QUANTUM COMPUTING

Principles of Quantum Information & Quantum Computing:
Introduction to Quantum Computing, Moore's law & its end, Differences between Classical & Quantum computing. Concept of qubit and its properties. Representation of qubit by Bloch sphere. Single and Two qubits. Extension to N qubits.

Dirac representation and matrix operations:
Matrix representation of 0 and 1 States, Identity Operator I, Applying I to$|0\rangle$and $|1\rangle$ states, Pauli Matrices and its operations on $|0\rangle$and $|1\rangle$states, Explanation of i) Conjugate of a matrix and ii) Transpose of a matrix. Unitary matrix U, Examples: Row and Column Matrices and their multiplication (Inner Product), Probability, and Quantum Superposition, normalization rule. Orthogonality, Orthonormality. Numerical Problems.

Quantum Gates:
Single Qubit Gates: Quantum Not Gate, Pauli – X, Y and Z Gates, Hadamard Gate, Phase Gate (or S Gate), T Gate.
Multiple Qubit Gates: Controlled gate, CNOT Gate, (Discussion for 4 different input states). Representation of Swap gate, Controlled -Z gate, Toffoli gate.

Introduction

Today's computers—both in theory and practice (PCs, HPCs, laptops, tablets, smartphone) are based on classical physics. They are limited by locality (operations have only local effects) and by the classical fact that systems can be in only one state at the time. However, modern quantum physics tells us that the world behaves quite differently. A quantum system can be in a superposition of many different states at the same time, and can exhibit interference effects during the course of its evolution. Moreover, spatially separated quantum systems may be entangled with each other and operations may have "non-local" effects because of this.
Quantum computation is the field that investigates the computational power and other properties of computers based on quantum-mechanical principles. It combines two of the most important strands of 20th-century science: quantum mechanics (developed by Planck, Einstein, Bohr, Heisenberg, Schrödinger and others in the period 1900–1925) and computer science . An important objective is to find quantum algorithms that are significantly faster than any classical algorithm solving the same problem.

Quantum computing is a type of computation whose operations can harness the phenomena of quantum mechanics, such as superposition, interference, and entanglement.
To store and manipulate the information, they use their own quantum bits also called **'*Qubits*'** unlike other classical computers which are based on classical computing that uses binary bits 0 and 1 individually
The computers using such type of computing are known as '***Quantum Computers'.***
In such small computers, circuits with transistors, logic gates, and Integrated Circuits are not possible. Hence, it uses the subatomic particles like atoms, electrons, photons, and ions as their bits along with their information of spins and states. Instead of 0 and 1, Quantum computers can choose two electronics states of an atom or two different polarization orientations of light for the two states They can be superposed and can give more combinations.

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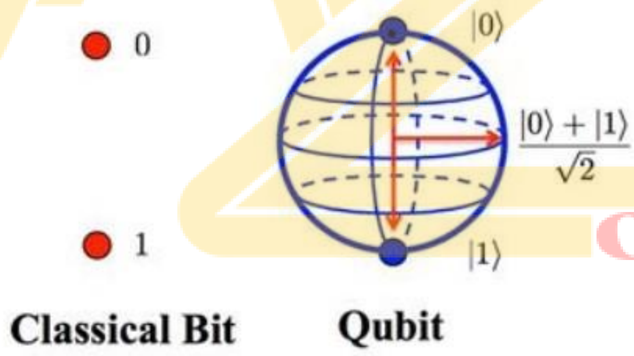
Moore's law & its end

In the year 1965, Gordon Moore, co-founder of Intel discovered that **"the number of transistors in a dense integrated circuit (IC) doubles about every two years".** Moore's law is an observation and projection of a historical trend. Rather than a law of physics, it is an empirical relationship linked to gains from experience in production.
Moore's Law is considerable because it means that computers and their computing power get smaller and faster over time. Though this law is putting the brakes on now and consequently, the improvement in classical computers is not like before it used to be. This shows that the size of the circuits of the classical computer after 3-2 nanometres has reached their limit. At these scales, controlling the flow of electrons becomes increasingly more difficult as all kinds of quantum effects play themselves within the transistor itself. Since, the transistor is approaching the point where it is simply as small as we can ever make it and have it still function.
The way we've been building and improving silicon chips is coming to its final iteration. Thus Quantum Computation is the option for the further generation.

Classical Bits :
It's a single unit of information that has a value of either 0 or 1 (off or on, false or true, low or high).

Quantum Bits :
In quantum computing, a qubit or quantum bit is the basic unit of quantum information. A quantum system like atom or electrons can exist in states as 0 and 1 or simultaneously both as 0 and 1.



Differences Between Classical and Quantum Computing

Classical Computing

1. .Used by large scale, multipurpose and devices.
2. Information is stored in bits. There is discrete number of possible states. Either 0 or 1.
3. Power increases in 1:1 relationship with number of transistor
4. They have low error rates and can operate at room temperature
5. Calculations are deterministic. This means repeating the same inputs results in the same output.
6. Data processing is carried out by logic and in sequential order
7. Operations are governed by Boolean Algebra.
8. Circuit behaviour is defined by Classical Physics.

Quantum Computing

1. Used by high speed, quantum mechanics-based computers.
2. Information is based on Quantum Bits. There is an infinite, continuous number of possible states. They are the result of quantum superposition.
3. Power increases in exponentially in proportion to the number of qubits
4. They have high error rates and need to be kept at ultracold
5. The calculations are probabilistic, meaning there are multiple possible outputs to the same inputs.
6. Data processing is carried out by quantum logic at parallel instances.
7. Operations are defined by linear algebra by Hilbert Space.
8. Circuit behaviour is defined by Quantum Mechanics.

