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MODULE - 05

NUMERICAL METHODS - III

* Solution of system of differential equations by Runge - Kutta Method

Consider, a system of equations $\frac{dy}{dx} = f(x,y,z)$, $\frac{dz}{dx} = g(x,y,z)$

subject to the initial conditions $y(x_0) = y_0$, $z(x_0) = z_0$.

$\Rightarrow$ To find y_1 & z_1

$k_1 = h.f(x_0, y_0, z_0)$ $\qquad l_1 = h.g(x_0, y_0, z_0)$

$k_2 = h.f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$ $\qquad l_2 = h.g\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$

$k_3 = h.f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$ $\qquad l_3 = h.g\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$

$k_4 = h.f(x_0 + h, y_0 + k_3, z_0 + l_3)$ $\qquad l_4 = h.g(x_0 + h, y_0 + k_3, z_0 + l_3)$

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$z_1 = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

① Use Runge - Kutta method to solve the system of equations $\frac{dy}{dx} = x + z$, $\frac{dz}{dx} = x - y^2$ at $x = 0.1$, given that $y(0) = 2$, $z(0) = 1$, $h = 0.1$

→ Given,

$f(x,y,z) = x + z$, $g(x,y,z) = x - y^2$.

$x_0 = 0$, $y_0 = 2$, $z_0 = 1$, $h = 0.1$

$x_1 = x_0 + h = 0 + 0.1 = 0.1$

To find y_1 & z_1,

$k_1 = h.f(x_0, y_0, z_0)$

$= 0.1 f(0,2,1)$

$= 0.1(0+1)$

$\boxed{k_1 = 0.1}$

$l_1 = h.g(x_0, y_0, z_0)$

$= 0.1 g(0,2,1)$

$= 0.1(0-2^2)$

$\boxed{l_1 = -0.4}$

$$k_2 = h.f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$= 0.1 f\left(0 + \frac{0.1}{2}, 2 + \frac{0.1}{2}, 1 + \left(\frac{-0.4}{2}\right)\right)$$

$$= 0.1 f(0.05, 2.05, 0.8)$$

$$= 0.1(0.05 + 0.8)$$

$\boxed{k_2 = 0.085}$

$$l_2 = h.g\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$= 0.1 g(0.05, 2.05, -0.8)$$

$$= 0.1(0.05 - (2.05)^2)$$

$\boxed{l_2 = -0.4153}$

$$k_3 = h.f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$= 0.1 f\left(0 + \frac{0.1}{2}, 2 + \frac{0.085}{2}, 1 - \frac{0.4153}{2}\right)$$

$$= 0.1 f(0.05, 2.0425, 0.7924)$$

$$= 0.1(0.05 + 0.7924)$$

$\boxed{k_3 = 0.0842}$

$$l_3 = h.g\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$= 0.1 g(0.05, 2.0425, 0.7924)$$

$$= 0.1(0.05 - (2.0425)^2)$$

$\boxed{l_3 = -0.4122}$

$k_4 = h.f(x_0+h, y_0+k_3, z_0+l_3)$

$= 0.1f(0+0.1, 2+0.0842, 1-0.4122)$

$= 0.1f(0.1, 2.0842, 0.5878)$

$= 0.1(0.1+0.5878)$

$\boxed{k_4 = 0.0688}$

$l_4 = h.g(x_0+h, y_0+k_3, z_0+l_3)$

$= 0.1g(0.1, 2.0842, 0.5878)$

$= 0.1(0.1-(2.0842)^2)$

$\boxed{l_4 = -0.4244}$

$$y_1 = y_0 + \frac{1}{6}(k_1+2k_2+2k_3+k_4)$$

$$= 2+\frac{1}{6}(0.1+2(0.085)+2(0.0842)+0.0688)$$

$\boxed{y_1 = 2.0845}$

$$z_1 = z_0 + \frac{1}{6}(l_1+2l_2+2l_3+l_4)$$

$$= 1+\frac{1}{6}(-0.4+2(-0.4153)+2(-0.4122)+(-0.4244))$$

$\boxed{z_1 = 0.5868}$

(2) Use Runge-Kutta method to solve $\frac{dy}{dx} = 1+zx, \frac{dz}{dx} = -xy$

at $x=0.3, y(0)=0, z(0)=1, h=0.3$.

→ Given,

$f(x,y,z) = 1+zx, g(x,y,z) = -xy$

$x_0=0, y_0=0, z_0=1, h=0.3$

$x_1 = x_0+h = 0.3$

To find y_1 & z_1,

$k_1 = h.f(x_0, y_0, z_0)$

$= 0.3 . f(0,0,1)$

$= 0.3(1+0)$

$\boxed{k_1 = 0.3}$

$l_1 = h.g(x_0, y_0, z_0)$

$= 0.3 g(0,0,1)$

$= 0.3(-0)$

$\boxed{l_1 = 0}$

$k_2 = h.f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$

$= 0.3 f\left(0 + \frac{0.3}{2}, 0 + \frac{0.3}{2}, 1 + \frac{0}{2}\right)$

$= 0.3 f(0.15, 0.15, 1)$

$= 0.3(1 + 0.15 \times 1)$

$\boxed{k_2 = 0.345}$

$l_2 = h.g\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$

$= 0.3 g(0.15, 0.15, 1)$

$= 0.3(-0.15 \times 0.15)$

$\boxed{l_2 = -0.0068}$

$k_3 = h.f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$

$= 0.3 f\left(0 + \frac{0.3}{2}, 0 + \frac{0.345}{2}, 1 - \frac{0.0068}{2}\right)$

