Section 2:

Q2. Let two such functions be f_1, f_2.

So if composition would be a binary operator, then $f_1 \circ f_2$ would be a continuous function from $[0,1]$ to $[2,3]$.

But this is not true because for values in $[0,1]$ f_2 would give a value in $[2,3]$ but for these values f_1 is not defined. Hence $[0,1]$ is not the domain for $f_1 \circ f_2$. Thus composition is not a binary operator here.

Q3. For being a binary operator, the function should satisfy $f: S \times S \to S$

Here we see that all outputs and inputs are elements of S. But as we are not given the outputs for $b \cdot a$, $a \cdot c$, $c \cdot b$ we can't comment whether this is a binary operator. This is due to the fact that binary operators aren't necessarily communicative.

Section 3:

Q1. Eg.1: Identity element: 0

Inverse of an element x: $-x$ and for 0, 0.
$x \neq 0$

Eg.2.: Identity element: $I_{n \times n}$: $n \times n$ Identity Matrix

Inverse for matrix A: A^{-1}

Eg. 3 : Identity Element : Identity function $f: f(x) = x$

Inverse Element of function $f(x)$: $f^{-1}(x)$

Eg. 4: Identity Element : R_0 : Rotation about the centre $0°$.

Inverse Element : For a reflection, itself, and for a $\theta°$ rotation, $(360-\theta)°$.

Q2. Let there be two identities e and f, then

$a \cdot e = a$ and $a \cdot f = a$

So $a \cdot e = a \cdot f$

Let a' be the inverse of a, then:

$$a' \cdot (a \cdot e) = a' \cdot (a \cdot f)$$

$$\Rightarrow (a' \cdot a) \cdot e = (a' \cdot a) \cdot f \quad \because \text{associativity}$$

$a' \cdot a$ is identity, because we don't know if identities are unique, let $a' \cdot a = g \rightarrow$ identity

$$\Rightarrow g \cdot e = g \cdot f$$

$$\Rightarrow \boxed{e = f} \quad \because g \text{ is identity.}$$

Hence proved that identity is unique.

Q3. Let there be two inverses a_1, a_2 of a, then

$$a_1 \cdot a = e$$

$$a_2 \cdot a = e \Rightarrow a_2 \cdot (a \cdot a_1) = e \cdot a_1$$

$$\Rightarrow a_2 \cdot e = a_1$$

$$\Rightarrow \boxed{a_2 = a_1}$$

Hence, inverse of every element is unique.

Q4. $a \cdot b = a \cdot c$ — ①

Let a' is inverse of a:

① $\Rightarrow (a' \cdot a) \cdot b = (a' \cdot a) \cdot c$

$\Rightarrow e \cdot b = e \cdot c$ where e is the identity

$\Rightarrow b = c$, Hence proved.

Section 4:

Q1 For this question we show that identity of H is also e.

Let, instead identity of H is f, then:

$f \cdot f = f = f \cdot e \quad \because f \in G$

Now $(f^{-1} \cdot f) f = (f^{-1} \cdot f) \cdot e$ where f^{-1} is inverse of f

$\Rightarrow f = e$. Hence $e \in H$.

Q2. The set of integers is a Group under addition.

But under addition odd numbers don't form a subgroup because on adding two odd integers we get an even integer. Hence odd integers aren't closed under addition.

The even integers form a subgroup over addition.

(1) They are closed under addition

(2) Associativity holds

(3) 0 is the identity element.

(4) For x, $-x$ is the inverse element.

All subgroups of $\mathbb{Z}$ are of the form $n\mathbb{Z}$ for all $n \in \mathbb{N}$.

$n\mathbb{Z}$ means multiples of n.

Proof: Let n be the smallest positive element of a subgroup of $\mathbb{Z}$.

Let k be another element of that subgroup.

Now $k = nq + r$, $q, r \in \mathbb{Z}$ and $0 \leq r < n$

As $n, k \in$ Subgroup, $r = k - nq \in$ Subgroup.

But $r < n$, so the only possibility for r is 0.

