MATHEMATICS AND STATISTICS

ES1043

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Basis

A subset $B = \{v_1, v_2, \dots, v_n\}$ of a vector space V is a basis of V iff

$i)$ B is linearly independent $ii)$ $Span(\mathrm{B}) = \mathrm{V}$.

Dimension of a vector space:

Number of elements in a basis is known as the dimension of the vector space.

If basis of vector space V contains finite number of vectors, the vector space is finite dimensional.

Check the following sets are linearly independent and spanning sets or not

1. $$S_1 = \left\{ \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ -3 \end{pmatrix}, \begin{pmatrix} -6 \\ 3 \\ 5 \end{pmatrix} \right\}$$

2. $$S_2 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

3. $$S_3 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

4. $$S_4 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Note ::

i) A given vector space may have more than one basis.

ii) A set having maximum number of linearly independent vectors is the basis.

iii) Minimum number of vectors which spans the vector space is the basis.

$iv)$ Let $B = \{v_1, v_2, \ldots, v_n\}$ is basis for vector space V, then every vector in V can be written in one and only one way as a linear combination of vectors in B.

$v)$ If a vector space V has one basis with n vectors, then every basis for V has n vectors.

Note: If basis of a vector space V has $'n'$ vectors then *dimension of* $\mathrm{V} = \dim(\mathrm{V}) = n$ & $\dim\{0\}$ is defined as 0.

Note:

1) In an $n-$dimensional vector space V, "$n+1$" vectors are linearly dependent.
2) In an $n-$dimensional vector space V, '$n-1$' vectors do not span vector space V.

Standard Bases

$$1.\ \mathrm{V} = R^n,\ B = \{e_1,\ e_2,\ \dots,\ e_n\}$$

where $e_1 = (1,\ 0,\ \dots,\ 0),\ e_2 = (0,\ 1,\ \dots,\ 0), \dots, \mathrm{e}_n = (0,\ 0,\ \dots,\ 1).$

$\dim(R^n) = n.$

$$2.\ \mathrm{V} = M_{2\times 2}(R), B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$\therefore \dim(M_{2\times 2}(R)) = 4.$ In general, $\dim(M_{m\times n}(R)) = m \cdot n$

$3.\ V = P_n(x)$ *space of polynomials in x at most degree n.*

Basis of $P_n(x)$ *is* $B = \{1,\ x,\ x^2,\ \dots,\ x^n\}.\ \therefore \dim\{P_n(x)\} = n + 1.$

Note :

If Basis of subsapce V of R^n contains

i) 1-vector, then geometrically it is a straight line in R^n through origin.

ii) 2-vectors, then geometrically it is a plane in R^n through origin.

$$\text{Consider B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

1. Is it basis of R^2? No, as $B \not\subset R^2$.

2. Is it basis of R^3?

No, as $Dim\left(R^3\right) = 3$ and B contains two linearly independent vectors.

3. Is $B = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ a basis of R^3?

No, as $Dim\left(R^3\right) = 3$ and B contains four vectors which can't be linearly independent.

4. Determine whether the set is a basis for $M_{2\times 2}(R)$?

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}$$

Since $\dim(M_{2\times 2}(R)) = 4$, and S contains 4 vectors, therefore S is a basis for $M_{2\times 2}(R)$ if and only if given set of vectors are linearly independent.

Let $v_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, v_2 = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, v_3 = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, v_4 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Consider $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0 \ldots (1)$

$$AC = 0 \ where \ A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

Reducing it to echelon form we get $A \sim \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

$$\rho(A) = 4$$

$\therefore$ System (1) has trivial solution only. $\therefore$ S is linearly independent.

$\therefore$ S is a basis for $M_{2\times 2}(R)$.

Summary

To check whether a given set is basis or not.

Consider, the given set of p vectors $\mathrm{B} = \{v_1, v_2, \cdots, v_p\}$

of vector spaces R^n *or* $\mathrm{M}_{m\times n}(R)$ *or* P_n.

Compare p with the dimension of given vector space,

i.e., n or mn or $n+1$

$$|B| = p \ \& \ \dim(V) = n$$

Test	$p < n$	$p > n$	$p = n$ Check for linear independance
Conclusion	It can't be a spanning set Not Basis	Always linearly dependent Not Basis	If yes, then is Basis

Method of determining the dimension of a subspace:

Dimension of a subspace can be determined by finding a set of linearly independent vectors that spans the subspace. This set is a basis for the subspace and its dimension is number of vectors in its basis.

4. $a)$ Show that $\mathrm{H}=\left\{\begin{bmatrix}0\\t\\t\end{bmatrix}: t\in R\right\}$ is a subspace of R^3. Find the basis and dimension.

b) Is $\mathrm{B}=\left\{v_1=\begin{bmatrix}0\\0\\1\end{bmatrix}, \mathrm{v}_2=\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$ a basis of for H?

