Engineering Mathematics (ES1032)
Partial Differentiation

In problems of maxima, minima, conduction of heat, vibration of strings, electromagnetic fields, theory of approximations, exact differential equations, vectors, multiple integrals, Boundary Value Problems, complex variables and some problems of electrical engineering (transmission lines) we need the study of partial derivatives, which are different from the ordinary differential coefficients.

In many applications, the values of the function under study are determined by the values of more than one independent variable. The function may be as simple as $V = \pi r^2 R$ for calculating the volume of a right circular cylinder from its radius r of its base and height R. Here V will undergo a change if either r or R changes. Here V is a function of two variables r (radius) and R (the height). So we can write $V = f(r, R)$. Here V is the dependent variable and r, R are independent variables.

Definition: If z has one definite value for each pair of values of x and y then z is called a function of two variables x and y. We denote it by $z = f(x, y)$ or $F(x, y)$ or $\phi(x, y)$

Here z is the dependent variable, x and y are independent variables. e.g. $z = x^3 + y^3 - 3axy$.

Similarly, $u = f(x, y, z)$ is a function of three variables x, y and z. e.g. $u = x^2 + y^2 + z^2$

Domain of definition of z:

Sometimes, z is defined only for those pairs of values of x and y for which the point (x, y) moves within a certain area in the $x-y$ plane, then that area is called the Domain of definition of z.

For example: Domain of the function $z = f(x, y) = \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$ is the inside of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Continuity of a function of two variables :

The function $z = f(x, y)$ is said to be continuous at a point (a, b) of its domain if the limit of the function is equal to the value of the function at that point, i. e.,

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b).$$

The limit being independent of the manner in which x approaches a and y approaches b. If the function is continuous at all points of some region R in the xy plane, then it is said to be continuous in the entire region R.

Partial derivatives or Partial differential coefficients :

Let $z = f(x, y)$ be a function of two variables x and y.

Definition: The partial derivative of $z = f(x, y)$ w. r. t. x is nothing but the ordinary derivative of $z = f(x, y)$ w. r. t. x treating y as constant. It is denoted by $\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$ or z_x or f_x and we read it as *dabba* z *dabba* x (or del z by del x)

$\frac{\partial z}{\partial x} = \lim_{h\to 0} \left\{ \frac{f(x+h, y) - f(x, y)}{h} \right\}$, where h is a small increment in x.

Similarly, $\frac{\partial z}{\partial y}$ or $\frac{\partial f}{\partial y}$ or f_y or z_y. $\frac{\partial z}{\partial y} = \lim_{k\to 0} \left\{ \frac{f(x, y+k) - f(x, y)}{k} \right\}$, where k is a small increment in y. $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are called <u>first order partial derivatives</u> of z.

Similarly, if f is a function of n variables $x_1, x_2, x_3, \dots, x_n$, the partial derivatives of f with respect to x_i, is the ordinary derivative of f when all the independent variables except x_i, are kept as constant, and is written as $\frac{\partial f}{\partial x_i}$.

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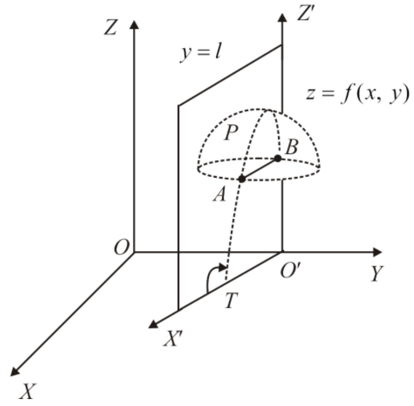
Partial Differentiation

Geometrical Interpretation of $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

If $P(x, y, z)$ be the coordinates of a point referred to rectangular area OX, OY, OZ then the equation $z = f(x, y)$ represents a surface.

Let a plane $y = l$ parallel to the XZ-plane pas through P cutting the surface along the curve APB given by $z = f(x, l)$.



As y remains equal to l and x varies then P moves along the curve APB and $\frac{\partial z}{\partial x}$ is the ordinary derivative of $f(x, l)$ w.r.t. x.

Hence, $\frac{\partial z}{\partial x}$ at P is the tangent of the angle which the tangent at P to the section of the surface $z = f(x, y)$ by a plane through P parallel to the plane XOZ, makes with a line parallel to the $x-$axis.

Similarly, $\frac{\partial z}{\partial y}$ at P is the tangent of the angle which the tangent at P to the curve of intersection of the surface $z = f(x, y)$ and the plane $x = a$, makes with a line parallel to the $y-$axis.

Illustrative Example

Q 1) If $z = x^3 + xy + y^3$, then find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

Sol^n. $\frac{\partial z}{\partial x} = 3x^2 + y,\ \frac{\partial z}{\partial y} = x + 3y^2$. $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are again functions of x and y and are in general different.

<u>Partial derivatives of Higher Order:</u>

If $z = f(x, y)$, then the first order partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ may also be function of x and y, hence can be differentiated again partially both with respect to x and y. Thus, there are four partial derivatives of the second order of a function of two variables.

These derivatives are denoted as follows:

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} \text{ or } \frac{\partial^2 f}{\partial x^2} \text{ or } z_{xx} \text{ or } f_{xx}\ ,\ \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 f}{\partial y \partial x} = z_{xy} = f_{xy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 f}{\partial x \partial y} = z_{yx} = f_{yx}\ ,\ \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 f}{\partial y^2} = z_{yy} = f_{yy}$$

Note: $\frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y \partial x}$ are called mixed partial derivatives.

Derivatives of the second order may again be differentiated both with respect to x and y. Then we get eight partial derivatives of third order, viz.

$z_{xxx}, z_{yxx}, z_{xxy}, z_{yxy}, z_{xyx}, z_{yyx}, z_{xyy}, z_{yyy}$.

<u>Comparative property of mixed partial derivative</u>

If $z = f(x, y)$ is continuous and possesses continuous partial derivatives, then $\boxed{\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}}$

Illustrative Examples

Q 1) If $u = \log(\tan x + \tan y + \tan z)$, show that $(\sin 2x)u_x + (\sin 2y)u_y + (\sin 2z)u_z = 2$.

Sol^n. Given $u = \log(\tan x + \tan y + \tan z)$. $u_x = \dfrac{1}{\tan x + \tan y + \tan z}\sec^2 z$

$u_y = \dfrac{1}{\tan x + \tan y + \tan z}\sec^2 y$, $u_z = \dfrac{1}{\tan x + \tan y + \tan z}\sec^2 z$

Now,

$$(\sin 2x)u_x + (\sin 2y)u_y + (\sin 2z)u_z = \frac{2\sin x\cos x\sec^2 x + 2\sin y\cos y\sec^2 y + 2\sin z\cos z\sec^2 z}{\tan x + \tan y + \tan z}$$

$$= \frac{2(\tan x + \tan y + \tan z)}{\tan x + \tan y + \tan z} = 2 \quad \text{[proved]}$$

Q 2) If $z(x+y) = x^2 + y^2$, show that $\left(\dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)^2 = 4\left(1 - \dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)$.

Sol^n. We have $z = \dfrac{x^2 + y^2}{x + y}$ ---------- (i)

Differentiating (i) partially w.r.t. x & y we get,

$$\frac{\partial z}{\partial x} = \frac{(x+y)2x - (x^2+y^2)}{(x+y)^2} = \frac{x^2 + 2xy - y^2}{(x+y)^2}, \quad \frac{\partial z}{\partial y} = \frac{y^2 + 2xy - x^2}{(x+y)^2}$$

$$\therefore \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = \frac{2(x^2 - y^2)}{(x+y)^2} = \frac{2(x-y)}{(x+y)} \ . \quad \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \frac{4xy}{(x+y)^2}$$

$$1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 1 - \frac{4xy}{(x+y)^2} = \frac{(x+y)^2 - 4xy}{(x+y)^2} = \frac{(x-y)^2}{(x+y)^2}$$

$$\therefore \left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\frac{(x-y)^2}{(x+y)^2} = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right) \quad \text{[proved]}$$

Q 3) If $z = f(x+ay) + \phi(x-ay)$, then show that $a^2\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial^2 z}{\partial y^2}$ (a is constant).

