MATH F113
Probability and Statistics

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Chapter # 5

Joint Distributions

Single Random Variables: (Univariate)

Discrete

Continuous

Two Dimensional Random variables (Bivariate)

Discrete

Continuous

Bivariate distribution occurs when we observe 2 nondeterministic quantities, one followed by another.

For example :

1) Record the atmospheric temperature T in celsius followed by atmospheric pressure P in pounds per square foot at a random place and time. This gives 2 dimensional r.v. (T, P).

2) For a randomly chosen student, record number of A grades he has received followed by the number of his B grades in the last semester.

n-dimensional random variable $\leftrightarrow$ we observe n nondeterministic quantities in sequence.

<u>Discrete Joint Density:</u>

Let X and Y be discrete r.v, the ordered pair (X,Y) is called a two dimensional discrete r.v, A function

$$f_{XY}(x, y) = P[X = x \; and \; Y = y]$$

is called the joint density for (X,Y) and is a probability density/probability mass function for two random variables.

Necessary and Sufficient Conditions:

Discrete Case:

$$1.\ f_{XY}(x,y) \geq 0 \ \forall (x,y) \in R^2$$

$$2. \sum_{all\ x} \sum_{all\ Y} f_{XY}(x,y) = 1$$

Continuous Case:

$$1.\ f_{XY}(x,y) \geq 0\ \forall (x,y) \in R^2$$

$$2.\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\ dy\ dx = 1$$

$$3.\ P[a \leq X \leq b \ and \ c \leq Y \leq d]$$

$$= \int_a^b \int_c^d f_{XY}(x,y)\ dy\ dx,$$

for $a \leq b, c \leq d$ real is called the joint density for (X,Y).

Note: In one dimensional continuous case, the probabilities correspond to areas under density curve while in the case of 2-D, they corresponds to volumes under density surfaces.

n-dimensional random variables

- The concept can be extended in an analogous manner to n dimensions.
- The joint density $f_{X_1 \ldots X_n}(x_1, \ldots, x_n)$ is a function of n random variables.
- Replicate the definition.

Example: In an automobile plant two tasks are performed by robots. The first entails welding two joints; the second, tightening 3 bolts. Let **X** denote the number of defective welds and **Y** the number of improperly tightened bolts produced per car. Since X and Y are each discrete, (X,Y) is a 2- dimensional discrete random variable.

Past data indicates that the joint density for (X,Y) is shown in Table. Note that each entry in the table is a number between 0 and 1 and therefore can be interpreted as a probability.

x \ y	0	1	2	3
0	0.840	0.030	0.020	0.010
1	0.060	0.010	0.008	0.002
2	0.010	0.005	0.004	0.001

Note : Sum of all entries =1

We see that:

$$\sum_{all\ x}\sum_{all\ y} f_{XY}(x,y)$$

$$= 0.840 + 0.030 + 0.020 + \ldots + 0.001 = 1$$

The probability that there will be no errors made by the robots is given by:

$$P[X = 0\ and\ Y = 0] = f_{XY}(0,0) = 0.840$$

The probability that there will be exactly one error made is:

$$P[X = 1 \ and \ \ Y = 0] + P[X = 0 \ and \ Y = 1]$$

$$= f_{XY}(1,0) + f_{XY}(0,1)$$

$$= 0.060 + 0.030 = 0.09$$

Marginal Distributions:

- Density of Y alone: Sum the joint densities over all values of X

- Density of X alone: Sum the joint densities over all values of Y

Marginal density:

Let (X,Y) be a two dimensional discrete random variable with joint density f_{XY}

The marginal density for X is defined to be

$$f_X(x) = \sum_{all\ y} f_{XY}(x, y)$$

The marginal density for Y, defined to be

$$f_Y(y) = \sum_{all\ x} f_{XY}(x, y)$$

x \ y	0	1	2	3	row sums $f_X(x)$
0	0.840	0.030	0.020	0.010	0.900
1	0.060	0.010	0.008	0.002	0.080
2	0.010	0.005	0.004	0.001	0.020
col. sums $f_Y(y)$	0.910	0.045	0.032	0.013	1

Continuous Case:

Let (X,Y) be a two dimensional continuous random variable with joint density f_{XY}.

The marginal density for X, denoted by f_X is

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dy$$

Similarly define f_Y, the marginal density for Y.

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dx$$

Verify f_X, f_Y are densities.

Characterizing properties of densities?

