ABES

Estd. 2000

DSTL (BCS-303) — Unit 2

Previous Year Questions with Step-by-Step Solutions

ABES Engineering College

Prepared for Practice and Revision

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Contents

1 2018–19 (III Semester Theory Exam) Unit 2

2018–19 Q1: Types of Functions

Define injective, surjective and bijective functions with examples.

Solution

Injective (One-to-One): A function $f : A \to B$ is injective if different inputs always give different outputs. Formally, if $f(a_1) = f(a_2)$ then $a_1 = a_2$.
Example: $f : \mathbb{R} \to \mathbb{R},\ f(x) = 2x+3$. - Suppose $f(x_1) = f(x_2) \Rightarrow 2x_1+3 = 2x_2+3 \Rightarrow x_1 = x_2$. - Hence, injective.
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Surjective (Onto): A function $f : A \to B$ is surjective if every element of the codomain B is mapped by at least one input from A.
Example: $g : \mathbb{Z} \to \{0, 1\},\ g(n) = n \bmod 2$. - Codomain $\{0, 1\}$ is covered: even $n \mapsto 0$, odd $n \mapsto 1$. - Hence, surjective. - Not injective since $g(2) = g(4) = 0$ but $2 \neq 4$.
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Bijective (One-to-One and Onto): A function is bijective if it is both injective and surjective. Thus, there is a perfect pairing between domain and codomain, and an inverse exists.
Example: $h : \mathbb{R} \to \mathbb{R},\ h(x) = x + 5$. - Injective: $h(x_1) = h(x_2) \Rightarrow x_1 + 5 = x_2 + 5 \Rightarrow x_1 = x_2$. - Surjective: For any $y \in \mathbb{R}$, choose $x = y - 5$, then $h(x) = y$. - Hence, bijective with inverse $h^{-1}(y) = y - 5$.
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Summary: - Injective: "Different inputs → different outputs." - Surjective: "Every output is hit by at least one input." - Bijective: "Perfect one-to-one matching (like exam hall seats assigned to students)."

2018–19 Q2: Boolean Function Expansions

Find the sum-of-products (SOP) and product-of-sums (POS) expansions of the Boolean function

$$F(x, y, z) = (x + y)z.$$

Solution

Step 1: Rewrite the function.

$$F(x, y, z) = (x + y)z = xz + yz$$

(using distributive law of Boolean algebra).
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Step 2: SOP Expansion. - SOP means the function must be written as a sum (OR) of product (AND) terms corresponding to minterms. - Expand $xz+yz$ into canonical SOP (list all minterms where $F = 1$).

$$F(x, y, z) = \Sigma m(3, 5, 6, 7).$$

Explanation: - $m_3 = x'yz$, - $m_5 = xy'z$, - $m_6 = xyz'$, - $m_7 = xyz$. All these satisfy $(x + y)z = 1$.
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Step 3: POS Expansion. - POS means the function is written as a product (AND) of sum (OR) terms corresponding to maxterms. - The maxterms are those inputs where $F = 0$. - Here $F = 0$ for minterms m_0, m_1, m_2, m_4.

$$F(x, y, z) = \Pi M(0, 1, 2, 4).$$

Explicitly,

$$F(x, y, z) = (x + y + z)(x + y + z')(x + y' + z)(x' + y + z).$$

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Final Answer:

$$\text{SOP: } F(x, y, z) = \Sigma m(3, 5, 6, 7).$$

$$\text{POS: } F(x, y, z) = \Pi M(0, 1, 2, 4).$$

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Student Tip: - SOP = where the function is 1. - POS = where the function is 0. - Always check using truth table if unsure.

2 2019–20 (III Semester Theory Exam) Unit 2

2019–20 Q1: Various Types of Functions

Define various types of functions with suitable examples.

Solution

Functions can be classified in many ways. The important types are:

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1. Identity Function

$$f(x) = x \quad \text{for all } x \in A.$$

Example: $f : \mathbb{R} \to \mathbb{R}, f(x) = x$. - Each element maps to itself. - Like "student roll number → the same roll number."

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2. Constant Function

$$f(x) = c \quad \text{for all } x \in A, \text{ with fixed } c.$$

Example: $f : \mathbb{R} \to \mathbb{R}, f(x) = 7$. - Every input maps to the same output. - Like "every student assigned the same grade C."

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3. Projection Function On a product set $A \times B$, projection picks one component. Example: $p_1(a, b) = a$, $p_2(a, b) = b$. - Used in databases: from a student record (Name, Roll, Marks), "projecting Name" gives only names.

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4. Inverse Function A function $f : A \to B$ is invertible if bijective. Its inverse $f^{-1} : B \to A$ reverses the mapping. Example: $f(x) = x + 5 \;\Rightarrow\; f^{-1}(y) = y - 5$. - "Encoding–decoding" is a real-life analogy.

