SC651 : Assignment 2

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Question 1

I have referred to the Wikipedia page on Wahba's problem to get the implementation of the algorithm using SVD. This is the code I used to apply the algorithm,

```
B = np.zeros((3,3))
for i in range(len(DATA)):
    w = DATA[i][1]
    v = DATA[i][0]
    w = w.reshape((len(w),1))
    v = v.reshape((len(v),1))
    B += w @ v.T
U, S , Vt = np.linalg.svd(B)
M = np.diag([1, 1, np.linalg.det(U) * np.linalg.det(Vt)])
R = (U @ M @ Vt)
```

Using the data given I obtained the rotation matrix to be,

$$R = \begin{bmatrix} 0.97307269 & -0.21660168 & 0.07882421 \\ 0.22659041 & 0.96160772 & -0.15481402 \\ -0.042265 & 0.16850611 & 0.98479407 \end{bmatrix}$$

The mean squared error(between v and w) in the original data was approximately 0.51, whereas after applying Wahba's algorithm, the mean squared error (calculated as the mean squared error between Rv and w) reduced to 0.492.

Question 2

$$\text{For a square matrix } A, exp(A) := \sum_{i=0}^{\infty} \frac{A^i}{i!} \tag{1}$$

Part a

$$\hat{x} = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}$$

Because $\hat{x}$ is 3×3 skew-symmetric matrix, $\hat{x}^3 = -(x_1^2 + x_2^2 + x_3^2)\hat{x}$. Let us define $a^2 := x_1^2 + x_2^2 + x_3^2$. This means that for all positive integers m,

$$\hat{x}^{2m+1} = (-1)^m a^{2m} \hat{x} \quad \text{and} \quad \hat{x}^{2m} = (-1)^{m-1} a^{2m-2} \hat{x}^2$$

Therefore,

$$exp(\hat{x}) = I + \sum_{m=0}^{\infty} \frac{\hat{x}^{2m+1}}{(2m+1)!} + \sum_{m=1}^{\infty} \frac{\hat{x}^{2m}}{(2m)!}$$

$$= I + \sum_{m=0}^{\infty} \frac{(-1)^m a^{2m}}{(2m+1)!}\hat{x} + \sum_{m=1}^{\infty} \frac{(-1)^{m-1} a^{2m-2}}{(2m)!}\hat{x}^2$$
$$= I + \frac{1}{a}\sum_{m=0}^{\infty} \frac{(-1)^m a^{2m+1}}{(2m+1)!}\hat{x} - \frac{1}{a^2}\sum_{m=1}^{\infty} \frac{(-1)^m a^{2m}}{(2m)!}\hat{x}^2$$

The two summations are the Taylor expansions of $sin(a)$ and $1 - cos(a)$ respectively. Thus,

$$exp(\hat{x}) = I + \frac{sin(a)}{a}\hat{x} + \frac{1 - cos(a)}{a^2}\hat{x}^2$$

Part b

$$\tilde{x} = \begin{pmatrix} x_3 & x_1 - \iota x_2 \\ x_1 + \iota x_2 & -x_3 \end{pmatrix}$$

Evaluating $\tilde{x}^2$, we get $\tilde{x}^2 = (x_1^2 + x_2^2 + x_3^2)I$ where I is the 2×2 identity matrix.
Let us define $a^2 := x_1^2 + x_2^2 + x_3^2$.

$$\tilde{x}^{2m+1} = a^{2m}\tilde{x} \quad \text{and} \quad \hat{x}^{2m} = a^{2m}I$$
$$exp(\tilde{x}) = \sum_{m=0}^{\infty} \frac{\tilde{x}^{2m}}{(2m)!} + \sum_{m=0}^{\infty} \frac{\tilde{x}^{2m+1}}{(2m+1)!}$$
$$= \sum_{m=0}^{\infty} \frac{a^{2m}}{(2m)!}I + \sum_{m=0}^{\infty} \frac{a^{2m}}{(2m+1)!}\tilde{x} = \sum_{m=0}^{\infty} \frac{a^{2m}}{(2m)!}I + \frac{1}{a}\sum_{m=0}^{\infty} \frac{a^{2m+1}}{(2m+1)!}\tilde{x}$$
$$exp(\tilde{x}) = cosh(a)I + \frac{sinh(a)}{a}\tilde{x}$$

where $cosh(a) = \frac{e^a + e^{-a}}{2}$ and $sinh(a) = \frac{e^a - e^{-a}}{2}$

Question 3

The elementary rotation matrices for rotations about X, Y and Z axes are,

$$R_X(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos(\theta) & -sin(\theta) \\ 0 & sin(\theta) & cos(\theta) \end{pmatrix} R_Y(\theta) = \begin{pmatrix} cos(\theta) & 0 & sin(\theta) \\ 0 & 1 & 0 \\ -sin(\theta) & 0 & cos(\theta) \end{pmatrix} R_Z(\theta) = \begin{pmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

