

I discussed this hw with William Plucknett (wbpluck) and David Abraham (dpabra)

$$a) \ell(\vec{w}) = \sum_{i=1}^N y^{(i)} \log h(\vec{x}^{(i)}) + (1 - y^{(i)}) \log (1 - h(\vec{x}^{(i)}))$$

$$h(\vec{x}) = \sigma(\vec{w}^\top \vec{x}^{(i)}) = \frac{1}{1 + \exp(-\vec{w}^\top \vec{x})} = \sigma^{(i)}$$

$$\begin{aligned} \nabla_{\vec{w}} \ell(\vec{w}) &= \sum_{i=1}^N \nabla_{\vec{w}} y^{(i)} \log h(\vec{x}^{(i)}) + (1 - y^{(i)}) \log (1 - h(\vec{x}^{(i)})) \\ &= \sum_{i=1}^N (y^{(i)} (1 - \sigma^{(i)}) - (1 - y^{(i)}) \sigma^{(i)}) \nabla_{\vec{w}} (\vec{w}^\top \vec{x}^{(i)}) \\ &= \sum_{i=1}^N (y^{(i)} - \sigma^{(i)}) \vec{x}^{(i)} \end{aligned}$$

$$H = \frac{\partial^2 \ell(\vec{w})}{\partial w_i \partial w_j} \Rightarrow \frac{\partial}{\partial \vec{w}} \nabla_{\vec{w}} \ell(\vec{w}) = \frac{\partial}{\partial \vec{w}} \sum_{i=1}^N (y^{(i)} - \sigma^{(i)}) \vec{x}^{(i)}$$

$$= - \sum_{i=1}^N \frac{\partial}{\partial \vec{w}} \sigma^{(i)} \vec{x}^{(i)}$$

$$= - \sum_{i=1}^N \frac{\partial}{\partial \vec{w}} \frac{1}{1 + \exp(\vec{w}^\top \vec{x}^{(i)})} \vec{x}^{(i)}$$

$$= \sum_{i=1}^N \underbrace{-\sigma(\vec{w}^\top \vec{x}^{(i)}) (1 - \sigma(\vec{w}^\top \vec{x}^{(i)}))}_{d} \vec{x}^{(i)} \vec{x}^{(i)}$$

$$H = X^\top D X$$

$$\text{where } D_{ii} = d = -\sigma^{(i)} (1 - \sigma^{(i)})$$

By definition of Gram matrix $X^\top X \succeq 0$

$$\& \because 0 < \sigma^{(i)} < 1 \quad -\sigma^{(i)} < 0 \Rightarrow d < 0$$

$\Rightarrow X^\top D X$ is negative semidefinite $\Rightarrow H$ is concave

For any \vec{z}

$$\vec{z}^\top H \vec{z} = \vec{z}^\top X^\top D X \vec{z}$$

$$= \sum_i \sum_j z_i x_{ji} d x_{ij} z_j$$

$$= \sum_i \sum_j d z_i x_{ji} x_{ij} z_j$$

$$= \sum_i d (\vec{x}^{(i)^\top} \vec{z})^2$$

$$d < 0 \text{ as shown above} \quad \& \quad (\vec{x}^{(i)^\top} \vec{z})^2 \geq 0$$

$$\Rightarrow \sum_i d (\vec{x}^{(i)^\top} \vec{z})^2 \leq 0$$

$$\Rightarrow \vec{z}^\top H \vec{z} \leq 0 \quad \underline{\text{Hence proven!}}$$

