COL333/671: Introduction to AI

Semester I, 2024-25

Solving Problems by Searching
Informed Search

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Outline

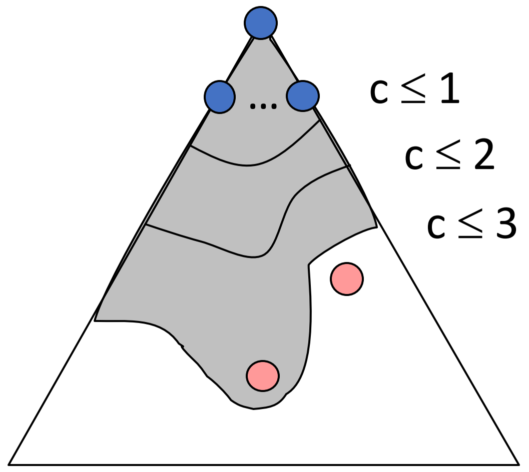
- Last Class
  - Uninformed Search
- This Class
  - Informed Search
    - Key idea behind Informed Search
    - Best First Search
    - Greedy Best First Search
    - A* Search: evaluation Function
- Reference Material
  - AIMA Ch. 3

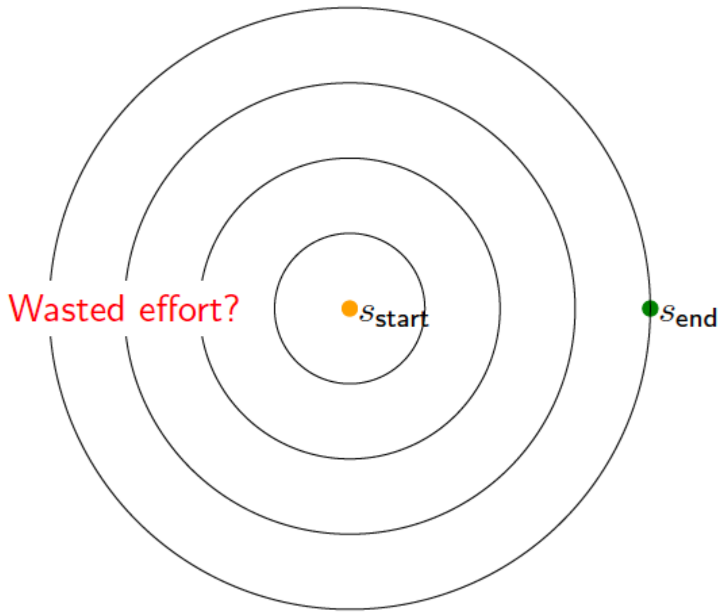
Acknowledgement

These slides are intended for teaching purposes only. Some material has been used/adapted from web sources and from slides by Doina Precup, Dorsa Sadigh, Percy Liang, Mausam, Dan Klein, Nicholas Roy and others.

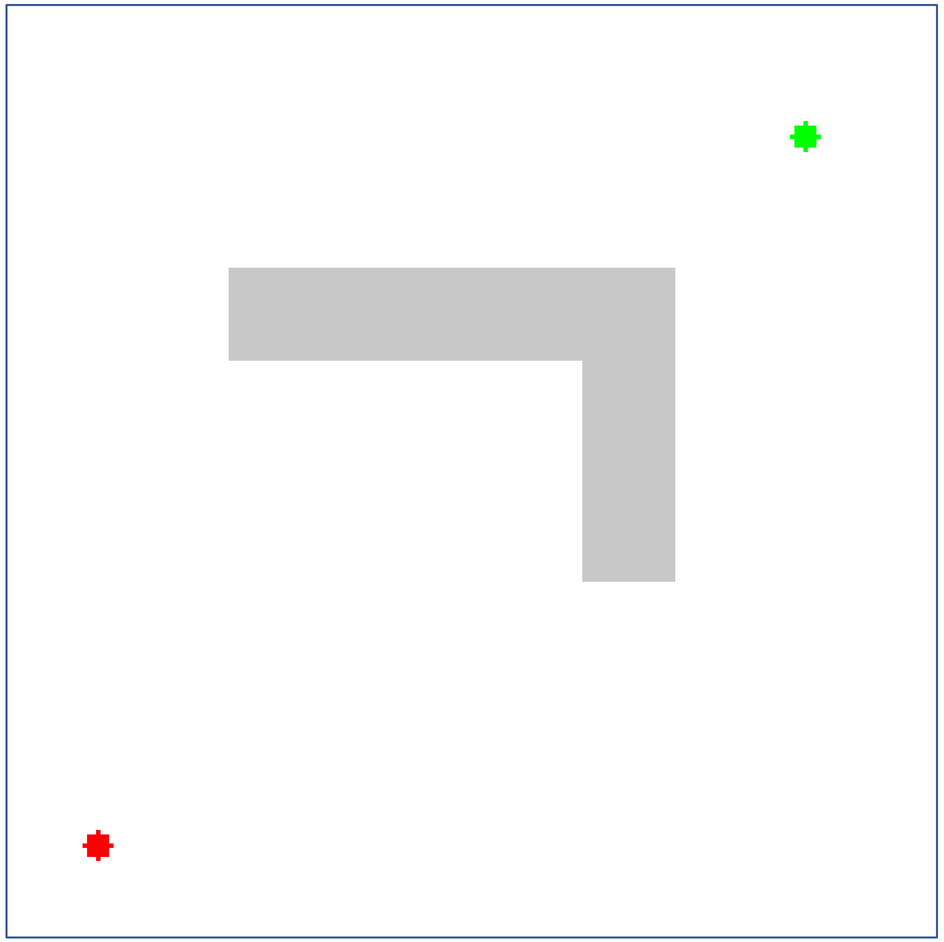
Informed Search

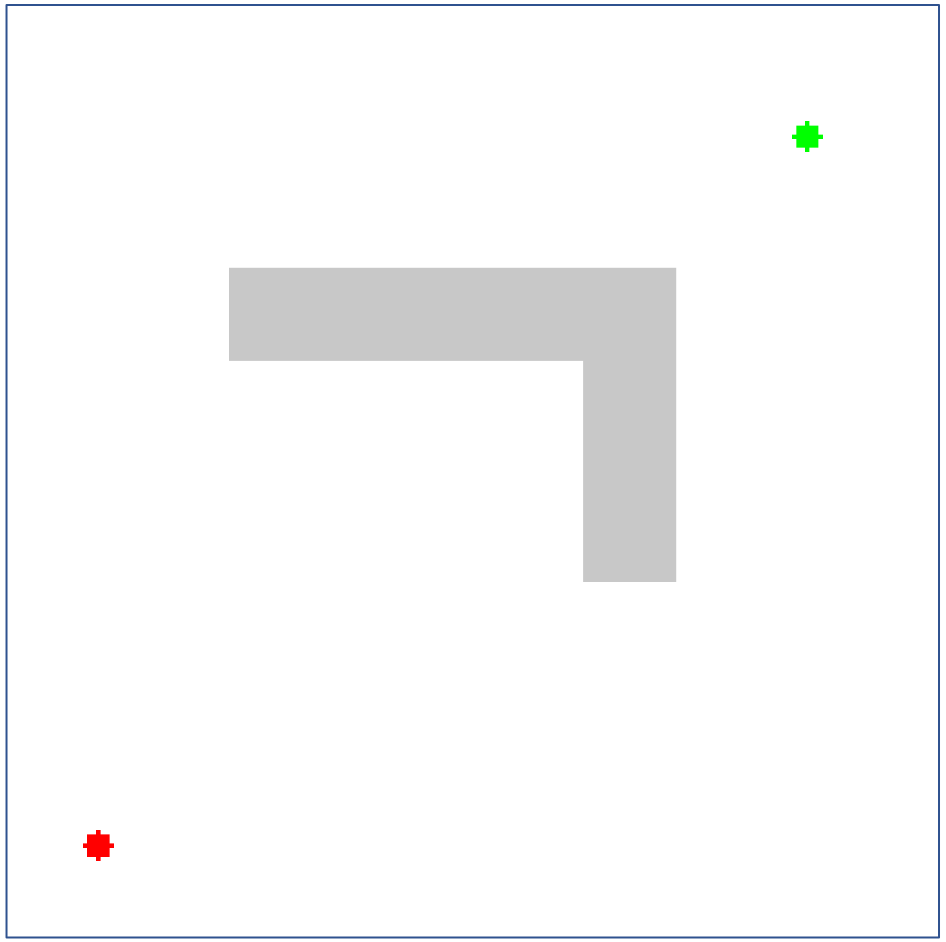
- Uniform Cost Search
  - Expand the lowest cost path
  - Complete
  - Optimal
- Problem
  - Explores options in every “direction”
  - No information about goal location
- Informed Search
  - Use problem-specific knowledge beyond the definition of the problem to guide the search towards the goal.





What other knowledge can be leveraged during search?



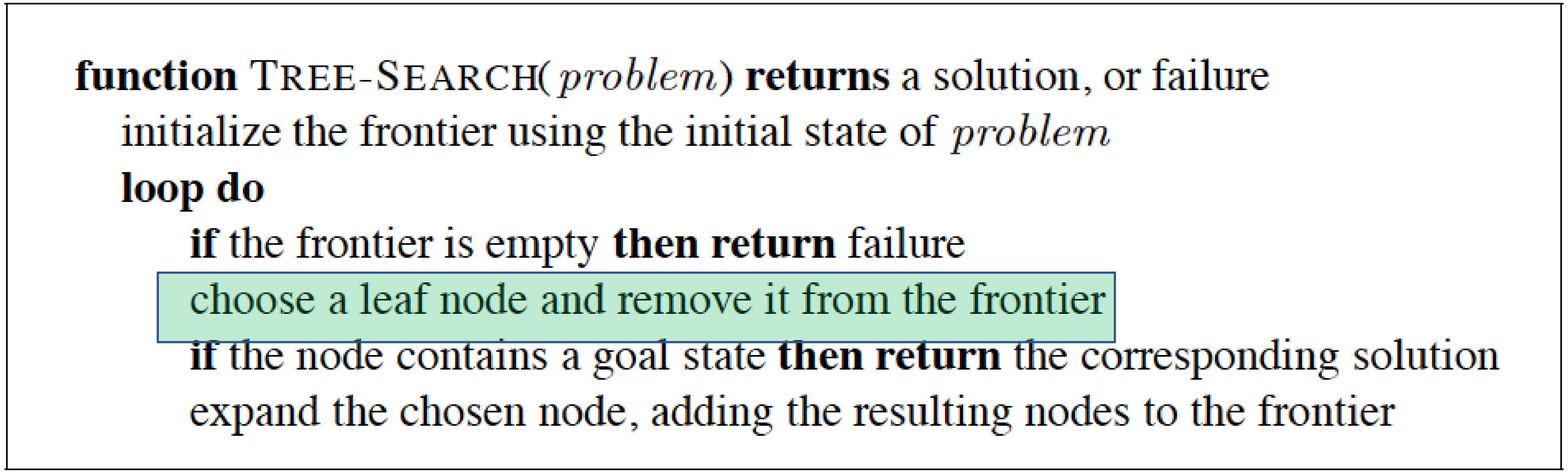


2D route finding problem: (left) uninformed search (right) an approach that uses approx. distance to goal information.

Recall: Tree Search

Note: Central to tree search is how nodes (partial plans) are kept in the frontier are expanded (prioritized)

Prioritization essentially means among many partial plans which one we should search first (over other options we have in the frontier).



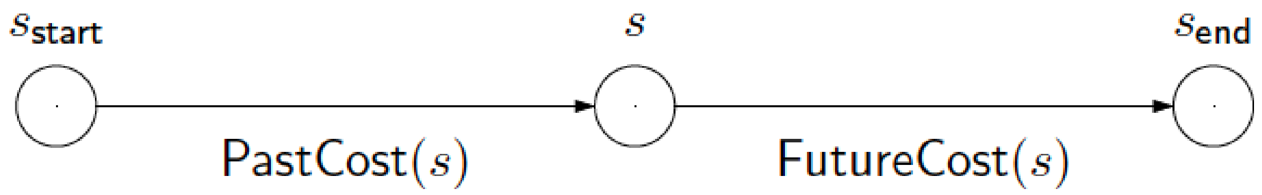
```
function TREE-SEARCH(problem) returns a solution, or failure
  initialize the frontier using the initial state of problem
  loop do
      if the frontier is empty then return failure
      choose a leaf node and remove it from the frontier
      if the node contains a goal state then return the corresponding solution
      expand the chosen node, adding the resulting nodes to the frontier
```

Evaluation of a node during search

- **Best First Search**
  - Always choose the node from frontier that has the best evaluation (according to a function).
  - The search orders nodes in the frontier (via priority queue) for expansion using this evaluation.

- **Incorporate an evaluation of every node**
  - Let f() denote the **evaluation** of a node.
  - Estimates the **desirability** of a node for the purposes of potentially reaching the goal. A search strategy is defined by picking the order of node expansion.
  - Expand **most desirable unexpanded node**. Order the nodes in frontier in decreasing order of desirability.

