Dr. Rupali Deshpande

Room No. 1120

rupali.deshpande@vit.edu

9960731502

Relation between solution of homogeneous and nonhomogeneous system:

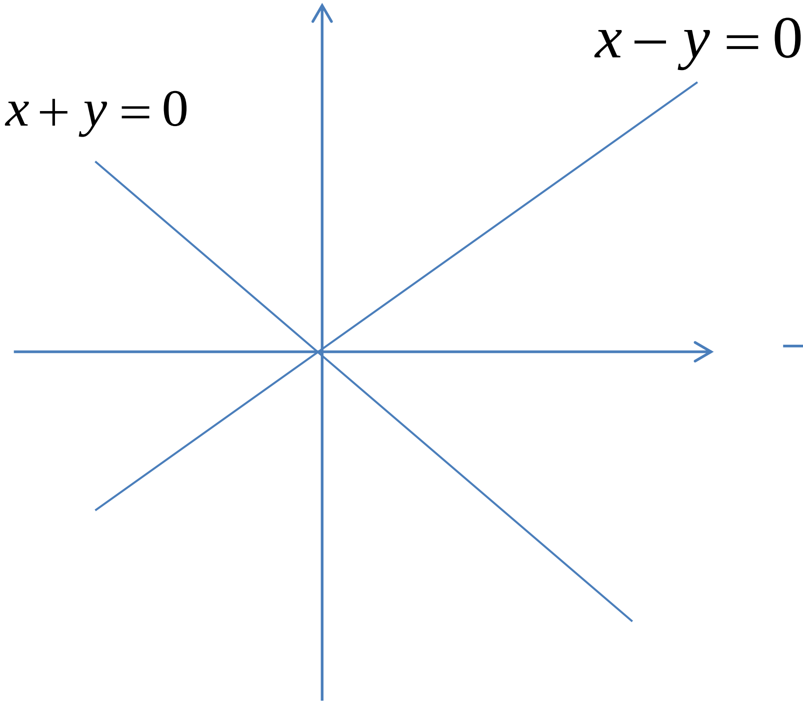
Consider The homogeneous system $x - y = 0$, $x + y = 0$

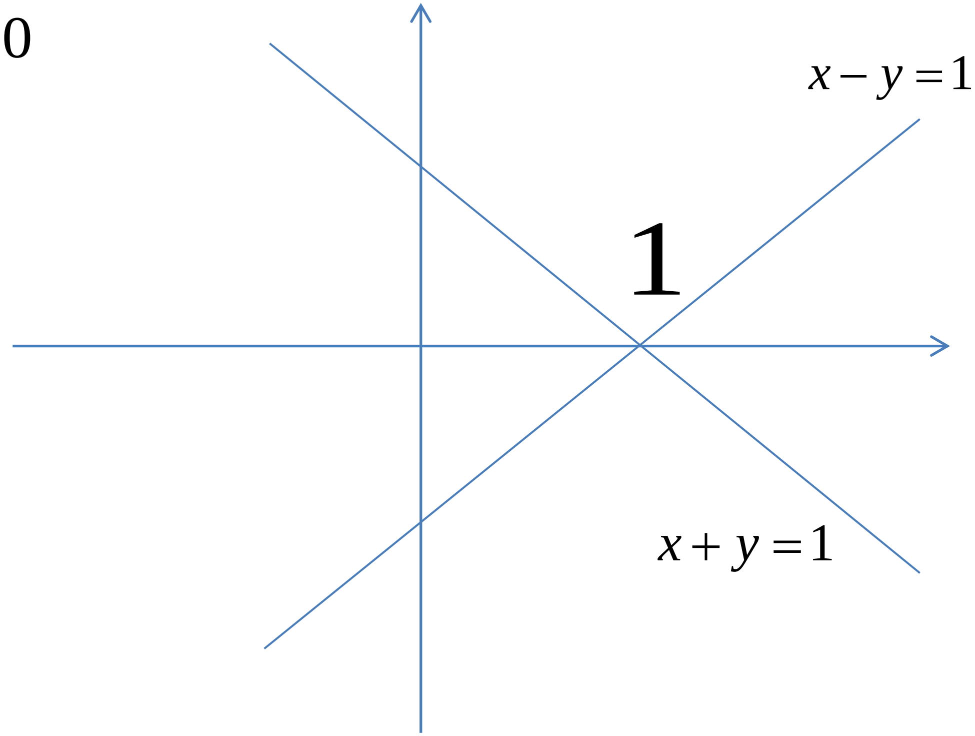
The solution of this is $x = y = 0, k \in \mathrm{R}$.

Let consider the corresponding nonhomogeneous system

$$x - y = 1, \quad x + y = 1$$

Solution of this system is $x = 1, \; y = 0$.





Solution of non-homgeneous system is obtained by shifting the origin of homogeneous system.

Illustrative Examples

Example : Check the consistency of following the systems.

If consistent, find the solution.

$$3x_1 - 6x_2 - x_3 - x_4 = 0, x_1 - 2x_2 + 5x_3 - 3x_4 = 0,$$
$$2x_1 - 4x_2 + 3x_3 - x_4 = 0.$$

The system is homogeneous, it is consistent.

$$\text{Coefficient matrix A} = \begin{bmatrix} 3 & -6 & -1 & -1 \\ 1 & -2 & 5 & -3 \\ 2 & -4 & 3 & -1 \end{bmatrix}$$

$$R_1 - R_3 \Rightarrow \begin{bmatrix} 1 & -2 & -4 & 0 \\ 1 & -2 & 5 & -3 \\ 2 & -4 & 3 & -1 \end{bmatrix}, \begin{matrix} R_2 - R_1, \\ R_3 - 2R_1 \end{matrix} \Rightarrow \begin{bmatrix} 1 & -2 & -4 & 0 \\ 0 & 0 & 9 & -3 \\ 0 & 0 & 11 & -1 \end{bmatrix}'$$

$$R_3 - \frac{11}{9}R_2 \Rightarrow \begin{bmatrix} 1 & -2 & -4 & 0 \\ 0 & 0 & 9 & -3 \\ 0 & 0 & 0 & \frac{8}{3} \end{bmatrix}$$

$\rho(A) = 3 < 4,$ system has a non-trivial solution.

