NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQAUTIONS OF FIRST ORDER AND FIRST DEGREE

4.1 Numerical methods for initial value problems

Consider a differential equation of first order and first degree in the form

$\frac{dy}{dx} = f(x, y)$ with the initial condition $y(x_0) = y_0$, that is $y = y_0$ at $x = x_0$.

This problem of finding y is called an ***initial value problem.***

We discuss five numerical methods for solving an initial value problem.

In these numerical methods, we compute y (x) in the neighbourhood of the value of 'x_0'.

Equivalently, we compute $y\ (x_0 \pm h)$ where h is small enough. Lesser the value of h, results in greater the accuracy of $y(x_0 \pm h)$.

4.11 Taylor's series method

Consider the initial value problem : $\frac{dy}{dx} = f(x, y)$ and $y(x_0) = y_0$.

The solution $y\ (x)$ is approximated to a power series in $(x - x_0)$ using Taylor's series. Then we can find the value of y for various values of x in the neighbourhood of x_0.

We have Taylor's series expansion $y\ (x)$ about the point x_0 in the form :

$$y(x) = y(x_0) + (x - x_0)y'(x_0) + \frac{(x - x_0)^2}{2!}y''(x_0) + \frac{(x - x_0)^3}{3!}y'''(x_0) + \cdots$$

Here $y'(x_0), y''(x_0), \cdots$ denote the value of the derivatives $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots$ at x_0 which can be found by making use of the data.

WORKED PROBLEMS

[1] (a) *Use Taylor's series method to find y at* $x = 0.1, 0.2, 0.3$ *considering terms upto the third degree given that* $\dfrac{dy}{dx} = x^2 + y^2$ *and* $y(0) = 1$

(b) *Compute* $y(0.1)$ **[Dec 2018]**

☞ Taylor's series expansion of $y(x)$ is given by

$$y(x) = y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2!} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots$$

By data, $x_0 = 0$, $y_0 = 1$ and $y' = x^2 + y^2$

$$\therefore \quad y(x) = y(0) + x\, y'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) \qquad \ldots (1)$$

We need to compute $y'(0), y''(0), y'''(0)$.

Consider, $y' = x^2 + y^2$; $y'(0) = 0^2 + [y(0)]^2 = 0 + 1 = 1$

Differentiating y' w.r.t x we have,

$$y'' = 2x + 2yy' \; ;$$

$$y''(0) = (2)(0) + 2 \cdot y(0) \cdot y'(0) = (2)(1)(1) = 2$$

Differentiating y'' w.r.t x we have,

$$y''' = 2 + 2[y\, y'' + (y')^2]$$

$$\therefore \quad y'''(0) = 2 + 2[(1)(2) + 1^2] = 8$$

Substituting these values in (1) we have,

$$y(x) = 1 + x \cdot 1 + \frac{x^2}{2} \cdot 2 + \frac{x^3}{6} \cdot 8 = 1 + x + x^2 + \frac{4x^3}{3}$$

This is called as Taylor's series approximation upto the third degree and we need to put $x = 0.1, 0.2, 0.3$ in the same. Thus we have,

$$y(0.1) = 1 + 0.1 + (0.1)^2 + \frac{4(0.1)^3}{3} = 1.1113$$

Note : *Solution of* (b) *ends here.*

$$y(0.2) = 1 + 0.2 + (0.2)^2 + \frac{4(0.2)^3}{3} = 1.2507$$

$$y(0.3) = 1 + 0.3 + (0.3)^2 + \frac{4(0.3)^3}{3} = 1.426$$

Thus, $\boxed{y(0.1) = 1.1113,\ y(0.2) = 1.2507,\ y(0.3) = 1.426}$

[2] *Find y at $x = 1.02$ correct to five decimal places given $dy = (xy-1)\,dx$ and $y = 2$ at $x = 1$ applying Taylor's series method.*

Taylor's series expansion is given by

$$y(x) = y(x_0) + (x - x_0)\,y'(x_0) + \frac{(x-x_0)^2}{2!}\,y''(x_0) + \frac{(x-x_0)^3}{3!}\,y'''(x_0) + \cdots$$

By data, $x_0 = 1,\ y_0 = 2$ and $y' = \dfrac{dy}{dx} = xy - 1$

Since the number of derivatives for approximation is not specifically mentioned, we shall have the approximation upto third degree.
Hence we have,

$$y(x) = y(1) + (x-1)\,y'(1) + \frac{(x-1)^2}{2!}\,y''(1) + \frac{(x-1)^3}{3!}\,y'''(1) \qquad \ldots(1)$$

Consider, $y' = xy - 1$; $y'(1) = (1)(2) - 1 = 1$

$y'' = xy' + y$; $y''(1) = (1)(1) + 2 = 3$

$y''' = xy'' + y' + y'$; $y'''(1) = (1)(3) + 1 + 1 = 5$

To find $y(1.02)$, we shall substitute these values along with $x = 1.02$ in (1).

$$\therefore \quad y(1.02) = 2 + (1.02 - 1)1 + \frac{(1.02-1)^2}{2}\cdot 3 + \frac{(1.02-1)^3}{6}\cdot 5$$

$$= 2 + (0.02) + \frac{(0.02)^2(3)}{2} + \frac{(0.02)^3(5)}{6}$$

Thus, $\boxed{y(1.02) = 2.02061}$

[3] *From Taylor's series method, find $y(0.1)$ considering upto fourth degree term if $y(x)$ satisfies the equation $\dfrac{dy}{dx} = x - y^2,\ y(0) = 1$.*

Taylor's series expansion is given by

$$y(x) = y(x_0) + (x - x_0)\,y'(x_0) + \frac{(x - x_0)^2}{2!}\,y''(x_0) + \frac{(x - x_0)^3}{3!}\,y'''(x_0) + \cdots$$

By data, $x_0 = 0,\ y_0 = 1,\ y' = x - y^2$

$$\therefore \quad y(x) = y(0) + x\,y'(0) + \frac{x^2}{2!}\,y''(0) + \frac{x^3}{3!}\,y'''(0) + \frac{x^4}{4!}\,y^{(4)}(0) + \cdots \quad \ldots (1)$$

Consider, $y' = x - y^2$; $y'(0) = 0 - 1^2 = -1$

$y'' = 1 - 2yy'$; $y''(0) = 1 - (2)(1)(-1) = 3$

$y''' = 0 - 2\,[\,y\,y'' + (y')^2\,]$; $y'''(0) = -2\,[\,(1)(3) + (-1)^2\,] = -8$

$y^{(4)} = -2[\,y\,y''' + y''\,y' + 2y'\,y''\,] = -2[\,y\,y''' + 3y'\,y''\,]$ *(fourth derivative)*

$\therefore \quad y^{(4)}(0) = -2[(1)(-8) + (3)(-1)(3)] = 34$

To find $y\,(0.1)$, we shall substitute these values along with $x = 0.1$ in (1).

$$y(0.1) = 1 + (0.1)(-1) + \frac{(0.1)^2}{2}(3) + \frac{(0.1)^3}{6}(-8) + \frac{(0.1)^4}{24}(34)$$

Thus, $\boxed{y(0.1) = 0.9138}$

[4] *Use Taylor's series method to obtain a power series in* $(x - 4)$ *for the equation* $5x\dfrac{dy}{dx} + y^2 - 2 = 0$; $x_0 = 4,\ y_0 = 1$ *and use it to find* y *at* $x = 4.1,\ 4.2,\ 4.3$ *correct to four decimal places.*

☞ Taylor's series expansion is given by

$$y(x) = y(x_0) + (x - x_0)\,y'(x_0) + \frac{(x - x_0)^2}{2!}\,y''(x_0) + \cdots$$

Since $x_0 = 4,\ y_0 = 1$ by data, the series becomes

$$y(x) = y(4) + (x - 4)\,y'(4) + \frac{(x - 4)^2}{2}\,y''(4) + \cdots \quad \ldots (1)$$

Consider, $5x\,y' + y^2 - 2 = 0$

Substituting the initial values we obtain, [**Note :** $y' = y'(x)$]

$$(5)(4)\,y'(4) + 1^2 - 2 = 0 \text{ or } 20\,y'(4) = 1$$

$$y'(4) = \frac{1}{20} = 0.05$$

Differentiating the given equation w.r.t. x ,

$$5[xy'' + y'] + 2yy' = 0 \quad [\ \textbf{Note :}\ y'' = y''(x)]$$

Substituting the initial values and the value of $y'(4)$ we have,

$$5[4y''(4) + 0.05] + (2)(1)(0.05) = 0$$

ie., $$20y''(4) + 0.25 + 0.1 = 0$$

$$\therefore \quad y''(4) = -\frac{0.35}{20} = -0.0175$$

Since the value of the second derivative itself is small enough we shall approximate Taylor's series as in (1) upto second degree terms only. Substituting these values in (1) we have,

$$\boxed{y(x) = 1 + (x-4)(0.05) + \frac{(x-4)^2}{2}(-0.0175)}$$

We now find $y\ (4.1)$, $y\ (4.2)$ and $y\ (4.3)$ from this expression.

$$y(4.1) = 1 + (4.1-4)(0.05) + \frac{(4.1-4)^2}{2}(-0.0175)$$

Thus we have,

$$y(4.1) = 1 + (0.1)(0.05) + \frac{(0.1)^2}{2}(-0.0175) = 1.0049$$

$$y(4.2) = 1 + (0.2)(0.05) + \frac{(0.2)^2}{2}(-0.0175) = 1.0097$$

$$y(4.3) = 1 + (0.3)(0.05) + \frac{(0.3)^2}{2}(-0.0175) = 1.0142$$

Thus, $\boxed{y(4.1) = 1.0049,\ y(4.2) = 1.0097,\ y(4.3) = 1.0142}$

[5] *Use Taylor's series method to find* $y\ (4.1)$ *given that* $\dfrac{dy}{dx} = \dfrac{1}{x^2 + y}$ *and* $y\ (4) = 4.$ **[Dec 2017]**

☞ Taylor's series expansion is given by

$$y(x) = y(x_0) + (x - x_0)y'(x_0) + \frac{(x - x_0)^2}{2!}y''(x_0) + \frac{(x - x_0)^3}{3!}y'''(x_0) + \ldots$$

By data, $y' = \dfrac{1}{x^2 + y}$; $x_0 = 4,\ y_0 = 4$

$$\therefore\quad y(x) = y(4) + (x - 4)y'(4) + \frac{(x - 4)^2}{2!}y''(4) + \frac{(x - 4)^3}{3!}y'''(4) \quad \ldots(1)$$

by approximating upto the third degree terms.

