Chapter 2: K&K

<u>Chapter 2</u>: Problems related to Newton's equations in polar coordinates are 2.29 to 2.35 excluding 2.31

Lecture: example 2.5, 2.6 & problems 2.33,2. 34

Tutorial: 2.35, 2 problems from old question papers (will be given later).

Suggested problems: 29, 30, 32

$$2.11. Let, T_1 = T_{up}; T_2 = T_{low}$$

$$With\ reference\ to\ the\ \ figure: T_1 \sin 45 + T_2 \sin 45 = m\omega^2 r$$

$$\Rightarrow T_1 + T_2 = \sqrt{2} m\omega^2 r ----(1)$$

$$With\ reference\ to\ the\ \ figure: T_1 \cos 45 = T_2 \cos 45 = mg$$

$$\Rightarrow T_1 - T_2 = \sqrt{2} mg -----(2)$$

$$Solving: T_1 = \frac{m\left(\omega^2 r + g\right)}{\sqrt{2}}\ and\ T_2 = \frac{m\left(\omega^2 r - g\right)}{\sqrt{2}}$$

$$where, r = l \sin 45^0 = \frac{l}{\sqrt{2}}; When, l\omega^2 = \sqrt{2} g, T_1 = \frac{m\left(\frac{\omega^2 l}{\sqrt{2}} + g\right)}{\sqrt{2}} = \sqrt{2} mg$$

2.35.

(a) *Equation of motion along radial and tangential direction:*

$$F_r = m\ddot{r} - mr\dot{\theta}^2 = -N \Rightarrow N = mr\dot{\theta}^2 = ml\omega^2 - - - - (1)$$

$$F_\theta = -f \Rightarrow mr\ddot{\theta} + 2m\dot{r}\dot{\theta} = -f = -\mu N = -\mu ml\omega^2$$

$$or, ml\ddot{\theta} = -\mu ml\omega^2 \Rightarrow \dot{\omega} = -\mu\omega^2 - - - - - (2)$$

$$\therefore \int_{\omega_0}^{\omega(t)} \frac{d\omega}{\omega^2} = -\mu \int_0^t dt \Rightarrow \frac{1}{\omega_0} - \frac{1}{\omega} = -\mu t \Rightarrow \frac{l}{\upsilon_0} - \frac{l}{\upsilon} = -\mu t$$

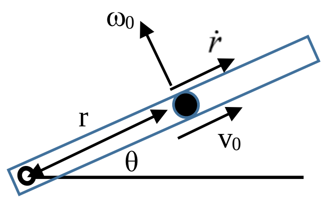
$$\therefore \boxed{\upsilon = \frac{\upsilon_0}{\left[1 + \frac{\mu \upsilon_0 t}{l}\right]}}$$

$$(b)\, \upsilon = \omega l = l\frac{d\theta}{dt} = \frac{\upsilon_0}{\left[1 + \frac{\mu \upsilon_0 t}{l}\right]}$$

$$\therefore \theta = \int_0^t \frac{\upsilon_0 dt}{[l + \mu\upsilon_0 t]} = \frac{1}{\mu}\ln(l + \mu\upsilon_0 t)\Big|_0^t = \frac{1}{\mu}\ln\left(\frac{l + \mu\upsilon_0 t}{l}\right)$$

$$\Rightarrow \boxed{\theta(t) = \frac{1}{\mu}\ln\left(\frac{l + \mu\upsilon_0 t}{l}\right)}$$

1. A bead slides on a long bar with constant speed v_0 relative to the bar and the bar is rotates about the axis with a constant angular speed ω_0. At time t=0, r=r_0 and $\omega=\omega_0$. Find (a) the velocity and acceleration of the bead at any instant of time, (b) the locus of the in terms of r, θ, v_0 and ω_0. [***The problem is taken from one of my old book***]



$$Here,\ \dot{r} = v_0;\ \dot{\theta} = \omega_0 \Rightarrow \boxed{\vec{v} = \hat{r}v_r + \hat{\theta}v_\theta = \hat{r}v_0 + \hat{\theta}r\omega_0}$$

$$Again,\ a_r = \left(\ddot{r} - r\dot{\theta}^2\right)\ and\ a_\theta = \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) \Rightarrow \boxed{\vec{a} = a_r\hat{r} + a_\theta\hat{\theta} = -r\omega_0^2\hat{r} + 2v_0\omega_0\hat{\theta}}$$

$$Now, \dot{r} = \frac{dr}{dt} = v_0 \Rightarrow r = r_0 + v_0 t\ and\ \dot{\theta} = \frac{d\theta}{dt} = \omega_0 \Rightarrow \theta = \theta_0 + \omega_0 t \therefore \frac{r - r_0}{\theta - \theta_0} = \frac{v_0}{\omega_0} \Rightarrow \boxed{r - r_0 = \frac{v_0}{\omega_0}(\theta - \theta_0)}$$

