INTRODUCTION TO POLAR COORDINATES.

Q. If the gravitational attraction of Sun is pulling Earth radially inwards, why Earth does not collapse onto the Sun. Why does Earth rotate in an elliptical orbit, ~~tangential~~ normal to the radially inward force?

Ans. • Why do we expect Earth to collapse on to Sun in the first place? Because as we push the duster on the table it moves in the direction of force.

• Now why shouldn't we expect things to not necessarily displace in the direction of force?

Because $\vec{F} \propto \vec{a} = \frac{d^2\vec{x}}{dt^2}$.

Force is proportional to accelaration which is second derivative of displacement.

• Thus if we explore the meaning of the derivative of a vector, it will clarify two things

→ Under what circumstance, force is linearly related to displacement i.e., duster being pushed on table

→ Force is orthogonal to displacement i.e., motion of planet around the sun, or a mass whirled on string.

→ A more general motion which is a combination of above two possibilities.

• ~~SCALAR~~ DERIVATIVE OF A SCALAR.

Consider a scalar function of time $f(t)$. i.e., for every value of time t, it gives some number. $f(t)$ could be temperature recorded by a thermometer in your room.

dt/dt then tells you instantaneous time rate of temperature and is given as

$$\frac{dt}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}.$$

The temperature can either increase, decrease or stay constant with time, and dt/dt is accordingly >0, <0, or $=0$, but a number nonetheless. Thus, in general a scalar can only change in magnitude with time and hence its time derivative is a scalar.

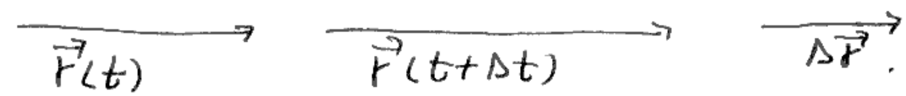
DERIVATIVE OF A VECTOR:

A vector has two attributes – magnitude and direction, and hence can change in time in two different and independent ways, and in general, both the ways simultaneously.

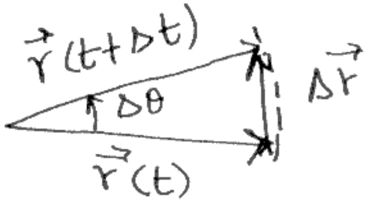
A VECTOR CHANGING PURELY IN MAGNITUDE: When a vector changes purely in magnitude without changing its direction, its derivative is very much akin to that of a scalar. Imagine going on a highway in a straight line with a constant speed. The displacement vector ~~$d\vec{x}$~~ $d\vec{r}$ changes only in magnitude.

$$\frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t}$$

Here, all the three vectors $\vec{r}(t+\Delta t)$, $\vec{r}(t)$ and ~~$\vec{r}(t+\Delta$~~ $\Delta\vec{r}$, point in the same direction and hence the vector difference $\Delta\vec{r}$ is trivial to obtain.



VECTOR CHANGING PURELY IN DIRECTION: The only way a vector can change in direction while staying constant in magnitude is for it to rotate keeping its tail fixed. As shown in the figure, position vector $\vec{r}(t)$ has rotated by an angle $\Delta\theta$. Now



$$\frac{d\vec{r}(t)}{dt} = \lim_{\Delta t\to 0} \frac{\vec{r}(t+\Delta t)-\vec{r}(t)}{\Delta t} = \lim_{\Delta t\to 0} \frac{\Delta\vec{r}}{\Delta t}$$

Now, unlike the previous case, $\vec{r}(t)$, $\vec{r}(t+\Delta t)$, and $\Delta\vec{r}$, all point in different direction and hence $\vec{r}(t+\Delta t)$ is obtained by vectorial addition of $\vec{r}(t)$ & $\Delta\vec{r}$.

. In the limit $\Delta t \to 0$, $\Delta\theta \to 0$, and we can approximate the magnitude $|\Delta\vec{r}| \approx |\vec{r}|\Delta\theta$. The direction of $\Delta\vec{r}$ in this limit is tangent to the arc and hence orthogonal to $\vec{r}(t)$. Thus,

$$\frac{d\vec{r}}{dt} = \lim_{\Delta t\to 0} \frac{\Delta\vec{r}}{\Delta t} = \lim_{\Delta t\to 0} |\vec{r}(t)| \frac{\Delta\theta}{\Delta t}\hat{\theta}$$

$$\boxed{\frac{d\vec{r}}{dt} = |\vec{r}|\frac{d\theta}{dt}\hat{\theta}}$$

This is a very important result. It says that the time derivative of a vector that is constant in magnitude is

a) $\propto |\vec{r}|$ that is magnitude of vector

b) $\propto \frac{d\theta}{dt}$ that is angular speed because the vector is rotating

c) Is orthogonal to the initial direction of the vector.

So a vector can change in three ways

1) Pure scaling: Changing only in magnitude w/o changing direction

2) Pure rotation: Changing only in direction w/o changing magnitude.

3) A general change where it changes in both magnitude and direction.

