Mathematics & Statistics ES1043
Linear Transformation

Mapping/Transformation: Let V and W be two vector spaces. A mapping T from V to W is a function that assigns to each vector $v \in \mathrm{V}$ a unique vector $w \in \mathrm{W}$. In this case we say that T maps V into W and is written as $\mathrm{T}:\mathrm{V} \to \mathrm{W}$. For each $v \in \mathrm{V}$ the vector $w = \mathrm{T}(v) \in \mathrm{W}$ is the image of v under T.

Linear Mapping/Linear Transformation: Let V and W be two vector spaces. A mapping $\mathrm{T}:\mathrm{V} \to \mathrm{W}$ is called linear transformation or linear mapping if

i) $\mathrm{T}(u+v) = \mathrm{T}(u) + \mathrm{T}(v),\ u, v \in \mathrm{V}$ (Additivity)

ii) $\mathrm{T}(\alpha u) = \alpha \mathrm{T}(u),\ \alpha \in \mathrm{R},\ u \in \mathrm{V}$ (Homogeneity)

When $\mathrm{V} = \mathrm{W}$, T is called as linear operator.

Note That : 1**.** Substituting , in condition (ii), we get $\mathrm{T}(0) = 0$, thus every linear mapping maps zero vector into zero vector.

2. If for transformation $\mathrm{T}:\mathrm{U} \to \mathrm{V},\ \mathrm{T}(0) = 0$, then T may or may not be linear.

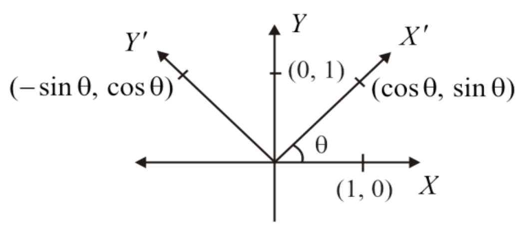
- **Some Examples of Linear Transformation**
  1) Any matrix transformation is a linear transformation.
  2) Derivative operator, integration operator are linear operators.
  3) $\mathrm{T}:\mathrm{R}^3 \to \mathrm{R}^2$ is projection of mapping is a linear transformation from R^3 to R^2.

$$A_{2\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

  4) Multiplication by a fixed polynomial is a linear transformation.
  5) Rotation Matrix is a linear transformation –

- Rotation matrix in 2 dimensions for anticlockwise rotation through an angle θ is

$$A(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



- Rotation matrix in 3 dimension

a) Rotation about X - axis

$$A(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

b) Rotation about Y-axis $A(y) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$

c) Rotation about Z-axis $A(z) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Illustrative Examples

Q 1) Let A be $n \times n$ matrix. Define $\mathrm{T}:\mathrm{R}^n \to \mathrm{R}^m$ by $\mathrm{T}(x) = \mathrm{A}x,\ x \in \mathrm{R}^n$.

i) Show that T is a linear transformation.

ii) Let $A=\begin{bmatrix}1 & 2 & -1\\ -1 & 3 & 2\end{bmatrix}$. Find the image of $\begin{bmatrix}1\\1\\1\end{bmatrix}$ & $\begin{bmatrix}7\\-1\\5\end{bmatrix}$ under the mapping $T(x)=Ax$.

Sol^n. i) Let $u,v\in\mathbb{R}^n$ & $\alpha\in\mathbb{R}$. Then $T(u+v)=A(u+v)=Au+Av=T(u)+T(v)$ and $T(\alpha u)=A(\alpha u)=\alpha Au=\alpha T(u)$. Hence T is linear.

ii) $T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right)=\begin{bmatrix}1 & 2 & -1\\ -1 & 3 & 2\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}2\\4\end{bmatrix}$ and $T\left(\begin{bmatrix}7\\-1\\5\end{bmatrix}\right)=\begin{bmatrix}1 & 2 & -1\\ -1 & 3 & 2\end{bmatrix}\begin{bmatrix}7\\-1\\5\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$.

Note That: Every matrix of order $m\times n$ determines a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Q 2) Define $T:\mathbb{R}^3\to\mathbb{R}^3$ by $T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{bmatrix}x+2y+z\\ -x+5y+z\end{bmatrix}$

i) Show that T is a linear transformation. ii) Find all vectors that are mapped to 0 of $\mathbb{R}^3$.

Sol^n. i) Let $u,v\in\mathbb{R}^3$ & $\alpha\in\mathbb{R}$. Then

a) $$T\left(\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}+\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}\right)=T\left(\begin{bmatrix}u_1+v_1\\u_2+v_2\\u_3+v_3\end{bmatrix}\right)=\begin{bmatrix}(u_1+v_1)+2(u_2+v_2)+(u_3+v_3)\\ -(u_1+v_1)+5(u_2+v_2)+(u_3+v_3)\end{bmatrix}$$

$$=\begin{bmatrix}(u_1+2u_2+u_3)+(v_1+2v_2+v_3)\\ (-u_1+5u_2+u_3)+(-v_1+5v_2+v_3)\end{bmatrix}=\begin{bmatrix}(u_1+2u_2+u_3)\\ (-u_1+5u_2+u_3)\end{bmatrix}+\begin{bmatrix}(v_1+2v_2+v_3)\\ (-v_1+5v_2+v_3)\end{bmatrix}=T(u)+T(v).$$

b) $$T\left(\alpha\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}\right)=T\left(\begin{bmatrix}\alpha u_1\\ \alpha u_2\\ \alpha u_3\end{bmatrix}\right)=\begin{bmatrix}(\alpha u_1+2\alpha u_2+\alpha u_3)\\ (-\alpha u_1+5\alpha u_2+\alpha u_3)\end{bmatrix}=\begin{bmatrix}\alpha(u_1+2u_2+u_3)\\ \alpha(-u_1+5u_2+u_3)\end{bmatrix}$$

$$=\alpha\begin{bmatrix}(u_1+2u_2+u_3)\\ (-u_1+5u_2+u_3)\end{bmatrix}=\alpha T\left(\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}\right)=\alpha T(u).$$

ii) To find u such that $T(u)=0$, *i.e.*, $\begin{bmatrix}(u_1+2u_2+u_3)\\ (-u_1+5u_2+u_3)\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$,

i.e., $u_1+2u_2+u_3=0$ & $-u_1+5u_2+u_3=0$.

