Module-3
Numerical Methods - I

* Finite Difference:

. Interpolation:

Given a set of data points of (x,y) i.e. $x_0, x_1, x_2 \dots x_n$ are values of x. $y_0, y_1, \dots y_n$ are values of y. The method of finding/estimating y at some value of x, where $x \in (x_0, x_n)$ is called interpolation.

The method of finding y at x where x lies outside the range (x_0, x_n) is called extrapolation.

* Methods of estimating y:

1. NFIF (Newton Forward Interpolation Formula).
2. NBIF (" Backward " ").
3. Lagrange's Interpolation Formula.
4. NDIF (Newton's Divided Interpolation Formula).

i) NFIF :-

x_0 1
x_1 3 } 3-1=2 (h)
x_2 5
$\vdots$
x_n

Finite differences: (Forward Difference)

1st F.D → $\Delta f(x) = f(x+h) - f(x)$ ...... (1)

forward difference

h is the common difference b/w values of x.

→ If we add 'h' to x_0, we get x_1

→ $y = f(x)$

$$\Delta f(x_0) = f(x_0+h) - f(x_0)$$
$$\Delta y_0 = f(x_1) - f(x_0) = y_1 - y_0$$
$$\Delta f(x_1) = f(x_1+h) - f(x_1)$$
$$\Delta y_1 = f(x_2) - f(x_1)$$
$$\therefore \boxed{\Delta y_1 = y_2 - y_1}$$

2nd F.D: $\Delta[\Delta f(x_0)] = \Delta[y_1 - y_0]$

$$\boxed{\Delta^2 y_0 = \Delta y_1 - \Delta y_0}$$

∴ The above relⁿ are called forward differences.

Eg:

	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
x_0	2	$28 \to y_0$	Δy_0 $\downarrow$ 4	$\Delta^2 y_0 = 14-4 = 10$	$\Delta^3 y_0 = -2-10 = -12$
x_1	4	$32 \to y_1$			
x_2	6	$46 \to y_2$	$\Delta y_1 = 14$	$\Delta^2 y_1 = 12-14 = -2$	
x_3	8	$58 \to y_3$	$\Delta y_2 = 12$		

→ If we get same value in a column of table, we can stop.

⊙ NFIF:

$$y(x_r) = y_0 + \frac{r}{1!}\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \cdots\cdots + \frac{r(r-1)\cdots(r-(n-1))\Delta^n y_0}{n!}$$

where,

$$\boxed{r = \frac{x_r - x_0}{h}}$$

→ x_r is value of x at which we have to compute y.

→ x_0 is initial value of x.

→ h is common difference.
$\Rightarrow x_1 - x_0$

* Backward Difference (B.D):

1st B.D:

$$\nabla f(x) = f(x) - f(x-h)$$

$$\nabla f(x_1) = f(x_1) - f(x_1 - h)$$

$h = x_1 - x_0$

$\Rightarrow \boxed{x_0 = x_1 - h}$

$$\nabla y_1 = f(x_1) - f(x_0)$$

$$\boxed{\nabla y_1 = y_1 - y_0}$$

$$\boxed{\nabla y_2 = y_2 - y_1}$$

2nd B.D: $\nabla^2 f(x) = \nabla[\nabla f(x)]$

$$\nabla^2 f(x_2) = \nabla[\nabla f(x_2)]$$

$$= \nabla[y_2 - y_1]$$

$$\boxed{\nabla^2 y_2 = \nabla y_2 - \nabla y_1}$$

∴ The above relations are called backward differences.

Eg:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
$x_0=2$	$y_0=28$			
$x_1=4$	$y_1=32$	$\nabla y_1 = 4$		
$x_2=6$	$y_2=46$	$\nabla y_2 = 14$	$\nabla^2 y_2 = 10$	
$x_3=8$	$y_3=58$	$\nabla y_3 = 12$	$\nabla^2 y_3 = -2$	$\nabla^3 y_3 = -12$

⊙ NBIF:

$$y(x_r) = y_n + \frac{r}{1!}\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \cdots + \frac{r(r+1)\ldots(r+(n-1))\nabla^n y_n}{n!}$$

where,

$x_r \rightarrow$ Value of x at which we find y.

$x_n \rightarrow$ final value of x.

$h \rightarrow$ common difference $= x_1 - x_0$.

* Note:

- If x_r is closer to x_0, we apply NFIF.
- If x_r " " " x_n, we apply NBIF.

