Introduction to Renormalization Group Techniques

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4 November 2018

Role of Minimum Length Λ^{-1}

One of the characteristic features of Renormalization Groups is characteristic minimum length scale. The importance or significance of which is the following:

- At length scales below this Λ^{-1} all physics happening is irrelevant to our consideration.
- This cutoff describes the length scale of physical phenomena.
- Λ^{-1} hence becomes an integral part of equations describing the system in the form of parameters or limits of those parameters.

Keep in mind that Λ^{-1} has to somehow show up in our equations :)

How Λ^{-1} influences the system?

Since we are changing length scale of Physics, the Hamiltonian should change. This is because we are now ignoring contributions at lower length scale. Also the Heisenberg equation changes (But we will not consider that here. It only influences the "Uncertainty").

The other influenced parameters are coupling constants. They undergo changes as new interactions will not have same dependence as it did previously.

To summarize:

- Form of Hamiltonian changes.
- Constants of Hamiltonian changes.

What is Renormalization Group?

Now we looked into the effect of Λ on our equations. Renormalization Group in a way are transformations of coupling parameters under the changes of Λ.(Please wait for relevance of these transformations)

So where are we stuck?

- Defining Transformations not easy. (As we will see later)
- Everything below length scale Λ^{-1} behaves as a single unit. So defining quantities which describe the unit is a necessity.
- After defining average quantities then we need to define the **"effective"** interaction terms.(They are not easy to obtain I assure you)

Is it useful?

Changing block size or Λ doesn't make any sense as far as physics is concerned. We just obtain new constants in the equation. So what do we learn?

It's true we do not get anything **"NEW"**. However we study the the problems where we can trace the so called **"TRANSFORMATIONS"** very easily, so that life becomes simple. Particularly in case of Phase Transitions, a very simple pattern exists.

Critical Phenomena

In a very simple language this theory explains the **"Diverging Quantities"** with changing parameters. These diverging parameters are described using the a simple model, where they are expressed as:

$$P = f(A - A_c)$$

where P is diverging parameter, A is any variable and A_c represents critical point. Since it is diverging, it will have exponents less than 0.

Here comes the role of Scaling Hypothesis.

Scaling Hypothesis

Correlation Length ξ is the length scale at which the parameters are correlated.

What Scaling Hypothesis says:

- There is a correlation length which is the **"ONLY"** relevant length scale in explaining critical phenomena.(Other lengths are ignored such as inter-atomic distances).
- ξ accounts the dominating temperature dependence of all quantities. Quantities or parameters depend on $(T - T_c)$ only through ξ.

Model and Notation

- Each point is represented by n-component vector $\phi(x)$.
- We transform them in Fourier components of $\phi(x)$.
- We represent Probability P_{micro} in terms of H_{micro}.
- Then we have correlation function $G(k)$.

Effective Hamiltonian

Hamiltonian includes all interactions (That is why it is Hamiltonian). But we expect the fluctuations at long scales(which we consider for critical phenomena) to be independent of low level interactions. Hence, even if we remove the low scale terms we should get the same answer.

Therefore we propose some sort of effective Hamiltonian which integrates out irrelevant random variables. (remember trivial pre-requisite?)

Some considerations:

- $H(\Lambda)$ will have no low scale singularities as now those variables are marginalized.
- We will be able to see major characteristics of Critical Behaviour independent of low scale interactions.

The New Set of Parameters

Any probability distribution is defined by some parameters. We consider each parameter to be a point in parameter space. These parameters will define how the system will behave.

Parameter space is huge, but some considerations reduce the parameter space. For Example:

- Low scale interaction parameters are already thrown out of picture due to marginalization.
- Symmetry considerations reduce parameters. (Like in class we studied symmetry in m caused odd terms to vanish.)

Of course these parameters depend on Length scale and Correlation length that is why we are studying them.

Coming back to Renormalization Group

Let us look into the "Transformation" we have been talking about so far till now. Consider two probability distributions P and $P^{'}$ which have parameter space as μ and $\mu^{'}$. Our transformation looks something just like this:

$$\mu = R_{\zeta}\mu$$

Steps to be followed:

- Integrate out ϕ_k with k between Λ and Λ/s
- Relabel random variables by enlarging k
- We multiply random variables by constant.

