Sec 30 - 32 : The Logarithmic Function :

The natural logarithm of $z = x + iy$

is denoted by $\log z$,

i.e. $\quad w = \log z$,

and $\log z$ is defined for $z \neq 0$

by the relation

$$e^{w} = z \quad \ldots\ldots\ldots\ldots\ldots (i)$$

$i.e.$ if $e^{w} = z$, then we write

$$w = \mathbf{log}\, z$$

Let $w = u + iv$,

$$z = x + iy = r\cos\Theta + i\,r\sin\Theta$$

$$= r\,e^{i\Theta}, \quad \text{where}$$

$$-\pi < \Theta \le \pi,\ \Theta = Arg\ z$$

$$Then(i) \Rightarrow e^{u+iv} = r\,e^{i\Theta}$$

$$\Rightarrow e^{u}.e^{iv} = r\,e^{i\Theta}$$

$$\Rightarrow e^{u} = r = \left|z\right|,$$

$$v = \Theta + 2n\pi,$$

$$n = 0, \pm 1, \pm 2, \ldots\ldots$$

$$\Rightarrow u = \ln r = \ln|z|,$$

$$v = \Theta + 2n\pi$$

$$\therefore w = \log z = u + iv$$

$$= \ln|z| + i(\Theta + 2n\pi)$$

$$Since\ Arg\, z = \Theta,\ -\pi < \Theta \leq \pi$$

$$and \quad \mathbf{arg}\, z = \Theta + 2n\pi,$$

n is any integer

$$\therefore \mathbf{log}\, z = \mathbf{ln}|z| + i\,\mathbf{arg}\, z, \qquad z \neq 0$$

When $n = 0$, then $\arg z = \operatorname{Arg} z$

When $n = 0$, then the value of $\log z$ is called the principal value of $\log z$ and is denoted by $\operatorname{Log} z$, i.e.

$$\operatorname{Log} z = \ln|z| + i\, Arg\, z,\ z \neq 0.$$

$$\therefore \mathbf{log}\, z = \mathbf{ln}|z| + i\,\mathbf{arg}\, z$$

$$= \mathbf{ln}|z| + i\left(\Theta + 2n\pi\right)$$

$$= \left(\mathbf{ln}|z| + i\Theta\right) + i\,2n\pi$$

$$\Rightarrow \mathbf{log}\, z = Log\, z + i\,2n\pi,$$

$$n = 0, \pm 1, \pm 2, \ldots$$

$$\text{Derivatives of } \log z \text{ and } \operatorname{Log} z:$$

$$\text{Remark 1:}$$

$$\text{Since } \log z = \ln|z| + i \arg z$$

$$= \ln|z| + i(\Theta + 2n\pi),$$

$$n = 0, \pm 1, \pm 2, \ldots\ldots$$

$$\Rightarrow \log z \text{ is a multivalued function.}$$

Remark 2 :

Since $\mathrm{Log}\, z = \ln|z| + i\,\Theta$,

$$\Theta = Arg\ z$$

$\Rightarrow Log\ z$ is a single - valued function.

Remark 3:

$$\ln|z| = \frac{1}{2}\ln\left(x^2 + y^2\right)$$

is continuous everywhere except at $(0,0)$.

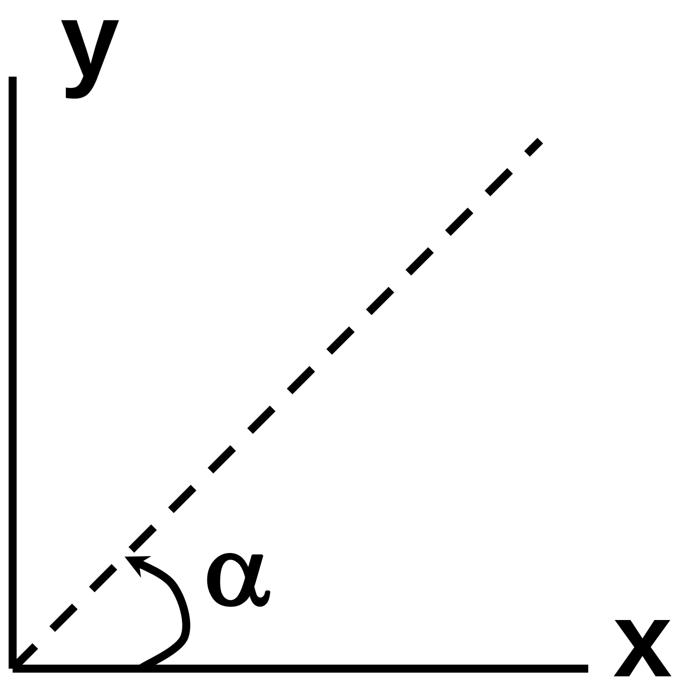
$Remark\ 4:$ Let α be any real number, and consider

