Lattices: A Comprehensive Chapter

Prepared for 2^{nd}-Year Students

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Chapter 1

Introduction to Lattices

1.1 Historical Background and Motivation

The study of lattices emerged from the intersection of algebra, logic, and order theory in the early twentieth century. Mathematicians noticed that many familiar operations—union and intersection of sets, greatest common divisors and least common multiples of integers, logical AND and OR—share a common order-theoretic pattern: given two objects, there is a *largest* thing below both (a greatest lower bound) and a *smallest* thing above both (a least upper bound). The abstract structure that captures this phenomenon is a *lattice*. The name evokes the crisscross structure that appears when these relationships are drawn as diagrams.

Lattice theory now plays a central role across discrete mathematics, algebra, topology, and computer science. It unifies reasoning across set theory (intersection/union), number theory (gcd/lcm), logic (AND/OR and Boolean algebra), and linear algebra (intersection/span of subspaces), and provides a natural language for hierarchies and fixed-point constructions used in program analysis.

1.2 Order and Structure in Everyday Life

Order relations appear in many real-world settings:

- **Task scheduling.** If Task A must precede Task B, we have an order. The *latest common prerequisite* for two tasks behaves like a meet; the *earliest common milestone* behaves like a join.

- **Hierarchies.** The lowest common supervisor of two employees is analogous to a meet; the least senior position that subsumes both roles is analogous to a join.

- **Sets.** The intersection of two sets is their common part (meet), while the union is the combined cover (join).

Lattices formalize this style of reasoning so that the same tools apply across different domains.

1.3 Posets and Lattices

Definition 1.1 (Poset)**.** *A* partially ordered set (poset) *is a set P equipped with a relation $\leq$ that is reflexive ($a \leq a$), antisymmetric (if $a \leq b$ and $b \leq a$, then $a = b$), and transitive (if $a \leq b$ and $b \leq c$, then $a \leq c$).*

Remark 1.1 (Antisymmetric $\neq$ asymmetric)**.** *Antisymmetry does* not *mean $a \neq b$. It means that* if *both $a \leq b$ and $b \leq a$ hold, then $a = b$. Asymmetry would forbid $b \leq a$ whenever $a \leq b$; that is a* different *property and not intended here.*

Definition 1.2 (Lattice)**.** *A poset $(L, \leq)$ is a* lattice *if every pair $a, b \in L$ has both a greatest lower bound (the* meet*, $a \wedge b$) and a least upper bound (the* join*, $a \vee b$).*

1.4 Hasse Diagrams

A *Hasse diagram* is a simplified drawing of a poset that makes the order visually clear:

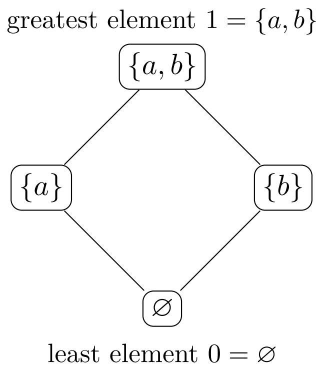
- Place the least element at the bottom and the greatest at the top (when they exist).
- Draw an (undirected) edge x–y only when $x < y$ and there is no z with $x < z < y$ (i.e., y *covers* x).
- Edges are implicitly oriented upward: higher nodes are greater.
- Transitive edges are omitted to reduce clutter.

1.5 First Examples with Diagrams

We present two canonical lattices and their Hasse diagrams.

1.5.1 Power Set of $\{a, b\}$ under Inclusion

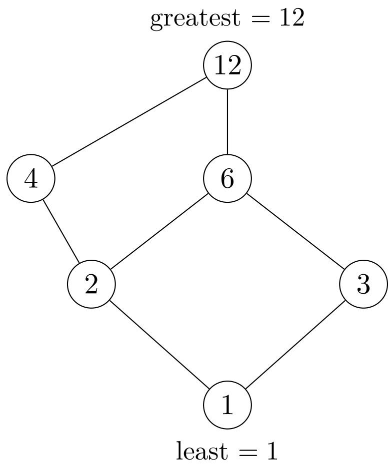
Here the order is $\subseteq$. Meet is intersection, join is union. The least element is $\varnothing$ and the greatest is $\{a, b\}$.



Meet and join example. $\{a\} \wedge \{b\} = \varnothing$ (largest set contained in both); $\{a\} \vee \{b\} = \{a, b\}$ (smallest set containing both).

