BITS Pilani
Pilani Campus

MATH F112 (Mathematics-II)

Complex Analysis

Lecture 39-40
Applications of Residues
(Improper Integrals)

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Evaluation of Improper Integrals

(1) Let $f(x)$ is continuous for all $x \geq 0$, then

$$\int_0^\infty f(x)dx = \lim_{R\to\infty} \int_0^R f(x)\,dx$$

provided the limit on RHS exists.

Evaluation of Improper Integrals

(2) Let $f(x)$ is continuous for all x.

then $\int_{-\infty}^{\infty} f(x)\,dx$

$$= \lim_{R_1\to\infty} \int_{-R_1}^{0} f(x)\,dx + \lim_{R_2\to\infty} \int_{0}^{R_2} f(x)\,dx,$$

provided both the limits on RHS exist.

Evaluation of Improper Integrals

Cauchy principal value (P.V.) of the integral (2) is the number

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx,$$

provided the limit on RHS exist.

Evaluation of Improper Integrals

Remark :

(1) Existence of improper integral

$\int_{-\infty}^{\infty} f(x)\, dx$ implies the existence of

P.V. $\int_{\infty}^{\infty} f(x)\, dx$

But converse is not true.

Evaluation of Improper Integrals

Ex. Let $f(x) = x$. Then

P.V. $\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} x\,dx$

$$= \lim_{R\to\infty} \left[\frac{x^2}{2}\right]_{-R}^{R} = 0$$

Evaluation of Improper Integrals

But $\int_{-\infty}^{\infty} f(x)\,dx$

$$= \lim_{R_1 \to \infty} \int_{-R_1}^{0} x\,dx + \lim_{R_2 \to \infty} \int_{0}^{R_2} x\,dx$$

$$= -\lim_{R_1 \to \infty} \frac{R_1^2}{2} + \lim_{R_2 \to \infty} \frac{R_2^2}{2}$$

Evaluation of Improper Integrals

$\therefore$ Limit on RHS fails to exist

$\Rightarrow$ The improper integral

$\int_{-\infty}^{\infty} f(x)\,dx$ fails to exist.

Evaluation of Improper Integrals

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If the function $f(x)$ $(-\infty < x < \infty)$ is an even function i.e. $f(-x) = f(x)$ for all x, then the symmetry of the graph of $y = f(x)$ with respect to y axis leads to

$$\int_{-R}^{R} f(x)dx = 2\int_{0}^{R} f(x)dx$$

Evaluation of Improper Integrals

When $f(x)$ is an even function and the Cauchy principal value exists, then

$$\text{P.V.}\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} f(x)dx = 2\int_{0}^{\infty} f(x)dx$$

Evaluation of Improper Integrals

To evaluate improper integral of ***Even Rational Functions*** $f(x) = p(x)/q(x)$

- $p(x)$ and $q(x)$ are polynomials with real coefficients and no factors in common

- $q(z)$ has no real zeros but has at least one zero above the real axis.

Evaluation of Improper Integrals

- Identify all distinct zeros of the polynomial $q(z)$ that lie above the real axis
- They will be finite in number
- May be labeled as $z_1, z_2, ...z_n$ where n is less than or equal to the degree of $q(z)$
- Now, integrate the quotient $f(z) = p(z)/q(z)$ around the positively oriented boundary of the semicircular region.

