MATHEMATICS AND STATISTICS

ES1043

DR. R. S. DESHPANDE

Spanning Set

Let $\mathrm{H} = \{v_1, v_2, \ldots, v_n\}$ be a subset of vector space V.

H is said to be a spanning set of V if every element of

V is expressible as linear combination of elements of H.

$v \in \text{span}\ \{H\}$ for all $v \in \mathrm{V}$.

$i.e.\ v = c_1v_1 + c_2v_2 + \ldots + c_nv_n$ is consistant for all $v \in V$.

Let $\mathrm{H} = \{e_1 = (1, 0),\ e_2 = (0, 1)\}$ spanning set of R^2 ?

Yes, as for every $v \in R^2$ the system $v = c_1 e_1 + c_2 e_2$ is consistant.

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow [A \mid v] = \left[\begin{array}{cc|c} 1 & 0 & v_1 \\ 0 & 1 & v_2 \end{array}\right]$$

$\therefore\ \rho[A \mid v] = \rho[A]$ for all $v \in R^2$.

Is $H=\left\{1+x,\ x+2x^2,\ 2-3x+2x^2\right\}$ a spanning subset of P_2?

Let $p \in P_2 \qquad \therefore\ p(x)=a+bx+cx^2$

Consider $p=c_1v_1+c_2v_2+c_3v_3$

$a+bx+cx^2=c_1(1+x)+c_2(x+2x^2)+c_3(2-3x+2x^2)$

Equating the coefficients, we get

$a = c_1+2c_3;\quad b=c_1+c_2-3c_3;\ c=2c_2+2c_3$

consider the Augmented matrix

$$[A \mid p]=\left[\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 1 & 1 & -3 & b \\ 0 & 2 & 2 & c \end{array}\right]$$

$$R_2 - R_1 \Rightarrow \left[\begin{array}{ccc:c} 1 & 0 & 2 & a \\ 0 & 1 & -5 & b-a \\ 0 & 2 & 2 & c \end{array}\right]$$

$$R_3 - 2R_2 \Rightarrow \left[\begin{array}{ccc:c} 1 & 0 & 2 & a \\ 0 & 1 & -5 & b-a \\ 0 & 0 & 12 & c-2b+2a \end{array}\right]$$

$as\ \rho[A \,\vdots\, p] = \rho[A]$ for every $p \in P_2$

$\therefore$ System is consistent for every $p \in P_2$

$\therefore$ H is a spanning subset of P_2.

Note : If $\{v_1, v_2, ..., v_n\}$ a spanning subset of a vector space V, then $\{v_1, v_2, ..., v_n, v_{n+1}\}$ will also be a spanning subset of a vector space V.

Note : If $\rho[v_1, v_2, ..., v_n] = n$, then $\{v_1, v_2, ..., v_n\}$ is a spanning subset of vector space V.

$$\left\{1+x^2, 1-x, 2+2x^2\right\} \subseteq P_2$$

Linear Dependence / Independence

Let $\mathrm{H} = \{v_1, v_2, v_3, \ldots, v_n\}$ be a subset of a vector space V, H is said to be linearly dependent if there exist scalars $\left(\text{real numbers}\right)$ $c_1, c_2, \cdots, c_n$ not all zero such that

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + \ldots + c_n v_n = 0.$$

i.e system of equation has nontrivial solution also.

Otherwise they are said to be linearly independent.

H is said to be linearly independent if

$$c_1v_1 + c_2v_2 + c_3v_3 + \ldots + c_nv_n = 0 \Rightarrow$$

$$c_1 = c_2 = \cdots = c_n = 0$$

i.e system has Trivial solution only.

Note :

- A set containing zero vector is linearly dependent.
- Set consists of single non zero vector is linearly independent.

Theorem: - If the set of vectors are linearly dependent then one of the vectors is expressible as a linear combination of the remaining.

Method of checking Linearly Dependent / Independent Set

Step 1: Consider,

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + \ldots + c_n v_n = 0$$

$$AC = 0 \text{ where } A = \begin{bmatrix} v_1 & v_2 & v_3 & \ldots & \mathrm{v}_n \end{bmatrix}, \quad C = \begin{bmatrix} c_1 \\ c_2 \\ \ldots \\ c_n \end{bmatrix}.$$

This is a homogenous system of linear equation if has trivial solution only

Step 2 : Find rank of A. Let $\rho(A) = r$

Step 3 : $i)$If $\rho(A) = r = n$(Number of unknowns), then set is linearly independent.

$ii)$ If $\rho(A) = r < n$(Number of unknowns), then set is linearly dependent.

Step 4 : If dependent find relation between the vectors.

Determine whether $S = \{1-t,\ 2t+3t^2,\ t^2-2t^3,\ 2+t^3\}$ is linearly dependent or independent. If dependent, find the relation between them.

$v_1 = 1-t,\ v_2 = 2t+3t^2,\ v_3 = t^2-2t^3,\ v_4 = 2+t^3$

Consider $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0$

$c_1(1-t) + c_2(2t+3t^2) + c_3(t^2-2t^3) + c_4(2+t^3) = 0 + 0t + 0t^2 + 0t^3$

Equating coefficints of powers of t,

$c_1 + 2c_4 = 0,\ -c_1 + 2c_2 = 0, 3c_2 + c_3 = 0,\ -2c_3 + c_4 = 0$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ -1 & 2 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix} \text{Reducing to echelon form}$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$

$$\Rightarrow \rho(A) = 4 = \text{Number of vectors}$$

$$\therefore \text{Set is Linearly independent.}$$

Determine whether the set of vectors

$$H=\left\{\begin{bmatrix}1 & 0\\ 2 & 3\end{bmatrix},\begin{bmatrix}-2 & 1\\ -1 & 0\end{bmatrix},\begin{bmatrix}3 & 4\\ 2 & 3\end{bmatrix},\begin{bmatrix}-3 & -3\\ 1 & 3\end{bmatrix}\right\}$$

linearly dependent or independent?