{for midpoint we apply NFIF.}

* Problems :

① Find $y(x=4)$.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
$x_0 = 1$	$22 = y_0$			
		$\Delta y_0 = 5$		
$x_1 = 3$	$27 = y_1$		$\Delta^2 y_0 = 0$	
		$\Delta y_1 = 5$		$\Delta^3 y_0 = 2$
$x_2 = 5$	$32 = y_2$		$\Delta^2 y_1 = 2$	
		$\Delta y_2 = 7$		
$x_3 = 7$	$39 = y_3$			

Given :

$x_r = 4 \,;\; x_0 = 1 \,;\; h = 2 = x_1 - x_0$

We apply NFIF (x_r is closer to x_0).

$$r = \frac{x_r - x_0}{h}$$

$$= \frac{4-1}{2} = 3/2$$

Now;

$$\text{NFIF}: y(x_r) = y_0 + \frac{r}{1!}\Delta y_0 + \frac{r(r-1)\Delta^2 y_0}{2!} + \frac{r(r-1)(r-2)\Delta^3 y_0}{3!}$$

$$= 22 + \frac{(3/2)5}{1} + \frac{(3/2)(1/2)}{2}(0) + \frac{(3/2)(1/2)(-1/2)}{6} \times 2$$

$$= 22 + \frac{15}{2} + \left(\frac{-1}{8}\right)$$

$$= 29.3750.$$

(2) Find the interpolating polynomial for the following data and hence find $y(x=0.75)$, $y(x=0.5)$.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
$x_0 = 0$	$1 = y_0$			
		$\Delta y_0 = 1$		
$x_1 = 1$	$2 = y_1$		$\Delta^2 y_0 = -2$	
		$\Delta y_1 = -1$		$\Delta^3 y = 12.$
$x_2 = 2$	$1 = y_2$		$\Delta^2 y_1 = 10$	
		$\Delta y_2 = 9$		
$x_3 = 3$	$10 = y_3$			

Leading entities.

Soln:

$h = 1$; $x_r = 0.75$, $x_r = 0.5$

Both values of x_r are closer to $x_0 = 0$, we apply NFIF.

Now:

NFIF: $$y(x_r) = y_0 + \frac{r}{1!}\Delta y_0 + \frac{r(r-1)\Delta^2 y_0}{2!} + \frac{r(r-1)(r-2)\Delta^3 y_0}{3!}$$

$$= 1 + \frac{r}{1}(1) + \frac{r(r-1)}{2}(-2) + \frac{r(r-1)(r-2)}{6}.12$$

$$= 1 + r - \{r^2 - r\} + 2(r^3 - 3r^2 + 2r)$$

$$= 2r^3 - 7r^2 + 6r + 1. \quad \cdots\cdots (1)$$

Case (i): $x_r = 0.75$

$$r = \frac{x_r - x_0}{h} = \frac{0.75 - 0}{1} = 0.75$$

$$\therefore y(0.75) = 2(0.75)^3 - 7(0.75)^2 + 6(0.75) + 1$$

$$= 2.40625$$

Case (ii): $x_r = 0.5$

$$r = \frac{x_r - x_0}{h} = \frac{0.5 - 0}{1} = 0.5$$

$$\therefore y(0.5) = 2(0.5)^3 - 7(0.5)^2 + 6(0.5) + 1$$

$$= 2.5000.$$

3) Find the no. of students who have scored:

ⓐ less than 45 marks.

ⓑ bet^{n} 40 and 45 marks? $\Rightarrow f(45)-f(40)$ for the following data.

x	y	$X(\le)$	Y (students scoring less than)	Δy	$\Delta^2 y$	$\Delta^3 y$
0-40	31	$x_0=40$	$31=y_0$	$\Delta y_0=42$	$\Delta^2 y_0=9$	$\Delta^3 y_0=-25$
40-50	42	$x_1=50$	$42+31=73=y_1$	$\Delta y_1=51$	$\Delta^2 y_1=-16$	$\Delta^3 y_1=12$
50-60	51	$x_2=60$	$73+51=124=y_2$	$\Delta y_2=35$	$\Delta^2 y_2=-4$	
60-70	~~33~~ 35	$x_3=70$	$124+35=159=y_3$	$\Delta y_3=31$		
70-80	31	$x_4=80$	$159+31=190=y_4$			

Data is continuous $\Rightarrow$ Discrete.

$\rightarrow$ Data is continuous
$\hookrightarrow$ make it discrete.

Case (i): $x_r=45$ is closer to $x_0=40$, we apply NFIF. $r=\frac{x_r-x_0}{h}=\frac{45-40}{10}=1/2$.

$$y(x_r)=31+\frac{(1/2)}{1}\times 42+\frac{(1/2)(-1/2)}{2}\cdot 9+\frac{(1/2)(-1/2)(-3/2)}{6}(-25)+\frac{(1/2)(-1/2)(-3/2)(-5/2)}{24}(12)$$

~~$y(45)=56.8679\simeq 57$~~

~~ii) $y(45)-y(40)=57-31=26$~~

i) $y(45)=47.86\simeq 48$

ii) $y(45)-y(40)=48-31=17$.