Evaluation of Improper Integrals

The simple closed contour consists of

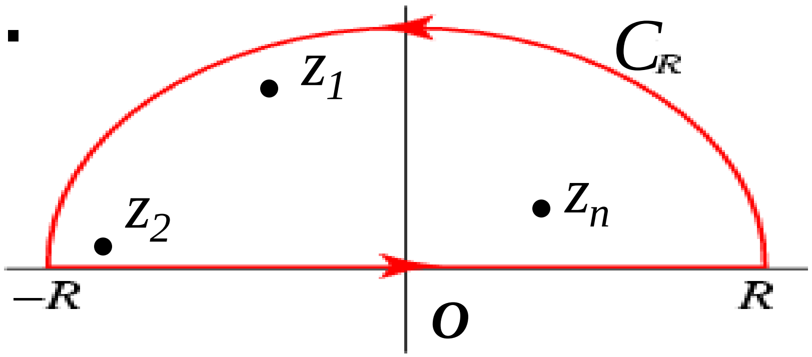
- The segment of the real axis from $z = -R$ to $z = R$ and

- The top half of the circle $|z| = R$ described counterclockwise and denoted by C_R.

Evaluation of Improper Integrals

Remark:

The positive number R is large enough that the points $z_1, z_2, \ldots z_n$ all lie inside closed path.



Evaluation of Improper Integrals

From Cauchy Residue theorem

$$\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$$

$$\text{If } \lim_{R\to\infty} \int_{C_R} f(z)dz = 0,$$

Evaluation of Improper Integrals

then it follows

$$\text{P.V.}\int_{-\infty}^{\infty} f(x)dx = 2\pi i\sum_{k=1}^{n} \underset{z=z_k}{\text{Res}}\, f(z)$$

If $f(x)$ is even, then

Evaluation of Improper Integrals

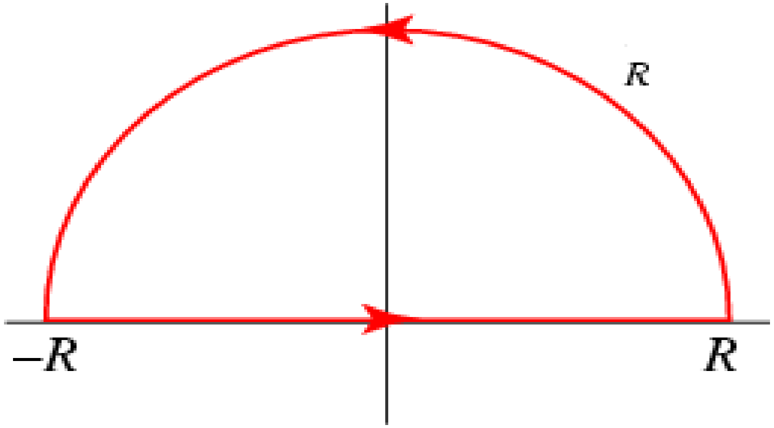
$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$$

and

$$\int_{0}^{\infty} f(x)dx = \pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$$

Evaluation of Improper Integrals

Q.4,p.267:Evaluate $I = \int_0^\infty \frac{x^2\,dx}{(x^2+1)(x^2+4)}$



Evaluation of Improper Integrals

Let $f(z) = \dfrac{z^2}{(z^2+1)(z^2+4)}$ & $C = [-R, R] \cup C_R$

then

$$\int_{-R}^{R} \frac{x^2 dx}{(x^2+1)(x^2+4)} + \int_{C_R} f(z)\, dz = 2\pi i \sum \operatorname*{Res}_{z=z_k} f(z)$$

clearly $z = \pm i,\ \pm 2i$ are poles of order of 1 of $f(z)$ but $z = -i,\ -2i$ lie outside the region C.

Evaluation of Improper Integrals

$$\therefore \operatorname*{Res}_{z=i} f(z) = \left.\frac{z^2}{(z+i)(z^2+4)}\right|_{z=i} = \frac{i^2}{2i\times 3} = \frac{i}{6}$$

$$\operatorname*{Res}_{z=2i} f(z) = \left.\frac{z^2}{(z^2+1)(z+2i)}\right|_{z=2i} = \frac{-4}{-3\times 4i} = \frac{-i}{3}$$