Concept of Qubit and its properties

Concept of Qubit

- In quantum computing, a **qubit** or **quantum bit** is a basic unit of quantum information
- A quantum system like atom or electrons can exist in states as 0 and 1 or simultaneously both as 0 and 1.
- It will not be know definitely, in which states they would be.
- But the number called probability factor associated with the corresponding state provide the probability of the atoms/electrons existence in each of these states.
- Since, they follow the quantum principles, it becomes a quantum system and called as quantum bits or Qubit.
- Similarly in case of light, a Qubit may correspond to superposed state of horizontal and vertical polarization of photons apart from two individual states. .

Properties of Qubits

- A Qubit can be physically implemented by the two states of an electron or horizontal and vertical polarizations of photons as $|\downarrow\rangle$ and $|\uparrow\rangle$

- **Superposition**

  A Qubit can be in a superposed state of the two states /0> and /1> **(Ket notation) .** If measurements are carried out with a qubit in superposed state then the results that we get will be probabilistic

  It is represented as /ψ> = α/0> +β/1>

  Where α and β are complex numbers and $|\alpha|^2 + |\beta|^2 = 1$

- **No Cloning Theorem**

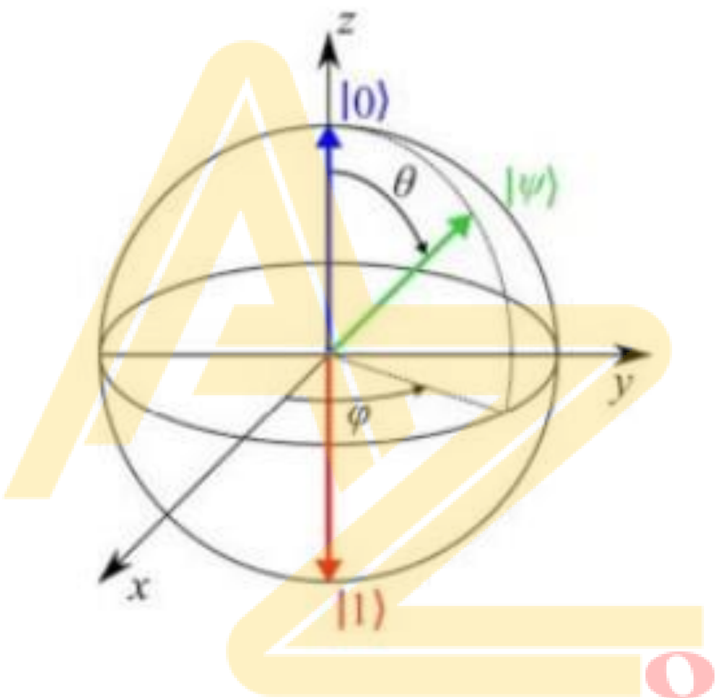
  The Qubit changes its state once it is subjected to the measurement. It means, one cannot copy the information from the qubit the way we do it in classical computers, as there will be no similarity between the copy and the original.This is known as "**no cloning principle"**

- **Entanglement**

  Two Qubits can strongly correlated with each other. Changing state of one of the qubits will instantaneously change the the state of the other one in predictable way. This happens even if they are separated by very long distances.

Representation of Qubits by Bloch Sphere

- The pure state space qubits (Two Level Quantum Mechanical Systems) can be visualized using an imaginary sphere called Bloch Sphere. It has a unit radius
- This Bloch sphere picture is elegant and powerful for the single qubit.



- The Arrow on the sphere represents the state of the Qubit. The north and south poles are used to represent the basis states $|0\rangle$ and $|1\rangle$ respectively. The other locations on the bloch sphere represents the superpositions state i.e $|0\rangle$ and $|1\rangle$ states and represented by $\alpha|0\rangle + \beta|1\rangle$ with $|\alpha|^2 + |\beta|^2 = 1$. Thus a Qubit can be any point on the Bloch Sphere.
- The Bloch sphere allows the state of the qubit to be represented in unit spherical co-ordinates. They are the polar angle θ and the azimuth angle ϕ. The block sphere is represented by the equation

$$|\psi\rangle = cos\,\frac{\theta}{2}\,|0\rangle + e^{i\,\phi} sin\,\frac{\theta}{2}\,|1\rangle$$

here $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$. The normalization constraint is given by

$$|cos\frac{\theta}{2}|^2 + |\sin\frac{\theta}{2}|^2 = 1$$

- For any Gate operation, taking an initial state to the final state of the single-qubit, is equivalent to a composition of one or more simple rotations on the Bloch sphere.

