Assignment 1 - SoC

To the Quantum Future

June 4, 2023

Here is the assignment for the second week of the SoC, after much delay. We hope you have fun and gain new perspectives of concepts you have studied this week. As usual, you are expected to submit your attempts to these questions on your forked/personal git repo.

The question

Everything is an operator! (Note: This question is heavily formal and notation overloaded. This is just so that you become more comfortable with writing in symbols; this is a valuable skill ahead in the course and beyond. You are encouraged to ask anything that strikes you as ununderstandable/weird/not convincing here. Also if you do not follow some part, it is perfectly okay, you will pick it up sooner or later in your academic journey) Buckle up for the ride, and keep your hands and feet inside the ride at all times. Recall that $A \cong B$ denotes that vector spaces A and B are isomorphic, and that $\mathcal{L}(A, B)$ denotes the set of all linear operators from A to B.

(a) (The dual space) Prove that $V \otimes F \cong V$ for arbitrary space V over F (For the pedantic ones: yes, the F in the isomorphism refers to the vector space F over field F). From assignment 0, we then have $\mathcal{L}(V, F) \cong V$. The former space is called the *dual space* - dual, since it parallels vectors in V but with a 'transpose conjugate' (you'll know what we mean when you find the isomorphism) - of V, denoted $V^\dagger$, and its elements are called *linear functionals* on V. The isomorphism that you should have found is called the *Riesz representation* of linear functionals.

(*Bonus!* - just for your reading, nothing to submit here) Fix an inner product $\mathcal{I} = \langle \cdot | \cdot \rangle$ on V. For unit $|v\rangle \in V$, the operator $\langle v| \in \mathcal{L}(V, F)$ (what one calls the dual vector of $|v\rangle \in V$) is the functional taking $|v\rangle \to 1_F$ (or $||v||^2_{\mathcal{I}}$, if $|v\rangle$ is not unit) and the rest of an orthogonal (wrt $\mathcal{I}$) basis to 0_F. Note that the Riesz representation of $\langle v|$ is $|v\rangle$ (hence the notation for the dual vector). Thus verify that $\langle v| \cdot |w\rangle = \langle v|w\rangle$, where the latter term is the same inner product. Indeed, you will agree with us in that the notation $\langle v|$ for the dual vector/functional is a masterpiece of Physics notation, since the action of the dual vector follows almost 'obviously' from the notation. You can now feel good about falling back to the notation and assume some inner product manipulation working out if it 'looks like it does' - you now know why. Finally, since $\mathcal{L}(V, V) \cong V \otimes \mathcal{L}(V, F)$, which (i.e. describe A's action) $A \in \mathcal{L}(V, V)$ do you usually call as $|v\rangle\langle w|$?

(b) (The partial trace) The trace is a linear functional $\text{tr} : \mathcal{L}(V, V) \to F$ defined by $\text{tr} A_{ij} \equiv \delta_{ij}$ for the standard basis A_{ij} of $\mathcal{L}(V, V)$. Show that $\text{tr} \in \mathcal{L}(V, V)^\dagger$ has the Riesz representation $\langle I|$ wrt the inner product $\mathcal{I}$ = Hilbert-Schmidt on $\mathcal{L}(V, V)$ (Here I is the identity on V). Using Q5 of Assignment 0 (HS inner product is the standard inner product on the vectorized space), show that the Riesz form of the trace operator when it is described as a function $\text{tr} : \text{vec}(\mathcal{L}(V, V)) \to F$ is $\langle \text{vec}(I)|$. The partial trace $\text{tr}_B \in \mathcal{L}(A \otimes B, A)$ is defined by

$$\text{tr}_B(A_i \otimes B_j) \equiv \text{tr}(B_j) A_i$$

i.e. by its action on basis $A_i \otimes B_j$ of $A \otimes B$ (where A_i and B_j are bases of A, B resp.). Prove that its Riesz representation wrt the inner product being HS inner product + standard inner product for tensor spaces (you know what that means!) is $|I\rangle\langle I|$. Finally, show that $\mathcal{L}(A \otimes B, A) \cong \mathcal{L}(A \otimes B, A \otimes F) \cong \mathcal{L}(A, A) \otimes \mathcal{L}(B, F)$. Prove that tr_B maps to $I_A \otimes \text{tr}$ under the above isomorphism.