NUMERICAL SOLUTION OF SECOND ORDER ORDINARY DIFFERENTIAL EQAUTIONS

5.1 Introduction and pre-amble

The given second order ODE with two initial conditions will reduce to two first order simultaneous ODEs which can be solved.

We present the method explicitly.

Let $y'' = g(x, y, y')$ with the initial conditions $y(x_0) = y_0$ and $y'(x_0) = y'_0$ be the given second order DE.

Now, let $y' = \dfrac{dy}{dx} = z$. This gives $y'' = \dfrac{d^2y}{dx^2} = \dfrac{dz}{dx}$

The given second order DE assumes the form : $\dfrac{dz}{dx} = g(x, y, z)$ with the conditions $y(x_0) = y_0$ and $z(x_0) = z_0$ where y'_0 is denoted by z_0.

Hence, we now have two first order simultaneous ODEs.

$$\frac{dy}{dx} = z \text{ and } \frac{dz}{dx} = g(x, y, z) \text{ with } y(x_0) = y_0 \text{ and } z(x_0) = z_0$$

Taking $f(x, y, z) = z$, we now have the following system of equations for solving:

$$\frac{dy}{dx} = f(x, y, z),\ \frac{dz}{dx} = g(x, y, z)\ ;\ y(x_0) = y_0 \text{ and } z(x_0) = z_0$$

5.11 Runge - Kutta Method

We have to compute $y(x_0 + h)$ and if required $y'(x_0 + h) = z(x_0 + h)$.

We need to first compute the following :

(Recall the formulae in the case of first order ODE)

$$k_1 = h\,f(x_0, y_0, z_0) \quad ; \quad l_1 = h\,g(x_0, y_0, z_0)$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right); \quad l_2 = h\,g\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right); \quad l_3 = h\,g\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$k_4 = h\,f(x_0 + h, y_0 + k_3, z_0 + l_3) \; ; \; l_4 = h\,g(x_0 + h, y_0 + k_3, z_0 + l_3)$$

The required, $y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

and $\quad y'(x_0 + h) = z(x_0 + h) = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

WORKED PROBLEMS

[1] *Given* $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - 2xy = 1, y(0) = 1, y'(0) = 0$. *Evaluate* $y(0.1)$ *using Runge-Kutta method of order* 4.

☞ By data, $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - 2xy = 1, y = 1, y' = 0$ at $x = 0$.

Putting, $\frac{dy}{dx} = z$ and differentiating w.r.t x we obtain $\frac{d^2y}{dx^2} = \frac{dz}{dx}$ so that the given equation assumes the form : $\frac{dz}{dx} - x^2z - 2xy = 1$

Hence, we have a system of equations,

$$\frac{dy}{dx} = z \; ; \; \frac{dz}{dx} = 1 + 2xy + x^2z \text{ where } y = 1,\ z = 0,\ x = 0.$$

Let, $\quad f(x, y, z) = z,\ g(x, y, z) = 1 + 2xy + x^2z$

$x_0 = 0,\ y_0 = 1,\ z_0 = 0$ and let us take $h = 0.1$

We shall first compute the following :

$$k_1 = h\,f(x_0, y_0, z_0) = (0.1)\,f(0,1,0) = (0.1)(0) = 0$$

$$l_1 = (0.1)[1 + (2)(0)(1) + (0)^2(0)] = 0.1$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$k_2 = (0.1)\,f(0.05, 1, 0.05) = (0.1)(0.05) = 0.005$$

$$l_2 = (0.1)[1 + (2)(0.05)(1) + (0.05)^2(0.05)] = 0.11$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$k_3 = (0.1)\,f(0.05, 1.0025, 0.055) = (0.1)(0.055) = 0.0055$$

$$l_3 = (0.1)[1 + (2)(0.05)(1.0025) + (0.05)^2(0.055)] = 0.11004$$

$$k_4 = h\,f(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$k_4 = (0.1)\,f(0.1, 1.0055, 0.11004) = (0.1)(0.11004) = 0.011$$

$$l_4 = (0.1)[1 + (2)(0.1)(1.0055) + (0.1)^2(0.11004)] = 0.12022$$

We have, $y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$\therefore \quad y(0.1) = 1 + \frac{1}{6}[0 + 2(0.005) + 2(0.0055) + 0.011]$$

Thus, $\boxed{y(0.1) = 1.0053}$

Remark : *The computation of* l_1, l_2, l_3, l_4 *is just the application of the expression associated with* $g(x, y, z)$.