Thus all elements of the subgroup are multiples of n.

Hence proved.

Q4. Yes $H \cap K$ is also a subgroup of G.

(1) We know that e the identity element of G is in H and K, so $e \in H \cap K$.

(2) If $a, b \in H \cap K$, then $a, b \in H$ and $a, b \in K$

So $a \cdot b \in H$ and $a \cdot b \in K$, so $a \cdot b \in H \cap K$

Thus the binary operator of G is closed in $H \cap K$.

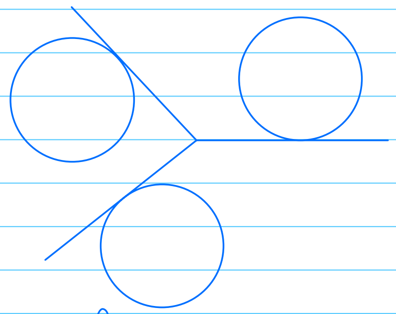
(3) If $c \in H \cap K$ then $c \in H$ and $c \in K \Rightarrow c^{-1} \in H, c^{-1} \in K$

So $c^{-1} \in H \cap K$ hence inverses also exist in $H \cap K$.

Thus $H \cap K$ is also a subgroup of G.

Section 5:

Q1.



The above figure has a 3 element group as its group of symmetries $\rightarrow$ 0°, 120°, 240° rotation.
↳ identity element.

Q2. The mappings can only be of the form x to $x+k$ for all x, because otherwise if x goes to $x+k$ and $x+1$ goes to $x+k'$ where $k' > k+1$, then no number can be mapped to $x+k+1$ because we need to maintain the rightward increasing order.

Now under composition, this is closed because if x to $x+k$ and x to $x+m$ are composed we get

$x+k+m \rightarrow$ which is also in this group.

Section 6:

Q1. $\phi(a \cdot_U a^{-1}) = \phi(a) \cdot_V \phi(a^{-1})$

$\Rightarrow \phi(e_U) = \phi(a) \cdot_V \phi(a^{-1})$

$\Rightarrow e_V = \phi(a) \cdot_V \phi(a^{-1})$

$\Rightarrow (\phi(a))^{-1} \cdot_V e_V = ((\phi(a))^{-1} \cdot_V \phi(a)) \cdot_V \phi(a^{-1})$

$\Rightarrow (\phi(a))^{-1} = \phi(a^{-1})$

Hence proved that: a^{-1} is mapped to inverse of image of a.

Q2. K is the set of elements of U s.t. $\phi(x) = e_V$.

We need to show that K is a subgroup of U.

(1) We know that $\phi(e_U) = e_V$, so $e_U \in K$.

(2) If $a_1, a_2 \in K$, then $\phi(a_1) = e_V$, $\phi(a_2) = e_V$

$\phi(a_1 \cdot_U a_2) = \phi(a_1) \cdot_V \phi(a_2)$
$= e_V$

So $a_1 \cdot_U a_2 \in K$

(3) If $a \in K$, then $\phi(a) = e_V$

$\phi(a^{-1}) = (\phi(a))^{-1} = e_V^{-1} = e_V$

So $a^{-1} \in K$.

Thus K is a subgroup.

Q3. Let's show that H is a subgroup of V.

(1) $\phi(e_U) = e_V$

So $e_V \in H$.

(2) If $a_1, a_2 \in H$, then $\exists\ x_1, x_2$ s.t. $\phi(x_1) = a_1$
$\phi(x_2) = a_2$

$$\phi(x_1 \cdot_U x_2) = \phi(x_1) \cdot_V \phi(x_2)$$
$$= a_1 \cdot_V a_2$$

So $a_1 \cdot_V a_2 \in H$

(3) Let $a \in H$ then $\exists x$ s.t. $\phi(x) = a$

$$\phi(x^{-1}) = (\phi(x))^{-1} = a^{-1}$$

So $\phi(x^{-1}) = a^{-1}$

Thus $a^{-1} \in H$.

Hence H is a subgroup of V.