M-nacci Sequences with Chebyshev Polynomial of First Kind

Aditya Vishwakarma (adityav31121999@gmail.com)

Abstract

This research paper explores the relationship between M-nacci sequences and Chebyshev polynomials of the first kind. The author examines how Chebyshev polynomial coefficients can be used to generate M-nacci sequences, where the value of m determines the number of previous terms used to calculate the next term. Furthermore, the paper investigates M-nacci sequences with random variables, demonstrating how these sequences can be constructed using square matrices containing coefficients based on the main M-nacci sequence. The paper ultimately shows that both Chebyshev polynomials and random variables can be used to create variations of M-nacci sequences, highlighting the versatile nature of these sequences.

Keywords: M-nacci Sequence, Generalization, Chebyshev Polynomial, Random Variables

Introduction

Fibonacci sequence is a sequence that progresses by taking sum of previous consecutive two terms to create new term. The Fibonacci numbers were first described in Indian mathematics as early as 200 BC in work by Pingala on enumerating possible patterns of Sanskrit poetry formed from syllables of two lengths. This sequence is based on recurrence relation. The Fibonacci sequence is: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, …

$$f_{n+1} = f_n + f_{n-1} \qquad \textit{eqn. 1}$$

The ratio of consecutive terms is called golden ratio $\boldsymbol{\varphi}$ which has value of 1.618033988749894…

$$\varphi = \frac{f_{n+1}}{f_n} \approx \mathbf{1.618033988749894} \ldots$$

There is another sequence called as Lucas Sequence which can is given as 2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, 2207, 3571, 5778, 9349, … This follows the same recurrence relation as Fibonacci sequence. The ratio of consecutive terms is φ. There are various such sequences based on recurrence relationship and hence this can be considered as generalised Fibonacci or M-nacci sequence. These sequences namely are Tribonacci, Tetranacci, Pentabonacci, Hexabonacci, and so on. These sequences also have their respective constants given by the same relationship as of golden ratio, but different values.

M-nacci Sequence

Following the path of Fibonacci sequence, different other sequences can be made with **m**-values. M-values are number of terms to be added recursively. These sequences will have $m\text{-}1$ 0s and just one 1. These series are 3-, 4-, 5-, 6-, … m-nacci. Some such series are as follows:

- *Tribonacci: 0, 0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, …*
- *Tetranacci: 0, 0, 0, 1, 1, 2, 4, 8, 15, 29, 56, 108, 208, 401, …*
- *Pentanacci: 0, 0, 0, 0, 1, 1, 2, 4, 8, 16, 31, 61, 120, 236, … And so on.*

A general form of these is given as

$$m_n = \sum_{i=n-m-1}^{n-1} m_i \qquad - eqn.2$$

$$M - nacci\ Constant = \frac{m_{i+1}}{m_i} \qquad - eqn.3$$

Considering m-1 0s and 1, we can create a sequence which will follow eqn. 2. This sequence is given by:

$$0, \dots (m-1) \dots 0,1,1,2,4,8,16, \dots, 2^{m-1},\ 2^m - 1, 2^{m+1} - 3, 2^{m+2} - 8, \dots$$

Following are the trends observed in m-nacci sequence:

1. First m terms are m-1 0s and one 1. From (m+1) th terms, new terms are created via recurrence through summation.
2. Second m terms are powers of 2, starting from 1 (2^0) to 2^{m-1}.
3. These are important terms for calculating m-nacci terms. $S_0 = 2^{i-m-1}, i \geq m+1$
4. Third m terms of main m-nacci are little less than 2^m but this difference start increasing as the values increase.
5. The difference follows a sequence which is $1, 3, 8, 20, 48, 112, 256, \dots$ $S_1 = (n+2) * 2^{n-1}$ (A001792).
6. For next m terms from 2m+1 to 3m and onwards, $m_i = S_0 - S_1 = 2^{i-m-1} - (i+2)2^{i-1}$
7. After 3m, the values differ and are lower compared to main m-nacci values, new values are needed to be added.
8. These new are in the sequence of 0, 1, 5, 18, 56, 160, … $S_2 = n(n+3) * 2^{n-3}$ (A001793)
9. For 3m+1 to 4m and onwards, $m_i = S_0 - S_1 + S_2 = 2^{2m+i-1} - (m+i+2)2^{m+i-1} + i(i+3)2^{i-3}$
10. Since the new values are higher from 4m+1, new values need to be subtracted which are $1, 7, 32, 120, 400, 1232, \dots \dots$ $S_3 = 2^{(n-2)}(n+1)(n+2)(n+6)/3$ (A001794).
11. For 4m+1 to 5m and onwards, $m_i = S_0 - S_1 + S_2 - S_3 = 2^{3m+i-1} - (2m+i+2)2^{2m+i-1} + (m+i)(m+i+3)2^{m+i-3} - (2^{(i-2)}(i+1)(i+2)(i+6)/3)$.