* Backward

① Find the

x	y
$x_0=10$	21
$x_1=11$	23
$x_2=12$	27
$x_3=13$	33

Given:

$h=x_1-$

~~Case ①~~

$x_r=12$

x_r is c

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$y(x_r$

$y(x$

Case ①:

$x_r=1$

Substitu

$y(12.5)$

$\therefore y(12.5$

Case ②:

① $\Rightarrow y($

$\therefore y$

* Backward:

(NBIF) {x_r is closer to x_n}

① Find the interpolating polynomial and hence find $y(12.5)$, $y(13.1)$

x	y	∇y	$\nabla^2 y$
$x_0 = 10$	21		
		$\nabla y_1 = 2$	
$x_1 = 11$	23		$\nabla^2 y_2 = 2$
		$\nabla y_2 = 4$	
$x_2 = 12$	27		$\nabla^2 y_3 = 2$
		$\nabla y_3 = 6$	
$x_3 = 13$	33		

Given:

$h = x_1 - x_0 = 11 - 10 = 1$, (common difference)

~~Case ①~~:

$x_r = 12.5$, $x_r = 13.1$

x_r is closer to $x_n = x_3$

We apply NBIF.

$$y(x_r) = 33 + \frac{r}{1!}\cdot 6 + \frac{r(r+1)}{2!}\cdot 2$$

$$= 33 + 6r + r^2 + r$$

$$y(x_r) = r^2 + 7r + 33 \quad \cdots\cdots ①$$

$x_r \neq r$

↳ they have a rel^n but not same

$$r = \frac{x_r - x_n}{h}$$

Case ①:

$x_r = 12.5 \Rightarrow r = \frac{12.5 - 13}{1}$

$\therefore r = -0.5$

Substitute in ①;

$y(12.5) = (-0.5)^2 + 7(-0.5) + 33$

$\therefore y(12.5) = 29.75000$

Case ②: $x_r = 13.1 \Rightarrow r = \frac{13.1 - 13}{1} = 0.1$

① $\Rightarrow y(13.1) = (0.1)^2 + 7(0.1) + 33$

$\therefore y(13.1) = 33.71000$

① $(x+a)(x+b)(x+c)$

$\Rightarrow x^3 + (a+b+c)x^2 + (ab+bc+ca)x + abc.$

② Find the interpolating polynomial and hence find $y(105)$

$\Rightarrow$ Find area where diameter is 105 cm.

Diameter x	Area y	∇y_n	$\nabla^2 y_n$	$\nabla^3 y_n$	$\nabla^4 y_n$
$x_0 = 80$	$5026 = y_0$				
		$\nabla y_1 = 648$			
$x_1 = 85$	$5674 = y_1$		$\nabla^2 y_2 = 40$		
		$\nabla y_2 = 688$		$\nabla^3 y_3 = -2$	
$x_2 = 90$	$6362 = y_2$		$\nabla^2 y_3 = 38$		$\nabla^4 y_n = 4$
		$\nabla y_3 = 726$		$\nabla^3 y_4 = 2$	
$x_3 = 95$	$7088 = y_3$		$\nabla^2 y_4 = 40$		
$x_4 = 100$	$7854 = y_4$	$\nabla y_4 = 766$			

Given:

$h = x_1 - x_0 = 85 - 80 = 5$

$x_k = 105$ (x_k is closer to $x_n = x_4$).

NBIF.

$k = \dfrac{x_k - x_n}{h} = \dfrac{105-100}{5}$

$\boxed{k = 1}.$

$y(x_k) = y_n + \dfrac{k}{1!}\nabla y_n + \dfrac{k(k+1)}{2!}\nabla^2 y_n + \cdots$

$= 7854 + \dfrac{k}{1!}\cdot 766 + \dfrac{k(k+1)}{2!}\cdot 40 + \dfrac{k(k+1)(k+2)}{\cancel{3!}\ 3}\cdot 2 + \dfrac{k(k+1)(k+2)(k+3)}{\cancel{4!}\ 6}\cdot \cancel{4}$

$y(x_k) = 7854 + 766\cdot k + 20k^2 + 20k + \frac{1}{3}(k^3 + 3k^2 + 2k) + \frac{1}{6}(k^4 + 6k^3 + 11k^2 + 6k)$

$y(x_k) = \frac{1}{6}\cdot k^4 + \frac{4}{3}k^3 + \left(21 + \frac{11}{6}\right)k^2 + \left(787 + \frac{2}{3}\right)k + 7854$