Sol^n. We have, $z = f(x+ay) + \phi(x-ay)$.

$z_x = f'(x+ay) + \phi'(x-ay),\ z_y = af'(x+ay) - a\phi'(x-ay)$

$z_{xx} = f''(x+ay) + \phi''(x-ay),\ z_{yy} = a^2 f''(x+ay) + a^2\phi''(x-ay) = a^2 z_{xx}$ [proved]

Q 4) If $f = \log\left(\dfrac{x^2 + y^2}{xy}\right)$, prove that $f_{xy} = f_{yx}$.

Sol^n. Differentiating f partially w.r.t. x,

$$f_x = \frac{xy}{x^2 + y^2}\left\{\frac{xy2x - (x^2+y^2)y}{(xy)^2}\right\} = \frac{x^2 - y^2}{x(x^2+y^2)}$$

Now, differentiating partially w.r.t. y,

$$f_{yx} = \frac{\partial f_x}{\partial y} = \frac{1}{x}\frac{(x^2+y^2)(-2y) - (x^2-y^2)(2y)}{(x^2+y^2)^2} = \frac{1}{x}\frac{-2x^2y - 2y^3 - 2x^2y + 2y^3}{(x^2+y^2)^2} = \frac{-4xy}{(x^2+y^2)}$$

$$f_y = \frac{xy}{x^2 + y^2}\left\{\frac{xy2y - (x^2+y^2)x}{(xy)^2}\right\} = \frac{y^2 - x^2}{y(x^2+y^2)}$$

$$\therefore f_{xy} = \frac{1}{y}\frac{(x^2+y^2)(-2x) - (y^2-x^2)(2x)}{(x^2+y^2)^2} = \frac{1}{y}\frac{-2x^3 - 2xy^2 - 2xy^2 + 2x^3}{(x^2+y^2)^2} = \frac{-4xy}{(x^2+y^2)}$$

From (i) & (ii), $f_{xy} = f_{yx}$

Q 5) If $f(x,\ y) = a\tan^{-1}\left(\dfrac{x}{y}\right)$, then prove that $f_{xy} = f_{yx}$.

Sol^n. Differentiating f partially w.r.t. x,

$$\frac{\partial f}{\partial x}=a\frac{1}{1+\frac{x^2}{y^2}}\frac{1}{y}=\frac{ay}{x^2+y^2}\ ,\ f_{yx}=\frac{(x^2+y)a-2ay^2}{(x^2+y^2)^2}=a\frac{x^2-y^2}{(x^2+y^2)^2}$$

$$f_y=a\frac{1}{1+\frac{x^2}{y^2}}\left(-\frac{x}{y^2}\right)=\frac{-ax}{x^2+y^2},\ f_{xy}=a\frac{-(x^2+y^2)+2x^2}{(x^2+y^2)^2}=a\frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\therefore\ f_{xy}=f_{yx}$$

Q 6) If $f(x,\ y)=x^y+y^x$, then prove that $f_{xy}=f_{yx}$.

Sol^n.

$$f_x=yx^{y-1}+y^x\log y$$

$$f_{yx}=x^{y-1}+y\,x^{y-1}\log x+x\,y^{x-1}\log y=x^{y-1}+y^{x-1}+y\,x^{y-1}\log x+x\,y^{x-1}\log y$$

$$f_y=x^y\log x+x\,y^{x-1},\ f_{xy}=y\,x^{y-1}\log x+x^{y-1}+y^{x-1}+x\,y^{x-1}\log y\ \therefore\ f_{xy}=f_{yx}$$

Q 7) If $u(x,\ t)=Ae^{-gx}\sin(nt-gx)$ and if $\frac{\partial u}{\partial t}=k^2\frac{\partial^2 u}{\partial x^2}$, then show that $n=2k^2g^2$, where $A,\ g$ are constants.

Sol^n. $$\frac{\partial u}{\partial t}=Ae^{-gx}\cos(nt-gx)\,n$$

$$\frac{\partial u}{\partial x}=-A\,g\,e^{-gx}\sin(nt-gx)-A\,g\,e^{-gx}\cos(nt-gx)=-A\,g\,e^{-gx}[\sin(nt-gx)+\cos(nt-gx)]$$

$$\frac{\partial^2 u}{\partial x^2}=A\,g^2e^{-gx}[\sin(nt-gx)+\cos(nt-gx)]-A\,g\,e^{-gx}[-g\cos(nt-gx)+g\sin(nt-gx)]$$

$$=A\,g^2e^{-gx}2\cos(nt-gx)$$

Now, $$\frac{\partial u}{\partial t}=k^2\frac{\partial^2 u}{\partial x^2}\Rightarrow A\,n\,e^{-gx}\cos(nt-gx)=k^2A\,g^2\,e^{-gx}2\cos(nt-gx)\ \Rightarrow\quad n=2k^2g^2$$

Q 8) If $\theta=t^ne^{-\frac{r^2}{4t}}$, what value of n will make $\frac{1}{r^2}\frac{\partial}{\partial x}\left(r^2\frac{\partial\theta}{\partial r}\right)=\frac{\partial\theta}{\partial t}$?

Sol^n. $$\frac{\partial\theta}{\partial r}=t^n-\frac{2r}{4t}e^{-\frac{r^2}{4t}},r^2\frac{\partial\theta}{\partial r}=-\frac{1}{2}r^3t^{n-1}e^{-\frac{r^2}{4t}}\ ,\ \frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=-\frac{1}{2}3r^2t^{n-1}e^{-\frac{r^2}{4t}}+\frac{1}{2}r^3t^{n-1}e^{-\frac{r^2}{4t}}\frac{2r}{4t}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=-\frac{3}{2}t^{n-1}e^{-\frac{r^2}{4t}}+\frac{1}{4}r^2t^{n-2}e^{-\frac{r^2}{4t}},\frac{\partial\theta}{\partial t}=n\,t^{n-1}e^{-\frac{r^2}{4t}}+t^ne^{-\frac{r^2}{4t}}\frac{r^2}{4t}=n\,t^{n-1}e^{-\frac{r^2}{4t}}+\frac{1}{4}r^2t^{n-1}e^{-\frac{r^2}{4t}}$$

Now, $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=\frac{\partial\theta}{\partial t}\quad\Rightarrow\quad n=-\frac{3}{2}.$$

Composite Function

1) If u be a function of two independent variables x and y, where x and y are separately functions of a single independent variable t, then u is called a composite function of t.
For example, if $u=y\cos(x+y)$ where $x=e^t$ and $y=\log t$, then u becomes a composite function of t, i.e. $u\to x,\ y\to t$.

2) If $u=f(x,\ y)$, where $x=\phi_1(t_1,\ t_2)$ and $y=\phi_2(t_1,\ t_2)$, then u is called the composite function of two independent variables t_1 & t_2.
For example, if $u=x^2+y^2-2xy$ where $x=t_1+t_2$ and $y=t_1-t_2$, then u become a composite function of two independent variables t_1 and t_2. i.e. $u\to x,\ y\to t_1,\ t_2$.

3) If $u=f(x_1,\ x_2,\ x_3,\ \dots,\ x_n)$ where $x_1,\ x_2,\ x_3,\ \dots,\ x_n$ are functions of a single independent variable t, then u is called a composite function of a single variable t.

i.e. $u \to x_1, x_2, x_3, \dots, x_n \to t$

4) If $u = f(x_1, x_2, x_3, \dots, x_n)$, where $x_1, x_2, x_3, \dots, x_n$ are functions of m independent variables $t_1, t_2, \dots, t_m$, then u is called the composite function of several independent variables $t_1, t_2, \dots, t_m$, i.e. $u \to x_1, x_2, \dots, x_n \to t_1, t_2, \dots, t_m$.

