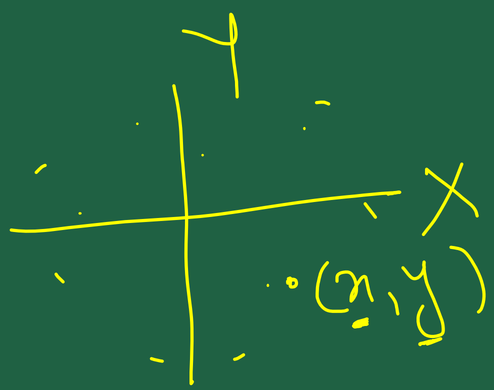
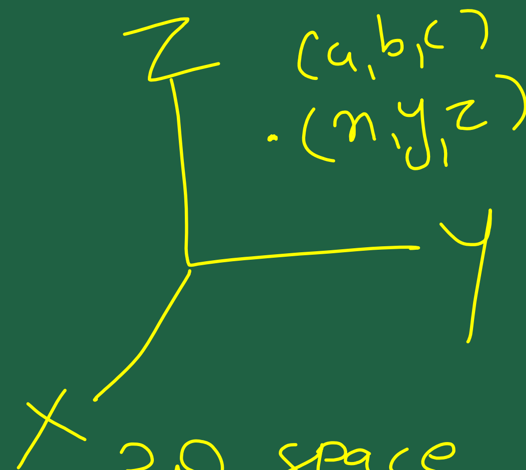
Vector Spaces

Cartesian product of two sets A & B :-



XY plane



3D-space

Cartesian product of two sets A & B

$$A \times B = \{ (a, b) \mid a \in A,\ b \in B \}$$

$$B \times A = \{ (b, a) \mid b \in B,\ a \in A \}$$

Generalised as

$$A \times B \times C = \{ (a, b, c) \mid a \in A,\ b \in B,\ c \in C \}$$

$\therefore$ XY plane is

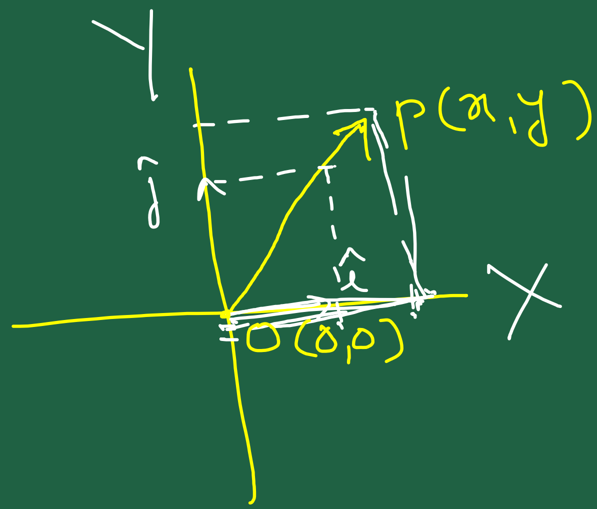
$$\mathbb{R} \times \mathbb{R} = \{ (x, y) \mid x \in \mathbb{R},\ y \in \mathbb{R} \}$$

Similarly 3D space is

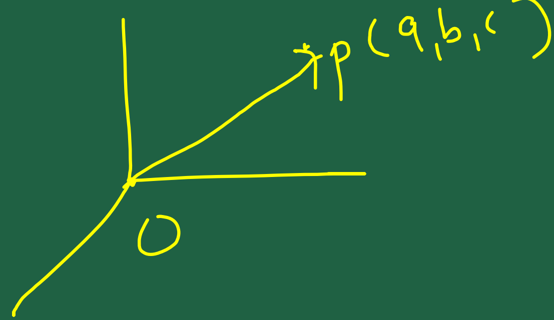
$$\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{ (x, y, z) \mid x, y, z \in \mathbb{R} \}$$

Henceonwards $\underbrace{\mathbb{R} \times \mathbb{R}}_{2\text{ times}} = \mathbb{R}^2$

$$\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^3$$



$$\vec{OP} = x\hat{i} + y\hat{j}$$



$$\vec{OP} = a\hat{i} + b\hat{j} + c\hat{k}$$

$$\underbrace{\text{Matrix, Polynomial, functions, Numbers}}_{\text{mathematical quantities}}$$

Satisfy Certain rules

Vector Spaces

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \longrightarrow (1 \quad 2 \quad 3 \quad 4)$$

array
list

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longrightarrow (1 \; 2 \; 0 \; 0 \; 0 \; 0 \; 1 \; 0)$$

8 elts

$$[\quad]_{m\times n} \longrightarrow (\cdot \; - \; - \; - \; - \; - \; -)$$

mn elements

$x^2 + 3x - 1 \longrightarrow (1 \quad 3 \quad -1)$

$x^4 + 3x^2 + 9 \longrightarrow (1 \quad 0 \quad 3 \quad 0 \quad 9)$

$\longrightarrow (9 \quad 0 \quad 3 \quad 0 \quad 1)$

1. $A_{2\times 2} + B_{2\times 2} = C_{2\times 2}$ Closure property '+'
2. $A + B = B + A$ commutative
3. $(A+B)+C = A+(B+C)$ Asso
4. $(-A)+A = A + (-A) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2}$ Additive Inverse
5. $A + O = A = O + A$ Additive Identity
6. $B_{2\times 2} = c A_{2\times 2}$ closure prop. '.' , $c \in \mathbb{R}$
7. $k(A+B) = kA + kB$, $k \in \mathbb{R}$ } Dist
8. $(k+l)A = kA + lA$, $k, l \in \mathbb{R}$
9. $(kl)A = k(lA) = l(kA)$ Asso.
10. $(1)A = A$ scalar identity

S - Non-empty set, '+' Addition, '.' Scalar mult.

$\forall$ (for all) $s \in S$ & $\forall k \in \mathbb{R}$

Definition (Vector Space)

Vector space is an Algebraic Structure s.t.

Let V be a non-empty set on which two operations i.e. vector addition denoted by '+' & scalar multiplication denoted by '.' are defined.

If for every vector u, v, w in V and for every scalar c, d in R [Here R is known as field F] following properties are satisfied then V is said to be the vector space over R

1. $u+v \in V$ i.e. closure prop of '+'
2. $u+v = v+u$ Commutative prop. of '+'
3. $u+(v+w) = (u+v)+w$ Associative prop. of '+'
4. $u+\underline{0} = u = 0+u$ Additive identity exists in V

   0 - Zero vector
5. $u+(\underline{-u}) = \underline{0} = (-u)+u$ Additive inverse exists in V

   -u Additive inverse
6. $c \cdot u \in V$ closure prop. of '.'
7. $(c+d)\cdot u = c\cdot u + d\cdot u$
8. $c\cdot(u+v) = c\cdot u + c\cdot v$

   } Distributive prop. of '+' & '.'
9. $(cd)\cdot u = c\cdot(d\cdot u)$ Associative prop. of '.'
10. $\underline{1}\cdot u = u$ Scalar Identity w.r.t. '.'

    ↳ Need not be always the no. 1

Refer to above ex. of matrices

Ex. let $M_{2\times 2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \, a, b, c, d \in \mathbb{R} \right\}$

and define vector addition &

scalar multiplication as below

Let $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$, $B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$

$C = \begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix} \in M_{2\times 2}$

& $c, d \in \mathbb{R}$

$$A+B = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}$$

and $c \cdot A = \begin{bmatrix} ca_1 & cb_1 \\ cc_1 & cd_1 \end{bmatrix}$

Prove that V is a vectorspace over $\mathbb{R}$

Sol^n ① $A+B = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}$

is a 2x2 matrix with

real entries.

Because a_1, a_2, b_1, b_2 — — —

all are real.

& addition of real no. is real.

$\Rightarrow \forall A, B \in V, A + B \in V$

$\Rightarrow$ Closure prop. is satisfied.

(2) $A+B = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}$

$= \begin{bmatrix} a_2+a_1 & b_2+b_1 \\ c_2+c_1 & d_2+d_1 \end{bmatrix}$ (real no.s addn is commutative)

$= B+A$

$\Rightarrow$ Commutative prop of '+' is satisfied.

(3) $A+(B+C) = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2+a_3 & b_2+b_3 \\ c_2+c_3 & d_2+d_3 \end{bmatrix}$

$= \begin{bmatrix} a_1+(a_2+a_3) & b_1+(b_2+b_3) \\ c_1+(c_2+c_3) & d_1+(d_2+d_3) \end{bmatrix}$

$$= \begin{bmatrix} (a_1+a_2)+a_3 & (b_1+b_2)+b_3 \\ (c_1+c_2)+c_3 & (d_1+d_2)+d_3 \end{bmatrix}$$

(Real no's add^n is Asso.
or Asso. of real no.)

$$= \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix} + \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}$$

$$= (A+B) + C$$

$\Rightarrow$ Vector addition is associative / Associative prop of '+' is satisfied

(4) $\exists$ $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in M_{2\times 2}$

s.t. $\forall$ $A \in M_{2\times 2}$

$$A + O = A = O + A$$

$\Rightarrow \exists$ zero vector $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ in $M_{2\times 2}$

⑤ $\exists \; -A = \begin{bmatrix} -a_1 & -b_1 \\ -c_1 & -d_1 \end{bmatrix} \in \underline{\underline{M_{2\times 2}}}$

for $\forall\, A = \begin{bmatrix} \underline{a_1} & \underline{b_1} \\ \underline{c_1} & \underline{d_1} \end{bmatrix} \in M_{2\times 2}$

s.t.

$$A + (-A) = \underline{\underline{O}} = (-A) + A$$

$\exists$ additive inverse for all vectors of V

⑥ $\underline{c} \cdot A = \begin{bmatrix} ca_1 & cb_1 \\ cc_1 & cd_1 \end{bmatrix} \in M_{2\times 2}$, $\forall\, A \in M_{2\times 2}$

$\because\; c \in \mathbb{R},\; a_1, b_1, c_1, d_1 \in \mathbb{R}$

$\Rightarrow\; ca_1, cb_1, cc_1, cd_1 \in \mathbb{R}$

$\Rightarrow$ Scalar multiplication is closed.

/ closure prop. of '$\cdot$' is satisfied.

⑦ $(c+d) \cdot A = (c+d) \cdot \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$

$$= \begin{bmatrix} (c+d)a_1 & (c+d)b_1 \\ (c+d)c_1 & (c+d)d_1 \end{bmatrix}$$

$$= \begin{bmatrix} ca_1+da_1 & cb_1+db_1 \\ cc_1+dc_1 & cd_1+dd_1 \end{bmatrix}$$ ($\because$ Real no. satisfies Distributive prop.)

$$= \begin{bmatrix} ca_1 & cb_1 \\ cc_1 & cd_1 \end{bmatrix} + \begin{bmatrix} da_1 & db_1 \\ dc_1 & dd_1 \end{bmatrix}$$

$$= c \cdot \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + d \cdot \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$$

$$= c \cdot A + d \cdot A$$

$\Rightarrow$ Distributive prop. satisfied.

8. $$c \cdot (A+B) = c \cdot \left\{ \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \right\}$$

$$= c \cdot \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}$$

$$= \begin{bmatrix} c(a_1+a_2) & c(b_1+b_2) \\ c(c_1+c_2) & c(d_1+d_2) \end{bmatrix}$$

$$= \begin{bmatrix} ca_1+ca_2 & cb_1+cb_2 \\ cc_1+cc_2 & cd_1+cd_2 \end{bmatrix}$$ ($\because$ Real no. satisfies Dist prop.)

$$= \begin{bmatrix} ca_1 & cb_1 \\ cc_1 & cd_1 \end{bmatrix} + \begin{bmatrix} ca_2 & cb_2 \\ cc_2 & cd_2 \end{bmatrix}$$

$$= c \cdot \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + c \cdot \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$$

$$= c \cdot A + c \cdot B$$

$\Rightarrow$ Dist. prop. is satisfied.

