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Design Optimization
Homework 4

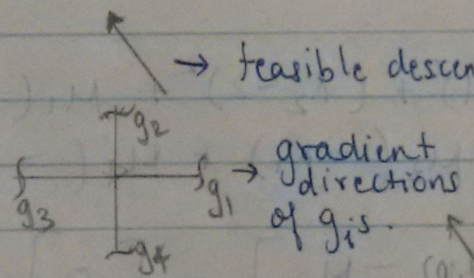
1. $\min f(x) = (x_1+1)^2 + (x_2-2)^2$

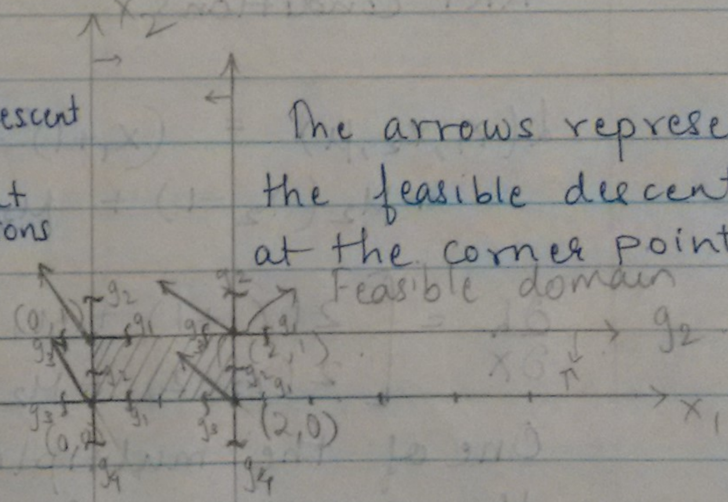
s.t. $g_1 = x_1 - 2 \le 0$, $g_3 = -x_1 \le 0$
$g_2 = x_2 - 1 \le 0$, $g_4 = -x_2 \le 0$.

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 2(x_1+1) \\ 2(x_2-2) \end{bmatrix}$$

$$\frac{\partial g_1}{\partial x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \frac{\partial g_2}{\partial x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \frac{\partial g_3}{\partial x} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\frac{\partial g_4}{\partial x} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$





The arrows represent the feasible descent at the corner points.

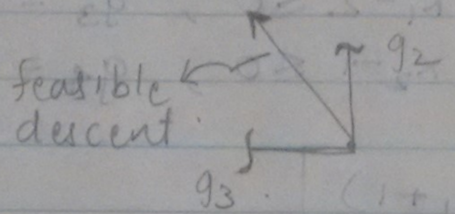
$$\left(\frac{\partial f}{\partial x}\right)_{(0,0)} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

$$\left(\frac{\partial f}{\partial x}\right)_{(2,0)} = \begin{bmatrix} 6 \\ -4 \end{bmatrix}$$

$$\frac{\partial f}{\partial x}_{(2,1)} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$$

$$\frac{\partial f}{\partial x}_{(0,1)} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

As obtained from the graph, g_2 and g_3 constraints are active for the considered feasible descent direction.



$\therefore x_2 - 1 = 0$

& $-x_1 = 0$

$\therefore x_1 = 0$ & $x_2 = 1$ is the optimum.

$f(0,1) = \underline{2}$

Verification:

KKT conditions.

$$\mathcal{L}(x_1, x_2, \mu) = (x_1+1)^2 + (x_2-2)^2 + \mu_1(x_1-2) + \mu_2(x_2-1) + \mu_3(-x_1) + \mu_4(-x_2)$$

$$\frac{\partial L}{\partial x} = \begin{bmatrix} 2(x_1+1) + \mu_1 - \mu_3 \\ 2(x_2-2) + \mu_2 - \mu_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

One of the multipliers in either of the equations of $\frac{\partial L}{\partial x}$ should be greater than zero so that we get an active constraint for x_1 & x_2 each.

Case 1: $\mu_1 > 0$, $\mu_2 > 0$ & $\mu_3 = \mu_4 = 0$.