Total Derivative

If $u = f(x, y)$, where $x = \phi(t)$ and $y = \psi(t)$, then u can be expressed as a function of a single variable t, substituting for x & y in $u = f(x, y)$, the derivative of u w.r.t. t is the ordinary differential coefficient $\frac{du}{dt}$. This $\frac{du}{dt}$ is called the Total Derivative.

We shall now establish a relation between $\frac{du}{dt}$ and the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$.

Theorem on Total Differential Coefficient

1) If u be a composite function of t given by the relation $u = f(x, y),\ x = \phi(t),\ y = \psi(t)$ where u possesses continuous partial derivative with respect to x and y and x, y possess derivatives w.r.t. t, then $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$.

   The required formula in terms of differentials only, can be written as:

   $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dt$, is called as total differential.

2) If u be a composite function of t given by the relation, $u = f(x, y, z),\ x = \phi(t),\ y = \psi(t),$ $z = \xi(t)$, where u possesses continuous partial derivatives with respect to x, y, z and x, y, z possess derivatives with respect to t, then $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$.

   We have, $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$ as total differential.

3) If $u = f(x, y)$ where $x = f_1(r, \theta),\ y = f_2(r, \theta)$, then the differential coefficients of u with respect to r and θ, will be partial derivatives $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}$ and are given by

   $$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}, \quad \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}$$

4) If $u = f(x, y, z)$, where $x = f_1(r, \theta),\ y = f_2(r, \theta)$ and $z = f_3(r, \theta)$ then the differential coefficients of u w.r.t. r & θ will be partial derivatives $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}$ and are given by

   $$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial r}, \quad \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \theta}$$

5) If $u = f(x, y, z)$ where $x = f_1(r, \theta, \phi),\ y = f_2(r, \theta, \phi),\ z = f_3(r, \theta, \phi)$ then the differential coefficients of u w.r.t. r, θ, ϕ will be

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial r}, \quad \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \theta}, \quad \frac{\partial u}{\partial \phi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \phi} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \phi}$$

Partial differentiation of Function of a Function

If $z = f(u)$ and $u = \phi(x, y)$, i.e. z is a function of u & u is a function of x, y, then z is a function of x & y. Then, $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x} = f'(x)\frac{\partial u}{\partial x}$. Similarly, $\frac{\partial z}{\partial y} = f'(u)\frac{\partial u}{\partial y}$.

Illustrative Examples

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Q 1) If $u = f(r)$ where $x = r\cos\theta,\ y = r\sin\theta$, then prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f''(r) + \dfrac{1}{r} f'(r)$.

Sol^n. Given $u = f(r)$ and $r^2 = x^2 + y^2$. $\dfrac{\partial u}{\partial r} = f'(r)$ & $\dfrac{\partial r}{\partial x} = \dfrac{x}{r},\ \dfrac{\partial r}{\partial y} = \dfrac{y}{r}$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} = f'(r)\frac{x}{r},$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(f'(r)\frac{x}{r}\right) = \frac{f'(r)}{r} + x\frac{\partial}{\partial x}\left(\frac{f'(r)}{r}\right) = \frac{f'(r)}{r} + x\frac{r f''(r) - f'(r)}{r^2}\frac{\partial r}{\partial x}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{f'(r)}{r} + \frac{x}{r^2}(r f''(r) - f'(r))\frac{x}{r} = \frac{x^2}{r^2} f''(r) + \frac{r^2 - x^2}{r^3} f'(r)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} = f'(r)\frac{y}{x}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{f'(r)}{r} + y\frac{\partial}{\partial y}\left(\frac{f'(r)}{r}\right) = \frac{f'(r)}{r} + y\frac{\partial}{\partial r}\left(\frac{f''(r)}{r}\right)\frac{\partial r}{\partial y} = \frac{y^2}{r^2} + f''(r) + \frac{r^2 - y^2}{r^3} f'(r)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{x^2 + y^2}{r^2} f''(r) + \frac{2r^2 - (x^2 + y^2)}{r^3} f'(r) = f''(r) + \frac{1}{r} f'(r) \quad [\because r^2 = x^2 + y^2]$$

Q 2) If $v = x\log(x + r) - r$ where $r^2 = x^2 + y^2$, then prove that $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = \dfrac{1}{x + r}$.

Sol^n. $r^2 = x^2 + y^2 \quad \Rightarrow \quad \dfrac{\partial r}{\partial x} = \dfrac{x}{r},\ \dfrac{\partial r}{\partial y} = \dfrac{y}{x}$. Now, $v = x\log(x + r) - r$.

$$\frac{\partial v}{\partial x} = \log(x + r) + \frac{x}{x + r}\left(1 + \frac{\partial r}{\partial x}\right) - \frac{\partial r}{\partial x}$$

$$= \log(x + r) + \frac{x}{x + r}\left(1 + \frac{x}{r}\right) - \frac{x}{r} = \log(x + r) + \frac{x}{r} - \frac{x}{r} = \log(x + r)$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\log(x + r) = \frac{1}{x + r}\left(1 + \frac{\partial x}{\partial r}\right) = \frac{1}{x + r}\left(1 + \frac{x}{r}\right) = \frac{1}{r}$$

Q 3) If $u = f(r)$ where $r = \sqrt{x^2 + y^2 + z^2}$, prove that $u_{xx} + u_{yy} + u_{zz} = f''(r) + \dfrac{2}{r} f'(r)$.

Sol^n. $r^2 = x^2 + y^2 + z^2 \ \therefore\ \dfrac{\partial r}{\partial x} = \dfrac{x}{r},\ \dfrac{\partial r}{\partial y} = \dfrac{y}{r},\ \dfrac{\partial r}{\partial z} = \dfrac{z}{r}$, $u_x = f'(r)\dfrac{\partial r}{\partial x} = f'(r)\dfrac{x}{r}$

$$u_{xx} = f''(r)\frac{x}{r}\frac{\partial r}{\partial x} + f'(r)\left(\frac{1}{r} - \frac{x}{r^2}\frac{\partial r}{\partial x}\right) = f''(r)\frac{x^2}{r^2} + \frac{1}{r} f'(r) - \frac{x^2}{r^3} f'(r)$$

Similarly, $u_{yy} = f''(r)\dfrac{y^2}{r^2} + \dfrac{1}{r} f'(r) - \dfrac{y^2}{r^3} f'(r)$ & $u_{zz} = f''(r)\dfrac{z^2}{r^2} + \dfrac{1}{r} f'(r) - \dfrac{z^2}{r^3} f'(r)$.

$$\therefore u_{xx} + u_{yy} + u_{zz} = \frac{f''(r)}{r^2}(x^2 + y^2 + z^2) + \frac{3}{r} f'(r) - \frac{f'(r)}{r^3}(x^2 + y^2 + z^2)$$

$$= f''(r) + \frac{2}{r} f'(r) \qquad [\because\ x^2 + y^2 + z^2 = r^2]$$

Q 4) If $z = e^{ax+by} f(ax - by)$, prove that $b\dfrac{\partial z}{\partial x} + a\dfrac{\partial z}{\partial y} = 2abz$.

Sol^n. $\dfrac{\partial z}{\partial x} = ae^{ax+by} f(ax - by) + ae^{ax+by} f'(ax - by)$, $b\dfrac{\partial z}{\partial x} = abe^{ax+by}[f(ax - by) + f'(ax - by)]$

$\dfrac{\partial z}{\partial y} = be^{ax+by} f(ax - by) - be^{ax+by} f'(ax - by)$, $a\dfrac{\partial z}{\partial y} = abe^{ax+by}[f(ax - by) + f'(ax - by)]$

$$\therefore b\frac{\partial z}{\partial x}+a\frac{\partial z}{\partial y}=2abe^{ax+by}f(ax-by)=2abz$$

Q 5) If $v=\frac{1}{\sqrt{t}}e^{-\frac{x^2}{4a^2t}}$, prove that $\frac{\partial v}{\partial t}=a^2\frac{\partial^2 v}{\partial x^2}$.

