A1. (a) Discuss the convergence of the series $\sum_{n=0}^{\infty}(e^x-4)^n$ and what values of x series converges absolutely/conditionally. Also find its sum. **[06]**

Solution: Given that

$$a_n = (e^x-4)^n$$

$$a_{n+1} = (e^x-4)^{n+1}$$

Now,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(e^x-4)^{n+1}}{(e^x-4)^n}\right| = |e^x-4|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |e^x-4| < 1 \quad \text{(converges absolutely)}$$

(2M)

or

$$-1 < e^x - 4 < 1$$

$$3 < e^x < 5$$

$$\ln 3 < x < \ln 5$$

Hence the converge region is $\ln 3 < x < \ln 5$

(2M)

End Points:

Put $x = \ln 3$

$\sum_{n=0}^{\infty}(-1)^n$ - divergent

Put $x = \ln 5$

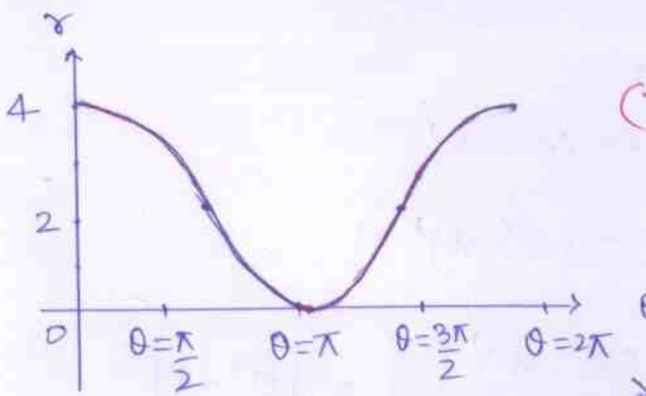
$\sum_{n=0}^{\infty}(1)^n$ - divergent (1M)

Sum: sum of the given series is (geometric series)

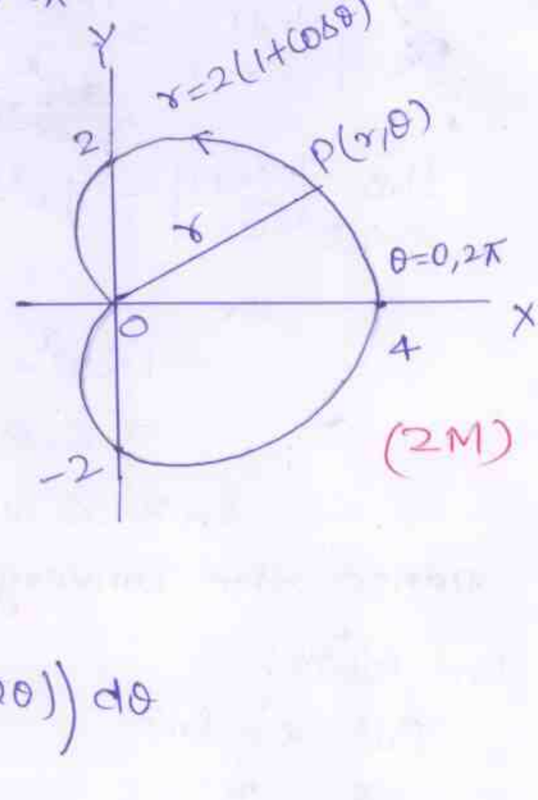
(where $a=1$

A1. **(b)** Trace the polar curve $r = 2(1 + \cos\theta)$ by faster graphing method and find its area. **[09]**

Solution:



(3M)



(2M)

Area of curve

$$A = \frac{1}{2}\int_0^{2\pi} r^2 \, d\theta \qquad (2M)$$

$$= \frac{1}{2}\int_0^{2\pi} 4(1+\cos\theta)^2 d\theta$$

$$= 2\int_0^{2\pi} (1+\cos^2\theta + 2\cos\theta)\, d\theta$$

$$= \int_0^{2\pi} \left(2 + 4\cos\theta + 2\cdot\frac{(1+\cos 2\theta)}{2}\right) d\theta$$

$$= \int_0^{2\pi} (2 + 4\cos\theta + \cos 2\theta + 1)\, d\theta$$

$$= \int_0^{2\pi} (3 + 4\cos\theta + \cos 2\theta)\, d\theta$$

$$= \left[3\theta + 4\sin\theta + \frac{\sin 2\theta}{2}\right|_0^{2\pi}$$

$$= 6\pi - 0 = 6\pi \quad \#$$

(2M)