Chebyshev Polynomial of 1st kind and its coefficient triangle

On closer inspections, though all these sequences look different and arise while taking differences when deviation is observed when compared with main m-nacci sequence, it becomes clear that some of these terms are part of Chebyshev T Polynomial of first kind. By recurrence definition, Chebyshev T Polynomials are given by:

$$T_0(x) = 1, T_1(x) = x$$

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \qquad \text{-eqn. 4}$$

$$\begin{aligned}
T_0(x) &= 1 \\
T_1(x) &= x \\
T_2(x) &= 2x^2 - 1 \\
T_3(x) &= 4x^3 - 3x \\
T_4(x) &= 8x^4 - 8x^2 + 1 \\
T_5(x) &= 16x^5 - 20x^3 + 5x \\
T_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1 \\
T_7(x) &= 64x^7 - 112x^5 + 56x^3 - 7x \\
T_8(x) &= 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1 \\
T_9(x) &= 256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x \\
T_{10}(x) &= 512x^{10} - 1280x^8 + 1120x^6 - 400x^4 + 50x^2 - 1
\end{aligned}$$

Fig. 1: First few Chebyshev polynomials of 1st kind [1]

The values in ith column are the coefficients of $(-1)^{i+1}\frac{1-x}{(1-2x)^i}$. Each of these columns starts at 2(i-1) row starting from 1st column itself.

T	Coefficient Triangle								
0	1								
1	1								
2	2	-1							
3	4	-3							
4	8	-8	1						
5	16	-20	5						
6	32	-48	18	-1					
7	64	-112	56	-7					
8	128	-256	160	-32	1				
9	256	-576	432	-120	9				
10	512	-1280	1120	-400	50	-1			
11	1024	-2816	2816	-1232	220	-11			
12	2048	-6144	6912	-3584	840	-72	1		
13	4096	-13312	16640	-9984	2912	-364	13		
14	8192	-28672	39424	-26880	9408	-1568	98	-1	
15	16384	-61440	92160	-70400	28800	-6048	560	-15	
16	32768	-131072	212992	-180224	84480	-21504	2688	-128	1	
17	65536	-278528	487424	-452608	239360	-71808	11424	-816	17	
18	131072	-589824	1105920	-1118208	658944	-228096	44352	-4320	162	-1
19	262144	-1245184	2490368	-2723840	1770496	-695552	160512	-20064	1140	-19

Table 1: Coefficient Triangle of 1st kind (OEIS: A028297)

M-nacci Terms with Coefficient

This equation gives jth term of the m-nacci term, $m_j = \sum S_{ji}$. Here, i is the column of coefficient triangle and j is the row. M-nacci terms can be made with the columns of coefficient triangle of Chebyshev Polynomial of 1st kind. The columns are adjusted below their original locations, and these adjustments are done as per the m-value. If its Fibonacci sequence, then m-value is 2, and each column will be shifted down by 1. For Tribonacci sequence it will be 2, Tetranacci will have 3 adjustments and so on. So, it can be said that each column will be shifted by m-1 steps below. While creating new terms, signs of values are considered as it is.

Following are the illustrations are the examples of such sequences, along with original sequence and then difference column to check whether the adjustments are needed or not, if 0 no adjustment. For original sequence, leading terms that are 0 are skipped, only m^{th} term 1 is included as the 0^{th} term formed due to T_0.

