Report

on

"Commuting graph associated with algebraic structure"

Special Project (MATH F491)

Submitted by-

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COMMUTING GRAPH OF GROUP STRUCTURE V_{8n}

STATEMENT OF PURPOSE

Purpose of this project was to explore more about the algebraic group structure V_{8n} and similar structures. We started with searching out for the previous work done in the group by scholars. With very little work done in the field, we began finding the elements of V_8 (for n=1) and moved on to discover the general form of its elements. Later we started working on finding the centre of the group with the help of some predetermined relations between elements of V_{8n}. After multiple incorrect and unsatisfactory results, we came out with results which would differ for odd and even. Center elements help us in getting a broad idea of the commuting graph of V_{8n}. We then worked on finding the neighbours of the elements of V_{8n} (general form) for both n odd and n even with the help of the centre we obtained before. Thereafter its commuting graph. We aimed to discover various properties with the help of neighbours and were quite successful in doing so for most property.

TERMINOLOGIES:

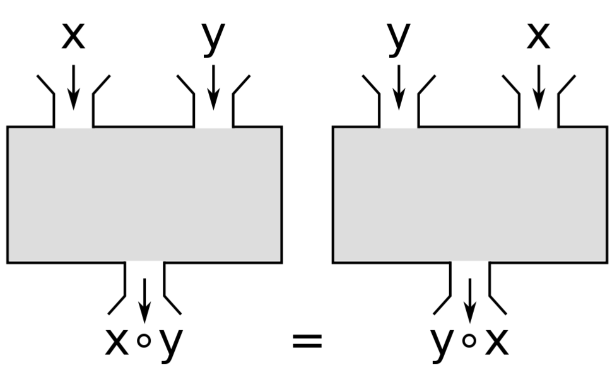
In this section, we will define the basic terminologies and results which will be useful in the sequel.

Graph: A graph is a structure amounting to a set of objects in which some pairs of the objects are in some sense "related". The objects correspond to a mathematical abstraction called vertices (also nodes or points) and each of the related pairs of vertices is called an edge and the relation is referred to as adjacency.

Simple Graph: A simple graph, also called a strict graph, is an unweighted, undirected graph containing no loops or multiple edges.

Commuting Graph: Given a finite group G and a subset X of G, the commuting graph of G on X, denoted by C(G, X) is the graph that has X as its vertex set with x, y $\in$ X joined by an edge whenever x $\neq$ y and x o y = y o x (i.e. elements commute w.r.t. the binary operation o)

Commutative property: Two elements (x and y) of any group G are said to commute iff x o y = y o x where "o" is the binary operation associated with the group.



Subgraph: A subgraph G of a graph G is a graph G' whose vertex set and edge set are subsets of those of G.

Edge connectivity: The minimum number of edges whose detention from a graph disconnects the graph

Vertex connectivity: The minimum number of vertices whose deletion from a graph disconnects the graph.

Clique number: The size of the largest complete subgraph(clique) in the given graph.

Independent set: A set of vertices of the graph, no two of which are adjacent.

Independence number: The cardinality of the largest independent vertex set.

Vertex covering number: The size of the minimum vertex cover (A vertex cover is a set of vertices such that each edge of the graph is incident to at least one vertex of the set)

Edge covering number: The size of the minimum edge cover (An edge cover of a graph is a set of edges such that every vertex of the graph is incident to at least one edge of the set)

Matching number: The size of maximum vertex cover (A vertex cover is a set of vertices such that each edge of the graph is incident to at least one vertex of the set)

Perfect graph: A perfect graph is a graph in which chromatic number (minimum number of colours required to colour the graph such that no two adjacent vertices are coloured with the same colour of every induced subgraph equals the size of the largest clique of that subgraph.

Hamiltonian graph: A Hamiltonian path (or traceable path) is a path in an undirected or directed graph that visits each vertex exactly once. A Hamiltonian cycle (or Hamiltonian circuit) is a Hamiltonian path that is a cycle. A Hamiltonian Graph is a graph that contains Hamiltonian cycles.

