DSTL Unit 3 - Propositional Logic

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Today's Roadmap

Why Propositional Logic in CS?

- **Specification & Verification:** Reason about program correctness.
- **Circuit Design:** Logic gates implement propositional connectives.
- **Databases:** SQL predicates mirror logical formulas.
- **AI/SE:** SAT/SMT solvers power compilers, verification, planning, testing.

Examples You'll Recognize

- If the buffer is full **then** drop packet **else** enqueue.
- A request must be authenticated **and** authorized.
- A transaction commits **only if** all checks pass.

We will formalize such statements to reason unambiguously.

Propositions & Truth Values

Proposition: Declarative statement that is either true or false, but not both.

Examples

- "New Delhi is the capital of India." (**T**)
- "What time is it?" (Not a proposition)
- $2 + 2 = 5$ (**F**)

Connectives (Operators)

- Negation: $\neg p$
- Conjunction: $p \wedge q$
- Disjunction: $p \vee q$
- Exclusive-or: $p \oplus q$
- Implication: $p \rightarrow q$
- Biconditional: $p \leftrightarrow q$

Operator Precedence (High $\to$ Low)

$$\neg \;>\; \wedge \;>\; \oplus \;>\; \vee \;>\; \to \;>\; \leftrightarrow$$

Tip: Use parentheses liberally in teaching code/specs.

Practice: Parse and evaluate

$$p \vee \neg q \to p \wedge q$$

for all four truth assignments of (p, q).

Truth Tables: Core Connectives

Negation

p	$\neg p$
T	F
F	T

Conjunction

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Implication

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Implication: Intuition & Rephrasings

$p \to q$ means: if p holds, q must hold.

- "p implies q", "q if p", "p is sufficient for q", "q is necessary for p".
- **Contrapositive:** $\neg q \to \neg p$ (logically equivalent to $p \to q$).
- **Converse:** $q \to p$ (not equivalent).
- **Inverse:** $\neg p \to \neg q$ (not equivalent).

Biconditional (Bi-implication)

Notation: $p \leftrightarrow q$ (read: "p iff q")

Meaning: p and q have the same truth value.

Truth table:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Natural language: "p is necessary *and* sufficient for q".

Biconditional: Useful Rewrites

$$p \leftrightarrow q \;\equiv\; (p \rightarrow q) \wedge (q \rightarrow p)$$

$$p \leftrightarrow q \;\equiv\; (p \wedge q) \;\vee\; (\neg p \wedge \neg q)$$

Tip: In proofs or SAT/CNF conversions, expand $p \leftrightarrow q$ using one of the above forms.

Terminology: SAT / UNSAT

- **SAT** (Satisfiable): There exists some assignment of truth values that makes the formula TRUE.
- **UNSAT** (Unsatisfiable): No assignment makes the formula TRUE.
- **Valid / Tautology**: Formula is TRUE under all assignments.

Logical Equivalence ($\equiv$)

Two formulas φ, ψ are equivalent if they have the same truth value under every valuation.

Examples

- $p \to q \equiv \neg p \vee q$
- $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
- **Contrapositive:** $p \to q \equiv \neg q \to \neg p$

De Morgan & Basic Laws

De Morgan's

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Double Negation

$$\neg\neg p \equiv p$$

Distributivity

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

Idempotent

$$p \wedge p \equiv p, \quad p \vee p \equiv p$$

Tautology, Contradiction, Contingency

- **Tautology:** Always true (e.g. $p \vee \neg p$).
- **Contradiction:** Always false (e.g. $p \wedge \neg p$).
- **Contingency:** Sometimes true, sometimes false (most formulas).

Normal Forms: DNF & CNF

Disjunctive Normal Form (DNF): OR of AND-clauses (literals).

$$(p \land q \land \neg r) \lor (p \land \neg q \land r)$$

Conjunctive Normal Form (CNF): AND of OR-clauses.

$$(p \lor q \lor \neg r) \land (\neg p \lor r) \land (q \lor r)$$

Why care? SAT solvers expect CNF; truth-table to DNF is straightforward.

