Birla Institute of Technology and Science, Pilani (Raj.)
Second Semester, 2017-2018
MATH F112 (Mathematics II)
Part-B (Open Book) Comprehensive Examination

Note: (i) Write **Part-B** on top right corner of the answer sheet.
(ii) Answer each sub-part of a question together.

Max. Marks: 69 **Max. Time: 90 min.** **Date: 1 May, 2018 (Tuesday)**

Q.1 Let $B = ([1,1,0],[0,1,1],[1,0,1])$ and $C = ([1,2],[2,1])$ be ordered bases of $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively. Suppose $L: \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation and the matrix of L with respect to bases B and C is

$$A_{BC} = \begin{bmatrix} 0 & 1 & 1/3 \\ 0 & -1 & 1/3 \end{bmatrix}.$$

Find $L([x,y,z])$ for all $[x,y,z] \in \mathbb{R}^3$. Find basis of $ker(L)$ and $range(L)$ and hence compute $dim(ker(L))$ and $dim(range(L))$. **[10+7]**

Q.2 (a) Suppose that $f(z) = u + iv$ is entire such that $u_x + v_y = 0$ for all $x,\ y \in \mathbb{R}$. Show that f has the form $f(z) = az + b$ where a, b are some complex constants with $Re(a) = 0$. **[10]**

(b) Show that $u(x,y) = e^{-2xy}\cos(x^2 - y^2)$ is harmonic in $\mathbb{R}^2$. Find its harmonic conjugate $v(x,y)$ in $\mathbb{R}^2$. Write $f(z) = u + iv$ as a function of z with $f(0) = 1$. **[6+8+2]**

Q.3 (a) Let $f(z) = u + iv$ be an entire function such that $au + bv \geq ln(ab), a > 1, b > 1$. Then evaluate the integral

$$\int_C \frac{f(z)}{(z-1)^{2018}}\, dz$$

where C is an equilateral triangle of side 1 with centroid at $z = 1$. **[13]**

(b) Using Laurent's series expansion of $e^{-\frac{1}{z}}$ evaluate the integral

$$\int_{-1}^{1} \frac{e^{-t}}{\sqrt{1-t^2}} \cos[2\cos^{-1}(t) + \sqrt{1-t^2}]\, dt$$

[13]

**********End of Part-B*********

Open book

Solution-1 Given $A_{BC} = \begin{bmatrix} 0 & 1 & 1/3 \\ 0 & -1 & 1/3 \end{bmatrix}$

implies $L[1,1,0] = 0[1,2] + 0[2,1] = [0,0]$

$L[0,1,1] = 1[1,2] - 1[2,1] = [-1,1]$

$L[1,0,1] = \frac{1}{3}[1,2] + \frac{1}{3}[2,1] = [1,1]$ [3M]

Since B is an ordered basis of $\mathbb{R}^3$, for $[x,y,z] \in \mathbb{R}^3$, we have [1M]

$[x,y,z] = a[1,1,0] + b[0,1,1] + c[1,0,1]$

$\Rightarrow [x,y,z] = [a+c, a+b, b+c]$

$\Rightarrow a+c = x, \quad a+b = y, \quad b+c = z$

On solving we get [3M]

$a = \frac{x+y-z}{2}, \quad b = \frac{y+z-x}{2}, \quad c = \frac{x-y+z}{2}$

Thus, $[x,y,z] = a[1,1,0] + b[0,1,1] + c[1,0,1]$ gives

$L[x,y,z] = \frac{x+y-z}{2} L[1,1,0] + \frac{y+z-x}{2} L[0,1,1] + \frac{x-y+z}{2} L[1,0,1]$

($\because$ L is a linear transformation) — 1M

$= \frac{x+y-z}{2}[0,0] + \frac{y+z-x}{2}[-1,1] + \frac{x-y+z}{2}[1,1]$

[2M]

$L[x,y,z] = [x-y, z] \quad \forall [x,y,z] \in \mathbb{R}^3$

$\ker L = \{[x, y, z] \in \mathbb{R}^3 : L[x, y, z] = 0\}$

$= \{[x, y, z] : [x-y, z] = 0\}$

$= \{[x, x, 0] : x \in \mathbb{R}\}$

$\ker L = \text{span}\{[1, 1, 0]\}$

$\because \{[1, 1, 0]\}$ contains a nonzero vector, the set $\{[1, 1, 0]\}$ is L-I. Thus $\{[1, 1, 0]\}$ is a basis of $\ker L$. Hence, $\dim(\ker L) = 1$ [3M]

$\text{range}(L) = \{L[x, y, z] : [x, y, z] \in \mathbb{R}^3\}$.