Evaluation of Improper Integrals

$$\therefore (1) \Rightarrow$$

$$\int_{-R}^{R} \frac{x^2 dx}{(x^2+1)(x^2+4)} + \int_{C_R} f(z)\,dz = 2\pi i\left(\frac{i}{6} - \frac{i}{3}\right)$$

$$= \pi/3$$

$$|f(z)| = \frac{|z^2|}{|z^2+1||z^2+4|} \le \frac{|z^2|}{(|z^2|-1)(|z^2|-4)}$$

Evaluation of Improper Integrals

Hence

$$\left|\int_{C_R} f(z)dz\right| \leq \frac{R^2 . \pi R}{\left|R^2-1\right|\left|R^2-4\right|}$$

$$= \frac{\pi}{R\left|1-\frac{1}{R^2}\right|\left|1-\frac{4}{R^2}\right|} \to 0 \text{ as } R \to \infty$$

Evaluation of Improper Integrals

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which yields

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)(x^2+4)} = \frac{\pi}{3}$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{x^2 dx}{(x^2+1)(x^2+4)} = \pi/3$$

$$\Rightarrow \int_{0}^{\infty} \frac{x^2 dx}{(x^2+1)(x^2+4)} = \frac{\pi}{6}$$

Evaluation of Improper Integrals

Evaluation of improper integral of form $\int_{-\infty}^{\infty} f(x)\cos ax\,dx$ and $\int_{-\infty}^{\infty} f(x)\sin ax\,dx$

$$\int_{-R}^{R} f(x)e^{iax}\,dx = \int_{-R}^{R} f(x)\cos ax\,dx + i\int_{-R}^{R} f(x)\sin ax\,dx$$

(together with the fact that the modulus

$$\left|e^{iaz}\right| = \left|e^{ia(x+iy)}\right| = \left|e^{-ay}.e^{iax}\right| \Rightarrow \left|e^{iaz}\right| = e^{-ay}$$

is bounded in the upper half plane $y \geq 0$)

Evaluation of Improper Integrals

Jordan's Lemma: Suppose that

(i) a function $f(z)$ is analytic at all points z in the upper half plane $y \geq 0$ that are exterior to the circle $|z| = R_0$;

Evaluation of Improper Integrals

$(ii)\ C_R: z = Re^{i\theta}, 0 \le \theta \le \pi, R > R_0;$

(iii) for all points z on C_R, there is a positive constant M_R such that $|f(z)| \le M_R$, where $\lim_{R\to\infty} M_R = 0$.

Evaluation of Improper Integrals

Then, for every positive constant a,

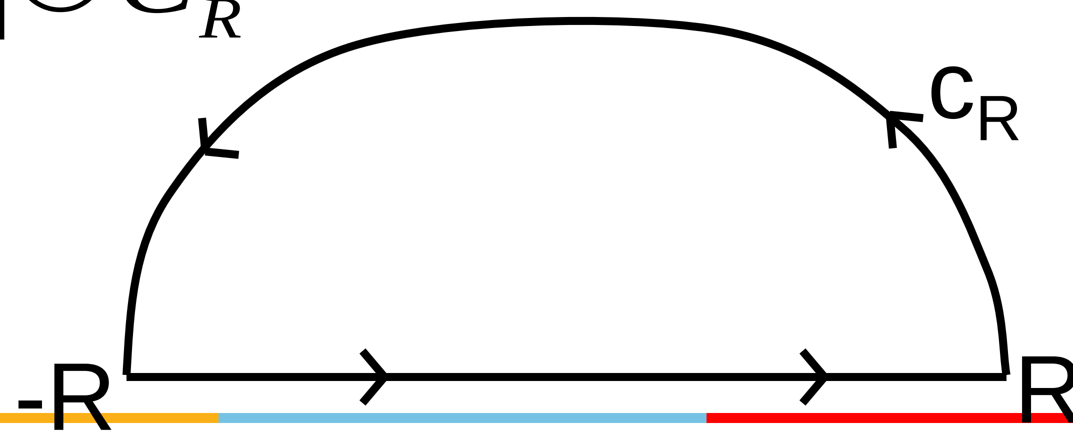
$$\lim_{R\to\infty} \int_{C_R} f(z)e^{iaz}\,dz = 0$$