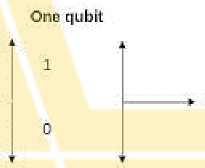
Single and Two qubits and Extension to N qubits

Single qubit

A Single Qubit has two computational basis states $|0\rangle$ and $|1\rangle$. In single qubit the state of qubit will be either in |0> and |1> where $|0> = \begin{bmatrix}1\\0\end{bmatrix}$ and $|1> = \begin{bmatrix}0\\1\end{bmatrix}$

Above basis vectors are written as column matrix in the form of identity matrix

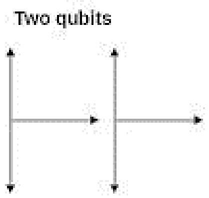
$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ Which represents the single qubit state. The pictorial representation of the single qubit is as follows. $\alpha\,|0\rangle + \beta\,|1\rangle$



Two qubit

A two-qubit system has 4 computational basis states denoted as $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$.The matrix representation of two qubit state is $\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$

The pictorial representation of two qubit is as follows. $\alpha\,|00\rangle + \beta\,|01\rangle + \gamma\,|10\rangle + \delta\,|11\rangle$



Extension to N qubits

A multi-qubit system of N qubits has 2^N computational basis states. For example a state with 3 qubits has 2^3 computational basis states. Thus for N qubits the computational basis states are denoted as $|00\cdots00\rangle$, $|00\cdots01\rangle$, $|00\cdots10\rangle$, $|00\cdots11\rangle\cdots|11\cdots11\rangle$. The block diagram of representation of N qubits is as follows

Dirac Representation and Matrix Operations

Linear Algebra

Linear Algebra is the study of vector spaces and operations on vector spaces. The Standard quantum mechanical notation for a quantum state ψ in a vector space is $|\psi\rangle$. The notation $|\rangle$ indicates that the object is a vector rand is called a ket vector. The examples of ket vectors are $|\psi\rangle$, $|\phi\rangle$ and $|u\rangle$ etc.

Matrix Representation of 0 and 1 States

The wave function could be expressed in ket notation as $|\psi\rangle$ (ket Vector), ψ is the wave function. The $|\psi\rangle = \begin{bmatrix} \alpha 1 \\ \alpha 2 \end{bmatrix}$. The matrix for of the states $|0\rangle$ and $|1\rangle$. $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

a. Identity Operator

The operator of type $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is called identity operator. When an identity operator acts on a state vector its keeps the state intact. By analogy we study identity operator as an identity matrix.

Let us consider the operation of Identity operator on $|0\rangle$ and $|1\rangle$ states. As per the principle of identity operation $I\,|0\rangle = |0\rangle$ and $I\,|1\rangle = |1\rangle$

$$I\,|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$I\,|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Thus the operation of identity matrix(operator) on $|0\rangle$ and $|1\rangle$ leaves the states unchanged.

b. Pauli Matrices and Their operation on $|0\rangle$ and $|1\rangle$ States

There are four extreamly useful matrices called Pauli Matrices that are often used in quantum computers. The Pauli matrices of the following form

$$\sigma_0 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This is an identity matrix.

$\sigma_1 = \sigma_X = \text{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\sigma_2 = \sigma_Y = \text{Y} = \begin{pmatrix} 0 & -i \\ 1 & 0 \end{pmatrix}$

$\sigma_3 = \sigma_Z = \text{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Here X,Y,Z are known as Pauli Matrices

Pauli Matrices operating on |0⟩ and |1⟩ States

1. $\sigma_0|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$

   $\sigma_0|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$

2. $\sigma_x|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$

   $\sigma_x|1\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$

3. $\sigma_y|0\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = i|1\rangle$

   $\sigma_y|1\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -i|0\rangle$

4. $\sigma_z|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$

   $\sigma_z|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -|1\rangle$

c. <u>Conjugate of a Matrix</u>

It is possible to find the conjugate for a given matrix by replacing each element of the matrix with its complex conjugate for example consider a matrix A as given below.

$\text{A} = \begin{pmatrix} i & 1 \\ 0 & 2-3i \end{pmatrix}$

The conjugate of the matrix A is given by

$A^* = \begin{pmatrix} -i & 1 \\ 0 & 2+3i \end{pmatrix}$

Thus A^* is the conjugate of A.

d. Transpose of a Matrix

The transpose of a matrix is found by interchanging its rows into columns or columns into rows. The Transpose of a matrix A is denoted by using the superscript as A^T. Consider a matrix A as given below.

$$A = \begin{pmatrix} i & 1 \\ 0 & 2-3i \end{pmatrix}$$

The Transpose of the matrix A is given by

$$A^T = \begin{pmatrix} i & 0 \\ 1 & 2-3i \end{pmatrix}$$

Thus A^T is the Transpose of A

e. The Conjugate Transpose of a Matrix

The complex conjugate transpose of a matrix interchanges the row and column index for each element, reflecting the elements across the main diagonal. The operation also negates the imaginary part of any complex numbers. It is denoted by a † symbol as a super script.

$$A = \begin{pmatrix} i & 1 \\ 0 & 2-3i \end{pmatrix}$$

The Conjugate Transpose of the matrix A is given by

$$A^\dagger = (A^*)^T = \begin{pmatrix} -i & 0 \\ 1 & 2+3i \end{pmatrix}$$

Thus $A^\dagger$ is the Conjugate-Transpose of A

d. Hermitian

The matrix that is equal to its conjugate-transpose is called Hermitian. Thus If $A\dagger = A$ then it is called Hermitian or Self-Adjoint matrix

$$A = \begin{pmatrix} 3 & 3+i \\ 3-i & 2 \end{pmatrix}$$

The complex conjugate of A is given by

$$A^* = \begin{pmatrix} 3 & 3-i \\ 3+i & 2 \end{pmatrix}$$

The transpose of A^* is given by

$$A^\dagger = (A^*)^T = \begin{pmatrix} 3 & 3+i \\ 3-i & 2 \end{pmatrix}$$

Hence $A^\dagger$= A

e. Unitary Matrix

A matrix is said to be Unitary if the condition $U^{\dagger}U = I$ is satisfied. Thus, an operator is said to be Unitary if each of its matrix representations are unitary. Consider an operator in matrix form U.