Solving the above homogeneous system of 2 equations in 3 unknowns, the possible set of vectors is $\left\{\begin{bmatrix}11\\-2\\7\end{bmatrix}t \;/\; t\in\mathbb{R}\right\}$.

Q 3) Determine whether the function $T:\mathbb{R}^3\to\mathbb{R}^2$ defined by $T(x,\ y,\ z)=(2x,\ x+y)$ is a linear transformation.

Sol^n. $T(x_1,\ y_1,\ z_1)=(2x_1,\ x_1+y_1)=T(u)$, $T(x_2,\ y_2,\ z_2)=(2x_2,\ x_2+y_2)=T(v)$

$T(u+v)=(x_1+x_2,\ y_1+y_2,\ z_1+z_2)$

$=\{2(x_1+x_2),\ [(x_1+x_2)+(y_1+y_2)]\}=(2x_1,\ x_1+y_1)+(2x_2,\ x_2+y_2)=T(u)+T(v)$

$T(\alpha u)=T(\alpha x_1,\ \alpha y_1,\ \alpha z_1)=(2\alpha x_1,\ \alpha x_1+\alpha y_1)=\alpha(2x_1,\ x_1+y_1)=\alpha T(u)$

$\Rightarrow$ It is linear transformation.

Examples of Non-linear Transformations

Q 4) Show that $\mathrm{T}:\mathrm{R}\to\mathrm{R}$ defined by $\mathrm{T}(x)=x+1$ is not linear, i. e., translation is not a linear transformation.

Sol$^{\mathbf{n}}$. Let $x_1,\ x_2\in\mathrm{R}$ & $\alpha\in\mathrm{R}$ then $\mathrm{T}(x_1+x_2)=(x_1+x_2)+1=x_1+x_2+1$ while $\mathrm{T}(x_1)+\mathrm{T}(x_2)=x_1+1+x_2+1=(x_1+x_2)+2$. Thus, $\mathrm{T}(x_1+x_2)\neq\mathrm{T}(x_1)+\mathrm{T}(x_2)$.

Also, $\mathrm{T}(\alpha\,x)=(\alpha\,x)+1$ and $\alpha\,\mathrm{T}(x)=\alpha\,(x+1)$ Thus, $T(\alpha\,x)\neq\alpha\,T(x)$.

$\therefore T$ is not a linear operation.

Or equivalently $T(0)=1\neq0$. Therefore T is not a linear transformation.

Q 5) Show that $T:\mathrm{R}\to\mathrm{R}$ defined by $T(x)=x^2$ is not linear.

Sol$^{\mathbf{n}}$. Let $x_1,\ x_2\in\mathrm{R}$. $T(x_1+x_2)=(x_1+x_2)^2$ $\quad T(x_1)+T(x_2)=x_1^2+x_2^2$

Thus, $T(x_1+x_2)\neq T(x_1)+T(x_2)$. $\therefore$ T is not additive.

$T(\alpha\,x)=(\alpha\,x)^2=\alpha^2x^2$ while $\alpha\,T(x)=\alpha\,x^2\neq T(\alpha\,x)$

$\therefore$ T is not satisfying homogeneity also. $\therefore$ T is not linear.

Q 6) Let $\mathrm{T}:\mathrm{M}_{n\times n}\to\mathrm{R}$ be the transformation that maps an $n\times n$ matrix to a number set by $\mathrm{T}(A)=\det(A).$ Show that the transformation is not linear.

Sol$^{\mathbf{n}}$. $\mathrm{T}(\mathrm{A}+\mathrm{B})=\det(\mathrm{A}+\mathrm{B})\neq\det(\mathrm{A})+\det(\mathrm{B})$ and $\det(\alpha\mathrm{A})=\alpha^n\det(\mathrm{A})\neq\alpha\det(\mathrm{A})$.

Therefore T is not linear transformation.

- **Matrices For Linear Transformation**

  Consider the transformation $\mathrm{T}:\mathrm{R}^3\to\mathrm{R}^3$ defined by

$$\mathrm{T}(x,\ y,\ z)=\left(x-2y+3z,\ -2x+3y-2z,\ x-y-z\right).$$

  This can be expressed as $\mathrm{T}(x,\ y,\ z)=\begin{bmatrix}1&-2&3\\-2&3&-2\\1&-1&-1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\mathrm{A}X.$

- **Method of Finding Standard Matrix for Linear Transformation**

  Consider the linear transformation $T:\mathrm{R}^n\to\mathrm{R}^m$ such that

  $\mathrm{T}(\mathrm{e}_1)=\begin{bmatrix}a_{11}\\a_{21}\\.\\.\\.\\a_{m1}\end{bmatrix},\ \mathrm{T}(\mathrm{e}_2)=\begin{bmatrix}a_{12}\\a_{22}\\.\\.\\.\\a_{m2}\end{bmatrix},\ ...,\ \mathrm{T}(\mathrm{e}_n)=\begin{bmatrix}a_{1n}\\a_{2n}\\.\\.\\.\\a_{mn}\end{bmatrix},$ where $\{e_1,\ e_2,\cdots,e_n\}$ is a standard basis for R^n. Then the $m\times n$ matrix whose n correspond to images of $e_1,\ e_2,\cdots,e_n$ under T, i.e., $\mathrm{T}(e_1),\mathrm{T}(\mathrm{e}_2),\ \ ...\ \ \mathrm{T}(\mathrm{e}_n)$ is called the standard matrix of T. Thus

$$\mathrm{A}=\left[\mathrm{T}(\mathrm{e}_1)\quad\mathrm{T}(\mathrm{e}_2)\quad...\quad\mathrm{T}(\mathrm{e}_n)\right]=\begin{bmatrix}a_{11}&a_{12}&...&a_{1n}\\a_{21}&a_{22}&...&a_{2n}\\.&.&.&.\\.&.&.&.\\.&.&.&.\\a_{m1}&a_{m2}&...&a_{mn}\end{bmatrix}.$$

  Hence for every $v\in\mathrm{R}^n$, $\mathrm{T}(v)=\mathrm{A}v.$

- **Method of Finding Standard Matrix for general vector spaces**

Consider the linear transformation $T : \mathrm{V} \to \mathrm{W}$ from a n-dimensional vector space V to a m-dimensional vector space W such that $\mathrm{T}(\mathrm{e}_1) = w_1,\ \mathrm{T}(\mathrm{e}_2) = w_2,\ ..., \mathrm{T}(\mathrm{e}_n) = w_n$, where $\{e_1, e_2, \cdots, e_n\}$ is a standard basis for V. Let $\{f_1, f_2, \cdots, f_m\}$ be standard basis of W.

