Problem Set 1

Question 1 Let m and n be positive integers with no common factor. Prove that if $\sqrt{m/n}$ is rational, then m and n are both perfect squares, that is to say there exists integers p and q such that $m = p^2$ and $n = q^2$.

Proof. We are given that $\sqrt{m/n}$ is rational, meaning $\sqrt{m/n} = p/q$ where p, q are uniquely determined as the irreducible rational equal to $\sqrt{m/n}$. Squaring both sides gives:

$$\left(\sqrt{\frac{m}{n}}\right)^2 = \left(\frac{m}{n}\right)^2 \implies \frac{m}{n} = \frac{p^2}{q^2}$$

Unfortunately, this is not enough to say that $m = p^2$ and $n = q^2$. We have to reason that p^2/q^2 is also irreducible. Notice that due to the fact that we can factor integers into unqiue primes, we have $p = p_1 p_2 \cdots p_n$ and $q = q_1 q_2 \cdots q_n$. Let the set of primes that compose each number be P and Q. Since P and Q have no common factors, $P \cap Q = \emptyset$. Squaring either of these numbers yields these prime factors being seen twice in the lists PQ and QS, but $PS \cap QS = \emptyset$. This helps to conclude that p^2/q^2 is irreducible and therefore $m = p^2$ and $n = q^2$. ■

Question 2 Prove that no order can be defined in the complex field that turns it into an ordered field

Proof. Let there exist an ordering on the complex field. Since all squares are positive, $i^2 = -1 \geq 0$. This means we can say the following:

$$1 = 0 + 1 \leq -1 + 1 = 0 \leq 1$$

This says that $1 \leq 0$ and also $1 \geq 0$ which means $1 = 0$, so we have reached a contradiction and an ordering cannot exist. ■

Question 3 Suppose $z = a + bi, w = c + di$. Define $z < w$ is $a < c$, and also if $a = c$ but $b < d$. Prove that this turns the set of all complex numbers into an ordered set. Does this ordered set have the least-upper-bound property?

Proof. To show that this is an ordered set, we need to show that any distinct numbers are either greater or less than the other. If we take $z = a + bi$ and $w = c + di$ where $z \neq w$ then this means either $a > c$ or $a < c$ if $a \neq c$. Further, if $a = c$, then $b \neq d$ meaning either $b > d$ or $b < d$. Either way, $z \neq w \implies z > w$ or $z < w$.

Further, this is transitive as well since if $x = a+bi, y = c+di, z = e+fi$ and $x < y$ and $y < z$, then $x < z$ since if $x < y$, it means $a \leq c$ and $c \leq e$ so $a \leq e$. In the case $a = e$, then $a = c$ meaning that $b < d$ and $d < e$ meaning $b < e$. In either case, it means that $x < z$. Hence, it means $\mathbb{C}$ is an ordered set.

For the least-upper-bound property, we can consider an imaginary line with $a = 0$ and notice that this is bounded, but there exists no least upper bound since for any $a > 0$, there is always a value a closer to 0 then what we think is the least upper bound. ■

Question 4 Let $\mathbb{R}$ be the set of real numbers and suppose $f : \mathbb{R} \to \mathbb{R}$ is a function such that for all real numbers x and y the following two equations hold:

$$f(x + y) = f(x) + f(y)$$

$$f(xy) = f(x)f(y)$$

(a) Prove that $f(0) = 0$ and that $f(1) \in \{0, 1\}$

Proof. $f(0) = 0$ as $f(0) = f(0 + 0) = f(0) = f(0) = 2f(0)$ which is only possible if $f(0) = 0$. $f(1)$ is either 0 or 1 since $f(1) = f(1)f(1) = f(1)^2$ and $x = x^2$ is only true when $x = 0, 1$ so $f(1) \in \{0, 1\}$ ■

(b) Prove that $f(n) = nf(1)$ for every integer n and that $f(n/m) = (n/m)f(1)$ for all integers n, m such that $m \neq 0$. Conclude that either $f(q) = 0$ for all rational numbers q or $f(q) = q$ for all rational numbers q.

Proof. Notice that $f(n) = f(1 + 1 + \cdots + 1) = f(1) + f(1) + \cdots + f(1) = nf(1)$. Now, assume that $f(n/m) = (n/m)f(1)$. Multiply both sides by $mf(1) = f(m)$ like so: $f(n/m)mf(1) = (n/m)f(1)mf(1)$ and remember that $f(1)^2 = f(1)$ since $f(1) = 0$ or 1. Leaving $f(n/m)mf(1) = nf(1) \implies f(n/m)f(m) = nf(1)$. Since $f(xy) = f(x)f(y)$, this means $f(n/m)f(m) = f(n)$ on the left hand side and the right hand side is $f(n)$ and indeed, $f(n) = f(n)$ so there is no contradiction. ■

(c) Prove that f is nondecreasing, that is to say that $f(x) \geq f(y)$ whenever $x \geq y$ for any $x, y \in \mathbb{R}$

Proof. Notice that every positive number has a real number square root so we can say that $\forall x \geq 0\ \exists y : x = y^2$. Also, notice that $f(x) + f(-x) = f(0) = 0$ so $f(x) = f(-x)$. Since we are given that $x \geq y$, then $x - y \geq 0$, then $f(x) - f(y) = f(x) + f(-y) = f(x - y)$ and since $x - y \geq 0$, then $f(x - y) \geq 0$ meaning $f(x) - f(y) \geq 0$ so $f(x) \geq f(y)$. ■

(d) Prove that if $f(1) = 0$ then $f(x) = 0$ for all real numbers x. Prove that if $f(1) = 1$ then $f(x) = x$ for all real numbers x.

Proof. In the case that $f(1) = 0$ then $f(x) = 0$ for all rational numbers as $f(n/m) = (n/m)f(1) = 0$. Notice any real number is neighbored by two real numbers infinitely close, hence $\exists p, q \in \mathbb{Q}$ such that $p \leq x \leq q$ (since f is strictly non-decreasing). Since $f(p) = 0$ and $f(q) = 0$, then $0 \leq f(x) \leq q$ leaves $f(x) = 0$.

When $f(1) = 1$, then $f(x) = x$ for all rational numbers as $f(n/m) = (n/m)$. There exist $p, q \in Q$ that are infinitely close to x that are between $x \pm \frac{1}{n}$ and x $(p \leq q)$. As $n \to \infty$, this means that p, q converges to x. Hence, $f(x) = x$. ■