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Design Optimization
Homework 4

1. $\min f(x) = (x_1+1)^2 + (x_2-2)^2$

s.t. $g_1 = x_1 - 2 \leq 0$, $g_3 = -x_1 \leq 0$

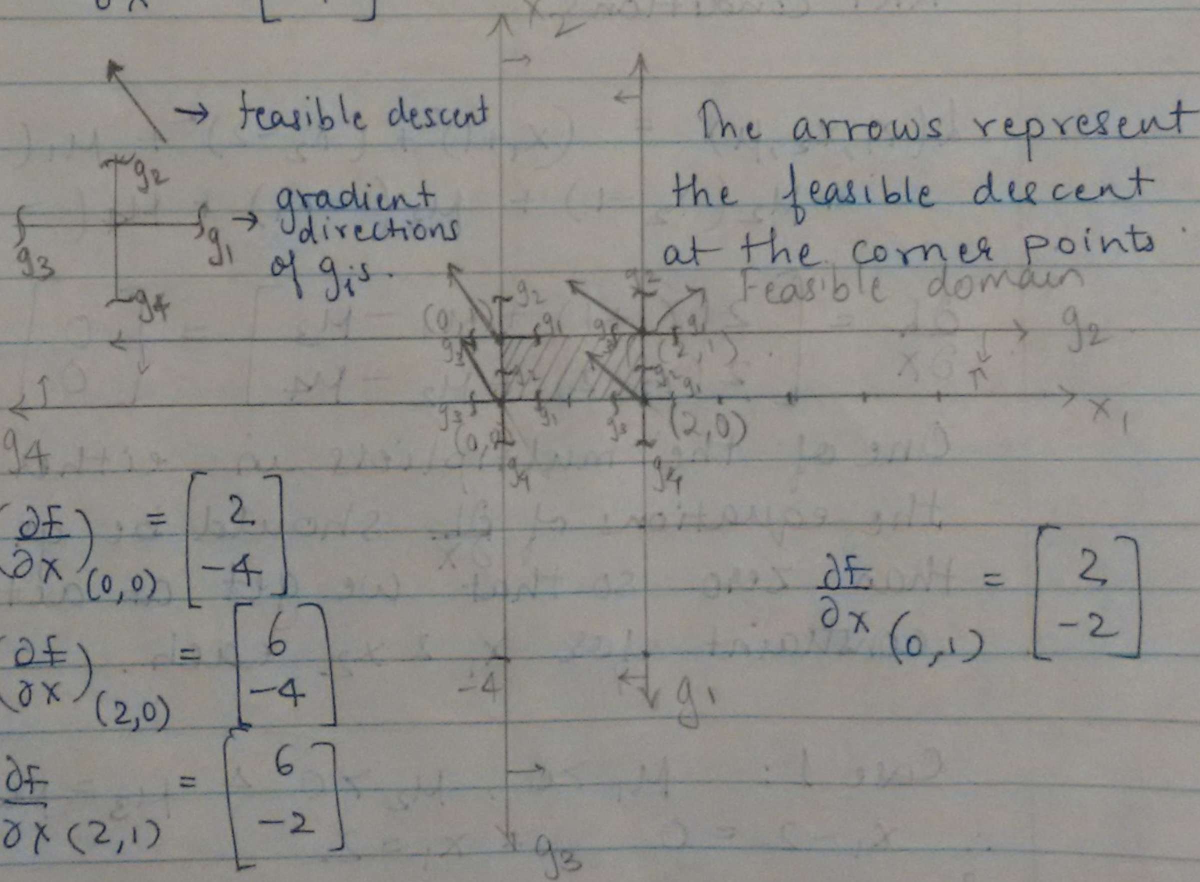
$g_2 = x_2 - 1 \leq 0$, $g_4 = -x_2 \leq 0$.

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 2(x_1+1) \\ 2(x_2-2) \end{bmatrix}$$

$$\frac{\partial g_1}{\partial x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \frac{\partial g_2}{\partial x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \frac{\partial g_3}{\partial x} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\frac{\partial g_4}{\partial x} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

The arrows represent the feasible descent at the corner points.



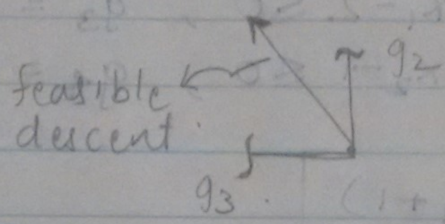
$$\left(\frac{\partial f}{\partial x}\right)_{(0,0)} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

$$\left(\frac{\partial f}{\partial x}\right)_{(2,0)} = \begin{bmatrix} 6 \\ -4 \end{bmatrix}$$

$$\frac{\partial f}{\partial x}_{(2,1)} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$$

$$\frac{\partial f}{\partial x}_{(0,1)} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

As obtained from the graph, g_2 and g_3 constraints are active for the considered feasible descent direction.



