ENGINEERING MATHEMATICS

ES1032

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Eigen Values and Eigen Vectors

$$A = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$$

Find the eigen values and eigen vectors of $A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$.

Note that given matrix is symmetric.

Char. eqn $\quad \lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$

$S_1 = 6,\ S_2 = -4 - 7 - 4 = -15,$

$|A| = 3(-4) - 2(-2) + 4(4) = -12 + 4 + 16 = 8$

$\lambda^3 - 6\lambda^2 - 15\lambda - 8 = 0$

Eigen values are $\lambda = 8, -1, -1$

To find eigen vector for $\lambda = 8$,

$$[A - 8I]X_1 = 0 \Rightarrow \begin{bmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{c} x_1 - 4x_2 + x_3 = 0 \\ -2x_2 + x_3 = 0 \end{array}$$

The solution is $x_3 = 2x_2, x_1 = 2x_2$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ t/2 \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}\frac{t}{2}, t \neq 0 \in \mathbb{R} \therefore X_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

AM *and* GM of $\lambda = 8$ is 1

Consider $\lambda = -1$

$$[A + I]\ X = 0 \Rightarrow \begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow 2x_1 + x_2 + 2x_3 = 0. \text{Let } x_2 = t, x_3 = s$$

$$\therefore x_1 = \frac{-x_2 - 2x_3}{2}, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \dfrac{-t-2s}{2} \\ t \\ s \end{bmatrix} = \left(-\frac{t}{2}\right) \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$where\ t, s \neq 0 \in \mathbb{R}$. $X_2 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, X_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. AM and GM of $\lambda = -1$ are 2

Note That $X_1 \perp X_2,\ X_1 \perp X_3$ but $X_2 \not\perp X_3$

To find $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that v is orthogonal to

X_1 & X_2 *OR* X_1 & X_3.

$$\langle X_1, v \rangle = 0 \ \& \ \langle X_2, v \rangle = 0 \Rightarrow 2x + y + 2z = 0 \ \& \ x - 2y = 0$$

$$v = \begin{bmatrix} 4 \\ 2 \\ -5 \end{bmatrix} . \therefore \left\{ X_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, X_2 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, v = \begin{bmatrix} 4 \\ 2 \\ -5 \end{bmatrix} \right\}$$

is a set of orthogonal eigen vectors.

If the choice is $\langle X_1, v \rangle = 0$ & $\langle X_3, v \rangle = 0$

then $v = \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix}$. Therefore $\left\{ X_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, X_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, v = \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} \right\}$

is a set of orthogonal eigen vectors.

Apply Gram-Schmidt orthogonalization process to

$$\left\{ X_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, X_2 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, X_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Find the Eigen values and Eigen vectors of $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$.

Eigen values are $\lambda_1 = 0,\ \lambda_2 = \lambda_3 = 3.$

Eigen vector for $\lambda_1 = 0,\ X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Eigen vectors for repeated eigen values $\lambda_2 = \lambda_3 = 3$

$X_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $X_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Char eqn $\lambda^3 - s_1\lambda^2 + s_2\lambda - |\mathrm{A}| = 0,\ \ S_1 = 6,\ S_2 = 9,\ \ |\mathrm{A}| = 0$

To obtained the orthgognal basis for $\mathbb{R}^3$, by Gram Schmidt process

Observe that $\langle X_1, X_2 \rangle = 0$ and $\langle X_1, X_3 \rangle = 0$ but $\langle X_2, X_2 \rangle \neq 0$.

$$v_1 = X_1, v_2 = X_2 - \frac{\langle v_1, \ X_2 \rangle}{\langle v_1, \ v_1 \rangle} v_1 = X_2 \ as \ \langle v_1, \ X_2 \rangle = \langle X_1, \ \ X_2 \rangle = 0.$$

$$v_3 = X_3 - \frac{\langle v_1, \ X_3 \rangle}{\langle v_1, \ v_1 \rangle} v_1 - \frac{\langle v_2, \ X_3 \rangle}{\langle v_2, \ v_2 \rangle} v_2, \langle v_1, \ X_3 \rangle = 0, \ \langle v_2, \ X_3 \rangle = 1, \langle v_2, \ v_2 \rangle = 2.$$

$$\therefore v_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} \approx \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

$\therefore$ Orthoganal eigen vectors are $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$..

Consider, $A = \begin{bmatrix} 5 & 6 \\ 3 & -2 \end{bmatrix}$

$Char.eqn\ \ is \quad \lambda^2 - 3\lambda - 28 = 0.$

Eigen values are $\lambda = 7, -4.$

Eigen vectors corresponding to $\lambda = 7$ and $\lambda = -4$ are $X_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ are $X_2 = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ respectively.

