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Roll No : 12345 **Dept.**:
e.g., 190040001 e.g., CSE

IIT Bombay
CS 6002: SAMD
Quiz 1, 2024-25-II
Date: January 31, 2025

CS 6002: Selected Areas of Mechanism Design

Total: **25 (+bonus) marks,** *Duration:* **1 hour, ATTEMPT ALL QUESTIONS**

Instructions:

1. This question-and-answersheet booklet contains a total of 5 sheets of paper (10 pages, page 2 is blank). Please verify.
2. Write your roll number and department on **every side of every sheet** (except the blank sheet) of this booklet. Use only **black/blue ball-point pen**. The first 5 minutes of additional time is given exclusively for this activity.
3. Write final answers neatly with a pen **only in the given boxes**.
4. Use the rough sheets for scratch works / attempts to solution. **Write only the final solution (which may be a sequence of logical arguments) in a precise and succinct manner in the boxes provided**. Do not provide unnecessarily elaborate steps. The space within the boxes are sufficient for the correct and precise answers.
5. Submit your answerscripts to the teaching staff when you leave the exam hall or the time runs out (whichever is earlier). **Your exam will not be graded if you fail to return the paper**.
6. **This is a closed book, notes, internet exam. No communication device, e.g., cellphones, iPad, etc., is allowed**. Keep it switched off in your bag and keep the bag away from you. If anyone is found in possession of such devices during the exam, that answerscript may be disqualified for evaluation and DADAC may be invoked.
7. Two A4 assistance sheets (**text on both sides**) are allowed for the exam.
8. **After you are done with your exam or the exam duration is over, please DO NOT rush to the desk for submitting your paper**. Please remain seated until we collect all the papers, count them, and give a clearance to leave your seat.

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Problem 1 (5 points). Balanced two-sided matching. In class, we have seen the structure of all possible stable matchings for a given profile of a **balanced** two-sided matching problem (the number of men and women are the same). We saw that there exists an order over the stable matchings from the men (or equivalently women) side. Now, answer the following questions.

(a) Can there be two stable matchings μ and μ' that are incomparable for the men, i.e., μ is preferred to μ' by some man but the opposite is true for another, such that μ (similarly μ') is at least as good as every other stable matching for all men except μ' (similarly μ)? (Yes/No) **1 point.**

No.

(b) Explain your answer above. If your answer is 'yes', then provide a small example (preferably with 3 agents on both sides). Otherwise, prove it formally and succinctly from first principles, i.e., not using the results derived in class (however, you are free to use the arguments that are similar). [Note: this part of the question will get no credit if the previous part is incorrect] **4 points.**

Proof is as done in the class (not repeated here).
The argument is if there indeed existed μ and μ', then using the $\max_{\mu,\mu'}$ function which is shown to be a stable matching (this part needs to be done from the class), we conclude that $\max_{\mu,\mu'}$ is at least as good as both μ and μ' and strictly better for at least one man. Since $\max_{\mu,\mu'}$ is different from both μ and μ', it contradicts the fact that μ (or μ') is at least as good as any other stable matching except μ' (or μ).

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Problem 2 (10 points). Unbalanced two-sided matching. Consider the setting of an **unbalanced** two-sided matching market, where the number of men is more than that of women, i.e., $|M| > |W|$. Answer the following questions.

(a) For any given preference profile P, every woman is always matched in a stable matching. (True/False) **1 point.**

True

(b) Explain your answer. If your answer is 'true', then prove it. Otherwise, provide a small counterexample (preferably with at most 4 agents on either side). [Note: this part of the question will get no credit if the previous part is incorrect] **4 points.**

Suppose not, then at least one woman is unmatched. Since the number of men is larger, there will always be some man who is unmatched. Since every individual prefers to be matched rather than stay unmatched, matching this woman to any of the unmatched men is better for both the man and woman. Hence, that man-woman pair is a blocking pair of the earlier matching, yielding that it is not stable, a contradiction.

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(c) For a given preference profile P, there may be two stable matchings μ and μ', where the set of men matched are M and M' respectively and $M \neq M'$. (True/False) [Hint: think about the proofs we did in the class for that do not make a balanced matching market assumption. Also, you may use the properties of men-optimal matching.] **1 point.**

False.

