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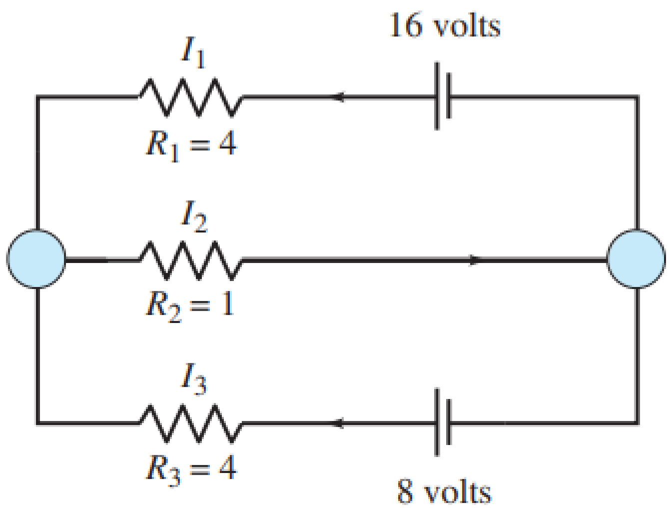
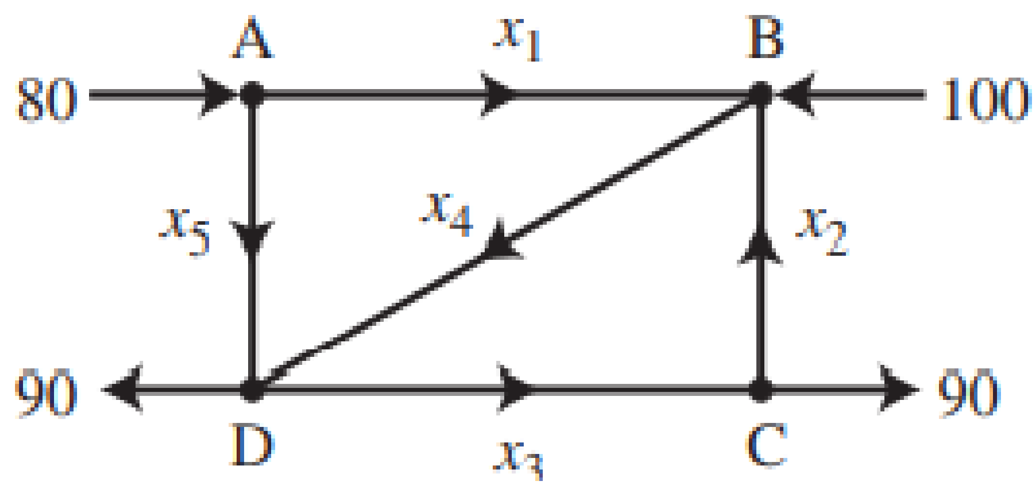
9960731502

Wassily Liontief ,
Harvard Professor

1973 Nobel Prize in
Economic Science



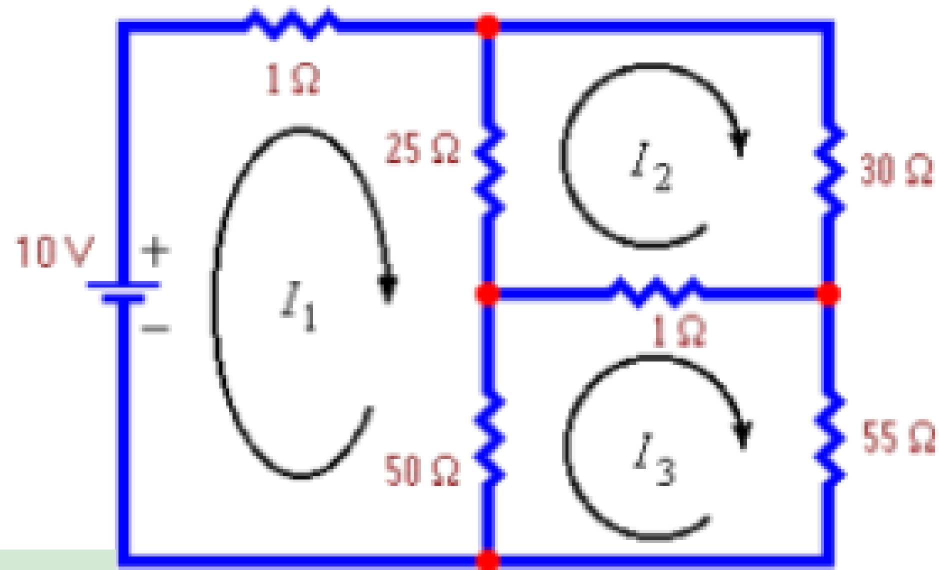


$$(x_1)C_3H_8 + (x_2)O_2 \rightarrow (x_3)CO_2 + (x_4)H_2O$$

Uniform motion problems

	Distance =	Rate ×	Time
Travel by car	$60x$	60 mph	x
Travel by air	$350y$	350 mph	y
Total	1,930 mi		8 hours