T	Coefficients Adjusted																	Sum	Fibonacci	diff
0	1																	1	1	0
1	1																	1	1	0
2	2																	2	2	0
3	4	-1																3	3	0
4	8	-3																5	5	0
5	16	-8																8	8	0
6	32	-20	1															13	13	0
7	64	-48	5															21	21	0
8	128	-112	18															34	34	0
9	256	-256	56	-1														55	55	0
10	512	-576	160	-7														89	89	0
11	1024	-1280	432	-32														144	144	0
12	2048	-2816	1120	-120	1													233	233	0
13	4096	-6144	2816	-400	9													377	377	0
14	8192	-13312	6912	-1232	50													610	610	0
15	16384	-28672	16640	-3584	220	-1												987	987	0
16	32768	-61440	39424	-9984	840	-11												1597	1597	0
17	65536	-131072	92160	-26880	2912	-72												2584	2584	0
18	131072	-278528	212992	-70400	9408	-364	1											4181	4181	0
19	262144	-589824	487424	-180224	28800	-1568	13											6765	6765	0
20	524288	-1245184	1105920	-452608	84480	-6048	98											10946	10946	0
21	1048576	-2621440	2490368	-1118208	239360	-21504	560	-1										17711	17711	0
22	2097152	-5505024	5570560	-2723840	658944	-71808	2688	-15										28657	28657	0
23	4194304	-11534336	12386304	-6553600	1770496	-228096	11424	-128										46368	46368	0
24	8388608	-24117248	27394048	-15597568	4659200	-695552	44352	-816	1									75025	75025	0
25	16777216	-50331648	60293120	-36765696	12042240	-2050048	160512	-4320	17									121393	121393	0
26	33554432	-104857600	132120576	-85917696	30638080	-5870592	549120	-20064	162									196418	196418	0
27	67108864	-218103808	288358400	-199229440	76873728	-16400384	1793792	-84480	1140	-1								317811	317811	0
28	134217728	-452984832	627048448	-458752000	190513152	-44843008	5637632	-329472	6600	-19								514229	514229	0
29	268435456	-939524096	1358954496	-1049624576	466944000	-120324096	17145856	-1208064	33264	-200								832040	832040	0
30	536870912	-1946157056	2936012800	-2387607552	1133117440	-317521920	50692096	-4209920	151008	-1540	1							1346269	1346269	0
31	1073741824	-4026531840	6325010432	-5402263552	2724986880	-825556992	146227200	-14057472	631488	-9680	21							2178309	2178309	0
32	2147483648	-8321499136	13589544960	-12163481600	6499598336	-2118057984	412778496	-45260800	2471040	-52624	242							3524578	3524578	0
33	4294967296	-17179869184	29125246976	-27262976000	15386804224	-5369233408	1143078912	-141213696	9152000	-256256	2024	-1						5702887	5702887	0
34	8589934592	-35433480192	62277025792	-60850962432	36175872000	-13463453696	3111714816	-428654592	32361472	-1144000	13728	-23						9227465	9227465	0
35	17179869184	-73014444032	1.32876E+11	-1.35291E+11	84515225600	-33426505728	8341487616	-1270087680	109983744	-4759040	80080	-288						14930352	14930352	0
36	34359738368	-1.50324E+11	2.82931E+11	-2.99708E+11	1.96293E+11	-82239815680	22052208640	-3683254272	361181184	-18670080	416416	-2600	1					24157817	24157817	0
37	68719476736	-3.09238E+11	6.01295E+11	-6.61693E+11	4.53438E+11	-2.00656E+11	57567870976	-10478223360	1151016960	-69701632	1976832	-18928	25					39088169	39088169	0
38	1.37439E+11	-6.35655E+11	1.27561E+12	-1.45626E+12	1.04217E+12	-4.85826E+11	1.48562E+11	-29297934336	3572121600	-249387008	8712704	-117936	338					63245986	63245986	0
39	2.74878E+11	-1.30567E+12	2.70153E+12	-3.19546E+12	2.38404E+12	-1.16795E+12	3.79364E+11	-80648077312	10827497472	-859955200	36095488	-652288	3276	-1				102334155	102334155	0
40	5.49756E+11	-2.68006E+12	5.71231E+12	-6.99221E+12	5.42978E+12	-2.78933E+12	9.59384E+11	-2.18864E+11	32133218304	-2870927360	141892608	-3281408	25480	-27				165580141	165580141	0
41	1.09951E+12	-5.49756E+12	1.20603E+13	-1.526E+13	1.23158E+13	-6.62083E+12	2.40459E+12	-5.8629E+11	93564370944	-9313976320	533172224	-15275520	168896	-392				267914296	267914296	0
42	2.19902E+12	-1.127E+13	2.54262E+13	-3.32216E+13	2.78271E+13	-1.56257E+13	5.97713E+12	-1.55194E+12	2.67777E+11	-29455450112	1926299648	-66646528	990080	-4060	1			433494437	433494437	0
43	4.39805E+12	-2.30897E+13	5.35325E+13	-7.21555E+13	6.26464E+13	-3.66812E+13	1.47436E+13	-4.06327E+12	7.54418E+11	-91044118528	6723526656	-275185664	5261568	-33600	29			701408733	701408733	0
44	8.79609E+12	-4.7279E+13	1.12563E+14	-1.56371E+14	1.40553E+14	-8.56782E+13	3.6108E+13	-1.05311E+13	2.09513E+12	-2.75653E+11	22761029632	-1083543552	25798656	-236096	450			1134903170	1134903170	0
45	1.75922E+13	-9.6757E+13	2.36395E+14	-3.38169E+14	3.14327E+14	-1.99183E+14	8.78417E+13	-2.70394E+13	5.7422E+12	-8.19082E+11	74977509376	-4093386752	118243840	-1462272	4960	-1		1836311903	1836311903	0
46	3.51844E+13	-1.97912E+14	4.9588E+14	-7.2987E+14	7.0081E+14	-4.61013E+14	2.12365E+14	-6.88224E+13	1.55477E+13	-2.39258E+12	2.40999E+11	-14910300160	511673344	-8186112	43520	-31		2971215073	2971215073	0
47	7.03687E+13	-4.0462E+14	1.03904E+15	-1.5723E+15	1.55799E+15	-1.06258E+15	5.10407E+14	-1.73753E+14	4.16265E+13	-6.88029E+12	7.57651E+11	-52581629952	2106890240	-42170880	323136	-512		4807526976	4807526976	0
48	1.40737E+14	-8.26833E+14	2.17483E+15	-3.381E+15	3.45415E+15	-2.43949E+15	1.22E+15	-4.35348E+14	1.10292E+14	-1.95028E+13	2.33438E+12	-1.80141E+11	8307167232	-202585600	2108544	-5984	1	7778742049	7778742049	0
49	2.81475E+14	-1.68885E+15	4.54758E+15	-7.25788E+15	7.63817E+15	-5.57978E+15	2.90101E+15	-1.08306E+15	2.89407E+14	-5.45532E+13	7.06135E+12	-6.01281E+11	31524634624	-916844544	12403200	-55488	33	12586269025	12586269025	0
50	5.6295E+14	-3.44807E+15	9.49978E+15	-1.55548E+16	1.68486E+16	-1.27176E+16	6.8646E+15	-2.67653E+15	7.52567E+14	-1.50733E+14	2.1003E+13	-1.96021E+12	1.15631E+11	-3940579328	66977280	-434112	578	20365011074	20365011074	0

