Mathematics for IT - Assignment 05
Linear Algebra

August 31, 2018

1. True or False ? Justify your answer.

   (a) If Q is an orthogonal matrix, then Q is invertible.

   (b) If x̂ is the orthogonal projection of x on W, then x̂ is orthogonal to x.

   (c) If û is the orthogonal projection of u on Span v, then: û = $\left(\frac{u.v}{v.v}\right).u$

   (d) Suppose P_1 and P_2 are projection matrices. Then, $(P1 - P2)^2$ + $(I - P1 - P2)^2$ = I.

   (e) Every orthonormal set of vectors in R^4 must be a basis for R^4.

   (f) In R^9, we can find a subspace W such that dim W = dim $W^\perp$.

2. Let P_1 and P_2 be n $\times$ n projection matrices. Then, which of the following statements is false?

   (a) P_1(P$_1$ - P_2)^2 = (P$_1$ - P_2)$^2 P_1$.

   (b) If P_1 and P_2 have the same rank, then they are similar.

   (c) rank (P_1) + rank (P_1 - I) $\neq$ rank (P_2) + rank (P_2 - I).

3. Let S = $\{$(1, -1, 1, 1),(1, 0, 1, 0),(0, 1, 0, 1)$\} \subseteq R^4$. Find an orthonormal set T such that LS(S) = LS(T). *LS stands for linear span.*

4. Let S = { e_1 + e_4, - e_1 + 3e_2 - e_3 } $\subset R^4$. Find $S^\perp$.

5. (a) Find the matrix P that projects every vector b in R^3 onto the line in the direction of a = (2, 1, 3).

   (b) What are the column space and nullspace of P ? Describe them geometrically and also give a basis for each space.

6. Let W be a subspace of R^n and $W^\perp$ denote its orthogonal complement. If W^1 is subspace of R^n such that if x ϵ W^1, then x^Tu = 0, for all u ϵ $W^\perp$. Then,

   (a) dim ${W_1}^\perp \leq$ dim $W^\perp$

   (b) dim ${W_1}^\perp \leq$ dim W

   (c) dim ${W_1}^\perp \geq$ dim $W^\perp$

   (d) dim ${W_1}^\perp \geq$ dim W

7. Consider a subspace S of R^4 spanned by the following vectors:

   $$u_1 = \begin{bmatrix}1\\0\\1\\0\end{bmatrix}, u_2 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix}, u_3 = \begin{bmatrix}1\\0\\1\\1\end{bmatrix}$$

   Using the usual dot product on R^4 , do the following:

   (a) Convert { u_1,u_2,u_3 } to an orthonormal basis for S.

   (b) Explain why Gram Schmidt algorithm fails when the input set of vectors is linearly dependent.

8. (a) p = A$\hat{x}$ is the vector in C(A) nearest to a given vector b. If A has independent columns, what equation determines $\hat{x}$? What are all the vectors perpendicular to the error e = b - A$\hat{x}$? What goes wrong if the columns of A are dependent?

   (b) Suppose A = QR where Q has orthonormal columns and R is upper triangular invertible. Find $\hat{x}$ and p in terms of Q and R and b (not A).

   (c) If q_1 and q_2 are any orthonormal vectors in R^5, give a formula for the projection p of any vector b onto the plane spanned by q_1 and q_2 (write p as a combination of q_1 and q_2).

9. What is the projection of v = e_1 + $2e_2$ - $3e_3$ on H : x_1 + $2x_2$ + $4x_4$ = 0?

10. The matrix Q has orthonormal columns q_1 , q_2 , q_3 :

$$Q = \begin{bmatrix} 0.1 & 0.5 & a \\ 0.7 & 0.5 & b \\ 0.1 & -0.5 & c \\ 0.7 & -0.5 & d \end{bmatrix}$$

   (a) What equations must be satisfied by the numbers a, b, c, d? Is there a unique choice for those (real) numbers, apart from multiplying them all by 1?

   (b) Suppose Gram-Schmidt starts with those same first two columns and with the third column a_3 = $(1, 1, 1, 1)^T$.What third column would it choose for q_3. (You can leave a square root as $\sqrt{\cdots}$ if you want to.)