Equivalent system is

$$x_1 - 2x_2 - 4x_3 = 0 ------------- (\text{i})$$

$$9x_3 - 3x_4 = 0 \qquad ------------- (\text{ii})$$

$$\frac{8}{3}x_4 = 0 \qquad ------------- (\text{iii})$$

From $(\text{iii}), x_4 = 0$ $------------- (\text{iv})$

From (ii) & $(\text{iv}), x_3 = 0.$

Here x_2 is a free variable(as x_2 is a non-pivot element)

Let $\quad x_2 = k \quad \therefore x_1 = 2k$

Solution vector

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2k \\ k \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} k$$

k is known as parameter.

This system has 1-parametric solution.

Example : Check whether the system have nontrivial solution, if it has find.

$$3x + 4y - z - 6w = 0; \qquad 2x - 3y + 2z - 3w = 0;$$

$$2x + y - 14z - 9w = 0; \quad x + 3y + 13z + 3w = 0.$$

This is a homogeneous sysyem of equation.

$\therefore$ it is always consistent.

$\therefore$ To check whether it has a non trivial solution are not

consider $AX = 0$, where $A = \begin{bmatrix} 3 & 4 & -1 & -6 \\ 2 & -3 & 2 & -3 \\ 2 & 1 & -14 & -9 \\ 1 & 3 & 13 & 3 \end{bmatrix}$

$$\begin{bmatrix} 3 & 4 & -1 & -6 \\ 2 & -3 & 2 & -3 \\ 2 & 1 & -14 & -9 \\ 1 & 3 & 13 & 3 \end{bmatrix} \xrightarrow{R_{14}} \begin{bmatrix} 1 & 3 & 13 & 3 \\ 2 & -3 & 2 & -3 \\ 2 & 1 & -14 & -9 \\ 3 & 4 & -1 & -6 \end{bmatrix} \xrightarrow[R_4-3R_1]{R_2-2R_1,\ R_3-2R_1,} \begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & -9 & -24 & -9 \\ 0 & -5 & -40 & -15 \\ 0 & -5 & -40 & -15 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{3}R_2,\,-\frac{1}{5}R_3,\,-\frac{1}{5}R_4} \begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & 3 & 8 & 3 \\ 0 & 1 & 8 & 3 \\ 0 & 1 & 8 & 3 \end{bmatrix} \xrightarrow{R_{23}} \begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & 1 & 8 & 3 \\ 0 & 3 & 8 & 3 \\ 0 & 1 & 8 & 3 \end{bmatrix} \xrightarrow[R_4-R_2]{R_3-3R_2} \begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & -16 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A\begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & -16 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\left(-\frac{1}{16}\right)R_3} \begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & 1 & 3/8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \rho(A) = 3$$

$\therefore$ System has non trivial solution as $\rho(A)=3<$ no of unknowns

$\therefore$ number of free variable is 1 and w being non-pivote variable is a free variable Equivalent system *is*

$$x+3y+13z+3w=0,\ y+8z+3w=0,\ z+\frac{3}{8}w=0.$$

Let $w=k$ $\therefore z=-\frac{3}{8}k,\ y=-8z-3w=-8\left(-\frac{3}{8}\right)-3k=3k-3k=0$

$$x=-3y-13z-3w=-3(0)+\frac{39}{8}k-3k=\frac{15}{8}k$$

Solution is $x=\begin{bmatrix}\frac{15}{8}k\\0\\-\frac{3}{8}k\\k\end{bmatrix}=\frac{k}{8}\begin{bmatrix}15\\0\\-3\\8\end{bmatrix},k\in\mathrm{R}$

Example : Determine for real values of a so that the system of equations have non-trivial solution $x+2y+3z=ax$, $3x+y+2z=ay$, and $2x+3y+z=a\,z$

$$\text{Consider the system}$$

$$(1-a)x+2y+3z=0$$

$$3x+(1-a)y+2z=0$$

$$2x+3y+(1-a)z=0$$

$$\text{This is a homogeneous system of linear equation}$$

$$\therefore \text{ has a non trivial solution iff } |\mathrm{A}|=0$$

$$\text{Consider } A = \begin{vmatrix} 1-a & 2 & 3 \\ 3 & 1-a & 2 \\ 2 & 3 & 1-a \end{vmatrix} = 0$$

$$a^3 - 3a^2 - 15a - 18 = 0 \Rightarrow a = 6 \ (\textit{only real root})$$

For $a = 6$, the equivalent system is

$$AX = 0 \Rightarrow \begin{bmatrix} -5 & 2 & 3 \\ 3 & -5 & 2 \\ 2 & 3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\mathrm{A}=\begin{bmatrix}-5 & 2 & 3\\ 3 & -5 & 2\\ 2 & 3 & -5\end{bmatrix}\xrightarrow{R_1+2R_2}\begin{bmatrix}1 & -8 & 7\\ 3 & -5 & 2\\ 2 & 3 & -5\end{bmatrix}\xrightarrow{R_2-3R_1,\ R_3-2R_1}$$

$$\begin{bmatrix}1 & -8 & 7\\ 0 & 19 & -19\\ 0 & 19 & -19\end{bmatrix}\xrightarrow{R_3-R_2}\begin{bmatrix}1 & -8 & 7\\ 0 & 19 & -19\\ 0 & 0 & 0\end{bmatrix}\xrightarrow{\left(\frac{1}{19}\right)R_2}$$

$$\begin{bmatrix}1 & -8 & 7\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix}\Rightarrow\ \rho(A)=2<3$$

Equivalent system is $x-8y+7z=0,\ y-z=0$

$Let\ \ z=k,\ y=k,\ x=k$

Solution is $X=\begin{bmatrix} k \\ k \\ k \end{bmatrix}=k\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$

Example : show that the system of equations
ax + by + cz = 0,bx + cy + az = 0 and cx + ay + bz = 0
have a non trivial solution if
(i) a + b + c = 0 (ii) a = b = c and solve them completely.