1.5.2 Divisors of 12 under Divisibility

Let $D_{12} = \{1, 2, 3, 4, 6, 12\}$ ordered by $a \preceq b$ iff a divides b. The least element is 1 and the greatest is 12. Meet is gcd, join is lcm.



Meet and join example. For 4 and 6 we have $\gcd(4, 6) = 2$ (GLB) and $\mathrm{lcm}(4, 6) = 12$ (LUB). In the Hasse diagram, 2 is the *highest* node below both 4 and 6, while 12 is the *lowest* node above both.

Chapter 2

Meet and Join as GLB and LUB

2.1 Definition and Motivation

In any partially ordered set $(P, \leq)$, two natural questions arise for elements $a, b \in P$:

- What is the *largest* element that is less than or equal to both a and b?
- What is the *smallest* element that is greater than or equal to both a and b?

These questions lead to the notions of **greatest lower bound (GLB)** and **least upper bound (LUB)**.

Definition 2.1 (Meet and Join). *Let $(P, \leq)$ be a poset and $a, b \in P$.*

- *The* meet *of a and b, denoted $a \wedge b$, is the* greatest lower bound *(GLB) of $\{a, b\}$.*
- *The* join *of a and b, denoted $a \vee b$, is the* least upper bound *(LUB) of $\{a, b\}$.*

2.2 Universal Property Formulation

- $m = a \wedge b$ if:
  1. $m \leq a$ and $m \leq b$, and
  2. For any x with $x \leq a$ and $x \leq b$, we have $x \leq m$.
- $j = a \vee b$ if:
  1. $a \leq j$ and $b \leq j$, and
  2. For any x with $a \leq x$ and $b \leq x$, we have $j \leq x$.

Proposition 2.1 (Uniqueness). *If a GLB or LUB exists, it is unique.*

Proof. Suppose m and m' are both GLBs of $\{a, b\}$. Then $m \leq m'$ (since m' is a lower bound) and $m' \leq m$ (since m is a lower bound). By antisymmetry, $m = m'$. A similar argument applies to LUBs. $\square$

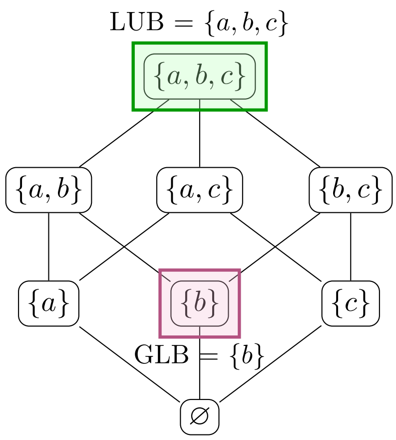
2.3 Examples in Different Contexts

- **Sets.** $A \wedge B = A \cap B$, $A \vee B = A \cup B$.
- **Numbers under divisibility.** $a \wedge b = \gcd(a, b)$, $a \vee b = \operatorname{lcm}(a, b)$.
- **Logic.** $p \wedge q = p \,\text{AND}\, q$, $p \vee q = p \,\text{OR}\, q$.
- **Vector spaces.** $U \wedge V = U \cap V$, $U \vee V = \operatorname{span}(U \cup V)$.

2.4 Worked Example: Power Set $\mathcal{P}(\{a, b, c\})$

Consider $X = \{a, b\}$ and $Y = \{b, c\}$. Then

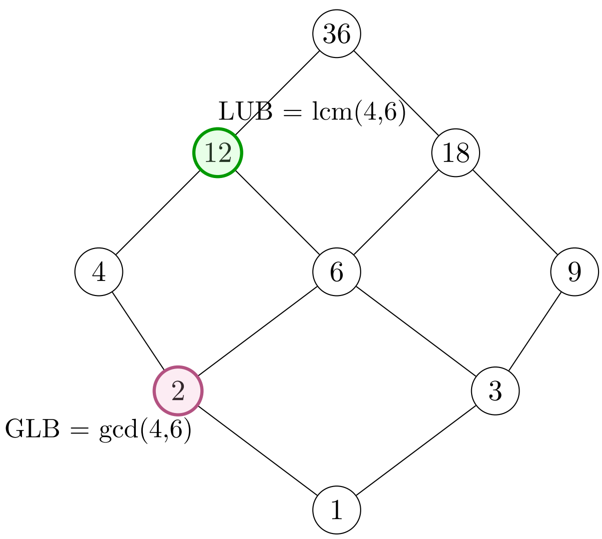
$$X \wedge Y = X \cap Y = \{b\}, \quad X \vee Y = X \cup Y = \{a, b, c\}.$$