$= \{[x-y, z] : x, y, z \in \mathbb{R}\}$

$= \{x[1, 0] + y[-1, 0] + z[0, 1] : x, y, z \in \mathbb{R}\}$

$\text{range}(L) = \text{span}\{[1, 0], [-1, 0], [0, 1]\}$

$= \text{span}\{[1, 0], [0, 1]\}$ $\because [-1, 0] = (-1)[1, 0]$ [2M]

$\because \{[1, 0], [0, 1]\}$ is a L-I subset of $\mathbb{R}^2$. Thus, [1M] $\{[1, 0], [0, 1]\}$ is a basis of $\text{range}(L)$. Hence

$\dim(\text{range}(L)) = 2$. [1M]

<u>Remark</u> No marks is given for $\ker L$, $\text{range}(L)$ and their dimension If $L[x, y, z]$ is incorrect.

Q.2 (O.B.)

Q.2(a) Suppose that $f = u + iv$ is entire such that $u_x + v_y = 0 \ \forall\ x, y$. Show that f has the form $f(z) = az + b$ where a, b are some complex constants with $Re(a) = 0$. — $\boxed{10}$

Soln. $f(z) = u + iv$

$$f'(z) = u_x + iv_x \quad \text{— (i)}$$

As $f(z)$ is analytic

Using C-R Equations

$$f'(z) = v_y - iu_y \quad \text{— (ii)} \qquad (2)$$

From (i) & (ii)

$$2f'(z) = u_x + v_y + i(v_x - u_y)$$

$$f'(z) = \frac{i}{2}(v_x - u_y) \qquad (\because u_x + v_y = 0)$$

$$\Rightarrow Re\, f'(z) = 0. \quad \text{—} \quad (3)$$

$$\Rightarrow |e^{f'(z)}| = 1. \quad \text{—} \quad (2)$$

$$\Rightarrow e^{f'(z)} = e^{it}$$

$f(z) = itz + b$, b is some complex constant.

— (3)

with $Re(a) = 0$. (Alternate Solution)

Alternate Soln Q2(a) $f = u + iv$ — (i).

Given Condition $u_x + v_y = 0$.

From C-R. $u_x = v_y$

$u_y = -v_x$

& $u_x = -v_y = 0$. ——— (2)

$\Rightarrow$ u is function of y only

& v is " " x only.

$\Rightarrow$ $u_x = -v_y = c$, $(c \in \mathbb{R})$

$u_x = c \Rightarrow u = c\cancel{y} + c_1$ $(c_1 \in \mathbb{R})$ ——— (2)

$v_y = -c \Rightarrow v = -c\cancel{x} + c_2$. $(c_2 \in \mathbb{R})$

From (i).

$f = c\cancel{y} + c_1 + i(-c\cancel{x} + c_2)$

$= c(y - ix) + c_1 + ic_2$ where $c \in \mathbb{R}$ —(2)

$= -ci(x + iy) + c_1 + ic_2$

$= -ciz + c_1 + ic_2 = az + b$, $a = -ci$, $b = c_1 + ic_2$ —(1) —(3)