(9) $(cd)\cdot A = (cd)\cdot\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$

$= \begin{bmatrix} (cd)a_1 & (cd)b_1 \\ (cd)c_1 & (cd)d_1 \end{bmatrix}$

$= \begin{bmatrix} c(da_1) & c(db_1) \\ c(dc_1) & c(dd_1) \end{bmatrix}$

($\because$ Real no multiplication is associative)

$= c\cdot\begin{bmatrix} da_1 & db_1 \\ dc_1 & dd_1 \end{bmatrix}$

$= c\cdot(d\cdot A)$

$= c(dA)$

$\Rightarrow$ Associativity of scalar mult is satisfied.

(10) $\exists\ 1 \in \mathbb{R}$ s.t.

$1\cdot A = 1\cdot\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} = \begin{bmatrix} 1a_1 & 1b_1 \\ 1c_1 & 1d_1 \end{bmatrix}$

$= \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} = A$

$\Rightarrow$ Scalar Identity exists in V

$\therefore$ V is a vector Space over $\mathbb{R}$

Note :- Vector addition and scalar multiplication defined in the above example is known as Standard vector addition & Standard scalar multiplication.

OR

Usual vector addition & Usual scalar multiplication

Note :- Refer above ex.

$$V = M_{2\times 2}(\mathbb{R})$$
$$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; a, b, c, d \in \mathbb{R} \right\}$$

is a vector space over $\mathbb{R}$ w.r.t. Standard operations of '+' & '.'

Thus if generalised

$$\boxed{V = M_{m\times n}(\mathbb{R})}$$

is a vector space over $\mathbb{R}$ w.r.t. standard operation of '+' & '.'

Note :- Vector spaces w.r.t. standard operation are known as Standard vector space

Thus $V = M_{m\times n}(\mathbb{R})$ is a standard vector space.

—— x ——

Ex. Let
$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \; a, b, c, d \in \mathbb{R} \right\}$$

define $A+B = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}$

& $c \cdot A = \begin{bmatrix} ca_1 & cb_1 \\ c_1 & d_1 \end{bmatrix}$, $\forall c \in \mathbb{R}$

where $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$, $B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in V$

(Here '+' is standard vector addition
'·' is non-standard scalar mult.)

Is V a vector space over $\mathbb{R}$

Solⁿ Refer to above example for property 1 to 5

⑥ $c \cdot A = \begin{bmatrix} ca_1 & cb_1 \\ c_1 & d_1 \end{bmatrix} \in V$

($\because$ Real no.s multiplication is real no.)

$\Rightarrow$ '·' is closed

(7) $(c+d)\cdot A = \underline{(c+d)}\cdot \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$

$$= \begin{bmatrix} (c+d)a_1 & (c+d)b_1 \\ c_1 & d_1 \end{bmatrix}$$

$$= \begin{bmatrix} ca_1+da_1 & cb_1+db_1 \\ c_1 & d_1 \end{bmatrix} \quad \text{— (1)}$$

$$c\cdot A + d\cdot A = c\cdot \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + d\cdot \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$$

$$= \begin{bmatrix} ca_1 & cb_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} da_1 & db_1 \\ c_1 & d_1 \end{bmatrix}$$

$$c\cdot A + d\cdot A = \begin{bmatrix} ca_1+da_1 & cb_1+db_1 \\ 2c_1 & 2d_1 \end{bmatrix} \quad \text{— (2)}$$

$\Rightarrow (c+d)\cdot A \neq c\cdot A + d\cdot A$

$\Rightarrow$ Dist. prop. not satisfied

$\Rightarrow V$ is not a vector space over $\mathbb{R}$.

— x —

Ex. $V = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\} = \mathbb{R}\times\mathbb{R}$

XY plane

Let $u = (x_1, x_2)$; $V = (y_1, y_2) \in V$
& $c \in \mathbb{R}$
and define
$u + v = (x_1 + y_1, x_2 + y_2)$ Vector add^n
$c \cdot u = (cx_1, cx_2)$ Scalar multi.
Prove that V is a vector space over $\mathbb{R}$

Soln $V \neq \phi$

① $u + v = (x_1 + y_1, x_2 + y_2) \in V$
$\because x_1 + y_1, x_2 + y_2 \in \mathbb{R}$
$\Rightarrow$ Closure prop. of '+' is satisfied

② $u + v = (x_1 + y_1, x_2 + y_2)$
$= (y_1 + x_1, y_2 + x_2)$ $\because$ Add^n of real no. is comm.
$= (y_1, y_2) + (x_1, x_2)$
$= v + u$
$\Rightarrow$ Comm. prop. of '+' is satisfied

③ Let $u = (x_1, x_2), v = (y_1, y_2)$
$w = (z_1, z_2) \in V$
Consider $u + (v + w) = (x_1, x_2) + (y_1 + z_1, y_2 + z_2)$
$= (x_1 + (y_1 + z_1), x_2 + (y_2 + z_2))$
$= ((x_1 + y_1) + z_1, (x_2 + y_2) + z_2)$
($\because$ Associative prop. of add^n of real no.)

$= (x_1+y_1,\ x_2+y_2) + (\underline{\underline{z_1, z_2}})$

$= (u+v) + w$

i.e. Associativity of '+' is satisfied.

(4) $\exists\ 0 = (0,0) \in V$ s.t.

$u + 0 = u = 0 + u$

i.e. $(x_1, x_2) + (0,0) = (x_1, x_2)$

$\therefore$ zero vector of V is $(0,0)$

$\underline{\underline{\text{or}}}$ Additive id. of V is $(0,0)$

(5) For every $u = (x_1, x_2) \in V$

$\exists\ -u = (-x_1, -x_2) \in V$

s.t. $u + (-u) = (x_1, x_2) + (-x_1, -x_2)$

$= (0,0) = \underline{\underline{0}}$

$= (-u) + u$

$\therefore$ Additive inverse exist for every $u \in V$.

(6) $\underline{k} \cdot \underline{u} = k \cdot (x_1, x_2) = (kx_1, kx_2) \in V$

$\because$ product of real nos is real.

Closure prop. of '$\cdot$' is satisfied.

(7) $(c+d) \cdot u = (c+d) \cdot (x_1, x_2)$

$= ((c+d)x_1,\ (c+d)x_2)$

$= (cx_1 + dx_1, cx_2 + dx_2)$

(Dist prop. of real nos)

$= (cx_1, cx_2) + (dx_1, dx_2)$

$= c \cdot u + d \cdot u$

(8) $c \cdot (u+v) = c \cdot (x_1+y_1, x_2+y_2)$

$= (c(x_1+y_1), c(x_2+y_2))$

$= (cx_1+cy_1, cx_2+cy_2)$

(Dist prop of real nos)

$= (cx_1, cx_2) + (cy_1, cy_2)$

$= c \cdot u + c \cdot v$

$\Rightarrow$ Dist prop is satisfied

(9) $(cd) \cdot u = (cd) \cdot (x_1, x_2)$

$= ((cd)x_1, (cd)x_2)$

$= (c(dx_1), c(dx_2))$

($\because$ prod. of real no. satisfies asso. prop.)

$= c \cdot (dx_1, dx_2)$

$= c \cdot (d \cdot u)$

(10) $\exists \ 1 \in \mathbb{R}$ s.t.

$1 \cdot u = 1 \cdot (x_1, x_2) = (x_1, x_2)$

$= u, \quad \forall u \in V$

$\Rightarrow V$ is a vector space over $\mathbb{R}$

—×—

Ex. Consider same ex. as above except

$$c \cdot u = \left(\frac{cx_1}{2}, \frac{cx_2}{2}\right),$$

Here scalar multiplication is not standard

Identify scalar identity of '$\cdot$'

Sol^n Suppose $K \in \mathbb{R}$ is the scalar identity

then

$$K \cdot u = u$$

$$\left(\frac{Kx_1}{2}, \frac{Kx_2}{2}\right) = (x_1, x_2)$$

$$\Rightarrow \frac{Kx_1}{2} = x_1, \quad \frac{Kx_2}{2} = x_2$$

$$\Rightarrow \boxed{K = 2} \text{ is the scalar identity}$$

—×o×—

Note:- Let $V = \underbrace{\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n\text{-times}}$

$$= \{ \underbrace{(x_1, x_2, x_3, \ldots, x_n)}_{n\text{-tuple}} \mid x_1, x_2, \ldots, x_n \in \mathbb{R} \}$$

$= \mathbb{R}^n$

i.e. $V = \mathbb{R}^n$ is a vector space over $\mathbb{R}$ w.r.t. standard operation of '+' & '.'

particularly

$n = 1$ $\quad V = \mathbb{R}$ is a vector sp. over $\mathbb{R}$

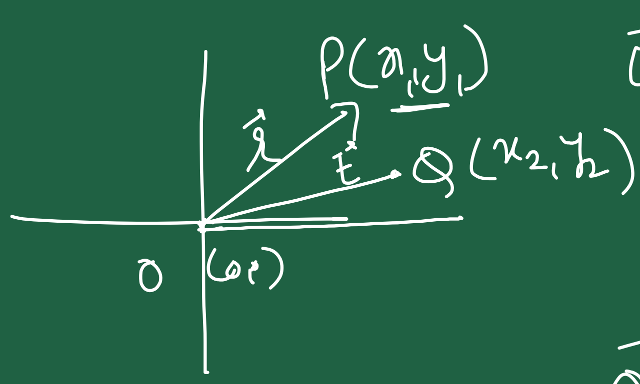
$n = 2$ $\quad V = \mathbb{R}^2$ is a $\underbrace{\qquad\qquad}$ $\mathbb{R}$

$n = 3$ $\quad V = \mathbb{R}^3$ $\underbrace{\qquad\qquad\qquad\qquad}$

$V = \mathbb{R}^n$ w.r.t. standard operations of '+' & '.' is a vector space over $\mathbb{R}$ & they are known as

Euclidean Spaces

$V = \mathbb{R}^2$ an Euclidean Space.

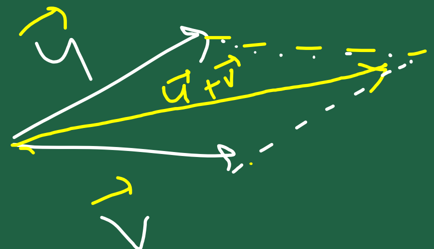


$\vec{OP} = \vec{r} = x_1\hat{i} + y_1\hat{j}$

$\vec{OQ} = \vec{t} = x_2\hat{i} + y_2\hat{j}$

$\vec{OP} + \vec{OQ} = \vec{r} + \vec{t}$

$= (x_1+x_2)\hat{i} + (y_1+y_2)\hat{j}$



Ex. $V = \mathbb{R}^5$ an Euclidean Space.

Write down

1. zero vector of $\mathbb{R}^5$
2. Additive inverse of $u = (1, 10, 100, 20, 50)$
3. whether $(1, 2, 3, 5) \in \mathbb{R}^5$?

Soln

1. $0 = (0, 0, 0, 0, 0)$
2. $-u = (-1, -10, -100, -20, -50)$
3. $(1, 2, 3, 5) \notin \mathbb{R}^5$

   $(1, 2, 3, 5) \in \mathbb{R}^4$

Ex. $V = \mathbb{R}^3$

& for $u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2)$

If $u + v = (x_1 + 2x_2, y_1 + 2y_2, z_1 + 2z_2)$

whether vector addition is commutative ?

is Associative ?

