Engineering Mathematics ES1032
Eigen Values and Eigen Vectors

Eigen Values and Eigen Vectors

Let A be an $n \times n$ matrix. A scalar (real number) λ is called **eigen value** of A if there is a **non-zero** vector X such that $AX = \lambda X$. The vector X is called an **eigen vector** of A corresponding to λ.

Geometrically, eigen vectors are those non zero vectors which get mapped on to their scalar multiples by matrix A. Now, By definition λ is eigen value then for non-zero vector X, $AX = \lambda X$ which implies $(A - \lambda I)X = 0$. This is a homogeneous system of linear equations in components of X and it will have a non trivial solution if and only if determinant of $A - \lambda I$, $|A - \lambda I| = 0$.

This equation is known as **characteristic equation.** If A is a $n \times n$ matrix then this is n^{th} degree polynomial in λ and will have n roots which are called as **Eigen values** of A. Eigen values are also called as **characteristic values / latent roots / proper values.**

The set of all eigen values of matrix A is known as **Spectrum of** A.

- **Characteristic equation for** 2×2 **matrix.**

  $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ Then $|A - \lambda I| = 0 \Rightarrow \lambda^2 - S_1\lambda + |A| = 0$, where

  $S_1 =$ sum of diagonal elements $=$ trace of $A = a_{11} + a_{22}$, $|A| = \det(A) = a_{11}a_{22} - a_{21}a_{12}$.

If λ_1, λ_2 are the roots of characteristic equation , then $\lambda_1 + \lambda_2 = S_1 =$ trace of A,
$\lambda_1 \cdot \lambda_2 = |A| =$ determinant of A.

- **Characteristic equation for** 3×3 **equation**

  $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Then $|A - \lambda I| = 0 \Rightarrow \lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$, where

  $S_1 =$ sum of diagonal elements=trace of $A = a_{11} + a_{22} + a_{33}$.

  $S_2 =$ sum of minors of diagonal elements of $A = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$.

  $\det(A) = |A| = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{23})$.

If $\lambda_1, \lambda_2, \lambda_3$ are the roots of characteristic equation, then $S_1 = \lambda_1 + \lambda_2 + \lambda_3$,
$S_2 = \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1$ and $\det(A) = |A| = \lambda_1 \lambda_2 \lambda_3$.

Example :

1. Is $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ a eigen vector of $A = \begin{bmatrix} 4 & -2 \\ -3 & 9 \end{bmatrix}$? if so find the corrosponding eigen value.

   Yes, because $\begin{bmatrix} 4 & -2 \\ -3 & 9 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 4+6 \\ -3-27 \end{bmatrix} = \begin{bmatrix} 10 \\ -30 \end{bmatrix} = 10\begin{bmatrix} 1 \\ -3 \end{bmatrix}$. Therefore the corresponding eigen value is 10.

2. Are $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ eigen vectors of $A = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}$ corrosponding to same eigen value? If so find the corrosponding eigen value.

$$\begin{bmatrix}4 & 2 & 3\\ -1 & 1 & -3\\ 2 & 4 & 9\end{bmatrix}\begin{bmatrix}-2\\ 1\\ 0\end{bmatrix}=\begin{bmatrix}-8+2\\ 2+1\\ -4+4\end{bmatrix}=\begin{bmatrix}-6\\ 3\\ 0\end{bmatrix}=3\begin{bmatrix}-2\\ 1\\ 0\end{bmatrix} \text{ and } \begin{bmatrix}4 & 2 & 3\\ -1 & 1 & -3\\ 2 & 4 & 9\end{bmatrix}\begin{bmatrix}-3\\ 0\\ 1\end{bmatrix}=\begin{bmatrix}-12+3\\ 3-3\\ -6+9\end{bmatrix}=\begin{bmatrix}-9\\ 0\\ 3\end{bmatrix}=3\begin{bmatrix}-3\\ 0\\ 1\end{bmatrix}.$$

Therefore $\begin{bmatrix}-2\\ 1\\ 0\end{bmatrix}$ and $\begin{bmatrix}-3\\ 0\\ 1\end{bmatrix}$ are eigen vectors of A corresponding to eigen value 3.

