Vector Spaces

Introduction: In this topic you will learn vector spaces, linear dependence and independence, basis, dimension.

Vector Space

Definition: Let V be non-empty set. Define two operations namely 'addition' and 'scalar multiplication' on V. V is said to be a vector space if for every u, v, w in V and any real, complex number α, following axioms hold.

1. $u+v \in \mathrm{V}$ (closed under addition)
2. $u+v = v+u$ (addition is commutative)
3. $u+(v+w) = (u+v)+w$ (addition is associative)
4. There exists $0 \in V$ such that $u+0 = 0+u = u$. (Existence of zero element for Addition known as additive identity)
5. There exists $-u \in \mathrm{V} \rightarrow -u+u = u+(-u) = 0$. (Existence of additive inverse)
6. $\alpha u \in \mathrm{V}, \alpha \in \mathrm{R}$ {closed under scalar multiplication}
7. $(\alpha+\beta)u = \alpha u + \beta u$
8. $\alpha(u+v) = \alpha u + \alpha v$ {Scalar multiplication is distributive}
9. $\alpha(\beta u) = (\alpha\beta)u$ {Scalar multiplication is associative}
10. There exists $1 \in \mathrm{R}$ such that $1 \cdot u = u$.

Elements / members of *V* are called as 'vectors'.

Theorem 1: For a vector space V

i) Zero vectors are unique. ii) $-u \in \mathrm{V}$ is unique such that $u+(-u) = 0$.

Theorem 2: Let V be a vector space and let x, y be vectors in V, then

i) $x+y = x \Rightarrow y = 0$ ii) $0 \cdot x = 0$ iii) $k \cdot 0 = 0$ for any $k \in \mathrm{R}$.

iv) $-x$ is unique and $-x = (-1)x$ v) If $kx = 0$, then $k = 0$ or $x = 0$.

Examples of Vector Spaces

1) $\mathrm{V} = \{0\}$ (Set consisting of zero vector only)

2) $\mathrm{R}^n = \{(x_1, x_2, \ldots\ldots\ldots, x_n) / x_1, x_2, x_3, \ldots, x_n \in \mathrm{R}\}$ (Set of all ordered n-tuple of real or complex numbers)is a vector space under component wise addition and scalar multiplication, i. e., $(x_1, x_2, \cdots, x_n) + (y_1, y_2, \cdots, y_n) = (x_1+y_1, x_2+y_2, \cdots, x_n+y_n)$

$\alpha(x_1, x_2, \cdots, x_n) = (\alpha x_1, \alpha x_2, \cdots, \alpha x_n)$.

3) $P_n(x) =$ Set of all polynomials in x up-to degree 'n'

$= \{a_0 x^n + a_1 x^{n-1} + \ldots + a_n / a_0, a_1, \ldots, a_n \in \mathrm{R}\}$

is a vector space under usual addition and scalar multiplication of polynomials, i.e.

$$(a_0 x^n + a_1 x^{n-1} + \cdots + a_n) + (b_0 x^n + b_1 x^{n-1} + \cdots + b_n) = (a_0+b_0)x^n + (a_1+b_1)x^{n-1} + \cdots + (a_n+b_n)$$

$$\alpha(a_0 x^n + a_1 x^{n-1} + \cdots + a_n) = (\alpha a_0)x^n + (\alpha a_1)x^{n-1} + \cdots + (\alpha a_n).$$

4) $M_{m\times n}(\mathrm{R}) = \{[a_{ij}]_{m\times n} / a_{ij} \in \mathrm{R}, i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n\}$ is a vector space with usual addition and scalar multiplication of matrices, i. e.,

$[a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$ and $\alpha[a_{ij}] = [\alpha a_{ij}]$.

5) $V = C[a,b] = \{f : [a,b] \rightarrow \mathrm{R}\}$ set of all real valued continuous functions is a vector space under addition and scalar multiplication of functions, i.e.,

$(f+g)(x) = f(x) + g(x)$ and $(\alpha f)(x) = \alpha \cdot f(x)$.

Examples of sets which are not vector spaces

1. $P_2 =$ Set of polynomials of exactly degree 2 is not closed with respect to vector addition.

For Example: *Let* $p(x) = -x^2 + 2x + 5,\ q(x) = x^2 + 3$

$\therefore p(x) + q(x) = 2x + 8 \notin P_2$ *as* $2x+8$ is not a polynomial of degeree 2.

2. The set of integers is not a vector space. Because the set is not closed with respect to scalar multiplication. For example $\frac{1}{3}(2)=\frac{2}{3}\rightarrow$ is not an integer.

3. Let $\mathrm{V}=\{(x,\ y,\ z):\ x, y,\ z\in R\}$**,** with addition as
$(x_1, y_1, z_1)+(x_2, y_2, z_2)=(x_1+x_2, y_1+y_2, z_1+z_2)$ and scalar multiplication as
$c(x,\ y,\ z)=(cx,\ 0,\ 0)$ [**Ans.:** $1(x,\ y,\ z)=(x,\ 0,\ 0)\neq\ (x,\ y,\ z)$]

4. Let $\mathrm{V}=\{(x,\ y):\ x, y\in \mathrm{R}\}$, with standard operation of addition and nonstandard operation scaler multiplication defined as $c(x,\ y)=(cx,\ y)$ is not a vector space.
[Ans.: In this example all the axioms are satisfied except axiom number 10.
$0\cdot(1,\ 1)=(0\cdot 1, 1)=(0,\ 1)\neq(0,0)$ V is not a vector space.**]**

Subspace: - Let V be a vector space and $\mathrm{V}\neq 0,\ \mathrm{U}\subset\mathrm{V}$, U is said to be a subspace of V if U itself is a vector space under the same 'addition' and 'scalar multiplication' operations as defined on V.