[2] *By Runge-Kutta method, solve* $\dfrac{d^2y}{dx^2} = x\left(\dfrac{dy}{dx}\right)^2 - y^2$ *for* $x = 0.2$ *correct to four decimal places, using the initial conditions* $y = 1$ *and* $y' = 0$ *when* $x = 0$

[June & Dec 2017].

☞ By data, $\dfrac{d^2y}{dx^2} = x\left(\dfrac{dy}{dx}\right)^2 - y^2$

Putting, $\dfrac{dy}{dx} = z$ and differentiating w.r.t x, we obtain $\dfrac{d^2y}{dx^2} = \dfrac{dz}{dx}$

The given equation becomes,

$$\frac{dz}{dx} = xz^2 - y^2 \text{ with } y = 1,\ z = 0 \text{ at } x = 0.$$

Hence, we have a system of equations $\dfrac{dy}{dx} = z,\ \dfrac{dz}{dx} = xz^2 - y^2$

Let, $f(x, y, z) = z,\ g(x, y, z) = xz^2 - y^2,\ x_0 = 0,\ y_0 = 1,\ z_0 = 0$ and $h = 0.2$

We shall first compute the following.

$$k_1 = h\,f(x_0, y_0, z_0) = (0.2)\,f(0, 1, 0) = (0.2)(0) = 0$$

$$l_1 = (0.2)[(0)(0)^2 - (1)^2] = -0.2$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$k_2 = (0.2)\,f(0.1, 1, -0.1) = (0.2)(-0.1) = -0.02$$

$$l_2 = (0.2)[(0.1)(-0.1)^2 - (1)^2] = -0.1998$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$k_3 = (0.2)\,f(0.1, 0.99, -0.0999) = (0.2)(-0.0999) = -0.01998$$

$$l_3 = (0.2)[(0.1)(-0.0999)^2 - (0.99)^2] = -0.1958$$

$$k_4 = h\,f(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$k_4 = (0.2)\,f(0.2, 0.98002, -0.1958) = (0.2)(-0.1958) = 0.03916$$

$$l_4 = (0.2)[(0.2)(-0.1958)^2 - (0.98002)^2] = -0.19055$$

We have $y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$\therefore \quad y(0.2) = 1 + \frac{1}{6}[\,0 + 2(-0.02) + 2(-0.01998) - 0.03916\,]$

Thus, $\boxed{y(0.2) = 0.9801}$

[3] *Compute $y\,(\,0.1\,)$ given $\frac{d^2y}{dx^2} = y^3$ and $y = 10$, $\frac{dy}{dx} = 5$ at $x = 0$ by Runge-Kutta method of fourth order.*

☞ Putting $\frac{dy}{dx} = z$ and differentiating w.r.t x we obtain $\frac{d^2y}{dx^2} = \frac{dz}{dx}$ so that the given equation assumes the form $\frac{dz}{dx} = y^3$. Hence we have a system of equations:

$$\frac{dy}{dx} = z\,;\ \frac{dz}{dx} = y^3 \text{ where } y = 10, z = 5,\ x = 0.$$

Let, $f(x, y, z) = z$, $g(x, y, z) = y^3$, $x_0 = 0$, $y_0 = 10$, $z_0 = 5$ and $h = 0.1$

We shall first compute the following :

$$k_1 = h\,f(x_0, y_0, z_0) = (0.1)\,f(0, 10, 5) = (0.1)5 = 0.5$$

$$l_1 = (0.1)[10^3] = 100$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$k_2 = (0.1)\,f\,(0.05,\ 10.25,\ 55) = (0.1)(55) = 5.5$$

$$l_2 = (0.1)[(10.25)^3] = 107.7$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2},\ z_0 + \frac{l_2}{2}\right)$$

$$k_3 = (0.1)\,f\,(0.05,\ 12.75,\ 58.85) = (0.1)(58.85) = 5.885$$

$$l_3 = (0.1)(12.75)^3 = 207.27$$

$$k_4 = h\,f\,(x_0 + h,\ y_0 + k_3,\ z_0 + l_3)$$

$$k_4 = (0.1)\,f\,(0.1,\ 15.885,\ 212.27) = (0.1)(212.27) = 21.227$$

$$l_4 = (0.1)(15.885)^3 = 400.83$$

We have, $y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$\therefore \quad y(0.1) = 10 + \frac{1}{6}[\,0.5 + 2(5.5) + 2(5.885) + 21.227]$