$= 0.3 f(0.15, 0.1725, 0.9966)$

$= 0.3(1 + 0.15 \times 0.9966)$

$\boxed{k_3 = 0.3448}$

$l_3 = h.g\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$

$= 0.3 g(0.15, 0.1725, 0.9966)$

$= 0.3(-0.15 \times 0.1725)$

$\boxed{l_3 = -0.0078}$

$k_4 = h \cdot f(x_0+h, y_0+k_3, z_0+l_3)$

$= 0.3 f(0+0.3, 0+0.3448, 1+(-0.0078))$

$= 0.3 f(0.3, 0.3448, 0.9922)$

$= 0.3(1+0.3 \times 0.9922)$

$\boxed{k_4 = 0.3893}$

$l_4 = h \cdot g(x_0+h, y_0+k_3, z_0+l_3)$

$= 0.3 g(0.3, 0.3448, 0.9922)$

$= 0.3(-0.3 \times 0.3448)$

$\boxed{l_4 = -0.0310}$

$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 0 + \frac{1}{6}(0.3 + 2(0.345) + 2(0.3448) + 0.3893)$

$\boxed{y_1 = 0.3448}$

$z_1 = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

$= 1 + \frac{1}{6}(0 + 2(-0.0068) + 2(-0.0078) + (-0.031))$

$\boxed{z_1 = 0.9900}$

③ Given, a system of differential equations, $\frac{dx}{dt} = y - t$, $\frac{dy}{dt} = x + t$, with the conditions $x=1, y=1$ at $t=0$. Obtain the values of x & y at $t=0.1$ by taking $h=0.1$.

→ Given,

$f(t,x,y) = y - t$, $g(t,x,y) = x + t$.

$t_0 = 0, y_0 = 1, x_0 = 1, h = 0.1$

$t_1 = t_0 + h = 0.1$

To find y_1 & z_1,

$k_1 = h.f(t_0, x_0, y_0)$ $\qquad$ $l_1 = h.g(t_0, x_0, y_0)$

$= 0.1 f(0,1,1)$ $\qquad$ $= 0.1 g(0,1,1)$

$= 0.1(1-0)$ $\qquad$ $= 0.1(0+1)$

$\boxed{k_1 = 0.1}$ $\qquad$ $\boxed{l_1 = 0.1}$

$$k_2 = h.f\left(t_0 + \frac{h}{2}, x_0 + \frac{k_1}{2}, y_0 + \frac{l_1}{2}\right)$$

$$= 0.1 f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}, 1 + \frac{0.1}{2}\right)$$

$$= 0.1 f(0.05, 1.05, 1.05)$$

$$= 0.1(1.05 - 0.05)$$

$\boxed{k_2 = 0.1}$

$$l_2 = h.g\left(t_0 + \frac{h}{2}, x_0 + \frac{k_1}{2}, y_0 + \frac{l_1}{2}\right)$$

$$= 0.1 g(0.05, 1.05, 1.05)$$

$$= 0.1(1.05 + 0.05)$$

$\boxed{l_2 = 0.11}$

$$k_3 = h.f\left(t_0 + \frac{h}{2}, x_0 + \frac{k_2}{2}, y_0 + \frac{l_2}{2}\right)$$

$$= 0.1 f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}, 1 + \frac{0.11}{2}\right)$$

$$= 0.1 f(0.05, 1.05, 1.055)$$

$$= 0.1(1.055 - 0.05)$$

$\boxed{k_3 = 0.1005}$

$$l_3 = h.g\left(t_0 + \frac{h}{2}, x_0 + \frac{k_2}{2}, y_0 + \frac{l_2}{2}\right)$$

$$= 0.1 g(0.05, 1.05, 1.055)$$

$$= 0.1(1.05 + 0.05)$$

$\boxed{l_3 = 0.1100}$

$k_4 = h \cdot f(t_0 + h, x_0 + k_3, y_0 + l_3)$

$= 0.1 \cdot f(0 + 0.1, 1 + 0.1005, 1 + 0.1100)$

$= 0.1 \cdot f(0.1, 1.1005, 1.1100)$

$= 0.1(1.1100 - 0.1)$

$\boxed{k_4 = 0.101}$

$l_4 = h \cdot g(t_0 + h, x_0 + k_3, y_0 + l_3)$

$= 0.1\, g(0 + 0.1, 1 + 0.1005, 1 + 0.1100)$

$= 0.1\, g(0.1, 1.1005, 1.1100)$

$= 0.1(0.1 + 1.1005)$

$\boxed{l_4 = 0.1201}$

$x_1 = x_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= x_0 1 + \frac{1}{6}(0.1 + 2(0.1) + 2(0.1005) + 0.101)$

$\boxed{x_1 = 1.1003}$

$y_1 = y_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

$= 1 + \frac{1}{6}(0.1 + 2(0.11) + 2(0.11) + 0.1201)$

$\boxed{y_1 = 1.1100}$

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* Solutions of second-order differential equations by Runge-Kutta method

① Using Runge-Kutta method, find $y(0.2)$. Given that y satisfies the differential equation $\frac{d^2y}{dx^2} = x.\left(\frac{dy}{dx}\right)^2 - y^2$

under the condition $y(0)=1, y'(0)=0, h=0.2$.

→ Let $\frac{dy}{dx} = z$

$$\frac{d^2y}{dx^2} = \frac{dz}{dx} = xz^2 - y^2$$

Given,

$$f(x,y,z) = z, \quad g(x,y,z) = xz^2 - y^2$$

$$x_0 = 0, y_0 = 1, z_0 = 0, h = 0.2$$

$$x_1 = x_0 + h = 0.2$$

To find y_1 & z_1,

$$K_1 = h.f(x_0, y_0, z_0)$$
$$= 0.2f(0,1,0)$$
$$= 0.2(0)$$
$$\boxed{K_1 = 0}$$

$$l_1 = h.g(x_0, y_0, z_0)$$
$$= 0.2g(0,1,0)$$
$$= 0.2(0-1)$$
$$\boxed{l_1 = 0.2}$$

$$K_2 = h.f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{l_1}{2}\right)$$
$$= 0.2f\left(0 + \frac{0.2}{2}, 1 + \frac{0}{2}, 0 + \frac{(-0.2)}{2}\right)$$
$$= 0.2f(0.1, 1, -0.1)$$
$$\boxed{K_2 = -0.02}$$

$$l_2 = h.g\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{l_1}{2}\right)$$
$$= 0.2g(0.1, 1, -0.1)$$
$$= 0.2(0.1(-0.1)^2 - 1^2)$$
$$\boxed{l_2 = -0.1998}$$