$a)$ Note that $\begin{bmatrix}0\\t\\t\end{bmatrix}=t\begin{bmatrix}0\\1\\1\end{bmatrix}, t\in R.$ This means $H=\left\{t\begin{bmatrix}0\\1\\1\end{bmatrix}, t\in R\right\}\Rightarrow \mathrm{H=span}\left\{\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}.$

Every span is a subspace, hence H is a subspace. Further $B=\left\{\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$ is a

spanning set of H.

Also set B contains a single non-zero vector, therefore it is a linearly independent set. Thus $B = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is linearly independent spanning set of H. Hence B is a basis of H.

Therefore dim(H)=1.

b) $\mathrm{B} = \left\{ v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \mathrm{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$, is a linearly independent set with two vectors and dim(H)=1, hence B is not a basis of H.

1. Determine the dimension of each subspace of R^4.

a) $S = \{(a,\ a+b,\ b,\ a-c) : a,\ b, c \in R\}$

$$(a,\ a+b,\ b,\ a-c)$$

$$= a(1,\ 1,\ 0,\ 1) + b(0,\ 1,\ 1,\ 0) + c(0,\ 0,\ 0,\ -1)$$

we can see that S is spanned by $(1,\ 1,\ 0,\ 1)$, $(0,\ 1,\ 1,\ 0)$ and $(0,\ 0,\ 0,\ -1)$.

The set of vectors are linearly independent.

$$\therefore \; B = \{(1, \; 1, 0, 1), (0, \; 1, 1, 0), \; (0, 0, 0, -1)\}$$

is a basis for S.

$$\therefore \; \dim(S) = 3.$$

b) $S=\left\{\left(3a,\ a,\ b,\ 0\right):\ a,\ b\in R\right\}$

$$\left(3a,\ a,\ b,\ 0\right)=a(3,\ 1,\ 0,\ 0)+\mathrm{b}(0,\ 0,\ 1,\ 0)$$

we can see that S is spanned by (3, 1, 0, 0), (0, 0, 1, 0)

The set of vectors are linearly independent.

$\therefore\ B=\left\{(3,\ 1,\ 0,\ 0),\ (0,\ 0,\ 1,\ 0)\right\}$ is a basis for S.

$\therefore\ \dim(S)=\ 2.$

Determine the dimension of each subspace of R^3.

$$\mathrm{S}=\left\{\begin{pmatrix} a+c \\ a+2b-c \\ a+b \end{pmatrix} / \ a,b,c\in R\right\}\subseteq R^3$$

$$\begin{pmatrix} a+c \\ a+2b-c \\ a+b \end{pmatrix}=\begin{pmatrix} a \\ a \\ a \end{pmatrix}+\begin{pmatrix} 0 \\ 2b \\ b \end{pmatrix}+\begin{pmatrix} c \\ -c \\ 0 \end{pmatrix}=a\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}+b\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}+c\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\therefore \mathrm{S}=span\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix},\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\right\}\Rightarrow \text{S is a subspace.}$$

Note that $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$ is a spanning set but

$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \Rightarrow$ vectors in above set are not linearly independent.

Hence $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$ is not a basis.

To find the basis, note that if $u = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, v = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

and $w = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ then we have $u - v = w$.

Hence $\{u, v\}$ OR $\{v, w\}$ are linearly independent.

Further note that $\{u, v\}$ is also a spanning set.

Because any $x \in \mathrm{S}$ is

$$x = au + bv + cw = au + bv + c\left(u - v\right) = (a + c)u + (b - c)v.$$

Similarly, $\{v, w\}$ is also a spanning set.

Thus basis of S is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right\}$ OR $\left\{ \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$.

Therefore dimension of S is 2.

2. If $\mathrm{W}=\left\{\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in \quad \right\}$ is a subspace of $\mathrm{M}_{2\times 2}$.

what is a dimension of W?

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathrm{W} = span\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

The set $B=\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ is a basis for W.

as B is linearly independent set. $\therefore \dim(\mathrm{W}) = 3.$

Find a basis and dimension of the solution space of the homogeneous system of equations.

$$x_1 + 2x_2 - 5x_3 + 11x_4 + 3x_5 = 0,\ \ 2x_1 + 4x_2 - 5x_3 + 15x_4 + 2x_5 = 0$$

$$x_1 + 2x_2 + 4x_4 + 5x_5 = 0,\ 3x_1 + 6x_2 - 5x_3 + 19x_4 - 2x_5 = 0$$

$$\text{Consider } AX = 0 \text{ where } A = \begin{bmatrix} 1 & 2 & -5 & 11 & 3 \\ 2 & 4 & -5 & 15 & 2 \\ 1 & 2 & 0 & 4 & 5 \\ 3 & 6 & -5 & 19 & -2 \end{bmatrix}$$

