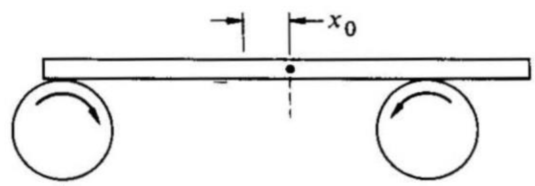
Chapter 6

Chapter 6: Calculation of moment of inertia of rigid bodies (related problems 6.7 & 6.8) are not in course.

Lecture: 6, 18, 22, 27, 35, *example* 6.13; *Tutorial*: 9, 17, 20, 23, 32, 38, 39

Suggested problems: 2, 3, 5, 10, 11, 13, 14, 15, 16, 24, 29, 31, 33, 37

6.9 A heavy uniform bar of mass M rests on top of two identical rollers which are continuously turned rapidly in opposite directions, as shown. The centers of the rollers are a distance 2l apart. The coefficient of friction between the bar and the roller surfaces is J.I, a constant independent of the relative speed of the two surfaces. Initially the bar is held at rest with its center at distance Xo from the midpoint of the rollers. At time t = 0 it is released. Find the subsequent motion of the bar.



Since the CM of the rod is not at the center of the rod, so, normal reactions would be different on the roller. Let N_1 and N_2 are the normal reaction on the left and on the right roller. Therefore, torque about the contact point of the rod and the left roller:

$$\tau_{left} = -(l + x_0)Mg + N_2.2l = 0 \text{ (because there is no resultant torque)}$$

$$\therefore \boxed{N_2 = \frac{Mg}{2}\left(1+\frac{x_0}{l}\right)}; Now,\ N_1 + N_2 = Mg \Rightarrow \boxed{N_1 = \frac{Mg}{2}\left(1-\frac{x_0}{l}\right)}$$

$$\text{Net force acting on the rod}: \mu N_1 - \mu N_2 = -\frac{\cancel{M} g\mu}{l}x_0 = \cancel{M}\frac{d^2x_0}{dt^2} \Rightarrow \boxed{\omega = \sqrt{\frac{g\mu}{l}}}$$

6.10.

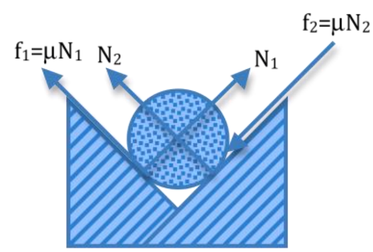
$$Here, \sum F_x = \mu N_1 \cos 45^0 + \mu N_2 \cos 45^0 + N_2 \cos 45^0 = N_1 \cos 45^0 = 0$$

$$\Rightarrow \mu(N_1 + N_2) = N_1 - N_2 \Rightarrow \mu = \frac{N_1 - N_2}{N_1 + N_2} ----(1) \text{and} \Rightarrow N_1(\mu - 1) + N_2(\mu+1) = 0 ----(2)$$

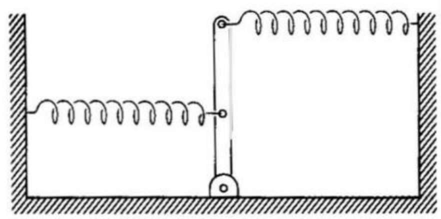
$$\sum F_y = \mu N_1 \cos 45^0 - \mu N_2 \cos 45^0 + (N_1 + N_2)\cos 45^0 = mg$$

$$\Rightarrow N_1(\mu+1) - N_2(\mu - 1) = \sqrt{2}mg ----(3)$$

$$Solving, N_1 = \frac{mg(\mu+1)}{\sqrt{2}(\mu^2+1)} \text{and } N_2 = \frac{mg(1-\mu)}{\sqrt{2}(1+\mu^2)}; Torque: \mu(N_1+N_2)R = \frac{\sqrt{2}mgR}{(\mu^2+1)} = 5.7$$



6.17 A rod of length l and mass m, pivoted at one end, is held by a spring at its midpoint and a spring at its far end, both pulling in opposite directions. The springs have spring constant k, and at equilibrium their pull is perpendicular to the rod. Find the frequency of small oscillations about the equilibrium position.



$$Moment\ of\ inertia\ of\ the\ rod\ about\ the\ pivot: I=\frac{1}{3}Ml^2$$

$$So,\ torque\ equation: \tau=I\alpha=-k\theta\frac{l}{2}-k\theta l+Mg\frac{l}{2}\theta=-\frac{l}{2}(3k-Mg)\theta$$

