Thermoelasticity model of a hollow sphere

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In this project we will be considering a spherical pressure vessel. The vessel consists of two layers with two different materials. The 1st material is Tungsten Carbide and the other is steel. Considering this as a problem of thermoelastic analysis we will encounter with how the vessel reacts to thermal effects as a body force. The vessel is subjected to different outer and inner temperature changes. The problem is solved by governing equations for conduction and convection heat transfer in each of the two materials, the thermal diffusivity of the material leading to radial displacement in the shell. The vessel as subjected to different temperature and stress distribution is shown by contour plots using MATLAB. The Successive Over Relaxation iterative method is used for to solve the equations at each step. We have also seen the displacement throughout the vessel for analysis of results.

1 | INTRODUCTION

The project includes a spherical pressure vessel having Tungsten Carbide as its inner layer and stainless steel as the outer layer. For the calculation of temperature variation the governing equation is solved by implicit method. Backward difference finite method approach is used to obtain the matrix equation and the method was proved to be unconditionally stable. We have been given that the inner layer has an inner radius of a and outer radius of b. The outer layer has an inner radius of b and outer radius of c.

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Properties	Tungsten	Steel
Inner Radius (m)	0.07	0.08
Outer Radius (m)	0.08	0.13
E (GPa)	620	210
ν	0.2	0.3
K (W/mK)	28	16.8
D (m^2/s)	0.01	$4.2x10^{-6}$

2 | TEMPERATURE DISTRIBUTION IN THE SPHERICAL VESSEL

We have initial condition given

$$T(a,t) = T_0 + (T_f - T_0)(1 - e^{-\beta t})$$

where

$$T_f = 300^{\circ}C \; and \; \beta = 10 \; s^{-1}.$$

The outer surface loses heat to the surroundings through convection. The provided ambient temperature is 30 degrees and the convective heat transfer coefficient is

$$\lambda = 200W/(m^2 - K)$$

.

We have been given the initial condition which changes with respect to time that will act as a heat source inside the shell. From the inner layer the heat will transfer throughout the shell of tungsten Carbide and then to steel via conduction and through steel to the surroundings via convection. At the interface of the two shells since the material is changing it will have the temperature at the outer surface of Tungsten Carbide and inner surface of steel the same. Following is discussed the matrix formation for the temperature distribution case from the governing equation provided.

$$\frac{\partial T}{\partial r} = \frac{D}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial T}{\partial r}) \tag{1}$$

Differentiating the equation partially which gives out =

$$= \frac{D}{r^2}[2r\frac{\partial T}{\partial t} + r^2\frac{\partial^2 T}{\partial r^2}]$$

For the above equation by using forward difference in time and centered difference in space we derive the following equation :

$$\frac{U_i^m - U_i^{m-1}}{\Delta t} = \frac{D}{r^2}[2r(\frac{U_{i+1}^m - U_{i-1}^m}{2\Delta r} + r^2(\frac{U_{i-1}^m - 2U_i^m + U_{i+1}^m}{(\Delta r)^2}]$$

Now by simplification,

$$U_i^m - U_i^{m-1} = \frac{\Delta t * D}{(\Delta r)^2}[\frac{\Delta r}{r}][U_{i+1}^m - U_{i-1}^m] + [U_{i-1}^m - 2U_i^m + U_{i+1}^m]$$

Now assigning,

$$\frac{\Delta t * D}{(\Delta r)^2} = F$$

After simplification we get,

$$[U_i^{m-1} = [\frac{\Delta r}{r} * F - F]U_{i-1}^m + (1 + 2F)U_i^m - (\frac{F\Delta r}{r} + F)U_{i+1}^m] \tag{2}$$

Let us consider the interface as p th node so the above equation will be used for solving from node 2 to p-1 that is for Tungsten Carbide shell and from node p+1 to n that is for steel shell. Further now we will calculate for interface condition and boundary conditions.