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix}$$

$$U^{*} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}$$

$$U^{\dagger} = (U^{*})^{T} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}$$

$$U^{\dagger}U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}+\frac{1}{2} & \frac{-i}{2}+\frac{i}{2} \\ \frac{i}{2}-\frac{i}{2} & \frac{1}{2}+\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{I}$$

Hence U is unitary

Column and Row Matrices

The Column Vectors are called ket Vectors denoted by $|\psi\rangle$ and are represented by Column Matrices. The Row Vectors are called Bra Vectors denoted by $\langle\phi|$ and are represented by Row Matrices. Let us consider a ket vector represented in the form of a column matrix

$$|\psi\rangle = \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix}$$

The Row Matrix is represented as

$$\langle\psi| = [\alpha_1^* \quad \beta_1^*]$$

$$\begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix}^{\dagger} = [\alpha_1^* \quad \beta_1^*]$$

Thus the Bra is the complex conjugate of ket and viceversa. For example

$$\begin{bmatrix} 1 \\ \mathrm{i} \end{bmatrix}^{\dagger} = [1 \quad -i]$$

Flipping between kets and bras is called "Taking the Dual".

Thus for $|0\rangle$ state the corresponding $\langle 0|$ is given by

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\langle 0| = [1 \quad 0]$$

and similarly for and $|1\rangle$ states we have $\langle 1|$ as follows.

$$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\langle 1| = [0 \quad 1]$$

Inner Product - Multiplication of Row and Column Matrices

Let us consider two states $|\psi\rangle$ and $|\phi\rangle$ as follows

$$|\psi\rangle = \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix}$$

$$|\phi\rangle = \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix}$$

$$\langle\psi| = [\alpha_1^* \quad \beta_1^*]$$

The multiplication of the $|\psi\rangle$ and $|\phi\rangle$ is possible only by taking the inner product and is given by $\langle\psi|\phi\rangle$

$$\langle\psi|\phi\rangle = [\alpha_1^* \quad \beta_1^*] \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix}$$

$$\langle\psi|\phi\rangle = \alpha_1^*\alpha_2 + \beta_1^*\beta_2$$

The inner product always results in a scalar product.

Probability

Let us consider a Quantum State

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

The above equation represents the Quantum Superposition of states $|0\rangle$ and $|1\rangle$

$$|\psi\rangle = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

The inner product $\langle\psi|\psi\rangle$ is given by

$$\langle\psi|\psi\rangle = [\alpha^* \quad \beta^*] \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha\alpha^* + \beta\beta^*$$

Thus,

$\alpha\alpha^* + \beta\beta^* = |\alpha|^2 + |\beta|^2$

This could also be written as

$|\psi|^2 = \psi\psi*$

Thus the above equation represents **Probability Density**. As per the principle of Normalization

$|\psi|^2 = \psi\psi* = \langle\psi|\psi\rangle = 1 = |\alpha|^2 + |\beta|^2$

Thus it implies $|\psi\rangle$ is normalized.

Orthogonality

Two states $|\psi\rangle$ and $|\phi\rangle$ are said to be orthogonal if their inner product is Zero.

Mathematically

$\langle\phi|\psi\rangle = 0$

The two states are orthogonal means they are mutually exclusive. Like Spin Up and Spin Down of an electron.

Consider $\langle 0|1\rangle$

$\langle 0|1\rangle = [1 \quad 0]\begin{bmatrix}0\\1\end{bmatrix} = (0+0) = (0)$

Consider $\langle 1|0\rangle$

$\langle 1|0\rangle = [0 \quad 1]\begin{bmatrix}1\\0\end{bmatrix} = (0+0) = (0)$

Thus the states $|1\rangle$ and $|0\rangle$ are Orthogonal

Orthonormality

The states $|\psi\rangle$ and $|\phi\rangle$ are said to be orthonormal if

1. $|\psi\rangle$ and $|\phi\rangle$ are normalized.

   i.e $<\psi|\psi> = <\phi|\phi> = 1$

2. $|\psi\rangle$ and $|\phi\rangle$ are orthogonal to each other.

   i.e $\langle\phi|\psi\rangle = 0$

Quantum Gates

In quantum computing a quantum logic gate is a basic quantum circuit operating on a small number of qubits. A qubit is useless unless it is used to carry out a quantum calculation. The quantum calculations are achieved by performing a series of fundamental operations, known as quantum logic gates. They are the building blocks of quantum circuits similar to the classical logic gates in conventional digital circuits.

Single Qubit Gates

a. Quantum Not Gate

In Quantum Computing the quantum NOT gate for qubits takes the state $|0\rangle$ to $|1\rangle$ and vice versa. It is analogous to the classical not gate.

The Matrix representation of Quantum Not Gate is given by

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$X|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$$

$$X|1\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

A Quantum State is given by $\alpha\,|0\rangle + \beta\,|1\rangle$ and its matrix representation is given by $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$.
Hence the operation of Quantum Not Gate on quantum state is given by

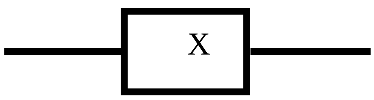
$$X\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$$

Thus the quantum state becomes $\alpha\,|1\rangle + \beta\,|0\rangle$. Similarly, The input $\alpha\,|1\rangle + \beta\,|0\rangle$ to the quantum not gates changes the state to $\alpha\,|0\rangle + \beta\,|1\rangle$. The quantum not gate circuit and the truth table are as shown below.