Theorem1.4: - A non-empty subset U of vector space V is a subspace of V if and only if
1) U is closed under addition, i. e., $u_1+u_2\in\mathrm{U}$ for all $u_1,\ u_2\in\mathrm{U}$
2) U is closed under scalar multiplication, i. e., $\alpha u\in\mathrm{U}$ for every $\alpha\in\mathrm{R}$ and $u\in\mathrm{U}$

Note: If $0\in\mathrm{V}$ is not a member of $\mathrm{U}\subseteq\mathrm{V}$ then U is not a subspace of V.

Illustrative Examples:

Q 1) Let A *be* $m\times n$ matrix, then $\mathrm{V}=\{X\in\mathrm{R}^n:\mathrm{AX}=0\}$, is a subspace of R^n.

Sol$^{\mathbf{n}}$: Let $x,\ y\in\mathrm{V}\ \therefore \mathrm{A}x=0,$ and $\mathrm{A}y=0$`
Consider, $\mathrm{A}(x+y)=\mathrm{A}x+\mathrm{A}y=0,\ \therefore x+y\in\mathrm{V}$ and *Let* $\alpha\in\mathrm{R}$, then $\mathrm{A}(\alpha x)=\alpha\mathrm{A}(x)=0,$
$\therefore\ \alpha x\in\mathrm{V}.$ Therefore V is closed with respect to addtion and scalar multilication
$\therefore$ V is a subsapce of R^n.

Q 2) List all the subspaces of $\mathrm{R}^2=\{(x,\ y)\mid x,\ y\in R\}$

Sol$^{\mathbf{n}}$. 1) $\{(0,\ 0)\}$ 2) Set of all lines passing through origin $\{(x,\ y)\mid y=mx,\ m\in\mathrm{R}\}$
3) R^2 itself.

Q 3) List all the subspaces of $\mathrm{R}^3=\{(x,\ y)\mid x,\ y,\ z\in\mathrm{R}\}$

Sol$^{\mathbf{n}}$. 1) $\{(0,\ 0,\ 0)\}$ 2) Any plane passing through origin.
3) Any line passing through origin. 4) R^3 itself.

Q 4) Is R^2 a subspace of R^3?

Sol$^{\mathbf{n}}$. No, because R^2 is not even a subset of R^3 (R^2 has ordered pairs while R^3 has ordered triples)

Q 5) Let U and W are subspaces of a vector space V. Sum of U and W is defined as $\mathrm{U}+\mathrm{W}=\{u+w\in\mathrm{V}:\ u\in\mathrm{U}\ \mathit{and}\ v\in\mathrm{V}\}$ Show that $\mathrm{U}+\mathrm{W}$ is a subspace of V.

Sol$^{\mathbf{n}}$. Let $x,\ y\in\mathrm{U}+\mathrm{W}\ \therefore x=u_1+w_1$ and $y=u_2+w_2$ where $u_1,\ u_2\in\mathrm{U}$ and $w_1,\ w_2\in\mathrm{W}$.

i) $x+y=\left(u_1+w_1\right)+\left(u_2+w_2\right)$

$=\left(u_1+u_2\right)+\left(w_1+w_2\right)$ (by associativity and commutativity of V)

but $\left(u_1+u_2\right)\in \mathrm{U}$ and $\left(w_1+w_2\right)\in \mathrm{W}$ (as U and W are subspaces of V)

$\therefore\ x+y\in \mathrm{U}+\mathrm{W}$.

ii) Let α be any real number, $\alpha x=\alpha\left(u_1+w_1\right)=\alpha u_1+\alpha w_1$ (Distributive property of V)

but $\alpha u_1\in \mathrm{U}$ *and* $\alpha w_1\in \mathrm{W}$ (as U and W are subspaces of V)

$\therefore\ \alpha x\in \mathrm{U}+\mathrm{W}$ $\therefore\ \mathrm{U}+\mathrm{W}$ is a subspace of V.

Examples of sets which are not subspaces

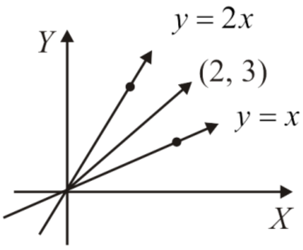
Q 1) Is M the set of all singular matrices of order 2×2 a subspace of $M_{2\times 2}(\mathrm{R})$?

Sol^n. M is not a subspace because it is not closed under addition.

$\begin{bmatrix}1&1\\0&0\end{bmatrix}+\begin{bmatrix}0&0\\1&2\end{bmatrix}=\begin{bmatrix}1&1\\1&2\end{bmatrix}\notin M$. Since addition matrix is non-singular

Q 2) Is $\{(x,\ y)\mid y=mx,\ m\in \mathrm{R}\}$ a subspace of R^2?

Sol^n. $(x_1,\ y_1),\ (x_2,\ y_2)\Rightarrow y_1=m_1x_1,\ y_2=m_2x_2,\ y_1+y_2=m_1x_1+m_2x_2\neq m(x_1+x_2)$



Hence it is not closed under addition**.** It is not a subspace of R^2.

Q 3) Is $H=\{(x,\ y)\mid y=mx,\ m$ is fixed real number$\}$ a subspace of R^2?

Sol^n. Yes, because fixed real number $\Rightarrow$ a single line passing through origin.

Let $(x_1,y_1),(x_2,y_2)\in H\Rightarrow y_1=mx_1,y_2=mx_2$.

Consider, $\left(x_1+x_2,\ y_1+y_2\right)=\left(x_1+x_2,\ mx_1+mx_2\right)=\left(x_1+x_2,\ \ m(x_1+x_2)\right)\in H$

and

$\alpha\left(x_1,\ y_1\right)=\left(\alpha x_1,\ \alpha(mx_1)\right)=\left(\alpha x_1,\ m(\alpha x_1)\right)\in H$ as m,α both are real numbers.

Q 4) Is $H_1=\{(x,\ y)\mid x^2+y^2=1\}$ a subspace of R^2?

Sol^n. H_1 is not a subspace because $(0,0)\notin H_1$.

Q 5) Is $H_2=\{(x,\ y)\mid x^2+y^2\leq 1\}$ a subspace of R^2?