The above discussion already hints towards the fact that in general time derivative of a vector need not point in the same direction as the vector. Thus, accelaration being second derivative of displacement, ~~has no nec~~ need not necessarily be parallel to displacement. When we are pushing a duster, the displacement is ~~pu~~ only changing in magnitude w/o changing direction. But for the motion of mass being tied to a string and whirled in a circle, the displacement stays constant in magnitude but continuously changes in direction. Thus velocity ($d\vec{r}/dt$) is orthogonal to $\vec{r}$ and, accelaration ($d\bar{v}/dt$) is orthogonal to velocity.

DECOMPOSING A VECTORIAL CHANGE INTO PURE SCALING AND PURE ROTATION LOOKS INTERESTING.

Q. Normally we resolve a 2-D vector along its x and y cartesian components.

a) Do we have a coordinate system to which resolving a vectorial change into pure scaling and pure rotation is native?

b) What physical expectations shall guide the construction of such a coordinate system?

c) How different would it be from the conventional cartesian system.

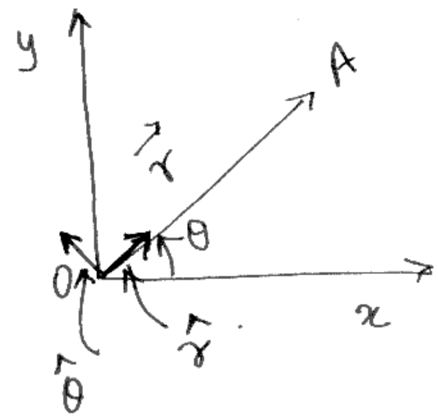
<u>ANSWER</u> a) Yes, plane polar coordinates are tailor made for such a resolution

b) CONSTRUCTION OF POLAR COORDINATE SYSTEM

→ In 2-dim we shall require two coordinates and hence two coordinate axis.

→ Once we settle on one of the axis, the other axis choice is trivially determined by orthogonality.

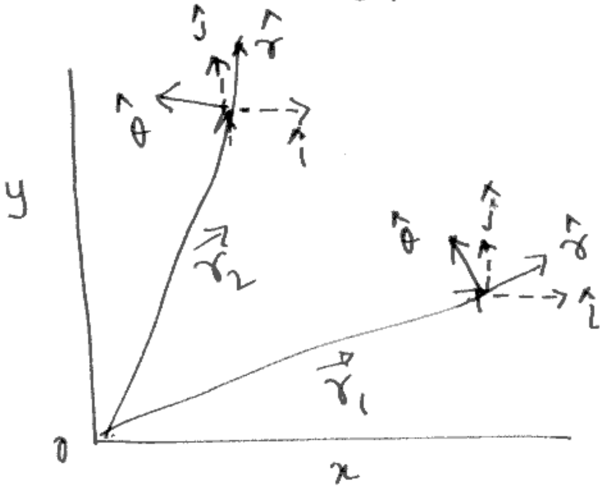
→ Since it should describe the scaling of an arbitrary vector which can be pointing in an arbitrary direction, it cannot be hinged to a fixed direction like $\hat{i}$ and $\hat{j}$ of cartesian system. Multiple of a unit vector pointing in the direction of a given vector, is the only way we can describe pure scaling. Thus if line segment $\vec{OA}$ describes the length r of a particular position vector $\vec{r}$ that makes



an angle θ to the x-axis, then the unit vector $\hat{r}$ pointing away from the origin and in the direction of vector $\vec{r}$ is capable of describing pure scaling of vector $\vec{r}$.

→ Having made a choice of a unit vector $\hat{r}$, the choice of other unit vector is trivial. It should be orthogonal to $\hat{r}$ and describe pure rotation of position vector $\vec{r}$ without scaling. Now let us call such a vector $\hat{\theta}$. Now we can rotate the vector clockwise or counter-clockwise. By convention we choose $\hat{\theta}$ to be positive when it describes counter-clockwise rotation and negative when it describes clockwise rotation. $\hat{r}$ is +ve radially outwards & -ve inwards.

c) POLAR COORDINATES ARE FUNDAMENTALLY DIFFERENT FROM CARTSIAN SYSTEM.



The figure on the left compares two position vectors $\vec{r}_1$ and $\vec{r}_2$ ~~are~~ resolved in cartesian and polar coordinates.

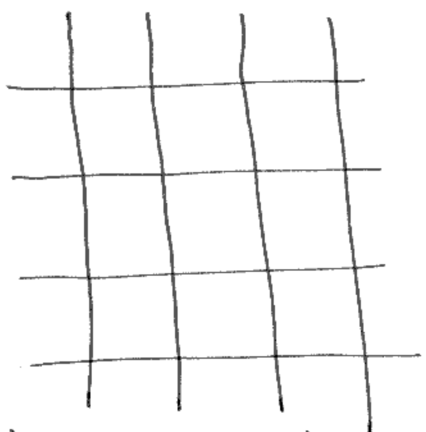
As can be clearly seen, unit vectors $\hat{i}$ and $\hat{j}$ have fixed directions once we decide upon cartesian system. Polar unit vectors $\hat{r}$ and $\hat{\theta}$ on the other hand have their directions defined by the directions of vectors they are describing.