Table 2: Fibonacci Sequence with Coefficients

T	Coefficients Adjusted											SUM	Tetranacci	diff
0	1											1	1	0
1	1											1	1	0
2	2											2	2	0
3	4											4	4	0
4	8											8	8	0
5	16	-1										15	15	0
6	32	-3										29	29	0
7	64	-8										56	56	0
8	128	-20										108	108	0
9	256	-48										208	208	0
10	512	-112	1									401	401	0
11	1024	-256	5									773	773	0
12	2048	-576	18									1490	1490	0
13	4096	-1280	56									2872	2872	0
14	8192	-2816	160									5536	5536	0
15	16384	-6144	432	-1								10671	10671	0
16	32768	-13312	1120	-7								20569	20569	0
17	65536	-28672	2816	-32								39648	39648	0
18	131072	-61440	6912	-120								76424	76424	0
19	262144	-131072	16640	-400								147312	147312	0
20	524288	-278528	39424	-1232	1							283953	283953	0
21	1048576	-589824	92160	-3584	9							547337	547337	0
22	2097152	-1245184	212992	-9984	50							1055026	1055026	0
23	4194304	-2621440	487424	-26880	220							2033628	2033628	0
24	8388608	-5505024	1105920	-70400	840							3919944	3919944	0
25	16777216	-11534336	2490368	-180224	2912	-1						7555935	7555935	0
26	33554432	-24117248	5570560	-452608	9408	-11						14564533	14564533	0
27	67108864	-50331648	12386304	-1118208	28800	-72						28074040	28074040	0
28	134217728	-104857600	27394048	-2723840	84480	-364						54114452	54114452	0
29	268435456	-218103808	60293120	-6553600	239360	-1568						104308960	104308960	0
30	536870912	-452984832	132120576	-15597568	658944	-6048	1					201061985	201061985	0
31	1073741824	-939524096	288358400	-36765696	1770496	-21504	13					387559437	387559437	0
32	2147483648	-1946157056	627048448	-85917696	4659200	-71808	98					747044834	747044834	0
33	4294967296	-4026531840	1358954496	-199229440	12042240	-228096	560					1439975216	1439975216	0
34	8589934592	-8321499136	2936012800	-458752000	30638080	-695552	2688					2775641472	2775641472	0
35	17179869184	-17179869184	6325010432	-1049624576	76873728	-2050048	11424	-1				5350220959	5350220959	0
36	34359738368	-35433480192	13589544960	-2387607552	190513152	-5870592	44352	-15				10312882481	10312882481	0
37	68719476736	-73014444032	29125246976	-5402263552	466944000	-16400384	160512	-128				19878720128	19878720128	0
38	1.37439E+11	-1.50324E+11	62277025792	-12163481600	1133117440	-44843008	549120	-816				38317465040	38317465040	0
39	2.74878E+11	-3.09238E+11	1.32876E+11	-27262976000	2724986880	-120324096	1793792	-4320				73859288608	73859288608	0
40	5.49756E+11	-6.35655E+11	2.82931E+11	-60850962432	6499598336	-317521920	5637632	-20064	1			1.42368E+11	1.42368E+11	0
41	1.09951E+12	-1.30567E+12	6.01295E+11	-1.35291E+11	15386804224	-825556992	17145856	-84480	17			2.74424E+11	2.74424E+11	0
42	2.19902E+12	-2.68006E+12	1.27561E+12	-2.99708E+11	36175872000	-2118057984	50692096	-329472	162			5.28969E+11	5.28969E+11	0
43	4.39805E+12	-5.49756E+12	2.70153E+12	-6.61693E+11	84515225600	-5369233408	146227200	-1208064	1140			1.01962E+12	1.01962E+12	0
44	8.79609E+12	-1.127E+13	5.71231E+12	-1.45626E+12	1.96293E+11	-13463453696	412778496	-4209920	6600			1.96538E+12	1.96538E+12	0
45	1.75922E+13	-2.30897E+13	1.20603E+13	-3.19546E+12	4.53438E+11	-33426505728	1143078912	-14057472	33264	-1		3.78839E+12	3.78839E+12	0
46	3.51844E+13	-4.7279E+13	2.54262E+13	-6.99221E+12	1.04217E+12	-82239815680	3111714816	-45260800	151008	-19		7.30237E+12	7.30237E+12	0
47	7.03687E+13	-9.6757E+13	5.35325E+13	-1.526E+13	2.38404E+12	-2.00656E+11	8341487616	-141213696	631488	-200		1.40758E+13	1.40758E+13	0
48	1.40737E+14	-1.97912E+14	1.12563E+14	-3.32216E+13	5.42978E+12	-4.85826E+11	22052208640	-428654592	2471040	-1540		2.71319E+13	2.71319E+13	0
49	2.81475E+14	-4.0462E+14	2.36395E+14	-7.21555E+13	1.23158E+13	-1.16795E+12	57567870976	-1270087680	9152000	-9680		5.22984E+13	5.22984E+13	0
50	5.6295E+14	-8.26833E+14	4.9588E+14	-1.56371E+14	2.78271E+13	-2.78933E+12	1.48562E+11	-3683254272	32361472	-52624	1	1.00808E+14	1.00808E+14	0