(d) Explain your answer above. If your answer is 'true', then provide a small example (preferably with at most 4 agents on either side). Otherwise, prove it formally and succinctly from first principles, i.e., not using the results derived in class (however, you are free to use the arguments that are similar). [Note: this part of the question will get no credit if the previous part is incorrect] **4 points.**

Note that in the proof that the men-optimal function yields a matching and is a stable matching, we never used the property of balanced market. Hence, the claims of men-optimal matching being women-pessimal hold even when the markets are unbalanced.

Now let's get to the question.

Suppose μ^* is the men-optimal stable matching and μ is any arbitrary stable matching in this setting.

Let (M^*, W^*) and $(\tilde{M}, \tilde{W})$ are the set of men and women matched in these two matchings.

Clearly, $M^* \supseteq \tilde{M}$, since M^* is men optimal and every man wants to be matched than not.

By the same argument, since W^* is women-pessimal, $\tilde{W} \supseteq W^*$

But, $|W^*| = |M^*| \geqslant |\tilde{M}| = |\tilde{W}| \geqslant |W^*|$. Hence $M^* = \tilde{M}$ and $W^* = \tilde{W}$.

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Problem 3 (10 points). Random priority and random endowment. Consider the following **house allocation** setting:

- There are **three** agents, $N = \{1, 2, 3\}$, without any initial endowments.
- There are **three** houses, $H = \{A, B, C\}$. Each agent wants exactly one house.
- Agents' preferences over houses are as follows:

$$1 : A \succ B \succ C; \quad 2 : B \succ C \succ A; \quad 3 : B \succ A \succ C.$$

In this setting, consider two different allocation mechanisms.

Uniformly random priority. In this mechanism, a priority order over the agents is chosen uniformly at random from all possible orders over the agents. For each priority order, the agents pick their favorite unallocated house. The final (randomized) allocation is a probability distribution for each agent and each house that can be represented in the form of an $|N| \times |H|$ matrix.

Uniformly random endowments and top-trading cycle. In this mechanism, an initial endowment over the houses is chosen uniformly at random from all possible orders over the houses. For each initial endowment, the top-trading cycle mechanism is run for the allocation. The final (randomized) allocation is a probability distribution for each agent and each house that can again be represented in the form of an $|N| \times |H|$ matrix.

(a) Write down the **deterministic** priority allocation for each of the possible priorities under the **uniformly random priority** mechanism.

Priorities			Deterministic Allocation		
			1	2	3
1	2	3	A	B	C
1	3	2	A	C	B
2	1	3	A	B	C
2	3	1	C	B	A
3	1	2	A	C	B
3	2	1	A	C	B

$6 \times \frac{1}{3}$ points.

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(b) Write down the **randomized** priority allocation matrix for each agent and house under the **uniformly random priority** mechanism. **$9 \times \frac{1}{3}$ points.**

Agents	**Randomized Allocation**		
	A	B	C
1	5/6	0	1/6
2	0	1/2	1/2
3	1/6	1/2	1/3

(c) Write down the **deterministic** TTC allocation for each of the following possible endowments under the **uniformly random endowment and TTC** mechanism. **$6 \times \frac{1}{3}$ points.**

Endowments			**TTC Allocation**		
1	2	3	1	2	3
A	B	C	A	B	C
A	C	B	A	C	B
B	A	C	A	B	C
B	C	A	A	C	B
C	A	B	A	C	B
C	B	A	C	B	A

(d) Write down the **randomized** priority allocation matrix for each agent and house under the **uniformly random endowment and TTC** mechanism. **$9 \times \frac{1}{3}$ points.**

Agents	**Randomized Allocation**		
	A	B	C
1	5/6	0	1/6
2	0	1/2	1/2
3	1/6	1/2	1/3

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(e) **(Bonus question)** The following questions ask about some patterns in the above setting. Note that if there are multiple cycles in a given round of TTC, we resolve **all** of them in that round.

