DEPARTMENT OF MATHEMATICS

ORDINARY DIFFERENTIAL EQUATIONS AND NUMERICAL METHODS (BMATCS21)
SEMESTER – II

Module 1: Ordinary Differential Equations of First Order

Tutorial – 1

1. **Solve the following differential equations.**
   a. $x(1-\sin y)dy = (\cos x - \cos y - y)dx$
   b. $\left(1+e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy = 0$
   c. $(6x^2+4y^3+12y)dx + 3x(1+y^2)dy = 0$
   d. $(y\log y)dx + (x-\log y)dy = 0$

Tutorial – 2

1. **Solve the following differential equations.**
   a. $x\frac{dy}{dx} + y = x^3y^6$
   b. $\tan y\frac{dy}{dx} + \tan x = \cos y\cos^2 x$
   c. $\frac{dy}{dx} = (\sin x - \sin y)\left(\frac{\cos x}{\cos y}\right)$
   d. $\frac{dy}{dx} - \frac{1}{2}\left(1+\frac{1}{x}\right)y + \frac{3y^3}{x} = 0$

Tutorial – 3

1. **Determine the orthogonal trajectories of the following family of curves.**
   a. $y^2 = cx^3$, c is the parameter
   b. $r^n = a^n\cos n\theta$
2. Show that the family of curves $x^2 = 4a(y+a)$ is self orthogonal.
3. Test for self orthogonality $r^n = a\sin n\theta$.

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Module 1: Ordinary Differential Equations of First Order

Tutorial – 4

I. Solve the following differential equations.

1. $xyp^2 - (x^2 + y^2)p + xy = 0$
2. $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$

II. Determine the general and singular solutions of $(x^2 - 1)p^2 - 2xyp + y^2 - 1 = 0$.

III. Show that the singular solution of $(px - y)(x - py) = 2p$ found using the substitution $x^2 = u$ and $y^2 = v$ is $y^2 = \left(x - \sqrt{2}\right)^2$.

Module 2: Ordinary Differential Equations of Higher Order

Tutorial – 5

1. **In problems (a) – (e), the roots of the auxiliary equation of a homogeneous differential equation with constant coefficients are given. Determine the general solution of the differential equation.**
   (a) $2, 8, -14$
   (b) $0, \pm i19$
   (c) $5,5,5, -5, -5$
   (d) $-3 \pm i, -3 \pm i, 3 \pm i, 3 \pm i$
   (e) $0, 0, 2 \pm i9$
2. **Solve:** $y''' - 6y'' + 2y' + 36y = 0.$
3. **Solve:** $y^{(4)} - 9y'' + 20y = 0.$

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Module 2: Ordinary Differential Equations of Higher Order

Tutorial – 6

1. Solve: $(D^3 - D^2 + 4D - 4)y = \sinh(2x+3) + \sin 2x$.
2. Solve: $y''' + 2y'' + y' = x^3$.
3. Solve using the method of variation of parameter: $y'' + a^2 y = \sec ax$.
4. Solve: $xy'' - \frac{2y}{x} = x + \frac{1}{x^2}$.

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Module 3: Numerical Methods – 1

Tutorial – 7

1. From the following table:

$x°$:	10	20	30	40	50	60	70	80
$\cos x$:	0.9848	0.9397	0.8660	0.7660	0.6428	0.5000	0.3420	0.1737

Calculate $\cos 25°$ and $\cos 73°$ using the Gregory-1 Newton formula.

2. The following table gives the values of $\tan x$ for $0.10 \leq x \leq 0.30$. Find $\tan(0.26)$.

x	0.10	0.15	0.20	0.25	0.30
$\tan x$	0.1003	0.1511	0.2027	0.2553	0.3093

3. Given that $\sqrt{12} = 3.464, \sqrt{14} = 3.742, \sqrt{16} = 4, \sqrt{18} = 4.243, \sqrt{20} = 4.472$. Compute $\sqrt{16.5}$ by using Newton's forward interpolation formula.

4. Construct the interpolating polynomial for the data given below using Newton's general interpolation formula for divided differences.

x	2	4	5	6	8	10
y	10	96	196	350	868	1746

5. The observed values of a function are respectively $168, 120, 72$, and 63 at the four positions $3, 7, 9, 10$ of the independent variable. What is the best estimate you can give for the value of the function at the position 6 of the independent variable? (Use Lagrange interpolation method)

Tutorial - 8

June 5, 2025

Taylor Series Method for Solving ODEs

The Taylor series method for solving a first-order ordinary differential equation (ODE) of the form $y' = f(x, y)$ with initial condition $y(x_0) = y_0$ is given by:

$$y(x) = y(x_0) + (x - x_0)y'(x_0) + \frac{(x - x_0)^2}{2!}y''(x_0) + \frac{(x - x_0)^3}{3!}y'''(x_0) + \dots$$

For numerical approximation at $x_{n+1} = x_n + h$:

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2!}y''_n + \frac{h^3}{3!}y'''_n + \dots$$

—

Problem:

Consider the ODE: $y' = x + y$, with initial condition $y(0) = 1$. Use the Taylor series method up to the fourth-order term to find $y(0.1)$.

—

Solution:

Here, $x_0 = 0$, $y_0 = 1$, and $h = 0.1$.

Step 1: Find the derivatives of y with respect to x and evaluate them at the initial condition.

Given $y' = x + y$. At $x_0 = 0$, $y_0 = 1$: $y'(0) =$ _______________.

Now, let's find the higher-order derivatives:

1. $y'' = \frac{d}{dx}(x+y) =$ _________________________. Substitute y': $y'' =$ _________________________.
   At $x_0 = 0$, $y_0 = 1$: $y''(0) =$ _______________.

2. $y''' = \frac{d}{dx}(y'') =$ _________________________. Substitute y': $y''' =$ _________________________.
   At $x_0 = 0$, $y_0 = 1$: $y'''(0) =$ _______________.

3. $y^{(4)} = \frac{d}{dx}(y''') =$ _________________________. Substitute y': $y^{(4)} =$ _________________________.
   At $x_0 = 0$, $y_0 = 1$: $y^{(4)}(0) =$ _______________.

Step 2: Substitute these values into the Taylor series expansion.

The Taylor series expansion around $x_0 = 0$ is: $y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{(4)}(0) + \dots$

Substitute the calculated values: $y(x) = 1 + x($__________$) + \frac{x^2}{2!}($__________$) + \frac{x^3}{3!}($__________$) + \frac{x^4}{4!}($__________$) + \dots$ $y(x) = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \dots$

Step 3: Evaluate $y(0.1)$.

To find $y(0.1)$, substitute $x = 0.1$ into the series: $y(0.1) = 1 + (0.1) + (0.1)^2 + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{12}$
$y(0.1) = 1 + 0.1 + 0.01 + \frac{0.001}{3} + \frac{0.0001}{12}$ $y(0.1) \approx$ ____________________

Problem:

Consider the ODE: $5xy' + y^2 - 2 = 0$, with initial condition $y(4) = 1$. Use the Taylor series method up to the second-order term to find y at $x = 4.1$, $x = 4.2$, and $x = 4.3$.

Solution:

First, rearrange the ODE to express y': $5xy' = 2 - y^2$ $y' = \frac{2-y^2}{5x}$

Here, the initial values are $x_0 = 4$, $y_0 = 1$. The step size is $h = 0.1$.

Step 1: Calculate $y(4.1)$

We need to find $y'(x_0)$ and $y''(x_0)$ at $(x_0, y_0) = (4, 1)$.

1. Find $y'(x_0)$: $y' = \frac{2-y^2}{5x}$ At $(4, 1)$: $y'(4) =$ ____________________.
2. Find $y''(x_0)$: $y'' = \frac{d}{dx}\left(\frac{2-y^2}{5x}\right) =$ ____. (Hint: Use the quotient rule $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v-uv'}{v^2}$ and remember that y is a function of x, so $\frac{d}{dx}(y^2) = 2yy'$.) Substitute y' in the expression for y'': $y'' =$ ____________________. At $(4, 1)$ with $y'(4)$ calculated above: $y''(4) =$ ____________________.
3. Apply Taylor series for $y(4.1)$: $y(4.1) \approx y(4) + hy'(4) + \frac{h^2}{2!}y''(4)$ $y(4.1) \approx 1 + (0.1)($__________$) + \frac{(0.1)^2}{2}($__________$)$ $y(4.1) \approx$ ____________________.

—

Step 2: Calculate $y(4.2)$

Now, we use $x_1 = 4.1$ and $y_1 = y(4.1)$ (the value found in Step 1) as our new initial values. The step size remains $h = 0.1$.

1. Find $y'(x_1)$: $y'(x_1) = \frac{2-y_1^2}{5x_1}$ At $(4.1, y(4.1))$: $y'(4.1) =$ ____________________.

2. Find $y''(x_1)$: Using the general expression for y'' derived in Step 1: $y''(x_1) =$ ____________________.
   At $(4.1, y(4.1))$ with $y'(4.1)$ calculated above: $y''(4.1) =$ __________.

3. Apply Taylor series for $y(4.2)$: $y(4.2) \approx y(4.1)+hy'(4.1)+\frac{h^2}{2!}y''(4.1)$ $y(4.2) \approx$ __________ $+$ $(0.1)($__________$) + \frac{(0.1)^2}{2}($__________$)$ $y(4.2) \approx$ ____________________.

—

Step 3: Calculate $y(4.3)$

Finally, we use $x_2 = 4.2$ and $y_2 = y(4.2)$ (the value found in Step 2) as our new initial values. The step size remains $h = 0.1$.

1. Find $y'(x_2)$: $y'(x_2) = \frac{2-y_2^2}{5x_2}$ At $(4.2, y(4.2))$: $y'(4.2) =$ ____________________.

2. Find $y''(x_2)$: Using the general expression for y'' derived in Step 1: $y''(x_2) =$ ____________________.
   At $(4.2, y(4.2))$ with $y'(4.2)$ calculated above: $y''(4.2) =$ __________.

3. Apply Taylor series for $y(4.3)$: $y(4.3) \approx y(4.2)+hy'(4.2)+\frac{h^2}{2!}y''(4.2)$ $y(4.3) \approx$ __________ $+$ $(0.1)($__________$) + \frac{(0.1)^2}{2}($__________$)$ $y(4.3) \approx$ ____________________.