Approaches for evaluating a node



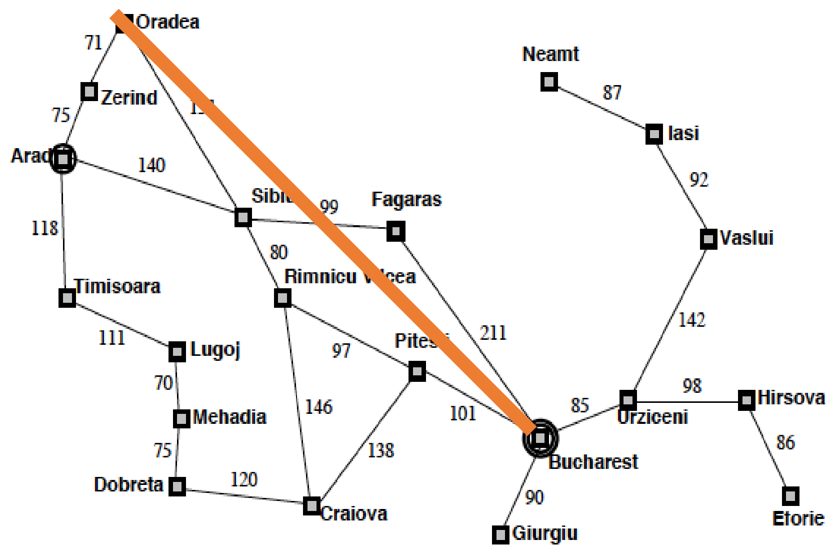
- **Central Idea**
  - To evaluate a need we need two things: cost so far and cost to go.
  - **Uninformed** search methods expand nodes based on the cost (or distance) from the start state to the the current state, **$d(s_0, s)$**
    - *Evaluation based on the exact cost so far.*
  - **Informed** search methods additionally **estimate** of the cost (or distance) from the current state to the goal state, **$d(s, s_g)$** and **use it** in deciding which node to expand next.
    - *Evaluation based on the exact cost so far* + an estimate of cost to go
  - *Note: What if we knew the exact distance to goal $d(s, s_g)$?*
    - *Then there is no need to search, we could just be greedy!* In practice, we do not know that exactly and must make an "estimate".

A heuristic *approximates the cost to goal*

- **An *intuition* about "approximate cost to goal"**
  - Even if we do not know d(s, s_g) exactly, we often have some ***intuition*** about this distance. This intuition is called a heuristic, **h(n).**
- **Heuristic function h(n)**
  - Assigns an estimate of the actual cost to go for each state.
    - Formally, h(n) = *estimated* cost of the *cheapest* path from the state at node n to a goal state.
  - Heuristic function can be *arbitrary*, *non-negative*, *problem-specific* functions.
    - Constraint, *h(n) = 0* if n is a goal. If you are at the goal, then there is no more cost to be incurred.

Example Heuristic – Path Planning

- Consider a path along a road system
- What is a reasonable heuristic?
  - The straight-line Euclidean distance from one place to another
- Is it exact?
  - Not always. Actual paths are rarely straight.



Straight−line distance to Bucharest

Arad	366
Bucharest	0
Craiova	160
Dobreta	242
Eforie	161
Fagaras	178
Giurgiu	77
Hirsova	151
Iasi	226
Lugoj	244
Mehadia	241
Neamt	234
Oradea	380
Pitesti	98
Rimnicu Vilcea	193
Sibiu	253
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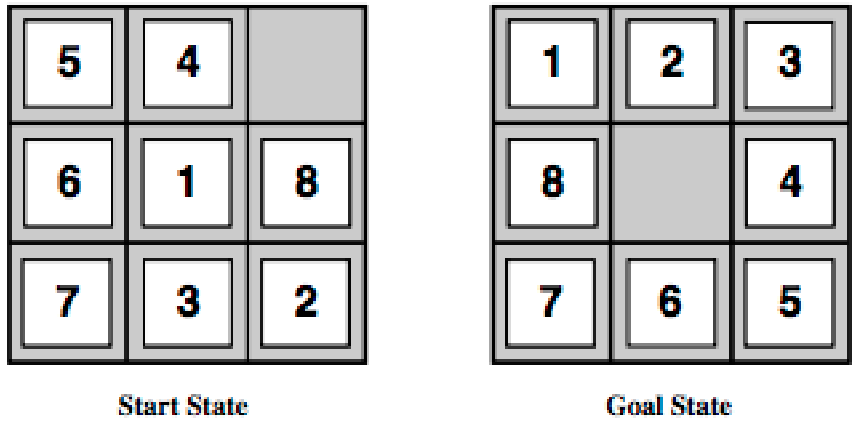
h(x)

Example Heuristic – 8 Puzzle

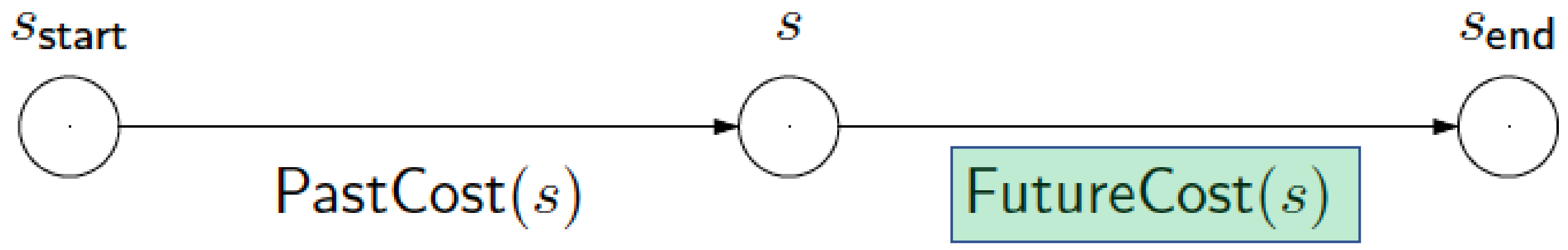
Intuitively, heuristics are trying to estimate how much more effort is needed from the current state to the goal.



Consider the following heuristics:

- h_1 = number of misplaced tiles (=7 in example)
- h_2 = total Manhattan distance (i.e., no. of squares from desired location of each tile) (= 2+3+3+2+4+2+0+2 = 18 in example)

Greedy Best-First Search (only guided by heuristic)

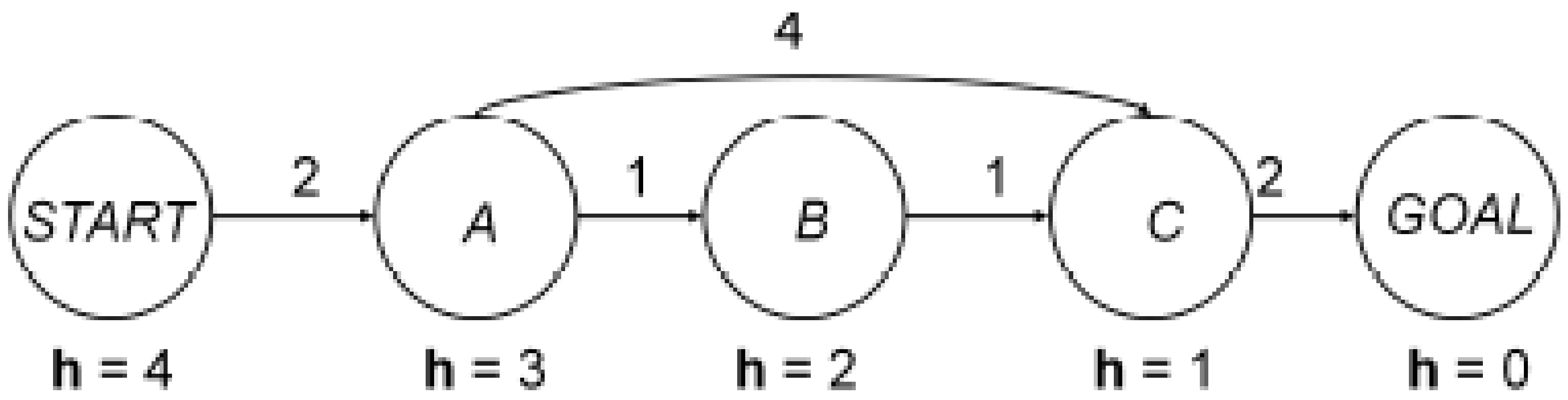


- **Best-First Search**
  - At any time, expand the most promising node on the frontier according to the evaluation function f(n).

- ***Greedy*** Best-First Search
  - Best-first search that uses h(n) as the evaluation function, Only guided by “cost to go” (not “cost so far”).
  - The evaluation function is, **f(n) = h(n)**, the estimated cost from a node n to the goal.

Greedy Best-First Search

- Which path does Greedy Best-First Search return?

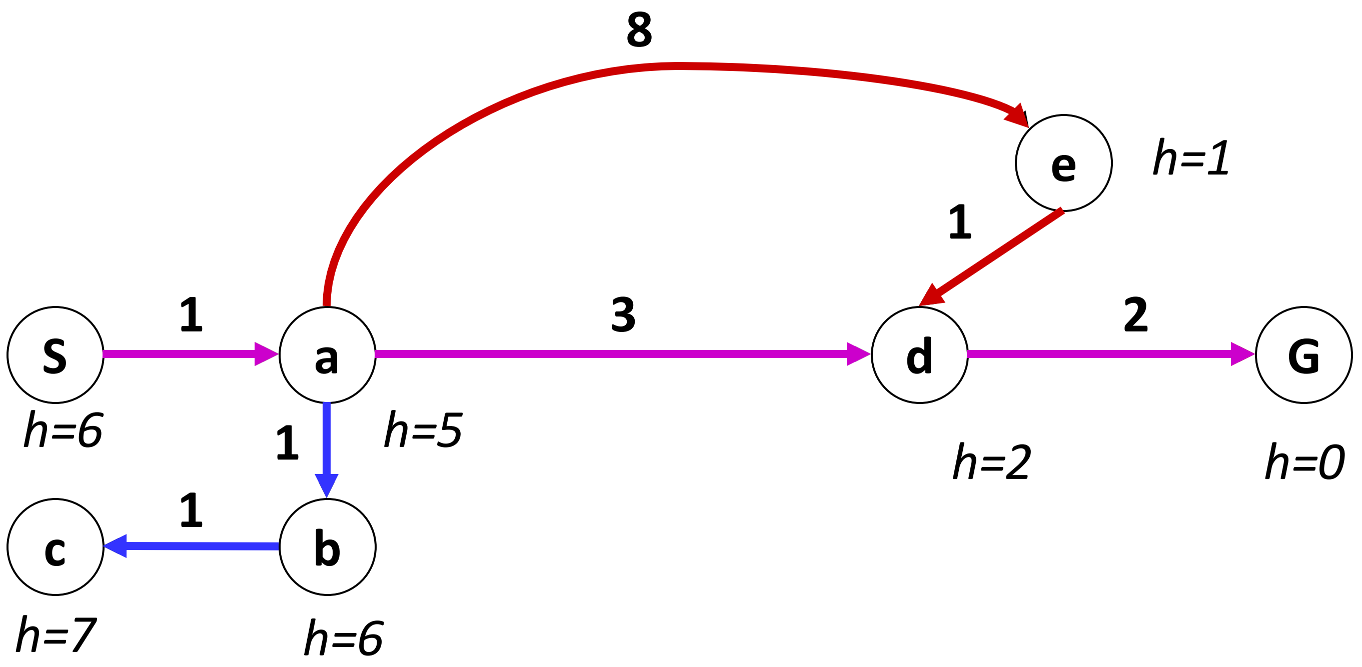


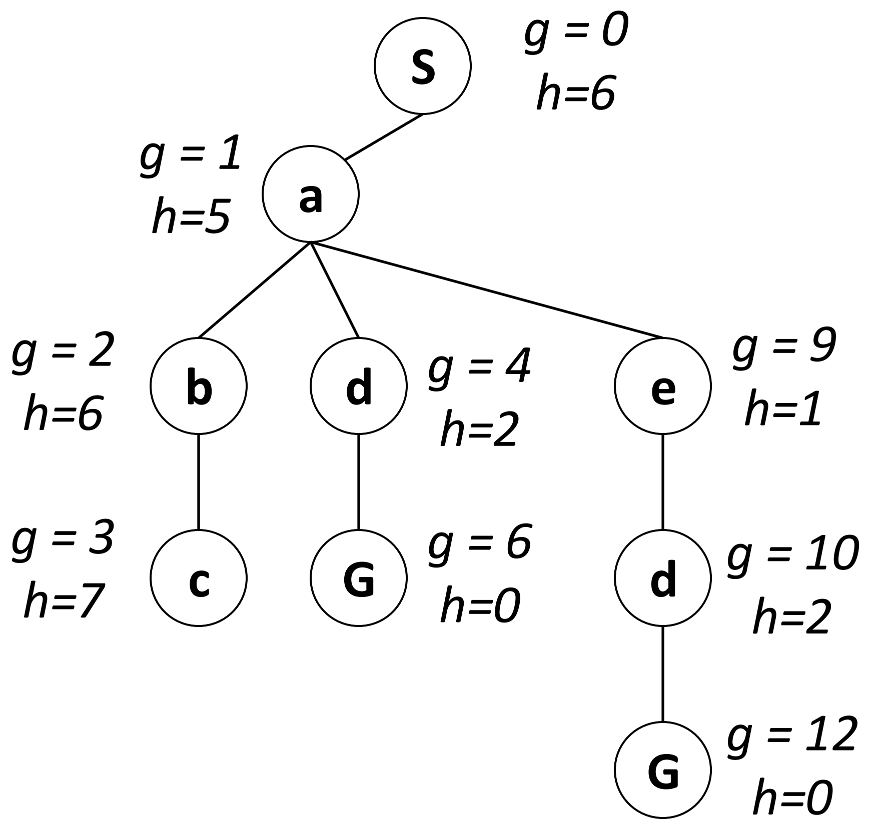
A* Search

- **Core Idea**
  - Combine the greedy search (*the estimated cost to go*) with the uniform-search strategy (*the cost incurred so far*).
  - Minimize estimated path costs. Avoid expanding paths that are already expensive.
- **Always expand node with lowest f(n) first, where**
  - g(n) = **actual cost** from the initial state to n.
  - h(n) = **estimated cost** from n to the next goal.
  - **f(n) = g(n) + h(n)**, the estimated cost of the cheapest solution through n.
- **Can I use any heuristic?**
  - Any heuristic will *not* work. [properties soon]

