# MATHEMATICS AND STATISTICS

# ES1043

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# FUNDAMENTAL SUBSPACES

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ 4 & 3 & 0 \\ 2 & 1 & 1 \end{bmatrix}$$

$$R_1 \approx (1, -2, 3), R_2 \approx (0, 1, -1),$$

$$R_3 \approx (4, 3, 3), R_4 \approx (2, 1, 1)$$

$$C_1 \approx \begin{bmatrix} 1 \\ 0 \\ 4 \\ 2 \end{bmatrix}, C_2 \approx \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, C_3 \approx \begin{bmatrix} 3 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Given a matrix A of order $m \times n$, its rows are vectors in $\mathbb{R}^n$ and columns of  are vectors in $\mathbb{R}^m$.

# Row Space

The collection of all possible linear combinations of rows of A, i.e., span of rows of A, denoted as row(A)/ Row(A) is defined as Row space.

$\text{row(A)=span}\left\{\text{R}_1, \text{R}_2, \cdots, \text{R}_m\right\}$.

The set of all linearly independent rows of A form basis of row(A) and dim(row(A))=rank of A=$\rho$(A).

# Column Space

The collection of all possible linear combinations of columns of A, i.e., span of columns of A, denoted as col(A)/ Col(A) is defined as Column space.

$\text{col(A)=span}\left\{C_1, C_2, \cdots, C_n\right\}.$

The set of all linearly independent columns of A form basis of col(A) and dim(col(A))=rank of A=$\rho$(A).

Note that :

column of $A^t$ = rows of A and rows of $A^t$ = column of A

Hence row($A^t$)=col(A) and col($A^t$)=row(A).

Hence to find basis of col(A)

1. Reduce $A^t$ to echelon form and the non-zero rows in echelon form of $A^t$ constitute basis of col(A).

OR

2. Reduce A to echelon form, the columns of A with pivot positions constitute basis of col(A).

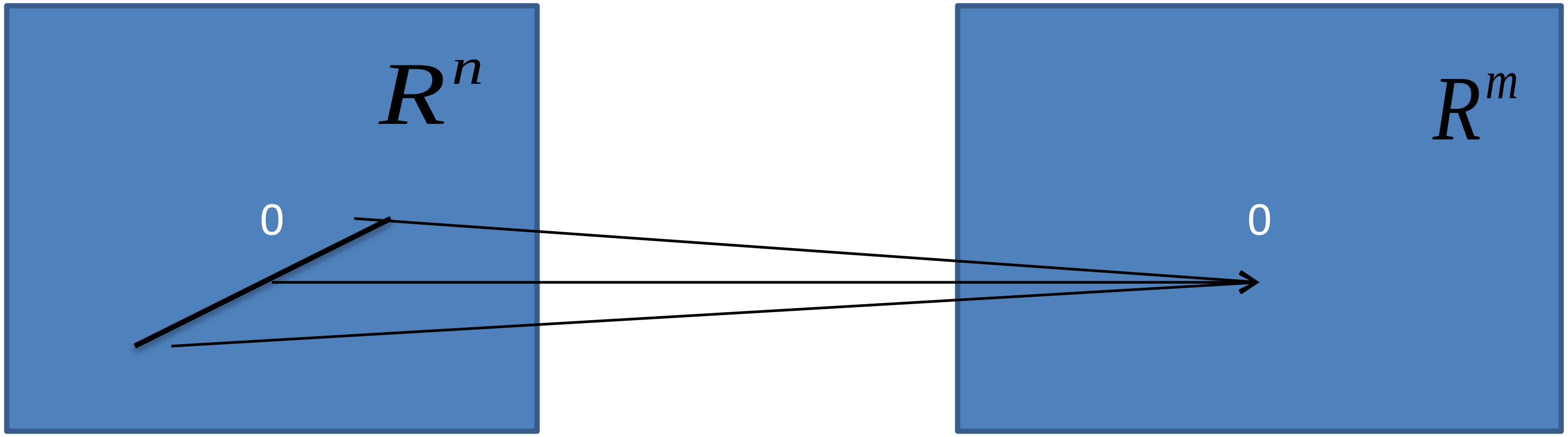
# Null Space

Let A be $m \times n$ matrix

Null space of A= $\text{Null}(A) = \left\{ X \in R^n : AX = 0 \right\}$

, *i.e.*, Null(A) is the solution space of

homogeneous system of linear equation AX=0.