Consider, $y' = \dfrac{1}{x^2 + y}$ [**Note:** $y' = y'(x), y'' = y''(x)$ etc.]

or $\quad y'(x^2 + y) = 1 \quad \ldots(2)$

Substituting the initial values we have,

$$y'(4)[4^2 + 4] = 1 \text{ or } y'(4) = \frac{1}{20} = 0.05$$

Differentiating (2) w.r.t. x,

$$y'(2x + y') + (x^2 + y)y'' = 0 \quad \ldots(3)$$

Substituting the initial values and the value of $y'(4)$ we have,

$$0.05[(2)(4) + 0.05] + [4^2 + 4]y''(4) = 0$$

ie., $\quad 0.05[8 + 0.05] + 20y''(4) = 0$

ie., $\quad 0.4025 + 20y''(4) = 0$ or $y''(4) = -0.020125$

We observe that the value of the derivatives are small enough and the third degree term can also be neglected. Substituting these values in (1) for computing $y(4.1)$ we have,

$$y(4.1) = 4 + (4.1 - 4)(0.05) + \frac{(4.1 - 4)^2}{2}(-0.020125)$$

Thus, $\quad \boxed{y(4.1) = 4.0049}$

[6] *Use Taylor's series method to solve* $y' = x^2 + y$ *in the range* $0 \le x \le 0.2$ *by taking step size* $h = 0.1$ *given that* $y = 10$ *at* $x = 0$ *initially considering terms upto the fourth degree.*

☞ In this problem, since the step size is specified as 0.1, the problem has to be done in two stages. We have to first find $y(0.1)$ and use this as the initial condition to compute $y(0.2)$.

Taylor's series expansion is given by,

$$y(x) = y(x_0) + (x - x_0) y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0)$$

$$+ \frac{(x-x_0)^3}{3!} y'''(x_0) + \frac{(x-x_0)^4}{4!} y^{(4)}(x_0) + \cdots \quad \ldots (1)$$

I Stage : By data, $y' = x^2 + y$, $x_0 = 0$, $y_0 = 10$

$$y'(0) = 0^2 + 10 = 10 \quad \text{or} \quad y'(0) = 10$$

Differentiating y' w.r.t. x successively we have,

$$y'' = 2x + y' \quad ; \quad y''(0) = (2)(0) + y'(0) = 0 + 10 = 10$$

$$y''' = 2 + y'' \quad ; \quad y'''(0) = (2) + y''(0) = 2 + 10 = 12$$

$$y^{(4)} = y''' \quad ; \quad y^{(4)}(0) = 12$$

With $x = 0.1$ and $x_0 = 0$, (1) becomes,

$$y(0.1) = y(0) + (0.1) y'(0) + \frac{(0.1)^2}{2} y''(0)$$

$$+ \frac{(0.1)^3}{6} y'''(0) + \frac{(0.1)^4}{24} y^{(4)}(0)$$

$$= 10 + (0.1)10 + \frac{0.01}{2}(10) + \frac{0.001}{6}(12) + \frac{0.0001}{24}(12)$$

Thus, $$\boxed{y(0.1) = 11.05205 \approx 11.052}$$

II Stage : Now taking $x_0 = 0.1$, $y_0 = 11.052$, we have,

$$y' = x^2 + y \quad ; \quad y'(0.1) = (0.1)^2 + 11.052 = 11.062$$

$$y'' = 2x + y' \quad ; \quad y''(0.1) = 2(0.1) + 11.062 = 11.262$$

$$y''' = 2 + y'' \quad ; \quad y'''(0.1) = 2 + 11.262 = 13.262$$

$$y^{(4)} = y''' \quad ; \quad y^{(4)}(0.1) = 13.262$$

With $x = 0.2$ and $x_0 = 0.1$, (1) becomes,

$$y(0.2) = y(0.1) + (0.1)\,y'(0.1) + \frac{0.01}{2}\,y''(0.1) + \frac{0.001}{6}\,y'''(0.1) + \frac{0.0001}{24}\,y^{(4)}(0.1)$$

$$= 11.052 + (0.1)(11.062) + \frac{0.01}{2}(11.262) + \frac{0.001}{6}(13.262) + \frac{0.0001}{24}(13.262)$$

Thus, $\boxed{y(0.2) = 12.216776 \approx 12.2168}$

[7] **(a)** *Employ Taylor's series method to find y at* $x = 0.1$ *and* 0.2 *correct to four places of decimal in step size of* 0.1 *given the linear differential equation* $\dfrac{dy}{dx} - 2y = 3e^x$ *whose solution passes through the origin. Also find* $y\,(0.1)$ *and* $y\,(0.2)$ *by analytical method.*

(b) *Compute* $y\,(0.1)$ **[Dec 2017]**

☞ By data, $y' = 2y + 3e^x$ and $y(0) = 0$. That is $x_0 = 0,\ y_0 = 0$.

Taylor's series expansion is given by

$$y(x) = y(x_0) + (x - x_0)\,y'(x_0) + \frac{(x - x_0)^2}{2!}\,y''(x_0) + \cdots \qquad \ldots (1)$$

Step - 1 : We shall compute $y\,(0.1)$

Consider, $y' = 2y + 3e^x$; $y'(0) = 2(0) + 3e^0 = 3$

$\therefore$ $y'' = 2y' + 3e^x$; $y''(0) = 2(3) + 3 = 9$

$y''' = 2y'' + 3e^x$; $y'''(0) = 2(9) + 3 = 21$

From (1) we have,

$$y(0.1) = y(0) + (0.1)\,y'(0) + \frac{(0.1)^2}{2}\,y''(0) + \frac{(0.1)^3}{6}\,y'''(0)$$

by considering terms upto third degree. Further we have

$$y(0.1) = 0 + (0.1)3 + \frac{0.01}{2}(9) + \frac{0.001}{6}(21)$$

Thus, $\boxed{y(0.1) = 0.3485}$

Note : *Solution of* (b) *ends here.*

Step - 2 : We shall compute $y\ (0.2)$

Consider, $y' = 2y + 3e^x$ and let $x_0 = 0.1,\ y_0 = 0.3485$

Now, $y'(0.1) = 2y(0.1) + 3e^{0.1}$; $y'(0.1) = 4.0125$

$y'' = 2y' + 3e^x$

$y''(0.1) = 2(4.0125) + 3e^{0.1}$; $y''(0.1) = 11.3405$

$y''' = 2y'' + 3e^x$; $y'''(0.1) = 2(11.3405) + 3\,e^{0.1}$; $y'''(0.1) = 25.9965$

We have from (1),

$$y(0.2) = y(0.1) + (0.1)\,y'(0.1) + \frac{0.01}{2}\,y''(0.1) + \frac{0.001}{6}\,y'''(0.1)$$

$$\therefore\ y(0.2) = 0.3485 + (0.1)(4.0125) + \frac{0.01}{2}(11.3405) + \frac{0.001}{6}(25.9965)$$

Thus, $\boxed{y(0.2) = 0.8108}$

Solution by analytical method

$\dfrac{dy}{dx} - 2y = 3e^x$ is of the form $\dfrac{dy}{dx} + Py = Q$ where $P = -2, Q = 3e^x$

Solution : $y\,e^{\int P\,dx} = \int Q\,e^{\int P\,dx}\,dx + c$

ie., $ye^{-2x} = \int 3\,e^x\,e^{-2x}\,dx + c$ or $y\,e^{-2x} = 3\int e^{-x}\,dx + c$

ie., $ye^{-2x} = -3e^{-x} + c$ or $y = -3e^x + c\,e^{2x}$ is the general solution.

Applying the condition $y\ (\ 0\ ) = 0$, the general solution becomes

$0 = -3 + c$ or $c = 3$.

Hence, $y = 3(e^{2x} - e^x)$ is the solution. Let us substitute $x = 0.1$ & 0.2.

Thus $\boxed{y(0.1) = 0.3487}$ and $\boxed{y(0.2) = 0.8113}$ by the analytical method

[8] *Using Taylor's series method, obtain the values of y at x = 0.1, 0.2, 0.3 to four significant figures if y satisfies the equation $y'' = -xy$ given that $y' = 0.5$ and $y = 1$ when $x = 0$ taking the first five terms of the Taylor's series expansion.*

☞ Taylor's series expansion given by

$$y(x) = y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2!}\, y''(x_0) + \cdots \qquad \ldots (1)$$

By data, $y'' = -xy$; $y(0) = 1,\ y'(0) = 0.5$

Consider, $y'' = -xy$; $y''(0) = 0$

$y''' = -xy' - y$; $y'''(0) = -1$

$y^{(4)} = -xy'' - 2y'$; $y^{(4)}(0) = -1$

From (1), the first five terms of the Taylor's series expansion is given by

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}\, y''(0) + \frac{x^3}{3!}\, y'''(0) + \frac{x^4}{24!}\, y^{(4)}(0)$$

Now, $y(0.1) = 1 + (0.1)(0.5) - \dfrac{0.001}{6} - \dfrac{0.0001}{24} = 1.0498$

$y(0.2) = 1 + (0.2)(0.5) - \dfrac{0.008}{6} - \dfrac{0.0016}{24} = 1.0986$

$y(0.3) = 1 + (0.3)(0.5) - \dfrac{0.027}{6} - \dfrac{0.0081}{24} = 1.1452$

Thus, $\boxed{y(0.1) = 1.0498,\ y(0.2) = 1.0986,\ y(0.3) = 1.1452}$

ASSINGMENT

Use Taylor's series method to solve the following initial value problems.

1. $\dfrac{dy}{dx} = x - y,\ y(0) = 1$. Compute $y(0.1)$ considering terms upto fourth degree.

2. $\dfrac{dy}{dx} = x^2 y - 1,\ y(0) = 1$. Compute $y(0.03)$ using the expansion of y upto second degree terms.

3. $\dfrac{dy}{dx} = x\, y^{1/3},\ y(1) = 1$. Compute $y(1.1)$ & $y(1.2)$ by taking step size 0.1

4. $\frac{dy}{dx} = x + y$; $x = 1$, $y = 0$. Find the third order approximation of the solution and use it compute $y(1.1)$, $y(1.2)$ and $y(1.3)$.

5. $\frac{dy}{dx} = x^2 y - 1, y(0) = 1$ in the range $0 \le x \le 2$ taking step size 0.1

ANSWERS

1. 0.8373
2. 0.97
3. 1.107, 1.228
4. $y = (x-1) + (x-1)^2 + \frac{1}{2}(x-1)^3$; 0.1105, 0.244, 0.4035
5. 0.9003, 0.8023

4.12 Modified Euler's Method

Consider the initial value problem $\frac{dy}{dx} = f(x, y)$; $y(x_0) = y_0$.

We need to find y at $x_1 = x_0 + h$.

We first obtain $y(x_1) = y_1$ by applying *Euler's formula* and this value is regarded as the initial approximation for y_1 usually denoted by $y_1^{(0)}$, also called as the predicted value of y_1.

Euler's formula is given by

$$y_1^{(0)} = y_0 + h\, f(x_0, y_0)$$

Since the accuracy is poor in this formula this value y_1 is successively improved (corrected) to the desired degree of accuracy by the following *Modified Euler's formula*, where the successive approximations are denoted by $y_1^{(1)}, y_1^{(2)}, y_1^{(3)}, \cdots$ etc.

