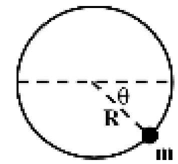
BIRLA INSTITUTE OF TECHNOLOGY AND SCIENCE PILANI, RAJASTHAN

Comprehensive Examination (Closed-Book): 2017-18, 2nd Semester

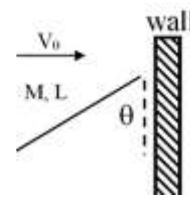
Mechanics Oscillations and Waves (MEOW): PHY F111, 5th May 2018, Duration: 3 hrs., Full Marks:105

Instruction(s): All questions are compulsory. Answer all parts of a question together. Write your final answer of each sub-part inside a box.

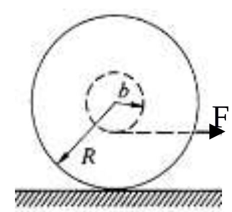
Q.1(A) A bead of mass 'm' is at rest at the top of a fixed frictionless hoop of radius R that lies in a vertical plane. The bead is given infinitesimal push so that it slides down and around the hoop in clockwise direction. Find all points on the hoop (angular position as θ w.r.t. horizontal) where the bead's acceleration is horizontal. **(10)**



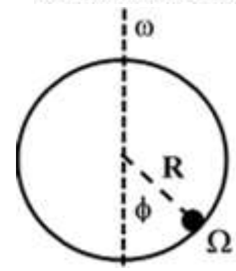
Q.1(B) On a frictionless floor (x-y plane) a uniform stick of mass M and length L moves with a linear speed V_0 (without rotation) in the direction perpendicular to a wall. It makes an angle θ with the wall as shown. If the stick collide elastically with a rigidly fixed wall, what should cosθ be so that the speed of the center of mass of the stick immediately after the collision is zero? **(10)**



Q.2(A) A Yo-Yo of mass M has an axle of radius b and a spool of radius R. Its moment of inertia can be taken to be $MR^2/2$. The Yo-Yo is placed upright on a table and the string is pulled with a horizontal force F as shown. The coefficient of friction between the Yo-Yo and the table is μ. What is the maximum value of F in terms of given parameters for which the Yo-Yo will roll without slipping? **(7)**



Q.2(B) Consider a thin hoop of radius R, spinning with constant angular speed ω about its diameter. A small bug of mass m walks with constant angular speed Ω on its inside surface as shown in figure. Let $\vec{F}$ be the total force the hoop applies on the bug when it is making an angle φ with the vertical. Taking the plane of hoop to coincide with the plane of paper at this instant as shown in figure, answer the following questions. **(Neglect gravity).**



(a) Find $\left(\vec{F}_r\right)$ and $\left(\vec{F}_\theta\right)$ (with directions) in terms of m, R, φ, ω and Ω. [Hint. Resolve $\vec{F}$ into components towards the axis of rotation $\left(\vec{F}_r\right)$ and perpendicular to the plane of hoop $\left(\vec{F}_\theta\right)$ write down the differential equation of motion for bug using polar coordinates. **(6)**

(b) Using the result of (a) find the torque applied by the hoop on the bug about the axis of rotation. **(2)**

(c) Find the $d\vec{L}/dt$ about the axis of rotation. **(2)**

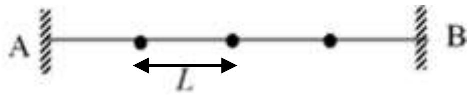
(d) Now analyze the problem from the rotating frame of hoop and answer the following questions.

(i) What is the net acceleration of the bug in the rotating frame of hoop? **(4)**

(ii) What are the magnitudes and describe the directions of fictitious forces. **(4)**

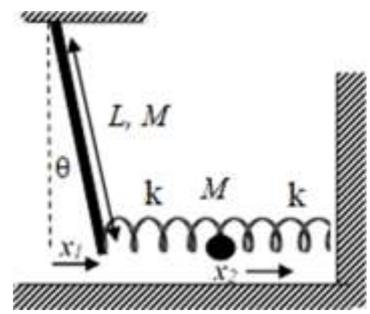
Q.3(A) The phase velocity of waves is given by $V_{ph} = \sqrt{(a/k)+(k/b)}$, where a and b are constants and k is the wave number.**(a)** Calculate the group velocity V_g of the waves. **(b)** Find the value of k at which phase and group velocities are equal. **(3+2)**

Q.3(B) An elastic string of negligible mass, stretched so as to have a tension T, is attached to fixed points A and B, a distance $4L$ apart, and carries three equally spaced particles of mass M, as shown in figure below.