Behavior at Large ζ

- Let $\mu(T)$ be the point in the parameter space representing the canonical ensemble at temperature T
- $\mu(T)$ represents the physical probability distribution
- We shall argue that, if T is very close to T_c, $R_\zeta \mu(T)$ will become close to the fixed point μ_*
- Critical behaviors, in particular the critical exponents, will then be related to the properties of R_ζ operating near the fixed point
- Near the fixed point, we can imagine a linearized R_ζ
- R_ζ is a complicated nonlinear transformation
- Once linearized, we can make a lot of guesses

Linearization 1

- If μ is near μ_*, we write formally

$$\mu = \mu_* + \delta\mu$$

- Now, we can write $\mu' = R_\zeta^L \delta\mu$ with $R_\zeta \mu_* = \mu_*$ as:

$$\delta\mu' = R_\zeta^L \delta\mu$$

- R_ζ becomes a linear operator when $O((\delta\mu)^2)$ terms are dropped
- We can then determine the eigenvalues $\lambda_j(\zeta)$ and correspoding eigenvectors e_j
- We label them in the order $\lambda_1 \geqslant \lambda_2 \geqslant \ldots$
- Since $R_\zeta R_{\zeta'} e_j = R_{\zeta\zeta'} e_j$, we have $\lambda_j(\zeta)\lambda_j(\zeta') = \lambda_j(\zeta\zeta')$

Linearization 2

- Define $\Delta_t^{(j)}$ by $\lambda_j(\zeta) = \zeta^{\Delta_t^{(j)}}$ with $\Delta_t \geqslant \Delta_t' \geqslant \Delta_t'' \geqslant \ldots$ since $\zeta > 1$
- We write $\delta\mu$ as linear combination

$$\delta\mu = \sum_j t_j e_j$$

- It follows:

$$\delta\mu' = \sum_j t_j \zeta^{\Delta_t^{(j)}} e_j$$

- Apparently, we have made no progress since we do not know $\Delta_t^{(j)}$ nor e_j

Linearization 3

- Simplicity appears if it turns out that only $\Delta_t > 0$ and all other $\Delta_t', \Delta_t''... < 0$
$$\delta\mu' = R^L_\zeta \delta\mu = t_1 \zeta^{\Delta_t} e_1 + O(\zeta^{\Delta_t'})$$
if ζ is so large that the first term dominates but $t_1\zeta^{\Delta_t}$ is still small enough so that the linear approximation for R_ζ is valid
- If $t_1 = 0$ to start with, then $R^L_\zeta \delta\mu \to 0$ as ζ increases, i.e. we push towards the critical point
- t_1 is called relevant and $t_2, t_3...$ are called irrelevant

Critical Surface

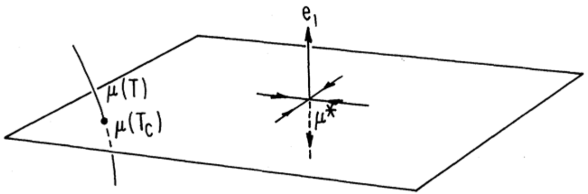

FIG. 1. Qualitative picture of a critical surface and a fixed point μ^* in the parameter space. The arrows point in directions of motion of $R_s\mu$ as s increases. The trajectory on the left is $\mu(T)$ for a continuous range of T, and $\mu(T_c)$ is the intersection of the trajectory and the critical surface.

Reference: Shang-Keng, Ma: Introduction to the Renormalization Group (Reviews of Modern Physics, 1973)

Temperature dependence of μ

- Now, we shall examine the effect of R_ζ on our system
- The particular probability distribution is represented by a certain point $\mu(T)$ in the parameter space
- Because we have integrated out long wave numbers $k > \Lambda$ in the microscopic Hamiltonian, $H(\Lambda)$ would be dependent on T.
- If we vary T continuously, we would trace a trajectory in the parameter space, which hits the critical surface at $T = T_c$