Soln

$u + v = (x_1 + 2x_2, y_1 + 2y_2, z_1 + 2z_2)$

& $v + u = (x_2 + 2x_1, y_2 + 2y_1, z_2 + 2z_1)$

$\Rightarrow u + v \neq v + u$ (i.e. '+' is not commutative)

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}_{3\times 3} \in M_{3\times 3}(\mathbb{R}) \longrightarrow (1,2,3,4,5,6,7,8,9) \in \mathbb{R}^9$$

9-list

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 100 & 1000 \end{bmatrix}_{2\times 3} \longrightarrow (1,2,3,0,100,1000) \in \mathbb{R}^6$$

$\in M_{2\times 3}(\mathbb{R})$

Note :- Every matrix $A \in M_{m\times n}(\mathbb{R})$ represents a unique vector of $\mathbb{R}^{mn}$

Standard vector Space of Polynomials :-

$p(x) = x^2 + 2x + 1$ $\qquad$ $\deg(p) = 2$

$q(x) = -x^2$ $\qquad$ $\deg(q) = 2$

$p(x) + q(x) = 2x + 1$ $\qquad$ $\deg(p+q) = 1$

$r(x) = x^2 + 1$

$p(x) + r(x) = 2x^2 + 2x + 2$ $\qquad$ $\deg(p+r) = 2$

Sum of two n-deg. polynomial need not be n-deg. polynomial always

$\leq n$

Let P_n = set of all polynomials with real coefficients of deg. $\leq n$

$$= \left\{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n \;\middle|\; a_0, a_1, \ldots a_n \in \mathbb{R} \right\}$$

$1 \in P_2$

$1 + x \in P_2$

$x^3 \notin P_2$

$x^3 \in P_3$

$x^3 \in P_4$

$P_1 \subset P_2 \subset P_3 \subset P_4 \subset P_5 \subset \cdots \subset P_{n-1} \subset P_n$

Suppose $n = 2$

$$P_2 = \left\{ a_0 + a_1 x + a_2 x^2 \;\middle|\; a_0, a_1, a_2 \in \mathbb{R} \right\}$$

there are infinite no. of polynomials in P_2.

—x—

Let $P_n = \left\{ a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \;\middle|\; a_i \in \mathbb{R},\ 0 \leq i \leq n \right\}$

for every $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ &
$q(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n$
in P_n define vector add^n as

$p(x)+q(x) = (a_0+b_0)+(a_1+b_1)x+(a_2+b_2)x^2$
$+ \cdots + (a_n+b_n)x^n$

& scalar multiplication as

$c \cdot p(x) = ca_0 + (ca_1)x + (ca_2)x^2 + \cdots$
$+ (c\,a_n)x^n.$

Here P_n forms a vector space over $\mathbb{R}$.

Ex. let $P_2 = \{ a_0 + a_1x + a_2x^2 \mid a_0, a_1, a_2 \in \mathbb{R} \}$ be a v.sp. over $\mathbb{R}$.
w.r.t. standard operation of '+' & '·'

1) Identify zero vector of P_2
2) Does $x^2+2x \in P_2$?
3) Does $2 \in P_2$
4) Additive Inverse of $-x^2-3x$ is ____

Soln 1) $O(x) = 0 + 0.x + 0x^2$ is the zero vector of P_2
e.g. $(x^2+9x-9) + (0+0x+0x^2) = x^2+9x-9$

2) Yes. 3) Yes 4) x^2+3x.

—x—

Standard vector spaces :-

1) $V = \mathbb{R}^n$ w.r.t. standard operations (Euclidean Space.)
2) $V = M_{m\times n}(\mathbb{R})$ ——"——
3) $V = P_n$ ——"——

Subspace :-

Euclidean Space $V = \{ (x,y) \mid x,y \in \mathbb{R} \} = \mathbb{R}^2$

✗ $W_1 = \{ (x,y) \mid x,y \in \mathbb{R}^+ \}$

✓ $W_2 = \{ (x,0) \mid x \in \mathbb{R} \} = X$-axis

✓ $W_3 = \{ (0,y) \mid y \in \mathbb{R} \} = Y$-axis

✗ $W_4 = \phi$, $W_5 = \{(0,0)\}$ ✓ $W_8 = \{(0,1)\}$ ✗

✗ $W_6 = \{(x,y) \in \mathbb{R}^2 \mid x+y=3\}$

✓ $W_7 = \{(x,y) \in \mathbb{R}^2 \mid 2x+y=0\}$

$W_1, W_2, \ldots, W_7$ all are subsets of $V = \mathbb{R}^2$

Subspaces of $V = \mathbb{R}^2$?

Subspace :-

Let V be a vector over R w.r.t. vector addition '+' & scalar multiplication '.'

A non-empty subset W of V is said to be subspace of V iff W is itself a vector space over R w.r.t same operations '+' & '.' defined on V.

Alternative:-

Let V be a vector space over R w.r.t. operations '+' & '.'

A non-empty subset W of V is subspace of V iff

$$\boxed{\alpha \cdot u + \beta \cdot v \in W, \ \forall \ u, v \in W, \ \alpha, \beta \in \mathbb{R}}$$

Note:- If we set $\alpha = \beta = 1 \Rightarrow u+v \in W \ \forall \ u, v \in W$

$\beta = 0 \Rightarrow \alpha \cdot u \in W, \ \forall \ u \in W, \ \alpha \in \mathbb{R}$

If we set $\alpha \cdot u + \beta \cdot v = 0 \Rightarrow \underline{0} \in W$ (Here 0 is zero vector of V)

Alternative :-

Let V be a vector space over $\mathbb{R}$ w.r.t. operations '+' & '.'

A non-empty subset W of V is said to be subspace of V iff

(I) $0 \in W$ (zero vector of V must belong to W)

(II) $u+v \in W, \forall u,v \in W$

(III) $\alpha \cdot u \in W, \forall u \in W, \alpha \in \mathbb{R}$

Ex. Determine which of the following are subspaces of the indicated vector space V
Consider standard operations '+' & '.' in each example.

(1) $W = \{(x,y) \in \mathbb{R}^2 \mid x+y=0\}, \quad V = \mathbb{R}^2$

(2) $T = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1\}, \quad V = \mathbb{R}^2$

(3) $H = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 2x_1+3x_2-x_3 = 0\}, \quad V = \mathbb{R}^3$

Sol^n (1) $W \neq \phi, \because (1,-1) \in W$

Now $(0,0)$ is the zero vector of R^2
& satisfies the cond^n $x+y=0$

$\Rightarrow \underline{0 \in W}$

Let $u = (x_1, y_1), \; v = (x_2, y_2) \in W$

$\Rightarrow \quad x_1+y_1 = 0, \quad x_2+y_2 = 0$ — (I)

Consider $u+v = (x_1+x_2, y_1+y_2)$

Consider $(x_1+x_2) + (y_1+y_2)$

$= (x_1+y_1) + (x_2+y_2)$

$= 0+0$ Using (I)

$= 0$

$\Rightarrow u+v \in W, \forall u,v \in W$

Consider $\alpha \cdot u = \alpha \cdot (x_1, y_1)$

$= (\alpha x_1, \alpha y_1)$

Now $\alpha x_1 + \alpha y_1 = \alpha(x_1 + y_1)$

$= \alpha \cdot 0 = 0$ from (1)

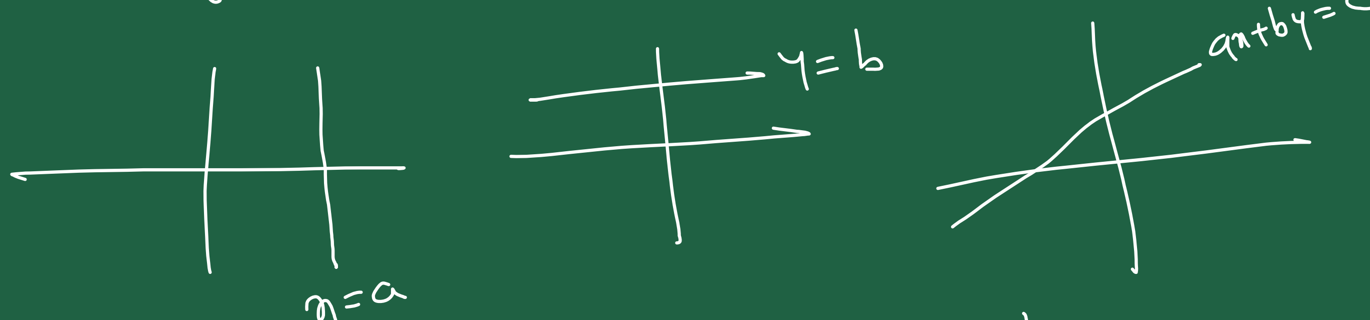
$\Rightarrow W$ is a subspace of $\mathbb{R}^2$.

Notation :- $W \leq V$

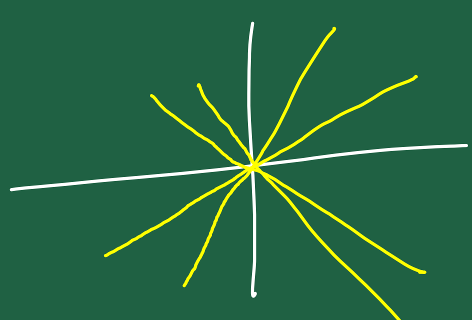
Note :- (1) If $V = \mathbb{R}^2$
& W = set of all points on the line passing through origin.

then $W < \mathbb{R}^2$

Otherwise line that do not pass through origin is not a subspace of $\mathbb{R}^2$.



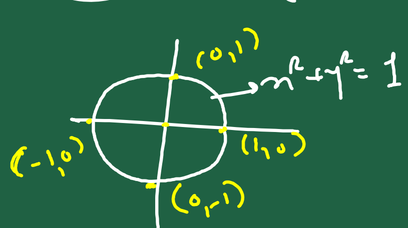
Not subspaces of $\mathbb{R}^2$



$ax + by = 0$
forms a subspace of $\mathbb{R}^2$

(2) Subspaces of $\mathbb{R}^3$:-
lines and planes passing through origin

(2) $T = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$, $V = \mathbb{R}^2$



Clearly from fig.
$(0, 0) \notin T$

OR $\quad 0^2+0^2 \neq 1 \Rightarrow (0,0) \notin T$

$\Rightarrow T \not\leq \mathbb{R}^2$

(3) $H = \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid 2x_1+3x_2-x_3=0\}$

Sol^n Clearly $H \neq \phi$

(1) zero vector of $\mathbb{R}^3$ is $(0,0,0)$

& $\because 2(0)+3(0)-0=0$

$\Rightarrow (0,0,0) \in H$

(2) Let $u=(x_1,x_2,x_3), V=(y_1,y_2,y_3) \in H$

$$\Rightarrow \left.\begin{array}{l} 2x_1+3x_2-x_3=0 \\ 2y_1+3y_2-y_3=0 \end{array}\right\} \text{ --- (I)}$$

Consider $u+V=(x_1+y_1, x_2+y_2, x_3+y_3)$

$\therefore 2(x_1+y_1)+3(x_2+y_2)-(x_3+y_3)$

$=(2x_1+3x_2-x_3)+(2y_1+3y_2-y_3)$

$=0+0 \qquad$ (using (I))

$=0$

$\Rightarrow u+V \in H$

(3) Now $\alpha\cdot u=(\alpha x_1, \alpha x_2, \alpha x_3)$

Consider $2(\alpha x_1)+3(\alpha x_2)-\alpha x_3$

$=\alpha[2x_1+3x_2-x_3]$

$=\alpha\cdot 0=0 \qquad$ (using (I))

$\Rightarrow \alpha\cdot u \in H, \forall \alpha \in \mathbb{R}, u \in H$

Thus $H \leq \mathbb{R}^3$

(4) $K=\left\{A=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) \middle| Tr(A)=1\right\}$

$V=M_{2\times 2}(\mathbb{R})$

Sol^n Clearly $K \neq \phi$ as $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in K$

$Tr(B) = 1$

Zero vector of $V = M_{2\times 2}(\mathbb{R})$ is

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ \& } \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin K$$

$\because Tr(\underline{O}) \neq 1$

$\Rightarrow$ K is not a subspace of $M_{2\times 2}(\mathbb{R})$

(5) $L = \{ A \in M_{2\times 2}(\mathbb{R}) \mid Tr(A) = 0 \}$, $V = M_{2\times 2}(\mathbb{R})$

Sol^n Clearly $L \neq \phi$

Zero vector of $M_{2\times 2}(\mathbb{R})$ i.e $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in L$

as $Tr(O) = 0$

Let A & $B \in L$

$\Rightarrow Tr(A) = 0$, $Tr(B) = 0$

Now $Tr(A+B) = Tr(A) + Tr(B)$

$= 0$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow a + d = 0$

$B = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \Rightarrow e + h = 0$

$A + B = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$ Here $a + e + d + h = 0$

Consider $\alpha \cdot A = \begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{bmatrix}$

$\because Tr(A) = 0$ i.e. $a + d = 0$

Now $Tr(\alpha \cdot A) = \alpha a + \alpha d = \alpha(a+d)$

$= 0$

Thus closure cond^{ns} are satisfied

$\therefore$ L is a subspace of $M_{2\times 2}(\mathbb{R})$

6) $V = M_{n\times n}(\mathbb{R})$

W_1 = set of all upper triangular matrices of order $n\times n$

W_2 = set of all lower triangular matrices of order $n\times n$

W_3 = Set of all invertible matrices of order $n\times n$

W_4 = Set of all matrices of order n with determinant 1

Determine which of the above is/are subspace of V

1) W_1 & W_2 correct option

2) W_1 & W_3

3) W_1, W_2 & W_3

4) All of them

Ex. $V = P_2$

$$W = \{ p(x) \in P_2 \mid p(1) = -1 \}$$

$$X = \{ p(x) \in P_2 \mid p(1) = 0 \}$$

Which of the above is a subspace of P_2.