→ Put another way cartesian unit vectors $\hat{i}$ and $\hat{j}$ are truly constant vectors. That is they are constant in both magnitude and direction. Thus their time derivative is always zero. This leads to immense simplicity in the expression for velocity and accelaration in cartesian coordinates.

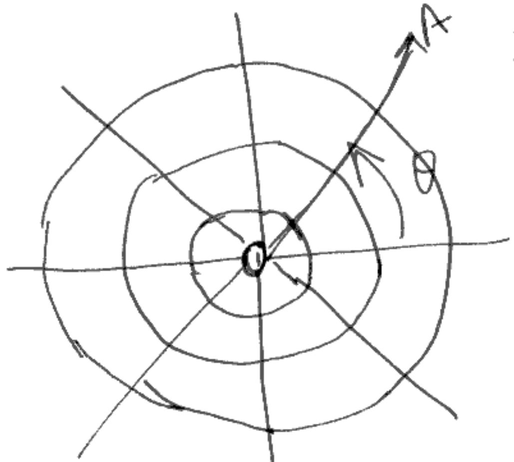
In contrast to this, plane polar unit vectors, though constant in magnitude (unit magnitude) vary in direction from point to point. This has the consequence that the time derivative of vectors in polar coordinates must also appropriately factor in non-vanishing time derivative of unit vectors $\hat{r}$ and $\hat{\theta}$. Thus, the expression for velocity and accelaration which were very straight forward in cartesian coordinates, now look very complicated in polar coordinates.

Q Why would one abandon the simplicity of cartesian coordinates in favour of more fancy but complicated polar coordinates?

ANSWER: One should abandon neither. It is rather a matter of choosing horses for courses. As Kleppner and Kolenkov put it - it is not that polar coordinates are more complicated but cartesian coordinates are simpler than they have the right to be - atleast for certain situations.



DOWNTOWN
CHICAGO



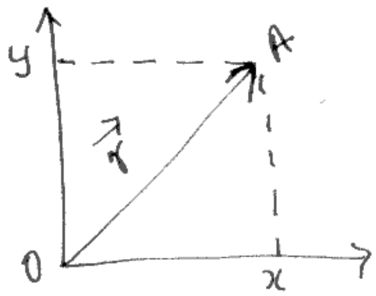
CONNOUGHT PLACE
NEW DELHI

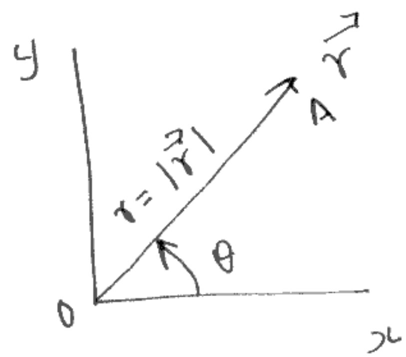
Figure on the left shows the street layout for downtown Chicago and Connought place, New Delhi. It is not difficult to imagine that polar coordinates are more suitable to describe the geometry of CP, New Delhi. If a car is going along the segment OA, the displacement is completely described as pure scaling of position vector without any change in angle θ. When viewed in cartesian system both x and y coordinates are changing. However, not both of them are independent as they are related by

$$\frac{y}{x} = \tan\theta = \text{constant}.$$

Such a constraint is already built into polar coordinate and hence it has only one free variable - r, distance from origin.

EXPRESSION FOR DISPLACEMENT VELOCITY AND ACCELARATION IN CARTESIAN & POLAR COORDINATES





→ polar coordinate r is defined as length of vector $\vec{r}$, i.e. $r = |\vec{r}|$.

→ polar coordinate θ is defined as the angle between $\vec{r}$ and x-axis. θ is measured +ve, away and in counterclockwise direction from x-axis.

Note: → Here the position vector $\vec{r}$ represents a generic vector. Like any vector, $\vec{r}$ has an existence INDEPENDENT of any coordinate system. Letter r in vector $\vec{r}$ has no affiliation whatsoever to any coordinate system.

→ Cartesian components of a vector $\vec{r}\,(= x\hat{i} + y\hat{j})$ are obtained by taking projections of $\vec{r}$ on x and y axis.

CARTESIAN SYSTEM

$$\vec{r} = x\hat{i} + y\hat{j}$$

$$\vec{V} = \frac{d\vec{r}}{dt} = \underbrace{\dot{x}}_{V_x}\hat{i} + \underbrace{\dot{y}}_{V_y}\hat{j} + \underbrace{x\dot{\hat{i}} + y\dot{\hat{j}}}_{=0}$$

$$\vec{a} = \frac{d\vec{V}}{dt} = \frac{d^2\vec{r}}{dt^2} = \underbrace{\ddot{x}}_{a_x}\hat{i} + \underbrace{\ddot{y}}_{a_y}\hat{j}$$

$\hat{i}$ and $\hat{j}$ being truly constant unit vectors, their differentiation yields zero, resulting in very simple expression for $\vec{v}$ and $\vec{a}$.