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5. Injective, Surjective, Bijective - Already defined in 2018–19 Q1. - Quick recall: - Injective: unique outputs, no collisions. - Surjective: codomain fully covered. - Bijective: perfect one-to-one mapping.

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6. Even and Odd Functions - Even: $f(-x) = f(x)$. Example: $f(x) = x^2$. - Odd: $f(-x) = -f(x)$. Example: $f(x) = x^3$. These classifications matter in symmetry and simplification.

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Student Summary: - Identity → "do nothing." - Constant → "same output always." - Projection → "select one part." - Inverse → "reverse the mapping." - Injective/surjective/bijective → "ways of pairing domain–codomain." - Even/odd → "symmetry properties."

2019–20 Q2: Lattice but not Boolean Algebra

Give an example of a lattice with 5 elements which is not a Boolean algebra.

Solution

Step 1. Recall definitions. - A *lattice* is a poset where every pair has a join ($\vee$) and meet ($\wedge$). - A *Boolean algebra* is a distributive, complemented lattice with least (0) and greatest (1). So, to find an example: pick a lattice that fails distributivity or complement property.

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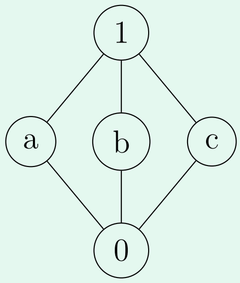
Step 2. Example: M_3 (diamond lattice).
Elements: $\{0, a, b, c, 1\}$, with order: - $0 < a, b, c < 1$; - a, b, c are incomparable.



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Step 3. Check properties. - It is a lattice: any two elements have meet and join. - It is not Boolean algebra because: - Not distributive: $a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$. - Complements fail: a has no unique complement.

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Conclusion. The lattice M_3 with 5 elements is a valid example of a lattice that is not a Boolean algebra.

2019–20 Q3: Equivalence Relation Check

Check whether the following relations are equivalence relations: (i) $R_1 = \{(a,a),(b,b),(c,c),(a,b),(b,a)\}$ (ii) $R_2 = \{(a,a),(b,b),(c,c),(a,b),(b,a),(a,c),(c,a),(b,c),(c,b)\}$

Solution

Recall. A relation R on a set A is an *equivalence relation* if it is: 1. Reflexive: $(x,x) \in R$ for all $x \in A$. 2. Symmetric: $(x,y) \in R \implies (y,x) \in R$. 3. Transitive: $(x,y),(y,z) \in R \implies (x,z) \in R$.

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(i) For R_1. - Reflexive: $(a,a),(b,b),(c,c)$ are all present $\Rightarrow$ Yes. - Symmetric: (a,b) and (b,a) included $\Rightarrow$ Yes. - Transitive: (a,b) and $(b,a) \Rightarrow (a,a)$ (already in R_1). But no pair links c with a or b. Transitivity still holds $\Rightarrow$ Yes.
So R_1 is an equivalence relation. Equivalence classes: $\{a,b\},\{c\}$.

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(ii) For R_2. - Reflexive: $(a,a),(b,b),(c,c)$ present $\Rightarrow$ Yes. - Symmetric: for each (x,y), (y,x) is present $\Rightarrow$ Yes. - Transitive: (a,b) and $(b,c) \Rightarrow (a,c)$, which is in R_2. All similar cases check out $\Rightarrow$ Yes.
So R_2 is also an equivalence relation. Equivalence class: $\{a,b,c\}$.

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Conclusion. - R_1 is an equivalence relation with two classes $\{a,b\},\{c\}$. - R_2 is an equivalence relation with one class $\{a,b,c\}$.

3 2020–21 (III Semester Theory Exam) Unit 2

2020–21 Q1: Injectivity of $f(x) = x^2 - 1$

Check whether the function $f(x) = x^2 - 1$ for $f : \mathbb{R} \to \mathbb{R}$ is injective or not.

Solution

Step 1. Recall definition. A function $f : A \to B$ is *injective* (one-to-one) if

$$f(x_1) = f(x_2) \implies x_1 = x_2.$$

Step 2. Apply to $f(x) = x^2 - 1$. Suppose $f(x_1) = f(x_2)$:

$$x_1^2 - 1 = x_2^2 - 1 \quad \Rightarrow \quad x_1^2 = x_2^2.$$

This implies $x_1 = \pm x_2$.
Step 3. Counterexample. Take $x_1 = 2,\ x_2 = -2$:

$$f(2) = 2^2 - 1 = 3, \quad f(-2) = (-2)^2 - 1 = 3.$$

So $f(2) = f(-2)$ but $2 \neq -2$.
Conclusion. $f(x) = x^2 - 1$ is *not injective* on $\mathbb{R} \to \mathbb{R}$.
Student note: Quadratic functions fail injectivity over all reals because $+x$ and $-x$ give the same output. If the domain were restricted to $[0, \infty)$, then f would be injective.