(i) Consider a preference profile P and a given initial endowment μ (which is an initial matching of the houses to agents). Suppose TTC runs for k_μ rounds on this profile and endowment. Let $N(\mu) := \{N^1(\mu), N^2(\mu), \dots, N^{k_\mu}(\mu)\}$ be the partition of the agents where $N^t(\mu)$ is the set of agents who got allocated a house in round t of TTC on this instance. Suppose, $H^t(\mu)$ is the set of houses that are owned by agents in $N^t(\mu)$ under the endowment μ. Consider the remaining houses at round $(t-1)$ of TTC. In that round, what conclusion can you draw about the favorite house of an agent in $N^t(\mu)$ among the remaining houses, i.e., which $H^\tau(\mu)$'s can that belong to? Explain why.

(ii) Consider the remaining houses at round $(t-1)$ of TTC. In that round, can every agent in $N^t(\mu)$ have his favorite house in $H^t(\mu)$ among these remaining houses? Explain why.

(iii) At round $(t-1)$ of TTC, what is the structure of the favorite house pointing graph of the agents $N^t(\mu)$?

2 + 2 + 1 points.

(i) In round $(t-1)$, the favorite house of an agent in $N^t(\mu)$ lies either in $H^{(t-1)}(\mu)$ or $H^t(\mu)$.

At $(t-1)$, the available houses are $H^{(t-1)}(\mu)$, $H^t(\mu)$, $H^{(t+1)}(\mu)$, ..., $H^{k_\mu}(\mu)$. By definition the agents in $N^t(\mu)$ get their houses allocated in round t.

Then they can't point to houses beyond t.

Hence their favorite houses can either be in $H^{(t-1)}(\mu)$ or $H^t(\mu)$.

(ii) No. If all agents in $N^t(\mu)$ had their favorite houses in $H^t(\mu)$, which is the endowments of the agents $N^t(\mu)$, in round $(t-1)$, then they would have allocated houses in round $(t-1)$ itself. TTC resolves all cycles in a round. Hence there must exist

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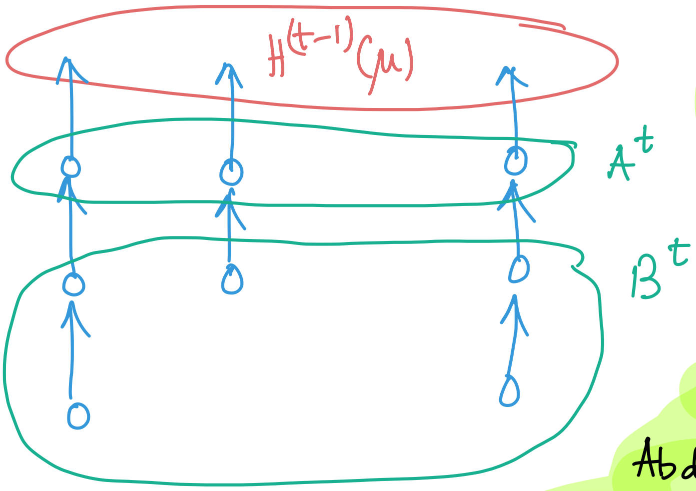
at least one agent in $N^t(\mu)$ who points to a house in $H^{(t-1)}(\mu)$.

(iii) Note that in round $(t-1)$, none of the agents in $N^t(\mu)$ will get a house, i.e., form a cycle. Also, all these agents will point to a house in $H^{(t-1)}(\mu)$ or $H^t(\mu)$. Also, we showed that some agent points to a house in $H^{(t-1)}(\mu)$.
Collect all the agents who point to the houses in $H^{(t-1)}(\mu)$ call this set A^t and all the $N^t(\mu) \setminus A^t = B^t$, who point to $H^t(\mu)$.
It is clear that the pointing graph will have a structure of a collection of chains where the head of the chain will have an agent from A^t and all others will be from B^t.



To probe further (and to know why the randomized allocation will always be identical)
read
Abdulkadiroglu, Sonmez 1998
Random Serial Dictatorships and the core from Random Endowments
"Econometrica".

END OF QUESTION PAPER. GOOD LUCK!