Let $\quad \mathrm{P} = \begin{bmatrix} 3 & -2 \\ 1 & 3 \end{bmatrix}$, $P^{-1} = \dfrac{1}{11}\begin{bmatrix} 3 & 2 \\ -1 & 3 \end{bmatrix}$.

$$P^{-1}\mathrm{AP} = \frac{1}{11}\begin{bmatrix} 3 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 1 & 3 \end{bmatrix}$$

$$= \frac{1}{11}\begin{bmatrix} 77 & 0 \\ 0 & -44 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & -4 \end{bmatrix}.$$

$$A = \begin{bmatrix} 2 & -1 & 5 \\ 0 & 2 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$

DIAGONALIZATION

Similarity of Matrices

A matrix A is said to be similar to B if there exists a non-singular matrix P such that A is expressible as $A = P^{-1}BP$.

Notation is $A \approx B$

A matrix A is said to be **diagonalizable** if A is similar to a diagonal matrix.

A matrix A is said to be diagonalizable if A is similar to a diagonal matrix. Diagonalization is the process of finding the non singular matrix P and the diagonal matrix D, such that $\mathrm{D} = P^{-1}\mathrm{A}P$.

Method of diagonalization

Let A be a matrix of order 3×3. Let X_1, X_2, X_3 be linearly independent eigen vectors corresponding to eigen values $\lambda_1, \lambda_2, \lambda_3$.

Let $P = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix}$ consist of columns as linearly independent eigen vectors X_1, X_2, X_3.

P is invertible as eigen vectors are linearly independent. Find P^{-1}.

$$P^{-1}\mathrm{A}P = D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

Matrix $P = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix}$ consist of columns as linearly independent eigen vectors is called as **Modal** matrix.

The diagonal matrix $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

containing eigen values is called as **Spectral** matrix.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\begin{cases} 3 \text{ linearly independent eign vectors if}, & \rho(A - \lambda_1 I) = 0 \\ 2 \text{ linearly independent eign vectors if}, & \rho(A - \lambda_2 I) = 1 \\ 1 \text{ linearly independent eigen vectors if}, & \rho(A - \lambda_2 I) = 2 \end{cases}$$

Results:

1. If a matrix has distinct eigen values, then it is always diagonalizable.
2. A matrix of order n is diagonalizable if and only if it has n linearly independent eigen vectors.
3. A matrix is diagonalizable if and only if algebraic multiplicity (AM) of each eigen value equals its geometric multiplicity (GM).

4. A matrix is diagonalizable if its eigen vectors form basis of $\mathbb{R}^n$

5. If is symmetric matrix then eigen vectors corresponding to distinct eigen values are always orthogonal. Further if eigen values are repeated then we can find orthogonal eigen vectors. Hence every symmetric matrix is orthogonally diagonalizable.

6. Note that the sequence of eigen vectors selected to construct P, the eigen values also appear with same sequence as diagonal elements in D. This means, X_1, X_2, X_3, are eigen vectors corresponding to $\lambda_1, \lambda_2, \lambda_3$ respectively.

If $P=\begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix}$ then $D=\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$.

If the choice of P is $P=\begin{bmatrix} X_3 & X_1 & X_2 \end{bmatrix}$ then

$$D=\begin{bmatrix} \lambda_3 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}.$$

$$A = \begin{bmatrix} 8 & 0 & 3 \\ 2 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

Characteristic eq^n is $\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$

$$S_1 = 8 + 2 + 3 = 13, S_2 = \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 3 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 0 \\ 2 & 2 \end{vmatrix} = 40$$

$$|A| = 8(6) - 0(6-2) + 3(0-4) = 48 - 12 = 36.$$

$$\therefore Char\ eq^n\ is\ \lambda^3 - 13\lambda^2 + 40\lambda - 36 = 0.$$

Eigen values are $2,\ 2$ and 9.

Eigen vectors are

$$\lambda = 9 \text{ eigen vector is } X_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}.$$

$\lambda = 2$ two independent eigen vectors are

$$X_2 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \,\&\, X_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