$\therefore x_1 - 2 = 0 \qquad \therefore x_1 = 2$

& $x_2 - 1 = 0 \qquad \therefore x_2 = 1$

After substitution, we get

$\mu_1 = -6$ & $\mu_2 = 2$.

This contradicts the assumption of $\mu_1 > 0$.
Thus, case 1 is not feasible.

Case 2: $\mu_1 > 0$, $\mu_4 > 0$, & $\mu_2 = \mu_3 = 0$.

$\therefore$ $x_1 - 2 = 0$ $\therefore x_1 = 2$

$-x_2 = 0$ $\therefore$ $x_2 = 0$

$\therefore$ $\mu_1 = -6$ & $\mu_4 = -4$.

This contradicts the assumption.

Case 3: $\mu_3 > 0$, $\mu_4 > 0$. & $\mu_1 = \mu_2 = 0$

$\therefore$ $-x_1 = 0$ $\therefore$ $x_1 = 0$

$-x_2 = 0$ $\therefore$ $x_2 = 0$

$\therefore$ $\mu_3 = 2$ & $\mu_4 = -4$.

This contradicts the assumption.

Case 4: $\mu_2 > 0$, $\mu_3 > 0$ & $\mu_1 = \mu_4 = 0$

$\therefore$ $x_2 - 1 = 0$ $\therefore$ $x_2 = 1$

$-x_1 = 0$ $\therefore$ $x_1 = 0$.

$\therefore$ $\mu_2 = 2$ & $\mu_3 = 2$. This satisfies the assumption

$\therefore$ $x_1 = 0$ & $x_2 = 1$ is the optimum.

Monotonicity analysis.

$\frac{\partial f}{\partial x_1} = 2(x_1+1)$, $\frac{\partial f}{\partial x_2} = 2(x_2-2)$.

$\frac{\partial g_1}{\partial x_1} = 1$, $\frac{\partial g_2}{\partial x_2} = 1$, $\frac{\partial g_3}{\partial x_1} = -1$, $\frac{\partial g_4}{\partial x_2} = -1$

$\therefore$

	x_1	x_2
f	$+$	$-$ (x_2 is between 0 & 1)
g_1	$+$	
g_2		$+$
g_3	$-$	
g_4		$-$

$\therefore$ g_3 is active for x_1 & g_2 is active for x_2

g_3: $-x_1 = 0$ $\therefore$ $x_1 = 0$

g_2: $x_2 - 1 = 0$ $\therefore$ $x_2 = 1$.

$\therefore$ $x_1 = 0$ & $x_2 = 1$ is the optimum.

Hence, the graphical results are analytically verified using KKT conditions and monotonicity analysis.

