Equivalence Relations, Equivalence Classes, and Partial Orders

With Hasse Diagrams and Poset Elements

Prepared for 2^{nd}-Year Students

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Contents

1 Relations: A Quick Recall

A *binary relation* R on a set A is any subset $R \subseteq A \times A$. We write $a\,R\,b$ when $(a, b) \in R$.

2 Equivalence Relations

Definition 1 (Equivalence Relation)**.** *A relation R on a set A is an* equivalence relation *if it is*

(i) ***reflexive:*** $\forall a \in A,\ aRa,$

(ii) ***symmetric:*** $\forall a, b \in A,\ aRb \Rightarrow bRa,$

(iii) ***transitive:*** $\forall a, b, c \in A,\ (aRb \land bRc) \Rightarrow aRc.$

Example 1 (Same Parity on $\mathbb{Z}$)**.** *Let $a \sim b$ if a and b have the same parity (both even or both odd). Then $\sim$ is an equivalence relation.*

Example 2 (Congruence Modulo 3 on $\mathbb{Z}$). *Define* $a \equiv b \pmod 3$ *if* $3 \mid (a-b)$. *This is an equivalence relation with three equivalence classes*

$$[0] = \{\dots, -6, -3, 0, 3, 6, \dots\}, \quad [1] = \{\dots, -5, -2, 1, 4, 7, \dots\}, \quad [2] = \{\dots, -4, -1, 2, 5, 8, \dots\}.$$

2.1 Equivalence Classes and Partitions

Definition 2 (Equivalence Class). *Given an equivalence relation R on A and $a \in A$, its* equivalence class *is*

$$[a] = \{x \in A \mid xRa\}.$$

The set A is partitioned into pairwise disjoint classes $\{[a] \mid a \in A\}$ whose union is A.

3 Partial Order Relations and Posets

Definition 3 (Partial Order and Poset). *A relation $\leq$ on a set P is a* partial order *if it is*

(i) ***reflexive:*** $a \leq a$,

(ii) ***antisymmetric:*** $(a \leq b \wedge b \leq a) \Rightarrow a = b$,

(iii) ***transitive:*** $(a \leq b \wedge b \leq c) \Rightarrow a \leq c$.

Then $(P, \leq)$ is called a partially ordered set *(poset).*

Remark 1 (Important Correction About Antisymmetry). *In a previous class, the antisymmetric property was misstated as "$a \neq b$". The correct statement is:*

If $a \leq b$ and $b \leq a$, then necessarily $a = b$.

Equivalently, for two distinct elements $a \neq b$, it cannot *happen that both $a \leq b$ and $b \leq a$ hold.*

Example 3 (Divisibility on Positive Integers). *On $\mathbb{N}$, define $a \preceq b$ iff a divides b (written $a \mid b$). Then $\preceq$ is a partial order.*

Example 4 (Subset Inclusion on Power Set). *For any set S, $(\mathcal{P}(S), \subseteq)$ is a poset.*

4 Hasse Diagrams

A *Hasse diagram* is a streamlined drawing of a finite poset that shows only the *covering* relations.

Construction Rules

R1. Omit reflexive loops ($a \leq a$).

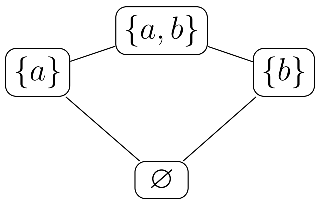
R2. Omit transitive edges: draw an edge a–b only if $a < b$ and there is no c with $a < c < b$; then b *covers* a.

R3. Draw edges vertically upward: place b above a if $a < b$.

4.1 Example: Divisibility on $\{1, 2, 4, 8\}$

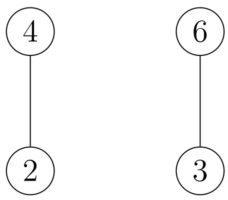
Here 1 is the least element and 8 is the greatest element of this poset.

4.2 Example: Subsets of $\{a, b\}$ ordered by $\subseteq$



This is a lattice: LUB of $\{a\}$ and $\{b\}$ is $\{a, b\}$; GLB is $\varnothing$.

4.3 Example: Divisibility on $\{2, 3, 4, 6\}$ (No least or greatest)



Here 2 and 3 are minimal; 4 and 6 are maximal; there is no single least or greatest element.

5 Extremal Elements and Bounds in Posets

Let $(P, \leq)$ be a poset.

Definition 4 (Minimal and Maximal Elements)**.** *An element $m \in P$ is* minimal *if there is no $x \in P$ with $x < m$. Dually, $M \in P$ is* maximal *if there is no $y \in P$ with $M < y$.*

Definition 5 (Least and Greatest Elements)**.** *The* least element *(also called the* zero element*, denoted* 0*) is an element $0 \in P$ such that*

$$0 \leq x \quad \textit{for all } x \in P.$$

The greatest element *(also called the* unit element*, denoted* 1*) is an element* $1 \in P$ *such that*

$$x \leq 1 \quad \text{for all } x \in P.$$

If they exist, the least and greatest elements are unique.

Definition 6 (Bounds, Infimum and Supremum)**.** *For a subset* $S \subseteq P$*:*

- u *is an* upper bound *of* S *if* $s \leq u$ *for all* $s \in S$*.*
- ℓ *is a* lower bound *of* S *if* $\ell \leq s$ *for all* $s \in S$*.*

The least upper bound *(LUB), also called the* supremum *(*$\sup S$*), is the smallest upper bound of* S*. The* greatest lower bound *(GLB), also called the* infimum *(*$\inf S$*), is the largest lower bound of* S*.*

Summary of Different Types of Elements in a Poset (Hasse Diagram Context)

- **Minimal element:** No element strictly below it.
- **Maximal element:** No element strictly above it.
- **Least element (0, zero element):** Below every other element.
- **Greatest element (1, unit element):** Above every other element.
- **Greatest Lower Bound (GLB, Infimum):** Largest element below all in a subset.
- **Least Upper Bound (LUB, Supremum):** Smallest element above all in a subset.

Example 5 (Bounds in a Chain)**.** *In* $P = \{1, 2, 3, 4, 5, 6\}$ *with the usual* $\leq$*, for* $S = \{2, 3, 5\}$*, lower bounds are* $\{1, 2\}$ *so* $\inf S = 2$*; upper bounds are* $\{5, 6\}$ *so* $\sup S = 5$*.*

Example 6 (Bounds in Divisibility Poset)**.** *In* $P = \{1, 2, 3, 4, 6, 12\}$ *with* $|$*, for* $S = \{4, 6\}$*, GLB is* 2 *(their* gcd *in* P*) and LUB is* 12 *(their* lcm *in* P*).*

6 Concept Map and Takeaways

- Equivalence relations group elements into **equivalence classes** (partitions).
- Partial orders organize elements into a **hierarchy** (posets), visualized by **Hasse diagrams**.
- Know the difference: *symmetric* (equivalence) vs. *antisymmetric* (partial order).
- Extremal elements: minimal, maximal, least, greatest; bounds: upper/lower, LUB/GLB.

Instructor's note (correction): In an earlier session, antisymmetry was mistakenly stated as "$a \neq b$". The correct definition is:
If $a \leq b$ *and* $b \leq a$ *then* $a = b$*.* Distinct elements cannot be mutually related in both directions in a partial order.