The coefficient matrix is $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$

To reduce it to echelon form

$$\xrightarrow{R_3 + R_2} \begin{bmatrix} a & b & c \\ b & c & a \\ c+b & a+c & b+a \end{bmatrix} \xrightarrow{R_3 + R_1} \begin{bmatrix} a & b & c \\ b & c & a \\ c+b+a & a+c+b & b+a+c \end{bmatrix}$$

Case I :: Let $a+b+c \neq 0$

$$\xrightarrow{\frac{1}{a+b+c} R_3} \begin{bmatrix} a & b & c \\ b & c & a \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_{13}} \begin{bmatrix} 1 & 1 & 1 \\ b & c & a \\ a & b & c \end{bmatrix} \xrightarrow{R_2 - bR_1,\, R_3 - cR_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & c-b & a-b \\ 0 & a-c & b-c \end{bmatrix}$$

but to have nontrivial solution $\rho(A) <$ number of unknowns

Therfore at least one of the row has to be zero

$a-c=0$ and $b-c=0 \Rightarrow a=b=c$

$$\therefore \; A=\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} and \quad \rho(A)=1<3$$

System has non-trivial solution with 2 free variables.

Equivalent system is $x+y+z=0$

let $y=s,\ z=t \Rightarrow x=-s-t$

$$x=\begin{bmatrix} -s-t \\ s \\ t \end{bmatrix}=s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}+t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Case II:: $Let\ a+b+c=0$

$$\begin{bmatrix} a & b & c \\ b & c & a \\ c+b+a & a+c+b & b+a+c \end{bmatrix} \sim \begin{bmatrix} a & b & c \\ b & c & a \\ 0 & 0 & 0 \end{bmatrix}$$

If $a \neq 0,\ \xrightarrow{\frac{1}{a}R_1} \begin{bmatrix} 1 & b/a & c/a \\ b & c & a \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2-bR_1} \begin{bmatrix} 1 & b/a & c/a \\ 0 & c-\dfrac{b^2}{a} & a-\dfrac{bc}{a} \\ 0 & 0 & 0 \end{bmatrix}$

Assuming $c-\dfrac{b^2}{a} \neq 0, i.e., ac \neq b^2$ and $a-\dfrac{bc}{a} \neq 0, i.e., bc \neq a^2$,

a, b, c are not in geometric progression

Equivalent system is $x+\frac{b}{a}y+\frac{c}{a}z=0,$

$$\left(c-\frac{b^2}{a}\right)y+\left(a-\frac{bc}{a}\right)z=0$$

There is only one variable is free

let $z=k\in\mathrm{R}\Rightarrow y=\frac{a^2-bc}{ac-b^2}k,$

$$x=-\frac{b}{a}y-\frac{c}{a}z=-\frac{b}{a}\left(\frac{a^2-bc}{ac-b^2}k\right)-\frac{c}{a}k=\frac{\left(ab-c^2\right)}{ac-b^2}k$$

Example : Investigate for consistency of the following equations and if possible find solutions

$$w - x + 3y - 3z = 3; \quad -5w + 2x - 5y + 4z = -5;$$

$$-3w - 4x + 7y - 2z = 7; \quad 2w + 3x + y - 11z = 1;$$

Given system can be written as $AX = B$

$$[A \,\vdots\, B] = \left[\begin{array}{cccc|c} 1 & -1 & 3 & -3 & 3 \\ -5 & 2 & -5 & 4 & -5 \\ -3 & -4 & 7 & -2 & 7 \\ 2 & 3 & 1 & -11 & 1 \end{array}\right]$$

$$\xrightarrow{\substack{R_2+5R_1,\\ R_3+3R_1,\\ R_4-2R_1}} \left[\begin{array}{cccc:c} 1 & -1 & 3 & -3 & 3 \\ 0 & -3 & 10 & -11 & 11 \\ 0 & -7 & 16 & -11 & 16 \\ 0 & 5 & -5 & -5 & -5 \end{array}\right] \xrightarrow{\frac{1}{5}R_4} \left[\begin{array}{cccc:c} 1 & -1 & 3 & -3 & 3 \\ 0 & -3 & 10 & -11 & 11 \\ 0 & -7 & 16 & -11 & 16 \\ 0 & 1 & -1 & -1 & -1 \end{array}\right]$$

$$\xrightarrow{R_{24}} \left[\begin{array}{cccc:c} 1 & -1 & 3 & -3 & 3 \\ 0 & 1 & -1 & -1 & -1 \\ 0 & -7 & 16 & -11 & 16 \\ 0 & -3 & 10 & -11 & 10 \end{array}\right] \xrightarrow{\substack{R_3+7R_2,\\ R_4+3R_2}} \left[\begin{array}{cccc:c} 1 & -1 & 3 & -3 & 3 \\ 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 9 & -18 & 9 \\ 0 & 0 & 7 & -14 & 7 \end{array}\right]$$

$$\xrightarrow{\frac{1}{9}R_3,\,\frac{1}{7}R_4} \left[\begin{array}{cccc:c} 1 & -1 & 3 & -3 & 3 \\ 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 & 1 \end{array}\right] \xrightarrow{R_4-R_3} \left[\begin{array}{cccc:c} 1 & -1 & 3 & -3 & 3 \\ 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rho\left[A \,\vdots\, B\right] = \rho\left[A\right] \Rightarrow \text{System is consistant.}$$

$\rho\left[A \,\vdots\, B\right] = \rho\left[A\right] = 3 < 4 \Rightarrow$ System possesses infinite solutions.

Reduced system is $y - 2z = 1, x - y - z = -1, w - x + 3y - 3z = 3$

Non pivot column is 4th column, therefore z is free variable.

Let $z = k \in R$, $y = 1 + 2k, x = -1 + y + z = -1 + 1 + 2k + k = 3k$,

$w = x - 3y + 3z + 3 = 3k - 3(2k+1) + 3k + 3 = 0.$

Solution is $X = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 3k \\ 2k+1 \\ k \end{bmatrix} = \begin{bmatrix} 0 \\ 3k \\ 2k \\ k \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

