CPAT on L(S) for the transformation L^{-1}, we get

$$L^{-1}\phi\left(L(S), L((0,0))\right) = L^{-1}((x^*, x^*))$$

$$\Rightarrow \quad \phi(S, (0,0)) = (x_1, x_2) = x_0. \qquad \square$$

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Problem 4 (10 points). Balanced two-sided matching. Answer the following questions.

(a) Consider a two-sided balanced matching instance (M, W, P) where the set of men and women are represented by the sets M and W respectively ($|M| = |W|$) and P represents their linear preference profile. Suppose a man is matched with his **least preferred woman** in the men-proposing deferred acceptance (mp-DA) algorithm. What is the maximum number of such men in any instance?

1 point.

1 (one)

(b) Prove your answer above. To prove your claim, write your arguments precisely and succinctly. You may use any result(s) derived in class, but state the result clearly and how you are applying it in your arguments. [Note: this part of the question will get no credit if the previous part is incorrect]

4 points.

Suppose not. There exists at least two men m_1, m_2 who are matched to their least preferred women w_1 and w_2 respectively in mp-DA. This implies that all women except w_1, w_2 rejected both m_1 and m_2 and therefore prefer their current partners more than both m_1 and m_2. Also all those men prefer their current partners more than w_1, w_2.

Consider the matching $m_1 - w_2$ and $m_2 - w_1$. This is a stable matching because no other men or women can form a blocking pair with these men or women (m_1, m_2, w_1, w_2) and $w_2 P_{m_1} w_1$ and $w_1 P_{m_2} w_2$.

But this is a better matching than the men optimal given by mp-DA, which is a contradiction.

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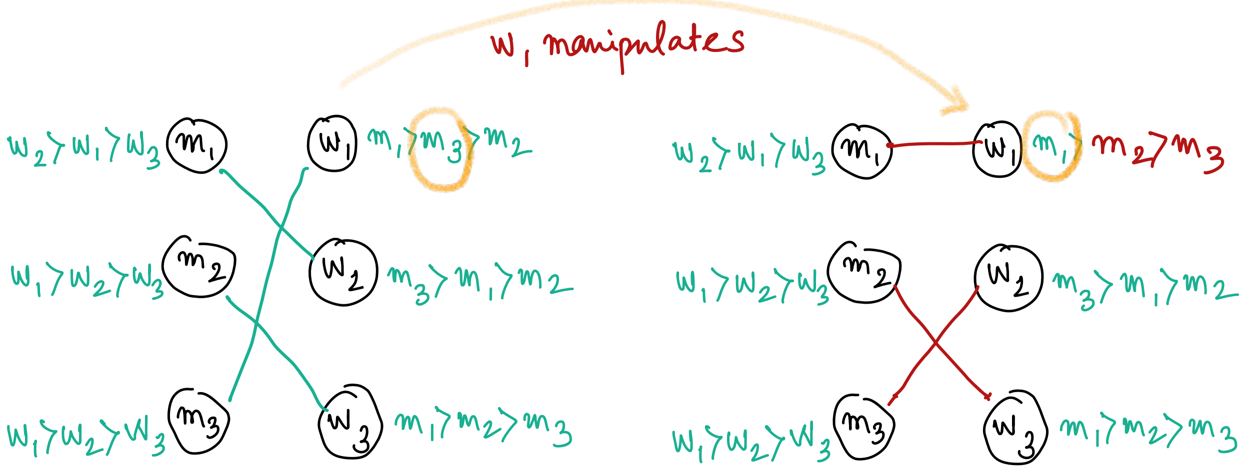
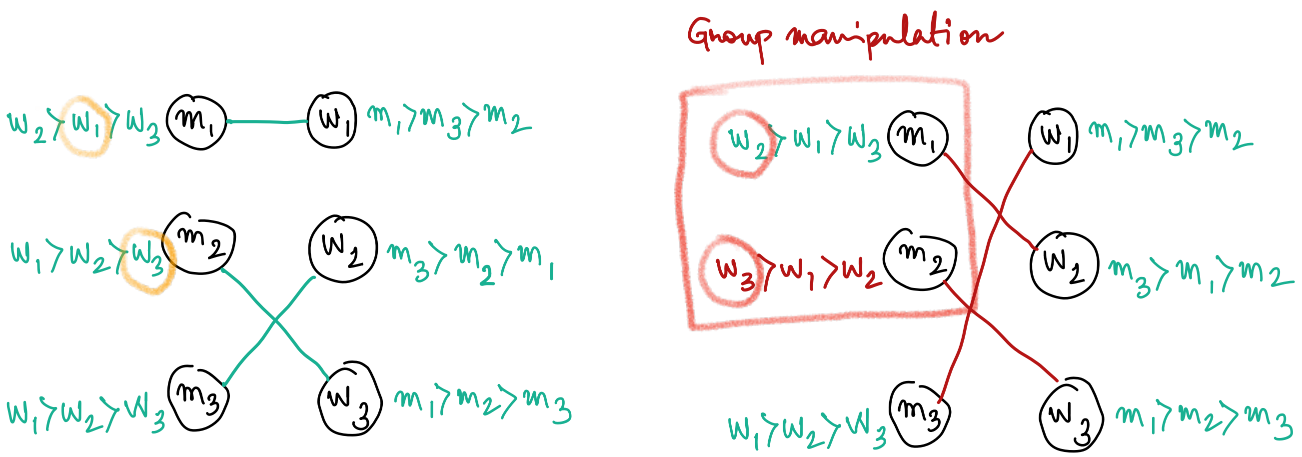
(c) Find a small instance (preferably with at most 3 agents on each side) in which a woman can improve her match in the mp-DA algorithm by misreporting her preference list. **2 points.**



(d) Find a small instance (preferably with at most 3 agents on each side) in which two men can collude, i.e., both work together, in such a way that one of them does better and the other does no worse than in the mp-DA algorithm. **3 points.**



m_1, m_2 gets better off (m_1 strictly and m_2 weakly) by collusion.

END OF QUESTION PAPER. GOOD LUCK!