Birla Institute of Technology and Science, Pilani (Raj.)
Second Semester, 2017-2018
MATH F112 (Mathematics II)
Part-A (Closed Book) Comprehensive Examination

Note: (i) This question paper has two parts Part-A (Closed Book) and Part-B (Open Book)
(ii) Write **Part-A** on top right corner of the answer sheet.
(iii) Answer each sub-part of a question together.

Max. Marks: 66 **Max. Time: 90 min.** **Date: 1 May, 2018 (Tuesday)**

Q.1 (a) Let V and W are subspaces of $\mathbb{R}^5$ spanned by $X = \{[1,3,-2,2,3],\ [2,7,-5,6,5],\ [3,6,-3,0,13]\}$ and $Y = \{[1,3,0,2,1],\ [5,16,\ -3,12,6],\ [3,8,3,4,2]\}$ respectively. Prove that $V + W = \{v + w\colon v \in V,\ w \in W\}$ is a subspace of $\mathbb{R}^5$ and find its basis with justification. **[9]**

(b) Find the homogeneous system of equations whose solution set U is spanned by $\{[-4,1,2],\ [2,1,0],\ [6,-3,-4],\ [12,-6,-8]\}$. **[9]**

Q.2 (a) Consider the complex numbers $z_1 = 1-\sqrt{3}i$ and $z_2 = -1-\sqrt{3}i$

(i) Determine the principle values (P.V.) of z_1^i and z_2^i

(ii) Using the P.V. prove/disprove that $z_1^i z_2^i = (z_1 z_2)^i$. **[6+5]**

(b) Find all the roots of the equation $\sin(z) = i\sinh(1)$. **[11]**

Q.3 (a) Consider $f(z) = \dfrac{(\log z)^3}{z^2+1}$ where $\log z = \ln r + i\theta$ $(r > 0,\ 0 < \theta < 2\pi)$. Find and classify all the singularities of $f(z)$ in finite complex plane $\mathbb{C}$ with justification and hence, find the residues of $f(z)$ at isolated singularities. **[3+3+2+2]**

(b) Use residues to show that $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}\,d\theta = \frac{2\pi}{b^2}\left\{a-\sqrt{a^2-b^2}\right\}$ where $a > b > 0$. **[16]**

************* **End of Part-A** *************

P.T.O.

1) (a) Let V & W are subspaces of $\mathbb{R}^5$ spanned by

$X = \{[1,3,-2,2,3], [2,7,-5,6,5], [3,6,-3,0,13]$ and

$Y = \{[1,3,0,2,1], [5,16,-3,12,6], [3,8,3,4,2]\}$ resp.

Prove that $V+W = \{v+w : v \in V, w \in W\}$ is a subspace of $\mathbb{R}^5$ and find its basis with justification.

Ans. $V+W = \{v+w : v \in V, w \in W\}$

since, $0 \in V, 0 \in W \Rightarrow 0+0 = \boxed{0 \in V+W} \Rightarrow V+W$ is non-empty.

Let $x, y \in V+W \Rightarrow x = v_1 + w_1$ for $v_1 \in V, w_1 \in W$

$y = v_2 + w_2$ for $v_2 \in V, w_2 \in W$

so, $x+y = \underbrace{v_1+v_2}_{\in V} + \underbrace{w_1+w_2}_{\in W}$ as V, W are subspaces of $\mathbb{R}^5$.

$\therefore \boxed{x+y \in V+W}$

Taking $\alpha \in \mathbb{R}$ then

$\alpha x = \underbrace{\alpha v_1}_{\in V} + \underbrace{\alpha w_1}_{\in W} \Rightarrow \boxed{\alpha x \in V+W}$

Hence, $V+W$ is a subspace of $\mathbb{R}^5$. — 3

For finding basis of $V+W$,

we use

$V+W = \langle \underbrace{v_1, v_2, v_3}_{X}, \underbrace{w_1, w_2, w_3}_{Y} \rangle$ — 1

$$\begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 2 & 7 & -5 & 6 & 5 \\ 3 & 6 & -3 & 0 & 13 \\ 1 & 3 & 0 & 2 & 1 \\ 5 & 16 & -3 & 12 & 6 \\ 3 & 8 & 3 & 4 & 2 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$R_4 \to R_4 - R_1$

