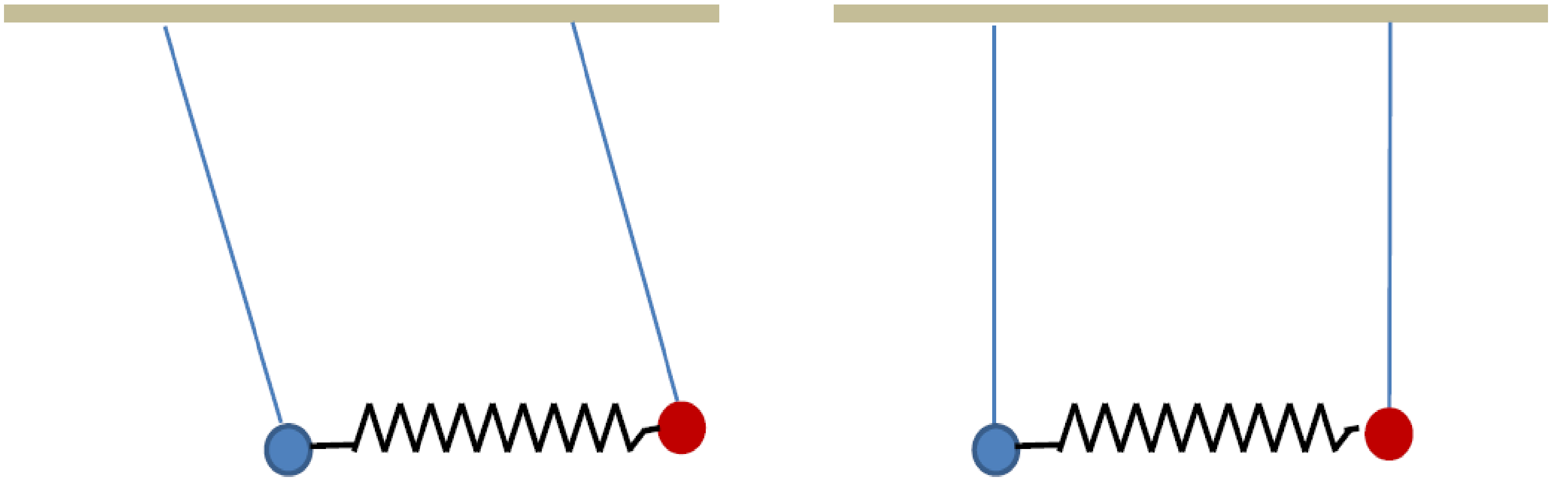
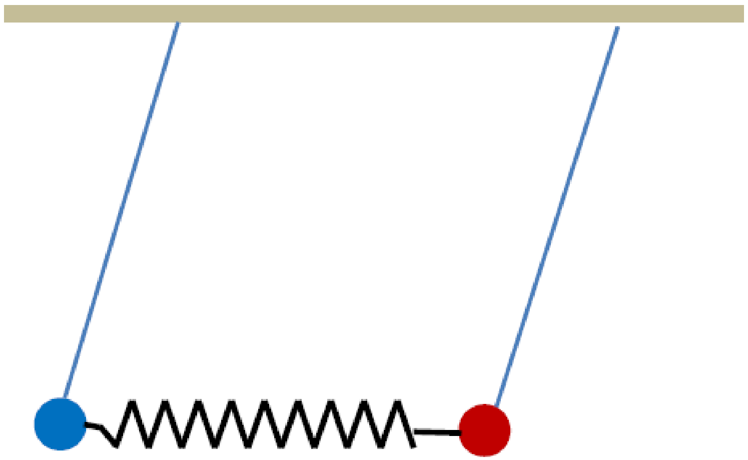
Chapter 5:
Coupled Oscillators and Normal Modes