3. Is $\lambda=-2$ a eigen value of $A=\begin{bmatrix}4 & 2 & 3\\ -1 & 1 & -3\\ 2 & 4 & 9\end{bmatrix}$? Why or why not?

No, because $\det(A+2I)\neq 0$ or $\rho(A+2I)=3=$ order of matrix A.

- **Eigen space:** The set of all eigen vectors corresponding to eigen value λ, is a subspace of $\mathbb{R}^n$ called as eigen space of λ.
- **Algebraic Multiplicity:** Let A be a matrix of order n. The number of times the eigen value λ is repeated is known as algebraic multiplicity. e.g. If A has eigen values $\lambda_1=\lambda_2=\lambda_3,\ \underbrace{\lambda_4,\ \lambda_5,\ \ldots\ \lambda_n}_{\text{distinct}}$ then algebraic multiplicity of λ_1 is 3.
- **Geometric Multiplicity: is the number of linearly independent eigen vectors corresponding** to Eigen value λ.
- **Relation between Algebraic Multiplicity and Geometric Multiplicity :** $\boxed{\text{AM}\geq\text{GM}}$.

<u>Example</u> :

<u>1.</u> Find eigen values and eigen vectors of the matrix, $A=\begin{bmatrix}8 & 0 & 3\\ 2 & 2 & 1\\ 2 & 0 & 3\end{bmatrix}$.

Characteristic equation is $|A-\lambda I|=0$ .i.e. $\lambda^{3.}-S_1\lambda^2+S_2\lambda-|A|=0$, where

$S_1=$ sum of diagonal elenents of $A=Trace(A)=13$,

$S_2=$ sum of minors of diagonal elements of $A=\begin{vmatrix}2 & 1\\ 0 & 3\end{vmatrix}+\begin{vmatrix}8 & 3\\ 2 & 3\end{vmatrix}+\begin{vmatrix}8 & 0\\ 2 & 2\end{vmatrix}=6+18+16=40$,

$|A|=\begin{vmatrix}8 & 0 & 3\\ 2 & 2 & 1\\ 2 & 0 & 3\end{vmatrix}=36$. Therefore characteristic of matrix A is $\lambda^3-13\lambda^2+40\lambda-36=0$

$\Rightarrow(\lambda-2)^2(\lambda-9)=0\Rightarrow\lambda=2,\ 2,\ 9.$ Therefore Eigen values of A are 2, 2, 4.

- **Methods of finding Eigen vectors**

Let $X_1=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$ be the eigen vector of A for $\lambda=9$. $[A-9I]X_1=0\ \Rightarrow$

$\begin{bmatrix} -1 & 0 & 3 \\ 2 & -7 & 1 \\ 2 & 0 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$. We observed that 3^{rd} row of $A-9I$ is -3 times 1^{st}. Therefore considering independent equations $-x_1+0x_2+3x_3=0$ and $2x_1-7x_2+x_3=0$, we have

$x_1=3t,\ x_2=t,\ x_3=t$. Thus, solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$. $\therefore\ X_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$.

Note that : Algebraic Multiplicity of $(\lambda=9)=1$ and Geometric multiplicity of $(\lambda=9)=1$.

The eigen space corresponding to eigen value $\lambda=9$ is $E_{\lambda=9} = span\left\{\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}\right\}$ and the basis is

$\left\{\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}\right\}$. Hence dimension of $E_{\lambda=9}$ is **one**.

Eigen vector for $\lambda_2=\lambda_3=2$, $[A-2I]X_1=0$ i.e. $\begin{bmatrix} 6 & 0 & 3 \\ 2 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$, $\begin{bmatrix} 6 & 0 & 3 \\ 2 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

$\therefore \rho[A-2I]=1<3$. Therefore we have only one equation

$2x_1+0x_2+1x_3=0$. Here $\rho[A-2I]=1$ and no. Of unknowns is 3, hence there are 2 free variables.

Let $x_2=s$ & $x_3=t$ $\quad \therefore 2x_1=-t \quad i.e. \quad x_1=\frac{-t}{2}$. Thus the solution is

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{-t}{2} \\ s \\ t \end{bmatrix} = \frac{-t}{2}\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Therefore eigen vectors are $X_2 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ and $X_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Algebraic Multiplicity of $(\lambda=2)=2$ and Geometric multiplicity of $(\lambda=2)=2$.