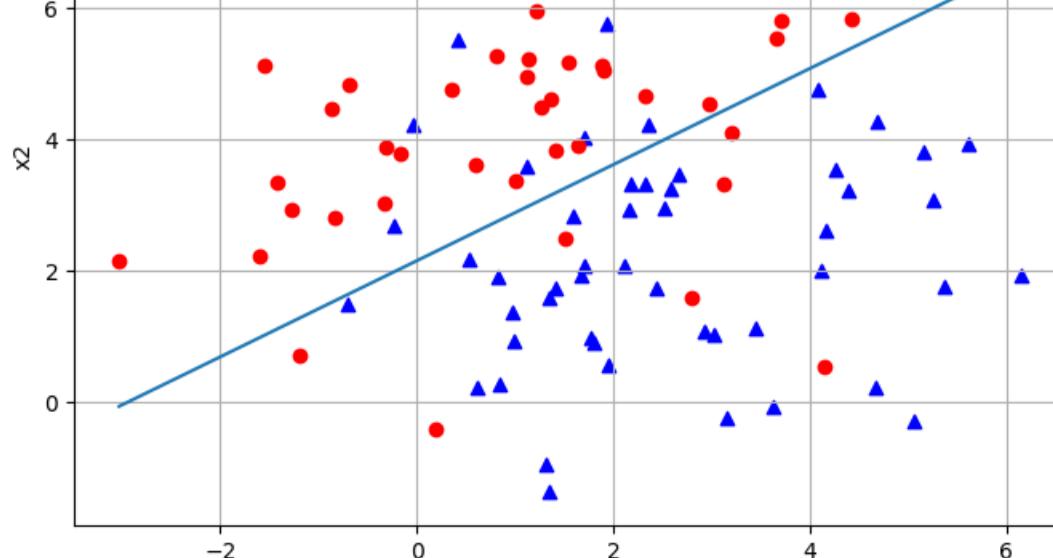
(b) The \vec{w} update under Newton-Raphson is given by $\vec{w} := \vec{w} - H^{-1} \nabla_w E$

$$\text{where } H_{ij}(\vec{w}) = \frac{\partial^2 E(\vec{w})}{\partial w_i \partial w_j}$$

The trained weights from the newton method

$$\mathbf{w} = [[-1.84922892] \\ [-0.62814188] \\ [0.85846843]]$$

True labels and decision boundary



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$$(a) \nabla_{\vec{w}_m} \ell(\vec{w}) = \sum_{i=1}^N \phi(\vec{x}^{(i)}) \left[\mathbb{I}(y^{(i)} = m) - \frac{\exp(\vec{w}_m^\top \phi(\vec{x}^{(i)}))}{1 + \sum_{j=1}^{K-1} \exp(\vec{w}_j^\top \phi(\vec{x}^{(i)}))} \right]$$

$$\ell(\vec{w}) = \sum_{i=1}^N \sum_{k=1}^K \log \left([p(y^{(i)} = k | \vec{x}^{(i)}, \vec{w})]^{1(y^{(i)} = k)} \right)$$

$$\ell(\vec{w}) = \sum_{i=1}^N \sum_{k=1}^K \mathbb{I}(y^{(i)} = k) \log (p(y^{(i)} = k | \vec{x}^{(i)}, \vec{w}))$$

$$\ell(\vec{w}) = \sum_{i=1}^N \sum_{k=1}^{K-1} (\mathbb{I}(y^{(i)} = k) \log [p(y^{(i)} = k | \vec{x}^{(i)}, \vec{w})]) + \mathbb{I}(y^{(i)} = K) \log [p(y^{(i)} = K | \vec{x}^{(i)}, \vec{w})]$$

$$\ell(\vec{w}) = \sum_{i=1}^N \sum_{k=1}^{K-1} \mathbb{I}(y^{(i)} = k) \left[\log [\exp(\vec{w}_k^\top \phi(\vec{x}^{(i)}))] - \log (1 + \sum_{j=1}^{K-1} \exp(\vec{w}_j^\top \phi(\vec{x}^{(i)}))) \right] + \mathbb{I}(y^{(i)} = K) \left[\log (1) - \log (1 + \sum_{j=1}^{K-1} \exp(\vec{w}_j^\top \phi(\vec{x}^{(i)}))) \right]$$

$$\ell(\vec{w}) = \sum_{i=1}^N \left\{ \mathbb{I}(y^{(i)} = 1) \left[\vec{w}_1^\top \phi(\vec{x}^{(i)}) - \log (1 + \sum_{j=1}^{K-1} \exp(\vec{w}_j^\top \phi(\vec{x}^{(i)}))) \right] \right.$$

$$+ \mathbb{I}(y^{(i)} = m) \left[\vec{w}_m^\top \phi(\vec{x}^{(i)}) - \log (1 + \sum_{j=1}^{K-1} \exp(\vec{w}_j^\top \phi(\vec{x}^{(i)}))) \right]$$

$$+ \mathbb{I}(y^{(i)} = K-1) \left[\vec{w}_{K-1}^\top \phi(\vec{x}^{(i)}) - \log (1 + \sum_{j=1}^{K-1} \exp(\vec{w}_j^\top \phi(\vec{x}^{(i)}))) \right]$$