$$f(z) = \log z = \ln|z| + i\theta$$

$$= \ln r + i\theta,$$

$$(r > 0, \alpha < \theta < \alpha + 2\pi)$$

$$\Rightarrow u(r,\theta) = \ln r, \quad v(r,\theta) = \theta$$



Then log z is single - valued and continuous in the domain

$$D = \{z : |z| > 0, \alpha < \theta < \alpha + 2\pi\}$$

Remark 5: The function log z is NOT continuous on the line $\theta = \alpha$ as arg z is NOT continuous on the line $\theta = \alpha$.

For if z is a point on the ray $\theta=\alpha$ then there are points arbitrary close to z at which the values of v are nearer to α, and also there are points such that the values of v are nearer to $\alpha+2\pi$.

$$\Rightarrow \lim_{z\to\alpha} \arg z \text{ does not exist.}$$

Remark 6 :

(i) $\log z = \ln r + i\theta$ is analytic

in domain

$$D_1 = \{z : |z| = r > 0, \alpha < \theta = \arg z < \alpha + 2\pi\}$$

(ii) $\mathrm{Log}\, z = \ln r + i\Theta$ is analytic in the domain

$$D_2 = \{z : |z| = r > 0, -\pi < \Theta = Arg\, z < \pi\}$$

$$As, u(r,\theta) = \ln r, \quad v(r,\theta) = \theta$$

$$\Rightarrow u_r = \frac{1}{r}, u_\theta = 0$$

$$v_r = 0, \; v_\theta = 1$$

$\Rightarrow$ CR-equations in polar form

$$u_r = \frac{1}{r} v_\theta, \quad v_r = -\frac{1}{r} u_\theta$$

are satisfied and first-order partial derivatives are continuous.

$$\Rightarrow f'(z) = \frac{d}{dz}(\mathbf{log}\, z) = e^{-i\theta}(u_r + i\,v_r)$$

$$= \frac{1}{r\,e^{i\theta}} = \frac{1}{z} \text{ in } \quad D_1$$

In particular, when $\alpha = -\pi$

$$\frac{d}{dz}(Log\, z) = \frac{1}{z} \text{ in } \ D_2.$$

Remark : 7

Log z is analytic on the whole complex plane except at $(0,0)$ and on the ray $\theta = -\pi$, *i.e. on negative* real axis.

i.e. singularties of $\operatorname{Log} z$ are given by

$$\operatorname{Re} z \leq 0 \quad \text{and} \quad \operatorname{Im} z = 0.$$

Definition:

A branch of a multiple-valued function $f(z)$ defined on a set S is any single valued function $F(z)$ that is analytic in some domain $D \subseteq S$ such that for all $z \in D$, $F(z)$ is one of the values of $f(z)$.

Ex. For each fixed α,

$$\log z = \ln|z| + i\theta,$$

$$\left(|z| > 0, \alpha < \theta < \alpha + 2\pi\right)$$

is a branch of

$$\log z = \ln|z| + i \arg z$$

$$Log\, z = \ln |z| + i\,\Theta,$$

$$\left(|z| > 0, -\pi < \Theta < \pi\right)$$

is called the principal branch.

Q.9(a) p.97

Show that the function

$$\text{Log}\,(z - i)$$

is analytic everywhere except on the half line $y = 1\ (x \leq 0)$.

Solution :

