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BITS Pilani
Pilani Campus

MATH F112 (Mathematics-II)

Complex Analysis

BITS Pilani
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Lecture 28
Elementary Functions

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Elementary Functions

1. Exponential Functions
2. Trigonometric Functions
3. Hyperbolic Functions
4. Logarithmic Functions
5. Complex Exponents

Elementary Functions

Self Study (Sec 36, p.112-115)

6. **Inverse Trigonometric Functions**

7. **Inverse Hyperbolic Functions**

Exponential Function

(1) Let $z = x + iy$, then

$$\exp(z) = e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots + \frac{z^n}{n!} + \ldots$$

is called Maclaurin' series of e^z

$$e^{z_1} . e^{z_2} = e^{z_1 + z_2}, \quad \frac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2}$$

Exponential Function

(2) Let $f(z) = e^z = e^{x+iy}$

$$= e^x . e^{iy}$$

$$= e^x(\cos y + i \sin y)$$

$$\equiv u + iv$$

$$\Rightarrow u = e^x \cos y, \; v = e^x \sin y,$$

Exponential Function

$$\Rightarrow u_x = e^x \cos y,\ u_y = -e^x \sin y$$

$$v_x = e^x \sin y,\ v_y = e^x \cos y$$

$$\Rightarrow u_x = v_y,\ u_y = -v_x$$

Thus CR - equations are satisfied and clearly u_x, u_y, v_x, v_y are continuous

Exponential Function

$\Rightarrow f(z)$ is differentiable and

$$f'(z) = u_x + i v_x$$

$$= e^x \cos y + i e^x \sin y = e^x . e^{iy} = e^z$$

$$\Rightarrow \frac{d}{dz}\left(e^z\right) = e^z$$

Exponential Function

$$(3)\ e^{z} = e^{x}.e^{iy}, \qquad e^{iy} = \cos y + i \sin y$$

$$\Rightarrow \left|e^{iy}\right| = \sqrt{\cos^{2} y + \sin^{2} y} = 1$$

$$\therefore\ \left|e^{z}\right| = \left|e^{x}\right| = e^{x} \text{ as } e^{x} > 0\ \forall\ x \in R$$

$$\Rightarrow\ e^{z} \neq 0 \text{ for any complex number } z.$$

Exponential Function

We may write $e^{z} = e^{x}.e^{iy} = \rho\, e^{i\phi}$,

when $\rho = e^{x} = \left|e^{z}\right| > 0$ & $\phi = y$

$$\therefore \arg\left(e^{z}\right) = y + 2n\pi,$$

$$n = 0, \pm 1, \pm 2 \ldots$$

Exponential Function

(4) $\because \cos 2\pi = 1 \;\&\; \sin 2\pi = 0$

Hence $e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1$

$$e^{\pi i} = \cos \pi + i \sin \pi = -1$$

$$\Rightarrow e^{i\pi} + 1 = 0$$

$$e^{-\pi i} = \cos(-\pi) + i \sin(-\pi) = -1$$

Exponential Function

$$e^{pi/2} = \cos p/2 + i\sin p/2 = i$$

$$e^{-pi/2} = \cos\left(-p/2\right) + i\sin\left(-p/2\right)$$

$$= -i$$

Exponential Function

(5). $$e^{z+2\pi i} = e^z . e^{2\pi i} = e^z$$

Þ e^z is periodic with imaginary period $2pi$.

\ $e^{z\pm 2npi} = e^z$, " $n = 0,1,2,3,\ldots$

Exponential Function

(6). $e^x > 0 \;\forall\; x \in \mathbb{R}$

But e^z may be negative if $z \in C$

Example: Find z such that $e^z = -1$

Solution:

$$e^z = -1$$

$$\Rightarrow\; e^x . e^{iy} = 1.e^{\pi i}$$

Exponential Function

$$\Rightarrow \quad e^x = 1, \quad \text{and}$$

$$y = \pi + 2n\pi, n = 0, \pm 1, \pm 2 \ldots$$

$$\Rightarrow x = 0 \;\; \& \;\; y = \pi + 2n\pi$$

Thus, if $z = x + iy$

$$= (2n+1).i\pi \,,$$

$$n = 0, \pm 1, \pm 2, \ldots$$

then $e^z = -1$

Exponential Function

Excercise:

(7) $e^{\bar{z}}$ is not analytic anywhere.