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)}) \right]$$

Each of the succeeding approximation is better than the preceding ones. They are called corrected values. Euler's formula and modified Euler's formula jointly are also called as ***Euler's predictor and corrector formulae.***

WORKED PROBLEMS

[9] (a) *Given* $\dfrac{dy}{dx} = 1 + \dfrac{y}{x}$, $y = 2$ *at* $x = 1$, *find the approximate value* y *at* x = 1.4 *by taking step size* $h = 0.2$ *applying modified Euler's method. Also find the value of* y *at* $x = 1.2$ *and* 1.4 *from the analytical solution of the equation.*

(b) *Compute* $y(1.2)$ *in two steps.* **[Dec 2018]**

☞ The problem has to be worked in two stages.

I Stage : $x_0 = 1,\ y_0 = 2,\ f(x, y) = 1 + (y/x),\ h = 0.2$

$$x_1 = x_0 + h = 1.2,\ y(x_1) = y_1 = y(1.2) = \ ?$$

Now, $f(x_0, y_0) = 1 + (2/1) = 3$

We have Euler's formula : $y_1^{(0)} = y_0 + h\, f(x_0, y_0)$ ... (1)

$$\therefore \quad y_1^{(0)} = 2 + (0.2)(3) = 2.6$$

Further we have modified Euler's formula :

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)}) \right] \quad \text{... (2)}$$

$$= 2 + (0.1)[3 + (1 + y_1^{(0)}/x_1)]$$

$$= 2 + (0.1)[4 + y_1^{(0)}/1.2]$$

$$= 2 + (0.1)[4 + 2.6/1.2]$$

$$\therefore \quad y_1^{(1)} = 2.6167$$

Next approximation $y_1^{(2)}$ is got just by replacing the value of $y_1^{(1)}$ in place of $y_1^{(0)}$.

Now, $y_1^{(2)} = 2 + (0.1)[4 + 2.6167/1.2] = 2.6181$

Again, $y_1^{(3)} = 2 + (0.1)[4 + 2.6181/1.2] = 2.6182$. Also $y_1^{(4)} = 2.6182$

Thus, $\boxed{y(1.2) = 2.6182}$

Note : *The required solution for* (b) *is,*

$$y_1^{(2)} = y(1.2) = 2.6181 \text{ in two steps.}$$

II Stage : We repeat the process by taking $y\ (1.2) = 2.6182$ as the initial condition.

$$x_0 = 1.2,\ y_0 = 2.6182\ ;\ f(x_0, y_0) = 1 + (y_0/x_0) = 3.1818$$

$$x_1 = x_0 + h = 1.4,\ y(x_1) = y_1 = y(1.4) = ?$$

We have from (1),

$$y_1^{(0)} = 2.6182 + (0.2)(3.1818) = 3.2546$$

Now from (2),

$$y_1^{(1)} = 2.6182 + (0.1)[3.1818 + (1 + y_1^{(0)}/x_1)]$$

ie., $$y_1^{(1)} = 2.6182 + (0.1)[4.1818 + 3.2546/1.4] = 3.2689$$

$$y_1^{(2)} = 2.6182 + (0.1)[4.1818 + 3.2689/1.4] = 3.2699$$

$$y_1^{(3)} = 2.6182 + (0.1)[4.1818 + 3.2699/1.4] = 3.2699$$

Thus, $\boxed{y(1.4) = 3.2699 \approx 3.27}$

Now, let us find the analytical solution of the equation :

$$\frac{dy}{dx} = 1 + \frac{y}{x} \quad \text{or} \quad \frac{dy}{dx} - \frac{y}{x} = 1$$

This is a linear DE of the form $\frac{dy}{dx} + Py = Q$ whose solution is given by

$$y\,e^{\int P\,dx} = \int Q\,e^{\int P\,dx}\,dx + c$$

Here, $P = -1/x$ and $Q = 1$.

$$\therefore \quad e^{\int P\,dx} = e^{-\int 1/x\,dx} = e^{-\log x} = 1/e^{\log x} = 1/x$$

The solution becomes, $y \cdot 1/x = \int 1 \cdot 1/x\,dx + c$

ie., $y/x = \log x + c.$

Applying the initial condition that $y = 2$ and $x = 1$ we have,

$$2/1 = \log 1 + c \quad \therefore \ c = 2 \text{ since } \log 1 = 0.$$

The solution now becomes,

$$y/x = \log x + 2 \ \text{ or } \ y = x(\log x + 2)$$

This is the analytical solution of the given initial value problem.

Now by putting $x = 1.2$ and 1.4 we obtain,

$$y(1.2) = 1.2(\log_e 1.2 + 2) = 2.6188 \ \& \ y(1.4) = 1.4(\log_e 1.4 + 2) = 3.2711$$

Solutions are tabulated for comparison.

Solution $y(x)$	By modified Euler's method	By analytical method
$y(1.2)$	2.6182	2.6188
$y(1.4)$	3.2699	3.2711

[10] (a) *Using modified Euler's method find y at x = 0.2 given* $\dfrac{dy}{dx} = 3x + \dfrac{1}{2}y$ *with* $y(0) = 1$ *taking h = 0.1. Perform three iterations at each step.*

(b) *Compute y (0.1) taking h = 0.1. (Two iterations)* **[Dec 2017]**

☞ We need to find $y(0.2)$ by taking $h = 0.1$

This implies that the problem has to be done in two stages.

I Stage : By data $x_0 = 0,\ y_0 = 1,\ h = 0.1,\ f(x, y) = 3x + (y/2)$

$$f(x_0, y_0) = 0.5,\ x_1 = x_0 + h = 0.1$$

$$y(x_1) = y_1 = y(0.1) = ?$$

From Euler's formula : $y_1^{(0)} = y_0 + h f(x_0, y_0)$ we obtain,

$$y_1^{(0)} = 1 + (0.1)(0.5) = 1.05$$

We have modified Euler's formula,

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$= 1 + \frac{0.1}{2}\left[0.5 + 3x_1 + \frac{y_1^{(0)}}{2}\right]$$

$$= 1 + 0.05\left[0.5 + 3(0.1) + \frac{y_1^{(0)}}{2}\right]$$

$$y_1^{(1)} = 1 + 0.05\left[0.8 + \frac{y_1^{(0)}}{2}\right]$$

ie., $$y_1^{(1)} = 1 + 0.05\left[0.8 + \frac{1.05}{2}\right] = 1.06625$$

The second iterative value is got simply by replacing $y_1^{(0)}$ by $y_1^{(1)}$. That is by replacing 1.06625 inplace of 1.05

$$\therefore \quad y_1^{(2)} = 1 + 0.05\left[0.8 + \frac{1.06625}{2}\right] = 1.0667$$

$$y_1^{(3)} = 1 + 0.05\left[0.8 + \frac{1.0667}{2}\right] = 1.0667$$

Thus, $\boxed{y(0.1) = 1.0667}$

Note : *The required solution for* (b) *is* $y_1^{(2)} = y(0.1) = 1.0667$.

II Stage : Now, let $x_0 = 0.1,\ y_0 = 1.0667$

We have, $f(x, y) = 3x + (y/2)$

$$\therefore \quad f(x_0, y_0) = 3(0.1) + \frac{1.0667}{2} = 0.83335$$

$$x_1 = x_0 + h = 0.2\ ;\ y_1 = y(x_1) = y(0.2) = ?$$

From Euler's formula we obtain

$$y_1^{(0)} = 1.0667 + 0.1(0.83335) = 1.15$$

Next, from modified Euler's formula

$$y_1^{(1)} = 1.0667 + \frac{0.1}{2}\left[0.83335 + 3x_1 + \frac{y_1^{(0)}}{2}\right]$$

$$= 1.0667 + 0.05\left[0.83335 + 3(0.2) + \frac{y_1^{(0)}}{2}\right]$$

$$= 1.0667 + 0.05\left[1.43335 + \frac{1.15}{2}\right] = 1.1671$$

$$y_1^{(2)} = 1.0667 + 0.05\left[1.43335 + \frac{1.1671}{2}\right] = 1.1675$$

$$y_1^{(3)} = 1.0667 + 0.05\left[1.43335 + \frac{1.1675}{2}\right] = 1.1676$$

Thus, $\boxed{y(0.2) = 1.1676}$

[11] (a) *Using modified Euler's method find $y(0.2)$ correct to four decimal places solving the equation $\frac{dy}{dx} = x - y^2$, $y(0) = 1$ taking $h = 0.1$*

(b) *Compute $y(0.1)$ taking $h = 0.1$.* **[June 2018]**

☞ We shall first compute $y(0.1)$ and use this value to compute $y(0.2)$

I Stage : By data $x_0 = 0$, $y_0 = 1$, $h = 0.1$, $f(x,y) = x - y^2$

$$f(x_0, y_0) = 0 - 1^2 = -1, x_1 = x_0 + h = 0.1$$

$$y(x_1) = y_1 = y(0.1) = ?$$

From Euler's formula : $y_1^{(0)} = y_0 + h f(x_0, y_0)$ we obtain

$$y_1^{(0)} = 1 + (0.1)(-1) = 0.9$$

We have modified Euler's formula,

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$= 1 + \frac{0.1}{2}\left[-1 + x_1 - \left\{y_1^{(0)}\right\}^2\right]$$

$$= 1 + 0.05[-1 + 0.1 - (0.9)^2]$$

$$= 1 + 0.05[-0.9 - (0.9)^2] = 0.9145$$

$$y_1^{(2)} = 1 + 0.05[-0.9 - (0.9145)^2] = 0.9132$$

$$y_1^{(3)} = 0.9133$$

Thus, $\boxed{y(0.1) = 0.9133}$ *Solution of* (b) *ends here.*

II Stage : Now, let $x_0 = 0.1$, $y_0 = 0.9133$. $f(x,y) = x - y^2$

$$f(x_0, y_0) = 0.1 - (0.9133)^2 = -0.7341$$

$$x_1 = x_0 + h = 0.2, y(x_1) = y(0.2) = ?$$

Substituting in the Euler's formula,

$$y_1^{(0)} = 0.9133 + (0.1)(-0.7341) = 0.8399$$

Now from the modified Euler's formula,

$$y_1^{(1)} = 0.9133 + \frac{0.1}{2}\left[-0.7341 + x_1 - \left\{y_1^{(0)}\right\}^2\right]$$

$$y_1^{(1)} = 0.9133 + 0.05[-0.7341 + 0.2 - (0.8399)^2]$$

$$= 0.9133 + 0.05[-0.5341 - (0.8399)^2] = 0.8513$$

$$y_1^{(2)} = 0.9133 + 0.05[-0.5341 - (0.8513)^2] = 0.8504$$

$$y_1^{(3)} = 0.9133 + 0.05[-0.5341 - (0.8504)^2] = 0.8504$$

Thus, $\boxed{y(0.2) = 0.8504}$

[12] *Using modified Euler's method find y (20.2) and y (20.4) given that* $\dfrac{dy}{dx} = \log_{10}\left(\dfrac{x}{y}\right)$ *with* $y(20) = 5$ *taking* $h = 0.2$ **[June 2017]**