We have $\quad f(z) = Log(z - i)$

singularity of $\quad f(z)$

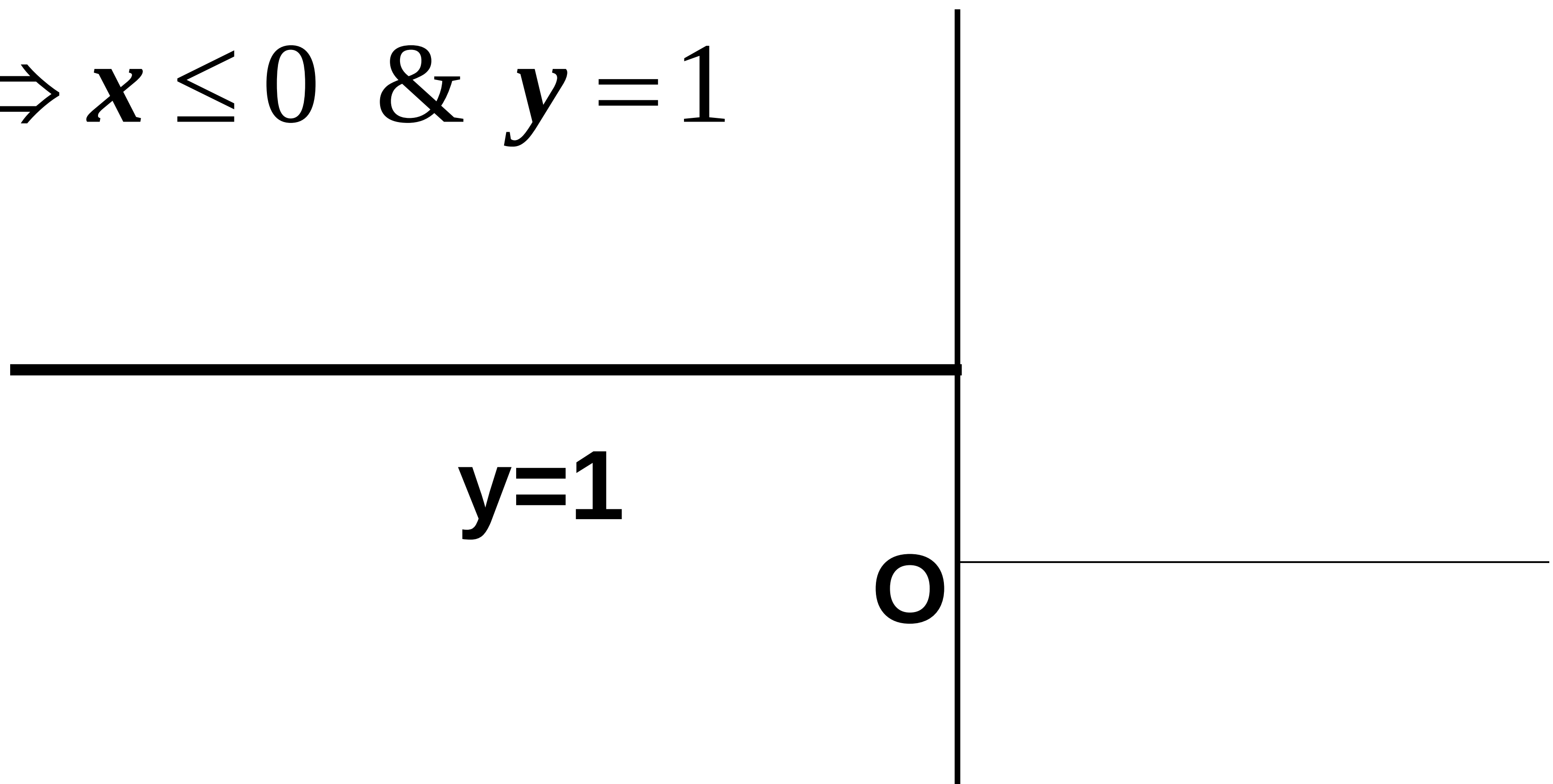
is given by

$$\mathrm{Re}\left(\boldsymbol{z}-\boldsymbol{i}\right)\le 0 \;\&\; \mathrm{Im}\left(\boldsymbol{z}-\boldsymbol{i}\right)=0$$

$$\Rightarrow \mathrm{Re}\left(\boldsymbol{x}+\boldsymbol{i}\left(\boldsymbol{y}-1\right)\right)\le 0 \;\&$$

$$\mathrm{Im}\left(\boldsymbol{x}+\boldsymbol{i}\left(\boldsymbol{y}-1\right)\right)=0$$

$$\Rightarrow \boldsymbol{x}\le 0 \;\;\&\;\; \boldsymbol{y}=1$$



$Q9(b)$ Show that the function

$$f(z) = \frac{Log(z+4)}{z^2+i}$$

is analytic everywhere except at the points $\pm(1-i)/\sqrt{2}$ *and* on the portion $x \le -4$ of the real axis.