AM=GM for each eigen value, hence matrix is diagonalizable

$$\text{Construct P}=\begin{bmatrix}3 & 1 & 0\\ 1 & 0 & 1\\ 1 & -2 & 0\end{bmatrix}, \det(P)=7 \Rightarrow P^{-1}=\frac{1}{7}\begin{bmatrix}2 & 0 & 1\\ 1 & 0 & -3\\ -2 & 7 & -1\end{bmatrix}$$

$$\text{Now } P^{-1}\text{AP}=\frac{1}{7}\begin{bmatrix}2 & 0 & 1\\ 1 & 0 & -3\\ -2 & 7 & -1\end{bmatrix}\begin{bmatrix}8 & 0 & 3\\ 2 & 2 & 1\\ 2 & 0 & 3\end{bmatrix}\begin{bmatrix}3 & 1 & 0\\ 1 & 0 & 1\\ 1 & -2 & 0\end{bmatrix}$$

$$=\begin{bmatrix}9 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}.$$

$$\text{Modal matrix is } P = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -2 & 0 \end{bmatrix}$$

$$\text{Spectral matrix is } \mathrm{D} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -1 & 4 & 3 \\ 1 & -2 & -1 \end{bmatrix}$$

Is $\begin{bmatrix} -14 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ diagonalizable?

Characteristic equation $\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$

$\lambda^3 + 10\lambda^2 - 52\lambda + 56 = 0 \Rightarrow \lambda = -14, 2, 2.$

AM =GM of $\lambda = -14 = 1.$

AM of $\lambda = 2 = 2.$

Consider, $A - 2I = \begin{bmatrix} -16 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \therefore \rho\left(A - 2I\right) = 2.$

GM of $\left(\lambda = 2\right) = 1$. Therefore AM $\neq$ GM of $\lambda = 2.$

Note That : In case of a symmetric matrix, the matrix P is formed by consideringorthonormal Eigen vectors as its columns.

Procedure to diagonalize symmetric matrix.

Step 1 : Find eigen values of A

Step 2 : Find eigen vectors of A

Step 3 : If eigen values are distinct A, eigen vectors will be orthogonal.

If eigen values are repeated, find orthogonal eigen vectors as discussed in Lecture 8.

Step 4 : Consider orthogonal eigen vectors.

Divide each vector by its norm.

Step 5 : Constructby taking these orthonormal eigen

vectors as columns, i.e.,

$$P = \left[\frac{X_1}{\|X_1\|} \quad \frac{X_2}{\|X_2\|} \quad \frac{X_3}{\|X_3\|} \right].$$

Step 6 : Find inverse of P. As P is orthogonal, $P^{-1} = P^t$.

Step 7: $P^{-1}AP = P^t AP = D$, diagonal matrix with diagonal

entries as eigen values of A and P is a orthogonal

modal matrix

Find the Eigen values and Eigen vectors of $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$.

Char eqn $\lambda^3 - s_1\lambda^2 + s_2\lambda - |A| = 0$, $S_1 = 6$, $S_2 = 9$, $|A| = 0$

Eigen values are $\lambda_1 = 0$, $\lambda_2 = \lambda_3 = 3$.

Eigen vector for $\lambda_1 = 0$, $X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Eigen vectors for repeated eigen values $\lambda_2 = \lambda_3 = 3$

$X_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $X_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. Here AM=GM for every eigen value.

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}. \right\}$$ is a orthogonal set of eigen vectors.

Normalize each eigen vector

$$W_1 = \frac{X_1}{\|X_1\|} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, W_2 = \frac{X_2}{\|X_2\|} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \text{ and } W_3 = \frac{X_3}{\|X_3\|} = \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}.$$

Orthogonal Modal matrix $P = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}$

$$P^{-1} = P^t = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix}.$$

$$P^{-1}AP = P^tAP = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \text{Spectral matrix}$$

Application of Diagonalization

If A is diagonalizable with P as modal matrix and as D spectral matrix then

$$A^n = (PDP^{-1})(PDP^{-1})\cdots(PDP^{-1}) = (PD)(P^{-1}P)(DP^{-1})\cdots(P^{-1}P)(DP^{-1})$$

$$\boxed{\therefore A^n = PD^nP^{-1}}$$

e. g. $\quad A = \begin{bmatrix} 5 & 6 \\ 3 & -2 \end{bmatrix}$, $P = \begin{bmatrix} 3 & -2 \\ 1 & 3 \end{bmatrix}$, $P^{-1} = \dfrac{1}{11}\begin{bmatrix} 3 & 2 \\ -1 & 3 \end{bmatrix}$ and

$D = \begin{bmatrix} 7 & 0 \\ 0 & -4 \end{bmatrix}$.

$$A^5 = PD^5P^{-1} = \begin{bmatrix} 3 & -2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 7^5 & 0 \\ 0 & (-4)^5 \end{bmatrix}\frac{1}{11}\begin{bmatrix} 3 & 2 \\ -1 & 3 \end{bmatrix}$$

$$= \frac{1}{11}\begin{bmatrix} 145013 & 119592 \\ 59796 & 5409 \end{bmatrix}.$$