ENGINEERING MATHEMATICS

ES1032

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Eigen Values and Eigen Vectors

- **Communication systems :** To determine a threshold for transmission of information through a communication medium
- **Designing bridges :** The natural frequency of the bridge is the eigen value of smallest magnitude of a system that models the bridge.
- **Electrical Engineering :** For decoupling three-phase systems through symmetrical component transformation

- **Designing car stereo system :** Design of the car stereo systems, where it helps to reproduce the vibration of the car due to the music
- **Mechanical Engineering :** Vectors in the principle directions are the eigenvectors and the percentage deformation in each principle direction is the corresponding eigen value
- **Oil companies :** to explore land for oil

Consider, $Y=\begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix} X$.

$$\begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}=\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}=\begin{bmatrix} -2 \\ 5 \end{bmatrix},$$

$$\begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}=\begin{bmatrix} -1 \\ 7 \end{bmatrix}.$$

But $\begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}=\begin{bmatrix} 3 \\ -3 \end{bmatrix}=3\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Let A be an $n \times n$ matrix.

A scalar $\left(\text{real number}\right)\lambda$ is called eigen value of A if there is a non-zero vector X such that

$$AX=\lambda X.$$

The vector X is called an eigen vector of A corresponding to λ.

Geometrically, eigen vectors are those non zero vectors which get mapped on to their scalar multiples by matrix A

λ eigen value with eigen vector X

$\mathrm{AX}=\lambda\mathrm{X} \Rightarrow \mathrm{AX}-\lambda\mathrm{X}=0, i.e., \left(\mathrm{A}-\lambda I\right) X = 0.$

For non-trivial solution $\det\left(\mathrm{A}-\lambda I\right)=0$

Characteristic equation :

$$\det\left(\mathrm{A}-\lambda I\right)=0$$

Eigen values

Roots of $\det(A - \lambda I) = 0$

characteristic values / latent roots / proper values.

Eigen Vectors

Solutions of $(A - \lambda I)X = 0$, for particular choice of λ.

Spectrum

Set of all eigen values

Characteristic equation for 2×2 matrix

$$\mathrm{A}=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \mathrm{A}-\lambda I=\begin{bmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{bmatrix}$$

$$|\mathrm{A}-\lambda I| = 0 \quad \Rightarrow \quad \left(a_{11}-\lambda\right)\left(a_{22}-\lambda\right)-a_{12}a_{21}=0$$

$$\lambda^2 - S_1\lambda + |\mathrm{A}| = 0, \text{where}$$

$$S_1 = trace\left(\mathrm{A}\right) = a_{11}+a_{22}, |\mathrm{A}| = a_{11}a_{22}-a_{21}a_{12}$$

$$A = \begin{bmatrix} 4 & -2 \\ -3 & 9 \end{bmatrix}$$

Characteristic equation is

$$\lambda^2 - S_1\lambda + |\mathrm{A}| = 0$$

$$S_1 = tr(\mathrm{A}) = a_{11} + a_{22} = 13,$$

$$|\mathrm{A}| = a_{11}a_{22} - a_{21}a_{12} = 36 - 6 = 30$$

$$\lambda^2 - 13\lambda + 30 = 0$$

Characteristic equation for 3×3 matrix

$$\text{Consider, } A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \text{A-}\lambda\text{I}=\begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{bmatrix}$$

$$|A-\lambda I| = 0 \Rightarrow \lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

$$S_1 = \text{sum of diagonal elements=trace of A}= a_{11}+a_{22}+a_{33}$$

$$S_2 = \text{sum of minors of diagonal elements of A}$$

$$= \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$\det(A) = |A| = a_{11}(a_{22}a_{33}-a_{32}a_{23}) - a_{12}(a_{21}a_{33}-a_{31}a_{23}) + a_{13}(a_{21}a_{32}-a_{31}a_{23})$$

$$A = \begin{bmatrix} 8 & 0 & 3 \\ 2 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

Characteristic eq^n is $\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$

$$S_1 = 8 + 2 + 3 = 13, S_2 = \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 3 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 0 \\ 2 & 2 \end{vmatrix} = 40$$

$$|A| = 8(6) - 0(6-2) + 3(0-4) = 48 - 12 = 36.$$

$$\therefore \text{Char}\, eq^n \text{ is } \lambda^3 - 13\lambda^2 + 40\lambda - 36 = 0.$$

$$\text{Consider } A = \begin{bmatrix} 8 & 0 & 3 \\ 2 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

$$\text{Char eq}^{n} \text{ is } \lambda^3 - 13\lambda^2 + 40\lambda - 36 = 0$$

$$\text{Eigen values are } 2,\ 2 \text{ and } 9.$$

Algebraic Multiplicity(AM): The number of times the eigen value is repeated as root of characteristic equation.