Sol^n for W

$\underline{\underline{W \neq \phi}}$ $\because$ $p(x) = x-2$, $q(x) = x^2 - x - 1$ $\in W$

zero vector of P_2 is $\underline{\underline{0 + 0x + 0 \cdot x^2}}$ and it does not satisfy the condition $p(1) = -1$

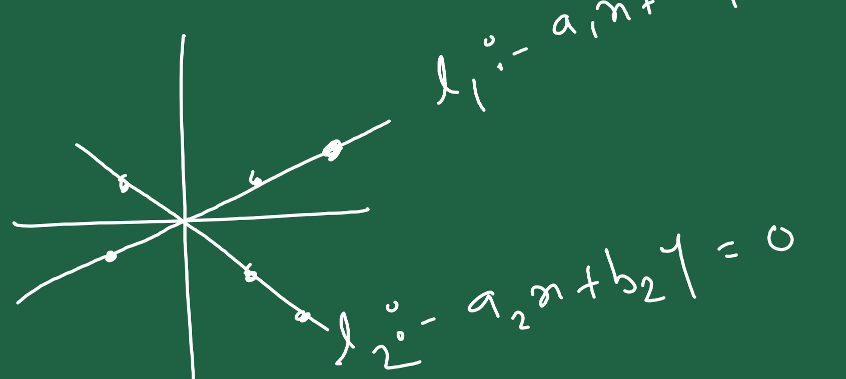
$\therefore$ $0(x) \notin W$

$\Rightarrow$ W is not a subspace of P_2

$$p(x) + q(x)\Big|_{at\ x=1} = p(1) + q(1)$$
$$= (-1) + (-1)$$
$$= -2$$

$\Rightarrow$ $p(x) + q(x) \notin \underline{\underline{W}}$

<u>Note</u> :- $V = \mathbb{R}^2$ an Euclidean Space



l_1

l_2

l_1 & l_2

$W_1 = \{(x,y) \in \mathbb{R}^2 \mid a_1x + b_1y = 0\}$

$W_2 = \{(x,y) \in \mathbb{R}^2 \mid a_2x + b_2y = 0\}$

W_1 & W_2 are subspaces of $V = \mathbb{R}^2$

Is $W_1 \cup W_2$ and $W_1 \cap W_2$ a subspace of $\mathbb{R}^2$?

$$W_1 \cup W_2 = \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; \text{either } a_1x + b_1y = 0 \text{ or } a_2x + b_2y = 0 \right\}$$

$$W_1 \cap W_2 = \{0\}$$

Conclusion :-

If V is a vector space over $\mathbb{R}$ w.r.t. '+' & '.'

& W_1 & W_2 are subspaces of V

then

$W_1 \cap W_2$ is always the subspace of V

whereas $W_1 \cup W_2$ need not be the subspace of V

Note:- Let V be a vector space over $\mathbb{R}$ then

1) V is subspace of itself

2) $W = \{0\}$, 0 = zero vector of V is a subspace of V (Infact it is the smallest possible subspace of V)

These two are trivial subspaces.

Ex. If $V = \mathbb{R}^3$ an Euclidean Space

$$W = \{ (u_1, u_2, u_3) \in \mathbb{R}^3 \mid u_1^2 + u_2^2 + u_3^2 = 0 \}$$

Is W a subspace of $\mathbb{R}^3$

Soln Here $W = \{(0,0,0)\}$

Trivial subspace.

—— xox ——

Linear Combination of vectors :-

$V = \mathbb{R}^2$

$u = (x_1, y_1)$, $v = (x_2, y_2)$, $w = (x_3, y_3)$

$\alpha \cdot u + \beta \cdot v + \lambda \cdot w$ Linear combination.

$\alpha, \beta, \lambda \in \mathbb{R}$

V

$v_1, v_2, \ldots, v_n$

$c_1 v_1 + c_2 v_2 + c_3 v_3 + \cdots + c_n v_n$ L.C.

$c_i \in \mathbb{R}$

$v_i \in V$

Let V be a vector space over $\mathbb{R}$ w.r.t. operations '+' & '.'

then linear combination of n-vectors of V is given by

$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$

where $v_i \in V$ $1 \le i \le n$

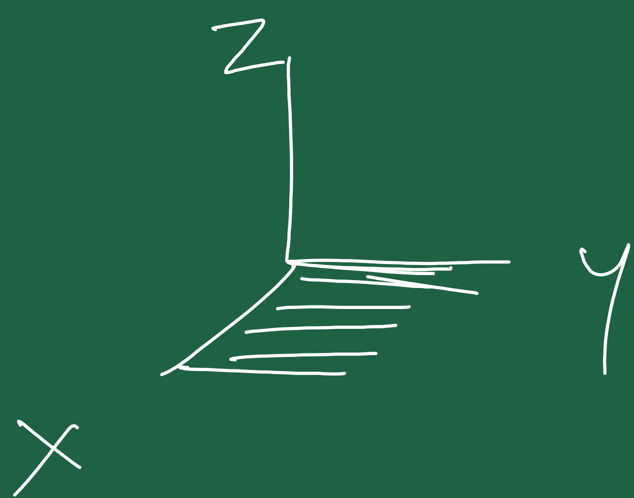
$c_i \in \mathbb{R}$

Span of a set :-

1) $S = \{(1,0)\}$ $\quad \alpha(1,0) \longrightarrow$ X-axis

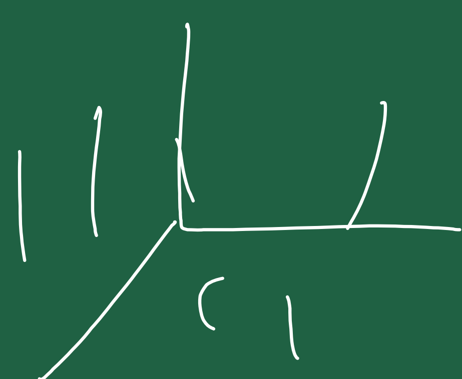
2) $S = \{(1,0,0), (0,1,0)\} \subset \mathbb{R}^3$, with $V_1 = (1,0,0)$ and $V_2 = (0,1,0)$

$u = c_1 V_1 + c_2 V_2$



$V = \mathbb{R}^3$

3) $S = \{(1,0,0), (0,1,0), (0,0,1)\}$



$\mathbb{R}^3$

Span of a set :-

Let V be a vector space w.r.t. operations '+' & '.'
and let $S = \{v_1, v_2, \dots, v_n\}$ be the subset of V then we define Span of a set S denoted by $\langle S \rangle$ or Span$\{S\}$

$$\langle S \rangle = \left\{ c_1 v_1 + c_2 v_2 + c_3 v_3 + \cdots + c_n v_n \;\middle|\; \begin{array}{l} c_i \in \mathbb{R} \\ v_i \in S \end{array} \right\}$$

ex① If $S = \{(1,0)\}$, $V = \mathbb{R}^2$

find Span S

<u>Solⁿ</u> $\text{Span}\{S\} = \{ \alpha \cdot (1,0) \}$

$= \{ (\alpha, 0) \mid \alpha \in \mathbb{R} \}$

$=$ X-axis.

② If $S = \{(0,1)\}$ then

$\langle S \rangle =$ Y-axis

③ $S = \{(0,2)\}$

$\langle S \rangle =$ Y-axis

④ $S = \{(0,0)\}$

$\langle S \rangle = S$

⑤ $S = \{(1,0), (0,3)\}$

$\langle S \rangle = \{ c_1(1,0) + c_2(0,3) \mid c_1, c_2 \in \mathbb{R} \}$

$= \{ (c_1, 3c_2) \mid c_1, c_2 \in \mathbb{R} \}$

$= \mathbb{R}^2$

(6) $S = \{ \underset{v_1}{1}, \underset{v_2}{x} \}$, $V = P_2$

$W = \langle S \rangle = \{ c_1 + c_2 x \mid c_1, c_2 \in \mathbb{R} \}$

(7) $S = \{ \underline{1, x, x^2} \}$, $V = P_2$

$\langle S \rangle = \{ c_1 + c_2 x + c_3 x^2 \mid c_1, c_2, c_3 \in \mathbb{R} \}$

$= P_2$

Note :- $|S| =$ finite

$|\langle S \rangle| =$ Infinite

\# Span of a set $S \subseteq V$, i.e. $\langle S \rangle$ is always the subspace of V

Spanning Set :-

$S_1 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$

$S_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$

$S_3 = \left\{ \underset{E_{11}}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \underset{E_{12}}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \underset{E_{21}}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}, \underset{E_{22}}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \right\}$

$$S_4 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \right\}$$

$$S_1, S_2, S_3, S_4 \subset M_{2\times 2}(\mathbb{R})$$

For all the sets given above Is there any set s.t. $\langle S \rangle = M_{2\times 2}(\mathbb{R})$

$$\langle S_1 \rangle = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \middle| \, a \in \mathbb{R} \right\} \neq M_{2\times 2}(\mathbb{R})$$

Here $\langle S_1 \rangle$ is proper subspace of $M_{2\times 2}(\mathbb{R})$

$$\langle S_2 \rangle = \left\{ c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \middle| \, c_1, c_2, c_3 \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} c_1 & c_2 \\ c_3 & 0 \end{bmatrix} \middle| \, c_1, c_2, c_3 \in \mathbb{R} \right\}$$

Here $\langle S_2 \rangle$ is proper subsp. of $M_{2\times 2}(\mathbb{R})$

$$\langle S_2 \rangle \neq M_{2\times 2}(\mathbb{R})$$

$$\langle S_3 \rangle = \left\{ c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \middle| \, c_1, c_2, c_3, c_4 \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} \middle| \, c_1, c_2, c_3, c_4 \in \mathbb{R} \right\}$$

$$= M_{2\times 2}(\mathbb{R})$$

$$\langle S_4 \rangle = \left\{ c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + c_5 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \middle| \, c_i \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} c_1 + c_5 & c_2 + 2c_5 \\ c_3 + 3c_5 & c_4 + 4c_5 \end{bmatrix} \middle| \, c_i \in \mathbb{R} \right\}$$

$$= \underline{M_{2\times 2}(\mathbb{R})}$$

S_1 ✗

S_2 ✗

S_3 ✓

S_4 ✓

S_3 and S_4 are spanning sets of Vector Space $M_{2\times2}(\mathbb{R})$

Spanning Set :-

Let V be a vector space over $\mathbb{R}$ w.r.t. operations '+' & '.' and $S = \{v_1, v_2, \dots, v_n\} \subseteq V$ then S is said to be spanning set of V iff $\boxed{\langle S \rangle = V}$

<u>OR</u> we say S spans V iff $\langle S \rangle = V$

e.g. 1) $S = \{(1,0), (0,1)\}$ spans $\mathbb{R}^2$

<u>OR</u> $\langle S \rangle = \{c_1(1,0) + c_2(0,1) \mid c_1, c_2 \in \mathbb{R}\}$

$= \{(c_1, c_2) \mid c_1, c_2 \in \mathbb{R}\}$

$= \mathbb{R}^2$

Here S is spanning set of $\mathbb{R}^2$

2) Spanning set of $\mathbb{R}^3$:-

$S = \{(1,0,0), (0,1,0), (0,0,1)\}$

$T = \{(1,0,0), (0,1,0), (0,0,1), (2,10,100)\}$

Here S & T both spans $\mathbb{R}^3$.

