5.51 Regula Falsi Method or Method of False Position

The formula to find the first approximation x_1 to the real root lying in (a, b) is as follows.

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

The process is continued till we get consistency in two consecutive approximations.

ILLUSTRATIVE EXAMPLES

Ex-1. *We shall find a real root of the equation* $x^3 - 2x - 5 = 0$ *correct to three decimal places.*

☞ Let, $f(x) = x^3 - 2x - 5$

$$f(0) = -5,\ f(1) = -6,\ f(2) = -1 < 0\ f(3) = 16 > 0$$

A real root lies in (2, 3).

It may be observed that the value of $f(x)$ at $x = 2$ being -1 is nearer to zero compared to $f(3) = 16$ and we expect the root in the neighbourhood of 2. We shall have the interval (a, b) for applying the method such that $(b - a)$ is small enough.

Now, $f(2.1) = (2.1)^3 - 2(2.1) - 5 = +0.061 > 0$

$\therefore$ the root lies in $(2, 2.1)$

I Step : $a = 2,\ f(a) = f(2) = -1$

$b = 2.1,\ f(b) = f(2.1) = +0.061$

First approximation, $$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)} \quad \cdots (1)$$

$$x_1 = \frac{2(0.061) - (2.1)(-1)}{0.061 - (-1)} = 2.0942$$

II Step :

$$f(2.0942) = (2.0942)^3 - 2(2.0942) - 5 = -0.00392 < 0$$

$\therefore$ the root lies in $(2.0942, 2.1)$

Now, $a = 2.0942 \quad f(a) = -0.00392$

$b = 2.1 \quad f(b) = 0.061$

Substituting in the RHS of (1) we obtain the second approximation.

$$x_2 = \frac{(2.0942)(0.061) - (2.1)(-0.00392)}{0.061 - (-0.00392)} = 2.09455$$

x_1 and x_2 are close enough. x_2 is a better approximation than x_1.
Thus the required approximate root correct to 3 decimal places is $\boxed{2.095}$

Ex-2. *We shall compute the real root of* $x \log_{10} x - 1.2 = 0$, *by correct to four decimals.*

☞ Let, $f(x) = x \log_{10} x - 1.2$

$$f(1) = -1.2,\ f(2) = -0.6 < 0,\ f(3) = 0.23 > 0$$

The real root lies in $(2, 3)$ and from the values of $f(x)$ at $x = 2, 3$ we expect the root in the neighbourhood of 3 and let us find (a, b) for applying the method such that $(b - a)$ is small enough.

$$\left.\begin{array}{l} f(2.7) = 2.7\log_{10} 2.7 - 1.2 = -0.0353 \\ f(2.8) = 2.8\log_{10} 2.8 - 1.2 = +0.052 \end{array}\right\}$$ The root lies in (2.7, 2.8)

I iteration : $a = 2.7 \quad f(a) = -0.0353$

$b = 2.8 \quad f(a) = +0.052$

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)} \qquad \ldots (1)$$

$$\therefore \quad x_1 = \frac{(2.7)(0.052) - (2.8)(-0.0353)}{0.052 + 0.0353} = 2.7404$$

II iteration : $f(2.7404) = -0.00021 < 0$

$\therefore$ the root lies in (2.7404, 2.8)

Now, $a = 2.7404 \quad f(a) = -0.00021$

$b = 2.8 \quad f(b) = 0.052$

Substituting in (1) we have,

$$x_2 = \frac{(2.7404)(0.052) + (2.8)(0.00021)}{0.052 + 0.00021} = 2.7406$$

III iteration : $f(2.7406) = -0.00004 < 0$

$\therefore$ the root lies in (2.7406, 2.8)

Now $a = 2.7406 \quad f(a) = -0.00004$

$b = 2.8 \quad f(b) = 0.052$

Again substituting in (1) we have,

$$x_3 = \frac{(2.7406)(0.052) + (2.8)(0.00004)}{0.052 + 0.00004} = 2.740646$$

Thus the required approximate root correct to four decimal places is $\boxed{2.7406}$

Ex-3. *We shall compute the fourth root of* 12 *correct to three decimal places.*

☞ Let $x = \sqrt[4]{12} \quad \therefore \; x^4 = 12$ or $x^4 - 12 = 0$

Taking, $f(x) = x^4 - 12$ we have,

$$f(0) = -12 < 0,\ f(1) = -11 < 0,\ f(2) = 4 > 0$$

A real root of $f(x) = 0$ lies in $(1, 2)$ and will be in the neighbourhood of 2.

