ENGINEERING MATHEMATICS

ES1032

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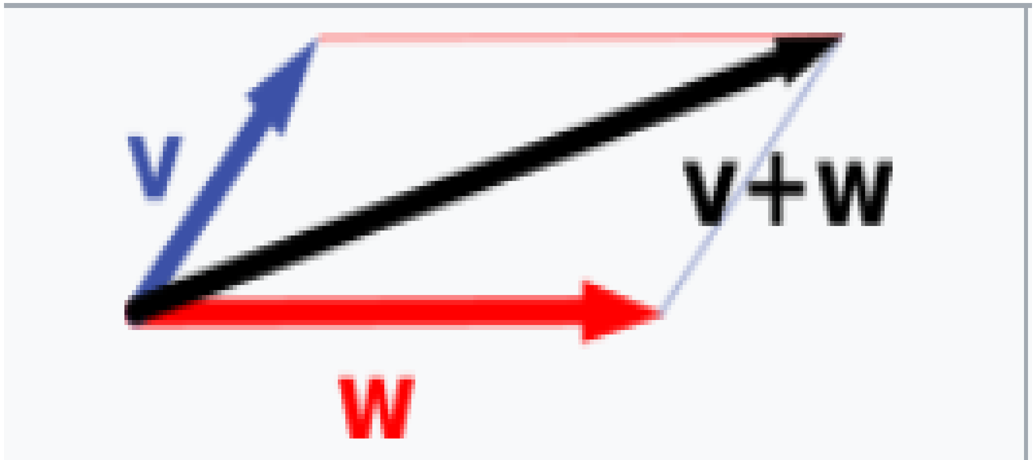
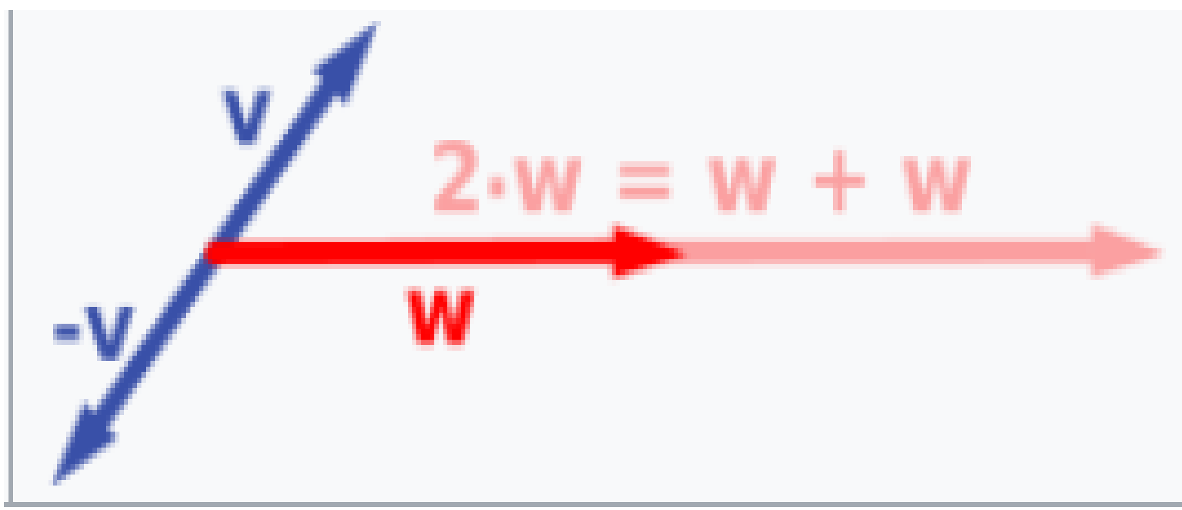
Algebraic operations on vectors



Addition

Multiplication by a real number

Opposite



Vector Space

Let V be non-empty set. Define two operations namely 'addition' and 'scalar multiplication' on V. V is said to be a vector space if for every u, v, w in V and any real, or complex number α following axioms hold.

1. Closed under addition

$$u, v \in \mathrm{V},\ u + v \in \mathrm{V}$$

2. Addition is commutative

$$u + v = v + u$$

3. Addition is associative

$$u + (v + w) = (u + v) + w.$$

4. Existence of zero element for addition known as additive identity

There exists $0 \in \mathrm{V}$ such that $u+0=0+u=u$

5. Existence of additive inverse

There exists $-u \in \mathrm{V} \to -u+u=u+(-u)=0$

6. Closed under scalar multiplication

$$\alpha \in \mathbb{R},\, u \in \mathrm{V} \Rightarrow \alpha\, u \in \mathrm{V}$$

7. Scalar multiplication is distributive

$$(\alpha+\beta)u=\alpha\, u+\beta\, u \ \ and$$

$$\alpha\,(u+v)=\alpha\, u+\alpha\, v$$

8. Scalar multiplication is associative

$$\alpha(\beta u) = (\alpha\beta)u$$

9. There exists $1 \in \mathbb{R}$ such that $1 \cdot u = u$.

Elements/members of V are called as 'vectors'.