Evaluation of Improper Integrals

Q1,p.275: $I=\int_{-\infty}^{\infty}\frac{\cos x\,dx}{\left(x^2+a^2\right)\left(x^2+b^2\right)}, a>b>0$

Let $f(z)=\frac{1}{\left(z^2+a^2\right)\left(z^2+b^2\right)}$

& $C=[-R,R]\cup C_R$



Evaluation of Improper Integrals

$f(z)$ has singularity at $z = \pm\, ai, \ \pm\ bi$ out of which $z = ai, bi$ are inside C & they are simple poles

$$\operatorname*{Res}_{z=ai} f(z)e^{iz} = \frac{e^{iz}}{(z+ai)(z^2+b^2)}\Big|_{z=ai}$$

$$= \frac{e^{-a}}{2ai(b^2-a^2)} = -\frac{ie^{-a}}{2a(b^2-a^2)}$$

Evaluation of Improper Integrals

$$\operatorname*{Res}_{z=bi} f(z)e^{iz} = \left.\frac{e^{iz}}{\left(z^2+a^2\right)\left(z+bi\right)}\right|_{z=bi}$$

$$= \frac{e^{-b}}{2bi\left(a^2-b^2\right)} = \frac{i\,e^{-b}}{2b\left(b^2-a^2\right)}$$

$$\therefore \int_{-R}^{R} \frac{e^{ix}dx}{\left(x^2+a^2\right)\left(x^2+b^2\right)} + \int_{C_R} f(z)e^{iz}dz$$

Evaluation of Improper Integrals

$$= 2\pi i \sum \operatorname{Res}\left(f(z)e^{iz}\right)$$

$$= \frac{2\pi\, i.i}{2\left(b^2 - a^2\right)}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

$$= \frac{\pi}{a^2 - b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

Evaluation of Improper Integrals

Taking real parts, we get

$$\int_{-R}^{R} \frac{\cos x \, dx}{\left(x^2+a^2\right)\left(x^2+b^2\right)} + \mathrm{Re} \int_{C_R} f(z) e^{iz}$$

$$= \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right) \qquad (1)$$

Evaluation of Improper Integrals

On C_R, we have

$$|f(z)| = \left| \frac{1}{(z^2+a^2)(z^2+b^2)} \right|$$

$$\leq \frac{1}{|R^2-a^2||R^2-b^2|}$$

$$\Rightarrow M_R \to 0 \quad \text{as} \quad R \to \infty$$

Evaluation of Improper Integrals

Hence by Jordan's Lemma:

$$\lim_{R\to\infty}\int_{C_R} e^{iz} f(z)\,dz = 0$$

Note :

$$\left|\operatorname{Re}\int_{C_R} e^{iz} f(z)\,dz\right| \le \left|\int_{C_R} e^{iz} f(z)\,dz\right|$$

Evaluation of Improper Integrals

$\therefore (1)$ yields

$$\int_{-\infty}^{\infty} \frac{\cos x \, dx}{\left(x^2 + a^2\right)\left(x^2 + b^2\right)}$$

$$= \frac{\pi}{a^2 - b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

Evaluation of Improper Integrals

Q.6,p.276: $I=\int_{-\infty}^{\infty}\frac{x^3\sin ax}{x^4+4}dx, a>0$

$$\text{Let } f(z)=\frac{z^3}{z^4+4}$$

Evaluation of Improper Integrals

$$z^4 + 4 = 0 \Rightarrow z^4 = -4 \Rightarrow z = (-4)^{\frac{1}{4}}$$

$$= (4(-1))^{\frac{1}{4}}$$

$$= \left(4e^{(\pi + 2k\pi)i}\right)^{\frac{1}{4}}, \ k = 0, \ 1, 2, 3$$

$$\Rightarrow z_k = \sqrt{2}\ e^{\left(\frac{\pi}{4} + \frac{k\pi}{2}\right)i}, \ k = 0, 1, 2, 3$$