Sol^n. H_2 is not a subspace because H_2 is not closed under addition. e.g. $(1,0),(0,1)\in H_2$ but

$(1,0)+(0,1)=(1,1)\notin H_2$.

Linear Combination (L.C.) of Vectors

Let $H=\{v_1,\ v_2,\ v_3,\ \ldots,\ v_n\}$ be a subset of a vector space V, then the sum $c_1v_1+c_2v_2+\ldots+c_nv_n$, where $c_1,\ c_2,\ \ldots,\ c_n\in \mathrm{R}$ is defined as a linear combination of $v_1,\ v_2,\ \ldots,\ v_n$.

 # Span of a Set

Let $S=\{v_1,\ v_2,\ v_3,\ \ldots,\ v_n\}$ be a subset of vector space V.

$\text{span}(S) = \{c_1v_1 + c_2v_2 + \ldots + c_nv_n \mid c_1, c_2, \ldots, c_n \in \mathbb{R}\}$ = Set of all possible linear combination of members of H

Theorem1.5: Let $S = \{v_1, v_2, v_3, \ldots, v_n\}$ be a subset of vector space V. Show that span S is a subspace of V containing S .

Proof. i) Let $h_1, h_2 \in \text{span } H \Rightarrow \left.\begin{array}{l} h_1 = \sum_{i=1}^{n} a_i v_i \\ h_2 = \sum_{i=1}^{n} b_i v_i \end{array}\right\} a_i, b_i \in \mathbb{R}, h_1 + h_2 = \sum_{i=1}^{n} (a_i + b_i) v_i = \sum_{i=1}^{n} c_i v_i, c_i \in \mathbb{R}$.

Therefore $h_1 + h_2 \in$ span(S)

ii) Let $\alpha \in \mathbb{R}$ and $\alpha h_1 = \sum_{i=1}^{n} (\alpha a_i) v_i = \sum_{i=1}^{n} d_i v_i, d_i \in \mathbb{R}$. Therefore $\alpha h_1 \in$ span(S)

$\therefore$ Span S is a subspace of a vector space V.

Note That : Span S is a smallest subspace of V containing S .

Illustrative Examples

Q 1) Show that the set M of all symmetric matrices of order 2×2 is a subspace of $M_{2\times 2}(\mathbb{R})$.

Sol^n. Consider any symmetric matrix $A \in M$. Then

$$A = \begin{bmatrix} a & c \\ c & b \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + c\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \text{ where } a, b, c \in \mathbb{R} .$$

Here A is a linear combination of $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ of $M_{2\times 2}$.

$\therefore$ Set of symmetric matrices of order 2 is a subspace of $M_{2\times 2}(\mathbb{R})$.

Q 2) Show that $H = \left\{ \begin{bmatrix} a-b+c \\ 2a-b \\ a-b-5c \\ 2a-c \end{bmatrix} \right\}$ is a subspace of $\mathbb{R}^4$.

Sol^n. $\begin{bmatrix} a-b+c \\ 2a-b \\ a-b-5c \\ 2a-c \end{bmatrix} = a\begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix} + b\begin{bmatrix} -1 \\ -1 \\ -1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 1 \\ 0 \\ -5 \\ -1 \end{bmatrix} \Rightarrow H = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -5 \\ -1 \end{pmatrix} \right\} = \{v_1, v_2, v_3\}$

where $v_1, v_2, v_3 \in \mathbb{R}^4$. $\therefore$ H is subspace of $\mathbb{R}^4$.

Q 3) Is $H = \left\{ \begin{bmatrix} a-b+c \\ 5 \\ a-b-5c \\ 2a-c \end{bmatrix} \right\}$ a subspace of $\mathbb{R}^4$?

Sol^n. Since $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is not a member of H, it is not a subspace of $\mathbb{R}^4$.

Spanning Set

Let $S = \{v_1, v_2, \ldots, v_n\}$ be a subset of vector space V. S is said to be a spanning set of V if every element of V is expressible as linear combination of elements of S, i.e., $v \in \text{span}\,\{S\}$ for all $v \in \text{V}$, i.e., $v = c_1v_1 + c_2v_2 + \ldots + c_nv_n$ for some $c_1, c_2, \ldots, c_n$ is a consistent system of linear equations.

Illustrative Examples

Q 1) Let $S = \{e_1 = (1, 0),\ e_2 = (0, 1)\}$. Is S spanning set of R^2?

Sol^n. Let $(x, y) \in \text{R}^2$. $(x, y) = xe_1 + ye_2$ is always consistent. $\therefore S$ spans R^2.

Q 2) Is $S = \{1, x, x^2\}$ spans P_2?

Sol^n. Let $p(x) \in P_2$ then $p(x) = a_0x^2 + a_1x + a_2$; $a_0, a_1, a_2 \in R$. $\therefore$ S spans P_2.

Q 3) i) For what value of h, will y be in the subspace of R^3 spanned by v_1, v_2, v_3

where $v_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}$, $v_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$ and $y = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$.

ii) Is v_1, v_2, v_3 span R^3?

Sol^n. i) Let $y \in \text{span}\,\{v_1, v_2, v_3\}$. Therefore $y = c_1v_1 + c_2v_2 + c_3v_3$.

i.e. $\begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix} = c_1\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + c_2\begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix} + c_3\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \Rightarrow$ $\begin{aligned} c_1 + 5c_2 - 3c_3 &= -4 \\ -c_1 - 4c_2 + c_3 &= 3 \\ -2c_1 - 7c_2 + 0c_3 &= h \end{aligned}$

Consider the augmented matrix $[\text{A} : y] = \begin{bmatrix} 1 & 5 & -3 & : & -4 \\ -1 & -4 & 1 & : & 3 \\ -2 & -7 & 0 & : & h \end{bmatrix} R_2 + R_1,\ R_3 + 2R_1 \Rightarrow$

$$\sim \begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & h-8 \end{bmatrix} R_3 - 3R_2 \Rightarrow \begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{bmatrix}.$$

For $h = 5$, $\rho[\text{A}] = \rho[\text{A} : y]$. $\therefore$ The system of equations is consistent. Hence,

$y \in \text{span}\,\{v_1, v_2, v_3\}$ iff $h = 5$.

ii) $\{v_1, v_2, v_3\}$ does not span R^3 because the system of equations

$v = c_1v_1 + c_2v_2 + c_3v_3$

is not consistent for every $v \in \text{V}$.