Equivalence Practice (Worked)

Simplify:

$$(p \to q) \land (p \to r)$$

Solution outline

$$(p \to q) \equiv (\neg p \lor q), \quad (p \to r) \equiv (\neg p \lor r)$$

$$(\neg p \lor q) \land (\neg p \lor r) \equiv \neg p \lor (q \land r)$$

(using distributivity). **So:** $(p \to q) \land (p \to r) \equiv \neg p \lor (q \land r)$.

Satisfiability & Validity

- **Satisfiable:** Some assignment makes formula true.
- **Unsatisfiable:** No assignment makes it true.
- **Valid:** True under all assignments (tautology).

Consistency of specs reduces to satisfiability of their conjunction.

Inference Rules (Sound)

- **Modus Ponens:** $p,\ p \to q\ \vdash\ q$
- **Modus Tollens:** $\neg q,\ p \to q\ \vdash\ \neg p$
- **Hypothetical Syllogism:** $(p \to q), (q \to r)\ \vdash\ p \to r$
- **Disjunctive Syllogism:** $p \vee q,\ \neg p\ \vdash\ q$
- **Addition:** $p\ \vdash\ p \vee q$
- **Conjunction:** $p,\ q\ \vdash\ p \wedge q$

Proof Methods at a Glance

- **Truth tables:** brute force but certain.
- **Algebraic rewrites:** use equivalences to simplify.
- **Proof by contradiction / contrapositive.**

Translating Natural Language $\rightarrow$ Logic

"You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old."

p: "You can ride"; q: "You are under 4 feet tall"; r: "You are older than 16".

Formalization: $(q \wedge \neg r) \rightarrow \neg p$

System Specs: Consistency Check

1. Msg stored in buffer **or** retransmitted. $(B \lor R)$
2. Msg **not** stored in buffer. $(\neg B)$
3. If stored then retransmitted. $(B \to R)$
4. Msg **not** retransmitted. $(\neg R)$

Are these consistent? Check satisfiability of
$(B \lor R) \land (\neg B) \land (B \to R) \land (\neg R)$.

- From $\neg B$ & $B \lor R$ we get R.
- Contradicts $\neg R \Rightarrow$ **Unsatisfiable**. Specs are inconsistent.

Logic & Circuits

- Map $p \wedge q$ to AND gate, $p \vee q$ to OR, $\neg p$ to NOT.
- $p \rightarrow q \equiv \neg p \vee q$ helps implement implication using OR + NOT.
- **Optimization:** Use equivalences to reduce gate count.

DNF from Truth Table (Mini-Example)

Suppose $f(p, q)$ is true exactly when $p = \mathbf{T}, q = \mathbf{F}$ or $p = \mathbf{F}, q = \mathbf{T}$.

$$f \equiv (p \wedge \neg q) \;\vee\; (\neg p \wedge q) \equiv p \oplus q$$

Metro Gate Logic: Smart Card

Let:

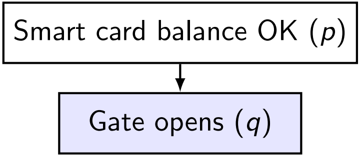
$$p = \text{"Smart card has sufficient balance"}, \quad q = \text{"Gate opens"}$$

Rule: $p \to q$.
Truth table:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Case (F, T) means gate opens without balance — a system error.

Metro Gate Logic Diagram



Rule: $p \to q$

Lab Entry Rule

Let:

$p =$ "You are a registered user", $q =$ "You wear lab coat", $r =$ "You

Policy: You can enter lab iff (registered and wearing lab coat).

$$r \leftrightarrow (p \land q)$$

Food Delivery Logic

$$\begin{aligned} p &= \text{“Restaurant is open”}, \\ q &= \text{“Area is serviceable”}, \\ s &= \text{“You have internet”}, \\ r &= \text{“You can place order”} \end{aligned}$$

Rule: You can place order only if all three conditions are true:

$$(p \wedge q \wedge s) \to r$$

If any of p, q, s is false (restaurant closed, area not serviceable, or no internet) $\Rightarrow$ you cannot place order.