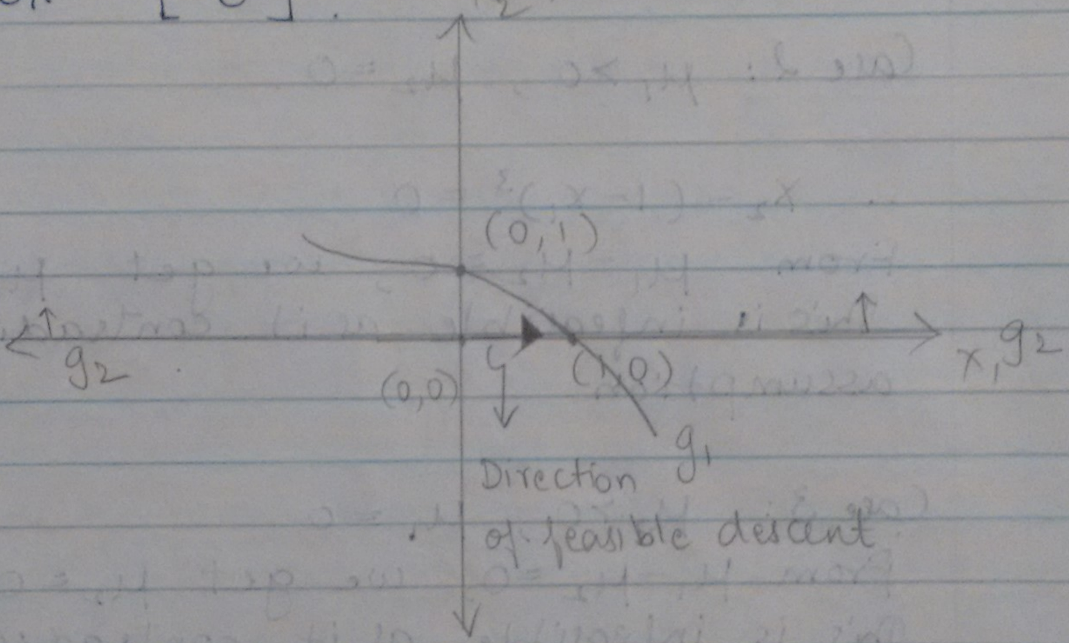
2) $\min f = -x_1$

s.t. $g_1 = x_2 - (1-x_1)^3 \leq 0$ &

$g_2 = x_2 \geq 0$. i.e. $-x_2 \leq 0$.

Plot :

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$



Considering the direction of feasible descent, the minima is obtained at $\underline{(1,0)}$. This is observed from the plot.

$f(1,0) = \underline{-1}$.

Apply the optimality KKT conditions.

$$\mathcal{L}(x_1, x_2, \mu) = -x_1 + \mu_1\left(x_2 - (1-x_1)^3\right) + \mu_2(-x_2)$$

$$\frac{\partial L}{\partial x} = \begin{bmatrix} -1 + 3\mu_1(1-x_1)^2 \\ \mu_1 - \mu_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Case 1: $\mu_1 = \mu_2 = 0$.

$\therefore -1 + 3\mu_1(1-x_1)^2 = 0$ gives $-1 = 0$

Thus, this is infeasible.

Case 2: $\mu_1 > 0$, $\mu_2 = 0$.

$\therefore x_2 - (1-x_1)^3 = 0$

From $\mu_1 - \mu_2 = 0$, we get $\mu_1 = 0$.

This is infeasible as it contradicts the assumption.

Case 3: $\mu_2 > 0$, $\mu_1 = 0$

From $\mu_1 - \mu_2 = 0$, we get $\mu_2 = 0$.

This is infeasible as it contradicts the assumption.

Case 4: $\mu_2 > 0$, $\mu_1 > 0$.

$\therefore \mu_1 = \mu_2$ (From $\mu_1 - \mu_2 = 0$)

$x_2 = 0$.

$\therefore$ From $x_2 - (1-x_1)^3 = 0$, we get $x_1 = 1$.

Substitute $x_1 = 1$ in $-1 + 3\mu_1(1-x_1)^2 = 0$,

we get $-1 = 0$. Hence, we cannot solve

for μ_1 & μ_2.

Therefore, the KKT optimality conditions cannot be used to solve the given optimization problem even though the correct values of x_1 & x_2 are obtained at optimum as μ_1 & μ_2 values cannot be found out.

Applying monotonicity rules,

	x_1	x_2
f	$-$	
g_1	$+$	$+$
g_2		$-$

$\frac{\partial f}{\partial x_1} = -1$

$\frac{\partial g_1}{\partial x_1} = 3(1-x_1)^2$

$\frac{\partial g_1}{\partial x_2} = 1$

$\frac{\partial g_2}{\partial x_2} = -1$. $[-x_2 \leq 0]$

g_1 is active for x_1

$\therefore\ x_2 - (1-x_1)^3 = 0$

$\therefore\ -x_1 = (x_2)^{1/3} - 1$

$\therefore$ The problem becomes,

min. $(x_2)^{1/3} - 1$

s.t. $g_2 = -x_2 \leq 0$.

$$\therefore \quad \begin{array}{c|c} & x_2 \\ f & + \\ g_2 & - \end{array} \qquad \frac{\partial f}{\partial x_2} = \frac{1}{3}(x_2)^{-2/3}$$

$$\frac{\partial g}{\partial x_2} = -1$$

g_2 is active for x_2

$\therefore -x_2 = 0 \quad \therefore x_2 = 0$

Substitute in g_1

$\therefore 0 - (1-x_1)^3 = 0$

$\therefore x_1 = 1.$

$\therefore$ The optimum minima obtained is $\underline{(1,0)}$.

