Globally Exponentially Convergent Continuous Observers for Velocity Bias and State for Invariant Kinematic Systems on Matrix Lie Groups

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***Abstract*—In this article globally exponentially convergent continuous observers for invariant kinematic systems on finite-dimensional matrix Lie groups has been proposed. Such an observer estimates, from measurements of landmarks, vectors, and biased velocity, both the system state and the unknown constant bias in velocity measurement, where the state belongs to the state-space Lie group and the velocity to the Lie algebra of the Lie group. The main technique is to embed a given system defined on a matrix Lie group into Euclidean space and build observers in the Euclidean space. The theory is illustrated with the special Euclidean group in three dimensions, and it is shown that the observer works well even in the presence of measurement noise.**

***Index Terms*—Estimation, Lie group, observer, velocity bias.**

I. Introduction

Consider an invariant kinematic system on a matrix Lie group G

$$\dot{g} = g\xi$$

with $(g, \xi) \in G \times \mathfrak{g}$, where G is embedded in $\mathbb{R}^{n\times n}$ and $\mathfrak{g}$ denotes the Lie algebra of G. Suppose that the velocity ξ is measured with an additive unknown constant bias as

$$\xi_{\mathrm{m}} = \xi + b$$

where $b \in \mathfrak{g}$ is the constant unknown bias. Suppose also that we measure landmarks and vectors such that an $n \times n$ matrix-valued signal A of the form

$$A = Fg$$

or

$$A = g^{-1}F$$

is available, where F is an $n \times n$ invertible matrix. In this article, we design continuous observers that globally and exponentially estimate (g, b) with ξ_{m} and A, where it is assumed that the value of F is available.

Relevant works are listed in the following. Khosravian [7] proposed continuous observers that estimate (g, b) with ξ_m and homogeneous outputs. Their observers are uniformly locally exponentially stable, but not globally exponentially stable. A similar work was carried out in [6], where a gradient-like innovation term was used in the observer design. The observers therein are not globally exponentially stable but only uniformly locally exponentially stable. Gradient-like observers were also proposed in [8], but these observers are not globally exponentially convergent either.

To the best of the author's knowledge, the observers in the present article are the first globally exponentially convergent *continuous* observers for velocity bias and state for kinematic systems on matrix Lie groups. One noticeable difference between the observers in [6]–[8] and this article is that the observers in this article are designed in $\mathbb{R}^{n\times n} \times \mathfrak{g}$ instead of $G \times \mathfrak{g}$, where $G \subset \mathbb{R}^{n\times n}$, such that the Euclidean structure of $\mathbb{R}^{n\times n}$ is fully utilized without being constrained to the group structure of G. This type of observers built in Euclidean space is called geometry-free and they have been widely used for the Lie group $\mathrm{SO}(3)$ (e.g., [1], [9]).

This article addresses the case of general matrix Lie groups embedded in Euclidean space, but for the sake of completeness of literature survey, we go over some important papers on observers for specific Lie groups such as $\mathrm{SO}(3)$ and $\mathrm{SE}(3)$ in addition to [1], [9]. There have been papers on estimation of pose and velocity measurement bias for $\mathrm{SE}(3)$, e.g., [5], [10], [11] and references therein. A globally exponentially convergent hybrid (not continuous) observer was proposed in [10], and a nonglobal exponentially convergent observer was proposed in [11]. A gradient-like observer design on $\mathrm{SE}(3)$ with system outputs on the real projective space was proposed in [5]. Refer to [4] for a global formulation of extended Kalman filter on $\mathrm{SE}(3)$ for geometric control of a drone. The observer on $\mathrm{SE}(3)$ proposed in [13] has the merit of using only one landmark but it makes a strong assumption of uniform observability along the system trajectory. It will be interesting to generalize these observers on $\mathrm{SE}(3)$ to the case of matrix Lie groups, which is left to the reader.

The article is organized as follows. In Section II, various forms of globally exponentially convergent continuous observers for velocity bias and state for kinematic systems on matrix Lie groups are proposed. In Section III, one of the observers proposed in Section II is illustrated by applying it to the special Euclidean group $\mathrm{SE}(3)$ and testing robustness to measurement noise. The article is concluded in Section IV.

II. Main Results

Let G be a matrix Lie group that is a subgroup of $\mathrm{GL}(n) = \{A \in \mathbb{R}^{n\times n} \mid \det A \neq 0\}$, and let $\mathfrak{g}$ denote the Lie algebra of G. Since G is a subgroup fo $\mathrm{GL}(n)$, we may assume that $\mathfrak{g}$ is a subalgebra of $(\mathbb{R}^{n\times n}, [\,,])$, where $[\,,]$ is the usual matrix commutator defined by $[A, B] = AB - BA$ for all $A, B \in \mathbb{R}^{n\times n}$. Let $\pi_{\mathfrak{g}} : \mathbb{R}^{n\times n} \to \mathfrak{g}$ denote the orthogonal projection onto $\mathfrak{g}$ with respect to the Euclidean inner product $\langle\,,\rangle$ that is defined by $\langle A, B\rangle = \mathrm{tr}(A^T B)$ for $A, B \in \mathbb{R}^{n\times n}$. Let $\|\|$ denote the Euclidean or Frobenius norm which is defined by $\|A\| = \sqrt{\langle A, A\rangle}$ for all $A \in \mathbb{R}^{n\times n}$. For a square matrix A, $\lambda_{\min}(A)$

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and $\lambda_{\max}(A)$ denote the minimum eigenvalue and the maximum eigenvalue of A, respectively. For any matrix A, $\sigma_{\min}(A)$ and $\sigma_{\max}(A)$ denote the minimum singular value and the maximum singular value of A, respectively. For any $A \in \mathbb{R}^{n\times n}$, $\|A\|^2 = \sum_{i=1}^n \sigma_i^2(A)$, where $\sigma_i(A)$'s are the singular values of A. We have $\sigma_{\min}(A)\|B\| \le \|AB\| \le \sigma_{\max}(A)\|B\|$ for all $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{m\times \ell}$. Refer to [2] for more about Lie groups in the context of geometric control and mechanics.

A. Observer I

The invariant kinematic equation on a matrix Lie group $G \subset \mathbb{R}^{n\times n}$ is given by

$$\dot{g} = g\xi \tag{1}$$

where $g \in G$ and $\xi \in \mathfrak{g}$. Suppose that there is given a trajectory of the system $(g(t), \xi(t)) \in G \times \mathfrak{g}, 0 \le t < \infty$. We make the following three assumptions.