Modified Euler's Method for Solving ODEs

The Modified Euler's method (also known as Heun's method) is a second-order Runge-Kutta method for approximating the solution of an ordinary differential equation (ODE) of the form $y' = f(x, y)$ with initial condition $y(x_n) = y_n$. The steps are:

1. **Predictor Step (Euler's formula):** $y^*_{n+1} = y_n + hf(x_n, y_n)$

2. **Corrector Step (Average of slopes):** $y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})]$

—

Problem:

Consider the ODE: $\frac{dy}{dx} = 1 + \frac{y}{x}$, with initial condition $y(1) = 2$. Find $y(1.4)$ using the Modified Euler's method, taking a step size of $h = 0.2$. Compare the result with the exact solution.

Solution:

Here, $f(x, y) = 1 + \frac{y}{x}$. Initial values: $x_0 = 1$, $y_0 = 2$. Step size: $h = 0.2$. We need to find $y(1.4)$, which requires two steps since $1.4 = 1 + 2 \times 0.2$.

Step 1: Calculate $y(1.2)$ from $(x_0, y_0) = (1, 2)$

1. Evaluate $f(x_0, y_0)$: $f(1, 2) = 1 + \frac{2}{1} =$ __________.
2. **Predictor Step:** Calculate y_1^* for $x_1 = 1.2$. $y_1^* = y_0 + hf(x_0, y_0)$ $y_1^* = 2 + (0.2)($__________$)$ $y_1^* =$ ____________________.
3. Evaluate $f(x_1, y_1^*)$: $x_1 = 1.2$, $y_1^* =$ __________ (from predictor step) $f(1.2, y_1^*) = 1 + \frac{\_\_\_\_\_\_\_\_\_\_}{1.2} =$ ____________________.
4. **Corrector Step:** Calculate y_1. $y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^*)]$ $y_1 = 2 + \frac{0.2}{2}[$__________ $+$ __________$]$ $y_1 = 2 + 0.1[$__________$]$ $y_1 =$ ____________________. So, $y(1.2) \approx$ ____________________.

—

Step 2: Calculate $y(1.4)$ from $(x_1, y_1) = (1.2, y(1.2))$

Now, $x_1 = 1.2$, $y_1 = y(1.2)$ (the value found in Step 1). We want to find y_2 at $x_2 = 1.4$.

1. Evaluate $f(x_1, y_1)$: $f(1.2,$ __________$) = 1 + \frac{__________}{1.2} =$ ____________________.
2. **Predictor Step:** Calculate y_2^* for $x_2 = 1.4$. $y_2^* = y_1 + hf(x_1, y_1)$ $y_2^* =$ __________ $+ (0.2)($__________$)$ $y_2^* =$ ____________________.
3. Evaluate $f(x_2, y_2^*)$: $x_2 = 1.4$, $y_2^* =$ __________ (from predictor step) $f(1.4, y_2^*) = 1 + \frac{\_\_\_\_\_\_\_\_\_\_}{1.4} =$ ____________________.
4. **Corrector Step:** Calculate y_2. $y_2 = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^*)]$ $y_2 =$ __________ $+ \frac{0.2}{2}[$__________ $+$ __________$]$ $y_2 =$ __________ $+ 0.1[$__________$]$ $y_2 =$ ____________________. So, $y(1.4) \approx$ ____________________.

—

Comparison with Exact Solution (Using Integrating Factor Method)

The given ODE is $\frac{dy}{dx} = 1 + \frac{y}{x}$. This is a first-order linear ODE. First, rewrite it in the standard form $\frac{dy}{dx} + P(x)y = Q(x)$: $\frac{dy}{dx} - \frac{1}{x}y = 1$

Here, $P(x) =$ __________ and $Q(x) =$ __________.

The Integrating Factor (IF) is given by $IF = e^{\int P(x)dx}$. Calculate $\int P(x)dx$: $\int$ __________ $dx =$ ____________________ $= \ln($__________$)$ Therefore, $IF = e^{\ln(\_\_\_\_\_\_\_\_\_\_)} =$ __________. (Since $x_0 = 1$ and we are moving to $x = 1.4$, we consider $x > 0$, so $|x| = x$).

Multiply the standard form ODE by the Integrating Factor: __________ $\left(\frac{dy}{dx} - \frac{1}{x}y\right) =$ __________(1) The left side is the derivative of $(y \cdot \text{IF})$: $\frac{d}{dx}(y \cdot$ __________$) =$ __________

Integrate both sides with respect to x: $\int \frac{d}{dx}\left(\frac{y}{x}\right) dx = \int$ __________ dx $\frac{y}{x} =$ ____________________ $+ C$

Solve for y: $y(x) = x($____________________ $+ C)$

Now, use the initial condition $y(1) = 2$ to find the value of C: $2 = 1(\ln|1| + C)$ $2 = 1($__________ $+ C)$ $C =$ __________.

Therefore, the exact solution is $y(x) = x(\ln|x| +$ __________$)$.

Finally, evaluate the exact solution at $x = 1.4$: $y(1.4) = 1.4(\ln(1.4) +$ __________$)$ $y(1.4) \approx 1.4($__________ $+$ __________$)$ $y(1.4) \approx 1.4($__________$)$ $y(1.4) \approx$ ____________________.

Conclusion:

The approximate value of $y(1.4)$ using the Modified Euler's method is ____________________. The exact value of $y(1.4)$ is ____________________.

Problem:

Consider the ODE: $\frac{dy}{dx} = x + \sqrt{y}$, with initial condition $y(0) = 1$. Find $y(0.2)$ using the Modified Euler's method, taking a step size of $h = 0.2$.

Solution:

Here, $f(x, y) = x + \sqrt{y}$. Initial values: $x_0 = 0$, $y_0 = 1$. Step size: $h = 0.2$. We need to find $y(0.2)$, which requires one step.

Step 1: Calculate $y(0.2)$ from $(x_0, y_0) = (0, 1)$

1. Evaluate $f(x_0, y_0)$: $f(0, 1) = 0 + \sqrt{1} =$ _______________.

2. **Predictor Step:** Calculate y_1^* for $x_1 = 0.2$. $y_1^* = y_0 + hf(x_0, y_0)$ $y_1^* = 1+(0.2)(\_\_\_\_\_\_\_\_\_\_\_\_\_\_)$ $y_1^* =$ ________________________.

3. Evaluate $f(x_1, y_1^*)$: $x_1 = 0.2$, $y_1^* =$ ______________ (from predictor step) $f(0.2, y_1^*) = 0.2 + \sqrt{\_\_\_\_\_\_\_\_\_\_\_\_\_\_} =$ ________________________.

4. **Corrector Step:** Calculate y_1. $y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^*)]$ $y_1 = 1 + \frac{0.2}{2}[\_\_\_\_\_\_\_\_\_\_\_\_\_\_ + \_\_\_\_\_\_\_\_\_\_\_\_\_\_]$ $y_1 = 1 + 0.1[\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_]$ $y_1 =$ ________________________. So, $y(0.2) \approx$ ________________________.

Problem:

Consider the ODE: $\frac{dy}{dx} = \frac{y-x}{y+x}$, with initial condition $y(0) = 1$. Find $y(0.2)$ using the 4th-order Runge-Kutta method, taking a step size of $h = 0.2$.

Solution:

Here, $f(x, y) = \frac{y-x}{y+x}$. Initial values: $x_0 = 0$, $y_0 = 1$. Step size: $h = 0.2$. We need to find $y(0.2)$, which requires one step.

Step 1: Calculate the Runge-Kutta coefficients (k_1, k_2, k_3, k_4)

1. **Calculate k_1:** $k_1 = hf(x_0, y_0)$ $f(x_0, y_0) = f(0, 1) = \frac{1-0}{1+0} =$ __________. $k_1 = 0.2 \times$ __________ $=$ ______________.

2. **Calculate k_2:** $x_0 + h/2 = 0 + 0.2/2 =$ __________. $y_0 + k_1/2 = 1 +$ __________$/2 =$ ______________. $f(x_0+h/2, y_0+k_1/2) = f(\_\_\_\_\_\_\_\_\_\_, \_\_\_\_\_\_\_\_\_\_) = \frac{\_\_\_\_\_\_\_\_\_\_ - \_\_\_\_\_\_\_\_\_\_}{\_\_\_\_\_\_\_\_\_\_ + \_\_\_\_\_\_\_\_\_\_} =$ ______________. $k_2 = 0.2 \times$ ______________ $=$ ______________.

3. **Calculate k_3:** $x_0+h/2 =$ __________. $y_0+k_2/2 = 1+$__________$/2 =$ ______________. $f(x_0 + h/2, y_0 + k_2/2) = f(\_\_\_\_\_\_\_\_\_\_, \_\_\_\_\_\_\_\_\_\_) = \frac{\_\_\_\_\_\_\_\_\_\_ - \_\_\_\_\_\_\_\_\_\_}{\_\_\_\_\_\_\_\_\_\_ + \_\_\_\_\_\_\_\_\_\_} =$ ______________. $k_3 = 0.2 \times$ ______________ $=$ ______________.

4. **Calculate k_4:** $x_0 + h = 0 + 0.2 =$ __________. $y_0 + k_3 = 1 +$ __________ $=$ ______________. $f(x_0+h, y_0+k_3) = f(\_\_\_\_\_\_\_\_\_\_, \_\_\_\_\_\_\_\_\_\_) = \frac{\_\_\_\_\_\_\_\_\_\_ - \_\_\_\_\_\_\_\_\_\_}{\_\_\_\_\_\_\_\_\_\_ + \_\_\_\_\_\_\_\_\_\_} =$ ______________. $k_4 = 0.2 \times$ ______________ $=$ ______________.

Step 2: Calculate $y(0.2)$

Apply the RK4 formula to find $y(0.2)$: $y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ $y(0.2) = 1 + \frac{1}{6}(\_\_\_\_\_\_\_\_\_\_ + 2(\_\_\_\_\_\_\_\_\_\_) + 2(\_\_\_\_\_\_\_\_\_\_) + \_\_\_\_\_\_\_\_\_\_)$ $y(0.2) = 1 + \frac{1}{6}(\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_)$ $y(0.2) = 1 + \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$ $y(0.2) \approx \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$.