Table 3: Tetranacci Sequence with Coefficients

Fibonacci Sequence with Random Variable

(0, 1) => (0, n)

Instead of 1, if a random number n is taken than the sequence that is generated is given by: $0, n, n, 2n, 3n, 5n, 8n, 13n, 21n, 34n, 55n, 89n, \dots$

$$\text{i.e., } F_n = f_n * n \qquad \text{-eqn. 6}$$

(0, 1) => (x, y)

If two random values that can be taken from Z (set of integers), say $x \text{ and } y$, we can create sequence like Fibonacci sequence:

$$x, y, x + y, x + 2y, 2x + 3y, 3x + 5y, 5x + 8y, 8x + 13y, 13x + 21y, \dots$$

This sequence is represented as:

$$F_{n+1} = f_{n-1}x + f_n y \qquad \text{eqn. 7}$$

Another thing is to notice in the change when y and x are taken and new series that is created is different when x and y are taken as first and second term.

$$y, x, y + x, y + 2x, 2y + 3x, 3y + 5x, 5y + 8x, 8y + 13x, 13y + 21x, \dots$$

This sequence is represented as:

$$F_{n+1} = f_{n-1}y + f_n x \qquad \textit{eqn. 8}$$

If x = 0 and y = 1, Fibonacci sequence is obtained, and if x = 1 and y = 0, as sequence which is 1, 0, 1, 1, 2, 3, 5, 8, … (1, Fibonacci terms). If $x = 2 \text{ and } y = 1$, the sequence obtained is 2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, … which is Lucas Sequence, and if x = 1 and y = 2, the sequence obtained is 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, …. And this is Fibonacci sequence minus 0 and 1. Is this is extrapolated to other terms, if two consecutive Fibonacci terms are taken, then the resulting sequence obtained is subset of Fibonacci sequence where terms before the initial terms x and y are not included i.e., if $x = f_{n-1} \text{ and } y = f_n$ then the series obtained is Fibonacci terms starting from f_{n-1}, terms before this are excluded. But for $x = f_n \text{ and } y = f_{n-1}$, a new sequence is obtained, but not consisting Fibonacci terms. This sequences, since they follow same recurrence relation, the ratio remains the same.

In Fig. 1, there are many Fibonacci variants which follow eqn. 3 and 4. From left to right it is eqn. 1 and from right to left it is eqn. 4. On closer look, eqn. 3 produces subset of Fibonacci sequence when inputs are consecutive terms are from original Fibonacci sequence. Vertically it mirrors across the input diagonal. With change in sign of values, at below there are positive values and above are negative values with triangle formed with rows having same values as differentiator or partition between these negative and positive values.

Here the row with 0 and 1 as x and y, is 0th row while the column with tow 0s is 0th column. The recurrence of values on the right-hand side of 0th column and below of 0th row is triangle of values in which all rows have same values, similarly on the left-hand side of it there is a similar triangle rising above the 0th row. Diagonally, values are in the form of Fibonacci sequence. This can be observed in coloured diagonal.