Assumption 1: A matrix-valued signal $A(t) \in \mathbb{R}^{n\times n}$ is available that can be expressed as

$$A = Fg \tag{2}$$

where F is a constant invertible matrix in $\mathbb{R}^{n\times n}$ and $g \in G$.

Assumption 2: A $\mathfrak{g}$-valued signal $\xi_{\rm m}(t)$ with bias is available and related to the true signal $\xi(t) \in \mathfrak{g}$ as follows:

$$\xi_{\rm m} = \xi + b$$

where $b \in \mathfrak{g}$ is an unknown constant bias.

Assumption 3: There are known constants $B_\xi > 0$ and $B_b > 0$ such that $\|\xi(t)\| \le B_\xi$ for all $t \ge 0$ and $\|b\| \le B_b$. There are numbers $L_g > 0$ and $U_g > 0$ such that

$$L_g \le \sigma_{\min}(g(t)) \le \sigma_{\max}(g(t)) \le U_g$$

for all $t \ge 0$, where the knowledge on the values of L_g and U_g is not assumed.

We propose the following observer:

$$\dot{\bar{A}} = \bar{A}\xi_{\rm m} - A\bar{b} + k_P(A - \bar{A}) \tag{3a}$$

$$\dot{\bar{b}} = -k_I\, \pi_{\mathfrak{g}}(A^T(A - \bar{A})) \tag{3b}$$

with $k_P > (B_\xi + B_b)$ and $k_I > 0$, where $(\bar{A}, \bar{b}) \in \mathbb{R}^{n\times n} \times \mathfrak{g}$ is an estimate of $(A, b) \in \mathbb{R}^{n\times n} \times \mathfrak{g}$. So, $(F^{-1}\bar{A}, \bar{b}) \in \mathbb{R}^{n\times n} \times \mathfrak{g}$ becomes an estimate of $(g, b) \in G \times \mathfrak{g}$ by Assumption 1. The global and exponentially convergent property of this observer is proven in the following theorem.

Theorem 1: Let

$$E_A = A - \bar{A}, \quad e_b = b - \bar{b}.$$

Under Assumptions 1–3, for any $k_P > (B_\xi + B_b)$ and $k_I > 0$, there exist numbers $a > 0$ and $C > 0$ such that

$$\|E_A(t)\| + \|e_b(t)\| \le C(\|E_A(0)\| + \|e_b(0)\|)e^{-at} \tag{4}$$

for all $t \ge 0$ and all $(\bar{A}(0), \bar{b}(0)) \in \mathbb{R}^{n\times n} \times \mathfrak{g}$.

Proof: See the Appendix. ■

Corollary 1: Suppose that Assumptions 1–3 hold, and let

$$E_g = g - F^{-1}\bar{A}, \quad e_b = b - \bar{b}.$$

Then, there exist numbers $a > 0$ and $C > 0$ such that

$$\|E_g(t)\| + \|e_b(t)\| \le C(\|E_g(0)\| + \|e_b(0)\|)e^{-at} \tag{5}$$

for all $t \ge 0$ and all $(\bar{A}(0), \bar{b}(0)) \in \mathbb{R}^{n\times n} \times \mathfrak{g}$.

Proof: Use $\|E_g\|/\|F^{-1}\| \le \|E_A\| \le \|F\|\|E_g\|$ and (4) with the constant C redefined appropriately. ■

Namely, the estimate $(F^{-1}\bar{A}(t), \bar{b}(t))$ converges globally and exponentially to the true value $(g(t), b)$ as t tends to ∞.

Remark 1: We can also build an observer that allows $\bar{b}$ to be in $\mathbb{R}^{n\times n}$ instead of $\mathfrak{g}$. The modified observer is given by

$$\dot{\bar{A}} = \bar{A}\xi_{\rm m} - A\bar{b} + k_P(A - \bar{A}) \tag{6a}$$

$$\dot{\bar{b}} = -k_I A^T(A - \bar{A}) \tag{6b}$$

where $(\bar{A}, \bar{b}) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$. Notice that the projection operator $\pi_{\mathfrak{g}}$ in (3b) is removed from (3b) to obtain (6b). Theorem 1 and Corollary 1 also hold for this observer, whose proof is almost identical to the proofs of Theorem 1 and Corollary 1, so it is left to the reader.

Remark 2: Two points are discussed here on observer (3). First, the assumption of a priori knowledge of bounds for $\|\xi(t)\|$ and $\|b\|$ can be lifted for $G = \mathrm{SO}(3)$, in which case we utilize, in the convergence proof, the fact that the set of 3×3 skew symmetric matrices, which is the Lie algebra of $\mathrm{SO}(3)$, is orthogonal to the set of 3×3 symmetric matrices with respect to the usual Euclidean norm on $\mathbb{R}^{3\times 3}$. Second, if the term $A\bar{b}$ in (3a) were replaced with $\bar{A}\bar{b}$, then the proof of convergence of the corresponding observer would not be feasible.

One may be concerned about the fact that the estimate $F^{-1}\bar{A}(t)$ may not lie in G for all $t \ge 0$ although it converges to $g(t) \in G$ as t tends to infinity. In our opinion, there can be two schools of thought regarding this. In the first school of thought, objects are viewed from a wide point of view. A matrix $g \in G \subset \mathbb{R}^{n\times n}$ can be regarded as a transformation that maps $\mathbb{R}^n$ vectors to $\mathbb{R}^n$. Hence, there is no reason to stop one from having an estimate of g in the ambient space $\mathbb{R}^{n\times n}$ of linear transformations. For example, suppose that for a given vector $x \in \mathbb{R}^n$, there is a $g_1 \in G$ and a $B_1 \in \mathbb{R}^{n\times n}$ such that $\|B_1x - gx\| < \|g_1x - gx\|$. In this case, one can say that B_1 approximates g better than g_1 does. More generally, if $\|B_1 - g\| < \|g_1 - g\|$, B_1 approximates g better than g_1 as an element in $\mathbb{R}^{n\times n}$. Hence, there is no reason to restrict ourselves to G in search of estimates of elements of G. Also, as demonstrated in [3], [14], for example, if one designs controllers in $\mathbb{R}^{n\times n}$ instead of G, then the direct use of $F^{-1}\bar{A}$ in feedback works very well.