For interface condition the temperature at interface will be same as mentioned earlier so we will use backward finite difference scheme for nodes of Tungsten Carbide shell and forward finite difference scheme for steel shell. These schemes will be equal to one another at the interface. We will be using 1st derivative of second order accurate scheme for which we get the following equation

$$-K_1 \frac{\partial T}{\partial r}_i^m = -K_2 \frac{\partial T}{\partial r}_i^m \tag{3}$$

$$-K_{11}[\frac{3U_i - 4U_{i-1} + U_{i-2}}{2h}] = -K_2[\frac{-U_{i-2} + 4U_{i+1} - 3U_i}{2h}]$$

After simplification,

$$-K_1 U_{i-2} + 4K_1 U_{i-1} - 3[K_1 + K_2]U_i - [K_2 U_{i+2}] + 4K_2 U_{i+1} = 0 \tag{4}$$

For end condition at outer surface of steel, convection is taking place so we will perform backward finite difference scheme for steel, since we have all the values known for the nodes to calculate last nodes value. The equation for end condition can be given as :

$$-K_2 \frac{\partial T}{\partial r}{}_i^m = h(U_{n+1}{}^m - U_a) \tag{5}$$

$$-K_2[\frac{3U_i^m - 4U_{i-1}^m + U_{i-2}^m}{2\Delta r}] = h[U_{n+1}^m - U_a]$$

$$i.e - K_2[\frac{3U_{n+1}{}^m - 4U_n^m + U_{n-1}{}^m}{2\Delta r}] = h[U_{n+1}{}^m - U_a]$$

$$[\frac{3K_2}{2h\Delta r} - 1]U_{n+1}{}^m - [\frac{2K_2}{h\Delta r}]U_n^m + [\frac{K_2}{2h\Delta r}]U_{n-1}{}^m = U_a \tag{6}$$

From the equations 2, 4, 6 the matrix can be formed as :

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ [\frac{\Delta r_1}{r_1} * F_1 - F_1] & [1+2F_1] & -[\frac{F_1 \Delta r_1}{r_1} + F_1] & 0 & 0 & 0 & 0 & 0 & 0 \\ . \\ . \\ . \\ . \\ . \\ . \\ 0 & 0 & -K_1 & 4K_1 & -3[K_1+K_2] & 4K_2 & -K_2 & b & 0 \\ . \\ . \\ . \\ . \\ . \\ 0 & 0 & 0 & 0 & 0 & 0 & [\frac{\Delta r_2}{r_2} * F_2 - F_2] & [1+2F_2] & -[\frac{F_2 \Delta r_2}{r_2} + F_2] \\ 0 & 0 & 0 & 0 & 0 & 0 & [\frac{K_2}{2h\Delta r_2}] & -[\frac{2K_2}{h\Delta r_2}] & [\frac{3K_2}{2h\Delta r_2 - 1}] \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ . \\ . \\ . \\ . \\ . \\ . \\ U_p \\ . \\ . \\ . \\ . \\ . \\ U_n \\ U_{n+1} \end{bmatrix} = \begin{bmatrix} U_{a,t}^{m-1} \\ U_2^{m-1} \\ . \\ . \\ . \\ . \\ . \\ . \\ 0 \\ . \\ . \\ . \\ . \\ . \\ U_n^{m-1} \\ U_a^{m-1} \end{bmatrix}$$

3 | RADIAL DISPLACEMENT IN THE SPHERICAL VESSEL

We have a governing equation given for calculating displacement as :

$$\frac{1+v}{1-v} \alpha \frac{\partial U}{\partial r} = \frac{d}{dr}[\frac{1}{r^2} \frac{d}{dr}[r^2 x]] \tag{7}$$

$$Here\ v = Poisson's\ ratio\ i.e\ for\ Tungsten\ Carbide\ v_t = 0.2. For\ steel\ v_s = 0.3$$

From the above equation we will get to solve for nodes 2 to p-1 and for nodes p+1 to n.

$$[\frac{1+v}{1-v}\alpha\frac{\partial U}{\partial r}] = [\frac{-2T}{r} + \frac{2}{r}\frac{dx}{dr} + \frac{d^2x}{dr^2}] \tag{8}$$

$$For\ Tungsten\ carbide = \alpha_t = 4.5E-6$$

$$and\ for\ steel = \alpha_S = 17.3E-6$$

For simplification, let us consider :

$$b = \frac{1+v}{1-v}\alpha$$

So for Tungsten,

$$b1 = \frac{1+v_t}{1-v_t}\alpha_t$$

For steel,

$$b2 = \frac{1+v_s}{1-v_s}\alpha_s$$

So from the displacement equation 8 we perform centered difference in space for temperature and forward difference in space for displacement.