$$60x + 350y = 1,930 \qquad x + y = 8$$



Stress analysis

Renting a car

	Jane	Charlie	Mary	Fred
J	0	1	1	1
C	1	0	1	0
M	1	1	0	1
F	0	1	1	0
Total	2	3	3	2

Application to Economics

Polynomial Interpolation

Mixture problems

Spring Mass Systems

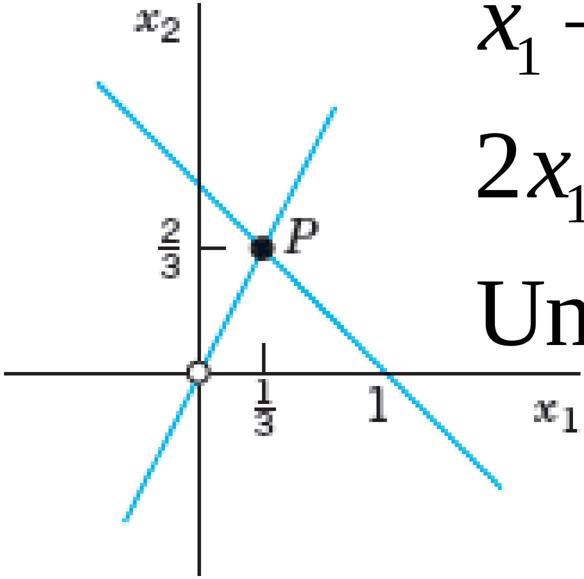
Aim

To Solve

System of Linear Equations

A **Linear Equation** can be in 2 dimensions (such as **x** and **y**)