Horizontal Relation

$$F_{n+1} = f_{n-1}x + f_n y$$

-55	**-34**	-89	-123	-212	-335	-547	-882	-1429	-2311	-3740	-6051	-9791	-15842	-25633	-41475	-67108	-108583	-175691	-284274	-459965
-55	**-34**	**-21**	-55	-76	-131	-207	-338	-545	-883	-1428	-2311	-3739	-6050	-9789	-15839	-25628	-41467	-67095	-108562	-175657
-55	-34	**-21**	**-13**	-34	-47	-81	-128	-209	-337	-546	-883	-1429	-2312	-3741	-6053	-9794	-15847	-25641	-41488	-67129
-55	-34	-21	**-13**	**-8**	-21	-29	-50	-79	-129	-208	-337	-545	-882	-1427	-2309	-3736	-6045	-9781	-15826	-25607
-55	-34	-21	-13	**-8**	**-5**	-13	-18	-31	-49	-80	-129	-209	-338	-547	-885	-1432	-2317	-3749	-6066	-9815
-55	-34	-21	-13	-8	**-5**	**-3**	-8	-11	-19	-30	-49	-79	-128	-207	-335	-542	-877	-1419	-2296	-3715
-55	-34	-21	-13	-8	-5	**-3**	**-2**	-5	-7	-12	-19	-31	-50	-81	-131	-212	-343	-555	-898	-1453
-55	-34	-21	-13	-8	-5	-3	**-2**	**-1**	-3	-4	-7	-11	-18	-29	-47	-76	-123	-199	-322	-521
-55	-34	-21	-13	-8	-5	-3	-2	**-1**	**-1**	-2	-3	-5	-8	-13	-21	-34	-55	-89	-144	-233
-55	-34	-21	-13	-8	-5	-3	-2	-1	**-1**	**0**	-1	-1	-2	-3	-5	-8	-13	-21	-34	-55
55	34	21	13	8	5	3	2	1	1	**0**	**1**	1	2	3	5	8	13	21	34	55
233	144	89	55	34	21	13	8	5	3	2	**1**	**1**	2	3	5	8	13	21	34	55
521	322	199	123	76	47	29	18	11	7	4	3	**1**	**2**	3	5	8	13	21	34	55
1453	898	555	343	212	131	81	50	31	19	12	7	5	**2**	**3**	5	8	13	21	34	55
3715	2296	1419	877	542	335	207	128	79	49	30	19	11	8	**3**	**5**	8	13	21	34	55
9815	6066	3749	2317	1432	885	547	338	209	129	80	49	31	18	13	**5**	**8**	13	21	34	55
25607	15826	9781	6045	3736	2309	1427	882	545	337	208	129	79	50	29	21	**8**	**13**	21	34	55
67129	41488	25641	15847	9794	6053	3741	2312	1429	883	546	337	209	128	81	47	34	**13**	**21**	34	55
175657	108562	67095	41467	25628	15839	9789	6050	3739	2311	1428	883	545	338	207	131	76	55	**21**	**34**	55
459965	284274	175691	108583	67108	41475	25633	15842	9791	6051	3740	2311	1429	882	547	335	212	123	89	**34**	**55**

Vertically Mirrored

$$F_{n+1} = f_{n-1}y + f_n x$$

Fig. 3: Row-wise fibonacci-like sequence using Fibonacci and its negative values

M-nacci Sequence with Random Variables

Like Fibonacci sequence with random variables, other sequences can be formed in same way. For Tribonacci sequence it will be x, y, and z as initial values. The new sequence will be x, y, z, x+y+z, x+2y+2z, 2x+3y+4z, 4x+6y+7z, 7x+11y+13z, … For Tetrabonacci sequence, the sequence will be p, q, r, s, p+q+r+s, p+2q+2r+2s, 2p+3q+4r+4s, 4p+6q+7r+8s, …

This can be done for m-nacci and for better coefficients of each variable for each term can be calculated in the forms of square matrices.

Let:

- x_i = i^{th} variable
- m = m-value/number of variables
- m_i = main m-nacci sequence
- M_j = j^{th} m-nacci term
- C_{ji} = coefficient of i^{th} variable of j^{th} term