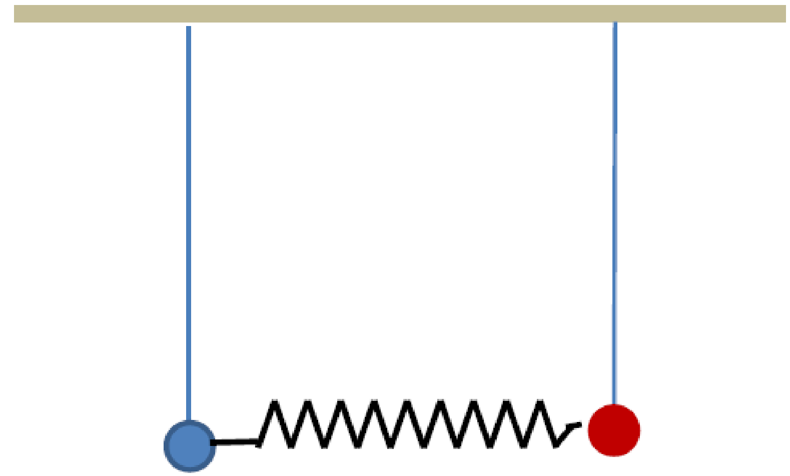
First Mode for the Coupled Pendulums



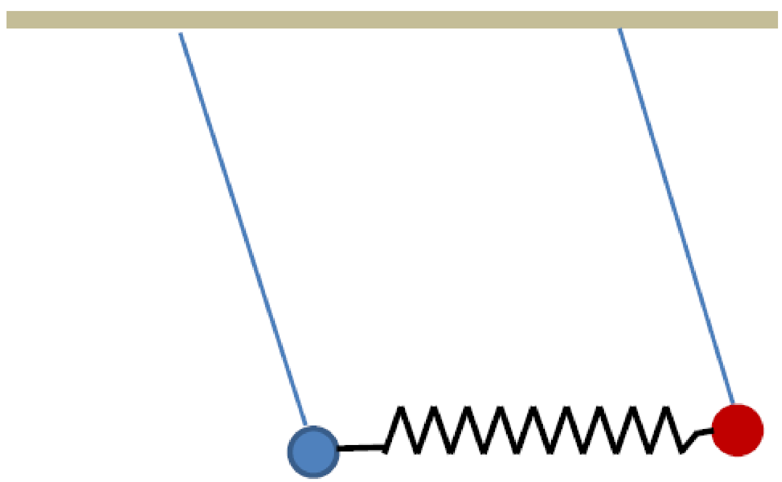
$t = 0$ $\qquad\qquad$ $t = T_0/4$



$t = T_0/2$

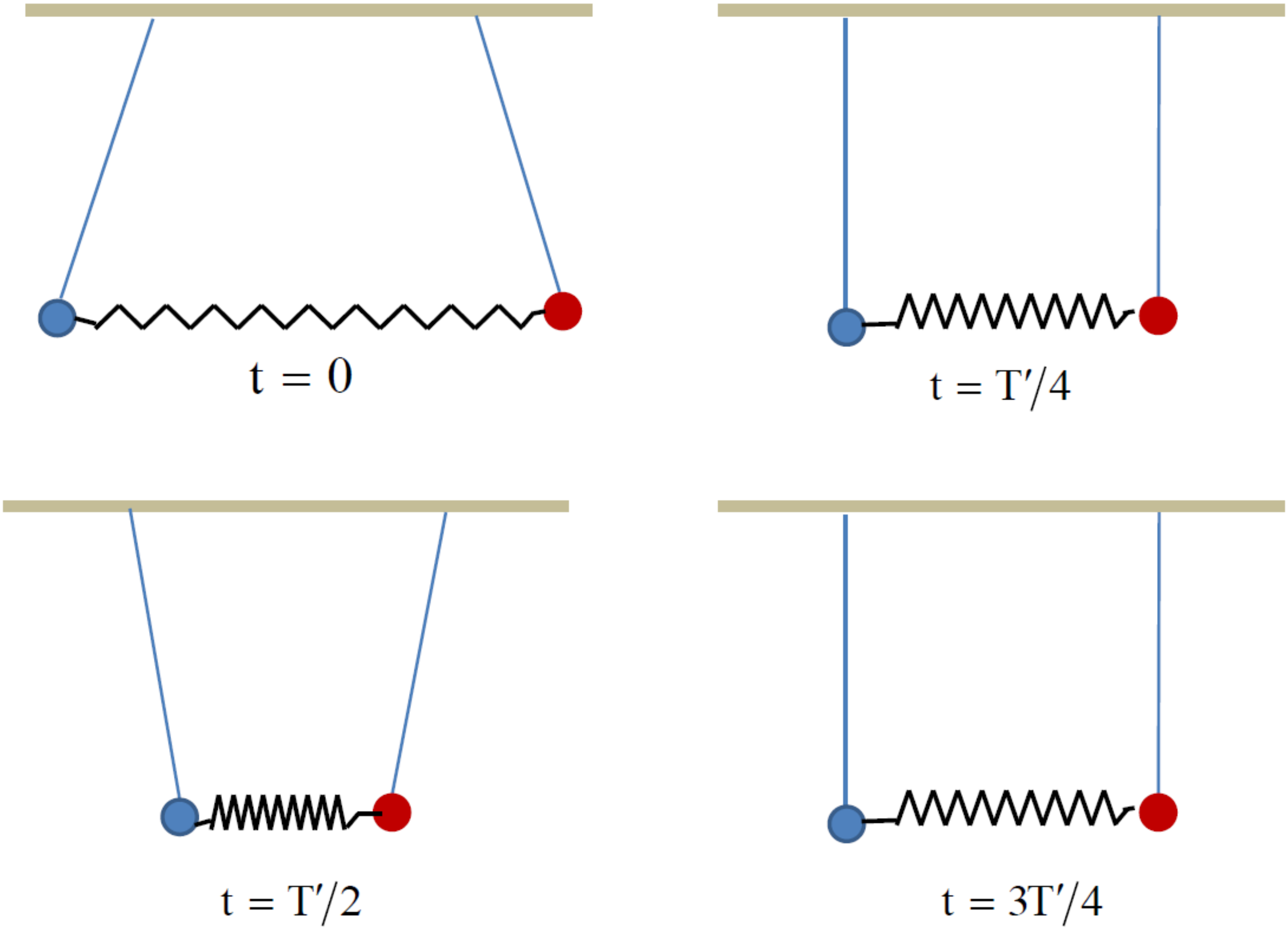


$t = 3T_0/4$

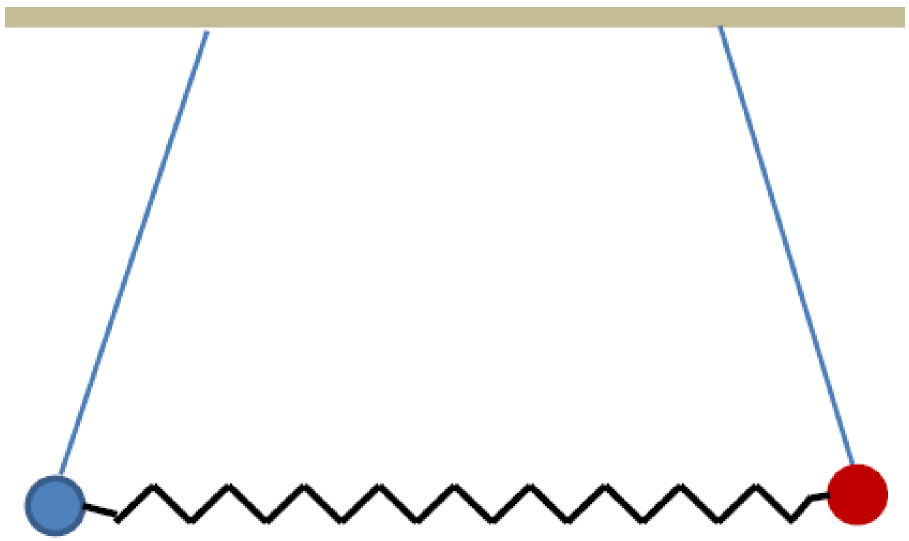


$t = T_0$

Second normal mode vibration :