$R_5 \to R_5 - 5R_1$, $R_6 \to R_6 - 3R_1$

$$\Rightarrow \begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & -3 & 3 & -6 & 4 \\ 0 & 0 & 2 & 0 & -2 \\ 0 & 0 & & & -9 \end{bmatrix}$$

$R_6 \to R_6 + R_2$

$R_3 \to R_3 + 3R_2$

$$\Rightarrow \begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & -2 \\ 0 & 0 & 8 & 0 & -8 \\ 0 & 0 & 8 & 0 & -8 \end{bmatrix}$$

$R_5 \to R_5 - 4R_4$
$R_6 \to R_6 - 4R_4$

$$\Rightarrow \begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_4 \to \frac{1}{2} R_4$
$R_3 \leftrightarrow R_4$

$$\Rightarrow \begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to 4$$

Hence the basis ~~is~~ :- $\{[1,3,-2,2,3], [0,1,-1,2,1], [0,0,1,0,-1], [0,0,0,0,1]\}$ — 1

—6—

(b):- Find the homogeneous system of eqns whose soln set U is spanned by

$$\{[-4,1,2]_{x_1},\ [2,1,0]_{x_2},\ [6,-3,-4]_{x_3},\ [12,-6,-8]_{x_4}\}$$

Ans- Let $(x,y,z) \in U \Rightarrow (x,y,z) = \alpha(x_1) + \beta(x_2) + \gamma(x_3) + \delta(x_4)$ $\longrightarrow$ 2

$$\Rightarrow x = -4\alpha + 2\beta + 6\gamma + 12\delta$$

$$y = \alpha + \beta - 3\gamma - 6\delta$$

$$z = 2\alpha + 0 - 4\gamma - 8\delta$$

$$\Rightarrow \left[\begin{array}{cccc|c} -4 & 2 & 6 & 12 & x \\ 1 & 1 & -3 & -6 & y \\ 2 & 0 & -4 & -8 & z \end{array}\right] \quad R_1 \to R_1/-4$$

$\longrightarrow$ 3

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & -\frac{1}{2} & -\frac{3}{2} & -3 & -\frac{x}{4} \\ 1 & 1 & -3 & -6 & y \\ 2 & 0 & -4 & -8 & z \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & -\frac{1}{2} & -\frac{3}{2} & -3 & -\frac{x}{4} \\ 0 & \frac{3}{2} & -\frac{3}{2} & -3 & y + \frac{x}{4} \\ 0 & 1 & -1 & -2 & z + \frac{x}{2} \end{array}\right] \quad R_2 \to \frac{2}{3}R_2$$

$\longrightarrow$ 1

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & -\frac{1}{2} & -\frac{3}{2} & -3 & -\frac{x}{4} \\ 0 & 1 & -1 & -2 & \frac{2}{3}\left(\frac{4y+x}{4}\right) \\ 0 & 1 & -1 & -2 & \frac{2z+x}{2} \end{array}\right] \quad R_3 \to R_3 - R_2$$

$\longrightarrow$ 1

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & -\frac{1}{2} & -\frac{3}{2} & -3 & -\frac{x}{4} \\ 0 & 1 & -1 & -2 & \frac{4y+x}{6} \\ 0 & 0 & 0 & 0 & \frac{x+2z}{2} - \frac{x+4y}{6} \end{array}\right]$$