$$+ \mathbb{I}(y^{(i)} = K) \left[-\log (1 + \sum_{j=1}^{K-1} \exp(\vec{w}_j^\top \phi(\vec{x}^{(i)}))) \right] \}$$

using the given hint and splitting into 2 cases

Case I : $y^{(i)} = k = m$

$$\nabla_{\vec{w}_m} \ell(\vec{w}) = \sum_{i=1}^N \left(\mathbb{I}\{y^{(i)} = k = m\} \phi(\vec{x}^{(i)}) - \mathbb{I}\{y^{(i)} = k = m\} \frac{\sum_{j=1}^{K-1} \exp(\vec{w}_j^\top \phi(\vec{x}^{(i)})) \nabla_{\vec{w}_m} (\vec{w}_j^\top \phi(\vec{x}^{(i)}))}{1 + \sum_{j=1}^{K-1} \exp(\vec{w}_j^\top \phi(\vec{x}^{(i)}))} \right)$$

terms become 0 : $\nabla_{\vec{w}_m} (\vec{w}_j^\top \phi(\vec{x}^{(i)})) = 0 \quad \forall j \neq m$

$$= \sum_{i=1}^N \left(\mathbb{I}\{y^{(i)} = m\} \phi(\vec{x}^{(i)}) - \mathbb{I}\{y^{(i)} = m\} \frac{\exp(\vec{w}_m^\top \phi(\vec{x}^{(i)})) \phi(\vec{x}^{(i)})}{1 + \sum_{j=1}^{K-1} \exp(\vec{w}_j^\top \phi(\vec{x}^{(i)}))} \right)$$

Case II : $y^{(i)} = k \neq m$

$$\ell(\vec{w}) = \sum_{i=1}^N \sum_{k: k \neq m} \mathbb{I}\{y^{(i)} = k\} \left[\vec{w}_k^\top \phi(\vec{x}^{(i)}) - \log (1 + \sum_{j=1}^{K-1} \vec{w}_j^\top \phi(\vec{x}^{(i)})) \right]$$

$y^{(i)}$ only has a single value, thus can only equal one k in the inner summation. For this value of k $\mathbb{I}\{y^{(i)} = k\} = 1$

$$\Rightarrow \nabla_{\vec{w}_m} \ell(\vec{w}) = \sum_{i=1}^N \frac{-\exp(\vec{w}_m^\top \phi(\vec{x}^{(i)})) \phi(\vec{x}^{(i)})}{1 + \sum_{j=1}^{K-1} \vec{w}_j^\top \phi(\vec{x}^{(i)})}$$

This shows $\nabla_{\vec{w}_m} \ell(\vec{w}) = \begin{cases} \sum_{i=1}^N \left(\mathbb{I}\{y^{(i)} = m\} \phi(\vec{x}^{(i)}) - \mathbb{I}\{y^{(i)} = m\} p(y^{(i)} = m | \vec{x}^{(i)}, \vec{w}) \phi(\vec{x}^{(i)}) \right) \\ \sum_{i=1}^N -p(y^{(i)} = m | \vec{x}^{(i)}, \vec{w}) \phi(\vec{x}^{(i)}) \end{cases}$

$$\Rightarrow \nabla_{\vec{w}_m} \ell(\vec{w}) = \sum_{i=1}^N \phi(\vec{x}^{(i)}) \left[\mathbb{I}\{y^{(i)} = m\} - p(y^{(i)} = m | \vec{x}^{(i)}, \vec{w}) \right]$$

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$$a) p(y=1 | \vec{x}) = \frac{p(\vec{x} | y=1) p(y=1)}{p(\vec{x} | y=1) p(y=1) + p(\vec{x} | y=0) p(y=0)}$$

$$= \frac{1}{1 + \frac{p(\vec{x} | y=0) p(y=0)}{p(\vec{x} | y=1) p(y=1)}} \quad \text{--- (1)}$$

$$\frac{p(\vec{x} | y=0) p(y=0)}{p(\vec{x} | y=1) p(y=1)} = \frac{e^{-\frac{1}{2} (\vec{x} - \mu_0)^T \Sigma^{-1} (\vec{x} - \mu_0)}}{e^{-\frac{1}{2} (\vec{x} - \mu_1)^T \Sigma^{-1} (\vec{x} - \mu_1)}} \cdot \frac{1 - \phi}{\phi}$$