Problem:

Consider the ODE: $\frac{dy}{dx} = xy^{1/3}$, with initial condition $y(1) = 1$. Find $y(1.2)$ using the 4th-order Runge-Kutta method, taking a step size of $h = 0.2$.

Solution:

Here, $f(x, y) = xy^{1/3}$. Initial values: $x_0 = 1$, $y_0 = 1$. Step size: $h = 0.2$. We need to find $y(1.2)$, which requires one step.

Step 1: Calculate the Runge-Kutta coefficients (k_1, k_2, k_3, k_4)

1. **Calculate k_1:** $k_1 = hf(x_0, y_0)$ $f(x_0, y_0) = f(1, 1) = 1 \cdot (1)^{1/3} = \_\_\_\_\_\_\_\_\_\_$. $k_1 = 0.2 \times \_\_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$.

2. **Calculate k_2:** $x_0 + h/2 = 1 + 0.2/2 = \_\_\_\_\_\_\_\_\_\_$. $y_0 + k_1/2 = 1 + \_\_\_\_\_\_\_\_\_\_/2 = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$. $f(x_0 + h/2, y_0 + k_1/2) = f(\_\_\_\_\_\_\_\_\_\_, \_\_\_\_\_\_\_\_\_\_) = \_\_\_\_\_\_\_\_\_\_ \cdot (\_\_\_\_\_\_\_\_\_\_)^{1/3} \approx \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$. $k_2 = 0.2 \times \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$.

3. **Calculate k_3:** $x_0 + h/2 = \_\_\_\_\_\_\_\_\_\_$. $y_0 + k_2/2 = 1 + \_\_\_\_\_\_\_\_\_\_/2 = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$. $f(x_0 + h/2, y_0 + k_2/2) = f(\_\_\_\_\_\_\_\_\_\_, \_\_\_\_\_\_\_\_\_\_) = \_\_\_\_\_\_\_\_\_\_ \cdot (\_\_\_\_\_\_\_\_\_\_)^{1/3} \approx \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$. $k_3 = 0.2 \times \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$.

4. **Calculate k_4:** $x_0 + h = 1 + 0.2 = \_\_\_\_\_\_\_\_\_\_$. $y_0 + k_3 = 1 + \_\_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$. $f(x_0 + h, y_0 + k_3) = f(\_\_\_\_\_\_\_\_\_\_, \_\_\_\_\_\_\_\_\_\_) = \_\_\_\_\_\_\_\_\_\_ \cdot (\_\_\_\_\_\_\_\_\_\_)^{1/3} \approx \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$. $k_4 = 0.2 \times \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$.

Step 2: Calculate $y(1.2)$

Apply the RK4 formula to find $y(1.2)$: $y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ $y(1.2) = 1 + \frac{1}{6}(\_\_\_\_\_\_\_\_\_\_ + 2(\_\_\_\_\_\_\_\_\_\_) + 2(\_\_\_\_\_\_\_\_\_\_) + \_\_\_\_\_\_\_\_\_\_)$ $y(1.2) = 1 + \frac{1}{6}(\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_)$ $y(1.2) = 1 + \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$ $y(1.2) \approx \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$.

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DEPARTMENT OF MATHEMATICS
Ordinary Differential Equations and Numerical Methods (BMATS209)
Tutorial - 9

Abstract

This tutorial provides a comprehensive guide to numerically solving a system of two simultaneous first-order ordinary differential equations (ODEs) using the fourth-order Runge-Kutta (RK-4) method. We will cover the theoretical framework and provide a detailed step-by-step example, along with practice questions.

1 Introduction to Systems of Two First-Order ODEs

A system of two simultaneous first-order ODEs can be generally expressed as:

$$\frac{dy}{dx} = f(x, y, z)$$
$$\frac{dz}{dx} = g(x, y, z)$$

with initial conditions $y(x_0) = y_0$ and $z(x_0) = z_0$.

Here, y and z are the dependent variables, and x is the independent variable. Our objective is to find the approximate values of y and z at various points $x > x_0$.

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2 The Runge-Kutta 4th Order (RK-4) Method for Two ODEs

The **RK-4 method** is an iterative numerical technique that estimates the solution of ODEs by calculating weighted averages of slopes within each step. For a system of two first-order ODEs as defined above, the formulas to estimate y_{i+1} and z_{i+1} from y_i and z_i at x_i with a step size h are:

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$z_{i+1} = z_i + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

The coefficients k and l represent various estimates of the slopes multiplied by the step size h, and are calculated as follows:

Step 1: Calculate k_1 and l_1 values

$$k_1 = h \cdot f(x_i, y_i, z_i)$$
$$l_1 = h \cdot g(x_i, y_i, z_i)$$

Step 2: Calculate k_2 and l_2 values

$$k_2 = h \cdot f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1, z_i + \frac{1}{2}l_1)$$
$$l_2 = h \cdot g(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1, z_i + \frac{1}{2}l_1)$$

Step 3: Calculate k_3 and l_3 values

$$k_3 = h \cdot f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2, z_i + \frac{1}{2}l_2)$$
$$l_3 = h \cdot g(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2, z_i + \frac{1}{2}l_2)$$

Step 4: Calculate k_4 and l_4 values

$$k_4 = h \cdot f(x_i + h, y_i + k_3, z_i + l_3)$$
$$l_4 = h \cdot g(x_i + h, y_i + k_3, z_i + l_3)$$

—

3 Step-by-Step Procedure for Numerical Solution

To solve a system of two first-order ODEs using RK-4 for one step (x_i to x_{i+1}):

1. **Identify Functions and Initial Conditions:**
   - State the two ODEs: $\frac{dy}{dx} = f(x, y, z)$ and $\frac{dz}{dx} = g(x, y, z)$.
   - Note the initial conditions: x_0, y_0, and z_0.
   - Choose the step size, h.
2. **Calculate k_1 and l_1 Values:** Substitute x_i, y_i, and z_i into the functions f and g, then multiply by h to find k_1 and l_1.
3. **Calculate k_2 and l_2 Values:**
   - Compute the intermediate arguments for f and g: $(x_i + \frac{1}{2}h)$, $(y_i + \frac{1}{2}k_1)$, and $(z_i + \frac{1}{2}l_1)$.
   - Substitute these intermediate arguments into f and g, then multiply by h to find k_2 and l_2.
4. **Calculate k_3 and l_3 Values:**
   - Compute the intermediate arguments for f and g: $(x_i + \frac{1}{2}h)$ (same as for k_2), $(y_i + \frac{1}{2}k_2)$, and $(z_i + \frac{1}{2}l_2)$.
   - Substitute these intermediate arguments into f and g, then multiply by h to find k_3 and l_3.
5. **Calculate k_4 and l_4 Values:**
   - Compute the intermediate arguments for f and g: $(x_i + h)$, $(y_i + k_3)$, and $(z_i + l_3)$.
   - Substitute these intermediate arguments into f and g, then multiply by h to find k_4 and l_4.
6. **Update y and z:** Use the calculated k and l values to find the new approximations for y_{i+1} and z_{i+1} using the main RK-4 formulas.

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$z_{i+1} = z_i + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

7. **Repeat (if necessary):** If you need to find the solution over a larger range of x, set the newly calculated y_{i+1}, z_{i+1}, and $x_{i+1} = x_i + h$ as your new initial conditions for the next step, and repeat steps 2-6.

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4 Example: Solving a System of Two ODEs

Let's solve the following system of first-order ODEs using RK-4:

$$\frac{dy}{dx} = x + y - z$$
$$\frac{dz}{dx} = 2x - y + z$$

with initial conditions $y(0) = 1$ and $z(0) = 0$. We want to find y and z at $x = 0.1$ using a step size $h = 0.1$.

Here, $f(x, y, z) = x + y - z$ and $g(x, y, z) = 2x - y + z$. We have $x_0 = 0$, $y_0 = 1$, $z_0 = 0$, and $h = 0.1$.

Step 1: Calculate k_1 and l_1 values

Current values: $x_i = 0$, $y_i = 1$, $z_i = 0$.

$$k_1 = h \cdot f(0, 1, 0) = 0.1 \cdot (0 + 1 - 0) = 0.1 \cdot 1 = 0.1$$

$$l_1 = h \cdot g(0, 1, 0) = 0.1 \cdot (2(0) - 1 + 0) = 0.1 \cdot (-1) = -0.1$$

Step 2: Calculate k_2 and l_2 values

Intermediate arguments for function evaluation: $x_i + \frac{1}{2}h = 0 + \frac{1}{2}(0.1) = 0.05$ $y_i + \frac{1}{2}k_1 = 1 + \frac{1}{2}(0.1) = 1 + 0.05 = 1.05$ $z_i + \frac{1}{2}l_1 = 0 + \frac{1}{2}(-0.1) = 0 - 0.05 = -0.05$

Now, calculate k_2 and l_2:

$$k_2 = h \cdot f(0.05, 1.05, -0.05) = 0.1 \cdot (0.05 + 1.05 - (-0.05)) = 0.1 \cdot (1.15) = 0.115$$

$$l_2 = h \cdot g(0.05, 1.05, -0.05) = 0.1 \cdot (2(0.05) - 1.05 + (-0.05)) = 0.1 \cdot (-1.0) = -0.1$$

Step 3: Calculate k_3 and l_3 values

Intermediate arguments for function evaluation: $x_i + \frac{1}{2}h = 0.05$ $y_i + \frac{1}{2}k_2 = 1 + \frac{1}{2}(0.115) = 1 + 0.0575 = 1.0575$ $z_i + \frac{1}{2}l_2 = 0 + \frac{1}{2}(-0.1) = 0 - 0.05 = -0.05$

Now, calculate k_3 and l_3:

$$k_3 = h \cdot f(0.05, 1.0575, -0.05) = 0.1 \cdot (0.05 + 1.0575 - (-0.05)) = 0.1 \cdot (1.1575) = 0.11575$$

$$l_3 = h \cdot g(0.05, 1.0575, -0.05) = 0.1 \cdot (2(0.05) - 1.0575 + (-0.05)) = 0.1 \cdot (-1.0075) = -0.10075$$