Express each image $T(e_i) = w_i$ as a linear combination of $\{f_1, f_2, \cdots, f_m\}$, basis of W.

Thus $w_1 = T(e_1) = a_{11} f_1 + a_{21} f_2 + \cdots + a_{m1} f_1$, $w_2 = T(e_2) = a_{12} f_1 + a_{22} f_2 + \cdots + a_{m2} f_m, \cdots$,
$w_j = T(e_j) = a_{1j} f_1 + a_{2j} f_2 + \cdots + a_{mj} f_m,\ \cdots\cdots,\ w_n = T(e_n) = a_{1n} f_1 + a_{2n} f_2 + \cdots + a_{mn} f_m$.

To obtain the matrix representation of T, arrange the coefficients in linear combination of each $w_j = T(e_j)$ as the j^{th} column of a $m \times n$ matrix, denoted as $[T]$. Thus,

$$[T] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & \cdots & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

Example

Consider the transformation $T : M_2(\mathrm{R}) \to P_3$ defined by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ax^3 + bx^2 + cx + d$.

T is a linear map. $T\left(k\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} ka_1 & kb_1 \\ kc_1 & kd_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} ka_1 + a_2 & kb_1 + b_2 \\ kc_1 + c_2 & kd_1 + d_2 \end{bmatrix}\right)$

$$= (ka_1 + a_2)x^3 + (kb_1 + b_2)x^2 + (kc_1 + c_2)x + (kd_1 + d_2)$$
$$= \left(ka_1 x^3 + kb_1 x^2 + kc_1 x + kd_1\right) + \left(a_2 x^3 + b_2 x^2 + c_2 x + d_2\right)$$
$$= k\left(a_1 x^3 + b_1 x^2 + c_1 x + d_1\right) + \left(a_2 x^3 + b_2 x^2 + c_2 x + d_2\right) = k\,T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$$

Standard basis of $M_2(\mathrm{R})$ is $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ and standard basis of P_3 is $\left\{x^3, x^2, x, 1\right\}$.

Now $T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = 1x^3 + 0x^2 + 0x + 0, T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = 0x^3 + 1x^2 + 0x + 0, T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = 0x^3 + 0x^2 + 1x + 0$ and

$T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0x^3 + 0x^2 + 0x + 1$. Hence the matrix of T, $[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Note That : If T is a linear transformation from a $n-$dimensional vector space V to a $m-$dimensional vector space W, then the matrix of T is of order $m \times n$.

- **Operations with Linear Transformations**

1. Let V and W be vector spaces and let $S,\ T : V \to W$ be linear transformations. The function $S + T$ defined by $(S + T)(v) = S(v) + T(v)$ is a linear transformation from

V into W.

Let $u, v \in V$ and $k \in \mathrm{R}$.

$$\begin{aligned}(S+T)(ku+v) &= S(ku+v)+T(ku+v)\\ &= \{kS(u)+S(v)\}+\{kT(u)+T(v)\}\\ &= k\{S(u)+T(u)\}+\{S(v)+T(v)\}\\ &= k\{(S+T)(u)\}+\{(S+T)(v)\}\end{aligned}$$

2. If c is any scalar, the function cS defined by $(cS)(v) = cS(v)$ is a linear transformation from V into W.

Let $u, v \in V$ and $k \in \mathrm{R}$.

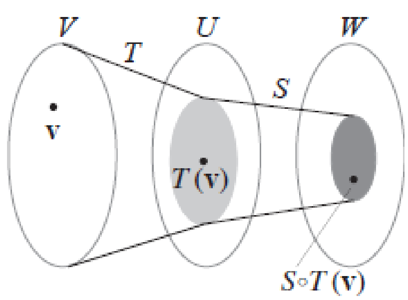
$$\begin{aligned}cS(ku+v) &= cS(ku+v) = c\{kS(u)+S(v)\}\\ &= ckS(u)+cS(v) = k(cS)(u)+(cS)(v)\end{aligned}$$

3. **Composite Linear Transformation**

Let U, V, and W be vector spaces. If $T: V \to U$ and $S: U \to W$ are linear transformations, then the composition map $S \circ T : V \to W$, defined by $(S \circ T)(\mathbf{v}) = S(T(\mathbf{v}))$ is a linear transformation.

Let $u, v \in V$ and $k \in \mathrm{R}$.

$$\begin{aligned}(S \circ T)(ku+v) &= S(T(ku+v)) = S\{kT(u)+T(v)\}\\ &= kS(T(u))+S(T(v)) = k(S \circ T)(u)+(S \circ T)(v)\end{aligned}$$



Note That : If $T: V \to U$ is a linear transformation from a n dimensional vector space V to a p dimensional vector space U and $S: U \to W$ is a linear transformation from a p dimensional vector space V to a m dimensional vector space W, then $S \circ T : V \to W$ is a linear transformation from a n dimensional vector space V to a m dimensional vector space W. Hence matrix of $S \circ T$ is of order $m \times n$, $[S \circ T] = [S][T]$.

<u>Example</u>

1. $S,\ T: \mathrm{R}^2 \to \mathrm{R}^2$ defined by $S\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x+y\\x\end{pmatrix}$ and $T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2x-y\\x-3y\end{pmatrix}$ then find $(S+T)$ and cS.

$(S+T)\begin{pmatrix}x\\y\end{pmatrix} = S\begin{pmatrix}x\\y\end{pmatrix} + T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x+y\\x\end{pmatrix} + \begin{pmatrix}2x-y\\x-3y\end{pmatrix} = \begin{pmatrix}3x\\2x-3y\end{pmatrix}$ and

$(cS)\begin{pmatrix}x\\y\end{pmatrix} = cS\begin{pmatrix}x\\y\end{pmatrix} = c\begin{pmatrix}x+y\\x\end{pmatrix} = \begin{pmatrix}cx+cy\\cx\end{pmatrix}$.