Evaluation of Improper Integrals

For $k = 0$,

$$z_0 = \sqrt{2}\, e^{i\pi/4}$$

$$= \sqrt{2}\left(\cos \pi/4 + i \sin \pi/4\right)$$

$$= 1 + i$$

Evaluation of Improper Integrals

For $k = 1$,

$$z_1 = \sqrt{2}\, e^{\left(\frac{\pi}{4}+\frac{\pi}{2}\right)i}$$

$$= \sqrt{2}\left[\cos\left(\frac{\pi}{4}+\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{4}+\frac{\pi}{2}\right)\right]$$

$$= \sqrt{2}\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = -1+i$$

Evaluation of Improper Integrals

For $k = 2$

$$z_2 = \sqrt{2}\, e^{\left(\frac{\pi}{4}+\pi\right)i}$$

$$= \sqrt{2}\left[\cos\left(\pi + \frac{\pi}{4}\right) + i\sin\left(\pi + \pi/4\right)\right]$$

$$= \sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = -1 - i$$

Evaluation of Improper Integrals

For $k=3$

$$z_3=\sqrt{2}\,e^{\left(\frac{\pi}{4}+\frac{3\pi}{2}\right)i}$$

$$=\sqrt{2}\left[\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right]$$

$$=1-i$$

Evaluation of Improper Integrals

simple poles are $z_0 = 1+i$ & $z_1 = -1+i$

which are inside C.

$$B_0 = \operatorname*{Res}_{z=z_0} f(z)e^{iaz} = \operatorname*{Res}_{z=1+i} \frac{z^3 e^{iaz}}{z^4+4}$$

$$= \frac{z^3 e^{iaz}}{4z^3}\Big|_{z=1+i} = \frac{1}{4}e^{ia(1+i)}$$

Evaluation of Improper Integrals

$$B_1 = \operatorname*{Res}_{z=z_1} f(z)e^{iaz} = \frac{z^3 e^{iaz}}{4z^3}\Big|_{z=-1+i}$$

$$= \frac{1}{4}e^{ia(-1+i)}$$

$$= \frac{1}{4}e^{-ai}.e^{-a}$$

Evaluation of Improper Integrals

$$\therefore B_0 + B_1 = \frac{1}{4} e^{-a}\left[e^{ai} + e^{-ai}\right]$$

$$= \frac{1}{4} e^{-a}\left(\cos a + i\sin a + \cos a - i\sin a\right)$$

$$= \frac{1}{2} e^{-a}\cos a$$

Evaluation of Improper Integrals

Now we have

$$\int_{-R}^{R} \frac{x^3 e^{iax}}{x^4+4}\,dx + \int_{C_R} f(z)e^{iaz}\,dz$$

$$= 2\pi i \sum \text{Res } f(z)e^{iaz}$$

$$= 2\pi i \,\frac{1}{2} e^{-a}\cos a$$

$$= \pi i e^{-a}\cos a$$

Evaluation of Improper Integrals

Taking Imaginary parts, we have

$$\mathrm{Im}\int_{-R}^{R}\frac{x^3 e^{iax}}{x^4+4}\,dx + \mathrm{Im}\int_{C_R} f(z)e^{iaz}\,dz$$

$$= \mathrm{Im}(2\pi i \sum \mathrm{Res}\ f(z)e^{iaz})$$

$$= \pi e^{-a}\cos a$$

Evaluation of Improper Integrals

Hence

$$\int_{-R}^{R} \frac{x^3 \sin ax}{x^4 + 4}\, dx + \operatorname{Im} \int_{C_R} f(z) e^{iaz}\, dz$$