Correlation function

- We identify t_1 as a function of T close to the critical point and expand:

$$t_1(T) = A(T - T_c) + B(T - T_c)^2 + ...$$

- Assume $A \neq 0$ and μ is close to μ_*
- We write $\mu(T) = \mu_* + \delta\mu$ and obtain:

$$R^L_\zeta \delta\mu(T) = A(T - T_c)\zeta^{1/\nu} e_1 + O(\zeta^{\Delta'_t})$$

  where ν is defined as $1/\nu = \Delta_t$
- We take $G(k, \mu) = \zeta^{2-\eta} G(\zeta k, R_\zeta \mu)$ and go to large ζ:

$$G(k, \mu(t)) = \zeta^{1-\eta}[G(\zeta k, \mu_* + A(T - T_c)\zeta^{1/\nu} e_1 + O(\zeta^{\Delta'_t}))]$$

Case $T = T_c$

- Keep: $G(k,\mu(t)) = \zeta^{1-\eta}[G(\zeta k, \mu_* + A(T - T_c)\zeta^{1/\nu} e_1 + O(\zeta^{\Delta'_t}))]$
- We choose our rescaling factor $\zeta = \Lambda/2k$

$$G(k,\mu(T_c)) = k^{-2+\eta}(\Lambda/2)^{2-\eta}[G(\Lambda/2,\mu_*) + O((\Lambda/2k)^{\Delta'_t})]$$

- In the limit of small k

$$G(k,\mu(T_c)) \propto k^{-2+\eta}$$

- The power law (3.11) for $G(k)$ at T_c is seen as a consequence of the fact that $R_\zeta \mu(T)$ approaches μ_* for large ζ
- How small must k be in order to be a good approximation?

Case $T = T_c$

- $(2k/\Lambda)$ must be small, say much smaller than $1/2$

$$2k/\Lambda \ll e^{1/\Delta_t'}$$

- This equation provides an estimation of the size of the critical surface in the k space
- Thus, it depends strongly on Δ_t'
- Recall: $\zeta^{\Delta_t'}$ was the second biggest eigenvalue in our linearization and the biggest eigenvalue in the critical surface

Case $T > T_c, k = 0$

- Keep: $G(k, \mu(t)) = \zeta^{1-\eta}[G(\zeta k, \mu_* + A(T - T_c)\zeta^{1/\nu} e_1 + O(\zeta^{\Delta'_t}))]$
- We choose our rescaling factor $\zeta = t_1^{-\nu}$ where $t_1 = T - T_c$

$$G(0, \mu(T)) = t_1^{-(2-\eta)\nu}[G(0, \mu_* + e_1) + O(t_1^{\nu\Delta'_t})]$$

- In the limit of small t_1

$$G(0, \mu(T)) \propto (T - T_c)^{-\gamma}$$

$$\gamma = \nu(2 - \eta)$$

- When does this approximation hold true?

Case $T > T_c$

- $t_1 \ll 2^{-\nu\Delta'_t}$ must be much smaller than order unity, say 1/2

$$t_1 << 2^{1/\nu\Delta'_t}$$

- This gives us another estimate of the size of the critical region in $T - T_c$
- Similar, but more involved calculations provide better estimates

Scaling hypothesis, revisited

- Correlation length: $\xi = |t_1|^{-\nu}$
- From slide before: $R^L_\zeta \delta\mu(T) = A(T - T_c)\zeta^{1/\nu} e_1 + O(\zeta^{\Delta'_t})$

$$R^L_\zeta \delta\mu = (\zeta/\xi)^{1/\nu} e_1 + O(s^{\Delta'_t})$$

- The effect of R_ζ thus decreases the correlation length by a factor ζ
- If we ignore the $O(\zeta^{\Delta'_t})$ term, we would then arrive at the scaling hypothesis

Case $T < T_c$

- In this case $t_1 < 0$ and we can simply set $\zeta = (-t_1)^{-\nu}$ and obtain a formula for $G(0, \mu(t))$ as before
- However, this is not helpful because $G(0, \mu) = \infty$ for $t_1 < 0$

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$$F_0[\phi] = \int d^d x \left[\frac{1}{2}\nabla\phi \cdot \nabla\phi + \frac{1}{2}\mu_0^2\phi^2\right] = \int^\Lambda \frac{d^d k}{(2\pi)^d}\frac{1}{2}(k^2 + \mu_0^2)\phi_{\mathbf{k}}\phi_{-\mathbf{k}}$$