Result 1:: For a vector space V

i) Zero vectors are unique.

ii) $-u \in V$ is unique such that $u + (-u) = 0$.

Result 2 :: Let V be a vector space and let $x,\ y$ be vectors in V, then

i) $x + y = x \Rightarrow y = 0$

ii) $0 \cdot x = 0$

iii) $k \cdot 0 = 0$ for any $k \in \mathbb{R}$.

iv) $-x$ is unique and $-x = (-1)x$

v) If $k\,x = 0$, then $k = 0$ or $x = 0$.

Standard Vector Spaces

1. $V = \{0\}$

2. $\mathbb{R}^n = \{(x_1, x_2, \ldots\ldots\ldots, x_n) / x_1,\ x_2,\ x_3,\ \ldots,\ x_n \in \mathbb{R}\}$

$$(x_1, x_2, \cdots, x_n) + (y_1, y_2, \cdots, y_n) = (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n)$$

$$\alpha(x_1, x_2, \cdots, x_n) = (\alpha x_1, \alpha x_2, \cdots, \alpha x_n), \alpha \in \mathbb{R}$$

3. $P_n(x) == \{a_0 x^n + a_1 x^{n-1} + \ldots + a_n / a_0,\ a_1,\ \ldots,\ a_n \in \mathbb{R}\}$

Set of all polynomials in x up-to degree 'n'

$$(a_0 x^n + a_1 x^{n-1} + \cdots + a_n) + (b_0 x^n + b_1 x^{n-1} + \cdots + b_n) =$$

$$(a_0 + b_0) x^n + (a_1 + b_1) x^{n-1} + \cdots + (a_n + b_n)$$

$$\alpha(a_0 x^n + a_1 x^{n-1} + \cdots + a_n) = (\alpha a_0) x^n + (\alpha a_1) x^{n-1} + \cdots + (\alpha a_n)$$

4. $M_{m\times n}(\mathbb{R}) = \{[a_{ij}]_{m\times n} / a_{ij} \in \mathbb{R},\ i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n\}$

collection of all matrices of order $m \times n$ with real entries

$$[a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}] \text{ and } \alpha[a_{ij}] = [\alpha a_{ij}]$$

5. $V = C[a,b] = \{f : [a,b] \to \mathbb{R}\}$

Set of all real valued continuous functions

$$(f+g)(x) = f(x) + g(x) \text{ and } (\alpha f)(x) = \alpha \cdot f(x)$$

Are the following sets vector spaces?

1. Set of polynomials of exactly degree 2 with respect to vector addition

2. The set of integers is not a vector space

3. $V = \left\{(x,\ y,\ z):\ \ x, y,\ z \in R\right\}$

$$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$$

$$c\,(x,\ y,\ z) = (cx,\ 0,\ 0)$$

4. $\mathrm{V} = \left\{(x,\ y):\ \ x, y \in \mathbb{R}\right\}$ with standard addition and

$$c\,(x,\ y) = (cx,\ y), c \in \mathbb{R}$$

Let $V = \{(a, b) \mid a, b \in \mathbb{R}\}$. Let $\mathbf{v} = (v_1, v_2)$ and $\mathbf{w} = (w_1, w_2)$. Define

$$(v_1, v_2) \oplus (w_1, w_2) = (v_1 + w_1 + 1, v_2 + w_2 + 1) \qquad \text{and}$$
$$c \odot (v_1, v_2) = (cv_1 + c - 1, cv_2 + c - 1)$$

Verify that V is a vector space.

$$(u_1, u_2) \oplus (e_1, e_2) = (u_1, u_2) \Rightarrow (u_1 + e_1 + 1, u_2 + e_1 + 1) = (u_1, u_2)$$
$$u_1 + e_1 + 1 = u_1,\ \ u_2 + e_2 + 1 = u_2 \Rightarrow e_1 = -1, e_2 = -1$$

$$(u_1, u_2) \oplus (v_1, v_2) = (e_1, e_2) \Rightarrow (u_1 + v_1 + 1, u_2 + v_1 + 1) = (-1, -1)$$
$$u_1 + v_1 + 1 = -1 \Rightarrow v_1 = -u_1 - 2, u_2 + v_2 + 1 = -1 \Rightarrow v_2 = -u_2 - 2$$

SUBSPACE

Let V be a vector space and $V \neq 0$,

$U \subset V, U$ is said to be a subspace of V

if U itself is a vector space under the

same 'addition' and 'scalar multiplication'

operations as defined on V.

Theorem

A non-empty subset U of vector space V is a subspace of V if and only if

1) U is closed under addition, i. e.,

$$u_1 + u_2 \in U \text{ for all } u_1, u_2 \in U$$

2) U is closed under scalar multiplication, i. e.,

$$\alpha u \in U \text{ for every } \alpha \in \mathbb{R} \text{ and } u \in U.$$

Note: If $0 \in V$ is not a member of U $\subseteq$ V then U is not a subspace of V.