Truth table of Quantum NOT Gate

Input	Output
$\|0>$	$\|1>$
$\|1>$	$\|0>$
$\alpha\|0> + \beta\|1>$	$\beta\|0> + \alpha\|1>$

Symbol:

$\alpha|0> + \beta|1>$  $\beta|0> + \alpha|1>$

b. Pauli-X,Y and Z Gates

- **X Gate**

The Pauli-X Gate is nothing but Quantum Not Gate

- **Y Gate**

Y Gate is represented by Pauli matrix σy or Y. This gate Maps $|0\rangle$ state to $i\,|1\rangle$ state and $|1\rangle$ state to $-i\,|0\rangle$ state. The Y Gate and its operation is as given below

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$Y\,|0\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix} = 0|0\rangle + i\,|1\rangle = i\,|1\rangle$$

$$Y\,|1\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -i \\ 0 \end{pmatrix} = -\,i\,|0\rangle + 0|1\rangle = -\,i\,|0\rangle$$

Thus the Y-Gate defines the transformation

$$Y\,(\alpha\,|0\rangle + \beta\,|1\rangle) = Y\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = -i\beta\,|0\rangle + i\alpha\,|1\rangle$$

Quantum Y-Gate is represented by

$$\alpha\,|0\rangle + \beta\,|1\rangle) \text{———} \boxed{\;Y\;} \text{———} -i\beta\,|0\rangle + i\alpha\,|1\rangle$$

Truth table of Y-gate

Input	Output
|0>	i|1>
|1>	-i|0>
α|0> + β|1>	-iβ|0> +iα |1>

- **Z-Gate**

The Z-gate is represented by Pauli Matrix σ_Z or Z. Z-Gate maps input state $|k\rangle$ to $(-1)^{k}|k\rangle$.

1. For input $|0\rangle$ the output remains unchanged.

2. For input $|1\rangle$ the output is $-\,|1\rangle$.

The Matrix representation and the operation of Z-Gate on $|0\rangle$ and $|1\rangle$ are as follows

$$Z\,|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

$Z\,|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -|1\rangle$

$Z(\alpha\,|0\rangle + \beta\,|1\rangle) = Z\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha\,|0\rangle - \beta\,|1\rangle$

The circuit symbol and the truth table of Z-Gate are as follows

$\alpha\,|0\rangle + \beta\,|1\rangle$ —— $\boxed{Z}$ —— $\alpha\,|0\rangle - \beta\,|1\rangle$

Truth Table of Z-gate is

$\|0>$	$\|0>$
$\|1>$	$-\|1>$
$\alpha\|0> + \beta\|1>$	$\alpha\|0> - \beta\|1>$

c. Hadamard Gate

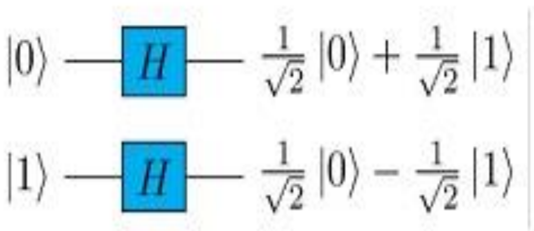
The Hadamard Gate is a truly quantum gate and is one of the most important in Quantum Computing. It is a self-inverse gate. It is used to create the superpositions of $|0\rangle$ and $|1\rangle$ states. The Matrix representation of Hadamard Gate is as follows

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

The Hadamard Gate and the output states for the $|0\rangle$ and $|1\rangle$ input states are represented as follows. The Hadamard Gate satisfies Unitary Condition.

$$H^{\dagger}H = 1$$

The truth-table for the Hadamard Gate is as follows



Input	Action of Hadamard gate	Output
$\|0\rangle$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix}$	$\frac{\|0\rangle + \|1\rangle)}{\sqrt{2}}$
$\|1\rangle$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\end{bmatrix}$	$\frac{\|0\rangle - \|1\rangle)}{\sqrt{2}}$
$\alpha\|0\rangle + \beta\|1\rangle$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}\alpha + \beta\\ \alpha - \beta\end{bmatrix}$	$\alpha\frac{\|0\rangle + \|1\rangle)}{\sqrt{2}} + \beta\frac{\|0\rangle - \|1\rangle)}{\sqrt{2}}$

d. Phase Gate or S Gate

The **S-gate** is an Rϕ gate with $\phi = \pi/2$ or in other terms, it rotates the vector $\pi/2$ radians around the z-axis. Unlike other gates, the S-gate is not its own inverse. Although it is still unitary.The phase gate turns a $|0\rangle$ into $|0\rangle$ and a $|1\rangle$ into $i\,|1\rangle$

$S^\dagger S = I$ and hence it is Unitary.