POLAR COORDINATES

Going by the analogy suggested by cartesian system, we might resolve a vector in polar coordinates as follows

$$\vec{r} = r\hat{r} + \theta\hat{\theta}$$

THIS IS HOWEVER WRONG! θ being dimensionless, the dimensions on two sides do not match. Note that, unit vectors being ratios of vectors and their magnitudes, are by definition, dimensionless in every coordinate system.

THE CORRECT REPRESENTATION IS

$$\vec{r} = r\hat{r}$$

A vector afterall is uniquely specified by its magnitude and direction. Here $r = |\vec{r}|$ specifies its magnitude and $\hat{r}$ specifies direction of $\vec{r}$. If you are wondering where is θ dependence, then you must remember that the direction $\hat{r}$ is indeed a function of θ. That is

$$\vec{r} = r\hat{r}(\theta).$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\underbrace{\dot{\hat{r}}}_{\neq 0.}$$

since $\hat{r}$ is constant but points in different directions for different values of θ, $\dot{\hat{r}} \neq 0$. Let us find $\dot{\hat{r}}$ by two methods.

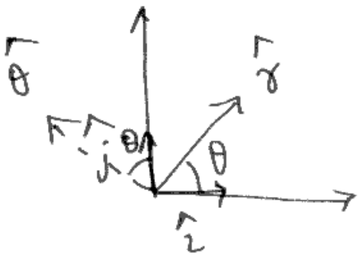
METHOD - 1.

Since $\hat{r}$ is of unit magnitude, the only way it can change in time is by pure rotation without scaling. We have already found the time derivative of such a vector and now we can borrow the result

$$\dot{\hat{r}} = |\hat{r}|\frac{d\theta}{dt}\hat{\theta}.$$ Thus,

$$\vec{v} = \dot{r}\hat{r} + r\underbrace{|\hat{r}|}_{1}\dot{\theta}\hat{\theta} = \underbrace{\dot{r}}_{v_r}\hat{r} + \underbrace{r\dot{\theta}}_{v_\theta}\hat{\theta}.$$

METHOD - 2



$$\hat{r} = \cos\theta\, \hat{i} + \sin\theta\, \hat{j}$$

$$\dot{\hat{r}} = \dot\theta\,(-\sin\theta\, \hat{i} + \cos\theta\, \hat{j})$$

$$\boxed{\dot{\hat{r}} = \dot\theta\, \hat\theta}$$ Thus

$$\vec{V} = \underbrace{\dot r}_{V_r}\,\hat r + \underbrace{r\dot\theta}_{V_\theta}\,\hat\theta$$

Now,

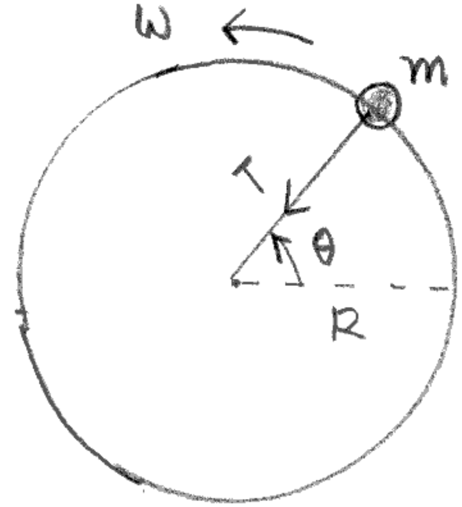
$$\vec{a} = \frac{d\vec V}{dt} = \ddot r\, \hat r + \dot r\, \dot{\hat r} + \dot r\dot\theta\, \hat\theta + r\ddot\theta\, \hat\theta + r\dot\theta\, \dot{\hat\theta}$$

We need to find $\dot{\hat\theta}$ and then substitute for $\dot{\hat r}$, $\dot{\hat\theta}$, and then collect the coefficients of $\hat r$ and $\hat\theta$ to identify a_r and a_θ. Following method 1 or 2 above, we find

$$\boxed{\dot{\hat\theta} = -\dot\theta\, \hat r}$$ Thus,

$$\vec{a} = \underbrace{(\ddot r - r\dot\theta^2)}_{a_r}\,\hat r + \underbrace{(r\ddot\theta + 2\dot r\dot\theta)}_{a_\theta}\,\hat\theta .$$

Example 2.5 Block on a string in absence of gravity.



Block of mass m is tied to a massless inextensible string of and rotated in a circular path of radius R. What is the tension T?

Sol. We all know the answer: $T = \frac{mv^2}{R}$ and is directed radially inwards, $v = R\omega$. Let us understand it in the context of polar coordinates:

1) Circular geometry ⇒ polar coordinates are more suitable.
2) Identify all the forces and resolve them in radial and tangential.
3) Component in radially outward (inward) is positive (negative). Similarly counterclockwise (clockwise) is +ve (-ve).