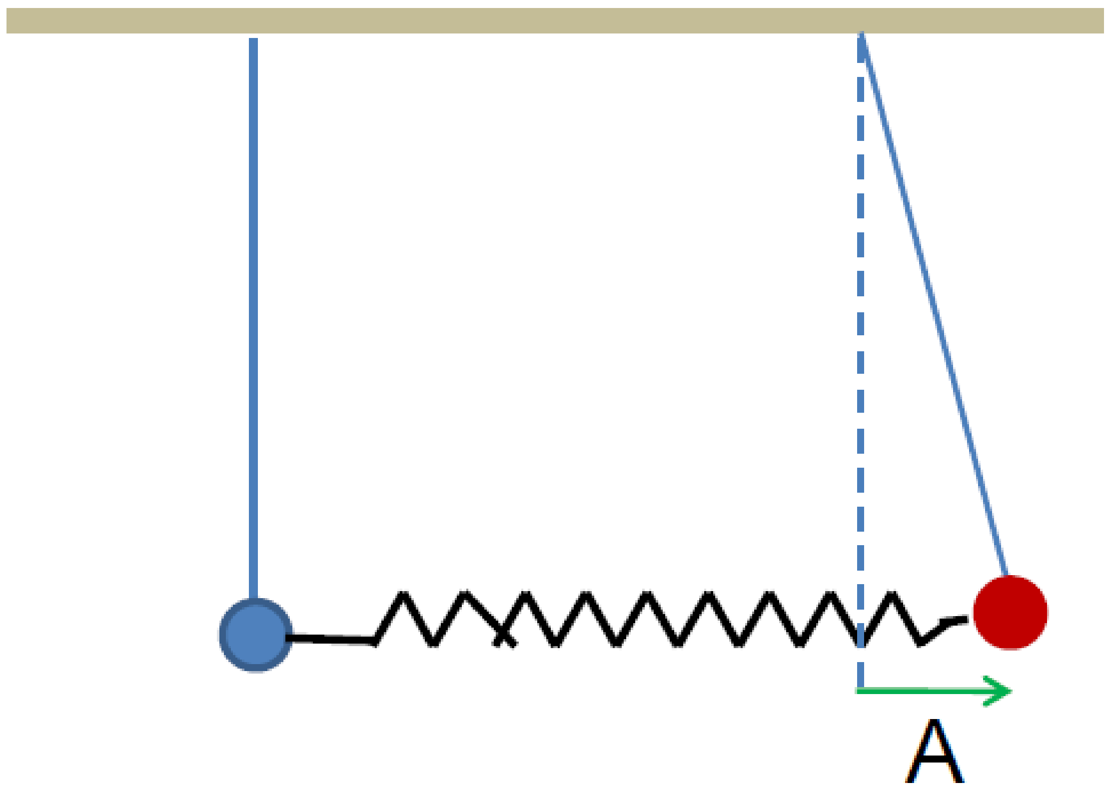


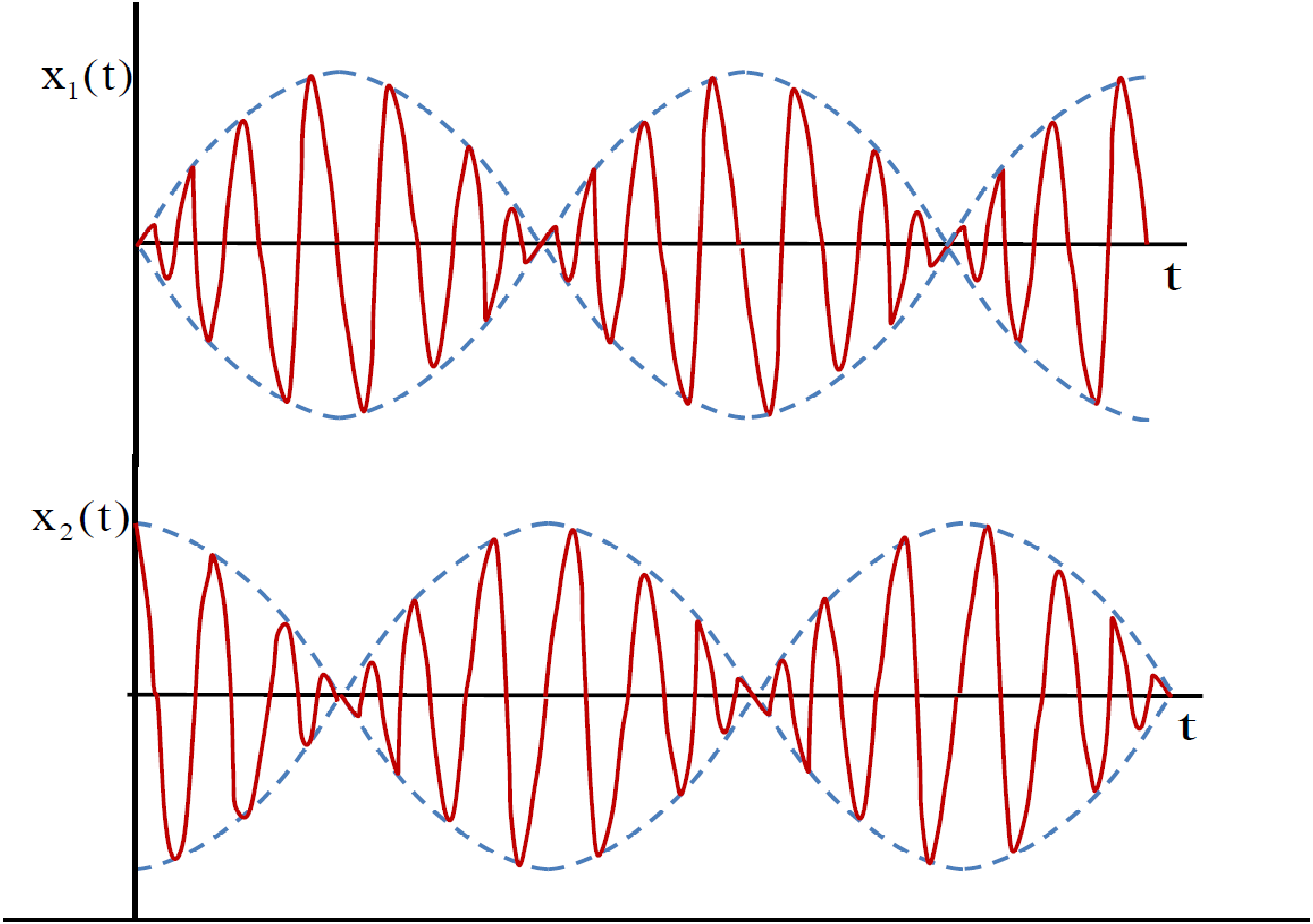
Second normal mode vibration :



$$t = T'$$

General Motion (Both Modes Excited)




$x_1(t)$
t
$x_2(t)$
t

Summary

1. There exist normal coordinates, which are such that, the equations of motion in them, are decoupled

2. Each normal coordinate behaves like a simple harmonic oscillation with its own frequency, the normal frequency

3. With appropriate initial conditions, one can excite only one normal coordinate, the other remaining dormant. Such vibrational modes are called normal mode vibrations.

4. In a normal mode vibration, each mass in the coupled system, executes a SHO with the same frequency, the corresponding normal frequency. The amplitudes of motion of the different masses, and their phases are in general different.

