Assignment 6

Group 25

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1.1

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 3 \\ 6 & -6 & -6 \end{bmatrix}$$

compute eigen values using $|A - \lambda I| = 0$
Eigen values(λ) = 2, 0, -3
Algebric multicity corresponding to λ=(2, 0, -3) are 1, 1, 1
Finding Eigen vector for $\lambda_1 = 2$:
$(A - \lambda_1\ I)X_1 = 0$

$$\begin{bmatrix} 2-2 & 0 & 0 \\ -1 & 3-2 & 3 \\ 6 & -6 & -6-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

reducing to row echelon form

$$\begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 3 \\ 6 & -6 & -8 \end{bmatrix}$$

$R_3 \rightarrow R_3 + 6R_2$

$$\begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 3 \\ 0 & 0 & 10 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 0 & 0 & 10 \\ -1 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} -1 & 1 & 3 \\ 0 & 0 & 10 \\ 0 & 0 & 0 \end{bmatrix}$$

Eigen vector $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Geometric multicity for (λ=2) = 1

Finding Eigen vector for $\lambda_2 = 2$:

$(A- \lambda_2\ I)X_2 = 0$

$$\begin{bmatrix} 2-0 & 0 & 0 \\ -1 & 3-0 & 3 \\ 6 & -6 & -6-0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

reducing to row echelon form

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Eigen vector $v_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

Geometric multicity for (λ=0) = 1

Finding Eigen vector for λ_3 = -3 :

$(A- \lambda_3\ I)X_3 = 0$

$$\begin{bmatrix} 2+3 & 0 & 0 \\ -1 & 3+3 & 3 \\ 6 & -6 & -6+3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

reducing to row echelon form

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 6 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Eigen vector $v_3 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$

Geometric multicity for (λ=-3) = 1 for every value of λ, AM = GM, so matrix A is diagonalizable.

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

$$X^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -2 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

AX = XΛ
A = X$\Lambda X^{-}1$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & -2 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

1.2

$$A = \begin{bmatrix} 1 & 4 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 5 & 3 & 1 & 1 \\ 4 & 7 & 2 & 2 \end{bmatrix}$$

compute eigen values using $|A{-}\lambda I| = 0$
Eigen values(λ) = 0, 1, 2, 3
Algebric multicity corresponding to λ=(0, 1, 2, 3) are 1, 1, 1, 1
Finding Eigen vector for $\lambda_1 = 0$:
$(A-\ \lambda_1\ I)X_1 = 0$

$$\begin{bmatrix} 1-1 & 4 & 0 & 0 \\ 0 & 2-1 & 0 & 0 \\ 5 & 3 & 1-1 & 1 \\ 4 & 7 & 2 & 2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

reducing to row echelon form

$$\begin{bmatrix} 1 & 4 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Eigen vector $v_1 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

Geometric multicity for (λ=0) = 1
Finding Eigen vector for $\lambda_2 = 1$:
$(A-\ \lambda_2\ I)X_2 = 0$

$$\begin{bmatrix} 1-2 & 4 & 0 & 0 \\ 0 & 2-2 & 0 & 0 \\ 5 & 3 & 1-2 & 1 \\ 4 & 7 & 2 & 2-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

reducing to row echelon form

$$\begin{bmatrix} 4 & 7 & 2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 10 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Eigen vector $v_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ -10 \end{bmatrix}$
Geometric multicity for (λ=1) = 1
Finding Eigen vector for $\lambda_3 = 2$:
$(A- \lambda_3\ I)X_3 = 0$

$$\begin{bmatrix} 1-3 & 4 & 0 & 0 \\ 0 & 2-3 & 0 & 0 \\ 5 & 3 & 1-3 & 1 \\ 4 & 7 & 2 & 2-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

reducing to row echelon form

$$\begin{bmatrix} -1 & 4 & 0 & 0 \\ 0 & 23 & 2 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Eigen vector $v_3 = \begin{bmatrix} 8 \\ 2 \\ -23 \\ -69 \end{bmatrix}$
Geometric multicity for (λ=2) = 1
Finding Eigen vector for $\lambda_4 = 3$:
$(A- \lambda_4\ I)X_4 = 0$

$$\begin{bmatrix} 1-0 & 4 & 0 & 0 \\ 0 & 2-0 & 0 & 0 \\ 5 & 3 & 1-0 & 1 \\ 4 & 7 & 2 & 2-0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

reducing to row echelon form

$$\begin{bmatrix} -2 & 4 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Eigen vector $v_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$
Geometric multicity for (λ=3) = 1
for every value of λ, AM = GM, so matrix A is diagonalizable.

$$X = \begin{bmatrix} 0 & 2 & 8 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 1 & -23 & 1 \\ 1 & -10 & -69 & 2 \end{bmatrix}$$

$$X^{-1} = \begin{bmatrix} 2 & -25/6 & -2/3 & 1/3 \\ 1/2 & -2 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 3/2 & 28/3 & 1/3 & 1/3 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

AX = XΛ
A = X$\Lambda X^{-}1$

$$A = \begin{bmatrix} 0 & 2 & 8 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 1 & -23 & 1 \\ 1 & -10 & -69 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & -25/6 & -2/3 & 1/3 \\ 1/2 & -2 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 3/2 & 28/3 & 1/3 & 1/3 \end{bmatrix}$$

1.3

$$A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 24 & -12 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

compute eigen values using $|A{-}\lambda I| = 0$
Eigen values(λ) = 2, 2, -2, -2
AM(λ=2) = 2
AM(λ=-2) = 2
Finding Eigen vector for λ_1 = 2 :
$(A- \ \lambda_1 \ I)X_1 = 0$

$$A = \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 24 & -12 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

reducing to row echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Eigen vector $v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$
Geometric multicity for (λ=2) = 2
Finding Eigen vector for λ_1 = -2 :
$(A- \ \lambda_2 \ I)X_2 = 0$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 24 & -12 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

reducing to row echelon form

$$\begin{bmatrix} 6 & -3 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Eigen vector $v_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$