$k_3 = h.f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$

$= 0.2 f\left(0 + \frac{0.2}{2}, 1 + \frac{(-0.002)}{2}, 0 - \frac{0.1998}{2}\right)$

$= 0.2 f(0.1, 0.99, -0.0999)$

$= 0.2(-0.0999)$

$\boxed{k_3 = -0.02}$

$l_3 = h.g\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$

$= 0.2\, g(0.1, 0.99, -0.0999)$

$= 0.2(0.1(0.0999)^2 - (0.99)^2)$

$\boxed{l_3 = -0.1958}$

$k_4 = h.f(x_0 + h, y_0 + k_3, z_0 + l_3)$

$= 0.2 f(0 + 0.2, 1 - 0.02, 0 - 0.1958)$

$= 0.2 f(0.2, -0.98, -0.1958)$

$\boxed{k_4 = -0.0392}$

$l_4 = h.g(x_0 + h, y_0 + k_3, z_0 + l_3)$

$= 0.2\, g(0.2, -0.98, -0.1958)$

$= 0.2(0.2(-0.1958)^2 - (0.98)^2)$

$\boxed{l_4 = -0.1905}$

$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 1 + \frac{1}{6}(0 + 2(-0.02) + 2(-0.02) + (-0.0392))$

$\boxed{y_1 = 0.9801}$

$z_1 = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

$= 0 + \frac{1}{6}(0.2 + 2(-0.1998) + 2(-0.1958) + (-0.1905))$

$\boxed{z_1 = -0.1303}$

② Using Runge-Kutta method at $x=0.1$ of the differential equation $\frac{d^2y}{dx^2} - x^2.\frac{dy}{dx} - 2xy = 1$ under the conditions $y(0)=1,\ y'(0)=0$, take $h=0.1$.

→ Given,

$$\frac{d^2y}{dx^2} - x^2.\frac{dy}{dx} - 2xy = 1.$$

Let, $\frac{dy}{dx} = z$, $\frac{d^2y}{dx^2} = \frac{dz}{dx} = 1+x^2z+2xy.$

$f(x,y,z) = z.$

$g(x,y,z) = 1+x^2z+2xy.$

$x_0 = 0,\ y_0 = 1,\ z_0 = 0,\ h = 0.1$

$x_1 = x_0 + h = 0+0.1 = 0.1$

To find $y_1 + z_1$,

$k_1 = h.f(x_0, y_0, z_0)$

$= 0.1 f(0,1,0)$

$\boxed{k_1 = 0}$

$l_1 = h.g(x_0, y_0, z_0)$

$= 0.1 g(0,1,0)$

$= 0.1(1+0)$

$\boxed{l_1 = 0.1}$

$k_2 = h.f\left(x_0+\frac{h}{2},\ y_0+\frac{k_1}{2},\ z_0+\frac{l_1}{2}\right)$

$= 0.1 f\left(0+\frac{0.1}{2},\ 1+\frac{0}{2},\ 0+\frac{0.1}{2}\right)$

$= 0.1 f(0.05, 1, 0.005)$

$\boxed{k_2 = 0.005}$

$l_2 = h.g\left(x_0+\frac{h}{2},\ y_0+\frac{k_1}{2},\ z_0+\frac{l_1}{2}\right)$

$= 0.1 g(0.05, 1, 0.05)$

$= 0.1(1+(0.05)^2(0.05)+2(0.05))$

$\boxed{l_2 = 0.11}$

$$k_3 = h.f\left(x_0+\frac{h}{2},\ y_0+\frac{k_2}{2},\ z_0+\frac{l_2}{2}\right)$$

$$=0.1f(0.05, 1.0025, 0.055)$$

$$=0.1\ (0.055)$$

$$\boxed{k_3 = 0.0055}$$

$$l_3 = h.g\left(x_0+\frac{h}{2},\ y_0+\frac{k_2}{2},\ z_0+\frac{l_2}{2}\right)$$

$$=0.1g\ (0.05, 1.0025, 0.055)$$

$$=0.1(1+(0.05)^2(0.055)+2(0.05)(1.0025))$$

$$\boxed{l_3 = 0.11}$$

$$k_4 = h.f(x_0+h, y_0+k_3, z_0+l_3)$$

$$=0.1f\ (0+0.1, 1+0.0055, 0+0.11)$$

$$=0.1f(0.1, 1.0055, 0.11)$$

$$=0.1(0.11)$$

$$\boxed{k_4 = 0.011}$$

$$l_4 = h.g\ (x_0+h, y_0+k_3, z_0+l_3)$$

$$=0.1g\ (0.1, 1.0055, 0.11)$$

$$=0.1(1+(0.1)^2(0.11)+2(0.1)(1.0055))$$

$$\boxed{l_4 = 0.1202}$$

$$y_1 = y_0+\frac{1}{6}(k_1+2k_2+2k_3+k_4)$$

$$=1+\frac{1}{6}(0+2(0.005)+2(0.0055)+(0.011))$$

$$\boxed{y_1 = 1.0053}$$

$$z_1 = z_0+\frac{1}{6}(l_1+2l_2+2l_3+l_4)$$

$$=0+\frac{1}{6}(0.1+2(0.11)+2(0.11)+0.1202)$$

$$\boxed{z_1 = 0.11}$$

③ The angular displacement 'θ' of a simple pendulum is given by the equation $\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$, where $l = 98$ cm, $g = 980$ cm/s² when $t=0$, $\theta = 0$ and $\frac{d\theta}{dt} = 4.472$.

Use Runge-Kutta method to find θ and $\frac{d\theta}{dt}$ when $t = 0.2s$, take $h = 0.2$.

→ $\frac{d\theta}{dt} = z$.

$$\frac{d^2\theta}{dt^2} = \frac{dz}{dt} = \frac{-g}{l}\sin\theta. = -\frac{980}{98}.\sin\theta = -10\sin\theta.$$

$f(t,\theta,z) = z$.

$g(t,\theta,z) = -10\sin\theta$.

$t_0 = 0,\ \theta_0 = 0,\ z_0 = 4.472,\ h = 0.2$.