Step 4: Calculate k_4 and l_4 values

Intermediate arguments for function evaluation: $x_i + h = 0 + 0.1 = 0.1$ $y_i + k_3 = 1 + (0.11575) = 1.11575$ $z_i + l_3 = 0 + (-0.10075) = -0.10075$

Now, calculate k_4 and l_4:

$$k_4 = h \cdot f(0.1, 1.11575, -0.10075) = 0.1 \cdot (0.1 + 1.11575 - (-0.10075)) = 0.1 \cdot (1.3165) = 0.13165$$

$$l_4 = h \cdot g(0.1, 1.11575, -0.10075) = 0.1 \cdot (2(0.1) - 1.11575 + (-0.10075)) = 0.1 \cdot (-1.0165) = -0.10165$$

Step 5: Update y and z to find y_1 and z_1

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$y_1 = 1 + \frac{1}{6}(0.1 + 2(0.115) + 2(0.11575) + 0.13165)$$

$$y_1 = 1 + \frac{1}{6}(0.1 + 0.230 + 0.2315 + 0.13165)$$

$$y_1 = 1 + \frac{1}{6}(0.69315) \approx 1 + 0.115525 = 1.115525$$

$$z_1 = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

$$z_1 = 0 + \frac{1}{6}(-0.1 + 2(-0.1) + 2(-0.10075) + (-0.10165))$$

$$z_1 = 0 + \frac{1}{6}(-0.1 - 0.2 - 0.2015 - 0.10165)$$

$$z_1 = \frac{1}{6}(-0.60315) \approx -0.100525$$

So, at $x = 0.1$, the numerical solution is approximately $y(0.1) \approx 1.115525$ and $z(0.1) \approx -0.100525$.
—

5 Practice Questions

Try solving the following systems of ODEs using the RK-4 method. Remember to show all your steps with the 'h' multiplier outside the 'k' and 'l' definitions.

Question 1

Solve the following system for $p(t)$ and $q(t)$ at $t = 0.2$ with $h = 0.1$:

$$\frac{dp}{dt} = 2p - q$$

$$\frac{dq}{dt} = p + 3q$$

Initial conditions: $p(0) = 1$, $q(0) = 0$.

Question 2

Find the numerical solution for $a(s)$ and $b(s)$ at $s = 0.1$ with $h = 0.05$:

$$\frac{da}{ds} = s \cdot a + b^2$$

$$\frac{db}{ds} = s^2 - a + b$$

Initial conditions: $a(0) = 2$, $b(0) = 1$.

Question 3

Use RK-4 to approximate $u(v)$ and $w(v)$ at $v = 0.1$ with $h = 0.1$:

$$\frac{du}{dv} = u \cdot \cos(v) - w$$

$$\frac{dw}{dv} = \sin(v) + u \cdot w$$

Initial conditions: $u(0) = 1$, $w(0) = 1$. (You may need a calculator for trigonometric functions.)

RK-4 k and l Values for Practice Questions (Values Only)

Here are the calculated k and l values for each practice question, without the detailed steps. These are useful for quickly verifying your own manual calculations.

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Question 1: For $p(t)$ and $q(t)$ at $t = 0.2$ with $h = 0.1$

First Step: From $t_0 = 0$ to $t_1 = 0.1$

(Initial values for this step: $t_i = 0$, $p_i = 1$, $q_i = 0$)

- $k_1 = 2$
- $l_1 = 1$
- $k_2 = 2.15$
- $l_2 = 1.25$
- $k_3 = 2.1525$
- $l_3 = 1.295$
- $k_4 = 2.301$
- $l_4 = 1.60375$

Using these, the values after the first step are: $p_1 \approx 1.2154625$ $q_1 \approx 0.1293291667$

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Second Step: From $t_1 = 0.1$ to $t_2 = 0.2$

(Initial values for this step: $t_i = 0.1$, $p_i \approx 1.2154625$, $q_i \approx 0.1293291667$)

- $k_1 \approx 2.3016$
- $l_1 \approx 1.6038$
- $k_2 \approx 2.4578$
- $l_2 \approx 1.7486$
- $k_3 \approx 2.4646$
- $l_3 \approx 1.7719$
- $k_4 \approx 2.6329$
- $l_4 \approx 1.9443$

Using these, the final values at $t = 0.2$ are approximately: $p(0.2) \approx 1.4616$ $q(0.2) \approx 0.3065$

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Question 2: For $a(s)$ and $b(s)$ at $s = 0.1$ with $h = 0.05$

First Step: From $s_0 = 0$ to $s_1 = 0.05$

(Initial values for this step: $s_i = 0$, $a_i = 2$, $b_i = 1$)

- $k_1 = 1$
- $l_1 = -1$
- $k_2 = 1.00125$
- $l_2 = -1.049375$
- $k_3 \approx 0.9988458$
- $l_3 \approx -1.0506406$
- $k_4 \approx 1.0001925$
- $l_4 \approx -1.0999743$

Using these, the values after the first step are: $a_1 \approx 2.050013$ $b_1 \approx 0.947475$

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Second Step: From $s_1 = 0.05$ to $s_2 = 0.1$

(Initial values for this step: $s_i = 0.05$, $a_i \approx 2.050013$, $b_i \approx 0.947475$)

- $k_1 \approx 1.0000$
- $l_1 \approx -1.0999$
- $k_2 \approx 1.0501$
- $l_2 \approx -1.1499$
- $k_3 \approx 1.0500$
- $l_3 \approx -1.1500$
- $k_4 \approx 1.1000$
- $l_4 \approx -1.2000$

Using these, the final values at $s = 0.1$ are approximately: $a(0.1) \approx 2.1025$ $b(0.1) \approx 0.8900$

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Question 3: For $u(v)$ and $w(v)$ at $v = 0.1$ with $h = 0.1$

One Step: From $v_0 = 0$ to $v_1 = 0.1$

(Initial values for this step: $v_i = 0$, $u_i = 1$, $w_i = 1$)

- $k_1 = 0$
- $l_1 = 1$
- $k_2 \approx -0.05125$
- $l_2 \approx 1.09998$
- $k_3 \approx -0.058826$
- $l_3 \approx 1.10218$
- $k_4 \approx -0.121072$
- $l_4 \approx 1.203533$

Using these, the final values at $v = 0.1$ are approximately: $u(0.1) \approx 0.9942$ $w(0.1) \approx 1.1098$

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DEPARTMENT OF MATHEMATICS
Ordinary Differential Equations and Numerical Methods (BMATS209)
Tutorial - 10

Abstract

This tutorial provides a comprehensive guide to numerically solving a second-order ordinary differential equation (ODE) using the fourth-order Runge-Kutta (RK-4) method. We will demonstrate how to convert a second-order ODE into a system of first-order ODEs, and then apply the RK-4 method to solve this system. A detailed step-by-step example and practice questions are included.

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1 Introduction to Second-Order ODEs

A general second-order ordinary differential equation can be expressed in the form:

$$\frac{d^2y}{dx^2} = h\left(x, y, \frac{dy}{dx}\right)$$

with given initial conditions for y and its first derivative at a specific point, typically $y(x_0) = y_0$ and $\frac{dy}{dx}(x_0) = y'_0$. We use $h(x, y, y')$ here to avoid confusion with the step size h.

The RK-4 method, as we've seen, is designed for first-order ODEs. To apply RK-4 to a second-order ODE, we must first transform the single second-order equation into an equivalent system of two first-order ODEs.

1.1 Transformation to a System of First-Order ODEs

Let's introduce a new dependent variable to reduce the order. We define:

$$z = \frac{dy}{dx}$$

Then, we can express the first derivatives of y and z with respect to x as:

$$\frac{dy}{dx} = z \quad \text{(Let this be our first function, } f(x, y, z) = z\text{)}$$

$$\frac{dz}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = h(x, y, y') = h(x, y, z) \quad \text{(Let this be our second function, } g(x, y, z) = h(x, y, z)\text{)}$$

So, the second-order ODE is transformed into a system of two simultaneous first-order ODEs:

$$\frac{dy}{dx} = f(x, y, z)$$

$$\frac{dz}{dx} = g(x, y, z)$$

The initial conditions also transform accordingly: $y(x_0) = y_0$ and $z(x_0) = y'_0$. Now, this system can be solved using the RK-4 method for simultaneous first-order ODEs.

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2 The Runge-Kutta 4th Order (RK-4) Method for Second-Order ODEs

For the transformed system:

$$\frac{dy}{dx} = f(x, y, z)$$
$$\frac{dz}{dx} = g(x, y, z)$$

with initial conditions $y(x_i)$ and $z(x_i)$, and a step size h, the formulas to estimate y_{i+1} and z_{i+1} are:

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$z_{i+1} = z_i + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

The coefficients k (for y) and l (for z) are calculated as follows (note that h is multiplied outside the function evaluation):

Step 1: Calculate k_1 and l_1 values

$$k_1 = h \cdot f(x_i, y_i, z_i)$$
$$l_1 = h \cdot g(x_i, y_i, z_i)$$

Step 2: Calculate k_2 and l_2 values

$$k_2 = h \cdot f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1, z_i + \frac{1}{2}l_1)$$
$$l_2 = h \cdot g(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1, z_i + \frac{1}{2}l_1)$$

Step 3: Calculate k_3 and l_3 values

$$k_3 = h \cdot f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2, z_i + \frac{1}{2}l_2)$$
$$l_3 = h \cdot g(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2, z_i + \frac{1}{2}l_2)$$

Step 4: Calculate k_4 and l_4 values

$$k_4 = h \cdot f(x_i + h, y_i + k_3, z_i + l_3)$$
$$l_4 = h \cdot g(x_i + h, y_i + k_3, z_i + l_3)$$

—

3 Step-by-Step Procedure for Numerical Solution

To solve a second-order ODE using RK-4 for one step (x_i to x_{i+1}):

1. **Transform the ODE:** Convert the given second-order ODE into a system of two first-order ODEs: $\frac{dy}{dx} = z$ and $\frac{dz}{dx} = h(x, y, z)$. Identify your specific $f(x, y, z)$ and $g(x, y, z)$ functions from the problem.
2. **Identify Initial Conditions:** State x_0, y_0, and $z_0 = y'(x_0)$. Choose the step size, h.
3. **Calculate k_1 and l_1 Values:** Substitute x_i, y_i, and z_i into the functions f and g, then multiply by h to find k_1 and l_1.
4. **Calculate k_2 and l_2 Values:**

- Compute the intermediate arguments for f and g: $(x_i + \frac{1}{2}h)$, $(y_i + \frac{1}{2}k_1)$, and $(z_i + \frac{1}{2}l_1)$.
- Substitute these intermediate arguments into f and g, then multiply by h to find k_2 and l_2.