$y(105) = \frac{1}{6} + \frac{4}{3} + 21 + \frac{11}{6} + 787 + \frac{2}{3} + 7854$

$y(105) = 8666\ cm^2$

3) Find the polynomial hence find $f(5)$, $f(0.5)$

$f(0) = -5$, $f(1) = -10$, $f(2) = -9$, $f(3) = 4$, $f(4) = 35$. $y = f(x)$.

x	y	Ist diff.	II diff.	III diff.
$x_0 = 0$	-5			
		$-5 = \Delta y_0$		
$x_1 = 1$	-10		$\Delta^2 y_0 = 6$	
		1		$6 = \Delta^3 y_0$
$x_2 = 2$	-9		12	
		13		$6 = \nabla^3 y_n$
$x_3 = 3$	4		$18 = \nabla^2 y_n$	
		$31 = \nabla y_n$		
$x_4 = 4$	35			

Since; $h = 1$;

Since, $x_r = 5$ is closer to $x_n = 4$.

Apply NBIF

$r = \frac{5-4}{1} = 1$

Case ①: $x_r = 5$

$$y(x_r) = 35 + \frac{r}{1!} \cdot 31 + \frac{r(r+1)}{2!} \cdot 18 + \frac{r(r+1)(r+2)}{3!} \cdot 6$$

$$= 35 + 31 + 9(2) + 6$$

$$y(5) = 90$$

Case ②:

$x_r = 0.5$

$x_0 = 0, h = 1$

$r = \frac{0.5 - 0}{1} = 0.5$

Apply NFIF;

$$y(x_r) = -5 + \frac{r}{1!} \times (-5) + \frac{r(r-1)}{2!}(\cancel{6}\,3) + \frac{r(r-1)(r-2)}{3!} \cdot \cancel{6}$$

$$y(0.5) = -5 - \frac{5}{2} - \frac{3}{4} + (+3/8)$$

$$y(0.5) = -7.87500$$

* Newton's Divided Difference formula : (unequal intervals) {h is constant}. ①

$f(x) = f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0, x_1, x_2) + (x-x_0)(x-x_1)(x-x_2) f(x_0, x_1, x_2, x_3) + \dots$

1st D.D : $f(x_0, x_1) = \dfrac{f(x_1) - f(x_2)}{x - x_0}$, $f(x_1, x_2) = \dfrac{f(x_1) - f(x_2)}{x_1 - x_2}$

2nd D.D : $f(x_0, x_1, x_2) = f$

* Problems :

① Find $f(9), f(18)$ for the given data.

x	$y = f(x)$
$x_0 = 5$	$150 = f(x_0)$
$x_1 = 7$	$392 = f(x_1)$
$x_2 = 11$	$1452 = f(x_2)$
$x_3 = 13$	$2366 = f(x_3)$
$x_4 = 17$	$5202 = f(x_4)$

1st D.D

$\rightarrow f(x_0, x_1)$
$392 - 150 / 7-5$

$\rightarrow f(x_1, x_2)$
$1452 - 392 / 11-7$

$h = x_1 - x_0$
$= 7 - 5 = 2$

$x_2 - x_1$
$= 11 - 7 = 4$

$\Rightarrow$ h is not constant.
Apply NDDF.

X	$f(x) = y$	1st D.D	2nd D.D	3rd D.D
$x_0 = 5$	$150 = f(x_0)$	$f(x_0, x_1)$ $\frac{392-150}{7-5} = 121$	$f(x_0, x_1, x_2)$ $\frac{265-121}{11-5} = 24$	$f(x_0, x_1, x_2, x_3)$ $\frac{32-24}{13-5} = 1$
$x_1 = 7$	$392 = f(x_1)$	$f(x_1, x_2)$ $\frac{1452-392}{11-7} = 265$	$f(x_1, x_2, x_3)$ $\frac{457-265}{13-7} = 32$	$f(x_1, x_2, x_3, x_4)$ $\frac{42-32}{17-7} = 1$
$x_2 = 11$	$1452 = f(x_2)$	$f(x_2, x_3)$ $\frac{2366-1452}{13-11} = 457$	$f(x_2, x_3, x_4)$ $\frac{709-457}{17-11} = 42$	
$x_3 = 13$	$2366 = f(x_3)$	$\frac{5202-2366}{17-13} = 709$		
$x_4 = 17$	$5202 = f(x_4)$			

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2) \cdot f(x_0, x_1, x_2, x_3)$$

$$f(x) = 150 + (x-5) \cdot 121 + (x-5)(x-7) \cdot 24 + (x-5)(x-7)(x-11) \cdot 1$$

$$= 150 + 121x - 605 + (x^2 - 12x + 35) 24 + x^3 - 23x^2 + 167x - 385$$

$$f(x) = x^3 + x^2$$

$\therefore f(9) = 9^3 + 9^2 = 810$

$\therefore f(18) = 18^3 + 18^2 = 6156$

2) Find $f(8)$, $f(15)$ for the following data.