Mathematics & Statistics ES1043
Linear Transformation

2. Define $T:\mathbb{R}^3 \to \mathbb{R}^2$ by $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+2y+z \\ -x+5y+z \end{pmatrix}$ and $S:\mathbb{R}^2 \to \mathbb{R}^4$ by $S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2y \\ x+y \\ 2x \\ x-y \end{pmatrix}$.

Find $S \circ T$ explicitly. Hence find matrix of $S \circ T$. Also verify $[S \circ T] = [S][T]$.

$T:\mathbb{R}^3 \to \mathbb{R}^2$ and $S:\mathbb{R}^2 \to \mathbb{R}^4$, therefore $S \circ T:\mathbb{R}^3 \to \mathbb{R}^4$.

$$(S \circ T)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = S\left(T\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = S\begin{pmatrix} x+2y+z \\ -x+5y+z \end{pmatrix} = \begin{pmatrix} 2(-x+5y+z) \\ (x+2y+z)+(-x+5y+z) \\ 2(x+2y+z) \\ (x+2y+z)-(-x+5y+z) \end{pmatrix} = \begin{pmatrix} -2x+10y+2z \\ 7y+2z \\ 2x+4y+2z \\ 2x-3y \end{pmatrix}.$$

To find matrix of $S \circ T$

Standard basis of $\mathbb{R}^3$ is $\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$ and $(S \circ T)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 2 \\ 2 \end{pmatrix}$, $(S \circ T)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 10 \\ 7 \\ 4 \\ -3 \end{pmatrix}$,

$(S \circ T)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 0 \end{pmatrix}$. Hence $[S \circ T] = \begin{bmatrix} -2 & 10 & 2 \\ 0 & 7 & 2 \\ 2 & 4 & 2 \\ 2 & -3 & 0 \end{bmatrix}_{4\times 3}$. To verify $[S \circ T] = [S][T]$

Standard basis of $\mathbb{R}^3$ is $\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$. $T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$, $T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Therefore $[T] = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 5 & 1 \end{bmatrix}_{2\times 3}$. Standard basis of $\mathbb{R}^2$ is $\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$.

$S\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}$, $S\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix}$. Therefore $[S] = \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 2 & 0 \\ 1 & -1 \end{bmatrix}_{4\times 2}$.

$$\text{Now } [S][T] = \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 2 & 0 \\ 1 & -1 \end{bmatrix}_{4\times 2} \begin{bmatrix} 1 & 2 & 1 \\ -1 & 5 & 1 \end{bmatrix}_{2\times 3} = \begin{bmatrix} -2 & 10 & 2 \\ 1-1 & 2+5 & 1+1 \\ 2 & 4 & 2 \\ 1+1 & 2-5 & 1-1 \end{bmatrix}_{4\times 3} \begin{bmatrix} -2 & 10 & 2 \\ 0 & 7 & 2 \\ 2 & 4 & 2 \\ 2 & -3 & 0 \end{bmatrix}_{4\times 3}.$$

- **Kernel and Range of a Linear transformation**

Let V and W be vector spaces. For a linear transformation $T:V \to W$ the **Kernel** or **null space** of T, denoted by $Ker(T)$ *or* $N(T)$, is the collection of all vectors in $v \in V$ which are map to zero vector of W. Thus $Ker(T) = N(T) = \{v \in V : T(v) = 0\}$.

The **range** of T, denoted by $R(T)$ is the collection of all vectors $w \in W$ which are images of vectors $v \in V$ under the map T. Thus $R(T) = \{w = T(v) : v \in V\}$.

Mathematics & Statistics ES1043
Linear Transformation

Note That : The linear transformation $T:\mathbb{R}^n \to \mathbb{R}^m$ is with the matrix representation $[T] = A$ then

- Range of $T = \{Y \in \mathbb{R}^m : \text{such that } AX = Y\} = col(A) = column\ space\ of\ A$.
- Kernel of $T = \{X \in \mathbb{R}^n : AX = 0\} = Null(A) = Null\ space\ of\ A$.

Result : 1) $Ker(T)$ *or* $N(T)$ is subspace of V.

Let $u, v \in Ker(T) \subseteq V \Rightarrow T(u) = 0, T(v) = 0$, $k \in \mathbb{R}$. Now $T(u+v) = T(u) + T(v) = 0$ and $T(ku) = kT(u) = 0$. Therefore $Ker(T)$ is closed under addition and scalar multiplication. Therefore $Ker(T)$ is subspace of V.

2) $R(T)$ is subspace of W.

Let $w_1, w_2 \in R(T) \subseteq W \Rightarrow \exists v_1, v_2 \in V$ such that $T(v_1) = w_1, T(v_2) = w_2$.
Now $w_1 + w_2 = T(v_1) + T(v_2) = T(v_1 + v_2)$. Thus $w_1 + w_2 \in W$ is image of $v_1 + v_2 \in V$.
Therefore $w_1 + w_2 \in R(T)$. For $k \in \mathbb{R}$, $kw_1 = kT(v_1) = T(kv_1)$. Thus kw_1 is image of kv_1.
Therefore
$kw_1 \in R(T)$. Therefore $R(T)$ is closed under addition and scalar multiplication.Therefore $R(T)$ is subspace of W.

Examples :

1) Let $T:\mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation defined by

$T(x_1, x_2, x_3, x_4) = (x_1 - x_2 + x_3 + x_4,\ 2x_1 - 2x_2 + 3x_3 + 4x_4,\ 3x_1 - 3x_2 + 4x_3 + 5x_4)$. Find

a) Basis and dimension of the range of T.
b) Basis and dimension of the kernel of T.

$T(1,0,0,0) = (1,2,3)$, $T(0,1,0,0) = (-1,-2,-3)$, $T(0,0,1,0) = (1,3,4)$ and

$T(0,0,0,1) = (1,4,5)$. Therefore the matrix of the transformation is $A = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & -2 & 3 & 4 \\ 3 & -3 & 4 & 5 \end{bmatrix}$.

a) To find the basis of image of T which is nothing but column space of A , we reduce A to

row Echelon form $A = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & -2 & 3 & 4 \\ 3 & -3 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Therefore the basis of range of T is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \right\}$ and the dimension of $R(T)$ the space is 2.

b) To find the basis for Kernel of T which is the null space of A, consider $AX = 0$.

