CS 3510 — Algorithms, Spring 2022

Homework 4B Solutions

1 Subset-Sum Pair

Theorem. *Subset-Sum Pair is NP-Complete.*

Proof. In order to show Subset-Sum Pair is NP-Complete, we must show that it is NP and NP-Hard. In order to show it is NP, we build a verifier for it. For some candidate solution, we can simply take the tuple-sum and verify it is equal to (x, y). This takes $O(n)$ time and hence verifying takes polynomial time.

Next, we must show Subset-Sum Pair is NP-Hard. We reduce from Subset-Sum. For some input of Subset-Sum a_i, g where s_i represents the ith element of the initial set and g being the goal. We reduce this to the inputs of Subset-Sum Pair by setting $s_i = a_i$, $t_i = 0$ and $x = g, y = 0$. This is simply making a new list with n elements which takes $O(n)$ time. Now, we must prove the reduction.

Proof. Subset-Sum has a solution if and only if Subset-Sum Pair has a solution.

($\implies$) If Subset-Sum has a solution, it must be the case that $\sum a_i = g$. Meanwhile, Subset-Sum Pair only has a solution when $\sum s_i = x$ and $\sum t_i = y$. Since we set $s_i = a_i$ and $x = g$, then $\sum s_i = x$ must be true as $\sum a_i = g$. Further, it is always true that $\sum t_i = y$ as $\sum 0 = 0$ is always true.

($\impliedby$) If Subset-Sum Pair has a solution, it must be the case that $\sum s_i = x$ and $\sum t_i = y$. Since we set $s_i = a_i$ and $x = g$, this implies $\sum a_i = g$. Hence, Subset-Sum Pair having a solution implies Subset-Sum pair has a solution.

Hence, since we have this is NP and NP-Hard, Subset-Sum Pair is NP-Complete. $\square$

2 ColoredPath

Theorem 2.1. *ColoredPathis NP-Complete.*

Proof. In order to show ColoredPath is NP-Complete, we must show that it is NP and NP-Hard. In order to show it is NP, we build a verifier for it. For some candidate solution, we want to verify it is in fact a colored path that is at most k length and traverses at least m colors. The candidate solution is a path, so we traverse the path incrementing a counter when we see a new color and increment a length counter for each new vertex traversed. We return true if the color counter is $\geq m$ and the length counter is $\leq k$.

Now, we must show ColoredPath is NP-Hard. We reduce from HamPath. For some input graph G of HamPath, we construct G' where each vertex is uniquely colored. We then set $m = k = |V|$. *Remark: for the sake of simplicity, here we assume that k refers to path vertex length, which is different than path edge length (by 1).*

Assigning each color a unique color takes $O(|V|)$ time, hence this reduction works in polynomial time. Now, we must prove the reduction.

Proof. G has HamPath if and only if G' has a ColoredPath with $\geq m$ colors, $\leq k$ length.

($\implies$) If G has a HamPath, there is some path of length $|V|$. Hence, this same path exists in G' as G' is only a colored variant of G. Further, since each vertex of G' is colored uniquely, this path of $|V|$ vertices also has $|V|$ unique colors. Hence, this is a ColoredPath of length $\leq |V|$ with $\geq |V|$ colors.

($\impliedby$) If G' has a ColoredPath, there is some path of length $\leq |V|$ with $\geq |V|$ colors. However, since each vertex in G' is colored with 1 color, it is impossible to have $\geq |V|$ colors without having exactly $|V|$ vertex path length. Further, since the path is of length $|V|$ but traverses every unique vertex color, it must be the case that no vertex is traversed twice. By definition, this is a HamPath, hence if G' has a ColoredPath with these parameters, then G has a HamPath.

Hence, since we have shown this is NP and NP-Hard, ColoredPath is NP-Complete. $\square$