Q 3) Let $v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $v_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$, $w = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

i) Is w belong to the set $\{v_1, v_2, v_3\}$? ii) How many vectors are in the set $\{v_1, v_2, v_3\}$?

iii) How many vectors are in $\text{span}\{v_1, v_2, v_3\}$? iv) Is $w \in \text{span}\,\{v_1, v_2, v_3\}$? Why?

v) Is $\text{span}\,\{v_1, v_2, v_3\} = \text{R}^3$?

Sol^n. i) No. w is not a member of the set $\{v_1, v_2, v_3\}$ because $w \neq v_1 \neq v_2 \neq v_3$.

ii) Total number of vectors in the set $\{v_1, v_2, v_3\}$ are 3.

iii) There are infinite numbers of vectors in $\text{span}\{v_1, v_2, v_3\}$.

iv) $w \in \text{span}\,\{v_1, v_2, v_3\}$

Consider, $w = c_1v_1 + c_2v_2 + c_3v_3$

$$\begin{bmatrix}3\\1\\2\end{bmatrix}=c_1\begin{bmatrix}1\\0\\-1\end{bmatrix}+c_2\begin{bmatrix}2\\1\\3\end{bmatrix}+c_3\begin{bmatrix}4\\2\\6\end{bmatrix}\Rightarrow \begin{array}{r}c_1+2c_2+c_3=3\\0c_1+c_2+2c_3=1\\-c_1+3c_2+6c_3=2\end{array}$$

Consider the augmented matrix, $[\mathrm{A}:\mathrm{B}]=\begin{bmatrix}1&2&4&3\\0&1&2&1\\-1&3&6&2\end{bmatrix}\xrightarrow{R_3+R_1}\begin{bmatrix}1&2&4&3\\0&1&2&1\\0&5&10&5\end{bmatrix}$

$\xrightarrow{R_3-5R_2}\begin{bmatrix}1&2&4&3\\0&1&2&1\\0&0&0&0\end{bmatrix}$

$\rho[\mathrm{A}:\mathrm{B}]=\rho[\mathrm{A}]\;\therefore$ The system of equations is consistent.

v) Consider $v=c_1v_1+c_2v_2+c_3v_3$

$v=\mathrm{A}C$, as $\rho[\mathrm{A}]=2\neq\rho[\mathrm{A},\,v]$ for every $\mathrm{v}\in\mathrm{R}^3$. $\therefore$ This system is not consistent for every $v\in\mathrm{R}^3$. $\quad\therefore$ Span $\{v_1,\,v_2,\,v_3\}\neq\mathrm{R}^3$.

Linearly Dependent/Independent Set

Let $H=\{v_1,\,v_2,\,v_3,\,\ldots,\,v_n\}$ be a subset of a vector space V. H is said to be linearly dependent if there exist scalars (real numbers) $c_1,c_2,\cdots,c_n$ not all zero such that

$$c_1v_1+c_2v_2+c_3v_3+\ldots+c_nv_n=0.$$

H is said to be linearly independent if $c_1v_1+c_2v_2+c_3v_3+\ldots+c_nv_n=0$ implies $c_1=c_2=\cdots=c_n=0$.

Note: i) H is linearly independent if and only if the system $c_1v_1+c_2v_2+c_3v_3+\ldots+c_nv_n=0$ has
trivial solution only, i.e. otherwise the set is linearly dependent.

ii) A set containing zero vector is linearly dependent.

iii) Set consists of single non – zero vector is linearly independent.

Theorem1.6: - If the set of vectors are linearly dependent then one of the vectors is expressible as a linear combination of the remaining.

Proof:- The set is linearly dependent, $c_1v_1+c_2v_2+c_3v_3+\ldots+c_nv_n=0$ has a non-trivial solution

$\Rightarrow$ At least one of $c_1,\,c_2,\,\ldots,\,c_n$ is non-zero.

Let us assume that $c_2\neq0$. Then $v_2=\dfrac{-1}{c_2}(c_1v_1+c_3v_3+\ldots+c_nv_n)$.

Method of checking Linearly Dependent / Independent Set

Step 1: Consider, $c_1v_1+c_2v_2+c_3v_3+\ldots+c_nv_n=0$

$$\mathrm{A}C=0 \text{ where } \mathrm{A}=\begin{bmatrix}v_1 & v_2 & v_3 & \ldots & \mathrm{v}_n\end{bmatrix},\; C=\begin{bmatrix}c_1\\c_2\\\cdots\\c_n\end{bmatrix}.$$

Step 2: Find rank of A. Let $\rho(\mathrm{A})=r$

Step 3: i) If $\rho(\mathrm{A})=r=n$ (Number of unknowns), then set is linearly independent.

ii) If $\rho(\mathrm{A})=r<n$ (Number of unknowns), then set is linearly dependent.

Step 3: If dependent find relation between the vectors.

Illustrative Examples

Q 1) Determine whether $S=\{1-t,\,2t+3t^2,\,t^2-2t^3,\,2+t^3\}$ is linearly dependent or independent. If dependent, find the relation between them.

Sol^n. Let $v_1 = 1-t,\ v_2 = 2t+3t^2,\ v_3 = t^2-2t^3,\ v_4 = 2+t^3$.

Consider, $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0$.

$c_1(1-t) + c_2(2t+3t^2) + c_3(t^2-2t^3) + c_4(2+t^3) = 0 + 0t + 0t^2 + 0t^3$

This is ideally zero for every 't' iff $\begin{aligned} c_1 + 2c_4 &= 0 \\ -c_1 + 2c_2 &= 0 \\ 3c_2 + c_3 &= 0 \\ -2c_3 + c_4 &= 0 \end{aligned}$ i.e. $\begin{bmatrix} 1 & 0 & 0 & 2 \\ -1 & 2 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = 0$

Let

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ -1 & 2 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix} \xrightarrow{R_2 + R_1} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 - 3R_2} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 2 & -1 \end{bmatrix} \xrightarrow{R_4 + 2R_3} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$\rho[A] = 4 =$ number of unknowns. $\therefore$ The system has a trivial solution only.