$$ax + by = c$$



$$x_1 + x_2 = 1$$

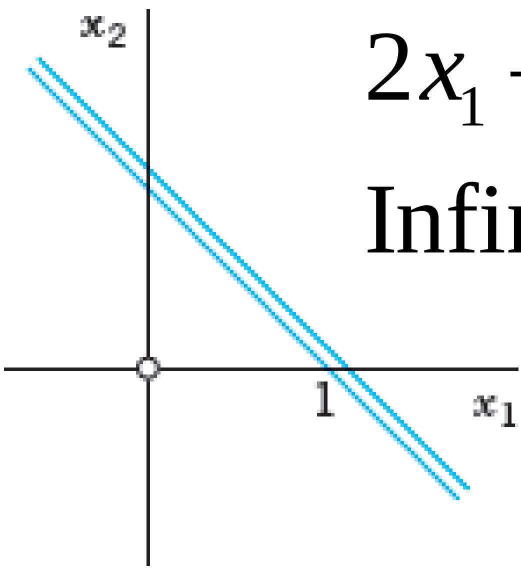
$$2x_1 - x_2 = 0$$

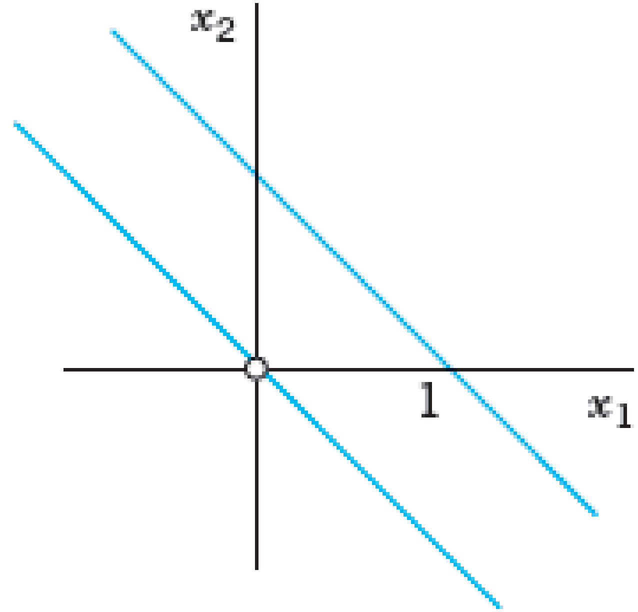
Unique soln

$$x_1 + x_2 = 1$$

$$2x_1 + 2x_2 = 2$$

Infinite soln

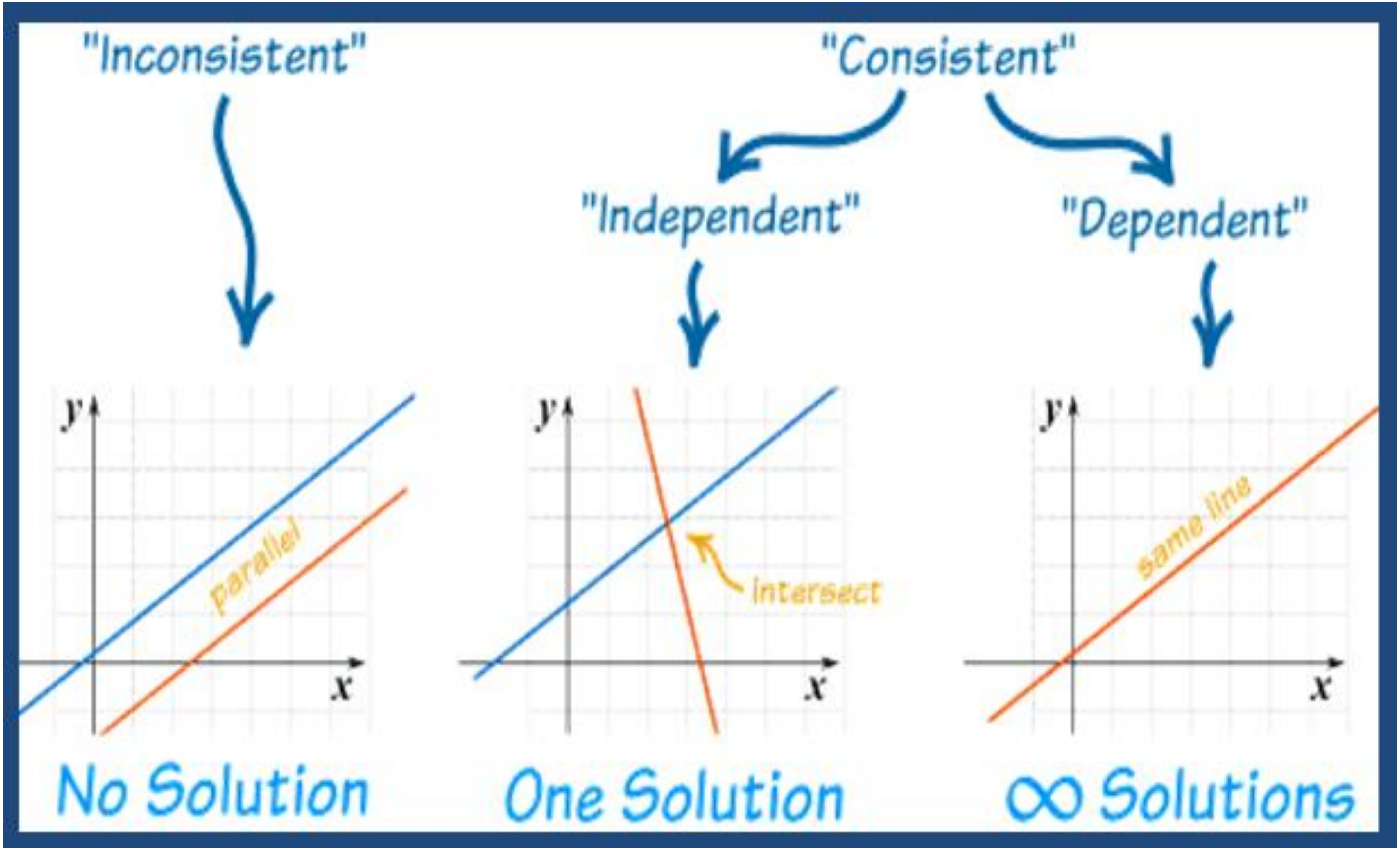




$$x_1 + x_2 = 1$$

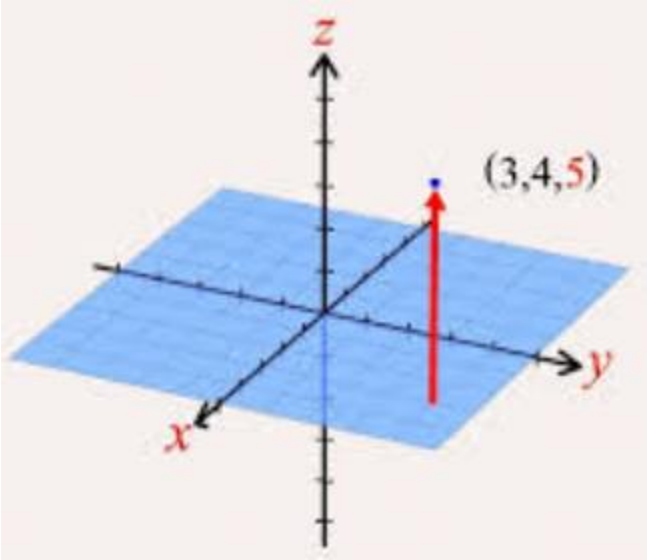
$$x_1 + x_2 = 0$$

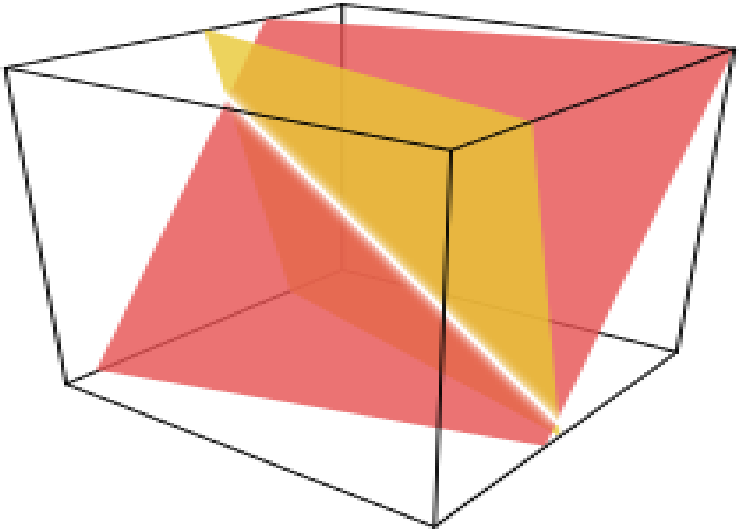
No soln


"Inconsistent"
"Consistent"
"Independent"
"Dependent"
y
x
parallel
intersect
same line
No Solution
One Solution
∞ Solutions

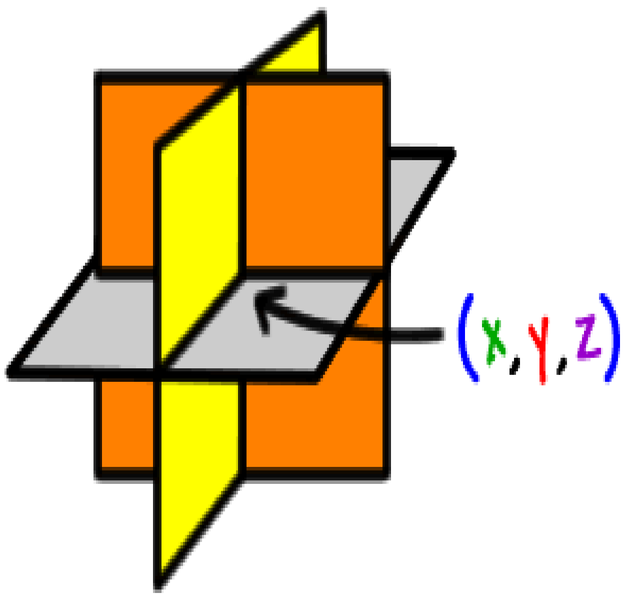