Eulerian graph: An Eulerian trail (or Eulerian path) is a trail in a finite graph that visits every edge exactly once. Similarly, a Eulerian circuit or Eulerian cycle is a Eulerian trail that starts and ends on the same vertex. A Eulerian graph is a graph that contains a Eulerian cycle

V_{8n} group

For positive integer n, the algebraic group V_{8n} of order 8n is defined as:

$$V_{8n} = \langle a, b \mid a^{2n} = b^4 = e, ba = a^{-1} b^{-1}, b^{-1} a = a^{-1} b \rangle$$

Note that $|V_{8n}| = 8n$.

V_{8n}: { a , a^2, ..., a^{2n-1},

ab , a^2b...........$a^{2n-1}b$,

ab2 , a^2b^2 ,............$a^{2n-1}b^2$,

ab^3 , a^2b^3 , ...$a^{2n-1}b^3$,

e, b , b^2 , b^3 }

Elements of V_8 = { e, a, b, b^2, b^3, ab, ab^2, ab^3 } (for n = 1)

Since, $ba = a^{-1}b^{-1}$ and $b^{-1}a = a^{-1}b$, one can observe that:

- $b^2a^i = a^ib^2$
- $ba^i = a^{2n-i}b$; if *i* is even

  $a^{2n-i}b^3$; if *i* is odd
- $b^3a^i = a^{2n-i}b^3$; if *i* is even

  $a^{2n-i}b$; if *i* is odd

Above shown results have played a crucial role in determining the neighbourhood of the elements of V_{8n} which in turn was used to determine properties of the group structure.

Thus, every element of $V_{8n} \setminus \langle a \rangle$ is of the form a^ib^j for some *i*, *j*, where $1 \leq i \leq 2n$ and $1 \leq j \leq 3$

Center for group structure V_{8n}

Center of a group: The centre of a group is the set of elements which commute with every element of the group. It is equal to the intersection of the centralizers of the group elements.

Examples:

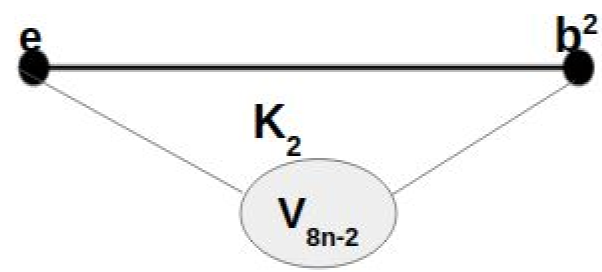
- **Thecentrer of an abelian group, *G*, is all of *G*.**
- **The centre of a nonabelian simple group is trivial.**

Center of group V_{8n}

Case1: if n is odd

$Z_{8n} = \{ e, b^2 \}$

(Where e is the identity element)



K_2 : Complete graph with two nodes

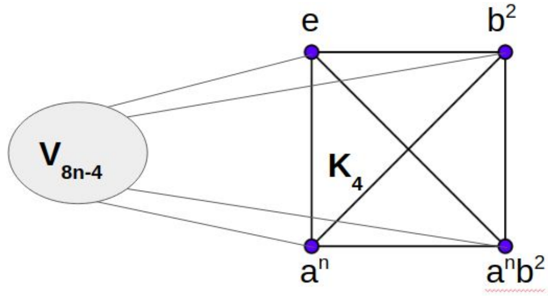
$V_{8n-2} = V_{8n} \sim Z_{8n}$

Case2: if n is even

$Z_{8n} = \{ e, b^2 , a^n, a^n b^2 \}$

(Where e is the identity element)

$V_{8n-4} = V_{8n} \sim Z_{8n}$



K_4 : Complete graph with four nodes

The neighbourhood of elements of V_{8n}

<u>The neighbourhood of a vertex:</u> The **neighbourhood** of a vertex *v* in a graph *G* is the subgraph of *G* induced by all vertices adjacent to *v*, i.e., the graph composed of the vertices adjacent to *v* and all edges connecting vertices adjacent to *v. (N[x])*

In case of V_{8n}, two elements are adjacents if and only if the elements commute among themselves that is if 'a' and 'b' are two elements of any group structure with '*' as the operator. Then they will be adjacent in the associated commuting graph if and only if a*b = b*a.