$$b\frac{U_{i+1}-U_{i-1}}{2\Delta r} = -2\frac{x_i}{r^2} + \frac{2}{r\Delta r}[x_{i+1}-x_i] + \frac{1}{\Delta r^2}[x_{i-1}-x_i+x_{i+1}]$$

$$U_{i+1}-U_{i-1} = [\frac{2}{\Delta r}]x_{i-1} - [\frac{-4\Delta r}{r^2} + \frac{4}{r} + \frac{4}{\Delta r}]x_i + [\frac{4}{r} + \frac{2}{\Delta r}]x_{i+1} \tag{9}$$

Equation 9 is to be used for calculating from nodes 2 to p-1 and from p+1 to n. For calculating the unknown end conditions and interface condition we will use the stress equation given as :

$$S_{rr} = \frac{E}{[1+v][1-2v]}[[1-v]\frac{\partial U}{\partial r} + 2v\frac{u}{r}] - \frac{ED[T-T_a]}{1-2v} \tag{10}$$

here E is the Young's modulus,

$$For\ Tungsten\ carbide = E_1 = 620GPa$$

$$and\ for\ steel = E_2 = 210GPa$$

At end condition at node 1 the stress is 0, here we use second order forward difference of first derivative and so equation 10 becomes;

$$0 = \frac{E_1}{[1+v_t][1-2_v t]}[[1-v_t]\frac{-3x_i + 4x_{i+1} - x_{i-1}}{2\Delta r1} + 2v_t \frac{u_i}{r1}] - \frac{E_1 D_1[U-U_a]}{1-2v_t}$$

After simplification,

$$[\frac{-3C_1}{2\Delta r_1} + \frac{2v_t}{r_1}]x_i + x_{i+1}\frac{2C_1}{\Delta r_1} - x_{i+2}[\frac{C_1}{2\Delta r_1}] = D_1[U-U_a] \tag{11}$$

Here ,

$$C_1 = \frac{1-v_t}{1+v_t}$$

Interface condition: At interface the stress from both the shells is same so we equate them to get the interface condition.

$$\frac{E_1}{[1+v_t][1-2v_t]}[[1-v_t]\frac{\partial x}{\partial r_1} + 2v_t\frac{x}{r_1}] - \frac{E_1 D_1[U-U_a]}{1-2v_t} = \frac{E_2}{[1+v_s][1-2v_s]}[[1-v_s]\frac{\partial x}{\partial r_2} + 2v_s\frac{x}{r_2}] - \frac{E_2 D_2[U-U_a]}{1-2v_s}$$

After simplifying we have,

$$x_{i-2}[\frac{c_2[1-v_t]}{2\Delta r_1} - x_{i-1}[\frac{2c_2[1-v_t]}{\Delta r_1}] + x_i\frac{[3c_2[1-v_t]}{2\Delta r_1} + \frac{3c_3[1-v_s]}{2\Delta r_2} + \frac{2v_t}{r_1} - \frac{2v_s}{r_2}] - x_{i+1}[\frac{[2c_3[1-v_s]}{\Delta r_2}] + x_{i+2}\frac{[c_3[1-v_s]}{2\Delta r_2} = [S_1 - S_2]U - U_a]]] \tag{12}$$

where,

$$\frac{E_1}{[1+v_t]+[1-2v_t]} = c_2$$

$$\frac{E_2}{[1+v_s][1-2v_s]} = c_3$$

$$\frac{E_1 D_1}{1-2v_t} = S_1$$

$$\frac{E_2 D_2}{1-2v_s} = S_2$$

For end condition the stress is zero and in here we use backward finite difference of second order first derivative we get following equation:

$$\frac{E_2}{1+v_s][1-2v_s]}[1-v_s]\frac{\partial U}{\partial r_2}+2v_s\frac{U_i}{r_2}-\frac{E_2D_2[U-U_a]}{1-2v_s}=0$$

By simplification,

$$\frac{1-v_s}{1+v_s}[\frac{3x_i-4x_{i-1}+x_{i-2}}{2\Delta r^2}+\frac{2v_sx_i}{r_2}]=D_2[U-U_a]$$