$$\Rightarrow 3x + 6z - x - 4y = 0$$

$$\Rightarrow 2x - 4y + 6z = 0$$

$$\boxed{\Rightarrow x - 2y + 3z = 0}$$

$\longrightarrow$ 2

Q2. (a)(i) P.V. $(1-\sqrt{3}i)^i = e^{i\,\mathrm{Log}(1-\sqrt{3}i)}$

$= e^{i\,\mathrm{Log}[2e^{-i\pi/3}]}$

$= e^{i\ln 2}\, e^{\pi/3} \longrightarrow$ 3

P.V. $(-1-\sqrt{3}i)^i = e^{i\,\mathrm{Log}(-1-\sqrt{3}i)}$

$= e^{i\,\mathrm{Log}(2e^{-i2\pi/3})}$

$= e^{i\ln 2} \cdot e^{2\pi/3} \longrightarrow$ 3

(b)(ii) $(z_1 z_2)^i = (-4)^i$

P.V. $(z_1 z_2)^i = e^{i\,\mathrm{Log}(-4)}$

$= e^{i\,\mathrm{Log}[4e^{i\pi}]}$

$= e^{i\ln 4}\, e^{-\pi} \longrightarrow$ 3

$z_1^i z_2^i = e^{2i\ln 2}\, e^{\pi} \neq (z_1 z_2)^i \longrightarrow$ 2

* Notes: Marks are deducted for writing:

(i) $\ln 2 \rightarrow \mathrm{Log}(2)$ or $\log(2)$

(ii) $e^{i\ln 2} \rightarrow 2^i$

(iii) unless z_1^i, z_2^i, $(z_1 z_2)^i$ calculated properly, $z_1^i z_2^i \neq (z_1 z_2)^i$ can't be proved/disproved.

— x —

Q2.(b) $\sin z = i \sinh(1)$

Solⁿ: $\sin x \cosh y + i \cos x \sinh y = i \sinh(1)$

$\Rightarrow \quad \sin x \cosh y = 0$

$\cos x \sinh y = \sinh(1) \longrightarrow$ (2)

$\because \cosh y \neq 0, \quad \sin x = 0 \Rightarrow x = n\pi$

$n = 0, \pm 1, \pm 2, \ldots$

$\Rightarrow (-1)^n \sinh y = \sinh(1) \longrightarrow$ (2)

Two cases:

(i) $n = 2k$ (even) $\Rightarrow \sinh y = \sinh(1)$

$\Rightarrow y = 1, \quad x = 2k\pi \longrightarrow$ (3)

$k = 0, \pm 1, \pm 2, \ldots$

(ii) $n = 2k+1$ (odd) $\Rightarrow -\sinh y = \sinh(1)$

$\Rightarrow y = -1, \quad x = (2k+1)\pi \longrightarrow$ (3)

$k = 0, \pm 1, \pm 2, \ldots$

Soln set: $z = 2n\pi + i$ and

$z = (2n+1)\pi - i \longrightarrow$ (1)

$n = 0, \pm 1, \pm 2, \ldots$

Notes: Marks are deducted if

(i) Mathematical steps are not shown

Q2(b) (Alternative solution)

$$\sin z = i \sinh(1)$$

$$\Rightarrow \frac{e^{iz} - e^{-iz}}{2i} = i \frac{e - e^{-1}}{2}$$

$$\Rightarrow e^{2iz} + \left(\frac{e^2-1}{e}\right) e^{iz} - 1 = 0 \longrightarrow 2$$

$$\Rightarrow e^{iz} = \frac{1}{e}, -e \longrightarrow 2$$

Two cases:

(i) $e^{iz} = \frac{1}{e} \Rightarrow e^{ix} \cdot e^{-y} = e^{-1}$

$$\Rightarrow x = 2n\pi, \; y = 1 \longrightarrow 3$$
$$n = 0, \pm 1, \ldots$$

(ii) $e^{iz} = -e \Rightarrow e^{ix} \cdot e^{-y} = -e$

$$\Rightarrow x = (2n+1)\pi, \; y = -1 \longrightarrow 3$$
$$n = 0, \pm 1, \ldots$$

Set of solutions:

$$z = 2n\pi + i \quad \text{and}$$

$$z = (2n+1)\pi - i \longrightarrow 1$$

$$n = 0, \pm 1, \pm 2, \ldots$$

Q.3 (C.B.)

Q: $f(z) = \dfrac{(\log z)^3}{(z^2+1)}$ $\log z = \ln(r) + i\theta$, $r>0,\ 0<\theta<2\pi$

Singular points are $z = i, -i$ and all the points in set $S = \{z = x+iy \in \mathbb{C} : x \geq 0 \text{ and } y = 0\}$ (3m)

$z = i$ is isolated as $f(z)$ is analytic in the neighborhood $\{z \in \mathbb{C} : |z-i| < 1/2\}$.