The Matrix representation of the S gate is given by

$$S = \begin{bmatrix}1 & 0\\ 0 & i\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & e^{\frac{i\pi}{2}}\end{bmatrix}$$

The effect of S gate on input $|0\rangle$ is given by

$$S|0\rangle = \begin{bmatrix}1 & 0\\ 0 & i\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \begin{bmatrix}1\\ 0\end{bmatrix} = |0\rangle$$

Similarly the effect of S gate on input $|1\rangle$ is given by

$$S|1\rangle = \begin{bmatrix}1 & 0\\ 0 & i\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} = \begin{bmatrix}0\\ i\end{bmatrix} = i|1\rangle$$

The transformation of state $|\psi\rangle$ is given by

$$S\,(\alpha\,|0\rangle + \beta\,|1\rangle) = S\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \alpha\,|0\rangle + i\beta\,|1\rangle$$

The S Gate and the Truth table are given by For S gate

$$\alpha\,|0\rangle + \beta\,|1\rangle \;—\boxed{S}—\; \alpha\,|0\rangle + i\,\beta\,|1\rangle$$

Truth Table of S Gate

Input	Output
$\|0\rangle$	$\|0\rangle$
$\|1\rangle$	$i\|1\rangle$
$\alpha\,\|0\rangle + \beta\,\|1\rangle$	$\alpha\,\|0\rangle + i\beta\,\|1\rangle$

e. T Gate or $\frac{\pi}{8}$ Gate

The **T-gate** is again a specific case of Rϕ gate with ϕ = π/4. It follows the same behaviour as that of the S-gate.

The T Gate is represented by the matrix as follows

$$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1+i}{\sqrt{2}} \end{bmatrix}$$

It is also called $\frac{\pi}{8}$ gate as it could be represented in the following form

$$T = \exp\frac{i\pi}{8}\begin{bmatrix} \exp\frac{-i\pi}{8} & 0 \\ 0 & \exp\frac{i\pi}{8} \end{bmatrix}$$

Another Important Feature of T gate is it could be related to S gate as

$$T^2 = S$$

The Operation of T gate on $|0\rangle$ and $|1\rangle$ are given by

$$T|0\rangle = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$$

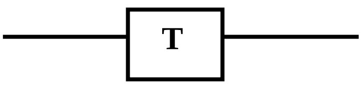
$$T|1\rangle = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ e^{\frac{i\pi}{4}} \end{bmatrix} = e^{\frac{i\pi}{4}}|1\rangle$$

The transformation of state $|\psi\rangle$ is given by

$$T\,(\alpha\,|0\rangle + \beta\,|1\rangle) = T\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha\,|0\rangle + e^{\frac{i\pi}{4}}\beta\,|1\rangle$$

The T Gate and the Truth Table are as follows

$\alpha\,|0\rangle + \beta\,|1\rangle$



$\alpha\,|0\rangle + e^{\frac{i\pi}{4}}\,\beta\,|1\rangle$

Truth Table of T Gate

Input	Output
$\|0\rangle$	$\|0\rangle$
$\|1\rangle$	$e^{\frac{i\pi}{4}}\|1\rangle$
$\alpha\ \|0\rangle + \beta\ \|1\rangle$	$\alpha\ \|0\rangle + e^{\frac{i\pi}{4}}\beta\ \|1\rangle$

Multiple Qubit Gates

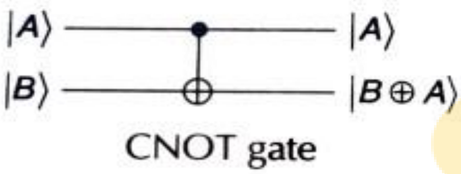
Multiple Qubit Gates operate on Two or More input Qubits. Usually on of them is a control qubit.

Controlled Gates

A Gate with operation of kind "If 'A' is True then do 'B'" is called Controlled Gate. The '$|A\rangle$' Qubit is called Control qubit and '$|B\rangle$' is the Target qubit. The target qubit is altered only when the control qubit is $|1\rangle$. The control qubit remains unaltered during the transformations.

a. Controlled Not Gate or CNOT Gate

The CNOT gate is a typical multi-qubit logic gate. The CNOT gate operates on a quantum register that consists of 2 qubits. The CNOT gate flips the second qubit (the target qubit) if and only if the first qubit (the control qubit) is |1> and the circuit is as follows.



Control Qubit

Target qubit

The Matrix representation of CNOT Gate is given by

$$U_{CN} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The Transformation could be expressed as

$$|A, B\rangle \rightarrow |A, B \oplus A\rangle$$

Consider the operations of CNOT gate on the four inputs $|00\rangle$,$|01\rangle$,$|10\rangle$ and $|11\rangle$.

Operation of CNOT Gate for input |00⟩

$U_{CN}|00\rangle = |00\rangle$

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix}=\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

Here in the inputs to the CNOT Gate the control qubit is $|0\rangle$. Hence no change in the state of Target qubit $|0\rangle$

$|00\rangle \rightarrow |00\rangle$

Operation of CNOT Gate for input |01⟩

$U_{CN}|01\rangle = |01\rangle$

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\begin{bmatrix}0\\1\\0\\0\end{bmatrix}=\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$$

Here in the inputs to the CNOT Gate the control qubit is $|0\rangle$. Hence no change in the state of Target qubit $|1\rangle$

$|01\rangle \rightarrow |01\rangle$

Operation of CNOT Gate for input |10⟩

$U_{CN}|10\rangle = |11\rangle$

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\begin{bmatrix}0\\0\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

Here in the inputs to the CNOT Gate the control qubit is $|1\rangle$. Hence the state of Target qubit flips from $|0\rangle$ to $|1\rangle$.

$|10\rangle \rightarrow |11\rangle$

Operation of CNOT Gate for input |11⟩

$U_{CN}|11\rangle = |10\rangle$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Here in the inputs to the CNOT Gate the control qubit is $|1\rangle$. Hence the state of Target qubit flips from $|1\rangle$ to $|0\rangle$.