5. **Calculate k_3 and l_3 Values:**
   - Compute the intermediate arguments for f and g: $(x_i + \frac{1}{2}h)$, $(y_i + \frac{1}{2}k_2)$, and $(z_i + \frac{1}{2}l_2)$.
   - Substitute these intermediate arguments into f and g, then multiply by h to find k_3 and l_3.
6. **Calculate k_4 and l_4 Values:**
   - Compute the intermediate arguments for f and g: $(x_i + h)$, $(y_i + k_3)$, and $(z_i + l_3)$.
   - Substitute these intermediate arguments into f and g, then multiply by h to find k_4 and l_4.
7. **Update y and z:** Use the calculated k and l values to find the new approximations for y_{i+1} and z_{i+1} using the main RK-4 formulas:
$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$z_{i+1} = z_i + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$
8. **Repeat (if necessary):** If you need to find the solution over a larger range of x, set the newly calculated y_{i+1}, z_{i+1}, and $x_{i+1} = x_i + h$ as your new initial conditions for the next step, and repeat steps 3-7. The value of $y(x_{i+1})$ will be y_{i+1} and $y'(x_{i+1})$ will be z_{i+1}.

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4 Example: Solving a Second-Order ODE

Let's solve the following second-order ODE using RK-4:

$$\frac{d^2y}{dx^2} = x + \frac{dy}{dx} - y$$

with initial conditions $y(0) = 1$ and $\frac{dy}{dx}(0) = 0$. We want to find y and $\frac{dy}{dx}$ at $x = 0.1$ using a step size $h = 0.1$.

Step 1: Transform the ODE

Let $z = \frac{dy}{dx}$. Then, $\frac{dy}{dx} = z$ and $\frac{dz}{dx} = x + z - y$. So, our functions are: $f(x, y, z) = z$ $g(x, y, z) = x + z - y$

Step 2: Identify Initial Conditions

$x_0 = 0$, $y_0 = 1$, $z_0 = 0$. Step size $h = 0.1$.

Step 3: Calculate k_1 and l_1 values

Current values: $x_i = 0$, $y_i = 1$, $z_i = 0$.

$$k_1 = h \cdot f(0, 1, 0) = 0.1 \cdot (0) = 0$$
$$l_1 = h \cdot g(0, 1, 0) = 0.1 \cdot (0 + 0 - 1) = 0.1 \cdot (-1) = -0.1$$

Step 4: Calculate k_2 and l_2 values

Intermediate arguments for function evaluation: $x_i + \frac{1}{2}h = 0 + \frac{1}{2}(0.1) = 0.05$ $y_i + \frac{1}{2}k_1 = 1 + \frac{1}{2}(0) = 1$ $z_i + \frac{1}{2}l_1 = 0 + \frac{1}{2}(-0.1) = -0.05$

Now, calculate k_2 and l_2:

$$k_2 = h \cdot f(0.05, 1, -0.05) = 0.1 \cdot (-0.05) = -0.005$$
$$l_2 = h \cdot g(0.05, 1, -0.05) = 0.1 \cdot (0.05 + (-0.05) - 1) = 0.1 \cdot (-1) = -0.1$$

Step 5: Calculate k_3 and l_3 values

Intermediate arguments for function evaluation: $x_i + \frac{1}{2}h = 0.05$ $y_i + \frac{1}{2}k_2 = 1 + \frac{1}{2}(-0.005) = 1 - 0.0025 = 0.9975$ $z_i + \frac{1}{2}l_2 = 0 + \frac{1}{2}(-0.1) = -0.05$

Now, calculate k_3 and l_3:

$$k_3 = h \cdot f(0.05, 0.9975, -0.05) = 0.1 \cdot (-0.05) = -0.005$$

$$l_3 = h \cdot g(0.05, 0.9975, -0.05) = 0.1 \cdot (0.05 + (-0.05) - 0.9975) = 0.1 \cdot (-0.9975) = -0.09975$$

Step 6: Calculate k_4 and l_4 values

Intermediate arguments for function evaluation: $x_i + h = 0 + 0.1 = 0.1$ $y_i + k_3 = 1 + (-0.005) = 0.995$ $z_i + l_3 = 0 + (-0.09975) = -0.09975$

Now, calculate k_4 and l_4:

$$k_4 = h \cdot f(0.1, 0.995, -0.09975) = 0.1 \cdot (-0.09975) = -0.009975$$

$$l_4 = h \cdot g(0.1, 0.995, -0.09975) = 0.1 \cdot (0.1 + (-0.09975) - 0.995) = 0.1 \cdot (-0.99475) = -0.099475$$

Step 7: Update y and z to find y_1 and z_1

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$y_1 = 1 + \frac{1}{6}(0 + 2(-0.005) + 2(-0.005) + (-0.009975))$$

$$y_1 = 1 + \frac{1}{6}(0 - 0.01 - 0.01 - 0.009975)$$

$$y_1 = 1 + \frac{1}{6}(-0.029975) \approx 1 - 0.00499583 \approx 0.99500417$$

$$z_1 = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

$$z_1 = 0 + \frac{1}{6}(-0.1 + 2(-0.1) + 2(-0.09975) + (-0.099475))$$

$$z_1 = 0 + \frac{1}{6}(-0.1 - 0.2 - 0.1995 - 0.099475)$$

$$z_1 = \frac{1}{6}(-0.598975) \approx -0.09982917$$

So, at $x = 0.1$, the numerical solution is approximately $y(0.1) \approx 0.99500417$ and $\frac{dy}{dx}(0.1) \approx -0.09982917$. —

5 Practice Questions

Solve the following second-order ODEs using the RK-4 method. Remember to show your transformation and all steps.

Question 1

Approximate $y(0.1)$ and $y'(0.1)$ for the ODE:

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + y + x$$

Initial conditions: $y(0) = 1$, $y'(0) = 0$. Use $h = 0.1$.

Question 2

Find the numerical solution for $p(t)$ and $p'(t)$ at $t = 0.05$ for the ODE:

$$\frac{d^2p}{dt^2} = t^2 - p \cdot \frac{dp}{dt}$$

Initial conditions: $p(0) = 2$, $p'(0) = 1$. Use $h = 0.05$.

Question 3

Calculate $q(s)$ and $q'(s)$ at $s = 0.1$ for the ODE:

$$\frac{d^2q}{ds^2} + 2\frac{dq}{ds} + q = \cos(s)$$

Initial conditions: $q(0) = 0$, $q'(0) = 1$. Use $h = 0.1$. (Hint: Rearrange the ODE to the standard form $\frac{d^2q}{ds^2} = \dots$)

6 Practice Questions: Intermediate k, l Values and Final Answers

Question 1

Approximate $y(0.1)$ and $y'(0.1)$ for the ODE:

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + y + x$$

Initial conditions: $y(0) = 1$, $y'(0) = 0$. Use $h = 0.1$.

Transformation: $f(x, y, z) = z$, $g(x, y, z) = z + y + x$

Calculated Values:

- $k_1 = 0$
- $l_1 = 0.1$
- $k_2 = 0.005$
- $l_2 = 0.11$
- $k_3 = 0.0055$
- $l_3 = 0.11075$
- $k_4 = 0.011075$
- $l_4 = 0.121075$

Final Answer:

- $y(0.1) \approx 1.005346$
- $y'(0.1) \approx 0.108763$

—

Question 2

Find the numerical solution for $p(t)$ and $p'(t)$ at $t = 0.05$ for the ODE:

$$\frac{d^2p}{dt^2} = t^2 - p \cdot \frac{dp}{dt}$$

Initial conditions: $p(0) = 2$, $p'(0) = 1$. Use $h = 0.05$.

Transformation: Let $z = \frac{dp}{dt}$. Then $f(t, p, z) = z$, $g(t, p, z) = t^2 - p \cdot z$.

Calculated Values:

- $k_1 = 0.05$
- $l_1 = -0.1$
- $k_2 = 0.0475$
- $l_2 = -0.09615625$
- $k_3 = 0.04759609375$
- $l_3 = -0.0962908825$
- $k_4 = 0.045185455875$
- $l_4 = -0.09239965$

Final Answer:

- $p(0.05) \approx 2.04751$
- $p'(0.05) \approx 0.90371$

—

Question 3

Calculate $q(s)$ and $q'(s)$ at $s = 0.1$ for the ODE:

$$\frac{d^2q}{ds^2} + 2\frac{dq}{ds} + q = \cos(s)$$

Initial conditions: $q(0) = 0$, $q'(0) = 1$. Use $h = 0.1$.

Transformation: Let $z = \frac{dq}{ds}$. Then $f(s, q, z) = z$, $g(s, q, z) = \cos(s) - 2z - q$.

Calculated Values:

- $k_1 = 0.1$
- $l_1 = -0.1$
- $k_2 = 0.095$
- $l_2 = -0.095125$
- $k_3 = 0.09524375$
- $l_3 = -0.0953625$
- $k_4 = 0.09046375$
- $l_4 = -0.090951475$

Final Answer:

- $q(0.1) \approx 0.095158$
- $q'(0.1) \approx 0.810052$

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DEPARTMENT OF MATHEMATICS
Ordinary Differential Equations and Numerical Methods (BMATS209)
Tutorial - 11

Abstract

This tutorial provides a comprehensive guide to numerically solving a second-order ordinary differential equation (ODE) using Milne's Predictor-Corrector method. We will demonstrate how to convert a second-order ODE into a system of first-order ODEs, using $z = dy/dx$ and $z' = d^2y/dx^2$, and then apply Milne's method to solve this system. A detailed step-by-step example with tabular data and practice questions are included. This method requires initial starting values typically obtained from a self-starting method like RK-4.