$$= \left(\frac{1 - \phi}{\phi} \right) \frac{\exp \left(-\frac{1}{2} \vec{x}^T \Sigma^{-1} \vec{x} + \frac{1}{2} \mu_0^T \Sigma^{-1} \vec{x} + \frac{1}{2} \vec{x}^T \Sigma^{-1} \mu_0 - \frac{1}{2} \mu_0^T \Sigma^{-1} \mu_0 \right)}{\exp \left(-\frac{1}{2} \vec{x}^T \Sigma^{-1} \vec{x} + \frac{1}{2} \mu_1^T \Sigma^{-1} \vec{x} + \frac{1}{2} \vec{x}^T \Sigma^{-1} \mu_1 - \frac{1}{2} \mu_1^T \Sigma^{-1} \mu_1 \right)}$$

$$= \frac{(1 - \phi)}{\phi} \exp \left(\frac{1}{2} \mu_0^T \Sigma^{-1} \vec{x} + \frac{1}{2} \vec{x}^T \Sigma^{-1} \mu_0 - \frac{1}{2} \mu_0^T \Sigma^{-1} \mu_0 - \frac{1}{2} \mu_1^T \Sigma^{-1} \vec{x} - \frac{1}{2} \vec{x}^T \Sigma^{-1} \mu_1 + \frac{1}{2} \mu_1^T \Sigma^{-1} \mu_1 \right)$$

$$= \frac{(1 - \phi)}{\phi} \exp \left(\frac{1}{2} \mu_0^T \Sigma^{-1} (\vec{x} - \mu_0) - \frac{1}{2} \mu_1^T \Sigma^{-1} (\vec{x} - \mu_1) + \frac{1}{2} \vec{x}^T \Sigma^{-1} \mu_0 - \frac{1}{2} \vec{x}^T \Sigma^{-1} \mu_1 \right)$$

$$= \frac{(1 - \phi)}{\phi} \exp \left(\frac{1}{2} \mu_0^T \Sigma^{-1} (\vec{x} - \mu_0) - \frac{1}{2} \mu_1^T \Sigma^{-1} (\vec{x} - \mu_1) + \frac{1}{2} \vec{x}^T \Sigma^{-1} (\mu_0 - \mu_1) \right)$$

plugging back in (1)

$$p(y=1 | \vec{x}) = \frac{1}{1 + \frac{(1 - \phi)}{\phi} \exp \left(\frac{1}{2} \mu_0^T \Sigma^{-1} (\vec{x} - \mu_0) - \frac{1}{2} \mu_1^T \Sigma^{-1} (\vec{x} - \mu_1) + \frac{1}{2} \vec{x}^T \Sigma^{-1} (\mu_0 - \mu_1) \right)}$$

$$= \frac{1}{1 + \exp \left(\log \left(\frac{1 - \phi}{\phi} \right) + \frac{1}{2} \mu_0^T \Sigma^{-1} (\vec{x} - \mu_0) - \frac{1}{2} \mu_1^T \Sigma^{-1} (\vec{x} - \mu_1) - \frac{1}{2} \vec{x}^T \underbrace{\Sigma^{-1} (\mu_1 - \mu_0)}_{\vec{w}_\alpha} \right)}$$

$$= \frac{1}{1 + \exp \left(\log \left(\frac{1 - \phi}{\phi} \right) - \frac{1}{2} \mu_0^T \Sigma^{-1} \mu_0 + \frac{1}{2} \mu_1^T \Sigma^{-1} \mu_1 + \frac{1}{2} \mu_0^T \Sigma^{-1} \vec{x} - \frac{1}{2} \mu_1^T \Sigma^{-1} \vec{x} - \frac{1}{2} \vec{x}^T \vec{w}_\alpha \right)}$$

$$= \frac{1}{1 + \exp \left(\log \left(\frac{1 - \phi}{\phi} \right) - \frac{1}{2} \mu_0^T \Sigma^{-1} \mu_0 + \frac{1}{2} \mu_1^T \Sigma^{-1} \mu_1 - \frac{1}{2} (\mu_1^T - \mu_0^T) \Sigma^{-1} \vec{x} - \frac{1}{2} \vec{w}_\alpha^T \vec{x} \right)} \quad \because \vec{x}^T \vec{w}_\alpha \text{ is a scalar} \\ \vec{x}^T \vec{w}_\alpha = \vec{w}_\alpha^T \vec{x}$$