2020–21 Q2: Compositions with $f(x) = 3x^2 + 2$, $g(x) = 7x - 5$, $h(x) = \dfrac{1}{x}$

Compute the following:

$$\text{(i) } (f \circ g \circ h)(x), \qquad \text{(ii) } (g \circ g)(x), \qquad \text{(iii) } (g \circ h)(x), \qquad \text{(iv) } (h \circ g \circ f)(x).$$

Solution

Given: $f(x) = 3x^2 + 2, \quad g(x) = 7x - 5, \quad h(x) = \dfrac{1}{x}.$

(i) $(f \circ g \circ h)(x) = f\big(g(h(x))\big).$

$$\begin{aligned}
h(x) &= \frac{1}{x} \qquad \text{(domain } x \neq 0\text{)},\\
g(h(x)) &= g\left(\frac{1}{x}\right) = \frac{7}{x} - 5,\\
(f \circ g \circ h)(x) &= f\left(\frac{7}{x} - 5\right) = 3\left(\frac{7}{x} - 5\right)^2 + 2\\
&= 3\left(\frac{49}{x^2} - \frac{70}{x} + 25\right) + 2\\
&= \frac{147}{x^2} - \frac{210}{x} + 77, \qquad (x \neq 0).
\end{aligned}$$

(ii) $(g \circ g)(x) = g(g(x)).$

$$\begin{aligned}
g(x) &= 7x - 5,\\
(g \circ g)(x) &= g(7x - 5) = 7(7x - 5) - 5 = 49x - 40.
\end{aligned}$$

(iii) $(g \circ h)(x) = g(h(x))$.

$$h(x) = \frac{1}{x} \quad (x \neq 0),$$
$$(g \circ h)(x) = g\left(\frac{1}{x}\right) = \frac{7}{x} - 5, \qquad (x \neq 0).$$

(iv) $(h \circ g \circ f)(x) = h\big(g(f(x))\big)$.

$$f(x) = 3x^2 + 2,$$
$$g(f(x)) = 7(3x^2 + 2) - 5 = 21x^2 + 9,$$
$$(h \circ g \circ f)(x) = h(21x^2 + 9) = \frac{1}{21x^2 + 9}.$$

Domain note: $21x^2 + 9 > 0$ for all real x, so (iv) is defined for every $x \in \mathbb{R}$; (i) and (iii) require $x \neq 0$ because h appears.

2020–21 Q3: Simplify Boolean Expressions

Simplify the following using Boolean algebra laws:

(i) $X = A\bar{B}+AB+\bar{A}B$, (ii) $Y = (A+B)(A+\bar{B})$, (iii) $Z = (A+B+C)(A+\bar{B}+C)(A+B+\bar{C})$.

Solution

Useful laws (symbols: + = OR, concatenation = AND, bar = NOT):
Idempotent $X + X = X$, $XX = X$; Complement $X + \bar{X} = 1$, $X\bar{X} = 0$;
Absorption $X + XY = X$, $X(X + Y) = X$; Distributive $X(Y + Z) = XY + XZ$;
Consensus $(X + Y)(X + \bar{Y}) = X$; $(X + Y)(X + Z) = X + YZ$.

(i) $X = A\bar{B} + AB + \bar{A}B$

$$\begin{aligned} A\bar{B} + AB &= A(\bar{B} + B) && \text{(factor } A\text{)} \\ &= A && \text{(complement law)} \\ X &= A + \bar{A}B \\ &= (A + \bar{A})(A + B) && \text{(distributive)} \\ &= 1 \cdot (A + B) = A + B. && \text{(identity)} \end{aligned}$$

Answer: $X = A + B$.

(ii) $Y = (A + B)(A + \bar{B})$

$$\begin{aligned} Y &= A + B\bar{B} && \text{(consensus: } (X + Y)(X + \bar{Y}) = X\text{)} \\ &= A + 0 = A. && \text{(complement, identity)} \end{aligned}$$

Answer: $Y = A$.

(iii) $Z = (A + B + C)(A + \bar{B} + C)(A + B + \bar{C})$

$$\begin{aligned} (A + B + C)(A + \bar{B} + C) &= A + (B + C)(\bar{B} + C) && ((X + Y)(X + Z) = X + YZ) \\ &= A + \big(C + B\bar{B}\big) && \text{(distribute, } C^2 = C\text{)} \\ &= A + C. \end{aligned}$$

Therefore

$$
\begin{aligned}
Z &= (A + C)(A + B + \bar{C}) \\
&= A + C(B + \bar{C}) && ((X + Y)(X + Z) = X + YZ) \\
&= A + (CB + C\bar{C}) \\
&= A + CB. && (C\bar{C} = 0)
\end{aligned}
$$

Answer: $Z = A + BC$.

4 2021–22 (III Semester Theory Exam) Unit 2

2021–22 Q1: Ackermann Function

Evaluate the Ackermann function value $A(2,1)$ given

$$A(0,n)=n+1,\quad A(m,0)=A(m-1,1)\ (m>0),\quad A(m,n)=A\big(m-1,\ A(m,n-1)\big)\ (m,n>0).$$

Solution

What is the Ackermann Function?