Null(A) is a subspace of $R^n$.

$$\mathrm{Null(A)} = \left\{ \mathrm{X} \in R^n : \mathrm{AX} = 0 \right\}$$

Let $\mathrm{X}, \mathrm{Y} \in \mathrm{Null(A)} \Rightarrow \mathrm{AX} = 0, \mathrm{AY} = 0$

$$\mathrm{A(X+Y)} = \mathrm{AX} + \mathrm{AY} = 0 + 0 = 0 \Rightarrow \mathrm{X} + \mathrm{Y} \in \mathrm{Null(A)}$$

Let $k \in R$ and $\mathrm{X} \in \mathrm{Null(A)}$

$$\mathrm{A}\left(k\mathrm{X}\right) = k\mathrm{AX} = k0 = 0 \Rightarrow k\mathrm{X} \in \mathrm{Null(A)}$$

Null(A) is closed under addition and scalar multiplication.

If $\rho(\mathrm{A}) = r$, then $\dim(\mathrm{Null}\,\mathrm{A}) = n - r$

and is called as nullity.

Null space of $\mathrm{A}^t$

$\mathrm{Null}(\mathrm{A}^t) = \left\{ \mathrm{Y} \in R^m : \mathrm{A}^t\mathrm{Y} = 0 \right\}$ is a subspace of $R^m$.

The dimension of this subspace is known as nullity of $\mathrm{A}^t$.

The dimension of $\mathrm{Null}(\mathrm{A}^t) = m - r$, where $r = \rho(\mathrm{A})$

Note that Null space of $\mathrm{A}^t$ is also known as left null space of A and denoted as LNull(A)/lNull(A).

$$1.\ \text{Is } u = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix} \in \text{Null(A)},\ \text{where A} = \begin{bmatrix} 3 & -5 & -3 \\ 6 & -2 & 0 \\ -8 & 4 & 1 \end{bmatrix}?$$

$$\text{Consider, A}u = \begin{bmatrix} 3 & -5 & -3 \\ 6 & -2 & 0 \\ -8 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.\ \therefore\ u \in \text{Null(A)}.$$

$$2.\ \text{Is } u = \begin{bmatrix} 5 \\ 3 \\ 0 \end{bmatrix} \in \text{Null(A)},\ \text{where A} = \begin{bmatrix} 3 & -5 & -3 \\ 6 & -2 & 0 \\ -8 & 4 & 1 \end{bmatrix}?$$

$$\text{A}u = \begin{bmatrix} 3 & -5 & -3 \\ 6 & -2 & 0 \\ -8 & 4 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 18 \\ 52 \end{bmatrix} \neq 0 \therefore\ u \notin \text{Null(A)}.$$

3. Find the basis and dimension for $\mathrm{row}(A)$, $\mathrm{col}(A)$ and $\mathrm{Null}(A)$ $\mathrm{Null}(A^t)$, where $A = \begin{bmatrix} 1 & 2 & -1 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$.

Consider, $A = \begin{bmatrix} 1 & 2 & -1 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$

Reduce it to echelon form $\begin{bmatrix} 1 & 2 & -1 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

From echelon form of A, $\rho\left(A\right)=2$.

Thus dimension of row space dim(row(A))=2.

row(A)=span$\left\{\left(1,2,-1,0,3\right),\left(0,0,1,0,1\right)\right\}$ and the basis is $\left\{\left(1,2,-1,0,3\right),\left(0,0,1,0,1\right)\right\}$.

To find Null space

By definition, Null(A)=$\left\{X : AX=0\right\}$, i.e., solution set of AX=0.

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{c} x_1 + 2x_2 - x_3 + 3x_5 = 0 \\ x_3 + x_5 = 0 \end{array}$$

Free variables are $x_2, x_4$ and $x_5$. Therefore

let $x_2 = r, x_4$=s and $x_5 = t, r, s, t \in R$.

Hence $x_3 = -t$ and $x_1 = -2r - 4t$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2r-4t \\ r \\ -t \\ s \\ t \end{bmatrix} = r\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -4 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow$$

$$\text{Null(A)} = span\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}\right\}$$

Note that $\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}\right\}$ is linearly independent.

$$\therefore \text{ Basis for Null (A) is } \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ and dimension}$$

of Null space ,i.e., nullity of A is 3.

$$\text{Consider, } A^t = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 0 \\ 3 & 1 & 1 \end{bmatrix} \text{. Reduce the matrix to echelon form}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 0 \\ 3 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \rho\left(A^t\right) = 2$$

Thus, col(A)=span$\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$ and basis of col(A) is $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$.

Dimension of col(A)=2.

$$\begin{bmatrix}1&0&0\\0&1&1\\0&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}\Rightarrow\begin{matrix}y_1=0\\y_2+y_3=0\end{matrix}$$

Here free variable is $y_3$, let $y_3 = t \in R$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \therefore \text{Null}(\text{A}^t) = \text{span}\left\{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right\}.$$

Further, $\left\{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right\}$ is also linearly independent.