5. In the most general motion, which results from arbitrary initial conditions, the motion of each mass is rather complicated. There is no definite frequency of vibration. However, the motion is a superposition of SHMs

General Approach for Finding Normal Modes

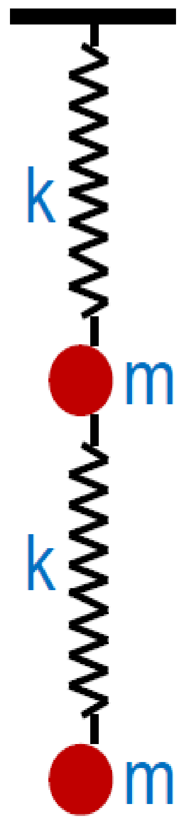
Back to Coupled Pendulums :

Since in a normal mode vibration, all masses execute SHM of a common frequency, put :

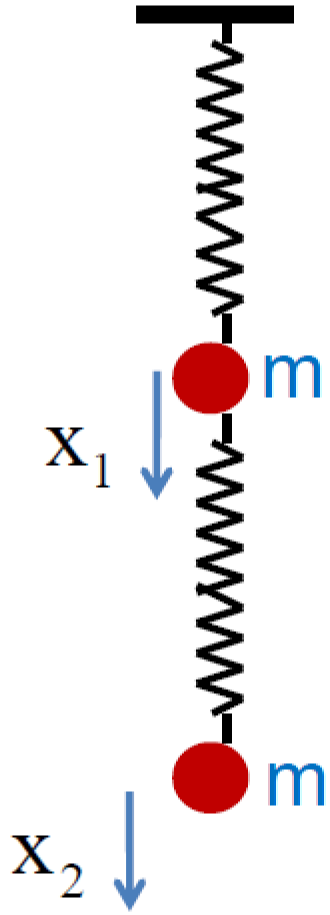
$$x_1 = A_1 \cos \omega t \quad ; \quad x_2 = A_2 \cos \omega t$$

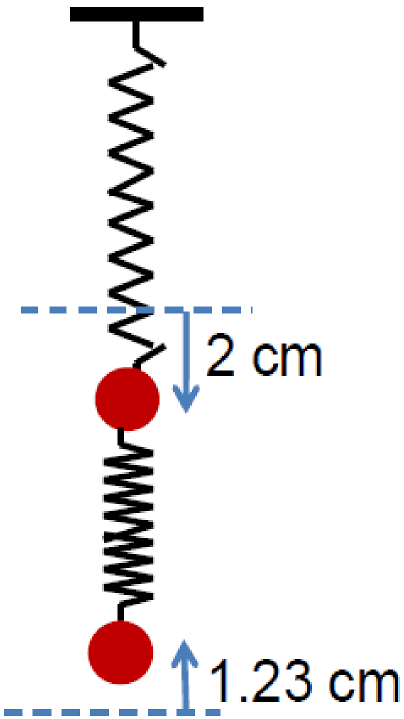
Prob. 5.10

Consider the vertical motion of the system.

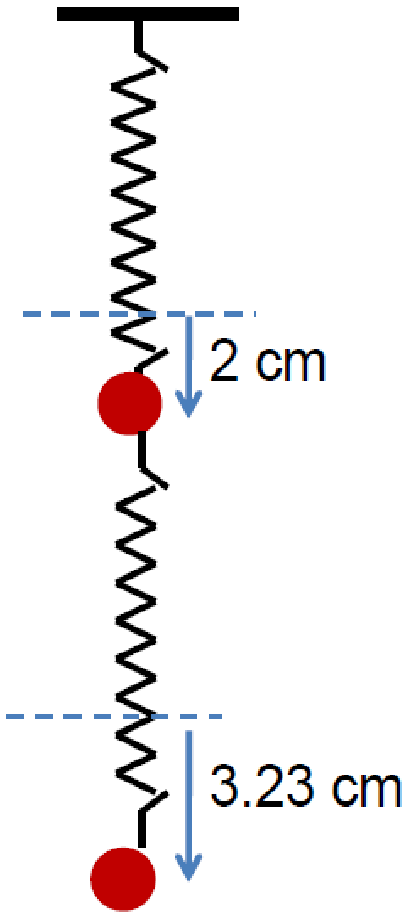


a) Find the normal mode frequencies and the ratio of the amplitude of the two masses in each mode


m
x1
m
x2



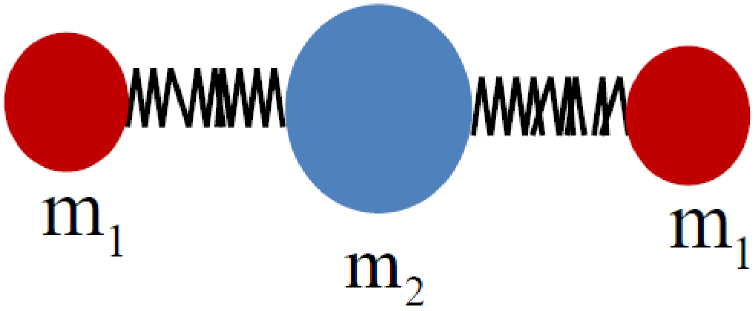
First Mode



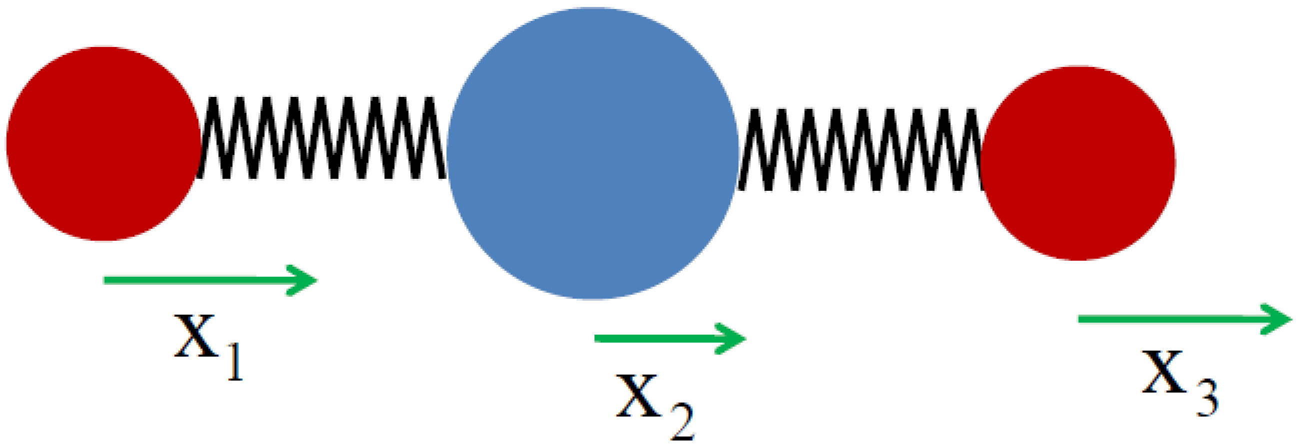
Second Mode

b*) Find the normal coordinates

Prob. 5.9 The carbon dioxide molecule can be likened to a central mass connected to two other identical masses, by identical springs of spring constant k.