Example: UCS , Greedy and A* Search

- Uniform-cost orders by path cost, or *backward cost* g(n)
- Greedy orders by goal proximity, or *forward cost* h(n)



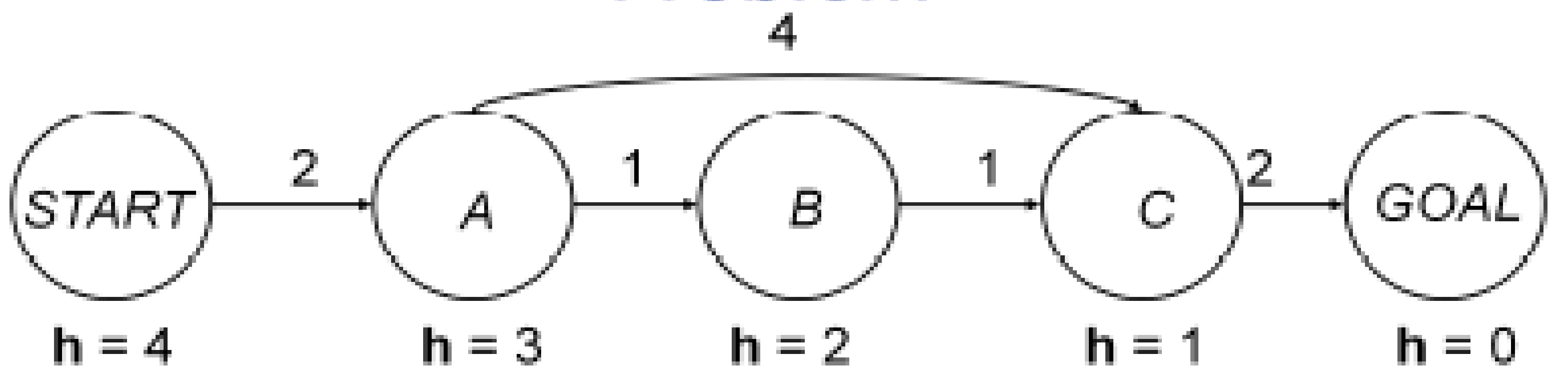


- A* Search orders by the sum: f(n) = g(n) + h(n)

Example: Teg Grenager

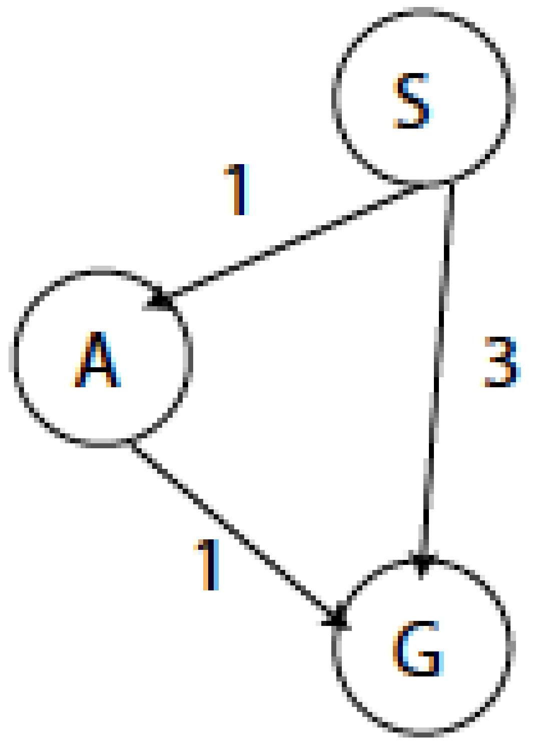
Example

Which path will A* search find?



Does any heuristic function work?

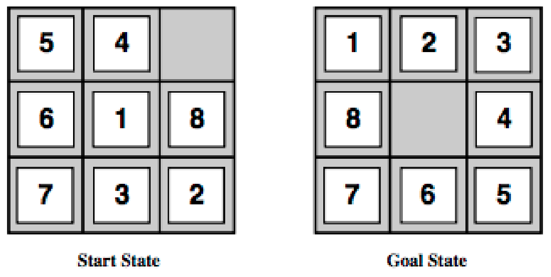
- For the following choices, would the optimal solution be found?
  - h(A) = 1
  - h(A) = 2
  - h(A) = 3
- Can we put conditions on the choice of heuristic to guarantee optimality?



Admissible Heuristics

- Let **h*(n)** be the actual **shortest path** from n to any goal state.
- Heuristic h is called ***admissible*** if **$h(n) \leq h^*(n)\ \forall n$**.
  - Admissible heuristics are ***optimistic*,** they often think that the cost to the goal is **less than the actual cost.**
- If h is admissible, then $h(g) = 0, \forall g \in G$
  - A **trivial** case of an admissible heuristic is $h(n) = 0, \forall n$.

Admissible or not admissible?



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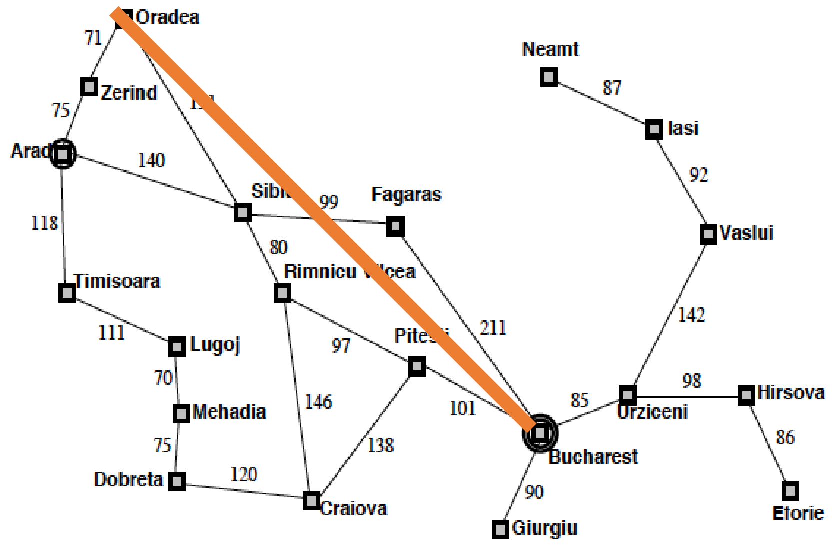
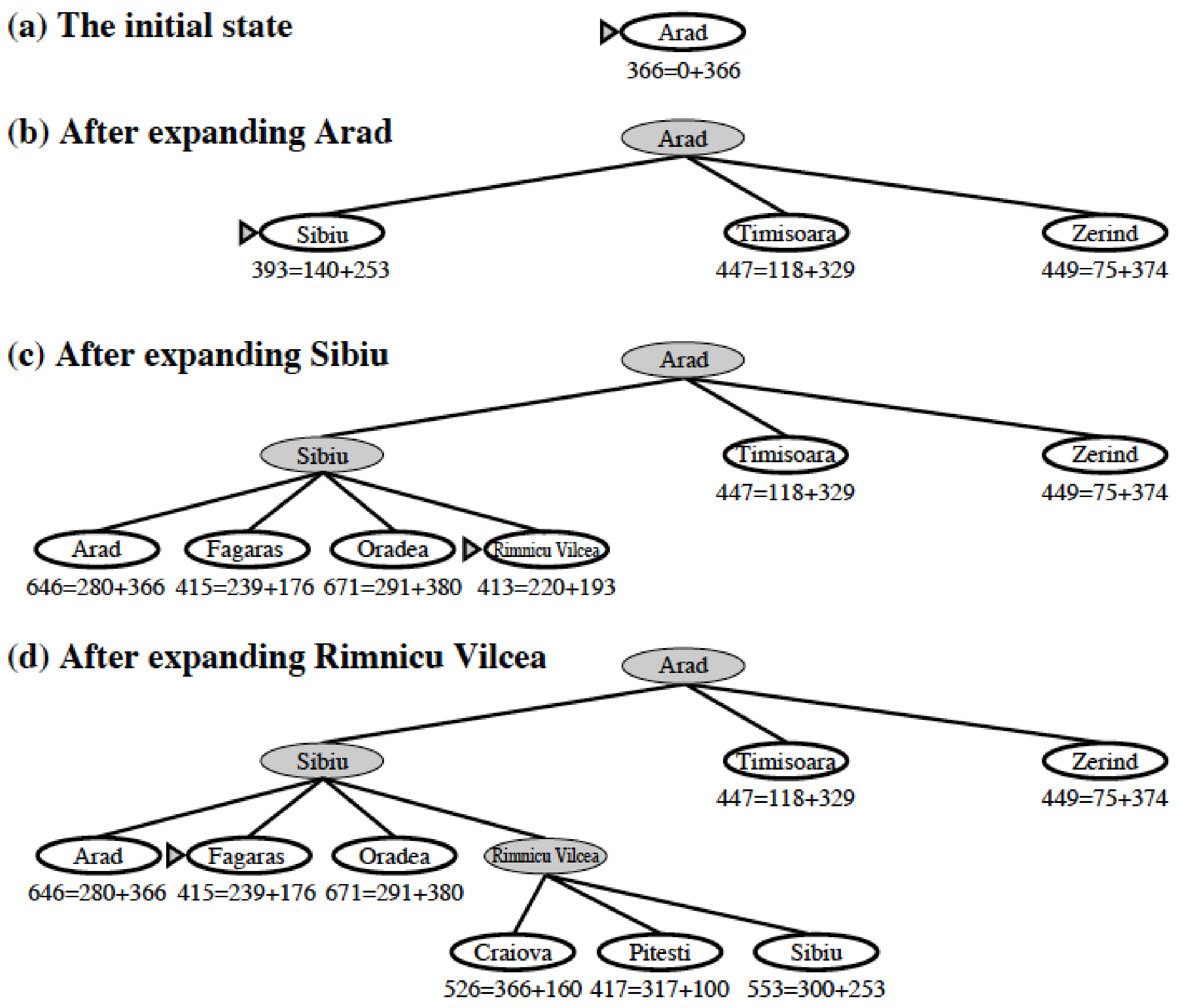
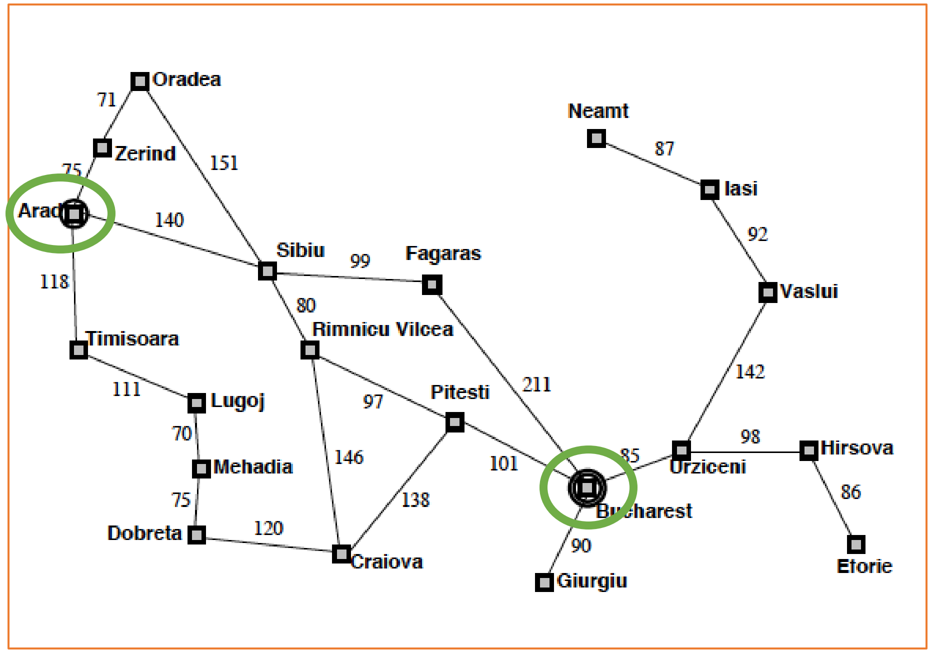