Q2(a) A2(a) → Identify & sketch the conic $r = \frac{25}{(10 - 5\cos\theta)}$ with justification. Also label all the vertices and foci with appropriate polar coordinates. [9]

Soln → given, $r = \frac{25}{(10 - 5\cos\theta)} = \frac{\left(\frac{25}{10}\right)}{\left(1 - \frac{5}{10}\cos\theta\right)} = \frac{\left(\frac{5}{2}\right)}{\left(1 - \frac{1}{2}\cos\theta\right)}$

we know, the standard form of conic with one focus at origin is,

$$r = \frac{ke}{(1 - e\cos\theta)}$$

so, on comparing, we get, $e = \frac{1}{2} < 1$ & $k = 5$

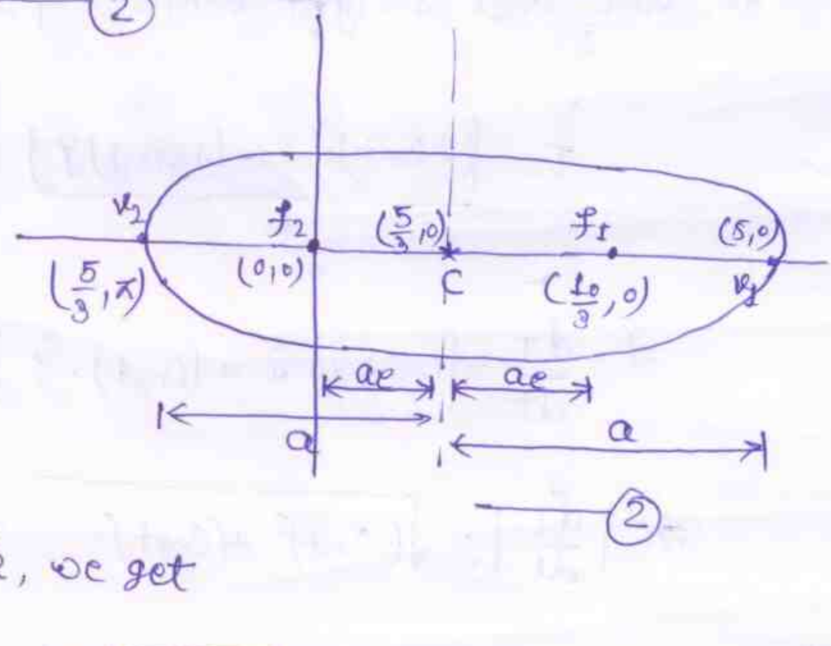
Here the value of e is strictly less than 1, so the given conic is ellipse. —— ②

Now, we know for ellipse,

$k = \left(\frac{a}{e} - ea\right)$

$\Rightarrow ke = (a - e^2 a)$

$\Rightarrow ke = a \cdot (1 - e^2)$



—— ②

On putting values of e & k, we get

$a \cdot (1 - \frac{1}{4}) = \frac{5}{2} \Rightarrow \boxed{a = \frac{10}{3}}$ & Hence, $\boxed{ae = \frac{5}{3}}$ —— ②

so, Coordinates of the Centre are, $(ae, 0) \equiv (\frac{5}{3}, 0)$

& Co-ordinates of vertices v_1 & v_2 are, $(5,0)$ & $(\frac{5}{3}, \pi)$ respectively. —— ②

& Co-ordinates of f_1 & f_2 are, $(\frac{10}{3}, 0)$ & $(0,0)$ respectively.

A2 (b) Find the unit tangent vector, principal unit normal and curvature for the space curve $\vec{r}(t) = [(\cos t + t\sin t)\hat{i} + (\sin t - t\cos t)\hat{j} + 3\hat{k}]$. [6]

Soln → given, $r(t) = [(t\sin t + \cos t)\hat{i} + (\sin t - t\cos t)\hat{j} + 3\hat{k}]$

$$\Rightarrow v = \frac{dr}{dt} = [(t\cdot\cos t + \cancel{\sin t} - \cancel{\sin t})\hat{i} + (\cancel{\cos t} + t\sin t - \cancel{\cos t})\hat{j}]$$

$$\Rightarrow v = [(t\cos t)\hat{i} + (t\sin t)\hat{j}]$$

$$\Rightarrow |v| = \sqrt{(t\cos t)^2 + (t\sin t)^2} = \sqrt{t^2} = t$$

so, the unit tangent vector, $T = \frac{v}{|v|}$, gives

$$T = \frac{[(t\cos t)\hat{i} + (t\sin t)\hat{j}]}{t} = (\cos t\,\hat{i} + \sin t\,\hat{j}) \quad \text{—— (2)}$$

$$\Rightarrow \frac{dT}{dt} = [(-\sin t)\hat{i} + (\cos t)\cdot\hat{j}]$$

$$\Rightarrow \left|\frac{dT}{dt}\right| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1$$