Sol^n. $\frac{\partial v}{\partial t}=\frac{-1}{2}t^{-\frac{3}{2}}e^{-\frac{x^2}{4a^2t}}+\frac{1}{\sqrt{t}}e^{-\frac{x^2}{4a^2t}}\frac{x^2}{4a^2t}$, $\quad \frac{\partial v}{\partial x}=\frac{-1}{\sqrt{t}}\frac{2x}{4a^2t}e^{-\frac{x^2}{4a^2t}}$

$$\frac{\partial^2 v}{\partial x^2}=\frac{-1}{2a^2t^{\frac{3}{2}}}e^{-\frac{x^2}{4a^2t}}+\frac{2x^2}{2a^2t^{\frac{3}{2}}}\frac{1}{4a^2t}e^{-\frac{x^2}{4a^2t}}\ ,\ a^2\frac{\partial^2 v}{\partial x^2}=-\frac{1}{2}t^{-\frac{3}{2}}e^{-\frac{x^2}{4a^2t}}+\frac{x^2}{4a^2t^{\frac{5}{2}}}e^{-\frac{x^2}{4a^2t}}$$

$$\therefore \frac{\partial v}{\partial t}=a^2\frac{\partial^2 v}{\partial x^2}$$

Q 6) If $u=x^2+y^2$, $v=2xy$ and $z=f(u,\ v)$, then show that $x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}=2\sqrt{u^2-v^2}\frac{\partial z}{\partial u}$.

Sol^n. $\frac{\partial u}{\partial x}=2x,\quad \frac{\partial u}{\partial y}=+2y,\quad \frac{\partial v}{\partial x}=2y,\quad \frac{\partial v}{\partial y}=2y$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=2\left(x\frac{\partial z}{\partial u}+y\frac{\partial z}{\partial v}\right),\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=2\left(+y\frac{\partial z}{\partial u}+x\frac{\partial z}{\partial v}\right)$$

$$x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}=2x^2\frac{\partial z}{\partial u}+2xy\frac{\partial z}{\partial v}-2y^2\frac{\partial z}{\partial v}-2xy\frac{\partial z}{\partial v}=2(x^2-y^2)\frac{\partial z}{\partial u}$$

$$x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}==2\sqrt{(x^2-y^2)^2}\frac{\partial z}{\partial u}=2\sqrt{(x^2+y^2)^2-4x^2y^2}\frac{\partial z}{\partial u}=2\sqrt{u^2-v^2}\frac{\partial z}{\partial u}$$

Q 7) If $z=f(x,\ y)$, where $x=u\cos\alpha-v\sin\alpha,\ y=u\sin\alpha+v\cos\alpha$ (α is a constant), show that $\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2$.

Sol^n. $\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}=\cos\alpha\frac{\partial z}{\partial x}+\sin\alpha\frac{\partial z}{\partial y},\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}=-\sin\alpha\frac{\partial z}{\partial x}+\cos\alpha\frac{\partial z}{\partial y}$

$$\left(\frac{\partial z}{\partial u}\right)^2=\cos^2\alpha\left(\frac{\partial z}{\partial x}\right)^2+\sin^2\alpha\left(\frac{\partial z}{\partial y}\right)^2+2\sin\alpha\cos\alpha\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}$$

$$\left(\frac{\partial z}{\partial v}\right)^2=\sin^2\alpha\left(\frac{\partial z}{\partial x}\right)^2+\cos^2\alpha\left(\frac{\partial z}{\partial y}\right)^2-2\sin\alpha\cos\alpha\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}$$

$$\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2=\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\quad \text{[proved]}$$

Q 8) If $z=f(x,\ y)$, where $x=e^u\cos v,\ y=e^v\sin v$, prove that

(a) $y\frac{\partial z}{\partial u}+x\frac{\partial z}{\partial v}=e^{2u}\frac{\partial z}{\partial y}$ (b) $\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2=e^{-2u}\left[\left(\frac{\partial f}{\partial u}\right)^2+\left(\frac{\partial f}{\partial v}\right)^2\right]$

Sol^n. (a) $\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}=\frac{\partial z}{\partial x}(e^u\cos v)+\frac{\partial z}{\partial y}(e^u\sin v)=x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}$

Also, $\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}=\frac{\partial z}{\partial x}(-e^u\sin v)+\frac{\partial z}{\partial y}(e^u\cos v)=-y\frac{\partial z}{\partial x}+x\frac{\partial z}{\partial y}$

$$\therefore y\frac{\partial z}{\partial u}+x\frac{\partial z}{\partial v}=xy\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}-xy\frac{\partial z}{\partial x}+x^2\frac{\partial z}{\partial y}=(y^2+x^2)\frac{\partial z}{\partial y}=e^{2u}\frac{\partial z}{\partial y}\quad \text{[proved]}$$

(b) $\left(\frac{\partial f}{\partial u}\right)^2+\left(\frac{\partial f}{\partial v}\right)^2=x^2\left(\frac{\partial z}{\partial x}\right)^2+y^2\left(\frac{\partial z}{\partial y}\right)^2+2xy\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}$

$$+x^2\left(\frac{\partial z}{\partial y}\right)^2+y^2\left(\frac{\partial z}{\partial x}\right)^2-2xy\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}$$

$$=(x^2+y^2)\left[\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]=e^{2u}\left[\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]$$

i.e. $\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=e^{-2u}\left[\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial v}\right)^2\right]$ [proved]

Q 9) If $V=e^{\frac{r-x}{l}}$ where $r^2=x^2+y^2$, l is a constant, show that $V_{xx}+V_{yy}+\frac{2}{l}V_x=\frac{V}{lr}$.

Sol^n. $V=e^{\frac{r-x}{l}}$. $\quad\therefore V_x=e^{\frac{(x-r)}{l}}\frac{1}{l}\left(\frac{\partial r}{\partial x}-1\right)=\frac{V}{l}\left(\frac{\partial r}{\partial x}-1\right)$

Now, $r^2=x^2+y^2 \quad\Rightarrow\quad \frac{\partial r}{\partial x}=\frac{x}{r},\ \frac{\partial r}{\partial y}=\frac{y}{r} \quad \therefore V_x=\frac{V}{l}\left(\frac{x}{r}-1\right)$

$$V_{xx}=\frac{V_x}{l}\left(\frac{x}{r}-1\right)+\frac{V}{l}\left(\frac{r-x\frac{\partial r}{\partial x}}{r^2}\right)=\frac{V_x}{l}\left(\frac{x}{r}-1\right)+\frac{V}{l}\left(\frac{1}{r}-\frac{x^2}{r^3}\right)$$

$$V_y=e^{\frac{(x-r)}{l}}\frac{1}{l}\left(\frac{\partial r}{\partial y}\right)=\frac{V}{l}\left(\frac{y}{r}\right),\ \therefore V_{yy}=\frac{V_y}{l}\left(\frac{y}{r}\right)+\frac{V}{l}\left(\frac{1}{r}-\frac{y}{r^2}\frac{\partial r}{\partial y}\right)=\frac{V_y}{l}\frac{y}{r}+\frac{V}{rl}-\frac{V}{r}\frac{y^2}{r^3}$$

$$\therefore V_{xx}+V_{yy}+\frac{2}{l}V_x=\frac{V}{l^2}\left(\frac{x}{r}-1\right)^2+\frac{V}{l}\left(\frac{1}{r}-\frac{x^2}{r^3}\right)+\frac{V}{l}\left(\frac{y}{r}\right)^2+\frac{V}{rl}-\frac{V}{l}\frac{y^2}{r^3}+\frac{2}{l^2}V\left(\frac{x}{r}-1\right)$$

$$=\frac{V}{l^2}\left(\frac{x^2}{r^2}+1-\frac{2x}{r}+\frac{2x}{r}-2+\frac{y^2}{x^2}\right)+\frac{V}{l}\left(\frac{1}{r}-\frac{x^2}{r^3}+\frac{1}{r}-\frac{y^2}{r^3}\right)$$

$$=\frac{V}{l}\left(\frac{2}{r}-\frac{1}{r}\right)=\frac{V}{lr}=\text{R.H.S.}$$

<u>Change of Independent Variables</u>

If $u=f(x,\ y)$, where $x=f_1(r,\ \theta)$ and $y=f_2(r,\ \theta)$, then $u\to x,\ y\to r,\ \theta$.
By the chain rule, differentiate w.r.t. r & θ partially, we get

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \qquad \text{---------- (1)}$$

or
$$\frac{\partial u}{\partial \theta}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} \qquad \text{---------- (2)}$$

Given terms are x and y in terms of $r,\ \theta$. $\therefore$ The known quantities are $\frac{\partial x}{\partial r},\ \frac{\partial x}{\partial \theta},\ \frac{\partial y}{\partial r},\ \frac{\partial y}{\partial \theta}$.