Q. Find all values of z such that

$$e^{2z-1} = 1 + i$$

Exponential Function

Solution:

$$e^{2z-1} = 1+i$$

$$\Rightarrow e^{2x-1} \cdot e^{2iy} = \sqrt{2}\, e^{\frac{\pi}{4}i}$$

$$\Rightarrow e^{2x-1} = \sqrt{2},$$

$$2y = \frac{\pi}{4} + 2n\pi; \;\; n = 0, \pm 1, \pm 2, \ldots$$

Exponential Function

$$\Rightarrow x = \frac{1}{2}\left(1 + \ln\sqrt{2}\right), \quad y = \frac{\pi}{8} + n\pi$$

$$\therefore z = x + iy$$

$$= \frac{1}{2}\left(1 + \ln\sqrt{2}\right) + i\left(\frac{\pi}{8} + n\pi\right),$$

$$n = 0, \pm 1, \pm 2, \ldots$$

Exponential Function

Problems done on board:

Q. Show that $\left|e^{z^2}\right| \leq e^{|z|^2}$

Q. Show in two ways that e^{z^2} is an entire function.

Q. $\overline{e^{iz}} = e^{i\bar{z}}$ iff $z = n\pi,\ n = 0, \pm 1, \pm 2, \dots \dots$

Q. Let $f(z) = u(x, y) + iv(x, y)$ be analytic function in domain D. Then show that $U = e^u \cos v,\ V = e^u \sin v$ are harmonic in D and V is harmonic conjugate of U.

Trigonometric Function

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(1) If x is real, then

$$\cos x = \frac{e^{ix} + e^{-ix}}{2},$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

Trigonometric Function

Similarly if z is complex, we define

$$\cos z = \frac{e^{iz} + e^{-iz}}{2},$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad ---(1)$$

$$\Rightarrow e^{iz} = \cos z + i \sin z,$$

Trigonometric Function

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z},$$

$$\sec z = \frac{1}{\cos z}, \quad \cos ec\, z = \frac{1}{\sin z}$$

Trigonometric Function

(2). Since e^{iz} is analytic $\forall\, z$ and linear combination of two analytic functions is again analytic, hence it follows that $\sin z$ and $\cos z$ are analytic functions.

Trigonometric Function

(3). Using (1) it is easy to prove :

$$(i) \quad \sin(-z) = -\sin z$$

$$(ii) \quad \cos(-z) = \cos z$$

$$(iii) \quad \frac{d}{dz}(\sin z) = \cos z$$

Trigonometric Function

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$$(iv) \quad \frac{d}{dz}\left(\cos z\right) = -\sin z$$

$$(v) \quad \frac{d}{dz}\left(\tan z\right) = \sec^2 z$$

Trigonometric Function

$$(vi)\quad \sin(z_1 \pm z_2)$$

$$= \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$$

$$(vii)\quad \cos(z_1 \pm z_2)$$

$$= \cos z_1 . \cos z_2 \mp \sin z_1 \sin z_2$$

Trigonometric Function

$$(4) \because \cos z = \frac{e^{iz} + e^{-iz}}{2},$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

Put $x = 0$, then

Trigonometric Function

$$\cos(iy) = \frac{e^{i(iy)} + e^{-i(iy)}}{2}$$

$$= \frac{e^{-y} + e^{y}}{2} = \cosh y$$

Trigonometric Function

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$$\sin(iy) = -\frac{1}{2i}\left(e^{y} - e^{-y}\right)$$

$$= i\,\frac{1}{2}\left(e^{y} - e^{-y}\right)$$

$$= i\sinh y$$

Trigonometric Function

$$\cos z = \cos(x + iy)$$

$$= \cos x \cos(iy) - \sin x . \sin(iy)$$

$$= \cos x . \cosh y - i \sin x . \sin hy$$

Trigonometric Function

$$\sin z = \sin(x + iy)$$

$$= \sin x.\cos iy + \cos x.\sin iy$$

$$= \sin x.\cosh y + i\cos x.\sinh y$$

Trigonometric Function

Hence (Exercise)

$$|\sin z|^2 = \sin^2 x + \sinh^2 y$$

$$|\cos z|^2 = \cos^2 x + \sin h^2 y$$

Hints : (Use)

$$\cos h^2 x - \sinh^2 x = 1$$

$$\cos^2 x + \sin^2 x = 1$$