Lattices: A Comprehensive Chapter

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Outline

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Historical Background and Motivation

The study of lattices emerged from algebra, logic, and order theory.

- Many familiar operations share a common structure: union/intersection, gcd/lcm, AND/OR.
- The abstract structure is a **lattice**.
- Lattices unify reasoning across set theory, number theory, logic, linear algebra, and CS.

Order and Structure in Everyday Life

- **Task scheduling:** latest common prerequisite $\Rightarrow$ meet; earliest common milestone $\Rightarrow$ join.
- **Hierarchies:** lowest common supervisor $\Rightarrow$ meet; combined responsibility $\Rightarrow$ join.
- **Sets:** intersection $=$ meet, union $=$ join.

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Posets and Lattices

Definition

A *poset* $(P, \leq)$: reflexive, antisymmetric, transitive.

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Antisymmetry $\neq$ asymmetry. Antisymmetric means: $a \leq b$ & $b \leq a \Rightarrow a = b$.

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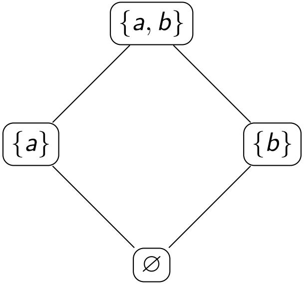
Definition

A *lattice* is a poset where every pair a, b has GLB $(a \wedge b)$ and LUB $(a \vee b)$.

Hasse Diagrams — Rules

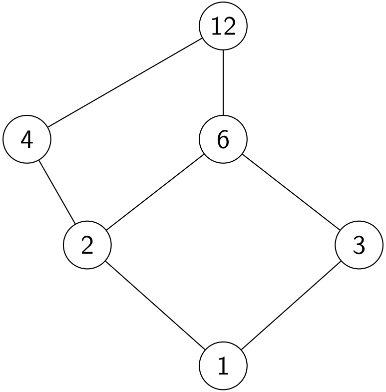
- Least element at bottom; greatest at top (if they exist).
- Draw edges only for cover relations; omit transitive edges.
- Edges are implicitly upward: higher nodes are greater.

Example: Power Set of {a,b}



Least $0 = \varnothing$, Greatest $1 = \{a, b\}$. Meet $= \cap$, Join $= \cup$.

Example: Divisors of 12 under Divisibility



Meet = gcd, Join = lcm; least = 1, greatest = 12.

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Definitions & Universal Properties

For a, b in a poset $(P, \leq)$:

- $m = a \wedge b$ if $m \leq a, b$ and for any $x \leq a, b$, $x \leq m$.
- $j = a \vee b$ if $a, b \leq j$ and for any x with $a, b \leq x$, $j \leq x$.

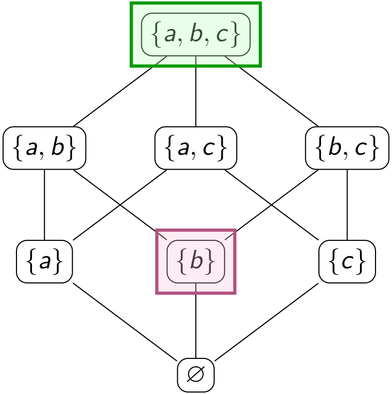
Uniqueness (Sketch)

If m and m' are both GLBs, then $m \leq m'$ and $m' \leq m$; by antisymmetry $m = m'$. (Similarly for LUBs.)

Contexts & Examples

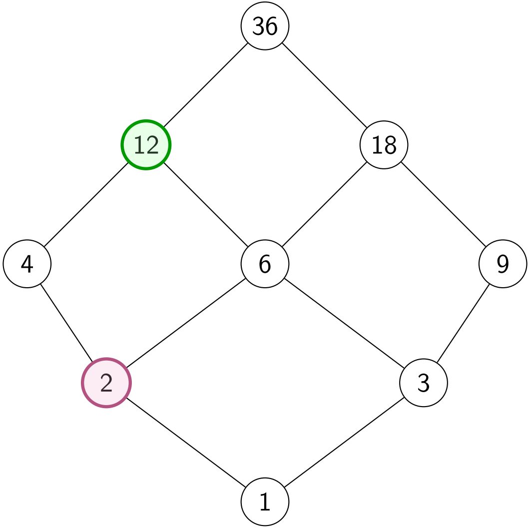
- **Sets:** $A \wedge B = A \cap B,\ A \vee B = A \cup B$.
- **Divisibility:** $a \wedge b = \gcd(a, b),\ a \vee b = \mathrm{lcm}(a, b)$.
- **Logic:** $p \wedge q$ (AND), $p \vee q$ (OR).
- **Vector spaces:** $U \wedge V = U \cap V,\ U \vee V = \mathrm{span}(U \cup V)$.