$$\because \Sigma \text{ is a covariance matrix } \Sigma^{-1} = (\Sigma^{-1})^T \\ \Rightarrow (\mu_1^T - \mu_0^T) \Sigma^{-1} = (\Sigma^{-1} (\mu_1 - \mu_0))^T = \vec{w}_\alpha^T$$

$$= \frac{1}{1 + \exp \left(\underbrace{\log \left(\frac{1 - \phi}{\phi} \right) + \frac{1}{2} \mu_0^T \Sigma^{-1} \mu_0 - \frac{1}{2} \mu_1^T \Sigma^{-1} \mu_1}_{w_0} - \frac{1}{2} \underbrace{(\mu_1^T - \mu_0^T) \Sigma^{-1} \vec{x}}_{\vec{w}_\alpha^T \vec{x}} \right)}$$

$$= \frac{1}{1 + \exp(-w_0 - \vec{w}_\alpha^T \vec{x})}$$

$$p(y=1 | \vec{x}) = \frac{1}{1 + \exp(-\vec{w}^T \vec{x})}$$

$$\text{where } \vec{w} = \begin{bmatrix} w_0 \\ \vec{w}_\alpha \end{bmatrix}$$

$$w_0 = \log \left(\frac{\phi}{1 - \phi} \right) + \frac{1}{2} \mu_0^T \Sigma^{-1} \mu_0 - \frac{1}{2} \mu_1^T \Sigma^{-1} \mu_1 \\ \vec{w}_\alpha = \Sigma^{-1} (\mu_1 - \mu_0)$$

& \vec{x} is redefined to be M+1 dimensional vectors
by adding an extra co-ordinate $x_0 = 1$

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3(c) let n_1 be the points where $y^{(i)} = 1$

$$\text{then } l(\phi, \mu_0, \mu_1, \Sigma) = \log \left(\prod_{i=1}^{n_1} p(x^{(i)} | y^{(i)}=1) p(y^{(i)}=1) \mathbb{I}\{y^{(i)}=1\} \right)$$

$$+ \log \left(\prod_{i=n_1+1}^N p(x^{(i)} | y^{(i)}=0) p(y^{(i)}=0) \mathbb{I}\{y^{(i)}=0\} \right)$$

$$= \log \left(\prod_{i=1}^{n_1} p(x^{(i)} | y^{(i)}=1) \phi \mathbb{I}\{y^{(i)}=1\} \right) + \log \left(\prod_{i=n_1+1}^N p(x^{(i)} | y^{(i)}=0) (1-\phi) \mathbb{I}\{y^{(i)}=0\} \right)$$

$$= \log \left(\prod_{i=1}^{n_1} \frac{1}{(2\pi)^{M/2} |\Sigma|^{1/2}} \exp \left(-\frac{1}{2} (x^{(i)} - \mu_1)^\top \Sigma^{-1} (x^{(i)} - \mu_1) \right) \right) + \sum_{i=1}^{n_1} \log (\phi \mathbb{I}\{y^{(i)}=1\})$$

$$+ \sum_{i=n_1+1}^N \log \left(\frac{1}{(2\pi)^{M/2} |\Sigma|^{1/2}} \exp \left(-\frac{1}{2} (x^{(i)} - \mu_0)^\top \Sigma^{-1} (x^{(i)} - \mu_0) \right) \right) + \sum_{i=n_1+1}^N \log ((1-\phi) \mathbb{I}\{y^{(i)}=0\})$$

$$l = \sum_{i=1}^{n_1} \left(-\frac{1}{2} (x^{(i)} - \mu_1)^\top \Sigma^{-1} (x^{(i)} - \mu_1) - \frac{M}{2} \log(2\pi) - \frac{1}{2} \log |\Sigma| \right) + \sum_{i=1}^{n_1} \log (\phi \mathbb{I}\{y^{(i)}=1\})$$

$$+ \sum_{i=n_1+1}^N \left(-\frac{1}{2} (x^{(i)} - \mu_0)^\top \Sigma^{-1} (x^{(i)} - \mu_0) - \frac{M}{2} \log(2\pi) - \frac{1}{2} \log |\Sigma| \right) + \sum_{i=n_1+1}^N \log ((1-\phi) \mathbb{I}\{y^{(i)}=0\})$$