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We go to the Fourier space to integrate out high k modes. To that end, we introduce the notation:

$$\phi^-_{\mathbf{k}} = \begin{matrix} \phi_{\mathbf{k}} & \text{if } k < \frac{\Lambda}{\zeta} \\ 0 & \text{otherwise} \end{matrix} \qquad \phi^+_{\mathbf{k}} = \begin{matrix} \phi_{\mathbf{k}} & \text{if } \frac{\Lambda}{\zeta} < k < \Lambda \\ 0 & \text{otherwise} \end{matrix}$$

The Gaussian Fixed Point

We can decompose this simple form of the free energy without any interaction terms (i.e. terms that mix $\phi_{\mathbf{k}}^-$ and $\phi_{\mathbf{k}}^+$).

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This means that we can easily integrate out the high-k contributions ϕ^+. Following it up with the scaling and field normalisation steps of RG, we end up with the following:

$$F_0'[\phi'] = \int^{\Lambda} \frac{d^d k}{(2\pi)^d} \frac{1}{2}(k^2 + \mu^2(\zeta))\phi'_{\mathbf{k}}\phi'_{-\mathbf{k}}$$

where $\mu^2(\zeta) = \zeta^2 \mu_0^2$

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$$F[\phi] = \int d^d x \left[\frac{1}{2} \nabla\phi \cdot \nabla\phi + \frac{1}{2}\mu_0^2 \phi^2 + \sum_{n=1}^{\infty} g_{0,n}\phi^n \right]$$

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$$F[\phi'] = \int d^d x' \left[\frac{1}{2} \zeta^{-2-2\Delta_\phi} |\nabla' \phi'|^2 + \frac{1}{2} \mu_0^2 \zeta^{-2\Delta_\phi} \phi'^2 + \sum_{n=1}^{\infty} g_{0,n} \zeta^{-n\Delta_\phi} \phi'^n \right]$$

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$$F_0'[\phi'] = \int^{\Lambda} d^d x \frac{1}{2}\left[|\nabla'\phi'|^2 + \mu^2(\zeta)\phi'^2 + \sum_{n=4}^{\infty} g_n(\zeta)\phi'^n\right]$$

where $g_n(\zeta) = \zeta^{(1-n/2)d+n} g_{0,n}$

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  Here, we see that all couplings $n = 6$ and above are irrelevant in dimensions $d > 3$.
- $g_4(\zeta) = \zeta^{4-d} g_{0,4}$, so at d=4, this coupling becomes "marginal".
  This is not the full story. We need to do perturbation theory to uncover first non-zero correction.
- Moreover, we will see that

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- The interaction term is

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RG with Interactions

- The effective free energy:

$$e^{-F'[\phi^-]} = e^{-F_0[\phi^-_{\mathbf{k}}]} \int \mathcal{D}\phi^+_{\mathbf{k}} e^{-F_0[\phi^+_{\mathbf{k}}]} e^{-F_I[\phi^-,\phi^+]}$$

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- Taking logarithms, we end up with

$$F'[\phi^-] = F_0[\phi^-_{\mathbf{k}}] - \log \left\langle e^{-F_I\left[\phi^-_{\mathbf{k}},\phi^+_{\mathbf{k}}\right]} \right\rangle_+$$

Feynman Diagrams and Perturbation Theory

- We can expand the *log* term in terms of cumulants

$$\left\langle e^{-F_I\left[\phi_{\mathbf{k}}^{-},\phi_{\mathbf{k}}^{+}\right]}\right\rangle_{+} = -\left\langle F_I\right\rangle_{+} + \frac{1}{2}\left[\left\langle F_I^2\right\rangle_{+} - \left\langle F_I\right\rangle_{+}^2\right] + \dots$$

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- Feynman diagrams give us a simple way to calculate the cumulants

Feynman Diagram Rules

We look into terms of the expansion of the form $g_0^p(\phi^-)^n(\phi^+)^l$

Drawing rules:

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- Each factor of g_0 is represented as a vertex at which four lines meet
- Each line has an attached momentum k which is conserved at every vertex

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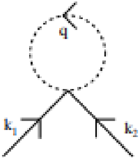
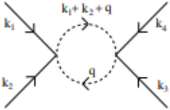
- For each internal loop, there is an integral $\int d^d q/(2\pi)^d$
- Each vertex comes with a factor of $g_0 (2\pi)^d \delta^d(\sum_i \mathbf{k}_i)$, the delta function ensuring momentum conservation.