$\therefore$ Basis of $\text{Null}(\text{A}^t)$ is $\left\{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right\}$ and dimension of

$\text{Null}(\text{A}^t)$, i.e., nullity of $\text{A}^t$ is 1.

**Another Method to find basis of column space of A**

$$\text{Consider, } A = \begin{bmatrix} 1 & 2 & -1 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

$$\text{Echelon form of A is } \begin{bmatrix} 1 & 2 & -1 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Here the pivot elements are observed in column 1 and column 3. Hence consider first and third columns of A.

Thus, the basis of col(A) is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Find the basis for Null(A), row(A) and determine it dimension.

$$\text{where A} = \begin{bmatrix} 1 & 2 & -1 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$\text{Let } u \in \text{Null(A)}, \therefore \text{Au} = \begin{bmatrix} 1 & 2 & -1 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = 0$$

Here $\rho(A) = 3$. Therefore there are 2 free variables.

here $u_2$ and $u_4$ are free variables. Let $u_2 = s$, $u_4 = t$

Equivalent system is $u_1 + 2u_2 - u_3 + 3u_5 = 0$

$u_3 - u_5 = 0, u_5 = 0 \Rightarrow u_3 = 0,\ u_1 = -2s$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} -2s \\ s \\ 0 \\ t \\ 0 \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\therefore \text{ Null space of A is } Null(\mathrm{A}) = span\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \; and$$

$$\text{basis is } \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}. \; Dim(Null(\mathrm{A})) = 2.$$

Find basis for $Null(\mathrm{A})$, $Col(\mathrm{A})$ and $Row(\mathrm{A})$ *where*

$$\mathrm{A} = \begin{bmatrix} 2 & -4 & 0 & 2 & 0 \\ -1 & 2 & 1 & 2 & 4 \\ 2 & -2 & 1 & 4 & 4 \end{bmatrix}$$

We will reduce A to echelon form

$$R_1 + R_2 \Rightarrow \begin{bmatrix} 1 & -2 & 1 & 4 & 4 \\ -1 & 2 & 1 & 2 & 4 \\ 2 & -2 & 1 & 4 & 4 \end{bmatrix}$$

$$\begin{matrix} R_2 + R_1 \\ R_3 - 2R_1 \end{matrix} \Rightarrow \begin{bmatrix} 1 & -2 & 1 & 4 & 4 \\ 0 & 0 & 2 & 6 & 8 \\ 0 & 2 & -1 & -4 & -4 \end{bmatrix}$$

$$R_{23} \Rightarrow \begin{bmatrix} 1 & -2 & 1 & 4 & 4 \\ 0 & 2 & -1 & -4 & -4 \\ 0 & 0 & 2 & 6 & 8 \end{bmatrix}$$

$$\frac{1}{2}R_3 \Rightarrow \begin{bmatrix} 1 & -2 & 1 & 4 & 4 \\ 0 & 2 & -1 & -4 & -4 \\ 0 & 0 & 1 & 3 & 4 \end{bmatrix}$$

$$\text{Basis for } Col(\mathrm{A}) = \left\{ \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ and } \dim\left(Col(\mathrm{A})\right) = 3.$$

Basis for $Row(\mathrm{A})$

$$\left\{ \begin{bmatrix} 1 & -2 & 1 & 4 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 2 & -1 & -4 & -4 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 3 & 4 \end{bmatrix} \right\}$$

and $\dim\left(Row(\mathrm{A})\right) = 3.$

$$\text{Let } x = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} \in Null(\mathrm{A}) \Rightarrow \begin{bmatrix} 1 & -2 & 1 & 4 & 4 \\ 0 & 2 & -1 & -4 & -4 \\ 0 & 0 & 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This gives $u_3 + 3u_4 + 4u_5 = 0, 2u_2 - u_3 - 4u_4 - 4u_5 = 0$

$u_1 - 2u_2 + u_3 + 4u_4 + 4u_5 = 0.$

Let $u_4 = t, u_5 = s, t, s \in R \Rightarrow u_3 = -3t - 4s, u_2 = \dfrac{t}{2}, u_1 = 0$

The solution is

$$\begin{bmatrix} 0 \\ t/2 \\ -3t - 4s \\ t \\ s \end{bmatrix} = \frac{t}{2}\begin{bmatrix} 0 \\ 1 \\ -6 \\ 2 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 0 \\ -4 \\ 0 \\ 1 \end{bmatrix} \Rightarrow Null(\mathrm{A}) = span\left\{\begin{bmatrix} 0 \\ 1 \\ -6 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -4 \\ 0 \\ 1 \end{bmatrix}\right\}$$

$\dim(Null(\mathrm{A})) = 2.$

Let $A = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}$ and $w = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$.

