

Design optimization.

29/02/2016.

(1)

MAE 598 / 494 Homework 2.

$$1. \text{ maximize } f = x_1 - x_2$$

$$\text{i.e. minimize } f = x_2 - x_1$$

$$\text{subject to: } g_1 = 2x_1 + 3x_2 - 10 \leq 0;$$

$$g_2 = -5x_1 - 2x_2 + 2 \leq 0$$

$$g_3 = -2x_1 + 7x_2 - 8 \leq 0.$$

$$\frac{\partial f}{\partial x_1} = -1, \quad \frac{\partial f}{\partial x_2} = 1, \quad \frac{\partial g_1}{\partial x_1} = 2, \quad \frac{\partial g_1}{\partial x_2} = 3$$

$$\frac{\partial g_2}{\partial x_1} = -5, \quad \frac{\partial g_2}{\partial x_2} = -2, \quad \frac{\partial g_3}{\partial x_1} = -2, \quad \frac{\partial g_3}{\partial x_2} = 7$$

Creating the monotonicity principle 1 table.

	x_1	x_2
f	-1	+1
g_1	+	+

It is observed that,
 g_1 is an active constraint
 for x_1 and g_2 is an
 active constraint for
 x_2 . g_3 is a redundant
 constraint.

As g_1 is active for x_1 , from g_1 ,

$$2x_1 + 3x_2 - 10 = 0$$

$$\therefore 2x_1 = 10 - 3x_2$$

$$x_1 = \frac{10}{2} - \frac{3}{2}x_2 = 5 - \frac{3}{2}x_2$$

Substitute in f and g_2 .

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Maximize $Z = 5x_1 + 3x_2$

Subject to $x_1 + x_2 \leq 7$

$$\begin{aligned} f &= x_1 - 5 + \frac{3}{2}x_2 \\ &= \frac{5}{2}x_2 - 5. = 7 \text{ minimum} \end{aligned}$$

$x_1 \geq 0, x_2 \geq 0$: at boundaries

$$g_2 = -5\left(5 - \frac{3}{2}x_2\right) - 2x_2 + 2 \leq 0.$$

$$\therefore g_2 = -25 + \frac{15}{2}x_2 - 2x_2 + 2 \leq 0$$

$$x_1 = 0, x_2 = 10, l = 70, l^- = 30$$

$$\therefore g_2 = \frac{11}{2}x_2 - 23 \leq 0. \quad x_2 = 2.18$$

$$x_1 = 0, x_2 = 10, l = 70, l^- = 30$$

Apply monotonicity principle 1 again,

eliminating f from constraint

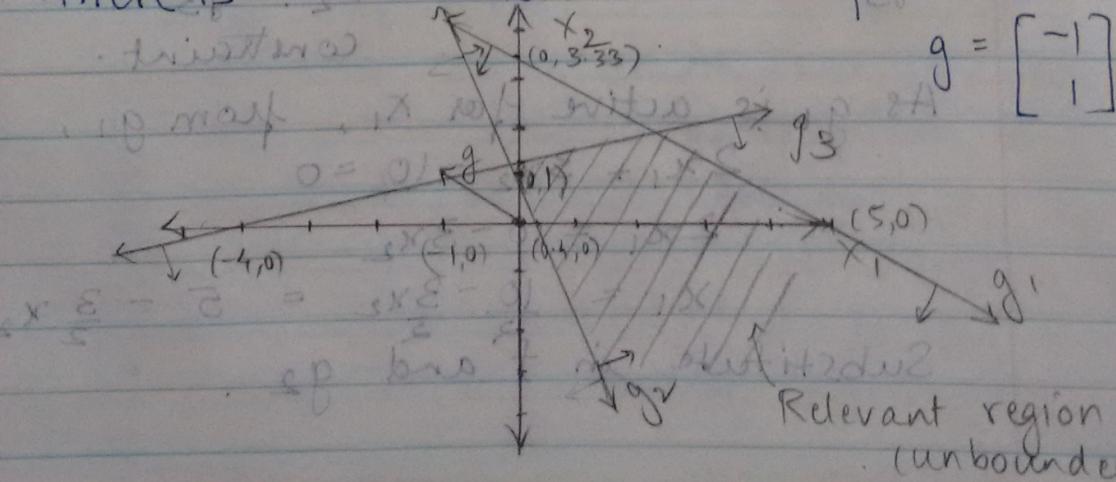
$$g_2 +$$

test boundary 2: $x_1 + 8$

$$\text{at } x_1 = 0, \frac{\partial f}{\partial x_2} = \frac{5}{2}, \quad \frac{\partial g_2}{\partial x_2} = \frac{11}{2}$$

$$\text{at } x_2 = 0, \frac{\partial f}{\partial x_1} = 1, \quad \frac{\partial g_2}{\partial x_1} = 1$$

Thus, x_2 has no active constraint. Hence, there is no solution to this problem.