$\therefore$ Set of vectors are linearly independent.

Q 2) Determine whether the set of vectors $H = \left\{ \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} -3 & -3 \\ 1 & 3 \end{bmatrix} \right\}$ is linearly dependent or independent? If dependent, find the relation.

Sol^n.: Let $v_1 = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix},\ v_2 = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix},\ v_3 = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix},\ v_4 = \begin{bmatrix} -3 & -3 \\ 1 & 3 \end{bmatrix}$.

Consider $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0$. ---------- (i)

$$c_1\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} + c_2\begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix} + c_3\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} + c_4\begin{bmatrix} -3 & -3 \\ 1 & 3 \end{bmatrix} = 0 \Rightarrow \begin{aligned} c_1 - 2c_2 + 3c_3 - 3c_4 &= 0 \\ c_2 + 4c_3 - 3c_4 &= 0 \\ 2c_1 - c_2 + 2c_3 + c_4 &= 0 \\ 3c_1 + 3c_3 + 3c_4 &= 0 \end{aligned}$$

$$A = \begin{bmatrix} 1 & -2 & 3 & -3 \\ 0 & 1 & 4 & -3 \\ 2 & -1 & 2 & 1 \\ 3 & 0 & 3 & 3 \end{bmatrix} \xrightarrow{R_3 - 2R_1,\ R_4 - 3R_1} \begin{bmatrix} 1 & -2 & 3 & -3 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & -16 & 16 \\ 0 & 0 & -30 & 30 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{(-16)}R_3,\ \frac{1}{30}R_4} \begin{bmatrix} 1 & -2 & 3 & -3 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \xrightarrow{R_4 + R_3} \begin{bmatrix} 1 & -2 & 3 & -3 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho[A] = 3 <$ number of unknowns $= 4$. $\therefore$ The system has a non-trivial solution.

$\therefore$ Vectors are linearly dependent. Equivalent system is

$c_1 - 2c_2 + 3c_3 - 3c_4 = 0, c_2 + 4c_3 - 3c_4 = 0, c_3 - c_4 = 0$

Let $c_4 = k,\ k \neq 0 \Rightarrow \quad c_3 = k$

$c_2 + 4k - 3k = 0, \quad c_2 = -k, \quad c_1 + 2k + 3k - 3k = 0, \quad c_1 = -2k$

Put values of c_1, c_2, c_3, c_4 in (i), $-2kv_1 - kv_2 + kv_3 + kv_4 = 0$.

$-2kv_1 - kv_2 + kv_3 + kv_4 = 0$, $i.e., 2v_1 + v_2 - v_3 - v_4 = 0$

Q 3) Let $H = \{v_1, v_2, \dots, v_k\}$ spans *V*. Show that if H is linearly dependent then $H_1 = \{v_1, v_2, \dots, v_{k-1}\}$ also spans *V*.

Sol^{n}. Let $v \in V$ then as H spans V, $v = d_1v_1 + d_2v_2 + \dots + d_kv_k$.

If H is linearly dependent, then any vector in H is expressible as the linear combination of others. Therefore let $v_k = c_1v_1 + c_2v_2 + \dots + c_{k-1}v_{k-1}$. Substituting this in v

$$v = (d_1 + d_kc_1)v_1 + (d_2 + d_kc_2)v_2 + \dots + (d_{k-1} + d_kc_{k-1})v_k$$

Thus v is expressible as linear combination of vectors in H_1. Hence H_1 spans *V*.

Q 4) Determine by inspection if the given set is linearly independent

a) $\begin{bmatrix} 1 \\ -1 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 7 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$ b) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

c) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -4 \\ -6 \\ -8 \end{bmatrix}$ d) $\begin{bmatrix} -2 \\ 4 \\ 6 \\ 10 \end{bmatrix}, \begin{bmatrix} 5 \\ -10 \\ -15 \\ 25 \end{bmatrix}$

Sol^{n}. a) Set is linearly dependent, because it contains 4-vectors in $\mathbb{R}^3$.

b) Set is linearly dependent, because it contain zero vector.

c) Set is linearly dependent, because $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} -4 \\ -6 \\ -8 \end{bmatrix}$.

d) Set is linearly independent as there is no relation between them.

Basis of a Vector Space

A subset $B = \{v_1, v_2, \dots, v_n\}$ of a vector space V is a basis for V if

1) B is linearly independent. 2) $V = \text{span}\ \{B\}$, i.e. *B* spans *V*.

Note That: i) A given vector space may have more than one bases.
e.g. $B_1 = \{e_1 = (1, 0)\ ,\ e_2 = (0, 1)\}$, $B_2 = \{v_1 = (1, 1), v_2 = (-1, 1)\}$
Both B_1 & B_2 are bases of $\mathbb{R}^2$.

ii) A set having maximum number of linearly independent vectors is the basis.

iii) Minimum number of vectors which spans the set is the basis.

iv) If $B = \{v_1, v_2, \dots, v_n\}$ is basis for vector space *V*, then every vector in *V* can be written in one and only one as a linear combination of vectors in *B*.

v) If a vector space *V* has one basis with n vectors, then every basis for *V* has *n* vectors.

Dimension of a vector space

Number of elements in a basis is known as the dimension of the vector space.

If basis of vector space *V* contains finite number of vectors, the vector space is finite dimensional. Otherwise it is said to be infinite dimensional.

\# If basis of a vector space *V* has <u>*n*-vectors</u> then dimension of $V = \dim(V) = n$ & $\dim\{0\} = 0$.

Note That: 1) In an n-dimensional vector space *V*, "$n+1$" vectors are linearly dependent.
2) In an n-dimensional vector space *V*, "$n-1$" vectors do not span vector space *V*.

