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Design optimization.



09/02/2016. MAE 598/494 Homework 2.

1. maximize $f = x_1 - x_2$

i.e. minimize $f = x_2 - x_1$

subject to: $g_1 = 2x_1 + 3x_2 - 10 \leq 0$;

$g_2 = -5x_1 - 2x_2 + 2 \leq 0$

$g_3 = -2x_1 + 7x_2 - 8 \leq 0$.

$\frac{\partial f}{\partial x_1} = -1$, $\frac{\partial f}{\partial x_2} = 1$, $\frac{\partial g_1}{\partial x_1} = 2$, $\frac{\partial g_1}{\partial x_2} = 3$

$\frac{\partial g_2}{\partial x_1} = -5$, $\frac{\partial g_2}{\partial x_2} = -2$, $\frac{\partial g_3}{\partial x_1} = -2$, $\frac{\partial g_3}{\partial x_2} = 7$

Creating the ~~moto~~ monotonicity principle 1 table,

| | $x_1$ | $x_2$ |
|---|---|---|
| $f$ | $-$ | $+$ |
| $g_1$ | $+$ | $+$ |
| $g_2$ | $-$ | $-$ |
| $g_3$ | $-$ | $+$ |

It is observed that, $g_1$ is an active constraint for $x_1$ and $g_2$ is an active constraint for $x_2$. $g_3$ is a redundant constraint.

As $g_1$ is active for $x_1$, from $g_1$,

$$2x_1 + 3x_2 - 10 = 0$$

$$\therefore 2x_1 = 10 - 3x_2$$

$$x_1 = \frac{10}{2} - \frac{3x_2}{2} = 5 - \frac{3}{2}x_2.$$

Substitute in $f$ and $g_2$.

$$\therefore f = x_2 - 5 + \frac{3}{2}x_2$$

$$= \frac{5}{2}x_2 - 5.$$

$$g_2 = -5\left(5 - \frac{3}{2}x_2\right) - 2x_2 + 2 \leq 0.$$

$$\therefore g_2 = -25 + \frac{15}{2}x_2 - 2x_2 + 2 \leq 0$$

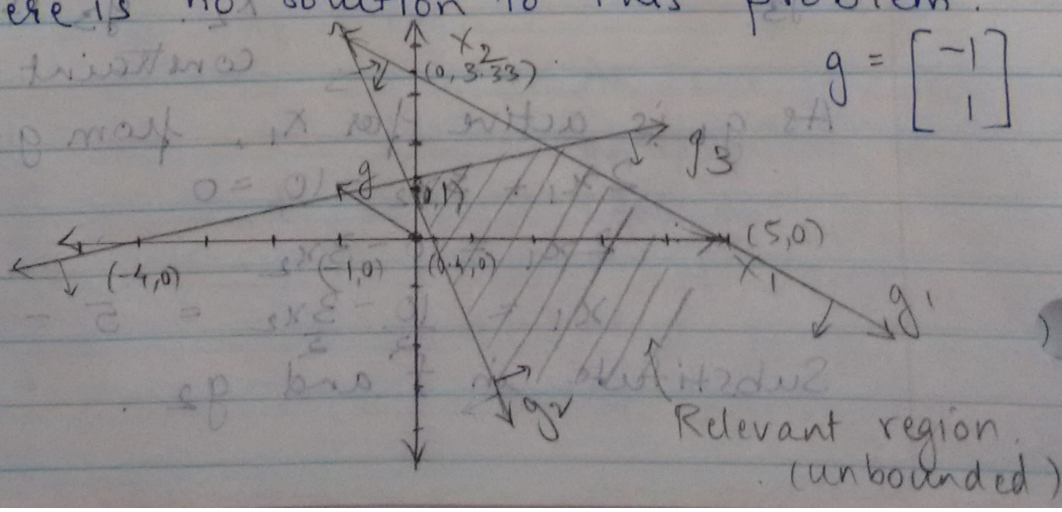
$$\therefore g_2 = \frac{11}{2}x_2 - 23 \leq 0.$$

Apply monotonicity principle 1 again,

| | $x_2$ |
|---|---|
| $f$ | + |
| $g_2$ | + |

$$\frac{\partial f}{\partial x_2} = \frac{5}{2}, \quad \frac{\partial g_2}{\partial x_2} = \frac{11}{2}$$

Thus, $x_2$ has no active constraint. Hence, there is no solution to this problem.