(2)

As observed from the monotonicity principle 1, to the drawn figure, the optimization problem has no solution as it is not well constrained. The problem is thus not well constrained.

$$0 \Rightarrow e^{x_1} - e^{x_3} = 0$$

2) $x_i, i = 1, 2, 3$.

$$\therefore x_1 > 0, x_2 > 0 \text{ and } x_3 > 0$$

$$\text{i.e. } -x_1 < 0, -x_2 < 0 \text{ and } -x_3 < 0.$$

maximizing $x_1 + x_2 + x_3$

$$\text{i.e. minimize: } -x_1 - x_2 - x_3$$

$$\text{subject to: } g_1 = \exp(x_1) \leq \exp(x_2), \text{ i.e. } e^{x_1} - e^{x_2} \leq 0,$$

$$g_2 = \exp(x_2) \leq \exp(x_3), \text{ i.e. } e^{x_2} - e^{x_3} \leq 0,$$

$$g_3 = x_3 \leq 10, \text{ i.e. } x_3 - 10 \leq 0.$$

$$g_4 = -x_1 < 0$$

$$g_5 = -x_2 < 0$$

$$g_6 = -x_3 < 0$$

Using the monotonicity principle 1 table,

$$x_1 \quad x_2 \quad x_3$$

$$f - \text{ As seen, } g_1 \text{ is the}$$

g_1 , $\textcircled{1}$ $\textcircled{2}$ $\textcircled{3}$ $\textcircled{4}$ $\textcircled{5}$ $\textcircled{6}$ in substitutive constraint

$$g_2 \text{ is } \textcircled{1} \text{ $\textcircled{2}$ } \therefore -01 = \text{for } x_1, \text{ i.e. }$$

$$g_3 \text{ is } \textcircled{1} \text{ $\textcircled{2}$ } \text{ for } x_2, \text{ i.e. } e^{x_1} - e^{x_2} = 0$$

$$g_4 - \text{ is } \textcircled{1} \text{ for } x_3, \text{ i.e. } e^{x_1} = x_3 - \textcircled{1}$$

$$\{01, \textcircled{1} \text{ for } x_1, \text{ i.e. } x_1 = \ln x_2\}$$

$$g_5 -$$

According to monotonicity principle 2,
 x_2 is bounded by g_1 (below) and g_2 (above)
 x_3 is bounded by g_2 (below) and g_3 (above).

i.e. $\min f = -\ln x_2$ (express x_1 in x_2 terms)

$$g_2 = e^{x_2} - x_3 \leq 0$$

$$x_2 \quad . \quad \text{e.g., } 1 = i, j, x \quad (8)$$

$$f = -\ln x_2 \leq 0 \leq x_2 \quad 0 \leq x_2 \quad \therefore$$

$$g_2 = e^{x_2} - x_3 \geq 0 \geq x_2 \quad 0 \geq x_2 \quad \therefore g_2$$

g_2 is active constraint for x_2 .

$$\therefore e^{x_2} - x_3 = 0 \quad \text{minimum g_2}$$

$$e^{x_2} = x_3 \quad \text{from ②, p: at boundary}$$

$$x_2 = \ln x_3 \quad \text{(express f in } x_3 \text{ terms)}$$

$$\min f = -\ln(\ln x_3) \quad \frac{\partial f}{\partial x_3} = -\frac{1}{\ln x_3} \times \frac{1}{x_3}$$

$$\therefore f = -\ln(\ln x_3) = 2.303$$

$$g_3 = e^{x_3} - 10 = 2.303$$

g_3 is active constraint for x_3 .