Now $f(1.7) = -3.6479,\ f(1.8) = -1.5024 < 0,\ f(1.9) = 1.0321 > 0$

$\therefore$ the root in the neighbourhood of 2 lies in (1.8, 1.9)

I Step : Let $a = 1.8 \quad f(a) = -1.5024$

$b = 1.9 \quad f(b) = 1.0321$

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)} \qquad \ldots (1)$$

$$\therefore \quad x_1 = \frac{(1.8)(1.0321) + (1.9)(1.5024)}{1.0321 + 1.5024} = 1.8593$$

II Step : $f(1.8593) = -0.0492 < 0$

$\therefore$ the root lies in (1.8593, 1.9)

Now, $a = 1.8593 \quad f(a) = -0.0492$

$b = 1.9 \quad f(b) = 1.0321$

Substituting in (1) we obtain $x_2 = 1.8612$

III Step : $f(x_2) = f(1.8612) = -0.00025 < 0$

$\therefore$ the root lies in (1.8612, 1.9)

Now, $a = 1.8612 \quad f(a) = -0.00025$

$b = 1.9 \quad f(b) = 1.0321$

Substituting in (1) we obtain $x_3 = 1.86121 \approx 1.861$

Thus the required fourth root of 12 correct to 3 decimal places is $\boxed{1.861}$

Ex-4. *Let us find the real root of the equation,* $\cos x = 3x - 1$ *correct to three decimals.*

☞ Let $f(x) = \cos x + 1 - 3x$.

In radians, $f(0) = 2 > 0,\ f(1) = -1.46 < 0$

A real root lies in (0, 1) and we expect the root in the neighbourhood of 1.

Consider, $f(0.6) = -0.0253 > 0,\ f(0.7) = -0.3352 < 0$

$\therefore$ the root lies in (0.6, 0.7)

I iteration : $a = 0.6 \quad f(a) = -0.0253$

$b = 0.7 \quad f(b) = -0.3352$

$$x_1 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)}$$

$\therefore \quad x_1 = 0.607$ (*on substitution and simplification*)

II iteration : $f(x_1) = f(0.607) = 0.00036 > 0$

$\therefore$ the root lies in $(0.607, 0.7)$

Now, $a = 0.607 \quad f(a) = 0.00036$

$b = 0.7 \quad f(b) = -0.3352$

$\therefore x_2 = 0.607$ (*on substitution and simplification*)

Hence the real root correct to 3 decimals is $\boxed{0.607}$

$\boxed{5.52}$ Newton - Raphson method

In this method we locate an approximate real root x_0 of the given equation and improve its accuracy by an iterative process.
The first approximation to the root x_0 is given by the following formula.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \quad \text{[Newton - Raphson formula]}$$

The second approximation is obtained by replacing x_0 by x_1 in the RHS of this expression.
The process is continued till we get consistency in two consecutive approximations.

ILLUSTRATIVE EXAMPLES

Ex-1. *Let us find a real root of the equation* $x^3 - 2x - 5 = 0$ *correct to three decimal places.*

☞ We shall find an interval (a, b) where a real root of the equation lies and then locate the approximate root.