The Ackermann function is a famous example of a function that grows very fast and is *not primitive recursive*. It is defined using simple rules:

$$A(0,n)=n+1,\quad A(m,0)=A(m-1,1),\quad A(m,n)=A(m-1,A(m,n-1)).$$

- Think of it like a puzzle: to find $A(m,n)$, you either reduce m, or reduce n, until you hit the base case. - It is used to show the difference between **primitive recursive** and more general **computable** functions. - For small inputs like $A(2,1)$, we can compute step by step. But as m,n grow, values explode quickly!

Step 1. $A(2,1)=A\big(1,\ A(2,0)\big)$.
Step 2. $A(2,0)=A(1,1)$.
Step 3. $A(1,1)=A(0,\ A(1,0))$.
Step 4. $A(1,0)=A(0,1)=2\ \Rightarrow\ A(1,1)=A(0,2)=3$.
Step 5. So $A(2,0)=3$ and $A(2,1)=A(1,3)$.
Step 6. $A(1,3)=A(0,\ A(1,2))$ with $A(1,2)=A(0,3)=4$.
Step 7. Hence $A(1,3)=A(0,4)=5$.
Answer: $\boxed{A(2,1)=5}$.

2021–22 Q2: De Morgan's Law and Absorption Law

State De Morgan's laws and the Absorption laws (for sets / Boolean algebra), and justify them.

Solution

De Morgan's Laws

$$\text{Sets:}\qquad (A\cup B)^c=A^c\cap B^c,\qquad (A\cap B)^c=A^c\cup B^c.$$

$$\text{Boolean:}\qquad (x+y)'=x'y',\qquad (xy)'=x'+y'.$$

Element method proof for sets:

1. $x\in(A\cup B)^c\ \Leftrightarrow\ x\notin A\cup B\ \Leftrightarrow\ (x\notin A\ \wedge\ x\notin B)$
   $\Leftrightarrow\ (x\in A^c\ \wedge\ x\in B^c)\ \Leftrightarrow\ x\in A^c\cap B^c$.
2. $x\in(A\cap B)^c\ \Leftrightarrow\ x\notin A\cap B\ \Leftrightarrow\ (x\notin A\ \vee\ x\notin B)$
   $\Leftrightarrow\ (x\in A^c\ \vee\ x\in B^c)\ \Leftrightarrow\ x\in A^c\cup B^c$.

Absorption Laws

$$\text{Sets:}\qquad A\cup(A\cap B)=A,\qquad A\cap(A\cup B)=A.$$

$$\text{Boolean:}\qquad x+xy=x,\qquad x(x+y)=x.$$

Set proof by double inclusion:

1. $A \cup (A \cap B) \subseteq A$: if $x \in A$ done; if $x \in A \cap B$ then $x \in A$. Also $A \subseteq A \cup (A \cap B)$ trivially. Hence $A \cup (A \cap B) = A$.
2. $A \cap (A \cup B) \subseteq A$: if $x \in A \cap (A \cup B)$ then $x \in A$. Also $A \subseteq A \cap (A \cup B)$ since $A \subseteq A \cup B$. Hence equality.

Boolean algebra one-liners (using 1 as identity):

$$x + xy = x(1 + y) = x \cdot 1 = x, \qquad x(x + y) = x.$$

Thus, De Morgan's and Absorption laws hold for both sets and Boolean expressions.

2021–22 Q3: Justify Set–Difference Identities

Prove the identities for any sets A, B, C in a common universe U:

1. $A - (A \cap B) = A - B$.
2. $A - (B \cap C) = (A - B) \ \cup \ (A - C)$.

Solution

We use the **element method**: for each identity show both sets contain exactly the same elements.
(i) $A - (A \cap B) = A - B$.

1. Take arbitrary x.
2. $x \in A - (A \cap B) \ \Leftrightarrow \ (x \in A) \ \wedge \ \neg(x \in A \cap B) \ \Leftrightarrow \ (x \in A) \ \wedge \ \neg(x \in A \wedge x \in B) \ \Leftrightarrow \ (x \in A) \ \wedge \ (x \notin B) \ \Leftrightarrow \ x \in A - B$.
3. Hence the two sets are equal.

(ii) $A - (B \cap C) = (A - B) \cup (A - C)$.

1. Take arbitrary x.
2. $x \in A - (B \cap C) \ \Leftrightarrow \ (x \in A) \ \wedge \ \neg(x \in B \cap C) \ \Leftrightarrow \ (x \in A) \ \wedge \ \neg(x \in B \wedge x \in C) \ \Leftrightarrow \ (x \in A) \ \wedge \ \big((x \notin B) \ \vee \ (x \notin C)\big)$ (De Morgan)
3. Distribute $x \in A$ over $\vee$:
$$\big((x \in A) \wedge (x \notin B)\big) \ \vee \ \big((x \in A) \wedge (x \notin C)\big).$$
4. This is exactly
$$x \in (A - B) \ \vee \ x \in (A - C) \ \Leftrightarrow \ x \in (A - B) \cup (A - C).$$
5. Hence equality holds.