$N[e] = V_{8n}$ (for both n even and odd)

(Where e is the identity element)

<u>Case 1: n is ODD</u>

Center of V_{8n} for n odd: $Z_{8n} = \{ e, b^2 \}$

- $N[e] = V_{8n}$
- $N[b] = \{e, b, b^2, b^3\}$
- $N[b^2] = V_{8n}$ (Since, $b^2 \in Z_{8n}$)
- $N[b^3] = \{e, b, b^2, b^3\}$
- $N[a^i b] = \{ e, b^2 \} \cup \{a^i b\} \cup \{a^i b^3\}$
- $N[a^i b^2] = \{ e, b^2 \} \cup \{a^i\} \cup \{a^i b^2\}$
- $N[a^i b^3] = \{ e, b^2 \} \cup \{a^i b^3\} \cup \{a^i b\}$
- $N[a^i] = \{ e, b^2 \} \cup \{ a^i \} \cup \{ a^i b^2 \}$

<u>Case 2: n is EVEN</u>

Center of V_{8n} for n even: $Z_{8n} = \{ e, b^2, a^n, a^n b^2 \}$

- $N[e] = V_{8n}$
- $N[b] = \{e, b, b^2, b^3, a^n b, a^n b^2, a^n b^3, a^n\}$
- $N[b^2] = V_{8n}$ (Since, $b^2 \in Z_{8n}$)

- $N[b^3] = \{e, b, b^2, b^3, a^nb, a^nb^2, a^nb^3, a^n\}$
- $N[a^ib] = \{e, b^2, a^n, a^nb^2\} \cup \{a^jb\} \cup \{a^jb^3\}$
- $N[a^ib^2] = V_{8n}$; for $i = n$

  $= \{e, b^2\} \cup \{a^j\} \cup \{a^jb^2\}$; otherwise
- $N[a^ib^3] = \{e, b^2, a^n, a^nb^2\} \cup \{a^jb\} \cup \{a^jb^3\}$
- $N[a^i] = V_{8n}$; for $i = n$

  $= \{e, b^2, a^n, a^nb^3\} \cup \{a^jb^2\} \cup \{a^j\}$; otherwise

PROPERTIES OF V_{8n}

1. **Edge connectivity:**

Case 1: n even

<u>Edge connectivity = 4</u>

Proof: Minimum degree for any vertex an element V_{8n} is 4 for $\{a^i b\}$. So, we have to remove at least 4 edges to make the graph disconnected from remaining connected. Hence, the edge connectivity is 4.

Case 2: n odd

<u>Edge connectivity = 3</u>

Proof: Minimum degree for any vertex (an element V_{8n}) is 3 for $\{a^i b\}$ and $\{a^i b^3\}$. So, we have to remove at least 3 edges to make the graph disconnected from remaining connected. Hence, the edge connectivity is 3.

2. **Vertex connectivity**

Case 1: n even

<u>Vertex connectivity = 4</u>

Proof: For n even, there are 4 elements in the centre of V_{8n} which are $\{e, b^2, a^n b^2, a^n\}$. Hence, we have to remove at least four elements to make the graph disconnected.

Case 2: n odd

<u>Vertex connectivity = 2</u>

Proof: For n even, there are 2 elements in the centre of V_{8n} which are $\{e, b^2\}$. Hence, we have to remove at least four elements to make the graph disconnected.

3. Clique Number

<u>Clique number = 4n (for both odd and even)</u>

Proof: Complete subgraphs are formed by center elements among themselves as they are adjacent to every vertex. $\langle a \rangle$ and $\langle b \rangle$ also form complete subgraphs.

$\langle b \rangle = \{e, b, b^2, b^3\}$, hence $|\langle b \rangle|=4$

$\langle a \rangle = \{e, a, a^2, \dots a^{2n-1}\}$, hence $|\langle a \rangle|=2n$

Central elements for n odd are $\{e, b^2\}$ and n even are $\{e, b^2, a^n, a^nb^2\}$

Since center is incident to both $\langle a \rangle$, $\langle b \rangle$ elements are largest complete subgraph formed by $Z_{8n} \cup \langle a \rangle$ also for elements of the form $\langle a^ib^2 \rangle$.

For n even clique number is 2n + 2n = 4n

Also for n odd is 4n.

4. Independence Number

Independence number is the size of the largest independent set. On observing the elements of V_{8n} for both n odd and even, we can clearly see that elements of form $\langle a^ib \rangle$ and $\langle a^ib^3 \rangle$ are not adjacent to any other elements except centre and $\langle a^ib^3 \rangle$ and $\langle a^ib \rangle$ respectively.

