Functions: A Comprehensive Chapter

Prepared for 2nd-Year Students

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Chapter 1

Functions

1.1 Introduction

A **function** is a rule that assigns to every element of a set (called the domain) exactly one element of another set (called the codomain). Functions are fundamental in mathematics and computer science because they describe transformations, processes, and mappings.

Examples:

- A vending machine button $\rightarrow$ a drink.
- A student roll number $\rightarrow$ a student record.
- $f(x) = 2x + 1$, mapping real numbers to real numbers.

1.2 Formal Definition

A function f from set A to set B, denoted $f : A \rightarrow B$, is a relation such that:

1. Each $a \in A$ is associated with exactly one $b \in B$.

- **Domain:** The set A.
- **Codomain:** The set B.
- **Range:** The set of all actual outputs.

1.3 Types of Functions

1.3.1 Injective (One-to-One) Functions

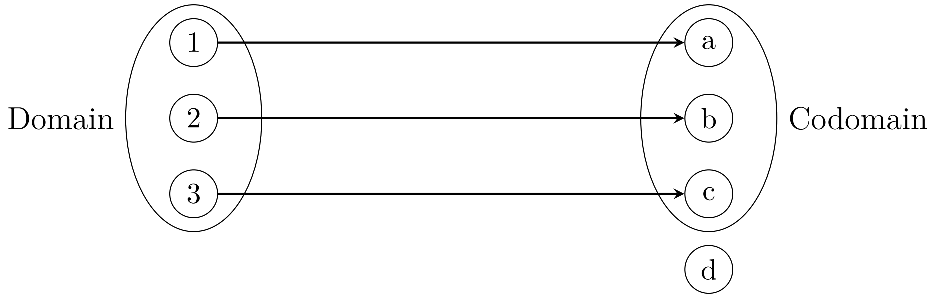
A function is **injective** if different inputs map to different outputs. Formally:

$$f(a_1) = f(a_2) \implies a_1 = a_2$$

Examples:

1. $f(x) = 2x + 3$ from $\mathbb{R}$ to $\mathbb{R}$.
2. Roll number $\rightarrow$ student (unique).

3. $f : \{1, 2, 3\} \to \{a, b, c, d\}$ with $f(1) = a, f(2) = b, f(3) = c$.



1.3.2 Surjective (Onto) Functions

A function is **surjective** if every element of the codomain has at least one preimage. Formally, for each $y \in B$, there exists $x \in A$ such that $f(x) = y$.

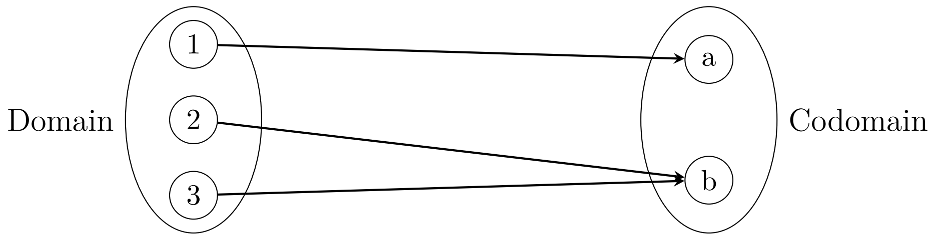
Surjectivity ensures that the codomain is completely covered. In practice, surjective mappings arise in cases like mapping students to their birth months, where every month must appear at least once. Such functions are important in linear algebra (onto transformations) and logic.

Examples:

1. $f(x) = x^3 : \mathbb{R} \to \mathbb{R}$.

2. Students $\to$ their birth month (all months covered).

3. $f : \mathbb{Z} \to \{0, 1\}, f(n) = n \bmod 2$.



1.3.3 Bijective Functions

A function is **bijective** if it is both injective and surjective. That is, every element of the codomain has exactly one preimage, and no two inputs map to the same output.