Solving this homogeneous system of linear equations the solution is

$x=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} l+k \\ l \\ -2k \\ k \end{bmatrix}=\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}l+\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}k$. Therefore the basis for kernel is $\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}\right\}$

and the dimension of the kernel is 2.

2) Let $T:\mathrm{R}^3 \to \mathrm{R}^3$ be a linear operator and $B=\{v_1,v_2,v_3\}$ a standard basis for R^3. Suppose That $T(v_1)=\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, T(v_2)=\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, T(v_3)=\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$. a) Is $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \in R(T)$? b) Find basis and dimension of $R(T)$. c) Find basis and dimension of null space $N(T)=Ker(T)$.

$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \in R(T)$ if there exist $k_1,k_2,k_3 \in \mathrm{R}$ such that $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}=k_1T(v_1)+k_2T(v_2)+k_3T(v_3)$.

,i.e., $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}=k_1\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}+k_2\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}+k_3\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \Rightarrow k_1+k_2+2k_3=1, k_1+k_3=2, -k_2-k_3=1$.

Augmented matrix

$$(\mathrm{A{:}B})=\begin{bmatrix} 1 & 1 & 2 & \vdots & 1 \\ 1 & 0 & 1 & \vdots & 2 \\ 0 & -1 & -1 & \vdots & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & \vdots & 1 \\ 0 & -1 & -1 & \vdots & 1 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix} \therefore \rho(\mathrm{A{:}B})=\rho(\mathrm{A}).$$

Thus the system is consistent. Therefore $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \in R(T)$.

b) To find basis and dimension of $R(T)$

As images of basis vectors are given, matrix of $T:\mathrm{R}^3 \to \mathrm{R}^3$ is $\mathrm{A}=\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}$.

As $R(T)=Col(\mathrm{A})$, Reduce A to Echelon form. $\mathrm{A} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$. Pivot columns of reduce matrix are 1st and 2nd . Therefore $R(T)=span\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\right\}$. Further vectors $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ are not scalar multiples of each other, hence are linearly independent. Thus basis of $R(T)$ is

$\left\{\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\-1\end{pmatrix}\right\}$ and dimension of $R(T)$ is 2.

c) $Ker(T)=\{v\in \mathrm{R}^3:\ T(v)=0\}$. Now every $v\in \mathrm{R}^3$ can be expressed as $v=k_1v_1+k_2v_2+k_3v_3$, as $B=\{v_1,v_2,v_3\}$ is basis. Therefore $0=T(v)=k_1T(v_1)+k_2T(v_2)+k_3T(v_3)$

i.e., $\begin{pmatrix}0\\0\\0\end{pmatrix}=k_1\begin{pmatrix}1\\1\\0\end{pmatrix}+k_2\begin{pmatrix}1\\0\\-1\end{pmatrix}+k_3\begin{pmatrix}2\\1\\-1\end{pmatrix}\Rightarrow \begin{matrix}k_1+k_2+2k_3=0\\ k_1+k_3=0\\ -k_2-k_3=0\end{matrix}$. This homogeneous system has reduce form $\begin{bmatrix}1&1&2\\0&-1&-1\\0&0&0\end{bmatrix}$. Hence the system possesses 1-parametric solution, $\begin{pmatrix}k_1\\k_2\\k_3\end{pmatrix}=\begin{pmatrix}-1\\-1\\1\end{pmatrix}t, t\in \mathrm{R}$.

Therefore $Ker(T)=span\left\{\begin{pmatrix}-1\\-1\\1\end{pmatrix}\right\}$. Basis of $Ker(T)$ is $\left\{\begin{pmatrix}-1\\-1\\1\end{pmatrix}\right\}$. Dimension of $Ker(T)$ is 1.

Also note that : $\dim\left(Ker(T)\right)+\dim\left(R(T)\right)=1+3=\dim(\mathrm{R}^3)$

Results : 1) Dimension of $Ker(T)$ is known as nullity.

2) Dimension of $R(T)$ is known as rank.

3) Rank-Nullity Theorem : Let $T:V\rightarrow W$ be a linear then

$$\dim\left(Ker(T)\right)+\dim\left(R(T)\right)=\dim(V).$$

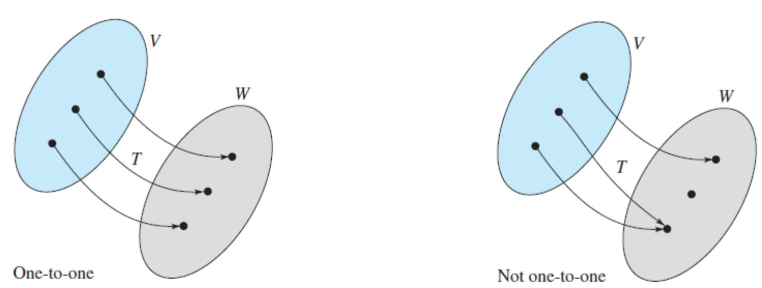
dim(range)+ dim(kernel)= dim(domain)

One-to-One and Onto :

Let V and W be vector spaces and let $T:V\rightarrow W$ be linear transformation.

1. The mapping T is called **one-to-one** or **injective** if distinct elements of V must have distinct images in W.
   i. e. $T(u)=T(v)\Rightarrow u=v$ OR $u\neq v\Rightarrow T(u)\neq T(v)$.
2. The mapping T is called **onto** or **surjective** if the range of T is W.
   i.e. given any $w\in W$ there is $v\in V$ such that $w=T(v)$ OR $\mathrm{W}=\mathrm{T(V)}$.

A mapping is called **bijective** if it is both injective and surjective.



Example : Show that $T:\mathrm{R}^2\rightarrow \mathrm{R}^2$ defined by $T(v)=\mathrm{A}v, \mathrm{A}=\begin{bmatrix}1&1\\-1&0\end{bmatrix}$. Show that T is bijective.

