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BITS Pilani
Pilani Campus

CS F214
Logic in Computer Science

MODULE: PROPOSITIONAL LOGIC

Proof System - Natural Deduction

Propositional Logic

- ***Propositional Logic*** is essentially Boolean Logic: i.e.
  - it is a logic of ***formulas*** **(i.e. *propositions*)**
    - that evaluate to TRUE or FALSE
  - made from ***atomic propositions*** and ***logical operations*** (AND, OR, NOT ...)

    e.g. raining AND sunny

    e.g. raining AND sunny OR NOT humid AND NOT hot

Propositional Logic - Propositions

- **An *atomic proposition* is *a*** statement that is atomic (i.e. *understood as is without structure)**:***
  - It may be ***true*** or ***false,***
    - but its truth or falsity is
      - *not based on any structure*
      - *nor is it derived from something else*
  - ***e.g.*** (from the formulas in the last slide) :
    - raining, sunny, humid, hot

- In Boolean Logic *atomic propositions* are referred to as *variables*.

Proof Rules

- Proofs in (propositional) logic can be written down in various formats:
  - one of them is (Gentzen's) ***natural deduction***
- Proofs are based on rules (of inference):
  - i.e. rules that state "one can infer S_1 from S_2"

Proof Rules – Generic Format

- In natural deduction, rules are written in the form:

$$\frac{\phi 1 \qquad\qquad \phi 2}{\psi}$$

premises (one or more)

conclusion

•Each ***premise*** is a (class of) propositional formula(s).
•Similarly a ***conclusion*** is a (class of) propositional formula(s)

Proof Rules - Format - Example

- Consider this rule (labeled $\wedge$ $\mathbf{e_2}$):

$$\frac{\phi 1 \wedge \phi 2}{\phi 2} \wedge \mathbf{e_2}$$

This rule is read as:

given the premise

$\boldsymbol{\phi 1 \wedge \phi 2}$

one can conclude

$\boldsymbol{\phi 2}$

Proof Rules – Different Kinds

- In natural deduction, typically, <u>for each operation there are two kinds of rules</u>:
  - one kind for ***elimination***
    - i.e. to prove a conclusion by <u>*eliminating the operation*</u>
      - e.g. see previous slide ($\wedge$ elimination)
  - one kind for ***introduction***
    - i.e. to prove a conclusion by <u>*introducing the operation*</u>
      - e.g.

$$\frac{\phi 1 \qquad \phi 2}{\phi 1 \wedge \phi 2} \quad \wedge \mathbf{i}$$

Proof Technique: Natural Deduction: Sequents

- In natural deduction, the statement (intended to be proved) is referred to as ***a sequent***:
  - $\phi 1, \phi 2, \ldots, \phi n \quad |\text{--} \quad \psi$

which is read as

  - (the set of premises) "$\phi 1, \phi 2, \ldots,$ and ϕn" ***entails*** (the conclusion) "ψ"

- e.g. (sequent):
  - $p \wedge q, r \ |\text{--}\ q \wedge r$
- *To prove the sequent* one has to apply (appropriate) proof rules.
  - Q: Which rule(s) is/are required to prove the example sequent given above?

Proofs in Natural Deduction

- Typically a ***(deduction) step*** in a proof involves:
  - applying a rule on one or more premises to derive a conclusion:
    - for instance, from applying the rule $\wedge$**e$_2$** on the premise **p** $\wedge$ **q** results in the conclusion, say, **q**
- Typically, then, a (natural deduction) proof is a sequence of such (deduction) steps:
  - e.g. proof for the sequent **p** $\wedge$ **q, r |-- q** $\wedge$ **r**
    - Step 1: apply $\wedge$**e$_2$** on <u>premise **p** $\wedge$ **q**</u> to result in **q**
    - Step 2: apply $\wedge$**i** on <u>inference **q**</u> and <u>premise **r**</u> to result in **q** $\wedge$ **r**

Proofs in Natural Deduction

- Proofs are written in a particular format:
  - as a table with one column of premises and results along with corresponding explanations in the next column
- For instance, the 2-step proof from the previous slide i.e.
  - Step 1: apply $\wedge \mathbf{e_2}$ on $\mathbf{p} \wedge \mathbf{q}$ to result in $\mathbf{q}$
  - Step 2: apply $\wedge$ **i** on premises **q** and **r** to result in $\mathbf{q} \wedge \mathbf{r}$

is written out as:

refers to premise in row 1

	Deduction	Explanation
1	$\mathbf{p} \wedge \mathbf{q}$	**Premise**
2	**q**	$\wedge e_2$ **1**
3	**r**	**Premise**
4	$\mathbf{q} \wedge \mathbf{r}$	$\wedge$**i** **2,3**

refers to result in row 2 and premise in row 3

Note that the explanations may refer to the entries in the deduction column by using the row number.