3) Do not touch the signs of accelarations.

Thus, $F_r \hat{r} = m a_r \hat{r}$

$F_\theta \hat{\theta} = m a_\theta \hat{\theta}$.

radial| $\underbrace{-T}_{\substack{\text{radially}\\ \text{inwards}\\ \text{hence -ve}}} \hat{r} = m\underbrace{(\ddot{r} - r\dot{\theta}^2)}_{a_r}\hat{r}$

do not touch any sign here

Since $r = R \Rightarrow \dot{r} = 0,\ \ddot{r} = 0,\ \dot{\theta} = \omega$

$$\boxed{T = mR\omega^2 = \frac{mv^2}{R}} \quad \because v = R\omega$$

tangential| $\underbrace{0}_{\substack{\text{No force}\\ \text{in } \hat{\theta}}} = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$

$$\Rightarrow R\ddot{\theta} = -2\underbrace{\dot{r}}_{\to 0}\dot{\theta}$$

$$\Rightarrow \ddot{\theta} = \frac{d\omega}{dt} = 0 \Rightarrow \omega = \text{const}.$$

Remarks:

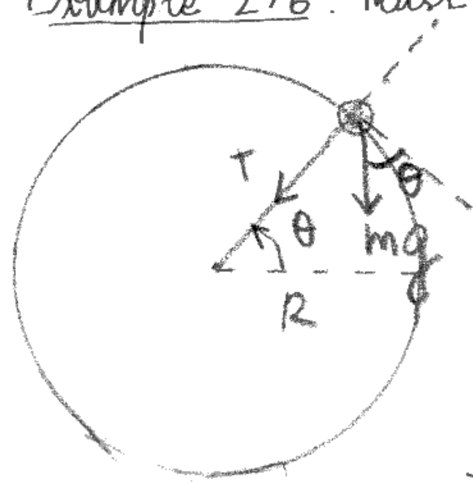
→ How did we use $V = R\omega$?
General expression for $\vec{V}$ in polar coordinates

$$\vec{V} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

Since $\dot{r} = 0$ here $|\vec{V}| = R\dot{\theta} = R\omega$.

→ In your class XII when you used centripetal accelaration to be v^2/R you were implicitly using polar coordinates.

Example 2.6. Mass on a string under gravity.



$\hat{r}|\ (-T - mg\sin\theta)\hat{r} = m(\underbrace{\ddot{r}}_{\to 0} - r\dot{\theta}^2)\hat{r}$

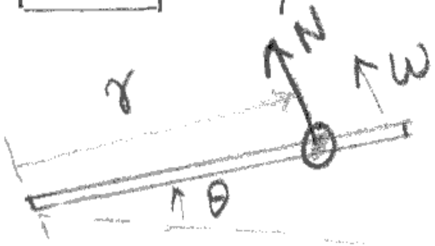
$\hat{\theta}|\ -mg\cos\theta\,\hat{\theta} = m(r\ddot{\theta} + 2\underbrace{\dot{r}}_{\to 0}\dot{\theta})\hat{\theta}$

Please note the signs of the components of forces along radial and tangential directions.

$T = mR\omega^2 - mg\sin\theta$

Since T can never be -ve, $mR\omega^2$ must always be greater than mg (maximum value of $\sin\theta = 1$). When this condition fails $\dot{r} \neq 0$

2.33 A particle of mass m is free to slide on a frictionless thin rod. The rod rotates in a plane about one end at constant angular speed ω. Show that the motion is given by $r = Ae^{-\beta t} + Be^{+\beta t}$, where β is a constant which you must find, and A and B are arbitrary constants. Neglect gravity. Show that for a particular choice of initial conditions (that is $r(t=0)$ and $\dot{r}(t=0)$), it is possible to obtain a solution such that r decreases continually in time, but that for any other choice r will eventually increase.



Solution: Note: This problem as well as the next problem beautifully illustrate some of the peculiarities of polar coordinates

→ Since the rod is frictionless, there is no force ~~in~~ at all in the radial direction. The particle is ofcourse subject to normal reaction due to rod but it is in tangential direction and has no component in the radial direction.

MYSTERY: Despite the absence of radial force, the centripetal accelaration is non-zero.

$\hat{r}$] $\quad 0 = m\underbrace{(\ddot{r} - r\dot{\theta}^2)}_{a_r} \Rightarrow a_r = 0.$

$\hat{\theta}$] $\quad N\hat{\theta} = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$, $\quad \ddot{\theta} \to 0$

Let us solve radial equation to find $r(t)$.

$$\ddot{r} = r\dot{\theta}^2 \quad \text{here } \dot{\theta} = \omega = \text{const}$$
$$\text{Thus } \ddot{\theta} = 0$$

Hence $r(t)$ is that function whose second derivative is constant times itself. What we have is a second order ordinary differential equation whose general solution will have two arbitrary constants, to be determined by specific initial conditions.