Solution :

Singularit ies of $f(z)$ *are* given by

$\mathrm{Re}(z+4) \leq 0,\ \mathrm{Im}(z+4) = 0$ &

$z^2 + i = 0$

$\Rightarrow x + 4 \leq 0,\ y = 0\ \&\ z^2 = -i$

$$Now\, z^2 = -i = e^{\left(\frac{-\pi}{2} + 2n\pi\right)i},$$

$$n = 0, 1$$

$$\Rightarrow z = e^{\left(\frac{-\pi}{2} + 2n\pi\right)\frac{i}{2}}$$

$$\Rightarrow z = e^{\left(\frac{-\pi}{4} + n\pi\right)i}, \quad n = 0,1$$

When n = 0, then

$$z = e^{\frac{-\pi}{4}i} = \cos\frac{\pi}{4} - i\sin\frac{\pi}{4}$$

$$= \frac{1}{\sqrt{2}}(1-i)$$

When $n=1$, then

$$z=e^{\left(\pi-\frac{\pi}{4}\right)i}$$

$$=\cos\left(\frac{3\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}\right)$$

$$=-\frac{1}{\sqrt{2}}(1-i)$$

Hence singularit ies of $f(z)$ *are*

$$\pm\frac{1}{\sqrt{2}}(1-i),\; x \le -4.$$

$\underline{Sec\,32}:$

If z_1 & z_2 be any two non-zero complex numbers, then

$$(1)\log\left(z_1 z_1\right)=\log z_1+\log z_2$$

$$(2)\log\left(\frac{z_1}{z_2}\right)=\log z_1-\log z_2$$

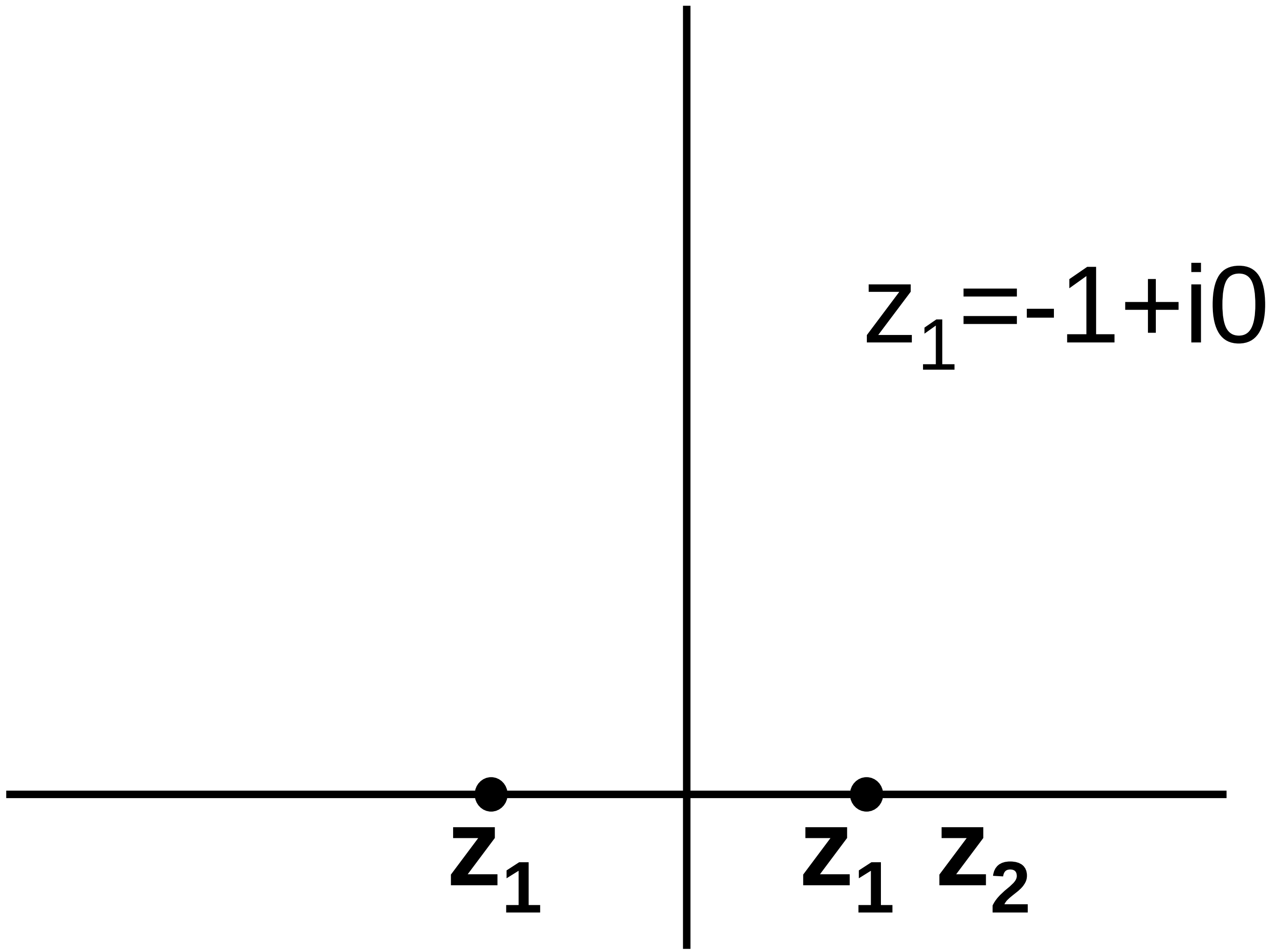
But

$$Log(z_1 z_2) \neq Log\, z_1 + Log\, z_2$$

$$Log\left(\frac{z_1}{z_2}\right) \neq Log\, z_1 - Log\, z_2$$

$$Log\, z^n \neq n\, Log\, z$$

$$Ex(1)\ Let\ z_1 = -1,\ z_2 = -1$$



$$\therefore Log\left(z_1\right) = \ln\left|z_1\right| + i\, Arg\ z_1$$