Thus, $\boxed{y(0.1) = 17.4162}$

[4] *Given $y'' - xy' - y = 0$ with the initial conditions $y(0) = 1,\ y'(0) = 0$, compute $y(0.2)$ and $y'(0.2)$ using fourth order Runge-Kutta method.*

[June 2018]

☞ Putting $y' = z$, we obtain $y'' = \frac{dz}{dx}$. The given equation becomes

$$\frac{dz}{dx} = xz + y \ ;\ y(0) = 1,\ z(0) = 0$$

Hence we have a system of equations,

$$\frac{dy}{dx} = z \; ; \; \frac{dz}{dx} = xz + y \text{ where } y = 1, \; z = 0, \; x = 0$$

Let, $f(x, y, z) = z, \; g(x, y, z) = xz + y, x_0 = 0, y_0 = 1, \; z_0 = 0$ and $h = 0.2$

We shall first compute the following.

$$k_1 = h\,f(x_0, y_0, z_0) = (0.2)\,f(0,1,0) = (0.2)0 = 0$$

$$l_1 = (0.2)[0 \times 0 + 1] = 0.2$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2}, \; y_0 + \frac{k_1}{2}, \; z_0 + \frac{l_1}{2}\right)$$

$$k_2 = (0.2)\,f(0.1, 1, 0.1) = (0.2)(0.1) = 0.02$$

$$l_2 = (0.2)[0.1 \times 0.1 + 1] = 0.202$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2}, \; y_0 + \frac{k_2}{2}, \; z_0 + \frac{l_2}{2}\right)$$

$$k_3 = (0.2)\,f(0.1, \; 1.01, 0.101) = (0.2)(0.101) = 0.0202$$

$$l_3 = (0.2)[0.1 \times 0.101 + 1.01] = 0.204$$

$$k_4 = h\,f(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$k_4 = (0.2)\,f(0.2, \; 1.0202, 0.204) = (0.2)(0.204) = 0.0408$$

$$l_4 = (0.2)[0.2 \times 0.204 + 1.0202] = 0.2122$$

We have, $y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$z(x_0 + h) = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

Substituting the appropriate values we obtain $y(0.2) = 1.0202$ and $z(0.2) = 0.204$

Thus, $\boxed{y(0.2) = 1.0202 \text{ and } y'(0.2) = 0.204}$

[5] *Obtain the value of x and* $\frac{dx}{dt}$ *when* $t = 0.1$ *given that x satisfies the equation* $\frac{d^2x}{dt^2} = t\frac{dx}{dt} - 4x$ *and* $x = 3$, $\frac{dx}{dt} = 0$ *when* $t = 0$ *initially. Use fourth order Runge-Kutta method.*

☞ Putting, $y = \frac{dx}{dt}$ we obtain $\frac{dy}{dt} = \frac{d^2x}{dt^2}$ The given equation becomes

$$\frac{dy}{dt} = ty - 4x,\ x = 3,\ y = 0 \text{ when } t = 0.$$

Hence we have a system of equations,

$$\frac{dx}{dt} = y,\ \frac{dy}{dt} = ty - 4x,\ x = 3,\ y = 0 \text{ when } t = 0.$$

Let $f(t,x,y) = y$, $g(t,x,y) = ty - 4x$, $t_0 = 0$, $x_0 = 3$, $y_0 = 0$ and $h = 0.1$