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MODULE: **PROPOSITIONAL LOGIC**

Natural Deduction: Rules for Conjunction

ND: Rules for Conjunction

- Conjunction refers to the operation "AND":

$$\frac{\phi 1 \qquad \phi 2}{\phi 1 \wedge \phi 2} \wedge i$$

The introduction rule for AND

The elimination rules for AND

$$\frac{\phi 1 \wedge \phi 2}{\phi 1} \wedge e_1 \qquad \frac{\phi 1 \wedge \phi 2}{\phi 2} \wedge e_2$$

ND: Rules for Conjunction – Example 1

- Prove the following sequent:
  - hot $\wedge$ humid, sleepy $\wedge$ dull |-- hot $\wedge$ sleepy
- The proof evolves via the following steps:

	Deduction	Explanation
1	**hot $\wedge$ humid**	**Premise**
2	**hot**	$\wedge e_1$ **1**

The first $\wedge$ -elimination rule is applied on the premise in line 1 i.e. *on hot $\wedge$ humid to obtain the result hot*

ND: Rules for Conjunction – Example 1 (continued)

	Deduction	Explanation
1	**hot $\wedge$ humid**	**Premise**
2	**hot**	**$\wedge e_1$ 1**
3	**dull $\wedge$ sleepy**	**Premise**
4	**sleepy**	**$\wedge e_2$ 3**

The second $\wedge$-elimination rule is applied on premise in line 3
i.e. *on dull $\wedge$ sleepy to obtain the result sleepy*

ND: Rules for Conjunction – Example 1 (continued)

	Deduction	Explanation
1	**hot $\wedge$ humid**	**Premise**
2	**hot**	**$\wedge e_1$ 1**
3	**dull $\wedge$ sleepy**	**Premise**
4	**sleepy**	**$\wedge e_2$ 3**
5	**hot $\wedge$ sleepy**	**$\wedge i$ 2,4**

The $\wedge$-introduction rule is applied on <u>results in line 2 and 4</u> to obtain the result *hot $\wedge$ sleepy*

Natural Deduction: Proofs

Observations (about proofs):

- Typically the first step of a proof is *a premise* from the given sequent

  and the last step is *the conclusion* from the given sequent.

  i.e. if the sequent (to be proved) is of the form

  $\phi_1, \phi_2, \ldots \phi_n \mid\!-\!- \psi$

  then the proof will (typically) be of the form

ϕ_k	**Premise**
...	
ψ	**Conclusion**

Natural Deduction: Proofs

Observations (about proofs):

- Typically the rule (to be) applied is

  (i*)* *an elimination rule* for an operation if

  the (desired) result should be devoid of the operation and one of the premises contains the operation.

  (ii) *an introduction rule* for an operation if

  the (desired) result should involve the operation and the premises may not

Conjunction - Exercises

- Prove the following sequent:
  - english_summer ∧ green_top ∧ fast_bowling, indian_batsmen_fail

    |--

    english_summer ∧ fast_bowling ∧ indian_batsmen_fail

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MODULE: **PROPOSITIONAL LOGIC**

Proof System - Natural Deduction – Implication Rules

Natural Deduction: Proof Rules: Implication

- **Modus Ponens** (Implication Elimination)

$$\frac{\phi \dashrightarrow \psi \qquad \phi}{\psi} \; \dashrightarrow e$$

This rule is read as:

if we know ϕ implies ψ and we know ϕ is true then we can conclude ψ

ND: Implication – Example 1

Prove using “modus ponens”:

rains, rains-->wet, wet-->slippery |-- slippery

The proof proceeds as follows:

	Deduction	Explanation
1	**rains**	**Premise**
2	**rains-->wet**	**Premise**
3	**wet**	**-->e 1,2**
		...