$$\Rightarrow Log\left(-1\right) = \ln\left(1\right) + i\ \ Arg\ z_1$$

$$= 0 + i\,\pi$$

$$\therefore Log\left(z_1\right) + Log\left(z_2\right) = 2\pi\, i$$

But $z_1 \; z_2 = 1$

$$\Rightarrow Log(z_1 z_2) = \ln|z_1 z_2| + i\,Arg\,(z_1\,z_2)$$

$$= 0 + i.0 = 0$$

Thus

$$Log\,(z_1 z_2) \neq Log\,z_1 + Log\,z_2$$

$Q.3(b)\ p.97$

$$Log(-1+i)^2 \neq 2\,Log(-1+i)$$

$$\text{L.H.S.} = Log(-1+i)^2$$

$$= Log\left[1+i^2-2i\right]$$

$$= Log(-2i)$$

$$= \ln\left|-2i\right| + iArg\left(-2i\right)$$

$$= \ln 2 + i\left(-\frac{\pi}{2}\right)$$

$$= \ln 2 - i\frac{\pi}{2}$$

$$RHS = 2\,Log\left(-1+i\right)$$

$$= 2\left[\ln\left|-1+i\right| + i\,Arg\left(-1+i\right)\right]$$

$$= 2\left[\ln\sqrt{2} + i\,\frac{3\pi}{4}\right]$$

$$= 2[\frac{1}{2}\ln 2 + i\frac{3\pi}{4}]$$

$$= \ln 2 + i\frac{3\pi}{2}$$

$$\therefore LHS \neq RHS$$

$$Q.4(EX, p.97) Show that$$

$$(a) \log\left(i^2\right) \neq 2\log i, \quad \text{when}$$

$$\log z = \ln r + i\theta,$$

$$r = |z| > 0,$$

$$\frac{3\pi}{4} < \theta < \frac{11\pi}{4}$$

$$(b)\ \log\left(i^2\right) = 2\log i, \quad \text{when}$$

$$\log z = \ln r + i\theta,$$

$$r = |z| > 0, \frac{\pi}{4} < \theta < \frac{9\pi}{4}$$

Soln (a):

$$\text{LHS} = \log(i^2) = \log(-1)$$

$$= \ln|-1| + i\theta,\ \theta = \arg(-1)$$

$$= 0 + (\pi + 2n\pi)i$$

NOTE:

$$3\pi / 4 < \theta < 11\pi / 4,$$

$$\theta = \arg(-1) = \pi + 2n\pi$$

and hence $\mathrm{n} = 0.$

Hence, $\mathrm{LHS} = \pi i$

We have

$$\log \mathrm{i} = \ln|i| + i \arg i$$

$$= \ln|1| + i\left(\frac{\pi}{2} + 2n\pi\right), \quad \text{where}$$

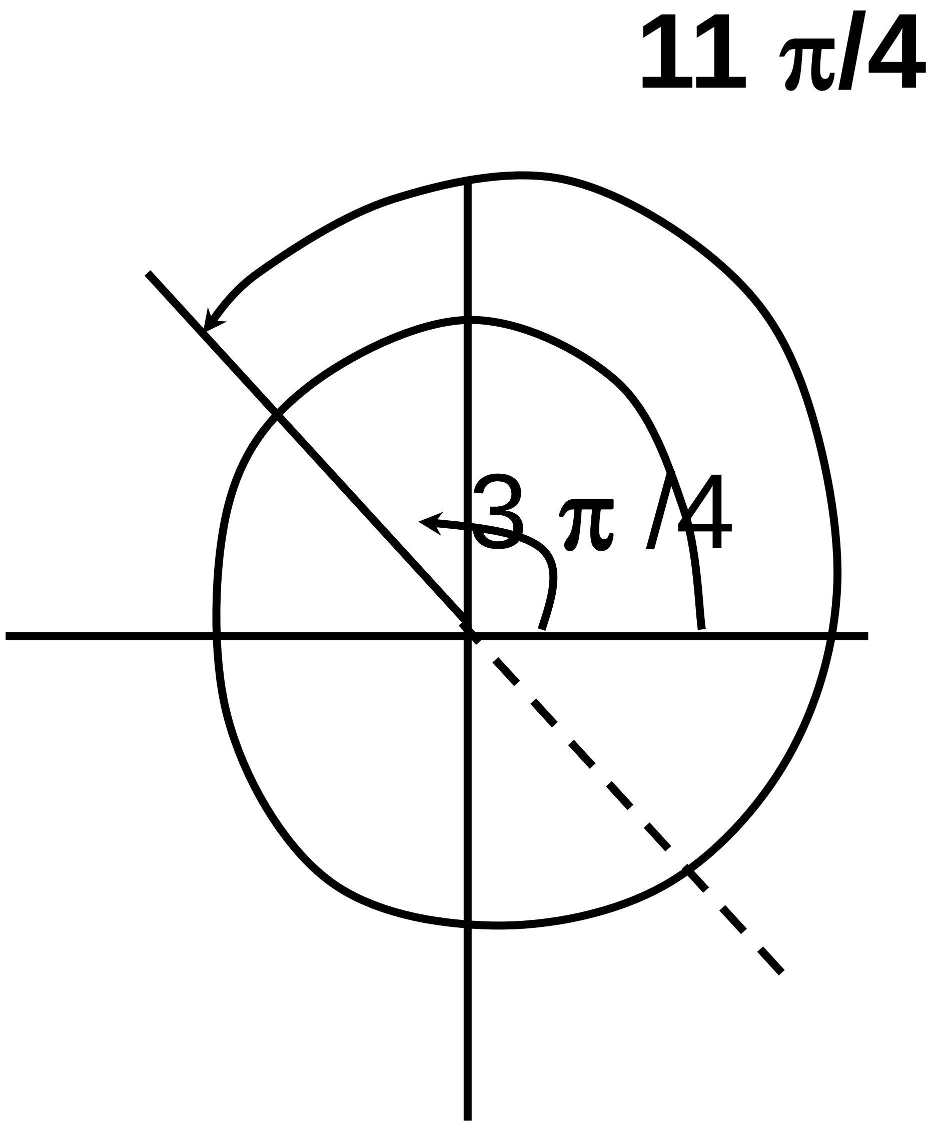
n is an integer

$$= \mathrm{i}\pi\left(2\mathrm{n} + \frac{1}{2}\right)$$

$$\because \frac{3\pi}{4} < \theta = \left(2n + \frac{1}{2}\right)\pi < \frac{11\pi}{4}$$