A **Linear Equation** can be in 3 dimensions (such as **x, y** and **z**)

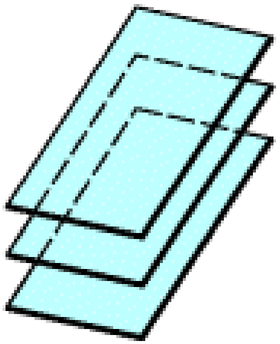
$$ax + by + cz = d$$



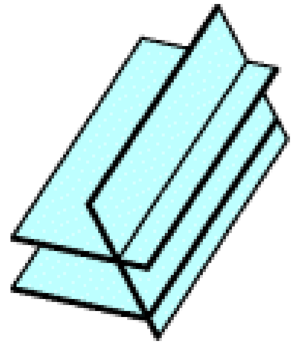


A linear system in three variables determines a collection of planes. The intersection point is the solution

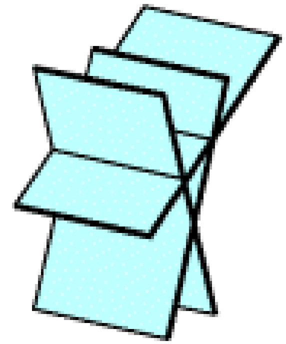

(x,y,z)



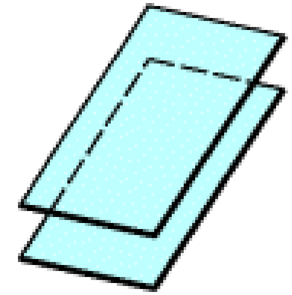
No solutions
(three parallel planes;
no common intersection)



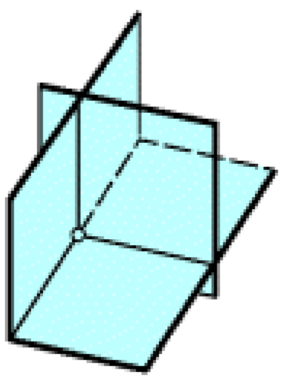
No solutions
(two parallel planes;
no common intersection)