To find θ_1 & z_1,

$k_1 = h.f(t_0, \theta_0, z_0)$

$= 0.2 f(0, 0, 4.472)$

$= 0.2(4.472)$

$\boxed{k_1 = 0.8944}$

$l_1 = h.g(t_0, \theta_0, z_0)$

$= 0.2g(0, 0, 4.472)$

$= 0.2(-10\sin(0))$

$\boxed{l_1 = 0}$

$$k_2 = h.f\left(t_0 + \frac{h}{2}, \theta_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$= 0.2 f\left(0 + \frac{0.2}{2}, 0 + \frac{0.8944}{2}, 4.472 + \frac{0}{2}\right)$$

$= 0.2(4.472)$

$\boxed{k_2 = 0.8944}$

$$l_2 = h.g\left(t_0 + \frac{h}{2}, \theta_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$= 0.2(0.1, 0.4472, 4.472)$

$= 0.2(-10\sin(0.4472))$

$\boxed{l_2 = -0.8649}$

$$k_3 = h \cdot f\left(t_0 + \frac{h}{2}, \theta_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$= 0.2 f\left(0 + \frac{0.2}{2}, 0 + \frac{0.8944}{2}, 4.472 + \frac{(0.8649)}{2}\right)$$

$$= 0.2 f(0.1, 0.4472, 4.0396)$$

$$= 0.2(4.0396)$$

$$\boxed{k_3 = 0.8079}$$

$$l_3 = h \cdot g\left(t_0 + \frac{h}{2}, \theta_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$= 0.2 g(0.1, 0.4472, 4.0396)$$

$$= 0.2(-10 \sin(0.4472))$$

$$\boxed{l_3 = -0.8649}$$

$$k_4 = h \cdot f(t_0 + h, \theta_0 + k_3, z_0 + l_3)$$

$$= 0.2 f(0 + 0.1, 0 + 0.8079, 4.472 + (-0.8649))$$

$$= 0.2 f(0.1, 0.8079, 3.6071)$$

$$= 0.2(3.6071)$$

$$\boxed{k_4 = 0.7214}$$

$$l_4 = h \cdot g(t_0 + h, \theta_0 + k_3, z_0 + l_3)$$

$$= 0.2 g(0.1, 0.8079, 3.6071)$$

$$= 0.2(-10 \sin(0.8079))$$

$$\boxed{l_4 = -1.4457}$$

$$\theta_1 = \theta_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 0 + \frac{1}{6}(0.8944 + 2(0.8944) + 2(0.8079) + 0.7214)$$

$$\boxed{\theta_1 = 0.8367}$$

$$z_1 = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

$$= 4.472 + \frac{1}{6}(0 + 2(-0.8649) + 2(-0.8649) - 1.4457)$$

$$\boxed{z_1 = 3.6545}$$

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* Milne's Method for II order Differential Equation

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_2)$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}(2z_1' - z_2' - 2z_3')$$

$$y_4^{(C)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4)$$

$$z_4^{(C)} = z_2 + \frac{h}{3}(z_2' + 4z_3' + z_4')$$

① Apply Milne's Predictor Corrector method to compute $y(0.8)$. Given that $\frac{d^2y}{dx^2} = 1 - 2y \cdot \frac{dy}{dx}$ and the following table of initial values.

x	0	0.2	0.4	0.6
y	0	0.02	0.0795	0.1762
y'	0	0.1996	0.3937	0.5689

→ Let $\frac{dy}{dx} = z$.

$$\frac{d^2y}{dx^2} = \frac{dz}{dx} = 1 - 2yz.$$

$x_0 = 0$, $x_1 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$

$y_0 = 0$, $y_1 = 0.02$, $y_2 = 0.0795$, $y_3 = 0.1762$

$z_0 = 0$, $z_1 = 0.1996$, $z_2 = 0.3937$, $z_3 = 0.5689$

We have,

$z' = 1 - 2yz$

$z_1' = 1 - 2y_1z_1 = 0.9920$

$z_2' = 1 - 2y_2z_2 = 0.9374$

$z_3' = 1 - 2y_3z_3 = 0.7995$

$y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3)$

$y_4^{(P)} = 0 + \frac{4(0.2)}{3}(2(0.1996) - 0.3937 + 2(0.5689))$

$\boxed{y_4^{(P)} = 0.3049}$

$z_4^{(P)} = z_0 + \frac{4h}{3}(2z_1' - z_2' + 2z_3')$

$= 0 + \frac{4(0.2)}{3}(2(0.9920) - (0.9374) + 2(0.7995))$

$\boxed{z_4^{(P)} = 0.7055}$

$z_4 = z_4^{(P)} = 0.7055$

$z_4' = 1 - 2y_4^{(P)} \cdot z_4^{(P)} = 0.5698$

$y_4^{(c)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4)$

$= 0.0795 + \frac{0.2}{3}(0.3937 + 4(0.5689) + 0.7055)$

$\boxed{y_4^{(c)} = 0.3045}$

$z_4^{(c)} = z_2 + \frac{h}{3}(z_2' + 4z_3' + z_4')$

$= 0.3937 + \frac{0.2}{3}(0.9374 + 4(0.7995) + 0.5698)$

$\boxed{z_4^{(c)} = 0.7074}$

To find refined solution, take $y_4^{(P)} = 0.3045$, $z_4^{(P)} = 0.7074$

$$z_4 = z_4^{(P)} = 0.7074$$

$$z_4' = 1 - 2z_4^{(P)} y_4^{(P)} = 0.5692$$

$$y_4^{(c)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4)$$

$$= 0.0795 + \frac{0.2}{3}(0.3937 + 4(0.5689) + 0.7074)$$

$$\boxed{y_4^{(c)} = 0.3046}$$

$$z_4^{(c)} = z_2 + \frac{h}{3}(z_2' + 4z_3' + z_4')$$

$$= 0.3937 + \frac{0.2}{3}(0.9374 + 4(0.7995) + 0.5692)$$

$$\boxed{z_4^{(c)} = 0.7073}$$

(2) obtain the solution of $2\frac{d^2y}{dx^2} = 4x + \frac{dy}{dx}$ at $x = 1.4$ by applying Milne's method and by using the data.