W$=\left\{(x, y, z) / x+y+z=0 \in \mathbb{R}\right\} \subseteq \mathbb{R}^3$ is subspace of $\mathbb{R}^3$.

$$u=(x_1, y_1, z_1) \in W \Rightarrow x_1+y_1+z_1=0,$$

$$v=(x_2, y_2, z_2) \in W \Rightarrow x_2+y_2+z_2=0$$

$$u+v=(x_1, y_1, z_1)+(x_2, y_2, z_2)=(x_1+x_2, y_1+y_2, z_1+z_2).$$

But $\left(x_1+x_2\right)+\left(y_1+y_2\right)+\left(z_1+z_2\right)=\left(x_1+y_1+z_1\right)+\left(x_2+y_2+z_2\right)=0$

$$\therefore u+v \in W.$$

$\alpha \in \mathbb{R}, u=(x_1, y_1, z_1) \in W, \alpha u=\alpha(x_1, y_1, z_1)=(\alpha x_1, \alpha y_1, \alpha z_1).$

Further $\alpha x_1+\alpha y_1+\alpha z_1=\alpha\left(x_1+y_1+z_1\right)=0$

Thus $\alpha u \in W$.

$$W = \{(x, y, z) / 2x + 3y + z = 5\}$$

Zero element of $\mathbb{R}^3$ is $(0,0,0)$. Given plane does not pass through $(0,0,0)$.

Thus zero element of $\mathbb{R}^3$ is not memeber of W.

$\therefore$ W is not a subspace.

$$W = \left\{(x, y) / y = x^2\right\} \subseteq \mathbb{R}^2$$

$$u = (x_1, y_1), v = (x_2, y_2) \in W \Rightarrow y_1 = x_1^2, y_2 = x_2^2.$$

$$u + v = (x_1 + x_2, y_1 + y_2) \text{ and}$$

$$\left(x_1 + x_2\right)^2 = x_1^2 + x_2^2 + 2x_1x_2 \neq x_1^2 + x_2^2 \; always.$$

$\therefore$ W is not closed under addition, hence not a subspace.

$$\mathrm{W}=\{(x,y)\,/\,y=mx,m\text{ fixed}\}\subseteq\mathbb{R}^2 \text{ is a subspace of } \mathbb{R}^2.$$

Let $u=(x_1,y_1),v=(x_2,y_2)\in \mathrm{W}\Rightarrow y_1=mx_1,y_2=mx_2$

$u+v=(x_1+x_2,y_1+y_2)$. But $y_1+y_2=mx_1+mx_2=m(x_1+x_2)$.

$\Rightarrow u+v\in \mathrm{W}$.

W is closed w.r.t addition.

Let $u=(x_1,y_1)\in \mathrm{W}$ and $\alpha\in\mathbb{R}$ then $\alpha u=(\alpha x_1,\alpha y_1)$

But $\alpha y_1=\alpha\left(mx_1\right)=m\left(\alpha x_1\right)\Rightarrow \alpha u\in \mathrm{W}$.

W is closed w.r.t scalar multiplication.

$\mathrm{W} = \{ p(x) / p(1) = 0 \} \subseteq P_n$ is a subspace of P_n.

Let $p(x), q(x) \in \mathrm{W} \Rightarrow p(1) = 0, q(1) = 0., i.e.,$

1 is root of both

$p(x)$ and $q(x)$ or $(x - 1)$ is factor of both $p(x)$ and $q(x)$.

To check 1 is also root of $p + q$ and αp for some $\alpha \in \mathbb{R}$.

Now p+q(1)=p(1)+q(1)=0 and $(\alpha p)(1) = \alpha p(1) = \alpha 0 = 0$

Thus 1 is root of both p+q and αp.

So W is closed w.r.t. addition as well as

scalar multiplication.

Let A *be* $m \times n$ matrix, then $V = \left\{ X \in \mathbb{R}^n : AX = 0 \right\}$,

is a subspace of $\mathbb{R}^n$.

Let $X, Y \in W \Rightarrow AX=0, AY=0.$

To check $X+Y \in W$.

$A(X+Y) = AX+AY=0+0=0.$

W is closed w.r.t. addition.

Similarly, $A(\alpha X) = \alpha AX = \alpha 0 = 0.$

W is closed w.r.t. scalar multiplication.

$$\mathrm{V} = P_2,\ \mathrm{W} = \left\{ax^2 + bx + c : a + b + c = 0\right\}$$

Let $p,\ q \in \mathrm{W} \therefore\ p(x) = ax^2 + bx + c,$ where $a + b + c = 0$

$q(x) = rx^2 + sx + t$ where $r + s + t = 0$

Consider $p + q = \left(ax^2 + bx + c\right) + \left(rx^2 + sx + t\right)$

$= (a + r)x^2 + (b + s)x + (c + t)$

where $(a + r) + (b + s) + (c + t)$

$= \left(a + b + c\right) + \left(r + s + t\right) = 0 + 0 = 0$

$\therefore\ p + q \in \mathrm{W}.$

Next $\alpha p = \alpha\left(ax^2 + bx + c\right) = \left(\alpha ax^2 + \alpha bx + \alpha c\right)$

where $\alpha a + \alpha b + \alpha c = \alpha\left(a + b + c\right) = \alpha 0 = 0$

$\therefore \alpha p \in \mathrm{W} \quad \therefore$ W is a subspace of P_2.