Case 1: n even

<u>Independence number = 2n -1</u>

Proof: Independent set = $\{\langle a^ib \rangle\}$, since for i=n, a^nb is adjacent to all elements in $\langle b \rangle$. Hence the independence number is 2n-1.

Case 2: n odd

<u>Independence number = 2n+1</u>

Proof: For n odd, the independence set consists of b, $<a^ib>$ and a. Hence the total number of elements is 1 + 2n-1 + 1. Hence the independence number for n odd is 2n+1.

5. Vertex covering number

Vertex cover number = 6n (for both odd and even)

Proof:

For n even, to cover all the edges, firstly we need to take all centre elements except one, all generators of a (<a>) and b(<b>) is either a^ib or a^ib^3 and the whole of a^ib^2.

$|Z_{8n}| - 1 + |<a>| + |<b>| + |< a^ib>| + |< a^ib^2>|$

$\Rightarrow 2n + 2 + 2n\text{-}1 + 2n\text{-}1 = 6n$

For n odd, Similarly

$|<a>| + |<b>| + |< c>|+ |< a^ib^2>|$,

$\Rightarrow 2n + 2 + 2n\text{-}1 + 2n\text{-}1 = 6n$

6. Matching Number

We need to remove the central vertex and edge associated with them to accurately calculate the matching number, as central elements are adjacent to all other elements.

Case 1: n even

Matching number = 4n-2

Proof: $V_{8n} - Z_{8n} = V_{8n} – 4$

$<a> \cup |< a^ib^2>| - \{e, a^n, a^nb^2, b^2\}$ form a complete subgraph with a cycle of length 4n – 4. Hence, 2n – 2 non adjacent edges.

Also, $\forall$ j in {1, 2, ... ,2n}, each $<a^jb^3>$ is adjacent to { a^jb } . Hence the matching number = 2n-2 + 2n = 4n-2.

Case 2: n odd: 4n-1

<u>Matching number = 4n-1</u>

Proof: $V_{8n} - Z_{8n} = V_{8n} - 2$

$<a> \cup |< a^ib^2>| - \{e, b^2\}$ form a complete subgraph with a cycle of length 4n – 2. Hence, there are 2n – 1 nonadjacent edge.

Also, $\forall$ j in {1, 2, ... ,2n}, each $<a^jb^3>$ is adjacent to { a^jb } . Hence the matching number = 2n-1 + 2n = 4n-1.

7. **Edge cover number**

<u>Edge cover number = 4n (for both n even and odd)</u>

By definition, edge cover number is the minimum edge covering that can be the minimum of the total number of vertices, that is $\lceil$number of vertices / 2$\rceil$.

For n even

$Z_{8n} = \{e, b^2, a^nb^2, a^n\}$

$\forall$ j in {1, 2, ... ,2n} a^jb is adjacent to a^jb^3, remaining elements (4n) form a regular complete subgraph. Hence, 2n edges join all 4n vertices.

Hence the edge covering number is 4n.

Similarly, for n odd

$Z_{8n} = \{e, b^2\}$

From the above explanation, the edge cover number is 4n.

8. Eulerian

For any connecting graph to be Eulerian, the degree of every vertex needs to be even (>0). Since V_{8n} for both n even and n odd has non – empty set of central elements.

Since, the graph of V_{8n} is connected.

For both n even and n odd there exist vertices of odd degree, Hence commuting graph of V_{8n} is not Eulerian.

CONCLUSION

The project described our contribution in the field of graph-theoretic algebraic structure. V_{8n} was a comparatively new group structure and very less work has been done in the field in turn proved to be a boon for us as it helped us stay focused on whatever limited resource was available and making our own hands dirty rather than googling out stuff. This project helped us in understanding how pure research is carried out and role patience and continuous effort plays.

Proofs to almost all the results are provided in the report with detailed explanation. Brute Force and Backtracking approach were used to determine the centre and neighbours of the elements taking help from some predetermined results on the elements of V_{8n}. We were successful in determining most properties of the group with the help of centre elements, neighbours and our prior knowledge of graph theory through the graph and network course.

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