-->elimination on premises in row 1 and 2

1st step

ND: Implication – Example 1 [continued]

Proof using “modus ponens”:

rains, rains-->wet, wet-->slippery |-- slippery

	Deduction	Explanation
1	**rains**	**Premise**
2	**rains-->wet**	**Premise**
3	**wet**	**-->e 1,2**
4	**wet-->slippery**	**Premise**
5	**slippery**	**-->e 3,4**

2nd step

-->-elimination on premises in row 3 and 4

ND: Implication – Example 2

Prove:

rains, rains-->wet, wet-->slippery |-- wet $\wedge$ slippery

	Deduction	Explanation
1	rains	Premise
2	rains-->wet	Premise
3	wet	-->e 1,2
4	wet-->slippery	Premise
5	slippery	-->e 1,4
6	wet $\wedge$ slippery	$\wedge$i 3,5

-->-elimination on premises in row 1 and 2

-->-elimination on premises in row 1 and 4

$\wedge$ -introduction on premises in row 3 and 5

Natural Deduction – Proofs

- Questions:
  - How do you identify the next step in a proof?
    - How do you identify which rule is to be applied?
- Observations:
  - An ND proof is usually driven bottom up i.e.
    - one identifies the steps (deductions) of the proof by starting with the conclusion
    - The structure of the conclusion (i.e. the top level operation(s) used) leads to the identification of the appropriate rule
      - for instance the choice of &i rule vs. the choice of -->e rule in the two proofs in the last two slides.

ND – Structure of Proofs - Example

Prove: **rains, rains-->wet, wet-->slippery |-- wet $\wedge$ slippery**

Deduction	Explanation
wet $\wedge$ slippery	$\wedge$i

$\wedge$ -introduction

ND: Implication – Example 2

Prove:

rains, rains-->wet, wet-->slippery |-- wet $\wedge$ slippery

	Deduction	Explanation
2	slippery	-->e
1	wet $\wedge$ slippery	$\wedge$i ?,2

-->elimination

$\wedge$ introduction on premise in row 2 and ?

ND: Implication – Example 2

Prove:

rains, rains-->wet, wet-->slippery |-- wet $\wedge$ slippery

	Deduction	Explanation
4	wet	?
3	wet --> slippery	Premise
2	slippery	-->e 3,4
1	wet $\wedge$ slippery	$\wedge$i ?,2

-->-elimination on Premise in row 3 and result in row 4

$\wedge$ -introduction on premise in row 2 and ?

ND: Implication – Example 2

Prove:

rains, rains-->wet, wet-->slippery |-- wet ∧ slippery

	Deduction	Explanation
6	rains	Premise
5	rains-->wet	Premise
4	wet	-->e 5,6
3	wet --> slippery	Premise
2	slippery	-->e 3,4
1	wet ∧ slippery	∧i ?,2

-->-elimination on Premises in rows 5 and 6

ND: Implication – Example 2

Prove:

rains, rains-->wet, wet-->slippery |-- wet ∧ slippery

	Deduction	Explanation
6	**rains**	**Premise**
5	**rains-->wet**	**Premise**
4	**wet**	**-->e 5,6**
3	**wet --> slippery**	**Premise**
2	**slippery**	**-->e 3,4**
1	**wet ∧ slippery**	**∧i 4,2**

-->-elimination on Premise in row 4 and result in row 3

∧ -introduction on premise in row 2 and result in row 4

ND: Proof Rules: Implication Introduction

- A "Proof" introduces implication

Assume $\phi 1$
.
.
.
$\phi 2$
$\phi 1 \dashrightarrow \phi 2$

-->i

i.e. *if we can assume $\phi 1$ and prove $\phi 2$, then $\phi 1$ --> $\phi 2$ can be inferred*

Question:
How is this rule different from the ones seen before?
[Hint: *Note that there is a sub-proof*. End of Hint.]

ND: Proofs using Implication Introduction

- Example:
  - Prove the sequent $|\text{--}\ p \wedge q \text{-->} q$
  - Proof:

	Deduction	Explanation
1	$p \wedge q$	Assumption
2	q	$\wedge e_2$ - 1
3	$p \wedge q \text{-->} q$	-->i - 1-2

These two rows constitute a proof of the sequent $p \wedge q \ |\text{--}\ q$

Observation:

Proof of a sequent of the form $\phi \ |\text{--}\ \psi$ can be treated as the proof of a sequent of the form $|\text{--}\ \phi \text{-->} \psi$

ND: Proofs using Implication Introduction

- **Observation:**
  - **Proof of a sequent of the form ϕ |-- ψ can be treated as the proof of a sequent of the form |-- ϕ --> ψ**
  - **Inductively, proof of $\phi 1, \phi 2, \ldots \phi n$ |-- ψ is proof of**

    |-- $\phi 1$ --> ($\phi 2$ --> (... --> (ϕn --> ψ)...))

Exercise:

Prove the following sequent :

p-->q, q-->r |-- p --> r

and thereby prove the following sequent, using the observation mentioned above:

|-- (p-->q) --> ((q-->r)-->(p-->r))