List all the subspaces of $\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$.

Solution : $(i)\ W = \{(0,0)\}$

$(ii)\ W = \{(x, y) : y = mx,\ m \in \mathbb{R}\}$

$(iii)\ W = \mathbb{R}^2.$

2. List all the subspaces of $\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$

Solution : $(i)\ W = \{(0,0,0)\}$

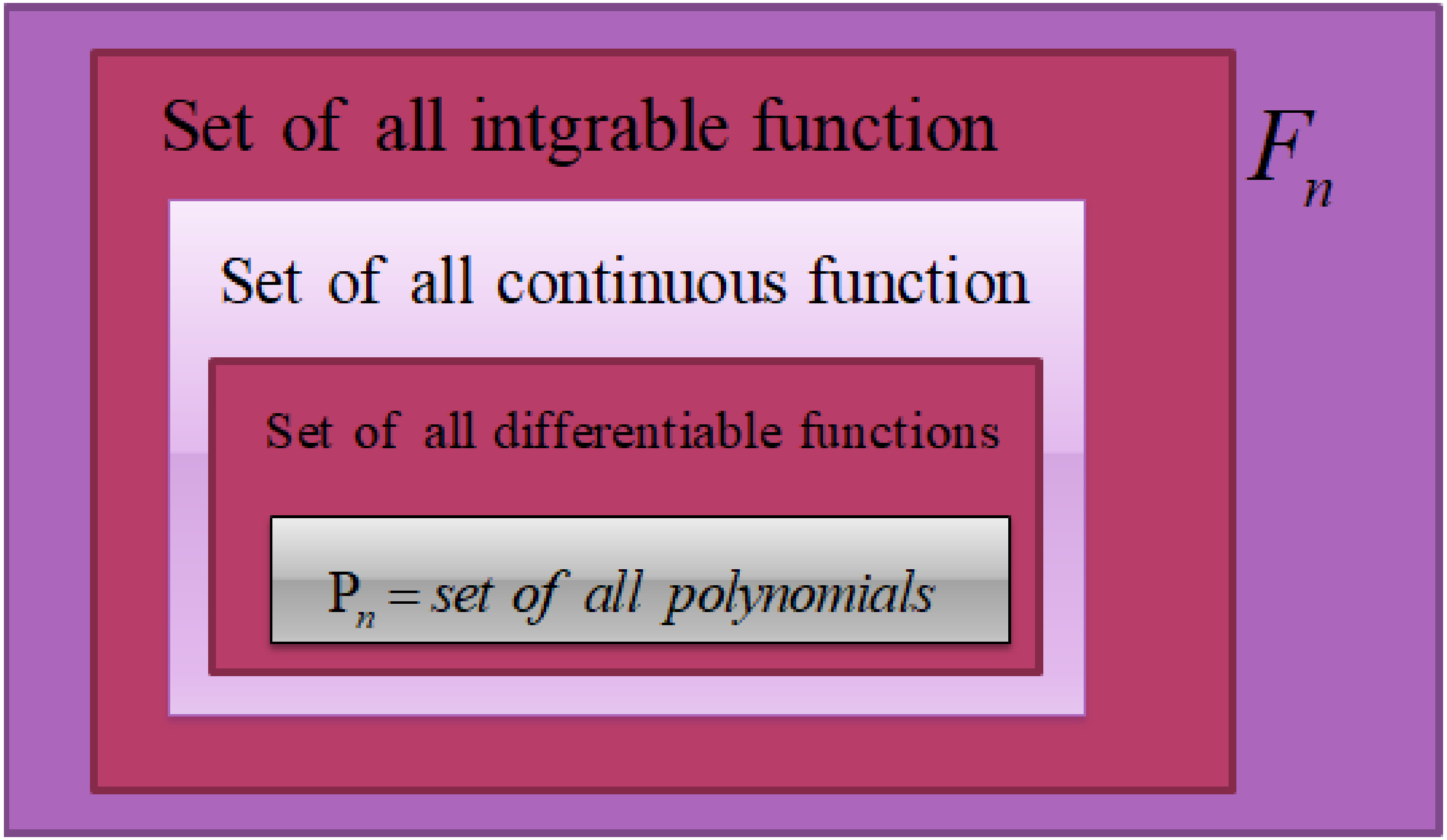
$(ii)\ W =$ Any line passing through origin

$(iii)\ W =$ Any plane passing through origin

$(iv)\ W = \mathbb{R}^3.$

List possible subspaces of

$F_n =$ Set of all function defined on $\mathbb{R}$.