Independent Random Variables:

Def: Let X and Y be random variables with joint density f_{XY} and marginal densities f_X and f_Y, respectively. X and Y are independent if and only if

$$f_{XY}(x, y) = f_X(x)\ f_Y(y)$$

$$for\ \ all\ \ x\ \ and\ \ y.$$

In general for n random variables,

$X_1,\ldots,X_n$ are indep iff

$$f_{X_1 \ldots X_n}(x_1,\ldots,x_n) = \prod_{i=1}^{n} f_{X_i}(x_i)$$

Here $f_{X_1 \ldots X_n}(x_1,\ldots,x_n)$ is the joint density

and $f_{X_i}(x_i)$ are the marginal densities of X_i.

Exercise 5.1:

3. The joint density for (X,Y) is given by:

$$f_{XY}(x,y) = \frac{1}{n^2}, x = 1,2,3,..n$$

$$y = 1,2,3,..n$$

and $f_{XY}(x,y) = 0$ elsewhere.

(a) Verify that $f_{XY}(x,y)$ satisfies the conditions necessary to be a density.

(b) Find the marginal densities for X and Y

(c) Are X and Y independent?

Sol:

4. The joint density for (X,Y) is given by

$$f_{XY}(x,y) = \frac{2}{n(n+1)}, 1 \le y \le x \le n$$

x, y **integers** $and\ n$ is a positive integer

(a) Verify that $f_{XY}(x,y)$ satisfies the conditions necessary to be a density.

(b) Find the marginal densities for X and Y

(c) Are X and Y independent?

(d) Assume that n = 5, use the density to find

$$P[X \le 3 \ \text{ and } \ Y \le 2]$$

Find also

$$P[X \le 3] \ \text{ and } \ P[Y \le 2]$$

(9) An engineer is studying early morning traffic pattern at a particular intersection. The observation period begins at 5.30 a.m. Let **X** denote the time of arrival of the first vehicle from the north – south direction. Let **Y** denote the first arrival time from the the east – west direction.

Time is measured in fractions of an hour after 5.30 a.m. Assume that density for (X,Y) is given by

$$f_{XY}(x,y) = \frac{1}{x},\ 0 < y < x < 1$$

(a)Verify that this is a valid density for a two dimensional random variable.

(b) $Find \;\; P[X \leq 0.5 \; and \; Y \leq 0.25]$

$$(d)\ Find\ \ P[X > 0.5\ and\ Y \geq 0.5]$$

$$= \int_{0.5}^{1}\int_{y}^{1}\frac{1}{x}\,dx\,dy = 0.15345$$

Example: Suppose that the two dimensional continuous random variable (X,Y) has joint pdf given by:

$$f(x,y)=\begin{cases} x^2+\dfrac{xy}{3}, 0\le x\le 1, 0\le y\le 2 \\ 0, \ \ elsewhere \end{cases}$$

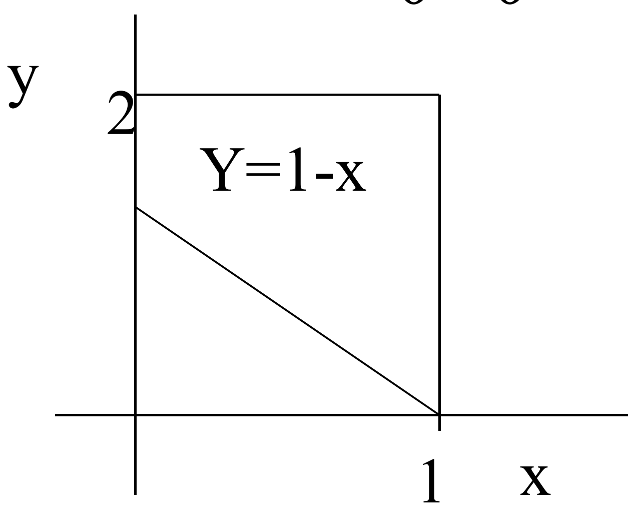
$$\text{Check that } \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)$$

$$= \int_{0}^{2}\int_{0}^{1} \left(x^2 + \frac{xy}{3}\right) dxdy$$

$$= 1$$

Let $B=\{X+Y\geq 1\}$, compute $P(B)$.

$$P(B) = 1 - \int_0^1 \int_0^{1-x} \left(x^2 + \frac{xy}{3}\right) dy\, dx = \frac{65}{72}$$



$P[B] = 1 - P(B')$, where

$B' = (X+Y<1\}$, Hence

Cumulative distribution function

Def: Let (X,Y) be a two dimensional random variable. The cdf **F** of the two dimensional random variable (X,Y) is defined by:

$$F(x, y) = P[X \le x \ and \ Y \le y]$$

If F is the cdf of a two dimensional random variable with joint density, f, then

$$\frac{\partial^2}{\partial x\, \partial y} F(x, y) = f(x, y), \text{ wherever}$$

$$F \text{ is differentiable in case of continuous case.}$$

Ex. 12 : Items coming off an assembly line are classified as being either non-defective, defective but salvageable, or defective but non salvageable. The probabilities of observing items in each of these categories are 0.9, 0.08 and 0.02 respectively. The probabilities do not change from trial to trial. Twenty items are randomly selected and classified. Let X_1 denote the number of non-defective items, X_2 the number of non-defective but salvageable and

X_3 the number of non-defective but non-salvageable items obtained.