$$= \pi e^{-a} \cos a$$

Evaluation of Improper Integrals

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We have

$$f(z) = \frac{z^3}{z^4+4}$$

Evaluation of Improper Integrals

$$\text{On } C_R : |z| = R, \quad \text{we get}$$

$$|f(z)| \leq \frac{R^3}{R^4 - 4} \to 0 \text{ as } R \to \infty$$

$$\text{Hence by Jordan's Lemma}$$

$$\lim_{R\to\infty} \int_{C_R} f(z) e^{iaz} dz = 0$$

Evaluation of Improper Integrals

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Thus, on taking limit when

$R \to \infty$, we get

$$\int_{-\infty}^{\infty} \frac{x^3 \sin ax}{x^4 + 4}\, dx = \pi e^{-a} \cos a$$

Evaluation of Improper Integrals

Definite integrals involving sines and cosines:

Consider the integral

$$I = \int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$$

Let $z = e^{i\theta}, \quad 0 \le \theta \le 2\pi$

$$\Rightarrow dz = i.e^{i\theta} d\theta \Rightarrow \frac{dz}{iz} = d\theta$$

Evaluation of Improper Integrals

$$C: \quad |z| = 1$$

$$\frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \cos\theta$$

$$\frac{1}{2i}\left(z - \frac{1}{z}\right) = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) = \sin\theta$$

Evaluation of Improper Integrals

$$\therefore I = \int_{0}^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$$

$$= \int_{C:|z|=1} f(z)\, dz = 2\pi i \sum \operatorname{Res} f(z)$$

Evaluation of Improper Integrals

Q1, $p.290$: Evaluate $I=\int_0^{2\pi}\frac{d\theta}{5+4\sin\theta}$

$\therefore$ Let $z=e^{i\theta}$

then $I=\int_{C:|z|=1}\frac{dz}{iz\left(5+4\frac{1}{2i}\left(z-\frac{1}{z}\right)\right)}$

$$=\int_C\frac{dz}{5iz+2z^2-2}$$

Evaluation of Improper Integrals

$$2z^2 + 5iz - 2$$

$$= 2z^2 + 4iz + iz - 2$$

$$= 2z(z + 2i) + i(z + 2i)$$

$$= (z + 2i)(2z + i)$$

$$= 2(z + 2i)\left(z + \frac{i}{2}\right)$$

Evaluation of Improper Integrals

$$\therefore I = \int_C f(z)\,dz,$$

where $f(z) = \dfrac{1}{2(z+2i)(z+i/2)}$

$z = -2i, \dfrac{-i}{2}$ are simple poles of $f(z)$

but $z = \dfrac{-i}{2}$ is the only pole which inside C.

Evaluation of Improper Integrals

$$\operatorname*{Res}_{z=-i/2} f(z) = \left.\frac{1}{2(z+2i)}\right|_{z=-i/2} = \frac{1}{2\left(\frac{-i}{2}+2i\right)} = \frac{1}{3i}$$

$$\therefore I = 2\pi i \times \frac{1}{3i} = \frac{2\pi}{3}$$

Evaluation of Improper Integrals

Q.5,p.291: $I = \int_0^{\pi} \frac{\cos 2\theta \, d\theta}{1 - 2a\cos\theta + a^2}, |a| < 1$

we have $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right),$

$$\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right), z = e^{i\theta}$$

Evaluation of Improper Integrals

$$\therefore 1-2a\cos\theta+a^2$$

$$=1-2a\frac{1}{2}\left(z+\frac{1}{z}\right)+a^2$$

$$=-\frac{a}{z}(z-a)\left(z-\frac{1}{a}\right)$$

Evaluation of Improper Integrals

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$= 2.\frac{1}{4}\left(z + \frac{1}{z}\right)^2 - 1$$

$$= \frac{1}{2z^2}\left(z^4 + 1\right)$$

Evaluation of Improper Integrals

$$\therefore I=\int_0^{\pi}\frac{\cos 2\theta.d\theta}{1+a^2-2a\cos\theta}$$

$$=\frac{1}{2}\int_0^{2\pi}\frac{\cos 2\theta.d\theta}{1+a^2-2a\cos\theta}$$

$$=-\frac{1}{4ai}\int_c\frac{\left(z^4+1\right)dz}{z^2\left(z-a\right)\left(z-\frac{1}{a}\right)}$$