We solve equations (1) and (2) simultaneously. We get the values of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ in terms of $\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial \theta}$, i.e., $\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$ and $\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}$

Differentiate equations (1) and (2) w.r.t. $r,\ \theta$ partially, we get

$$\frac{\partial^2 u}{\partial r^2}=\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\right)=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\right)\frac{\partial x}{\partial r}+\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial r}\right)\frac{\partial y}{\partial r}$$

$$\frac{\partial^2 u}{\partial \theta^2}=\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right)=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \theta}\right)\frac{\partial x}{\partial \theta}+\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial \theta}\right)\frac{\partial y}{\partial \theta}$$

Illustrative Examples

Q 1) If $u=\sin(x^2+y^2)$, where $a^2x^2+b^2y^2=e^z$, find $\dfrac{du}{dx}$.

Sol^n. $a^2x^2+b^2y^2=e^z$, $2a^2x+2b^2y\dfrac{dy}{dx}=0 \quad \Rightarrow \quad \dfrac{dy}{dx}=-\dfrac{a^2x}{b^2y}$

$$\frac{du}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\frac{dy}{dx}=2x\cos(x^2+y^2)+2y\cos(x^2+y^2)\left(-\frac{a^2x}{b^2y}\right)=2\cos(x^2+y^2)\left[x-\frac{a^2}{b^2}x\right]$$

Q 2) If $u=x^2+y^2+z^2$ and $x=e^{2t},\ y=e^{2t}\cos 3t,\ z=e^{2t}\sin 3t$. Find $\dfrac{du}{dt}$ as a total derivative and verify the result by direct substitution.

Sol^n. $u=e^{4t}+e^{4t}\cos^2 3t+e^{4t}\sin^2 3t=2e^{4t} \quad \therefore \dfrac{du}{dt}=8e^{4t}$

Also, $\dfrac{du}{dt}=\dfrac{\partial u}{\partial x}\dfrac{dx}{dt}+\dfrac{\partial u}{\partial y}\dfrac{dy}{dt}+\dfrac{\partial w}{\partial z}\dfrac{dz}{dt}$

$$=2x2e^{2t}+2y[2e^{2t}\cos 3t-3e^{2t}\sin 3t]+2[2e^{2t}\sin 3t+3e^{2t}\cos 3t]$$

$$=4e^{4t}+4e^{4t}\cos^2 3t-6e^{4t}\cos 3t\sin 3t+4e^{4t}\sin^2 3t+6e^{4t}\sin 3t\cos 3t=8e^{4t}$$

Q 3) If $z=\log(u^2+v^2),\ u=e^{x^2+y^2},\ v=x^2+y$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

Sol^n. $\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial x}+\dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial x}=\dfrac{2u}{u^2+v}2xe^{x^2+y^2}+\dfrac{1}{u^2+v}2x=\dfrac{2x}{u^2+v}(2u^2+1)$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\frac{2u}{u^2+v}2ye^{x^2+y^2}+\frac{1}{u^2+v}=\frac{4yu^2+1}{u^2+v}$$

Q 4) If $u=f(2x-3y,\ 3y-4z,\ 4z-2x)$, prove that $\dfrac{1}{2}\dfrac{\partial u}{\partial x}+\dfrac{1}{3}\dfrac{\partial u}{\partial y}+\dfrac{1}{4}\dfrac{\partial u}{\partial z}=0$.

Sol^n. $r=2x-3y,\ s=3y-4z,\ t=4z-2x$

$\therefore u=f(r,\ s,\ t)$, $\therefore \dfrac{\partial u}{\partial x}=\dfrac{\partial u}{\partial r}\dfrac{\partial r}{\partial x}+\dfrac{\partial u}{\partial s}\dfrac{\partial s}{\partial x}+\dfrac{\partial u}{\partial t}\dfrac{\partial t}{\partial x}=2\dfrac{\partial u}{\partial r}-2\dfrac{\partial u}{\partial t}$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial y}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial y}=-3\frac{\partial u}{\partial r}+\frac{\partial u}{\partial s}3$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial z}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial z}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial z}=4\frac{\partial u}{\partial t}-4\frac{\partial u}{\partial s}$$

$$\frac{1}{2}\frac{\partial u}{\partial x}+\frac{1}{3}\frac{\partial u}{\partial y}-\frac{1}{4}\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}-\frac{\partial u}{\partial t}-\frac{\partial u}{\partial r}+\frac{\partial u}{\partial s}+\frac{\partial u}{\partial t}-\frac{\partial u}{\partial s}=0$$

Q 5) If $x=u+v+w,\ y=vw+wu+uv,\ z=uvw$ and F is a function of $x,\ y,\ z$, show that $u\dfrac{\partial F}{\partial u}+v\dfrac{\partial F}{\partial v}+w\dfrac{\partial F}{\partial w}=x\dfrac{\partial F}{\partial x}+2y\dfrac{\partial f}{\partial y}+3z\dfrac{\partial F}{\partial z}$.

Sol^n. $\dfrac{\partial F}{\partial u}=\dfrac{\partial F}{\partial x}\dfrac{\partial x}{\partial u}+\dfrac{\partial F}{\partial y}\dfrac{\partial y}{\partial u}+\dfrac{\partial F}{\partial z}\dfrac{\partial z}{\partial u}=\dfrac{\partial F}{\partial x}+(u+w)\dfrac{\partial F}{\partial y}+vw\dfrac{\partial F}{\partial z}$

$$\frac{\partial F}{\partial v}=\frac{\partial F}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial F}{\partial y}\frac{\partial y}{\partial v}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial v}=\frac{\partial F}{\partial x}+(u+w)\frac{\partial F}{\partial y}+uw\frac{\partial F}{\partial z}$$

$$\frac{\partial F}{\partial w}=\frac{\partial F}{\partial x}\frac{\partial x}{\partial w}+\frac{\partial F}{\partial y}\frac{\partial y}{\partial w}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial w}=\frac{\partial F}{\partial x}+(u+v)\frac{\partial F}{\partial y}+uv\frac{\partial F}{\partial z}$$

Now,

$$\frac{\partial F}{\partial x}(u+v+w)+(uv+uw+uv+vw+uw+vw)\frac{\partial F}{\partial y}+3uvw\frac{\partial F}{\partial z}=x\frac{\partial F}{\partial x}+2y\frac{\partial F}{\partial y}+3z\frac{\partial F}{\partial z}$$

Q 6) Given that $z^3+xy-y^2z=6$, obtain the expressions for $\frac{\partial y}{\partial x}, \frac{\partial z}{\partial x}$ in terms of x, y, z and find their values at the point (0, 1, 2).