Here, the principal unit normal, $N = \frac{\left(\frac{dT}{dt}\right)}{\left|\frac{dT}{dt}\right|}$

$$\Rightarrow N = \frac{[(-\sin t)\hat{i} + (\cos t)\hat{j}]}{1} = [(-\sin t)\hat{i} + (\cos t)\cdot\hat{j}] \quad \text{—— (2)}$$

& the curvature $\kappa = \frac{1}{|v|}\cdot\left|\frac{dT}{dt}\right| = \frac{1}{t}\cdot 1 = \frac{1}{t}$ —— (2)

$\underline{\text{or}}$ we have, $r = [(t\cos t)\hat{i} + (t\sin t)\hat{j}]$

$$\Rightarrow a = [(-t\sin t + \cos t)\hat{i} + (t\cos t + \sin t)\hat{j}]$$

$$\Rightarrow (r \times a) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t\cos t & t\sin t & 0 \\ (-t\sin t + \cos t) & (t\cos t + \sin t) & 0 \end{vmatrix}$$

$$= \hat{k}\cdot[t^2\cos^2 t + \cancel{t\cos t\sin t} + t^2\sin^2 t - \cancel{t\sin t\cos t}]$$

$$= \hat{k}\cdot[t^2\cdot 1] = t^2\cdot\hat{k}$$

$$\Rightarrow |r \times a| = t^2$$

& Hence, $k = \dfrac{|r \times a|}{|r|^3} = \dfrac{\cancel{t^2}}{\cancel{t^2}t} = \dfrac{1}{t}$

Sol: (A3)

$u = i + j$

$\frac{u}{\|u\|} = \frac{i+j}{\sqrt{2}}$

$$\left(\frac{df}{ds}\right)_{u,P_0} = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$$

$$= \lim_{s \to 0} \frac{f(1 + s/\sqrt{2}, 2 + s/\sqrt{2}) - f(1,2)}{s}$$

$$= \lim_{s \to 0} \frac{\left(1 + \frac{2s}{\sqrt{2}} + \frac{s^2}{2}\right) + \left(2 + \frac{3s}{\sqrt{2}} + \frac{s^2}{2}\right) - 3}{s}$$

$$= \lim_{s \to 0} \frac{\frac{5s}{\sqrt{2}} + s^2}{s}$$

$$= \lim_{s \to 0} \left(\frac{5}{\sqrt{2}} + s\right) = \frac{5}{\sqrt{2}}$$

(6 m)

The function increases most rapidly in the direction of

$$\nabla f|_{(1,2)} = \nabla(x^2 + xy)|_{(1,2)}$$

$$= 4i + j$$

The function decreases most rapidly in the direction of $-\nabla f|_{(1,2)} = -4i - j$

(3 m)

and the rate of change in the direction is

$-|\nabla f| = -\sqrt{17}$ — (1 m)

linearization

$$f(x,y) = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2)$$

$$= 3 + 4(x-1) + 1(y-1)$$

$f(x,y) = 4x + y - 3$

$4x + y - z - 3 = 0$ in the plane

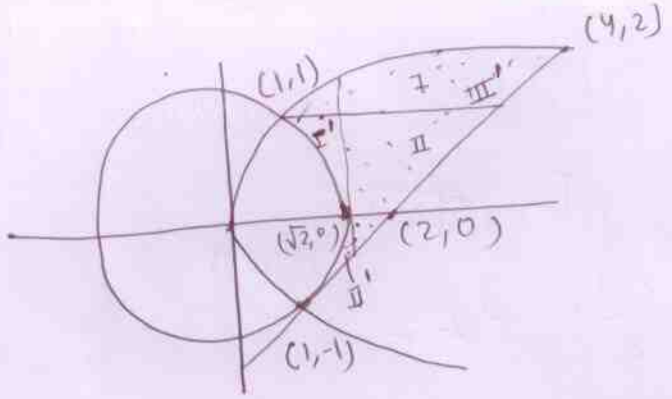
(3 m)

$$\hat{n} = \frac{4i + j - k}{\sqrt{16+1+1}}$$

$$\hat{n} = \frac{1}{\sqrt{18}}(4i + j - k)$$

(2 m)

Qus.4.