Let U and W are subspaces of a vector space V.

Sum of U and W is defined as

$$U+W=\{u+w\in V:\ u\in U\ \ and\ \ v\in V\}$$

Show that $U+W$ is a subspace of V.

Let $x,\ y\in \mathrm{U}+\mathrm{W}\ \ \therefore x=u_1+w_1\ \ and\ \ y=u_2+w_2$

where $u_1,\ u_2\ \in$ U and $w_1\ ,\ w_2\ \in$ W.

$$x+y=\left(u_1+w_1\ \right)+\left(u_2+w_2\right)$$

$$=\left(u_1+u_2\right)+\left(w_1+w_2\right)$$

but $\left(u_1+u_2\right)\in U$ and $\left(w_1+w_2\right)\in W$

(as U and W are subspaces of V) $\therefore\ x+y\in U+W$.

Let α be any real number, $\alpha x=\alpha\left(u_1+w_1\right)=\alpha u_1+\alpha w_1$

but $\alpha u_1\in U$ and $\alpha w_1\in W$

(as U and W are subspaces of V) $\therefore\ \alpha x\in U+W$

$\therefore\ U+W$ is a subspace of V.

Is M_1 the set of all nonsingular matrices of order 2 a subspace of $M_{2\times 2}$?

$$\text{Consider } \mathrm{A}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ \mathrm{B}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\in M_1$$

$$\mathrm{A}+\mathrm{B}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\notin M_1, \text{as } \det\left(\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\right)=0$$

$\therefore M_1$ is not a subspace of V.

Are the following sets subspaces of $\mathrm{M}_{n\times n}(\mathbb{R})$?

1. $\mathrm{W}_1=$ Set of all $\mathrm{n}\times\mathrm{n}$ symmetric matrices with real entries
2. $\mathrm{W}_2=$ Set of all $\mathrm{n}\times\mathrm{n}$ skew-symmetric matrices with real entries

Is $W=\left\{\begin{bmatrix} a+2b \\ a-b+2 \end{bmatrix}: a,b \in R\right\} \subseteq \mathbb{R}^2$ a subspace *of* $\mathbb{R}^2$?

Is zero element of $\mathbb{R}^2$ member of W?

Yes $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in$ W because for a= $\frac{4}{3}, b=-\frac{2}{3} \Rightarrow \begin{matrix} a+2b=0, \\ a-b+2=0 \end{matrix}$

But W is not a subspace.

$u=\begin{bmatrix} a+2b \\ a-b+2 \end{bmatrix}, v=\begin{bmatrix} c+2d \\ c-d+2 \end{bmatrix} \in \mathrm{W}, a,b,c,d \in \mathbb{R}.$

$u+v=\begin{bmatrix} (a+c)+2(b+d) \\ (a+c)-(b+d)+4 \end{bmatrix}=\begin{bmatrix} p+2q \\ p-q+4 \end{bmatrix}, p=a+c, q=b+d \in \mathbb{R}$

$u+v \notin$ W as this vector does not follow the pattern of W.

W is not closed under addition.

Is $H_2 = \{(x, y) \mid x^2 + y^2 \le 1\}$ a subspace of $\mathbb{R}^2$?

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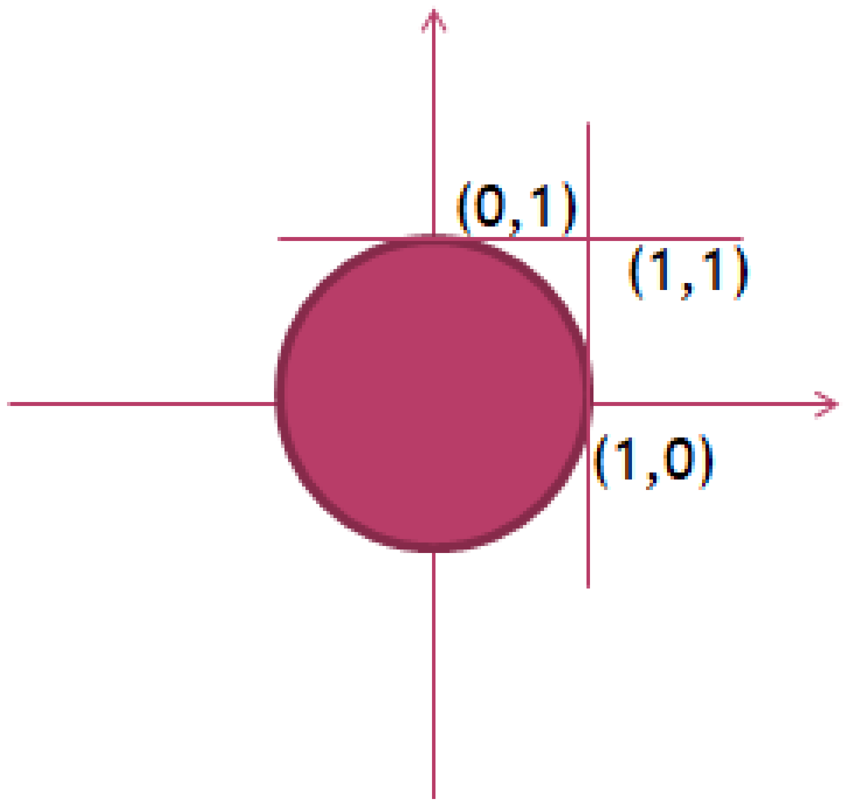