$r = e^{\beta t}$ fits the bill. Since it must satisfy our d.e.,

$$\dot{r} = \beta e^{\beta t}$$

$$\ddot{r} = \beta^2 \underbrace{e^{\beta t}}_{r(t)} = \omega^2 r(t)$$

$$\Rightarrow \quad \beta = \pm \omega .$$

Thus, for a 2^{nd} order linear d.e. there are two linearly independent solutions. A general solution is linear superposition of two. Thus

$$r(t) = A e^{-\omega t} + B e^{\omega t} .$$

~~To~~ To determine A and B, we need two equations which are obtained by specific initial conditions stating position and velocity at $t=0$.

Say $r(t=0) = r_0$ $v(t=0) = v_0$. Thus,

$$r(t) = A e^{-\omega t} + B e^{\omega t} \Rightarrow r_0 = A + B .$$

$$\dot{r}(t) = -A\omega e^{-\omega t} + B\omega e^{\omega t} \Rightarrow v_0 = \omega(B - A) .$$

Thus

$$\omega r_0 = (A+B)\omega$$

$$v_0 = (-A+B)\omega .$$

Adding, we get $B = \dfrac{\omega r_0 + v_0}{2\omega}$

Subtracting, we get $A = \dfrac{\omega r_0 - v_0}{2\omega}$.

Thus,

$$r(t) = \left(\frac{\omega r_0 - v_0}{2\omega}\right) e^{-\omega t} + \left(\frac{\omega r_0 + v_0}{2\omega}\right) e^{\omega t} .$$

For r to constantly decrease in time, dr/dt should be -ve.

$$\frac{dr}{dt} = -\frac{(\omega r_0 - v_0)}{2} e^{-\omega t} + \frac{(\omega r_0 + v_0)}{2} e^{\omega t}$$

Thus for $\frac{dr}{dt} < 0$, $(\omega r_0 + v_0) < 0$.

THE MYSTERIOUS PART OF 2.33

The mystery stems from the expectation that since $F_r = m a_r$ and since $F_r = 0$, a_r should be zero and hence there should be no dynamics in radial direction. We are shocked that despite $F_r = 0$, both the radial terms, $\ddot{r} \neq 0$ and $r\dot{\theta}^2 \neq 0$.

RESOLUTION: Problem lies with our intuition that borrows heavily from our cartesian coordinate experience. There, ~~if~~ $F_x = 0 \Rightarrow a_x = 0 \Rightarrow \ddot{x} = 0$, because $a_x = \ddot{x}$.

In polar coordinates, $a_r \neq \ddot{r}$, rather

$$a_r = \ddot{r} - r\dot{\theta}^2.$$

Thus, $F_r = m a_r$, sure $\Rightarrow a_r = 0$, but a_r can be zero in two ways

1) $\ddot{r} = 0$, $r\dot{\theta}^2 \Rightarrow$ Trivial ~~dy~~ case as no dynamics

2) $\ddot{r} = r\dot{\theta}^2 \Rightarrow$ Non-trivial dynamics

so let us try to understand how $\ddot{r} \neq 0$ and $r\dot{\theta}^2 \neq 0$ despite $F_r = 0$.

My contention is that only way to logically accomodate the observed fact that $\dot{\theta} \neq 0$ and $F_\theta = N \neq 0$ is to have $\dot{r} \neq 0$, $r\dot{\theta}^2 \neq 0$ and $\ddot{r} \neq 0$.

→ Suppose at $t = 0$, the position vector of ~~pos~~ the particle is

$$\vec{r} = r\hat{r}$$

Some undeniable observations

1) The mass is compelled to rotate with the rod hence
   - → $\dot{\theta} \neq 0$
   - → $\hat{r}$ is changing direction thus $\dot{\hat{r}} = \dot{\theta}\hat{\theta} \neq 0$
   - → The only force on the mass is normal reaction: $F_\theta = N$.

Now $\vec{V} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$.

Naively, since $F_r = 0$ we do not expect any motion in radial direction so let us ~~take~~ put $\dot{r} = 0$, and where does it lead us.

so $\vec{v} = r\dot{\theta}\hat{\theta}$

$$\vec{a} = \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\dot{\hat{\theta}}$$

$$= \underbrace{-r\dot{\theta}^2}_{a_r}\hat{\theta} + \underbrace{(\dot{r}\dot{\theta} + r\ddot{\theta})}_{a_\theta}\hat{\theta}$$

$\dot{r}\dot{\theta} \to 0$ (we assumed). $r\ddot{\theta} \to 0$ ($\because \dot{\theta} = \omega = \text{const.}$)

Thus, we ~~we~~ have $a_r = -r\dot{\theta}^2 \neq 0$ and ~~$a_\theta = \dot{r}\dot{\theta} \neq$~~ $a_\theta = 0$.

Now $F_r = m a_r$ hence $F_r \neq 0$.

