

Sec 21: Cauchy - Riemann Equations

Suppose that

$$f(z) = u(x, y) + iv(x, y)$$

and that $f'(z)$ exists at a

point $z_0 = x_0 + iy_0$

Then

- (i) the first - order partial derivatives u_x, u_y, v_x and v_y must exist at (x_0, y_0) ,
- (ii) they satisfy the CR - eqns
$$u_x = v_y, u_y = -v_x \quad \text{at } (x_0, y_0).$$

(iii)

$f'(z_0)$ can be written as

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0).$$

Proof

Since $f(z)$ is differentiable at z_0

$$\Rightarrow f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \dots\dots (1)$$

Note that $z = x + iy$, $z_0 = x_0 + iy_0$

$$\Delta z = \Delta x + i\Delta y$$

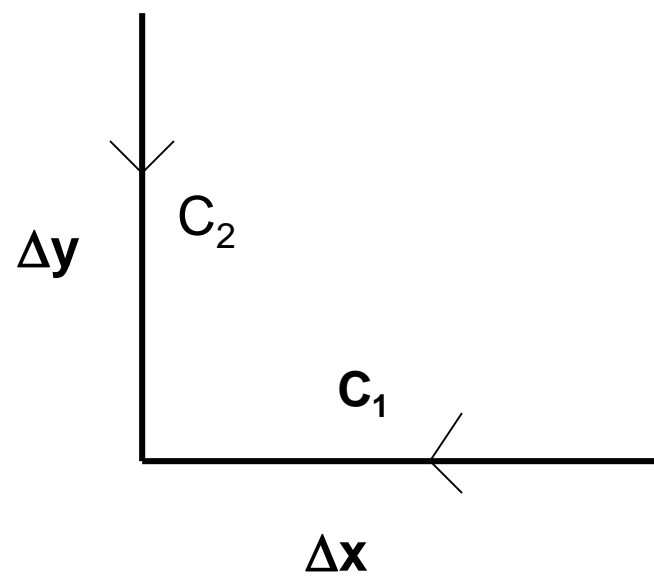
$$f(z_0) = u(x_0, y_0) + i v(x_0, y_0)$$

$$\begin{aligned} \Rightarrow f(z_0 + \Delta z) = & u(x_0 + \Delta x, y_0 + \Delta y) \\ & + i v(x_0 + \Delta x, y_0 + \Delta y) \end{aligned}$$

$\therefore \textit{Eq. (1) gives}$

$$f'(z_0) =$$

$$\lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} \left[\frac{u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)}{\Delta x + i\Delta y} + i \frac{v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)}{\Delta x + i\Delta y} \right]$$



$$f'(z_0) =$$

$$\left\{ \begin{array}{l} u_x(x_0, y_0) + i v_x(x_0, y_0), \\ \text{along the path } C_1 \\ -i u_y(x_0, y_0) + v_y(x_0, y_0), \\ \text{along the path } C_2 \end{array} \right.$$

$$\Rightarrow u_x = v_y, \quad u_y = -v_x \text{ at } (x_0, y_0),$$

$$\text{and } f'(z_0) = u_x + i v_x \text{ at } (x_0, y_0)$$

WHY ???

Sec 22. Sufficient conditions for differentiability

Let $f(z) = u(x, y) + i v(x, y)$ be any
function defined throughout in
some nbd. of the point

$z_0 = x_0 + iy_0$ such that

(i) u_x, u_y, v_x, v_y exist in that
nbd of z_0 ,

(ii) u_x, u_y, v_x, v_y are
continuous at (x_0, y_0)

(iii) the first order derivative s
satisfy the CR - equations

$$u_x = v_y, \quad u_y = -v_x \text{ at } (x_0, y_0).$$

Then $f'(z)$ exists at z_0 .