However, if one does not belong to the first school of thought, then he would or should project $F^{-1}\bar{A}(t)$ to G as output of the observer, but the projected image is never fed back into observer (3). For example, if $G = \mathrm{SO}(3)$, then the polar decomposition via singular decomposition, which does not require matrix invertibility [12], can be used to define a projection from $\mathbb{R}^{3\times 3}$ to $\mathrm{SO}(3)$. This point will be demonstrated with $\mathrm{SE}(3) = \mathrm{SO}(3) \rtimes \mathbb{R}^3$ in Section III.

B. Observer II

Recall the kinematic equation in (1). We now consider a case where the measurement matrix A is related to the true signal $g(t)$ as $A = g^{-1}(t)F$ instead of $A = Fg(t)$. Consequently, in place of Assumption 1, we make the following assumption.

Assumption 4: A matrix-valued signal $A(t) \in \mathbb{R}^{n\times n}$ is available that can be expressed as

$$A = g^{-1}F \tag{7}$$

where F is a constant invertible matrix in $\mathbb{R}^{n\times n}$ and $g \in G$.

By (1), A defined in (7) satisfies

$$\dot{A} = -\xi A. \tag{8}$$

Under Assumptions 4, 2, and 3, we propose the following observer:

$$\dot{\bar{A}} = -\xi_{\mathrm{m}}\bar{A} + \bar{b}A + k_P(A - \bar{A}) \tag{9a}$$

$$\dot{\bar{b}} = k_I\, \pi_{\mathfrak{g}}((A - \bar{A})A^T) \tag{9b}$$

with $k_P > (B_\xi + B_b)$ and $k_I > 0$, where $(\bar{A}, \bar{b}) \in \mathbb{R}^{n\times n} \times \mathfrak{g}$ is an estimate of $(A, b) \in \mathbb{R}^{n\times n} \times \mathfrak{g}$.

Theorem 2: For observer (9), let

$$E_A = A - \bar{A}, \quad e_b = b - \bar{b}.$$

Under Assumptions 2, 3, and 4 for any $k_P > (B_\xi + B_b)$ and $k_I > 0$, there exist numbers $a > 0$ and $C > 0$ such that

$$\|E_A(t)\| + \|e_b(t)\| \le C(\|E_A(0)\| + \|e_b(0)\|)e^{-at}$$

for all $t \ge 0$ and all $(\bar{A}(0), \bar{b}(0)) \in \mathbb{R}^{n\times n} \times \mathfrak{g}$.

Proof: See the Appendix. ■

Since $\bar{A}(t)$ converges to $A(t)$ as $t \to \infty$ and $A(t)$ is invertible for all $t \ge 0$, one can see that after some time, the estimate $(F\bar{A}(t)^{-1}, \bar{b}(t))$ converges to the true value $(g(t), b)$ as t tends to infinity.

Remark 3: Notice that $\bar{A}(t)^{-1}$ is never used in observer (9). However, since it is used in the estimation of $g(t)$ as $F\bar{A}(t)^{-1}$, one may worry about the case when $\bar{A}(t)$ is not invertible for some t, for which we provide two solutions, here. The first solution is that we temporarily use $FA(t)^{-1}$ instead of $F\bar{A}(t)^{-1}$ as an estimate of $g(t)$ until $\bar{A}(t)$ eventually becomes invertible after some time on. Notice, however, that this operation does not affect the process of observer (9) because $\bar{A}(t)$ is fed back to the observer regardless of its invertibility. Also, notice that observer (9) is essentially an observer for the velocity bias because the information on $g(t)$ is essentially given through the measurement of $A(t)$ as in Assumption 4. Hence, the second solution is that we can always choose the initial condition $\bar{A}(0)$ close or equal to $A(0)$. Then, $\bar{A}(t)$ will remain invertible since it converges to $A(t)$ exponentially in time. Therefore, with these two practical solutions, the possible singularity of $\bar{A}(t)$ over some finite time interval cannot be a problem in practice.

We now derive from (3) various observers of *concrete* form that estimate (R, b) from vector measurements. Assume that there is a set $\mathcal{S} = \{s_i, 1 \le i \le m\}$ of m known fixed inertial vectors, where each s_i in $\mathcal{S}$ is a vector in $\mathbb{R}^n$, such that the rank of $\mathcal{S}$ is n. Assume also that measurements of the vectors are made in the body-fixed frame and the set of the measured vectors is denoted by $\mathcal{C} = \{c_i, 1 \le i \le m\}$ and related to $\mathcal{S}$ as follows:

$$c_i = g^{-1}s_i, \quad i = 1, \ldots, m$$

where $g \in G$. Let

$$S = \begin{bmatrix} s_1 & \cdots & s_m \end{bmatrix}, \quad C = \begin{bmatrix} c_1 & \cdots & c_m \end{bmatrix} \tag{10}$$

be $n \times m$ matrices made of the *column* vectors from $\mathcal{S}$ and $\mathcal{C}$, respectively.

Corollary 2: Let S and C be given in (10). If there is a matrix $W \in \mathbb{R}^{m\times n}$ such that $F := SW$ has rank n, then (7) is satisfied by $A = CW$ and observer (9) is applicable.

Proof: Trivial. ■

Remark 4: Corollary 2 can be applied in several ways. For example, the substitution of WS^T into W in Corollary 2 would yield

$$F = SWS^T, \quad A = CWS^T$$

where it is assumed that W is an $m \times m$ matrix such that F has rank n. Likewise, W in Corollary 2 can be chosen such that F depends more nonlinearly on S.

C. Variants

We here propose an observer that is a variant of observer (3) with A^T replaced by A^{-1} in (3b). Recall the kinematic equation (1), and under Assumptions 1–3, we propose the following new observer:

$$\dot{\bar{A}} = \bar{A}\xi_{\mathrm{m}} - A\bar{b} + k_P(A - \bar{A}) \tag{11a}$$

$$\dot{\bar{b}} = -k_I\, \pi_{\mathfrak{g}}(A^{-1}(A - \bar{A})) \tag{11b}$$

with $k_P > (2B_\xi + B_b)$ and $k_I > 0$, where $(\bar{A}, \bar{b}) \in \mathbb{R}^{n\times n} \times \mathfrak{g}$ is an estimate of $(A, b) \in \mathbb{R}^{n\times n} \times \mathfrak{g}$. So, $(F^{-1}\bar{A}, \bar{b}) \in \mathbb{R}^{n\times n} \times \mathfrak{g}$ becomes an estimate of $(g, b) \in G \times \mathfrak{g}$ by Assumption 1.