Straight line distance

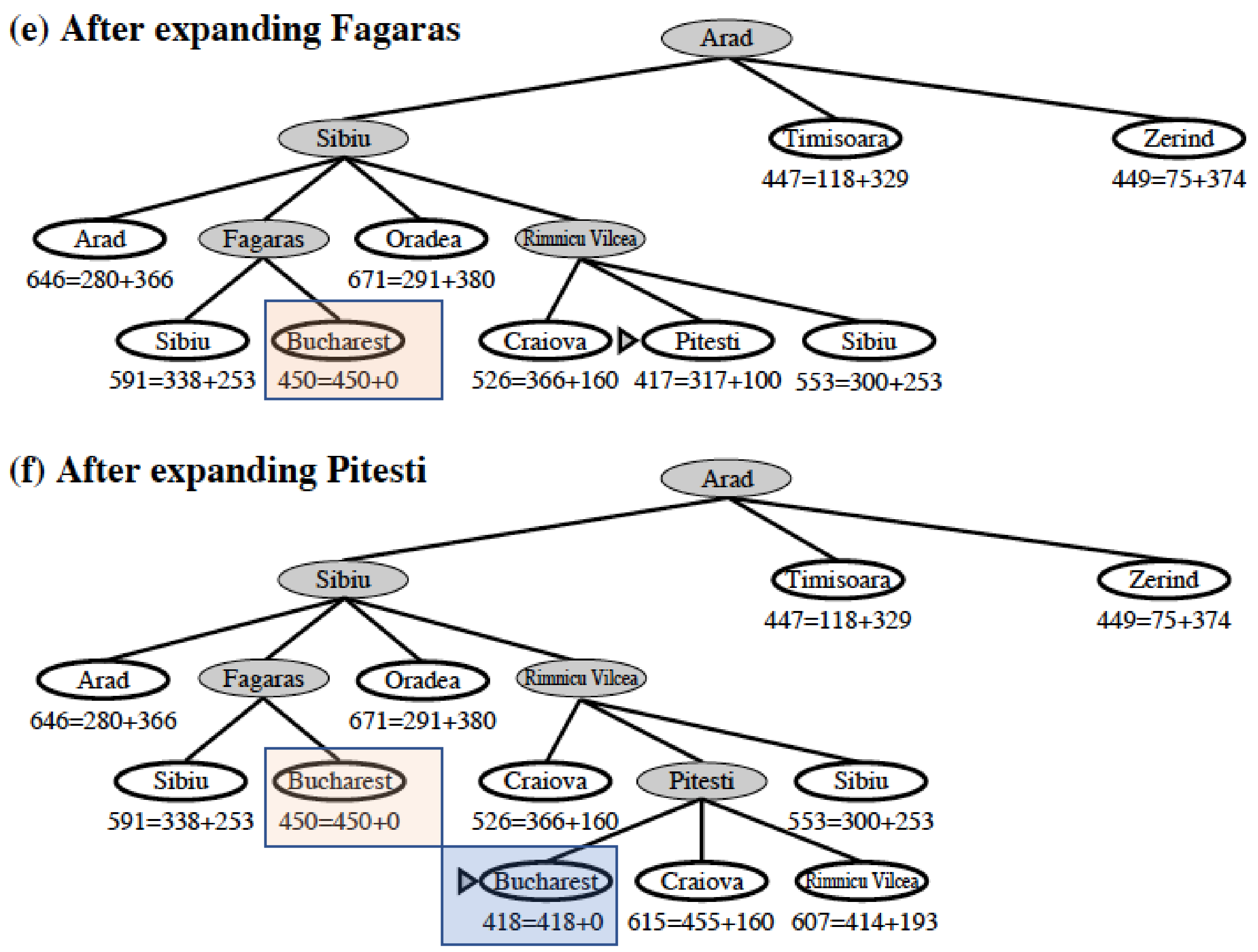
A* Tree Search: Route Finding Example

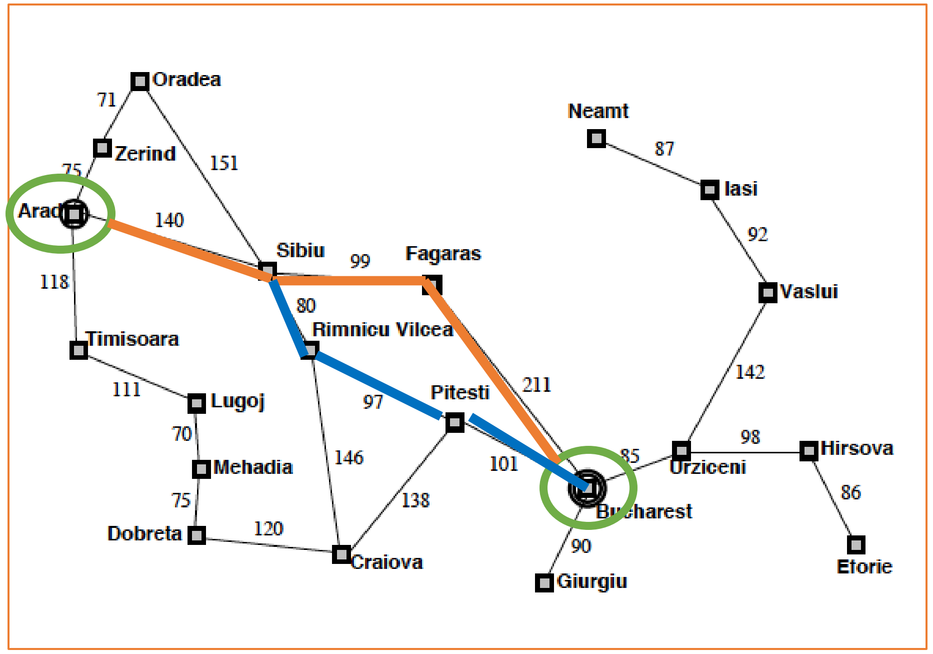




Use of both cost so far and straight line heuristic.

A* Tree Search: Route Finding Example



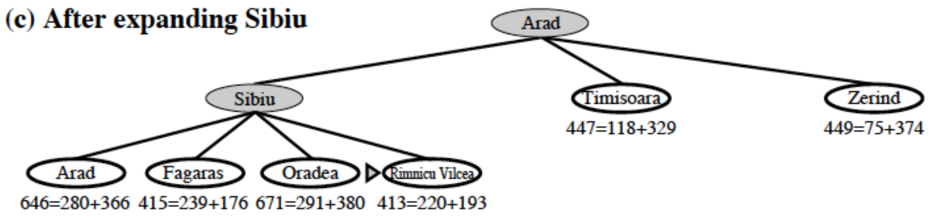


- Bucharest remains on the frontier with cost 450.
- Two paths. One via Fagaras (cost 450) and the other via Rimnicu (cost 418).
- The goal is not popped and the second path is explored.
- A* Tree Search will find the optimal path if the heuristic is admissible.

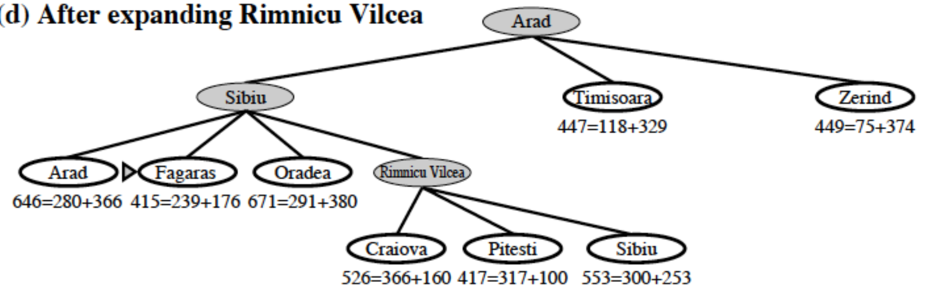
Using an inadmissible h

Search with original heuristic

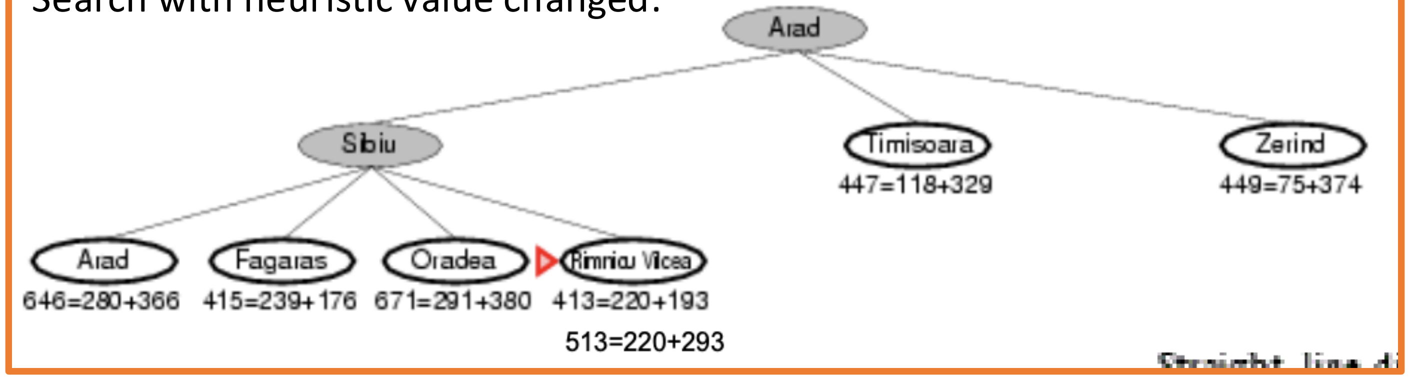
(c) After expanding Sibiu



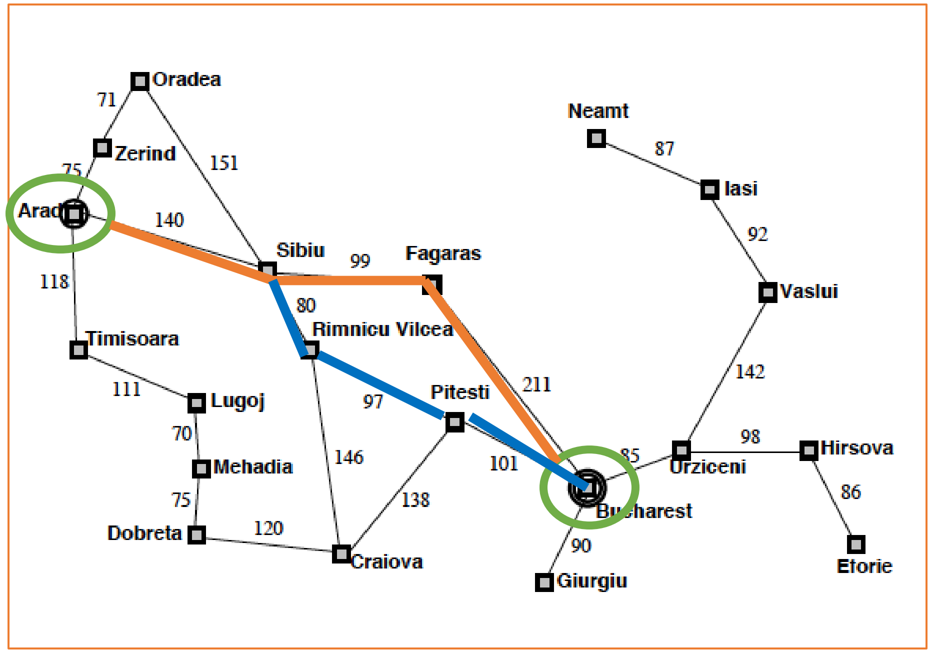
(d) After expanding Rimnicu Vilcea



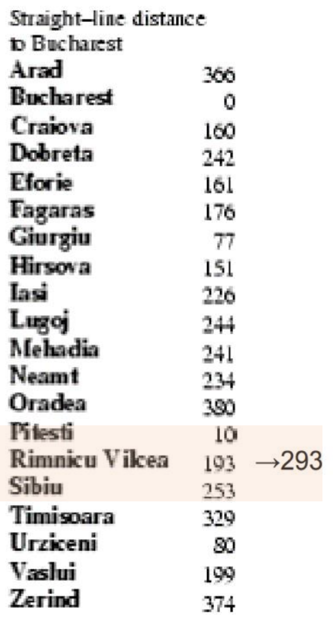
Search with heuristic value changed.



State space graph



Straight line heuristic, we change one value.



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A* Tree Search with Admissible Heuristic leads to an optimal path

- Suppose it finds a suboptimal path, ending in goal state G_1 where $f(G_1) > f^*$ where $f^* = h^*(start)$ = cost of optimal path.
- There must exist a node n which is
  - Unexpanded
  - The path from start to n (stored in the BackPointers(n) values) is the start of a true optimal path

Why must such a node exist? Consider any optimal path $s,n1,n2\ldots$goal. If all along it were expanded, the goal would've been reached along the shortest path.