X	$y = f(x)$	1st D.D	2nd D.D	3rd D.D
$x_0 = 4$	$48 = f(x_0)$			
		$f(x_0,x_1) \rightarrow \frac{100-48}{5-4} = 52$		
$x_1 = 5$	$100 = f(x_1)$		$f(x_0,x_1,x_2) \rightarrow \frac{97-52}{7-4} = 15$	
		$f(x_1,x_2) \rightarrow \frac{294-100}{7-5} = 97$		$f(x_0,x_1,x_2,x_3) \rightarrow \frac{21-15}{10-4} = 1$
$x_2 = 7$	$294 = f(x_2)$		$f(x_1,x_2,x_3) \rightarrow \frac{202-97}{10-5} = 21$	
		$f(x_2,x_3) \rightarrow \frac{900-294}{10-7} = 202$		$f(x_1,x_2,x_3,x_4) \rightarrow \frac{27-21}{11-5} = 1$
$x_3 = 10$	$900 = f(x_3)$		$f(x_2,x_3,x_4) \rightarrow \frac{310-202}{11-7} = 27$	
		$f(x_3,x_4) \rightarrow \frac{1210-900}{11-10} = 310$		$f(x_2,x_3,x_4,x_5) \rightarrow \frac{33-27}{13-7} = 1$
$x_4 = 11$	$1210 = f(x_4)$		$f(x_3,x_4,x_5) \rightarrow \frac{409-310}{13-10} = 33$	
		$f(x_4,x_5) \rightarrow \frac{2028-1210}{13-11} = 409$		
$x_5 = 13$	$2028 = f(x_5)$			

$$f(x) = f(x_0) + (x-x_0)f(x_0,x_1) + (x-x_0)(x-x_1)f(x_0,x_1,x_2) + (x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3) \cdots$$

$$f(x) = \cancel{150}\,48 + (x-4).52 + (x-4)(x-5).15 + (x-4)(x-5)(x-7).1$$

$$= 48 + 52x - 208 + ((x^2 - 5x - 4x + 20)15) + (x^3 - 16x^2 + 83x - 140)$$

$$= 48 + 52x - 208 + 15x^2 - 135x + 300 + x^3 - 16x^2 + 83x - 140$$

$\therefore f(x) = x^3 - x^2$

$\therefore f(8) = 8^3 - 8^2 = 448$

$\therefore f(15) = 15^3 - 15^2 = 3150$

3) Find $U(2)$, when ~~$U(10)$~~ $U_{10}=355$, $U_0=-5$, $U_1=-14$, $U_3=-21$, $U_4=-125$

x	$f(x)=U_x$	1st D.D	2nd D.D	3rd D.D
0	-5			
1	-14	$f(x_0,x_1)$ $\frac{-14+5}{1-0}=-9$		
4	-125	$f(x_1,x_2)$ $\frac{-125+14}{4-1}=-37$	$f(x_0,x_1,x_2)$ $\frac{-37+9}{4-0}=-7$	
8	-21	$f(x_2,x_3)$ $\frac{-21+125}{8-4}=26$	$f(x_1,x_2,x_3)$ $\frac{26+37}{8-1}=9$	$f(x_0,x_1,x_2,x_3)$ $\frac{9+7}{8-0}=2$
10	355	$f(x_3,x_4)$ $\frac{355+21}{10-8}=188$	$f(x_2,x_3,x_4)$ $\frac{188-26}{10-4}=27$	$f(x_1,x_2,x_3,x_4)$ $\frac{27-9}{10-1}=2$

$U_2 \Rightarrow x=2.$

$$\therefore f(x) = -5+(2-0)\cdot(-9)+(2-0)(2-1)(-7)+(2-0)(2-1)(2-4)\cdot 2$$

$$\therefore f(2) = -45$$

* Lagrange's Interpolation Formula (LIF)

{ for both equal and non-equal intervals }:

$$y = \frac{(x-x_1)(x-x_2)\cdots\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots\cdots(x_0-x_n)} \cdot y_0 + \frac{(x-x_0)(x-x_2)\cdots\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots\cdots(x_1-x_n)} y_1 + \cdots + \frac{(x-x_0)(x-x_1)\cdots\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots\cdots(x_n-x_{n-1})} \cdot y_n$$

x_0	y_0
x_1	y_1
x_2	y_2
x_3	y_3
$\vdots$	$\vdots$
x_n	y_n

$y(x = ?)$

Note:

1. By above formula, we find y at some point of x.
2. If we need estimate x at some point of y, we apply Inverse Lagrange's formula: by interchanging roles of x & y in (1).
3. Finding a root of an equation at $y=0$, we find x.