☞ We shall first compute $y(20.2)$ and use this value to compute $y(20.4)$

I Stage : By data $x_0 = 20$, $y_0 = 5$ and $h = 0.2$

$$f(x,y) = \log_{10}\left(\frac{x}{y}\right) ; \; f(x_0, y_0) = \log_{10}(4) = 0.6021$$

$$x_1 = x_0 + h = 20.2 ; \; y(x_1) = y_1 = y(20.2) = ?$$

From Euler's formula : $y_1^{(0)} = y_0 + h\,f(x_0, y_0)$ we obtain,

$$y_1^{(0)} = 5 + (0.2)(0.6021) = 5.1204$$

Next by modified Euler's formula,

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$= 5 + \frac{0.2}{2}\left[0.6021 + \log_{10}\left(\frac{x_1}{y_1^{(0)}}\right)\right]$$

$$\therefore \quad y_1^{(1)} = 5 + 0.1\left[0.6021 + \log_{10}\left(\frac{20.2}{5.1204}\right)\right] = 5.1198$$

$$y_1^{(2)} = 5 + 0.1\left[0.6021 + \log_{10}\left(\frac{20.2}{5.1198}\right)\right] = 5.1198$$

Thus, $\boxed{y(20.2) = 5.1198}$

II stage : Now, let $x_0 = 20.2,\ y_0 = 5.1198$

$$f(x, y) = \log_{10}\left(\frac{x}{y}\right);\ f(x_0, y_0) = 0.5961$$

$$x_1 = x_0 + h = 20.4,\ y(x_1) = y_1 = y(20.4) = ?$$

Substituting in the Euler's formula,

$$y_1^{(0)} = 5.1198 + (0.2)(0.5961) = 5.239$$

Now by modified Euler's formula,

$$y_1^{(1)} = 5.1198 + \frac{0.2}{2}\left[0.5961 + \log_{10}\left(\frac{x_1}{y_1^{(0)}}\right)\right]$$

$$= 5.1198 + 0.1\left[0.5961 + \log_{10}\left(\frac{20.4}{5.239}\right)\right] = 5.2384$$

$$y_1^{(2)} = 5.1198 + 0.1\left[0.5961 + \log_{10}\left(\frac{20.4}{5.2384}\right)\right] = 5.2385$$

Thus, $\boxed{y(20.4) = 5.2385}$

[13] *Use modified Euler's method to solve* $\dfrac{dy}{dx} = x + \left|\sqrt{y}\right|$ *in the range* $0 \le x \le 0.4$ *by taking* $h = 0.2$ *given that* $y = 1$ *at* $x = 0$ *initially.* **[June, 2017, Dec 18]**

☞ We need to compute $y(0.2)$ and $y(0.4)$ with $h = 0.2$

I Stage : By data $x_0 = 0,\ y_0 = 1,\ f(x, y) = x + \sqrt{y},\ h = 0.2$ where the modulus sign indicates that we have to take only the positive value of $\sqrt{y}$.

$$f(x_0, y_0) = 0 + \sqrt{1} = 1.\ x_1 = x_0 + h = 0.2$$

$$y(x_1) = y_1 = y(0.2) = ?$$

From Euler's formula : $y_1^{(0)} = y_0 + h\,f(x_0, y_0)$ we obtain

$$y_1^{(0)} = 1 + 0.2(1) = 1.2$$

We have modified Euler's formula,

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)}) \right]$$

$$= 1 + \frac{0.2}{2}\left[1 + x_1 + \sqrt{y_1^{(0)}} \right]$$

$$= 1 + 0.1[1 + 0.2 + \sqrt{1.2}] = 1.2295$$

$$y_1^{(2)} = 1 + 0.1[1.2 + \sqrt{1.2295}] = 1.2309$$

$$y_1^{(3)} = 1 + 0.1[1.2 + \sqrt{1.2309}] = 1.2309$$

Thus, $\boxed{y(0.2) = 1.2309}$

II Stage : Now let $x_0 = 0.2$, $y_0 = 1.2309$

$$f(x, y) = x + \sqrt{y} \ ; \ f(x_0, y_0) = 0.2 + \sqrt{1.2309} = 1.3095$$

$$x_1 = x_0 + h = 0.4 \ ; \ y(x_1) = y_1 = y(0.4) = ?$$

Substituting in the Euler's formula,

$$y_1^{(0)} = 1.2309 + 0.2(1.3095) = 1.4928$$

Next from modified Euler's formula,

$$y_1^{(1)} = 1.2309 + \frac{0.2}{2}\left[1.3095 + x_1 + \sqrt{y_1^{(0)}} \right]$$

$$= 1.2309 + 0.1\left[1.3095 + 0.4 + \sqrt{1.4928} \right] = 1.524$$

$$y_1^{(2)} = 1.2309 + 0.1\left[1.7095 + \sqrt{1.524} \right] = 1.5253$$

$$y_1^{(3)} = 1.2309 + 0.1\left[1.7095 + \sqrt{1.5253} \right] = 1.5254$$

Also $y_1^{(4)} = 1.5254$

Thus, $\boxed{y(0.4) = 1.5254}$

[14] *Use modified Euler's method to compute $y(0.1)$ given that $\dfrac{dy}{dx} = x^2 + y$, $y(0) = 1$ by taking $h = 0.05$ considering the accuracy upto two approximations in each step.*

☞ We need to compute $y(0.05)$ first and use this value to compute $y(0.1)$

I Stage : By data $x_0 = 0,\ y_0 = 1,\ f(x, y) = x^2 + y,\ h = 0.05$

$$f(x_0, y_0) = 0^2 + 1 = 1.\ \ x_1 = x_0 + h = 0.05$$

$$y(x_1) = y_1 = y(0.05) = ?$$

From Euler's formula : $y_1^{(0)} = y_0 + h\,f(x_0, y_0)$ we obtain,

$$y_1^{(0)} = 1 + (0.05)(1) = 1.05$$

Next by modified Euler's formula,

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$= 1 + \frac{0.05}{2}\left[1 + x_1^2 + y_1^{(0)}\right]$$

$$= 1 + 0.025[1 + (0.05)^2 + 1.05]$$

$$= 1 + 0.025[1.0025 + 1.05] = 1.0513$$

$$y_1^{(2)} = 1 + 0.025[1.0025 + 1.0513] = 1.0513$$

Thus, $\boxed{y(0.05) = 1.0513}$

II stage : Now, let $x_0 = 0.05,\ y_0 = 1.0513$

$$f(x, y) = x^2 + y\ ;\ f(x_0, y_0) = (0.05)^2 + 1.0513 = 1.0538$$

$$x_1 = x_0 + h = 0.1,\ y(x_1) = y_1 = y(0.1) = ?$$

Substituting in the Euler's formula,

$$y_1^{(0)} = 1.0513 + 0.05(1.0538) = 1.104$$

Next from the modified Euler's formula,

$$y_1^{(1)} = 1.0513 + \frac{0.05}{2}\left[1.0538 + x_1^2 + y_1^{(0)}\right]$$

$$= 1.0513 + 0.025\left[1.0538 + (0.1)^2 + 1.104\right]$$

$$= 1.0513 + 0.025\left[1.0638 + 1.104\right] = 1.1055$$

$$y_1^{(2)} = 1.0513 + 0.025\left[1.0638 + 1.1055\right] = 1.1055$$

Thus the required $\boxed{y\,(\,0.1\,) = 1.1055}$

[15] *Using Euler's predictor and corrector formulae solve* $\dfrac{dy}{dx} = x + y$ *at* $x = 0.2$ *given that* $y\,(\,0\,) = 1$.

☞ We need to compute $y\,(\,0.2\,)$ and since the step size is not specified we shall take it be 0.2 itself.

By data we have, $x_0 = 0,\ y_0 = 1,\ f(x, y) = x + y$ and $h = 0.2$ *(assumed)*

$$f\,(x_0\,,\ y_0\,) = 0 + 1 = 1.\ x_1 = x_0 + h = 0.2$$

$$y\,(x_1\,) = y_1 = y\,(0.2\,) = ?$$

We have Euler's formula *(predictor formula)*

$$y_1^{(0)} = y_0 + h\,f\,(x_0\,, y_0\,) = 1 + (0.2\,)1 = 1.2$$

Next consider modified Euler's formula *(corrector formula)*

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[\,f\,(x_0\,,\ y_0\,) + f\,(x_1\,,\ y_1^{(0)}\,)\right]$$

$$= 1 + \frac{0.2}{2}\left[1 + x_1 + y_1^{(0)}\right]$$

$$= 1 + 0.1[\,1 + 0.2 + (1.2\,)\,] = 1.24$$

$$y_1^{(2)} = 1 + 0.1[\,1.2 + (\,1.24\,)\,] = 1.244$$

$$y_1^{(3)} = 1 + 0.1[\,1.2 + 1.244\,] = 1.2444$$

$$y_1^{(4)} = 1 + 0.1[\,1.2 + 1.2444\,] = 1.24444$$

Thus the required $\boxed{y(0.2) = 1.2444}$

Remark : *If we had worked the problem in two stages (Taking* $h = 0.1$ *) we would have got more accurate answer. It may be noted that lesser is the step size, greater is the accuracy.*

[16] *Using Euler's predictor and corrector formula compute y (1.1) correct to five decimal places given that* $\dfrac{dy}{dx}+\dfrac{y}{x}=\dfrac{1}{x^2}$ *and* $y=1$ *at* $x=1$*. Also find the analytical solution.*

☞ By data, $\dfrac{dy}{dx}=\dfrac{1}{x^2}-\dfrac{y}{x}$ or $\dfrac{dy}{dx}=\dfrac{1-xy}{x^2}$

We have, $f(x,y)=\dfrac{1-xy}{x^2}$; $x_0=1,\ y_0=1$. Let us take $h=0.1$

$$f(x_0, y_0)=0,\ x_1=x_0+h=1.1$$

$$y(x_1)=y_1=y(1.1)=?$$

From Euler's formula : $y_1^{(0)}=y_0+hf(x_0,y_0)$, we obtain $y_1^{(0)}=1$

We have modified Euler's formula,

$$y_1^{(0)}=y_0+\frac{h}{2}\left[f(x_0,y_0)+f(x_1,y_1^{(0)})\right]$$

$$=1+\frac{0.1}{2}\left[0+\frac{1-x_1\,y_1^{(0)}}{x_1^2}\right]$$

$$=1+0.05\left[\frac{1-1.1(1)}{(1.1)^2}\right]=0.9959$$

$$y_1^{(2)}=1+0.05\left[\frac{1-1.1(0.9959)}{(1.1)^2}\right]=0.99605$$

$$y_1^{(3)}=1+0.05\left[\frac{1-1.1(0.99605)}{(1.1)^2}\right]=0.99605$$

Thus, $\boxed{y(1.1)=0.99605}$

Analytical Solution

$\dfrac{dy}{dx}+\dfrac{y}{x}=\dfrac{1}{x^2}$ *is* of the form $\dfrac{dy}{dx}+Py=Q$ where $P=1/x$ and $Q=1/x^2$

whose solution is given by

$$y\, e^{\int P\,dx} = \int Q\, e^{\int P\,dx}\, dx + c$$

$$e^{\int P\,dx} = e^{\int 1/x\,dx} = e^{\log x} = x$$

Solution : $y \cdot x = \int \frac{1}{x^2} \cdot x\, dx + c$

ie., $xy = \log x + c \cdot$ Using $y(1) = 1,\ 1 = \log 1 + c \quad \therefore\ c = 1.$

The befitting solution is $xy = \log x + 1$ or $y = \dfrac{\log x + 1}{x}$

Now $y(1.1) = \dfrac{\log(1.1) + 1}{1.1} = 0.99574$

Thus, $\boxed{y(1.1) = 0.99574}$ is the analytical solution.