Note :- 1) Let V be a vector space, then V can have more than one spanning set.

2) Spanning Set is also known as "Generating Set"

Alternative defn

Spanning Set :- A subset $S \subseteq V$ is spanning set / generating set iff every vector $u \in V$ can be expressed as L.C. of vectors of set S.

Note:- Spanning sets of various std. V.spaces.

① $V = \mathbb{R}^n$

$S = \{ (\underbrace{1,0,0,\ldots,0}_{n\text{-tuple}}), (0,1,0,0,\ldots 0), (0,0,1,0,0,\ldots 0), \ldots, (0,0,0,\ldots,1) \}$

$= \{ e_1, e_2, \ldots, e_n \}$

where $e_1 = (1, 0, 0 \ldots, 0)$
$e_2 = (0, 1, 0, \ldots, 0)$
$\ldots\ e_i = (0, \ldots \underset{i^{th}\text{ position}}{1}, \ldots 0)$

② $V = P_n$

$S = \{ 1, x, x^2, \ldots, x^n \}$

$T = \{ 2, 2x, x^2, \ldots, x^n \}$

$K = \{ 2, 2x+1, x^2, \ldots, x^n \}$

③ $V = M_{m\times n}(\mathbb{R})$

$S = \{ E_{11}, E_{12}, E_{13}, \ldots, E_{ij}, \ldots, E_{mn} \mid 1 \le i \le m, 1 \le j \le n \}$

where E_{ij} = matrix of order $m\times n$ whose ij^{th} entry '1', rest of the entries are '0'

e.g. $E_{11} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ \vdots & & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix}_{m\times n}$

Ex. $V = M_{2\times 2}(\mathbb{R})$

$S = \{ E_{11}, E_{12}, E_{21}, E_{22} \}$

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}$$

$$\underline{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Ex. Let $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 9 \end{bmatrix} \right\}$

Whether S is a spanning set of $V = M_{2\times 2}(\mathbb{R})$

Sol^n Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R})$

For S to be spanning set, we must have some $c_1, c_2, c_3, c_4 \in \mathbb{R}$ s.t.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \underline{c_1}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \underline{c_2}\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} + \underline{c_3}\begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} + \underline{c_4}\begin{bmatrix} 0 & 0 \\ 0 & 9 \end{bmatrix}$$

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$$\left.\begin{aligned} c_1 + c_2 + c_3 &= a \\ 2c_2 + 3c_3 &= b \\ -c_3 &= c \\ 2c_1 + 3c_2 + 2c_3 + 9c_4 &= d \end{aligned}\right\} \text{System of lin. eq}^{ns}$$

Here $[A|B] = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & a \\ 0 & 2 & 3 & 0 & b \\ 0 & 0 & -1 & 0 & c \\ 2 & 3 & 2 & 9 & d \end{array}\right]$

$$\xrightarrow{R_4 \to R_4 - 2R_1} \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & a \\ 0 & 2 & 3 & 0 & b \\ 0 & 0 & -1 & 0 & c \\ 0 & \underline{1} & 0 & 9 & d-2a \end{array}\right] \xrightarrow{R_4 \to 2R_4 - R_2} \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & a \\ 0 & 2 & 3 & 0 & b \\ 0 & 0 & -1 & 0 & c \\ 0 & 0 & -3 & 18 & 2(d-2a)-b \end{array}\right]$$

$$\underset{R_4\to R_4-3R_3}{\sim}\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & a \\ 0 & 2 & 3 & 0 & b \\ 0 & 0 & -1 & 0 & c \\ 0 & 0 & 0 & 18 & 2(d-2a)-b-3c \end{array}\right]$$

Here $\rho(A) = \rho(A|B) = 4$ = No. of unknown

$\Rightarrow$ System is consistent with unique solⁿ

i.e. we can express $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R})$

as a unique linear combination of vectors of set S.

$\Rightarrow$ S spans V / S is a spanning set of V.

—xox—

How to determine whether $S = \{V_1, V_2, \ldots, V_n\} \subseteq V$ spans V ?

Consider a vector equation

$V = c_1V_1 + c_2V_2 + \cdots + c_nV_n$, $V_i \in S$, $c_i \in \mathbb{R}$ — (I)

V → arbitrary vector of vectorspace V

Reduce (I) to System of linear equations

$AX = B$ (Non-homogeneous system)

with n-unknowns $c_1, c_2, \ldots, c_n$.

If above system is consistent then S spans V.

Ex. $S = \{\underset{V_1}{(1,0,0)}, \underset{V_2}{(0,2,0)}, \underset{V_3}{(1,4,0)}, \underset{V_4}{(1,6,0)}\}$

whether S spans $\mathbb{R}^3$?

Soln Let $v = (a, b, c) \in \mathbb{R}^3$

and consider $v = c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4$

$$(a, b, c) = c_1(1, 0, 0) + c_2(0, 2, 0) + c_3(1, 4, 0) + c_4(1, 6, 0)$$

above vector eqn reduces to the below system of lin. eqns in the unknowns c_1, c_2, c_3, c_4

$$c_1 + c_3 + c_4 = a$$

$$2c_2 + 4c_3 + 6c_4 = b$$

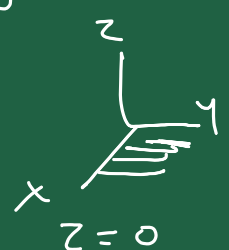
$$0 = c$$

$$[A|B] = \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & a \\ 0 & 2 & 4 & 6 & b \\ 0 & 0 & 0 & 0 & c \end{array}\right]$$

Here $\rho(A) = 2$, $\rho(A|B) = 2$ or 3
$\quad c = 0 \quad c \neq 0$

$\rho(A) \neq \rho(A|B)$ in general

$\Rightarrow S$ is not a spanning set of $\mathbb{R}^3$



H.W. Ex. Let $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} \right\} \subseteq M_{4\times 1}(\mathbb{R})$

OR

$S = \{ (1, 0, 0, 0), (2, 1, 0, 0), (5, 2, 0, 0), (1, 2, 3, 0) \} \subseteq \mathbb{R}^4$

Is S a spanning set of $M_{4\times 1}(\mathbb{R})$ / $\mathbb{R}^4$

—x—

Linearly Dependent Sets and Linearly Independent Sets.

(Linear Dependence & Linear Independence of vectors)

L.D → S_1, S_3

L.I → S_2, S_4

$V = \mathbb{R}^2$

$S_1 = \{ \overset{v_1}{(1,2)}, \overset{v_2}{(-2,-4)}, \overset{v_3}{(3,6)} \}$

$S_2 = \{ \underset{v_1}{(1,2)}, \underset{v_2}{(0,1)} \}$, $S_3 = \{(1,2), (2,3), (3,5)\}$

$S_4 = \{ (1,0), (0,1) \}$ $\qquad S_1, \dots S_4 \subseteq \mathbb{R}^2$

for S_1:- relation v_1, v_2, v_3 ?

$0 \cdot v_1 = 0 \cdot v_2$, $\quad v_2 = -2v_1$, $\quad v_3 = 3v_1$, $\quad -\frac{v_2}{2} = \frac{v_3}{3}$

$0 \cdot v_2 = 0 \cdot v_3$, $\quad \boxed{-2v_1 - v_2 = 0}$ $\quad \boxed{3v_1 - v_3 = 0}$ $\quad \boxed{2v_3 + 3v_2 = 0}$

$0 \cdot v_1 = 0 \cdot v_3$

S_2:- $\quad 0 \cdot v_1 = 0 \cdot v_2 \Rightarrow \boxed{0 \cdot v_1 - 0 \cdot v_2 = 0}$

S_3:- $\quad v_1 = v_3 - v_2$ $\quad \boxed{0 \cdot v_1 = 0 \cdot v_2} \Rightarrow 0 \cdot v_1 - 0 \cdot v_2 = 0$

$\boxed{v_1 + v_2 - v_3 = 0}$

S_4:- $\quad 0 \cdot v_1 = 0\, v_2$

$S = \{ v_1 - - - v_n \} \subset V$

$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0$

$c_1 = c_2 = \dots = c_n = 0$ $\quad$ L.I. ?

O.W $\quad$ L.D.

Linearly dependent & Linearly Independent Sets :-

Let V be a vector space over $\mathbb{R}$ w.r.t. '+' & '.'

Let $S = \{ v_1, v_2, v_3, \dots, v_n \} \subseteq V$, & consider the vector equation $c_1 v_1 + c_2 v_2 + c_3 v_3 + \dots + c_n v_n = \underline{0}$ — (*)

(zero vector of V)

Set $S \subseteq V$ is said to be linearly independent iff the vector eq^n $c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0$ is true for $c_1 = c_2 = \dots = c_n = 0$

Otherwise S is linearly dependent.

Note :- Vector eq^n (*) is equivalent to the homogeneous system $AX = O$

Recall $\qquad AX = O$ (n-unknowns)

Trivial sol^n / Unique sol^n / $\rho(A) = n$ → S is L.I.

Non-Trivial sol^n / Infinitely many sol^n / $\rho(A) < n$ → S is L.D.

$\rho(A) \quad \rho(A|O)$

Ex. Determine whether $S = \{(1,2), (-2,-4), (3,6)\}$ is
Ⅰ L.D. or L.I. in $\mathbb{R}^2$ Ⅱ Is S a spanning set of $\mathbb{R}^2$

Soln Consider $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$ (vector eqn)

$\Rightarrow c_1(1,2) + c_2(-2,-4) + c_3(3,6) = (0,0)$

$\Rightarrow$ $c_1 - 2c_2 + 3c_3 = 0$
$2c_1 - 4c_2 + 6c_3 = 0$ $AX = 0$

S is not a spanning set of $\mathbb{R}^2$

$\therefore [A|B] = [A|0] = \left[\begin{array}{ccc|c} 1 & -2 & 3 & \cancel{0}\ a \\ 2 & -4 & 6 & \cancel{0}\ b \end{array}\right]$

$A_{2\times 3}$
$\rho(A) \le 2$
$n = 3$
$\rho(A) < 3$
$\Rightarrow$ Non-trivial
$\Rightarrow$ L.D.

$R_2 \to R_2 - 2R_1 \sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & \cancel{0}\ a \\ 0 & 0 & 0 & \cancel{0}\ b-2a \end{array}\right]$

$\Rightarrow \rho(A) = \rho(A|B) = 1 < 3$

$\Rightarrow$ System is consistent with non-trivial soln

$\Rightarrow$ S is L.D. in $\mathbb{R}^2$

(Vectors of S are linearly dependent)

Ex. $S = \{\underset{v_1}{1}, \underset{v_2}{x^2}, \underset{v_3}{2x^2+3}\} \subset P_2$. Is S L.D. or L.I

Soln S is L.D. because
$3(v_1) + 2v_2 = v_3$

$\boxed{c_1 v_1 + c_2 v_2 + c_3 v_3 = 0}$
$3v_1 + 2v_2 - v_3 = 0$
for this vector eqn
$\exists$ non-trivial soln
$\Rightarrow$ S is L.D.

Ex. $S = \{\underset{v_1}{1}, \underset{v_2}{x}, \underset{v_3}{x^2}\} \subset P_2$, Is S L.D. or L.I.

Soln Let $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$
$c_1 + c_2 x + c_3 x^2 = 0 + 0\cdot x + 0\cdot x^2$

$\Rightarrow$ $c_1 = 0$, $c_2 = 0$, $c_3 = 0$ $\to$ Trivial soln

$\therefore$ S is L.I

$[A|B] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$\Rightarrow \rho(A) = \rho(A|0) = 3 =$ no. of unknowns

$\Rightarrow$ System is consistent with trivial soln

$\Rightarrow$ S is L.I.

Consider above set $S = \{1, x, x^2\} \subset P_2$

Is S a spanning set of P_2

Yes It is a spanning set of P_2.