Let $u=\begin{pmatrix}u_1\\u_2\end{pmatrix}, v=\begin{pmatrix}v_1\\v_2\end{pmatrix}\in \mathrm{R}^2$,

$T\begin{pmatrix}u_1\\u_2\end{pmatrix}=T\begin{pmatrix}v_1\\v_2\end{pmatrix}\Rightarrow\begin{bmatrix}1&1\\-1&0\end{bmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}=\begin{bmatrix}1&1\\-1&0\end{bmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}\Rightarrow\begin{pmatrix}u_1+u_2\\-u_1\end{pmatrix}=\begin{pmatrix}v_1+v_2\\-v_1\end{pmatrix}$

This implies $u_1=v_1 \,\&\, u_2=v_2, i.e., \begin{pmatrix}u_1\\u_2\end{pmatrix}=\begin{pmatrix}v_1\\v_2\end{pmatrix}\Rightarrow u=v$. Therefore T is one-one.

To check T is onto. Let $w=\begin{pmatrix}a\\b\end{pmatrix}\in\mathrm{R}^2$ be any general vector, to find $u=\begin{pmatrix}u_1\\u_2\end{pmatrix}$ such that $w=T(u)$. $T(u)=\begin{bmatrix}1&1\\-1&0\end{bmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}=w\Rightarrow\begin{pmatrix}u_1+u_2\\-u_1\end{pmatrix}=\begin{pmatrix}a\\b\end{pmatrix}$. This solves to $u_1=-b \,\&\, u_2=a+b$. Thus given $w=\begin{pmatrix}a\\b\end{pmatrix}\in\mathrm{R}^2$, there is $u=\begin{pmatrix}-b\\a+b\end{pmatrix}$ such that $T\begin{pmatrix}-b\\a+b\end{pmatrix}=\begin{pmatrix}a\\b\end{pmatrix}$. Therefore T is onto.

As T is one-one as well as onto, T is bijective.

<u>Results</u> : 1) Let $T:V\to W$ be a linear transformation. Then T is one-to-one if and only if $Ker(T)=\{0\}$,i.e., $T:V\to W$ is One-one if and only if $Ker(T)$ only contains zero or null vector of W.

2) Let $T:V\to W$ be a linear transformation, where W is finite dimensional. Then T is onto if and only if rank of T equals dimension of W, i. e, $\dim(R(T))=rank\ of\ T=\dim(W)$.

3) Let $T:V\to W$ be a linear transformation, where $\dim(V)=\dim(W)=n\ finite$. Then T is one-one if and only if T is onto.

Example :

The linear transformation $T:\mathrm{R}^n\to\mathrm{R}^m$ is represented by $T(X)=AX$. Find the nullity and rank of and determine whether is one-to-one, onto, or neither.

a) $T:\mathrm{R}^3\to\mathrm{R}^3, \mathrm{A}=\begin{bmatrix}1&2&0\\0&1&1\\0&0&1\end{bmatrix}$. Here matrix of T is in echelon form, hence $rank\ of\ T=3$.

Further $\dim(V)=3$. By rank-nullity thm, nullity= $\dim(V)-rank=0$. Hence $Ker(T)=\{0\}$.
Therefore T is one-one. By result 3 above, $\dim(V)=\dim(W)=3$, T is onto also.

b) $T:\mathrm{R}^2\to\mathrm{R}^3, \mathrm{A}=\begin{bmatrix}1&2\\0&1\\0&0\end{bmatrix}$. Here matrix of T is in echelon form, hence $rank\ of\ T=2$.

Further $\dim(V)=2$. By rank-nullity thm, nullity= $\dim(V)-rank=0$. Hence $Ker(T)=\{0\}$.
Therefore T is one-one. By result 2 above, $\dim(W)=3\neq rank=2$, T is not onto also.

c) $T:\mathrm{R}^3\to\mathrm{R}^2, \mathrm{A}=\begin{bmatrix}1&2&0\\0&1&-1\end{bmatrix}$. Here matrix of T is in echelon form, hence $rank\ of\ T=2$.

Further $\dim(V)=3$. By rank-nullity thm, nullity= $\dim(V)-rank=1$. Hence $Ker(T)\neq\{0\}$.
Therefore T is not one-one. By result 2 above, $\dim(W)=2=rank$, T is onto also.

d) $T:\mathrm{R}^3 \to \mathrm{R}^3, \mathrm{A} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Here matrix of T is in echelon form, hence $rank\, of\ T = 2$.

Further $\dim(V) = 3$. By rank-nullity thm, nullity$= \dim(V) - rank = 1$. Hence $Ker(T) \neq \{0\}$.

Therefore T is not one-one. By result 2 above, $\dim(W) = 3 \neq rank = 2$, T is not onto also.

Illustrative Example:

Q 1) Find the standard matrix of the linear transformation $T:\mathrm{R}^3 \to \mathrm{R}^2$ defined by $T(\mathrm{x},\ \mathrm{y},\ \mathrm{z}) = (x+3y,\ 2x-y)$.

Sol$^{\mathbf{n}}$. $T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $T(e_1) = T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Thus $A = [T(e_1)\ \ \mathrm{T}(\mathrm{e}_2)\ \ T(e_3)] = \begin{bmatrix} 1 & 3 & 0 \\ 2 & -1 & 0 \end{bmatrix}$.

Q 2) Let $T:\mathrm{R}^4 \to \mathrm{R}^3$ be the linear transformation defined by $T(x_1,\ x_2,\ x_3,\ x_4) = (x_1 - x_2 + x_3 + x_4,\ 2x_1 - 2x_2 + 3x_3 + 4x_4,\ 3x_1 - 3x_2 + 4x_3 + 5x_4)$**.** Find

c) Basis and dimension of the image of T.

d) Basis and dimension of the kernel of T.