(3m)
1 → identification
2 → justification

$z = -i$ is isolated as $f(z)$ is analytic in the neighborhood $\{z \in \mathbb{C} : |z+i| < 1/2\}$.

All the singular points in S are not isolated. $z_0 \in S$, the every neighborhood of z_0 contains at least one point in S, ~~thus~~ Thus singularities in S are non isolated.

(i) Residue at $z = i$:

$$f(z) = \frac{(\log z)^3/(z+i)}{(z-i)} = \frac{\phi_1(z)}{(z-i)}$$

$$\operatorname*{Res}_{z=i} f(z) = \frac{(\log i)^3}{(\cancel{z}i+i)} = \frac{(\ln 1 + i\pi/2)^3}{2i}$$

(3m)

$$= -\frac{\pi^3}{16}.$$

(ii) Residue at $z = -i$: $f(z) = \dfrac{(\log z)^3/z-i}{(z+i)} = \dfrac{\phi_2(z)}{(z+i)}$

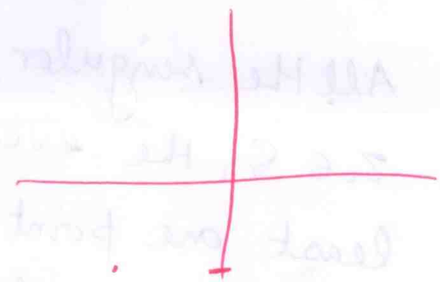
$$\operatorname*{Res}_{z=-i} f(z) = \frac{(\log(-i))^3}{(-2i)}$$

$$= \frac{\left(\ln 1 + i\,3\pi/2\right)^3}{-2i}$$

$$= \frac{27\pi^3(-i)}{-16i} = \frac{27\pi^3}{16}. \quad (3m)$$

$2\pi - \frac{\pi}{2} = \frac{3\pi}{2}$

Alternate Sol (at $z=-i$) $\dfrac{\left(\ln 1 + i(-\pi/2)\right)^3}{-2i}$



$$= \frac{-i^3\pi^3}{-16i}$$

$$= \frac{i^2\pi^3}{16} = \frac{-\pi^3}{16}$$

Q2 Prove that $\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}\,d\theta = \frac{2\pi}{b^2}\left\{a-\sqrt{a^2-b^2}\right\}$
$a>b>0$

Let $I = \int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}\,d\theta$ putting $z=e^{i\theta}$

$$I = \frac{1}{i}\int_C \frac{(1/2i)\left(z-\frac{1}{z}\right)^2}{a+\frac{1}{2}b\left(z+\frac{1}{z}\right)}\cdot\frac{dz}{z} \qquad C:\ |z|=1 \qquad \text{— [2]}$$

$$= -\frac{1}{2i}\int_C \frac{(z^2-1)^2}{z^2(2az+bz^2+b)}\,dz = -\frac{1}{2ib}\int_C \frac{(z^2-1)^2\,dz}{z^2\left(z^2+\left(\frac{2a}{b}\right)z+1\right)}$$

$$= -\frac{1}{2ib}\int_C f(z)\,dz \qquad \text{— [2]}$$

$\Rightarrow$ $f(z)$ has a double pole at $z=0$ and simple poles at $z=\alpha$ and $z=\beta$ where α & β are the roots of — [1]
$z^2+\frac{2a}{b}z+1=0$

$$\Rightarrow z = \frac{1}{2}\left[-\frac{2a}{b}+\sqrt{\frac{4a^2}{b^2}-4}\right] \qquad [2]$$

Let $\alpha = \frac{-a+\sqrt{a^2-b^2}}{b}$; $\beta = \frac{-a-\sqrt{a^2-b^2}}{b}$ — [1]

since $a>b>0$, $|\beta|>1$ but $|\alpha\beta|=1$ so that $|\alpha|<1$

Therefore the pole inside the circle $|z|=1$ are a double pole at the origin and a simple pole at $z=\alpha$

$$\text{Residue at } z=\alpha = \lim_{z\to\alpha}(z-\alpha)\left[-\frac{1}{2ib}\frac{(z^2-1)^2}{z^2(z-\alpha)(z-\beta)}\right]$$