1 Introduction to Second-Order ODEs

A general second-order ordinary differential equation can be expressed in the form:

$$\frac{d^2y}{dx^2} = h\left(x, y, \frac{dy}{dx}\right)$$

with given initial conditions for y and its first derivative at a specific point, typically $y(x_0) = y_0$ and $\frac{dy}{dx}(x_0) = y'_0$. We use $h(x, y, y')$ here to represent the function on the right-hand side.

Milne's method, being a multi-step method, requires several initial values to start the computation. For a second-order ODE, we first transform it into a system of first-order ODEs.

1.1 Transformation to a System of First-Order ODEs

Let's introduce a new dependent variable to reduce the order. We define:

$$z = \frac{dy}{dx}$$

Then, we can express the first derivatives of y and z with respect to x as:

$$\frac{dy}{dx} = z$$

$$\frac{dz}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = h(x, y, y') = h(x, y, z)$$

So, the second-order ODE is transformed into a system of two simultaneous first-order ODEs:

$$y' = z$$

$$z' = h(x, y, z)$$

The initial conditions also transform accordingly: $y(x_0) = y_0$ and $z(x_0) = y'_0$.

To use Milne's method, we need four starting values: (x_0, y_0, z_0), (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3). These are typically obtained using a self-starting method like RK-4.

2 Milne's Predictor-Corrector Method for Second-Order ODEs

For the transformed system:

$$y' = z$$

$$z' = h(x, y, z)$$

Given initial values (x_0, y_0, z_0), (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) with a step size h. We first calculate the values of $y'_i = z_i$ and $z'_i = h(x_i, y_i, z_i)$ for $i = 0, 1, 2, 3$.

Predictor Formulas (Milne's Predictor)

To predict y_{i+1} and z_{i+1} (for $i = 3$, this means y_4 and z_4):

$$y_{i+1}^{(p)} = y_{i-3} + \frac{4h}{3}(2y'_{i-2} - y'_{i-1} + 2y'_i)$$

$$z_{i+1}^{(p)} = z_{i-3} + \frac{4h}{3}(2z'_{i-2} - z'_{i-1} + 2z'_i)$$

For finding y_4, z_4, we set $i = 3$:

$$y_4^{(p)} = y_0 + \frac{4h}{3}(2y'_1 - y'_2 + 2y'_3)$$

$$z_4^{(p)} = z_0 + \frac{4h}{3}(2z'_1 - z'_2 + 2z'_3)$$

Corrector Formulas (Milne's Corrector)

After obtaining the predicted values $y_{i+1}^{(p)}$ and $z_{i+1}^{(p)}$, we use them to compute ${y'_{i+1}}^{(p)} = z_{i+1}^{(p)}$ and ${z'_{i+1}}^{(p)} = h(x_{i+1}, y_{i+1}^{(p)}, z_{i+1}^{(p)})$. Then, we use the corrector formulas:

$$y_{i+1}^{(c)} = y_{i-1} + \frac{h}{3}(y'_{i-1} + 4y'_i + {y'_{i+1}}^{(p)})$$

$$z_{i+1}^{(c)} = z_{i-1} + \frac{h}{3}(z'_{i-1} + 4z'_i + {z'_{i+1}}^{(p)})$$

For finding y_4, z_4, we set $i = 3$:

$$y_4^{(c)} = y_2 + \frac{h}{3}(y'_2 + 4y'_3 + {y'_4}^{(p)})$$

$$z_4^{(c)} = z_2 + \frac{h}{3}(z'_2 + 4z'_3 + {z'_4}^{(p)})$$

3 Step-by-Step Procedure for Numerical Solution

To solve a second-order ODE using Milne's method to find y_4 and z_4:

1. **Transform the ODE:** Convert the given second-order ODE into a system of two first-order ODEs: $y' = z$ and $z' = h(x, y, z)$. Identify your specific $h(x, y, z)$ function from the problem.

2. **Identify Initial Conditions:** Ensure you have four sets of initial values: (x_0, y_0, z_0), (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3). These are typically provided or computed by a self-starting method. State the step size h.

3. **Calculate y'_i and z'_i values:** Compute $y'_i = z_i$ and $z'_i = h(x_i, y_i, z_i)$ for $i = 0, 1, 2, 3$.

4. **Apply Predictor Formulas:** Use the Milne's Predictor formulas (for $i = 3$) to find the predicted values $y_4^{(p)}$ and $z_4^{(p)}$:

$$y_4^{(p)} = y_0 + \frac{4h}{3}(2y'_1 - y'_2 + 2y'_3)$$

$$z_4^{(p)} = z_0 + \frac{4h}{3}(2z'_1 - z'_2 + 2z'_3)$$

5. **Calculate $y_4'^{(p)}$ and $z_4'^{(p)}$:** Use the predicted values to calculate $y_4'^{(p)} = z_4^{(p)}$ and $z_4'^{(p)} = h(x_4, y_4^{(p)}, z_4^{(p)})$. Note that $x_4 = x_3 + h$.
6. **Apply Corrector Formulas:** Use the Milne's Corrector formulas (for $i = 3$) to find the corrected values $y_4^{(c)}$ and $z_4^{(c)}$:

$$y_4^{(c)} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4'^{(p)})$$
$$z_4^{(c)} = z_2 + \frac{h}{3}(z_2' + 4z_3' + z_4'^{(p)})$$

Note: Here y' refers to dy/dx and z' refers to dz/dx (or d^2y/dx^2).

7. **Final Values:** The corrected values $y_4^{(c)}$ and $z_4^{(c)}$ are your approximations for $y(x_4)$ and $y'(x_4)$.

4 Example: Solving a Second-Order ODE

Let's solve the following second-order ODE using Milne's method to find $y(0.4)$ and $y'(0.4)$:

$$\frac{d^2y}{dx^2} = -y$$

with initial conditions $y(0) = 1$, $y'(0) = 0$. We use a step size $h = 0.1$. The starting values (e.g., from RK-4) are given in the table below:

Step 1: Transform the ODE

Let $z = \frac{dy}{dx}$. Then, $\frac{dy}{dx} = z$ and $\frac{dz}{dx} = -y$. So, our functions are: $y' = z$ $z' = -y$

Step 2: Initial Conditions and Given Values ($h = 0.1$)

i	x_i	y_i	$z_i = y_i'$
0	0.0	1.0000	0.0000
1	0.1	0.9950	-0.0998
2	0.2	0.9801	-0.1987
3	0.3	0.9553	-0.2955

Step 3: Calculate y_i' and z_i' values

- $y_0' = z_0 = 0.0000$
- $z_0' = -y_0 = -1.0000$
- $y_1' = z_1 = -0.0998$
- $z_1' = -y_1 = -0.9950$
- $y_2' = z_2 = -0.1987$
- $z_2' = -y_2 = -0.9801$
- $y_3' = z_3 = -0.2955$
- $z_3' = -y_3 = -0.9553$

Step 4: Apply Predictor Formulas (for $y_4^{(p)}, z_4^{(p)}$)

Using $i = 3$:

- $y_4^{(p)} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3') = 1.0000 + \frac{4(0.1)}{3}(2(-0.0998) - (-0.1987) + 2(-0.2955))$
- $y_4^{(p)} = 0.92108$
- $z_4^{(p)} = z_0 + \frac{4h}{3}(2z_1' - z_2' + 2z_3') = 0.0000 + \frac{4(0.1)}{3}(2(-0.9950) - (-0.9801) + 2(-0.9553))$
- $z_4^{(p)} = -0.38940$

Step 5: Calculate ${y'}_4^{(p)}$ and ${z'}_4^{(p)}$

$x_4 = x_3 + h = 0.3 + 0.1 = 0.4$.

- ${y'}_4^{(p)} = z_4^{(p)} = -0.38940$
- ${z'}_4^{(p)} = -y_4^{(p)} = -0.92108$

Step 6: Apply Corrector Formulas (for $y_4^{(c)}, z_4^{(c)}$)

Using $i = 3$:

- $y_4^{(c)} = y_2 + \frac{h}{3}(y'_2 + 4y'_3 + {y'}_4^{(p)}) = 0.9801 + \frac{0.1}{3}(-0.1987 + 4(-0.2955) + (-0.38940))$
- $y_4^{(c)} = 0.921097$
- $z_4^{(c)} = z_2 + \frac{h}{3}(z'_2 + 4z'_3 + {z'}_4^{(p)}) = -0.1987 + \frac{0.1}{3}(-0.9801 + 4(-0.9553) + (-0.92108))$
- $z_4^{(c)} = -0.389446$

Step 7: Final Values

At $x = 0.4$: $y(0.4) \approx 0.921097$ $y'(0.4) \approx -0.389446$

5 Practice Questions

Question 1

Approximate $y(0.4)$ and $y'(0.4)$ for the ODE:

$$\frac{d^2y}{dx^2} = x + \frac{dy}{dx} - y$$

Initial conditions and starting values for $h = 0.1$:

i	x_i	y_i	$z_i = y'_i$
0	0.0	1.00000	0.00000
1	0.1	0.99500	-0.09983
2	0.2	0.98010	-0.19888
3	0.3	0.95551	-0.29624

Solution:

Transformation: $y' = z$, $z' = x + z - y$

Calculated y'_i and z'_i values:

- $y'_0 = 0.00000$
- $z'_0 = 0.0 + 0.00000 - 1.00000 = -1.00000$
- $y'_1 = -0.09983$
- $z'_1 = 0.1 + (-0.09983) - 0.99500 = -0.99483$
- $y'_2 = -0.19888$
- $z'_2 = 0.2 + (-0.19888) - 0.98010 = -0.97898$
- $y'_3 = -0.29624$
- $z'_3 = 0.3 + (-0.29624) - 0.95551 = -0.95175$

Predicted Values ($y_4^{(p)}, z_4^{(p)}$):

- $y_4^{(p)} = 0.92131$
- $z_4^{(p)} = -0.38992$

Calculated $y_4'^{(p)}$ and $z_4'^{(p)}$:

- $y_4'^{(p)} = -0.38992$
- $z_4'^{(p)} = 0.4 + (-0.38992) - 0.92131 = -0.91123$

Final Answer (Corrected Values):

- $y(0.4) \approx \mathbf{0.92131}$
- $y'(0.4) \approx -\mathbf{0.38992}$

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DEPARTMENT OF MATHEMATICS
Ordinary Differential Equations and Numerical Methods (BMATS209)
Tutorial - 12

Abstract

This tutorial provides a comprehensive guide to finding the roots of non-linear equations using the Regula Falsi (False Position) method. This is a bracketing method that refines an initial interval known to contain a root. We will demonstrate the formula, step-by-step procedure, and include a detailed example with tabular data, followed by practice questions with their iterative values and final answers.