The eigen space corresponding to eigen value $\lambda=9$ is $E_{\lambda=2} = span\left\{\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$ and the basis is

$\left\{\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$. Hence dimension of $E_{\lambda=2}$ is **two**.

<u>Problem Session</u>

Q.1		**Attempt the following**
	1.	Is $\lambda=-2$ a eigen value of $A=\begin{bmatrix} 1 & 0 & -1 \\ 1 & -3 & 0 \\ 4 & -13 & 1 \end{bmatrix}$? Justify. If so find the corresponding eigen vectors.

	2.	Find eigen value of $\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$ whose eigen vector is $\begin{bmatrix} 4 \\ 2 \\ -5 \end{bmatrix}$.
	3.	Find the eigen values and eigen vectors of $A = \begin{bmatrix} -14 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$. State algebraic and geometric multiplicities of each eigen value. Also find the eigen space of each eigen value and state the dimension of each.
	4.	Find a basis for the eigen space of $A = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$.
	5.	Find all eigen values and corresponding eigen vectors of $\begin{bmatrix} 3 & -4 \\ 2 & -6 \end{bmatrix}$.

<u>Note That</u> :

1) If matrix is singular, one of its eigen value is zero.
2) The eigen values of upper and lower triangular matrices are diagonal elements themselves.
3) Eigen vector is a null space of $A - \lambda I$ for every eigen value λ.
4) Eigen vector cannot be zero.
5) λ is an eigen value of A, if and only if the system of homogeneous equations $(A - \lambda I)X = 0$ has a non-trivial solution. This implies $\det(A - \lambda I)$ has to be zero, i.e., rank of $A - \lambda I$ must be less than n.
6) X is an eigen vector of A, if $AX = \lambda X$, $X \neq 0$.

Consider, A be a matrix of order 3×3. Let $\lambda_1, \lambda_2, \lambda_3$ be its eigen values.

1. If all eigen values are distinct, i.e., $\lambda_1 \neq \lambda_2 \neq \lambda_3$ then there are 3 linearly independent eigen vectors.
2. If one of the eigen values is repeated, say, $\lambda_1 \neq \lambda_2 = \lambda_3$, then

$$\begin{cases} 2 \text{ linearly independent eign vectors if } rank,\ \rho(A - \lambda_1 I) = 1 \\ 1 \text{ linearly independent eigen vectors if } rank,\ \rho(A - \lambda_1 I) = 2 \end{cases}.$$

Further eigen vectors corresponding to distinct eigen values λ_1 and λ_3 are linearly independent.

3. If all the eigen values are repeated, i. e., $\lambda_1 = \lambda_2 = \lambda_3$, then number of linearly independent eigen vectors $= 3 - r$, where $r = \rho[A - \lambda_1 I]$

<u>Note That</u> : Number of linearly independent eigen vectors corresponding to each eigen value is the dimension of the null space of $A - \lambda I$, i. e., dimension of null space of $A - \lambda I$, $\dim \text{Null}(A - \lambda I)$, i.e., nullity of $(A - \lambda I)$.

<u>Properties of eigen values and eigen vectors</u>

If X is an eigen vector of A, corresponding to eigen value λ, then

1. λ^k is eigen value of A^k with same eigen vector X.
2. If all eigen values of A are non-zero the eigen values of A^{-1} are $\frac{1}{\lambda}$.
3. Eigen value of kA is $k\lambda, k \in R$ with same eigen vector X.

4. Eigen value of $Adj\,\mathrm{A}$ is $\frac{|\mathrm{A}|}{\lambda}$.
5. Eigen value of $\mathrm{A}^3+k_1\mathrm{A}^2+k_2\mathrm{A}+k_3I$ is $\lambda^3+k_1\lambda^2+k_2\lambda+k_3$, where k_1, k_2 and k_3 are real numbers.

Example

1. If 3 is eigen value of A then find the eigen value of $\mathrm{A}^2+5\mathrm{A}$.

By above property 1 and 3**, eigen value of** A^2 **is** $3^2=9$ **and eigen value of** $5\mathrm{A}$ **is** $5\times 3=15$**.**
Therefore eigen value of $\mathrm{A}^2+5\mathrm{A}$ is $9+15=24$.