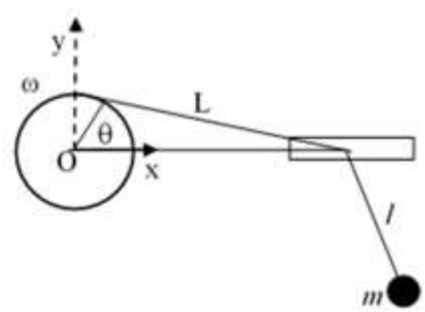


(a) Suppose that the particles have small transverse displacements y_1, y_2 and y_3 respectively, at some instant. Derive the differential equations of motion for each particle. **(b)** The appearance of the normal modes can be found by the sine curves that pass through A and B. Find the relative values and sign of the amplitudes A_1, A_2 and A_3 in each of the possible modes of the system. **(c)** Now take $T = 10\ N,\ M = 4\ grams$ and $4L = 1\ m$. Find the angular frequencies of all the possible modes by putting $y_1 = A_1 \sin\omega t,\ y_2 = A_2 \sin\omega t\ and\ y_3 = A_3 \sin\omega t$ in the equations obtained in part **(a)** and using the ratios $A_1 : A_2 : A_3$ obtained in part **(b). (3+3+4)**

Q.3(C) W.r.t. the figure, in a coupled oscillator system, a physical pendulum of length L and mass M pivoted at the upper end, and its lower end is coupled to a mass-spring system. A particle of mass M is at the junction of two identical springs of spring constant k. The springs are massless. Using the approximation of small oscillations. **(a)** Write down the total energy of the system at some instant when angular displacement of physical pendulum is θ, linear displacement of lowest end of the pendulum is x_1 and linear displacement of the particle is x_2. **(b)** Derive the differential equations of motion for x_1 and x_2. **(c)** If $g/L = 2k/M$, determine the angular frequencies of the normal modes of the system in terms of ***g* and *L*.** Neglect damping in this problem. **(4+5+6).**



Q.4(A) The point of suspension of a simple pendulum of mass m length l is driven (strictly horizontally) by a motor which consists of a disk of radius R rotating with an angular speed ω as shown. The point of suspension of the pendulum is connected to the motor-wheel by a rod of negligible mass of length L (L>>R) and also it can move inside a fixed frictionless groove. In free vibration (when the motor is off) the amplitude of the pendulum drops by a factor of 'e' in 50 swings. Assume the whole system is in x-y plane. **(a)** Write down the equation of motion of the pendulum for forced oscillation choosing the origin of your axis at the center of the disc. **(b)** Find out $A(\omega)$ and $\delta(\omega)$ from **(a)**. **(c)** Calculate the maximum mean power absorbed by the forced oscillator for the given data: m=0.1 kg, l=0.5 m, R= 0.01 m (Take, g = 10 m/s^2). **(8+8+4)**



Q.4(B). In the forced oscillation, the power input to maintain forced vibration can be calculated by recognizing that this power is the mean rate of doing work against the resistive force $-b\upsilon$. **(a)** Find out the instantaneous rate of doing work against the resistive force. **(b)** Find out the mean rate of doing work in the forced oscillation assuming $x = A\cos(\omega t - \delta)$. **(4+6)**

End

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MODEL SOLUTIONS

Q.1(A). Let at an angle θ *below w.r.t horizontal diameter the acceleration is purely horizontal.*

While moving along the circle, the bead has two acceleration always, tangential acceleration (a_θ) *and radial acceleration* (a_r). *At a position* θ *below horizontal:*

From energy conservation: $\frac{1}{2}m\upsilon^2 = mg\left(R + R\sin\theta\right) \Rightarrow \boxed{\upsilon = \sqrt{2gR\left(1+\sin\theta\right)}} ----(3)$

$\therefore$ *Radial acceleration:* $a_r = \frac{\upsilon^2}{R} = 2g\left(1+\sin\theta\right) ----(1)$

and tangential acceleration : $a_\theta = g\cos\theta ----(1)$

Condition for horizontal acceleration: Vertical components of a_r *and* a_θ *must be equal and oppisite.*

i.e. $a_r \sin\theta = a_t\cos\theta \Rightarrow 2g\left(1+\sin\theta\right)\sin\theta = g\cos\theta.\cos\theta \Rightarrow 3\sin^2\theta + 2\sin\theta - 1 = 0$

$$\sin\theta = \frac{-2\pm\sqrt{16}}{6} = \frac{-2\pm 4}{6} = \frac{1}{3} \text{ or } -1$$