$(a,b,c) = c_1 + c_2 x + c_3 x^2$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right]$$

$\rho(A) = 3 = \rho(A|B)$

$\Rightarrow$ System is consistent with unique soln

$\Rightarrow$ S is spanning set of $\mathbb{R}^3$.

Note:- $S = \{1, x, x^2\} \subset P_2$

is ✓ Linearly Independent in P_2 &

✓ Spanning set of P_2

Ex. $S = \{\overset{v_1}{1}, \overset{v_2}{x^2}, \overset{v_3}{2x^2+3}\} \subset P_2$.

Is S a spanning set of P_2

Soln

$$x+1 = c_1 v_1 + c_2 v_2 + c_3 v_3$$

$$\neq 1 \cdot v_1 + 0 v_2 + 0 v_3$$

$\exists$ $x+1 \in P_2$ which can not be expressed as L.C. of vectors of set S.

$\Rightarrow$ S is not a spanning set of P_2

Let $u = a_0 + a_1 x + a_2 x^2 \in P_2$

Consider $u = c_1 v_1 + c_2 v_2 + c_3 v_3$, for unknowns c_1, c_2, c_3

$\Rightarrow a_0 + a_1 x + a_2 x^2 = c_1 + c_2 x^2 + c_3 (2x^2 + 3)$

On comparison

$$\left.\begin{aligned} c_1 + 3c_3 &= a_0 \\ 0 &= a_1 \\ c_2 + 2c_3 &= a_2 \end{aligned}\right\} \text{Non-homogeneous system } AX = B$$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 0 & 3 & a_0 \\ 0 & 0 & 0 & a_1 \\ 0 & 1 & 2 & a_2 \end{array}\right]$$

$$\underset{R_{23}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 3 & a_0 \\ 0 & 1 & 2 & a_2 \\ 0 & 0 & 0 & a_1 \end{array}\right]$$

leading 1 / pivot elements

$\Rightarrow \rho(A) = 2$

$\rho(A|B)$ is dependent on a_1 value

$\Rightarrow S$ is not a spanning set of P_2.

$\boxed{\text{S is L.D. \& not spanning}}$

S

- L.D & Not spanning set
- $\boxed{\text{L.I. \& Spanning set}}$
- L.D. & Spanning set
- L.I. & Not spanning set

$V = \mathbb{R}^3$

(1) $S = \{(1,0,0),(0,1,0)\}$ L.I. & Not spaning set

(2) $S = \{(1,0,0),(0,1,0),(0,0,1),(1,2,3)\}$

L.D. & Spanning set

(3) $S = \{(2,0,0),(0,-1,0),(0,0,9)\}$

Basis of $\mathbb{R}^3$

$$[A|B] = \left[\begin{array}{ccc|c} 2 & 0 & 0 & a/0 \\ 0 & -1 & 0 & b/0 \\ 0 & 0 & 9 & c/0 \end{array}\right]$$

$\boxed{\text{L.I. \& Spanning Set}}$

(4) $S = \{(1,0,0),(2,0,0),(4,0,0)\}$

L.D. & Not spanning set

Basis of a Vector Space :-

Let V be a vector space over $\mathbb{R}$ w.r.t '+' & '.'

The set $S = \{v_1, v_2, ---, v_n\} \subseteq V$ is said to be Basis of a vector space V iff

S is Linearly Independent & Spanning set of V

Note:- (1) Vector space has more than one basis.

(2) We denote the basis set by $\underline{\underline{B}}$

Examples of Vector spaces with their <u>Standard</u> <u>basis</u>

(1) $V = \mathbb{R}^2$

$B = \{(1,0), (0,1)\}$ $\quad \dim(\mathbb{R}^2) = 2$

(2) $V = \mathbb{R}^n$

$B = \{e_1, e_2, e_3, ---, e_n\}$ $\quad \dim(\mathbb{R}^n) = n$

where $e_i = (-----, \underset{i^{th}\text{ entry}}{1}, 0 ---)$

(3) $V = P_n$

$B = \{1, x, x^2, ----, x^n\}$ $\quad \dim(P_n) = n+1$

(4) $V = M_{2\times 2}(\mathbb{R})$

$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$\dim(M_{2\times 2}) = 4$

(5) $V = M_{m\times n}(\mathbb{R})$

$B = \{ E_{ij} \mid E_{ij}$ is the matrix of order $m\times n$ whose ij^{th} entry '1', rest entries are '0' $\}$

$\dim(M_{m\times n}(\mathbb{R})) = \underline{mn}$

Dimension of a vector Space V :-

The no. of vectors in the basis of a vector space V is its dimension.

Note :- Dimension is Unique.

Notation :- $\dim(V) = n$
$=$ no. of vectors in the basis

Ex. Determine whether following are bases for Indicated V.sp. V

① $S = \{(1,2), (3,4), (5,6)\}$, $V = \mathbb{R}^2$

This is not a basis of $\mathbb{R}^2$
because $\dim(\mathbb{R}^2) = 2$
but in S there are 3 vectors, hence S cannot be basis of $\mathbb{R}^2$.

② $S = \left\{ \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$, $V = M_{2\times2}(\mathbb{R})$

$\because \dim(M_{2\times2}(\mathbb{R})) = 4$
S is not a basis of $M_{2\times2}(\mathbb{R})$.

Ex. let $S = \{\underset{v_1}{(1,2,0,1)}, \underset{v_2}{(-1,0,0,5)}, \underset{v_3}{(0,0,3,4)}, \underset{v_4}{(1,2,3,5)}\} \subseteq \mathbb{R}^4$

Is S a basis of $\mathbb{R}^4$.

Soln S is a basis iff ① S is L.I.
② S spans $\mathbb{R}^4$.

To check 2^{nd} condn
Let $u = (a,b,c,d) \in \mathbb{R}^4$
Consider the vector eqn
$$u = c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4$$

i.e. $(a,b,c,d) = c_1(1,2,0,1) + c_2(-1,0,0,5) + c_3(0,0,3,4) + c_4(1,2,3,5)$

for L.D. w.L.I $\leftarrow (0,0,0,0)$

This vector eq^n is equivalent to the system

$$\begin{aligned} c_1 - c_2 + c_4 &= a \\ 2c_1 + 2c_4 &= b \\ 3c_3 + 3c_4 &= c \\ c_1 + 5c_2 + 4c_3 + 5c_4 &= d \end{aligned}$$

$$[A|B] = \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & a \\ 2 & 0 & 0 & 2 & b \\ 0 & 0 & 3 & 3 & c \\ 1 & 5 & 4 & 5 & d \end{array}\right]$$

$R_2 \to R_2 - 2R_1$, $R_4 \to R_4 - R_1$

$$\sim \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & a \\ 0 & 2 & 0 & 0 & b-2a \\ 0 & 0 & 3 & 3 & c \\ 0 & 6 & 4 & 4 & d-a \end{array}\right] \underset{R_{24}}{\sim} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & a \\ 0 & 6 & 4 & 4 & d-a \\ 0 & 0 & 3 & 3 & c \\ 0 & 2 & 0 & 0 & b-2a \end{array}\right]$$

$$\underset{R_{34}}{\sim} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & * \\ 0 & 6 & 4 & 4 & * \\ 0 & 2 & 0 & 0 & * \\ 0 & 0 & 3 & 3 & * \end{array}\right] \underset{R_3 \to 3R_3 - R_2}{\sim} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & * \\ 0 & 6 & 4 & 4 & * \\ 0 & 0 & -4 & -4 & * \\ 0 & 0 & 3 & 3 & * \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & * \\ 0 & 6 & 4 & 4 & * \\ 0 & 0 & 1 & 1 & * \\ 0 & 0 & 0 & 0 & * \end{array}\right]$$

$*$ – Some combination of a,b,c,d

Here $\rho(A) = 3$
& $\rho(A|B) \neq 3$ Always

$\Rightarrow \exists$ at least one one vector $u = (a,b,c,d) \in \mathbb{R}^4$ which can not be expressed as L.C. of $v_1, \ldots, v_4$.

$\Rightarrow S$ does not span $\mathbb{R}^4$.

from Above row-echelon form,

$\rho(A) = 3, \quad n = 4$

$\Rightarrow$ System has Non-trivial sol^n $\Rightarrow S$ is L.D.

$\Rightarrow S$ is not a basis of $\mathbb{R}^4$.

Observations :-

Let V be a n-dimensional vector space over $\mathbb{R}$ w.r.t. '+' & '.'

Let $B = \{v_1, v_2, \ldots, v_n\}$ be a basis of V.

1. Any set $S \subseteq V$ which contains more than n-vectors is always L.D.

2. $S = \{0\} \subset V$, 0 = zero vector of V is always L.D.

   $c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$

   $cv = 0$

   $c \neq 0$, $c = 0$

3. $S = \{v\} \subseteq V$, where $v \neq 0$ is always L.I.

4. Any set $S \subseteq V$, which contains the zero vector of V is always L.D.

5. Subset of L.I. set is L.I.

6. Any set containing the L.D. set is always L.D.

$S = \{\underset{v_1}{(1,0)}, \underset{v_2}{(0,0)}\} \subset \mathbb{R}^2$

L.D.

$\alpha v_1 + \beta v_2 = (0,0)$

$\alpha = 0$, $\beta \neq 0$ Nontrivial Soln

$\alpha = 0$, $\beta = 0$

$c_i = 0$, $c_j \neq 0$

Ex. $S = \{(1,2), (3,5), (4,6)\}$ L.D.

$\begin{bmatrix} 1 & 4 \\ 2 & 6 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{bmatrix} 1 & 4 \\ 0 & -2 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$S_1 = \{(1,2), (4,6)\}$ L.I.

$S_2 = \{(1,2), (3,5)\}$ L.I.

$S_3 = \{(3,5), (4,6)\}$ L.I.

$S_4 = \{(1,2), (3,5), (4,6), (5,6)\}$ L.D.

$S_1 \subseteq S$
L.D. $\quad$ L.D.

Basis of vector space V is $\quad \dim(V) = n$
maximal Linearly Independent set &
minimal Spanning / Generating set.

$B = \{(1,0,0),(0,1,0),\cancel{(0,0,1)}\}$ $\quad V = \mathbb{R}^3$, $\dim(\mathbb{R}^3) = 3$

L.I. / Span $\mathbb{R}^3$ ×

L.I. $\quad \{(1,0,0),\cancel{(0,1,0)}\}$

L.I. $\quad \{(1,0,0)\}$

L.D. $\{(1,0,0),(0,1,0),(0,0,1),(1,2,5)\}$
Spanning $\mathbb{R}^3$

3 vectors
L.I.

Ex: $S = \{(1,2,3),\cancel{(-1,-2,-3)},\cancel{(0,0,1)},(1,2,4),(3,4,5)\}$

I. Is it a basis of $\mathbb{R}^3$. No

II. Is it a spanning set of $\mathbb{R}^3$? Yes

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 1 & 3 & a \\ 2 & -2 & 0 & 2 & 4 & b \\ 3 & -3 & 1 & 4 & 5 & c \end{array}\right]$$

$\rho(A) = 3$

III. Is it L.D. or L.I.? L.D.

IV. Construct a subset $B \subseteq S$, so that B is a basis of $\mathbb{R}^3$

Eliminate 2-vectors

$B = \{\underset{v_1}{(1,2,3)}, \underset{v_2}{(1,2,4)}, \underset{v_3}{(3,4,5)}\}$

$c_1 v_1 + c_2 v_2 + c_3 v_3 = (a, b, c) \Rightarrow c_1(1,2,3) + c_2(1,2,4)$
$+ c_3(3,4,5) = (a,b,c)$

$$[AB] = \left[\begin{array}{ccc|c} 1 & 1 & 3 & a \\ 2 & 2 & 4 & b \\ 3 & 4 & 5 & c \end{array}\right]$$

$$\underset{\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1}}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 3 & a \\ 0 & 0 & 2 & b-2a \\ 0 & 1 & 2 & c-3a \end{array}\right] \underset{R_{23}}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 3 & a \\ 0 & 1 & 2 & c-3a \\ 0 & 0 & 2 & b-2a \end{array}\right]$$

$\Rightarrow \quad \rho(A) = 3 = \rho(A|B) = (\text{no. of var})$

B is L.I. & Spanning set.