- **Standard Bases**

1) $V = R^n$ then the standard basis is $B = \{e_1, e_2, \dots, e_n\}$, where, $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, \dots, 0), \cdots, \ e_n = (0, 0, \dots, 0, 1)$. $\therefore \dim(R^n) = n$.

In particular, for $\mathbb{R}^2$ standard basis is $\{(1,0),(0,1)\}$ and , for $\mathbb{R}^3$ standard basis is $\{(1,0,0),(0,1,0),(0,0,1)\}$.

2) $V = M_{2\times 2}(R)$. $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ $\therefore \dim(M_{2\times 2}(R)) = 4$.

(**Note :** If $V = M_{m\times n}(R)$ then $\dim(M_{m\times n}(R)) = mn$)

3) $V = P_n(x)$: space of polynomials in x at most degree n or degree $\le n$. Standard basis
of $P_n(x)$ is $B = \{1, x, x^2, \dots, x^n\}$ $\therefore \dim(P_n(x)) = n+1$.

4) $C[a,b]$: space of all continuous functions defined on $[a,b]$ is an infinite dimensional
vector space.

Illustrative Examples

1. Is $B = \{v_1, v_2\}$ a basis of R^3, where $v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$?

Sol^n^. No. Because $\dim(\mathrm{R}^3) = 3$.

2. Is $B = \{v_1, v_2\}$ a basis of R^2, where $v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$?

Sol^n^. No. Because v_1 & v_2 are not even the members of R^2.

3. What is the span $\{B\}$ in the above example?

Sol^n^. It is a plane through origin. Since 3 non-collinear points viz origin, point corresponding to vector v_1 and point corresponding to vector v_2 forms a plane.

Note: If Basis of subsapce V of R^n contains

i) 1-vector, then geometrically it is a straight line in R^n through origin.

ii) 2-vectors, then geometrically it is a plane in R^n through origin.

Illustrative Examples

Q 1) Determine whether the set is a basis for $M_{2\times 2}(\mathrm{R})$?

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}$$

Sol^n^. Since $\dim(M_{2\times 2}(\mathrm{R})) = 4$ and S contains 4 vectors, therefore S is a basis for $M_{2\times 2}(\mathrm{R})$ if and only if given set of vectors are linearly independent.

Let $v_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, v_2 = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, v_3 = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, v_4 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Consider $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0$. i.e. $AC = 0$ where $A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$.

$$\xrightarrow{R_4 - R_1} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & -2 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 2 & -2 & 0 \end{bmatrix}$$

$$\xrightarrow{R_4 - 2R_2} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -4 & 0 \end{bmatrix} \xrightarrow{R_4 + 4R_3} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$\rho(A) = 4 =$ number of unknowns. Hence given set of vectors are linearly independent.

Hence S is a basis for $M_{2\times 2}(R)$.

Q 2) Determine whether the set $S = (1,\ t,\ 1+t^2)$ a basis for P_2 ?

Sol^n. $\because$ $\dim(P_2(x)) = 3$ and S has exactly 3 elements, it forms a basis for $P_2(x)$ iff set of vectors are linearly independent.

Let $v_1 = 1,\ v_2 = t,\ v_3 = 1+t^2$

Consider $c_1v_1 + c_2v_2 + c_3v_3 = 0,$ where $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \rho(A) = 3 =$ number of unknowns. Hence given set of vectors are linearly independent.

$\therefore S$ forms a basis for P_2.

Note: If W is a subspace of an n-dimensional vector space, then dimension of W is less than or equal to n.

Method of determining the dimension of a subspace: Dimension of a subspace can be determined by finding a set of linearly independent vectors that spans the subspace. This set is a basis for the subspace and its dimension is number of vectors in its basis.

Illustrative Examples

1. Determine the dimension of each subspace of R^4.

a) $S = \{(a,\ a+b,\ b,\ a-c) : a,\ b, c \in R\}$ *b)* $S = \{(3a,\ a,\ b,\ 0) : a,\ b \in R\}$

Sol^n. *a)* $(a,\ a+b,\ b,\ a-c) = a(1,\ 1,\ 0,\ 1) + b(0,\ 1,\ 1,\ 0) + c(0,\ 0,\ 0,\ -1)$

we can see that S is spanned by $(1,\ 1,\ 0,\ 1)$, $(0,\ 1,\ 1,\ 0)$ and $(0,\ 0,\ 0,\ -1)$

and can easily be shown that the set of vectors are linearly independent.

$\therefore$ $B = \{(1,\ 1,\ 0,\ 1),\ (0,\ 1,\ 1,\ 0),\ (0,\ 0,\ 0,\ -1)\}$ is a basis for S. $\therefore$ $\dim(S) =$ 3.

b) $(3a,\ a,\ b,\ 0) = a(3,\ 1,\ 0,\ 0) + b(0,\ 0,\ 1,\ 0)$

we can see that S is spanned by $(3,\ 1,\ 0,\ 0)$, $(0,\ 0,\ 1,\ 0)$

and can easily be shown that the set of vectors are linearly independent.

$\therefore$ $B = \{(3,\ 1,\ 0,\ 0),\ (0,\ 0,\ 1,\ 0)\}$ is a basis for S. $\therefore$ $\dim(S) =$ 2.

2. If $W = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a,\ b,\ c \in R \right\}$ is a subspace of $M_{2\times 2}$. What is a dimension of W?