As observed from the monotonicity principle 1, & the drawn figure, the optimization problem has no solution as it is not well constrained. The problem is thus not well constrained.

2) $x_i$, $i = 1, 2, 3$.

$\therefore$ $x_1 > 0$, $x_2 > 0$ & $x_3 > 0$

i.e. $-x_1 < 0$, $-x_2 < 0$ & $-x_3 < 0$.

maximize: $x_1$

i.e. minimize: $-x_1$

subject to: $g_1 = \exp(x_1) \leq x_2$; i.e. $e^{x_1} - x_2 \leq 0$,

$g_2 = \exp(x_2) \leq x_3$; i.e. $e^{x_2} - x_3 \leq 0$,

$g_3 = x_3 \leq 10$. i.e. $x_3 - 10 \leq 0$.

$g_4 = -x_1 < 0$

$g_5 = -x_2 < 0$

$g_6 = -x_3 < 0$.

Creating the monotonicity principle 1 table,

| | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|
| $f$ | $-$ | | |
| $g_1$ | $+$ | $-$ | |
| $g_2$ | | $+$ | $-$ |
| $g_3$ | | | $+$ |
| $g_4$ | $-$ | | |
| $g_5$ | | $-$ | |
| $g_6$ | | | $-$ |

As seen, $g_1$ is the active constraint for $x_1$:

$e^{x_1} - x_2 = 0$

$\therefore e^{x_1} = x_2$ — (1)

$x_1 = \ln x_2$

According to monotonicity principle 2,
$x_2$ is bounded by $g_1$ (below) and $g_2$ (above)
and $x_3$ is bounded by $g_2$ (below) and $g_3$ (above).

i.e. $\therefore$ min $f: -\ln x_2$ (express $f$ ~~$x_1$~~ in $x_2$ terms)

$g_2 = e^{x_2} - x_3 \le 0$

| | $x_2$ |
|---|---|
| $f$ | $-$ |
| $g_2$ | $+$ |

$g_2$ is active constraint for $x_2$.

$\therefore e^{x_2} - x_3 = 0$

$\therefore e^{x_2} = x_3$ — (2)

$\therefore x_2 = \ln x_3$ (express $f$ in $x_3$ terms)

i.e. min $f = -\ln(\ln x_3)$. $\frac{\partial f}{\partial x_3} = \frac{-1}{\ln x_3} \times \frac{1}{x_3}$.

$\therefore$

| | $x_3$ |
|---|---|
| $f$ | $-$ |
| $g_3$ | $+$ |

$g_3$ is active constraint for $x_3$.

$\therefore x_3 = 10$. — (3)

Back substitute in (2) & (1),

$\therefore e^{x_2} = 10 \quad \therefore x_2 = 2.303$.

& $e^{x_1} = x_2 \quad \therefore e^{x_1} = 2.303$

$\therefore x_1 = 0.834$

$\therefore$ The solution is $\{x_1, x_2, x_3\} = \{0.834, 2.303, 10\}$

3) $f = 2x_1^2 - 4x_1x_2 + 1.5x_2^2 + x_2$

$$g = \begin{bmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \end{bmatrix} = \begin{bmatrix} 4x_1 - 4x_2 \\ -4x_1 + 3x_2 + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{matrix} -① \\ -② \end{matrix}$$

Solve ① & ②.

$\therefore$ the stationary point $[x_1^*, x_2^*] = [1, 1]$

$$H = \begin{bmatrix} \partial^2 f/\partial x_1^2 & \partial^2 f/\partial x_1 \partial x_2 \\ \partial^2 f/\partial x_2 \partial x_1 & \partial^2 f/\partial x_2^2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix}$$

$|H| = 12 - 16 = -4$ $\quad |H| < 0.$

$|H| = \lambda_1 \lambda_2$ ($\lambda_1, \lambda_2$ are eigenvalues of $H$)

$(\lambda_1 \lambda_2) < 0.$

$\therefore \lambda_1 > 0$ & $\lambda_2 < 0.$

Therefore, the function has positive curvature (for $\lambda_1$) in one direction and a negative curvature (for $\lambda_2$) in another direction. Hence, the stationary point (1,1) is a saddle.