Sol^n. $f(x,\ y,\ z)=z^3+xy-y^2z-6 \qquad f\to x,\ y,\ z\to x,\ y$

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial x}=0 \quad\Rightarrow\quad y+(3z^2-y^2)\frac{\partial z}{\partial x}=0 \quad\Rightarrow\quad \frac{\partial z}{\partial x}=-\frac{y}{3z^2-y^2}$$

Q 7) At a given instant the sides of a rectangle are 4 ft and 3 ft respectively and they are increasing at the rate of 1.5 ft / sec and 0.5 ft / sec respectively. Find the rate at which the area is increasing at that instant.

Sol^n. $A=xy$, $\frac{dA}{dt}=\frac{\partial A}{\partial x}\frac{dx}{dt}+\frac{\partial A}{\partial y}\frac{dy}{dt}=y\frac{dx}{dt}+x\frac{dy}{dt}$. At the given instant,

$$\frac{dA}{dt}=3\times 1.5+4\times 0.5=4.5+2=6.5 \text{ sq. ft}/\text{sec}$$

Q 8) If $z=2xy^2-3x^2y$ and if x is increasing at the rate of 2 cm / sec and it passes through the value $x=3$ cm , show that if y is passing through the value $y=1$ cm , y must be decreasing at the rate of $2\frac{2}{15}$ cm / sec in order that z shall remain constant.

Sol^n. $\frac{dx}{dt}=2,\ \frac{dy}{dt}=-\frac{32}{15}$. $\quad z$ must remain constant $\quad\Rightarrow\quad \frac{dz}{dt}=0$

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}=0 \quad\Rightarrow\quad (2y^2-6xy)2+(4xy-3x^2)\frac{dy}{dt}=0$$

Putting $x=3,\ y=1,\quad \frac{dy}{dt}=\frac{36-4}{12-27}=-\frac{32}{15}=2\frac{2}{15}$ cm / sec

Q 9) If $u=\tan^{-1}\left(\frac{y}{x}\right)$ where $x=e^t-e^{-t}$ and $y=e^t+e^{-t}$, find $\frac{du}{dt}$.

Sol^n. $u\to x,\ y\to t$. $\frac{du}{dx}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}$

$$\therefore\ \frac{du}{dx}=\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right)[e^t+e^{-t}]+\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right)[e^t-e^{-t}]=\frac{-y^2}{x^2+y^2}+\frac{x^2}{x^2+y^2}$$

$$\therefore\ \frac{du}{dx}=\frac{(e^t-e^{-t})^2-(e^t+e^{-t})^2}{(e^t-e^{-t})^2+(e^t+e^{-t})^2}=\frac{e^{2t}+e^{-2t}-2-e^{-2t}-2-e^{2t}}{e^{2t}+e^{-2t}-2+e^{2t}+2+e^{-2t}}=\frac{-2}{e^{2t}+e^{-2t}}$$

Q 10) Find the total differential coefficient of x^2y w.r.t. x, where x and y are connected by the relation $x^2+xy+y^2=1$.

Sol^n. $x^2+xy+y^2-1=0=f(x,\ y) \quad f_x(x,\ y)=2x+y,\quad f_y(x,\ y)=x+2y$

$$\therefore\ \frac{dy}{dx}=-\frac{f_x}{f_y}=-\frac{2x+y}{x+2y} \qquad z=x^2y \quad\Rightarrow\quad \frac{\partial z}{\partial x}=2xy,\ \frac{\partial z}{\partial y}=x^2$$

$z\to x,\ y\to x$

$$\frac{dz}{dx}=\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\frac{dy}{dx}=2xy+x^2\left(-\frac{2x+y}{x+2y}\right)=\frac{2x^2y+4xy^2-2x^3-x^2y}{x+2y}=\frac{x[xy+4y^2-2x^2]}{x+2y}$$

Q 11) If $\phi(x-az,\ cy-bz)=0$, show that $ac\frac{\partial z}{\partial x}+b\frac{\partial z}{\partial y}=c$.

Sol^n. Let $u=x-az,\ v=cy-bz \quad\therefore\ \phi\to u,\ v\to x.\ y,\ z\to x,\ y$

$$\therefore\ \frac{\partial\phi}{\partial x}=\phi_u u_x+\phi_u u_z z_x+\phi_v u_z z_x=\phi_u 1-az_x\phi_u-bz_x\phi_v$$

$$\frac{\partial\phi}{\partial y}=\phi_u u_y+\phi_u u_z z_y+\phi_v u_y+\phi_v u_z z_y=-az_y\phi_u+c\phi_v-bz_y\phi_v$$

Now, $\phi(u,\ v)=0 \ \Rightarrow\ d\phi=0 \ \Rightarrow\ \phi_x dx+\phi_y dy=0 \ \Rightarrow\ \phi_x=0\ \&\ \phi_y=0$

$\Rightarrow\ (1-az_x)\phi_u-bz_x\phi_v=0\ \&\ -az_y\phi_u+(c-bz_y)\phi_v=0$

Eliminating ϕ_u & ϕ_v we get, $\begin{vmatrix}1-az_x & -bz_x\\ -az_y & (c-bz_y)\end{vmatrix}=0$

$\Rightarrow\ c-bz_y-acz_x+abz_xz_y-abz_xz_y=0 \ \Rightarrow\ ac\dfrac{\partial z}{\partial x}+b\dfrac{\partial z}{\partial y}=c$

Q 12) If $f(x,\ y)=0,\ \phi(y,\ z)=0$, show that $\dfrac{\partial f}{\partial y}\dfrac{\partial\phi}{\partial z}\dfrac{dz}{dx}=\dfrac{\partial f}{\partial x}\dfrac{\partial\phi}{\partial y}$.

Sol^n. $f(x,\ y)=0 \ \Rightarrow\ \dfrac{dy}{dx}=-\dfrac{f_x}{f_y}$, $\quad \phi(y,\ z)=0 \ \Rightarrow\ \dfrac{dy}{dz}=-\dfrac{\phi_z}{\phi_y}$

Now, $\dfrac{dy}{dx}=\dfrac{dy}{dz}\dfrac{dz}{dx} \ \Rightarrow\ -\dfrac{dz}{dy}\dfrac{dz}{dx} \ \Rightarrow\ \dfrac{\partial f}{\partial x}\dfrac{\partial\phi}{\partial y}=\dfrac{\partial f}{\partial y}\dfrac{\partial\phi}{\partial z}\dfrac{dz}{dx}$ [proved]

Q 13) If the curves $f(x,\ y)=0$ and $\phi(y,\ z)=0$ touch, show that at the point of contact, $\dfrac{\partial f}{\partial x}\dfrac{\partial\phi}{\partial y}=\dfrac{\partial f}{\partial y}\dfrac{\partial\phi}{\partial z}$.

Sol^n. At the point of contact both the curves will have common tangent.

$\therefore\ \dfrac{dy}{dx}=\dfrac{dy}{dz} \ \Rightarrow\ -\dfrac{f_x}{f_y}=-\dfrac{\phi_z}{\phi_y} \ \Rightarrow\ \dfrac{\partial f}{\partial x}\dfrac{\partial\phi}{\partial y}=\dfrac{\partial f}{\partial y}\dfrac{\partial\phi}{\partial z}$

Differentiation of Implicit Functions

Implicit Function: $f(x,\ y)=0$ represents an implicit function.

e.g. $x^3+y^3-3axy=0$

Note that when y is directly expressed in terms of x i.e. $y=f(x)$, then y is called an explicit function of x.

