$Sol^n 1$: (a)

Given matrix A,

$$A = \begin{bmatrix} 5 & -2 & 5 \\ 11 & 4 & -8 \\ 5 & 9 & 8 \\ 1 & 11 & 23 \end{bmatrix}$$

Apply Row Echelon Form,

$R_3 \leftarrow R_3$ - R_1

$$\begin{bmatrix} 5 & -2 & 5 \\ 11 & 4 & -8 \\ 0 & 11 & 3 \\ 1 & 11 & 23 \end{bmatrix}$$

$R_4 \leftarrow R_4$ - $R_1/5$

$$\begin{bmatrix} 5 & -2 & 5 \\ 11 & 4 & -8 \\ 0 & 11 & 3 \\ 0 & 57/5 & 22 \end{bmatrix}$$

$R_2 \leftarrow R_2$ - $11/5R_1$

$$\begin{bmatrix} 5 & -2 & 5 \\ 0 & 42/5 & -19 \\ 0 & 11 & 3 \\ 0 & 57/5 & 22 \end{bmatrix}$$

$R_4 \leftarrow R_4$ - $57/55R_1$

$$\begin{bmatrix} 5 & -2 & 5 \\ 0 & 42/5 & -19 \\ 0 & 11 & 3 \\ 0 & 0 & 1039/55 \end{bmatrix}$$

$R_3 \leftarrow R_3$ - $55/42R_2$

$$\begin{bmatrix} 5 & -2 & 5 \\ 0 & 42/5 & -19 \\ 0 & 0 & 1171/42 \\ 0 & 0 & 1039/55 \end{bmatrix}$$

$R_4 \leftarrow R_4$ - $42 * 1039/1171 * 55R_1$

$$\begin{bmatrix} 5 & -2 & 5 \\ 0 & 42/5 & -19 \\ 0 & 0 & 1171/42 \\ 0 & 0 & 0 \end{bmatrix}$$

Number of Non zero rows = 3

so, Rank of Matrix A = 3

(b)
Given matrix B,

$$B = \begin{bmatrix} 2 & 5 & 4 & 6 \\ 8 & 5 & 6 & 9 \\ 4 & 5 & 6 & 8 \end{bmatrix}$$

Apply Row Echelon Form,

$R_3 \leftarrow R_3$ - $R_2/2$

$$\begin{bmatrix} 2 & 5 & 4 & 6 \\ 8 & 5 & 6 & 9 \\ 0 & 5/2 & 3 & 7/2 \end{bmatrix}$$

$R_2 \leftarrow R_2$ - $4R_1$

$$\begin{bmatrix} 2 & 5 & 4 & 6 \\ 0 & -15 & -10 & -15 \\ 0 & 5/2 & 3 & 7/2 \end{bmatrix}$$

$R_2 \leftarrow R_2/$

$$\begin{bmatrix} 2 & 5 & 4 & 6 \\ 0 & -3 & -2 & -3 \\ 0 & 5/2 & 3 & 7/2 \end{bmatrix}$$

$R_3 \leftarrow R_3 + 5/6R_2$

$$\begin{bmatrix} 2 & 5 & 4 & 6 \\ 0 & -3 & -2 & -3 \\ 0 & 0 & 4/3 & 1 \end{bmatrix}$$

Number of non rows = 3

Rank of B = 3

$Sol^n 2:$

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 3 & 4 & -1 & 2 \\ -1 & -2 & 5 & 4 \end{bmatrix}$$

Rank Nullity Theorem,

Rank(A) + Nullity(A) = Columns(A)

Nullity(A) = Basis of Null Space

Basis of Null Space,

Step1: Apply Reduced Row Echelon form on A,

$R_3 \leftarrow R_3 + R_1$

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 3 & 4 & -1 & 2 \\ 0 & -1 & 7 & 7 \end{bmatrix}$$

$R_2 \leftarrow R_2$ - $3R_1$

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & -7 & -7 \\ 0 & -1 & 7 & 7 \end{bmatrix}$$

$R_3 \leftarrow R_3 + R_2$

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & -7 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 2: Write RREF(A) in equation form,