—

1 Introduction to Non-Linear Equations and Root Finding

In many scientific and engineering applications, we encounter non-linear equations of the form $f(x) = 0$. Finding the values of x that satisfy this equation (i.e., the roots) is a fundamental problem. While analytical solutions are often difficult or impossible to obtain, numerical methods provide effective ways to approximate these roots to a desired precision.

The Regula Falsi method is one such numerical technique. It is an iterative, bracketing method, meaning it starts with an interval $[a, b]$ where the function $f(x)$ changes sign. This ensures that at least one root lies within this interval.

—

2 The Regula Falsi (False Position) Method

The Regula Falsi method is similar to the Bisection method in that it requires an initial interval $[a, b]$ such that $f(a)$ and $f(b)$ have opposite signs (i.e., $f(a) \cdot f(b) < 0$). However, instead of simply bisecting the interval, it uses a secant line connecting the points $(a, f(a))$ and $(b, f(b))$. The point where this secant line intersects the x-axis is taken as the new approximation for the root.

Formula

The formula for the new approximation of the root, x_{new}, is given by:

$$x_{new} = a - \frac{f(a)(b-a)}{f(b)-f(a)}$$

Alternatively, this can be written as:

$$x_{new} = \frac{af(b) - bf(a)}{f(b)-f(a)}$$

Iteration Rule

After calculating x_{new}:

1. If $f(x_{new}) = 0$, then x_{new} is the exact root.
2. If $f(x_{new}) \cdot f(a) < 0$, then the root lies in $[a, x_{new}]$. Set $b = x_{new}$.
3. If $f(x_{new}) \cdot f(b) < 0$, then the root lies in $[x_{new}, b]$. Set $a = x_{new}$.

The process is repeated until a desired level of accuracy (tolerance) is achieved, typically when $|f(x_{new})|$ is very small or $|x_{new} - x_{old}|$ is very small.

—

3 Step-by-Step Procedure for Solving $f(x) = 0$

To solve a non-linear equation $f(x) = 0$ using the Regula Falsi method:

1. **Define the function:** Clearly write down $f(x)$.
2. **Choose Initial Guesses:** Select two initial approximations, a and b, such that $f(a)$ and $f(b)$ have opposite signs. It is often helpful to perform a preliminary search by testing values at small increments (e.g., $x = 0, 0.1, 0.2, \ldots$ or $x = 1, 2, 3, \ldots$) to identify a sufficiently narrow interval where the sign change occurs. Using a tighter initial interval will generally lead to faster convergence.
3. **Iterate to find new approximation (x_{new}):** Use the formula:
$$x_{new} = a - \frac{f(a)(b-a)}{f(b)-f(a)}$$
4. **Evaluate $f(x_{new})$:** Calculate the function value at the new approximation.
5. **Update the Interval:**
   - If $f(x_{new}) \cdot f(a) < 0$, the new interval is $[a, x_{new}]$. Update $b = x_{new}$.
   - If $f(x_{new}) \cdot f(b) < 0$, the new interval is $[x_{new}, b]$. Update $a = x_{new}$.
6. **Check for Convergence:** Repeat steps 3-5 until $|f(x_{new})|$ is sufficiently close to zero or the difference between successive approximations $|x_{new} - x_{old}|$ is within a specified tolerance.

—

4 Example: Finding a Root using Regula Falsi Method

Find a real root of the equation $f(x) = x^3 - 2x - 5 = 0$ using the Regula Falsi method. We will perform 3 iterations.

Step 1: Define the function

$f(x) = x^3 - 2x - 5$

Step 2: Choose Initial Guesses

Let's test some values to find a tight interval: $f(2) = 2^3 - 2(2) - 5 = 8 - 4 - 5 = -1$ $f(2.1) = (2.1)^3 - 2(2.1) - 5 = 9.261 - 4.2 - 5 = 0.061$ Since $f(2) = -1$ and $f(2.1) = 0.061$ have opposite signs, a root lies between 2 and 2.1. Initial interval: $[a_0, b_0] = [2, 2.1]$.

Iterative Table

Iteration	a	b	$f(a)$	$f(b)$	x_{new}	$f(x_{new})$
1	2.0000	2.1000	-1.0000	0.0610	2.0942	-0.0039
2	2.0942	2.1000	-0.0039	0.0610	2.0945	-0.0000
3	2.0945	2.1000	-0.0000	0.0610	2.0946	-0.0000

Calculations for Iteration 1: $a = 2, b = 2.1, f(a) = -1, f(b) = 0.061$ $x_{new} = 2 - \frac{(-1)(2.1-2)}{0.061-(-1)} = 2 - \frac{-0.1}{1.061} \approx 2 + 0.09425 = 2.09425$ (rounded to 4 decimal places) $x_{new} \approx 2.0942$ $f(2.0942) = (2.0942)^3 - 2(2.0942) - 5 \approx -0.0039$ Since $f(x_{new})$ is negative, and $f(a)$ is negative, the root must be between x_{new} and b. So, new a becomes x_{new}. New interval: $[2.0942, 2.1000]$.

Calculations for Iteration 2: $a = 2.0942, b = 2.1, f(a) = -0.0039, f(b) = 0.061$ $x_{new} = 2.0942 - \frac{(-0.0039)(2.1-2.0942)}{0.061-(-0.0039)} = 2.0942 - \frac{(-0.0039)(0.0058)}{0.0649} = 2.0942 - \frac{-0.00002262}{0.0649} \approx 2.0942 + 0.0003485 = 2.0945485$ $x_{new} \approx 2.0945$ $f(2.0945) \approx -0.0000$ (very close to zero) Since $f(x_{new})$ is negative, update $a = x_{new}$. New interval: $[2.0945, 2.1000]$.

Calculations for Iteration 3: $a = 2.0945, b = 2.1, f(a) \approx -0.0000, f(b) = 0.061$ $x_{new} = 2.0945 - \frac{(-0.0000)(2.1-2.0945)}{0.061-(-0.0000)} \approx 2.0945 + 0.0000 = 2.0945$ More precise calculations would show a tiny change leading to 2.0946. $x_{new} \approx 2.0946$ $f(2.0946) \approx -0.0000$

Final Answer after 3 iterations: The approximate root is **2.0946**.

—

5 Practice Questions

Solve the following non-linear equations using the Regula Falsi method. Provide the iterative table for 3 iterations for each.

Question 1

Find a root of $f(x) = x\log_{10}(x) - 1.2 = 0$. Use initial interval $[2, 3]$.

Function: $f(x) = x\frac{\ln x}{\ln 10} - 1.2$

Initial Values: $a_0 = 2$, $b_0 = 3$ $f(2) = 2\log_{10}(2) - 1.2 \approx -0.5980$ $f(3) = 3\log_{10}(3) - 1.2 \approx 0.2313$

Iterative Table:

Iteration	a	b	$f(a)$	$f(b)$	x_{new}	$f(x_{new})$
1	2.0000	3.0000	-0.5980	0.2313	2.7208	-0.0171
2	2.7208	3.0000	-0.0171	0.2313	2.7402	-0.0004
3	2.7402	3.0000	-0.0004	0.2313	2.7406	-0.0000

Final Answer after 3 iterations: The approximate root is **2.7406**.

—

Question 2

Find a root of $f(x) = x\sin(x) + \cos(x) = 0$. Use initial interval $[2, 3]$ (radians).

Function: $f(x) = x\sin(x) + \cos(x)$

Initial Values: $a_0 = 2$, $b_0 = 3$ $f(2) = 2\sin(2) + \cos(2) \approx 1.4025$ $f(3) = 3\sin(3) + \cos(3) \approx -0.5666$

Iterative Table:

Iteration	a	b	$f(a)$	$f(b)$	x_{new}	$f(x_{new})$
1	2.0000	3.0000	1.4025	-0.5666	2.7465	0.0653
2	2.7465	3.0000	0.0653	-0.5666	2.8029	0.0016
3	2.8029	3.0000	0.0016	-0.5666	2.8039	0.0000

Final Answer after 3 iterations: The approximate root is **2.8039**.

—

Question 3

Find a root of $f(x) = xe^x - 2 = 0$. Use initial interval $[0, 1]$.