Sol^n. $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

So, the set $B=\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\ \begin{bmatrix}0&1\\0&0\end{bmatrix},\ \begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$ is a basis for W. $\therefore \dim(\mathrm{W})=3.$

3. Find the dimension of the subsapce $\mathrm{W}=\left\{\begin{bmatrix}a-3b+2c\\b-4c-3d\\a-3c-2d\\2a+5b-2c+d\end{bmatrix}:\ a,\ b,\ c,d\in\mathrm{R}\right\}$

Sol$^{\mathbf{n}}$. $\begin{bmatrix}a-3b+2c\\b-4c-3d\\a-3c-2d\\2a+5b-2c+5d\end{bmatrix}=a\begin{bmatrix}1\\0\\1\\2\end{bmatrix}+b\begin{bmatrix}-3\\1\\0\\5\end{bmatrix}+c\begin{bmatrix}2\\-4\\-3\\-2\end{bmatrix}+d\begin{bmatrix}0\\-3\\-2\\5\end{bmatrix}$

W is spanned by $\begin{bmatrix}1\\0\\1\\2\end{bmatrix},\begin{bmatrix}-3\\1\\0\\5\end{bmatrix},\begin{bmatrix}2\\-4\\-3\\-2\end{bmatrix},\begin{bmatrix}0\\-3\\-2\\5\end{bmatrix}$,i.e., these vectors is a basis for W if vectors are

Linearly independent.

$\therefore$ consider $\mathrm{A}=\begin{bmatrix}1&-3&2&0\\0&1&-4&-3\\1&0&-3&-2\\2&5&-2&5\end{bmatrix}\xrightarrow{R_3-R_1,\ R_4-2R_1}\begin{bmatrix}1&-3&2&0\\0&1&-4&-3\\0&3&-5&-2\\0&11&-6&5\end{bmatrix}$

$\xrightarrow{R_3-3R_2,\ R_4-11R_2}\begin{bmatrix}1&-3&2&0\\0&1&-4&-3\\0&0&7&7\\0&0&38&38\end{bmatrix}\xrightarrow{R_4-\frac{38}{7}R_3}\begin{bmatrix}1&-3&2&0\\0&1&-4&-3\\0&0&7&7\\0&0&0&0\end{bmatrix}\therefore \rho(A)=3$

$\therefore$basis for subspace W of R^4 pivote columns of A

$\therefore$basis for subspace W $=B=\left\{\begin{bmatrix}1\\0\\1\\2\end{bmatrix},\begin{bmatrix}-3\\1\\0\\5\end{bmatrix},\begin{bmatrix}2\\-4\\-3\\-2\end{bmatrix}\right\}$ $\therefore \dim(\mathrm{W})=3.$

Column space, Row space and Null space of a Matrix

 # Column Space of a Matrix

Let A be $m\times n$ matrix. Column space of matrix A is the set of all possible linear combinations
of columns of A, denoted as $\mathrm{col}(\mathrm{A})$, i.e., *If* $\mathrm{A}=\begin{bmatrix}v_1 & v_2 & \cdots & v_n\end{bmatrix}$ then

$col(\mathrm{A})=span\{v_1\quad v_2\quad \cdots\quad v_n\}$

Note : 1) Number of columns of $\mathrm{A}=n$

2) Number of vectors in $\mathrm{col}\ (\mathrm{A})=$ infinite.

3) $\mathrm{col}\ (\mathrm{A})$ is a subspace of R^m.

4) $y\in\mathrm{col}\ (\mathrm{A})$ means the system of linear equations $y=\mathrm{A}X$ for some $X\in\mathrm{R}^n$ is consistent. i. e. $col(\mathrm{A})=\left\{y\in\mathrm{R}^m : y=\mathrm{A}X\ \ for\ some\ X\in\mathrm{R}^n\right\}$

5) Elementary row operation on a matrix does not affect the linear dependence

relation among the columns of the matrix.

6) The Pivot columns of matrix A form a basis for Col(A).

 # Row Space of a Matrix

Let A be $m \times n$ matrix. Row space of matrix A is the set of all possible linear combinations of rows of A, denoted as $\text{row}(A)$.

Note : 1) number of rows of $A = m$

2) number of vectors in $\text{row}\ (A) =$ infinite

3) $\text{row}\ (A)$ is a subspace of R^n

4) $y \in \text{row}\ (A)$ means this system of linear equations $y = A^T X$ for some $X \in R^m$ is consistent.

 # Null Space of a Matrix

Let A be $m \times n$ matrix. Null space of A is the set of all those vectors in R^n which get map on to zero by A , denoted as $\text{Null}(A)$, i. e. , $\text{Null}(A) = \{x \in R^n \mid AX = 0\}$.

Note : 1) $\text{Null}\ (A)$ is a subspace of R^n. $0 \in \text{Null}\ (A)$. Let $x_1, x_2 \in \text{Null}\ (A)$, then

$$A x_1 = 0, A x_2 = 0 \Rightarrow A(x_1 + x_2) = A x_1 + A x_2 = 0 \Rightarrow x_1 + x_2 = \text{Null}(A)$$

$$\alpha(A x_1) = \alpha A x_1 = 0 \quad \Rightarrow \quad \alpha A = \text{Null}\ (A)$$

Null (A) is the subspace of R^n (solution space of the homogeneous system $AX = 0$.)

2) If $\rho(A) = r$, then $\dim(\text{Null}\ A) = n - r =$ Nullity of A.

3) $\dim(\text{col}\ A) = \dim(\text{row}\ A) = r$.

Theorem : For any matrix $A_{m\times n}$, $\dim(\text{Null}\ A) + \dim(\text{col}\ A) =$ number of columns of A.

Illustrative Examples

1. Determine the basis for $\text{Null}\ (A), \text{col}\ (A)$ and $\text{row}\ (A)$ and hence determine their dimensions. Assume that A is row equivalent to B.

$$A = \begin{bmatrix} 1 & -3 & 4 & -1 & 9 \\ -2 & 6 & -6 & -11 & -10 \\ -3 & 9 & -6 & -6 & -3 \\ 3 & -9 & 4 & 9 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & -3 & 0 & 5 & -7 \\ 0 & 0 & 2 & -3 & 8 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution: Given : $A \sim B$.

Each non-pivot columns of B is a linear combination of pivot columns.