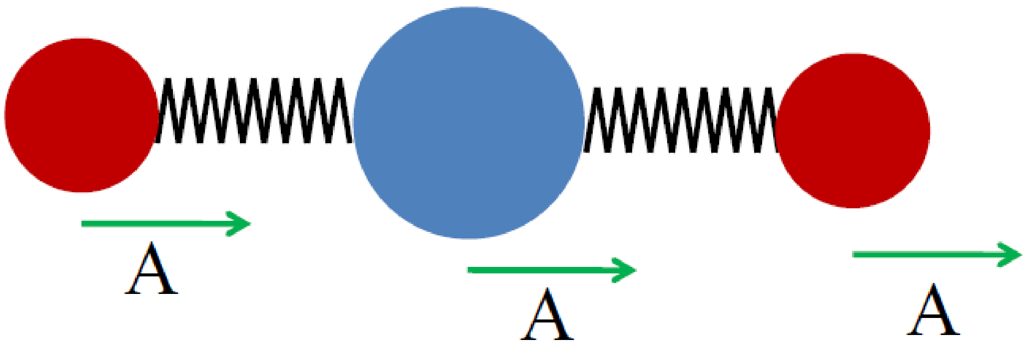
a) Set up the equations of motion, find the normal frequencies and ratios of the amplitudes in the normal modes

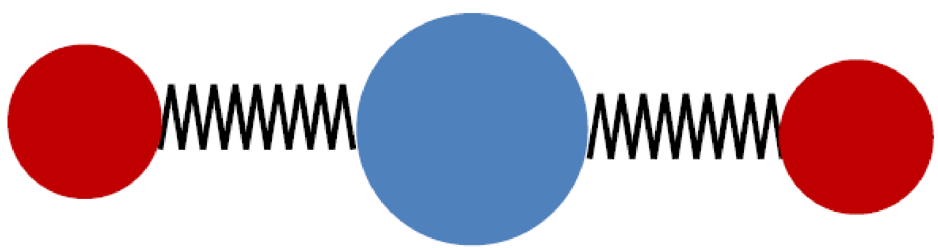

x_1
x_2
x_3

First mode

$$A_1 : A_2 : A_3 = 1 : 1 : 1$$

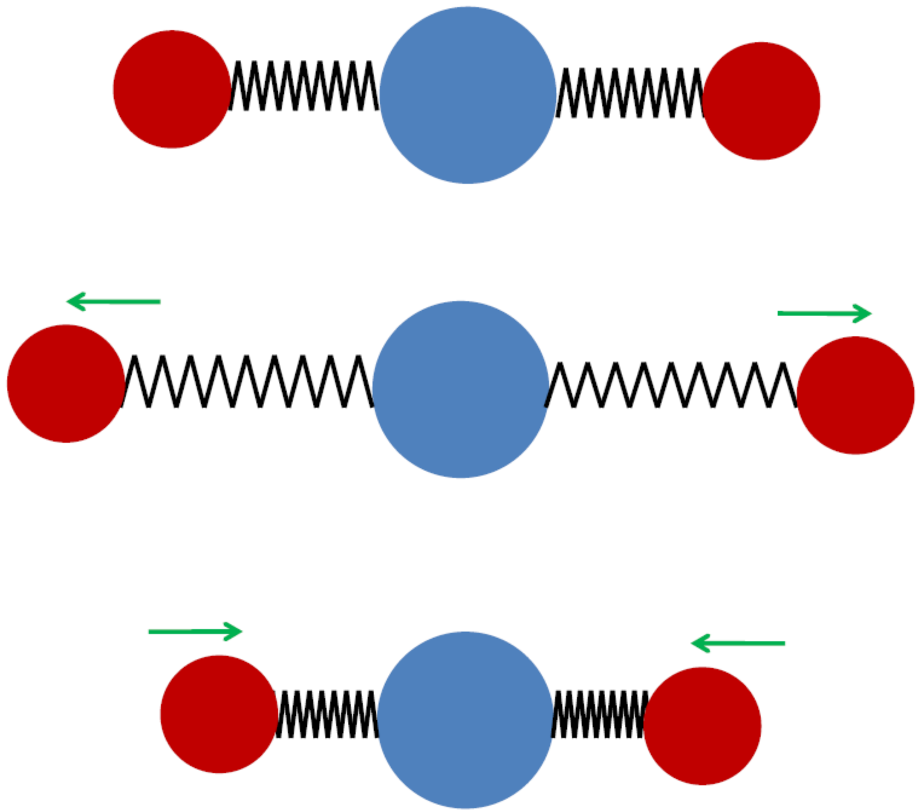
No oscillations, rigid shift of the molecule