(a)Find $P[X_1=15, X_2=3, X_3=2]$.

(b)Find the general formula for joint density of (X_1, X_2, X_3).

$$\boldsymbol{Soln}:(a)\frac{20!}{(15!)(3!)(2!)}0.9^{15}0.08^{3}0.02^{2}$$

$$(b)\frac{n!}{n_1!n_2!n_3!}p_1^{n_1}p_2^{n_2}p_3^{n_3}$$

(This is called multinomial distribution).

Expectation and Covariance

Def: Let (X,Y) be a 2-D r.v with joint density f_{XY}. Let H(X,Y) be a real valued function of (X, Y).

The expected value of H(X,Y), denoted by E[H(X,Y)] is given by:

$$E[H(X,Y)] = \sum_{all\ x} \sum_{all\ y} H(x,y)\ f_{XY}(x,y)$$

provided

$$\sum_{all\ x} \sum_{all\ y} |H(x,y)|\ f_{XY}(x,y) \quad \text{converges.}$$

For (X,Y) continuous

$$E[H(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x,y)\ f_{XY}(x,y)dxdy$$

provided

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\,H(x,y)\,|\ f_{XY}(x,y)\ dy\ dx \text{ exists.}$$

Variance

- Var[H(X,Y)]=E[(H(X,Y)-E[H(X,Y)])^2] if all expectations considered exist.
- Recall : E[aX+bY+c]=aE[X]+bE[Y]+c.
- More generally, for n-dimensional r.v.

$$E[a_0 + a_1 X_1 + \ldots + a_n X_n]$$
$$= a_0 + a_1 E[X_1] + \ldots + a_n E[X_n]$$

Univariate Average found via the joint density

$$E[X] = \sum_{all\ x} \sum_{all\ y} x \ \ f_{XY}(x, y) \quad for\,(X, Y)\ discrete$$

$$E[Y] = \sum_{all\ x} \sum_{all\ y} y \ \ f_{XY}(x, y) \quad for\,(X, Y)\ discrete$$

$$E[X] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\, f_{XY}(x, y) \ \ dy\, dx$$

$$E[Y] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y\, f_{XY}(x, y) \ \ dx\, dy$$

$$for \ \ (X, Y)\ continuous$$

Covariance

Def: Let X and Y be random variables with means μ_X and μ_Y respectively. The Covariance between X and Y, denoted by Cov(X,Y) or σ_{XY} is given by:

$$\mathrm{Cov}(X, Y) = E[(X - \mu_x)(Y - \mu_Y)]$$

Remark: If small values of X tend to be associated with small values of Y and large values of X with large values of Y, Then $(X-\mu_X)(Y-\mu_Y)$ will be positive, yields a positive covariance and the reverse is true, that is small values of X tend to associated with large values of Y and vice versa, then $(X-\mu_X)(Y-\mu_Y)<0$ yields a negative covariance. In essence covariance is an indication of how X and Y are related.

Theorem: $\text{Cov}(X,Y) = E[XY] - E[X]\, E[Y]$

Theorem: Let (X,Y) be a two dimensional random variable with joint density f_{XY}. If X and Y are independent then

$$E[XY] = E[X]\, E[Y].$$

Thus if X, Y are independent, then Cov(X,Y)=0.

Proof: We shall prove for a continuous case (of course, discrete case is similar)

Remark: An immediate consequence of this theorem is that, if X and Y are independent, then Cov(X,Y) =0. But the **Converse is not true**: that is, ` zero covariance implies independence' is **not true**..

EX : $f(x,y) = 1$ if $|x|<1$ and $|y|<|x|/2$
$f(x,y)=0$ otherwise.
Show : Cov(X,Y) = 0 but X, Y are not indep.