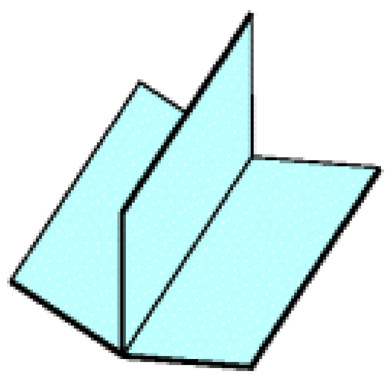
No solutions
(no common intersection)



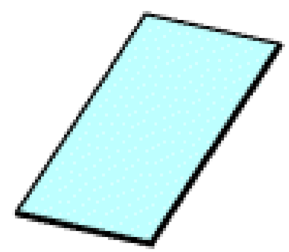
No solutions
(two coincident planes
parallel to the third;
no common intersection)



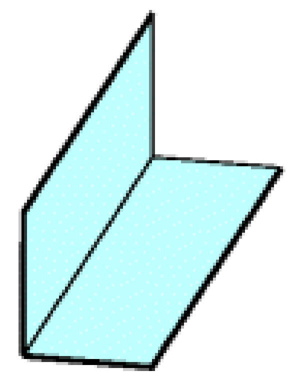
One solution
(intersection is a point)



Infinitely many solutions
(intersection is a line)

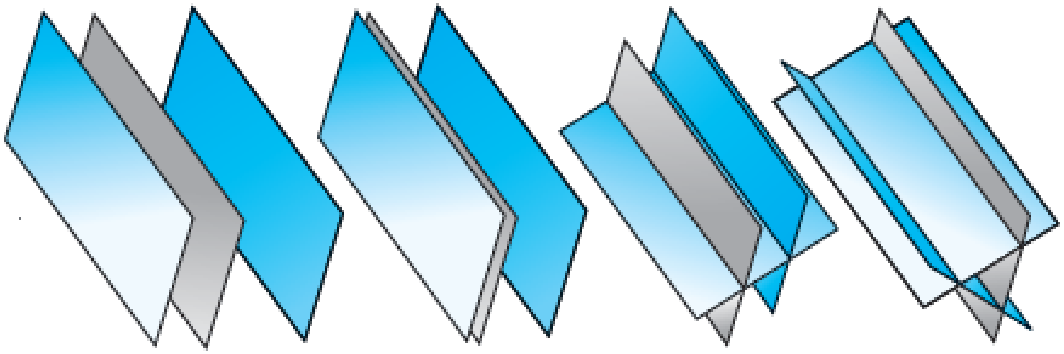


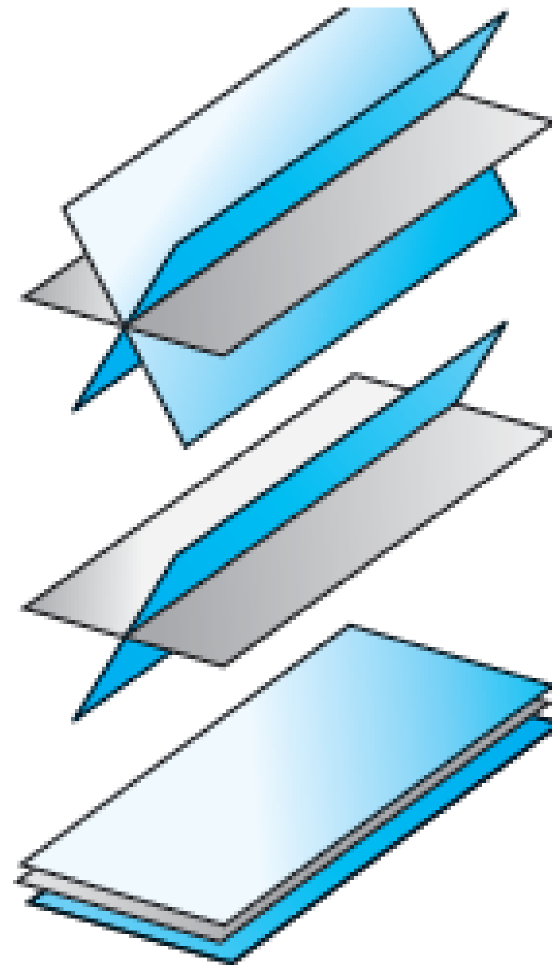
Infinitely many solutions
(planes are all coincident;
intersection is a plane)