Thus, monotonicity analysis helps in determining the minima.

3) $\max f = x_1x_2 + x_2x_3 + x_1x_3$

i.e $\min f = -x_1x_2 - x_2x_3 - x_1x_3$

s.t. $h = x_1 + x_2 + x_3 - 3 = 0$

I) <u>Direct elimination</u> :

From h, $x_3 = 3 - x_1 - x_2$

Substitute in f.

$\therefore f = -x_1x_2 - x_2(3 - x_1 - x_2) - x_1(3 - x_1 - x_2)$

$\therefore f = -x_1x_2 - 3x_2 + x_1x_2 + x_2^2 - 3x_1 + x_1^2 + x_1x_2$

$\therefore \min f = -3x_2 + x_2^2 - 3x_1 + x_1^2 + x_1 x_2$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -3 + 2x_1 + x_2 \\ -3 + 2x_2 + x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$\therefore \quad x_1 + 2x_2 = 3$$
$$x_2 + 2x_1 = 3$$

As $x_1 + x_2 + x_3 = 3$,
$\therefore x_1 = 1, \; x_2 = 1, \; x_3 = 1.$
is the minima

$\frac{\partial^2 f}{\partial x^2}$ can be calculated by the Hessian.

$$H = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

$\therefore \; |H| = 3$

$\therefore$ The eigenvalues of H are 1 & 3.
Therefore, this H is a positive definite matrix which makes the problem strictly convex. Hence, there exists only one minima. Therefore, (1,1) is the global minima.

II) Reduced gradient.

Number of state variables is 1.

$\therefore \quad s = x_3$. (one constraint).

Number of decision variables is 2

$d_1 = x_1$ & $d_2 = x_2$.

$\therefore$ we have,

$$\frac{\partial z}{\partial d_1} = \frac{\partial f}{\partial d_1} - \frac{\partial f}{\partial s}\left(\frac{\partial h}{\partial s}\right)^{-1}\frac{\partial h}{\partial d_1} = 0. \quad \text{—} ①$$

$$\cancel{\frac{\partial f}{\partial}} \quad \therefore f = -d_1 d_2 - d_2 s - d_1 s$$

$$h = d_1 + d_2 + s - 3 = 0.$$

$$\therefore \frac{\partial f}{\partial d_1} = -d_2 - s$$

$$\frac{\partial f}{\partial s} = -d_2 - d_1$$

$$\frac{\partial h}{\partial s} = 1, \quad \frac{\partial h}{\partial d_1} = 1.$$

Substitute in ①

$$\therefore -d_2 - s - (-d_2 - d_1)(1)(1) = 0$$

$$\therefore -d_2 - s + d_2 + d_1 = 0$$

$$\therefore \underline{d_1 = s} \quad \text{—} ③$$

Also,

$$\frac{\partial z}{\partial d_2} = \frac{\partial f}{\partial d_2} - \frac{\partial f}{\partial s}\left(\frac{\partial h}{\partial s}\right)^{-1}\frac{\partial h}{\partial d_2} = 0. \quad - 2$$

$$\therefore \frac{\partial f}{\partial d_2} = -d_1 - s$$

$$\frac{\partial h}{\partial d_2} = 1.$$

substitute previous and new relations in 2

$$\therefore \quad -d_1 - s - (-d_2 - d_1)(1)(1) = 0$$

$$\therefore \quad -d_1 - s + d_2 + d_1 = 0.$$

$$\therefore \quad \underline{d_2 = s}. \quad - 4$$

$\therefore$ we have from 3 & 4,

$$d_1 = d_2 = s.$$

substitute in h.

$$\therefore \quad d_1 = d_2 = s = 1.$$

i.e $d_1 = 1$, $d_2 = 1$ & $s = 1$ is the global minimum. $(x_1, x_2, x_3) \equiv \underline{(1, 1, 1)}$

III) <u>Lagrange multipliers.</u>

$$L(x_1, x_2, x_3, \lambda) = (-x_1x_2 - x_2x_3 - x_1x_3) +$$

$\lambda(x_1 + x_2 + x_3 - 3)$

$$\frac{\partial L}{\partial X} = \begin{bmatrix} -x_2 - x_3 + \lambda \\ -x_1 - x_3 + \lambda \\ -x_2 - x_1 + \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\frac{\partial L}{\partial \lambda} = x_1 + x_2 + x_3 - 3 = 0.$$