Second Mode

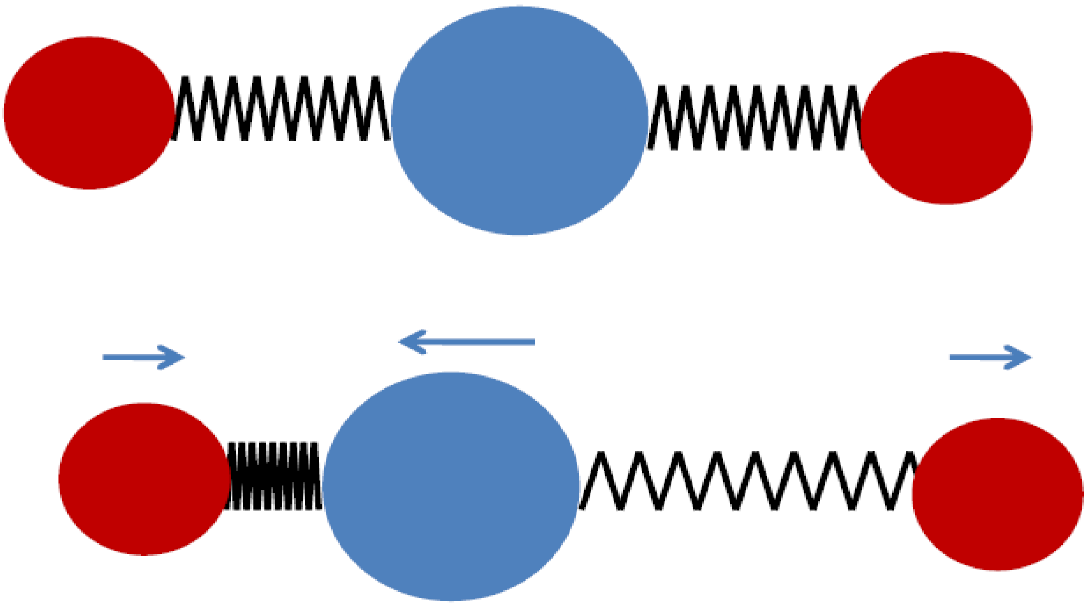
$$A_1 : A_2 : A_3 = 1 : 0 : -1$$



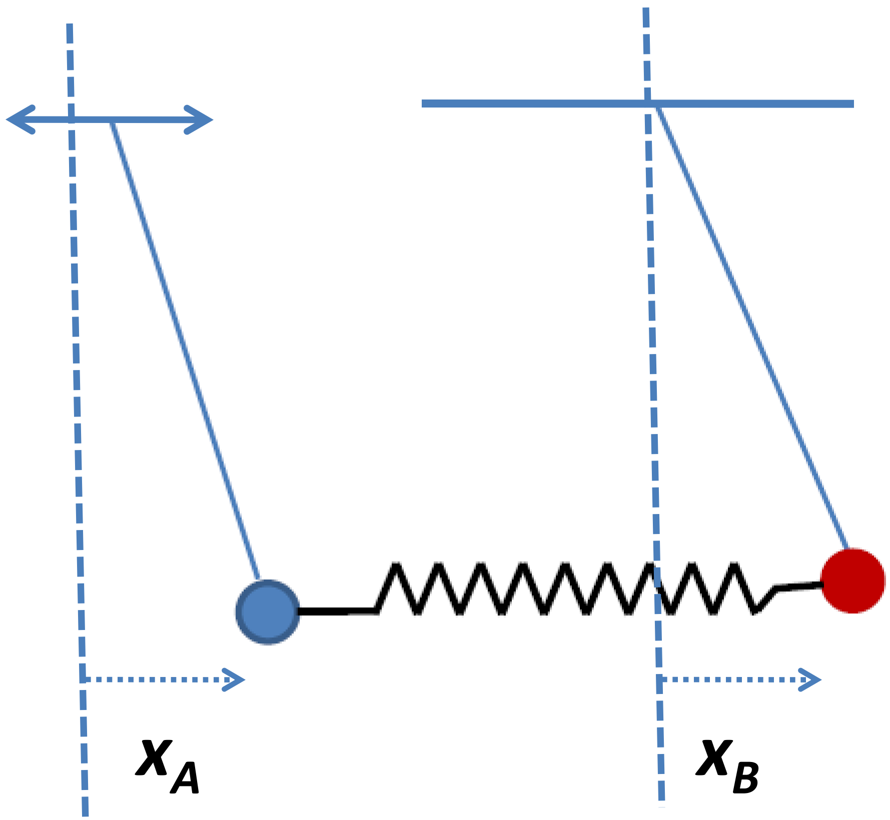
Third Mode

$$A_1 : A_2 : A_3 = 1 : -2\frac{m_1}{m_2} : 1$$

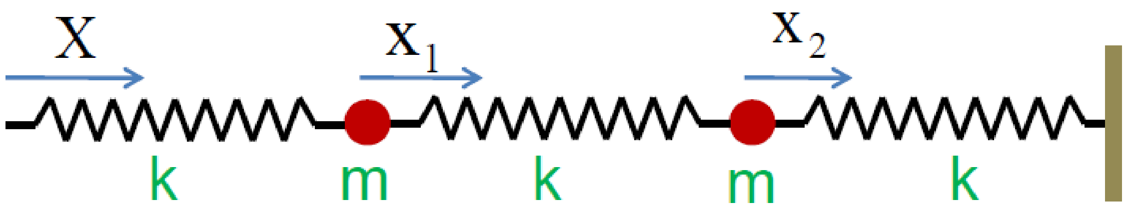
$$= 1 : -2.7 : 1$$



Coupled and Driven Oscillators



Prob. 5.12 Two identical masses are connected to three identical springs on a frictionless surface as shown

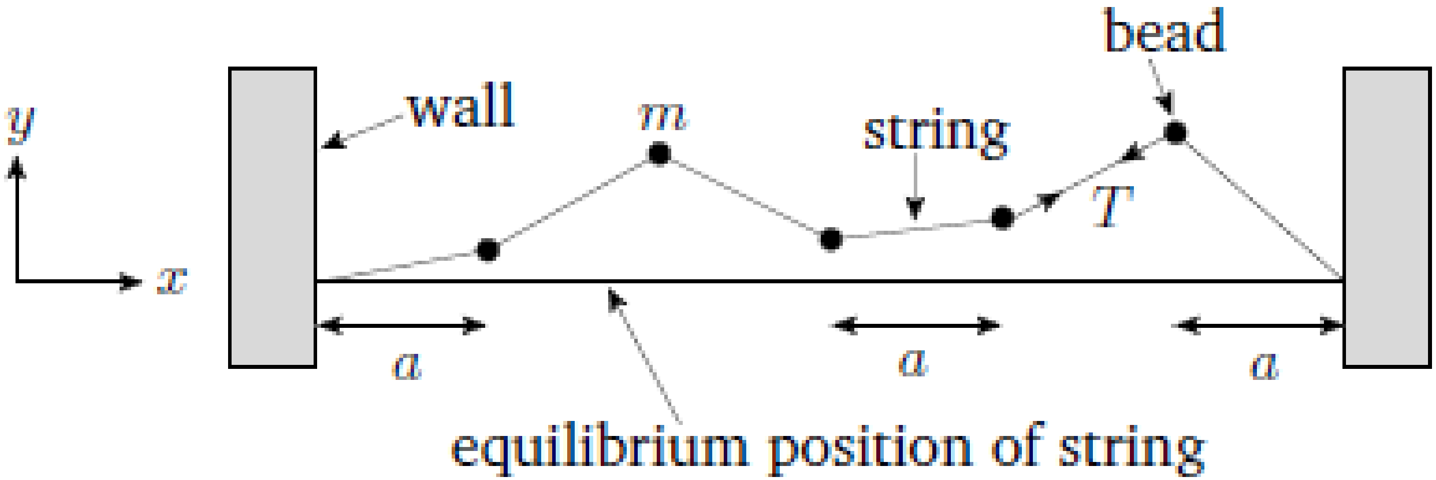


The free end is driven with a displacement :