Function: $f(x) = xe^x - 2$

Initial Values: $a_0 = 0$, $b_0 = 1$ $f(0) = 0 \cdot e^0 - 2 = -2.0000$ $f(1) = 1 \cdot e^1 - 2 \approx 0.7183$

Iterative Table:

Iteration	a	b	$f(a)$	$f(b)$	x_{new}	$f(x_{new})$
1	0.0000	1.0000	-2.0000	0.7183	0.7358	-0.4646
2	0.7358	1.0000	-0.4646	0.7183	0.8229	-0.0645
3	0.8229	1.0000	-0.0645	0.7183	0.8504	-0.0080

Final Answer after 3 iterations: The approximate root is **0.8504**.
—

Question 4

Find a root of $f(x) = \cos(x) - xe^x = 0$. Use initial interval $[0, 1]$ (radians).
Function: $f(x) = \cos(x) - xe^x$
Initial Values: $a_0 = 0$, $b_0 = 1$ $f(0) = \cos(0) - 0 \cdot e^0 = 1.0000$ $f(1) = \cos(1) - 1 \cdot e^1 \approx 0.5403 - 2.7183 = -2.1780$

Iterative Table:

Iteration	a	b	$f(a)$	$f(b)$	x_{new}	$f(x_{new})$
1	0.0000	1.0000	1.0000	-2.1780	0.3146	0.3601
2	0.3146	1.0000	0.3601	-2.1780	0.3541	0.0093
3	0.3541	1.0000	0.0093	-2.1780	0.3551	0.0000

Final Answer after 3 iterations: The approximate root is **0.3551**.
—

Question 5: Finding the Cube Root of 9

Find a real root of $x^3 - 9 = 0$. This is equivalent to finding the cube root of 9. Use initial interval $[2, 2.1]$.
Function: $f(x) = x^3 - 9$
Initial Values: $a_0 = 2$, $b_0 = 2.1$ $f(2) = 2^3 - 9 = 8 - 9 = -1.0000$ $f(2.1) = (2.1)^3 - 9 = 9.261 - 9 = 0.2610$

Iterative Table:

Iteration	a	b	$f(a)$	$f(b)$	x_{new}	$f(x_{new})$
1	2.0000	2.1000	-1.0000	0.2610	2.0785	-0.0093
2	2.0785	2.1000	-0.0093	0.2610	2.0799	-0.0001
3	2.0799	2.1000	-0.0001	0.2610	2.0800	-0.0000

Final Answer after 3 iterations: The approximate cube root of 9 is **2.0800**.
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Question 6: Finding the Fourth Root of 64

Find a real root of $x^4 - 64 = 0$. This is equivalent to finding the fourth root of 64. Use initial interval $[2.8, 2.9]$.
Function: $f(x) = x^4 - 64$
Initial Values: $a_0 = 2.8$, $b_0 = 2.9$ $f(2.8) = (2.8)^4 - 64 = 61.4656 - 64 = -2.5344$ $f(2.9) = (2.9)^4 - 64 = 70.7281 - 64 = 6.7281$

Iterative Table:

Iteration	a	b	$f(a)$	$f(b)$	x_{new}	$f(x_{new})$
1	2.8000	2.9000	-2.5344	6.7281	2.8273	-0.0610
2	2.8273	2.9000	-0.0610	6.7281	2.8279	-0.0011
3	2.8279	2.9000	-0.0011	6.7281	2.8280	-0.0000

Final Answer after 3 iterations: The approximate fourth root of 64 is **2.8280**.

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Ordinary Differential Equations and Numerical Methods (BMATS209)
Tutorial - 13

Abstract

This tutorial provides a comprehensive guide to finding the roots of non-linear equations using the Newton-Raphson method. This is an open method known for its rapid convergence when an initial guess close to the root is available. We will demonstrate the formula, step-by-step procedure, and include a detailed example with tabular data, followed by practice questions with their iterative values and final answers.

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1 Introduction to Non-Linear Equations and Root Finding

Finding the roots of non-linear equations $f(x) = 0$ is a common task in various scientific and engineering fields. Numerical methods are essential tools for approximating these roots when analytical solutions are not feasible.

The Newton-Raphson method is a powerful and widely used iterative technique for finding roots. Unlike bracketing methods (like Regula Falsi or Bisection) that require an interval enclosing the root, Newton-Raphson is an open method that requires only a single initial guess. Its strength lies in its fast convergence rate, typically quadratic, meaning the number of correct decimal places roughly doubles with each iteration. However, it requires the calculation of the derivative of the function, $f'(x)$.

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2 The Newton-Raphson Method

The Newton-Raphson method is based on the idea of approximating the function $f(x)$ by its tangent line at a given point. If x_i is an approximation to the root, then the next approximation, x_{i+1}, is the x-intercept of the tangent line to $f(x)$ at $(x_i, f(x_i))$.

Formula

The formula for the next approximation of the root, x_{i+1}, is given by:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

where $f'(x_i)$ is the derivative of $f(x)$ evaluated at x_i. For the method to work, $f'(x_i)$ must not be zero or very close to zero at any iteration.

Iteration Rule

Starting with an initial guess x_0, the formula is applied iteratively to generate a sequence of approximations $x_1, x_2, x_3, \ldots$ that ideally converge to the root. The process stops when $|f(x_{i+1})|$ is sufficiently small or when the change between successive approximations $|x_{i+1} - x_i|$ is below a predefined tolerance.

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3 Step-by-Step Procedure for Solving $f(x) = 0$

To solve a non-linear equation $f(x) = 0$ using the Newton-Raphson method:

1. **Define the function and its derivative:** Clearly write down $f(x)$ and its first derivative, $f'(x)$.
2. **Choose an Initial Guess:** Select an initial approximation, x_0. A good initial guess (one close to the actual root) significantly improves convergence.
3. **Iterate to find new approximation (x_{i+1}):** Use the formula:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

4. **Check for Convergence:** Repeat step 3 until $|f(x_{i+1})|$ is sufficiently close to zero or the absolute difference between successive approximations $|x_{i+1} - x_i|$ is within a specified tolerance.

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4 Example: Finding a Root using Newton-Raphson Method

Find a real root of the equation $f(x) = x^3 - 2x - 5 = 0$ using the Newton-Raphson method. We will perform 3 iterations.

Step 1: Define the function and its derivative

$f(x) = x^3 - 2x - 5$ $f'(x) = 3x^2 - 2$

Step 2: Choose an Initial Guess

Based on previous analysis (e.g., from Regula Falsi example, or graphical inspection), a root is near $x = 2$. Let's choose $x_0 = 2$.

Iterative Table

Iteration	x_i	$f(x_i)$	$f'(x_i)$	x_{i+1}
1	2.00000	-1.00000	10.00000	2.10000
2	2.10000	0.06100	11.23000	2.09457
3	2.09457	0.00006	11.16100	2.09455

Calculations for Iteration 1: $x_0 = 2$ $f(2) = 2^3 - 2(2) - 5 = 8 - 4 - 5 = -1$ $f'(2) = 3(2)^2 - 2 = 12 - 2 = 10$ $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{-1}{10} = 2 + 0.1 = 2.10000$

Calculations for Iteration 2: $x_1 = 2.10000$ $f(2.1) = (2.1)^3 - 2(2.1) - 5 = 9.261 - 4.2 - 5 = 0.06100$ $f'(2.1) = 3(2.1)^2 - 2 = 3(4.41) - 2 = 13.23 - 2 = 11.23000$ $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.1 - \frac{0.061}{11.23} = 2.1 - 0.0054318 = 2.0945682$ $x_2 \approx 2.09457$

Calculations for Iteration 3: $x_2 = 2.09457$ $f(2.09457) = (2.09457)^3 - 2(2.09457) - 5 \approx 0.00006$ $f'(2.09457) = 3(2.09457)^2 - 2 \approx 11.16100$ $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.09457 - \frac{0.00006}{11.16100} = 2.09457 - 0.00000537 = 2.0945463$ $x_3 \approx 2.09455$

Final Answer after 3 iterations: The approximate root is **2.09455**.

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5 Practice Questions

Solve the following non-linear equations using the Newton-Raphson method. Provide the iterative table for 3 iterations for each.

Question 1

Find a root of $f(x) = x\sin(x) + \cos(x) = 0$. Use initial guess $x_0 = 3$ (radians).

Function: $f(x) = x\sin(x) + \cos(x)$ **Derivative:** $f'(x) = \sin(x) + x\cos(x) - \sin(x) = x\cos(x)$

Initial Values: $x_0 = 3$

Iterative Table:

Iteration	x_i	$f(x_i)$	$f'(x_i)$	x_{i+1}
1	3.00000	-0.56660	-2.96996	2.81001
2	2.81001	0.00833	-2.80975	2.80705
3	2.80705	0.00001	-2.80705	2.80705

Final Answer after 3 iterations: The approximate root is **2.80705**.

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Question 2

Find a root of $f(x) = e^{-x} - x = 0$.

Function: $f(x) = e^{-x} - x$ **Derivative:** $f'(x) = -e^{-x} - 1$

Initial Values: $x_0 = 0.5$

Iterative Table:

Iteration	x_i	$f(x_i)$	$f'(x_i)$	x_{i+1}
1	0.50000	0.10653	-1.60653	0.56631
2	0.56631	0.00131	-1.56762	0.56714
3	0.56714	0.00000	-1.56714	0.56714

Final Answer after 3 iterations: The approximate root is **0.56714**.

Question 3: Finding the Cube Root of 15

Find a real root of $x^3 - 15 = 0$. This is equivalent to finding the cube root of 15. **Function:** $f(x) = x^3 - 15$

Derivative: $f'(x) = 3x^2$

Initial Values: $x_0 = 2.5$

Iterative Table:

Iteration	x_i	$f(x_i)$	$f'(x_i)$	x_{i+1}
1	2.50000	0.62500	18.75000	2.46667
2	2.46667	0.00551	18.25333	2.46637
3	2.46637	0.00000	18.24888	2.46637

Final Answer after 3 iterations: The approximate cube root of 15 is **2.46637**.

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Question 4: Finding the Square Root of 7

Find a real root of $x^2 - 7 = 0$. This is equivalent to finding the square root of 7.

Function: $f(x) = x^2 - 7$ **Derivative:** $f'(x) = 2x$

Initial Values: $x_0 = 2.5$

Iterative Table:

Iteration	x_i	$f(x_i)$	$f'(x_i)$	x_{i+1}
1	2.50000	-0.75000	5.00000	2.65000
2	2.65000	0.02250	5.30000	2.64575
3	2.64575	0.00000	5.29150	2.64575

Final Answer after 3 iterations: The approximate square root of 7 is **2.64575**.

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Question 5

Find a real root of $f(x) = x^2 - e^x = 0$. Use initial guess $x_0 = -0.7$.

Function: $f(x) = x^2 - e^x$ **Derivative:** $f'(x) = 2x - e^x$

Initial Values: $x_0 = -0.7$

Iterative Table:

Iteration	x_i	$f(x_i)$	$f'(x_i)$	x_{i+1}
1	-0.70000	-0.00650	-1.89650	-0.70343
2	-0.70343	-0.00001	-1.89966	-0.70343
3	-0.70343	0.00000	-1.89966	-0.70343

Final Answer after 3 iterations: The approximate root is $-\mathbf{0.70343}$.