$$\text{Reducing A to Echelon form we get } A \sim \begin{bmatrix} 1 & 2 & -5 & 11 & 3 \\ 0 & 0 & 5 & -7 & -4 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Equivalent system is

$$x_1 + 2x_2 - 5x_3 + 11x_4 - 3x_5 = 0, 5x_3 - 7x_4 - x_5 = 0, x_5 = 0$$

Let $x_4 = s,\ x_2 = t, x_3 = \frac{7}{5}s,\ x_1 = -2t - 4s$

$$\therefore \text{Solution is} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2t-4s \\ t \\ \frac{7}{5}s \\ s \\ 0 \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \frac{s}{5}\begin{bmatrix} -20 \\ 0 \\ 7 \\ 5 \\ 0 \end{bmatrix}$$

$$B = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -20 \\ 0 \\ 7 \\ 5 \\ 0 \end{bmatrix} \right\} \text{ is a basis for solution space of homgeneous system.}$$

$$\dim(\textit{solution space}) = 2.$$

3. Let $\mathrm{H} = Span\{v_1,\ v_2\}$ and $\mathrm{W} = \{u_1,\ u_2\}$, where

$$v_1 = \begin{bmatrix} 5 \\ 3 \\ 8 \end{bmatrix},\ v_2 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix},\ u_1 = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \text{ and } u_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and $\mathrm{K} = \mathrm{H} \cap \mathrm{W}$, find the basis for K.

Geometrically H and W are the planes in R^3.

$\therefore$ $\mathrm{H} \cap \mathrm{W}$ is a line of intersection of the planes H and W.

$\therefore$ K can be written *as* $c_1 v_1 + c_2 v_2$ and also as $c_3 u_1 + c_4 u_2$.

K is a solution space of $c_1v_1 + c_2v_2 = c_3u_1 + c_4u_2$.

i. e. K is a solution space of $c_1v_1 + c_2v_2 - c_3u_1 - c_4u_2 = 0$

$$\text{Coefficient matrix } A = \begin{bmatrix} 5 & 1 & -2 & 0 \\ 3 & 3 & 1 & 0 \\ 8 & 4 & -4 & -1 \end{bmatrix}$$

$$\text{By reducing it to echelon form } A \sim \begin{bmatrix} 1 & 5 & 4 & 0 \\ 0 & -12 & -11 & 0 \\ 0 & 0 & -3 & -1 \end{bmatrix}$$

$\therefore \rho(A) = 3$ Let $c_4 = k$ (free variable)

$$c_3 = -\frac{1}{3}k, \ \ c_2 = \frac{11}{36}k, \quad c_1 = -\frac{7}{36}k.$$

Thus $c_1 = -\dfrac{7}{36}k, c_2 = \dfrac{11}{36}k, \ c_3 = -\dfrac{1}{3}k, \ c_4 = k$

Every vector in K is either $c_1 v_1 + c_2 v_2 \ \ or \ \ c_3 u_1 + c_4 u_2$.

$$-\frac{7}{36}k\begin{bmatrix}5\\3\\8\end{bmatrix}+\frac{11}{36}k\begin{bmatrix}1\\3\\4\end{bmatrix}=\begin{bmatrix}-24k/36\\12k/36\\-12k/36\end{bmatrix}=\frac{k}{3}\begin{bmatrix}-2\\1\\-1\end{bmatrix}$$

OR $-\dfrac{1}{3}k\begin{bmatrix}2\\-1\\4\end{bmatrix}+k\begin{bmatrix}0\\0\\1\end{bmatrix}=\dfrac{k}{3}\begin{bmatrix}-2\\1\\-1\end{bmatrix}$ $\therefore$ Basis for $K = \left\{\begin{bmatrix}-2\\1\\-1\end{bmatrix}\right\}$.

Exercise

1. Determine whether the set is a basis for the vector space if it is avector space determine its dimension.

$$a)\ \mathrm{V}=\mathrm{M}_{2\times 2},\ \mathrm{B}=\left\{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix},\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix},\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\right\}.$$

$$b)\ \mathrm{V}=\mathrm{P}_2,\ \left\{1-x,\ 1-x^2,\ x-x^2\right\}$$

2.Find the basis and dimension of vector space V.

$a)\ \mathrm{V} = \{ p(x) \in \mathrm{P}_2 :\ p(1) = 0 \}.$

$b)\ \mathrm{V} = \{ p(x) \in \mathrm{P}_2 : xp'(\mathrm{x}) = p(x) \}.$

$c)\ \mathrm{V} = \left\{ \mathrm{A} \in \mathrm{M}_{2\times 2} : \mathrm{A} = \mathrm{A}^T \right\}.$

3. Find the basis for span $\left\{ 1-2x, 2x-x^2, 1-x^2, 1+x^2 \right\}.$

4. Find the basis for span $\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \right\}.$