—— (2)

Area

$$= \int_1^2 \int_{y^2}^{2+y} dx\,dy + \int_{-1}^{1} \int_{\sqrt{2-y^2}}^{2+y} dx\,dy \quad \text{—— (4)}$$

$$= \int_1^2 [2+y-y^2]\,dy + \int_{-1}^{1} [2+y-\sqrt{2-y^2}]\,dy$$

$$= \left[2y + \frac{y^2}{2} - \frac{y^3}{3}\right]_1^2 + \left[2y + \frac{y^2}{2} - \frac{y\sqrt{2-y^2}}{2} - \sin^{-1}\frac{y}{\sqrt{2}}\right]_{-1}^{1}$$

$$= \left[2y + \frac{y^2}{2} - \frac{y^3}{3}\right]_1^2 + \left[2y + \frac{y^2}{2} - \frac{y\sqrt{2-y^2}}{2} - \sin^{-1}\frac{y}{\sqrt{2}}\right]_{-1}^{1}$$

$$= \left[\left(\frac{10}{3} - \frac{13}{6}\right)\right] + \left[2 - \frac{\pi}{4} + 1 - \frac{\pi}{4}\right]$$

$$= \frac{7}{6} + 3 - \pi/2$$

$$= \left(\frac{25}{6} - \frac{\pi}{2}\right)$$

$$= \left(\frac{25 - 3\pi}{6}\right) \quad \text{—— (3)}$$

$dy\,dx$

$$\int_{1}^{\sqrt{2}} \int_{\sqrt{2-x^2}}^{\sqrt{x}} dy\,dx + \int_{1}^{\sqrt{2}} \int_{x-2}^{\sqrt{2-x^2}} dy\,dx$$

$$+ \int_{\sqrt{2}}^{4} \int_{x-2}^{\sqrt{x}} dy\,dx \quad \text{——} \ (6)$$

OR, Solve it by limits

$$\int_{1}^{4} \int_{x-2}^{\sqrt{x}} dy\,dx - 2\int_{1}^{\sqrt{2}} \sqrt{2-x^2}\,dx \quad \text{——} \ (6)$$

and

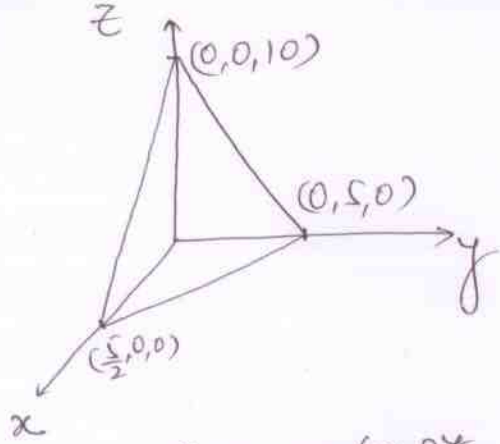
$$\int_{-1}^{2} \int_{y^2}^{y+2} dx\,dy - 2.\int_{0}^{1} \int_{y^2}^{\sqrt{2-y^2}} dx\,dy \quad \text{——} \ (4)$$

5(a):- Sketch the region bounded by the planes

$$4x+2y+z=10,\ y=0,\ z=0 \text{ and } x=0$$

and find its volume by using triple integral.

Sol:-



→ 1.5 M

$$\text{Volume} = \int_0^{\frac{5}{2}}\int_0^{5-2x}\int_0^{10-4x-2y} dz\,dy\,dx \longrightarrow 2.5\text{ M}$$

$$= \int_0^{\frac{5}{2}}\int_0^{5-2x} (10-4x-2y)\,dy\,dx$$

$$= \int_0^{\frac{5}{2}} 10y-4xy-y^2 \Big|_0^{5-2x} dx$$

$$= \int_0^{\frac{5}{2}} \left[10(5-2x)-4x(5-2x)-(5-2x)^2\right] dx$$

$$= \int_0^{\frac{5}{2}} (4x^2-20x+25)\,dx$$

$$= \frac{4x^3}{3} - \cancel{20}^{10}\frac{x^2}{\cancel{2}} + 25x \Big|_0^{\frac{5}{2}} = \frac{\cancel{4}}{3}\times\frac{125}{\cancel{8}_2} - \cancel{10}^{5}\times\frac{25}{\cancel{4}_2} + 25\times\frac{5}{2}$$

$$= \frac{125}{6} - \frac{125}{2} + \frac{125}{2} = \frac{125}{6}$$

→ 3 M

Ans.

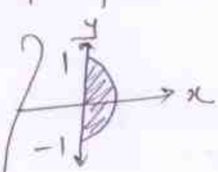
5(b):- Convert the following integral

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz\,dz\,dx\,dy$$

to an equivalent integral in cylindrical coordinates and evaluate it.