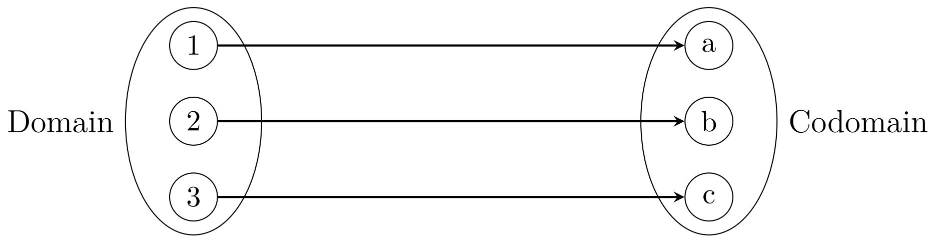
Bijective mappings create a perfect "pairing" between the domain and codomain. They are invertible: every bijective function has an inverse. Bijective functions appear in seat allocations (one seat per student, one student per seat), cryptographic keys, and database records.

Examples:

1. $f(x) = x + 5$ on $\mathbb{R}$.

2. Students $\leftrightarrow$ Seats (one-to-one).

3. $f : \{1, 2, 3\} \to \{a, b, c\}, f(1) = a, f(2) = b, f(3) = c$.

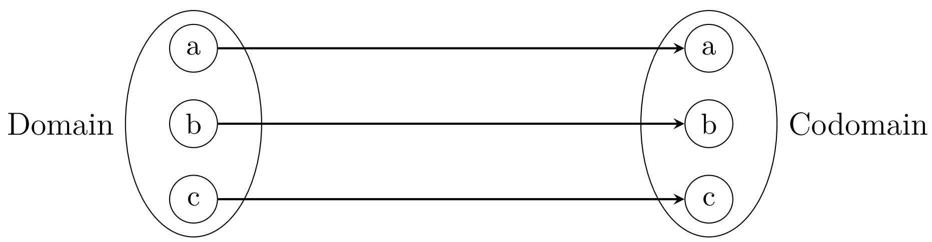


1.3.4 Identity Function

The **identity function** is defined by $f(x) = x$ for all x in the domain. It simply returns each input unchanged. It is important because it acts as the neutral element under function composition, much like 0 for addition or 1 for multiplication.

Examples:

1. $f(x) = x$ on $\mathbb{R}$.
2. Student $\rightarrow$ same student.
3. On $\{a, b, c\}$: $f(a) = a, f(b) = b, f(c) = c$.

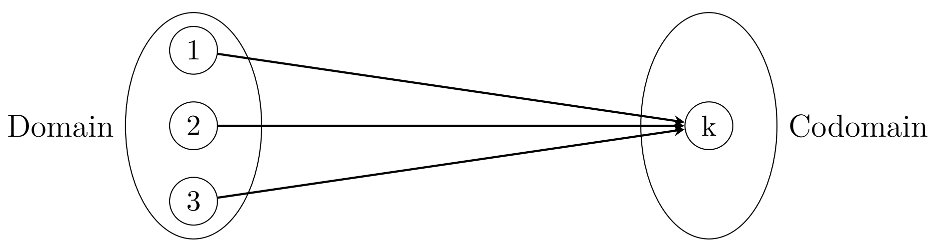


1.3.5 Constant Function

A **constant function** maps every element of the domain to the same codomain element. For instance, $f(x) = 5$ returns 5 for all x. While simple, constant functions appear in default values, uniform assignments, and flat-rate systems. They collapse all inputs into one output.

Examples:

1. $f(x) = 5$ for all x.
2. Every student $\rightarrow$ grade "A".
3. $\{1, 2, 3\} \rightarrow \{k\}$.



1.3.6 Projection Function

A **projection function** extracts one component from a tuple. For example, $\pi_1(x, y) = x$ returns the first coordinate. Projections are useful in computer science (database queries), geometry (dimensional reduction), and ordered data handling.

Examples:

1. $\pi_1(x, y) = x$.
2. Projection $(x, y, z) \mapsto x$.
3. $(roll, name) \mapsto roll$.

1.3.7 Inverse Function

If f is bijective, it has an **inverse function** f^{-1} that reverses its action. Inverse functions are critical in solving equations, cryptography, and unit conversions.

Examples:

1. $f(x) = 2x + 3$, inverse $f^{-1}(y) = (y - 3)/2$.
2. Celsius $\leftrightarrow$ Fahrenheit.
3. Seat number $\leftrightarrow$ student.