2. The problem is based on equal acceleration components in 2-D: Not included in list of problems

An object moves in a circular path of radius R. At time t=0, it has speed v_0. From this point on, the magnitude of the radial and tangential accelerations is arranged to be equal at all times. (a) As functions of time, find the speed and the distance travelled by the object. (b) If the tangential acceleration is positive (i.e. if the object is speeding up), there is a special value for t. What is the value and why it is special? [***Problem is taken from old book***]

$$(a)\, a_r = \frac{v^2}{R} \text{ and } a_t = \frac{dv}{dt};\text{Following the circular path}: a_r = \left|\frac{dv}{dt}\right| = \frac{v^2}{R}.$$

$$\text{Now, assuming } \frac{dv}{dt} = \text{positive (object is speeding up) and integrating } a_r: \int_{v_0}^{v} \frac{dv}{v^2} = \int_0^t \frac{dt}{R} \Rightarrow v(t) = 1 \Big/ \left(\frac{1}{v_0} - \frac{t}{R}\right)$$

$$\text{If } \frac{dv}{dt} = \text{negative (object is slowing down)},\ v(t) = 1 \Big/ \left(\frac{1}{v_0} + \frac{t}{R}\right)$$

$$\text{Using positive case}: s(t) = \text{distance travelled} = \int_0^t v(t)\,dt = \int_0^t 1 \Big/ \left(\frac{1}{v_0} - \frac{t}{R}\right) dt = -R\ln\left(1 - \frac{v_0 t}{R}\right)$$

$$\text{If } \frac{dv}{dt} \text{ is negative}, s(t) = R\ln\left(1 + \frac{v_0 t}{R}\right); \text{For, } \frac{dv}{dt} \text{ is positive, then at, } t = \frac{R}{v_0}, \text{ both } v \text{ and } s \text{ go to}$$

$$\text{infinity, so after this time, the stated motion is impossible. In case where } \frac{dv}{dt} \text{ is negative,}$$

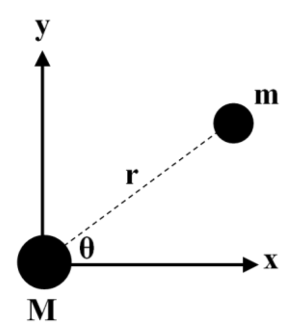
$$v \text{ goes to zero as } t \to \infty, \text{ but it goes to zero slowly enough so that the } s \text{ diverges as } t \to \infty.$$

3. Consider a mass m moving under the gravitational attraction of another mass M, fixed at the origin. The motion of m is confined to the xy plane, as shown in figure (3a).

i) Write down the equations of motion of m in polar coordinates

ii) Using the -equation, show that the quantity remains constant

iii) What is the physical meaning of the conserved quantity in part (ii)?



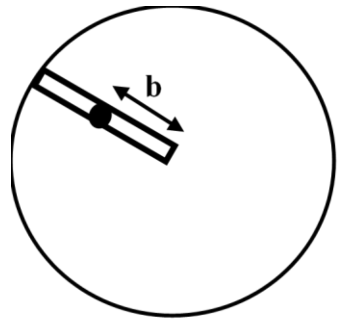
$$(a)\, m\left(\ddot{r}-r\dot{\theta}^2\right)=-\frac{GMm}{r^2}----(1) \text{ and } m\left(r\ddot{\theta}+2\dot{r}\dot{\theta}\right)=0----(2)$$

$$(b) \text{ From eq.}(2)\ m\left(r\ddot{\theta}+2\dot{r}\dot{\theta}\right)=0 \Rightarrow m\left(r^2\ddot{\theta}+2r\dot{r}\dot{\theta}\right)=0$$

$$\Rightarrow \frac{d}{dt}\left(mr^2\dot{\theta}\right)=0 \Rightarrow mr^2\dot{\theta}=const. \Rightarrow \text{Orbital angular momentum.}$$

$$(c)\frac{d\vec{L}}{dt}=const \text{ when there is no external torque / force acting.}$$