Soln Q.2(b) $u = e^{-2xy}\cos(x^2-y^2)$

$$u_x = -2e^{-2xy}\left[y\cos(x^2-y^2)+x\sin(x^2-y^2)\right]$$

$$u_{xx} = 4ye^{-2xy}\left[y\cos(x^2-y^2)+x\sin(x^2-y^2)\right]$$

$$-2e^{-2xy}\left[-2xy\sin(x^2-y^2)+\sin(x^2-y^2)+2x^2\cos(x^2-y^2)\right]$$

(1)

$$u_y = 2e^{-2xy}\left[y\sin(x^2-y^2)-x\cos(x^2-y^2)\right]$$

$$u_{yy} = -4xe^{-2xy}\left[y\sin(x^2-y^2)-x\cos(x^2-y^2)\right]$$

(1)

$$+2e^{-2xy}\left[\sin(x^2-y^2)-2y^2\cos(x^2-y^2)-2xy\sin(x^2-y^2)\right]$$

$\because$ $u(x,y)$ has continuous partial derivatives of the first & second order and satisfy the partial differential equation — (1)

$$u_{xx}+u_{yy}=0.$$

— (2)

$u_{xy} = -2e^{-2xy}\left[\cos(x^2-y^2)+2y^2\sin(x^2-y^2)+2xy\cos(x^2-y^2)\right]$

$-2e^{-2xy}(-2x)\left[y\cos(x^2-y^2)+x\sin(x^2-y^2)\right]$

— (1)

$\Rightarrow$ u is Harmonic.

To find $v(x,y)$:

As $u(x,y) = e^{-2xy}\cos(x^2-y^2)$

$u_{xy} =$??

As v is Harmonic conjugate of u

$$\because \; v_y = u_x = -2y\, e^{-2xy}\cos(x^2-y^2) - 2x\sin(x^2-y^2)e^{-2xy}$$

$$v = -2\int \left[e^{-2xy}\left[y\cos(x^2-y^2) + x\sin(x^2-y^2)\right]\right] dy$$

$$v = -2x\left\{\left[\sin(x^2-y^2)\cdot\frac{e^{-2xy}}{-2x}\right] - \int (-2y)\cos(x^2-y^2)\frac{e^{-2xy}}{-2x}\,dy\right.$$

$$\left. -2\int e^{-2xy}\, y\cos(x^2-y^2)\,dy \right\}$$

$$v = -2x\left\{\left[\frac{e^{-2xy}\sin(x^2-y^2)}{-2x} - \int \frac{y}{x}\cos(x^2-y^2)\,e^{-2xy}\,dy\right.\right.$$

$$= \cancel{\frac{x}{x}}\, e^{-2xy}\sin(x^2-y^2) + \int 2y\cos(x^2-y^2)\,e^{-2xy}\,dy$$

$$- \int 2y\, e^{-2xy}\cos(x^2-y^2)\,dy$$

$$v = e^{-2xy}\sin(x^2-y^2) + \phi(x) \quad (3)$$

$$\frac{\partial v}{\partial x} = -2y\, e^{-2xy}\sin(x^2-y^2) + 2x\cos(x^2-y^2)$$

$$+\phi'(x).$$

From C-R Equation:

$$-2y\, e^{-2xy} \sin(x^2-y^2) + 2x\, e^{-2xy} \cos(x^2-y^2)$$

$$= -2y\, e^{-2xy} \sin(x^2-y^2) + 2x\, e^{-2xy} \cos(x^2-y^2) + \phi'(x)$$

$$\Rightarrow \phi'(x) = 0.$$

2

$$\phi(x) = c.$$

$$v = e^{-2xy} \sin(x^2 - y^2) + c$$

$$f(z) = e^{-2xy} \cos(x^2-y^2) + i\left[e^{-2xy} \sin(x^2-y^2) + c\right]$$

$$f(0) = 0.$$

3

$$\Rightarrow c = 0.$$

$$f(z) = e^{-2xy} \cos(x^2-y^2) + i\left(e^{-2xy} \sin(x^2-y^2)\right)$$

$$f(z) = e^{-2xy}\left[\cos(x^2-y^2) + i \sin(x^2-y^2)\right]$$

$$= e^{-2xy}\left[e^{i(x^2-y^2)}\right]$$

$$= e^{i(x^2-y^2+2xy)}$$

$$= e^{i(x+iy)^2}$$

$$= e^{iz^2}$$

2

Q.3 (O.B.)