$$M_j = \sum_{k=j-m-1}^{j-1} M_k = \sum_{i=1}^{m} C_{ji} x_i \qquad - eqn.$$

Instead of putting these in same sequence, a matrix is formed with "m" columns and every row will have a coefficient for the respective term. This term is than calculated by taking sum of the respective products of coefficient and i^{th} variable of that respective row. For first m rows, the matrix will be like a square identity matrix, since the first m terms of sequence are nothing but the variables themselves.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	⋯	x_{m-3}	x_{m-2}	x_{m-1}	x_m
1	1	0	0	0	0	0	0	0	0	⋯	0	0	0	0
2	0	1	0	0	0	0	0	0	0	⋯	0	0	0	0
3	0	0	1	0	0	0	0	0	0	⋯	0	0	0	0
4	0	0	0	1	0	0	0	0	0	⋯	0	0	0	0
5	0	0	0	0	1	0	0	0	0	⋯	0	0	0	0
6	0	0	0	0	0	1	0	0	0	⋯	0	0	0	0
7	0	0	0	0	0	0	1	0	0	⋯	0	0	0	0
8	0	0	0	0	0	0	0	1	0	⋯	0	0	0	0
9	0	0	0	0	0	0	0	0	1	⋯	0	0	0	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋮
m-3	0	0	0	0	0	0	0	0	0	⋯	1	0	0	0
m-2	0	0	0	0	0	0	0	0	0	⋯	0	1	0	0
m-1	0	0	0	0	0	0	0	0	0	⋯	0	0	1	0
m	0	0	0	0	0	0	0	0	0	⋯	0	0	0	1

Fig. 4: Initial coefficients of variables for Sequence with $m \times m$ identity matrix

As the series progresses, the coefficients follow same recurrence, but the values are as similar in relationship as given previously in Tribonacci and Tetrabonacci. The second m terms form following matrix:

i =>	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	… … … …							*m-1*	*m*
	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	… … … …							x_{m-1}	x_m
m+1	**1**	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1		1	1	**1**
m+2	1	**2**	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2		2	2	**2**
m+3	2	3	**4**	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	…	4	4	**4**
m+4	4	6	7	**8**	8	8	8	8	8	8	8	8	8	8	8	8	8	8		8	8	**8**
m+5	8	12	14	15	**16**	16	16	16	16	16	16	16	16	16	16	16	16	16		16	16	**16**
m+6	16	24	28	30	31	**32**	32	32	32	32	32	32	32	32	32	32	32	32		32	32	**32**
m+7	32	48	56	60	62	63	**64**	64	64	64	64	64	64	64	64	64	64	64	$\mathbf{2^{i-1}}$	64	64	**64**
m+8	64	96	112	120	124	126	127	**128**	128	128	128	128	128	128	128	128	128	128		128	128	**128**
m+9	128	192	224	240	248	252	254	255	**256**	256	256	256	256	256	256	256	256	256		256	256	**256**
m+10	256	384	448	480	496	504	508	510	511	**512**	512	512	512	512	512	512	512	512		512	512	**512**
m+11	512	768	896	960	992	1008	1016	1020	1022	1023	**1024**	1024	1024	1024	1024	1024	1024	1024	…	1024	1024	**1024**
m+12	1024	1536	1792	1920	1984	2016	2032	2040	2044	2046	2047	**2048**	2048	2048	2048	2048	2048	2048		2048	2048	**2048**
m+13	2048	3072	3584	3840	3968	4032	4064	4080	4088	4092	4094	4095	**4096**	4096	4096	4096	4096	4096		4096	4096	**4096**
m+i				⋮																		
2m	$\mathbf{2^{m-2}}$	$\mathbf{2^{m-1} - 2^{m-3}}$						… … … …						$\mathbf{2^{m-1} - 2^7}$	$\mathbf{2^{m-1} - 2^6}$	$\mathbf{2^{m-1} - 2^5}$	$\mathbf{2^{m-1} - 2^4}$	$\mathbf{2^{m-1} - 2^3}$	$\mathbf{2^{m-1} - 2^2}$	$\mathbf{2^{m-1} - 2^1}$	$\mathbf{2^{m-1} - 2^0}$	$\mathbf{2^{m-1}}$

Fig. 5: $m \times m$ Coefficient Matrix for second m terms

Coefficients of x_m of j^{th} term is equal to 2^{i-1}, likewise, x_i has the value of 1 and 2^{i-2} starting from $i \geq 2$. For diagonal values and coefficients on its right have the value as of 2^{i-1} and forms a triangle of such values. On the left-hand side of diagonal, start to change and corresponds to the equation

$$C_{ji} = \sum_{j=0}^{i-1} 2^{m-j-2} = 2^{m-1} - 2^{m-i-1},\ i = 2\ to\ m-1, j = 1\ to\ m$$