Let U_1 and U_2 be two subspaces of a vector space V then is $U_1 \cap U_2$ also a subspace? Justify

Let $u, v \in U_1 \cap U_2 \Rightarrow u, v \in U_1 \; and \; u, v \in U_2$

But U_1 and U_2 are subspaces, so are closed w.r.t addition and scalar multiplication.

$\therefore u, v \in U_1$ and $u, v \in U_2 \Rightarrow u + v \in U_1$ and $u + v \in U_2$

Also $u \in U_1$ and $u \in U_2, k \in \mathbb{R} \Rightarrow ku \in U_1$ and $k\mathrm{u} \in U_2$

Thus $u + v \in U_1 \cap U_2$ and $ku \in U_1 \cap U_2$

Let U_1 and U_2 be two subspaces of a vector space V then is $U_1 \cup U_2$ also a subspace? Justify

$U_1 = \{(x, y) \mid y = 2x\}$ is a subspace of $\mathbb{R}^2$.

$u = (x_1, y_1), v = (x_2, y_2) \in U_1 \Rightarrow y_1 = 2x_1, y_2 = 2x_2$

$u + v = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and

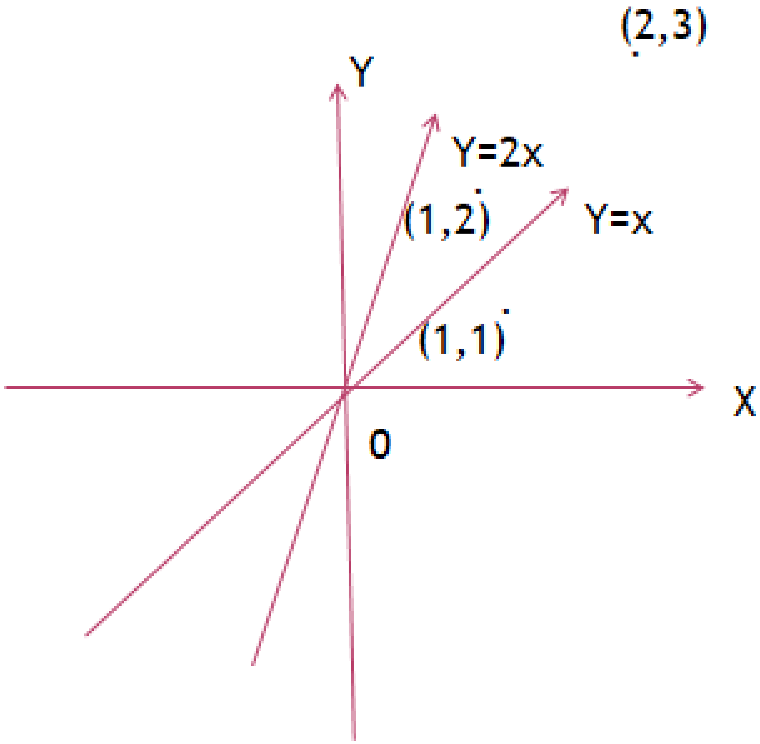
$y_1 + y_2 = 2x_1 + 2x_2 = 2\left(x_1 + x_2\right)$.

Also $\alpha u = \alpha(x_1, y_1) = (\alpha x_1, \alpha y_1)$ and $\alpha y_1 = \alpha 2 x_1 = 2\left(\alpha x_1\right)$

Thus $u + v, \alpha u \in U_1$. Therefore U_1 is a subspace of $\mathbb{R}^2$.

Similarly, $U_2 = \{(x, y) \mid y = x\}$ is also a subspace of $\mathbb{R}^2$.

But $U_1 \cup U_2 = \{(x,\ y) \mid y = 2x\ or\ \ y = x\}$ is not a subspace of $\mathbb{R}^2$



$(1,1)$ lies on line $y = x$.

$(1,2)$ lies on line $y = 2x$.

Now (1,1)+(1,2)=(2,3).

But lines $y = x$ and $y = 2x$ do not pass through (2,3).

$U_1 \cup U_2$ is not closed under addition.