- $f(n) >= f(G_1)$ (else search wouldn't have ended)
- Also $f(n) = g(n) + h(n)$

  $= g^*(n) + h(n)$ because it's on optimal path

  $<= g^*(n) + h^*(n)$ By the admissibility assumption

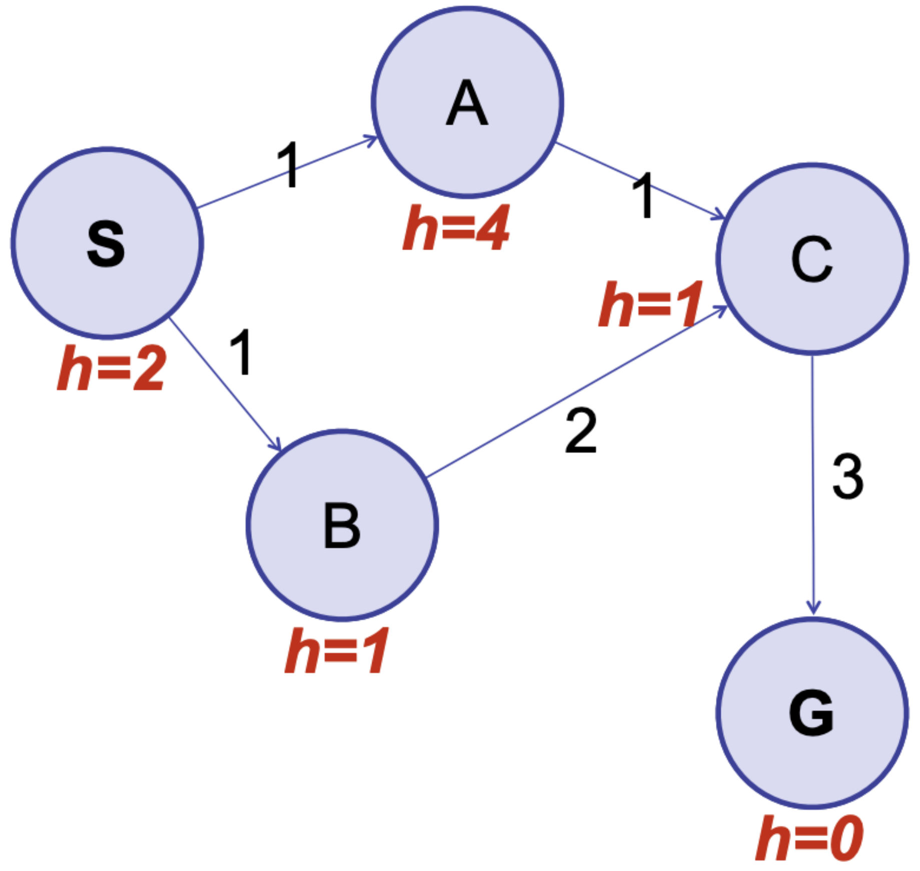
  $= f^*$ Because n is on the optimal path

So $f^* >= f(n) >= f(G_1)$ contradicting top of slide

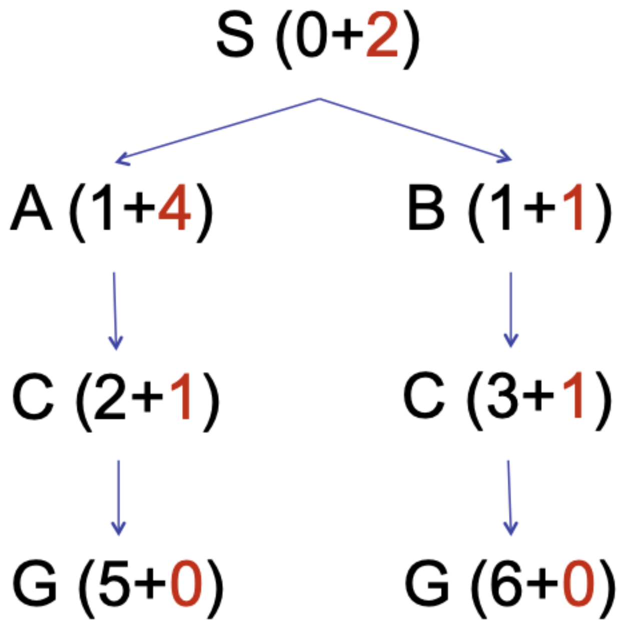
Slide: Andrew Moore

Is admissibility enough?

State Space Graph



Is the heuristic given admissible?
What path does A* tree search give?
What path does A* graph search give?



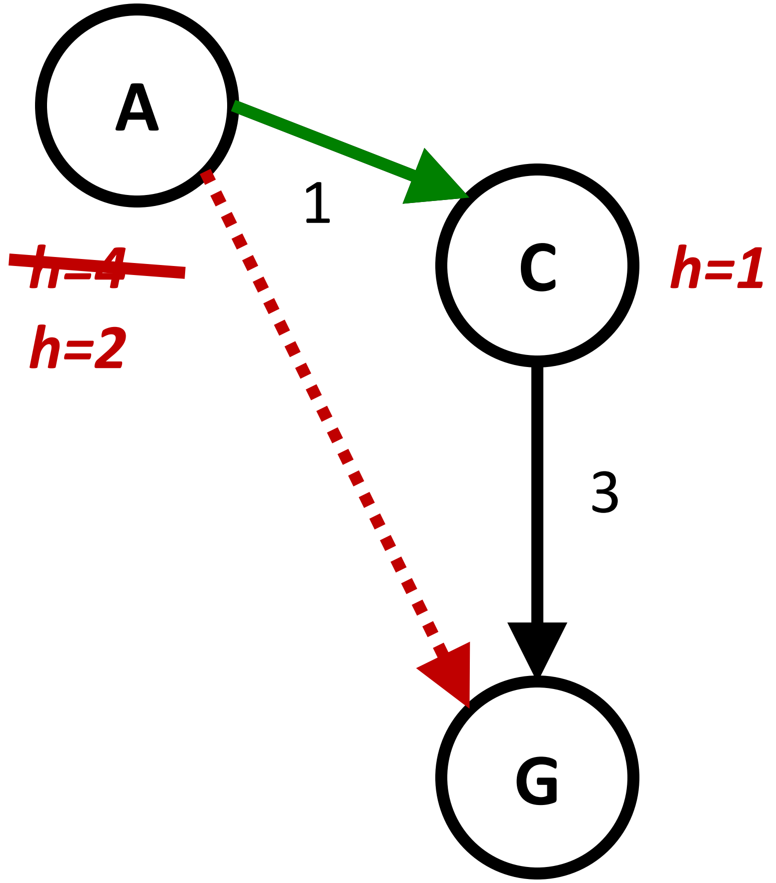
Condition on h beyond admissibility needed.

Consistency (monotonicity)

- **Consistency**
  - An admissible heuristic h is called consistent if for every state s and for every successor s', **$h(s) \leq c(s, s') + h(s')$**
  - This is a version of triangle inequality
  - Consistency is a **stricter requirement** than admissibility.

- **Property**
  - If h is a consistent heuristic and all costs are non-zero, then f values **cannot decrease** along any path:
  - Claim $f(n') >= f(n)$, where n' is the successor of n.
    - $g(n') = g(n) + c(n, a, n')$
    - $f(n') = g(n) + c(n, a, n') + h(n') >= f(n)$

Admissibility and Consistency



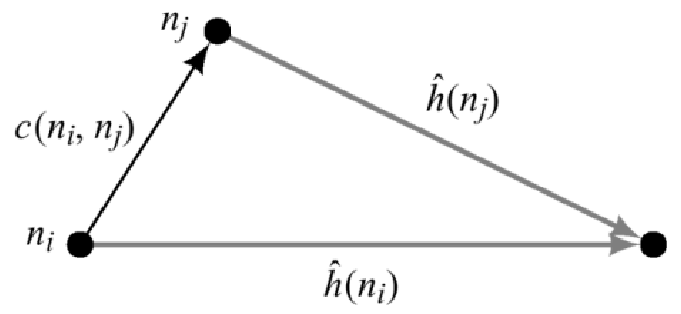
- Main idea: estimated heuristic costs ≤ actual costs
  - **Admissibility:** heuristic cost ≤ actual cost to goal

    h(A) ≤ actual cost from A to G
  - **Consistency:** heuristic "arc" cost ≤ actual cost for each arc

    h(A) – h(C) ≤ cost(A to C)

Consistency (monotonicity)

- h is consistent if the heuristic function satisfies triangle inequality for every n and its child node n': $h(n_i) <= h(n_j) + c(n_i,n_j)$



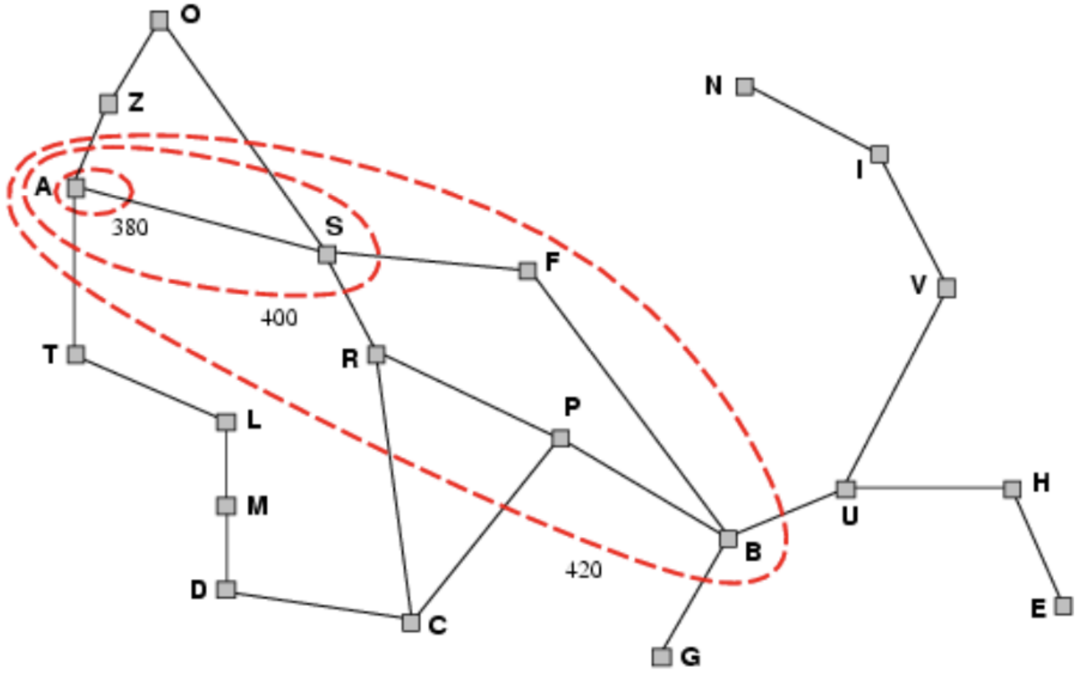
$$\hat{h}(n_i) \le c(n_i, n_j) + \hat{h}(n_j)$$

- When h is consistent, the f values of nodes expanded by A* are never decreasing.
- When A* selected n for expansion it already found the shortest path to it.
- When h is consistent every node is expanded once.
- Normally the heuristics we encounter are consistent
  - the number of misplaced tiles
  - Manhattan distance
  - straight-line distance

Search proceeds as f-value contours

- A^* expands nodes in order of increasing *f* value

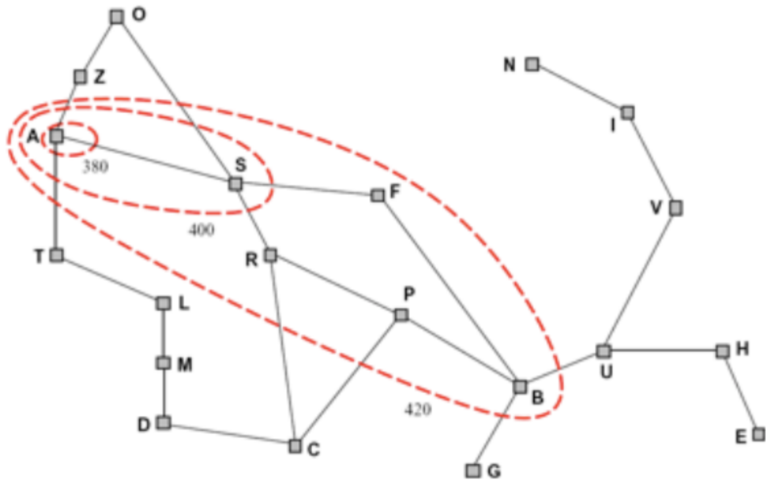
- Gradually adds "*f*-contours" of nodes
- Contour *i* has all nodes with $f=f_i$, where $f_i < f_{i+1}$



Optimality of A* Graph Search

Proof:

- Main idea: Show nodes are popped with non-decreasing f-scores
  - for n' popped after n :
    - f(n') ≥ f(n)
  - is this enough for optimality?