Food Delivery Logic Diagram

Restaurant open (p)
$\downarrow$
Area serviceable (q)
$\downarrow$
Internet available (s)
$\downarrow$
Order placed (r)

Rule: $(p \wedge q \wedge s) \rightarrow r$

Street Light with Motion Sensor

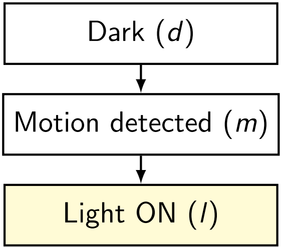
Let:

$$d = \text{“It is dark”}, \quad m = \text{“Motion detected”}, \quad l = \text{“Light ON”}$$

Rule: If it is dark and motion is detected, then light turns on:

$$(d \wedge m) \to l$$

Street Light Logic Diagram



Rule: $(d \wedge m) \rightarrow l$

Quick Check 1 (Precedence)

Evaluate $p \vee \neg q \to p \wedge q$ when $(p, q) = (\mathbf{T}, \mathbf{F})$.

Choices: A. T B. F

Quick Check 2 (Equivalence)

Which is equivalent to $p \to q$?

A) $q \to p$ **B)** $\neg p \vee q$ **C)** $p \wedge q$

Quick Check 3 (Consistency)

Are the following jointly satisfiable?

$\neg P \vee Q, \quad \neg Q, \quad P$

Choices: **A**. Yes **B**. No

Exercise 1 (Translation)

Formalize: "Access is granted if and only if the user is authenticated and not blacklisted."

G: granted, A: authenticated, B: blacklisted.

Exercise 2 (CNF Practice)

Convert $(p \to q) \land (r \lor \neg q)$ to CNF.

Exercise 3 (Tautology?)

Is $\big[(p \to q) \land (p \to \neg q)\big] \to \neg p$ a tautology?

Exercise 4 (SAT Warm-up)

Is $(p \vee q) \wedge (\neg p \vee r) \wedge (\neg q \vee r) \wedge (\neg r)$ satisfiable?

Recap

- Syntax/semantics of propositional logic; precedence.
- Truth tables, equivalence, laws, DNF/CNF.
- Implication variants (converse, inverse, contrapositive).
- Reasoning methods, satisfiability, consistency.
- Applications: specs, puzzles, circuits.

What's Next

- Predicate Logic: quantifiers $\forall, \exists$
- Normal forms for predicates, prenex form (later)
- Proof systems & strategies

Answers (Quick Checks)

QC1: $(p, q) = (\mathbf{T}, \mathbf{F})$: compute $\neg q = \mathbf{T}$, then $p \lor \neg q = \mathbf{T}$, and $p \land q = \mathbf{F}$, so $\mathbf{T} \rightarrow \mathbf{F}$ is $\mathbf{F}$. **Ans: B**.

QC2: $p \rightarrow q \equiv \neg p \lor q$. **Ans: B**.

QC3: From $\neg Q$ and $\neg P \lor Q$, we infer $\neg P$. But also P. Contradiction. **Ans: No**.

Exercise 1 (Model Answer)

"Access is granted iff authenticated and not blacklisted."

$G \leftrightarrow (A \land \neg B)$

Exercise 2 (CNF Sketch)

$(p \rightarrow q) \wedge (r \vee \neg q) \equiv (\neg p \vee q) \wedge (r \vee \neg q)$ (already CNF).

Exercise 3 (Tautology)

Yes. If p implies q and also p implies $\neg q$, then p cannot be true; thus $\neg p$ must hold. Hence the implication is always **T**.

Exercise 4 (Unsat)

$(\neg r)$ forces $r = \mathbf{F}$. Then $(\neg p \vee r)$ becomes $\neg p$, and $(\neg q \vee r)$ becomes $\neg q$. But $(p \vee q)$ requires p or q true. Contradiction. Unsatisfiable.

End

Thank you!