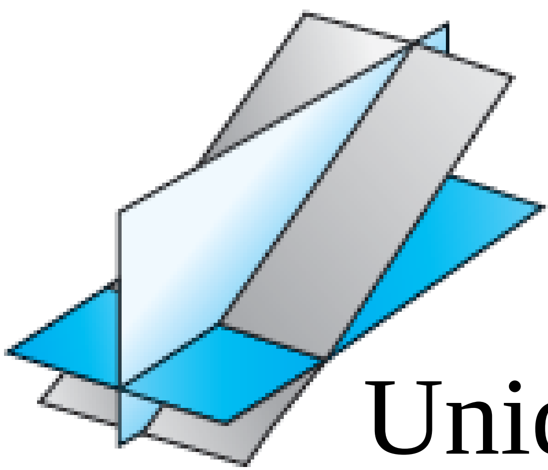
Infinitely many solutions
(two coincident planes;
intersection is a line)

No solution





Infinite soln



Unique soln

System of Linear Equations

Consider a system of m equations and n unknowns.

$$a_{11}x_1 + a_{12}x_2 + \dots\dots\dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots\dots\dots + a_{2n}x_n = b_2$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots\dots\dots + a_{mn}x_n = b_m$$

The above system of equation can be written in the matrix form as AX=B

$$A = [a_{ij}]_{m \times n}$$

$$= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n} \rightarrow \text{coefficient matrix}$$

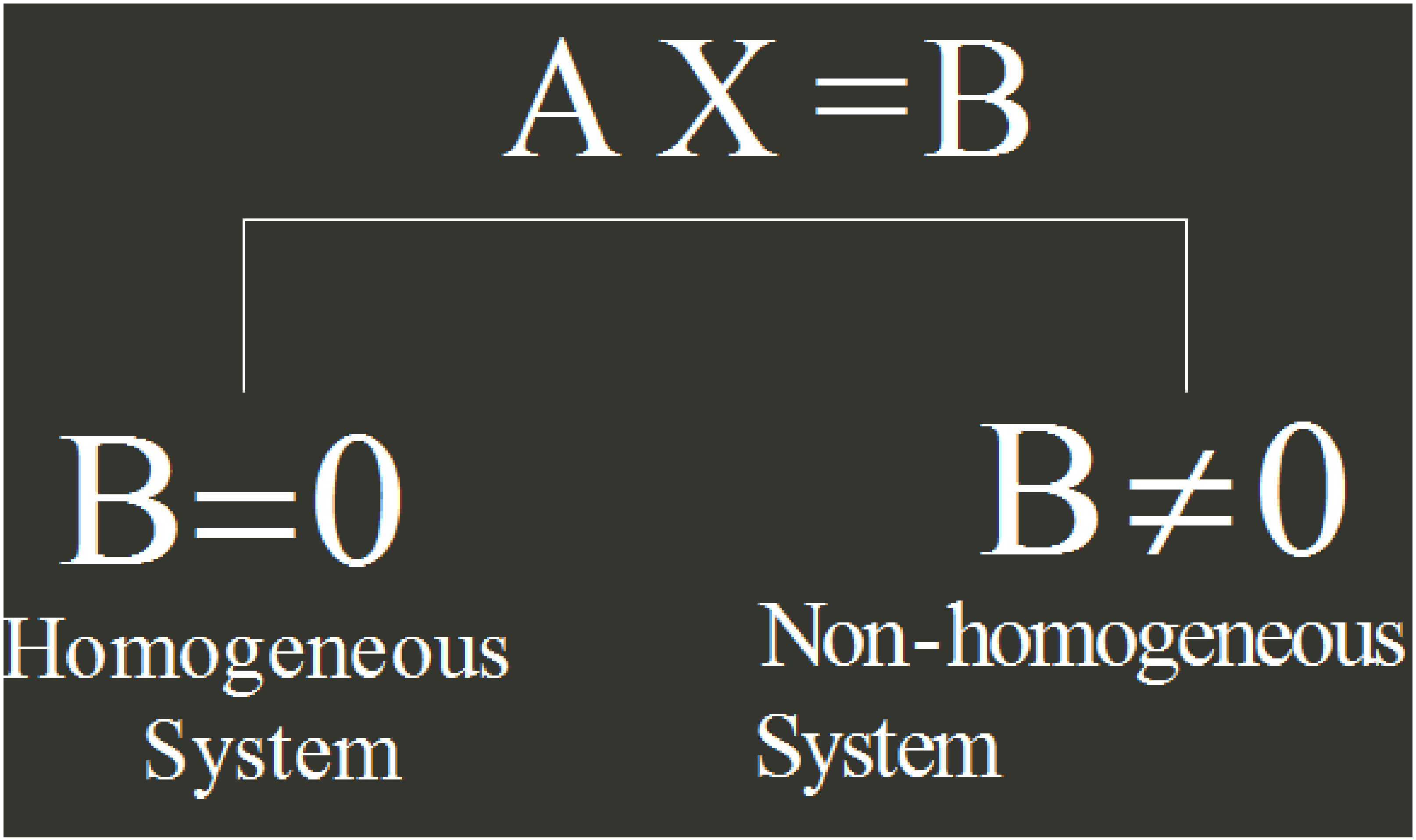
$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \rightarrow \text{Matrix of unknowns} \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m \times 1} \rightarrow \text{Matrix of constants}$$