Problems:

① Find $f(4)$ for given data.

	x_0	x_1	x_2	x_3
x	0	2	3	6
$f(x) = y$	-4	2	14	158
	y_0	y_1	y_2	y_3

Sol:

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \cdot y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \cdot y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \cdot y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \cdot y_3$$

$\boxed{x=4}$

$$f(4) = y = \frac{(4-2)(4-3)(4-6)}{(-2)(-3)(-6)} \cdot (-4) + \frac{(4-0)(4-3)(4-6)}{(2)(-1)(-4)} \cdot 2 + \frac{(4-0)(4-2)(4-6)}{(3)(1)(-3)} \cdot 14$$

$$+ \frac{(4-0)(4-2)(4-3)}{(6)(+4)(3)} \cdot 158$$

$$y = \frac{2(1)(-2)(-4)}{-36} + \frac{(4)(1)(-2)(2)}{-8} + \frac{(4)(2)(-2) \cdot 14}{-9} + \frac{4(2)(1) \cdot 158}{72}$$

$\therefore y = 40.$

① Using LIF, find interpolating polynomial and hence find y at $x=2$, for the following data.

$(x+a)(x+b)(x+c)$
$= x^3 + (a+b+c)x^2 + (ab+bc+ca)x + abc$

	x_0	x_1	x_2	x_3
x	0	1	3	4
y	-12	0	6	12
	y_0	y_1	y_2	y_3

Solⁿ

$$y = \frac{(x-1)(x-3)(x-4)}{(-1)(3)(4)}\cdot(-12) + \frac{(x-0)(x-3)(x-4)}{(1)(-2)(-3)}\cdot(0) + \frac{(x-0)(x-1)(x-4)}{(3)(2)(-1)}\cdot 6$$

$$+ \frac{(x-0)(x-1)(x-3)}{4(3)(1)}\cdot 12$$

$$= \frac{(x^3-8x^2+19x-12)\cancel{-12}}{\cancel{(-12)}} + \cancel{\frac{(x^3-7x^2+12x)}{6}}\cdot 0 + \frac{(x^3-5x^2+4x)\cancel{6}}{-\cancel{6}}$$

$$+ \frac{(x^3-4x^2+3x)}{\cancel{12}}\cdot\cancel{12}$$

$$\cancel{= -x^3 + x^2 + 12x - 12}$$

$$= x^3 - 7x^2 + 18x - 12$$

$$y(x=2) = 8 - 28 + 36 - 12$$
$$= 4.$$

③ Find real root of the function $f(x)=0$ for following data by applying appropriate Lagrange's formula.

x	30	34	38	42
$f(x)=$ y	-30	-13	3	18

{Real root of $f(x) \xRightarrow{\text{where}} f(x)=0$, we find $x=?$}

Apply Inverse Lagrange's formula:

$$x = \frac{(y-y_1)(y-y_2)(y-3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)} \cdot x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)} \cdot x_1$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_3)}{(y_2-y_0)(y_2-y_1)(y_2-y_1)} \cdot x_2 + \frac{(y-y_0)(y-y_1)(y-y_2)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)} \cdot x_3$$

$$x = \frac{(+13)(-3)(-18)}{(-17)(-33)(-48)} \cdot 30 + \frac{(30)(-3)(-18)}{(17)(-16)(-31)} \cdot 34 + \frac{(30)(+13)(-18)}{(33)(16)(-15)} \cdot 38$$

$$+ \frac{(30)(13)(-3)}{(48)(31)(15)} \cdot (42)$$

$$x(y=0) = 37.23038.$$

* Numerical Integration:

$$I = \int_a^b y \cdot dx$$

$\Rightarrow \int_0^1 e^{-x^2}$ are non-integrable f^n, they can be integrated using these formulas.

Step size $(h) = \frac{b-a}{n}$ ↳ no. of divisions

*) Methods to integrate:

ⓐ Simpson's $1/3^{rd}$ Rule; (n is a Multiple of 2)

$$I = \frac{h}{3}\left[(y_0+y_n) + 2(y_2+y_4+y_6+\cdots\cdots) + 4(y_1+y_3+\cdots\cdots)\right]$$

ⓑ Simpson's $\frac{3}{8}$ th Rule; (n is a Multiple of 3)

$$I = \frac{3h}{8}\left[(y_0+y_n) + 2(y_3+y_6+\cdots\cdots) + 3(y_1+y_2+y_4+y_5+\cdots\cdots)\right]$$

Note:

① If $(n+1)$ ordinates are given then we can make n divisions.