2.5 Worked Example: Divisors of 36

Let $D_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$ ordered by divisibility.

$$4 \wedge 6 = \gcd(4, 6) = 2, \qquad 4 \vee 6 = \operatorname{lcm}(4, 6) = 12.$$



2.6 Remarks

- In analysis and general posets, the GLB is often called the *infimum* and the LUB the *supremum*.
- The symbols $\wedge$ and $\vee$ are chosen by analogy with logical AND/OR and set intersection/union.
- A lattice may have infinitely many elements; the idea of meet and join still applies.

2.7 Summary

- Meet = greatest lower bound (GLB) = infimum.
- Join = least upper bound (LUB) = supremum.
- Examples: $\cap/\cup$, gcd/lcm, AND/OR, intersection/span.
- Existence and uniqueness of meet and join are what distinguish lattices from general posets.

Chapter 3

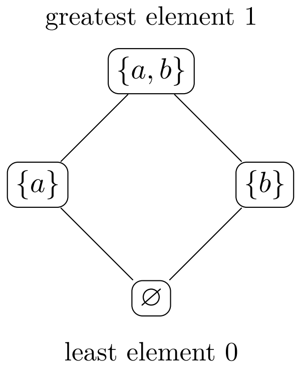
Types of Lattices

3.1 Bounded Lattices

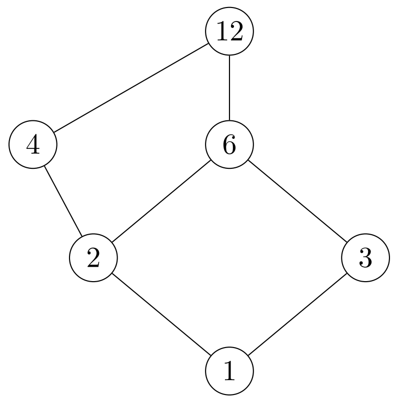
Definition 3.1. *A lattice* $(L, \wedge, \vee)$ *is called* bounded *if it contains both:*

- *a* least element 0 *such that* $0 \leq x$ *for all* $x \in L$*, and*
- *a* greatest element 1 *such that* $x \leq 1$ *for all* $x \in L$*.*

Example 1: Power set $\mathcal{P}(\{a, b\})$ The least element is $\varnothing$, the greatest is $\{a, b\}$.



Example 2: Divisors of 12 under divisibility $D_{12} = \{1, 2, 3, 4, 6, 12\}$ with $0 = 1$ (least) and $1 = 12$ (greatest).



3.2 Complemented Lattices

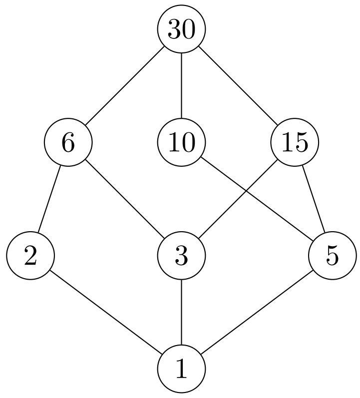
Definition 3.2. *A bounded lattice is* complemented *if for every element a there exists b such that:*

$$a \wedge b = 0 \quad \textit{and} \quad a \vee b = 1.$$

b is called a complement *of a.*

Example 1: Power set $\mathcal{P}(\{a, b, c\})$ Complement of A is $S \setminus A$.

Example 2: Divisors of 30 Under divisibility, d and $30/d$ are complements.



Non-example: Divisors of 12 Not complemented; e.g. 2 has no complement.

3.3 Distributive Lattices

Definition 3.3. *A lattice is* distributive *if for all a, b, c:*

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c),$$

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

Example: Power set $\mathcal{P}(\{a, b\})$ Union and intersection distribute.

Non-example: M_3 (diamond lattice) Fails distributivity; worked out in Chapter 4.

3.4 Modular Lattices

Definition 3.4. *A lattice is* modular *if whenever $a \leq c$, we have:*

$$a \vee (b \wedge c) = (a \vee b) \wedge c.$$

Example: Subspaces of a vector space Always modular, but not always distributive.