Theorem 3: For observer (11), let

$$E_A = A - \bar{A}, \quad e_b = b - \bar{b}.$$

Under Assumptions 1–3, for any $k_P > (2B_\xi + B_b)$ and $k_I > 0$, there exist numbers $a > 0$ and $C > 0$ such that

$$\|E_A(t)\| + \|e_b(t)\| \le C(\|E_A(0)\| + \|e_b(0)\|)e^{-at}$$

for all $t \ge 0$ and all $(\bar{A}(0), \bar{b}(0)) \in \mathbb{R}^{n\times n} \times \mathfrak{g}$.

Proof: See the Appendix. ■

We also propose a variant of observer (9) with A^T replaced by A^{-1} in (9b). Under Assumptions 2, 3, and 4, we propose the following new observer:

$$\dot{\bar{A}} = -\xi_{\mathrm{m}}\bar{A} + \bar{b}A + k_P(A - \bar{A}) \tag{12a}$$

$$\dot{\bar{b}} = k_I\, \pi_{\mathfrak{g}}((A - \bar{A})A^{-1}) \tag{12b}$$

with $k_P > (2B_\xi + B_b)$ and $k_I > 0$, where $(\bar{A}, \bar{b}) \in \mathbb{R}^{n\times n} \times \mathfrak{g}$ is an estimate of $(A, b) \in \mathbb{R}^{n\times n} \times \mathfrak{g}$. So, $(F\bar{A}^{-1}, \bar{b}) \in \mathbb{R}^{n\times n} \times \mathfrak{g}$ becomes an estimate of $(g, b) \in G \times \mathfrak{g}$ by Assumption 1.

Theorem 4: For observer (12), let

$$E_A = A - \bar{A}, \quad e_b = b - \bar{b}.$$

Under Assumptions 4, 2 and 3, for any $k_P > (B_\xi + B_b)$ and $k_I > 0$, there exist numbers $a > 0$ and $C > 0$ such that

$$\|E_A(t)\| + \|e_b(t)\| \le C(\|E_A(0)\| + \|e_b(0)\|)e^{-at}$$

for all $t \ge 0$ and all $(\bar{A}(0), \bar{b}(0)) \in \mathbb{R}^{n\times n} \times \mathfrak{g}$.

Proof: Omitted since it is similar to the proof of Theorem 3. ■

III. Example: Application to SE(3)

We now illustrate the theory presented in Section II with the special Euclidean group on $\mathbb{R}^3$. The group can be expressed in homogeneous coordinates as

$$\mathrm{SE}(3) = \left\{ \begin{bmatrix} R & x \\ 0 & 1 \end{bmatrix} \mid R \in \mathrm{SO}(3), x \in \mathbb{R}^3 \right\}$$

where $\mathrm{SO}(3) = \{R \in \mathbb{R}^{3\times 3} \mid R^TR = I, \det R = 1\}$ is the special orthogonal group whose Lie algebra is $\mathfrak{so}(3) = \{A \in \mathbb{R}^{3\times 3} \mid A^T = -A\}$. It is easy to see that SE(3) is a subgroup of $\mathrm{GL}(4) = \{A \in \mathbb{R}^{4\times 4} \mid \det A \ne 0\}$. The Lie algebra of SE(3) is then given by

$$\mathfrak{se}(3) = \left\{ \begin{bmatrix} \hat{\Omega} & v \\ 0 & 0 \end{bmatrix} \text{ for some } \Omega \in \mathbb{R}^3, x \in \mathbb{R}^3 \right\}$$

where the hat map $\wedge : \mathbb{R}^3 \to \mathfrak{so}(3)$ is defined such that $\hat{x}y = x \times y$ for all $x, y \in \mathbb{R}^3$. In homogeneous coordinates, landmarks to be measured with sensors are expressed in the form

$$\begin{bmatrix} x \\ 1 \end{bmatrix}, \quad x \in \mathbb{R}^3 \tag{13}$$

and such vectors *at infinity* as the gravity or the Earth's magnetic field are expressed in the form

$$\begin{bmatrix} x \\ 0 \end{bmatrix}, \quad x \in \mathbb{R}^3. \tag{14}$$

The orthogonal projection $\pi_{\mathfrak{se}(3)} : \mathbb{R}^{4\times 4} \to \mathfrak{se}(3)$ is given as follows: for $A \in \mathbb{R}^{4\times 4}$ given by

$$A = \begin{bmatrix} B & x \\ y^T & z \end{bmatrix}, \quad B \in \mathbb{R}^{3\times 3}, x \in \mathbb{R}^{3\times 1}, y \in \mathbb{R}^{3\times 1}, z \in \mathbb{R}$$

we have

$$\pi_{\mathfrak{se}(3)}(A) = \begin{bmatrix} \frac{1}{2}(B - B^T) & x \\ 0_{1\times 3} & 0 \end{bmatrix} \in \mathfrak{se}(3).$$

Suppose that we measure in the body frame the following inertial vectors:

$$s_1 = (e_1, 1), s_2 = (e_2, 1), s_3 = (e_3, 1), s_4 = (-e_3, 0)$$

where $\{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{R}^3$ and s_4 represents the gravity direction or it could be regarded as the vector computed as $(e_2 - e_3) \times (e_1 - e_3)$, from s_1, s_2, s_3. Suppose the measured signal matrix $A(t)$ is given by

$$A(t) = g(t)^{-1} F$$

with

$$F = \begin{bmatrix} s_1 & s_2 & s_3 & s_4 \end{bmatrix}.$$

Here, each column in the $A(t)$ matrix is a quantity measured in the body-fixed frame. Suppose that a set of true trajectories $(R(t), x(t)) \in \mathrm{SE}(3)$ and $(\Omega(t), V(t)) \in \mathbb{R}^3 \times \mathbb{R}^3$ are given as follows:

$$R(t) = \exp(t\hat{e}_1)\exp(t\hat{e}_3)\exp(t\hat{e}_1) \tag{15}$$

$$x(t) = (\cos t, \sin t, \cos t) \tag{16}$$

$$\Omega(t) = (1 + \cos t, \sin t - \sin t \cos t, \cos t + \sin^2 t) \tag{17}$$

$$V(t) = R^T(t)\dot{x}(t) \tag{18}$$

where $\Omega(t)$ satisfies $\hat{\Omega}(t) = R^T(t)\dot{R}(t)$. Assume that the unknown constant gyro bias b_Ω and the unknown constant velocity bias b_v are, respectively, given by

$$b_\Omega = (1, 0.5, -1), \quad b_v = (0.5, -0.5, 0.5). \tag{19}$$

We use the observer of the form (9). The gains are chosen as $k_P = k_I = 4$, and the initial state of the observer is given by