Example : Investigate for consistency of the following equations and if possible find solutions

$$w - x + 3y - 3z = 3; \quad -5w + 2x - 5y + 4z = 5;$$
$$-3w - 4x + 7y - 2z = 7; \quad 2w + 3x + y - 11z = 1$$

$$(A|B) = \begin{bmatrix} 1 & -1 & 3 & -3 & : & 3 \\ -5 & 2 & -5 & 4 & : & 5 \\ -3 & -4 & 7 & -2 & : & 7 \\ 2 & 3 & 1 & -11 & : & 1 \end{bmatrix} \xrightarrow{\substack{R_2 + 5R_1, \\ R_3 + 3R_1, \\ R_4 - 2R_1}} \begin{bmatrix} 1 & -1 & 3 & -3 & : & 3 \\ 0 & -3 & 10 & -11 & : & 20 \\ 0 & -7 & 16 & -11 & : & 16 \\ 0 & 5 & -5 & -5 & : & -5 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{5}R_4} \begin{bmatrix} 1 & -1 & 3 & -3 & : & 3 \\ 0 & -3 & 10 & -11 & : & 20 \\ 0 & -7 & 16 & -11 & : & 16 \\ 0 & 1 & -1 & -1 & : & -1 \end{bmatrix} \xrightarrow{R_{24}} \begin{bmatrix} 1 & -1 & 3 & -3 & : & 3 \\ 0 & 1 & -1 & -1 & : & -1 \\ 0 & -7 & 16 & -11 & : & 16 \\ 0 & -3 & 10 & -11 & : & 20 \end{bmatrix}$$

$$\xrightarrow[R_4+3R_2]{R_3+7R_2}\begin{bmatrix}1 & -1 & 3 & -3 & : & 3\\ 0 & 1 & -1 & -1 & : & -1\\ 0 & 0 & 9 & -18 & : & 9\\ 0 & 0 & 7 & -14 & : & 17\end{bmatrix}\xrightarrow{\frac{1}{9}R_3}\begin{bmatrix}1 & -1 & 3 & -3 & : & 3\\ 0 & 1 & -1 & -1 & : & -1\\ 0 & 0 & 1 & -2 & : & 1\\ 0 & 0 & 7 & -14 & : & 17\end{bmatrix}$$

$$\xrightarrow{R_4-7R_3}\begin{bmatrix}1 & -1 & 3 & -3 & : & 3\\ 0 & 1 & -1 & -1 & : & -1\\ 0 & 0 & 1 & -2 & : & 1\\ 0 & 0 & 0 & 0 & : & 10\end{bmatrix}\Rightarrow \rho\left(A\,|\,B\right)=4\,\text{but}\,\rho\left(A\right)=3$$

$$\rho\left(A\,|\,B\right)\neq\rho\left(A\right).$$

Hence the system is inconsistent.

Example : Investigate the values of a & b so that the equations

$$2x+3y+5z=9, 7x+3y-2z=8, 2x+3y+az=b$$

have (i) no solution (ii) unique solution (iii) infinite solutions.

$$\text{Agumented matrix is } [A : B] = \begin{bmatrix} 2 & 3 & 5 & : & 9 \\ 7 & 3 & -2 & : & 8 \\ 2 & 3 & a & : & b \end{bmatrix}$$

$$R_2 - 3R_1 \Rightarrow \begin{bmatrix} 2 & 3 & 5 & : & 9 \\ 1 & -6 & -17 & : & -19 \\ 2 & 3 & a & : & b \end{bmatrix} R_{12} \Rightarrow \begin{bmatrix} 1 & -6 & -17 & : & -19 \\ 2 & 3 & 5 & : & 9 \\ 2 & 3 & a & : & b \end{bmatrix}$$

$$R_2 - 2R_1,\ R_3 - 2R_1 \Rightarrow \begin{bmatrix} 1 & -6 & -17 & : & -19 \\ 0 & 15 & 39 & : & 47 \\ 0 & 15 & a+34 & : & b+38 \end{bmatrix}$$

$$R_3 - R_2 \Rightarrow \begin{bmatrix} 1 & -6 & -17 & : & -19 \\ 0 & 15 & 39 & : & 47 \\ 0 & 0 & a-5 & : & b-9 \end{bmatrix}$$

i) Condition for no solution is $\rho[A] \neq \rho[A : B]$.

Thus if $a-5=0$ & $b-9 \neq 0$ then $\rho[A]=2,\ \rho[A : B]=3$.

ii) Condition for unique solution is $\rho[A]=\rho[A : B] = number\ of\ unknowns$.

Thus if $a-5 \neq 0$ *then* $\rho[A]=\rho[A : B]=3=number\ of\ unknowns$.

iii) Condition for infinite solutions is $\rho[A]=\rho[A : B] < number\ of\ unknowns$.

Thus if $a-5=0$ *and* $b-9=0$ then $\rho[A]=\rho[A : B]=2<3$

Conclusion

$\mathrm{i)}\, a-5=0 \;\&\; b-9\neq 0 \Rightarrow\ No\ solution$

$ii)\ a-5\neq 0, b\in \mathrm{R} \Rightarrow Unique\ solution$

$iii)\, a-5=0 \,\&\, b-9=0 \Rightarrow Infinite\, Solutions$

Do the three planes $2x_1 + 4x_2 + 4x_3 = 4$, $x_2 - 2x_3 = -2$, and $2x_1 + 3x_2 = 0$ have at least one common point of intersection? Explain.

$$2x_1 + 4x_2 + 4x_3 = 4, x_2 - 2x_3 = -2, 2x_1 + 3x_2 = 0.$$

System can be written as $\begin{bmatrix} 2 & 4 & 4 \\ 0 & 1 & -2 \\ 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix}$

Augumented matrix $(A\,|\,B) = \begin{bmatrix} 2 & 4 & 4 & \vdots & 4 \\ 0 & 1 & -2 & \vdots & -2 \\ 2 & 3 & 0 & \vdots & 0 \end{bmatrix}$

$$\xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 2 & 2 & \vdots & 2 \\ 0 & 1 & -2 & \vdots & -2 \\ 2 & 3 & 0 & \vdots & 0 \end{bmatrix} \xrightarrow{R_3 - 2R_1} \begin{bmatrix} 1 & 2 & 2 & \vdots & 2 \\ 0 & 1 & -2 & \vdots & -2 \\ 0 & -1 & -4 & \vdots & -4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 2 & \vdots & 2 \\ 0 & 1 & -2 & \vdots & -2 \\ 0 & -1 & -4 & \vdots & -4 \end{bmatrix} \xrightarrow{R_3+R_2} \begin{bmatrix} 1 & 2 & 2 & \vdots & 2 \\ 0 & 1 & -2 & \vdots & -2 \\ 0 & 0 & -6 & \vdots & -6 \end{bmatrix}$$

$\rho(\mathrm{A} \mid \mathrm{B}) = \rho(\mathrm{A}) = 3 =$ no. of unknowns

$\therefore$ System possesses unique solution.