$\therefore x_3 = 3 - x_1 - x_2$ — 4

Substitute in all the equations of

$\frac{\partial L}{\partial X} = 0$

we get,

$-x_2 + x_1 + x_2 + \lambda - 3 = 0$

$\therefore x_1 + \lambda = \cancel{0} + 3$ — 1

$-x_1 + x_1 + x_2 + \lambda - 3 = 0$

$\therefore x_2 + \lambda = \cancel{0} + 3$ — 2

$-x_2 - x_1 + \lambda = 0$ — 3

Solve 1, 2, 3, we get,

$x_1 = 1$, $x_2 = 1$ & $\lambda = 2$.

From 4,

$\therefore x_3 = 1$

$\therefore (x_1, x_2, x_3) \equiv (1, 1, 1)$

The minima values obtained from all the three methods are the same

$f(1,1,1) = -3$ (min)
OR $f(1,1,1) = 3$ (max).
As shown after method I, positive definite condition of Hessian H makes the problem convex and hence, $(1,1,1)$ is the global minimum.

4). $x_1 = 1$ & $x_2 = 2$ is the solution to the following problem:

$$\max f = 2x_1 + bx_2.$$

~~s.t.~~

i.e. $\min f = -2x_1 - bx_2$
s.t. $g_1 = x_1^2 + x_2^2 - 5 \leq 0.$
& $g_2 = x_1 - x_2 - 2 \leq 0.$

The given solution satisfies g_1 constraint only when $g_1 = 0$. It does not satisfy $g_2 = 0$. Therefore, for this problem, only g_1 constraint is active according to monotonicity arguments. Hence, the problem reduces to

$\min f = -2x_1 - bx_2$
s.t. $h = x_1^2 + x_2^2 - 5 = 0.$

Using constrained derivative approach, as there is only one constraint,

$$d = x_1 \;\&\; s = x_2 .$$

$$\therefore \frac{\partial z}{\partial d} = \frac{\partial f}{\partial d} - \frac{\partial f}{\partial s}\left(\frac{\partial h}{\partial s}\right)^{-1}\frac{\partial h}{\partial d} = 0 \quad -①$$

$\cancel{\partial f}$

$\min f = -2d - bs$

s.t $h = d^2 + s^2 - 5 = 0$

$$\frac{\partial f}{\partial d} = -2 , \quad \frac{\partial f}{\partial s} = -b ,$$

$$\frac{\partial h}{\partial s} = 2s , \quad \frac{\partial h}{\partial d} = 2d .$$

substitute in ①

$$\therefore -2 - (-b)(2s)^{-1} 2d = 0 .$$

$$\therefore -2 + \frac{b}{2s} 2d = 0$$

$$\therefore \frac{bd}{s} = 2 .$$

Here, as $x_1 = 1$ & $x_2 = 2$, $d = 1$ & $s = 2$.

$$\therefore \frac{b(1)}{2} = 2 \qquad \therefore \underline{b = 4} .$$

Problem 5
Generalized Reduced Gradient Method

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The given optimization problem with two equality constraints is solved using the GRG method in MATLAB. The initial guess for x1, x2 and x3 is taken to be 0.7, 0.5 and 0.6 respectively. The Levenberg-Marquardt method is implemented to remove singularity observed in the matrices. The stopping criterion tolerance for the reduced gradient is considered to be 0.001. For the given problem, the minimum is obtained at (x1, x2, x3) = (-1.5445, 1.4195, -0.12499) and the corresponding function value is 4.4161 as observed in the following plot where the red star mark denotes the minimum. The function value logarithmic error plot is also shown below.