$|11\rangle \longrightarrow |10\rangle$

The Truth Table of operation of CNOT gate is as follows

Input	Output
|00>	|00>
|01>	|01>
|10>	|11>
|11>	|10>

b. Swap Gate

The SWAP gate is two-qubit operation. Expressed in basis states, the SWAP gate swaps the state of the two qubits involved in the operation. The Matrix representation of the Swap Gate is as follows

$$U_{SWAP} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operation of SWAP Gate for input $|00\rangle$

$U_{SWAP}|00\rangle = |00\rangle$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$|00\rangle \longrightarrow |00\rangle$

Operation of SWAP Gate for input $|01\rangle$

$U_{SWAP}|01\rangle = |10\rangle$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$|01\rangle \rightarrow |10\rangle$

Operation of SWAP Gate for input |10⟩

$U_{SWAP}|10\rangle = |01\rangle$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$|10\rangle \rightarrow |01\rangle$

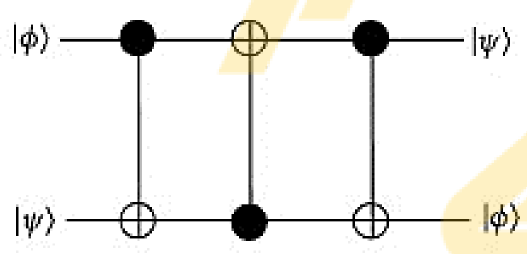
Operation of SWAP Gate for input |11⟩

$U_{SWAP}|11\rangle = |11\rangle$

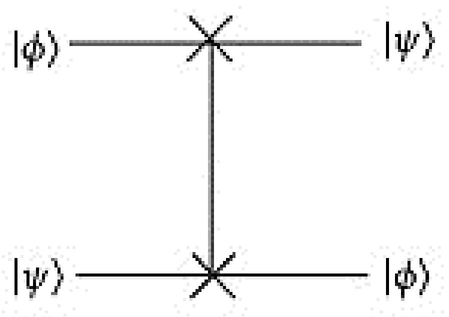
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$|11\rangle \rightarrow |11\rangle$

The schematic symbol of swap gate circuit is as follows



equivalent circuit symbol



The swap gate is a combined circuit of 3 CNOT gates and the over all effect is that two input qubits are swapped at the output. The Action and truth table of the swap gate is as follows.

Gate	Input	Output
1	$\|A, B\rangle$	$\|A, A \oplus B\rangle$
2	$\|A, A \oplus B\rangle$	$\|B, A \oplus B\rangle$
3	$\|B, A \oplus B\rangle$	$\|B, A\rangle$

The Truth table of SWAP Gate

Input	Output
$\|00\rangle$	$\|00\rangle$
$\|01\rangle$	$\|01\rangle$
$\|10\rangle$	$\|01\rangle$
$\|11\rangle$	$\|11\rangle$

c. <u>Controlled Z Gate</u>

In Controlled Z Gate, The operation of Z Gate is controlled by a Control Qubit. If the control Qubit is $|A\rangle = |1\rangle$ then only the Z gate transforms the Target Qubit $|B\rangle$ as per the Pauli-Z operation. The action of Controlled Z-Gate could is specified by a matrix as follows.

$$U_Z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Operation of C-Z Gate for input |00⟩

$U_Z|00\rangle = |00\rangle$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$|00\rangle \longrightarrow |00\rangle$

Operation of C-Z Gate for input |01⟩

$U_Z|01\rangle = |01\rangle$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$|01\rangle \longrightarrow |01\rangle$

Operation of C-Z Gate for input |10⟩

$U_Z|10\rangle = |10\rangle$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$|10\rangle \rightarrow |10\rangle$

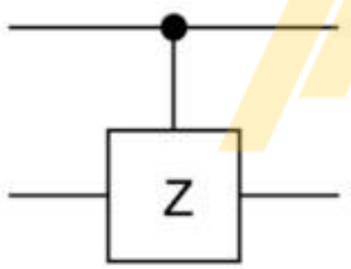
Operation of C-Z Gate for input |11⟩

$U_{SWAP}|11\rangle =$ -$|11\rangle$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

$|11\rangle \rightarrow$ -$|11\rangle$

The controlled Z gate and the truth table are as follows



Input	Output
|00>	|00>
|01>	|01>
|10>	|10>
|11>	-|11>

d. Toffoli Gate

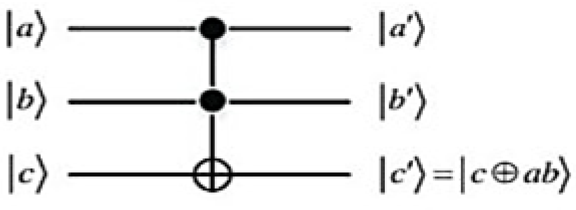
The Toffoli Gate is also known as CCNOT Gate (Controlled-Controlled-Not). It has three inputs out of which two are Control Qubits and one is the Target Qubit. The Target Qubit flips only when both the Control Qubits are $|1\rangle$. The two Control Qubits are not altered during the operation. The matrix representation, Gate Circuit and the Truth Table of Toffoli Gate are as follows.

$$U_T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Truth Table of Toffoli gate

INPUT			OUTPUT		
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	1	0	0
1	0	1	1	0	1
1	1	0	1	1	1
1	1	1	1	1	0

Representation of Toffoli gate



The Toffolli matrix is unitary. The Toffolli Gate is its own inverse. It could be used for NAND Gate Simulation.

Module-3:Quantum Computing

1.1 Linear Operators and Matix Operations

1. A Linar Operator 'X' operates such that $X|0\rangle = |1\rangle$ and $X|1\rangle = |0\rangle$. Find the matrix representation of 'X'.