4.13 Runge-Kutta method of fourth order

Consider the initial value problem $\dfrac{dy}{dx} = f(x, y), y(x_0) = y_0$.

We need to find $y(x_0 + h)$, where h is the step size.

We have to first compute k_1, k_2, k_3, k_4 by the following formulae.

$$k_1 = h\, f(x_0,\ y_0)$$

$$k_2 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = h\, f(x_0 + h,\ y_0 + k_3)$$

The required $y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

WORKED PROBLEMS

[17] *Given* $\dfrac{dy}{dx} = 3x + \dfrac{y}{2}, y(0) = 1$, *compute* $y(0.2)$ *by taking* $h = 0.2$ *using Runge-Kutta method of fourth order. Also find the analytical solution.*

☞ By data, $f(x, y) = 3x + \dfrac{y}{2}, x_0 = 0, y_0 = 1, h = 0.2$

We shall first compute k_1, k_2, k_3, k_4.

$$k_1 = h f(x_0, y_0) = (0.2) f(0, 1) = (0.2)\left[3 \times 0 + \frac{1}{2}\right] = 0.1$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.2) f\left(0 + \frac{0.2}{2}, 1 + \frac{0.1}{2}\right)$$

$$k_2 = (0.2) f(0.1, 1.05) = (0.2)\left[3 \times 0.1 + \frac{1.05}{2}\right] = 0.165$$

$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.2) f\left(0 + \frac{0.2}{2}, 1 + \frac{0.165}{2}\right)$$

$$k_3 = (0.2) f(0.1, 1.0825) = (0.2)\left[3 \times 0.1 + \frac{1.0825}{2}\right] = 0.16825$$

$$k_4 = h f(x_0 + h, y_0 + k_3) = (0.2) f(0.2, 1.16825)$$

$$k_4 = (0.2)\left[3 \times 0.2 + \frac{1.16825}{2}\right] = 0.236825$$

We have, $y(x_0 + h) = y_0 + \dfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$\therefore \quad y(0.2) = 1 + \frac{1}{6}(0.1 + 2 \times 0.165 + 2 \times 0.16825 + 0.236825)$$

Thus, $\boxed{y(0.2) = 1.1672208 \approx 1.1672}$

We shall find the analytical solution of the given equation by writing in the form $\dfrac{dy}{dx} + Py = Q$ whose solution is $y e^{\int P\,dx} = \int Q e^{\int P\,dx} dx + c$

We have, $\frac{dy}{dx} - \frac{y}{2} = 3x$. Here $P = \frac{-1}{2}$, $Q = 3x$; $e^{\int P\,dx} = e^{-x/2}$

The solution becomes, $y\,e^{-x/2} = 3\int x\,e^{-x/2}\,dx + c.$

Integrating RHS by parts we have,

$$y\,e^{-x/2} = 3\left[x\,e^{-x/2}(-2) - \int e^{-x/2}(-2)\cdot 1\,dx\right] + c$$

$$y\,e^{-x/2} = 3[-2x\,e^{-x/2} - 4\,e^{-x/2}] + c$$

Multiplying with $e^{x/2}$ we obtain, $y = -6x - 12 + c\,e^{x/2}$

Applying the initial condition that $y = 1$ at $x = 0$ the solution becomes,

$$1 = 0 - 12 + c \quad \therefore \quad c = 13$$

The analytical solution of the initial value problem is given by,

$$y = -6x - 12 + 13\,e^{x/2}$$

Now by putting $x = 0.2$ we have,

$$y(0.2) = -6(0.2) - 12 + 13\,e^{0.1} = 1.1672219$$

Thus, $\boxed{y(0.2) = 1.1672219 \approx 1.1672}$ by analytical method.

[18] *Use fourth order Runge-Kutta method to solve,* $(x+y)\frac{dy}{dx} = 1, y(0.4) = 1$ *at* $x = 0.5$ *correct to four decimal places.*

☞ We have, $\frac{dy}{dx} = \frac{1}{x+y}$ and $y = 1$ at $x = 0.4$

$$f(x,y) = \frac{1}{x+y},\ x_0 = 0.4,\ y_0 = 1,\ y(0.5) = ?$$

Here, $x_0 + h = 0.5 \quad \therefore\ h = 0.5 - x_0 = 0.5 - 0.4 = 0.1$

We shall first compute k_1, k_2, k_3, k_4.

$$k_1 = h\,f(x_0, y_0) = (0.1)\,f(0.4, 1) = (0.1)\left[\frac{1}{0.4+1}\right] = 0.0714$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.1)\,f\left(0.4 + \frac{0.1}{2}, 1 + \frac{0.0714}{2}\right)$$

$$k_2 = (0.1)\,f(0.45, 1.0357) = (0.1)\left[\frac{1}{0.45 + 1.0357}\right] = 0.0673$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.1)\,f\,(0.45, 1.03365)$$

$$k_3 = (0.1)\left[\frac{1}{0.45 + 1.03365}\right] = 0.0674$$

$$k_4 = h\,f(x_0 + h,\ y_0 + k_3) = (0.1)\,f(0.5, 1.0674)$$

$$k_4 = (0.1)\left[\frac{1}{0.5 + 1.0674}\right] = 0.0638$$

We have, $\quad y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$\therefore \quad y(0.5) = 1 + \frac{1}{6}[\,0.0714 + 2(0.0673) + 2(0.0674) + 0.0638]$$

Thus, $\quad \boxed{y(0.5) = 1.0674}$

[19] *Using Runge-Kutta method of fourth order, find* $y\ (\ 0.2\ )$ *for the equation* $\frac{dy}{dx} = \frac{y - x}{y + x}$, $y(0) = 1$ *taking* $h = 0.2$. **[Dec 2017, June 18]**

☞ By data, $f(x, y) = \frac{y - x}{y + x}, x_0 = 0,\ y_0 = 1,\ h = 0.2$

We shall first compute k_1, k_2, k_3, k_4.

$$k_1 = h\,f(x_0, y_0) = (0.2)\,f(0,1) = (0.2)\left[\frac{1-0}{1+0}\right] = 0.2$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.2)\,f(0.1, 1.1)$$

$$k_2 = (0.2)\left[\frac{1.1-0.1}{1.1+0.1}\right] = 0.1667$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.2)\,f\,(0.1, 1.0835)$$

$$k_3 = (0.2)\left[\frac{1.0835-0.1}{1.0835+0.1}\right] = 0.1662$$

$$k_4 = h\,f\,(x_0 + h\,,\, y_0 + k_3) = (0.2)\,f\,(0.2, 1.1662)$$

$$k_4 = (0.2)\left[\frac{1.1662-0.2}{1.1662+0.2}\right] = 0.1414$$

We have, $y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$\therefore \quad y(0.2) = 1 + \frac{1}{6}[0.2 + 2(0.1667) + 2(0.1662) + 0.1414]$$

Thus, $\boxed{y(0.2) = 1.1679}$

[20] *Use fourth order Runge-Kutta method to find y at x = 0.1 given that*

$$\frac{dy}{dx} = 3e^x + 2y\,,\, y(0) = 0 \quad \textit{and } h = 0.1.$$

☞ By data, $f(x, y) = 3e^x + 2y,\ x_0 = 0, y_0 = 0,\ h = 0.1$

We shall first compute k_1, k_2, k_3, k_4.

$$k_1 = h\,f\,(x_0, y_0) = (0.1)\,f\,(0,0) = (0.1)[3e^0 + 2\times 0] = 0.3$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.1)\,f\,(0.05, 0.15)$$

$$k_2 = (0.1)\left[3e^{0.05} + 2(0.15)\right] = 0.3454$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.1)\,f\,(0.05, 0.1727)$$

$$k_3 = (0.1)\left[3e^{0.05} + 2(0.1727)\right] = 0.3499$$

$$k_4 = h\,f(x_0 + h,\ y_0 + k_3) = (0.1)\,f(0.1, 0.3499)$$

$$k_4 = (0.1)\left[3e^{0.1} + 2(0.3499)\right] = 0.4015$$

We have, $y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$\therefore \quad y(0.1) = 0 + \frac{1}{6}[0.3 + 2(0.3454) + 2(0.3499) + 0.4015]$

Thus, $\boxed{y(0.1) = 0.3487}$

[21] *Use fourth order Runge-Kutta method to compute* $y(1.1)$ *given that*

$\frac{dy}{dx} = xy^{1/3}$, $y(1) = 1$.

☞ By data, $f(x, y) = xy^{1/3}$, $x_0 = 1$, $y_0 = 1$

We need to compute $y(1.1)$. This implies that $x_0 + h = 1.1 \quad \therefore h = 0.1$

We shall first compute k_1, k_2, k_3, k_4.

$$k_1 = h\,f(x_0, y_0)$$

$$k_1 = (0.1)\,f(1,1) = (0.1)[(1)(1)^{1/3}] = 0.1$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.1)\,f(1.05, 1.05)$$

$$k_2 = (0.1)[(1.05)(1.05)^{1/3}] = 0.1067$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.1)\,f(1.05, 1.05335)$$

$$k_3 = (0.1)[(1.05)(1.05335)^{1/3}] = 0.1068$$

$$k_4 = h\,f(x_0 + h,\ y_0 + k_3) = (0.1)\,f(1.1, 1.1068)$$

$$k_4 = (0.1)[(1.1)(1.1068)^{1/3}] = 0.1138$$

We have, $y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$\therefore \quad y(1.1) = 1 + \frac{1}{6}[0.1 + 2(0.1067) + 2(0.1068) + 0.1138]$$

Thus, $\boxed{y(1.1) = 1.1068}$

[22] *Using Runge-Kutta method of fourth order solve* $\dfrac{dy}{dx} + y = 2x$ *at* $x = 1.1$ *given that* $y = 3$ *at* $x = 1$ *initially.*

☞ We have, $\dfrac{dy}{dx} = 2x - y,\ x_0 = 1,\ y_0 = 3$

$$f(x,y) = 2x - y,\ x_0 + h = 1.1 \quad \therefore h = 0.1$$

We shall first compute k_1, k_2, k_3, k_4.

$$k_1 = h\,f(x_0, y_0) = (0.1)\,f(1,3) = (0.1)[2(1) - 3] = -0.1$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.1)\,f\left(1.05, 3 - \frac{0.1}{2}\right)$$

$$k_2 = (0.1)\,f(1.05, 2.95) = (0.1)[2(1.05) - 2.95] = -0.085$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.1)\,f\left(1.05, 3 - \frac{0.085}{2}\right)$$

$$k_3 = (0.1)\,f(1.05, 2.9575) = (0.1)[2(1.05) - 2.9575] = -0.08575$$

$$k_4 = h\,f(x_0 + h,\ y_0 + k_3) = (0.1)\,f(1.1, 3 - 0.08575)$$

$$k_4 = (0.1)\,f(1.1, 2.91425) = (0.1)[2(1.1) - 2.91425]$$

$$k_4 = -0.071425$$

We have, $y(x_0 + h) = y_0 + \dfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$\therefore \quad y(1.1) = 3 + \frac{1}{6}[-0.1 + 2(-0.085) + 2(-0.08575) - 0.071425]$$

Thus, $\boxed{y(1.1) = 2.9145125 \approx 2.9145}$

[23] (a) *Using Runge-Kutta method of fourth order find y (0.2) for the equation* $\frac{dy}{dx} = \frac{y-x}{y+x}$, $y(0) = 1$ *taking* $h = 0.1$

(b) *Compute* $y(0.1)$. **[Dec 2018]**

☞ The problem has to be done in two stages.