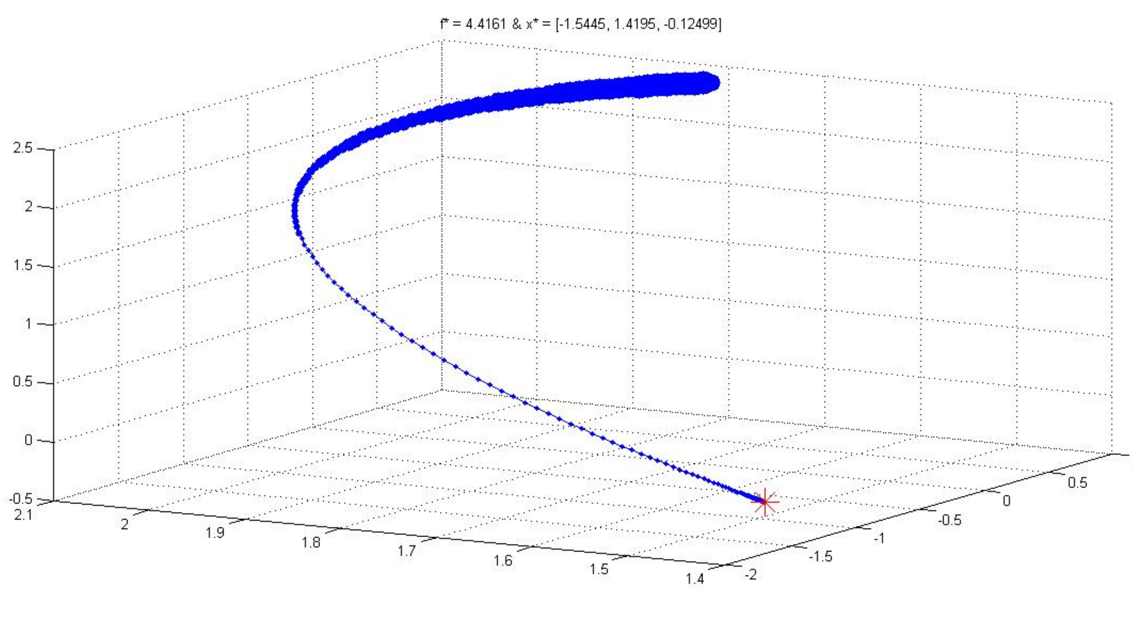

Figure 1: Minimum (red star) obtained from GRG method

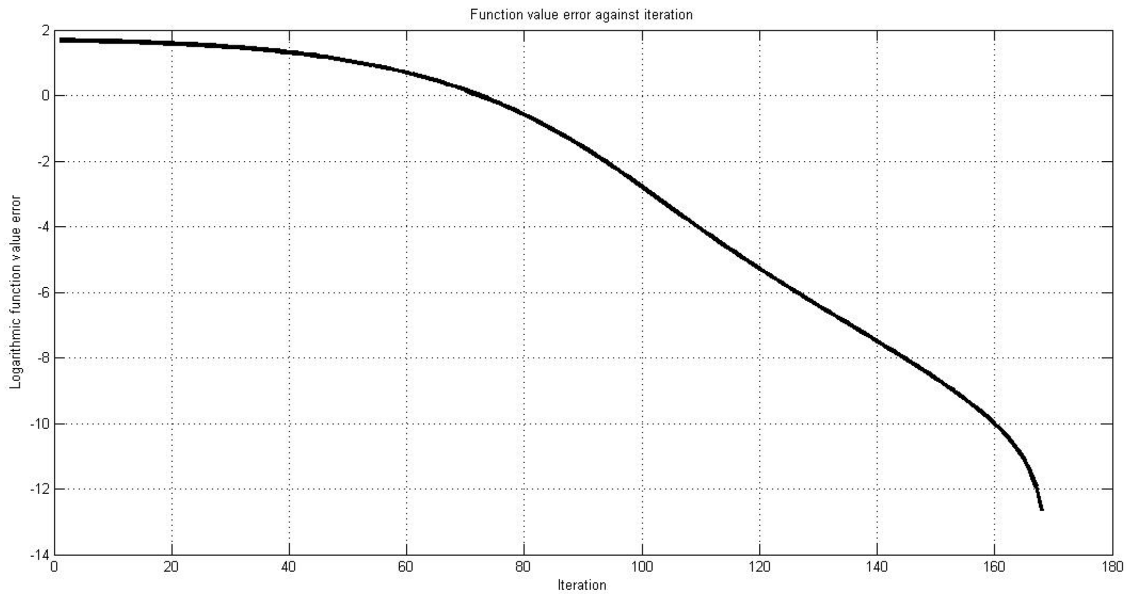

Figure 2: Function error plot

```
%%%%%%%%%%%%%% Generalized Reduced Gradient Implementation %%%%%%%%%%%%%%%
%%%%%%%%%%%%%% By Max Yi Ren and Emrah Bayrak %%%%%%%%%%%%%%%%%%%%%%%%%%%%

function solution = gradient(f, g, h, delh, d0, d_id, s_id, opt)
    % Set initial conditions

    d = d0; % Set current solution to the initial guess
    s0 = [0.5;0.6]; % Set an arbitrary guess for s corresponding to your d0 (column
vector)

    x = zeros(numel(s_id)+numel(d_id),1); % Create a zero column vector for x

    x([d_id, s_id]) = [d;s0]; % Set initial guess to current solution.

    s = solveh(x, h, delh, s_id); % Find the corresponding s to d0 starting from s0

    x([d_id, s_id]) = [d;s]; % Save corrected s and d to current solution.

    % Initialize a structure to record search process
    solution = struct('x',[]);
    solution.x = [solution.x, x]; % save current solution to solution.x

    % Set the termination criterion:
    % Remember that in GRG reduced gradient is used to find stationary points.

    dfdv = g(x);

    dhds = delh(x);

    m = dhds(:,2:3); % current dh/ds

        % Modify dh/ds when it is singular
        %%% KEEP THIS %%%
    dhds_inv = correctH(m);

    delzdeld = dfdv(1) - dfdv(2:3)*dhds_inv*dhds(:,1); % reduced gradient


    gnorm = norm(delzdeld,2); % norm of reduced gradient

    while gnorm>opt.eps % if not terminated

        % opt.linesearch switches line search on or off.
        % You can first set the variable "a" to different constant values and see how
it
        % affects the convergence.
        if opt.linesearch
            a = lineSearch(f, g, delh, x, d_id, s_id);
        else
            a = 0.01;
        end

        % Gradient descent:
        dstep = a*delzdeld;      % step for decision variables (make sure it is column