x	1	1.1	1.2	1.3
y	2	2.2156	2.4649	2.7514
y'	2	2.3178	2.6725	3.0657

→ Given,

$$\frac{dy}{dx} = z$$

$$\frac{d^2y}{dx^2} = \frac{dz}{dx} = 2x + \frac{z}{2}$$

$x_0 = 1$, $x_1 = 1.1$, $x_2 = 1.2$, $x_3 = 1.3$

$y_0 = 2$, $y_1 = 2.2156$, $y_2 = 2.4649$, $y_3 = 2.7514$

$z_0 = 2$, $z_1 = 2.3178$, $z_2 = 2.6725$, $z_3 = 3.0657$

We have,

$$z' = 2x + \frac{z}{2}$$

$$z_1' = 2x_1 + \frac{z_1}{2} = 3.3589$$

$$z_2' = 2x_2 + \frac{z_2}{2} = 3.7363$$

$$z_3' = 2x_3 + \frac{z_3}{2} = 4.1329$$

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3)$$

$$= 2 + \frac{4(0.1)}{3}(2(2.3178) - 2.6725 + 2(3.0657))$$

$$\boxed{y_4^{(P)} = 3.0793}$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}(2z_1' - z_2' + 2z_3')$$

$$= 2 + \frac{4(0.1)}{3}(2(3.3589) - 3.7363 + 2(4.1329))$$

$$\boxed{z_4^{(P)} = 3.4996}$$

$$z_4 = z_4^{(P)} = 3.4996$$

$$z_4' = 2x_4^{(P)} + \frac{z_4^{(P)}}{2}$$

$$\boxed{z_4' = 4.5498}$$

$$y_4^{(c)} = y_2 + \frac{h}{3}(z_2 + 4z_2 + z_4)$$

$$= 2.4649 + \frac{0.1}{3}(2.6725 + 4(\cancel{0}\,3.0657) + 3.4996)$$

$$\boxed{y_4^{(c)} = 3.0794}$$

$$z_4^{(c)} = z_2 + \frac{h}{3}(z_2' + 4z_3' + z_4')$$

$$= 2.6725 + \frac{0.1}{3}(3.7363 + 4(4.1329) + 4.5498)$$

$$\boxed{z_4^{(c)} = 3.4998}$$

To refined solution, we have, $y_4^{(p)} = 3.0794$.
find

$$z_4 = z_4^{(p)} = 3.4998$$

$$z_4' = 2x_4 + \frac{z_4^{(p)}}{2} = 4.5499.$$

$$y_4^{(c)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4)$$

$$= 2.4649 + \frac{0.1}{3}(2.6725 + 4(3.0657) + 3.4998)$$

$$\boxed{y_4^{(c)} = 3.0794}$$

$$z_4^{(c)} = z_2 + \frac{h}{3}(z_2' + 4z_3' + z_4')$$

$$= 2.6725 + \frac{0.1}{3}(3.7363 + 4(4.1329) + 4.5499)$$

$$\boxed{z_4^{(c)} = 3.4998}$$

③ Obtain the approximate solution at $x = 0.4$ of the differential equation $\frac{d^2y}{dx^2} + 3x.\frac{dy}{dx} - 6y = 0$. Given that

x	0	0.1	0.2	0.3
y	1	1.034	1.138	1.2987
y'	0.1	0.6955	1.2580	1.8730.