1.4 Composition of Functions

If $f : A \to B$ and $g : B \to C$, then $(g \circ f)(x) = g(f(x))$.

Examples:

1. $f(x) = 2x,\ g(x) = x + 3,\ (g \circ f)(x) = 2x + 3$.
2. Student $\to$ Roll $\to$ Marks.
3. km $\to$ m $\to$ cm.

1.5 Special Properties

- Even: $f(-x) = f(x)$, e.g., $x^2, \cos x$.
- Odd: $f(-x) = -f(x)$, e.g., $x^3, \sin x$.
- Periodic: repeats, e.g., sine, cosine.

1.6 Applications

- Computer Science: hashing, encryption, programming functions.
- Mathematics: modeling growth/decay, probability distributions.
- Real Life: ticket booking, authentication, currency conversion.

1.7 Summary Table

Type	Definition	Example
Injective	Distinct inputs $\to$ distinct outputs	$f(x) = 2x + 1$
Surjective	Every codomain element has preimage	$f(n) = n \bmod 2$
Bijective	One-to-one and onto	$f(x) = x + 5$
Identity	$f(x) = x$	Student $\to$ Student
Constant	All inputs map to same output	$f(x) = 7$
Projection	Returns one component	$\pi_1(x, y) = x$
Inverse	Reverses a bijection	Celsius $\leftrightarrow$ Fahrenheit

Chapter 2

Functions — Exercises and Practice

Part A: Solved Questions

Q1. Verify whether $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x + 3$ is injective.

Solution. Assume $f(x_1) = f(x_2)$. Then $2x_1 + 3 = 2x_2 + 3 \Rightarrow x_1 = x_2$. Hence injective. □

Q2. Show that $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ is surjective.

Solution. Given $y \in \mathbb{R}$, choose $x = \sqrt[3]{y}$. Then $f(x) = y$. Hence surjective. □

Q3. Prove $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x + 5$ is bijective.

Solution. Injective: $x_1 + 5 = x_2 + 5 \Rightarrow x_1 = x_2$. Surjective: for any y, take $x = y - 5$. Hence bijective. □

Q4. Prove identity $f(x) = x$ on any set A is bijective.

Solution. Injective: $x_1 = x_2$ if $f(x_1) = f(x_2)$. Surjective: for any $y \in A$, $f(y) = y$. Hence bijective. □

Q5. Check injectivity of $f : \mathbb{Z} \to \{0, 1\}$, $f(n) = n \bmod 2$.

Solution. $f(2) = 0 = f(4)$ with $2 \neq 4$. Not injective. □

Q6. Check surjectivity of the same f in Q5.

Solution. Range is $\{0, 1\}$ (even $\mapsto 0$, odd $\mapsto 1$). Hence surjective. □

Q7. Find the inverse of $f(x) = 2x + 3$ on $\mathbb{R}$.

Solution. $y = 2x + 3 \Rightarrow x = \dfrac{y-3}{2}$. So $f^{-1}(y) = \dfrac{y-3}{2}$. □

Q8. Show constant $f(x) = 5$ on $\mathbb{R}$ is not injective.

Solution. $f(2) = 5 = f(3)$ but $2 \neq 3$. Not injective. $\square$

Q9. Show the same constant $f(x) = 5$ is not surjective onto $\mathbb{R}$.

Solution. Range is $\{5\} \neq \mathbb{R}$. Not surjective. $\square$

Q10. Verify $f : \{1, 2, 3\} \to \{a, b\}$ with $f(1) = a, f(2) = b, f(3) = b$ is surjective.

Solution. Range is $\{a, b\}$, which equals the codomain. Hence surjective (not injective). $\square$

Q11. Compute $(g \circ f)(x)$ for $f(x) = 2x$, $g(x) = x + 3$.

Solution. $(g \circ f)(x) = g(2x) = 2x + 3$. $\square$

Q12. Is $f(x) = x^2$ injective on $\mathbb{R}$?

Solution. No: $f(2) = f(-2) = 4$ with $2 \neq -2$. $\square$

Q13. Is $f(x) = x^2$ injective on $[0, \infty)$?