vector)
```

```
        d = d - dstep;          % update d with dstep
        sstep = dhds_inv*dhds(:,1)*a*delzdeld;      % find approximate step for state
variables (column vector)
        s_approx = s - sstep;   % calculate approximate values for s using the the
approximate step

        x([d_id,s_id]) = [d; s_approx]; % save the decision and approximate state
variables to current solution

        % State variable correction
        s = solveh(x, h, delh, s_id); % Calculate the actual state variables using
linear approximation of h

        x([d_id,s_id]) = [d; s];      % Save the corrected variables to current
solution

        % Update termination criterion:
        dfdv = g(x);

        dhds = delh(x);

        m = dhds(:,2:3); % current dh/ds

            % Modify dh/ds when it is singular
            %%% KEEP THIS %%%
        dhds_inv = correctH(m);

        delzdeld = dfdv(1) - dfdv(2:3)*dhds_inv*dhds(:,1); % reduced gradient


        gnorm = norm(delzdeld,2); % norm of reduced gradient


        % save current solution to solution.x
        solution.x = [solution.x, x];
    end
    %disp(solution.x);
end
```

```
function s = solveh(x, h, delh, s_id)

eps = 1e-8;  % Set a tolerance for convergence
s = x(s_id);     % Save the current state variables

iter = 0;        % Set initial iteration to 0
% Set termination criterion
hnorm = norm(h(x),2); % norm of the constraint vector

    while(hnorm > eps)
        iter = iter+1; % Increase iteration by 1

        dhds = delh(x);
        m = dhds(:,2:3); % current dh/ds

        % Modify dh/ds when it is singular
        %%% KEEP THIS %%%
        dhds_inv = correctH(m);
        %%%%%%%%%%%%%%%%%

        s = s - dhds_inv*h(x); % Use modified dh/ds to calculate new s
        x(s_id) = s;         % Save new s to the current solution

        hnorm = norm(h(x),2); % Update termination critetion
    end
end
```