8

Soln: $-1 \le y \le 1$, $0 \le x \le \sqrt{1-y^2}$, $x^2+y^2 \le z \le \sqrt{x^2+y^2}$



For cylindrical coordinates,

$$x = r\cos\theta,\ y = r\sin\theta,\ z = z$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

2M

Ranges in cylindrical format:-

$$-\pi/2 \le \theta \le \pi/2,\quad 0 \le r \le 1,\quad r^2 \le z \le r$$

$$\therefore I = \int_{-\pi/2}^{\pi/2}\int_{0}^{1}\int_{r^2}^{r} (r\cos\theta\, r\sin\theta\, z)\,dz\, r\,dr\,d\theta$$

3M

$$= \int_{-\pi/2}^{\pi/2}\int_{0}^{1} r^3\sin\theta\cos\theta \left.\frac{z^2}{2}\right|_{r^2}^{r} dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\int_{0}^{1} \frac{r^5-r^7}{2}\,\frac{\sin 2\theta}{2}\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left.\left(\frac{r^6}{12}-\frac{r^8}{16}\right)\right|_{0}^{1} \sin 2\theta\,d\theta = \frac{1}{2}\times\frac{1}{48}\int_{-\pi/2}^{\pi/2}\sin 2\theta\,d\theta$$

$$= \frac{1}{96}\left.\frac{-\cos 2\theta}{2}\right|_{-\pi/2}^{\pi/2} = \frac{-1}{192}\left[\cos(\pi)-\cos(-\pi)\right]$$

$$= 0$$ Ans.

3M

B1.

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0 \qquad \text{(1M)}$$

$$\begin{aligned} f_x(0,k) &= \lim_{h\to 0} \frac{f(h,k)-f(0,k)}{h} = \lim_{h\to 0} \frac{h^2\tan^{-1}\left(\frac{k}{h}\right)+k^2\tan^{-1}\left(\frac{h}{k}\right)-0}{h} \\ &= \lim_{h\to 0} h\tan^{-1}\left(\frac{k}{h}\right) + \lim_{h\to 0} k\frac{\tan^{-1}\left(\frac{h}{k}\right)}{h/k} \\ &= k \qquad \text{(4M)} \end{aligned}$$

$$f_{xy}(0,0) = \lim_{k\to 0} \frac{f_x(0,k)-f_x(0,0)}{k} = \lim_{k\to 0} \frac{k-0}{k} = 1. \qquad \text{(1M)}$$

We claim that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$. Changing into polar form:

$$f(r,\theta) = r^2\theta\cos^2\theta + r^2\sin^2\theta\tan^{-1}\cot\theta = r^2\theta\cos^2\theta + r^2\sin^2\theta(\pi/2-\theta) = r^2\theta\cos 2\theta + \frac{\pi}{2}r^2\sin^2\theta.$$

Thus

$$|f(r,\theta)-0| \le r^2\theta + \frac{\pi}{2}r^2 \to 0 \text{ as } r\to 0. \qquad \text{(3M)}$$

As $f(0,0)=0$ therefore f is continuous at $(0,0)$. (1M)

Now for $(x,y)\neq(0,0)$, we have

$$f_y = \frac{x^3}{x^2+y^2} - \frac{xy^2}{x^2+y^2} + 2y\tan^{-1}(x/y).$$

Thus,

$$-|x|-|x|-2|y|\pi/2 \le f_y \le |x|+|x|+2|y|\pi/2$$

or

$$-2|x|-2|y|\pi/2 \le f_y \le 2|x|+2|y|\pi/2. \qquad \text{(3M)}$$

As $(x,y)\to(0,0)$ it gives $0 \le \lim_{(x,y)\to(0,0)} f_y(x,y) \le 0$. Hence using Sandwich theorem $\lim_{(x,y)\to(0,0)} f_y(x,y) = 0$. (1M)

Now using symmetry we have $f_y(0,0)=0$ therefore f_y is continuous at $(0,0)$. (1M)

~~Ashok~~

Q B2(a)

An $A+B+C=\pi$ (for a plane triangle)

So, $C=\pi-(A+B)$

$$f(A,B) = \cos A \cos B \cos(\pi-(A+B))$$
$$= -\cos A \cos B \cos(A+B)$$

$$\frac{\partial f}{\partial A} = -\cos B[-\sin A\cos(A+B) - \cos A\sin(A+B)]$$
$$= \cos B \sin(2A+B)$$

$$\frac{\partial f}{\partial B} = \cos A \sin(A+2B) \longrightarrow (2M)$$

$$\frac{\partial^2 f}{\partial A^2} = 2\cos B\cos(2A+B), \quad \frac{\partial^2 f}{\partial B^2} = 2\cos A\cos(A+2B)$$

$$\frac{\partial^2 f}{\partial A \partial B} = \cos(2A+2B) \longrightarrow (2M)$$