2. For what values of *a*, does the matrix $\begin{bmatrix} 0 & 1 \\ a & 1 \end{bmatrix}$ have the characteristics listed below.

i) A has an eigen value of multiplicity 2.
ii) A has – 1 and – 2 as eigen values.
iii) A has – 1 and 2 as eigen values.
iv) A has real eigen values.

Characteristic equation of A is $\lambda^2-S_1\lambda+|\mathrm{A}|=0$.
For given matrix $S_1=1$, $|\mathrm{A}|=-a$. Therefore equation is $\lambda^2-\lambda-a=0$. This will have repeated roots if $b^2-4ac=0$, i.e., $1+4a=0$. This gives $\boxed{a=\frac{-1}{4}}$.

A has -1 and -2 as eigen values. Therefore roots of characteristic equation are -1 and -2. Now $Trace=$sum of eigen values= -1-2= -3 , but for given matrix trace is one, so this is not possible. Hence there is no real value of a which satisfy given condition.

A has -1 and 2 as eigen values. With these eigen values $Trace=$sum of eigen values= -1+2=1
Now det =product of eigen values= -1×2= -2=-a $\therefore a=2$.

A has real eigen values, implies discriminant of $\lambda^2-S_1\lambda+|\mathrm{A}|=0$ must be positive.
Thus $1+4a\ge 0 \Rightarrow \boxed{a\ge \frac{-1}{4}}$

3. Find the eigen values of $\mathrm{A}=\begin{bmatrix} 3 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$. State geometric and algebraic multiplicities of each eigen value. Is A inverible? If so, find eigen values of A^{-1}.

A is a upper triangular matrix, therefore diagonal elements are eigen values. Therefore eigen values of A are $3,1\&3$. Algebraic multiplicity of eigen value 1 is One as it appears only once, while
3 appears twice, so algebraic multiplicity of eigen value 3 is Two.
As AM of $\lambda=1$ is One, there will be only one eigen vector, hence geometric multiplicity is also One.
Now GM of $\lambda=3$ =dimension of kernel of A-3I, so we simply check rank of A-3I.

$$\mathrm{A\text{-}3I}=\begin{bmatrix} 0 & 2 & 4 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 2 & 4 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$\rho(\mathrm{A\text{-}3I})=2<3.$ Therefore there will be only one eigen vector. Therefore geometric multiplicity of $\lambda=3$ is also One.

As all eigen values of A are non-zero, A is invertible. Eigen values of A^{-1} are $\frac{1}{3},1\&\frac{1}{3}$.

In particular, if A is a **symmetric matrix,** of order *n* then it has *n* linearly independent eigen vectors.

<u>Example</u>

1. Find the eigen values and eigen vectors of $A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$.

Note that given matrix is symmetric.

Characteristic equation is $|A - \lambda I| = 0 \Rightarrow \lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$

$S_1 = 6,\ S_2 = -4 - 7 - 7 = -15,\ |A| = 3(-4) - 2(-2) + 4(4) = -12 + 4 + 16 = 8$

Characteristic equation is $\lambda^3 - 6\lambda^2 - 15\lambda - 8 = 0$. $\therefore\ \lambda = 8, -1, -1$.

Consider eigen vector for $\lambda = 8$, $[A - 8I]X_1 = 0$ $\begin{bmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\begin{bmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} x_1 - 4x_2 + x_3 = 0 \\ -2x_2 + x_3 = 0 \end{matrix}$. The solution is $x_3 = 2x_2, x_1 = 2x_2$, i.e.,

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2t \\ t \\ 2t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} t, t \neq 0 \in R$. Therefore eigen vector for $\lambda = 8$ is $X_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

Note : AM *and* GM of $\lambda = 8 = 1$.

Consider eigen vector for $\lambda = -1\ [A + I]\ X = 0 \Rightarrow \begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$.

$\begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow 2x_1 + x_2 + 2x_3 = 0$. $\rho[A+I] = 1 < 3$. Therefore there are two linearly independent vectors. The solution is $x_2 = -2x_1 - 2x_3, x_1 = t, x_3 = s$, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -2t - 2s \\ s \end{bmatrix}$, i.e.,

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} s, t, s \neq 0 \in R$. Thus the two linearly independent eigen vectors are

$X_2 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, X_3 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$. Note AM and GM of $\lambda = -1$ are 2.