$x_1 + x_2 + 2x_3 + 3x_4 = 0$

x_2 - $7x_3$ - $7x_4 = 0$

Step 3: Express Pivot variables in terms of free variables,

Case1: $x_3 = 1$, $x_4 = 0$

$x_1 + x_2 =$ -2

$x_2 = 7$

$x_1 + x_2 =$ -2

$x_1 =$ -9

Case2: $x_3 = 0$, $x_4 = 1$,

$x_1 + x_2 =$ -3

x_2 = 7

x_1 = -10

Null space(A) = x_3 $\begin{bmatrix} -9 \\ 7 \\ 1 \\ 0 \end{bmatrix}$ + x_4 $\begin{bmatrix} -10 \\ 7 \\ 0 \\ 1 \end{bmatrix}$

Basis = $\begin{bmatrix} -9 \\ 7 \\ 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} -10 \\ 7 \\ 0 \\ 1 \end{bmatrix}$

Nullity(A) = Number of Basis vectors in Null Space

Nullity(A) = 2

Rank(A) = Cols(A) - Nullity(A)

Rank(A) = 2

$Sol^n 3:$

A = $\begin{bmatrix} 5 & 6 & 5/4 \\ 7 & 7 & 7/3 \\ 1 & 3 & 8 \end{bmatrix}$

Basis of Row space,

Apply Row Echelon form,

$R_3 \leftarrow R_3$ - $R_1/5$

$\begin{bmatrix} 5 & 6 & 5/4 \\ 7 & 7 & 7/3 \\ 0 & 9/5 & 31/4 \end{bmatrix}$

$R_2 \leftarrow R_2$ - 7/5R_1

$\begin{bmatrix} 5 & 6 & 5/4 \\ 0 & -7/5 & 7/12 \\ 0 & 9/5 & 3/4 \end{bmatrix}$

$R_3 \leftarrow R_3 + 9/7R_2$

$\begin{bmatrix} 5 & 6 & 5/4 \\ 0 & -7/5 & 7/12 \\ 0 & 0 & 17/2 \end{bmatrix}$

Basis for Row space(C(A^T)) = Non Zero rows of Echelon form of A

$$\text{Basis(C(A}^T\text{))} = \begin{bmatrix} 5 \\ 6 \\ 5/4 \end{bmatrix} \begin{bmatrix} 0 \\ -7/5 \\ 7/12 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 17/2 \end{bmatrix}$$

Basis of Col space,

Apply Row Echelon form,

$R_3 \leftarrow R_3$ - $R_1/5$

$$\begin{bmatrix} 5 & 6 & 5/4 \\ 7 & 7 & 7/3 \\ 0 & 9/5 & 31/4 \end{bmatrix}$$

$R_2 \leftarrow R_2$ - $7/5R_1$

$$\begin{bmatrix} 5 & 6 & 5/4 \\ 0 & -7/5 & 7/12 \\ 0 & 9/5 & 3/4 \end{bmatrix}$$

$R_3 \leftarrow R_3 + 9/7R_2$

$$\begin{bmatrix} 5 & 6 & 5/4 \\ 0 & -7/5 & 7/12 \\ 0 & 0 & 17/2 \end{bmatrix}$$

Basis of col space(C(A)) = Cols in the original matrix that contain pivot in the Row Echelon form

$$\text{Basis(C(A))} = \begin{bmatrix} 5 \\ 7 \\ 1 \end{bmatrix} \begin{bmatrix} 6 \\ 7 \\ 3 \end{bmatrix} \begin{bmatrix} 5/4 \\ 7/3 \\ 8 \end{bmatrix}$$

$Sol^n 4:$

$$\text{A} = \begin{bmatrix} 8 & 7 & 4 & 2 \\ 3 & 8 & 2 & 2 \\ 8 & 7 & 8 & 2 \end{bmatrix}$$

Null space for A

Apply Row reduced Echelon form,

$R_3 \leftarrow R_3$ - R_1

$$\begin{bmatrix} 8 & 7 & 4 & 2 \\ 3 & 8 & 2 & 2 \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

$R_1 \leftarrow R_1/8$

$$\begin{bmatrix} 1 & 7/8 & 1/2 & 1/4 \\ 3 & 8 & 2 & 2 \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