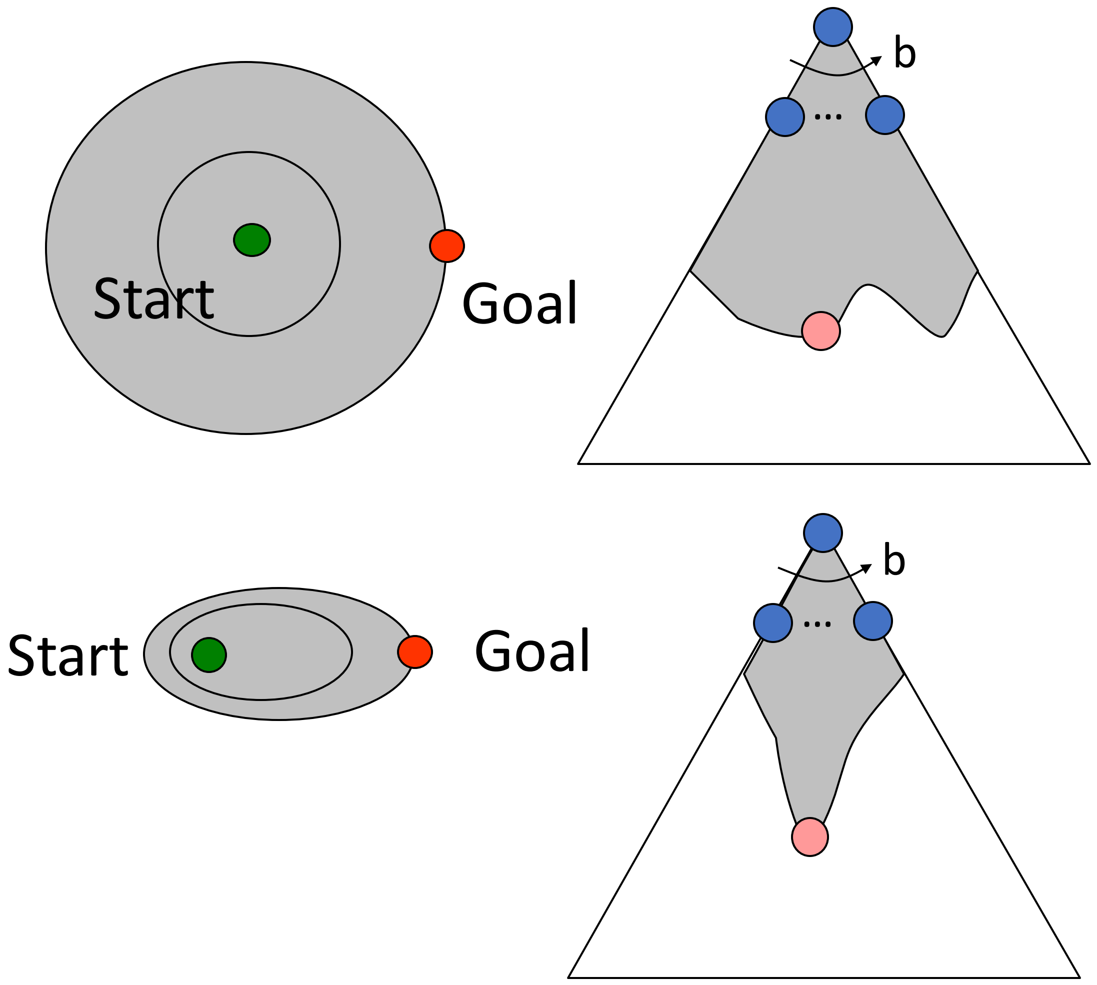


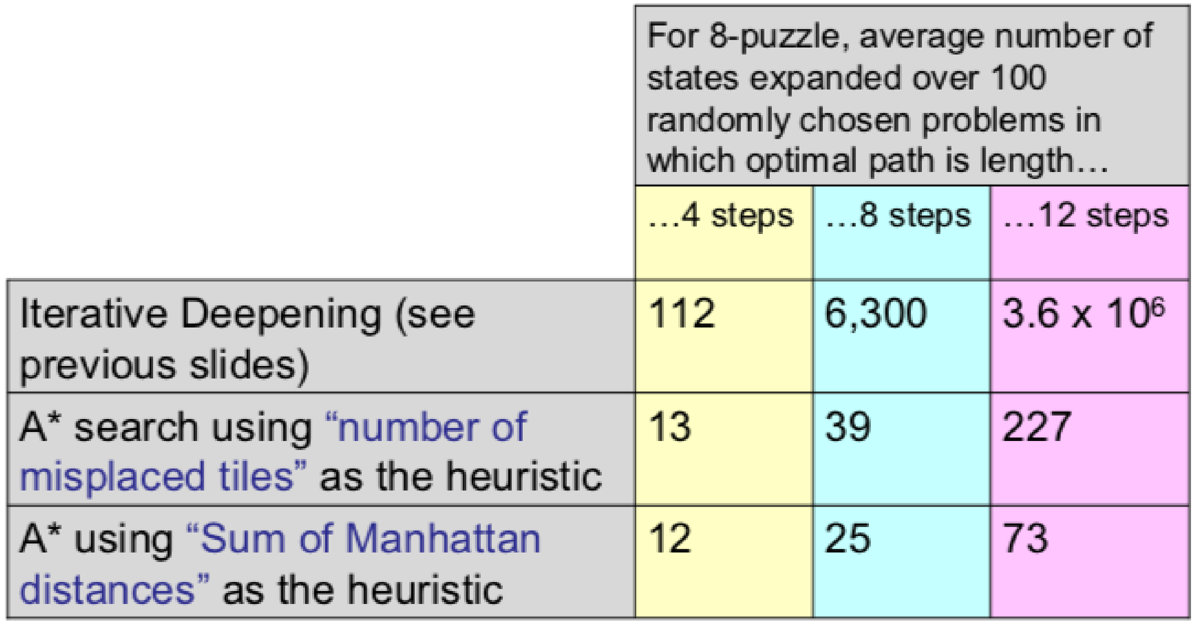
- Sketch:
- assume: f(n') ≥ f(n), for all edges (n,a,n') and all actions a
  - is this true?
- proof: A* never expands nodes with the cost f(n)>C*
- proof by induction(1) always pop the lowest f-score from the fringe, (2) all new nodes have larger (or equal) scores, (3) add them to the fringe, (4) repeat!

A* Search Properties

- **Optimality**
  - Tree search version of A* optimal if the heuristic is admissible.
  - Graph search version of A* is optimal if the heuristic is consistent.
- **Completeness**
  - If a solution exists, A* will find it provided that:
    - Every node has a finite number of successor nodes (b is finite).
    - There exists a positive constant $\delta > 0$ such that every step has at least cost δ
    - Then there exists only a finite number of nodes with cost less than or equal to C*.
- **Admissibility and Consistency**
  - Consistency implies admissibility
  - In general, natural admissible heuristics tend to be consistent

How heuristics effect nodes searched





	For 8-puzzle, average number of states expanded over 100 randomly chosen problems in which optimal path is length…		
	…4 steps	…8 steps	…12 steps
Iterative Deepening (see previous slides)	112	6,300	3.6×10^6
A* search using "number of misplaced tiles" as the heuristic	13	39	227
A* using "Sum of Manhattan distances" as the heuristic	12	25	73

Impact: reduction in the number of nodes expanded for reaching the goal.

Measuring the effect of a heuristic: *effective branching factor*

- Let A* generate *N* nodes to find a goal at depth *d*
- Let b* be the branching factor that a uniform tree of depth d would have, such that it contains (N+1) nodes.

$$N+1 = 1 + b^* + (b^*)^2 + \dots + (b^*)^d$$

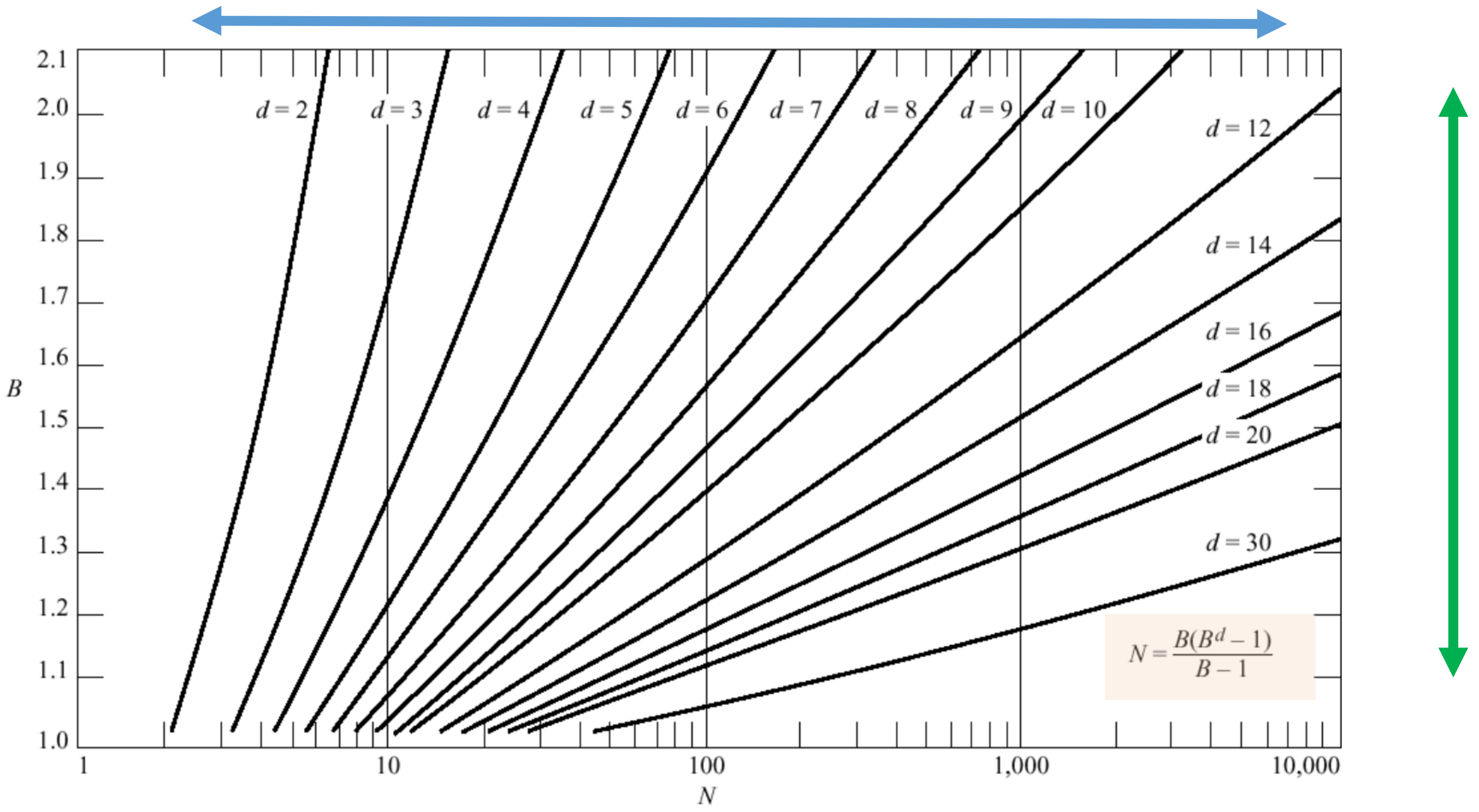
$$N+1 = ((b^*)^{d+1} - 1)/(b^* - 1)$$

$$N \approx (b^*)^d \Rightarrow b^* \approx \sqrt[d]{N}$$

- Acts as a measure of a heuristic's overall usefulness.
  - Provides a way to compare different heuristics.

Effective branching factor

Effective branching factor

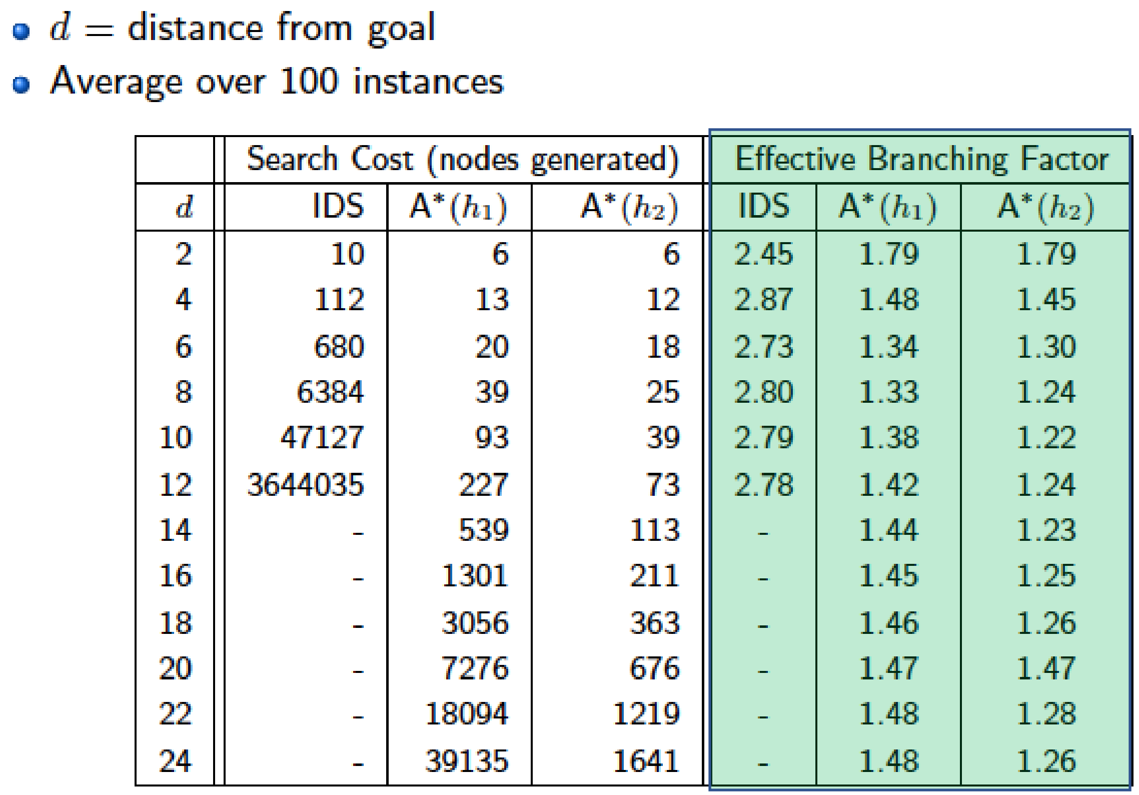


Number of nodes in the tree

Comparing Heuristics

Effective branching factors for A* search for the 8-puzzle.