Reduce form of system is $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -6 \end{bmatrix}.$

Thus $-6x_3 = -6 \Rightarrow x_3 = 1, x_2 - 2x_3 = -2 \Rightarrow x_2 = 0$

$x_1 + 2x_2 + 2x_3 = 2 \Rightarrow x_1 = 0$

Thus the given planes $2x_1 + 4x_2 + 4x_3 = 4, x_2 - 2x_3 = -2,$ and $2x_1 + 3x_2 = 0$ intersect at common point $(0,\ 0,1)$.

Example 2 : Determine the value of h so that the following matrix is the augmented matrix of a consistent linear system.

1. $\begin{bmatrix} 1 & h & 4 \\ 3 & 6 & 8 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 4 & -2 \\ 3 & h & -6 \end{bmatrix}$

3. Find the condition on unknowns so that the matrix is a augmented matrix of a consistent linear system. $\begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{bmatrix}$

1. Consider, $\begin{bmatrix} 1 & h & 4 \\ 3 & 6 & 8 \end{bmatrix}$.

Reduce the matrix to Euchelon form

$$\xrightarrow{R_2 - 3R_1} \begin{bmatrix} 1 & h & 4 \\ 0 & 6-3h & \overset{e_2}{-4} \end{bmatrix}$$

For consistent system, $\rho(A|B) = \rho(A)$

For 6-3$h \neq 0$, *i.e.*, $h \neq 2$, $\rho(A|B) = \rho(A) = 2$

$\therefore$ Above system is always consistent for $h \neq 2$.

$$2.\text{ Consider}, \begin{bmatrix} 1 & 4 & -2 \\ 3 & h & -6 \end{bmatrix}.$$

Reduce the matrix to Euchelon form

$$\xrightarrow{R_2 - 3R_1} \begin{bmatrix} 1 & 4 & -2 \\ 0 & h-12 & 0 \end{bmatrix}$$

For consistent system, $\rho(A|B)=\rho(A)$

If $h-12 \neq 0, i.e., h \neq 12,$ system possesses unique solution.

If $h-12 = 0, i.e., h = 12,$ system will have infinite solutions.

$\therefore$ For any real value of h, above system is always consistent.

$$3.\ \text{Consider, } \begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{bmatrix}.$$

Reduce the matrix to Euchelon form

$$\xrightarrow{R_3 + 2R_1} \begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & -3 & 5 & k+2g \end{bmatrix}$$

$$\xrightarrow{R_3 + R_2} \begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & 0 & 0 & k+2g+h \end{bmatrix}$$

For consistent system, $\rho(A|B) = \rho(A)$

$\therefore$ if $k+2g+h=0$,then and then only above system will be consistent. Further under this condition it will have infinite number of solutions.

$$\text{Ex}: 2x+3y+3z+6w=6,\ 4x-3y-9z+6w=0,\ 4x-21y-39z-6w=-24$$

$$(\mathrm{A},\mathrm{B})=\left[\begin{array}{cccc:c} 2 & 3 & 3 & 6 & 6 \\ 4 & -3 & -9 & 6 & 0 \\ 4 & -21 & -39 & -6 & -24 \end{array}\right] \xrightarrow[R_3-2R_1]{R_2-2R_1}$$

$$\left[\begin{array}{cccc:c} 2 & 3 & 3 & 6 & 6 \\ 0 & -9 & -15 & -6 & -12 \\ 0 & -27 & -45 & -18 & -36 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{3}R_2,\ \frac{1}{3}R_3} \left[\begin{array}{cccc:c} 2 & 3 & 3 & 6 & 6 \\ 0 & 3 & 5 & 2 & 4 \\ 0 & -9 & -15 & -6 & -12 \end{array}\right] \xrightarrow{R_3+3R_2}$$

$$\left[\begin{array}{cccc:c} 2 & 3 & 3 & 6 & 6 \\ 0 & 3 & 5 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \rho(\mathrm{A}:\mathrm{B})=\rho(\mathrm{A})=2<4$$

$$\begin{bmatrix} 2 & 3 & 3 & 6 \\ 0 & 3 & 5 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} 3y+5z+2w=4 \\ 2x+3y+3z+6w=6 \end{matrix}$$

$$y=\frac{4-5z-2w}{3} \text{ and } x=\frac{6-3y-3z-6w}{2}.$$

Let $z=t\, and\, w=s,\ t,s\in R$

$$y=\frac{4-5z-2w}{3}=\frac{4-5t-2s}{3},$$

$$x=\frac{6-3y-3z-6w}{2}=1+t-2s$$

Thus solution is

$$x = 1 + t - 2s, \; y = \frac{4 - 5t - 2s}{3}, \; z = t, \; w = s$$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 + t - 2s \\ \dfrac{4 - 5t - 2s}{3} \\ t \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 4/3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} t - 2s \\ \dfrac{5t - 2s}{3} \\ t \\ s \end{bmatrix}$$

Example: Check the following system for consistency.

If consistent, find the solution.

$$x_1 + 2x_2 + 3x_3 = 1, 2x_1 + x_2 - x_3 = 2, x_1 - x_2 - 4x_3 = 1.$$

In matrix form AX=B, where $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & -4 \end{bmatrix}, \mathrm{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \mathrm{B} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$

Consider Augmented matrix

$$[\mathrm{A} \mid \mathrm{B}] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 1 & -1 & 2 \\ 1 & -1 & -4 & 1 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -7 & 0 \\ 0 & -3 & -7 & 0 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rho[\mathrm{A} \mid \mathrm{B}] = \rho[\mathrm{A}] = 2 < 3$$

$\therefore$ system is consistant and has infinite solution.