For critical points, $\frac{\partial f}{\partial A} = 0 \Rightarrow \cos B\sin(2A+B) = 0$ — ①

$\frac{\partial f}{\partial B} = 0 \Rightarrow \cos A \sin(A+2B) = 0$ — ②

If $\cos B = 0 \Rightarrow B = \pi/2$ then from ② $\cos A\sin(A+\pi) = 0$

$\Rightarrow -\cos A\sin A = 0$

$\Rightarrow \sin 2A = 0 \Rightarrow 2A = 0$ or $\pi \Rightarrow A = 0$ or $\frac{\pi}{2}$

which is not possible, so $\cos B \neq 0$

similarly, $\cos A \neq 0$ so, $\sin(2A+B) = 0 \Rightarrow 2A+B = \pi$

& $\sin(A+2B) = 0 \Rightarrow A+2B = \pi$

$\Rightarrow A = B = \pi/3$ ⟵ (3M)

and $(\pi/3, \pi/3)$ $\frac{\partial^2 f}{\partial A^2} = 2\cos\frac{\pi}{3}\cdot\cos\pi = -1$, $\frac{\partial^2 f}{\partial B^2} = -1$

and $\frac{\partial^2 f}{\partial A\partial B} = -\frac{1}{2}$ and $\frac{\partial^2 f}{\partial A^2}\cdot\frac{\partial^2 f}{\partial B^2} - \left(\frac{\partial^2 f}{\partial A \partial B}\right)^2 = 1 - \frac{1}{4} = \frac{3}{4} > 0$

and $\frac{\partial^2 f}{\partial A^2} = -1 < 0$, so at $(\frac{\pi}{3}, \frac{\pi}{3})$ there is local max.

B2(b)

$$\cancel{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4} = \lambda(lx+my+nz)+\mu}$$

$$\frac{2x}{a^4}\hat{i}+\frac{2y}{b^4}\hat{j}+\frac{2z}{c^4}\hat{k} = \lambda(l\hat{i}+m\hat{j}+n\hat{k})+\mu\left(\frac{2x}{a^2}\hat{i}+\frac{2y}{b^2}\hat{j}+\frac{2z}{c^2}\hat{k}\right)$$

— (1M)

$$\frac{2x}{a^4}-\lambda l-\mu\cdot\frac{2x}{a^2} = 0 \quad \text{—(1)}$$

$$\frac{2y}{b^4}-\lambda m-\mu\cdot\frac{2y}{b^2} = 0 \quad \text{—(2)}$$

$$\frac{2z}{c^4}-\lambda n-\frac{\mu\cdot 2z}{c^2} = 0 \quad \text{—(3)}.$$

— (2M)

(1) $\times x$ + (2) $\times y$ + (3) $\times z$ gives

$$2u - \lambda\cdot 0 - \mu\cdot 2 = 0$$

$$\Rightarrow \mu = u$$

$$\frac{2x}{a^4}-\lambda l-\frac{2xu}{a^2} = 0 \quad \text{or} \quad \frac{2x}{a^4}(1-a^2u) = \lambda l$$

$$\text{or} \quad x = \frac{\lambda l a^4}{2(1-a^2u)}, \quad y = \frac{\lambda m b^4}{2(1-b^2u)}$$

$$\& \quad z = \frac{\lambda n c^4}{(1-c^2u)2}$$

— (3M)

Consider,

$$lx+my+nz = 0$$

$$\Rightarrow \frac{\lambda}{2}\left[\frac{l^2a^4}{1-a^2u}+\frac{m^2b^4}{1-b^2u}+\frac{n^2c^4}{1-c^2u}\right] = 0$$

$$\Rightarrow \frac{l^2a^4}{1-a^2u}+\frac{m^2b^4}{1-b^2u}+\frac{n^2c^4}{1-c^2u} = 0$$

— (2M)

B3 (a) $M = \ln(1+y^2+z^2)$, $N = \dfrac{(b-a^2)xy}{1+y^2+z^2}$,

$$P = \frac{axz}{1+y^2+z^2}$$

Since $\mathbb{R}^3$ is simply connected and M, N, P have continuous 1st partials on $\mathbb{R}^3$,

