COL352 Problem Sheet 5

April 23, 2025

Problem 1. *Show that the collection of decidable languages is closed under the operation of*

1. *union*
2. *complementation*
3. *concatenation*
4. *intersection*
5. *star*

Problem 2. *Show that the collection of Turing-recognizable languages is closed under the operation of*

1. *union*
2. *concatenation*
3. *intersection*
4. *star*

Problem 3. *Say that a write-once Turing machine is a single-tape TM that can alter each tape square at most once (including the input portion of the tape). Show that write-once Turing machine model is equivalent to the ordinary Turing machine model.*

Problem 4. *Let* $B = \{\langle M_1\rangle, \langle M_2\rangle, \dots\}$ *be a Turing-recognizable language consisting of TM descriptions. Show that there is a decidable language* C *consisting of TM descriptions such that every machine described in* B *has an equivalent machine in* C *and vice versa.*

Problem 5. *Consider a Turing machine* $M\tilde{}$ *with the following property: the head movements of* $M\tilde{}$ *are independent of the contents of its tapes and depend only on the input length (i.e.,* $M\tilde{}$ *always performs a sequence of left to right and back sweeps of the same form regardless of what is the input). A machine with this property is called oblivious Turing Machine. Prove that every Turing Machine can be simulated by an oblivious Turing Machine.*

Problem 6. *Define a RAM Turing machine to be a Turing machine that has random access memory. We formalize this as follows: the machine has additional two symbol on its alphabet we denote by R and W and an additional state we denote by q_access. We also assume that the machine has an infinite array A that is initialized to all blanks. Whenever the machine enters q_access, if its address tape contains* $i_B R$ *(where* i_B *denotes the binary representation of number i) then the value A[i] is written in the cell next to the R symbol. If its tape contains* $i_B W\sigma$ *(where* σ *is some symbol in the machine's alphabet) then A[i] is set to the value* σ*. Show that if a Boolean function f is computable within time* $T(n)$ *(for some time-constructible T) by a RAM TM, then it can be computed by a turing machine in time* $T(n) \times (T(n) + n)$ *).*

Problem 7. *** The* Busy Beaver [1] *function BB(n) is defined for an n-state Turing machine (TM) with a two-symbol alphabet* $\{0, 1\}$ *and a single infinite tape. A TM is an n-state Busy Beaver candidate if it halts when started on a blank tape (all 0s). BB(n) is the maximum number of 1s left on the tape by any such halting TM. For* $n = 1$*, a 1-state TM (states* $\{q_1, h\}$*) can, at best, write one 1 and halt, e.g., via* $\delta(q_1, 0) = (h, 1, R)$*, producing BB(1) = 1, as other configurations either loop or write no 1s.*

[1] You can find out more about Busy Beavers here and here.

1. *Compute BB(2). Find a 2-state TM (states $\{q_1, q_2, h\}$) that maximizes the number of 1s left on the tape when started on a blank tape. Can you prove it achieves BB(2) by showing no other 2-state TM can produce more 1s?*

2. *Show that BB(n) is non-computable, i.e., there is no TM that, given n as input, outputs BB(n).*

3. *Building on part (2), prove that the function $S(n)$, the maximum number of steps taken by any halting n-state Busy Beaver candidate, is also non-computable. Explain how a TM that computes $S(n)$ could be used to compute BB(n), and why this implies $S(n)$'s non-computability.*

Problem 8. *Let C be a language. Prove that C is Turing-recognizable if and only if there exists a decidable language D such that $C = \{x \mid \exists y\ (\langle x, y\rangle \in D)\}$.*