4. A circular table (of radius R) with a smooth horizontal surface is rotating at a constant angular velocity ω about its axis as shown in the figure. A groove is made on the surface along a radius and a particle is gently placed inside the groove at a distance b from the center. Find the velocity and the force on the particle (as vectors in polar coordinates), when its distance from the center becomes L (b<L<R).



$$\text{Here, } F_r=0 \text{ on the particle.}$$

$$\therefore m\left(\ddot{r}-r\dot{\theta}^2\right)\Rightarrow m\left(\ddot{r}-r\omega^2\right)=0\Rightarrow \frac{d^2r}{dt^2}=\frac{dv_r}{dt}=r\omega^2\Rightarrow \frac{dv_r}{dr}\frac{dr}{dt}\Rightarrow v_r\frac{dv_r}{dr}=r\omega^2$$

$$\therefore \int_0^{V_L} v_r dv_r=\int_b^L r\omega^2 dr\Rightarrow \frac{V_L^2}{2}=\frac{1}{2}\omega^2\left(L^2-b^2\right)\Rightarrow V_L=\omega\sqrt{L^2-b^2}$$

$$V_\theta=r\omega \text{ and } r=L, V_L=L\omega$$

$$\vec{V}_{Total}=\hat{r}\omega\sqrt{L^2-b^2}+\hat{\theta}L\omega \therefore |V|=\sqrt{\omega^2\left(L^2-b^2\right)+L^2\omega^2}=\omega\sqrt{2L^2-b^2}$$

$$\vec{F}=\hat{\theta}F_\theta=m\left(r\ddot{\theta}+2r\dot{\theta}^2\right)\hat{\theta}=m\left(r\dot{\omega}+2r\omega^2\right)\hat{\theta}=2mr\omega^2\hat{\theta}=2m\omega^2\sqrt{L^2-b^2}\hat{\theta}$$

Derivation of 2 – D velocity and acceleration in polar form

$x = r\cos\theta$ *and* $y = r\sin\theta$; $\hat{i} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$ *and* $\hat{j} = \hat{r}\sin\theta + \hat{\theta}\cos\theta$

$\therefore \dot{x} = \dot{r}\cos\theta - r\sin\theta\dot{\theta}$ *and* $\dot{y} = \dot{r}\sin\theta + r\cos\theta\dot{\theta}$

$\Rightarrow \ddot{x} = \ddot{r}\cos\theta - \dot{r}\dot{\theta}\sin\theta - \dot{r}\dot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta - r\ddot{\theta}\sin\theta = \ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta - r\ddot{\theta}\sin\theta$

and $\ddot{y} = \ddot{r}\sin\theta + \dot{r}\dot{\theta}\cos\theta + \dot{r}\dot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta + r\ddot{\theta}\cos\theta = \ddot{r}\sin\theta + 2\dot{r}\dot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta + r\ddot{\theta}\cos\theta$

$\therefore \vec{\upsilon}_r(r,\theta) = \textit{component of } \dot{x} \textit{ along } \hat{r} + \textit{component of } \dot{y} \textit{ along } \hat{r}$

$= \hat{r}\left(\dot{x}\cos\theta + \dot{y}\sin\theta\right) = \hat{r}\dot{r}$ *and similarly,* $\vec{\upsilon}_\theta(r,\theta) = \hat{\theta}\left(-\dot{x}\sin\theta + \dot{y}\cos\theta\right) = \hat{\theta}r\dot{\theta}$

$\therefore \boxed{\vec{\upsilon}(r,\theta) = \vec{\upsilon}_r(r,\theta) + \vec{\upsilon}_\theta(r,\theta) = \hat{r}\dot{r} + \hat{\theta}r\dot{\theta}}$

Similarly, $\vec{a}_r = \textit{component of } \ddot{x} \textit{ along } \hat{r} + \textit{component of } \ddot{y} \textit{ along } \hat{r}$

$= \hat{r}\left(\ddot{x}\cos\theta + \ddot{y}\sin\theta\right) = \hat{r}\left(\ddot{r} - r\dot{\theta}^2\right)$ *and similarly,* $\vec{a}_\theta = \hat{\theta}\left(-\ddot{x}\sin\theta + \ddot{y}\cos\theta\right) = \hat{\theta}\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)$

$\therefore \boxed{\vec{a}(r,\theta) = \hat{r}\left(\ddot{r} - r\dot{\theta}^2\right) + \hat{\theta}\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)}$