$\vec{F}$ is conservative if and only if (i), (ii), (iii) are true where

(i) $\dfrac{\partial M}{\partial z} = \dfrac{\partial P}{\partial x}$ i.e. $\dfrac{2z}{1+y^2+z^2} = \dfrac{az}{1+y^2+z^2}$

Thus ~~$b = a^2$~~ $a = 2$ (1 pt)

(ii) $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ ie $\dfrac{2y}{1+y^2+z^2} = \dfrac{(b-a^2)y}{1+y^2+z^2}$

Thus $b - a^2 = 2$. (1 pt)

$\therefore (a,b) = (2,6)$

(iii) $\dfrac{\partial P}{\partial y} = \dfrac{\partial N}{\partial z}$; i.e. $\dfrac{4xyz}{(1+y^2+z^2)^2} = \dfrac{4xyz}{(1+y^2+z^2)^2}$ (1 pt)

is automatically satisfied for $(a,b) = (2,6)$

$\therefore \vec{F}$ is conservative if and only if $(a,b) = (2,6)$ (1 pt)

Let $f(x,y,z)$ be a potential function for $\vec{F}$

$\therefore \dfrac{\partial f}{\partial x} = M = \ln(1+y^2+z^2)$

Integrate wrt x, $f(x,y,z) = x\ln(1+y^2+z^2) + g(y,z)$ (1 pt)

$\dfrac{\partial f}{\partial y} = N \Rightarrow \dfrac{2xy}{1+y^2+z^2} = \dfrac{2xy}{1+y^2+z^2} + g_y$ (1 pt)

$\therefore g_y = 0 \quad \therefore g(y,z) = h(z)$

$\therefore f(x,y,z) = x\ln(1+y^2+z^2) + h(z)$

$f_z = P \Rightarrow \dfrac{2xz}{1+y^2+z^2} = \dfrac{2xz}{1+y^2+z^2} + h'(z)$ (1 pt)

B3 (b) As M, N, P have continuous 1st partials on and inside C, by Green's Theorem,

$\oint M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$ where R is inside of C — (1 pt)

$M = \sqrt{1+x^2} - y e^{xy} + 3y$

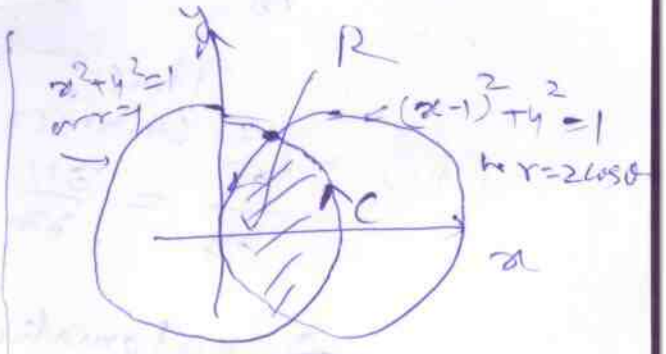
$\therefore \frac{\partial M}{\partial y} = -e^{xy} - xy e^{xy} + 3$ — (1 pt)

$N = x^2 - x e^{xy} + \ln(1+y^4)$

$\therefore \frac{\partial N}{\partial x} = 2x - xy e^{xy} - e^{xy}$ — (1 pt)

$\therefore$ counterclockwise circulation of $\vec{F}$ along C

$= \iint_R (2x-3)\,dA$ (1 pt)



(1 pt)

points of intersection are $\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$, corresponding values of $\theta = \pm\pi/3$ (1 pt)

$$\therefore \oint_C F\cdot dr = 2\left[\int_0^{\pi/3}\int_0^1 (2r\cos\theta - 3)\,r\,dr\,d\theta + \int_{\pi/3}^{\pi/2}\int_0^{2\cos\theta} (2r\cos\theta - 3)\,r\,dr\,d\theta\right]$$

(as region R and function $(2x-3)$ is sym wrt x-axis)

(1 pt)

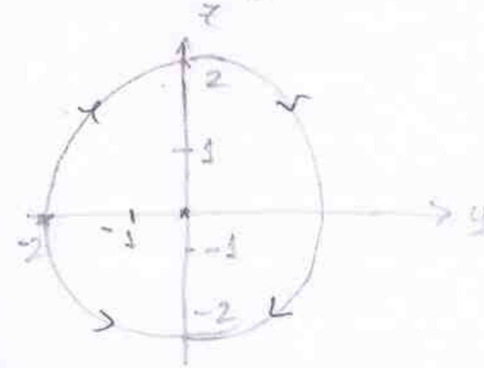
Q. B4(a)

Ans. From Stokes' theorem $\iint_S \vec{\nabla}\times\vec{F}\cdot n\,d\sigma = \int_C \vec{F}\cdot d\vec{r}$

C is the curve of intersection

$-2 = 6 - 4y^2 - 4z^2$

$\Rightarrow \quad y^2 + z^2 = 2$ — [1]



C is Circle in yz-plane with radius $\sqrt{2}$; circle is at $x = -2$.