I Stage : $f(x,y) = \frac{y-x}{y+x}$, $x_0 = 0, y_0 = 1, h = 0.1$

We shall first compute k_1, k_2, k_3, k_4.

$$k_1 = h f(x_0, y_0) = (0.1) f(0,1) = (0.1)\left[\frac{1-0}{1+0}\right] = 0.1$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.1) f(0.05, 1.05)$$

$$k_2 = (0.1)\left[\frac{1.05-0.05}{1.05+0.05}\right] = 0.091$$

$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.1) f(0.05, 1.0455)$$

$$k_3 = (0.1)\left[\frac{1.0455-0.05}{1.0455+0.05}\right] = 0.0909$$

$$k_4 = h f(x_0 + h, y_0 + k_3) = (0.1) f(0.1, 1.0909)$$

$$k_4 = (0.1)\left[\frac{1.0909-0.1}{1.0909+0.1}\right] = 0.0832$$

We have, $y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$\therefore \quad y(0.1) = 1 + \frac{1}{6}(0.1 + 0.182 + 0.1818 + 0.0832)$

Thus, $\boxed{y(0.1) = 1.091167 \approx 1.0912}$

Note : *Solution of* (b) *ends here.*

II Stage : $f(x,y) = \dfrac{y-x}{y+x}$; $x_0 = 0.1,\ y_0 = 1.0912, h = 0.1$

Again by using the same formulae for k_1, k_2, k_3, k_4 we have,

$$k_1 = (0.1)\ f(0.1, 1.0912) = (0.1)\left[\frac{1.0912-0.1}{1.0912+0.1}\right] = 0.0832$$

$$k_2 = (0.1)\ f(0.15, 1.1328) = 0.0766$$

$$k_3 = (0.1)\ f(0.15, 1.1295) = 0.07655$$

$$k_4 = (0.1)\ f(0.2, 1.16775) = 0.07075$$

Now, $y(x_0 + h) = y_0 + \dfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$y(0.1+0.1) = 1.0912 + \frac{1}{6}(0.0832 + 0.1532 + 0.1531 + 0.07075)$$

Thus, $\boxed{y(0.2) = 1.167908 \approx 1.1679}$

Remark : *Referring to Problem-[19], the problem has been worked in one stage with $h = 0.2$ and we have obtained $y\ (0.2) = 1.1679$.*

[24] (a) *Solve :* $(y^2 - x^2)\,dx = (y^2 + x^2)\,dy$ *for* $x = 0.2$ *and* 0.4 *given that* $y = 1$ *at* $x = 0$ *initially, by applying Runge-Kutta method of order 4.*
(b) *Compute* $y\ (0.2)$ *by taking* $h = 0.2$. **[June 2017]**

☞ We have $\dfrac{dy}{dx} = \dfrac{y^2 - x^2}{y^2 + x^2}$; $x_0 = 0,\ y_0 = 1,\ h = 0.2$

I Stage : $f(x,y) = \dfrac{y^2 - x^2}{y^2 + x^2}$ We shall compute k_1, k_2, k_3, k_4.

$$k_1 = h\,f(x_0, y_0) = (0.2)\,f(0,1) = (0.2)1 = 0.2$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.2)\,f(0.1, 1.1) = 0.1967$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.2)\,f(0.1, 1.0984) = 0.1967$$

$$k_4 = h\,f(x_0 + h,\ y_0 + k_3) = (0.2)\,f(0.2, 1.1967) = 0.1891$$

We have, $y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

Thus, $\boxed{y(0.2) = 1.19598 \approx 1.196}$

Note : *Solution of* (b) *ends here.*

II Stage : $f(x, y) = \dfrac{y^2 - x^2}{y^2 + x^2}$, $x_0 = 0.2$, $y_0 = 1.196$, $h = 0.2$

Again using the same formula for k_1, k_2, k_3, k_4 we have,

$$k_1 = (0.2)\, f(0.2, 1.196) = 0.1891$$

$$k_2 = (0.2)\, f(0.3, 1.29055) = 0.1795$$

$$k_3 = (0.2)\, f(0.3, 1.28575) = 0.1793$$

$$k_4 = (0.2)\, f(0.4, 1.3753) = 0.1688$$

Now, $y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

Thus, $\boxed{y(0.4) = 1.37525 \approx 1.3753}$

ASSIGNMENT

Apply modified Euler's method to solve the following initial value problems by considering the accuracy upto two approximations in every step.

1. $\dfrac{dy}{dx} = -xy^2, y(0) = 2.$ Compute $y\,(0.2)$ by taking $h = 0.1$

2. $\dfrac{dy}{dx} + y = 1, y = 0$ at $x = 0$. Compute y at $x = 0.4$ by taking $h = 0.1$

3. $\dfrac{dy}{dx} = x^2 + y$ in the range $0 \le x \le 0.06$ by taking $h = 0.02$ given that $y = 1$ at $x = 0$ initially.

4. $\dfrac{dy}{dx} = x + y, y(0) = 1.$ Compute $y\,(0.2)$ by taking $h = 0.1$

5. $\dfrac{dy}{dx} + x^2 = y, y(0) = 1.$ Compute y in the range $[\,0, 0.6\,]$ by taking $h = 0.2$.

6. $\dfrac{dy}{dx} = 2 + \left|\sqrt{xy}\right|, y = 1$ at $x = 1$. Compute y at $x = 2$ by taking $h = 0.2$.

7. $\dfrac{dy}{dx} = x + \left|\sqrt{y}\right|, y(0) = 1$. Compute $y(0.6)$ by taking $h = 0.2$

Use fourth order Runge-Kutta method to solve the following initial value problems.

8. $\dfrac{dy}{dx} = x + y, y(0) = 1$. Compute $y(0.2)$

9. $\dfrac{dy}{dx} = x + y^2, y(0) = 1$. Compute $y(0.2)$ by taking $h = 0.1$

10. $\dfrac{dy}{dx} = 3x + \dfrac{y}{2}, y(0) = 1$. Compute $y(0.2)$ with $h = 0.1$

11. $10\dfrac{dy}{dx} = x^2 + y^2, y(0) = 1$ for the interval $0 \le x \le 0.2$ by taking $h = 0.1$.

12. $\dfrac{dy}{dx} = xy, y(1) = 2$. Compute $y(1.2)$ by taking $h = 0.2$.

13. $\dfrac{dy}{dx} = 1 + y^2, x_0 = 0, y_0 = 0$. Compute the values of y corresponding to the values of x in the range $0 \le x \le 0.6$ by taking $h = 0.2$.

14. $\dfrac{dy}{dx} = x(1 + xy), y(0) = 1$ in the range $0 \le x \le 0.2$ by taking $h = 0.1$.

15. $\dfrac{dy}{dx} = \log_{10}(x/y)$ in the range $20 \le x \le 20.4$ by taking $h = 0.2$ given that $y = 5$ when $x = 20$.

ANSWERS

1. 1.923	**2.** 0.3292	**3.** 1.0202, 1.0408, 1.0619
4. 1.2427	**5.** 1.218, 1.467, 1.737	**6.** 5.051
7. 1.8851	**8.** 1.2428	**9.** 1.2736
10. 1.1678	**11.** 1.0101, 1.0205	**12.** 2.4921
13. 0.2027, 0.4228, 0.6841	**14.** 1.0053, 1.0227	**15.** 5.12, 5.24

4.2 Numerical Predictor and Corrector methods

In these methods the vlaue of y at a desired value of x is estimated from a set of four values of y corresponding to four equally spaced values of x. The four values may be readily available or be generated using the given initial condition by any numerical method discussed earlier. Taylor's series method would be appropriate to generate three more values of y given one value initially.

We discuss two predictor & corrector methods namely,

1. Milne's method 2. Adams - Bashforth method

Consider the differential equation $y' = \dfrac{dy}{dx} = f(x,y)$ with a set of four pre determined values of $y: y(x_0) = y_0, y(x_1) = y_1, y(x_2) = y_2$ and $y(x_3) = y_3$.

Here x_0, x_1, x_2, x_3 are equally spaced values of x with width h.

Also $x_4 = x_3 + h = x_0 + 4h$

Predictor and Corrector formulae to compute $y(x_4) = y_4$ are as follows.

4.21 Milne's predictor and corrector formulae

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3')\ldots \qquad \text{(Predictor formula)}$$

$$y_4^{(C)} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4')\ldots \qquad \text{(Corrector formula)}$$

Note : *These two formulae can be written inthe genral form as follows :*

$$y_{n+1}^{(P)} = y_{n-3} + \frac{4h}{3}\left[2y_{n-2}' - y_{n-1}' + 2y_n'\right]$$

$$y_{n+1}^{(C)} = y_{n-1} + \frac{h}{3}\left[y_{n-1}' + 4y_n' + y_{n+1}'\right]$$

4.22 Adams - Bashforth predictor and corrector formulae

$$y_4^{(P)} = y_3 + \frac{h}{24}(55y_3' - 59y_2' + 37y_1' - 9y_0') \qquad \text{(Predictor formula)}$$

$$y_4^{(C)} = y_3 + \frac{h}{24}(9y_4' + 19y_3' - 5y_2' + y_1') \qquad \text{(Corrector formula)}$$

Note : *We can write down the general form of these two formulae also as in Milne's method.*

Working procedure for problems

- We first prepare the table showing the values of y corresponding to four equidistant values of x and the computation of $y' = f(x, y)$.
- We compute y_4 from the predictor formula.
- We use this value of y_4 to compute $y'_4 = f(x_4, y_4)$
- We apply corrector formula to obtain the corrected value of y_4.
- This value is used for computing y'_4 to apply the corrector formula again.
- The process is continued till we get consistency in two consecutive values of y_4.