We shall first compute the following :

$$k_1 = hf(t_0, x_0, y_0) = (0.1)f(0,3,0) = (0.1)(0) = 0$$

$$l_1 = (0.1)[0 - 12] = -1.2$$

$$k_2 = hf\left(t_0 + \frac{h}{2},\ x_0 + \frac{k_1}{2},\ y_0 + \frac{l_1}{2}\right)$$

$$k_2 = (0.1)f(0.05, 3, -0.6) = (0.1)(-0.6) = -0.06$$

$$l_2 = (0.1)[(0.05)(-0.6) - 12] = -1.203$$

$$k_3 = hf\left(t_0 + \frac{h}{2},\ x_0 + \frac{k_2}{2},\ y_0 + \frac{l_2}{2}\right)$$

$$k_3 = (0.1)f(0.05, 2.97, -0.6015) = (0.1)(-0.6015) = -0.06015$$

$$l_3 = (0.1)[(0.05)(-0.6015) - 4 \times 2.97] = -1.191$$

$$k_4 = hf(t_0 + h,\ x_0 + k_3,\ y_0 + l_3)$$

$$k_4 = (0.1)\,f(0.1, 2.93985, -1.191) = (0.1)(-1.191) = -0.1191$$

$$l_4 = (0.1)[(0.1)(-1.191) - 4 \times 2.93985] = -1.18785$$

We have, $x(t_0 + h) = x_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$y(t_0 + h) = y_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) \text{ where } y = \frac{dx}{dt}$$

Substituting the appropriate values we obtain $x(0.1) = 2.9401$, $y(0.1) = -1.196$

Thus, $\boxed{x = 2.9401 \text{ and } \frac{dx}{dt} = -1.196 \text{ when } t = 0.1}$

5.12 Milne's method

Preamble : We recall [*Module-4, Article-4.21*] Milne's predictor and corrector formulae for solving first order ODE : $y' = f(x, y)$ with $y(x_0) = y_0$, $y(x_1) = y_1$, $y(x_2) = y_2$, $y(x_3) = y_3$. Here x_0, x_1, x_2, x_3 are equidistant values of x distant h.

We have to compute $y(x_4)$ where $x_4 = x_0 + 4h$.

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3') \qquad \text{[Predictor formula]}$$

$$y_4^{(C)} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4') \qquad \text{[Corrector formula]}$$

Method to solve the ODE $y'' = f(x, y, y')$ given a set of four initial values for y and y' :

- We put $y' = z$ which gives $y'' = \frac{dz}{dx} = z'$.

  The given DE becomes $z' = f(x, y, z)$

- We equip with the following table of values using the given data.

x	x_0	x_1	x_2	x_3
y	y_0	y_1	y_2	y_3
$y' = z$	$y'_0 = z_0$	$y'_1 = z_1$	$y'_2 = z_2$	$y'_3 = z_3$
$y'' = z'$	$y''_0 = z'_0$	$y''_1 = z'_1$	$y''_2 = z'_2$	$y''_3 = z'_3$

- We first apply predictor formula to compute $y_4^{(P)}$ and $z_4^{(P)}$ where,

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3), \text{ since } y' = z.$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}(2z'_1 - z'_2 + 2z'_3)$$

- We compute $z'_4 = f(x_4, y_4, z_4)$ and then apply corrector formula where,

$$y_4^{(C)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4)$$

$$z_4^{(C)} = z_2 + \frac{h}{3}(z'_2 + 4z'_3 + z'_4)$$

- Corrector formula can be applied repeatedly for better accuracy.

WORKED PROBLEMS

[6] *Apply Milne's method to solve* $\dfrac{d^2y}{dx^2} = 1 + \dfrac{dy}{dx}$ *given the following table of initial values. Compute* $y(0.4)$. **[June 2018]**

x	0	0.1	0.2	0.3
y	1	1.1103	1.2427	1.399
y'	1	1.2103	1.4427	1.699

☞ Putting $y' = \dfrac{dy}{dx} = z$, we obtain $y'' = \dfrac{d^2y}{dx^2} = \dfrac{dz}{dx}$

The given equation becomes, $\dfrac{dz}{dx} = 1 + z$ or $z' = 1 + z$.