Worked Example: $\mathcal{P}(\{a, b, c\})$



$$X = \{a, b\},\ Y = \{b, c\}:\ \text{GLB} = \{b\},\ \text{LUB} = \{a, b, c\}.$$

Worked Example: Divisors of 36



For 4 and 6: GLB $= 2$, LUB $= 12$.

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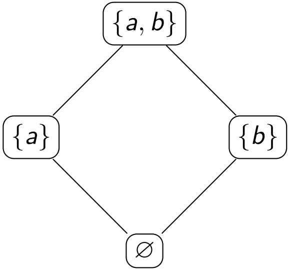
Bounded Lattices

Definition

A lattice is *bounded* if it has least 0 and greatest 1 with $0 \leq x \leq 1$ for all x.

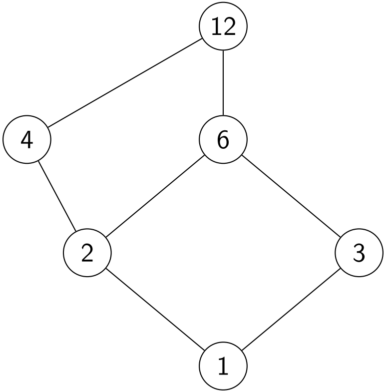
Bounded: Examples



Power set $\mathcal{P}(\{a, b\})$

Least $= \varnothing$, Greatest $= \{a, b\}$.

Bounded: Divisors of 12



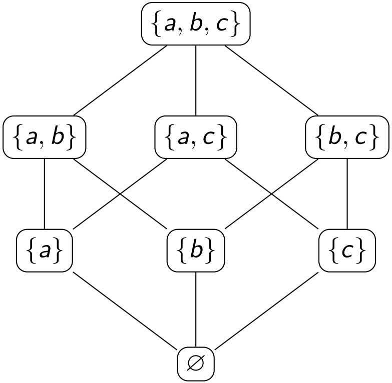
Least $= 1$, Greatest $= 12$.

Complemented Lattices

Definition

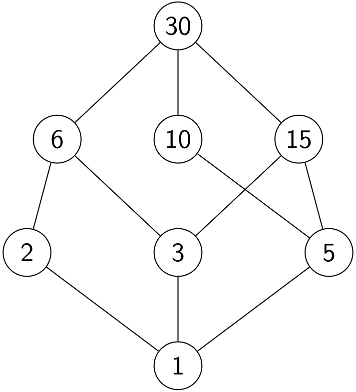
A bounded lattice is *complemented* if for every a there is b with $a \wedge b = 0$ and $a \vee b = 1$.

Complemented: $\mathcal{P}(\{a, b, c\})$



Complements are set-theoretic complements.

Complemented: Divisors of 30 (Boolean)



Complements: $d \leftrightarrow 30/d$ (e.g., $2 \leftrightarrow 15$).

Distributive Lattices

Definition

A lattice is **distributive** if for all a, b, c:

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c), \quad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

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Examples

- Power set lattices $\mathcal{P}(S)$ under $\subseteq$.
- Divisors of a square-free integer under divisibility.

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Non-example: M_3 (diamond lattice), N_5 (pentagon lattice).

Modular Lattices

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- Subspaces of a vector space under inclusion.
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Non-example: N_5 (fails modular law).

Complete Lattices

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A lattice is **complete** if every subset $S \subseteq L$ has:

$$\bigwedge S \quad \text{(infimum)} \quad \text{and} \quad \bigvee S \quad \text{(supremum)}.$$

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Examples

- $\mathcal{P}(S)$ (power set): arbitrary intersections/unions.
- $(\mathbb{R}, \leq)$ extended with $\pm\infty$.

Boolean Lattices

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- Bounded (has 0 and 1),
- Distributive,
- Complemented.

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Boolean lattices underpin **Boolean algebra** and **digital logic**.