Finally we have,

$$x_i[\frac{3c_4}{2r_2}+\frac{2c_4v_S}{r_2}]-x_{i-1}[\frac{2c_4}{\Delta r_2}]+x_{i-2}[\frac{c_4}{2\Delta r_2}]=D_2[U-U_a]$$

$$\begin{bmatrix}
[\frac{-3C_1}{2\Delta r_1}+\frac{2v_t}{r_1}] & \frac{2C_1}{\Delta r_1} & [\frac{C_1}{2\Delta r_1}] & 0 & 0 & 0 & 0 & 0 & 0 \\
[\frac{2}{b_1\Delta r_1}] & [\frac{-4}{b}(\frac{\Delta r_1}{r_1^2}+\frac{1}{r_1}+\frac{1}{\Delta r_1})] & [\frac{4}{r_1}+\frac{2}{\Delta r_1}] & 0 & 0 & 0 & 0 & 0 & 0 \\
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. & & & & & & & & \\
0 & 0 & \frac{C_2(1-v_t)}{2\Delta r_1} & \frac{-2C_2[1-v_t]}{\Delta r_1} & [\frac{3C_2(1-v)}{2\Delta r_1}+\frac{3C_3(1-v_s)}{2\Delta r_2}] & \frac{2C_3(1-v_s)}{\Delta r_2} & \frac{C_3(1-v_s)}{2\Delta r_2} & 0 & 0 \\
. & & & & & & & & \\
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. & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & [\frac{2}{b_2\Delta r_2}] & [-4(\frac{\Delta r_2}{r_2^2}+\frac{1}{r_2}+\frac{1}{\Delta r-2})] & [\frac{4}{r_2}+\frac{2}{\Delta r_2}] \\
0 & 0 & 0 & 0 & 0 & 0 & [\frac{C_4}{2\Delta r_2}] & -[\frac{2C_4}{\Delta r_2}] & [\frac{3C_4}{2\Delta r_2}+\frac{2C_4v_s}{r_2}]
\end{bmatrix}
\begin{bmatrix} X_1 \\ X_2 \\ . \\ . \\ . \\ . \\ . \\ . \\ X_p \\ . \\ . \\ . \\ . \\ . \\ X_n \\ X_{n+1} \end{bmatrix}
=
\begin{bmatrix} D_1(T-T_a) \\ U_{i+1}-U_{i-1} \\ . \\ . \\ . \\ . \\ . \\ . \\ (S_1-S_2)(T_1-T_a) \\ . \\ . \\ . \\ . \\ . \\ U_{i+1}-U_{i-1} \\ D_2(T_1-T_a) \end{bmatrix}$$

4 | STRESS DISTRIBUTION IN THE SPHERICAL VESSEL

Stress in the vessel is generated due to thermal expansion for both the materials the thermal properties are different so we have two different stress distribution which are governed by the following stress distribution equation:

$$S_{rr}=\frac{E}{[1+v][1-2v]}(\frac{[1-v]\partial x}{\partial r}+2v\frac{x}{r})-\frac{ED[U-U_a]}{1-2v}$$

As mentioned earlier the stresses at node 1 and node n+1 are zero. At the interface condition since the stresses for both the materials for the shell are same hence we can choose either of the material properties to get stress distribution at the interface. So, we will be using properties of material 1 to get stress distribution from node 2 to p and properties of material 2 to get stress distribution from node p+1 to n. We will be using the constants for simplification mentioned in earlier equations.

5 | SOR RELAXATION FACTOR STUDY

Sor means successive over relaxation.The generalized form of the equation by using relaxation factor is given as:

$$X_i^{k+1} = X_i^k + \frac{\Omega}{ai,i}[b_i - \sum a[i,j]x[j]^k + 1 - \sum a[i,j]x[j]^{k+1}$$

In here is the relaxation factor which takes values from 0 to 2, depending on what method is to be used to solve the equation. When it is a requirement to solve for Successive Under Relaxation takes value such that 0< <1. When the equation uses = 1 the equation method is a Gauss Seidel method. For Successive Over Relaxation takes values as 1< <2. The choosing of helps us to determine optimum value of the number of iterations that are needed to get the desired results for the problem. In given problem we solve the equations by SOR method hence, must be chosen from 1 to 2. As takes values between 1 to 2 the number of iterations it uses to solve the equation also changes. By choosing the right value of we are to solve the equation so we check with different values of . For calculating the optimum number of iterations the SOR solver mentioned was put in a loop to get values from 1 to 2. When the omega was < 1.7 the iteration numbers required to obtain conversions was in 3000 iterations to 600. For 1.7 the iteration number to get conversions was 250 iterations. For 1.8 value of omega the iterations were 150 and as the omega went from 1.9 to the iteration number slightly increased per conversion. From the above study we can say the optimum value for relaxation factor is 1.8 for which we get the conversions at least number of iterations.