$$\Rightarrow n = 1 \quad \& \text{hence}$$

$$\theta = \frac{5\pi}{2}$$



11 π/4
3 π /4

$$\therefore RHS = 2\log i = 2.i\frac{5\pi}{2} = 5\pi i$$

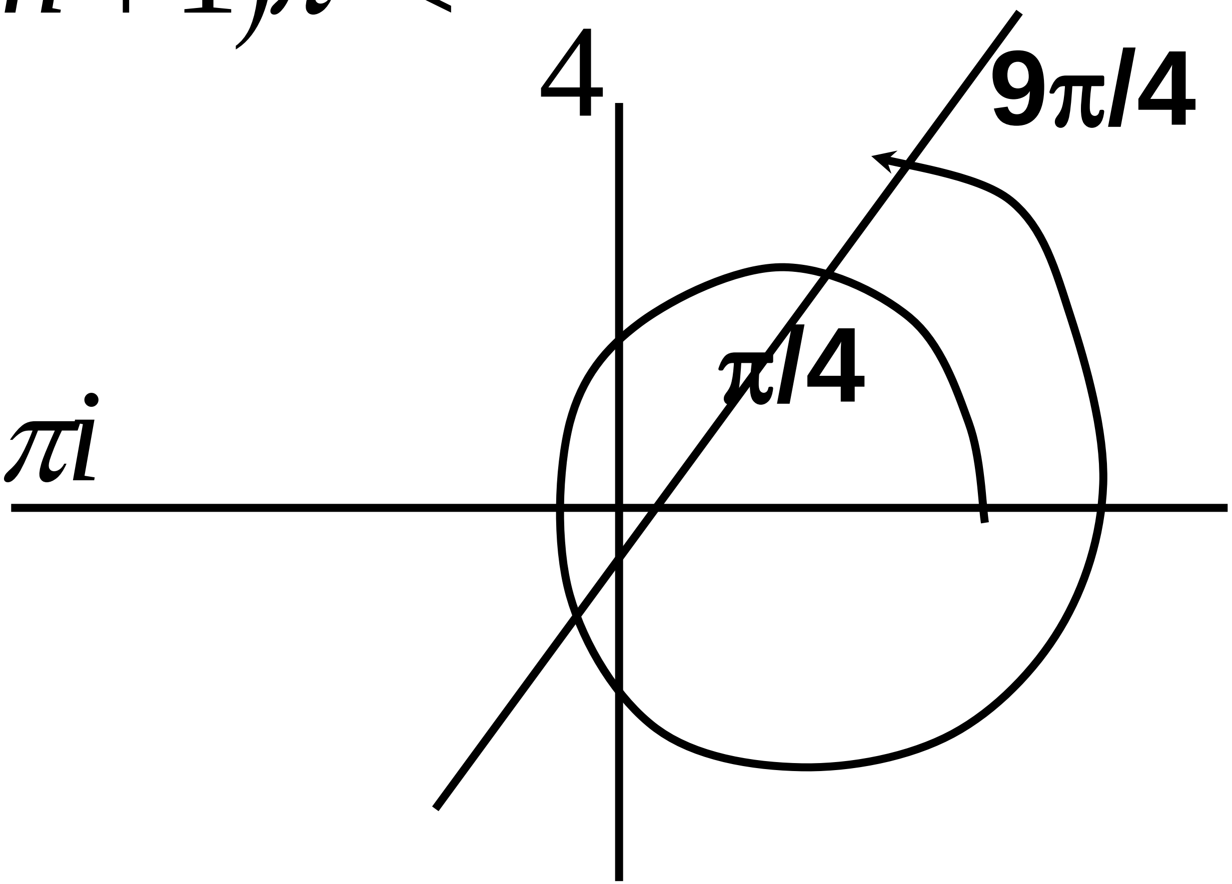
$$LHS \neq RHS$$

$\text{Soln(b)}:$

$$\frac{\pi}{4} < \theta = (2n+1)\pi < \frac{9\pi}{4}$$

$$\Rightarrow n = 0$$

$$\Rightarrow LHS = \pi i$$



But when

$$\frac{\pi}{4} < \theta = \left(2n + \frac{1}{2}\right)\pi < \frac{9\pi}{4}$$

$$\Rightarrow n = 0 \;\text{ \& hence }\; \theta = \frac{\pi}{2}$$

$$RHS = 2\log i = 2i\frac{\pi}{2} = \pi i$$

$$\therefore LHS = RHS$$

$$i.e. \log\left(i^2\right) = 2\log i$$

$$if \;\; \frac{\pi}{4} < \Theta < \frac{9\pi}{4}$$

Q. Solve:

$$(i)\ \ Log z = 1 - \pi i / 4$$

$$(ii)\ \ Log(z-1) = \pi i / 2$$

Sec 33 : Complex Exponents

(1) Let $z \neq 0$ be a complex no.,

and c is any complex no.

Then z^{c} *is* defined as

$$\mathrm{z}^{\mathrm{c}} = e^{c\log z}$$

If $\log z$ is replaced by $\operatorname{Log} z$, then

$$z^c = e^{c \operatorname{Log} z}$$

is called the principal value of z^c.

$Q.2(a)$ Show that i^i is real and find its principal value.

Soln: $i^i = e^{i\,\mathbf{log}\,i}$

$$\mathbf{log}(i) = \mathbf{ln}|i| + i\,\mathbf{arg}(i)$$

$$= 0 + i\left(\frac{\pi}{2} + 2n\pi\right) = \left(2n + \frac{1}{2}\right)\pi i$$

$$\therefore\ i^i = e^{-\left(2n+\frac{1}{2}\right)\pi}$$, which is real,

Principal value of i^i is

$$e^{-\frac{\pi}{2}}\quad (n=0).$$

EX. (b) Find P.V. of i^{-i}.