-eqn. 11

i	x1	x2	x3	x4	x5	x6	x7	x8	x9	x10
1	1	0	0	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0	0	0
3	0	0	1	0	0	0	0	0	0	0
4	0	0	0	1	0	0	0	0	0	0
5	0	0	0	0	1	0	0	0	0	0
6	0	0	0	0	0	1	0	0	0	0
7	0	0	0	0	0	0	1	0	0	0
8	0	0	0	0	0	0	0	1	0	0
9	0	0	0	0	0	0	0	0	1	0
10	0	0	0	0	0	0	0	0	0	1
11	1	1	1	1	1	1	1	1	1	1
12	1	2	2	2	2	2	2	2	2	2
13	2	3	4	4	4	4	4	4	4	4
14	4	6	7	8	8	8	8	8	8	8
15	8	12	14	15	16	16	16	16	16	16
16	16	24	28	30	31	32	32	32	32	32
17	32	48	56	60	62	63	64	64	64	64
18	64	96	112	120	124	126	127	128	128	128
19	128	192	224	240	248	252	254	255	256	256
20	256	384	448	480	496	504	508	510	511	512
21	512	768	896	960	992	1008	1016	1020	1022	1023
22	1023	1535	1791	1919	1983	2015	2031	2039	2043	2045
23	2045	3068	3580	3836	3964	4028	4060	4076	4084	4088
24	4088	6133	7156	7668	7924	8052	8116	8148	8164	8172
25	8172	12260	14305	15328	15840	16096	16224	16288	16320	16336
26	16336	24508	28596	30641	31664	32176	32432	32560	32624	32656
27	32656	48992	57164	61252	63297	64320	64832	65088	65216	65280
28	65280	97936	114272	122444	126532	128577	129600	130112	130368	130496
29	130496	195776	228432	244768	252940	257028	259073	260096	260608	260864
30	260864	391360	456640	489296	505632	513804	517892	519937	520960	521472
31	521472	782336	912832	978112	1010768	1027104	1035276	1039364	1041409	1042432
32	1042432	1563904	1824768	1955264	2020544	2053200	2069536	2077708	2081796	2083841
33	2083841	3126273	3647745	3908609	4039105	4104385	4137041	4153377	4161549	4165637
34	4165637	6249478	7291910	7813382	8074246	8204742	8270022	8302678	8319014	8327186
35	8327186	12492823	14576664	15619096	16140568	16401432	16531928	16597208	16629864	16646200
36	16646200	24973386	29139023	31222864	32265296	32786768	33047632	33178128	33243408	33276064
37	33276064	49922264	58249450	62415087	64498928	65541360	66062832	66323696	66454192	66519472
38	66519472	99795536	116441736	124768922	128934559	131018400	132060832	132582304	132843168	132973664
39	132973664	199493136	232769200	249415400	257742586	261908223	263992064	265034496	265555968	265816832
40	265816832	398790496	465309968	498586032	515232232	523559418	527725055	529808896	530851328	531372800

Table 8: Coefficient Table for M-nacci for $m = 10$

As seen in figure 5, from left of diagonal, values subtract from 2^{i-1} and a new triangle is formed on left side of diagonal with these values corresponding to above equation. By looking at the trends of Tribonacci and Tetrabonacci sequence with random variables, a more concise definition for coefficients can be created which involves terms of general m-nacci sequence.

$$C_{ji} = \sum_{k=1}^{i} m_{j-k}$$

-eqn. 12

$$C_{j1} = m_{j-1}$$

$$C_{j2} = m_{j-1} + m_{j-2}$$

$$C_{j3} = m_{j-1} + m_{j-2} + m_{j-3}$$

$$C_{j4} = m_{j-1} + m_{j-2} + m_{j-3} + m_{j-4}$$

$$C_{j5} = m_{j-1} + m_{j-2} + m_{j-3} + m_{j-4} + m_{j-5}$$

$$C_{j6} = m_{j-1} + m_{j-2} + m_{j-3} + m_{j-4} + m_{j-5} + m_{j-6}$$

$$\vdots$$

$$C_{j(i-1)} = m_{j-1} + m_{j-2} + m_{j-3} + \cdots\cdots\cdots\cdots\cdots + m_{j-i-2} + m_{j-i-1}$$

$$C_{jm} = m_{j-1} + m_{j-2} + m_{j-3} + \cdots\cdots\cdots + m_{j-k-1} \cdots\cdots\cdots + m_{j-m-2} + m_{j-m-1}$$

This can be further expressed as:

$$C_{ji} = C_{j(i-1)} + m_{j-i} \qquad \text{-eqn. 13}$$

$$C_{jm} = C_{(j+1)1} = m_j \qquad \text{-eqn. 14}$$

Summary

Following is the summary of all the results:

1. The sum of coefficients of any polynomial of Chebyshev T Polynomial of 1st kind results into 1.
2. M-nacci Sequences have Different sequences involved which are part of Chebyshev T Polynomials of 1st kind, and whole m-nacci sequences can be generated by these coefficients when each sequence is taken at regular interval and row-wise added.
3. C_{jm}, Coefficients of x_m, are main m-nacci sequence.
4. M-nacci Sequence Formed using random variables and the coefficients of each variable can be represented in square matrix of each m intervals. These coefficients are based on main M-nacci sequence.
5. Each Column in this m-nacci can be called as special cases of M-nacci with random variables or special case of main m-nacci sequence itself.

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