Examples



$$= 6g_0 \int_{\Lambda/\zeta}^{\Lambda} \frac{d^dq}{(2\pi)^d} \frac{1}{q^2+\mu_0^2} \int d^dx \, (\phi^-)^2$$

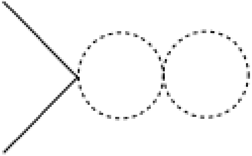
Figure 1: Order g_0



$$= 36g_0^2 \int_0^{\Lambda/\zeta} \prod_{i=1}^{4} \left[\frac{d^dk_i}{(2\pi)^d} \, \phi^-_{\mathbf{k}_i} \right] f(\mathbf{k}_1+\mathbf{k}_2) \, (2\pi)^d \delta^d \left(\sum_i \mathbf{k}_i\right)$$

Figure 2: Order g_0^2

Examples



$$= g_0^2 \int \frac{d^d k}{(2\pi)^d} \, \frac{1}{2} C(\Lambda) \phi_{\mathbf{k}}^- \phi_{-\mathbf{k}}^-$$

Figure 3: Order g_0^2

$$= g_0^2 \int \frac{d^d k}{(2\pi)^d} \, \frac{1}{2} A(k, \Lambda) \phi_{\mathbf{k}}^- \phi_{-\mathbf{k}}^-$$

Figure 4: Order g_0^2

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$$\mu^2(\zeta) = \zeta^2(\mu_0 + a g_0) \qquad\qquad g(\zeta) = \zeta^{4-d}(g_0 - b g_0^2) \tag{1}$$

where

$$a = 12 \int_{\Lambda/\zeta}^{\Lambda} \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + \mu_0^2} \qquad b = 36 \int_{\Lambda/\zeta}^{\Lambda} \frac{d^d q}{(2\pi)^d} \frac{1}{\left(q^2 + \mu_0^2\right)^2} \tag{2}$$

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- For the following analysis, we will focus on $d = 4$.
- To find the β function, all we need to do is to take the derivatives of the above equations with respect to $s = \log\zeta$.

Beta Functions (cont.)

- We will use one approximation here (assuming s is small):

$$\int_{\Lambda e^{-s}}^{\Lambda} dq f(q) \approx \left[\Lambda - \Lambda e^{-s}\right] f(\Lambda) \approx \Lambda f(\Lambda) s \tag{3}$$

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- We get our beta function equations:

$$\boxed{\frac{d\mu^2}{ds} = 2\mu^2 + \frac{12\Omega_3}{(2\pi)^4} \frac{\Lambda^4}{\Lambda^2 + \mu^2} g \quad \text{and} \quad \frac{dg}{ds} = -\frac{36\Omega_3}{(2\pi)^4} \frac{\Lambda^4}{(\Lambda^2 + \mu^2)^2} g^2 \tag{5}}$$

where Ω_3 is the surface area of sphere in 4 dimensions.

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However this epsilon expansion gives up some insight into what is going on in the system.

Epsilon Expansion (cont.)

- Generalizing the beta function gives,

$$\frac{d\mu^2}{ds} = 2\mu^2 + \frac{12\Omega_{3-\epsilon}}{(2\pi)^4}\frac{\Lambda^4}{\Lambda^2+\mu^2}\tilde{g} + \dots \tag{6}$$

$$\frac{d\tilde{g}}{ds} = \epsilon\tilde{g} - \frac{36\Omega_{3-\epsilon}}{(2\pi)^4}\frac{\Lambda^4}{(\Lambda^2+\mu^2)^2}\tilde{g}^2 + \dots \tag{7}$$

where $\tilde{g} = \Lambda^{-\epsilon} g$

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- The first term in the second equation above was missing in the beta function for $d = 4$. This comes from the scaling part of the formula, $g(\zeta) = \zeta^{\epsilon}(g_0 - bg_0^2)$
- We know the formula for the surface area in terms of the gamma function but it gives only $\mathcal{O}(\epsilon^2)$ corrections. So, $\Omega_{3-d} \approx \Omega_3 = 2\pi^2$

Epsilon Expansion (cont.)