Note : If $X_i, i = 1,...,n$ are random va

and $a_i, i = 0,1...,n$ are constants then

$$Var\left[a_0 + \sum_{i=1}^{n} a_i X_i\right]$$

$$= \sum_{i=1}^{n} a_i^2 Var[X_i] + 2\sum_{i<j} a_i a_j Cov[X_i, X_j]$$

In particular, if $X_1,..., X_n$ are independent, then

$$Var\left[a_0 + \sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} a_i^2 Var[X_i]$$

(25) Prove that $Var(X+Y) = Var(X) + Var(Y) + 2\, Cov(X,Y)$

Proof:

(27) Show that if X=Y, then

Cov(X,Y) =Var(X) = Var(Y)

Proof:

Exercise 5.2: (Page No: 184)

19. The joint density for (X,Y) , where X is the inside and Y is the outside Barometric pressure on an air support roof

is given by:

$$f_{XY}(x, y) = \frac{c}{x},$$

$$27 \le y \le x \le 33.$$

a) Find c.

$$c = \frac{1}{(6 - 27 \ln 33/27)} = 1.72$$

(b) Find E[X-Y].

Conditional Densities:

- The conditional density for X given Y=y denoted by $f_{X|y}$, For discrete joint density,

$$f_{X|y}(x) = \frac{P[X = x \text{ and } Y = y]}{P[Y = y]}$$

$$= \frac{f_{XY}(x, y)}{f_Y(y)}, \text{provided } f_Y(y) \neq 0.$$

Def: Let (X,Y) be a two dimensional random variable (discrete or continuous) with joint density f_{XY} and marginal densities f_X and f_Y. Then

1. The conditional density for X given Y= y, denoted by $f_{X|y}$ is given by:

$$f_{X|y}(x) = \frac{f_{XY}(x,y)}{f_Y(y)},\ f_Y(y) > 0$$

Note: The conditional densities satisfies all the requirements for a one dimensional pdf. Thus for a fixed y with $f_Y(y) \neq 0$, we have $f_{X|y}(x) \geq 0$ for all x and

$$\int_{-\infty}^{\infty} f_{X|y}(x)\,dx = \int_{-\infty}^{\infty} \frac{f(x,y)}{f_Y(y)}\,dx$$

$$= \frac{1}{f_Y(y)} \int_{-\infty}^{\infty} f(x,y)\,dx = \frac{f_Y(y)}{f_Y(y)} = 1$$

Example: Suppose that the two dimensional continuous random variable (X,Y) has joint pdf given by:

$$f(x,y) = x^2 + \frac{xy}{3}, 0 \le x \le 1, 0 \le y \le 2$$

$$= 0, \quad elsewhere$$

The marginal density of X is:

2. The conditional density for Y given

given X= x, denoted by $f_{Y|x}$ is given by:

$$f_{Y|x}(y) = \frac{f_{XY}(x,y)}{f_X(x)},\ f_X(x) > 0$$

Exercise 5.4: (Page No: 188)

54. An electronic device is designed to switch house lights on and off at random times after it has that the device is designed in such a way that it will be switched on and off exactly once in a 1- hour period.

Let Y denote the time at which the lights are turned on and X the time at which they are turned off been activated.
Assume that the joint density for (X,Y) is given by

$$f_{XY}(x,y)=8xy, \quad 0<y<x<1$$

(g) Find the conditional distribution of X given Y

(h) Find the probability that the lights will be switched off within 45 minutes of the system being activated given that they were switched on 10 minutes after the system was activated.

(j) Find the expected time that the lights will be switched off given that they were switched on 10 minutes after the system was activated.

56.Let X denote the number of “do loops” in a Fortran program and Y the number of runs needed to debug the program. Assume that the joint density for (X,Y) is given as:

x/y	1	2	3	4
0	0.059	0.100	0.050	0.001
1	0.093	0.120	0.082	0.003
2	0.065	0.102	0.100	0.010
3	0.050	0.075	0.070	0.020

(d) Find the probability that a randomly selected program requires at least two runs to debug given that it contains exactly one "do loop"

x/y	1	2	3	4	f_x
0	0.059	0.100	0.050	0.001	0.210
1	0.093	0.120	0.082	0.003	0.298
2	0.065	0.102	0.100	0.010	0.277
3	0.050	0.075	0.070	0.020	0.215
f_y	0.267	0.397	0.302	0.034	1.00

Note: If X, Y are independent then $f_{Y|x}(y) = f_Y(y)$ for all X

Example: The joint density of X and Y is given by:

$$f(x,y)=\begin{cases}\frac{1}{2}ye^{-xy}, & 0<x<\infty, 0<y<2\\ 0, & elsewhere\end{cases}$$

1) Find E[$X|y=1$]. (Also denoted by $\mu_{X|y=1}$. This is the mean of $f_{X|1}$)
2) Find P[X >2|y =1].