→ Given,

$$\frac{dy}{dx} = z.$$

$$\frac{d^2y}{dx^2} = \frac{dz}{dx} = 6y - 3xz$$

$x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$

$y_0 = 1$, $y_1 = 1.034$, $y_2 = 1.138$, $y_3 = 1.2987$

& $z_0 = 0.1$, $z_1 = 0.6955$, $z_2 = 1.2580$, $z_3 = 1.8730$

We have,

$z' = 6y - 3xz$

$z_1' = 6y_1 - 3x_1z_1 = 5.9954$

$z_2' = 6y_2 - 3x_2z_2 = 6.0732$

$z_3' = 6y_3 - 3x_3z_3 = 6.1065$

$y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3)$

$= 1 + \frac{4(0.1)}{3}(2(0.6955) - 1.2580 + 2(1.8730))$

$= \boxed{y_4^{(P)} = 1.5172}$

$z_4^{(P)} = z_0 + \frac{4h}{3}(2z_1' - z_2' + 2z_3')$

$= 0.1 + \frac{4(0.1)}{3}(2(5.9954) - 6.0732 + 2(6.1065))$

$\boxed{z_4^{(P)} = 2.5174}$

$z_4 = z_4^{(P)} = 2.5174$

$z_4' = 6y_4 - 3x_4z_4 = 6.0823.$

$y_4^{(c)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4)$

$= 1.138 + \frac{0.1}{3}(1.2580 + 4(1.8730) + 2.5174)$

$\boxed{y_4^{(c)} = 1.5136}$

$z_4^{(c)} = z_2 + \frac{h}{3}(z_2' + 4z_3' + z_4')$

$= 1.2580 + \frac{0.1}{3}(6.0732 + 4(6.1065) + 6.0823)$

$\boxed{z_4^{(c)} = 2.4774}$

To find refined solution, we have $y_4^{(P)} = 1.5136$

$z_4 = z_4^{(P)} = 2.4774$

$z_4' = 6y_4 - 3x_4z_4 = 6(1.5136) - 3(0.4)(2.4774)$

$\boxed{z_4' = 6.1087}$

$y_4^{(c)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4)$

$= 1.138 + \frac{0.1}{3}(1.2580 + 4(1.8730) + 2.4774)$

$\boxed{y_4^{(c)} = 1.5122}$

$z_4^{(c)} = z_2 + \frac{h}{3}(z_2' + 4z_3' + z_4')$

$= 1.2580 + \frac{0.1}{3}(6.0732 + 4(6.1065) + 6.1087)$

$\boxed{z_4^{(c)} = 2.4783}$

Let $y_4^{(P)} = 1.5122, z_4^{(P)} = 2.4783, z_4' = 6.0992$

$y_4^{(c)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4)$

$= 1.138 + \frac{0.1}{3}(1.2580 + 4(1.8730) + 2.4783)$

$\boxed{y_4^{(c)} = 1.5123}$

$z_4^{(c)} = z_2 + \frac{h}{3}(z_2' + 4z_3' + z_4')$

$= 1.2580 + \frac{0.1}{3}(6.0732 + 4(6.1065) + 6.0992)$

$\boxed{z_4^{(c)} = 2.4779}$

Let $y_4^{(P)} = 1.5123, z_4^{(P)} = 2.4779, z_4' = 6.1003$

$y_4^{(c)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4)$

$= 1.138 + \frac{0.1}{3}(1.2580 + 4(1.8730) + 2.4779)$

$\boxed{y_4^{(c)} = 1.5123}$

$z_4^{(c)} = z_2 + \frac{h}{3}(z_2' + 4z_3' + z_4')$

$= 1.2580 + \frac{0.1}{3}(6.0732 + 4(6.1065) + 6.1003)$

$\boxed{z_4^{(c)} = 2.4780}$

⊛ Solution of Algebraic & Transcendental equations

Transcendental ↓ The equations apart from algebraic.

a) Regula - Falsi Method

Consider the equation, $f(x)=0$.

Step 1 :- Choose two points x_0 & x_1, such that $f(x_0), f(x_1)$ are of opposite signs i.e., $f(x_0)<0, f(x_1)>0$.

Step 2 :- Then the root lies between x_0 & x_1.

Step 3 :- The first approximation is given by,

$$x_2 = \frac{x_0 . f(x_1) - x_1 . f(x_0)}{f(x_1) - f(x_0)}$$

Step 4 :- Find $f(x_2)$,

a) If $f(x_2)<0$, then the root lies between x_1 & x_2.

b) If $f(x_2)>0$, then the root lies between x_0 & x_2.

{Search for two consecutive opposite sign functions).

Step 5 :- Continue this process, till any two consecutive approximations are almost identical.

① By Regula-falsi method, find the real root of the equation, $x^4 - x - 10$. Carry out 3 approximations.

→ Let $f(x) = x^4 - x - 10$.

$x_0 = 1$ $\qquad$ $f(0) = -10<0$

$x_1 = 2$ $\qquad$ $f(1) = -10<0$

$f(2) = 4>0$.

The root lies b/w x_0 & x_1,

$f(x_0) = -10<0$

$f(x_1) = 4>0$

The I approximation is,

$$x_2 = \frac{x_0 . f(x_1) - x_1 . f(x_0)}{f(x_1) - f(x_0)}$$

$$= \frac{1(4) - 2(-10)}{4 - (-10)}$$ $\boxed{x_2 = 1.7143}$

$f(x_2) = -3.0776 < 0.$

Therefore, the root lies b/w x_1 & x_2.

The II approximation is,

$$x_3 = \frac{x_1 . f(x_2) - x_2 . f(x_1)}{f(x_2) - f(x_1)}$$

$$= \frac{2(-3.0776) - 1.7143(4)}{-3.0776 - 4}$$

$\boxed{x_3 = 1.8385}$

$f(x_3) = -0.4135 < 0.$

∴ The root lies b/w x_1 & x_3.

The III approximation is,

$$x_4 = \frac{x_1 . f(x_3) - x_3 . f(x_1)}{f(x_3) - f(x_1)}$$

$$= \frac{2(-0.4135) - 1.8385(4)}{-0.4135 - 4}$$

$\boxed{x_4 = 1.8536}$

The approximate root is 1.8536.

② Use Regula-Falsi method to find a real root of $\cos x = 3x - 1$ between 0.5 and 0.1.

→ $f(x) = \cos x - 3x + 1.$

$x_0 = 0.5$, $x_1 = 1.$

$f(x_0) = 0.3776 > 0$

$f(x_1) = -1.4597 < 0$

The root lies b/w x_0 & x_1.

The I approximation is,

$$x_2 = \frac{x_0 . f(x_1) - x_1 . f(x_0)}{f(x_1) - f(x_0)}$$

$$= \frac{0.5(-1.4597) - 1(0.3776)}{-1.4597 - 0.3776}$$

$$\boxed{x_2 = 0.6028}$$

$f(x_2) = 0.0154 > 0.$

Therefore, the root lies b/w x_1 & x_2.

The II approximation is,

$$x_3 = \frac{x_1 . f(x_2) - x_2 . f(x_1)}{f(x_2) - f(x_1)}$$

$$= \frac{1(0.0154) - (0.6028)(-1.4597)}{0.0154 + 1.4597}$$

$$\boxed{x_3 = 0.6069}$$

$f(x_3) = 0.0007 > 0$

Therefore, root lies b/w x_1 & x_3.

The III approximation is,

$$x_4 = \frac{x_1 . f(x_3) - x_3 . f(x_1)}{f(x_3) - f(x_1)}$$

$$= \frac{1(0.0007) - 0.6069(-1.4597)}{0.0007 + 1.4597}$$

$$\boxed{x_4 = 0.6071}$$

$f(x_4) = 0.$

∴ The approximate root is 0.6071.

③ $f(x) = x.\log_{10}x - 1.2$ b/w 2 & 3.

→ Given, $f(x) = x.\log_{10}x - 1.2$, $x_0 = 2$, $x_1 = 3$.

$f(x_0) = -0.5979 < 0$

$f(x_1) = 0.2314 > 0$.

∴ The root lies b/w x_0 & x_1.

The I approximation is,

$$x_2 = \frac{x_0.f(x_1) - x_1.f(x_0)}{f(x_1) - f(x_0)}$$

$$= \frac{2(0.2314) - 3(-0.5979)}{0.2314 - (-0.5979)}$$

$$\boxed{x_2 = 2.7210}$$

$f(x_2) = -0.0171 < 0$.

∴ The root lies b/w x_1 & x_2.

The II approximation is,

$$x_3 = \frac{x_1.f(x_2) - x_2.f(x_1)}{f(x_2) - f(x_1)}$$

$$= \frac{3(-0.0171) - 2.7210(0.2314)}{-0.0171 - 0.2314}$$

$$\boxed{x_3 = 2.7402}$$

$f(x_3) = -0.0004 < 0$.