Further, $z' = 1 + z$ will give us the following values

$$z'(0) = 1 + z(0) = 1 + 1 = 2$$

$$\therefore \quad z'(0.1) = 1 + z(0.1) = 2.2103$$

$$z'(0.2) = 1 + z(0.2) = 2.4427$$

$$z'(0.3) = 1 + z(0.3) = 2.699$$

Now we tabulate these values.

x	$x_0 = 0$	$x_1 = 0.1$	$x_2 = 0.2$	$x_3 = 0.3$
y	$y_0 = 1$	$y_1 = 1.1103$	$y_2 = 1.2427$	$y_3 = 1.399$
$y' = z$	$z_0 = 1$	$z_1 = 1.2103$	$z_2 = 1.4427$	$z_3 = 1.699$
$y'' = z'$	$z'_0 = 2$	$z'_1 = 2.2103$	$z'_2 = 2.4427$	$z'_3 = 2.699$

We first consider Milne's predictor formulae :

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3),$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}(2z'_1 - z'_2 + 2z'_3)$$

Hence, $y_4^{(P)} = 1 + \dfrac{4(0.1)}{3}[2(1.2103) - 1.4427 + 2(1.699)]$

$$z_4^{(P)} = 1 + \frac{4(0.1)}{3}[2(2.2103) - 2.4427 + 2(2.699)]$$

$$\therefore \quad y_4^{(P)} = 1.5835 \text{ and } z_4^{(P)} = 1.9835$$

Next we consider Milne's corrector formulae :

$$y_4^{(C)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4)$$

$$z_4^{(C)} = z_2 + \frac{h}{3}(z_2' + 4z_3' + z_4')$$

We have, $z_4' = 1 + z_4^{(P)} = 1 + 1.9835 = 2.9835$

Hence, $y_4^{(C)} = 1.2427 + \frac{0.1}{3}[1.4427 + 4(1.699) + 1.9835]$

$$z_4^{(C)} = 1.4427 + \frac{0.1}{3}[2.4427 + 4(2.699) + 2.9835]$$

$\therefore \quad y_4^{(C)} = 1.58344$ and $z_4^{(C)} = 1.98344$

Applying the corrector formula again for y_4 we obtain $y_4^{(C)} = 1.583438$

Thus the required, $\boxed{y(0.4) = 1.5834}$

[7] *Apply Milne's method to compute y (0.8) given that* $\frac{d^2y}{dx^2} = 1 - 2y\frac{dy}{dx}$ *and the following table of initial values.* **[Dec 2017, 18]**

x	0	0.2	0.4	0.6
y	0	0.02	0.0795	0.1762
y'	0	0.1996	0.3937	0.5689

Apply the corrector formula twice in presenting the value of y at x = 0.8

☞ Putting $y' = \frac{dy}{dx} = z$ we obtain, $y'' = \frac{d^2y}{dx^2} = z'$

The given equation becomes, $z' = 1 - 2yy' = 1 - 2yz$

Now, $z_0' = 1 - 2(0)(0) = 1$

$$z_1' = 1 - 2(0.02)(0.1996) = 0.992$$

$$z_2' = 1 - 2(0.0795)(0.3937) = 0.9374$$

$$z_3' = 1 - 2(0.1762)(0.5689) = 0.7995$$

We have the following table.

x	$x_0 = 0$	$x_1 = 0.2$	$x_2 = 0.4$	$x_3 = 0.6$
y	$y_0 = 0$	$y_1 = 0.02$	$y_2 = 0.0795$	$y_3 = 0.1762$
$y' = z$	$z_0 = 0$	$z_1 = 0.1996$	$z_2 = 0.3937$	$z_3 = 0.5689$
$y'' = z'$	$z_0' = 1$	$z_1' = 0.992$	$z_2' = 0.9374$	$z_3' = 0.7995$

We first consider Milne's predictor formulae,

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3),$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}(2z_1' - z_2' + 2z_3')$$

On substituting the appropriate values from the table we obtain,

$$y_4^{(P)} = 0.3049 \text{ and } z_4^{(P)} = 0.7055$$

Next we consider Milne's corrector formulae,

$$y_4^{(C)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4)$$

$$z_4^{(C)} = z_2 + \frac{h}{3}(z_2' + 4z_3' + z_4')$$

We have $z_4' = 1 - 2y_4^{(P)} z_4^{(P)} = 1 - 2(0.3049)(0.7055) = 0.5698$

Hence by substituting the appropriate values in the corrector formulae, we obtain,