From now I will use $c(\theta)$ for $cos(\theta)$ and $s(\theta)$ for $sin(\theta)$

XZY Euler angles

$$R(\alpha, \beta, \gamma) = R_Y(\gamma)R_Z(\beta)R_X(\alpha)$$
$$= \begin{pmatrix} c(\gamma) & 0 & s(\gamma) \\ 0 & 1 & 0 \\ -s(\gamma) & 0 & c(\gamma) \end{pmatrix} \begin{pmatrix} c(\beta) & -c(\alpha)s(\beta) & s(\alpha)s(\beta) \\ s(\beta) & c(\alpha)c(\beta) & -s(\alpha)c(\beta) \\ 0 & s(\alpha) & c(\alpha) \end{pmatrix}$$
$$= \begin{pmatrix} c(\beta)c(\gamma) & s(\alpha)s(\gamma) - s(\beta)c(\alpha)c(\gamma) & c(\alpha)s(\gamma) + s(\beta)c(\gamma)s(\alpha) \\ s(\beta) & c(\alpha)c(\beta) & -s(\alpha)c(\beta) \\ -s(\gamma)c(\beta) & s(\alpha)c(\gamma) + s(\beta)c(\alpha)s(\gamma) & c(\alpha)c(\gamma) - s(\beta)s(\alpha)s(\gamma) \end{pmatrix}$$
$$\text{at } \beta = \frac{\pi}{2}, \quad R(\alpha, \frac{\pi}{2}, \gamma) = \begin{pmatrix} 0 & -c(\alpha+\gamma) & s(\alpha+\gamma) \\ 1 & 0 & 0 \\ 0 & s(\alpha+\gamma) & c(\alpha+\gamma) \end{pmatrix}$$

This is a singularity as α and γ are not uniquely determinable. For a rotation of $\frac{\pi}{2}$ about the middle axis, the first and last axes become parallel.

YZY Euler angles

$$R(\alpha,\beta,\gamma) = R_Y(\gamma)R_Z(\beta)R_Y(\alpha)$$

$$= \begin{pmatrix} c(\gamma) & 0 & s(\gamma) \\ 0 & 1 & 0 \\ -s(\gamma) & 0 & c(\gamma) \end{pmatrix} \begin{pmatrix} c(\alpha)c(\beta) & -s(\beta) & s(\alpha)c(\beta) \\ c(\alpha)s(\beta) & c(\beta) & s(\alpha)s(\beta) \\ -s(\alpha) & 0 & c(\alpha) \end{pmatrix}$$

$$= \begin{pmatrix} c(\beta)c(\alpha)c(\gamma) - s(\alpha)s(\gamma) & -s(\beta)c(\gamma) & c(\beta)s(\alpha)c(\gamma) + c(\alpha)s(\gamma) \\ c(\alpha)s(\beta) & c(\beta) & s(\alpha)s(\beta) \\ -s(\alpha)c(\gamma) - c(\beta)s(\gamma)c(\alpha) & s(\beta)s(\gamma) & c(\alpha)c(\gamma) - c(\beta)s(\alpha)s(\gamma) \end{pmatrix}$$

$$\text{at } \gamma = 0, \quad R(\alpha,0,\gamma) = \begin{pmatrix} c(\alpha+\gamma) & 0 & s(\alpha+\gamma) \\ 0 & 1 & 0 \\ -s(\alpha+\gamma) & 0 & c(\alpha+\gamma) \end{pmatrix}$$

This is a singularity as α and γ are not uniquely determinable. For a rotation of 0 radians about the middle axis, the first and last axes become parallel.

Thus any euler angle representation of the form **ABC** or **ABA**, where **A**,**B**,**C** $\in$ {**X, Y, Z**} and $\mathbf{A} \neq \mathbf{B} \neq \mathbf{C}$, will have a singularity when the rotation is about the middle axis by either $\frac{\pi}{2}$ radians(**ABC**) or 0 radians(**ABA**).