Equivalent system is $x_1 + 2x_2 + 3x_3 = 1, -3x_2 - 7x_3 = 0$

Let $x_3 = k,\ x_2 = -\frac{7}{3}k,$

$$x_1 = 1 - 2x_2 - 3x_3 = 1 + \frac{14}{3}k - 3k = 1 + \frac{5}{3}k$$

$$\mathrm{X} = \begin{bmatrix} 1 + \frac{5}{3}k \\ -\frac{7}{3}k \\ k \\ \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + k \begin{bmatrix} \frac{5}{3} \\ -\frac{7}{3} \\ 1 \\ \end{bmatrix} = x_p + x_h$$

Example : For what vlaues of a, the system

$$x+2y-3z=4; \quad 3x-y+18z=2; \quad 4x+y+(a^2-1)z=a+2$$

has (i) No solution (ii) Unique solution (iii) Infinite solutions

The given system can be written as AX=B

$$\text{where A} = \begin{bmatrix} 1 & 2 & -3 \\ 3 & -1 & 18 \\ 4 & 1 & a^2-1 \end{bmatrix}, \text{ B} = \begin{bmatrix} 4 \\ 2 \\ a+2 \end{bmatrix}$$

Consider the augmented matrix

$$\left[\text{A} \,\vdots\, \text{B}\right] = \left[\begin{array}{ccc:c} 1 & 2 & -3 & 4 \\ 3 & -1 & 18 & 2 \\ 4 & 1 & a^2-1 & a+2 \end{array}\right] \xrightarrow[R_3-4R_1]{-3R_1,} \left[\begin{array}{ccc:c} 1 & 2 & -3 & 4 \\ 0 & -7 & 27 & -10 \\ 0 & -7 & a^2+11 & a-14 \end{array}\right]$$

$$\xrightarrow{-R_2} \left[\begin{array}{ccc:c} 1 & 2 & -3 & 4 \\ 0 & -7 & 27 & -10 \\ 0 & 0 & a^2-16 & a-4 \end{array}\right]$$

Case 1: For $a-4 \neq 0,\ \rho[\mathrm{A} \vdots \mathrm{B}] = \rho[\mathrm{A}] = 3$

System has a unique solution.

Case II: For $a+4=0,\ \rho[\mathrm{A} \vdots \mathrm{B}] \neq \rho[\mathrm{A}]$

System do not have a solution.

Case III: For infinite solution

$a-4=0,\ as\ \rho[\mathrm{A} \vdots \mathrm{B}] = \rho[\mathrm{A}] < 3.$

Example : Investigate for consistency of the following equations and if possible find solutions

$$w - x + 3y - 3z = 3; \quad -5w + 2x - 5y + 4z = 2;$$

$$-3w - 4x + 7y - 2z = 7; \quad 2w + 3x + y - 11z = 1;$$

Given system can be written as AX=B

$$[\mathrm{A} \mid B] = \left[\begin{array}{cccc|c} 1 & -1 & 3 & -3 & 3 \\ -5 & 2 & -5 & 4 & 2 \\ -3 & -4 & 7 & -2 & 7 \\ 2 & 3 & 1 & -11 & 1 \end{array}\right]$$

$$\xrightarrow{\substack{R_2+5R_1,\\ R_3+3R_1,\\ R_4-2R_1}} \left[\begin{array}{cccc:c} 1 & -1 & 3 & -3 & 3 \\ 0 & -3 & 10 & -11 & 17 \\ 0 & -7 & 16 & -11 & 25 \\ 0 & 5 & -5 & -5 & -5 \end{array}\right] \xrightarrow{\frac{1}{5}R_4} \left[\begin{array}{cccc:c} 1 & -1 & 3 & -3 & 3 \\ 0 & -3 & 10 & -11 & 17 \\ 0 & -7 & 16 & -11 & 25 \\ 0 & 1 & -1 & -1 & -1 \end{array}\right]$$

$$\xrightarrow{R_{24}} \left[\begin{array}{cccc:c} 1 & -1 & 3 & -3 & 3 \\ 0 & 1 & -1 & -1 & -1 \\ 0 & -7 & 16 & -11 & 25 \\ 0 & -3 & 10 & -11 & 17 \end{array}\right] \xrightarrow{\substack{R_3+7R_2,\\ R_4+3R_2}} \left[\begin{array}{cccc:c} 1 & -1 & 3 & -3 & 3 \\ 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 9 & -18 & 18 \\ 0 & 0 & 7 & -14 & 14 \end{array}\right]$$

$$\xrightarrow{\frac{1}{9}R_3,\frac{1}{7}R_4} \left[\begin{array}{cccc:c} 1 & -1 & 3 & -3 & 3 \\ 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -2 & 2 \end{array}\right] \xrightarrow{R_4-R_3} \left[\begin{array}{cccc:c} 1 & -1 & 3 & -3 & 3 \\ 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rho[A \,\vdots\, B] = \rho[A] = 3$$

$\therefore$ System of equations is consistant.

Equivalent system is

$$w - x + 3y - 3z = 3, x - y - z = -1, y - 2z = 2$$

Let $z = k$, be free variable

$$\therefore\; y = 2 + 2k,\; x = 1 + 3k,\; w = -5 - 3k$$

$$\therefore\; X = \begin{bmatrix} -5-3k \\ 1+3k \\ 2+2k \\ k \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \\ 2 \\ 0 \end{bmatrix} + k \begin{bmatrix} -3 \\ 3 \\ 2 \\ 1 \end{bmatrix} = x_p + x_h.$$

Example: Find A such that the only solution to

$$AX = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \; is \; \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

A must be 3×2 matrix such that

$$\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

$a_1, b_{1,} c_1$ *can be any real numbers and* $a_2 = 1, b_2 = 3,$ *and* $c_2 = 5.$

$$\therefore A = \begin{bmatrix} a_1 & 1 \\ b_1 & 3 \\ c_1 & 5 \end{bmatrix}$$

Example: Is there exist a matrix B such that the only

solution to $BX = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is $\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$? Justify

We want $B\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \Rightarrow$ B must be of order 2×3.

But then maximum possible rank of B, $\rho(B)$

will be 2, hence rank of augmented matrix will also

be atmost 2 which is always less than number of

unknowns=3. Therefore unique solution is not possible.

Hence such matrix does not exist.

Exercise :

1. Determine whether the nonhomogeneous system is consistent, and if the system is consistent, find the solution.

a)
$$\begin{aligned} x + 3y + 10z &= 18 \\ -2x + 7y + 32z &= 29 \\ -x + 3y + 14z &= 12 \\ x + y + 2z &= 8 \end{aligned}$$

b)
$$\begin{aligned} 3w - 2x + 16y - 2z &= -7 \\ -w + 5x - 14y + 18z &= 29 \\ 3w - x + 14y + 2z &= 1 \end{aligned}$$

2. Find the conditions on a_1, a_2, a_3 and a_4 such that

the system $\begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 2 & 5 \\ 4 & 12 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$ is consistant. Solve it.