Example: M_3 Modular but not distributive.

3.5 Complete Lattices

Definition 3.5. *A lattice is* complete *if every subset $S \subseteq L$ has an infimum $\bigwedge S$ and supremum $\bigvee S$.*

Example: Power set $\mathcal{P}(S)$ Infimum = intersection, supremum = union of arbitrary collections.

Example: Real numbers $(\mathbb{R}, \leq)$ Inf = min, Sup = max for any subset bounded above/below.

3.6 Boolean Lattices

Definition 3.6. *A lattice is* Boolean *if it is bounded, distributive, and complemented. Boolean lattices are isomorphic to power sets of finite sets.*

Example 1: $\mathcal{P}(\{a, b, c\})$

Example 2: Divisors of 30 under divisibility Boolean lattices are the algebraic backbone of Boolean algebra and digital circuit theory.

3.7 Summary

- **Bounded**: has least 0 and greatest 1.
- **Complemented**: every element has complement.
- **Distributive**: meet/join distribute over each other.
- **Modular**: weaker than distributive.
- **Complete**: arbitrary sup/inf exist.
- **Boolean**: bounded + complemented + distributive.

Chapter 4

Special Lattices M_3 and N_5

4.1 Introduction

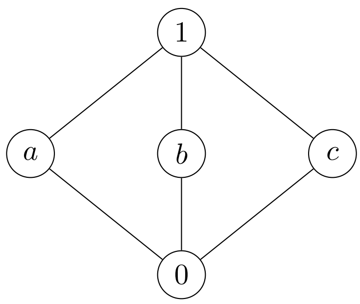
In lattice theory, two small but significant lattices frequently arise: the **diamond lattice** M_3 and the **pentagon lattice** N_5. They are important because:

- Both serve as *minimal counterexamples* to distributivity.
- M_3 is modular but not distributive.
- N_5 is neither distributive nor modular.

4.2 M_3 (Diamond Lattice)

Definition 4.1. *M_3 is the lattice with 5 elements: $\{0, 1, a, b, c\}$, where 0 is the least element, 1 is the greatest element, and a, b, c are incomparable elements between them.*

Hasse Diagram of M_3

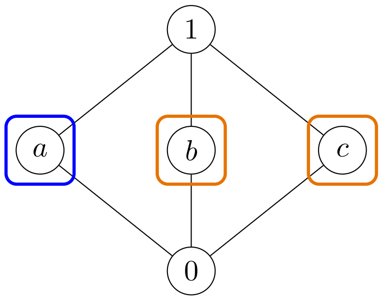


Failure of Distributivity in M_3

Choose a, b, c as the middle elements.

$$a \wedge (b \vee c) = a \wedge 1 = a, \qquad (a \wedge b) \vee (a \wedge c) = 0 \vee 0 = 0.$$

Since $a \neq 0$, distributivity fails.



Modularity of M_3

Even though distributivity fails, M_3 satisfies the modular law:

$$a \vee (b \wedge 1) = a \vee b = (a \vee b) \wedge 1.$$

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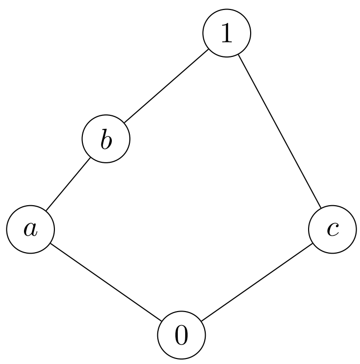
4.3 N_5 (Pentagon Lattice)

Definition 4.2. *N_5 is the lattice with 5 elements: $\{0, 1, a, b, c\}$ arranged in a pentagon shape, with order relations:*

$$0 < a < b < 1, \quad 0 < c < 1, \quad \text{and } c \text{ incomparable with } a, b.$$

Hasse Diagram of N_5

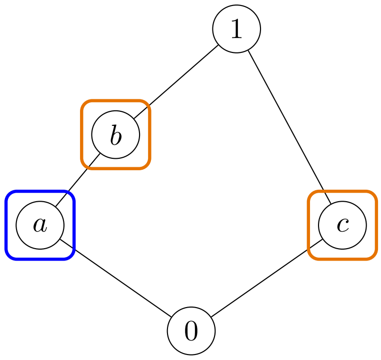


Failure of Modularity in N_5

Take $a \leq b$, and choose c:

$$a \vee (b \wedge c) = a \vee 0 = a, \qquad (a \vee b) \wedge (a \vee c) = b \wedge 1 = b.$$

Since $a \neq b$, the modular identity fails. Therefore, N_5 is not modular and hence not distributive.