② No terms are repeated more than once in any formula.

③ $x_1 = x_0 + h$

$x_2 = x_1 + h.$

Problems:

① Evaluate: $\int_0^1 \frac{dx}{1+x^2}$ by taking 7 ordinates also find the value of π comparing with analytical method.

7o → 6 divisions

Sol^n

$$I = \int_a^b y \cdot dx \Rightarrow y = \frac{1}{1+x^2}$$

$$a = 0, \; b = 1.$$

7 ordinates makes ⇒ No. of divisions (n) = 6

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
X	0	1/6	2/6	3/6	4/6	5/6	6/6 = 1
Y	1	36/37	9/10	4/5	9/13	36/61	1/2
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$\Rightarrow y_0 = \frac{1}{1+0^2} = 1$

$\Rightarrow y_1 = \frac{1}{1+(1/6)^2} = 36/37$

$\Rightarrow y_2 = \frac{1}{1+(2/6)^2} = 9/10$

$\Rightarrow y_3 = \frac{1}{1+(3/6)^2} = 4/5$

$\Rightarrow y_4 = \frac{1}{1+(4/6)^2} = 9/13$

$\Rightarrow y_5 = \frac{1}{1+(5/6)^2} = 36/61$

$\Rightarrow y_6 = \frac{1}{1+1^2} = 1/2$

Case (I): Simpson's 1/3rd Rule:

$$I = \frac{h}{3}\left[(y_0+y_n) + 2(y_2+y_4+y_6+\cdots\cdots) + 4(y_1+y_3+\cdots\cdots)\right]$$

$$= \frac{1/6}{3}\left[(1+1/2) + 2(9/10 + 9/13) + 4(36/37 + 4/5 + 36/61)\right]$$

$$= 0.785\bullet 40$$

Case (II): Simpson's 3/8th Rule:

$$I = \frac{3h}{8}\left[(y_0+y_6) + 2(y_3) + 3(y_1+y_2+y_4+y_5)\right]$$

y_6 can't be mentioned as it's y_n.

$$= \frac{3}{8}\left(\frac{1}{6}\right)\left[\left(1+\frac{1}{2}\right) + 2\left(\frac{4}{5}\right) + 3\left(\frac{36}{37} + \frac{9}{10} + \frac{9}{13} + \frac{36}{61}\right)\right]$$

$$= 0.78540$$

Direct integration:

$$I = \int_0^1 \frac{1}{1+x^2}\,dx = \tan^{-1}x\Big]_0^1 = \tan^{-1}(1) - \tan^{-1}(0)$$

$$\therefore I = \frac{\pi}{4} - 0$$

$$\Rightarrow \pi = 4I \Rightarrow \pi = 4(0.78540) = 3.1415$$

2) By taking 6 equal parts ~~hence~~ find value of $\log_e 2$.

~~dot~~ $I = \int_0^1 \frac{dx}{1+x}$

~~yc~~

Sol^n:

$y = \frac{1}{1+x}$ $\qquad h = \frac{b-a}{n}$

$a = 0$ $\qquad = 1/6$

$b = 1$

Divisions (n) = 6

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
x	0	1/6	2/6	3/6	4/6	5/6	6/6 = 1
y	1	6/7	3/4	2/3	3/5	6/11	1/2
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$\Rightarrow y_0 = \frac{1}{1+(1/6)} = 6/7$ $\qquad \Rightarrow y_3 = \frac{1}{1+1/2} = 2/3$ $\qquad \Rightarrow y_6 = \frac{1}{1+1} = 1/2$

$\Rightarrow y_1 = \frac{1}{1+(2/6)} = 3/4$ $\qquad \Rightarrow y_4 = \frac{1}{1+2/3} = 3/5$

$\Rightarrow y_2 = \frac{1}{1+(3/6)} = 2/3$ $\qquad \Rightarrow y_5 = \frac{1}{1+5/6} = 6/11$

Using Simpson's $\frac{3}{8}$ th Rule:

$$I = \frac{3h}{8}\left[(y_0+y_6) + 2(y_3) + 3(y_1+y_2+y_4+y_5)\right]$$

$$= \frac{3}{8}\left(\frac{1}{6}\right)\left[\left(1+\frac{1}{2}\right) + 2\left(2/3\right) + 3\left(\frac{6}{7}+\frac{3}{4}+\frac{3}{5}+\frac{6}{11}\right)\right]$$

$$= 0.693195$$

Direct integration:

$$I = \int_0^1 \frac{dx}{1+x} = \log_e(1+x)\Big|_0^1 = \log 2 - \cancel{\log 1}^{\,0}$$