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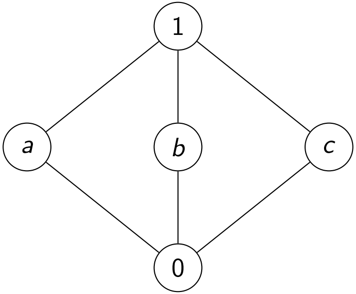
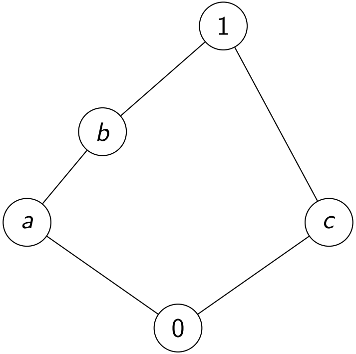
M_3 (Diamond): Modular, Not Distributive



$$a \wedge (b \vee c) = a \wedge 1 = a \neq 0 = (a \wedge b) \vee (a \wedge c).$$

N_5 (Pentagon): Not Modular (Not Distributive)



$$a \vee (b \wedge c) = a \vee 0 = a \neq b = (a \vee b) \wedge (a \vee c).$$

Outline

Hierarchy

$$\text{Boolean} \Rightarrow \text{Distributive} \Rightarrow \text{Modular} \Rightarrow \text{Lattice}.$$

- Not reversible in general.
- Counterexamples: M_3 (modular $\not\Rightarrow$ distributive), N_5 (lattice $\not\Rightarrow$ modular), divisors of 12 (distributive $\not\Rightarrow$ Boolean).

Counterexamples (Sketch)

- **Modular $\not\Rightarrow$ Distributive:** M_3 (already shown).
- **Lattice $\not\Rightarrow$ Modular:** N_5 (already shown).
- **Distributive $\not\Rightarrow$ Boolean:** A distributive lattice without complements (e.g., divisors of 12).

Summary Table

Class Implication	True	Counterexample to Reverse
Boolean $\Rightarrow$ Distributive	Yes	Distributive not Boolean: D_{12}
Distributive $\Rightarrow$ Modular	Yes	M_3
Modular $\Rightarrow$ Lattice	Yes	N_5

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Glossary (1/2)

- **Poset:** reflexive, antisymmetric, transitive relation.
- **Hasse diagram:** show cover relations upward; omit transitive edges.
- **Meet ($\wedge$):** GLB / infimum. Examples: $\cap$, gcd.
- **Join ($\vee$):** LUB / supremum. Examples: $\cup$, lcm.
- **Least/Greatest elements:** 0 and 1 if they exist.

Glossary (2/2)

- **Bounded:** has 0 and 1.
- **Complement/Complemented:** $a \wedge b = 0$, $a \vee b = 1$; every element has complement.
- **Distributive:** meet/join distribute.
- **Modular:** $a \leq c \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c$.
- **Complete:** arbitrary $\bigwedge, \bigvee$ exist.
- **Boolean:** bounded + distributive + complemented.
- M_3**:** modular, not distributive. N_5**:** not modular.

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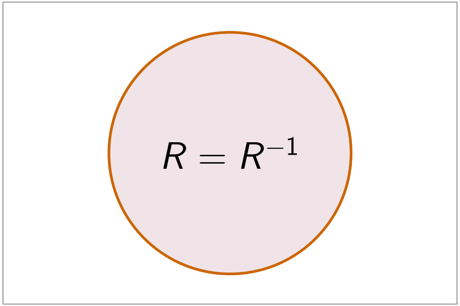
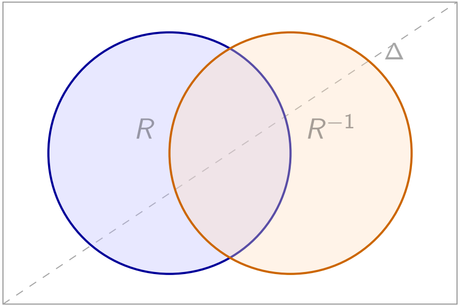
Symmetric Relation ($R = R^{-1}$)



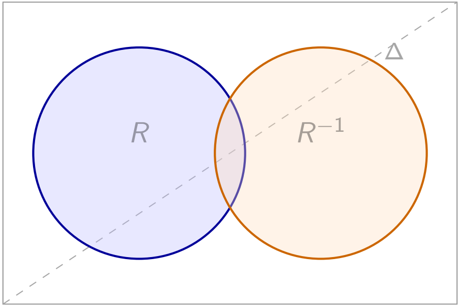
Definition: If $(a, b) \in R \Rightarrow (b, a) \in R$. **Real–world:** friendship / siblings / equality.

Anti-symmetric ($R \cap R^{-1} \subseteq \Delta$)



Rule: overlap only on the diagonal $\Delta = \{(x, x)\}$. **Examples:** $\subseteq$, $\leq$, divides.

Asymmetric ($R \cap R^{-1} = \varnothing$ & irreflexive)



Rule: $R \cap R^{-1} = \varnothing$ and no $(a, a) \in R$. **Examples:** parent-of, $<$.

Questions?