$$\bar{A}(0) = \bar{g}_0^{-1} F$$

where

$$\bar{g}_0 = \begin{bmatrix} \exp(\frac{\pi}{2}\hat{e}_1) & 0_{3\times 1} \\ 0_{1\times 3} & 1 \end{bmatrix}$$

and

$$\bar{b}_\Omega(0) = (0, 0, 0), \quad \bar{b}_v(0) = (0, 0, 0).$$

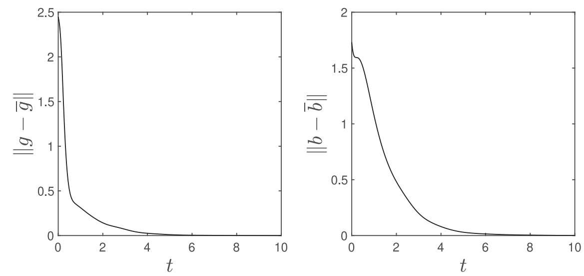

Fig. 1. Pose estimation error $\|g(t) - \bar{g}(t)\|$ and the velocity bias estimation error $\|b - \bar{b}(t)\|$ by observer (9).

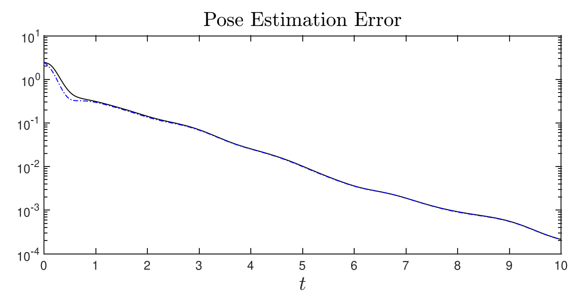

Fig. 2. Two pose estimation errors by observer (9): $\|g(t) - \bar{g}(t)\|$ by observer (solid) and $\|g(t) - \bar{g}_{\mathrm{SE}(3)}(t)\|$ by the $\mathrm{SE}(3)$ factor $\bar{g}_{\mathrm{SE}(3)}(t)$ of $\bar{g}(t)$ obtained through the projection (20) (dash-dot).

The simulation results are plotted in Fig. 1, where the pose estimation error $\|g(t) - \bar{g}(t)\|$ with $\bar{g}(t) := F\bar{A}(t)^{-1} \in \mathbb{R}^{4\times 4}$, and the bias estimation error $\|b - \bar{b}(t)\|$ are plotted. Both estimation errors converge well to zero as theoretically predicted.

To examine if the image trajectory of $\bar{g}(t)$ under a projection onto $\mathrm{SE}(3)$ also converges to $g(t)$, let us define a projection $\mathrm{proj} : \mathbb{R}^{4\times 4} \to \mathrm{SE}(3)$ as follows: for any

$$\bar{g} = \begin{bmatrix} \bar{g}_1 & \bar{g}_2 \\ \bar{g}_3 & \bar{g}_4 \end{bmatrix} \in \mathbb{R}^{4\times 4}$$

with $\bar{g}_1 \in \mathbb{R}^{3\times 3}$, $\bar{g}_2 \in \mathbb{R}^{3\times 1}$, $\bar{g}_3 \in \mathbb{R}^{1\times 3}$, and $\bar{g}_4 \in \mathbb{R}$

$$\mathrm{proj}(\bar{g}) := \begin{bmatrix} \bar{g}_{1,\mathrm{SO}(3)} & \bar{g}_2 \\ 0_{1\times 3} & 1 \end{bmatrix} \in \mathrm{SE}(3) \tag{20}$$

where $\bar{g}_{1,\mathrm{SO}(3)}$ denotes the $\mathrm{SO}(3)$ factor in polar decomposition of $\bar{g}_1$. For convenience, let

$$\bar{g}_{\mathrm{SE}(3)}(t) := \mathrm{proj}(\bar{g}(t)).$$

The pose estimation error $\|g(t) - \bar{g}_{\mathrm{SE}(3)}(t)\|$ by $\bar{g}_{\mathrm{SE}(3)}(t)$ is plotted in Fig. 2 along with the pose estimation error $\|g(t) - \bar{g}(t)\|$ by $\bar{g}(t)$ that was obtained in the simulation. It can be seen in the figure that $\bar{g}(t)$ stays very close to its $\mathrm{SE}(3)$ factor $\bar{g}_{\mathrm{SE}(3)}(t)$, and $\bar{g}_{\mathrm{SE}(3)}(t)$ also converges to the true pose $g(t)$ as time tends to infinity.

To test robustness of the observer, we assume that there are noises in measurements of attitude and position as follows:

$$Re^{\hat{v}_R}, \quad x + v_x$$

where $v_R \in \mathbb{R}^3$ is Gaussian with zero mean and standard deviation 1, and $v_x \in \mathbb{R}^3$ is Gaussian with zero mean and standard deviation 1. This will consequently affect $A(t) = g(t)^{-1}F$. To inspect any negative effect by high gains through k_P and k_I, we set $k_P = k_I = 40$, which

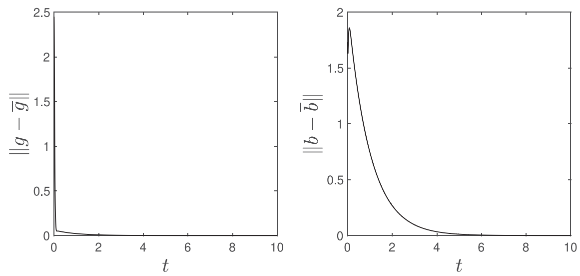

Fig. 3. Pose estimation error $\|g(t) - \bar{g}(t)\|$ and the velocity bias estimation error $\|b - \bar{b}(t)\|$ by observer (9) with high gains in the presence of noise.

