MATHEMATICS-II (MATH F112)

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Section 5.3

The Dimension Theorem

Kernel of a LT

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$$\ker(L) = \{v \in V | L(v) = \mathbf{0}_W\}$$

Example 1

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Result: If $L : V \to W$ is a LT, then $\ker(L)$ is a subspace of V.

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$$range(L) = \{L(v) | v \in V\}$$

Thus a vector $w \in range(L)$ if there exists some vector $v \in V$ such that $L(v) = w$.

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$$\begin{cases} \text{range}(L) = \{L([x, y, z]) | [x, y, z] \in \mathbb{R}^3\} \\ \\ \end{cases}$$

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$$\begin{cases} \text{range}(L) = \{L([x, y, z]) | [x, y, z] \in \mathbb{R}^3\} \\ \qquad\qquad\quad = \{[0, y] | y \in \mathbb{R}\} \\ \qquad\qquad\quad = \{y[0, 1] | y \in \mathbb{R}\} \end{cases}$$

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Also, $\text{range}(L)$ is a subspace of the vector space $\mathbb{R}^2$.

Note that range(L) = span$\{[0, 1]\}$. Since $\{[0, 1]\}$ is LI subset of $\mathbb{R}^2$ (Why?).Thus, the set $B = \{[0, 1]\}$ is a basis of range(L).

Also, range(L) is a subspace of the vector space $\mathbb{R}^2$.

Result: If $L : V \to W$ is a LT, then range(L) is a subspace of W.

$$\begin{cases} \ker(L) = \{[x, 0, z] | x, z \in \mathbb{R}\} \\ \\ \end{cases}$$

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Now, $\{[1,0,0],[0,0,1]\}$ is a LI subset of $\mathbb{R}^3$, hence, they form a basis for $\mathbf{ker}(L)$.

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Basis for range(L) is $\{[1,0],[0,1]\}$.

Exercises

Q:. Suppose $L : P_3 \to P_2$ is a LT given by $L_1(at^3 + bt^2 + ct + d) = 3at^2 + 2bt + c$. Find $\mathbf{ker}(L)$ and range(L).

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Q:. Let $L : \mathbb{R}^4 \to P_2$ be a LT given by $L([a,b,c,d]) = a + (b+c)x + (b-d)x^2$.

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Q:. Let $L : \mathbb{R}^4 \to P_2$ be a LT given by $L([a,b,c,d]) = a+(b+c)x+(b-d)x^2$. Find **ker**$(L)$ and range(L). Also, find basis for **ker**(L) and range(L).

Sol. ker$(L) = \{[0,b,-b,b]|b \in \mathbb{R}\} = \text{span}\{[0,1,-1,1]\}$ and range$(L) = \text{span}\{1, x+x^2, x, x^2\} = \text{span}\{1,x,x^2\}$.

Example 4

Q:. Let $L : \mathbb{R}^3 \to \mathbb{R}^4$ be a LT given by $L([x, y, z]) = [x, y - z, x - y + z, x + y - z]$. Find a basis for $\ker(L)$ and range(L).

Example 4

Q:. Let $L : \mathbb{R}^3 \to \mathbb{R}^4$ be a LT given by $L([x,y,z]) = [x, y-z, x-y+z, x+y-z]$. Find a basis for $\mathbf{ker}(L)$ and range(L).

Sol. $\{[0,1,1]\}$ is a basis for for $\mathbf{ker}(L)$ and $\{[1,0,1,1],[0,1,-1,1]\}$ is a basis for range(L).

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For Example 4, we have $L(X) = AX$ where

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

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$BX = 0$ gives

$x = 0, y = z \implies \textbf{ker}(L) = \{[0, z, z] | z \in \mathbb{R}\} = \text{span}\{[0, 1, 1]\}.$

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For Example 4, $\{[0, 1, 1]\}$ is a basis for for $\mathbf{ker}(L)$.

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Step 2: Column vectors in A corresponding to **pivot columns** of RREF(A) forms a basis for range(L). Note that, Columns I and II have leading entry. Thus, the corresponding column vector of A, i.e., $\{[1,0,1,1],[0,1,-1,1]\}$ is a basis of range (L).

Example 5

Q:. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a LT given by

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 5 & 1 & -1 \\ -3 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

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- Is $[2, -1, 4]^T \in \text{range}(L)$.

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- Is $[1, -2, 3]^T \in \ker(L)$.
- Is $[2, -1, 4]^T \in \text{range}(L)$.

Sol. Yes, $[1, -2, 3]^T \in \ker(L)$ because $L([1, -2, 3]^T) = [0, 0, 0]^T$.

No, $[2,-1,4]^T \notin \mathrm{range}(L)$.