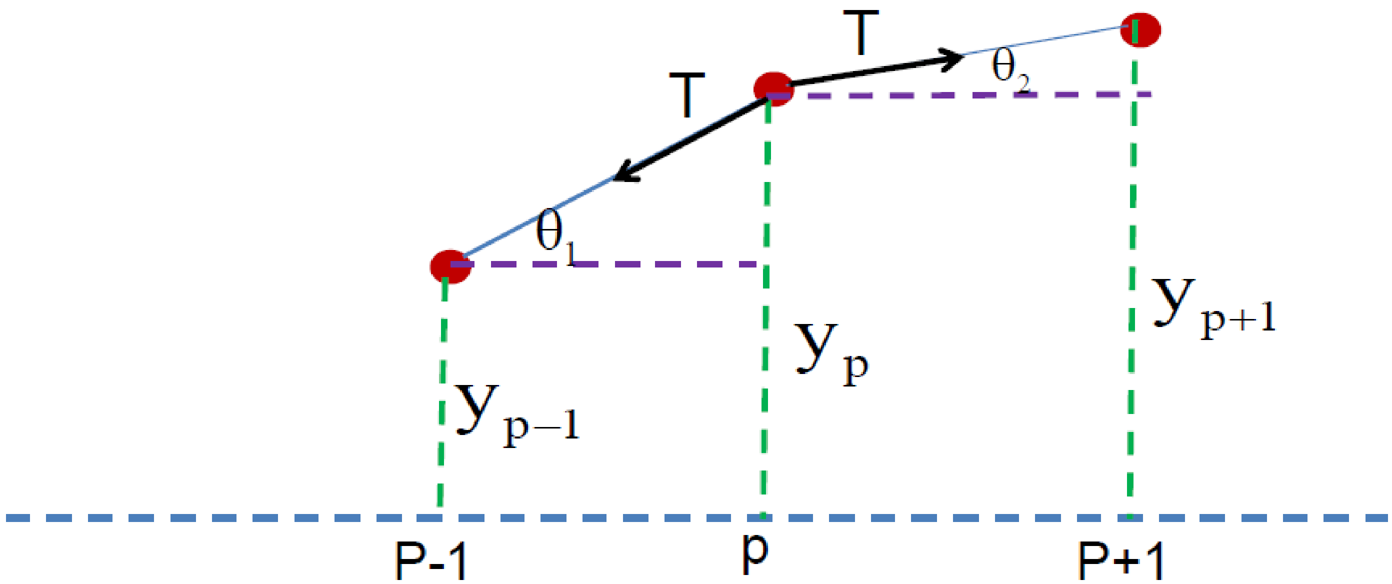
$$X = X_0 \cos \omega t$$

Find and draw the graphs of the displacements of the two masses.

N-Coupled Oscillator



Equations of Motion



Finding Normal Modes for N-Coupled Oscillator

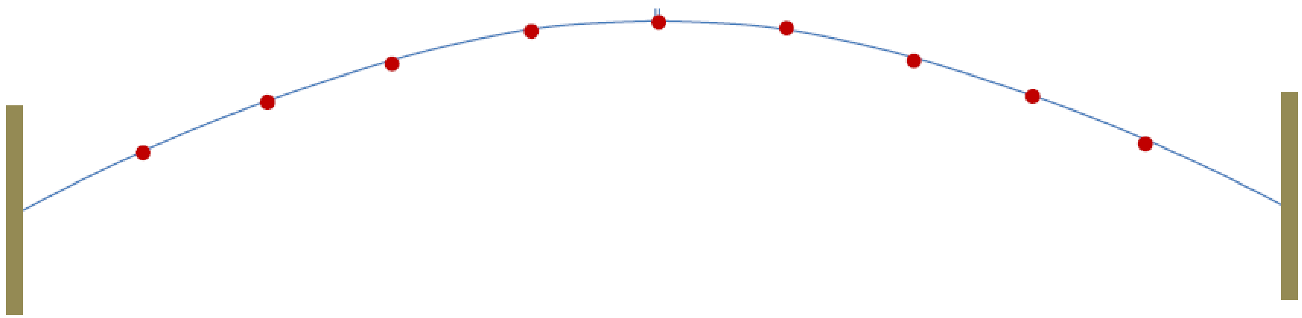
Equation of Motion:

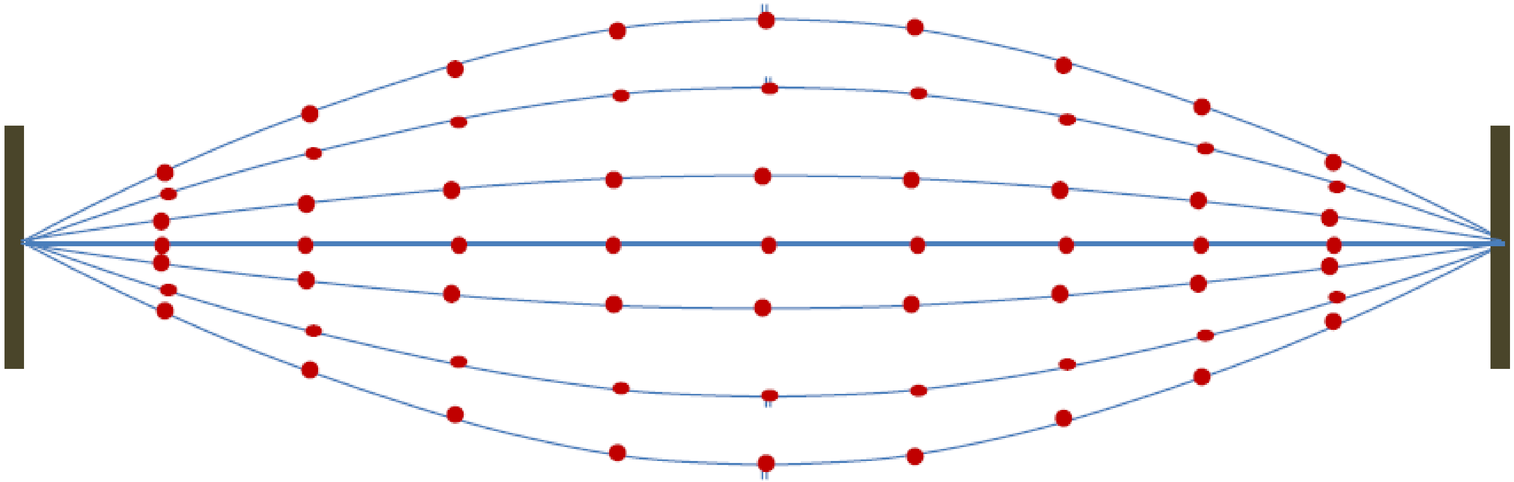
$$\frac{d^2 y_p}{dt^2} + 2\omega_0^2 y_p - \omega_0^2 \left(y_{p+1} + y_{p-1} \right) = 0$$

Finding Normal Modes for N-Coupled Oscillator

The First Mode

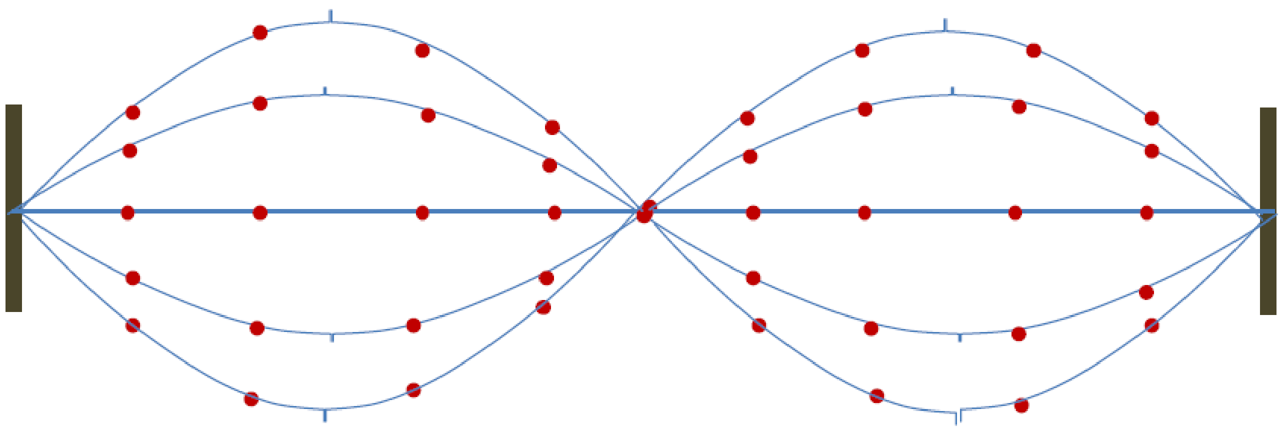
$$\omega_1 = 2\omega_0 \sin\left(\frac{\pi}{2(N+1)}\right) \qquad A_1(x) = C\sin\left(\frac{\pi x}{L}\right)$$