2021–22 Q4: Simplify Boolean functions using K-map

(i) $F(A, B, C, D) = \Sigma(m_0, m_1, m_2, m_4, m_5, m_6, m_8, m_9, m_{12}, m_{13}, m_{14})$.
(ii) $F(A, B, C, D) = \Sigma(0, 2, 5, 7, 8, 10, 13, 15)$.

Solution

K-map convention (Gray order). Columns for CD: $00, 01, 11, 10$; rows for AB: $00, 01, 11, 10$.

(i) $F = \Sigma(0, 1, 2, 4, 5, 6, 8, 9, 12, 13, 14)$

$AB\backslash CD$	00	01	11	10
00	**1**	**1**	0	**1**
01	**1**	**1**	0	**1**
11	**1**	**1**	0	**1**
10	**1**	**1**	0	0

Grouping (describe rectangles verbally):

- All cells with $C = 0$ (columns 00 and 01) form an 8-cell group $\Rightarrow$ term $\boxed{\overline{C}}$.
- Remaining 1's at m_2, m_6, m_{14} (column $CD = 10$ except m_{10}) are covered by two 2-cell groups:
  - Pair (m_2, m_6) (rows $AB = 00$ and 01) $\Rightarrow$ $\overline{A}\,C\,\overline{D}$.
  - Pair (m_6, m_{14}) (rows $AB = 01$ and 11) $\Rightarrow$ $B\,C\,\overline{D}$.

Minimal SOP:

$$\boxed{F = \overline{C} + \overline{A}\,C\,\overline{D} + B\,C\,\overline{D} = \overline{C} + C\,\overline{D}\,(\overline{A} + B)}$$

(ii) $F = \Sigma(0, 2, 5, 7, 8, 10, 13, 15)$

$AB\backslash CD$	00	01	11	10
00	**1**	0	0	**1**
01	0	**1**	**1**	0
11	0	**1**	**1**	0
10	**1**	0	0	**1**

Grouping:

- Quad across rows $AB = 00$ and 10 in columns $CD = 00$ and 10 (cells m_0, m_2, m_8, m_{10}) $\Rightarrow$ $\boxed{\overline{B}\,\overline{D}}$.
- Quad across rows $AB = 01$ and 11 in columns $CD = 01$ and 11 (cells m_5, m_7, m_{13}, m_{15}) $\Rightarrow$ $\boxed{B\,D}$.

Minimal SOP:

$$\boxed{F = \overline{B}\,\overline{D} + B\,D}$$

Quick check: The two terms are disjoint by construction (one needs $BD = 00$, the other $BD = 11$), so no further reduction is possible.

5 2022–23 (III Semester Theory Exam) Unit 2

2022–23 Q1: Does $\lceil x+y\rceil = \lceil x\rceil + \lceil y\rceil$ hold for all real x, y?

Identify whether the identity $\lceil x+y\rceil = \lceil x\rceil + \lceil y\rceil$ is true for all real numbers x, y.

Solution

Short answer: *No, not for all x, y.*
Counterexample.

$$x = \tfrac{1}{2}, \quad y = \tfrac{1}{2} \quad \Rightarrow \quad \lceil x+y\rceil = \lceil 1\rceil = 1, \qquad \lceil x\rceil + \lceil y\rceil = \lceil \tfrac{1}{2}\rceil + \lceil \tfrac{1}{2}\rceil = 1 + 1 = 2.$$

Hence $\lceil x+y\rceil \neq \lceil x\rceil + \lceil y\rceil$ in general.
What *is* always true. For all real x, y,

$$\boxed{\lceil x\rceil + \lceil y\rceil - 1 \ \leq\ \lceil x+y\rceil \ \leq\ \lceil x\rceil + \lceil y\rceil}.$$

Why the upper bound holds: Since $x \leq \lceil x\rceil$ and $y \leq \lceil y\rceil$, we have $x + y \leq \lceil x\rceil + \lceil y\rceil$, so taking ceilings on both sides gives $\lceil x+y\rceil \leq \lceil x\rceil + \lceil y\rceil$.
Why the lower bound holds: Write $x = \lceil x\rceil - \alpha$, $y = \lceil y\rceil - \beta$ with $0 < \alpha, \beta \leq 1$ (or $= 0$ if x or y is already an integer). Then

$$x + y = (\lceil x\rceil + \lceil y\rceil) - (\alpha + \beta).$$

If $\alpha + \beta \leq 1$, the ceiling is $\lceil x\rceil + \lceil y\rceil$; if $1 < \alpha + \beta \leq 2$, it is $(\lceil x\rceil + \lceil y\rceil) - 1$. Thus $\lceil x+y\rceil \geq \lceil x\rceil + \lceil y\rceil - 1$.
When equality happens.

