Inner Product Spaces and Orthogonality.

Inner Product Space :—

Let V be a real vector space. Suppose to each pair of vectors $u, v \in V$ there is assigned a real number, denoted by $\langle u, v \rangle$. This function is called a (real) inner product on V if it satisfies the following axioms:

(i) Linear Property : $\langle au_1 + bu_2, v \rangle = a\langle u_1, v\rangle + b\langle u_2, v \rangle$

(ii) Symmetric Property: $\langle u, v \rangle = \langle v, u \rangle$

(iii) Positive Definite Property : $\langle u, u \rangle \geq 0$; and $\langle u, u \rangle = 0$ if and only if $u = 0$.

Norm of a vector :

An inner product, $\langle u, u \rangle$ is nonnegative for any vector u.

$$\|u\| = \sqrt{\langle u, u \rangle} \quad \text{or} \quad \|u\|^2 = \langle u, u \rangle$$

This non negative number is called the norm or length of u.

* Every non zero vector v in V can be multiplied by the reciprocal of its length to obtain the unit vector $\hat{v} = \frac{1}{\|v\|} v$. which is a positive multiple of v.

This process is called normalizing v.

Orthogonality :-

Let V be an inner product space. The vectors $u, v \in V$ are said to be orthogonal and u is said to be orthogonal to v if

$$\langle u, v \rangle = 0.$$

Problems :-

1) Consider vectors $u = (1, 2, 4)$, $v = (2, -3, 5)$, $w = (4, 2, -3)$ in R^3. Find a) $\langle u \cdot v \rangle$ b) $\langle u \cdot w \rangle$ c) $\langle v \cdot w \rangle$ d) $\langle (u+v) \cdot w \rangle$

e) $\|u\|$ f) $\|v\|$

a) $\langle u \cdot v \rangle = 2 - 6 + 20 = 16$

b) $\langle u \cdot w \rangle = 4 + 4 - 12 = -4$

c) $\langle v \cdot w \rangle = 8 - 6 - 15 = -13$.

d) $\langle (u+v) \cdot w \rangle = (3, -1, 9) \cdot (4, 2, -3) = 12 - 2 - 27 = -17$.

e) $\|u\| = \sqrt{1^2 + 2^2 + 4^2} = \sqrt{1 + 4 + 16} = \sqrt{21}$.

f) $\|v\| = \sqrt{4 + 9 + 25} = \sqrt{38}$.

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2) Consider the vectors $u = (1, 5)$ and $v = (3, 4)$ in R^2.

Find : a) $\langle u, v \rangle$ with respect to the usual inner product in R^2.

b) $\| v \|$ using the inner product in R^2.

3) Consider the following polynomials in $P(t)$ and inner product:

$f(t) = t+2$, $g(t) = 3t-2$, $h(t) = t^2-2t-3$ and

$\langle f, g \rangle = \int_0^1 f(t)g(t)\,dt$.

(a) find $\langle f, g \rangle$ and $\langle f, h \rangle$ (b) find $\|f\|$ and $\|g\|$

(c) normalize f and g.

Sol^n:- (a) $\langle f, g \rangle = \int_0^1 (t+2)(3t-2)\,dt = \int_0^1 (3t^2+4t-4)\,dt$

$$\langle f, g \rangle = t^3 + 2t^2 - 4t \Big|_0^1 = -1.$$

$$\langle f, h \rangle = \int_0^1 (t+2)(t^2-2t-3)\,dt = \frac{t^4}{4} - \frac{7t^2}{2} - 6t \Big]_0^1 = \frac{-37}{4}$$

(b) $\langle f, f \rangle = \int_0^1 (t+2)(t+2)\,dt = \frac{19}{3}$; $\|f\| = \frac{\sqrt{19}}{\sqrt{3}} = \frac{\sqrt{57}}{3}$.

$\langle g, g \rangle = \int_0^1 (3t-2)(3t-2)\,dt = 1$; $\|g\| = \sqrt{1} = 1$

(c) Since $\|f\| = \frac{\sqrt{57}}{3}$ and g is already a unit vector,

$$\hat{f} = \frac{1}{\|f\|} f = \frac{3}{\sqrt{57}}(t+2)$$

$$\hat{g} = \frac{1}{\|g\|} g = 3t-2.$$

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4) Let $M = M_{2,3}$ with inner product. $\langle A, B \rangle = tr\langle B^T A \rangle$ and let $A = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ $C = \begin{bmatrix} 3 & -5 & 2 \\ 1 & 0 & -4 \end{bmatrix}$

Find (a) $\langle A, B \rangle$, $\langle A, C \rangle$, $\langle B, C \rangle$

(b) $\langle 2A+3B, 4C \rangle$ (c) $\|A\|$ and $\|B\|$

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(a) $\langle A, B\rangle = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} b_{ij}$

$\langle A, B\rangle = 9+16+21+24+25+24 = 119$

$\langle A, C\rangle = 27-40+14+6+0-16 = -9$

$\langle B, C\rangle = 3-10+6+4+0-24 = -21$

(b) $2A+3B = \begin{bmatrix} 21 & 22 & 23 \\ 24 & 25 & 26 \end{bmatrix}$ $\quad 4C = \begin{bmatrix} 12 & -20 & 8 \\ 4 & 0 & -16 \end{bmatrix}$

$\langle 2A+3B, 4C\rangle = 252-440+96+0-416 = -324$

(c) $\|A\|^2 = \langle A, A\rangle = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2$, the sum of the squares of all the elements of A.

$\|A\|^2 = \langle A, A\rangle = 9^2+8^2+7^2+6^2+5^2+4^2 = 271 \Rightarrow \|A\| = \sqrt{271}$

$\|B\|^2 = \langle B, B\rangle = 1^2+2^2+3^2+4^2+5^2+6^2 = 91 \Rightarrow \|B\| = \sqrt{91}$

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4) Verify the vectors $u=(1,1,1)$, $v=(1,2,-3)$ & $w=(1,-4,3)$ in R^3 are orthogonal or not.

Soln:- $\langle u, v\rangle = 1+2-3=0$, $\langle u, w\rangle = 1-4+3=0$, $\langle v, w\rangle = 1-8-9=-16$

Thus u is orthogonal to v and w,
v & w are not orthogonal.