Let, $\quad f(x) = x^3 - 2x - 5$

$f(0) = -5 < 0, f(1) = -6 < 0, f(2) = -1 < 0, f(3) = 16 > 0$

A real root lies in (2, 3). It will be in the neighbourhood of 2 and let the approximate root $x_0 = 2$.
The first approximation is given by,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)}$$

We have, $f(x) = x^3 - 2x - 5, f'(x) = 3x^2 - 2$

$$\therefore \quad x_1 = 2 - \frac{(-1)}{3(2^2)-2} = 2 + \frac{1}{10} = 2.1$$

Again, $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = 2.1 - \dfrac{f(2.1)}{f'(2.1)}$

$$x_2 = 2.1 - \frac{[(2.1)^3 - 2(2.1) - 5]}{[3(2.1)^2 - 2]} = 2.0946$$

Similarly, $x_3 = 2.0946 - \dfrac{[(2.0946)^3 - 2(2.0946) - 5]}{[3(2.0946)^2 - 2]} = 2.0946$

Thus the required approximate root correct to 3 decimal places is $\boxed{2.095}$

Ex-2. *Let us find a real root of the equation* $x^3 + x^2 + 3x + 4 = 0$, *by performing two iterations.*

☞ Let, $f(x) = x^3 + x^2 + 3x + 4$

$f(0) = 4, f(-1) = 1, f(-2) = -6 < 0$

$\therefore$ a real root lies in $(-2, -1)$ and let $x_0 = -1$

We also have, $f'(x) = 3x^2 + 2x + 3$

I iteration : $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$$x_1 = -1 - \frac{f(-1)}{f'(-1)} = -1 - \frac{1}{3-2+3} = -1.25$$

II iteration : $x_2 = -1.25 - \dfrac{f(-1.25)}{f'(-1.25)}$

$$x_2 = -1.25 - \frac{[(-1.25)^3 + (-1.25)^2 + 3(-1.25) + 4]}{[3(-1.25)^2 + 2(-1.25) + 3]}$$

$\therefore \quad x_2 = -1.2229$

Thus the required real root is $\boxed{-1.2229 \approx -1.223}$

Ex-3. *Let us find cube root of 37 correct to 3 decimal places.*

☞ Let, $x = \sqrt[3]{37} \quad \therefore \quad x^3 = 37$ or $x^3 - 37 = 0$

Taking, $f(x) = x^3 - 37$, we have $f(3) = -10 < 0$, $f(4) = 27 > 0$.

$\therefore$ a real root lies in (3, 4) and let $x_0 = 3$ be the initial approximation.

Also, $f'(x) = 3x^2$

The first approximation, $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$$x_1 = 3 - \frac{f(3)}{f'(3)} = 3 - \frac{(-10)}{3(3^2)} = 3.3704$$

Now, $x_2 = 3.3704 - \dfrac{f(3.3704)}{f'(3.3704)} = 3.3704 - \dfrac{[(3.3704)^3 - 37]}{3(3.3704)^2}$

$\therefore \quad x_2 = 3.3327$

Now, $x_3 = 3.3327 - \dfrac{[(3.3327)^3 - 37]}{3(3.3327)^2} = 3.3322$.

Again replacing 3.3322 in place of 3.3327 as earlier we obtain $x_4 = 3.3322$

Thus the required $\sqrt[3]{37}$ correct to 3 decimal places is 3.332.

Ex-4. *Let us find the real root of the equation* $x e^x - 2 = 0$ *correct to three decimal places.*

☞ Let, $f(x) = x e^x - 2$

$f(0) = -2 < 0$, $f(1) = 0.7183 > 0$

$\therefore$ A real root lies in $(0, 1)$ and let $x_0 = 1$.

Also, $f'(x) = x e^x + e^x = e^x (x+1)$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{f(1)}{f'(1)}$$

$$x_1 = 1 - \frac{0.1783}{e^1(2)} = 0.8679$$

$$x_2 = 0.8679 - \frac{f(0.8679)}{f'(0.8679)} = 0.8679 - \frac{[0.8679\, e^{0.8679} - 2]}{e^{0.8679}(0.8679+1)} = 0.8528$$

$$x_3 = 0.8528 - \frac{[0.8528\, e^{0.8528} - 2]}{e^{0.8528}(1.8528)} = 0.8526$$

Again replacing 0.8526 in place of 0.8528 as earlier, we obtain $x_4 = 0.8526$

Thus the required real root correct to three decimal places is $\boxed{0.853}$

5.6 Numerical Integration