1. For what values of a the system of equations have

$$x - y + z = 1; \quad x + 3y + az = 2; \quad 2x + ay + 3z = 3$$

i) No solutions ii) Unique solutions

iii) infinitely many solutions.

2. Investigate for consistency of the following equations and if possible find solutions:

(i) $x + y + z = 6; \quad x + y - 3z = -1: \quad 15x - 3y + 9z = 21$

(ii) $2x_1 + x_2 + 2x_3 + x_4 = 6,\ 6x_1 - 6x_2 + 6x_3 + 12x_4 = 36,$

$4x_1 + 3x_2 + 3x_3 - 3x_4 = -1,\ 2x_1 + 2x_2 - x_3 + x_4 = 10.$

3. Investigate the values of λ and μ such that the system

$$x + y + z = 6, \quad x + 2y + 3z = 10, \quad x + 2y + \lambda z = \mu$$

has i) No solution ii) A unique solution

iii) An infinite number of solutions.

Analysis of electrical Network

There are three basic quantities associated with electrical circuit. Electrical potential (E-Volts), resistance (R-ohms), and current (I-ampere).

Electrical potential (called as voltage drops between two points) is associated with two points in an electrical circuits and is measured by connecting those points to a device called a voltmeter.

The flow of current in an electrical circuits is governed by three basic laws.

The flow of current in an electrical circuits is governed by three basic laws.

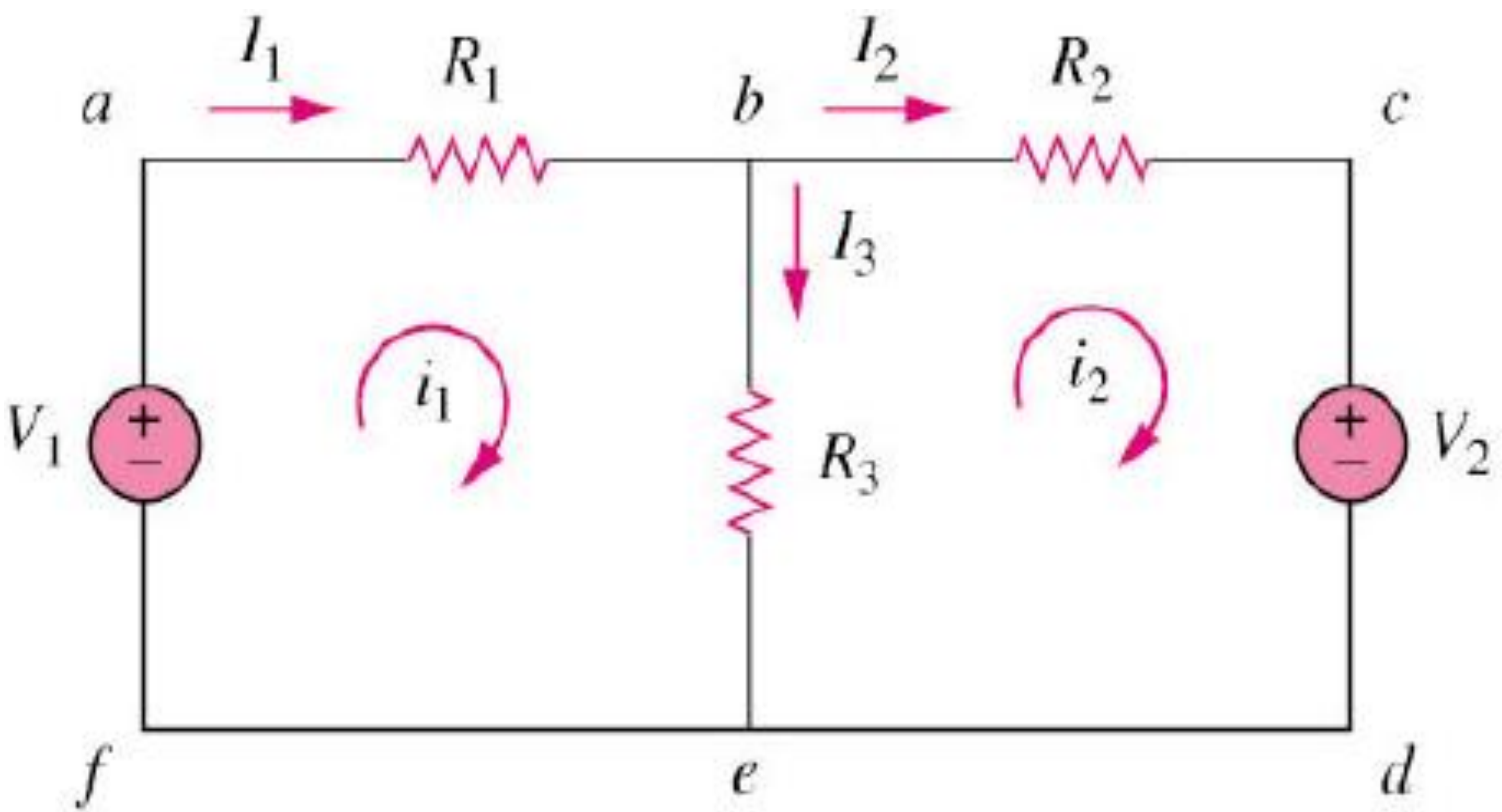
Ohms law: The voltage drop across a resistor is the product of current passing through it and its resistance

Kirchhoff's current law: The sum of currents flowing into any point equals the sum of the currents flowing out from the point.

Kirchhoff's voltage law: Around any closed loop, the algebraic sum of voltage drops is zero.

Note: A current passing through the resistor produces a positive voltage drop if it flows in the positive direction of the loop and negative voltage drop if it flows in the negative direction of the loop.

Find the unknown currents $I_1, I_2, and\ I_3$ in the circuit shown in the figure.