ML estimate ϕ :

$$\frac{\partial l}{\partial \phi} = \sum_{i=1}^{n_1} \frac{\mathbb{I}\{y^{(i)}=1\}}{\phi} - \sum_{i=n_1+1}^N \frac{\mathbb{I}\{y^{(i)}=0\}}{1-\phi}$$

$$\text{set } \frac{\partial l}{\partial \phi} = 0 \Rightarrow O = \sum_{i=1}^{n_1} \frac{(1-\phi) \mathbb{I}\{y^{(i)}=1\}}{\phi(1-\phi)} - \sum_{i=n_1+1}^N \frac{\phi \mathbb{I}\{y^{(i)}=0\}}{\phi(1-\phi)}$$

$$O = \sum_{i=1}^{n_1} (1-\phi) \mathbb{I}\{y^{(i)}=1\} - \sum_{i=n_1+1}^N \phi \mathbb{I}\{y^{(i)}=0\}$$

$$= \sum_{i=1}^{n_1} \mathbb{I}\{y^{(i)}=1\} - \sum_{i=1}^{n_1} \phi \mathbb{I}\{y^{(i)}=1\} - \sum_{i=n_1+1}^N \phi \mathbb{I}\{y^{(i)}=0\}$$

$$\Rightarrow N\phi = \sum_{i=1}^{n_1} \mathbb{I}\{y^{(i)}=1\}$$

$$\Rightarrow \phi = \frac{1}{N} \sum_{i=1}^{n_1} \mathbb{I}\{y^{(i)}=1\}$$

$$= \frac{1}{N} \left(\underbrace{\sum_{i=1}^{n_1} \mathbb{I}\{y^{(i)}=1\}}_{O} + \underbrace{\sum_{i=n_1+1}^N \mathbb{I}\{y^{(i)}=1\}}_{0} \right)$$

we know this term is always 0

$\because y^{(i)}=0$ in n_1+1 to N

$$\Rightarrow \phi = \frac{1}{N} \sum_{i=1}^{n_1} \mathbb{I}\{y^{(i)}=1\}$$

ML estimate μ_1

$$\frac{\partial l}{\partial \mu_1} = \frac{1}{2} \sum_{i=1}^{n_1} 2(x^{(i)} - \mu_1) \Sigma^{-1}, \text{ set } \frac{\partial l}{\partial \mu_1} = 0$$

$$\Rightarrow O = \sum_{i=1}^{n_1} (x^{(i)} - \mu_1) \Sigma^{-1}$$

$$\Rightarrow O = \sum_{i=1}^{n_1} \mathbb{I}\{y^{(i)}=1\} x^{(i)} \Sigma^{-1} - \sum_{i=1}^{n_1} \mathbb{I}\{y^{(i)}=1\} \mu_1 \Sigma^{-1}$$

$$\Rightarrow \mu_1 = \frac{\sum_{i=1}^{n_1} \mathbb{I}\{y^{(i)}=1\} x^{(i)}}{\sum_{i=1}^{n_1} \mathbb{I}\{y^{(i)}=1\}}$$

$$\mu_1 = \frac{\sum_{i=1}^{n_1} \mathbb{I}\{y^{(i)}=1\} x^{(i)}}{\sum_{i=1}^{n_1} \mathbb{I}\{y^{(i)}=1\}}$$

ML estimate for μ_0 : Similar to μ_1 but for $y^{(i)}=0$

$$\Rightarrow \mu_0 = \frac{\sum_{i=1}^N \mathbb{I}\{y^{(i)}=0\} x^{(i)}}{\sum_{i=1}^N \mathbb{I}\{y^{(i)}=0\}}$$

$$\sum_{i=1}^N \nabla_{\Sigma^{-1}} \log |\Sigma| = \sum_{i=1}^N (x^{(i)} - \mu_{y^{(i)}}) (x^{(i)} - \mu_{y^{(i)}})^\top$$

$$\Sigma = \frac{1}{N} \sum_{i=1}^N (x^{(i)} - \mu_{y^{(i)}}) (x^{(i)} - \mu_{y^{(i)}})^\top$$

3(b) As shown in 3(c) the ML estimates work for all values of M including $M=1$.

3(b) is just a special case of 3(c)

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- A) Error: 1.6250%
- B) Top 5 spam words : ['httpaddr' 'spam' 'unsubscrib' 'ebai' 'valet']
- C) Training size: 50
Error: 3.8750%