In this example it is to observe that

$$b_2 = -3b_1 \ and\ b_4 = 5b_1 - \frac{3}{2}b_3 \quad \therefore a_2 = -3a_1 \ and\ a_4 = 5a_1 - \frac{3}{2}a_3$$

by spanning set theorem we may discard $a_2\ and\ a_4$, to get Basis for column space of A, is the set containing pivot columns of A.

$$\text{Basis for } \text{col}\ (A) = \left\{ \begin{bmatrix} 1 \\ -2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ -6 \\ 4 \end{bmatrix}, \begin{bmatrix} 9 \\ -10 \\ -3 \\ 0 \end{bmatrix} \right\} \qquad \dim(\text{col}\ A) = 3$$

Basis for row space of A, is the set containing pivot rows of B.

Therefore Basis for $\text{row}\ (A)$ is $\{(1,-3,0,5,-7), (0,0,2,-3,8), (0,0,0,0,5)\}$.

Therefore $\dim(\text{row}\ A) = 3$.

Basis for $\text{Null}\ (A) -$ is the solution space of $AX = 0$.

Equivalent system is $x_1 - 3x_2 + 5x_4 - 7x_5 = 0,\ 2x_3 - 3x_4 + 8x_5 = 0, 5x_5 = 0 \Rightarrow \ x_5 = 0$

Let $x_4 = k_1$ & $x_2 = k_2$, $x_1 = 3k_2 - 5k_1$.

$\therefore$ Solution vector $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3k_2 - 5k_1 \\ k_2 \\ 3k_1 \\ k_1 \\ 0 \end{bmatrix} = k_1 \begin{bmatrix} -5 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Basis for Null (A) is

$\left\{ \begin{bmatrix} -5 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$. $\dim\ (\text{Null (A)}) = 2$.

2. Find k such that Null space of A is subsapce of R^k and column space of A is subsapce of R^k, where $\mathrm{A} = \begin{bmatrix} 1 & 2 & -1 & 5 \\ 3 & 1 & -1 & 1 \\ 7 & 6 & 5 & 2 \\ 1 & 2 & -3 & 1 \\ 2 & 0 & 1 & 3 \end{bmatrix}$.

Sol$^{\mathbf{n}}$. $i)$ Here A is 5×4 matrix and we know that Null space is a subspace of R^4, $\therefore k = 4$.

$ii)$ Here A is 5×4 matrix and we know that column space is a subspace of R^5, $\therefore k = 5$.

3. Let $\mathrm{A} = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}$ and $w = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$.

$i)$determine if w is in column space of A. $ii)$ Is w in null space of A? $iii)$ Is w in row space of A?

$ii)$ w is in $null(\mathrm{A})$ if and only if $\mathrm{A}w = 0$

$$\mathrm{A}w = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \therefore w \in null(\mathrm{A}).$$

$iii)$ w will be in row space of A, if there exists some vector $x \in \mathrm{R}^3$ such that $A^T x = w$ is consistant

$$\left[\mathrm{A}^T,\, w\right] = \begin{bmatrix} -8 & 6 & 4 & 2 \\ -2 & 4 & 0 & 1 \\ -9 & 8 & 4 & -2 \end{bmatrix} \xrightarrow{R_1 - R_3} \begin{bmatrix} 1 & -2 & 0 & 0 \\ -2 & 4 & 0 & 1 \\ -9 & 8 & 4 & -2 \end{bmatrix} \xrightarrow{R_2 + 2R_1,\ R_3 + 9R_1} \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -10 & 4 & -2 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & -10 & 4 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \therefore \rho\left[\mathrm{A},\, w\right] \neq \rho\left[\mathrm{A}\right]. \therefore w \notin Row(\mathrm{A}).$$

4. If $\mathrm{A}X = \mathrm{V}$ *and* $\mathrm{A}X = \mathrm{W}$ *are* both consistant. Is the equation $\mathrm{A}X = \mathrm{V} + \mathrm{W}$ consistant?

Sol$^{\mathbf{n}}$. $\mathrm{A}X = \mathrm{V}$ *is* consistant means $V \in col(A)$, and $AX = W$ *is* consistant means $W \in col(A)$. But $col(\mathrm{A})$ is a subspace.

$\therefore \mathrm{V} + \mathrm{W} \in Col(A) \therefore AX = V + W$ is consistant.

5. Let $H = Span\{v_1,\ v_2\}$ and $W = \{u_1,\ u_2\}$, where $v_1 = \begin{bmatrix} 5 \\ 3 \\ 8 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$, $u_1 = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$,

and $u_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $K = H \cap W$, find the basis for K.

Sol$^{\mathbf{n}}$. Geometrically H and W are the plains in R^3. $H \cap W$ is a line of intersection of the planes H and W. $\therefore$ *K can be written as* $c_1v_1 + c_2v_2$ *and also as* $c_3u_1 + c_4u_4$.

Coefficient matrix $A = \begin{bmatrix} 5 & 1 & -2 & 0 \\ 3 & 3 & 1 & 0 \\ 8 & 4 & -4 & -1 \end{bmatrix}$ $\qquad \therefore \rho(A) = 3$

By reducing it to echelon form $A = \begin{bmatrix} 1 & 5 & 4 & 0 \\ 0 & -12 & -11 & 0 \\ 0 & 0 & -3 & -1 \end{bmatrix}$

Let $c_4 = k$ (free variable)

$c_3 = -\dfrac{1}{3}k,\ c_2 = \dfrac{11}{36}k,\quad c_1 = -\dfrac{7}{36}k.$ Thus $c_1 = -\dfrac{7}{36}k, c_2 = \dfrac{11}{36}k,\ c_3 = -\dfrac{1}{3}k,\ c_4 = k$

Every vector in K is either $c_1v_1 + c_2v_2$ *or* $c_3u_1 + c_4u_2$.

$$-\frac{7}{36}k\begin{bmatrix} 5 \\ 3 \\ 8 \end{bmatrix} + \frac{11}{36}k\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -24k/36 \\ 12k/36 \\ -12k/36 \end{bmatrix} = \frac{k}{3}\begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$$

OR $-\dfrac{1}{3}k\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} + k\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \dfrac{k}{3}\begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$ $\therefore$ Basis for $K = \left\{\begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}\right\}$.