- Figure compares two heuristics: Misplaced tiles (h_1) and Manhattan distance (h_2)
- In effect, Heuristic (h_2) expands fewer nodes and has a lower effective branching factor



- d = distance from goal
- Average over 100 instances

	Search Cost (nodes generated)			Effective Branching Factor		
d	IDS	A*(h_1)	A*(h_2)	IDS	A*(h_1)	A*(h_2)
2	10	6	6	2.45	1.79	1.79
4	112	13	12	2.87	1.48	1.45
6	680	20	18	2.73	1.34	1.30
8	6384	39	25	2.80	1.33	1.24
10	47127	93	39	2.79	1.38	1.22
12	3644035	227	73	2.78	1.42	1.24
14	-	539	113	-	1.44	1.23
16	-	1301	211	-	1.45	1.25
18	-	3056	363	-	1.46	1.26
20	-	7276	676	-	1.47	1.47
22	-	18094	1219	-	1.48	1.28
24	-	39135	1641	-	1.48	1.26

Reference: AIMA

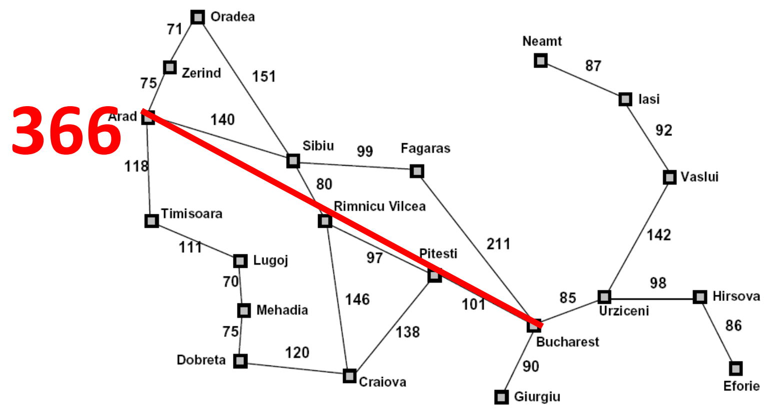
Ways to design heuristics

- Constructing Relaxations
  - Examples:
    - If the rules of the 8-puzzle are relaxed so that a tile can move anywhere, then h1 gives the shortest solution
    - If the rules are relaxed so that a tile can move to any adjacent square, then h_2 gives the shortest solution
    - Exact solution cost of a relaxed version of the problem

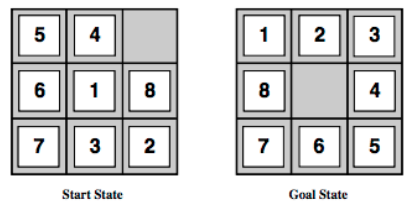
- Combining heuristics
- Prior experience in terms of seeing plans for problems encountered in the past

Method I: Creating admissible heuristics from relaxed problems

- Relaxation
    - Ignore constraints/rules.
    - Increase possibilities for actions.
- State space graph for the relaxed problem is a super-graph of the original state space
    - The removal of restrictions adds more edges.
    - Hope is that in the relaxed graph, it is easier to find a solution.



Permitting straight line movement adds edges to the graph.



Consider the following heuristics:

- h_1 = number of misplaced tiles (=7 in example)
- h_2 = total Manhattan distance (i.e., no. of squares from desired location of each tile) (= 2+3+3+2+4+2+0+2 = 18 in example)

Admissible Heuristics from Relaxed Problems

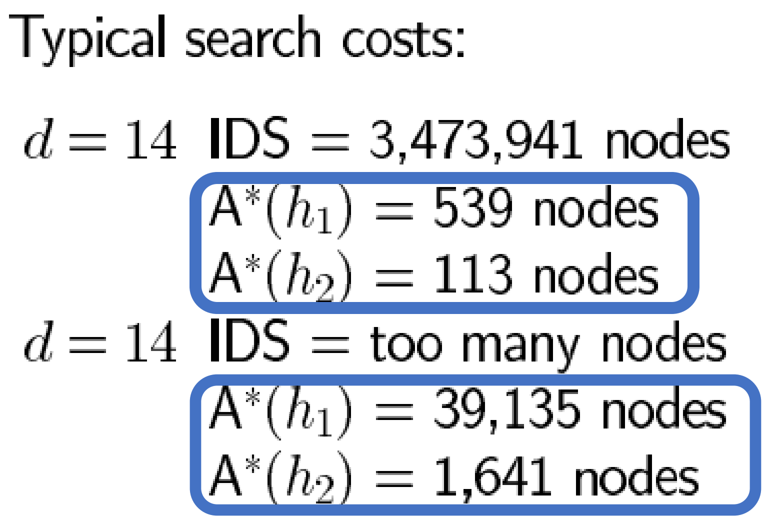
- Optimal solution in the original problem is also a solution for the relaxed problem.
- Cost of the optimal solution in the relaxed problem is an admissible heuristic in the original problem.
- Finding the optimal solution in the relaxed problem should be “easy”
  - Without performing search.
  - If decomposition is possible, it is easier to directly solve the problem.

Comparing heuristics: *dominance*

- Heuristic function h_2 (strictly) dominates h_1 if
  - Both are admissible and
  - for every node n, $h_2(n)$ is (strictly) greater than $h_1(n)$.

- Theorem (Hart, Nilsson and Raphale, 1968)
- An A* search with a dominating heuristic function h_2 has the property that any node it expands is also expanded by A* with h_1
- Equivalently, A* search with a dominating heuristic function h_2 will never expand more nodes that A* with $h_{1.}$

- Domination leads to efficiency
  - Prefer heuristics with higher values, they lead to fewer expansions and more goal-directedness during search.


Typical search costs:

$d = 14$ IDS $=$ 3,473,941 nodes
$\text{A}^*(h_1) =$ 539 nodes
$\text{A}^*(h_2) =$ 113 nodes
$d = 14$ IDS $=$ too many nodes
$\text{A}^*(h_1) =$ 39,135 nodes
$\text{A}^*(h_2) =$ 1,641 nodes

Method II: Combining admissible heuristics

- Heuristic design process
  - We may have a set of heuristics but not a single "clearly best" heuristic.
  - Have a set of heuristics for a problem and none of them dominates any of the other.

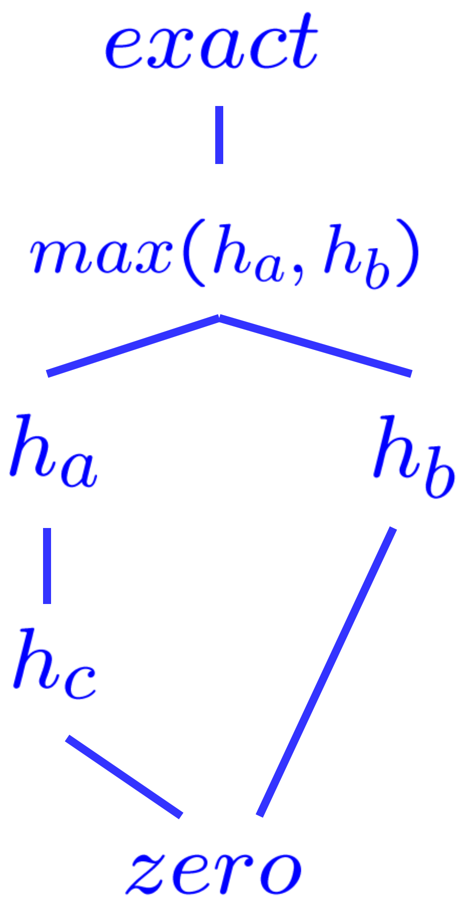
- Combining heuristics
  - Can use a composite heuristic
  - Max of admissible heuristics is admissible when the component heuristics are admissible.
  - The composite heuristic dominates the component heuristic.

$$h(n) = max(h_a(n), h_b(n))$$

Slide adapted from Dan Klein

Combining admissible heuristics

- Fundamentally, heuristic functions form a semi-lattice structure
  - Some heuristics can be compared to others via dominance.
  - There may be others not comparable.
  - Can create composites by combining component heuristics.
- Bottom of lattice is the zero heuristic
  - No or little computation effort
  - Not useful during search
- Top of lattice is the exact heuristic
  - A lot of computation effort
  - Really useful during search (give the exact cost)

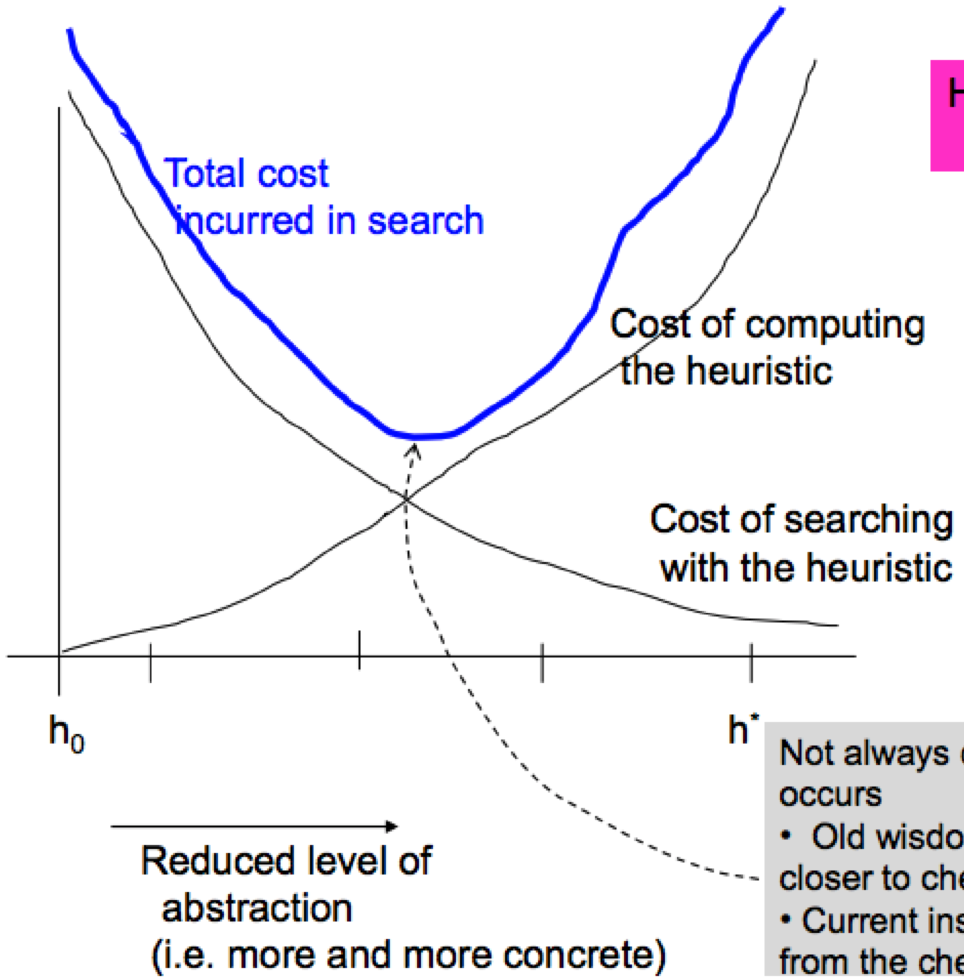


Slide adapted from Dan Klein

Trade-off in heuristic design

Good heuristics make search easier. But good heuristics may also need more time to compute.

Effectiveness of the heuristic (reduced search time with the heuristic) vs. effort required to compute the heuristic



How informed should the heuristic be?