$\therefore \sin\theta = \frac{1}{3}$ *is only the physically acceptable solution.i.e.* $\theta = 19.47^0$ *or* $167.53^0 ----(5)$

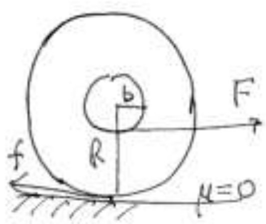
Q.1(B). Method 1: W.r.t. the collision point, $L_i = L_f \Rightarrow MV_0\frac{L}{2}\cos\theta = I\omega = \frac{ML^2}{12}\omega \Rightarrow \boxed{\omega = \frac{6V_0\cos\theta}{L} ----(5)}$

From energy conservation: $\frac{1}{2}MV_0^2 = \frac{1}{2}I\omega^2 = \frac{1}{2}.\frac{ML^2}{12}.\left(\frac{6V_0\cos\theta}{L}\right)^2 \Rightarrow \boxed{\cos\theta = \frac{1}{\sqrt{3}} ----(5)}$

Method 2 : $\frac{1}{2}MV_0^2 = \frac{1}{2}I\omega^2 \Rightarrow MV_0^2 = \frac{ML^2}{12}\omega^2 \Rightarrow \boxed{\omega = \frac{2\sqrt{3}V_0}{L}} ---(3)$

Now, $L_i = L_f \Rightarrow MV_0\frac{L}{2}\cos\theta = I\omega = \frac{ML^2}{12}.\frac{2\sqrt{3}V_0}{L} \Rightarrow \boxed{\cos\theta = \frac{1}{\sqrt{3}}} ----(8)$

Q-2 A



Translational equation of motion

$$F - f = ma \quad (1) \quad \boxed{2}$$

It is clear that Yo-Yo will translate to the right. For it to roll without slipping, the torque which makes it roll to the right (fR) will determine the sign of angular acceleration α.

Thus

$$fR - bF = \frac{mR^2}{2}\alpha \quad (2) \quad \boxed{2}$$

Rolling without slipping $\rightarrow$ $a = R\alpha$ $\boxed{1}$ (3)

Solving (1), (2) (3), we get,

$$F_{max} = \frac{3\mu mgR}{2b+R} \quad \boxed{2}$$ (For F_{max}, we need $f_{max} = \mu mg$).

c) $\vec{L} = I\omega = mR^2 \sin^2\phi\, \omega$

$$\boxed{\frac{dL}{dt} = 2mR^2\omega \sin\phi \cos\phi\, \Omega}$$ $\boxed{2}$

This matches torque in part b).

d) Equation of motion in rotating frame

i) of hoop is:

$$\vec{F}' = \vec{F} - m\vec{\omega}\times(\vec{\omega}\times\vec{r}) \cancel{- 2m\,\vec{v}_{rot}\times} - 2m\,\vec{\omega}\times\vec{v}_{rot} = m\vec{a}'$$

Here, $\vec{F}'$ = Forces on bug in rotating frame

$\vec{F}$ = Vector sum of real forces on bug

$\vec{a}'$ = acceleration of bug in rot. frame

the rest are fictitious forces.

$$\vec{F} = \vec{F}_r + \vec{F}_\theta$$

$|\vec{F}_r| = mR\sin\phi\,\omega^2$ axially inwards

$|F_\theta| = 2mR\cos\phi\,\Omega\,\omega$ pointing into the page.

$v = R\sin\phi\,\Omega$

$-|2m\,\vec{\omega}\times\vec{v}| = -|2m\,\omega\, v \sin(90-\phi)|$

$= |2m\omega R \cancel{\sin\phi} \cos\phi|$

$\boxed{\vec{F}_{cor} = -2m\,\vec{\omega}\times\vec{v} = 2m\,\omega R \cancel{\sin\phi} \cos\phi \text{ pointing out of page}}$ $\boxed{2}$

$\boxed{\vec{\omega}\times(\vec{\omega}\times\vec{r}) = m\omega^2 R\sin\phi \text{ pointing axially outwards}}$ $\boxed{2}$

Thus $\vec{F}' = 0 = m\vec{a}'$

$$\Rightarrow \boxed{\vec{a}' = 0}$$ $\boxed{4}$

Here we have neglected $-R\Omega^2\hat{R}$ = acceleration of bug.

