CS 3510 — Algorithms, Spring 2022

Homework 1A

Big-O

For the following list of functions, cluster the functions of the same order into one group, and then rank the groups in increasing order.

- (a) $n\log(n)$
- (b) $n^{1.01}$
- (c) $n\sqrt[3]{n}$
- (d) $2^{\log_3(n)}$
- (e) $n + \log(n)$
- (f) $2^{\log(\log(n))}$
- (g) $10n\log(10n) + 10$
- (h) $(\log(n))^{10}$
- (i) $\log(n^{10})$
- (j) $42n$

Solution: For ease of ordering, manipulate each function like so:

- (a) $n\log(n)$
- (b) $n^{1.01}$
- (c) $n\sqrt[3]{n} \to n^{4/3}$
- (d) $2^{\log_3(n)} \to n^{\log_2 3}$
- (e) $n + \log(n) \to n$
- (f) $2^{\log(\log(n))} \to \log n$
- (g) $10n\log(10n) + 10 \to 10n(\log(n) + \log(10)) \to n\log(n)$
- (h) $(\log(n))^{10} \to 2^{\log\log^{10}(n)}$ (manipulate other functions to compare exponents)
- (i) $\log(n^{10}) \to 10\log(n) = \log(n)$
- (j) $42n \to n$

Hence, we can group $\{f, i\} < \{h\} < \{d\} < \{e, j\}, \{a, g\} < \{b\} < \{c\}$.

Slow Multiplication
Consider the following multiplication algorithm:

```
def slowmult(x,y):
    result = 0
    for i from 1 to x (inclusive):
        result += y
    return result
```

Assume x and y are n-bit numbers. What is the running time of this algorithm in terms of n?

Solution: The runtime of this algorithm is $O(n2^n)$. This is because the for loop iterates a total of x times, and x has n bits meaning it can represent a number as large as 2^n meaning $x = O(2^n)$.

Within each iteration, we are adding $result$ and y together, which costs $O(n)$ time per iteration, giving a total runtime of $O(2^n) \times O(n) = O(n2^n)$.

Fast Multiplication

Let x and y be two n-bit numbers where n is divisible by 3. Let x_L, x_M, x_R consist of the first, middle, and last third of the bits in x and define y_L, y_M, y_R similarly.

(a) Express xy in terms of $x_L, x_M, x_R, y_L, x_M, y_R$.

Solution: See the following expansion and simplification:

$$\begin{aligned}
xy &= (2^{\frac{2n}{3}}x_L + 2^{\frac{n}{3}}x_M + x_R)(2^{\frac{2n}{3}}y_L + 2^{\frac{n}{3}}y_M + y_R) \\
&= \quad 2^{\frac{4n}{3}}x_Ly_L + 2^{\frac{3n}{3}}x_Ly_M + 2^{\frac{2n}{3}}x_Ly_R \\
&\quad + 2^{\frac{3n}{3}}x_My_L + 2^{\frac{2n}{3}}x_My_M + 2^{\frac{n}{3}}x_My_R \\
&\quad + 2^{\frac{2n}{3}}x_Ry_L + 2^{\frac{n}{3}}x_Ry_M + x_Ry_R \\
&= 2^{\frac{4n}{3}}x_Ly_L + 2^n(x_Ly_M + x_My_L) + 2^{\frac{2n}{3}}(x_Ly_R + x_My_M + x_Ry_L) \\
&\quad + 2^{\frac{n}{3}}(x_My_R + x_Ry_M) + x_Ry_R
\end{aligned}$$

(b) Give a recursive algorithm that calculates the above polynomial with a recurrence relation of:

$$T(n) = 6T\left(\frac{n}{3}\right) + O(n)$$

Solution: In the above algorith, we are required to make 9 recursive calls which would give an incorrect runtime. Through algebraic manipulation, we can reduce the total number of multiplications.

To design our algorithm, first create a base case table when multiplying 1-bit numbers (using Piazza clarification that n is a power of 3). This table is $0 \times 1, 1 \times 0, 0 \times 0, 1 \times 1$ which gives $O(1)$ resolution of base cases.

Further, we compute 3 multiplications recursively:

$$A = x_Ly_L \qquad B = x_My_M \qquad C = x_Ry_R$$

We also compute three more terms recursively given by:

$$\begin{aligned}
D &= (x_L + x_M)(y_L + y_M) = (x_L y_L + x_L y_M + x_M y_L + x_M y_M) \\
E &= (x_L + x_R)(y_L + y_R) = (x_L y_L + x_R y_L + x_R y_R) \\
F &= (x_M + x_R)(y_M + y_R) = (x_M y_M + x_M y_R + x_R y_M + x_R y_R)
\end{aligned}$$

None of these terms are immediately present in our polynomial, but by adding/removing copies of A, B, C, we can reconstruct the polynomial with only these 6 recursive calls.

$$\begin{aligned}
D - A - B &= (x_L + x_M)(y_L + y_M) - x_L y_L - x_M y_M = x_L y_M + x_M y_L \\
B + E - A - C &= x_M y_M + (x_L + x_R)(y_L + y_R) - x_L y_L - x_R y_R \\
&= x_M y_M + x_L y_R + x_R y_L \\
F - B - C &= (x_M + x_R)(y_M + y_R) - x_M y_M - x_R y_R = x_M y_R + x_R y_M
\end{aligned}$$

These three terms ARE present in the polynomial and only require a total of 6 recursive multiplications to construct. Hence, our polynomial can be written as:

$$= 2^{\frac{4n}{3}} A + 2^n (D - A - B) + 2^{\frac{2n}{3}} (B + E - A - C) + 2^{\frac{n}{3}} (F - B - C) + C$$

Now, we can describe the entire algorithm. The algorithm takes an input of bits for x and y and checks them against a 1 bit base case table (returning the base case). Otherwise, we split x and y into three parts by splicing the bits into 3 partitions (left, right, and middle).

Using the calls from above, we recursively call our algorithm to construct A, B, C, D, E, F and return the reconstructed polynomial with the equation given above.

The runtime of this is given by 6 recursive calls, all of which have size of $n/3$, and $O(n)$ non recursive work (bit partitioning and polynomial reconstruction). All in all, this combines to give the target runtime recurrence relation of $T(n) = 6T(n/3) + O(n)$.