Not always clear where the total minimum occurs

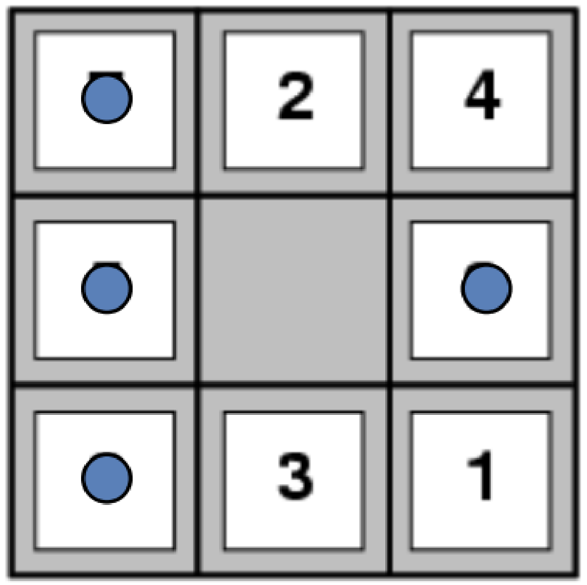
- Old wisdom was that the global min was closer to cheaper heuristics
- Current insights are that it may well be far from the cheaper heuristics for many problems

Slide adapted from Mausam

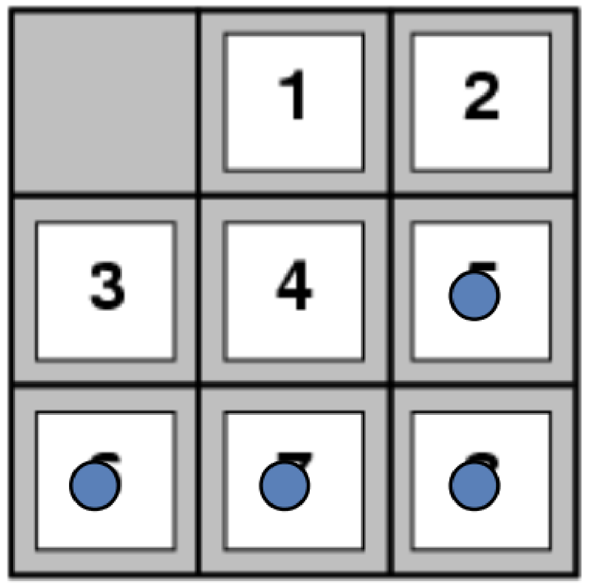
Pattern Databases

Optimally solve a sub-problem.
In the tiles example, mask out certain numbers. Consider a smaller-sized pattern.

Observing a pattern and making an estimate of the cost of solving this sub-problem.



Start State



Goal State

Pattern Databases: Example

- For sliding tiles
  - For a subset of the tiles compute shortest path to the goal (say using breadth-first search)
  - For 15 puzzles, if we have 7 fringe tiles and one blank, the number of patterns to store are 16!/(16-8)! = 518,918,400.
  - For each table entry we store the shortest number of moves to the goal from the current location.
- Use different subsets of tiles and take the max heuristic during IDA*search (we will study it shortly).

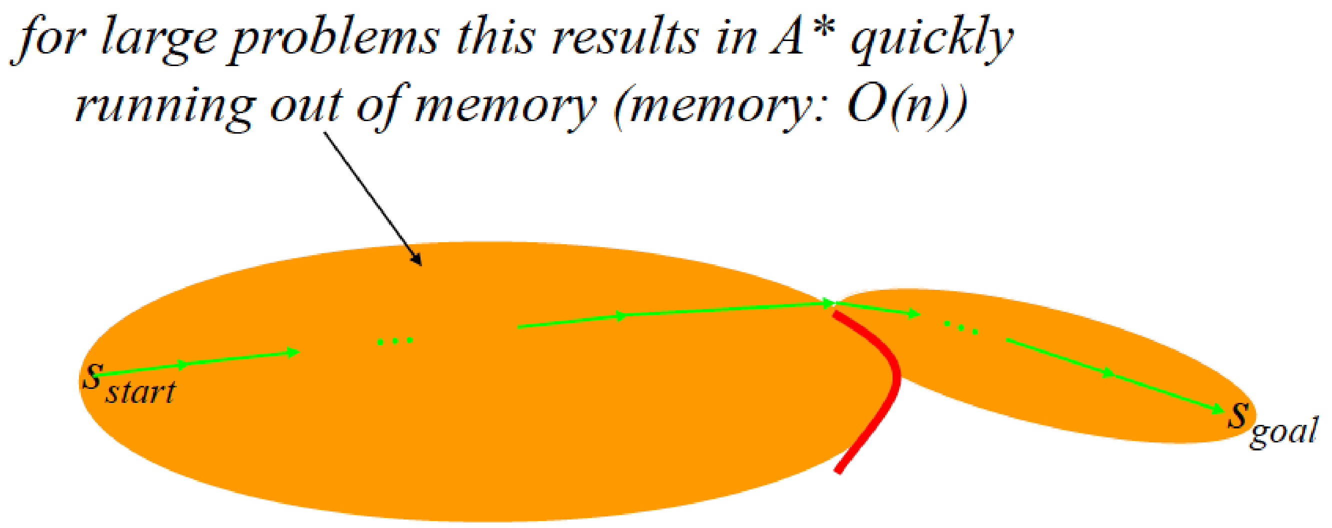
Slide from Kalev Kask

A* Search: Other Properties

- Exponential worst-case time and space complexity
  - Let $e = (h^* - h)/h^*$ (relative error)
  - Complexity **$O(b^{ed})$** where **b^e** is the effective branching factor.
  - With a good heuristic complexity is often sub-exponential
- Optimally efficient
  - With a given h, no other search algorithm will be able to expand fewer nodes
    - If an algorithm does not expand all nodes with $f(n) < C^*$ (the cost of the optimal solution) then there is a chance that it will miss the optimal solution.
- **Main Limitation: Space Requirement!**
  - **The number of states within the goal contour search space is still *exponential* in the length of the solution.**

How to improve space efficiency of A* search?

A* Search may still take a long time to find the optimal solution


for large problems this results in A quickly running out of memory (memory: O(n))*
...
s_{start}
...
s_{goal}

- The memory needed is O(total number of states). The frontier is O(b^d). Despite using the heuristic, it may be difficult to store the frontier.
- How to reduce memory requirement for A*?

Adapted from Maxim Likhachev

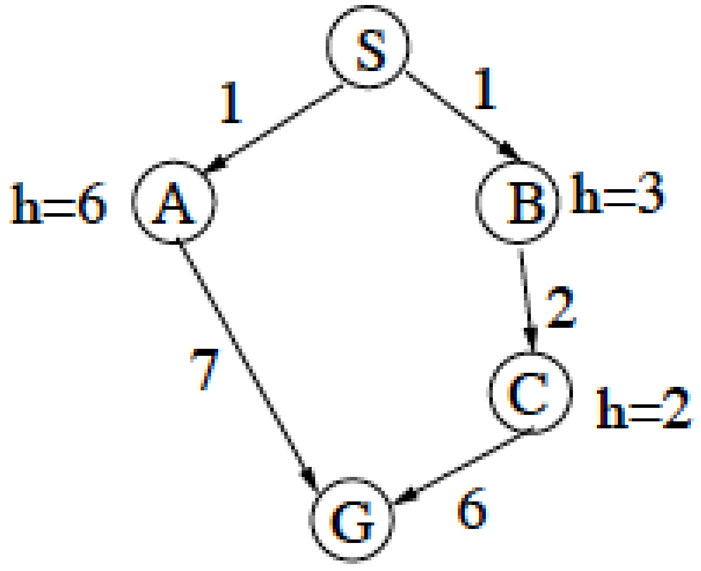
Iterative Deepening A* (IDA*)

- **Key Idea**
  - A* uses a lot of memory. Alternative: Don’t’ keep all the nodes, recompute them. Borrow idea from iterative deepening search (discussed before).
- **IDA***
  - Use an f-value limit, rather than a depth limit for search.
    - Expand all nodes up to f1, f2, ...
  - Keep track of the *next limit* to consider
    - so we will search at least one more node next time.
  - If the depth-bounded search fails, then the *next bound* is the *minimum* of the f-values that *exceeded the previous bound*.
  - **IDA* checks the same nodes as A* but recomputes them using a <u>depth-first search (DFS)</u> instead of *storing* them.**

Iterative Deepening A* (IDA*)

- Iterative deepening A*. Actually, pretty different from A*. Assume costs integer.
  1. Do loop-avoiding DFS, not expanding any node with *f*(*n*) > 0. Did we find a goal? If so, stop.
  2. Do loop-avoiding DFS, not expanding any node with *f*(*n*) > 1. Did we find a goal? If so, stop.
  3. Do loop-avoiding DFS, not expanding any node with *f*(*n*) > 2. Did we find a goal? If so, stop.
  4. Do loop-avoiding DFS, not expanding any node with *f*(*n*) > 3. Did we find a goal? If so, stop.

  …keep doing this, increasing the *f*(*n*) threshold by 1 each time, until we stop.



IDA* example

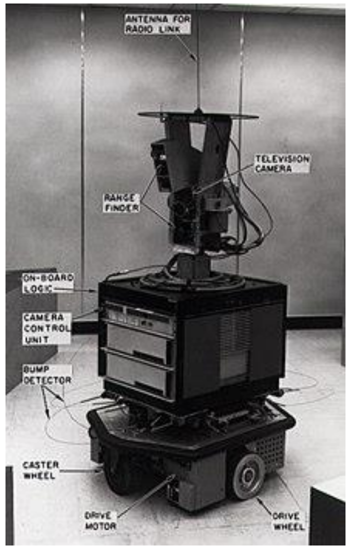
- If $f_1 = 4$, then which nodes are searched?
- If $f_2 = 8$, then which nodes are searched?

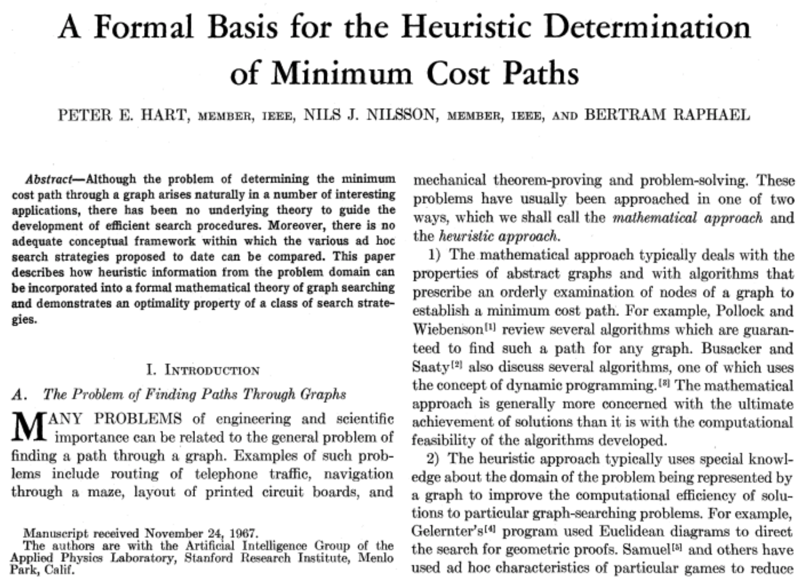
Iterative Deepening A* (IDA*)

- Key Idea: DFS in the inner loop is giving the space advantage. The depth is decided by the f value.
- Like A*, IDA* is guaranteed to find the shortest path leading from the given start node to any goal node in the problem graph, if the heuristic function *h* is admissible.
- Suited to memory constrained applications
- Also see: https://en.wikipedia.org/wiki/Iterative_deepening_A*

History of A* Search

- Origin
  - Shakey Experiment (AI Center at Stanford Research Institute)
  - https://www.youtube.com/watch?v=GmU7SimFkpU
- Peter Hart, Nils Nilsson and Bertram Raphael first published the algorithm in 1968.
  - Djikstra was too slow for path finding.
  - https://ieeexplore.ieee.org/stamp/stamp.jsp?tp=&arnumber=4082128





A Formal Basis for the Heuristic Determination of Minimum Cost Paths

PETER E. HART, MEMBER, IEEE, NILS J. NILSSON, MEMBER, IEEE, AND BERTRAM RAPHAEL

Abstract—Although the problem of determining the minimum cost path through a graph arises naturally in a number of interesting applications, there has been no underlying theory to guide the development of efficient search procedures. Moreover, there is no adequate conceptual framework within which the various ad hoc search strategies proposed to date can be compared. This paper describes how heuristic information from the problem domain can be incorporated into a formal mathematical theory of graph searching and demonstrates an optimality property of a class of search strategies.

I. INTRODUCTION

A. The Problem of Finding Paths Through Graphs

MANY PROBLEMS of engineering and scientific importance can be related to the general problem of finding a path through a graph. Examples of such problems include routing of telephone traffic, navigation through a maze, layout of printed circuit boards, and mechanical theorem-proving and problem-solving. These problems have usually been approached in one of two ways, which we shall call the *mathematical approach* and the *heuristic approach.*

1) The mathematical approach typically deals with the properties of abstract graphs and with algorithms that prescribe an orderly examination of nodes of a graph to establish a minimum cost path. For example, Pollock and Wiebenson[1] review several algorithms which are guaranteed to find such a path for any graph. Busacker and Saaty[2] also discuss several algorithms, one of which uses the concept of dynamic programming.[3] The mathematical approach is generally more concerned with the ultimate achievement of solutions than it is with the computational feasibility of the algorithms developed.

2) The heuristic approach typically uses special knowledge about the domain of the problem being represented by a graph to improve the computational efficiency of solutions to particular graph-searching problems. For example, Gelernter's[4] program used Euclidean diagrams to direct the search for geometric proofs. Samuel[5] and others have used ad hoc characteristics of particular games to reduce

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In 1964 Nils Nilsson invented a heuristic based approach to increase the speed of Dijkstra's algorithm. This algorithm was called A1. In 1967 Bertram Raphael made dramatic improvements upon this algorithm, but failed to show optimality. He called this algorithm A2. Then in 1968 Peter E. Hart introduced an argument that proved A2 was optimal when using a consistent heuristic with only minor changes. His proof of the algorithm also included a section that showed that the new A2 algorithm was the best algorithm possible given the conditions. He thus named the new algorithm in Kleene star syntax to be the algorithm that starts with A and includes all possible version numbers or A*