Linear Combination (L.C.) of Vectors

Let $H = \{v_1, v_2, v_3, \ldots, v_n\}$ be a subset of a vector space V,then the sum $c_1v_1 + c_2v_2 + \ldots + c_nv_n$, where $c_1, c_2, \ldots, c_n \in \mathbb{R}$ is defined as a linear combination of $v_1, v_2, \ldots, v_n$.

Span of a Set

Let $\mathrm{H} = \{v_1,\ v_2,\ v_3,\ \ldots\ ,\ v_n\}$ be a subset of a vector space V. Then span of H denoted by $Span\mathrm{H}$ is defined as

$$span\mathrm{H} = \{c_1 v_1 + c_2 v_2 + \ldots + c_n v_n \mid c_1,\ c_2,\ \ldots\ ,\ c_n \in \mathbb{R}\}$$

$= \text{Set of of all possible linear combinition of H}$

Theorem : Let $H=\{v_1, v_2, v_3, \ldots , v_n\}$ be a subset of vector space V. then spanH is a smallest subspace of V containing H.

Proof : Let $h_1, h_2 \in \text{span } H$.

$$h_1=\sum_{i=1}^{n} a_i v_i = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n , a_i \in \mathbb{R}$$

$$h_2=\sum_{i=1}^{n} b_i v_i = b_1 v_1 + b_2 v_2 + \cdots + b_n v_n ,\ b_i \in \mathbb{R}$$

$$h_1 + h_2 = (a_1 + b_1) v_1 + (a_2 + b_2) v_2 + \cdots + (a_n + b_n) v_n .$$

$$(i)\ h_1 + h_2 = \sum_{i=1}^{n} (a_i + b_i) v_i = \sum_{i=1}^{n} c_i\, v_i,\ c_i \in \mathbb{R},\ i = 1, 2, .., n$$

$$\therefore h_1 + h_2 \in span\,\mathrm{H}$$

$$(ii)\ Let\ \alpha \in \mathbb{R},\ \alpha\, h_1 = \sum_{i=1}^{n} (\alpha\, a_i) v_i = \sum_{i=1}^{n} d_i\, v_i,\ d_i \in \mathbb{R}$$

$$\therefore\ \alpha\, h_1 \in \mathrm{span\,H.}$$

$$\therefore \text{ span H is a subspace of V.}$$

To prove that $Span\mathrm{H}$ is a smallest subspace of V containing H.

We need to show that any subspace of V conatining $v_1, v_2, ..., v_n$ also contains $span\mathrm{H}$,i.e., $span\mathrm{H} \subseteq \mathrm{W}$

Let W be a subspace of V conatining $v_1, v_2, ..., v_n$.

$\because$ W is a subspace $\therefore c_1v_1 + c_2v_2 + ... + c_nv_n \in \mathrm{W}$

$\therefore span\mathrm{H} \subseteq \mathrm{W}$.

Note : (i) In $\mathbb{R}^2/\mathbb{R}^3$ $Span(v)$ is a line through origin.

(ii) In $\mathbb{R}^3$ $Span\{v_1, v_2\}$, $where\ v_1 \neq \alpha v_2$, represents a plane through origin.

Important : The spanning set theorem is most important tool to prove that given subset a subspace or not.

` Show that $W = \left\{ \begin{bmatrix} a+2b \\ a-b \end{bmatrix} : a,b \in R \right\} \subseteq \mathbb{R}^2$ is a subspace of $\mathbb{R}^2$.

Let $u,\ v \in W \therefore u = \begin{bmatrix} a+2b \\ a-b \end{bmatrix},\ v = \begin{bmatrix} r+2s \\ r-s \end{bmatrix}$

$\therefore u+v = \begin{bmatrix} a+2b \\ a-b \end{bmatrix} + \begin{bmatrix} r+2s \\ r-s \end{bmatrix} = \begin{bmatrix} (a+r)+2(b+s) \\ (a+r)-(b+s) \end{bmatrix} \in W$, as $a+r,\ b+s \in R$.

To show $\alpha u \in W$

$$\alpha \in \mathbb{R}, u = \begin{bmatrix} a+2b \\ a-b \end{bmatrix} \in W,\ \alpha u = \begin{bmatrix} \alpha a + \alpha 2b \\ \alpha a - \alpha b \end{bmatrix} = \begin{bmatrix} \alpha a + 2\alpha b \\ \alpha a - \alpha b \end{bmatrix}$$

$\therefore$ W is a subspace of $\mathbb{R}^2$.

` Show that $W = \left\{ \begin{bmatrix} a+2b \\ a-b \end{bmatrix} : a, b \in R \right\} \subseteq \mathbb{R}^2$ is a subspace of $\mathbb{R}^2$.

$$\begin{bmatrix} a+2b \\ a-b \end{bmatrix} = \begin{bmatrix} a \\ a \end{bmatrix} + \begin{bmatrix} 2b \\ -b \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\therefore \text{W=span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$$

Show that the set of all symmetric matrices of order 2×2 is a subspace of $M_{2\times 2}(\mathbb{R})$.