$$x_3 = 10 \quad - \quad \text{from ③}$$

Back substitute in ② & ①,

$$\therefore x_2 = 2.303 \quad \therefore x_2 = 2.303$$

$$e^{x_1} = x_2 \quad \therefore e^{x_1} = 2.303$$

$$x_1 = x_2 \quad \therefore x_1 = 0.834$$

i) The solution is $\{x_1, x_2, x_3\} = \{0.834, 2.303, 10\}$

(3)

$$3) f = 2x_1^2 - 4x_1x_2 + 1.5x_2^2 + x_2 \text{ i.e.}$$

$$g = \begin{bmatrix} \partial f / \partial x_1 \\ \partial f / \partial x_2 \end{bmatrix} = \begin{bmatrix} 4x_1 - 4x_2 \\ -4x_1 + 3x_2 + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{array}{l} \text{---(1)} \\ \text{---(2)} \end{array}$$

Solve ① & ②.

\therefore the stationary point $[x_1^*, x_2^*] = [1, 1]$

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix}$$

$$|H| = \begin{vmatrix} 4 & -4 \\ -4 & 3 \end{vmatrix} = 12 - 16 = -4 \quad |H| < 0$$

$|H| = \lambda_1, \lambda_2$ (λ_1, λ_2 are eigenvalues of H)

$\therefore \lambda_1 > 0 \& \lambda_2 < 0$

Therefore, the function has positive curvature (for λ_1) in one direction and a negative curvature (for λ_2) in another direction. Hence, the stationary point $(1, 1)$ is a saddle.

$$= 1 + x_1 - x_2 + \dots = (1-x)x - (1-x)s$$

$$= 1 - x - xs \quad \therefore \quad 0 = (1-x) - (1-x)s$$

(8)

(4)

Using Taylor's expansion, \dots (e)

$$f(x) = f(x^*) + g^T \partial x + \frac{1}{2} \partial x^T H_{x^*} \partial x$$

$$\text{For } x = x^*, \dots \quad (8) + (1) \text{ into}$$

$[1, 1] = [x_1, x_2]$ taking parallel set \dots

$$f(x) - f(x^*) = g_{x^*}^T \partial x + \frac{1}{2} \partial x^T H_{x^*} \partial x = 0.$$

$$\text{At } x^*, g^T = [0, 0]$$

$$\therefore \partial x^T H_{x^*} \partial x = 0. \quad \partial x =$$

$$\therefore [\partial x_1, \partial x_2] \begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} \partial x_1 \\ \partial x_2 \end{bmatrix} = 0.$$

$$\therefore \begin{bmatrix} 4\partial x_1 & -4\partial x_2 & -4\partial x_1 + 3\partial x_2 \end{bmatrix} \begin{bmatrix} \partial x_1 \\ \partial x_2 \end{bmatrix} = 0$$

$$\therefore 4\partial x_1^2 - 4\partial x_2 \partial x_1 - 4\partial x_1 \partial x_2 + 3\partial x_2^2 = 0$$

$$\therefore 4\partial x_1^2 - 8\partial x_1 \partial x_2 + 3\partial x_2^2 = 0.$$

$$\text{writing } 4\partial x_1^2 + 6\partial x_1 \partial x_2 + 2\partial x_1 \partial x_2 + 3\partial x_2^2 = 0$$

$$\text{but writing } 2\partial x_1(2\partial x_1 - 3\partial x_2) - 2\partial x_2(2\partial x_1 - 3\partial x_2) = 0$$