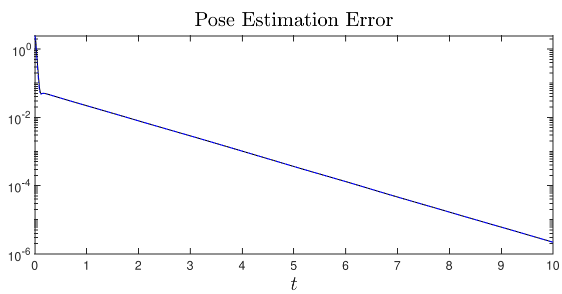

Fig. 4. Two pose estimation errors by observer (12): $\|g(t) - \bar{g}(t)\|$ by the observer (solid) and $\|g(t) - \bar{g}_{\mathrm{SE}(3)}(t)\|$ by the $\mathrm{SE}(3)$ factor $\bar{g}_{\mathrm{SE}(3)}(t)$ of $\bar{g}(t)$ obtained through the projection (20) (dash-dot), where the two signals are very close to each other.

are ten times larger than the values used in the preceding simulation. We use the standard fourth-order Runge–Kutta algorithm for numerical integration with time step size $h = 10^{-5}$. The measurement noises are generated with the Matlab function, normrnd, in the simulation. The reference trajectory, the bias value, and the initial state are the same as in the preceding simulation. The simulation results are plotted in Fig. 3, where the pose estimation error $\|g(t) - \bar{g}(t)\|$ with $\bar{g}(t) := F\bar{A}(t)^{-1} \in \mathbb{R}^{4\times 4}$ and the bias estimation error $\|b - \bar{b}(t)\|$ are plotted. It can be seen that even in the presence of noise, both estimation errors converge much faster with $k_P = k_I = 40$ than in the case with $k_P = k_I = 4$ and no noise. It implies that higher gains induce faster convergence and the observer is robust to measurement noise. The pose estimation error $\|g(t) - \bar{g}_{\mathrm{SE}(3)}(t)\|$ by $\bar{g}_{\mathrm{SE}(3)}(t)$ is plotted in Fig. 4 along with the pose estimation error $\|g(t) - \bar{g}(t)\|$ by $\bar{g}(t)$ that was obtained in the simulation with the measurement noise. It can be seen that the two signals $\bar{g}(t)$ and $\bar{g}_{\mathrm{SE}(3)}(t)$ stay closer together than in the case with $k_P = k_I = 4$ and no noise.

IV. Conclusion

We have successfully designed globally exponentially convergent continuous observers for kinematic invariant systems on finite-dimensional matrix Lie groups that estimate state and constant velocity bias from measurements of landmarks, vectors, and biased velocity. We have applied the result to the special Euclidean group $\mathrm{SE}(3)$ and carried out a simulation study to demonstrate an excellent performance of the observer for $\mathrm{SE}(3)$ with fast convergence and robustness to measurement noise.

One possible drawback of our observers is that an upper bound for $\xi(t)$ and b is assumed to be available for convergence proof. Our simulation study shows that high gains do not seem to amplify measurement noise much. Hence, the bounds assumption does not limit our observers in practice. However, from a theoretical point of view, there is room for improvement.

Appendix

Proof of Theorem 1

Proof: From (1) and Assumption 1, $A(t)$ satisfies

$$\dot{A} = A\xi. \tag{21}$$

By Assumption 2, observer (3) can be written as

$$\dot{\bar{A}} = \bar{A}(\xi + b) - A\bar{b} + k_P E_A \tag{22a}$$

$$\dot{\bar{b}} = -k_I\, \pi_{\mathfrak{g}}(A^T E_A). \tag{22b}$$

By Assumption 3, there is a number ϵ such that

$$0 < \epsilon < \min\left\{H, \frac{1}{\|F\|U_g\sqrt{k_I}}\right\}$$

where

$$H = \frac{4(k_P - B_\xi - B_b)L_g^2\lambda_{\min}(F^TF)}{(4k_I L_g^2\lambda_{\min}(F^TF) + (k_P + B_b + 2B_\xi)^2)U_g^2\|F\|^2}.$$

The following three quadratic functions of $(\|E_A\|, \|e_b\|)$ are then all positive definite:

$$V_1(\|E_A\|, \|e_b\|) = \frac{1}{2}\|E_A\|^2 + \frac{1}{2k_I}\|e_b\|^2 - \epsilon U_g\|F\|\|E_A\|\|e_b\|$$

$$V_2(\|E_A\|, \|e_b\|) = \frac{1}{2}\|E_A\|^2 + \frac{1}{2k_I}\|e_b\|^2 + \epsilon U_g\|F\|\|E_A\|\|e_b\|$$

$$\begin{aligned}V_3(\|E_A\|, \|e_b\|) &= (k_P - (B_\xi + B_b) - \epsilon k_I U_g^2\|F\|^2)\|E_A\|^2\\ &\quad + \epsilon\lambda_{\min}(F^TF)L_g^2\|e_b\|^2\\ &\quad - \epsilon(k_P + B_b + 2B_\xi)U_g\|F\|\|E_A\|\|e_b\|.\end{aligned}$$

Hence, there are numbers $\alpha > 0$ and $\beta > 0$ such that

$$V_2 \le \alpha V_1, \quad \beta V_2 \le V_3. \tag{23}$$

Let

$$V(E_A, e_b) = \frac{1}{2}\|E_A\|^2 + \frac{1}{2k_I}\|e_b\|^2 + \epsilon\langle E_A, Ae_b\rangle$$

which satisfies

$$V_1(\|E_A\|, \|e_b\|) \le V(E_A, e_b) \le V_2(\|E_A\|, \|e_b\|) \tag{24}$$

for all $(E_A, e_b) \in \mathbb{R}^{n\times n} \times \mathfrak{g}$ by the Cauchy–Schwarz inequality and $\|A\| = \|Fg\| \le \|F\|U_g$. From (21) and (22), and the assumption of the bias b being constant, it follows that the estimation error (E_A, e_b) obeys

$$\dot{E}_A = E_A(\xi + b) - Ae_b - k_P E_A$$

$$\dot{e}_b = k_I\, \pi_{\mathfrak{g}}(A^T E_A).$$

Along any trajectory of the composite system consisting of rigid body (1) and observer (3)

$$\begin{aligned}\frac{dV}{dt} &= \langle E_A, E_A(\xi + b) - Ae_b - k_P E_A\rangle\\ &\quad + \langle e_b, \pi_{\mathfrak{g}}(A^T E_A)\rangle\end{aligned}$$

$$
\begin{aligned}
&+\epsilon\langle E_A(\xi+b)-Ae_b-k_PE_A, Ae_b\rangle\\
&+\epsilon\langle E_A, A\xi e_b\rangle+\epsilon k_I\langle E_A, A\,\pi_{\mathfrak{g}}(A^TE_A)\rangle\\
&\le -(k_P-(B_\xi+B_b)-\epsilon k_I U_g^2\|F\|^2)\|E_A\|^2\\
&-\epsilon\lambda_{\min}(F^TF)L_g^2\|e_b\|^2\\
&+\epsilon(k_P+B_b+2B_\xi)U_g\|F\|\|E_A\|\|e_b\|\\
&=-V_3\le-\beta V_2\le-\beta V
\end{aligned}
$$