Sketch of C if we are in front of the paraboloid and look directly along the x-axis.

C can be parametrized as

$\vec{r}(t) = -2\hat{i} + \sqrt{2}\sin t\,\hat{j} + \sqrt{2}\cos t\,\hat{k}$ [1]

$0 \le t \le 2\pi$

$\vec{F}(\vec{r}(t)) = (2\cos^2 t - 1)\hat{i} + (\sqrt{2}\cos t + 4\sqrt{2}\sin^3 t)\hat{j} + 6\hat{k}$ — [1]

$\frac{d\vec{r}}{dt} = \sqrt{2}\cos t\,\hat{j} - \sqrt{2}\sin t\,\hat{k}$

$F(\vec{r}(t))\cdot\frac{d\vec{r}}{dt} = 2\cos^2 t + 8\sin^3 t\cos t - 6\sqrt{2}\sin t$

$\iint_S (\vec{\nabla}\times\vec{F})\cdot n\,d\sigma = \int_C \vec{F}\cdot d\vec{r}$

$= \int_0^{2\pi}\Big[(1-\sin 2t) + 8\cos t\sin^3 t - 6\sqrt{2}\sin t\Big]dt$ — [2]

$= t + \frac{\cos 2t}{2} + 2\sin^4 t + 6\sqrt{2}\cos t\Big|_0^{2\pi}$

$= 2\pi + \frac{1}{2}(\cos 2\pi - \cos 0) + 2(\sin^4 2\pi - \sin^4 0) + 6\sqrt{2}(\cos 2\pi - \cos 0)$

$= 2\pi + \frac{1}{2}(1-1) + 2\times 0 + 6\sqrt{2}(1-1) = 2\pi$ [3]

Q. B4(b)

Ans.: $\bar{\nabla}\cdot\bar{F} = \frac{\partial}{\partial x}(yx^2) + \frac{\partial}{\partial y}(xy^2 - 3z^4) + \frac{\partial}{\partial z}(x^3+y^2)$

$= 2xy + 2yx = 4xy$ [1]

From divergence theorem: $\iint_S \vec{F}\cdot n\, d\sigma = \iiint_D \bar{\nabla}\cdot\bar{F}\, dV$

$= \iiint_D 4xy\, dV$

Considering spherical polar coordinates:

the sphere: $y \leq 0 \Rightarrow$ portion of the circle of radius 4 that's below the x-axes

$\Rightarrow \pi \leq \theta \leq 2\pi$

$z \leq 0 \rightarrow$ portion of the sphere that is below the xy-plane — [2]

$\Rightarrow \pi/2 \leq \phi \leq \pi$

$0 \leq \rho \leq 4$

$$\iiint_D 4xy\, dV = \int_{\theta=\pi}^{2\pi}\int_{\phi=\pi/2}^{\pi}\int_0^4 4(\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)(\rho^2\sin\phi)\, d\rho\, d\phi\, d\theta$$

$$= \int_{\pi}^{2\pi}\int_{\pi/2}^{\pi}\int_0^4 4\rho^4\sin^3\phi\cos\theta\sin\theta\, d\rho\, d\phi\, d\theta$$ — [1]

$$= \int_{\pi}^{2\pi}\int_{\pi/2}^{\pi} \frac{4}{5}\rho^5 \sin^3\phi \cos\theta \sin\theta \Big|_0^4 \, d\phi \, d\theta$$

$$= \int_{\pi}^{2\pi}\int_{\pi/2}^{\pi} \frac{4096}{5} \sin\phi \left(1-\cos^2\phi\right) \sin\theta \cos\theta \, d\phi \, d\theta$$

$$= \int_{\pi}^{2\pi} -\frac{4096}{5} \left(\cos\phi - \frac{1}{3}\cos^3\phi\right) \cos\theta \sin\theta \Big|_{\pi/2}^{\pi} d\theta$$

$$= \int_{\pi}^{2\pi} -\frac{2}{3} \times \left(-\frac{4096}{5}\right) \cos\theta \sin\theta \, d\theta$$

$$= \int_{\pi}^{2\pi} \frac{4096}{15} \sin 2\theta \, d\theta = \frac{2048}{15} \cos 2\theta \Big|_{\pi}^{2\pi} = 0$$

[3]