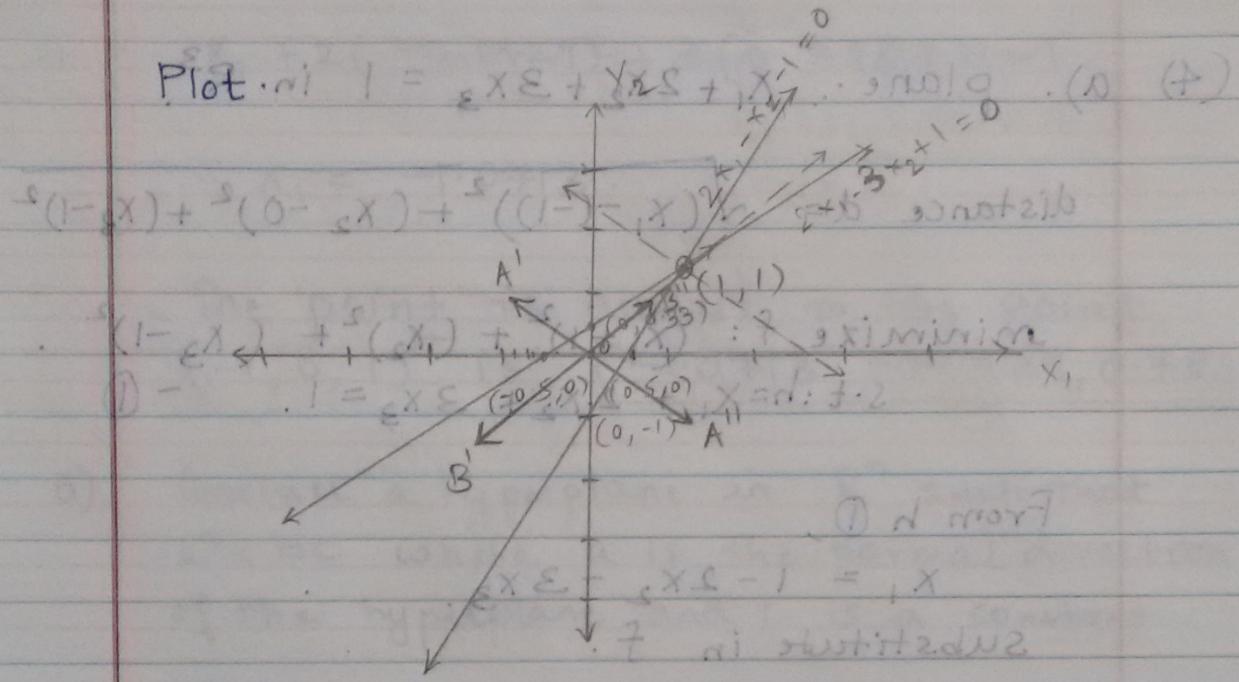
$$\text{writing in } (2\partial x_1 - 3\partial x_2) = 0 \text{ writing } \partial x_1 = x_1 - 1$$

$$\text{writing in } 2\partial x_1 - \partial x_2 = 0 \text{ writing } \partial x_2 = x_2 - 1.$$

$$\therefore 2(x_1 - 1) - 3(x_2 - 1) = 0. \quad \therefore 2x_1 - 3x_2 + 1 = 0$$

$$2(x_1 - 1) - 2(x_2 - 1) = 0. \quad \therefore 2x_1 - x_2 - 1 = 0$$

(4)



Finding eigenvalues and eigenvectors of

$$H = \begin{bmatrix} 4 & -4 \\ -4 & 2 \end{bmatrix}$$

$$\det(H - \lambda I) = \det \begin{bmatrix} 4 - \lambda & -4 \\ -4 & 2 - \lambda \end{bmatrix} = (\lambda - 2)^2 - 16 = \lambda^2 - 4\lambda - 12 = 0$$

$$\lambda_1 = 6, \lambda_2 = -2$$

$$\text{for } \lambda_1 = 6, \text{ eigenvector } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\text{for } \lambda_2 = -2, \text{ eigenvector } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

The eigenvector $B'OB''$ represents the direction that reduces f as the curvature along this vector is negative (-0.5311). Thus,

The function downslopes along $B'OB''$ direction i.e. $[-0.6618, -0.7497]$ eigenvector away from the saddle $[1, 1]$ is substituted.

(4) a). plane: $x_1 + 2x_2 + 3x_3 = 1$ in \mathbb{R}^3

$$\text{distance } d = \sqrt{(x_1 - (-1))^2 + (x_2 - 0)^2 + (x_3 - 1)^2}$$

$$\begin{aligned} & \text{minimize } f: (x_1 + 1)^2 + (x_2)^2 + (x_3 - 1)^2 \\ & \text{s.t.: } h = x_1 + 2x_2 + 3x_3 = 1. \quad -\textcircled{1} \end{aligned}$$

From h $\textcircled{1}$,

$$x_1 = 1 - 2x_2 - 3x_3.$$

Substitute in f .

$$\therefore \min f: [1 - 2x_2 - 3x_3 + 1]^2 + x_2^2 + (x_3 - 1)^2.$$

$$\therefore f: (2 - 2x_2 - 3x_3)^2 + x_2^2 + (x_3 - 1)^2.$$

Thus, the problem is converted to an unconstrained problem.