But there is no physical agency (since $\mu = 0$ and $g = 0$) that can provide $F_r \neq 0$. Hence, we reach a farcical conclusion that

$$0 = -mr\dot{\theta}^2.$$

The only way to save this is to assume that the other term in radial accelaration $\ddot{r} \neq 0$, so that we can have

$$0 = \ddot{r} - r\dot{\theta}^2 = 0 \text{. Makes sense!}$$

However, just a while ago that $\dot{r} = 0$ when wrote down $\vec{v}$. Now we are forced to admit that $\dot{r} \neq 0$ since $\ddot{r} \neq 0$.

Thus, the correct expression for $\vec{v}$ is

$$\vec{V} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

But there may still be skeptics amongst you who are not willing to accept $\dot{r} \neq 0$ since $F_r = 0$. Here is another arguement for them. We just now found that

$$a_\theta = \dot{r}\dot{\theta} + \cancel{2}\, r\ddot{\theta}$$

$\dot{r}\dot{\theta}$: you are forcing to be zero. $r\ddot{\theta} \to 0$ $\omega = \text{const}$

Thus $a_\theta = 0 \Rightarrow F_\theta = m a_\theta = 0$.

But we agreed that there is non-zero normal reaction $N = F_\theta$ providing tangential force and compelling the particle to move with the rod. The only term in a_θ, that can provide for $N \neq 0$ is $\dot{r}\dot{\theta}$ term as $\ddot{\theta} = 0$. Thus $\dot{r} \neq 0$.

Hence $\vec{V} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$

$$\bar{a} = \underbrace{(\ddot{r} - r\dot{\theta}^2)}_{\to 0}\hat{r} + (\underbrace{2\dot{r}\dot{\theta}}_{N/m} + \underbrace{r\ddot{\theta}}_{\to 0})\hat{\theta}.$$

ANGULAR MOMENTUM PERSPECTIVES ON 2.33.

→ Mass m is at a distance r from the pivot of the rod and has non-vanishing tangential force N. Thus, it is acted upon by a torque

$$\tau = r(t) N = \frac{dL}{dt} \qquad L = \text{Angular momentum}, \quad L = I\omega.$$

Naively, we would tend to equate

$$\tau = I\ddot{\theta} \qquad I = mr^2.$$

but $\dot{\theta} = \text{constant} \Rightarrow \ddot{\theta} = 0 \Rightarrow \tau = 0$.

But $L = I\omega$

$$\tau = \frac{dL}{dt} = \dot{I}\omega + \underbrace{\dot{\omega} I}_{\to 0}$$

$$\tau = \frac{d}{dt}(mr^2)\,\omega$$

$$\tau = 2mr\dot{r}\omega = rN$$

~~But~~ $\Rightarrow N = 2m\dot{r}\omega$.

Now you understand that naively putting $\dot{r} = 0 \Rightarrow N = 0 \Rightarrow \tau = 0 \Rightarrow L = \text{const} \Rightarrow r = \text{const} \Rightarrow r\dot{\theta}^2$ is nonzero despite $F_r = 0 = \ddot{r}$.

UPSHOT is once you have a rotating $\hat{r}$, $\Rightarrow \dot{\hat{r}} \propto \hat{\theta}$ and $\dot{\hat{\theta}} \propto \hat{r}$. Thus the time derivative of unit vectors feed the dynamics from $\hat{\theta}$ ($N\hat{\theta} \neq 0$) into $\hat{r}$ ($\ddot{r} \neq 0, r\dot{\theta}^2 \neq 0$). Can we have a situation in which real forces in ~~dyar~~ direction ~~$\hat{\theta}$~~ feed the dynamics into $\hat{\theta}$ direction? Yes, this is precisely the situation in problem 2.34. That is our next problem. But before that

2.33 FROM ~~ROTA~~ THE PERSPECTIVES OF NON INERTIAL FRAME

In the frame of someone rotating with the rod, the particle is not rotating at all. However there is pseudo-force $mr\dot{\theta}^2$ in radially outward direction. Thus, equation of motion is

$$mr\dot{\theta}^2 = ma = m\ddot{r} \Rightarrow \ddot{r} = r\dot{\theta}^2$$

Isn't this what we got from inertial frame perspective.

2.33 : A PHYSICAL PERSPECTIVE

aka, How to reset a mercury thermomter or drain clean a garden hose :

The mystefying crux of 2.33 was radial motion without radial motion. We confront a similar physical situation when we want to get rid of water from the garden hose or wants the mercury to reset to normal level after use. How do we do it? Not by shaking the tube longitudinally but by whirling the tube in a circular arc. Consider a water droplet on the frictionless inside wall of a standard garden hose. To expel it we whirl it in a arc as shown.

The water drop which was initially at A, under the influence of the normal force of the movement of hose, travels tangentially and reaches point C. Now, from the perspective of an inertial observer, it has rotated by an angle $\Delta\theta$, as well as gone radially outward by a distance $\Delta r = BC$. As viewed from the rotating frame of hose, it has only moved radially outward by $\Delta r = BC$.