$$Q.3(A)(a): \upsilon_{ph} = \nu/k \Rightarrow \nu = k\upsilon_{ph};$$

$$\upsilon_g = \frac{d\nu}{dk} = \upsilon_{ph} + k\frac{d\upsilon_{ph}}{dk} = \sqrt{\frac{a}{k}+\frac{k}{b}} + k\frac{d}{dk}\left(\sqrt{\frac{a}{k}+\frac{k}{b}}\right)$$

$$\Rightarrow \boxed{\upsilon_g = \frac{1}{\upsilon_{ph}}\left(\frac{a}{2k}+\frac{3k}{2b}\right)} ----(3)$$

$$(b)\,\upsilon_{ph} = \upsilon_g \Rightarrow \upsilon_{ph} = \frac{1}{\upsilon_{ph}}\left(\frac{a}{2k}+\frac{3k}{2b}\right) \Rightarrow \upsilon_{ph}^2 = \frac{a}{k}+\frac{k}{b} = \frac{a}{2k}+\frac{3k}{2b} \Rightarrow \boxed{k = \pm\sqrt{ab}} ----(2)$$

$$Q.3(B)(a)\,From\ Fig.:$$

$$\boxed{\begin{aligned} m\ddot{y}_1 &= -T\sin\theta_1 + T\sin\theta_2 = -T\frac{y_1}{l} + T\frac{y_2-y_1}{l} = -\frac{2T}{l}y_1 + \frac{T}{l}y_2 \\ m\ddot{y}_2 &= -T\sin\theta_2 - T\sin\theta_3 = -T\frac{y_2-y_1}{l} - T\frac{y_2-y_3}{l} = -\frac{2T}{l}y_2 + \frac{T}{l}y_1 + \frac{T}{l}y_3 \\ m\ddot{y}_3 &= T\sin\theta_3 - T\sin\theta_4 = T\frac{y_2-y_3}{l} - \frac{T}{l}y_3 = -\frac{2T}{l}y_3 + \frac{T}{l}y_2 \end{aligned}} ----(3)$$

$$(b)\,A_{pn} = C_n \sin\left(\frac{pn\pi}{N+1}\right); p = principle\ number, n = normal\ mode\ number; Here: N = 3$$

$$\therefore A_{pn} = C_n \sin\left(\frac{pn\pi}{4}\right); For, n=1: A_{p1} = C_1 \sin\left(\frac{p\pi}{N+1}\right)$$

$$\boxed{\begin{aligned} &For, n=1, A_{11}:A_{21}:A_{31} = 1:\sqrt{2}:1 \\ &For\ n=2, A_{12}:A_{22}:A_{32} = 1:0:-1 \\ &For, n=3, A_{13}:A_{23}:A_{33} = 1:-\sqrt{2}:1 \end{aligned}} ----(3)$$

(c) *Using the results of* (a) *and substituting* $y_1 = A_1\cos\omega t, y_2 = A_2\cos\omega t$ *and* $y_3 = A_3\cos\omega t$ *we have:*

$$\boxed{\begin{aligned} &In\,1st\ mode, \frac{A_1}{A_2} = \frac{T/ml}{\frac{2T}{ml}-\omega^2} \Rightarrow \frac{1}{\sqrt{2}} = \frac{T/ml}{\frac{2T}{ml}-\omega_1^2} \Rightarrow \omega_1 = 76.54\,rad/sec \\ &In\ 2nd\ mode, \frac{A_2}{A_1} = 0 = \frac{\frac{2T}{ml}-\omega_2^2}{T/ml} \Rightarrow \omega_2 = 141.4\ rad/sec \\ &In\ 3rd\ mode, \frac{A_1}{A_2} = -\frac{1}{\sqrt{2}} = \frac{T/ml}{\frac{2T}{ml}-\omega_3^2} \Rightarrow \omega_3 = 184.78\,rad/sec \end{aligned}} ----(4)$$

Q.3(C)(a) *Total energy E is constant.*

$$\boxed{E=\frac{1}{2}I\dot{\theta}^2+Mg\frac{L}{2}(1-\cos\theta)+\frac{1}{2}k\left(x_1-x_2\right)^2+\frac{1}{2}M\dot{x}_2^2+\frac{1}{2}kx_2^2}----(4)$$