∴ The root lies b/w x_1 & x_3.

The III approximation is,

$$x_4 = \frac{x_1.f(x_3) - x_3.f(x_1)}{f(x_3) - f(x_1)}$$

$$= \frac{3(-0.0004) - 2.7402(0.2314)}{-0.0004 - 0.2314}$$

$$\boxed{x_4 = 2.7406}$$

$f(x_4) = 0$.

Therefore, the approximate root is 2.7406.

④ $x \cdot e^x = \cos x$ b/w 0.4 & 0.6 (3 approximations).

→ Given, $f(x) = xe^x - \cos x$, $x_0 = 0.4, x_1 = 0.6$.

$f(x_0) = -0.3243 < 0$

$f(x_1) = 0.2679 > 0$.

∴ The root lies b/w x_0 & x_1.

The I approximation is,

$$x_2 = \frac{x_0 \cdot f(x_1) - x_1 \cdot f(x_0)}{f(x_1) - f(x_0)}$$

$$= \frac{0.4(0.2679) - 0.6(-0.3243)}{0.2679 - (-0.3243)}$$

$\boxed{x_2 = 0.5095}$

$f(x_2) = -0.0249 < 0$.

∴ The root lies b/w x_1 & x_2.

The II approximation is,

$$x_3 = \frac{x_1 \cdot f(x_2) - x_2 \cdot f(x_1)}{f(x_2) - f(x_1)}$$

$$= \frac{0.6(-0.0249) - 0.5095(0.2679)}{-0.0249 - 0.2679}$$

$\boxed{x_3 = 0.\overset{5172}{\cancel{4181}}}$

$f(x_3) = -0.\overset{0017}{\cancel{2864}} < 0$.

∴ The root lies b/w x_1 & x_3.

The III approximation is,

$$x_4 = \frac{x_1 \cdot f(x_3) - x_3 \cdot f(x_1)}{f(x_3) - f(x_1)}$$

$$= \frac{0.6(-0.\overset{0017}{\cancel{2864}}) - 0.\overset{5172}{\cancel{4181}}(0.2679)}{-0.\underset{0017}{\cancel{2864}} - 0.2679}$$

$\boxed{x_4 = 0.5177}$

The approximate root is 0.5177.

⑤ Find real root of $\sin x = \frac{1}{x}$ b/w 1 & 1.5. (3 approximations)

→ Given, $f(x) = x\sin x - 1$, $x_0 = 1$, $x_1 = 1.5$

$f(x_0) = -0.1585 < 0$

$f(x_1) = 0.4962 > 0$

∴ The root lies b/w x_0 & x_1.

The I approximation is,

$$x_2 = \frac{x_0 . f(x_1) - x_1 . f(x_0)}{f(x_1) - f(x_0)}$$

$$= \frac{1(0.4962) - 1.5(-0.1585)}{0.4962 - (-0.1585)}$$

$\boxed{x_2 = 1.1210}$

$f(x_2) = 0.0095 > 0$.

∴ The root lies b/w x_0 & x_2.

The II approximation is,

$$x_3 = \frac{x_0 . f(x_2) - x_2 . f(x_0)}{f(x_2) - f(x_0)}$$

$$= \frac{1(0.0095) - 1.1210(-0.1585)}{0.0095 - (0.1585)}$$

$\boxed{x_3 = 1.1142}$

$f(x_3) = 0.0001 > 0$.

∴ Root lies ~~lies~~ b/w x_0 & x_3.

The III approximation is,

$$x_4 = \frac{x_0 . f(x_3) - x_3 . f(x_0)}{f(x_3) - f(x_0)}$$

$$= \frac{1(0.0001) - 1.1142(-0.1585)}{0.0001 - (-0.1585)}$$

$\boxed{x_4 = 1.1141}$

The approximate root is 1.1141.

⑥ Find the 5^{th} root of 10 by using Regula falsi method.

→ Given, $\sqrt[5]{10} = x \Rightarrow x^5 = 10$.

$f(x) = x^5 - 10$,

$f(0) = -10 < 0$

$f(1) = -9 < 0$

$f(2) = 22 > 0$.

$f(x_0) = -9 < 0 \qquad x_0 = 1, x_1 = 2$.

$f(x_1) = 22 > 0$.

∴ Root lies b/w x_0 & x_1.

The I approximation is,

$$x_2 = \frac{x_0 . f(x_1) - x_1 . f(x_0)}{f(x_1) - f(x_0)}$$

$$= \frac{1(22) - 2(-9)}{22 - (-9)}$$

$$\boxed{x_2 = 1.2903}$$

$f(x_2) = -6.4235 < 0$.

∴ Root lies b/w x_1 & x_2.

The II approximation is,

$$x_3 = \frac{x_1 . f(x_2) - x_2 . f(x_1)}{f(x_2) - f(x_1)}$$

$$= \frac{2(-6.4235) - 1.2903(22)}{-6.4235 - 22}$$

$$\boxed{x_3 = 1.4507}$$

$f(x_3) = -3.5748 < 0$

∴ Root lies b/w x_1 & x_3.

The III approximation is,

$$x_4 = \frac{x_1 f(x_3) - x_3 . f(x_1)}{f(x_3) - f(x_1)}$$

$$= \frac{2(-3.5748) - 1.4507(22)}{-3.5748 - 22}$$

$$\boxed{x_4 = 1.5275}$$

The approximate root is 1.5275.

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b) Newton - Raphson Method

Consider any equation $f(x) = 0$. Let x_0, be the initial approximations to the root.

The first approximation is given by,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

The second approximation is given by,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

① Find a real root of the equation $x^3 - 2x - 5 = 0$ by using Newton Raphson method.

→ Given,

$f(x) = x^3 - 2x - 5$, $f'(x) = 3x^2 - 2$.