- $\lceil x+y\rceil = \lceil x\rceil + \lceil y\rceil$ holds iff the fractional parts satisfy $\{-x\} + \{-y\} \leq 1$ (equivalently, at least one of x, y is an integer, or their "gaps to the next integers" don't add past 1).
- $\lceil x+y\rceil = \lceil x\rceil + \lceil y\rceil - 1$ occurs when those gaps sum to > 1 (like $x = y = \frac{1}{2}$).

2022–23 Q2: Define Big–O in terms of growth of functions

Define *Big–O* notation formally for functions $f, g : \mathbb{N} \to \mathbb{R}$ (or $\mathbb{R}_{\geq 0} \to \mathbb{R}$) in the context of growth of functions.

Solution

Formal definition. We say that

$$\boxed{f(n) = O\big(g(n)\big)}$$

if there exist positive constants c and n_0 such that

$$\forall n \geq n_0 : \qquad 0 \leq f(n) \leq c\, g(n).$$

In words: *beyond some threshold n_0, f never exceeds a constant multiple of g.*
Interpretation (growth). Big–O gives an *asymptotic upper bound* on the rate of growth: g grows at least as fast as f up to a constant factor.
Example. Let $f(n) = 3n^2 + 7n + 10$ and $g(n) = n^2$. Choose $c = 4$ and $n_0 = 9$. For all $n \geq 9$,

$$3n^2 + 7n + 10 \ \leq\ 3n^2 + 7n^2 + 10n^2 \ =\ 20n^2 \ \leq\ 4n^2 \cdot n^2 \quad \text{(crude)}.$$

A cleaner check is:

$$\frac{f(n)}{g(n)} = \frac{3n^2+7n+10}{n^2} = 3+\frac{7}{n}+\frac{10}{n^2} \le 3+\frac{7}{9}+\frac{10}{81} < 4 \quad (n \ge 9),$$

so $f(n) \le 4\,g(n)$ for $n \ge 9$. Hence $f(n) = O(n^2)$.

Equivalent limit test (when limit exists/finite). If $\limsup_{n\to\infty} \frac{f(n)}{g(n)} < \infty$, then $f(n) = O(g(n))$.

Common facts.

- If $f(n) = a_k n^k + \cdots + a_0$ with $a_k > 0$, then $f(n) = O(n^k)$.
- If $f(n) = O(g(n))$ and $g(n) = O(h(n))$, then $f(n) = O(h(n))$ (transitivity).
- Constants don't matter: $c \cdot f(n) = O(f(n))$ for any $c > 0$.

What Big–O is not. It is not a tight equality; it hides constant factors and lower-order terms. For a *tight* bound one uses $\Theta(\cdot)$.

2022–23 Q3: Composite Mapping

Find the composite mapping $g \circ f$ if $f(x) = e^x$ and $g(x) = \sin x$.

Solution

Step 1. Recall definition. For functions $f : X \to Y$ and $g : Y \to Z$, the composition $g \circ f$ is defined by

$$(g \circ f)(x) = g\big(f(x)\big).$$

Step 2. Apply to the given functions. We are given:

$$f(x) = e^x, \qquad g(x) = \sin x.$$

So

$$(g \circ f)(x) = g\big(f(x)\big) = g(e^x).$$

Step 3. Substitute into g. Since $g(x) = \sin x$, we replace x by e^x:

$$(g \circ f)(x) = \sin(e^x).$$

Final Answer.

$$\boxed{(g \circ f)(x) = \sin(e^x)}$$

Note for Students. - Composition means "apply f first, then g". - Here, input x passes through f to become e^x, then g takes sin of that result. - Always check the order: $g \circ f \ne f \circ g$ in general. For example, $f \circ g$ here would be $e^{\sin x}$, which is a different function.

2022–23 Q4: Bijectivity and Inverse

Show that $f(x) = \dfrac{x-1}{x-3}$, $f : \mathbb{R} - \{3\} \to \mathbb{R} - \{1\}$, is bijective and compute its inverse.

Solution

Step 1. Domain and Codomain. - Domain: $\mathbb{R} - \{3\}$ (since denominator $x - 3 \ne 0$). - Codomain: $\mathbb{R} - \{1\}$ (since $f(x)$ never equals 1).

Step 2. Injectivity. Suppose $f(x_1) = f(x_2)$:

$$\frac{x_1 - 1}{x_1 - 3} = \frac{x_2 - 1}{x_2 - 3}.$$

Cross-multiplying:

$$(x_1 - 1)(x_2 - 3) = (x_2 - 1)(x_1 - 3).$$

Simplify:

$$x_1x_2 - 3x_1 - x_2 + 3 = x_1x_2 - x_1 - 3x_2 + 3.$$

Cancel x_1x_2 and 3:

$$-3x_1 - x_2 = -x_1 - 3x_2.$$

$$-2x_1 = -2x_2 \quad \Longrightarrow \quad x_1 = x_2.$$

Hence f is **injective**.