$$y_4^{(C)} = 0.3045 \text{ and } z_4^{(C)} = 0.7074$$

Applying the corrector formula again for y_4 we have,

$$y_4^{(C)} = 0.0795 + \frac{0.2}{3}[0.3937 + 4(0.5689) + 0.7074] = 0.3046$$

Thus the required, $\boxed{y(0.8) = 0.3046}$

[8] *Obtain the solution of the equation* $2\frac{d^2y}{dx^2} = 4x + \frac{dy}{dx}$ *by computing the value of the dependent variable corresponding to the value 1.4 of the independent variable by applying Milne's method using the following data.* **[June 2017]**

x	1	1.1	1.2	1.3
y	2	2.2156	2.4649	2.7514
y'	2	2.3178	2.6725	3.0657

☞ Dividing the given equation by 2 we have,

$$\frac{d^2y}{dx^2} = 2x + \frac{1}{2}\frac{dy}{dx} \quad \text{or} \quad y'' = 2x + \frac{y'}{2}$$

Putting, $y' = z$ we obtain $y'' = z'$ and the given equation becomes $z' = 2x + \frac{z}{2}$

$$\text{Now, } z_0' = 2(1) + \frac{2}{2} = 3$$

$$z_1' = 2(1.1) + \frac{2.3178}{2} = 3.3589$$

$$z_2' = 2(1.2) + \frac{2.6725}{2} = 3.73625$$

$$z_3' = 2(1.3) + \frac{3.0657}{2} = 4.13285$$

We have the following table.

x	$x_0 = 1$	$x_1 = 1.1$	$x_2 = 1.2$	$x_3 = 1.3$
y	$y_0 = 2$	$y_1 = 2.2156$	$y_2 = 2.4649$	$y_3 = 2.7514$
$y' = z$	$z_0 = 2$	$z_1 = 2.3178$	$z_2 = 2.6725$	$z_3 = 3.0657$
$y'' = z'$	$z'_0 = 3$	$z'_1 = 3.3589$	$z'_2 = 3.73625$	$z'_3 = 4.13285$

We first consider Milne's predictor formulae,

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3)$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}(2z'_1 - z'_2 + 2z'_3)$$

On substituting the appropriate values from the table we obtain,

$$y_4^{(P)} = 3.0793 \text{ and } z_4^{(P)} = 3.4996$$

Next we consider Milne's corrector formulae,

$$y_4^{(C)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4)$$

$$z_4^{(C)} = z_2 + \frac{h}{3}(z'_2 + 4z'_3 + z'_4)$$

We have, $z'_4 = 2x_4 + \dfrac{z_4^{(P)}}{2} = 2(1.4) + \dfrac{3.4996}{2} = 4.5498$

Hence by substituting the appropriate values in the corrector formulae we obtain

$$y_4^{(C)} = 3.0794 \text{ and } z_4^{(C)} = 3.4997$$

Thus the required, $\boxed{y(1.4) = 3.0794}$

[9] *Given the ODE* $y'' + xy' + y = 0$ *and the following table of initial values, compute* $y\,(0.4)$ *by applying Milne's method.*

x	0	0.1	0.2	0.3
y	1	0.995	0.9801	0.956
y'	0	−0.0995	−0.196	−0.2867

☞ Putting $y' = z$, we get $y'' = z'$.

Also we have, $z' = -(xz + y)$ from the given equation.

Further, $z'(0) = -[0+1] = -1$

$$z'(0.1) = -[(0.1)(-0.0995) + 0.995] = -0.985$$

$$z'(0.2) = -[(0.2)(-0.196) + 0.9801] = -0.941$$

$$z'(0.3) = -[(0.3)(-0.2867) + 0.956] = -0.87$$

We also have the following table.

x	$x_0 = 0$	$x_1 = 0.1$	$x_2 = 0.2$	$x_3 = 0.3$
y	$y_0 = 1$	$y_1 = 0.995$	$y_2 = 0.9801$	$y_3 = 0.956$
$y' = z$	$z_0 = 0$	$z_1 = -0.0995$	$z_2 = -0.196$	$z_3 = -0.2867$
$y'' = z'$	$z'_0 = -1$	$z'_1 = -0.985$	$z'_2 = -0.941$	$z'_3 = -0.87$

We first consider Milne's predictor formulae,

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3)$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}(2z'_1 - z'_2 + 2z'_3)$$