4.4 Summary of M_3 and N_5

- M_3: Modular but not distributive.
- N_5: Neither modular nor distributive.
- These two lattices are the canonical "forbidden sublattices" that characterize distributivity.

Chapter 5

Relations Among Lattice Types

5.1 Hierarchy of Lattice Classes

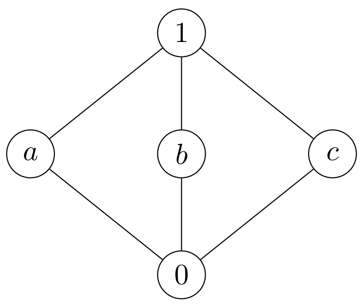
Lattices can be classified into increasingly restrictive subclasses. The relationships are summarized in the following implication chain:

$$\text{Boolean} \Rightarrow \text{Distributive} \Rightarrow \text{Modular} \Rightarrow \text{Lattice}.$$

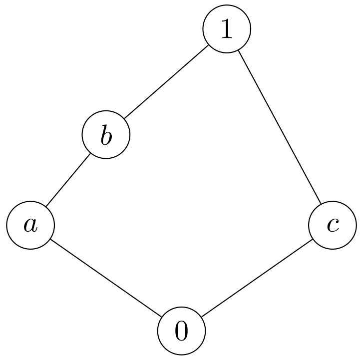
- Every **Boolean lattice** is distributive, modular, and of course a lattice.
- Every **Distributive lattice** is modular, but not necessarily Boolean.
- Every **Modular lattice** is a lattice, but not necessarily distributive.
- Every **Lattice** is a poset with meets and joins, but may fail modularity or distributivity.

5.2 Non-reversible Implications

- **Distributive $\not\Rightarrow$ Boolean.**
  Example: A distributive lattice without complements (e.g., divisors of 12 under divisibility).
- **Modular $\not\Rightarrow$ Distributive.**
  Example: M_3 is modular but not distributive.



- **Lattice $\not\Rightarrow$ Modular.**
  Example: N_5 is a lattice but not modular.



5.3 Applications of the Hierarchy

- In **Boolean algebra**, all laws of logic and circuit simplification are valid because Boolean lattices are distributive and complemented.
- In **vector spaces**, the lattice of subspaces is modular but not always distributive, explaining why distributive simplifications fail for spans and intersections.
- In **number theory**, the divisors of n form lattices under divisibility. If n is square-free, the lattice is Boolean; otherwise, it is merely distributive or modular.

5.4 Summary Table

Class	Property	Counterexample if reversed
Boolean $\Rightarrow$ Distributive	Every Boolean lattice is distributive	Divisors of 12 (distributive but not Boolean)
Distributive $\Rightarrow$ Modular	Every distributive lattice is modular	M_3 (modular but not distributive)
Modular $\Rightarrow$ Lattice	Every modular lattice is a lattice	N_5 (lattice but not modular)

Glossary of Lattice Theory

Poset (Partially Ordered Set) A set P with a binary relation $\leq$ that is reflexive, antisymmetric, and transitive. **Example:** Natural numbers under $\leq$.

Hasse Diagram A simplified diagram of a poset, drawn so that if $x < y$, y is placed above x. Only cover relations are drawn, transitive edges are omitted. **Example:** Power set $\mathcal{P}(\{a, b\})$.

Meet ($\wedge$) The **greatest lower bound (GLB)** of two elements a, b. The largest element $\leq a, b$. Also called *infimum.* **Examples:** $A \wedge B = A \cap B$ (sets), $\gcd(a, b)$ (numbers).

Join ($\vee$) The **least upper bound (LUB)** of two elements a, b. The smallest element $\geq a, b$. Also called *supremum.* **Examples:** $A \vee B = A \cup B$ (sets), $\operatorname{lcm}(a, b)$ (numbers).

Least Element (0) An element 0 with $0 \leq x$ for all x. **Example:** $\varnothing$ in $\mathcal{P}(S)$, 1 in divisors of n.