Note : *We can also find $y_5, y_6 \cdots$ by deducing expressions from the general form of the predictor and corrector formulae.*

WORKED PROBLEMS

[23] *Given that $\dfrac{dy}{dx} = x - y^2$ and the data $y(0) = 0, y(0.2) = 0.02$, $y(0.4) = 0.0795, y(0.6) = 0.1762$. Compute y at $x = 0.8$ by applying*

(a) Milne's method **[June 2018]** *(b) Adams - Bashforth method.*

☞ We prepare the following table using the given data which is essentially required for applying the predictor and corrector formulae.

x	y	$y' = x - y^2$
$x_0 = 0$	$y_0 = 0$	$y'_0 = 0 - 0^2 = 0$
$x_1 = 0.2$	$y_1 = 0.02$	$y'_1 = 0.2 - (0.02)^2 = 0.1996$
$x_2 = 0.4$	$y_2 = 0.0795$	$y'_2 = 0.4 - (0.0795)^2 = 0.3937$
$x_3 = 0.6$	$y_3 = 0.1762$	$y'_3 = 0.6 - (0.1762)^2 = 0.5689$
$x_4 = 0.8$	$y_4 = ?$	

(a) By Milne's method

We have the predictor formula

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2y'_1 - y'_2 + 2y'_3)$$

$$\therefore \quad y_4^{(P)} = 0 + \frac{4(0.2)}{3}[2(0.1996) - 0.3937 + 2(0.5689)] = 0.3049$$

Now, $y'_4 = x_4 - y_4^2 = 0.8 - (0.3049)^2 = 0.707$

Next we have the corrector formula,

$$y_4^{(C)} = y_2 + \frac{h}{3}(y'_2 + 4y'_3 + y'_4)$$

$$\therefore \quad y_4^{(c)} = 0.0795 + \frac{0.2}{3}[0.3937 + 4(0.5689) + 0.707] = 0.3046$$

Now, $y'_4 = x_4 - y_4^2 = 0.8 - (0.3046)^2 = 0.7072$

Substituting this value of y'_4 again in the corrector formula,

$$y_4^{(C)} = 0.0795 + \frac{0.2}{3}[0.3937 + 4(0.5689) + 0.7072] = 0.3046$$

Thus, $\boxed{y_4 = y(0.8) = 0.3046}$

(b) By Adams - Bashforth method

We have the predictor formula

$$y_4^{(P)} = y_3 + \frac{h}{24}(55y'_3 - 59y'_2 + 37y'_1 - 9y'_0)$$

$$\therefore \quad y_4^{(P)} = 0.1762 + \frac{0.2}{24}[55(0.5689) - 59(0.3937) + 37(0.1996) - 9(0)]$$

$$y_4^{(P)} = 0.3049$$

Now, $y'_4 = x_4 - y_4^2 = 0.8 - (0.3049)^2 = 0.707$

Next, we have the corrector formula,

$$y_4^{(C)} = y_3 + \frac{h}{24}(9y'_4 + 19y'_3 - 5y'_2 + y'_1)$$

$$\therefore \quad y_4^{(C)} = 0.1762 + \frac{0.2}{24}[9(0.707) + 19(0.5689) - 5(0.3937) + 0.1996]$$

$$y_4^{(C)} = 0.3046$$

Now, $y'_4 = x_4 - y_4^2 = 0.8 - (0.3046)^2 = 0.7072$

Applying the corrector formulae again with only change in the value of y'_4 we obtain,

$$y_4^{(C)} = 0.3046$$

Thus, $\boxed{y_4 = y(0.8) = 0.3046}$

[26] *Apply Milne's method to compute y (1.4) corrector to four decimal places given* $\dfrac{dy}{dx} = x^2 + \dfrac{y}{2}$ *and following data :*

$$y(1) = 2,\ y(1.1) = 2.2156,\ y(1.2) = 2.4649,\ y(1.3) = 2.7514$$

☞ First we shall prepare the following table.

x	y	$y' = x^2 + \dfrac{y}{2}$
$x_0 = 1$	$y_0 = 2$	$y'_0 = 1^2 + \dfrac{2}{2} = 2$
$x_1 = 1.1$	$y_1 = 2.2156$	$y'_1 = (1.1)^2 + \dfrac{2.2156}{2} = 2.3178$
$x_2 = 1.2$	$y_2 = 2.4649$	$y'_2 = (1.2)^2 + \dfrac{2.4649}{2} = 2.67245$
$x_3 = 1.3$	$y_3 = 2.7514$	$y'_3 = (1.3)^2 + \dfrac{2.7514}{2} = 3.0657$
$x_4 = 1.4$	$y_4 = ?$	

We have $y_4^{(P)} = y_0 + \dfrac{4h}{3}(2y'_1 - y'_2 + 2y'_3)$

$$\therefore \quad y_4^{(P)} = 2 + \frac{4(0.1)}{3}[2(2.3178) - 2.67245 + 2(3.0657)] = 3.0793$$

Hence, $y'_4 = x_4^2 + \dfrac{y_4}{2} = (1.4)^2 + \dfrac{3.0793}{2} = 3.49965$

Now consider, $y_4^{(C)} = y_2 + \dfrac{h}{3}(y'_2 + 4y'_3 + y'_4)$

$$\therefore \quad y_4^{(C)} = 2.4649 + \frac{0.1}{3}[2.67245 + 4(3.0657) + 3.49965] = 3.0794$$

Now, $y'_4 = (1.4)^2 + \dfrac{3.0794}{2} = 3.4997$

Substituting this value of y'_4 again in the corrector formula we obtain

$y_4^{(c)} = 3.0794$

Thus, $\boxed{y_4 = y(1.4) = 3.0794}$

[27] *Use Taylor's series method (upto third derivative term) to find y at $x = 0.1, 0.2, 0.3$ given that $\dfrac{dy}{dx} = x^2 + y^2$ with $y(0) = 1$.*

Apply Milne's predictor-corrector formulae to find $y(0.4)$ using the generated set of initial values.

☞ Referring to Problem - [1], we have obtained $y(0.1) = 1.1113$, $y(0.2) = 1.2507$, $y(0.3) = 1.426$. Using these values along with $y(0) = 1$ initially, we prepare the following table.

x	y	$y' = x^2 + y^2$
$x_0 = 0$	$y_0 = 1$	$y'_0 = 0^2 + 1^2 = 1$
$x_1 = 0.1$	$y_1 = 1.1113$	$y'_1 = (0.1)^2 + (1.1113)^2 = 1.245$
$x_2 = 0.2$	$y_2 = 1.2507$	$y'_2 = (0.2)^2 + (1.2507)^2 = 1.6043$
$x_3 = 0.3$	$y_3 = 1.426$	$y'_3 = (0.3)^2 + (1.426)^2 = 2.1235$
$x_4 = 0.4$	$y_4 = ?$	

Consider, $y_4^{(P)} = y_0 + \dfrac{4h}{3}(2y'_1 - y'_2 + 2y'_3)$

$$\therefore \quad y_4^{(P)} = 1 + \frac{4(0.1)}{3}[2(1.245) - 1.6043 + 2(2.1235)] = 1.6844$$

Hence, $y'_4 = x_4^2 + y_4^2 = (0.4)^2 + (1.6844)^2 = 2.9972$

Next we have, $y_4^{(C)} = y_2 + \dfrac{h}{3}(y'_2 + 4y'_3 + y'_4)$

$$\therefore \quad y_4^{(C)} = 1.2507 + \frac{0.1}{3}[1.6043 + 4(2.1235) + 2.9972] = 1.6872$$

Now, $y'_4 = (0.4)^2 + (1.6872)^2 = 3.0067$

Substituting this value of y'_4 in the corrector formula again,

$$y_4^{(C)} = 1.2507 + \frac{0.1}{3}[1.6043 + 4(2.1235) + 3.0067] = 1.6875$$

Now, $y_4' = (0.4)^2 + (1.6875)^2 = 3.0077$

Substituting again in the corrector formula we obtain $y_4^{(C)} = 1.6876$

Now $y_4' = (0.4)^2 + (1.6876)^2 = 3.008$

Substituting again in the corrector formula we obtain

$$y_4^{(C)} = 1.6875733 \approx 1.6876$$

Thus, $\boxed{y(0.4) = 1.6876}$

[28] *The following table gives the solution of* $5xy' + y^2 - 2 = 0$. *Find the value of y at* $x = 4.5$ *using Milne's predictor and corrector formulae. Use the corrector formula twice.*

x	4	4.1	4.2	4.3	4.4
y	1	1.0049	1.0097	1.0143	1.0187

[June 2017]

☞ By data $5xy' + y^2 - 2 = 0$ or $y' = \dfrac{2 - y^2}{5x}$

We prepare the following table.

x	y	$y' = \dfrac{2-y^2}{5x}$
$x_0 = 4$	$y_0 = 1$	$y_0' = \dfrac{2-1^2}{5\times 4} = 0.05$
$x_1 = 4.1$	$y_1 = 1.0049$	$y_1' = \dfrac{2-(1.0049)^2}{5\times 4.1} = 0.0483$
$x_2 = 4.2$	$y_2 = 1.0097$	$y_2' = \dfrac{2-(1.0097)^2}{5\times 4.2} = 0.0467$
$x_3 = 4.3$	$y_3 = 1.0143$	$y_3' = \dfrac{2-(1.0143)^2}{5\times 4.3} = 0.0452$
$x_4 = 4.4$	$y_4 = 1.0187$	$y_4' = \dfrac{2-(1.0187)^2}{5\times 4.4} = 0.0437$
$x_5 = 4.5$	$y_5 = ?$	

We have Milne's predictor and corrector formulae in the standard form

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3');$$

$$y_4^{(C)} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4')$$

Since we require y_5, the equivalent form of these formulae are given by

$$y_5^{(P)} = y_1 + \frac{4h}{3}(2y_2' - y_3' + 2y_4');$$

$$y_5^{(C)} = y_3 + \frac{h}{3}(y_3' + 4y_4' + y_5')$$

Hence, $y_5^{(P)} = 1.0049 + \frac{4(0.1)}{3}[2(0.0467) - 0.0452 + 2(0.0437)] = 1.023$

Now, $y_5' = \frac{2 - y_5^2}{5x_5} = \frac{2 - (1.023)^2}{5 \times 4.5} = 0.0424$

Hence, $y_5^{(c)} = 1.0143 + \frac{0.1}{3}[0.0452 + 4(0.0437) + 0.0424] = 1.023$

Thus, $\boxed{y(4.5) = 1.023}$

Remark : *Though we had a set of five pre determined values of y, we used only a set of four values to determine the fifth value in the sequence.*

[29] *If* $\frac{dy}{dx} = 2e^x - y$, $y(0) = 2$, $y(0.1) = 2.010$, $y(0.2) = 2.040$ *and* $y(0.3) = 2.090$, *find* $y(0.4)$ *correct to four decimal places by using*

(*a*) *Milne's predictor-corrector method.* **[Dec 2017]**

(*b*) *Adams - Bashforth predictor - corrector method.*
(*Apply the corrector formula twice*)

☞ We prepare the following table.