Cauchy - Reimann Equations in Polar Form

Let $f(z) = u(r, \theta) + i v(r, \theta)$

be differentiable at any given point

$$z_0 = r_0 e^{i\theta_0}.$$

Then the partial derivatives

$u_r, u_\theta, v_r, v_\theta$ exist at (r_0, θ_0)

and they satisfy

$$u_r = \frac{1}{r} v_\theta, \quad v_r = -\frac{1}{r} u_\theta$$

and

$$f'(z_0) = e^{-i\theta} \left(u_r + i v_r \right) \Big|_{(r_o, \theta_o)}$$

Example 1 : For the function

$$f(z) = z^2,$$

find out the points where the function is differentiable. Also find $f'(z)$

Consider

$$f(z) = z^2 = x^2 - y^2 + i2xy \equiv u + iv$$

$$\Rightarrow u(x, y) = x^2 - y^2, \quad v(x, y) = 2xy$$

$$\Rightarrow u_x = 2x, \quad u_y = -2y,$$

$$v_x = 2y, \quad v_y = 2x$$

$$\Rightarrow u_x = v_y \quad \& \quad u_y = -v_x$$

\Rightarrow (i) CR - equations are
satisfied for all x, y

(ii) u_x, u_y, v_x , and v_y are
continuous for all x, y

$\Rightarrow f(z) = z^2$ is differentiable
at any point z , and

$$f'(z) = u_x + iv_x = 2x + i2y = 2z$$

Example 2 : For the function

$$f(z) = |z|^2,$$

find out the points where the function is differentiable. Also find $f'(z)$

Consider $f(z) = |z|^2 = x^2 + y^2$

$$\Rightarrow u(x, y) = x^2 + y^2 \text{ \& } v(x, y) = 0$$

$$\Rightarrow u_x = 2x, u_y = 2y, v_x = 0, v_y = 0,$$

If CR - equations are satisfied,
then we must have

$$x = 0 = y.$$

$\Rightarrow f(z)$ is differentiable only at $(0,0)$
and nowhere else. Further

$$f'(0) = u_x(0,0) + iv_x(0,0) = 0$$

Page 72/Q.6 Let u & v denote the real & imaginary parts of the function f defined by

$$f(z) = \begin{cases} \frac{(\bar{z})^2}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

Show that CR - equations are
satisfied at $(0,0)$ although

f is NOT differentiable
at $(0,0)$.

Solution :

RECALL : f is not differentiable at $(0,0)$

(already done)

We have, when $z \neq 0$,

$$\begin{aligned} f(z) &= \frac{(\bar{z})^2}{z} = \frac{(x - iy)^2}{x + iy} = \frac{(x - iy)^3}{(x + iy)(x - iy)} \\ &= \frac{x^3 - 3xy^2}{x^2 + y^2} - i \frac{3x^2y - y^3}{x^2 + y^2} \end{aligned}$$

$$\Rightarrow u(x, y) = \frac{x^3 - 3xy^2}{x^2 + y^2},$$

$$v(x, y) = \frac{y^3 - 3x^2y}{x^2 + y^2}, (x, y) \neq (0,0)$$

When $z = 0$, then

$$u(x, y) = 0 = v(x, y)$$

Now

$$u_x(0,0) = \lim_{x \rightarrow 0} \frac{u(x,0) - u(0,0)}{x}$$

$$= \lim_{x \rightarrow 0} \frac{x - 0}{x} = 1$$

$$u_y(0,0) = \lim_{y \rightarrow 0} \frac{u(0,y) - u(0,0)}{y}$$

$$= \lim_{y \rightarrow 0} \frac{0 - 0}{y} = 0$$

$$v_x(0,0) = \lim_{x \rightarrow 0} \frac{v(x,0) - v(0,0)}{x}$$

$$= \lim_{x \rightarrow 0} \frac{0 - 0}{x} = 0$$

$$\begin{aligned}
 v_y(0,0) &= \lim_{y \rightarrow 0} \frac{v(0,y) - v(0,0)}{y} \\
 &= \lim_{y \rightarrow 0} \frac{y - 0}{y} = 1
 \end{aligned}$$

Thus $u_x = v_y$ & $u_y = -v_x$.

Hence, proved.