Let $z=f(x,\ y)=0$. $\therefore\ \dfrac{dz}{dx}=\dfrac{\partial z}{\partial x}\dfrac{dx}{dx}+\dfrac{\partial z}{\partial y}\dfrac{dy}{dx}=0 \Rightarrow \dfrac{\partial z}{\partial x}+\dfrac{\partial z}{\partial y}\dfrac{dy}{dx}=0 \Rightarrow \boxed{\dfrac{dy}{dx}=-\dfrac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}}=-\dfrac{p}{q}}$

where, $p=\dfrac{\partial f}{\partial x},\ q=\dfrac{\partial f}{\partial y},\ r=\dfrac{\partial^2 f}{\partial x^2},\ s=\dfrac{\partial^2 f}{\partial x\partial y},\ t=\dfrac{\partial^2 f}{\partial y^2}$ and $\dfrac{d^2y}{dx^2}=-\dfrac{q^2r-2pqs+p^2t}{q^3}$,

i.e., $q^3\dfrac{d^2y}{dx^2}=-(q^2r-2pqs+p^2t)=\begin{vmatrix}r & s & p\\ s & t & q\\ p & q & 0\end{vmatrix}$

Illustrative Examples

Q 1) Find $\dfrac{du}{dx}$, given $u=x\log xy$ and $x^3+y^3=-3xy$.

Sol^n. $\dfrac{du}{dx}=\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}\dfrac{dy}{dx}$, $\dfrac{\partial u}{\partial x}=\log xy+x\dfrac{1}{xy}y=1+\log xy \quad \dfrac{\partial u}{\partial y}=x\dfrac{1}{xy}x=\dfrac{x}{y}$

$f(x,\ y)=x^3+y^3+3xy,\ f_x(x,\ y)=3x^2+3y \quad f_y(x,\ y)=3y^2+3x$

$\therefore\ \dfrac{dy}{dx}=-\dfrac{f_x}{f_y}=-\dfrac{x^2+y}{y^2+x}$

$$\therefore \frac{du}{dx}=1+\log xy+\frac{x}{y}\left(-\frac{x^2+y}{y^2+x}\right)=\log xy+\frac{y^3+xy-x^3-xy}{y^3+xy}=\log xy+\frac{y^3-x^3}{y^3+xy}$$

Q 2) If $(\cos x)^y=(\sin y)^x$ then find $\frac{dy}{dx}$.

Sol^n^. Taking logarithm, we have, $y\log(\cos x)=x\log(\sin y)$.

Let $f(x,\ y)=y\log(\cos x)-x\log(\sin y)=0$.

Now, $\frac{\partial f}{\partial x}=y\frac{-\sin x}{\cos x}-\log(\sin y)=-y\tan x-\log(\sin y)$

$\frac{\partial f}{\partial y}=\log(\cos x)-x\frac{\cos y}{\sin y}=\log(\cos x)-x\cot y$ $\therefore \frac{dy}{dx}=-\frac{f_x}{f_y}=\frac{y\tan x+\log(\sin y)}{\log(\cos x)-x\cot y}$

Q 3) Find $\frac{dy}{dx}$ for $x^y+y^x=a^b$.

Sol^n^. Let $f(x,\ y)=x^y+y^x-a^b=0$ $f_x=yx^{y-1}+y^x\log y$ $f_y=x^y\log x+xy^{x-1}$.

$$\therefore \frac{dy}{dx}=-\frac{f_x}{f_y}=-\frac{yx^{y-1}+y^x\log y}{x^y\log x+xy^{x-1}}$$

Q 4) Find $\frac{dy}{dx}$ if $x^2y^4+\sin y=0$.

Sol^n^. $f(x,\ y)=x^2y^4+\sin y$. $f_x=2xy^4,\ f_y=4x^2y^3+\cos y$ $\therefore \frac{dy}{dx}=-\frac{2xy^4}{4x^2y^3+\cos y}$

Q.5) Find $\frac{dy}{dx}$ if $(\tan x)^y+y^{\cot x}=a$.

Sol^n^. $f(x,\ y)=(\tan x)^y+y^{\cot x}-a$, $f_x=y(\tan x)^{y-1}\sec^2 x+y^{\cot x}\log y(-\cos ec^2 x)$

$f_y=(\tan x)^y\log(\tan x)+\cot x y^{\cot x-1}$ $\therefore \frac{dy}{dx}=-\frac{y(\tan x)^{y-1}\sec^2 x-\cos ec^2 x y^{\cot x}\log y}{(\tan x)^y\log(\tan x)+\cot x y^{\cot x-1}}$

<u>Maxima and Minima of a Function of Two Independent Variables</u>

Working Rule:

Step 1: Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, and equate them to zero. i.e. $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$.
Solve these simultaneous equations for x & y.
Let $a_1,\ b_1,\ a_2,\ b_2,\ \dots$ be the pair of values.

Step 2: Calculate the values of $r=\frac{\partial^2 f}{\partial x^2}$, $s=\frac{\partial^2 f}{\partial x\partial y}$, $t=\frac{\partial^2 f}{\partial y^2}$ and substitute in them by terms $a_1,\ b_1,\ a_2,\ b_2,\ \dots$ for $x,\ y$.

Step 3:
(i) If $rt-s^2>0$ and $r<0$ at $(a_1,\ b_1)$, $f(x,\ y)$ is a maximum at $(a_1,\ b_1)$ and $f(a_1,\ b_1)$ is maximum value.
(ii) If $rt-s^2>0$ and $r>0$ at $(a_1,\ b_1)$, $f(x,\ y)$ is a minimum at $(a_1,\ b_1)$ and $f(a_1,\ b_1)$ is minimum value.
(iii) If $rt-s^2<0$ at $(a_1,\ b_1)$, $f(x,\ y)$ is neither maximum nor minimum at $(a_1,\ b_1)$ i.e. $f(a_1,\ b_1)$ is not an extreme value. Such a point is sometimes called a saddle point.
(iv) If $rt-s^2=0$ at $(a_1,\ b_1)$, the case is undecided.

Illustrative Examples

Q 1) Examine for minimum and maximum values of $\sin x+\sin y+\sin(x+y)$.

Sol^n. Here, $f(x,\ y)=\sin x+\sin y+\sin(x+y)$

Step 1: $\frac{\partial f}{\partial x}=0\Rightarrow \cos x+\cos(x+y)=0$, $\frac{\partial f}{\partial y}=0\Rightarrow \cos y+\cos(x+y)=0$

By (i) and (ii), $\cos x-\cos y=0\Rightarrow \cos x=\cos y\Rightarrow x=y$.

$\therefore \cos x+\cos 2x=0\Rightarrow \cos 2x=-\cos x=\cos(\pi-x)\Rightarrow 2x=\pi-x\Rightarrow x=\frac{\pi}{3}$

$\therefore\ y=\frac{\pi}{3}$. Thus, $\left(\frac{\pi}{3},\frac{\pi}{3}\right)$ is the stationary point of the function.

Step 2: Now, $\frac{\partial^2 f}{\partial x^2}=-\sin x-\sin(x+y)=r$, $\frac{\partial^2 f}{\partial x\partial y}=-\sin(x+y)=s$,

$\frac{\partial^2 f}{\partial y^2}=-\sin y-\sin(x+y)=t$. At $\left(\frac{\pi}{3},\frac{\pi}{3}\right)$, $r=-\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}=-\sqrt{3}$,

$s=-\frac{\sqrt{3}}{2}$, $t=-\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}=-\sqrt{3}$

Step 3: At $\left(\frac{\pi}{3},\frac{\pi}{3}\right)$, $rt-s^2=3-\frac{3}{4}=\frac{9}{4}>0$ and $r=-\sqrt{3}<0$.

$\therefore\ f(x,\ y)$ is maximum at $\left(\frac{\pi}{3},\frac{\pi}{3}\right)$ and the maximum value is

$$f\left(\frac{\pi}{3},\frac{\pi}{3}\right)=\sin\frac{\pi}{3}+\sin\frac{\pi}{3}+\sin\left(\frac{2\pi}{3}\right)=\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}=\frac{3\sqrt{3}}{2}$$

[Here we consider only values of $x,\ y$ lying between 0 & 1]

Illustrative Examples

Q 1) Divide 120 into three parts so that the sum of their products taken two at a time shall be maximum.