2. Find the Eigen values and Eigen vectors of $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$.

Charactristic equation of A is $\lambda^3 - s_1\lambda^2 + s_2\lambda - |A| = 0$. $S_1 = 6,\ S_2 = 9,\ |A| = 0$.

Eigen values are $\lambda_1 = 0,\ \lambda_2 = \lambda_3 = 3.$ Eigen vector for $\lambda_1 = 0$ is $X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$

Eigen vectors for repeated eigen values $\lambda_2 = \lambda_3 = 3$ are $X_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $X_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$

<u>Problem Session :</u>

Q.1		Attempt the following
	1)	Is $\lambda = -2$ eigen value of $\begin{bmatrix} 1 & 0 & -1 \\ 1 & -3 & 0 \\ 4 & -13 & 1 \end{bmatrix}$? If so find an eigen vector.
	2)	Find the values of a, b and c such that the chractistic polynomial of $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{bmatrix}$ is $-\lambda^3 + 4\lambda^2 + 5\lambda + 6.$
	3)	Find the values of a and b if eigen values of $A = \begin{bmatrix} 0 & 1 \\ a & b \end{bmatrix}$ are -4 and $7.$
	4)	Find orthogonal eigen vectors of $A = \begin{bmatrix} -2 & 5 & 4 \\ 5 & 7 & 5 \\ 4 & 5 & -2 \end{bmatrix}$.
	5)	Find orthogonal eigen vectors of $A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$.

- **Diagonalization of a Matrix**
  In this section, we are going to find the condition on square matrices so that they are similar to diagonal matrices.
- **Similarity of Matrices**
  A matrix A is said to be similar to B if there exists a non-singular matrix P such that A is expressible as $\mathrm{A} = P^{-1}\mathrm{B}P$. **Notation is** $\mathrm{A} \approx \mathrm{B}$
- **Note That :** If 2 matrices are similar, then their Eigen values are same.
- A matrix A is said to be **diagonalizable** if A is similar to a diagonal matrix.
- **Diagonalization** is the process of finding the non-singular matrix P and the diagonal matrix D such that $\mathrm{D} = P^{-1}\mathrm{A}P$.
- **Method of diagonalization :**

Let A be a matrix of order 3×3. Let $X_1,\ X_2,\ X_3$ be linearly independent eigen vectors corresponding to eigen values $\lambda_1,\ \lambda_2,\ \lambda_3$, i.e., $\mathrm{A}X_1 = \lambda_1 X_1, \mathrm{A}X_2 = \lambda_2 X_2, \mathrm{A}X_3 = \lambda_3 X_3$.

Let $P = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix}$, consist of columns as linearly independent eigen vectors $X_1,\ X_2,\ X_3$. P is invertible as eigen vectors are linearly independent. Find P^{-1}. Then

Engineering Mathematics ES1032
Eigen Values and Eigen Vectors

$P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$. Matrix P which diagonalizes A is called as **Modal** matrix and D the diagonal matrix containing eigen values of A as diagonal elements, is called as **Spectral** matrix.

Results :

1. If a matrix A has distinct eigen values, then it is always diagonalizable.
2. A matrix A of order n if and only if it has n linearly independent eigen vectors.
3. A matrix is diagonalizable if and only if algebraic multiplicity (AM) of each eigen value equals its geometric multiplicity(GM).
4. A matrix is diagonalizable if its eigen vectors form basis of R^n.
5. If A is symmetric matrix then eigen vectors corresponding to distinct eigen values are always orthogonal. Further if eigen values are repeated then we can find orthogonal eigen vectors. Hence every symmetric matrix is orthogonally diagonalizable.
6. Note that the sequence of eigen vectors selected to construct P, the eigen values also appear with same sequence as diagonal elements in D. This means, X_1, X_2, X_3 are eigen vectors corresponding to $\lambda_1, \lambda_2, \lambda_3$ respectively.

   If $P=[X_1 \quad X_2 \quad X_3]$ then $D=\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$. If the choice of P is $P=[X_3 \quad X_1 \quad X_2]$

   then $D=\begin{bmatrix} \lambda_3 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$.