Solution. Yes; monotone increasing on $[0, \infty)$, hence injective. $\square$

Q14. Check injectivity of $f(x) = |x|$ on $\mathbb{R}$.

Solution. $f(2) = f(-2)$; not injective. $\square$

Q15. Show $f : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$, $f(x) = 1/x$ is bijective.

Solution. Injective: $1/x_1 = 1/x_2 \Rightarrow x_1 = x_2$. Surjective: given $y \neq 0$, $x = 1/y$ works. Hence bijective. $\square$

Part B: Unsolved Questions

U1. Prove/disprove: $f(x) = x^2 + 1$ is surjective $\mathbb{R} \to \mathbb{R}$.

U2. Is $f(x) = 3x + 7$ injective on $\mathbb{R}$?

U3. Check if $\sin x$ is injective on $\mathbb{R}$.

U4. Restrict the domain of $\sin x$ to make it injective and find its inverse on that domain.

U5. Find f^{-1} for $f(x) = \dfrac{x-2}{3}$.

U6. Is $f : \{1, 2, 3\} \to \{a, b\}$ with $f(1) = a, f(2) = a, f(3) = a$ surjective?

U7. Is e^x surjective $\mathbb{R} \to \mathbb{R}$?

U8. Is e^x bijective $\mathbb{R} \to (0, \infty)$?

U9. Prove/disprove: $\tan x$ is injective on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

U10. Define a function Students $\to$ Classrooms; discuss injectivity/surjectivity under realistic constraints.

U11. Find $(f \circ g)(x)$ for $f(x) = x^2$, $g(x) = x + 1$.

U12. Show $\cos x$ is periodic and find its period.

U13. Is the projection $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$, $\pi_1(x, y) = x$ injective?

U14. Is $\pi_2 : \mathbb{R}^2 \to \mathbb{R}$, $\pi_2(x, y) = y$ surjective?

U15. Find f^{-1} of $\ln x$ (specify domain/codomain).

U16. Draw a mapping diagram of a surjective but not injective function between finite sets.

U17. Draw a mapping diagram of an injective but not surjective function between finite sets.

U18. Is $f(x) = \sqrt{x}$ injective on $[0, \infty)$?

U19. Is $f(x) = \sqrt{x}$ surjective $[0, \infty) \to \mathbb{R}$?

U20. If $f(x) = 2x$, $g(x) = x + 1$, find $(f \circ g)(x)$.

U21. For the same f, g, find $(g \circ f)(x)$.

U22. Prove the identity function on any set is bijective.

U23. Give an example of a constant function that is surjective (specify domain/codomain carefully).

U24. Define a bijection between digits $\{0, 1, \ldots, 9\}$ and letters $\{A, \ldots, J\}$.

U25. Show that $f(x) = \dfrac{1}{1 + e^{-x}}$ is neither injective nor surjective as a map $\mathbb{R} \to \mathbb{R}$.

Part C: Multiple Choice Questions (30)

M1. If $f(x) = 2x + 3$, then $f^{-1}(x) =$

a) $\frac{x-3}{2}$ **b)** $2x - 3$ **c)** $\frac{x+3}{2}$ **d)** None

M2. Which is surjective $\mathbb{R} \to \mathbb{R}$?

a) x^2 **b)** x^3 **c)** e^x **d)** $|x|$

M3. The identity function on a nonempty set is always:

a) injective only **b)** surjective only **c)** bijective **d)** constant

M4. A constant function on a nonempty domain is:

a) always injective **b)** never injective **c)** always bijective **d)** even

M5. $f : \mathbb{Z} \to \{0, 1\}$, $f(n) = n \bmod 2$ is:

a) injective only **b)** surjective only **c)** bijective **d)** neither

M6. If f is bijective, then f^{-1} exists and is:

a) injective only **b)** surjective only **c)** bijective **d)** constant

M7. If f and g are injective, then $g \circ f$ is:

a) injective **b)** surjective **c)** constant **d)** none

M8. If f and g are surjective, then $g \circ f$ is:

a) injective **b)** surjective **c)** constant **d)** none

M9. $(g \circ f)(x)$ for $f(x) = 2x$, $g(x) = x + 1$ equals:

a) $2x + 1$ **b)** $x + 3$ **c)** $2x - 1$ **d)** $x + 2$

M10. A function $f : \{1, 2, 3\} \to \{a, b\}$ with images $\{a, b, b\}$ is:

a) injective **b)** surjective **c)** bijective **d)** neither

M11. $f(x) = x^2$ on $\mathbb{R}$ is:

a) injective **b)** surjective **c)** neither **d)** bijective

M12. Restricting $f(x) = x^2$ to $[0, \infty)$ makes it:

a) injective **b)** surjective $\mathbb{R} \to \mathbb{R}$ **c)** bijective on $\mathbb{R}$ **d)** constant

M13. $f(x) = |x|$ on $\mathbb{R}$ is:

a) injective **b)** surjective $\mathbb{R} \to \mathbb{R}$ **c)** neither **d)** bijective

M14. The projection $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$, $\pi_1(x, y) = x$ is:
a) injective **b)** surjective **c)** bijective **d)** constant

M15. $f(x) = 1/x$ from $\mathbb{R} \setminus \{0\}$ to itself is:
a) injective only **b)** surjective only **c)** bijective **d)** constant

M16. If f is bijective, then $(f^{-1})^{-1}$ equals:
a) f **b)** f^{-1} **c)** identity **d)** constant

M17. If f is injective but not surjective, then f^{-1}:
a) exists and is injective **b)** exists and is surjective **c)** does not exist **d)** is constant

M18. The range of $f(x) = e^x$ is:
a) $\mathbb{R}$ **b)** $(0, \infty)$ **c)** $[0, \infty)$ **d)** $(-\infty, 0)$

M19. The function $f(x) = \ln x$ has domain:
a) $\mathbb{R}$ **b)** $(0, \infty)$ **c)** $[0, \infty)$ **d)** $(-\infty, 0)$

M20. If f is bijective, the unique g with $g \circ f = \mathrm{id}$ is:
a) f **b)** f^{-1} **c)** constant **d)** none

M21. $(f \circ g)(x)$ for $f(x) = x^2$, $g(x) = x + 1$ is:
a) $x^2 + 1$ **b)** $x^2 + 2x + 1$ **c)** $x^2 + 2x$ **d)** $x^2 - 1$

M22. $(g \circ f)(x)$ for the same f, g is:
a) $x^2 + 1$ **b)** $x^2 + 2x + 1$ **c)** $x^2 + 2x$ **d)** $x^2 - 1$

M23. A bijection between finite sets must have:
a) equal cardinalities **b)** domain larger **c)** codomain larger **d)** none

M24. The identity on any set A is:
a) constant **b)** bijective **c)** surjective only **d)** injective only

M25. A constant function from a nonempty set to a singleton is:
a) injective **b)** surjective **c)** bijective **d)** none

M26. The composition of two bijections is:
a) injective **b)** surjective **c)** bijective **d)** constant

M27. If f is surjective and g is injective, then $g \circ f$ is:
a) injective **b)** surjective **c)** may be neither **d)** always bijective

M28. $f : \mathbb{Z} \to \mathbb{Z}$, $f(n) = n + 1$ is:
a) injective only **b)** surjective only **c)** bijective **d)** neither

M29. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 + 1$ is:
a) injective only **b)** surjective only **c)** bijective **d)** neither

M30. If f is bijective, then the inverse of f^{-1} is:
a) f **b)** identity **c)** constant **d)** undefined

Answer Key for MCQs

M1	(a)	M16	(a)
M2	(b)	M17	(c)
M3	(c)	M18	(b)
M4	(b)	M19	(b)
M5	(b)	M20	(b)
M6	(c)	M21	(a)
M7	(a)	M22	(a)
M8	(b)	M23	(b)
M9	(a)	M24	(b)
M10	(b)	M25	(b)
M11	(c)	M26	(c)
M12	(a)	M27	(c)
M13	(c)	M28	(c)
M14	(b)	M29	(c)
M15	(c)	M30	(a)