$\therefore \; x_2 - 1 = 0$

$\& \; -x_1 = 0$

$\therefore \; x_1 = 0 \; \& \; x_2 = 1$ is the optimum.

$f(0,1) = \underline{2}$

Verification:

KKT conditions.

$$\mathcal{L}(x_1, x_2, \mu) = (x_1+1)^2 + (x_2-2)^2 + \mu_1(x_1-2) + \mu_2(x_2-1) + \mu_3(-x_1) + \mu_4(-x_2)$$

$$\frac{\partial L}{\partial x} = \begin{bmatrix} 2(x_1+1) + \mu_1 - \mu_3 \\ 2(x_2-2) + \mu_2 - \mu_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

One of the multipliers in either of the equations of $\frac{\partial L}{\partial x}$ should be greater than zero so that we get an active constraint for x_1 & x_2 each.

Case 1: $\mu_1 > 0$, $\mu_2 > 0$ & $\mu_3 = \mu_4 = 0$.

$\therefore \; x_1 - 2 = 0 \qquad \therefore \; x_1 = 2$

$\& \; x_2 - 1 = 0 \qquad \therefore \; x_2 = 1$

After substitution, we get

$\mu_1 = -6$ & $\mu_2 = 2$.

This contradicts the assumption of $\mu_1 > 0$.

Thus, case 1 is not feasible.

Case 2: $\mu_1 > 0$, $\mu_4 > 0$, & $\mu_2 = \mu_3 = 0$.

$\therefore$ $x_1 - 2 = 0$ $\therefore x_1 = 2$

$-x_2 = 0$ $\therefore$ $x_2 = 0$

$\therefore$ $\mu_1 = -6$ & $\mu_4 = -4$.

This contradicts the assumption.

Case 3: $\mu_3 > 0$, $\mu_4 > 0$. & $\mu_1 = \mu_2 = 0$

$\therefore$ $-x_1 = 0$ $\therefore$ $x_1 = 0$

$-x_2 = 0$ $\therefore$ $x_2 = 0$

$\therefore$ $\mu_3 = 2$ & $\mu_4 = -4$.

This contradicts the assumption.

Case 4: $\mu_2 > 0$, $\mu_3 > 0$ & $\mu_1 = \mu_4 = 0$

$\therefore$ $x_2 - 1 = 0$ $\therefore$ $x_2 = 1$

$-x_1 = 0$ $\therefore$ $x_1 = 0$.

$\therefore$ $\mu_2 = 2$ & $\mu_3 = 2$. This satisfies the assumption

$\therefore$ $x_1 = 0$ & $x_2 = 1$ is the optimum.

Monotonicity analysis.

$\frac{\partial f}{\partial x_1} = 2(x_1+1)$, $\frac{\partial f}{\partial x_2} = 2(x_2-2)$

$\frac{\partial g_1}{\partial x_1} = 1$, $\frac{\partial g_2}{\partial x_2} = 1$, $\frac{\partial g_3}{\partial x_1} = -1$, $\frac{\partial g_4}{\partial x_2} = -1$

$\therefore$

	x_1	x_2
f	+	−
g_1	+	
g_2		+
g_3	−	
g_4		−

(x_2 is between 0 & 1)

$\therefore$ g_3 is active for x_1 & g_2 is active for x_2

g_3: $-x_1 = 0$ $\therefore$ $x_1 = 0$

g_2: $x_2 - 1 = 0$ $\therefore$ $x_2 = 1$.

$\therefore$ $x_1 = 0$ & $x_2 = 1$ is the optimum.

Hence, the graphical results are analytically verified using KKT conditions and monotonicity analysis.