Let $v_1=\begin{bmatrix}1 & 0\\ 2 & 3\end{bmatrix}, v_2=\begin{bmatrix}-2 & 1\\ -1 & 0\end{bmatrix}, v_3=\begin{bmatrix}3 & 4\\ 2 & 3\end{bmatrix}, v_4=\begin{bmatrix}-3 & -3\\ 1 & 3\end{bmatrix}$

$$Let\, c_1v_1+c_2v_2+c_3v_3+c_4v_4=0-----------(1)$$

$$c_1\begin{bmatrix}1 & 0\\ 2 & 3\end{bmatrix}+c_2\begin{bmatrix}-2 & 1\\ -1 & 0\end{bmatrix}+c_3\begin{bmatrix}3 & 4\\ 2 & 3\end{bmatrix}+c_4\begin{bmatrix}-3 & -3\\ 1 & 3\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$$c_1 - 2c_2 + 3c_3 - 3c_4 = 0$$

$$c_2 + 4c_3 - 3c_4 = 0$$

$$2c_1 - c_2 + 2c_3 + c_4 = 0$$

$$3c_1 + 3c_3 + 3c_4 = 0$$

$$A = \begin{bmatrix} 1 & -2 & 3 & -3 \\ 0 & 1 & 4 & -3 \\ 2 & -1 & 2 & 1 \\ 3 & 0 & 3 & 3 \end{bmatrix}$$

Reducing A to Echelon form

$$A \sim \begin{bmatrix} 1 & -2 & 3 & -3 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho[A] = 3 <$ number of unknowns

$\therefore$ The system has a non-trivial solution.

$\therefore$ Vectors are linearly dependent.

Equivalent system is

$$c_1 - 2c_2 + 3c_3 - 3c_4 = 0, c_2 + 4c_3 - 3c_4 = 0, c_3 - c_4 = 0$$

Let $c_4 = k,\ k \neq 0 \Rightarrow \quad c_3 = k$

$$c_2 + 4k - 3k = 0 \Rightarrow c_2 = -k,$$

$$c_1 + 2k + 3k - 3k = 0 \Rightarrow c_1 = -2k$$

Substituting in (1) we get

$$-2kv_1 - kv_2 + kv_3 + kv_4 = 0\ , i.e.\ \ 2v_1 + v_2 - v_3 - v_4 = 0.$$

1. Determine by inspection if the given set is linearly independent

a. $\left\{\begin{bmatrix}1\\-1\\6\end{bmatrix}, \begin{bmatrix}4\\7\\-2\end{bmatrix}, \begin{bmatrix}1\\1\\-2\end{bmatrix}, \begin{bmatrix}3\\1\\4\end{bmatrix}\right\}$ **b.** $\left\{\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\5\\7\end{bmatrix}, \begin{bmatrix}2\\3\\1\end{bmatrix}\right\}$

c. $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\3\\4\end{bmatrix}, \begin{bmatrix}-4\\-6\\-8\end{bmatrix}\right\}$ **d.** $\left\{v_1 = \begin{bmatrix}-2\\4\\6\\10\end{bmatrix}, v_2 = \begin{bmatrix}5\\-10\\-15\\-25\end{bmatrix}\right\}$

a) Set is linearly dependent, because it contains 4-vectors in $\mathbb{R}^3$.

b) Set is linearly dependent, because it contain zero vector.

c) Set is linearly dependent, because $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} -4 \\ -6 \\ -8 \end{bmatrix}$.

d) Set is linearly dependent as there is relation between them.

as $-\frac{2}{5} = -\frac{4}{10} = -\frac{6}{15} = -\frac{10}{25}$ $\quad$ *i.e.* $v_1 = -\frac{2}{5}v_2$

Exercise

1. Are the following vectors linearly dependent? if so find the relation between them.

i) (2, -1, 3, 2), (1, 3, 4, 2) and (3, -5, 3, 2)

ii) [3 0 2 4 5], [7 2 6 1 0], [1 2 2 -7 -10].

iii) [9 0 9] , [0 6 6], [3 3 0].

iv) (1, 2, -1, 0), (1, 3, 1, 2), (4, 2, 1, 0), (6, 1, 0, 1).

2. Determine the given functions are linearly dependent or independent if dependent find the relation between them.

i) $p_1(x) = -1 - 3x + 3x^2,\ p_2(x) = 1 + 2x + x^2,\ p_3(x) = 2 + 5x,$

$p_4(x) = 3 + 8x - 2x^2.$

(ii). $\mathrm{H} = \left\{ \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} \right\}.$

4. Let $f(x) = x\, and\ g(x) = |x|$, determine wheather f(x) and g(x) are independent in

$$C[-1, 1] \text{ or in } C[0, 1].$$

5. Test for Linear dependence if dependent find relation among them.

i) $\{1, \sin x, \cos x\}$ $\qquad$ ii) $\{1, \sin^2 x, \cos^2 x\}$

iii) $\{\cos 2\text{x}, \sin^2 x, \cos^2 x\}$ $\qquad$ iv) $\{e^x, e^{-x}\}$