- Finally, the following are the beta function equations:

$$\frac{d\mu^2}{ds} \approx 2\mu^2 + \frac{3}{2\pi^2}\frac{\Lambda^4}{\Lambda^2+\mu^2}\tilde{g} \tag{8}$$

$$\frac{d\tilde{g}}{ds} \approx \epsilon\tilde{g} - \frac{9}{2\pi^2}\frac{\Lambda^4}{(\Lambda^2+\mu^2)^2}\tilde{g}^2 \tag{9}$$

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- The non-trivial fixed points of this transformation are:

$$\mu_\star^2 = -\frac{1}{6}\Lambda^2\epsilon \quad \text{and} \quad \tilde{g}_\star = \frac{2\pi^2}{9}\epsilon \tag{10}$$

These are known as the *Wilson–Fisher* fixed points.

Wilson–Fisher Fixed Point

- To understand the flow in the vicinity of the fixed point, we linearize the beta functions, which gives,

$$\frac{d}{ds}\begin{pmatrix} \delta\mu^2 \\ \delta\tilde{g} \end{pmatrix} = \begin{pmatrix} 2-\epsilon/3 & \frac{3}{2\pi^2}\Lambda^2\left(1+\frac{\epsilon}{6}\right) \\ 0 & -\epsilon \end{pmatrix}\begin{pmatrix} \delta\mu^2 \\ \delta\tilde{g} \end{pmatrix} \tag{11}$$

- This gives two eigenvalues–one positive and one negative: (one relevant and one irrelevant directions)

$$\Delta_t = 2-\frac{\epsilon}{3}+\mathcal{O}\left(\epsilon^2\right) \quad \text{and} \quad \Delta_g = -\epsilon+\mathcal{O}\left(\epsilon^2\right) \tag{12}$$

Note that $\epsilon = 0$ means we are working with a 4-dimensional system for which $\Delta_t = 2$ is an exact result.

Critical Exponents Revisited

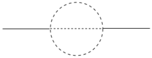
- From the relation $\Delta_t = 1/\nu$, we get

$$\nu = \frac{1}{2} + \frac{\epsilon}{12}$$

- From hyperscaling relation, $\alpha = 2 - d\nu$, we get,

$$\alpha = \frac{\epsilon}{6}$$

- To calculate the other exponents, we need to find the anomalous dimension η. It is related to the two-loop diagram,



$$\eta = \frac{\epsilon^2}{6}$$

Critical Exponents Revisited (cont.)

- The scaling dimension of the field is then given by,

$$\Delta_\phi = \frac{d-2+\eta}{2} \approx 1 - \frac{\epsilon}{6} \tag{13}$$

- β is found from the equation $\beta = (d-2+\eta)\nu/2$:

$$\beta = \frac{1}{2} - \frac{\epsilon}{6} \tag{14}$$

- From Fisher's identity, $\gamma = \nu(2-\eta)$, we get

$$\gamma = 1 + \frac{\epsilon}{6} \tag{15}$$

- Finally, we can find δ using $\delta = (d+2-\eta)/(d-2+\eta)$,

$$\delta = 3 + \epsilon \tag{16}$$

Critical Exponents Revisited (cont.)

- All the exponents are correct up to $\mathcal{O}(\epsilon^2)$.
- Although we are interested in only small ϵ, it is worth looking at the case when $\epsilon = 1$ or in other words, $d = 3$. The mean field results, ϵ expansion results, and the experimental results are given in the table below:

	α	β	γ	δ	η	ν
MF	0	$\frac{1}{2}$	1	3	0	$\frac{1}{2}$
$\epsilon = 1$	0.17	0.33	1.17	4	0	0.58
d = 3	0.1101	0.3264	1.2371	4.7898	0.0363	0.6300

Thank You

Questions?

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Thank you ¨⌣

Bibliography

- Ma, Shang-keng. "SK Ma, Rev. Mod. Phys. 45, 589 (1973)." Rev. Mod. Phys. 45 (1973): 589.
- Tong, David. Statistical Field Theory. DAMTP, 2017, www.damtp.cam.ac.uk/user/tong/sft/sft.pdf
- Wilson, Kenneth G., and Michael E. Fisher. "Critical exponents in 3.99 dimensions." Physical Review Letters 28.4 (1972): 240.