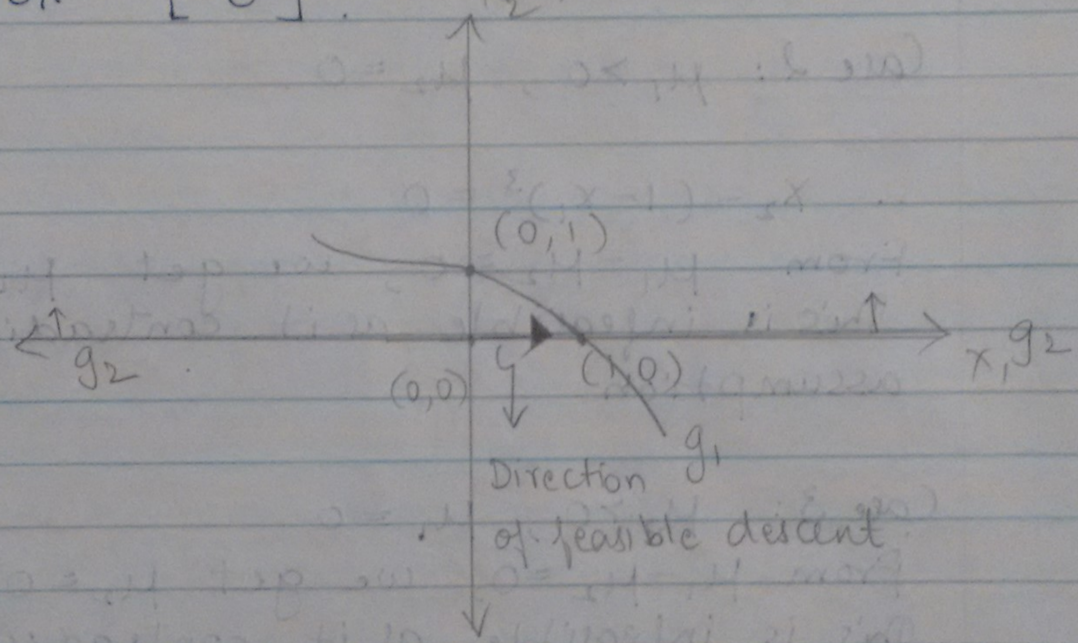
2) $\min f = -x_1$

s.t. $g_1 = x_2 - (1-x_1)^3 \leq 0$ &

$g_2 = x_2 \geq 0$. i.e. $-x_2 \leq 0$.

Plot :

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$



Considering the direction of feasible descent, the minima is obtained at (1, 0). This is observed from the plot.

$f(1,0) = -1$.

Apply the optimality KKT conditions.

$$\mathcal{L}(x_1, x_2, \mu) = -x_1 + \mu_1\left(x_2 - (1-x_1)^3\right) + \mu_2(-x_2)$$

(3)

$$\frac{\partial L}{\partial x} = \begin{bmatrix} -1 + 3\mu_1(1-x_1)^2 \\ \mu_1 - \mu_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Case 1: $\mu_1 = \mu_2 = 0$.

$\therefore -1 + 3\mu_1(1-x_1)^2 = 0$ gives $-1 = 0$

Thus, this is infeasible.

Case 2: $\mu_1 > 0$, $\mu_2 = 0$.

$\therefore x_2 - (1-x_1)^3 = 0$

From $\mu_1 - \mu_2 = 0$, we get $\mu_1 = 0$.

This is infeasible as it contradicts the assumption.

Case 3: $\mu_2 > 0$, $\mu_1 = 0$

From $\mu_1 - \mu_2 = 0$, we get $\mu_2 = 0$.

This is infeasible as it contradicts the assumption.

Case 4: $\mu_2 > 0$, $\mu_1 > 0$.

$\therefore \mu_1 = \mu_2$ (From $\mu_1 - \mu_2 = 0$)

$x_2 = 0$.

$\therefore$ From $x_2 - (1-x_1)^3 = 0$, we get $x_1 = 1$.

Substitute $x_1 = 1$ in $-1 + 3\mu_1(1-x_1)^2 = 0$,

we get $-1 = 0$. Hence, we cannot solve

for μ_1 & μ_2.

Therefore, the KKT optimality conditions cannot be used to solve the given optimization problem, even though the correct values of x_1 & x_2 are obtained at optimum as μ_1 & μ_2 values cannot be found out.

Applying monotonicity rules,

	x_1	x_2
f	$-$	
g_1	$+$	$+$
g_2		$-$

$\frac{\partial f}{\partial x_1} = -1$

$\frac{\partial g_1}{\partial x_1} = 3(1-x_1)^2$

$\frac{\partial g_1}{\partial x_2} = 1$

$\frac{\partial g_2}{\partial x_2} = -1$. $[-x_2 \leq 0]$

g_1 is active for x_1

$\therefore\ x_2 - (1-x_1)^3 = 0$

$\therefore\ -x_1 = (x_2)^{1/3} - 1$

$\therefore$ The problem becomes,

min. $(x_2)^{1/3} - 1$

s.t. $g_2 = -x_2 \leq 0$.

$$\therefore \quad \begin{array}{cc} & x_2 \\ f & + \\ g_2 & - \end{array} \qquad \frac{\partial f}{\partial x_2} = \frac{1}{3}(x_2)^{-2/3}$$

$$\frac{\partial g}{\partial x_2} = -1.$$

g_2 is active for x_2

$\therefore -x_2 = 0 \quad \therefore x_2 = 0$

Substitute in g_1

$\therefore 0 - (1-x_1)^3 = 0$

$\therefore x_1 = 1.$

$\therefore$ The optimum minima obtained is $\underline{(1,0)}$.

Thus, monotonicity analysis helps is determining the minima.

3) $\max f = x_1 x_2 + x_2 x_3 + x_1 x_3$

i.e $\min f = -x_1 x_2 - x_2 x_3 - x_1 x_3$

S.t. $h = x_1 + x_2 + x_3 - 3 = 0$

I) Direct elimination :

From h, $x_3 = 3 - x_1 - x_2$

Substitute in f.

$\therefore f = -x_1 x_2 - x_2(3 - x_1 - x_2) - x_1(3 - x_1 - x_2)$

$\therefore f = -x_1 x_2 - 3x_2 + x_1 x_2 + x_2^2 - 3x_1 + x_1^2 + x_1 x_2$