6 | ANALYSIS OF RESULTS

The problem provided was studied and coded to solve for the iterative method of successive over relaxation for the temperature distribution, the radial displacement and the stress distribution in the spherical vessel. Results were plotted for 6 different time instances and then a 2D contour for temperature with corresponding time and radial increment, radial displacement with corresponding time and radial increment and for stress distribution with corresponding time and radial increment. The temperature and displacement requires to use the SOR iterative method and stress is directly calculated from displacement.
The number of nodes chosen were 140 for time of 50secs in total this space and time nodes will give the mesh size of $9.09x10^-4$ for space and 50 for time. This chosen space and time nodes will give out easy results and calculations. The mesh size for both tungsten carbide and steel were chosen different but was it was sure they have same mesh size in space for easy calculations. Throughout there were constants allotted for complex values for simplification of code.

From the plot of temperature, as the inner temperature is at higher value it dissipated throughout the shell of tungsten carbide and steel. The whole vessel is at an initial temperature of 20 deg C i.e. 293 K. as we move from inside surface of tungsten carbide to outside surface of steel, we see the temperature drop suggesting that the temperature is dissipating and to support this we can see at various instances the plot for temperature vs radial increment all get overlapped slowly, which are at high temperature initially and then drop as the radial increment increases.

For the plot of displacement we see as we go from inside layer to the outer surface we see that there is a drop and then it rises as the temperature dissipates in the vessel, it also supports the statement that the temperature is slowly

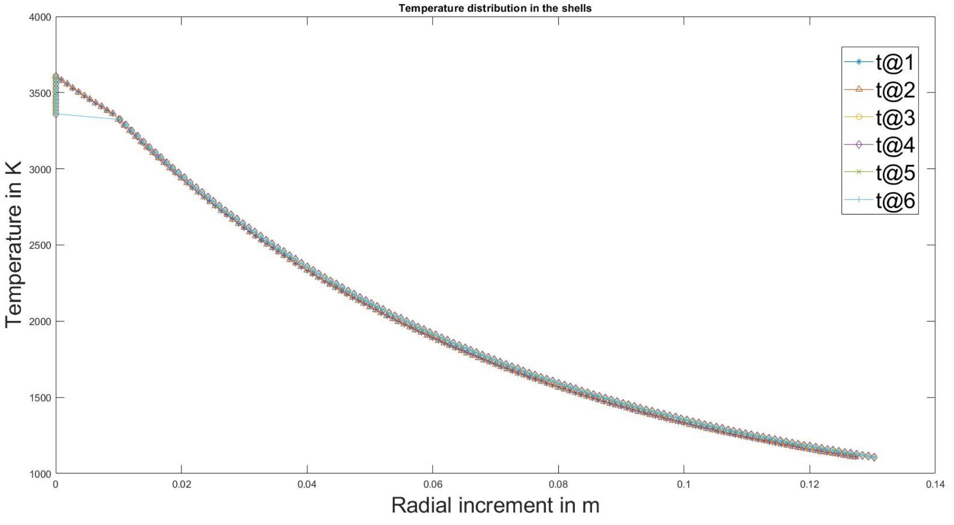

FIGURE 1 Temperature vs radial increment

distributing throughout the shell. Eventually, the temperature reaches steady state value. We can spot the interface of the two different materials by a sharp change which shows less number of nodes for tungsten and a lot more on steel shell. The contour plot shows how at a particular time and instance the radial displacement of materials occur as there is increase in temperature which causes material to expand and hence the radial displacement.

For the stress distribution we have the initial condition and final condition are zero. This is because with the increase in temperature as there is no resistance at the ends there is no stress but for the inner nodes there is thermal expansion which results in stress distribution throughout the vessel and also at the intersection of 2 materials.

From the plot of stress distribution v/s radial increment we can see for tungsten the stress is at peak value at the initial near the inner layer. We see that it gradually decreases at interface. At the interface there is a sudden drop in the plot showing the material change and from the next node of the material 2 shell the stress in increasing and keeps on increasing upto the end condition when it is free to expand. Looking at the contour it shows the stress distribution with respect to radial and time increments throughout the vessel showing very low stress distribution. Additionally, we see higher stress distribution at certain areas.