$\therefore I = \log_e 2$

$\therefore I = 0.693195$

3) $\int_0^{0.6} e^{-x^2} \cdot dx$, take 7 ordinates. Consider Simpson's 1/3rd Rule for the problem. (6 equal division)

Sol?:

$a=0, \quad b=0.6, \quad n=6$

$$h = \frac{b-a}{n} = \frac{0.6-0}{6} = 0.1$$

$y = e^{-x^2}$

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
x	0	0.1	0.2	0.3	0.4	0.5	0.6
y	1	0.99	0.96079	0.91393	0.85214	0.7788	0.69768
	y_0	y_1	y_2	y_3	y_4	y_5	

$\Rightarrow y_0 = e^{-x_0^2} = e^{-0} = 1$

$\Rightarrow y_1 = e^{-(0.1)^2} = 0.99$

$\Rightarrow y_2 = e^{-(0.2)^2} = 0.96079$

$\Rightarrow y_3 = e^{-(0.3)^2} = 0.91393$

$\Rightarrow y_4 = e^{-(0.4)^2} = 0.85214$

$\Rightarrow y_5 = e^{-(0.5)^2} = 0.77880$

$\Rightarrow y_6 = e^{-(0.6)^2} = 0.69768$

Using Simpson's 1/3rd Rule:

$$I = \frac{h}{3}\left[(y_0+y_6) + 2(y_2+y_4) + 4(y_1+y_3+y_5)\right]$$

$$= \frac{0.1}{3}\left[(1+0.6789) + 2(0.96079+0.85214) + 4(0.99+0.91393+0.7788)\right]$$

$$= 0.53451.$$

4) Evaluate $\int_0^{\pi/2} \cos x \, dx$ by taking 5 ordinates.

Sol^n:

$a = 0$, $b = \pi/2$, $n = 4$ (divisions)

$h = \frac{b-a}{n} = \frac{\pi/2 - 0}{4} = \pi/8$

$y = \cos x$

	x_0	x_1	x_2	x_3	x_4
x	0	$\pi/8$	$2\pi/8$	$3\pi/8$	$4\pi/8$
y	1	0.92388	0.70711	0.38268	0
	y_0	y_1	y_2	y_3	y_4

$\Rightarrow y_0 = \cos 0 = 1$

$\Rightarrow y_4 = \cos \pi/2 = 0$

$\Rightarrow y_1 = \cos\frac{\pi}{8} = \cancel{0.99998}$ 0.92388

$\Rightarrow y_2 = \cos\frac{2\pi}{8} = 0.70711$

$\Rightarrow y_3 = \cos\frac{3\pi}{8} = 0.38268$

Since, n is a multiple of 2.

Using Simpson's 1/3rd Rule:

$$I = \frac{h}{3}\left[(y_0 + y_4) + 2(y_2) + 4(y_1 + y_3)\right]$$

$$= \frac{\pi/8}{3}\left[(1+0) + 2(0.70711) + 4(0.92388 + 0.38268)\right]$$

$$I = 1.00054$$

Note:

$$\int \frac{1}{(2x+1)^2}\,dx = \frac{(2x+1)^{-2+1}}{(-2+1)\cdot 2} = \frac{-1}{2(2x+1)}$$

$$\int \frac{1}{x^n} = \frac{x^{-n+1}}{-n+1}$$

$$\int \frac{1}{(ax+b)^n}\,dx = \frac{(ax+b)^{-n+1}}{a(-n+1)}$$

5) Find
for t

Time

speed
(miles

sol

sp

a=

h=

Usi

I

5) Find distance travelled by a train between 8:20 AM to 9:00 AM for the following data.

	t_0	t_1	t_2	t_3	t_4
Time	8:20	8:30	8:40	8:50	9:00
Speed (miles/hr)	24.2	35	41.3	42.8	39.4
	v_0	v_1	v_2	v_3	v_4

Soln

$$\text{speed} = \frac{\text{distance travelled}}{\text{time taken}} \Rightarrow$$

$$S = \int_{t_0}^{t_n} v \cdot dt$$

$a = 8:20 \qquad b = 9:00 \qquad , n = 4$

$$h = \frac{b-a}{n} = \frac{40 \text{ min}}{4} = 10 \text{ min} = 1/6 \text{ hr}$$

Using Simpson's 1/3nd Rule:

$$I = \frac{h}{3}\left[(y_0 + y_4) + 2(y_2) + 4(y_1 + y_3)\right]$$

$$= \frac{(1/6)}{3}\left[(24.2 + 39.4) + 2(41.3) + 4(35 + 42.8)\right]$$

$$= 25.4 \text{ miles}.$$