Geometric multicity for (λ=-2) = 2

for every value of λ, AM = GM, so matrix A is diagonalizable.

$$X = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$X^{-1} = \begin{bmatrix} -1/2 & 1/6 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1/2 & 0 & 0 & 0 \\ 0 & 1/6 & 0 & 0 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

AX = XΛ

A = XΛX^-1

$$A = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} -1/2 & 1/6 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1/2 & 0 & 0 & 0 \\ 0 & 1/6 & 0 & 0 \end{bmatrix}$$

2.1

$$A = \begin{bmatrix} 7 & 1 \\ 5 & 5 \\ 0 & 0 \end{bmatrix}, A = U_{m*m}\Sigma_{m*n}V_{n*n}$$

$\text{AA}^T = \begin{bmatrix} 7 & 1 \\ 5 & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 7 & 5 & 0 \\ 1 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 50 & 40 & 0 \\ 40 & 50 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Eigen value for AA^T,

$\lambda = 0$, 10, 90

Singular values, $\sigma_1 = \sqrt{10}$, $\sigma_2 = \sqrt{90}$

$$\Sigma_{3*2} = \begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{90} \\ 0 & 0 \end{bmatrix}$$

$(\text{AA}^T$ - $\lambda\text{I}) = 0$

for $\lambda = 0$,

Reduced echelon form is

$$\begin{bmatrix} 5 & 4 & 0 \\ 0 & 9/5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Eigen vector $\text{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Orthogonal vector $\text{q}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

for $\lambda = 10$,

Reduced echelon form is

$$\begin{bmatrix} 40 & 40 & 0 \\ 0 & 0 & -10 \\ 0 & 0 & 0 \end{bmatrix}$$

Eigen vector $\text{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

Orthogonal vector $\text{q}_2 = 1/\sqrt{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

for $\lambda = 90$,
Reduced echelon form is

$$\begin{bmatrix} -40 & 40 & 0 \\ 0 & 0 & -90 \\ 0 & 0 & 0 \end{bmatrix}$$

Eigen vector $v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Orthogonal vector $q_3 = 1/\sqrt{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$$U_{3*3} = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$A^T A = \begin{bmatrix} 74 & 32 \\ 32 & 26 \end{bmatrix}$
Eigen value for $A^T A$,
$\lambda = 10, 90$
Singular values, $\sigma_1 = \sqrt{10}$, $\sigma_2 = \sqrt{90}$
for $\lambda = 10$,
Eigen vector $v_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$
Orthogonal vector $q_1 = 1/\sqrt{5} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$
for $\lambda = 90$,
Eigen vector $v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
Orthogonal vector $q_2 = 1/\sqrt{5} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$V_{2*2} = \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

$$V_{2*2}^T = \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

$$A = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{90} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

2.2

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}, A = U_{2*2}\Sigma_{2*3}V_{3*3}$$

$\mathrm{AA}^T = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$
Eigen value for AA^T,
$\lambda = 9$, 25
Singular values, $\sigma_1 = 3$, $\sigma_2 = 5$

$$\Sigma_{2*3} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix}$$

(AA^T - λI) = 0
for $\lambda = 9$,
Reduced echelon form is

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Eigen vector $\mathrm{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$
Orthogonal vector $\mathrm{q}_1 = 1/\sqrt{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
for $\lambda = 25$,
Reduced echelon form is

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

Eigen vector $\mathrm{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
Orthogonal vector $\mathrm{q}_2 = 1/\sqrt{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$U_{2*2} = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

calculating V matrix,
$v_i = 1/\sigma_i A^T u_1$

$$v_1 = 1/3 \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$v_1 = \begin{bmatrix} -1/3\sqrt{2} \\ 1/3\sqrt{2} \\ -4/3\sqrt{2} \end{bmatrix}$$

$$v_2 = 1/5 \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

taking v_3 as perpendicular to both v_1 and v_2,

$$v_3 = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

$$V_{3*3} = \begin{bmatrix} -1/3\sqrt{2} & 1/\sqrt{2} & -2/3 \\ 1/3\sqrt{2} & 1/\sqrt{2} & 2/3 \\ -4/3\sqrt{2} & 0 & 1/3 \end{bmatrix}$$

$$V_{3*3}^T = \begin{bmatrix} -1/3\sqrt{2} & 1/3\sqrt{2} & -4/3\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -2/3 & 2/3 & 1/3 \end{bmatrix}$$

$$A = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} -1/3\sqrt{2} & 1/3\sqrt{2} & -4/3\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -2/3 & 2/3 & 1/3 \end{bmatrix}$$

5.1 A = $\begin{bmatrix} -6 & 2 & 0 \\ 2 & -6 & 2 \\ 0 & 2 & -6 \end{bmatrix}$

A is square matrix and transpose is:

$A^T = \begin{bmatrix} -6 & 2 & 0 \\ 2 & -6 & 2 \\ 0 & 2 & -6 \end{bmatrix}$

A = A^T, so A is symmetric matrix.

5.2 A = $\begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 2 & 2 & 1 & 2 \end{bmatrix}$

A is not a square matrix so A is not symmetric.

7.

$$B = C^T AC$$

$X^T BX = X^T C^T ACX$

assume, CX = Y then $Y^T = X^T C^T$

$$X^T BX = Y^T AY,$$

A is symmetric positive definite and C is nonsingular

so, Y $\neq$ 0,

hence,

$$X^T BX > 0.$$

B is also symmetric positive definite