$R_2 \leftarrow R_2$ - $3R_1$

$$\begin{bmatrix} 1 & 7/8 & 1/2 & 1/4 \\ 0 & 43/8 & 1/2 & 5/4 \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

$R_2 \leftarrow R_2/(43/8)$

$$\begin{bmatrix} 1 & 7/8 & 1/2 & 1/4 \\ 0 & 1 & 4/43 & 10/43 \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

$R_3 \leftarrow R_3/4$

$$\begin{bmatrix} 1 & 7/8 & 1/2 & 1/4 \\ 0 & 1 & 4/43 & 10/43 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$R_2 \leftarrow R_2 + 4/43R_3$

$$\begin{bmatrix} 1 & 7/8 & 1/2 & 1/4 \\ 0 & 1 & 0 & 10/43 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$R_1 \leftarrow R_1$ - $7/8R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 2/43 \\ 0 & 1 & 0 & 10/43 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

x_1, x_2, x_3 = Pivot elements

x_4 = Free Variable
$x_1 + 2/43\ x_4 = 0$
$x_2 + 10/43\ x_4 = 0$
$x_3 = 0$

Express Null space in terms of free and Pivot variables,
$x_4 = 1$
x_1 = -2/43
x_2 = -10/43

Basis for Null space(A) = $\begin{bmatrix} -2/43 \\ -10/43 \\ 0 \\ 1 \end{bmatrix}$

Basis of Null space of A^T,

$$A^T = \begin{bmatrix} 8 & 3 & 8 \\ 7 & 8 & 7 \\ 4 & 2 & 8 \\ 2 & 2 & 2 \end{bmatrix}$$

Apply Row Reduced Echelon Form,

$R_1 \leftarrow R_1/8$

$$\begin{bmatrix} 1 & 3/8 & 1 \\ 7 & 8 & 7 \\ 4 & 2 & 8 \\ 2 & 2 & 2 \end{bmatrix}$$

$R_4 \leftarrow R_4$ - $R_3/2$

$$\begin{bmatrix} 1 & 3/8 & 1 \\ 7 & 8 & 7 \\ 4 & 2 & 8 \\ 0 & 1 & -2 \end{bmatrix}$$

$R_3 \leftarrow R_3$ - $4R_1$

$$\begin{bmatrix} 1 & 3/8 & 1 \\ 7 & 8 & 7 \\ 0 & 1/2 & 4 \\ 0 & 1 & -2 \end{bmatrix}$$

$R_2 \leftarrow R_2$ - $7R_1$

$$\begin{bmatrix} 1 & 3/8 & 1 \\ 0 & 43/8 & 0 \\ 0 & 1/2 & 4 \\ 0 & 1 & -2 \end{bmatrix}$$

$R_2 \leftarrow R_2/(43/8)$

$$\begin{bmatrix} 1 & 3/8 & 1 \\ 0 & 1 & 0 \\ 0 & 1/2 & 4 \\ 0 & 1 & -2 \end{bmatrix}$$

$R_4 \leftarrow R_4$ - R_2

$$\begin{bmatrix} 1 & 3/8 & 1 \\ 0 & 1 & 0 \\ 0 & 1/2 & 4 \\ 0 & 0 & -2 \end{bmatrix}$$

$R_3 \leftarrow R_3$ - $R_2/2$

$$\begin{bmatrix} 1 & 3/8 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & -2 \end{bmatrix}$$

$R_3 \leftarrow R_3/4$

$$\begin{bmatrix} 1 & 3/8 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

$R_4 \leftarrow R_4 + 2R_3$

$$\begin{bmatrix} 1 & 3/8 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

x_1, x_2, x_3 = Pivot elements

NO Free Variable

$x_1 + 3/8\ x_2 + x_3 = 0$
$x_2 = 0$
$x_3 = 0$

Basis(A^T) = $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$Sol^n 5$:

$3x_2$ - $6x_3$ + $6x_4$ + $4x_5$ = -5
$3x_1$ - $7x_2$ + $8x_3$ - $5x_4$ + $8x_5$ = 9
$3x_1$ - $9x_2$ + $12x_3$ - $9x_4$ + $6x_5$= 15

$$\left[\begin{array}{ccccc|c} 0 & 3 & -6 & 6 & 4 & 5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & 9 & 6 & 15 \end{array}\right]$$