Sol$^{\mathbf{n}}$. $T(1,\ 0, 0,\ 0) = (1,\ 2,\ 3)$, $T(0,\ 1, 0,\ 0) = (-1,\ -2,\ -3)$, $T(0,\ 0, 1,\ 0) = (1,\ 3,\ 4)$ and $T(0,\ 0, 0,\ 1) = (1,\ 4,\ 5)$. Therefore the matrix of the transformation is

$$A = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & -2 & 3 & 4 \\ 3 & -3 & 4 & 5 \end{bmatrix}.$$

a) To find the basis of image of T which is nothing but column space of A, we reduce A to row Echelon form

$$A = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & -2 & 3 & 4 \\ 3 & -3 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore the basis of image of T is $\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}\right\}$ and the dimension of the space is 2.

b) To find the basis for Kernel of T which is the null space of A, consider $AX = 0$. Solving this homogeneous system of linear equations the solution is

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} l+k \\ l \\ -2k \\ k \end{bmatrix} = l\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

Therefore the basis for kernel is $\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}\right\}$ and the dimension of the kernel is 2.

Q 3) Let $B = \{v_1,\ v_2\}$ be basis of R^2, where $v_1 = (1,\ 1)$ and $v_2 = (1,\ 2)$. Let $\mathrm{T}:\mathrm{R}^2 \to \mathrm{R}^3$ is such that $T(v_1)=(-1,\ 2,\ 0)$ and $T(v_2)=(0,\ -2,\ 4)$. Find $T(5,\ -1)$.

Sol^n. Since $B=\{v_1,\ v_2\}$ is basis of R^2, every vector in R^2 can be expressed as a linear combination of v_1 *and* v_2.

$\therefore\ (5,\ -1)=c_1(1,\ 1)+c_2(1,\ 2)\Rightarrow c_1+c_2=5$ and $c_1+2c_2=-1\ \therefore\ c_1=11$ and $c_2=-6$

Thus $(5,\ -1)=11\times(1,\ 1)-6\times(1,\ 2)$.

$$\therefore\ T(5,\ -1)=T\left(11\times(1,\ 1)-6\times(1,\ 2)\right)=11\times T(1,\ 1)-6\times T(1,\ 2)$$
$$=11\times(-1,\ 2,\ 0)-6\times(0,\ -2,\ 4)=(-11,\ 34,\ -24).$$

- **Invertible Linear Transformation**

A linear transformation $\mathrm{T}:\mathrm{R}^n\to\mathrm{R}^n$ defined by $T(X)=\mathrm{A}X=Y$ is said to be **invertible** or **non singular** or **regular** if the matrix of transformation A is non singular matrix, i.e., invertible. The corresponding inverse transformation $S:\mathrm{R}^n\to\mathrm{R}^n$ is given by $S(Y)=\mathrm{A}^{-1}Y=X$.

<u>Note That :</u> If $T:\mathrm{R}^n\to\mathrm{R}^n$ and its inverse is $S:\mathrm{R}^n\to\mathrm{R}^n$ then $S\circ T:\mathrm{R}^n\to\mathrm{R}^n$ is such that $S\circ T(X)=S(T(X))=S(\mathrm{A}X)=\mathrm{A}^{-1}\mathrm{A}X=X$. Also $T\circ S:\mathrm{R}^n\to\mathrm{R}^n$ is such that $T\circ S(Y)=T(S(Y))=T(\mathrm{A}^{-1}Y)=\mathrm{A}\mathrm{A}^{-1}Y=Y$. This implies $S\circ T$ and $T\circ S$ is identity map on R^n.

Example : Is $T:\mathrm{R}^2\to\mathrm{R}^2$ defined by $T\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}x_1-x_2\\2x_1+x_2\end{pmatrix}$ regular?

If regular, find the inverse transformation.

$T\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}x_1-x_2\\2x_1+x_2\end{pmatrix}=\begin{bmatrix}1&-1\\2&1\end{bmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$**.** Thus $T(X)=\mathrm{A}X$, where $\mathrm{A}=\begin{bmatrix}1&-1\\2&1\end{bmatrix}$.

Further $\det(\mathrm{A})\neq 0$. Therefore T is a regular transformation. The inverse transformation $S:\mathrm{R}^2\to\mathrm{R}^2$ is given by $S\begin{pmatrix}y_1\\y_2\end{pmatrix}=A^{-1}\begin{pmatrix}y_1\\y_2\end{pmatrix}=A^{-1}=\dfrac{1}{3}\begin{bmatrix}1&1\\-2&1\end{bmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix}=\begin{pmatrix}(y_1+y_2)/3\\(-2y_1+y_2)/3\end{pmatrix}$.

Orthogonal Transformation : A transformation $Y=\mathrm{A}X$ is said to orthogonal if the matrix A is orthogonal matrix.

Orthogonal Matrix : Matrix A is said to be orthogonal matrix if $\mathrm{A}\mathrm{A}^T=\mathrm{A}^T\mathrm{A}=I$.

Note That :1) Every rotation matrix is an orthogonal matrix & vice versa.

2) If A is orthogonal then $|A|=\pm 1$.

3) If for certain matrix A, $|A|=\pm 1$ then A may or may not be orthogonal.

4) Orthogonal matrix A is always invertible and $\mathrm{A}^{-1}=\mathrm{A}^T$.

Illustrative Examples

Q 1) Is $A=\begin{bmatrix}0&a&a\\2b&b&-b\\c&-c&c\end{bmatrix}$ orthogonal?

Sol^n. No.

Q 2) Find l,m,n so that the matrix $A=\begin{bmatrix}0&2m&n\\l&m&-n\\l&-m&n\end{bmatrix}$ is orthogonal**.**

Sol^n. A is an Orthogonal matrix, therefore $\sqrt{o^2+l^2+l^2}=1\Rightarrow 2l^2=1\Rightarrow l=\pm\dfrac{1}{\sqrt{2}}$,

$\sqrt{(2m)^2+m^2+m^2}=1 \Rightarrow 6m^2=1 \Rightarrow m=\pm\frac{1}{\sqrt{6}}$ and $\sqrt{n^2+(-n)^2+n^2}=1 \Rightarrow 3n^2=1 \Rightarrow n=\pm\frac{1}{\sqrt{3}}$.

Q 3) Is $T : \mathrm{R}^2 \to \mathrm{R}^2$ defined by $T(x,\ y)=(x-y,\ 2x+y)$ regular? If regular, find the inverse transformation.

Sol$^{\mathbf{n}}$. $T(x,\ y)=(x-y,\ 2x+y)=\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. Thus $TU = AU$, where $A=\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$.

