innovate achieve lead

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Logic in Computer Science

MODULE: PREDICATE LOGIC

Expressiveness

Compactness Theorem

Reachability is NOT EXPRESSIBLE

Predicate Logic – Results

- Soundness and Completeness
- Validity is Undecidable
  - Satisfiability?
  - Provability?

Predicate Logic - Expressiveness

- Propositional formulas vs. Predicate formulas
  - What can not be expressed in Predicate Logic?

Reachability in Graphs

- Consider a directed graph: G = (V,E)
- Define **Reachability** as follows:
  - *a vertex u is reachable from a vertex v, if there is a finite path (of edges) from v to u.*
- **Question:**
  - Is Reachability expressible in Predicate Logic?

Expressing Reachability in Graphs using Predicate Logic

- An attempt:
  - In English:

(u and v are the same vertices) |

(there is an edge from v to u) |

(there is an edge from v to a vertex X and an edge from X to u) |

...

  - In predicate logic: (assume **E** denotes the edge relation)

$$(u=v) \vee E(v,u) \vee (\exists X\, E(v,X) \wedge E(X,u)) \vee$$

$$(\exists X_1 \exists X_2\, E(v,X_1) \wedge E(X_1, X_2) \wedge E(X_2,u)) \vee \ldots$$

    - This approach does not work! Why?

- Note:
  - It turns out that Reachability is not expressible in First Order Predicate Logic. (*see Proof in following slides*)

Definitions

- Sentence:
  - A sentence is a formula without any free variable.
- Let Γ be a set of formulas:
  - Γ is said to be satisfiable if the conjunction of formulas in Γ is said to be satisfiable.

Compactness Theorem

- Theorem:
  - Let Γ be a set of sentences in predicate logic. If all finite subsets of Γ are satisfiable then so is Γ.
- Proof (by contradiction):
  - Assume that <u>Γ is not satisfiable</u> (but *<u>all its finite subsets are</u>*):
  - Then the semantic entailment $\Gamma \models \perp$ holds. (Why?)
  - By completeness of Predicate Logic, $\Gamma \vdash \perp$
    - i.e. there is a proof in Natural Deduction for this.
  - But <u>this proof is finite</u> and can use *<u>only finitely many premises</u>* Δ from Γ:
    - i.e. $\Delta \vdash \perp$
  - And from soundness, $\Delta \models \perp$ which is a contradiction.

Reachability is not expressible in Predicate Logic

Theorem:

- There is no formula ψ in predicate logic with u and v as its only free variables and E as its only predicate symbol (of arity 2) such that
  - *ψ holds in a directed graph (model) G = (U,V)* iff
  - *there is a path in that graph from node u to node v.*

Proof: ??

Reachability is not expressible in Predicate Logic

- Proof (***by contradiction***):
- Suppose there is such a formula ψ expressing *existence of a path from node **u** to node **v***. Let **c** and **c'** be constants.
- Define ϕ_n to be the formula expressing existence of a path of length n from **c** to **c'** :

  $\phi_0 =^{def} \mathbf{c=c'}$

  $\phi_1 =^{def} \mathbf{E(c, c')}$

  $\phi_n =^{def} \exists X_1 \ldots \exists X_{n-1} \; (E(c,X_1) \wedge E(X_1,X_2) \ldots \wedge E(X_{n-1},c'))$ for n>1
- Let $\Delta = \{ \neg\phi_i \mid i \in \mathbb{N} \} \cup \{\psi[c/u][c'/v]\}$
  - Δ is unsatisfiable. (Why?)
  - However every finite subset of Δ is satisfiable. (Why?)
  - ***This is a contradiction*** by Compactness Theorem.