where the following have been used:

$$
\begin{aligned}
&\langle E_A, E_A(\xi+b)\rangle \le \|E_A\|^2(B_\xi+B_b)\\
&\langle E_A, Ae_b\rangle = \langle A^TE_A, e_b\rangle = \langle \pi_{\mathfrak{g}}(A^TE_A), e_b\rangle\\
&\langle Ae_b, Ae_b\rangle \ge \lambda_{\min}(F^TF)\|ge_b\|^2 \ge \lambda_{\min}(F^TF)L_g^2\|e_b\|^2\\
&\langle E_A, A\,\pi_{\mathfrak{g}}(A^TE_A)\rangle = \|\,\pi_{\mathfrak{g}}(A^TE_A)\|^2\\
&\qquad\qquad \le \|A^TE_A\|^2 \le U_g^2\|F\|^2\|E_A\|^2.
\end{aligned}
$$

Hence, $V(t)\le V(0)e^{-\beta t}$ for all $t\ge 0$ and all $(\bar{A}(0),\bar{b}(0))\in\mathbb{R}^{n\times n}\times\mathfrak{g}$. It follows from (23) and (24) that

$$V_1(t)\le V(t)\le V(0)e^{-\beta t}\le V_2(0)e^{-\beta t}\le \alpha V_1(0)e^{-\beta t}$$

for all $t\ge 0$ and all $(\bar{A}(0),\bar{b}(0))\in\mathbb{R}^{n\times n}\times\mathfrak{g}$. Since $0<\epsilon<1/(\|F\|U_g\sqrt{k_I})$, the map defined by

$$(x_1,x_2)\mapsto\sqrt{\frac{1}{2}x_1^2+\frac{1}{2k_I}x_2^2-\epsilon U_g\|F\|x_1x_2}$$

is a norm on $\mathbb{R}^2$, where $(x_1,x_2)\in\mathbb{R}^2$, which is equivalent to the 1-norm on $\mathbb{R}^2$ since all norms are equivalent on a finite-dimensional vector space. Hence, $V_1(t)\le\alpha V_1(0)e^{-\beta t}$ implies that there exists $C>0$ such that (4) holds for all $t\ge 0$ and all $(\bar{A}(0),\bar{b}(0))\in\mathbb{R}^{n\times n}\times\mathfrak{g}$, where $a=\beta/2$. ∎

Proof of Theorem 2

Proof: By Assumption 3, there is a number ϵ such that

$$0<\epsilon<\min\left\{H,\frac{L_g}{\|F\|\sqrt{k_I}}\right\}$$

where

$$H=\frac{4(k_P-B_\xi-B_b)L_g^2\lambda_{\min}(F^TF)}{(4k_I\lambda_{\min}(F^TF)+(k_P+B_b+2B_\xi)^2U_g^2)\|F\|^2}.$$

The following three quadratic functions of $(\|E_A\|,\|e_b\|)$ are then all positive definite:

$$
\begin{aligned}
V_1(\|E_A\|,\|e_b\|)&=\frac{1}{2}\|E_A\|^2+\frac{1}{2k_I}\|e_b\|^2-\frac{\epsilon}{L_g}\|F\|\|E_A\|\|e_b\|\\
V_2(\|E_A\|,\|e_b\|)&=\frac{1}{2}\|E_A\|^2+\frac{1}{2k_I}\|e_b\|^2+\frac{\epsilon}{L_g}\|F\|\|E_A\|\|e_b\|\\
V_3(\|E_A\|,\|e_b\|)&=\left(k_P-(B_\xi+B_b)-\frac{\epsilon k_I\|F\|^2}{L_g^2}\right)\|E_A\|^2\\
&\quad+\frac{\epsilon\lambda_{\min}(F^TF)}{U_g^2}\|e_b\|^2\\
&\quad-\frac{\epsilon(k_P+B_b+2B_\xi)}{L_g}\|F\|\|E_A\|\|e_b\|.
\end{aligned}
$$

Hence, there are numbers $\alpha>0$ and $\beta>0$ such that (23) holds. Let

$$V(E_A,e_b)=\frac{1}{2}\|E_A\|^2+\frac{1}{2k_I}\|e_b\|^2-\epsilon\langle E_A,e_bA\rangle$$

which satisfies (24). Since b is constant by assumption, it follows from (8) and (9) that

$$
\begin{aligned}
\dot{E}_A&=-\xi_{\mathrm{m}}E_A+e_bA-k_PE_A\\
\dot{e}_b&=-k_I\,\pi_{\mathfrak{g}}(E_AA^T).
\end{aligned}
$$

Along any trajectory of the composite system consisting of rigid body (1) and observer (9)

$$
\begin{aligned}
\frac{dV}{dt}&=\langle E_A,-\xi_{\mathrm{m}}E_A+e_bA-k_PE_A\rangle-\langle e_b,\pi_{\mathfrak{g}}(E_AA^T)\rangle\\
&\quad-\epsilon\langle-\xi_{\mathrm{m}}E_A+e_bA-k_PE_A,e_bA\rangle\\
&\quad+\epsilon\langle E_A,e_b\xi A\rangle+\epsilon k_I\langle E_A,\pi_{\mathfrak{g}}(E_AA^T)A\rangle\\
&\le-\left(k_P-(B_\xi+B_b)-\frac{\epsilon k_I\|F\|^2}{L_g^2}\right)\|E_A\|^2\\
&\quad-\frac{\epsilon\lambda_{\min}(F^TF)}{U_g^2}\|e_b\|^2\\
&\quad+\frac{\epsilon(k_P+B_b+2B_\xi)\|F\|}{L_g}\|E_A\|\|e_b\|\\
&=-V_3\le-\beta V_2\le-\beta V.
\end{aligned}
$$

The rest of the proof is identical to the corresponding part in the proof of Theorem 1, so it is omitted. ∎

Proof of Theorem 3

Proof: The measured matrix $A=Fg$ obeys (21). Let

$$\mathcal{E}_A=I-A^{-1}\bar{A},\quad e_b=b-\bar{b}.$$

From (21) and (11),

$$
\begin{aligned}
\dot{\mathcal{E}}_A&=\mathcal{E}_A\xi_{\mathrm{m}}-\xi\mathcal{E}_A-e_b-k_P\mathcal{E}_A\\
\dot{e}_b&=k_I\,\pi_{\mathfrak{g}}(\mathcal{E}_A).
\end{aligned}
$$