A solution of the system is a list $\left(s_1, s_2, \cdots, s_n\right)$ of numbers that satisfies each and every equation when the values $s_1, s_2, \cdots, s_n$ are substituted for $x_1, x_2, \cdots, x_n$.

The set of all possible solutions is called the solution set of the linear system.

Two linear systems are called equivalent if they have the same solution set.

Classification



Homogeneous System

Always Consistent

$\rho(A) =$ number of unknows

UNIQUE SOLUTION

$X = 0$

Trivial Solution

Less than

$\rho(A) \leq$ number of unknows

An INFINITE number of solutions

Non -Trivial Solution

Augmented Matrix: denoted as $(\mathrm{A} \mid \mathrm{B})$ is the matrix obtained by attaching additional column B to matrix A. $\left[A : B\right] = [a_{ij} : b_i \;]_{m \times (n+1)}$

Non- homogeneous System

- Not always consistent

$$\rho(A \mid B) = \rho(A)$$

Consistent

System

$$\rho(A \mid B) \neq \rho(A)$$

Inconsistent

System

Consistent non-homogeneous system

* $\rho(A\ B) = \rho(A) =$ the number of unknows

System possess a UNIQUE solution

* $\rho(A\ B) = \rho(A) <$ the number of unknows

System has an INFINITE number of solutions

Inconsistent Non-homogeneous system

$$\rho(A\ B) \neq \rho(A).$$

System has ***NO*** *solution.*

The behavior of a linear system is determined by the relationship between the number of equations (m) and the number of unknowns (n):

- A system with fewer equations than unknowns has no solution then the system is known as a **underdetermined system**.
- Usually, a system with the same number of equations and unknowns has a single unique solution.
- Usually, a system with more equations than unknowns has no solution. Such a system is also known as an **overdetermined** system.

Summary

$\rho[A : B] = \rho[A] = r$ (say) $\Rightarrow$ the system is always consistent

If $\rho[A : B] = \rho[A] = r = n$ (number of unknowns)

$\Rightarrow$ the system has a unique solution.

In case of homogeneous system, <u>unique</u> solution is a **<u>Trivial</u> solution**.

$r < n \Rightarrow$ the system has infinite number of solutions.

$r < n \Rightarrow n - r$ variables are "free variables."

Non-pivot variable is considered as a free variable

If x_p is a particular solution of the nonhomogeneous system AX=B then every solution of this system can be written in the form $x = x_p + x_h$ where x_h is a solution of the corresponding homogeneous system AX= 0.

Observations :

1. If $AX = 0$, is such that A is $n \times n$ matrix, *i.e.*, we have $n-$equations in $n-$unknowns, then system will have

$a)$ nontrivial solution also iff $|A| = \det(A) = 0$.

$b)$ Trivial solution only iff $|A| = \det(A) \neq 0$.

2. For $A_{m \times n} X = 0$, and if $m < n$, then always have non-trivial solution also and there are at least $n - m$ variables are free.

Observations :

1. If $AX = 0$, is such that A is $n \times n$ matrix, *i.e.*, we have $n-$equations in $n-$unknowns, then system will have

   a) nontrivial solution also iff $|A| = \det(A) = 0$.

   b) Trivial solution only iff $|A| = \det(A) \neq 0$.

2. For $A_{m \times n}X = 0$, and if $m < n$, then always have non-trivial solution also (in this case at least $n - m$ variables are free).

3. If $AX = B$, $B \neq 0$ is such that A is $n \times n$ matrix, *i.e.*, we have $n-$equations in $n-$unknowns, then system will have unique solution iff $|A| = \det(A) \neq 0$ and the unique solution is $X = A^{-1}B$.

Rank of a Matrix

Let $A_{m \times n}$ be any matrix, $r \geq 0$ is said to be rank of the matrix A denoted as $\rho(A)$ if the matrix has

1. at least one non-zero minor of order 'r'
2. all the minors of order 'r+1' are zero

OR

The order of highest order non-vanishing minor is the rank of the matrix.

OR

The order of highest order non-zero determinant present in in the matrix is the rank of the matrix.

Theorem: Elementary transformations do not alter the rank of the matrix.

Echelon Form (Row reduced form)

(To reduce a matrix into an Echelon form, only row transformations are permitted.)

