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$\therefore \min f = -3x_2 + x_2^2 - 3x_1 + x_1^2 + x_1 x_2$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -3 + 2x_1 + x_2 \\ -3 + 2x_2 + x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$\therefore \quad x_1 + 2x_2 = 3$$
$$x_2 + 2x_1 = 3$$

As $x_1 + x_2 + x_3 = 3$,
$\therefore x_1 = 1, \ x_2 = 1, \ x_3 = 1.$
is the minima

$\frac{\partial^2 f}{\partial x^2}$ can be calculated by the Hessian.

$$H = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

$\therefore |H| = 3$

$\therefore$ The eigenvalues of H are 1 & 3.
Therefore, this H is a positive definite matrix which makes the problem strictly convex. Hence, there exists only one minima. Therefore, (1, 1) is the global minima.

II) Reduced gradient.

Number of state variables is 1.

$\therefore \quad s = x_3$. (one constraint).

Number of decision variables is 2

$d_1 = x_1$ & $d_2 = x_2$.

$\therefore$ we have,

$$\frac{\partial z}{\partial d_1} = \frac{\partial f}{\partial d_1} - \frac{\partial f}{\partial s}\left(\frac{\partial h}{\partial s}\right)^{-1}\frac{\partial h}{\partial d_1} = 0. \quad -①$$

$\therefore f = -d_1 d_2 - d_2 s - d_1 s$

$h = d_1 + d_2 + s - 3 = 0$.

$$\therefore \frac{\partial f}{\partial d_1} = -d_2 - s$$

$$\frac{\partial f}{\partial s} = -d_2 - d_1$$

$$\frac{\partial h}{\partial s} = 1, \quad \frac{\partial h}{\partial d_1} = 1.$$

Substitute in ①

$\therefore -d_2 - s - (-d_2 - d_1)(1)(1) = 0$

$\therefore -d_2 - s + d_2 + d_1 = 0$

$\therefore \quad \underline{d_1 = s} \quad -③$

Also,

$$\frac{\partial z}{\partial d_2} = \frac{\partial f}{\partial d_2} - \frac{\partial f}{\partial s}\left(\frac{\partial h}{\partial s}\right)^{-1}\frac{\partial h}{\partial d_2} = 0. \quad -(2)$$

$$\therefore \frac{\partial f}{\partial d_2} = -d_1 - s$$

$$\frac{\partial h}{\partial d_2} = 1.$$

substitute previous and new relations in (2)

$$\therefore \quad -d_1 - s - (-d_2 - d_1)(1)(1) = 0$$

$$\therefore \quad -d_1 - s + d_2 + d_1 = 0.$$

$$\therefore \quad \underline{d_2 = s}. \quad -(4)$$

$\therefore$ we have from (3) & (4),

$$d_1 = d_2 = s.$$

substitute in (h).

$$\therefore \quad d_1 = d_2 = s = 1.$$

i.e $d_1 = 1$, $d_2 = 1$ & $s = 1$ is the global minimum. $(x_1, x_2, x_3) \equiv \underline{(1, 1, 1)}$

III) <u>Lagrange multipliers.</u>

$$L(x_1, x_2, x_3, \lambda) = (-x_1x_2 - x_2x_3 - x_1x_3) +$$

$\lambda(x_1 + x_2 + x_3 - 3)$

$$\frac{\partial L}{\partial x} = \begin{bmatrix} -x_2 - x_3 + \lambda \\ -x_1 - x_3 + \lambda \\ -x_2 - x_1 + \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\frac{\partial L}{\partial \lambda} = x_1 + x_2 + x_3 - 3 = 0.$$

$\therefore x_3 = 3 - x_1 - x_2$ — ④

Substitute in all the equations of

$\frac{\partial L}{\partial x} = 0$

we get,

$$-x_2 + x_1 + x_2 + \lambda - 3 = 0$$
$$\therefore x_1 + \lambda = \cancel{0} + 3 \quad - ①$$
$$-x_1 + x_1 + x_2 + \lambda - 3 = 0$$
$$\therefore x_2 + \lambda = \cancel{0} + 3 \quad - ②$$
$$-x_2 - x_1 + \lambda = 0 \quad - ③$$

Solve ①, ②, ③, we get,

$x_1 = 1, \; x_2 = 1$ & $\lambda = 2$.

From ④,

$\therefore x_3 = 1$

$\therefore (x_1, x_2, x_3) \equiv \underline{(1, 1, 1)}$

The minima values obtained from all the three methods are the same

$f(1,1,1) = -3$ (min)
OR $f(1,1,1) = 3$ (max).
As shown after method I, positive definite condition of Hessian H makes the problem convex and hence, $(1,1,1)$ is the global minimum.

4). $x_1 = 1$ & $x_2 = 2$ is the solution to the following problem:

$$\max f = 2x_1 + bx_2.$$

~~s.t.~~

i.e. $\min f = -2x_1 - bx_2$
s.t. $g_1 = x_1^2 + x_2^2 - 5 \leq 0.$
& $g_2 = x_1 - x_2 - 2 \leq 0.$

The given solution satisfies g_1 constraint only when $g_1 = 0$. It does not satisfy $g_2 = 0$. Therefore, for this problem, only g_1 constraint is active according to monotonicity arguments. Hence, the problem reduces to

$\min f = -2x_1 - bx_2$
s.t. $h = x_1^2 + x_2^2 - 5 = 0.$

Using constrained derivative approach, as there is only one constraint,

$$d = x_1 \ \& \ s = x_2 .$$

$$\therefore \frac{\partial z}{\partial d} = \frac{\partial f}{\partial d} - \frac{\partial f}{\partial s}\left(\frac{\partial h}{\partial s}\right)^{-1}\frac{\partial h}{\partial d} = 0 \quad -(1)$$

∂f

$\min f = -2d - bs$

s.t $h = d^2 + s^2 - 5 = 0$

$$\frac{\partial f}{\partial d} = -2 , \quad \frac{\partial f}{\partial s} = -b ,$$

$$\frac{\partial h}{\partial s} = 2s , \quad \frac{\partial h}{\partial d} = 2d .$$

substitute in (1)

$$\therefore -2 - (-b)(2s)^{-1} 2d = 0 .$$

$$\therefore -2 + \frac{b}{2s} 2d = 0$$

$$\therefore \frac{bd}{s} = 2 .$$

Here, as $x_1 = 1$ & $x_2 = 2$, $d = 1$ & $s = 2$.

$$\therefore \frac{b(1)}{2} = 2 \quad \therefore \underline{b = 4} .$$