$$g = \begin{bmatrix} \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 2(2 - 2x_2 - 3x_3)(-2) + 2x_2 \\ 2(2 - 2x_2 - 3x_3)(-3) + 2(x_3 - 1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-8 + 8x_2 + 12x_3 + 2x_2 = 0$$

$$10x_2 + 12x_3 = 8 \quad -\textcircled{1}$$

$$(1) \times 2 + (2) \times 3 \Rightarrow 12x_2 + 18x_3 + 2x_3 - 2 = 0$$

$$12x_2 + 20x_3 = 14 \quad -\textcircled{2}$$

$$x_2 = -0.1429, \quad x_3 = 0.7857$$

Substitute in plane equation

(5)

$$2j) - x_1 + 2(-0.1429) + 3(0.7857) = 1.$$

$$\therefore x_1 = -1.0713.$$

\therefore The point is nearest to the point $(-1, 0, 1)^T$ is $(-1.0713, -0.1429, 0.7857)$.

- 5). Consider a hyperplane in R^n such that $a^T x = c$ where a is the normal direction of the hyperplane and c is a constant.

Now, consider two points in hyperplane x_1, x_2 . They lie in that hyperplane.

Then, $\alpha x_2 + (1-\alpha)x_1$ is any point on the line joining x_1, x_2 such that $0 \leq \alpha \leq 1$.

$$\begin{aligned} \therefore a^T x_1 &= c \quad a^T x_2 = c \\ a^T [\alpha x_2 + (1-\alpha)x_1] &= \alpha a^T x_2 + (1-\alpha)a^T x_1 \\ &= \alpha c + (1-\alpha)c \\ &= c \end{aligned}$$

\therefore All the points on the line $\overrightarrow{x_1 x_2}$ lie in the hyperplane $a^T x = c$ as they fulfill the criteria.

According to the definition of convex set, a set $S \subseteq R^n$ is convex if and for every point x_1, x_2 in S , the point $x(\lambda) = \lambda x_2 + (1-\lambda)x_1$, $0 \leq \lambda \leq 1$ belongs to the set.

②

Therefore, it ~~can't~~ a hyperplane $a^T x = c$ is a convex set.

$$\cdot \mathbf{E}[\mathbf{f}(0,1)] = \mathbf{x} \therefore$$

using set of two \Rightarrow using set:

$$(\mathbf{f}(2,0), \mathbf{f}(1,0), \mathbf{f}(0,1)) \quad \mathbf{z}_i^T (1, 0, 1)$$

test \mathbf{z}_0 in enveloped set (a
with \mathbf{z}_0 linear set \mathbf{z}_i a vector $c = \mathbf{x}^T_0$
- test \mathbf{z}_0 \rightarrow \mathbf{z}_0 enveloped set to

enveloped in strict out obvious, with

enveloped test in gil part. $s\mathbf{x}_1 + t\mathbf{x}_2$

set no triag and \mathbf{z}_i , $\mathbf{x}(x-1) + s\mathbf{x}_0$, with

$t \geq s \geq 0$ test \mathbf{z}_0 $s\mathbf{x}_1 + t\mathbf{x}_2$ strict and

$$c = s\mathbf{x}_1^T + t\mathbf{x}_0^T \therefore c = \mathbf{x}_0^T$$

$$\mathbf{x}_1^T(x-1) + s\mathbf{x}_0^T x_0 = [\mathbf{x}(x-1) + s\mathbf{x}_0]^T$$

$$c(x-1) + c x_0 =$$

$$c =$$

$s\mathbf{x}_1 + t\mathbf{x}_2$ strict \rightarrow line triag set HA:

part \mathbf{z}_0 $c = \mathbf{x}_0^T$ enveloped set in gil

- strict set linearly

- \mathbf{z}_0 vector for nonstrict, \mathbf{z}_0 of gibson

plane rep, with \mathbf{z}_0 vector $\mathbf{z}_1^T \mathbf{R} \mathbf{z}_2 + b$

$\mathbf{z}_0^T = (k)\mathbf{x}$ triag set, 2 in $s\mathbf{x}_1 + t\mathbf{x}_2$ triag

- \mathbf{z}_0 set of isolated $t \geq 0, x(x-1)$