$$= \frac{(\alpha^2-1)^2}{2ib\,\alpha^2(\alpha-\beta)} = -\frac{1}{2ib}\left(\frac{\left(\alpha-\frac{1}{\alpha}\right)^2}{(\alpha-\beta)}\right)$$

$$= -\frac{1}{2ib}\frac{(\alpha-\beta)^{\cancel{2}}}{\cancel{\alpha-\beta}} \qquad \text{since } \frac{1}{\alpha}=\beta$$

$$= -\frac{1}{2ib}\cdot\frac{2}{b}\sqrt{a^2-b^2} = \boxed{\frac{i}{b^2}\sqrt{a^2-b^2}} \qquad \text{— [3]}$$

Residue at origin = coeff. of $\frac{1}{z}$ in $-\frac{1}{2ibz^2}\frac{(z^2-1)^2}{z^2+\frac{2a}{b}z-1}$ $z\to 0$

$= \text{coeff of } \frac{1}{z} \text{ in } -\frac{1}{2ibz^2}(1-2z^2+z^4)\left(1+\frac{2a}{b}z+z^2\right)^{-1}$

$= \text{coeff of } \frac{1}{z} \text{ in } -\frac{1}{2ibz^2}(1-2z^2+\cdots)\left(1-\frac{2a}{b}z \cdots\right)$

$= \frac{a}{ib^2} = \boxed{-\frac{ai}{b^2}}$ —[3]

[2]

$$\therefore \int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}\,d\theta = 2\pi i\left[\frac{i}{b^2}\sqrt{a^2-b^2} - \frac{ai}{b^2}\right]$$

$$= \frac{2\pi}{b^2}\left[a-\sqrt{a^2-b^2}\right] \quad a>b>0$$

Alternate Sol^n

Q.3(b) $I = \int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}\,d\theta = \frac{1}{2}\int_0^{2\pi} \frac{2\sin^2\theta}{a+b\cos\theta}\,d\theta$

$$= \frac{1}{2}\int_0^{2\pi} \frac{(1-\cos 2\theta)}{a+b\cos\theta}\,d\theta \quad \text{——[1]}$$

$$= \frac{1}{2} Re\left(\int_0^{2\pi} \frac{(1-e^{+2i\theta})}{(a+b\cos\theta)}\,d\theta\right) \quad \text{——[2]}$$

Put $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$

$$I' = \int_0^{2\pi} \frac{(1-e^{2i\theta})}{(a+b\cos\theta)}\,d\theta = \int_{C:|z|=1} \frac{(1-z^2)}{\left(a+b\left(\frac{z^2+1}{2z}\right)\right)}\frac{dz}{iz}$$

$$= \frac{2}{i}\int_C \frac{(1-z^2)}{(bz^2+2az+b)}\,dz$$

$$= \frac{2}{ib}\int_C \frac{(1-z^2)}{(z^2+\frac{2a}{b}z+1)}\,dz$$

$$= \frac{2}{ib}\int_C \frac{(1-z^2)}{(z-\alpha)(z-\beta)}\,dz \quad \text{——[3]}$$

where $\alpha = \frac{-a+\sqrt{a^2-b^2}}{b}$, $\beta = \frac{-a-\sqrt{a^2-b^2}}{b}$ ——[2]

$$\operatorname*{Res}_{z=\alpha} f(z) = \lim_{z\to\alpha} (z-\alpha) \frac{2}{ib} \frac{(1-z^2)}{(z-\alpha)(z-\beta)}$$

$$= \frac{2}{ib} \frac{(1-\alpha^2)}{(\alpha-\beta)}$$

$$= \frac{2}{ib^2} \left(a - \sqrt{a^2-b^2}\right) \qquad \text{——— [4]}$$

$$I' = 2\pi i \operatorname*{Res}_{z=\alpha} f(z) = 2\pi i \frac{2}{ib^2} \left(a-\sqrt{a^2-b^2}\right)$$

$$= \frac{4\pi}{b^2}\left(a-\sqrt{a^2-b^2}\right) \qquad \text{——— [2]}$$

$$I = \frac{1}{2} I' = \frac{2\pi}{b^2}\left(a-\sqrt{a^2-b^2}\right) \qquad \text{——— [1]}$$