④

Using Taylor's expansion,

$$f(x) = f(x^*) + g^T_{x^*}\,\partial x + \frac{1}{2}\,\partial x^T H_{x^*}\,\partial x$$

For $x = x^*$,

$$f(x) - f(x^*) = g^T_{x^*}\,\partial x + \frac{1}{2}\,\partial x^T H_{x^*}\,\partial x = 0.$$

At $x^*$, $g^T = [0 \;\; 0]$.

$$\therefore \quad \partial x^T H_{x^*}\,\partial x = 0. \qquad \cancel{\partial x} =$$

$$\therefore [\partial x_1 \;\; \partial x_2]\begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} \partial x_1 \\ \partial x_2 \end{bmatrix} = 0.$$

$$\therefore \begin{bmatrix} 4\partial x_1 - 4\partial x_2 & -4\partial x_1 + 3\partial x_2 \end{bmatrix}\begin{bmatrix} \partial x_1 \\ \partial x_2 \end{bmatrix} = 0$$

$$\therefore 4\partial x_1^2 - 4\partial x_2\partial x_1 - 4\partial x_1\partial x_2 + 3\partial x_2^2 = 0$$

$$\therefore 4\partial x_1^2 - 8\partial x_1\partial x_2 + 3\partial x_2^2 = 0.$$

$$4\partial x_1^2 - 6\partial x_1\partial x_2 - 2\partial x_1\partial x_2 + 3\partial x_2^2 = 0$$

$$\therefore 2\partial x_1(2\partial x_1 - 3\partial x_2) - \partial x_2(2\partial x_1 - 3\partial x_2) = 0$$

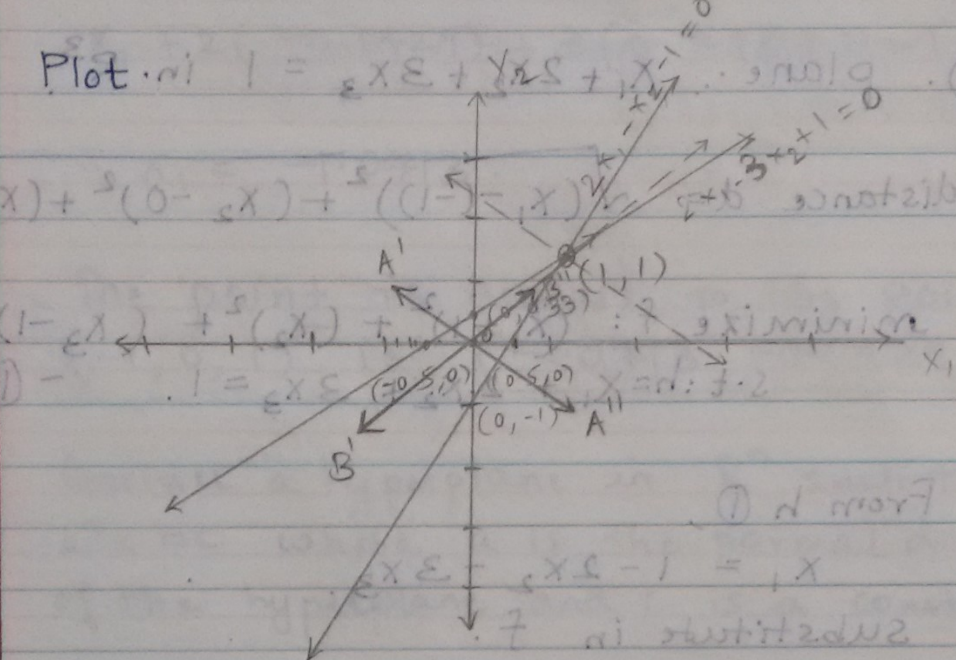
$$\therefore 2\partial x_1 - 3\partial x_2 = 0 \qquad \partial x_1 = x_1 - 1$$