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$$\begin{cases} 5x + y - z = 2 \\ -3x + z = -1 \\ x - y - z = 4 \end{cases}$$

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$$\begin{cases} 5x+y-z=2 \\ -3x+z=-1 \\ x-y-z=4 \end{cases}$$ which has no solution.

The Dimension Theorem

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If $L : V \to W$ is a LT and V is finite dimensional, then range(L) is finite dimensional, and

$$\dim(\ker(L)) + \dim(\text{range}(L)) = \dim(V)$$

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Note: $\dim(\ker(L))$ is called nullity(L) and $\dim(\text{range}(L))$ is called rank(L).

Example 6

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Q:. Consider a LT $L : P_2 \longrightarrow P_3$ given by $L(a+bx+cx^2) = x(a+bx+cx^2)$.

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Sol.

$$\begin{aligned}
\ker(L) &= \{(a+bx+cx^2) \in P_2 | L(a+bx+cx^2) = 0_{P_3}\} \\
&= \{(a+bx+cx^2) \in P_2 | ax+bx^2+cx^3 = 0\} \\
&= \{(a+bx+cx^2) \in P_2 | a=b=c=0\} \\
&= \{0_{P_2}\}
\end{aligned}$$

$$\implies \dim(\ker(L)) = 0.$$

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Also $\mathbf{dim}(\text{range}(L)) = 3$ (why). Since, $\mathbf{dim}(P_2) = 3$, clearly, we have $\mathbf{dim}(\mathbf{ker}(L)) + \mathbf{dim}(\text{range}(L)) = \mathbf{dim}(P_2)$.

Example 7

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Sol. $\ker(L) =$

$$\left\{ \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & -a-e \end{bmatrix} \in M_{33} | a,b,c,d,e,f,g,h \in \mathbb{R} \right\} \implies$$

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$\dim(\ker(L)) = 8$.

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Q:. Consider a LT $L : M_{33} \to \mathbb{R}$ given by $L(A) = \text{trace}(A)$. Find $\mathbf{ker}(L)$, $\mathbf{dim}(\mathbf{ker}(L))$, $\text{range}(L)$ and $\mathbf{dim}(\text{range}(L))$.

Sol. $\ker(L) =$

$$\left\{ \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & -a-e \end{bmatrix} \in M_{33} | a,b,c,d,e,f,g,h \in \mathbb{R} \right\} \implies$$

$\dim(\ker(L)) = 8$. Also, $\text{range}(L) = \mathbb{R}$

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$\dim(\ker(L)) = 8$. Also, $\text{range}(L) = \mathbb{R}$ and $\dim(\text{range}(L)) = 1$. Hence,

$$\dim(\ker(L)) + \dim(\text{range}(L)) = \dim(M_{33})$$

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Sol. $\mathbf{ker}(L)$ is precisely the set of all skew-symmetric 3×3 matrices.

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range(L) is the set of all symmetric 3×3 matrices.

$\mathbf{dim}(\mathbf{ker}(L)) = 3$ and $\mathbf{dim}(\text{range}(L)) = 6$.

True or False

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Q:. If $L : V \to W$ is a LT and $\dim(V) = 5, \dim(W) = 3$, then the Dimension Theorem implies that $\dim(\ker(L)) = 2$.

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Q:. If $L : V \to W$ is a LT, then range(L) is a subspace of V.

Sol. False, range(L) is a subspace of W.

Exercises

Q:. Let W be the vector space of all 2×2 symmetric matrices. Define a LT $L : W \to P_2$ by

$$L\left(\begin{bmatrix} a & b \\ b & c \end{bmatrix}\right) = (a-b)+(b-c)x+(c-a)x^2$$

Find **dim(ker**(L)) and **dim**(range(L)).

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Find **dim(ker(**L**))** and **dim**(range(L)).

Sol. **dim(ker(**L**))** $= 1$ and **dim**(range(L)) $= 2$.

Q:. Let $\{e_1, e_2, e_3, e_4\}$ be standard basis for $\mathbb{R}^4$ and $L : \mathbb{R}^4 \to \mathbb{R}^3$ be a LT given by

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Sol. $\textbf{dim}(\textbf{ker}(L)) = 1$ and $\textbf{dim}(\text{range}(L)) = 3$.

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Sol. $\mathbf{dim}(\mathbf{ker}(L)) = 1$ and $\mathbf{dim}(\text{range}(L)) = 3$.

Q:. Consider $L : P_2 \to P_4$ given by $L(p) = x^2 p$. Find $\mathbf{ker}(L)$, $\mathbf{dim}(\mathbf{ker}(L))$, range$(L)$ and $\mathbf{dim}(\text{range}(L))$.

Sol. $\mathbf{dim}(\mathbf{ker}(L)) = 0$ and $\mathbf{dim}(\text{range}(L)) = 3$.