To show that $\mathrm{H}=\left\{\begin{bmatrix} a & c \\ c & b \end{bmatrix} : a,b,c\in\mathbb{R}\right\}\subset M_{2\times 2}$ is a subspace of $M_{2\times 2}(\mathbb{R})$.

We will use spanning set theorem *i.e* we will show that $\begin{bmatrix} a & c \\ c & b \end{bmatrix}$ can be expressed as linear combinition of members of $M_{2\times 2}(\mathbb{R})$.

Consider $\begin{bmatrix} a & c \\ c & b \end{bmatrix}=a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}+b\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}+c\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\therefore \mathrm{H}=span\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}.$

Hence the result.

Show that $U=\left\{\begin{pmatrix} r-s \\ 2r-5s+t \\ s+t \end{pmatrix} / \ r,s,t\in\mathbb{R}\right\}$ subspace of $\mathbb{R}^3$.

$$\begin{pmatrix} r-s \\ 2r-5s+t \\ s+t \end{pmatrix}=\begin{pmatrix} r \\ 2r \\ 0 \end{pmatrix}+\begin{pmatrix} -s \\ -5s \\ s \end{pmatrix}+\begin{pmatrix} 0 \\ t \\ t \end{pmatrix}=r\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}+s\begin{pmatrix} -1 \\ -5 \\ 1 \end{pmatrix}+t\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\therefore U=span\left\{\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix},\begin{pmatrix} -1 \\ -5 \\ 1 \end{pmatrix},\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$$

Show that $H = \left\{ \begin{bmatrix} a-b+c \\ 2a-b \\ a-b-5c \\ 2a-c \end{bmatrix} : a,b,c \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^4$.

Consider $\begin{bmatrix} a-b+c \\ 2a-b \\ a-b-5c \\ 2a-c \end{bmatrix} = a\begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix} + b\begin{bmatrix} -1 \\ -1 \\ -1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 1 \\ 0 \\ -5 \\ -1 \end{bmatrix}$

$\Rightarrow \mathrm{H} = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -5 \\ -1 \end{pmatrix} \right\} = span\{v_1,\ v_2,\ v_3\}$

where $v_1,\ v_2,\ v_3 \in \mathbb{R}^4$. $\therefore$ H is a subspace of $\mathbb{R}^4$.

i) For what value of h, will y be in a subspace spanned by v_1, v_2, v_3.

$$\text{where } v_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, v_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}, v_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, y = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}.$$

ii) Is v_1, v_2, v_3 spans $\mathbb{R}^3$.

i) Let $y = c_1 v_1 + c_2 v_2 + c_3 v_3$

$$\begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix} + c_3 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

This is a nonhomgeneous *system of linear equations*

$$\therefore consider \ \left[A \;\vdots\; B\right] = \left[\begin{array}{ccc:c} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{array}\right]$$

Reducing to echelon form $\left[A \;\vdots\; B\right] \sim \begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{bmatrix}$

$\therefore$ The system will be consistent if $h = 5. \therefore$ for $h = 5, y \in span\{v_1, v_2, v_3\}$.

ii) Let $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 . v = c_1 \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix} + c_3 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$

$$\left[A \,\vdots\, B\right] = \left[\begin{array}{ccc|c} 1 & 5 & -3 & a \\ -1 & -4 & 1 & b \\ -2 & -7 & 0 & c \end{array}\right]$$

Reducing to echelon form $\left[A \,\vdots\, B\right] \sim \begin{bmatrix} 1 & 5 & -3 & a \\ 0 & 1 & -2 & b+a \\ 0 & 0 & 0 & c-3b-a \end{bmatrix}$

As $\rho[A] = 2$, $\therefore$ the system will not be consistant for every $v \in \mathbb{R}^3$.

$\therefore\ span\left\{v_1,\ v_2,\ v_3\right\} \neq \mathbb{R}^3$.

Summary

A non-empty subset $U \subseteq V$ is subspace of V

If zero element of V is not member of U, U can't be a subspace.

U is closed under addition as well as scalar multiplication operations same as defined on V.

If U is a linear combination of vectors in V then U is a subspace, i.e., U is a span of vectors in V.

EXERCISE

1. Find the value of k, for which $v = (3, 0, k)$ be in the subspace spanned by u_1, u_2, u_3 where $u_1 = (1, -1, 2)$, $u_2 = (2, 4, -2)$, $u_3 = (1, 2, -4)$.

2. Determine if y is in the subspace of $\mathbb{R}^4$ spanned by v_1, v_2, v_3, *where*

$$y = \begin{bmatrix} 6 \\ 7 \\ 1 \\ -4 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 5 \\ 8 \\ -5 \\ 3 \end{bmatrix}, \quad v_1 = \begin{bmatrix} -5 \\ 8 \\ -9 \\ -2 \end{bmatrix}, \quad v_1 = \begin{bmatrix} -9 \\ -6 \\ 3 \\ -7 \end{bmatrix}$$