Step 3. Surjectivity. Let $y \in \mathbb{R} - \{1\}$. Solve $y = \frac{x-1}{x-3}$ for x:

$$y(x - 3) = x - 1 \quad \Longrightarrow \quad yx - 3y = x - 1.$$

$$yx - x = 3y - 1 \quad \Longrightarrow \quad x(y - 1) = 3y - 1.$$

$$x = \frac{3y - 1}{y - 1}.$$

Since $y \neq 1$, this gives a valid preimage. Thus every $y \in \mathbb{R} - \{1\}$ has an $x \in \mathbb{R} - \{3\}$. So f is **surjective**.

Step 4. Bijectivity. Since f is both injective and surjective, it is **bijective**.

Step 5. Inverse function. From Step 3:

$$f^{-1}(y) = \frac{3y - 1}{y - 1}.$$

Final Answer.

$$f : \mathbb{R} - \{3\} \to \mathbb{R} - \{1\}, \quad f(x) = \frac{x - 1}{x - 3}$$

is bijective, and

$$\boxed{f^{-1}(y) = \frac{3y - 1}{y - 1}, \quad y \neq 1.}$$

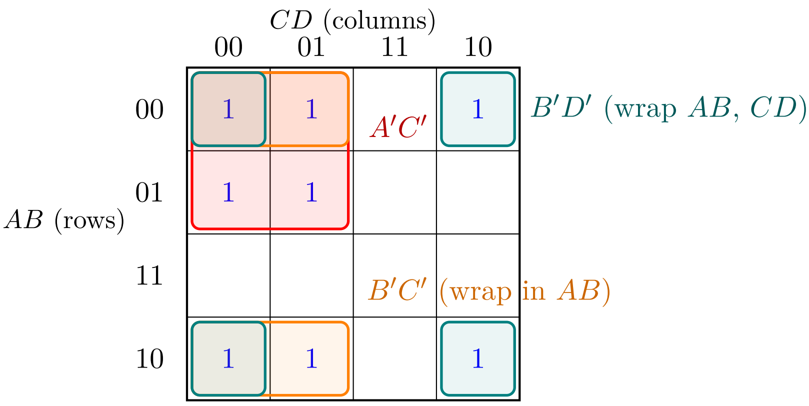
Note for Students. - *Injective*: No two x map to the same y. - *Surjective*: Every valid y has some x. - Inverse is found by solving $y = f(x)$ for x.

2022–23 Q5: K-map Minimization

Solve using a 4-variable K-map:

(i) $F(A, B, C, D) = \Sigma(m_0, m_1, m_2, m_4, m_5, m_8, m_9, m_{10})$

(ii) $F(A, B, C, D) = \Sigma m(0, 2, 5, 7, 8, 10, 13, 15)$



$$F = A'C' + B'C' + B'D'$$

Part (i) Solution

Step 1: Plot the 1's on a 4-variable K-map (Gray code).

$AB \backslash CD$	00	01	11	10
00	1	1	0	1
01	1	1	0	0
11	0	0	0	0
10	1	1	0	1

(Rows: $AB = 00, 01, 11, 10$; Columns: $CD = 00, 01, 11, 10$.)
The blue 1's correspond to the listed minterms:

$$\begin{aligned}
&00|00 \to m_0,\ 00|01 \to m_1,\ 00|10 \to m_2,\\
&01|00 \to m_4,\ 01|01 \to m_5,\\
&10|00 \to m_8,\ 10|01 \to m_9,\ 10|10 \to m_{10}.
\end{aligned}$$

Step 2: Make largest possible groups (powers of 2), allowing wrap-around.
We can cover all 1's efficiently with three prime implicants:

1. $\boxed{A'C'}$: group the column $CD = 00, 01$ in rows $AB = 00, 01$ (that is m_0, m_1, m_4, m_5).
   Fixed bits: $A = 0 \Rightarrow A'$, $C = 0 \Rightarrow C'$.

2. $\boxed{B'C'}$: group the column $CD = 00, 01$ in rows $AB = 00, 10$ (that is m_0, m_1, m_8, m_9).
   Fixed bits: $B = 0 \Rightarrow B'$, $C = 0 \Rightarrow C'$.

3. $\boxed{B'D'}$: group the column $CD = 00, 10$ in rows $AB = 00, 10$ (that is m_0, m_2, m_8, m_{10}).
   Fixed bits: $B = 0 \Rightarrow B'$, $D = 0 \Rightarrow D'$.

(Overlaps are allowed and often necessary to minimize the number of terms.)
Step 3: Write the minimized SOP.

$$F = A'C' + B'C' + B'D'.$$

Why this is minimal (intuition).