1. The first non-zero element in each row, called the **leading entry**, is 1.
2. All elements below the leading element are zero.
3. Each leading entry is in a column to the right of the leading entry in the previous row.
4. Rows with all zero elements, if any, are below rows having a non-zero element.

pivot elements

$$\begin{bmatrix} 1 & 2 & 3 & 5 & -7 \\ 0 & 0 & 1 & -2 & 5 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{6 \times 5}$$

Leading entry – First non-zero element in a row called as leading entry or **pivot element**.

Pivot elements are the elements whose position should not be altered by application of row transformations.

If matrix A is in Echelon form then, rank of A,

$\rho(A)$= number of non-zero rows in Echelon form.

Equivalent Matrices : - A matrix B is said to be an equivalent matrix of matrix A, shown as $B \sim A$ if and only if B is obtained by applying elementary transformations on A.

Example 1. $A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & 2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$

$$R_1 : R_1 + R_4 \Rightarrow \quad A \sim \begin{bmatrix} 1 & 1 & -1 & -5 & 2 \\ -1 & 2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

$$R_2 + R_1,\ R_3 + 2R_1,\ R_4 - R_1 \Rightarrow \sim \begin{bmatrix} 1 & 1 & -1 & -5 & 2 \\ 0 & 3 & -2 & -2 & 3 \\ 0 & -1 & -2 & -7 & 3 \\ 0 & 3 & 6 & -4 & -9 \end{bmatrix}$$

$$R_{23} \Rightarrow \qquad \sim \begin{bmatrix} 1 & 1 & -1 & -5 & 2 \\ 0 & -1 & -2 & -7 & 3 \\ 0 & 3 & -2 & -2 & 3 \\ 0 & 3 & 6 & -4 & -9 \end{bmatrix}$$

$$-R_2 \Rightarrow \quad \sim \begin{bmatrix} 1 & 1 & -1 & -5 & 2 \\ 0 & 1 & 2 & 7 & -3 \\ 0 & 3 & -2 & -2 & 3 \\ 0 & 3 & 6 & -4 & -9 \end{bmatrix}$$

$$R_3 - 3R_2,\ R_4 - 3R_2 \Rightarrow \quad \sim \begin{bmatrix} 1 & 1 & -1 & -5 & 2 \\ 0 & 1 & 2 & 7 & -3 \\ 0 & 0 & -8 & -23 & 12 \\ 0 & 0 & 0 & -25 & 0 \end{bmatrix}$$

$$
-\frac{1}{8}R_3 \sim \begin{bmatrix} 1 & 1 & -1 & -5 & 2 \\ 0 & 1 & 2 & 7 & -3 \\ 0 & 0 & 1 & 23/8 & -12/8 \\ 0 & 0 & 0 & -25 & 0 \end{bmatrix}
$$

$$
-\frac{1}{25}R_4 \sim \begin{bmatrix} 1 & 1 & -1 & -5 & 2 \\ 0 & 1 & 2 & 7 & -3 \\ 0 & 0 & 1 & 23/8 & -12/8 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}
$$

Consider, $A=\begin{bmatrix} 1 & -1 & 3 & 0 \\ 2 & 3 & 1 & 5 \\ 2 & -2 & h & 0 \end{bmatrix}$. What is the value of h for which rank of A, i) $\rho(A)>2$ ii) $\rho(A)<3$

iii) Is there any real value of h, for which $\rho(A)=1$?

$$A=\begin{bmatrix} 1 & -1 & 3 & 0 \\ 2 & 3 & 1 & 5 \\ 2 & -2 & h & 0 \end{bmatrix} \xrightarrow{R_2-2R_1,\, R_3-2R_1} \begin{bmatrix} 1 & -1 & 3 & 0 \\ 0 & 5 & -5 & 5 \\ 0 & 0 & h-6 & 0 \end{bmatrix}$$

i) If $h - 6 \neq 0 \Rightarrow$ No. of non zero rows in Echelon form =3

Hence ρ(A)=3.

ii) If $h - 6 = 0 \Rightarrow$ No. of non zero rows in Echelon form =2

Hence ρ(A)=2.

iii) There is no real value of h, for which ρ(A)=1.

$$A = \begin{bmatrix} 3 & 2 & 5 & 7 & 12 \\ 1 & 1 & 2 & 3 & 5 \\ 3 & 3 & 6 & 9 & 15 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2 \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 3 & 5 \\ 3 & 2 & 5 & 7 & 12 \\ 3 & 3 & 6 & 9 & 15 \end{bmatrix} R_2 - 3R_1,\ R_3 - 3R_1 \Rightarrow$$

$$\begin{bmatrix} 1 & 1 & 2 & 3 & 5 \\ 0 & -1 & -1 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \therefore\ \rho(A) = 2$$