Greatest Element (1) An element 1 with $x \leq 1$ for all x. **Example:** S in $\mathcal{P}(S)$, n in divisors of n.

Bounded Lattice A lattice that has both least and greatest elements. **Example:** $\mathcal{P}(S)$, divisors of 12.

Complement In a bounded lattice, b is a complement of a if $a \wedge b = 0$ and $a \vee b = 1$. **Example:** In $\mathcal{P}(\{a, b\})$, complement of $\{a\}$ is $\{b\}$.

Complemented Lattice A bounded lattice where every element has a complement. **Example:** $\mathcal{P}(S)$.

Distributive Lattice A lattice where meet distributes over join and vice versa. **Example:** Power sets. **Non-example:** M_3, N_5.

Modular Lattice A lattice where $a \leq c$ implies $a \vee (b \wedge c) = (a \vee b) \wedge c$. **Example:** Subspaces of a vector space. **Non-example:** N_5.

Complete Lattice A lattice where every subset S has both $\bigwedge S$ and $\bigvee S$. **Example:** $\mathcal{P}(S)$.

Boolean Lattice A lattice that is bounded, distributive, and complemented. **Example:** $\mathcal{P}(\{a, b, c\})$, divisors of 30.

M_3 **(Diamond Lattice)** 5-element lattice: modular but not distributive.

N_5 **(Pentagon Lattice)** 5-element lattice: neither modular nor distributive.

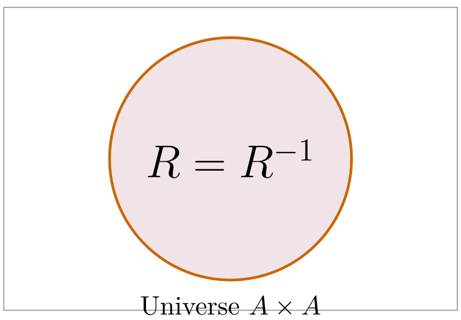
Infimum / Supremum Other names for GLB (infimum) and LUB (supremum).

Isomorphism of Lattices Two lattices are isomorphic if there exists a bijection between them that preserves meet and join.

Visual Guide: Symmetric, Asymmetric, and Antisymmetric Relations

Symmetric Relation ($R = R^{-1}$)

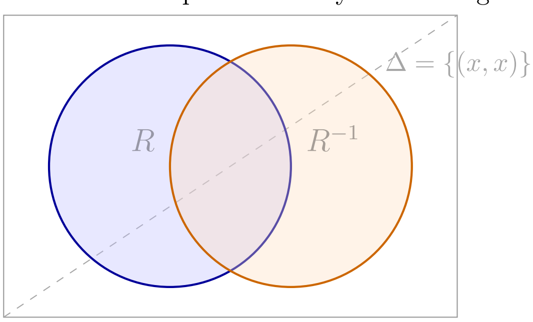
Symmetric: If $(a, b) \in R \Rightarrow (b, a) \in R$.



Real–world: friendship / sibling / equality.

Anti-symmetric Relation ($R \cap R^{-1} \subseteq \Delta$)

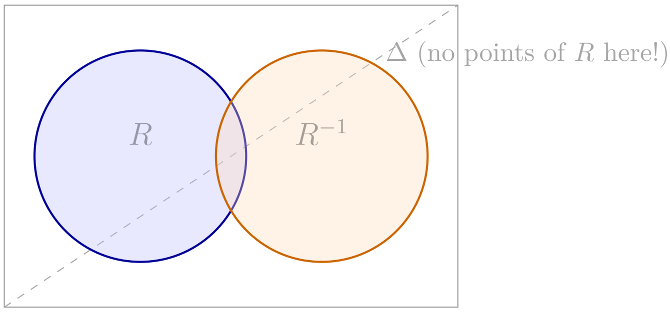
Anti-symmetric: overlap allowed only on the diagonal Δ.



Examples: $\subseteq$, $\leq$, divides. If (a, b) and (b, a) both in R, then $a = b$.

Asymmetric Relation ($R \cap R^{-1} = \varnothing$ and irreflexive)

Asymmetric: $R \cap R^{-1} = \varnothing$ and also R is irreflexive.



Examples: parent-of, $<$. If $(a, b) \in R$ then $(b, a) \notin R$, and $(a, a) \notin R$.