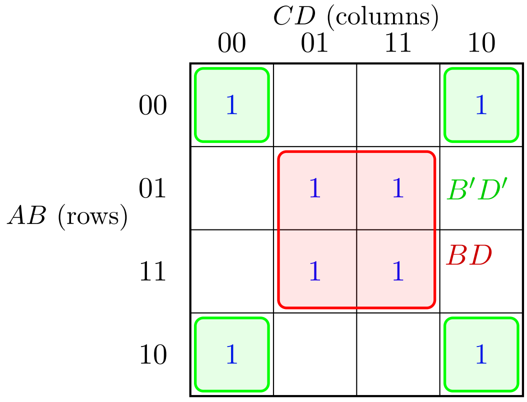
- The "$C = 0$" 1's split across three different AB rows. Two 4-cell groups $A'C'$ and $B'C'$ cover all of them, but $m_9(1001)$ forces the term $B'C'$.
- The two "$C = 1$" 1's (at m_2, m_{10}) both have $B = 0$ and $D = 0$, giving the third 4-cell group $B'D'$.

- No single term can replace any of these three without leaving an uncovered minterm, so three implicants is optimal here.

(Optional) A factored form.

$$F \;=\; C'(A' + B') \;+\; B'D'.$$

Both forms are equivalent; the SOP $A'C' + B'C' + B'D'$ is a clean, minimal answer.



$$F = B'D' \;+\; BD \;=\; \overline{B \oplus D}$$

Part (ii) Solution

Step 1: Plot the 1's on the 4×4 K-map.

$AB \backslash CD$	00	01	11	10
00	1	0	0	1
01	0	1	1	0
11	0	1	1	0
10	1	0	0	1

Positions:

$$m_0(0000),\; m_2(0010),\; m_5(0101),\; m_7(0111),$$
$$m_8(1000),\; m_{10}(1010),\; m_{13}(1101),\; m_{15}(1111).$$

Step 2: Make largest possible groups.

1. $\boxed{B'D'}$: a 4-cell group $\{m_0, m_2, m_8, m_{10}\}$ (rows $AB = 00, 10$, columns $CD = 00, 10$). Here $B = 0, D = 0$.
2. $\boxed{BD}$: a 4-cell group $\{m_5, m_7, m_{13}, m_{15}\}$ (rows $AB = 01, 11$, columns $CD = 01, 11$). Here $B = 1, D = 1$.

Step 3: Write minimized SOP.

$$F \;=\; B'D' + BD.$$

Step 4: Interpret the result.

- The function only depends on B and D.
- It outputs 1 when B and D are *equal* (both 0 or both 1).
- This is exactly the **XNOR** function:

$$F = \overline{B \oplus D}.$$

Equivalent POS (optional):

$$F = (B + \overline{D})(\overline{B} + D).$$

For students: This K-map shows how symmetry works: instead of a complicated 8-term SOP, the minimized form reveals the simple idea—"B and D must match." That's why we end up with just $B'D' + BD$.

2022–23 Q6: Boolean Algebra Proof

Define Boolean algebra. Show that

$$a' \cdot \Big[(b' + c)' + b \cdot c\Big] + \Big[(a + b')' \cdot c\Big] = a' \cdot b$$

using rules of Boolean Algebra. Here a' is the complement of element a.

Solution

Step 1: Definition of Boolean algebra. A Boolean algebra is an algebraic structure $(B, +, \cdot, ', 0, 1)$ with operations:

- $+$ (OR), $\cdot$ (AND), $'$ (NOT or complement),
- constants 0 (false), 1 (true),
- satisfying properties: commutativity, distributivity, identity, complement, absorption, De Morgan's laws.

Step 2: Simplify the LHS.

$$a' \cdot \Big[(b' + c)' + b \cdot c\Big] + \Big[(a + b')' \cdot c\Big]$$

Part A: Simplify $(b' + c)' + b \cdot c$

$$(b' + c)' = (b')' \cdot c' = b \cdot c'$$

So,

$$(b' + c)' + b \cdot c = (b \cdot c') + (b \cdot c) = b \cdot (c' + c) = b \cdot 1 = b$$

Thus,

$$a' \cdot \Big[(b' + c)' + b \cdot c\Big] = a' \cdot b$$

Part B: Simplify $[(a + b')' \cdot c]$

$$(a + b')' = a' \cdot (b')' = a' \cdot b$$

So,

$$[(a + b')' \cdot c] = (a' \cdot b) \cdot c$$

Step 3: Substitute back.

$$\text{LHS} = (a' \cdot b) + (a' \cdot b \cdot c)$$

Factorize:

$$= a' \cdot b \cdot (1 + c) = a' \cdot b \cdot 1 = a' \cdot b$$

Step 4: RHS.

$$\text{RHS} = a' \cdot b$$

Final Result:

$$a' \cdot \Big[(b' + c)' + b \cdot c\Big] + \Big[(a + b')' \cdot c\Big] = a' \cdot b$$

Hence proved ✓