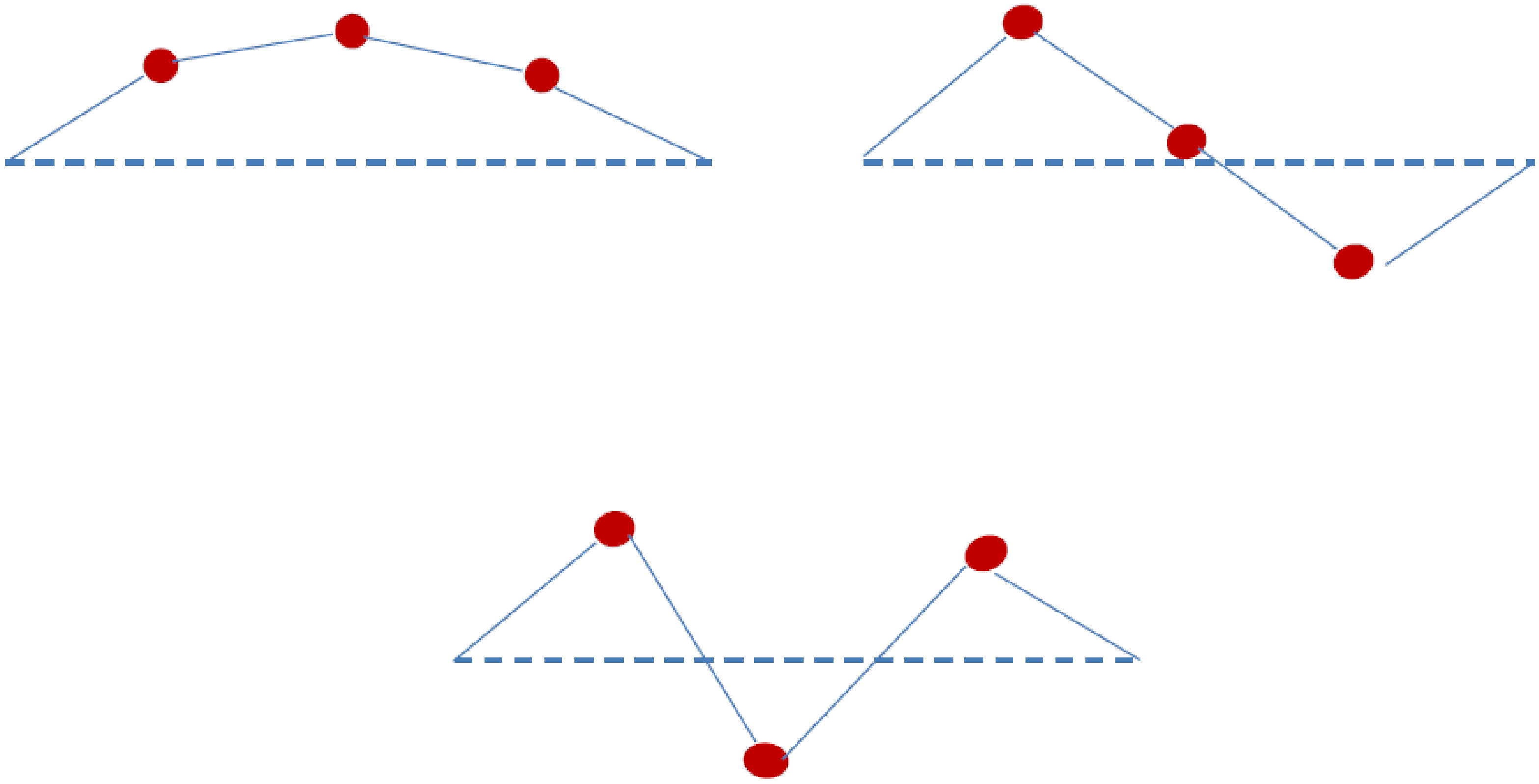


The Second Mode

$$\omega_2 = 2\,\omega_0 \sin\left(\frac{\pi}{(N+1)}\right) \qquad A_2(x) = C\sin\left(\frac{2\pi x}{L}\right)$$



Three Particles



General Solution

$$y_p(t) = \sum_{n=1}^{N} C_n \sin\frac{np\pi}{N+1}\cos(\omega_n t + \phi_n) \qquad p = 1,\ldots\ldots,N$$

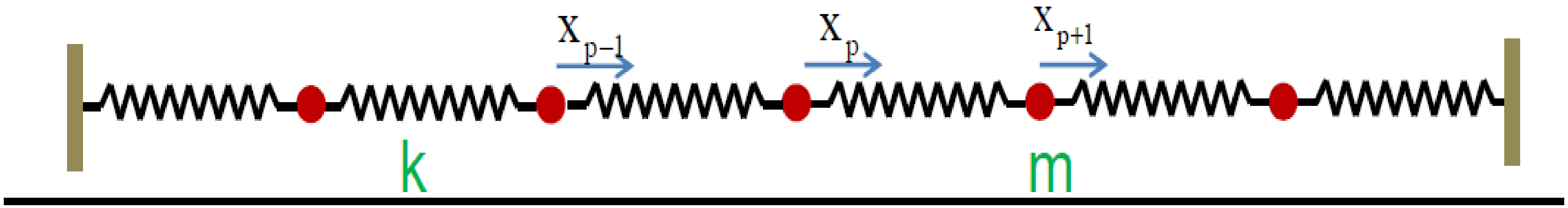
Prob. 5.16 (Resonance in N coupled Oscillators)

Consider a system of N coupled oscillators driven at a frequency ω . Forcing is done at the extreme end such that

$$y_{N+1}(t) = h\cos\omega t$$

Find the resulting amplitudes of the particles.

2. N Light Beads Connected by Springs



$$m\frac{d^2x_p}{dt^2} = k(x_{p+1} - x_p) - k(x_p - x_{p-1})$$

Or,

$$\frac{d^2x_p}{dt^2} + \omega_0^2(-x_{p-1} + 2x_p - x_{p+1}) = 0 \;; \qquad \omega_0^2 = \frac{k}{m}$$