$f(0) = -5 < 0$
$f(1) = -6 < 0$
$f(2) = -1 < 0$
$f(3) = 16 > 0$

Let $x_0 = 2$

$f(x_0) = -1$

$f'(x_0) = 10$

The I approximatⁿ is,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 2 - \frac{(-1)}{10}$$

$$\boxed{x_1 = 2.1}$$

$f(x_1) = 0.0610$

$f'(x_1) = 11.2300$

The II approximatⁿ is,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 2.1 - \frac{0.0610}{11.23}$$

$$\boxed{x_2 = 2.0946}$$

$f(x_2) = 0.0005$

$f'(x_2) = 11.1620$

The III approximation is,

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 2.0946 - \frac{0.0005}{11.1620}$$

$$\boxed{x_3 = 2.0946}$$

$x_2 \simeq x_3$

The approximate root is

2.0946.

② Use Newton Raphson method to find a real root of

$\cos x = 3x - 1.$

$f(0) = 2 > 0$

$f(1) = -1.4597 < 0.$

$\rightarrow f(x) = \cos x - 3x + 1$, $f'(x) = -\sin x - 3.$

Let $x_0 = 0.$

$f(x_0) = 2.$

$f'(x_0) = -3.$

The I approximat^n is,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0 - \frac{2}{-3}$$

$$\boxed{x_1 = 0.6667}$$

$f(x_1) = -0.2142$

$f'(x_1) = -3.6184.$

The II approximation is,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.6667 - \frac{(-0.2142)}{(-\cancel{0}3.6184)}$$

$$\boxed{x_2 = 0.6075}$$

$f(x_2) = -0.0014$

$f'(x_2) = -3.5708$

The III approximation is,

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.6075 - \frac{(-0.0014)}{-3.5708}$$

$$\boxed{x_3 = 0.6071}$$

$f(x_3) = 0.$

The approximate root is,

0.6071

③ $\cos x = xe^x$ near $x = 0.5.$

$\rightarrow$ Let $f(x) = \cos x - xe^x$, $f'(x) = -\sin x - x.e^x - e^x.$

$x_0 = 0.5$

$f(x_0) = 0.0532$

$f'(x_0) = -2.9525$

The I approximat^n is,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0.5 - \frac{0.0532}{-2.9525}$$

$$\boxed{x_1 = 0.5180}$$

$f(x_1) = -0.0007$

$f(x_2) = -3.0434$

The II approximation is,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.5180 - \frac{(-0.0007)}{-3.0434}$$

$$\boxed{x_2 = 0.5178}$$

$f(x_2) = -0.0001$

$f'(x_2) = -3.0423$

The III approximation is,

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.5178 - \frac{(-0.0001)}{-3.0423}$$

$$\boxed{x_3 = 0.5178}$$

$x_2 \approx x_3$.

The approximate root is 0.5178.

④ $x.\log_{10} x = 1.2$ near $x = 2.5$

→ $f(x) = x.\log_{10} x - 1.2$, $x_0 = 2.5$.

$$\log_b a = \frac{\log_e a}{\log_e b}$$

$$f'(x) = x.\frac{d}{dx}(\log_{10} x) + \log_{10} x$$

$$= x.\frac{d}{dx}\left(\frac{\log_e x}{\log_e 10}\right) + \log_{10} x$$

$$= \cancel{x} \log_{10} e \,.\, \cancel{x}\left(\frac{1}{\cancel{x}}\right) + \log_{10} x$$

$$f'(x) = 0.4343 + \log_{10} x$$

$f(x_0) = -0.2051$

$f'(x_0) = 0.8322$

The I approximat^n is,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 2.5 - \frac{(-0.2051)}{0.8322}$$

$$\boxed{x_1 = 2.7465}$$

$f(x_1) = 0.0051$

$f'(x_1) = 0.8731$

The II approximation is,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 2.7465 - \frac{0.0051}{0.8731}$$

$$\boxed{x_2 = 2.7407}$$

$f(x_2) = 0$

The III approximate root is

2.7407

⑤ $x \sin x + \cos x = 0$ near $x = \pi$.

→ Given,

$f(x) = x \sin x + \cos x.$

$x_0 = \pi.$

$f(x) = x \sin x + \cos x$

$f'(x) = x \cos x + \sin x - \sin x$

$= x \cos x$

$f(x_0) = -1$

$f'(x_0) = -\pi$

The I approximation is,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= \pi - \frac{(-1)}{-\pi}$$

$$\boxed{x_1 = 2.8233}$$

$f(x_1) = -0.0662$

$f'(x_1) = -2.6815$

The II approximation is,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 2.8233 - \frac{(-0.0662)}{-2.6815}$$

$$\boxed{x_2 = 2.7986}$$

$f(x_2) = -0.0006$

$f'(x_2) = -2.6356$

The III approximation is,

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 2.7986 - \frac{(-0.0006)}{-2.6356}$$

$$\boxed{x_3 = 2.7984}$$

$f(x_3) = 0.$

∴ The approximate root is 2.7984.

⑥ In calculating the height of a vertical column which will buckle under it's own weight, it is necessary to solve the equation, $\frac{x^3}{12960} - \frac{x^2}{180} + \frac{x}{6} - 1 = 0.$

Find one of the roots at $x = 8$ upto 3 decimal places by using Newton - Raphson method.

→ Given,

$$f(x) = \frac{x^3}{12960} - \frac{x^2}{180} + \frac{x}{6} - 1 = 0., \quad x_0 = 8.$$

$$f'(x) = \frac{3x^2}{12960} - \frac{2x}{180} + \frac{1}{6}$$

$$f'(x) = \frac{x^2}{4320} - \frac{x}{90} + \frac{1}{6}.$$

$f(x_0) = 0.0173$

$f'(x_0) = 0.0926.$

The I approximation is,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 8 - \frac{0.0173}{0.0926}$$

$$\boxed{x_1 = 7.8132}$$

$f(x_1) = -0.0001$

$f'(x_1) = 0.0940$

The II approximation is,

$$x_2 = x_1 - \frac{f(x_0)}{f'(x_0)}$$

$$= 7.8132 - \frac{(-0.0001)}{0.0940}$$

$$\boxed{x_2 = 7.8143}$$

$f(x_2) = 0.$

The approximate root is 7.8143.