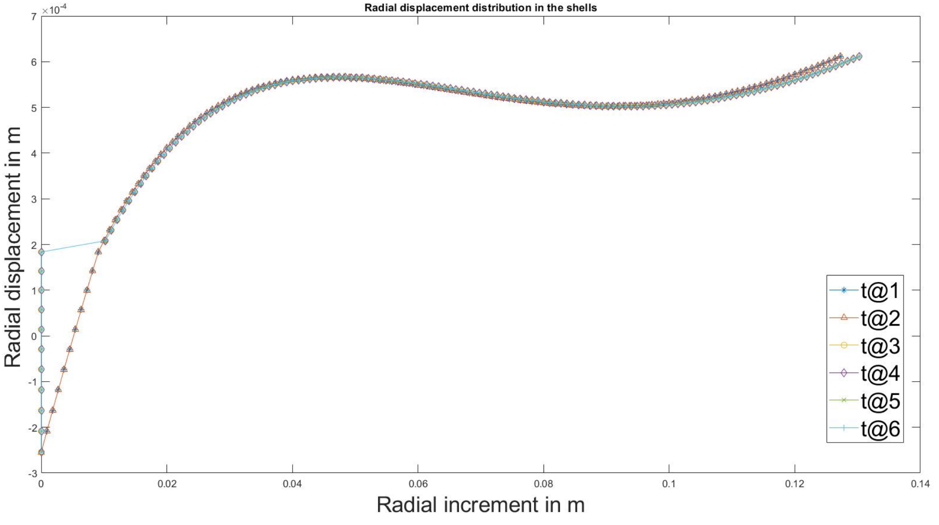

FIGURE 2 Radial Displacement vs Radial Increment

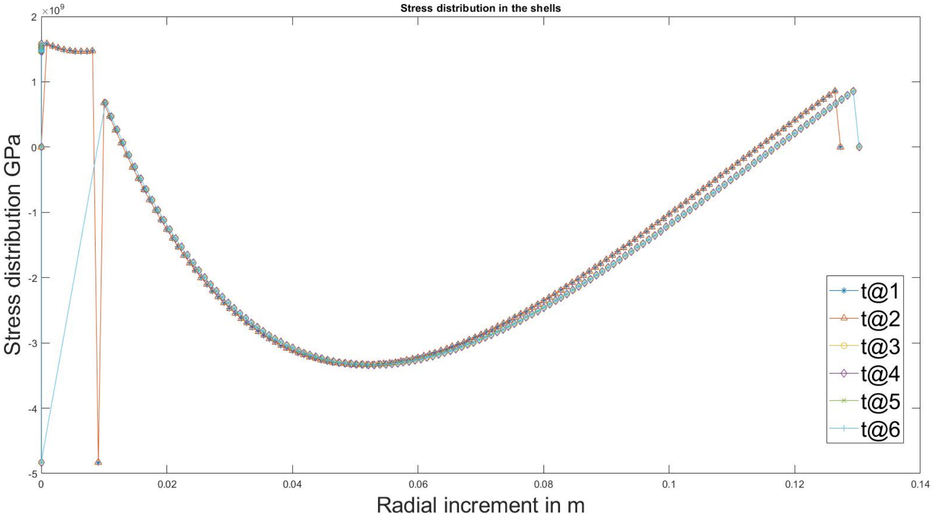

FIGURE 3 Stress Distribution vs Radial Increment

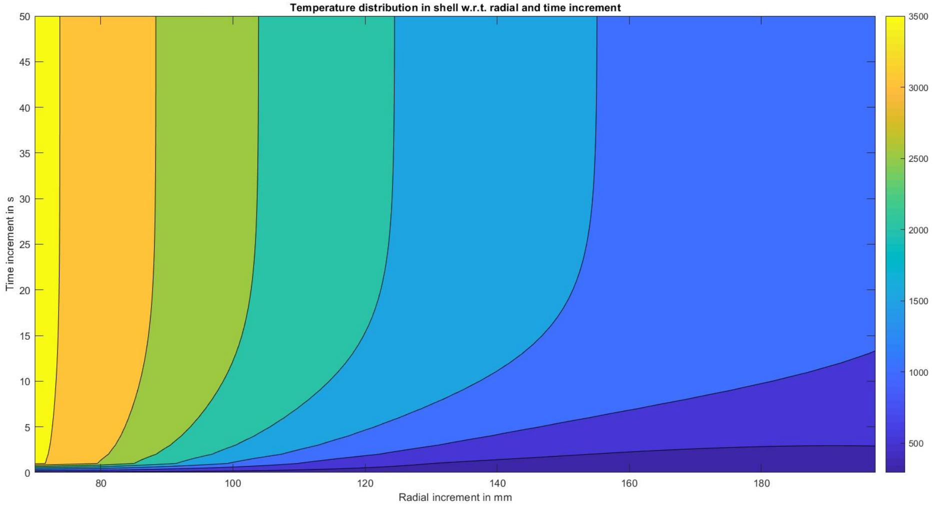