—— x0x ——

V.sp — Subspace is vector space.

Basis & Dim. — Basis & Dim

S — L.I, Spanning Set

Ex. $V = \mathbb{R}^2$

$W = \{(x,y) \in \mathbb{R}^2 \mid y = 2x\}$ = line passing thr origin

(I) Is $W < \mathbb{R}^2$?

(II) If yes find its basis & the dimension

Soln (I) W denotes line passing thr origin

$\therefore$ It is a subspace of $\mathbb{R}^2$.

(II) every vector u of w is of the form $u = (x, y)$

$= (x, 2x)$

$$\boxed{u = x \cdot (1,2)}$$

$W = \{a \cdot (1,2) \mid a \in \mathbb{R}\}$

$= \langle B \rangle$, $B = \{(1,2)\}$

Here B is L.I.

& also B is spanning set of W

$\Rightarrow B$ is a basis of W

$\Rightarrow \dim(W) = 1$

Note:- If V is a vector space & W is subspace of V then

$$\dim(W) \leq \dim(V)$$

Basis & dimension of the subspace :-

Let V be a vector space.

Let W be a subspace of V spanned by the set $S = \{v_1, v_2, v_3, \dots, v_k\} \subset W$

then the basis of W is the set of Linearly Independent vectors of S.

Thus dimension of W is no. of L.I. vectors of the set S.

Ex. Let $W = \left\{ \begin{bmatrix} a+b \\ 2a-b \\ 3a+b \end{bmatrix} \middle| \; a, b \in \mathbb{R} \right\}$

(I) Is it possible to write $W = \langle S \rangle$ for some subset S of $\mathbb{R}^3$.

(II) If yes, use (I) to prove that W is a subspace of $\mathbb{R}^3$.

(III) Using (II), determine the Basis & dim. of W.

Sol^n every vector of u of W can be written as

$$u = a\begin{bmatrix}1\\2\\3\end{bmatrix} + b\begin{bmatrix}1\\-1\\1\end{bmatrix} \quad , a,b \in \mathbb{R}$$

$\underbrace{\qquad\qquad\qquad}_{\text{L.C. of } v_1=\begin{bmatrix}1\\2\\3\end{bmatrix} \text{ \& } v_2=\begin{bmatrix}1\\-1\\1\end{bmatrix}}$

Recall
Span of set S
= All possible L.C. of vectors of S

$\Rightarrow W = \langle S \rangle$

where $S = \{v_1, v_2\}$

(I) Span of a set $S \subseteq V$ is always the subspace of V

$\therefore$ W is a subspace of V.

(II) Now $S = \{v_1 ; v_2\}$

To determine whether S is L.D. or L.I

Consider

$$c_1 v_1 + c_2 v_2 = 0 = (0,0,0)$$

$$c_1(1,2,3) + c_2(1,-1,1) = (0,0,0)$$

$$\Rightarrow c_1 + c_2 = 0$$
$$2c_1 - c_2 = 0$$
$$3c_1 + c_2 = 0$$

$$\Rightarrow [A|0] = \left[\begin{array}{cc|c}1 & 1 & 0\\2 & -1 & 0\\3 & 1 & 0\end{array}\right] \underset{\substack{R_2\to R_2-2R_1\\R_3\to R_3-3R_1}}{\sim} \left[\begin{array}{cc|c}1 & 1 & 0\\0 & -3 & 0\\0 & -2 & 0\end{array}\right]$$

$$\underset{\left(-\frac{1}{3}\right)R_2,\ \left(-\frac{1}{2}\right)R_3 \text{ then } R_3-R_2}{\sim} \left[\begin{array}{cc|c}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 0\end{array}\right]$$

$\Rightarrow \rho(A) = 2 = n$

$\Rightarrow$ System has trivial sol^n $\Rightarrow$ S is L.I.

& as we know that

$$W = \langle S \rangle$$

i.e. S is a spanning set of W

$\therefore$ S is basis of W.

$\therefore \dim(W) = 2.$

— xox —

Find the dimension of a subspace of $M_{2\times 2}(\mathbb{R})$ spanned by the vectors

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$$

Sol^n Consider

$$S = \Big\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}}_{v_2}, \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}}_{v_3}, \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}}_{v_4} \Big\}$$

Let W denote the subspace of $M_{2\times 2}(\mathbb{R})$ spanned by S.

i.e. $W = \langle S \rangle$

So here S is a spanning set of W

To determine whether S is L.D. or L.I.

$$[A|0] = \left[\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 4 & 9 & 0 \end{array}\right]$$

$$\underset{R_{24}}{\sim} \left[\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 0 \\ 3 & 2 & 4 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\underset{R_2 \to R_2 - 3R_1}{\sim} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & \\ \multicolumn{5}{c}{\left[\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 0 \\ 0 & 5 & 1 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]} \end{array}$$

Here $\rho(A) = 2$ L.I.

$\Rightarrow$ System has non-trivial sol^n

$\Rightarrow S$ is L.D.

$\therefore$ There are 2 L.I. vectors in S

$\therefore$ Basis of $W = \{v_1, v_2\}$

$$= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \right\}$$

$\Rightarrow \dim(W) = 2$

—xox—

Let $S = \{v_1, v_2, \ldots, v_k\}$ & $W = \langle S \rangle$

then Basis of W is L. I. vectors of set S

& $\dim(W)$ = no. of L. I. vectors of set S

How to find L. I. vectors of a set

$$A = \begin{array}{c} \begin{matrix} v_1 & v_2 & \cdots & v_n \end{matrix} \\ \begin{bmatrix} & & & \\ & & & \\ & & & \end{bmatrix} \end{array}$$

Vectors corresponding to columns containing leading 1 are L. I. vectors

OR

(Pivot Columns :- Columns containing 'leading 1')

$\therefore$ Vectors corresponding to Pivot Columns are L. I. vectors

—xox—

Note. If $W = \langle S \rangle$

$\dim(W) = \rho(A)$

where A is the matrix whose columns are vectors of set S.

Note:- Alternative def^n of Rank of matrix :-

Let A be a matrix of order $m \times n$

then $\rho(A)$ is defined as no. of L. I. rows / columns of A.

—— xox ——

Ex. Let $W = \left\{ \begin{bmatrix} l+m+n \\ 2l+m \\ 3l-2m+n \\ 5l-m \end{bmatrix} \middle| \; l, m, n \in \mathbb{R} \right\}$

Identify the v.sp. V s.t. $W \subset V = \mathbb{R}^4$

find a set S s.t. $\underline{\underline{W}} = \langle S \rangle$

Also find basis & dimension of W

Soln every vector u of W can be written as

$$u = \underbrace{l \begin{bmatrix} 1 \\ 2 \\ 3 \\ 5 \end{bmatrix} + m \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix} + n \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\text{L.C.}}$$

Let $S = \left\{ \underset{v_1}{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 5 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix}}, \underset{v_3}{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}} \right\}$

and we have $W = \langle S \rangle$

i.e. S is a spanning of W

Consider $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 3 & -2 & 1 \\ 5 & 1 & 3 \end{bmatrix}$$

$$\underset{\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 5R_1}}{\sim} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & -5 & -2 \\ 0 & -4 & -2 \end{bmatrix} \underset{\substack{R_3 \to R_3 - 5R_2 \\ R_4 \to R_4 - 4R_2}}{\sim} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 8 \\ 0 & 0 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Here $\rho(A) = 3$

i.e. S is L.I.

Basis of $W = \underline{\underline{S}}$

$\dim(W) = 3$

—xox—

Is $(1,2,-1) \in \langle S \rangle$?

where $S = \left\{ \underset{v_1}{(2,0,-1)}, \underset{v_2}{(3,4,2)} \right\}$

i.e. $c_1 \underline{v_1} + c_2 \underline{v_2} = (1,2,-1)$

$AX = B$ Does $\exists$ $c_1, c_2 \in \mathbb{R}$

Fundamental Subspaces :- Col(A), Row(A), Nul(A), Nul(A^T)

Column Space of A :- Col(A)

Row Space of A :- Row(A)

Null space of A :- Nul(A)

Null space of A^T :- Nul(A^T)

Ex. $A = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 4 & 0 \end{bmatrix}_{\underline{2 \times 3}}$

Row vectors (Rows of A) :- $\underline{(1,2,-1)} \in \mathbb{R}^3$

$(-2,4,0) \in \mathbb{R}^3$

Column vectors (Columns of A) :- $\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \in \mathbb{R}^2$

$\therefore$ Row vectors of A are the vectors of $\mathbb{R}^3$

Column vectors of A are the vectors of $\mathbb{R}^2$

$A = \begin{bmatrix} & & \end{bmatrix}_{\underset{m\times n}{5\times 6}}$

Row vectors of A are vectors of $\mathbb{R}^6 = \mathbb{R}^n$

Column vector 〃 of $\mathbb{R}^5 = \mathbb{R}^m$

$S = \{ \underbrace{(1,2,-1)}_{v_1}, \underbrace{(2,4,0)}_{v_2} \} \subseteq \mathbb{R}^3$ (S is L.I.)

$Row(A) = \langle S \rangle = \{ c_1 v_1 + c_2 v_2 \mid c_1, c_2 \in \mathbb{R} \}$

$\langle S \rangle$ is always subspace

$=$ L.C. of rows of A

$T = \left\{ \underset{v_1}{\begin{bmatrix} 1 \\ -2 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 2 \\ 4 \end{bmatrix}}, \underset{v_3}{\begin{bmatrix} -1 \\ 0 \end{bmatrix}} \right\}$ (T is L.D.)

$col(A) = \langle T \rangle = \{ c_1 v_1 + c_2 v_2 + c_3 v_3 \mid c_i \in \mathbb{R} \}$

$=$ L.C. of columns of A.

Let A be the $m\times n$ matrix.

Row Space of A :- The subspace of $\mathbb{R}^n$ spanned by rows of A is known as row space of A and it is denoted by $Row(A)$

Column Space of A :- The subspace of $\mathbb{R}^m$ spanned by columns of A, is known as Column space of A, denoted by $col(A)$

Ex. $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & -1 & -2 & 4 \end{bmatrix}$

Find the basis & dimension of $Row(A)$, $col(A)$.

Soln

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & -1 & -2 & 4 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -5 & 0 & -2 \end{bmatrix}$$

1. $\rho(A) = 2$
2. 1^{st} and 2^{nd} column contains the leading 1
3. there are 2 rows which are non-zero rows.

Here $S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$ is L.I.

$T = \{ (1, 2, -1, 3), (0, -5, 0, -2) \}$ is L.I.

$\therefore$ Basis of Row(A) = T

Basis of col(A) = S

dim of Row(A) = 2 = dim(Row(A))

dim of col(A) = 2 = dim(col(A))

Note:- If A is m×n matrix then

$\rho(A) = \dim(\text{Row}(A)) = \dim(\text{col}(A))$

—×○×—

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 4 & 6 \\ 1 & 3 & 5 \\ 0 & 1 & 3 \end{bmatrix}_{5\times3}$$

Colums $\in \mathbb{R}^5$

Rows $\in \mathbb{R}^3$

$2R_1 = R_3$, $R_4 = R_1 + R_2$,

$R_3 \to R_3 - 2R_1$

$R_4 \to R_4 - (R_1 + R_2)$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{array}{c} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{l} R_1 \\ R_2 \\ R_5 \\ \\ \\ \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_5 \\ \\ \\ \end{matrix}$$

Steps to find basis & dim of Row(A), col(A) :-

Given $A_{m \times n}$

1. Reduce A to echelon form
2. find $\rho(A)$
3. Identify the pivot columns (containing leading 1's)
4. To find basis of col(A)
   Consider to the columns of original A which corresponds to pivot columns in echelon form.
5. To find basis of Row(A)
   Consider non-zero rows / L. I. rows of A in it's echelon form.

Note :- If $A \sim B$ then row(A) is same as row(B)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -1 & -2 & -3 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

L.I. L.I.