On substituting the appropriate values from the table we obtain

$$y_4^{(P)} = 0.9231 \text{ and } z_4^{(P)} = -0.3692$$

Next we consider Milne's corrector formulae,

$$y_4^{(C)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4)$$

$$z_4^{(C)} = z_2 + \frac{h}{3}(z'_2 + 4z'_3 + z'_4)$$

We have, $z'_4 = -(x_4 z_4^{(P)} + y_4^{(P)}) = -[(0.4)(-0.3692) + 0.9231] = -0.7754$

Hence by substituting the appropriate values in the corrector formulae we obtain

$$y_4^{(C)} = 0.9230 \text{ and } z_4^{(C)} = -0.3692$$

Thus the required, $\boxed{y(0.4) = 0.923}$

[10] *Applying Milne's predictor and corrector formulae compute $y(0.8)$ given that y satisfies the equation $y'' = 2yy'$ and y & y' are governed by the following values.*

$$y(0) = 0,\ y(0.2) = 0.2027,\ y(0.4) = 0.4228,\ y(0.6) = 0.6841$$

$$y'(0) = 1,\ y'(0.2) = 1.041,\ y'(0.4) = 1.179,\ y'(0.6) = 1.468$$

Apply corrector formula twice.

☞ Putting $y' = z$ we obtain $y'' = \frac{dz}{dx} = z'$ & the given equation becomes $z' = 2yz$

Now, $z'(0) = 0, z'(0.2) = 2(0.2027)(1.041) = 0.422$

$$z'(0.4) = 2(0.4228)(1.179) = 0.997$$

$$z'(0.6) = 2(0.6841)(1.468) = 2.009$$

Now we tabulate all the values.

x	$x_0 = 0$	$x_1 = 0.2$	$x_2 = 0.4$	$x_3 = 0.6$
y	$y_0 = 0$	$y_1 = 0.2027$	$y_2 = 0.4228$	$y_3 = 0.6841$
$y' = z$	$z_0 = 1$	$z_1 = 1.041$	$z_2 = 1.179$	$z_3 = 1.468$
$y'' = z'$	$z'_0 = 0$	$z'_1 = 0.422$	$z'_2 = 0.997$	$z'_3 = 2.009$

We first consider Milne's predictor formulae,

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3)$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}(2z'_1 - z'_2 + 2z'_3)$$

On substituting the appropriate values from the table we obtain,

$$y_4^{(P)} = 1.0237 \text{ and } z_4^{(P)} = 2.0307$$

Next we consider Milne's corrector formulae,

$$y_4^{(C)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4)$$

$$z_4^{(C)} = z_2 + \frac{h}{3}(z'_2 + 4z'_3 + z'_4)$$

We have, $z'_4 = 2y_4^{(P)} z_4^{(P)} = 4.1577$

Hence by substituting the appropriate values in the corrector formulae, we obtain

$$y_4^{(C)} = 1.0282 \text{ and } z_4^{(C)} = 2.0584$$

Applying the corrector formula again we have,

$$y_4^{(C)} = 0.4228 + \frac{0.2}{3}[1.179 + 4(1.468) + 2.0584] = 1.03009$$

Thus the required, $\boxed{y(0.8) = 1.0301}$

ASSIGNMENT

1. Use fourth order Runge- Kutta method to solve the equation $\frac{d^2y}{dx^2} = x\frac{dy}{dx} + y$ given that $y = 1$ and $\frac{dy}{dx} = 0$ when $x = 0$.

   Compute y and $\frac{dy}{dx}$ at $x = 0.2$

2. Solve $y'' + 4y = xy$ given that $y(0) = 3$ and $y'(0) = 0$

Compute $y(0.1)$ using Runge-Kutta method of order 4.

3. Apply Milne's method to compute $y(0.4)$ given the equation $y'' + y' = 2e^x$ and the following table of initial values. Compare the result with theoretical value.

x	0	0.1	0.2	0.3
y	2	2.01	2.04	2.09
y'	0	0.2	0.4	0.6

4. Solve the equation $y'' + y' = 2x$ at $x = 0.4$ by applying Milne's method given that $y = 1$, $y' = -1$ at $x = 0$. Requisite initial values be generated from Taylor's series method.

ANSWERS

1. $y(0.2) = 1.0202$, $y'(0.2) = 0.204$
2. $y(0.1) = 2.94$
3. $y(0.4) = 2.16$
4. $y(0.4) = 0.6897$