Further $\det(A) \neq 0$. Therefore T is a regular transformation. The inverse transformation is $S : \mathrm{R}^2 \to \mathrm{R}^2$ is given by $S(x,y)=A^{-1}(x,y)^T$, where $A^{-1}=\frac{1}{3}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$.

- **Geometry of Linear Operators on** R^2

If $T : \mathrm{R}^2 \to \mathrm{R}^2$ is the matrix operator whose standard matrix is $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$X=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}.$$

It is natural question that Geometrically how can we view the above transformation?
We may view entries in the matrices as components of vectors or as co-ordinates of points.

- Compressions and Expansions

If T: $\mathrm{R}^2 \to \mathrm{R}^2$ is a compression or expansion in the $x-$direction with the factor k, then $T(e_1)=T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)=\begin{bmatrix} k \\ 0 \end{bmatrix}$ and $T(e_2)=T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)=\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. So the standard matrix for T is $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$. Similarly, the standard matrix for compression or expansion in the y$-$direction is $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$.

Mathematics & Statistics ES1043
Linear Transformation

- **Special operators in** R^2

Sr. No.	**Operator**	**Matrix Representation**
1.	Reflection about X -axis	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
2.	Reflection about Y -axis	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
3.	Reflection about the line $y = x$.	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
4.	Projection on x-axis	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$
5.	Projection on y-axis	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
6.	Counterclockwise Rotation through an angle θ.	$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$
7.	Compression $(0 < k < 1)$ or Expansion $(k > 1)$ in the x_1 - direction	$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$
8.	Compression $(0 < k < 1)$ or Expansion $(k > 1)$ in the x_2 - direction	$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$
9.	Contraction $(0 < k < 1)$ or Dilation $(k > 1)$	$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$
10.	Shear in the x_1 -direction with factor k.	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
11.	Shear in the x_2 -direction with factor k.	$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

Linear Transformation

- **Special operators in** $\mathbb{R}^3$

Sr. No.	**Operator**	**Matrix Representation**
1.	Reflection about XY -plane	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
2.	Reflection about XZ -plane	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
3.	Reflection about YZ -plane	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
4.	Projection on XY -plane	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
5.	Projection on YZ -plane	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
6.	Projection on XZ -plane	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
7.	Rotation about X -axis	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$
8.	Rotation about Y -axis	$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$
9.	Rotation about Z -axis	$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Illustrative Examples

Q 1) Find a transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that first shears in x_1 direction by a factor of 3 and then reflects about $y = x$.

Sol$^{\mathbf{n}}$. The standard shear matrix in x_1 direction by a factor of 3 is $A_1 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$.

The standard matrix of reflection about $y = x$ is $A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Hence the required matrix is $A_2 A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}$.

Q 2) Find a transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that first reflects about $y = x$ and then shears by a factor of 3 in x_1 direction.

Sol^n. The standard shear matrix in x_1 direction by a factor of 3 is $\mathrm{A}_1=\begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix}$.

The standard matrix of reflection about $y=x$ is $A_2=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$.

The required transformation is $A_1A_2=\begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}3 & 1\\ 1 & 0\end{bmatrix}$.

Q 3) Find the standard matrix for $\mathrm{T}:\mathrm{R}^3\to\mathrm{R}^3$, that first reflects about xy - plane, then rotates the resulting vector in counterclockwise direction through an angle θ, about z - axis and then finally resultant vector is projected on xz - plane.

Sol^n. The standard matrix for reflection about xy - plane is $A_1=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}$.

The standard matrix for rotation about z - axis $A_2=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$.

The standard matrix for projection onto xz - plane $A_3=\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$.

Therefore the required transformation is

$$A_3A_2A_1=\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{bmatrix}.$$

Note that elementary matrix represents shear, compression, expansion or reflection.
Thus every 2 by 2 invertible matrix geometrically represents the appropriate succession of shear, compression, expansion or reflection.

- **Geometric properties of Linear Operator on** R^2.

Theorem 2.3 : If $T:\mathrm{R}^2\to\mathrm{R}^2$ is multiplication by an invertible matrix A, then the geometric effect of T is the appropriate succession of shears, compressions, expansions and reflections.
Proof : Since A is invertible, it can be reduced to identity matrix by a finite sequence of elementary row transformation. An elementary row operations can be performed by multiplying on the left by elementary matrix and so there exist elementary matrices $E_1,\ E_2,\ \ldots,\ E_k$ such that $E_k\cdots E_2E_1A=I$. Therefore $A=E_1^{-1}E_2^{-1}\cdots E_n^{-1}I=E_1^{-1}E_2^{-1}\cdots E_n^{-1}$.

Illustrative Example:

Q 1) Express the following matrix as a product of elementary matrices. Describe the effect of multiplication by the given matrix in terms of compression, expression, reflection and shear. $\mathrm{A}=\begin{bmatrix}1 & 4\\ 2 & 9\end{bmatrix}$.

Sol^n. A can be reduced to identity as follows :

$$\begin{bmatrix}1 & 4\\ 2 & 9\end{bmatrix}\xrightarrow{R_2-2R_1}\begin{bmatrix}1 & 4\\ 0 & 1\end{bmatrix}\xrightarrow{R_1-4R_2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Two row operations can be performed on the left successively by

$$E_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}. \text{ Therefore } E_2E_1A = I. \quad A = E_1^{-1}E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}.$$

It follows that the effect of multiplication by A

i) Shearing by a factor 4 in the x_1-direction.

ii) Followed by shearing by a factor 2 in the x_2-direction.

Q 2) Express the following matrix as a product of elementary matrices. Describe the effect of multiplication by the given matrix in terms of compression, expression, reflection and shear. $A=\begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix}$.

Sol^n. A can be reduce to identity as follows :

$$\begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix} \xrightarrow{R_1-R_2} \begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix} \xrightarrow{R_2-3R_1} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1+R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Three row operations can be performed on the left successively by

$$E_1 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}. \text{ Therefore } E_3E_2E_1A = I.$$

$$A = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

Hence the effect of multiplication is

i) Shearing by a factor 1 in the negative x_1-direction.

ii) Shearing by a factor 3 in the negative x_2-direction.

iii) Shearing by a factor 1 in the x_1-direction.