x	y	$y' = 2e^x - y$
$x_0 = 0$	$y_0 = 2$	$y'_0 = 2e^0 - 2 = 0$
$x_1 = 0.1$	$y_1 = 2.010$	$y'_1 = 2e^{0.1} - 2.01 = 0.2003$
$x_2 = 0.2$	$y_2 = 2.040$	$y'_2 = 2e^{0.2} - 2.04 = 0.4028$
$x_3 = 0.3$	$y_3 = 2.090$	$y'_3 = 2e^{0.3} - 2.09 = 0.6097$
$x_4 = 0.4$	$y_4 = ?$	

(a) By Milne's predictor - corrector method

We have Milne's predictor formula,

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2y'_1 - y'_2 + 2y'_3)$$

$$\therefore \quad y_4^{(P)} = 2 + \frac{4(0.1)}{3}[2(0.2003) - 0.4028 + 2(0.6097)] = 2.1623$$

Now $y'_4 = 2e^{0.4} - 2.1623 = 0.8213$

Next, we have Milne's corrector formula,

$$y_4^{(C)} = y_2 + \frac{h}{3}(y'_2 + 4y'_3 + y'_4)$$

$$\therefore \quad y_4^{(C)} = 2.04 + \frac{0.1}{3}[0.4028 + 4(0.6097) + 0.8213] = 2.1621$$

Now, $y'_4 = 2e^{0.4} - 2.1621 = 0.8215$

Applying the corrector formula again we have,

$$y_4^{(C)} = 2.04 + \frac{0.1}{3}[0.4028 + 4(0.6097) + 0.8215] = 2.1621$$

Thus, $\boxed{y(0.4) = 2.1621}$

(b) By Adams - Bashforth predictor - corrector method

We have, $y_4^{(P)} = y_3 + \frac{h}{24}(55y'_3 - 59y'_2 + 37y'_1 - 9y'_0)$

$$\therefore \quad y_4^{(P)} = 2.09 + \frac{0.1}{24}[55(0.6097) - 59(0.4028) + 37(0.2003) - 9(0)] = 2.1616$$

Now, $y'_4 = 2e^{0.4} - 2.1616 = 0.822$

Next we have, $y_4^{(C)} = y_3 + \frac{h}{24}(9y'_4 + 19y'_3 - 5y'_2 + y'_1)$

$$\therefore \quad y_4^{(C)} = 2.09 + \frac{0.1}{24}[9(0.822) + 19(0.6097) - 5(0.4028) + 0.2003] = 2.1615$$

Now, $y'_4 = 2e^{0.4} - 2.1615 = 0.82215$

Substituting again in the corrector formula, we obtain $y_4^{(C)} = 2.1615$

Thus, $\boxed{y(0.4) = 2.1615}$

[30] *Apply Adams - Bashforth method to solve the equation $(y^2 + 1)\,dy - x^2\,dx = 0$ at $x = 1$ given $y(0) = 1$, $y(0.25) = 1.0026$, $y(0.5) = 1.0206$, $y(0.75) = 1.0679$ Apply the corrector formula twice.*

☞ By data, $\frac{dy}{dx} = y' = \frac{x^2}{y^2 + 1}$ We prepare the following table.

x	y	$y' = \frac{x^2}{y^2+1}$
$x_0 = 0$	$y_0 = 1$	$y'_0 = \frac{0^2}{1^2+1} = 0$
$x_1 = 0.25$	$y_1 = 1.0026$	$y'_1 = \frac{(0.25)^2}{(1.0026)^2+1} = 0.0312$
$x_2 = 0.5$	$y_2 = 1.0206$	$y'_2 = \frac{(0.5)^2}{(1.0206)^2+1} = 0.1225$
$x_3 = 0.75$	$y_3 = 1.0679$	$y'_3 = \frac{(0.75)^2}{(1.0679)^2+1} = 0.2628$
$x_4 = 1$	$y_4 = ?$	

We have the predictor formula

$$y_4^{(P)} = y_3 + \frac{h}{24}(55y_3' - 59y_2' + 37y_1' - 9y_0')$$

$$\therefore y_4^{(P)} = 1.0679 + \frac{0.25}{24}[55(0.2628) - 59(0.1225) + 37(0.0312) - 9(0)] = 1.1552$$

Now, $y_4' = \dfrac{x_4^2}{y_4^2+1} = \dfrac{1^2}{(1.1552)^2+1} = 0.4284$

Next we have the corrector formula,

$$y_4^{(C)} = y_3 + \frac{h}{24}(9y_4' + 19y_3' - 5y_2' + y_1')$$

$$y_4^{(C)} = 1.0679 + \frac{0.25}{24}[9(0.4284) + 19(0.2628) - 5(0.1224) + 0.0312]$$

$$y_4^{(C)} = 1.154$$

Now, $y_4' = \dfrac{1^2}{(1.154)^2+1} = 0.4289$

Applying the corrector formula again we obtain, $y_4^{(C)} = 1.1541$

Thus the required, $\boxed{y(1) = 1.1541}$

[31] *Find the value of y at x = 4.4 by applying Adams - Bashforth method given that*

$5x\dfrac{dy}{dx} + y^2 - 2 = 0$ *and* $y = 1$ *at* $x = 4$ *initially by generating the other required values from the Taylor's polynomial.*

☞ We need to generate the value of y at $x = 4.1, 4.2, 4.3$

[*Refer Problem - [4] for the generation of the required values*]

We have obtained $y(4.1) = 1.0049$, $y(4.2) = 1.0097$, $y(4.3) = 1.0142$ by using the given initial condition $y(4) = 1$.

We prepare the following table.

x	y	$y' = \dfrac{2-y^2}{5x}$
$x_0 = 4$	$y_0 = 1$	$y'_0 = \dfrac{2-1^2}{5\times 4} = 0.05$
$x_1 = 4.1$	$y_1 = 1.0049$	$y'_1 = \dfrac{2-(1.0049)^2}{5\times 4.1} = 0.0483$
$x_2 = 4.2$	$y_2 = 1.0097$	$y'_2 = \dfrac{2-(1.0097)^2}{5\times 4.2} = 0.0467$
$x_3 = 4.3$	$y_3 = 1.0142$	$y'_3 = \dfrac{2-(1.0142)^2}{5\times 4.3} = 0.0452$
$x_4 = 4.4$	$y_4 = ?$	

We have the predictor formula,

$$y_4^{(P)} = y_3 + \frac{h}{24}(55y'_3 - 59y'_2 + 37y'_1 - 9y'_0)$$

$$y_4^{(P)} = 1.0142 + \frac{0.1}{24}[55(0.0452) - 59(0.0467) + 37(0.0483) - 9(0.05)]$$

$$\therefore \quad y_4^{(P)} = 1.0187$$

Now, $y'_4 = \dfrac{2-y_4^2}{5x_4} = \dfrac{2-(1.0187)^2}{5\times 4.4} = 0.0437$

Next we have, $y_4^{(C)} = y_3 + \dfrac{h}{24}(9y'_4 + 19y'_3 - 5y'_2 + y'_1)$

$$y_4^{(C)} = 1.0142 + \frac{0.1}{24}[9(0.0437) + 19(0.0452) - 5(0.0467) + 0.0483]$$

$$\therefore \quad y_4^{(C)} = 1.0186$$

Now $y_4' = \dfrac{2-(1.0186)^2}{5\times 4.4} = 0.0437$ (*Same value as earlier*)

Thus, $\boxed{y(4.4) = 1.0186}$

[32] *Solve the differential equation* $y' + y + xy^2 = 0$ *with the initial values of* $y : y_0 = 1$, $y_1 = 0.9008$, $y_2 = 0.8066$, $y_3 = 0.722$ *corresponding to the values of* $x : x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$ *by computing the value of y corresponding to* $x = 0.4$ *applying Adams - Bashforth predictor and corrector formula.* **[Dec 2017]**

☞ We prepare the following table.

x	y	$y' = -(y + xy^2)$
$x_0 = 0$	$y_0 = 1$	$y_0' = -1$
$x_1 = 0.1$	$y_1 = 0.9008$	$y_1' = -0.9819$
$x_2 = 0.2$	$y_2 = 0.8066$	$y_2' = -0.9367$
$x_3 = 0.3$	$y_3 = 0.722$	$y_3' = -0.8784$
$x_4 = 0.4$	$y_4 = ?$	

We have the predictor formula,

$$y_4^{(P)} = y_3 + \frac{h}{24}(55y_3' - 59y_2' + 37y_1' - 9y_0')$$

On substitution we obtain, $y_4^{(P)} = 0.6371$

Now, $y_4' = -[y_4 + x_4 y_4^2] = -0.7995$

Next we have, $y_4^{(C)} = y_3 + \dfrac{h}{24}(9y_4' + 19y_3' - 5y_2' + y_1')$

On substitution we obtain, $y_4^{(C)} = 0.6379$

Now, $y_4' = -[0.6379 + (0.4)(0.6379)^2] = -0.8007$

Applying the corrector formula again we obtain $y_4^{(C)} = 0.6379$

Thus, $\boxed{y(0.4) = 0.6379}$

ASSIGNMENT

***Applying Milne's method compute y at the specified value of x for the following.* [1 to 4]**

1. $\dfrac{dy}{dx} + xy^2 = 0$

x	0	0.2	0.4	0.6
y	2	1.9231	1.7241	1.4706

Compute $y(0.8)$

2. $\dfrac{dy}{dx} = 1 + y^2$; $y(0) = 0$. Compute $y(0.8)$ correct to four decimal places by generating the initial values from Taylor's polynomial of order 2.

3. $\dfrac{dy}{dx} = x + y^2$

x	0	0.1	0.2	0.3
y	1	1.1	1.231	1.402

Compute $y(0.4)$

4. $\dfrac{dy}{dx} = \dfrac{1}{2}(x+y)$, $y(0) = 2$, $y(0.5) = 2.636$, $y(1) = 3.595$, $y(1.5) = 4.968$. Compute y at $x = 2$ correct to three decimal places.

5. Use Taylor's series method to obtain the solution as a power series in x (upto the third derivative terms) given that $\frac{dy}{dx} + y^2 = x, y(0) = 0$. Using this generate the values of y corresponding to $x = 0.2, 0.4, 0.6$ correct to four decimal places. Then apply Milne's predictor- corrector formulae to compute y at $x = 0.8$ and at $x = 1$.

6. Using Adams - Bashforth method, find $y(1.4)$ given that $\frac{dy}{dx} = x^2 + \frac{y}{2}$ with $y(1) = 2$. Obtain the initial values of y at $x = 1.1, 1.2, 1.3$ by Taylor's series method of order 4.

7. Given $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$ and the data :

x	1	1.1	1.2	1.3
y	1	0.996	0.986	0.972

Compute $y(1.4)$ correct to 3 decimal places by applying Adams - Bashforth predictor and corrector formulae.

8. Solve $\frac{dy}{dx} + y = x^2$; $y(0) = 1$ by obtaining Taylor polynomial of order 4. Evaluate y at $x = 0.1, 0.2, 0.3$ and by using these values obtain y at $x = 0.4$ applying Adams - Bashforth predictor and corrector formulae.

ANSWERS

1. 1.22

2. 1.0234

3. 1.7003

4. 6.873

5. 0.3046, 0.4555

6. 3.0793

7. 0.949

8. 0.6897