Applying Kirchhoff's current law to the points b, and e yields

$$I_1 = I_2 + I_3 \quad (\text{Point}\, b)$$

$$I_2 + I_3 = I_1 \quad (\text{Point e})$$

$$\therefore\ I_1 - I_2 - I_3 = 0 \qquad \text{...(1)}$$

Applying Kirchhoff's voltage law and Ohms law to loop 1 in the figure yields

$$\mathrm{R_1 I_1 + R_3 I_3} = V_1 \qquad \text{...(2)}$$

Now applying to loop (2), we get

$$\mathrm{R_2 I_2 - R_3 I_3} = V_2 \qquad \text{...(3)}$$

Expressing equations $(1),(2),$ and (3) in matrix form

$$\begin{bmatrix} 1 & -1 & -1 \\ R_1 & 0 & R_3 \\ 0 & R_2 & -R_3 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 0 \\ V_1 \\ V_2 \end{bmatrix}$$

$$(\mathrm{A}\,|\,\mathrm{B}) = \begin{bmatrix} 1 & -1 & -1 & \vdots & 0 \\ R_1 & 0 & R_3 & \vdots & V_1 \\ 0 & R_1 & -R_3 & \vdots & V_2 \end{bmatrix} \xrightarrow{\mathrm{R}_2 - R_1 \mathrm{R}_1} \begin{bmatrix} 1 & -1 & -1 & \vdots & 0 \\ 0 & R_1 & R_3 + R_1 & \vdots & V_1 \\ 0 & R_1 & -R_3 & \vdots & V_2 \end{bmatrix}$$

$$\xrightarrow{\mathrm{R}_3 - \mathrm{R}_2} \begin{bmatrix} 1 & -1 & -1 & \vdots & 0 \\ 0 & R_1 & R_3 + R_1 & \vdots & V_1 \\ 0 & 0 & -2R_3 - R_1 & \vdots & V_2 - V_1 \end{bmatrix} \Rightarrow \rho(\mathrm{A}\,|\,\mathrm{B}) = \rho(\mathrm{A}) = 3$$

Therefore system possesses unique solution.

From reduced form of system, we have

$$\left(-2R_3 - R_1\right)\mathrm{I}_3 = \left(V_2 - V_1\right) \Rightarrow \mathrm{I}_3 = \frac{\left(V_2 - V_1\right)}{\left(-2R_3 - R_1\right)}.$$

$$R_1\mathrm{I}_2 + \left(R_3 + R_1\right)\mathrm{I}_3 = V_1 \Rightarrow \mathrm{I}_2 = \frac{1}{R_1}\left(V_1 - R_3\mathrm{I}_3 - R_1\mathrm{I}_3\right)$$

$$\therefore\ \mathrm{I}_2 = \frac{-R_3\left(V_1 + V_2\right) - R_1V_2}{R_1\left(-2R_3 - R_1\right)} \quad \text{and}$$

$$\mathrm{I}_1 = \mathrm{I}_2 + \mathrm{I}_3 = \frac{-R_3\left(V_1 + V_2\right) - R_1V_1}{R_1\left(-2R_3 - R_1\right)}.$$

In particular if we set

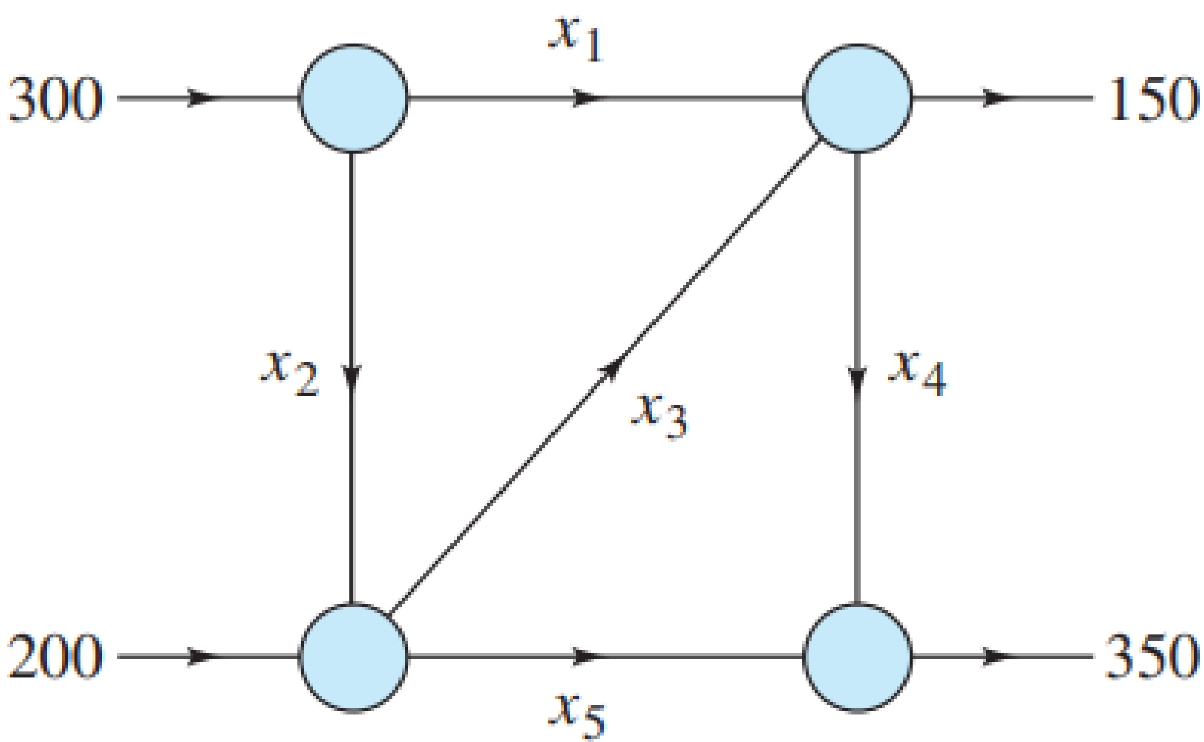
$V_1 = 30V$, $V_2 = 50V$. $R_1 = 7\Omega$, $R_2 = 11\Omega$, *and* $R_3 = 3\Omega$

we get

$$I_1 = \frac{570}{131}A, \quad I_2 = \frac{590}{131}A, \quad and \quad I_3 = -\frac{20}{131}A.$$

Analysis of Network

The total flow into a junction is equal to the total flow out of the junction.



Balancing Chemical Equations

Boron Sulphide B_2S_3 reacts violently with water to form Boric acid H_3BO_3 and Hydrogen Sulphide H_2S. For each compound, construct a vector that list number of atoms for each of Boron, Sulphur, Hydrogen and Oxygen. Constructing a system of linear equations for the number of molecules for each compound, balance the chemical equation.