$\rho(A) = 2$
= no. of non-zero rows in the echelon form

Basis of Row(A) = $\{(1,2,3), (0,1,2)\}$ $A \sim B$

Basis of Col(A) = $\{(1,2,-1,0), (2,4,-2,1)\}$

— xox —

Ex. Let $A = \begin{bmatrix} 1 & 0 & -3 & 3 & 2 \\ 3 & 1 & 0 & 4 & 0 \\ 1 & 1 & 6 & -2 & -4 \\ 3 & 0 & -1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 3 & 2 \\ 0 & 1 & 9 & -5 & -6 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(I) find the basis & dimension of Row(A)

(II) ———————————————— col(A).

(III) Specify the vector spaces s.t. Row(A) & col(A) are their subspaces.

$\underline{\underline{Soln}}$

$\dim(Row(A)) = 3$

$\dim(col(A)) = 3$

Basis of Row(A) = $\{(1,0,-3,3,2), (0,1,9,-5,-6), (0,0,1,-1,-1)\}$

Basis of col(A) = $\{(1,3,1,3), (0,1,1,0), (-3,0,6,-1)\}$

$Row(A) < \mathbb{R}^5$

$Col(A) < \mathbb{R}^4$

—XOX—

$\underline{\underline{Null}}$ space of A & A^T :-

(1) $A = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$

Does $\exists$ a vector $X = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$

s.t. $\underset{\text{Input}}{A\underline{X}} = \underset{\text{Output}}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}$ 'zero vector of $\mathbb{R}^2$

i.e. $\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$AX = 0$ Homo system

Yes $\exists$ only one vector $X=\begin{bmatrix}0\\0\end{bmatrix}$ s.t. $AX=0$

Null space of A $\Leftarrow$ W = set of all sol^{ns} of $AX=0$ = $\{(0,0)\}$ is a subspace of $\mathbb{R}^2$

(2) $\underbrace{\begin{bmatrix}1&2\\2&4\end{bmatrix}}_{m\times n}\underbrace{\begin{bmatrix}x\\y\end{bmatrix}}_{n\times 1}=\begin{bmatrix}0\\0\end{bmatrix}_{m\times 1}$

$\exists$ $X=\begin{bmatrix}x\\y\end{bmatrix}$ s.t. $AX=0$

Infact $\exists$ Infinitely many X's s.t. $AX=0$

Null space of A $\Leftarrow$ H = set of All possible sol^n of $AX=0$

$= \{k(2,-1) \mid k\in\mathbb{R}\}$

= line passing thr origin

i.e. H is a subspace of $\mathbb{R}^2$

Null space of A

$A_{m\times n}X_{n\times 1}=0_{m\times 1}$

Null space of A :-

Let A is $m\times n$ matrix

The null space of A is defined as the set of all possible solutions of $AX=0$

OR

set of all vectors of $\mathbb{R}^n$ satisfying $AX=0$

It is denoted by $\text{Nul}(A)$ & It is the subspace of $\mathbb{R}^n$

$$\text{Nul}(A)=\{X\in\mathbb{R}^n \mid AX=0\}$$

$\hookrightarrow$ zero vector of $\mathbb{R}^m$

1. $(0,0,0,\dots,0)\in\text{Nul}(A)$ ✓
2. $X_1,X_2\in\text{Nul}(A) \Rightarrow X_1+X_2\in\text{Nul}(A)$
3. $X_1\in\text{Nul}(A) \Rightarrow c\cdot X_1\in\text{Nul}(A)$

$AX=0$

$X=\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}$

$A(X_1+X_2)$
$=AX_1+AX_2$
$=0+0$
$=0$

Null space of $A^T_{n\times m}$:-

$$Nul(A^T) = \{ X \in \mathbb{R}^m \mid \underbrace{A^T_{n\times m} X_{m\times 1}}_{} = 0_{n\times 1} \}$$

—x—

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ -1 & 2 & 3 \end{bmatrix}$$ find nul(A), nul(A^T)

Solve $AX = 0$ $\quad Nul(A) = \{ X \in \mathbb{R}^n \mid AX = 0 \}$

Solve $A^T X = 0$ $\quad Nul(A^T)$

Note :- Null space of A is also known as Solution space (solution of $AX=0$)

$N(A)$ / $Nul(A)$

—xox—

Ex. Let
$$A = \begin{array}{c} \begin{bmatrix} 1 & 2 & -2 & 1 \\ 3 & 6 & -5 & 4 \\ 1 & 2 & 0 & 3 \end{bmatrix} \\ \begin{matrix} v_1 & v_2 & v_3 & v_4 \end{matrix} \end{array}$$

find the Basis and dim. of
1) Col(A)
2) Row(A)
3) Nul(A)
4) Nul(A^T)

Soln
$$A \sim \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} u_1 \\ u_2 \\ u_3 \end{matrix} \quad \text{— } (*)$$

① Basis of col(A) = $\{ v_1, v_3 \}$

dim (col(A)) = 2

10/2/22

② Basis of Row(A) = $\{ (1,2,-2,1), (0,0,1,1) \}$

dim (Row(A)) = 2

③ Basis & dim of Nul(A) :-

Consider $AX = 0_{3\times 1}$, where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4$

↳ zero vector of $\mathbb{R}^3$

$$[A|0] \sim \left[\begin{array}{cccc|c} 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

pivot columns :- column containing leading 1's

Non-pivot columns :-

equivalent Homo system of eq^{ns} is

$$x_1+2x_2-2x_3+x_4=0$$
$$x_3+x_4=0$$

$\because \rho(A)=2<\underline{\underline{4}}$

$\Rightarrow$ System has non-trivial sol^n & there are $\underline{2}$ free variables $(n-r)$

let $x_2=r, x_4=s$ be the free var.

$\boxed{x_3=-s}$ & $x_1=-2x_2+2x_3-x_4$
$$=-2r-2s-s$$
$$=-2r-3s.$$

$\therefore$ Required sol^n. vector is $X=\begin{bmatrix}-2r-3s\\ r\\ -s\\ s\end{bmatrix}=r\underbrace{\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}}_{v_1}+s\underbrace{\begin{bmatrix}-3\\0\\-1\\1\end{bmatrix}}_{v_2}$

$\therefore \underline{\underline{Nul(A)}}=\left\{X=\begin{bmatrix}-2r-3s\\ r\\ -s\\ s\end{bmatrix}\in\mathbb{R}^4 \;\middle|\; r,s\in\mathbb{R}\right\}$

Basis of $Nul(A)=\left\{\begin{bmatrix}-2\\1\\0\\0\end{bmatrix},\begin{bmatrix}-3\\0\\-1\\1\end{bmatrix}\right\}$.

$\Rightarrow \dim(Nul(A)) = \underline{2}$

—xox —

Dimension of Nul (A) = no. of free variables
$= n-r$

It is also known as Nullity of A i.e. Nullity (A)

i.e. $\dim(Nul(A)) = Nullity(A) = n-r$.

Rank - Nullity th^m :- Let A be $m\times n$ matrix

$$\boxed{rank(A)+Nullity(A)=n}$$

(no of columns of A / no. of var. of $AX=0$)

Example Continued :-

Sol^n (4) To find $Nul(A^T)$ where $A = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 3 & 6 & -5 & 4 \\ 1 & 2 & 0 & 3 \end{bmatrix}$

Here $A^T = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 6 & 2 \\ -2 & -5 & 0 \\ 1 & 4 & 3 \end{bmatrix} \underset{\substack{R_2-2R_1 \\ R_3+2R_1 \\ R_4-R_1}}{\sim} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \underset{R_4-R_3}{\sim} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$Nul(A^T)$, $A^T X = 0$

$\sim \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\rho(A^T) = 2 < 3 \Rightarrow$ there is 1 free var.

Here equivalent system $\quad x_1 + 3x_2 + x_3 = 0$
$\qquad x_2 + 2x_3 = 0$

Let $x_3 = K$, free var.

$x_2 = -2K$

$x_1 = -3x_2 - x_3$

$= 5K$

$\therefore$ Req. sol^n vector satisfying the system $A^T X = 0$ is given by $X = K \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$, $K \in \mathbb{R}$.

$\therefore \underline{Nul}(A^T) = \left\{ K \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} \middle| K \in \mathbb{R} \right\} \rightarrow$ Set containing Infinite vectors

and Basis of $Nul(A^T) = \left\{ \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} \right\} \rightarrow$ finite set

$\Rightarrow \dim(Nul(A^T)) = 1$

— xox —

Description of $\underline{Col(A)}$ / $Row(A)$:-

If $A = \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 & v_4 \end{matrix} \\ \begin{bmatrix} 2 & 4 & 3 & -6 \\ 1 & 2 & 2 & -5 \\ 3 & 6 & 5 & -11 \end{bmatrix} \end{array} \sim \begin{array}{cc} \begin{bmatrix} 1 & 2 & 2 & -5 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{matrix} u_1 \\ u_2 \\ u_3 \end{matrix} \end{array}$

Basis of $col(A) = \{ v_1, v_3 \}$

$Col(A) = \{ v = av_1 + bv_3 \mid a, b \in \mathbb{R} \}$

$= \left\{ v = \begin{bmatrix} 2a+3b \\ a+2b \\ 3a+5b \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}$

Basis of $Row(A) = \{ u_1, u_2 \}$

$$Row(A) = \{ u = l u_1 + m u_2 \mid l, m \in \mathbb{R} \}$$

$$= \{ u = l(1,2,2,-5) + m(0,0,-1,-4) \mid l, m \in \mathbb{R} \}$$

$$= \{ u = (l, 2l, 2l-m, -5l-4m) \mid l, m \in \mathbb{R} \}$$

—xox—

Fundamental Subspaces

$A_{m \times n}$, $\mathbb{R}^m$, $\mathbb{R}^n$

Row(A)

Nul(A)

$\mathbb{R}^n$

Col(A) = Row(A^T)

Nul(A^T)

$\mathbb{R}^m$

Recall Dot Product

$$\vec{a} \cdot \vec{b} = |a||b| \cos\theta$$

$$\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$$

$$\vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$$

$$\left.\begin{array}{l} \vec{a} = (a_1, a_2, a_3) \\ \vec{b} = (b_1, b_2, b_3) \end{array}\right\} \in \mathbb{R}^3$$

$$\boxed{\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3}$$

$= 0$

$\vec{a} \perp \vec{b}$

—xox—

Ex Determine whether $\omega = \begin{bmatrix} 6 \\ -1 \\ 3 \end{bmatrix}$ is in the

(1) $Nul(A)$ (2) $Col(A)$ where $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$

Sol^n (I) for ω to be in $Nul(A)$ it must satisfy $AX = 0$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3} \begin{bmatrix} 6 \\ -1 \\ 3 \end{bmatrix}_{3\times1} = \begin{bmatrix} 8 \\ 5 \\ 3 \end{bmatrix}_{3\times1} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \omega \notin Nul(A)$

OR $\rho(A) = 3 \Rightarrow AX = 0$ has trivial sol^n

$\Rightarrow \dim(Nul(A)) = 0$ $\Rightarrow Nul(A) = \{0\}$

$\Rightarrow \omega \notin Nul(A)$

(2) To check whether $\omega \in col(A)$?

$\omega \in col(A)$ if it is some L.C. of columns of A.

i.e. If $\exists$ $c_1, c_2, c_3 \in \mathbb{R}$ s.t.

$c_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \\ 3 \end{bmatrix}$

$c_1V_1 + c_2V_2 + c_3V_3 = \omega$ — (*)

where V_1, V_2, V_3 are columns of A.

eqn (*) is equivalent to non-homo. system $AX = \omega$ where $X = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

$$\Rightarrow [A|\omega] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right] \Rightarrow \rho(A) = \rho(A|\omega) = 3$$

Clearly $AX = \omega$ is consistent with unique sol^n

$\Rightarrow \exists$ c_1, c_2, c_3 s.t. ω is L.C. of columns of A and it is given by $\omega = c_1V_1 + c_2V_2 + c_3V_3$

$\Rightarrow \omega \in col(A)$

— x o x —

Note:- Vector $b \in \mathbb{R}^n$ is in the $col(A)$ iff $AX = b$ is consistent.

— xox —