FIGURE 4 Time increment vs radial increment

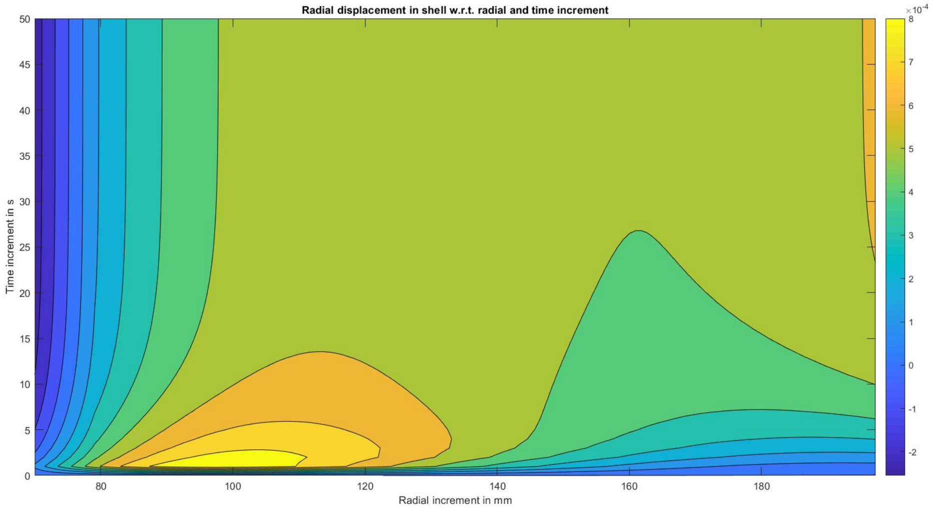

FIGURE 5 Time increment vs radial increment

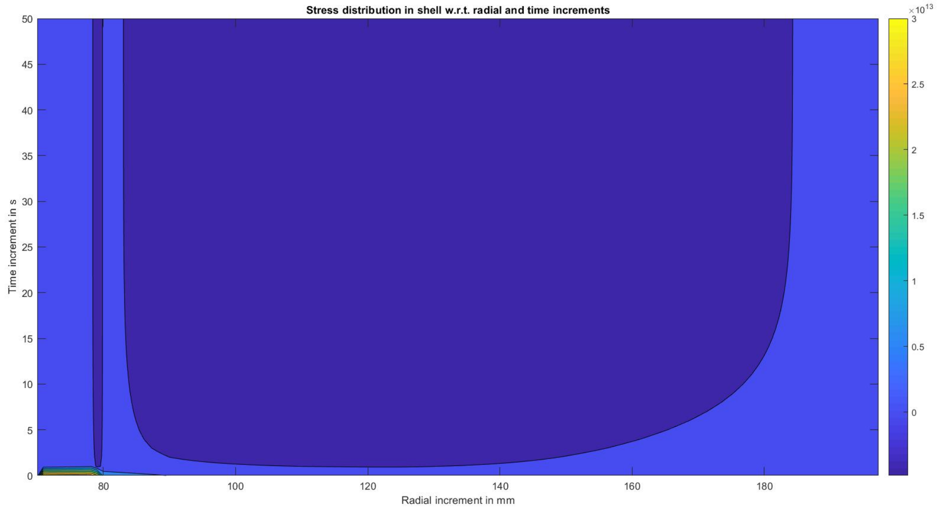

FIGURE 6 Time increment vs radial increment