COL352 Problem Sheet 4

March 24, 2025

Problem 1. *Show that the following languages are not context-free.*

1. $L_1 = \{a^m b^n \mid mn \text{ is the square of an integer } \}$.
2. $L_2 = \{x \in \{0,1\}^* \mid x \text{ is the binary representation of } 3^{n^2} \text{ for some } n \in \mathbb{N} \cup \{0\}\}$
3. $L_3 = \{a^m b^n \mid n \text{ is a multiple of } m\}$
4. $L_4 = \{x \in \{a,b\}^* \mid \# \text{ of } a\text{'s in } x \text{ is a multiple of } \# \text{ of } b\text{'s in } x\}$
5. $L_5 = \{xyx \mid x, y \in \{0,1\}^* \text{ and } |x| > 0,\ |y| > 0\}$

Problem 2. *The strict shuffle of two strings $x = x[1]\ldots x[n]$ and $y = y[1]\ldots y[n]$ of equal length is defined as*

$$\mathit{sshuffle}(x, y) = x[1]y[1]x[2]y[2]\ldots x[n]y[n]$$

The strict shuffle of two languages $L_1, L_2 \subseteq \Sigma^*$ is defined as

$$\mathit{sshuffle}(L_1, L_2) = \{\mathit{sshuffle}(x, y) \mid x \in L_1,\ y \in L_2,\ |x| = |y|\}$$

Is the class of context-free languages closed under the sshuffle operation? Prove your answer.

Problem 3. *We say that z is a shuffle of x and y if the characters in x and y can be interleaved, while maintaining their relative order within x and y, to get z. Formally, if $|x| = m$ and $|y| = n$, then $|z|$ must be $m + n$, and it should be possible to partition the set $\{1, 2, \ldots, m + n\}$ into two increasing sequences, $i_1 < i_2 < \cdots < i_m$ and $j_1 < j_2 < \cdots < j_n$, such that $z[i_k] = x[k]$ and $z[j_k] = y[k]$ for all k. Given two languages $L_1, L_2 \subseteq \Sigma^*$, define*

$$\mathit{shuffle}(L_1, L_2) = \{z \in \Sigma^* \mid z \text{ is a shuffle of some } x \in L_1 \text{ and some } y \in L_2\}.$$

Is the class of context-free languages closed under the shuffle operation? Prove your answer.

Problem 4. *Let $G = (NT, T, R, S)$ be a grammar. Prove that for every $x \in L(G)$, there exists a parse tree of G with root S, yield x, and height at most $|V| \cdot (|x| + 1)$.*

Problem 5. *Let us say that a PDA is a binary stack PDA if the size of its set of stack symbols Γ is 2; assume $\Gamma = \{0, 1\}$ for concreteness. Prove that binary stack PDAs and PDAs are equivalent in terms of computational power.*

Problem 6. *An n-stack PDA is like a regular PDA except that it has n stacks instead of one.*

1. *Show that for all n, an n-stack PDA could be simulated by a 2-stack PDA.*
2. *Show that anything that can be computed by a Turing machine can be computed by a 2-stack PDA.*

Problem 7. *Prove that each of the following functions is computable (You can assume that x, y are positive integers given in their binary representation, and you need the answer in binary representation too).*

1. $x - 1$ *(assuming $x > 0$)*
2. $x + y$
3. $x \times y$
4. x^y

Problem 8. *Show that every language in Problem 1 can be decided by a Turing machine.*