$$2\partial x_1 - \partial x_2 = 0. \qquad \partial x_2 = x_2 - 1.$$

$$\therefore 2(x_1 - 1) - 3(x_2 - 1) = 0. \qquad \therefore \underline{2x_1 - 3x_2 + 1 = 0}$$

$$2(x_1 - 1) - (x_2 - 1) = 0 \qquad \therefore \underline{2x_1 - x_2 - 1 = 0}$$

Plot




Finding eigenvalues and eigen vectors of

$$H = \begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix}$$

$$\lambda_1 = 7.5311, \quad \begin{bmatrix} x_1^1 \\ x_2^1 \end{bmatrix} = \begin{bmatrix} -0.7497 \\ 0.6618 \end{bmatrix}.$$

$$\lambda_2 = -0.5311, \quad \begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix} = \begin{bmatrix} -0.6618 \\ -0.7497 \end{bmatrix}.$$

The eigenvector B'OB'' represents the direction that reduces f as the curvature along this vector is negative (−0.5311). Thus, The function <u>downslopes along B'OB'' direction i.e. [−0.6618 −0.7497] eigenvector away from the saddle</u> [1, 1].

(4) a). plane: $x_1 + 2x_2 + 3x_3 = 1$ in $R^3$.

distance $d = \sqrt{(x_1-(-1))^2 + (x_2-0)^2 + (x_3-1)^2}$

minimize $f: (x_1+1)^2 + (x_2)^2 + (x_3-1)^2$.
s.t: $h = x_1 + 2x_2 + 3x_3 = 1$. — (1)

From h (1),

$x_1 = 1 - 2x_2 - 3x_3$.

substitute in f.

$\therefore$ min $f: \left[1 - 2x_2 - 3x_3 + 1\right]^2 + x_2^2 + (x_3-1)^2$.

$\therefore f: (2 - 2x_2 - 3x_3)^2 + x_2^2 + (x_3-1)^2$.

Thus, the problem is converted to an unconstrained problem.

$$g = \begin{bmatrix} \partial f/\partial x_2 \\ \partial f/\partial x_3 \end{bmatrix} = \begin{bmatrix} 2(2-2x_2-3x_3)(-2) + 2x_2 \\ 2(2-2x_2-3x_3)(-3) + 2(x_3-1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore -8 + 8x_2 + 12x_3 + 2x_2 = 0$

$\therefore 10x_2 + 12x_3 = 8$ — (1).

$-12 + 12x_2 + 18x_3 + 2x_3 - 2 = 0$

$\therefore 12x_2 + \cancel{18}^{\,20} x_3 = 14$. — (2)

$\therefore x_2 = -0.1429$, $x_3 = 0.7857$.

substitute in plane equation

$x_1 + 2(-0.1429) + 3(0.7857) = 1$.

$\therefore x_1 = -1.0713$.

$\therefore$ The point nearest to the point $(-1, 0, 1)^T$ is $\underline{(-1.0713, -0.1429, 0.7857)}$.

5). Consider a hyperplane in $R^n$ such that $a^T x = c$ where $a$ is the normal direction of the hyperplane and $c$ is a constant.

Now, consider two points in hyperplane $x_1$ & $x_2$. They lie in that hyperplane.
Then, $\alpha x_2 + (1-\alpha) x_1$ is any point on the line joining $x_1$ & $x_2$ such that $0 \leq \alpha \leq 1$.

$\therefore a^T x_1 = c$ & $a^T x_2 = c$

$$a^T\left[\alpha x_2 + (1-\alpha)x_1\right] = \alpha a^T x_2 + (1-\alpha) a^T x_1$$
$$= \alpha c + (1-\alpha) c$$
$$= \underline{c}$$

$\therefore$ All the points on the line joining $x_1$ & $x_2$ lie in the hyperplane $a^T x = c$ as they fulfill the criteria.
According to the definition of convex set, a set $S \subseteq R^n$ is convex if, for every point $x_1, x_2$ in $S$, the point $x(\lambda) = \lambda x^2 + (1-\lambda)x_1$, $0 \leq \lambda \leq 1$ belongs to the set.

Therefore, ~~it~~ ~~cant~~ a hyperplane $a^T x = c$ is a convex set.