Sol^n. Consider $x,\ y,\ z$ whose sum is 120. $\therefore\ x+y+z=120$

$u=xy+yz+zx=xy+y(120-x-y)+x(120-x-y)$

$\therefore u=xy+120y-xy-y^2+120x-x^2-xy=120(x+y)-x^2-y^2-xy$

$\frac{\partial u}{\partial x}=120-2x-y=0,\frac{\partial u}{\partial y}=120-2y-x=0$

$120-3y=0\Rightarrow y=40\Rightarrow x=40$. $\therefore\ (40,\ 40)$ is the stationary point.

$\frac{\partial^2 u}{\partial x^2}=-2<0,\frac{\partial^2 u}{\partial y^2}=-2,\frac{\partial^2 u}{\partial x\partial y}=-1.\ \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2}-\left(\frac{\partial^2 u}{\partial x\partial y}\right)^2=4-1=3>0$

$\therefore u$ is maximum at $x=40,\ y=40,\ z=40$.

Q 2) Find the extreme values of $f(x,\ y)=x^2+y^2+\frac{2}{x}+\frac{2}{y}$.

Sol^n. $f_x=2x-\frac{2}{x^2}=0\Rightarrow x^3=1\Rightarrow x=1$, $f_y=2y-\frac{2}{y^2}=0 \Rightarrow y=1$

$\therefore$ (1, 1) is the stationary point. $f_{xx}=2+\frac{4}{x^3}$ $f_{xx}(1,\ 1)=6>0$, $f_{yy}=2+\frac{4}{y^3}$, $f_{xy}=0$

$f_{xx}(1,\ 1)f_{yy}(1,\ 1)-\{f_{xy}(1,\ 1)\}^2=12>0\ \therefore$ (1, 1) is a point of minima.

$f(1,\ 1)=1+1+2+2=6$ is the minimum value.

Q 3) Find the extreme values of $f(x,\ y)=x^4+y^4-4a^2xy$, a is constant.

Sol^n. $f_x=4x^3-4a^2y=0 \Rightarrow x^3=a^2y \Rightarrow y=\frac{x^3}{a^2}$

$f_y = 4y^3 - 4a^2x = 0 \;\Rightarrow\; \left(\dfrac{x^3}{a^2}\right) - a^2x = 0 \;\Rightarrow\; x^8 - a^8 = 0 \;\Rightarrow\; x = a, -a,\; y = a, -a$

$f_{xx} = 12x^2,\; f_{xx}(a, a) = 12a^2 > 0,\; f_{xx}(-a, -a) = 12a^2 > 0,\; f_{yy} = 12y^2,\; f_{xy} = -a^2,\; f_{xx}f_{yy} - (f_{xy})^2 > 0$

$\therefore\; f(a, a) = a^4 + a^4 - 4a^4 = -2a^4$

Q 4) As the dimensions of a triangle ABC are varied, show that the maximum value of $\cos A \cos B \cos C$ is obtained when the triangle is equilateral.

Sol^n. $u = \cos A \cos B \cos C, \quad \phi = A + B + C - \pi = 0$, $F = \cos A \cos B \cos C + \lambda(A + B + C - \pi)$

$\dfrac{\partial F}{\partial A} = -\sin A \cos B \cos C + \lambda = 0, \dfrac{\partial F}{\partial B} = -\cos A \sin B \cos C + \lambda = 0,$

$\dfrac{\partial F}{\partial C} = -\cos A \cos B \sin C + \lambda = 0$

$\therefore \sin A \cos B \cos C = \cos A \sin B \cos C = \cos A \cos B \sin C$

Dividing by $\cos A \cos B \cos C$, we get $\tan A = \tan B = \tan C \quad \Rightarrow \quad A = B = C$

$\Rightarrow$ The triangle is equilateral. (0, 0) is a saddle point.

Q 11) Divide 24 into three parts such that the continued product of the first, square of the second and the cube of the third may be maximum.

Sol^n. $x + y + z = 24 \quad \Rightarrow \quad x = 24 - z - y$ $x + y + z = 24 \quad \Rightarrow \quad x = 24 - z - y$

$u = xy^2z^3 = (24 - z - y)y^2z^3 = 24y^2z^3 - y^3z^3 - y^2z^4$

$\dfrac{\partial u}{\partial y} = 48yz^3 - 3y^2z^3 - 2yz^4 = 0$, $\dfrac{\partial u}{\partial z} = 72y^2z^2 - 3y^3z^2 - 4y^2z^3 = 0$

$yz^3(48 - 3y - 2z) = 0$ & $y^2z^2(7z - 3y - 4z) = 0$

$r = u_{yy} = 48z^3 - 6yz^3 - 2z^4,\; s = u_{yz} = 144yz^2 - 9y^2z^2 - 8yz^3$

$t = u_{zz} = 144y^2z - 6y^3z - 12y^2z^3,\; H = rt - s^2 = 0$ at $(0, 0) \Rightarrow \quad y = 0,\; z = 0$

OR $3y + 2z = 48,\; 3y + 4z = 72 \quad \Rightarrow \quad 2z = 24 \quad \Rightarrow \quad z = 12$ and $3y = 48 - 24 = 24 \quad \Rightarrow \quad y = 8$

$\therefore$ Stationary points (0, 0), (8, 12).

$H = (-41472)(-36864) - (-27648)^2 > 0 \quad \Rightarrow \quad r < 0 \quad \Rightarrow \quad$ maxima

Practice Exercise

Q 1) If $z(x + y) = x^2 + y^2$, show that $\left(\dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)^2 = 4\left(1 - \dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)$.

Q 2) $z = u(x, y)e^{ax+by}$ where $u(x, y)$ is such that $\dfrac{\partial^2 u}{\partial x \partial y} = 0$,

if $\dfrac{\partial^2 z}{\partial x \partial y} - \dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y} + z = 0$ then find the constants a and b.

Q 3) If $u(x, t) = Ae^{-gx}\sin(nt - gx)$ satisfies the equation $\dfrac{\partial u}{\partial t} = k^2 \dfrac{\partial^2 u}{\partial x^2}$ where A, g, n and k are constants, show that $n = 2k^2g^2$.

Q 4) If $u = x\log(x + r) - r$ where $r^2 = x^2 + y^2$, find the value of $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$.

[**Ans.** : $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \dfrac{1}{(x + r)}$]

Q 5) If $f(xy^2, z - 2x) = 0$, prove that $x\dfrac{\partial z}{\partial x} - \dfrac{1}{2}y\dfrac{\partial z}{\partial y} = 2x$.

Q 6) If $x = \sqrt{vw},\; y = \sqrt{uw},\; z = \sqrt{uv}$, prove that

Engineering Mathematics (ES1032)

Partial Differentiation

$$x\frac{\partial\Phi}{\partial x}+y\frac{\partial\Phi}{\partial y}+z\frac{\partial\Phi}{\partial z}=u\frac{\partial\Phi}{\partial u}+v\frac{\partial\Phi}{\partial v}+w\frac{\partial\Phi}{\partial w}$$, where Φ is a function of x, y, z.

Q 7) If $x^n+y^n=a^n$ then prove that $\frac{d^2y}{dx^2}=-(n-1)a^n\frac{x^{n-2}}{y^{2n-1}}$.

Q 8) If $x^4+y^4+4a^2xy=0$, prove that $(y^3+a^2x)^3\frac{d^2y}{dx^2}=2a^2xy(x^2y^2+3a^4)$.

Q 9) Discuss the stationary values of $u=x^4+y^4-2x^2+4xy-2y^2$.

Q 10) Find the stationary points of the following functions.

$f(x, y)=y^2+4xy+3x^2+x^3$

Q 11) Find the stationary points of the following functions.

$f(xy)=x^3y^2(1-x-y)$

Q 12) Find the stationary points of the following functions

$f(xy)=x^3+3xy^2-15x^2-15y^2+72x$

[**Ans.** : (i) $\left(\frac{2}{3}, -\frac{4}{3}\right)$, minimum (ii) $\left(\frac{1}{2}, \frac{1}{3}\right)$, maximum (iii) (6, 0), (4, 0)]