Eigen value 9 is repeated only once, hence AM of 9 is **ONE.**

Eigen value 2 repeated twice, hence AM of 2 is **TWO**

$$AX = \lambda X \Rightarrow \left(A - \lambda I\right) X = 0$$

$$\lambda = 9, \ \ [A - 9I]X = 0$$

$$A - 9I = \begin{bmatrix} -1 & 0 & 3 \\ 2 & -7 & 1 \\ 2 & 0 & -6 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & 3 \\ 2 & -7 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{c} -x_1 + 0x_2 + 3x_3 = 0 \\ 2x_1 - 7x_2 + x_3 = 0 \end{array}$$

Let $x_1 = 3t, \ \ x_2 = t, \ \ x_3 = t.$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} t \ \therefore \ \ X_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}.$$

$$\lambda = 2, \quad [A - 2I]X = 0$$

$$A - 2I = \begin{bmatrix} 6 & 0 & 3 \\ 2 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow 2x_1 + 0x_2 + 1x_3 = 0$$

$$\text{Let} \quad x_1 = x_1 = \frac{-t}{2}, \quad x_2 = s, \quad x_3 = t.$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{-t}{2} \\ s \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \frac{-t}{2} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} s$$

$$\therefore \quad X_2 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \& X_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Eigen space: The set of all eigen vectors corresponding to given eigen value λ, which is a subspace of $\mathbb{R}^n$.

Denoted as $\mathrm{E}(\lambda)$ or E_λ

$$E_{\lambda=9} = span\left\{\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}\right\} \text{ and the basis is } \left\{\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}\right\}$$

$$E_{\lambda=2} = span\left\{\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$$

and the basis is $\left\{\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$.

$\left\{\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$ is linearly independent.

If all eigen values are distinct,i.e.,$\lambda_1 \neq \lambda_2 \neq \lambda_3$

then there are 3 linearly independent eigen vectors.

If one of the eigen values is repeated $\lambda_1 \neq \lambda_2 = \lambda_3$

OR all are repeated $\lambda_1 = \lambda_2 = \lambda_3$

$$\begin{cases} \text{3 linearly independent eign vectors if , } \rho(A - \lambda_1 I) = 0 \\ \text{2 linearly independent eign vectors if , } \rho(A - \lambda_2 I) = 1 \\ \text{1 linearly independent eigen vectors if , } \rho(A - \lambda_2 I) = 2 \end{cases}$$

Geometric Multiplicity :

the number of linearly independent eigen vectors corresponding to given eigen value.

$$\boxed{AM \geq GM}$$

Eigen value 9 has only once e.v., hence GM of 9 is **ONE. AM=GM=1**

Eigen value 2 has two lin. indep. e.vs., hence GM of 2 is **TWO. AM=GM=2**

Properties of eigen values and eigen vectors

If X is an eigen vector of A, corresponding to eigen value λ, then

1. λ^n is a eigen value of A^n with same eigen vector.
2. If all eigen values of A are non-zero the eigen values of A^{-1} are $\frac{1}{\lambda}$.
3. Eigen values of kA are $k\lambda, k \in \mathbb{R}$ with same eigen vector.
4. Eigen value of $A^3 + k_1 A^2 + k_2 A + k_3 I$ is, $\lambda^3 + k_1\lambda^2 + k_2\lambda + k_3$ where k_1, k_2 and k_3 are real numbers.
5. Determinant of A is product of eiegen values and trace of A is sum of eigen values.

$$A = \begin{bmatrix} 2 & 8 & -5 \\ 0 & 5 & -1 \\ 0 & 0 & 3 \end{bmatrix}$$

1) Find eigen values of A

2) Find eigen values of A^3

3) Find eigen values of A^{-1}

4) Find eigen values of $4A$

5) Find eigen values of $Adj(A)$

6) Find eigen values of $A^3 - 2A$

Is $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ a eigen vector of $A = \begin{bmatrix} 4 & -2 \\ -3 & 9 \end{bmatrix}$?

$$A = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$$

$$A = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$$

Is $\lambda = 4$ an eigen value of $\begin{bmatrix} 3 & 0 & 2 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$? If so find an eigen vector.

Find h in the matrix such that eigen space for

$\lambda = 5$ is two dimensional, $A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & h & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 2 & 5 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$$