Apply row echelon form,

$R_1 \leftrightarrow R_3$

$$\left[\begin{array}{ccccc|c} 3 & -9 & 12 & 9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & 5 \end{array}\right]$$

$R_2 \leftarrow R_2$ + R_1

$$\left[\begin{array}{ccccc|c} 3 & -9 & 12 & 9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & 5 \end{array}\right]$$

$R_3 \leftarrow R_3$ - $3/2R_1$

$$\left[\begin{array}{ccccc|c} 3 & -9 & 12 & 9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 14 \end{array}\right]$$

Rank(A) = Rank(A|B)

So for the given set of linear equations, Solution exist.

But R(A) ¡ No. of cols of A

So No unique solution exists. Multiple Solution exists for the given matrix.

$Sol^n 6$:

A = $\begin{bmatrix} 4 & 5 & 6 \\ 2 & -5 & 6 \\ 8 & 10 & 12 \end{bmatrix}$

Basis of Row space,

Apply Row Echelon form,

$R_3 \leftarrow R_3$ - $2R_1$

$$\begin{bmatrix} 4 & 5 & 6 \\ 2 & -5 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_2 \leftarrow R_2$ - $R_1/2$

$$\begin{bmatrix} 4 & 5 & 6 \\ 0 & -15/2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Basis of the row space = Number of Non zero rows in the Echelon form.

$$\text{Basis(C(A}^T\text{))} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \begin{bmatrix} 0 \\ -15/2 \\ 3 \end{bmatrix}$$

$Sol^n 7$:

Suppose a matrix,

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ - & - & - & - & - \\ - & - & - & - & - \end{bmatrix}$$

After Applying row echelon form, We have 3 pivots.

Rank(A) = Rank(A|B)

So for the given set of linear equations, Solution exist. System is consistent.

$Sol^n 8$:

$$\text{A} = \begin{bmatrix} 2 & 0 & 6 \\ -1 & 8 & 5 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\text{B} = \begin{bmatrix} 10 \\ 3 \\ 3 \end{bmatrix}$$

$$x_1 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 8 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ 3 \end{bmatrix}$$

Using Elimination,

$$\left[\begin{array}{rrr|r} 2 & 0 & 6 & 10 \\ -1 & 8 & 5 & 3 \\ 1 & -2 & 1 & 3 \end{array}\right]$$

$$\left[\begin{array}{rrr|r} 2 & 0 & 6 & 10 \\ -1 & 8 & 5 & 3 \\ 1 & -2 & 1 & 3 \end{array}\right]$$

$R_2 \leftarrow R_2 + R_1/2$

$$\left[\begin{array}{lrr|r} 2 & 0 & 6 & 10 \\ 0 & 8 & 8 & 8 \\ 1 & -2 & 1 & 3 \end{array}\right]$$

$R_3 \leftarrow R_3$ - $R_1/2$

$$\left[\begin{array}{lcr|r} 2 & 0 & 6 & 10 \\ 0 & 8 & 8 & 8 \\ 0 & -2 & -2 & -2 \end{array}\right]$$

$R_3 \leftarrow R_3 + R_2/4$

$$\left[\begin{array}{ccc|c} 2 & 0 & 6 & 10 \\ 0 & 8 & 8 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$2x_1 + 6x_3 = 10$
$8x_2 + 8x_3 = 8$

Rank(A) = Rank(A|B) < n

Infinite Sol^n exists.

i) YES, b is in w(linear combinations of cols of A)

$$x_1 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 8 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix}$$

If we put $x_1 = 0$, $x_2 = 0$, $x_3 = 1$ then the third column of A exists in the linear combination of A.

$Sol^n 9$:

AX = B

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix}$$

$$B = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

Applying Row Echelon form,

$R_2 \leftarrow R_2 + R_1$

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 6 & 1 & -8 & -4 \end{array}\right]$$

$R_3 \leftarrow R_3$ - $2R_1$

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 6 & -9 & 0 & -18 \end{array}\right]$$

$R_3 \leftarrow R_3 + 3R_2$

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

R(A) $<$ Number of equations. Infinite solution exists for the system.