There is an $\epsilon>0$ such that

$$0<\epsilon<\min\left\{\frac{1}{\sqrt{k_I}},\frac{4(k_P-2B_\xi-B_b)}{4k_I+(k_P+2B_\xi+B_b)^2}\right\}.$$

The following three quadratic functions of $(\|E_A\|,\|e_b\|)$ are then all positive definite:

$$
\begin{aligned}
V_1(\|\mathcal{E}_A\|,\|e_b\|)&=\frac{1}{2}\|\mathcal{E}_A\|^2+\frac{1}{2k_I}\|e_b\|^2-\epsilon\|\mathcal{E}_A\|\|e_b\|\\
V_2(\|\mathcal{E}_A\|,\|e_b\|)&=\frac{1}{2}\|\mathcal{E}_A\|^2+\frac{1}{2k_I}\|e_b\|^2+\epsilon\|\mathcal{E}_A\|\|e_b\|\\
V_3(\|\mathcal{E}_A\|,\|e_b\|)&=(k_P-(2B_\xi+B_b)-\epsilon k_I)\|\mathcal{E}_A\|^2\\
&\quad+\epsilon\|e_b\|^2-\epsilon(k_P+B_b+2B_\xi)\|E_A\|\|e_b\|.
\end{aligned}
$$

Hence, there are numbers $\alpha>0$ and $\beta>0$ such that (23) holds. Let

$$V(\mathcal{E}_A,e_b)=\frac{1}{2}\|\mathcal{E}_A\|^2+\frac{1}{2k_I}\|e_b\|^2+\epsilon\langle\mathcal{E}_A,e_b\rangle$$

which satisfies (24) for all $(\mathcal{E}_A, e_b) \in \mathbb{R}^{n\times n} \times \mathfrak{g}$. It is easy to show that along any trajectory of the composite system consisting of rigid body (1) and observer (11)

$$\dot{V} \leq -V_3 \leq -\beta V_2 \leq -\beta V.$$

As in the proof of Theorem 1, it is east to show that there are numbers $\tilde{C} > 0$ and $a > 0$ such that

$$\|\mathcal{E}_A(t)\| + \|e_b(t)\| \leq \tilde{C}(\|\mathcal{E}_A(0)\| + \|e_b(0)\|)e^{-at} \tag{25}$$

for all $(\bar{A}(0), \bar{b}(0)) \in \mathbb{R}^{n\times n} \times \mathfrak{g}$ and all $t \geq 0$. Since $\mathcal{E}_A = I - A^{-1}\bar{A} = A^{-1}E_A$ or $E_A = A\mathcal{E}_A = Fg\mathcal{E}_A$, we have

$$\sigma_{\min}(F)L_g\|\mathcal{E}_A\| \leq \|E_A\| \leq \sigma_{\max}(F)U_g\|\mathcal{E}_A\|. \tag{26}$$

It follows from (25) and (26) that there is a number $C > 0$ such that

$$\|E_A(t)\| + \|e_b(t)\| \leq C(\|E_A(0)\| + \|e_b(0)\|)e^{-at}$$

for all $(\bar{A}(0), \bar{b}(0)) \in \mathbb{R}^{n\times n} \times \mathfrak{g}$ and all $t \geq 0$. ∎

References

[1] P. Batista, C. Silvestre, and P. Oliveira, "Globally exponentially stable cascade observers for attitude estimation," *Control Eng. Pract.*, vol. 20, no. 2, pp. 148–155, Feb. 2012.

[2] A. M. Bloch, "Nonholonomic mechanics," in *Nonholonomic Mechanics and Control*. Berlin, Germany: Springer, 2003, pp. 207–276.

[3] D. E. Chang, "On controller design for systems on manifolds in euclidean space," *Int. J. Robust Nonlinear Control*, vol. 28, no. 16, pp. 4981–4998, Aug. 2018.

[4] F. A. Goodarzi and T. Lee, "Global formulation of an extended Kalman filter on SE(3) for geometric control of a quadrotor UAV," *J. Intell. Robot. Syst.*, vol. 88, nos. 2–4, pp. 395–413, Mar. 2017.

[5] M.-D. Hua, T. Hamel, R. Mahony, and J. Trumpf, "Gradient-like observer design on the special euclidean group SE(3) with system outputs on the real projective space," in *Proc. 54th IEEE Conf. Decis. Control*, Dec. 2015, pp. 2139–2145.

[6] A. Khosravian, J. Trumpf, R. Mahony, and C. Lageman, "Bias estimation for invariant systems on lie groups with homogeneous outputs," in *Proc. 52nd IEEE Conf. Decis. Control*, Dec. 2013, pp. 4454–4460.

[7] A. Khosravian, J. Trumpf, R. Mahony, and C. Lageman, "Observers for invariant systems on lie groups with biased input measurements and homogeneous outputs," *Automatica*, vol. 55, pp. 19–26, May 2015.

[8] C. Lageman, J. Trumpf, and R. Mahony, "Gradient-like observers for invariant dynamics on a lie group," *IEEE Trans. Autom. Control*, vol. 55, no. 2, pp. 367–377, Feb. 2010.

[9] P. Martin and I. Sarras, "A global observer for attitude and gyro biases from vector measurements," *IFAC-PapersOnLine*, vol. 50, no. 1, pp. 15409–15415, Jul. 2017.

[10] M. Wang and A. Tayebi, "Hybrid pose and velocity-bias estimation on $SE(3)$ using inertial and landmark measurements," *IEEE Trans. Autom. Control*, vol. 64, no. 8, pp. 3399–3406, Aug. 2019.

[11] J. F. Vasconcelos, R. Cunha, C. Silvestre, and P. Oliveira, "A nonlinear position and attitude observer on SE(3) using landmark measurements," *Syst. Control Lett.*, vol. 59, nos. 3/4, pp. 155–166, Mar. 2010.

[12] R. A. Horn and C. R. Johnson, "Matrix analysis," in *Matrix Analysis*, 2nd ed. Cambridge, U.K. University Press, 2009.

[13] A. Moeini and M. Namvar, "Global attitude/position estimation using landmark and biased velocity measurements," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 52, no. 2, pp. 852–862, Apr. 2016.

[14] D. E. Chang, K. S. Phogat, and J. Choi, "Model predictive tracking control for invariant systems on matrix lie groups via stable embedding into euclidean spaces," *IEEE Trans. Autom. Control*, vol. 65, no. 7, pp. 3191–3198, Jul. 2019.