Example

1. Consider $A = \begin{bmatrix} 5 & 6 \\ 3 & -2 \end{bmatrix}$. Is A diagonalizable? If yes, find the model and spectral matrices.

Characteristic equation of A is $\lambda^2 - 3\lambda - 28 = 0$. Eigen values of A are $\lambda = 7, -4$.

Eigen vector corresponding to $\lambda = 7$ is $X_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\lambda = -4$ is $X_2 = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$.

As algebraic multiplicity equal to the geometric multiplicity of both the eigen values, A is diagonalizable.

The model matrix $P = \begin{bmatrix} 3 & -2 \\ 1 & 3 \end{bmatrix}$. A has distinct eigen values, hence eigen vectors are linearly independent. $|P| \neq 0$, therefore P is invertible and $P^{-1} = \frac{1}{11}\begin{bmatrix} 3 & 2 \\ -1 & 3 \end{bmatrix}$.

$$P^{-1}AP = \frac{1}{11}\begin{bmatrix} 3 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 1 & 3 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 77 & 0 \\ 0 & -44 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & -4 \end{bmatrix}.$$

Thus the resulting matrix is a diagonal matrix with diagonal entries as eigen values of A.

Thus spectral matrix $D = \begin{bmatrix} 7 & 0 \\ 0 & -4 \end{bmatrix}$.

2. Is $\begin{bmatrix} -14 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ diagonalizable?

Characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$. $S_1 = -10$, $S_2 = 4 + (-28) + (-28) = -52$,

$|A| = -14(4) - 1(0) = -56 \;\therefore \lambda^3 + 10\lambda^2 - 52\lambda + 56 = 0 \Rightarrow \lambda = -14, 2, 2$

Algebraic multiplicity of $\lambda = -14$ is **One**, hence geometric multiplicity of $\lambda = -14$ is also **One**. Algebraic multiplicity of $\lambda = 2$ is **TWO**. We now check geometric multiplicity of $\lambda = 2$. Consider the matrix $A - \lambda I$.

$$A - 2I = \begin{bmatrix} -16 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \therefore \rho(A - 2I) = 2.$$ Therefore there is only one eigen vector for $\lambda = 2$. Hence geometric multiplicity of $\lambda = 2$ is **ONE**. Thus, AM of $\lambda = 2 \neq$ GM of $\lambda = 2$. Therefore given matrix is not diagonalizable.

Application of Diagonalization (powers of matrices)

To find higher powers of given matrix.

If A is diagonalizable with P as modal matrix and D as spectral matrix then

$$A^n = (PDP^{-1})(PDP^{-1})\cdots(PDP^{-1}) = (PD)(P^{-1}P)(DP^{-1})\cdots(P^{-1}P)(DP^{-1})$$

$$= PD^nP^{-1} \quad \boxed{\therefore A^n = PD^nP^{-1}}$$

Problem Session

Q. 1		**Attempt the following**
	1)	Is $A = \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}$ Diagonalizable, if yes diagonalize A.
	2)	Orthogonally diagonalize the matrix $A = \begin{bmatrix} 2 & -2 \\ -2 & -1 \end{bmatrix}$.
	3)	Diagaonalize $A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
	4)	Let $A_{2\times 2}$ be a symmetric matrix having Eigen values 1, 1 and one of its Eigen vector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Find A.
	5)	Which of the matrices cannot be diagonalized? a) $\begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix}$ b) $\begin{bmatrix} 3 & -3 \\ 3 & -3 \end{bmatrix}$ c) $\begin{bmatrix} 3 & 0 \\ 3 & -3 \end{bmatrix}$ d) $\begin{bmatrix} 3 & 0 \\ 3 & 3 \end{bmatrix}$
	6)	For the matrix $A = \begin{bmatrix} 3 & a & 0 & 0 \\ 0 & 3 & b & 0 \\ 0 & 0 & 3 & c \\ 0 & 0 & 0 & 3 \end{bmatrix}$ i) Find eigen values of A. ii) Find the condition on a, b, and c such that A is diagonalizable. iii) Under what condition does the eigen space of $\lambda = 3$ have dimensions 1? 2? 3?