Question: What is accelaration of water in rotating frame as it goes distance BC.

$$BC = OC - OB. \quad (OA = OB = r).$$

OAC being a right triangle

$$BC = r\sec\Delta\theta - r.$$

$$\text{Now } \sec\Delta\theta = (\cos\Delta\theta)^{-1} \approx \left[1 - \tfrac{1}{2}(\Delta\theta)^2\right]^{-1}$$

$$\approx 1 + \tfrac{1}{2}(\Delta\theta)^2$$

$$\Rightarrow BC \approx \frac{1}{2} r(\Delta\theta)^2. \qquad \Delta\theta = \omega\Delta t$$

$$BC \doteq \frac{1}{2} r\omega^2(\Delta t)^2.$$

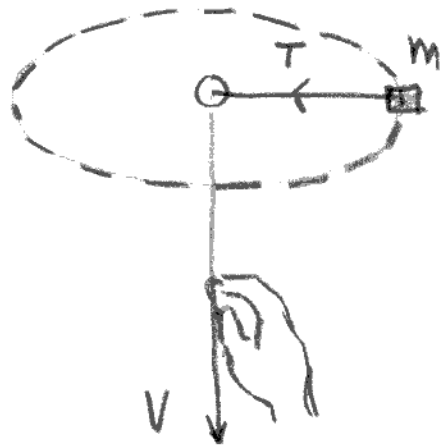
Comparing with $s = \frac{1}{2}at^2 \Rightarrow a = r\omega^2$.

Thus in rotating frame $a = \ddot{r} = r\omega^2$.

In an inertial frame: $\ddot{r} - r\omega^2 = 0$ as expected.

Lesson: If you strip a phenomena of superficial differences, underlying physics may be same.

2.34 Mass m whirls around on a string which passes through a ring as shown. Neglect gravity.



Initially the mass is at a distance r_0 and is revolving at a angular velocity ω_0. The string is pulled at a constant velocity V starting at $t=0$ so that the radial distance to the mass decreases. Draw a force diagram and obtain a differential equation for ω. Find a) $\omega(t)$, b) The force needed to pull the string.

sol: r] $\quad -T\hat{r} = m(\ddot{r} - r\dot{\theta}^2)\hat{r}$

Since the string is being pulled with constant speed V,

$$\frac{dr}{dt} = -V \Rightarrow r = r_0 - Vt$$
$$\ddot{r} = 0.$$

Thus, $T = m r(t)\omega^2$

The force needed to pull the string is this tension T. Is T increasing or decreasing with time? Since r is decreasing it seems to suggest that T is decreasing. But since we are pulling T must be increasing with time. This means, not only is $\omega(t)$ a function of t, but it is sufficiently increasing function of time, to ensure that T is rising despite decrease in time.

MYSTERY: It is clear that there is no force in tangential direction we ~~wo~~ 'd expect tangential acceleration to be zero. But we just concluded that $\dot{\omega} = \ddot{\theta} > 0$. How come we succeed in giving tangential acceleration by pulling radially inwards? This problem is an anti-thesis of 2.33, where $\ddot{r} \neq 0$ despite $F_r = 0$. Here $\ddot{\theta} \neq 0$, $F_\theta = 0$.

The resolution of mystery is also similar.

$F_\theta = 0 \implies a_\theta = 0$.

But $a_\theta = 0$ in two ways

1) Both $r\ddot{\theta} = 0$ and $\dot{r}\dot{\theta} = 0$
This is a trivial case with no dynamics

2) $r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$ Non-trivial dynamics
$\implies r\ddot{\theta} = -2\dot{r}\dot{\theta}$

The dynamics of radial equation already suggested the need for differential equation for ω. This is provided by tangential equation. Thus,

$$F_\theta = 0 = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$$

$$\text{or} \quad r\frac{d\omega}{dt} = -2\dot{r}\omega \qquad \left[\frac{dr}{dt} = -V\right]$$

$$\int_{\omega(r_0)}^{\omega(r)} \frac{d\omega}{\omega} = -2\int_{r_0}^{r} \frac{dr'}{r'} \qquad \left[dt = -\frac{dr}{V}\right.$$

$$\ln\left[\frac{\omega(r)}{\omega(r_0)}\right] = -2\ln\left[\frac{r(t)}{r_0}\right]$$

$$\omega(r) = \cancel{\frac{\omega(r_0)\, r^2(t)}{r_0^2}} = \frac{\omega(r_0)\, r_0^2}{r^2(t)}$$

$$\boxed{\omega(t) = \cancel{\frac{\omega(r_0)(r_0 - Vt)^2}{r_0^2}} \quad \boxed{\frac{\omega(r_0)\, r_0^2}{(r_0 - Vt)^2}}}$$

$$T(t) = m\, r(t)\, \omega^2(t) = \boxed{\frac{m\omega_0^2 r_0^4}{(r_0 - Vt)^3}}$$

$$= \cancel{\frac{m\omega^2(r_0)(r_0 - Vt)^5}{r_0^4}}$$

<u>Note</u>: The same result could have been obtained in two steps using principle of conservation of angular momentum (since the only force T passes through origin, has no moment arm, and hence cannot exert torque). Conservation laws, though powerful, hide a lot of details which are revealed by dynamical method. Both are crucial for complete understanding.