$i$) determine if $w$ is in column space of A.

$ii$) Is $w$ in null space of A? $iii$) Is $w$ in row space of A?

$i$**)** $w$ will be in column space of A, if there exists some vector $x \in R^3$ such that $Ax = w$ is consistant.

$$\text{let}\left[A,\ w\right] = \begin{bmatrix} -8 & -2 & -9 & 2 \\ 6 & 4 & 8 & 1 \\ 4 & 0 & 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 2 & -1 \\ 0 & 4 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho[A, w] = \rho[A] = 2$, $\therefore$ the system of linear equations is consistant.

$\therefore\ w \in col(A)$

$ii)$ $w$ is in $null(A)$ if and only if $Aw = 0$

$$Aw = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \therefore w \in null(A).$$

**iii)** $w$ will be in row space of $A$, if there exists some vector $x \in R^3$ such that $A^T x = w$ is consistant

$$\left[A^T, w\right] = \begin{bmatrix} -8 & 6 & 4 & 2 \\ -2 & 4 & 0 & 1 \\ -9 & 8 & 4 & -2 \end{bmatrix}$$

$$\left[A^T \;\vdots\; w\right] \sim \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & -10 & 4 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \therefore \rho[A,\ w] \neq \rho[A] \therefore\ w \notin Row(A).$$

Ex : Let $A_{3\times 3}$ be such that $|A| \neq 0$, find fundamental subspaces of A. Let $B = [A \quad A]$, also find Fundamental subspaces of B. Further, find bases for all subspaces.

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$. since $|A| \neq 0$, AX=0 has a trivial solution,

therefore $Null(A) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$. As A is in euchelon form, all columns of

A are pivot columns, $Col(A) = span\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix} \right\}$ and

$Row(A) = span\{(1,2,3),(0,1,5),(0,0,1)\}$.

Basis of $Col(A)$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix} \right\}$ *and* basis of $Row(A)$

is $\{(1,2,3),(0,1,5),(0,0,1)\}$.

$$B = \begin{bmatrix} A & A \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$ B is in echelon form,

first three columns of A are pivot columns. Therefore,

$$Col(A) = span\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$ Basis is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

$$Row(A) = span\left\{ (1,0,0,1,0,0), (0,1,0,0,1,0), (0,0,1,0,0,1) \right\}.$$

Basis of $Row(A)$ is $\left\{ (1,0,0,1,0,0), (0,1,0,0,1,0), (0,0,1,0,0,1) \right\}$.

To find null space of B, solve BX=0.

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x_1 + x_4 = 0 \\ x_2 + x_5 = 0 \\ x_3 + x_6 = 0 \end{matrix}$$

Free variables are $x_4, x_5 \; and \; x_6$.

Let $x_4 = t, x_5 = s \; and \; x_6 = r, t, s, r \in R$

$\Rightarrow x_1 = -t, \; x_2 = -s, \; x_3 = -r.$

$$\therefore \mathrm{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} t \\ s \\ r \\ -t \\ -s \\ -r \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} + r\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}.$$

$$\text{Thus } Null(\mathrm{A}) = span\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}\right\} and \text{ basis is } \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}\right\}.$$

Ex: If $AX = V$ and $AX = W$ are both consistant.

Is the equation $AX = V + W$ consistant?

$AX = V$ is consistant means $V \in col(A)$, and

$AX = W$ *is* consistant means $W \in col(A)$.

But $col(A)$ *is a subspace.* $\therefore V + W \in col(A) \therefore AX = V + W$ is consistant.

**Theorem:** *Let $A$ be $m \times n$ matrix and $\rho(A) = r$, then*

$$\dim(col(A)) + \dim(Null(A)) = number\ of\ Columns\ of\ A.$$

Determine if the following statements true or falls, and justify your answers.

a) A linearly independent set in a subspace H is a basis for H.

b) The columns of a nonsingular matrix forms a basis for Col(A).

c) The null space of an $m \times n$ matrix A is a subsapce of $R^m$.

$d)\, col(A)$ is a set of vectors that can be written as $AX$ for some $X$

$e)$ $A$ plane in $R^3$ is two dimensional subspace of $R^3$.

$f)$ The dimension of vector space $P_3$

(set of all polynomials at most of degree 3), is 3.

g) If the $6 \times 3$ matrix A rank 3, then dimension of $Null(A) = 0$.

$h)$ If the $4 \times 5$ matrix A has 4 pivote columns, then $col(A) = R^4$.