Evaluation of Improper Integrals

Let $f(z) = \dfrac{z^4 + 1}{z^2 (z-a)\left(z - \dfrac{1}{a}\right)}$

then $z = a, \dfrac{1}{a}$ are simple poles

& $z = 0$ is a pole of order 2 of $f(z)$

Evaluation of Improper Integrals

$$\therefore |a| < 1 \Rightarrow \frac{1}{|a|} > 1$$

$\therefore\ z = 0\ \&\ z = a$ are the only poles which are inside C.

Evaluation of Improper Integrals

$$B_0 = \operatorname*{Res}_{z=a} f(z) = \left. \frac{z^4+1}{z^2\left(z-\frac{1}{a}\right)} \right|_{z=a}$$

$$= \frac{a^4+1}{a(a^2-1)}$$

Evaluation of Improper Integrals

$$B_1 = \operatorname*{Res}_{z=0} f(z) = \frac{d}{dz}\left(\frac{z^4+1}{(z-a)\left(z-\frac{1}{a}\right)} \right)\Bigg|_{z=0}$$

$$= \frac{a^2+1}{a}$$

Evaluation of Improper Integrals

$$\therefore B_0 + B_1$$

$$= \frac{a^4+1}{a\left(a^2-1\right)} + \frac{a^2+1}{a}$$

$$= \frac{a^4+1+a^4-1}{a\left(a^2-1\right)} = \frac{2a^3}{a^2-1}$$

Evaluation of Improper Integrals

$$\therefore I = -\frac{1}{4ai} \times 2\pi i \times \frac{2a^3}{a^2 - 1}$$

$$= \frac{a^2 \pi}{1 - a^2}$$

Problem-1

Show $\int_0^{2\pi} \frac{\cos^2 3\theta \, d\theta}{1-2p\cos 2\theta + p^2} = \pi \frac{1-p+p^2}{1-p}, 0 < p < 1$

Sol. $I = \frac{1}{2}\int_0^{2\pi} \frac{(1+\cos 6\theta)\, d\theta}{1-2p\cos\theta + p^2} = \frac{1}{2}\mathrm{Re}\left\{\int_0^{2\pi} \frac{(1+e^{i6\theta})\, d\theta}{1-2p\cos\theta + p^2}\right\}$

$$I' = \frac{1}{i}\int_C \frac{z(1+z^6)\, dz}{(1-pz^2)(z^2-p)} = 2\pi \sum \mathrm{Res}\, f(z)$$

where $C : |z| = 1$

Problem-1

$f(z)$ has simple poles at $z=\pm\sqrt{p}$ and $z=\pm\sqrt{\dfrac{1}{p}}$

out of which $z=\pm\sqrt{p}$ are inside C.

$$\sum \text{Res}\, f(z)=\lim_{z\to\sqrt{p}}\frac{z(1+z^6)}{(1-pz^2)(z+\sqrt{p})}+\lim_{z\to-\sqrt{p}}\frac{z(1+z^6)}{(1-pz^2)(z-\sqrt{p})}$$