(b) *For small angle* : $\theta=\frac{x_1}{L}$ *and* $1-\cos\theta=\frac{x_1^2}{2L^2}$

$$\therefore\ \boxed{E=\frac{1}{6}M\dot{x}_1^2+\frac{Mgx_1^2}{4L}+\frac{1}{2}k\left(x_1-x_2\right)^2+\frac{1}{2}M\dot{x}_2^2+\frac{1}{2}kx_2^2}$$

$$Now,\frac{\delta E}{\delta t}=\frac{\delta E}{\delta x_1}.\frac{\delta x_1}{\delta t}=0\Rightarrow\boxed{\frac{1}{3}\ddot{x}_1+\frac{g}{2L}x_1+\frac{k}{M}\left(x_1-x_2\right)=0}----(3)$$

$$Similarly,\boxed{\ddot{x}_2+\frac{k}{M}x_2-\frac{k}{M}\left(x_1-x_2\right)=0}----(2)$$

(c) *Given* : $\frac{2k}{M}=\frac{g}{L}$; *So, above two equations become* :

$$\boxed{\ddot{x}_1+\frac{3g}{2L}x_1+\frac{3g}{2L}\left(x_1-x_2\right)=0}\ and\ \boxed{\ddot{x}_2-\frac{g}{2L}x_1+\frac{g}{L}x_2=0}$$

In normal modes: $x_1=A\cos\omega t, x_2=B\cos\omega t$

$$\therefore\ \ddot{x}_1+\frac{3g}{2L}x_1+\frac{3g}{2L}\left(x_1-x_2\right)=0\Rightarrow\left(\frac{3g}{L}-\omega^2\right)A-\frac{3g}{2L}B=0$$

$$\ddot{x}_2-\frac{g}{2L}x_1+\frac{g}{L}x_2=0\Rightarrow-\frac{g}{2L}A+\left(\frac{g}{L}-\omega^2\right)B=0$$

$$Solving:\boxed{\omega=\sqrt{\frac{g}{2L}\left(4\pm\sqrt{7}\right)}}----(6)$$

Solution:

Q4 (A) (a) The point of suspension is driven by the projection of rotating motion of the contact point of the disc. Let it be $x_p = A\cos\omega t$ from the equilibrium position (from the position in groove when the contact point is at $\theta = 90^0$).

The equation of motion is;

$$F = -mg\,\frac{(x-x_p)}{l} - b\,\dot{x} = -\,\frac{mg\,x}{l} + \frac{mg\,R\cos\,\omega t}{l} - b\,\dot{x}$$

Finally we get equation for simple pendulum;

$\ddot{x} + \gamma\,\dot{x} + \omega_0^2\,x = R\,\omega_0^2\cos\omega t$, here $\omega_0^2 = g/\,l$ and $\gamma = \frac{b}{m}$. (8)

(b) $E = E_0\,e^{-\gamma t} = E_0\,e^{-1}$, which gives $\gamma = \frac{1}{t} = \frac{\omega_0}{n2\pi} = \frac{\sqrt{\frac{g}{l}}}{100\pi}$. (2)

$$A(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}} = \frac{gR/l}{\sqrt{\left(\frac{g}{l} - \omega^2\right)^2 + \left(\frac{\sqrt{g}}{100\,\pi\,\sqrt{l}}\,\omega\right)^2}}$$

$$\delta(\omega) = tan^{-}\left(\frac{\gamma\omega}{\omega_0^2 - \omega^2}\right) = tan^{-}\left(\frac{\sqrt{\frac{g}{l}}\,\frac{\omega}{100\,\pi}}{\frac{g}{l} - \omega^2}\right) \quad (6)$$

(c) $P_{max} = \frac{Q\,F_0^2}{2\,m\,\omega_0} = \frac{2\,\pi\,n\left(\frac{mg\,l}{R}\right)^2}{2\,\times 0.1 \times \sqrt{\frac{g}{l}}} = 0.14\,Watts$ (4)

Q4 (B) (a) Instantaneous Work done:

$dW = F\,dx = F\,v\,dt = bv\,v\,dt = b\,v^2\,dt$. Therefore, $\frac{dW}{dt} = b\,v^2$.

(4)

(b) Mean rate of doing work

$W = \frac{1}{T}\int dW = \frac{1}{T}\int_0^T b\,v^2\,dt = \frac{1}{T}\int_0^T b\,A^2\,\omega^2\,sin^2\,(\omega t - \delta)\,dt = \frac{b\,\omega^2\,A^2}{2}$

(6)