Assignment 0 - SoC

To the Quantum Future

May 9, 2023

Here is the assignment for the first week of the SoC, you are expected to submit your attempts to these questions on your forked/personal git repo. (Would be good if you $\LaTeX$ed it)

1. (Elegance) This is just for practice, related to Linear Algebra but not Quantum Mechanics. Look at both the Oddtown and Eventown problems (hopefully not the solutions) and solve them. (You might need to go over the field $\mathbb{F}_2$ before this)

2. (Weaker suffices!) Suppose A is any linear operator on a Hilbert space, V. A is said to be Hermitian iff $\langle x, Ay \rangle = \langle Ax, y \rangle$ for all vectors $x, y \in V$.

   Now suppose V is finite dimensional. Show that a necessary and sufficient condition for an operator A in V to be Hermitian is:

   $$\langle x, Ax \rangle = \langle Ax, x \rangle$$

   for all vectors $x \in V$.

3. (Higher Dimensions!) We're going to give you some matrices, and your job is to find their eigenvalues (no calculator, and write down how you find them, because that is what we want to know)

   (a) $\begin{bmatrix} 0 & 5 & 0 & 4 \\ 5 & 0 & 4 & 0 \\ 0 & 3 & 0 & 2 \\ 3 & 0 & 2 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 & 5 & 4 \\ 0 & 0 & 3 & 2 \\ 5 & 4 & 0 & 0 \\ 3 & 2 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 25 & 20 & 20 & 16 \\ 15 & 10 & 12 & 8 \\ 15 & 10 & 12 & 8 \\ 9 & 6 & 6 & 4 \end{bmatrix}$

4. (Algebra and Technicalities) Look up the definitions of a norm and a metric. Show that $|x| = \sqrt{\langle x|x \rangle}$ is a valid norm and $d(x, y) = |x - y|$ is a valid metric.

   A Hilbert space is just a *complete* inner product space. This means that every Cauchy sequence in the vector space converges to some element **in the vector space**.

   Consider the set of all continuous real functions on $[-1, 1]$. Confirm that this is a vector space over $\mathbb{R}$ with the normal operations: $(f + g)(t) = f(t) + g(t)$, $(\alpha f)(t) = \alpha f(t)$.

   Define the inner product as $\langle f|g \rangle = \int_{-1}^{1} f(t)g(t)dt$. Now take the sequence of functions

   $$f_n(t) := \begin{cases} 1 & t \in [-1, 0] \\ 1 - nt & t \in [0, \frac{1}{n}] \\ 0 & \text{otherwise} \end{cases}$$

   Compute $\langle f_n|f_m \rangle$ and show that $\{f_n\}$ is Cauchy. Find where this sequence converges and conclude that this inner product space is not complete.

   Show the following remarkable result: *Finite dimensional vector space V under field $\mathbb{F}$ is a Hilbert space under any valid inner product on V for $\mathbb{F} = \mathbb{R}/\mathbb{C}$.*

Hint: You might be familiar with the result for $V = \mathbb{R}^n$. Try to show the same for $\mathbb{C}^n$. Next, note that any finite vector space V over field $\mathbb{F}$ is isomorphic (look up the definition) to $\mathbb{F}^n$ where $n = \dim V$. Use this to reduce the general problem to that of $\mathbb{F}^n$.

5. (Hilbert-Schmidt and Vectorization) Verify that $\mathcal{L}(A, B)$ is a vector space. Consider the following function $T : \mathcal{L}(A, B) \to A \otimes B$ defined by

$$U = \sum_{i,j} \alpha_{ji} |w_j\rangle\langle v_i| \longmapsto u = \sum_{i,j} \alpha_{ij} |v_i\rangle \otimes |w_j\rangle$$

where $\mathcal{B} = (\{v_i\}, \{w_i\})$ is a fixed input-output orthonormal basis pair. Check that T is a bijection. This establishes $A \otimes B \cong \mathcal{L}(A, B)$. Consider the inner product $\langle U|V\rangle_{HS} := \langle TU|TV\rangle = \langle u|v\rangle$ on $\mathcal{L}(A, B)$ (Note that the latter inner product refers to that of the tensor product space). Show that $\langle U|V\rangle_{HS}$ reduces to $\mathrm{tr}(U^\dagger V)$. This is called the Hilbert-Schmidt inner product and is quite useful once we see density operators later on.

One more connection between the two spaces: define the vectorization operator $\mathrm{vec} : \mathcal{M}_{m\times n}(\mathbb{F}) \to \mathcal{M}_{mn\times 1}(\mathbb{F})$ by:

$$\mathrm{vec}(A) := \left[a_{11}, a_{12}, ..., a_{1n}, a_{21}, ..., a_{2n}, ..., a_{mn}\right]^{\mathrm{T}}$$

that is, vec flattens the matrix A in row-major form into one long vector. This is exactly how C-style 2D arrays are stored in memory - as their vectorized forms. Vectorization operators are used extensively in Quantum Operations. Here's a neat fact: Consider $U \in \mathcal{L}(A, B)$. Show that

$$(TU)_{\mathcal{B}} = \mathrm{vec}(U_{\mathcal{B}}).$$

Here, $U_{\mathcal{B}}$ is the matrix form of U under bases $\mathcal{B}$, and $u_{\mathcal{B}}$ is the matrix form of $u \in A \otimes B$ under basis $\{v_i \otimes w_j\}_{i,j}$ (wait a minute, doesn't u also count as a linear operator? Yes! Then where are the two bases for matrix representation? Why does one suffice? Think about it.) This equality means that vec is a sort of 'matrix version' of the bijection T.