$$=\frac{1+p^3}{1-p^2}=\frac{1-p+p^2}{1-p}$$

$$\Rightarrow I'=2\pi\frac{1-p+p^2}{1-p}\Rightarrow I=\pi\frac{1-p+p^2}{1-p}$$

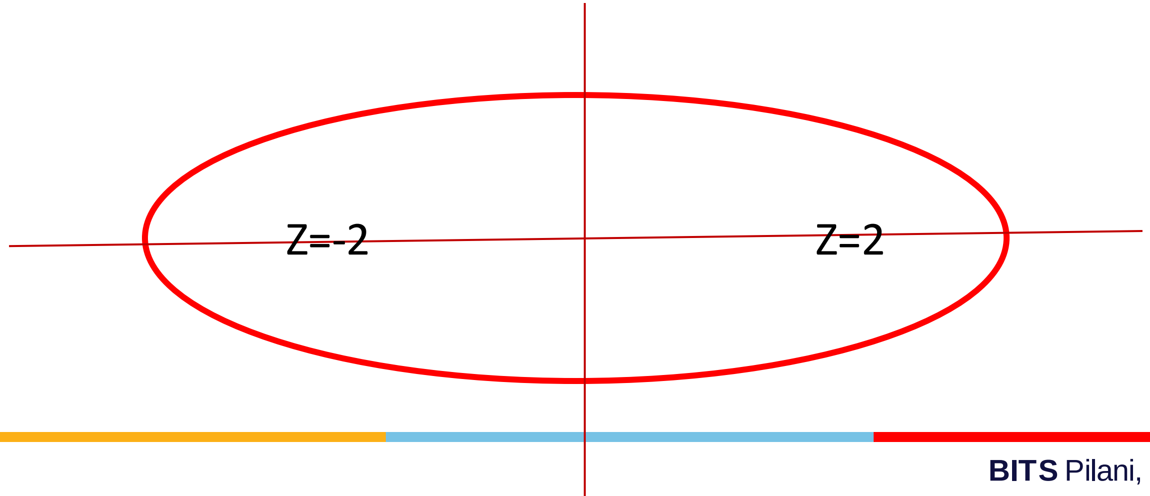
Problem-2

Evaluate $I = \int_C \frac{dz}{z(2z-5)(z-4)}$,

where $C = \{z : |z+2| + |z-2| = 6\}$

Solu. C is a ellipse with semi major axis 3

and minor axis $\sqrt{5}$, and foci are $z = 2$ and $z = -2$



Problem-2

only two poles $z = 0$ and $z = 5/2$ lie inside C.

$$\operatorname*{Res}_{z=0} = \lim_{z\to 0} zf(z) = \lim_{z\to 0} \frac{z}{2z(z-4)(z-5/2)} = 1/20$$

$$\operatorname*{Res}_{z=5/2} = \lim_{z\to 5/2} (z-5/2) f(z) = \lim_{z\to 0} \frac{z-5/2}{2z(z-4)(z-5/2)} = -2/15$$

$$I = 2\pi i\left(\frac{1}{20} - \frac{2}{15}\right) = \frac{-i\pi}{6}$$

Problem-3

Evaluate $\int_C \frac{\log z}{(z^3+z^2+z+1)} dz$, where $C = C_1 \cup C_2$,

C_1 : line $2x + 2y + 1 = 0$

C_2 : portion of the circle $|z| = 2$ lies below the line C_1

$\log z$ is a branch $|z| > 0,\ \pi/2 < \theta < 5\pi/2$

Solu. $f(z)$ has simple pole at $z = \pm i$, -1, where

$z = -i$ and -1 are interior to the C.

Problem-3

$$\int_C \frac{\log z}{(z^3+z^2+z+1)}dz = \int_{C_{-1}} \frac{(\log z)/(z^2+1)}{(z+1)}dz$$

$$+\int_{C_{-i}} \frac{(\log z)/(z-i)(z+1)}{(z+i)}dz$$

$$= 2\pi i\left[\frac{\log(-1)}{2}+\frac{\log(-i)}{(1-i)(-2i)}\right] = \frac{-\pi^2}{4}(1+3i)$$

ALL THE BEST

THANK YOU
FOR YOUR PATIENCE !!!