Solution :

$$i^{-i} = e^{-i\log i} = e^{-i\left(2n+\frac{1}{2}\right)\pi i}$$

$$= e^{\left(2n+\frac{1}{2}\right)\pi}, \; n = 0, \pm 1, \pm 2, \ldots$$

Principal value of $i^{-i} = e^{\pi/2}$

(c) Write log(Log i) in terms of a + ib

$$\text{We have } \mathrm{Log}\,\mathrm{i} = \frac{\pi}{2} i \quad (WHY\,\mathbf{??})$$

$$\Rightarrow \mathbf{log}(Log\, i) = \mathbf{log}\left(\frac{\pi}{2} i\right) = \mathbf{ln}\left|\frac{\pi}{2} i\right| + i\,\mathbf{arg}\left(\frac{\pi}{2} i\right)$$

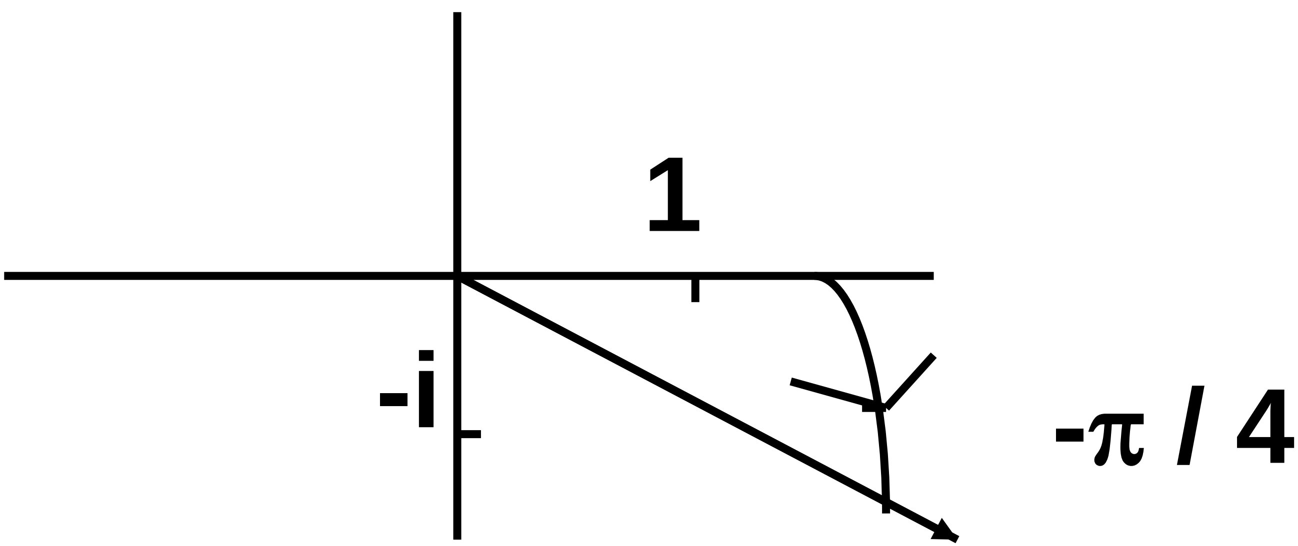
$$= \mathbf{ln}(\pi/2) + i\left(\frac{\pi}{2} + 2n\pi\right)$$

Principal value of

$$\mathrm{Log}(\mathrm{Log}\,\mathrm{i}) \ \ is \ \ \mathbf{ln}(\pi/2) + i\frac{\pi}{2}$$

$Q.Find$ the principal value of $(1\text{-}\mathrm{i})^{1+i}$

$$Solution:(1-i)^{1+i}=e^{(1+i)\log(1-i)}$$



Now,

$$\log(1 - i) = \ln|1-i| + i \arg(1-i)$$

$$= \ln\sqrt{2} + i\left(-\frac{\pi}{4} + 2n\pi\right)$$

$$\therefore\ (1-i)^{1+i} = e^{\log(1-i)+i\log(1-i)}$$

$$= e^{\log(1-i)} \cdot e^{i\log(1-i)}$$

$$= (1-i) \cdot e^{i\ln\sqrt{2}-\left(2n-\frac{1}{4}\right)\pi}$$

$$= (1\text{-}\mathrm{i})e^{i\ln\sqrt{2}} \cdot e^{-\left(2n-\frac{1}{4}\right)\pi}$$

Principal value of

$$(1\text{-}\mathrm{i})^{1+i} \; is \quad (1-i)e^{i\ln\sqrt{2}} \; e^{\frac{\pi}{4}}$$

Example 4, p. 102-103

P.V. $(z_1 z_2)^i \neq (P.V.z_1^i)(P.V.z_2^i)$

Ex: $z_1 = 1 + i, z_2 = 1 - i,\ z_3 = -1 - i$