   Solution : $X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}, |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

   Given $X|0\rangle = |1\rangle$

   $X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

   Multiplying the Matrices on LHS and equating with the Matrix on the RHS we get

   $x_{11} = 0 \; \& \; x_{21} = 1$

   Given $X|1\rangle = |0\rangle$

   $X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

   Multiplying the Matrices on LHS and equating with the Matrix on the RHS we get

   $x_{12} = 1 \; \& \; x_{22} = 0$

   Therefore $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

2. Given $A = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, Prove that $A^\dagger = A$.

   Soution : Given $A = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

   We Know That $A^\dagger = (A^*)^T$

   The Conjugate of Matrix A is Given by $A^* = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$

   Thus $(A^*)^T = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = A$

3. Show that the Matrix $U = \begin{bmatrix} \frac{1}{\sqrt{(2)}} & \frac{1}{\sqrt{(2)}} \\ \frac{i}{\sqrt{(2)}} & \frac{-i}{\sqrt{(2)}} \end{bmatrix}$ is Unitary.

   Solution : A Matrix is Unitary if $U^\dagger U = I$

   Therefore $U^\dagger = \begin{bmatrix} \frac{1}{\sqrt{(2)}} & \frac{-i}{\sqrt{(2)}} \\ \frac{1}{\sqrt{(2)}} & \frac{i}{\sqrt{(2)}} \end{bmatrix}$

   $\Longrightarrow$

   $$U^\dagger U = \begin{bmatrix} \frac{1}{\sqrt{(2)}} & \frac{-i}{\sqrt{(2)}} \\ \frac{1}{\sqrt{(2)}} & \frac{i}{\sqrt{(2)}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{(2)}} & \frac{1}{\sqrt{(2)}} \\ \frac{i}{\sqrt{(2)}} & \frac{-i}{\sqrt{(2)}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

   Here I is identity Matrix.

4. Find the inner product of states $|1\rangle$ *and* $|0\rangle$ and draw conclusions on the result.

   Solution : The inner product is given by $\langle\psi|\phi\rangle$ here $\langle\psi|$ is conjugate-transpose of $|\psi\rangle$.

   We know that

   $$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

   $$\langle 1| = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

   Therefore $\langle 1|0\rangle = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$

   Thus the states $|1\rangle$ and $|0\rangle$ are **Orthogonal**.

5. Given $|\psi\rangle = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$ and $|\phi\rangle = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$ Prove that $\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^*$

   Solution : $\langle\phi|\psi\rangle = \begin{pmatrix} \beta_1^* & \beta_2^* \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \beta_1^*\alpha_1 + \beta_2^*\alpha_2..........(1)$

   $$\langle\psi|\phi\rangle = \begin{pmatrix} \alpha_1^* & \alpha_2^* \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \beta_1\alpha_1^* + \beta_2\alpha_2^*$$

   $$\langle\psi|\phi\rangle^* = (\beta_1\alpha_1^* + \beta_2\alpha_2^*)^* = \beta_1^*\alpha_1 + \beta_2^*\alpha_2..........(2)$$

   Thus from (1) and (2)

   $$\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^*$$

1.2 Quantum Gates

1. Using Matrix multiplication show that on applying Hadamard gate twice to a $|0\rangle$ results in its original state.

   Solution : To show that $HH|0\rangle = |0\rangle$

   $$HH = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\implies \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Since $I\,|0\rangle = |0\rangle$
Thus applying two Hadamard gates result in the original state of $|0\rangle$.

2. Using two X-gates in series show that two not gates in serries are equivalent to a quantum wire.
Solution : To show that $XX\,|0\rangle = |0\rangle$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Since $I\,|0\rangle = |0\rangle$
Hence two Not gates in series result in Quantum Wire.

3. Show the Hadamard Gate is Unitary.

Solution : To Show that $H^{\dagger}H = I$

We Know That $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$H^{\dagger} = (H^{*})^{T} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = H$
Thus using the solution in problem number 1
$H^{\dagger}H = HH = I$
Thus the Hadamard Gate is Unitary.

4. Two Qbits are passed through CNOT gate.If the first qubit is the control qubit then what is the output for the following initial states 1. $|00\rangle$, 2. $|01\rangle$, and 3.$|11\rangle$.
Solution : The Operation of the CNOT gate could be represented as
$|x, y\rangle \rightarrow |x, x \oplus y\rangle$
$\oplus$ is analogous to Classical XOR gate. XOR gate out put is high when both the inputs are dissimilar.
$|00\rangle \rightarrow |0, 0 \oplus 0\rangle = |00\rangle$
$|01\rangle \rightarrow |0, 0 \oplus 1\rangle = |01\rangle$
$|11\rangle \rightarrow |1, 1 \oplus 1\rangle = |10\rangle$

5. Show that S gate can be formed by connecting two T gates in Series.

Solution : The T gate is is also called $\frac{\pi}{8}$ gate and is given by

$$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{(1+i)}{\sqrt{(2)}} \end{bmatrix}$$

To Prove that $T^2 = S$

$$T^2 = TT = \begin{bmatrix} 1 & 0 \\ 0 & \frac{(1+i)}{\sqrt{(2)}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{(1+i)}{\sqrt{(2)}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = S$$

$$\because \left(\frac{1+i}{\sqrt{2}}\right)\left(\frac{1+i}{\sqrt{2}}\right) = \left(\frac{(1+i)^2}{2}\right) = i$$