Q. $au + bv \geq \ln(ab)$, $a > 1, b > 1$.

Let $f(z) = u + iv$ be entire function

$af(z) = au + iav$ is " "

$-ibf(z) = -ibu + bv$ " " "

$(a-ib)f(z) = (au+bv) + i(av-bu)$ is " "

$\Rightarrow e^{-(a-ib)f(z)}$ is entire function —— 3

(Composition of entire functions is entire.)

Consider,

$$|e^{-(a-ib)f(z)}| = e^{-(au+bv)}$$
$$\leq e^{-\ln(ab)} = e^{\ln(1/ab)}$$
$$= \frac{1}{ab} < 1 \quad \text{—— 3}$$

(As $au+bv \geq \ln(ab) \Rightarrow -(au+bv) \leq -\ln(ab)$ and e^x is an increasing function.)

So, $e^{-(a-ib)f(z)}$ is entire and bounded so by Liouville's theo.

$$e^{-(a-ib)f(z)} = \text{Constant.}$$

$\Rightarrow |e^{-(a-ib)f(z)}| = \text{Constant} = e^{-(au+bv)}$

$\Rightarrow au+bv = \text{Constant.}$ (by taking log on both sides.) —— 3

$\Rightarrow \text{Re}[(a-ib)f(z)] = \text{Constant.}$

$\Rightarrow \text{Im}[(a-ib)f(z)] = \text{Constant.}$

(Using C-R equations ...

$\Rightarrow \quad (a-ib)f(z) = \text{Const.}$ ——— ②

$\Rightarrow \quad f(z) = \text{constant} = d \text{ (let.)}$

Now,

$$\int_C \frac{f(z)}{(z-1)^{1001}}\,dz = \frac{2\pi i}{(1000)!} \cdot f^{(1000)}(1) = 0$$

(By Cauchy's Integral formula for derivatives as f(z) is analytic on and inside simple closed contour C.)

——— ②

Q1: $e^{-\frac{1}{z}} = 1 - \frac{1}{z} + \frac{1}{\lfloor 3\, z^3} - - - - -$. Therefore $b_n = \frac{(-1)^n}{\lfloor n}$.

Also

$$b_n = \frac{1}{2\pi i}\int_C \frac{e^{-\frac{1}{z}}}{z^{-n+1}}\,dz$$ where c be any closed curve around $z=0$.

Let we take c as $|z|=1$. Then

$$b_n = \frac{1}{2\pi i}\int_{|z|=1} \frac{e^{-\frac{1}{z}}}{z^{-n+1}}\,dz$$

$$= \frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{e^{-e^{-i\theta}}}{e^{-i\theta(n-1)}}\cdot i e^{i\theta}\,d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n\theta+\sin\theta)}\cdot e^{-\cos\theta}\,d\theta \quad \text{———} \quad 2$$

On Comparing both values of b_n:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n\theta+\sin\theta)}\, e^{-\cos\theta}\,d\theta = \frac{(-1)^n}{\lfloor n} \quad \text{———} \quad 2$$

On Comparing ~~real~~ parts from both sides

$$\frac{1}{\pi}\int_{0}^{\pi} \cos(n\theta+\sin\theta)\, e^{-\cos\theta}\,d\theta = \frac{(-1)^n}{\lfloor n} \quad \text{———} \quad 2$$

put $\cos\theta = t \Rightarrow \sin\theta\,d\theta = -dt \Rightarrow d\theta = -\frac{dt}{\sqrt{1-t^2}}$

$$\therefore \frac{1}{\pi}\int_{-1}^{1} \frac{e^{-t}}{\sqrt{1-t^2}} \cos(n\cos^{-1}t + \sqrt{1-t^2})\,dt = \frac{(-1)^n}{\lfloor n} \quad \text{———} \quad 4$$

take $n=2$

$$\int_{-1}^{1} \frac{e^{-t}}{\sqrt{1-t^2}} \cos(2\cos^{-1}t + \sqrt{1-t^2})\,dt = \frac{\pi}{2} \quad \text{———} \quad 3$$