Q. Let $f(z) = \bar{z}$. Does $f'(z)$ exist ?

Method I

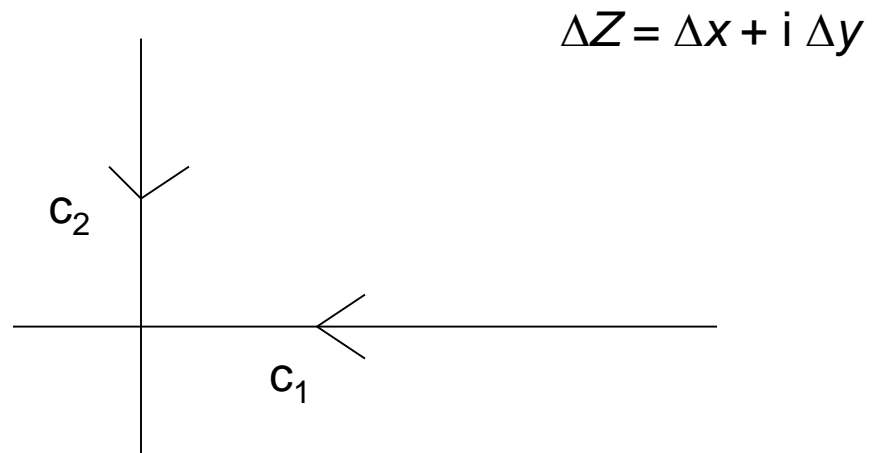
We have

$$\begin{aligned} f'(z) &= \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \\ &= \lim_{\Delta z \rightarrow 0} \frac{\overline{z + \Delta z} - \bar{z}}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{\overline{\Delta z}}{\Delta z} \end{aligned}$$

$$= \lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

$$= \begin{cases} 1 & \text{along the path } C_1 \\ -1 & \text{along the path } C_2 \end{cases}$$

$\Rightarrow f'(z)$ does not exist at any point.



Method II

$$\because f(z) = \bar{z} = x - iy$$

$$\Rightarrow u = x, v = -y$$

$$\therefore u_x = 1, \quad u_y = 0,$$

$$v_x = 0, \quad v_y = -1$$

$$\text{Thus } u_x \neq v_y, u_y = -v_x$$

\Rightarrow C.R.equations are not
satisfied

$\Rightarrow f(z)$ is not differentiable
at any point.

Q 2b.

Let $f(z) = e^{-z}$. Show that $f(z)$ is differentiable and find

$f'(z)$ and $f''(z)$.

Solution :

$$f(z) = e^{-z} = e^{-x-iy}$$

$$= e^{-x}(\cos y - i \sin y)$$

$$\Rightarrow u = e^{-x} \cos y, \quad v = -e^{-x} \sin y$$

$$\therefore u_x = -e^{-x} \cos y,$$

$$u_y = -e^{-x} \sin y,$$

$$v_x = e^{-x} \sin y,$$

$$v_y = -e^{-x} \cos y.$$

Clearly

$$(1) \quad u_x = v_y \text{ \& } u_y = -v_x$$

$$(2) \quad u_x, u_y, v_x, v_y$$

are continuous at any
point (x, y) .

$\therefore f'(z)$ exists and

$$f'(z) = u_x + i v_x$$

$$= -e^{-x} \cos y + i e^{-x} \sin y$$

$$= -e^{-x} \cdot e^{-iy} = -e^{-z}$$

To find $f''(z)$:

Let $F(z) = f'(z)$

$$= -e^{-x} \cos y + i e^{-x} \sin y$$

$$\equiv U + iV \quad (\text{say})$$

$$\therefore U = -e^{-x} \cos y,$$

$$V = e^{-x} \sin y$$

$$\Rightarrow U_x = e^{-x} \cos y,$$

$$U_y = e^{-x} \sin y,$$

$$V_x = -e^{-x} \sin y,$$

$$V_y = e^{-x} \cos y.$$

Thus,

$$(1) \quad U_x = V_y \quad \& \quad U_y = -V_x$$

$$(2) \quad U_x, U_y, V_x, V_y \text{ are}$$

continuous at any

point (x, y)

$\therefore F'(z)$ exists &

$$F'(z) = f''(z) = U_x + iV_x$$

$$= e^{-x} \cos y + i(-e^{-x} \sin y)$$

$$= e^{-x} \cdot e^{-iy}$$

$$= e^{-z}$$

Ex. Page 72, Q5.

Show that when

$$f(z) = x^3 + i(1 - y)^3$$

it is legitimate to write

$$f'(z) = 3x^2 \text{ only when } z = i.$$