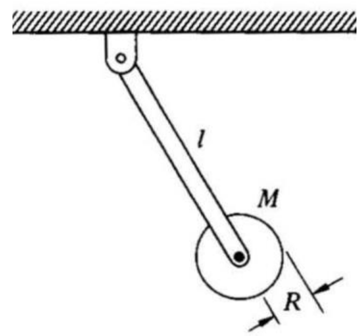
$$\Rightarrow\frac{1}{3}Ml^2\ddot{\theta}=-\frac{l}{2}(3k-Mg)\theta\Rightarrow\ddot{\theta}+\frac{3}{2Ml}(3k-Mg)\theta=0\ \therefore\ \boxed{\omega=\sqrt{\frac{3(3k-Mg)}{2Ml}}}$$

6.18.

Torque about the point of suspension:

$$\tau=\left(-\frac{mgl}{2}-Mgl\right)\sin\theta\approx\left(-\frac{mgl}{2}-Mgl\right)\theta=I_{total}\ddot{\theta}\Rightarrow I_{total}\ddot{\theta}+\left(\frac{mgl}{2}+Mgl\right)\theta=0$$

$$\therefore\ \boxed{\omega=\sqrt{\frac{\frac{mgl}{2}+Mgl}{I_{total}}}\ and\ T=2\pi\sqrt{\frac{I_{total}}{\frac{mgl}{2}+Mgl}};I_{total}=\frac{1}{3}ml^2+Ml^2+\frac{1}{2}MR^2}$$



When the disk is attached via a frictionless bearing it cannot totate about its own axis and hence is not contributing any moment of inertial. So, $I'_{total}=\frac{1}{3}ml^2+Ml^2$

$$\therefore\ \omega=\sqrt{\frac{\frac{mgl}{2}+Mgl}{I'_{total}}}\ and\ T=2\pi\sqrt{\frac{I'_{total}}{\frac{mgl}{2}+Mgl}}$$

6.3.

The ring is in contact to the table and vertical. If we consider bug is moving in anticlockwise direction, the ring will rotate in clockwise direction. Since there no external force / torque acting on the system,

so, $L_{net}=L_{bug}-L_{ring}$

$$L_{bug}\Big|_{Top\ point}=m(distance\ from\ contact\ point)(relative\ velocity\ of\ the\ bug)$$

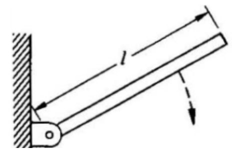
or, $L_{bug}\Big|_{Top\ point}=m.2R.(v-2\omega R)$ *where* $2R\times\omega$ *is the tangential velocity of the top most point of the ring. And* $L_{ring}\Big|_{About\ pivot}=-\omega\left(MR^2+MR^2\right)=-2\omega MR^2$ *Clockwise direction*

From conservation of $L\Rightarrow L_{bug}\Big|_{Top\ point}-L_{ring}\Big|_{About\ pivot}=0\Rightarrow\omega=\frac{mv}{(m+2M)R}$

When bug will come back to pivot, L_{bug} *about the pivot* $=0$ *and so,* $L_{ring}=0$

and so $\omega=0$. *Means the ring will be at rest at that time instant.*

6.20 A thin plank of mass M and length I is pivoted at one end (see 1 figure below). The plank is released at 60° from the vertical. What is the magnitude and direction of the force on the pivot when the plank is horizontal?



6.20. $Here, torque\ of\ the\ rod\ when\ it\ is\ horizontal: \tau = I\alpha = Mg.\frac{l}{2}$

$$\Rightarrow \alpha = \frac{Mgl}{2I} = \frac{Mgl}{2\left(\frac{Ml^2}{3}\right)} = \frac{3g}{2l}, Hence\ the\ linear\ acceleration\ a = \frac{l}{2}.\alpha = \frac{l}{2}.\frac{3g}{2l} = \frac{3g}{4}$$

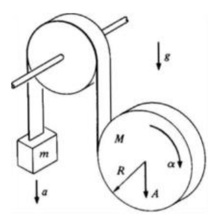
$$Now, \sum_j F_{jVertical} = 0 \therefore Mg - F_V = Ma \Rightarrow \boxed{F_V = M(g-a) = M\left(g - \frac{3g}{4}\right) = \frac{Mg}{4}}$$

$$Again, \sum_i F_{i\ Horizontal} = 0 \therefore F_H + centripetial\ force = 0 \Rightarrow \boxed{F_H = -\frac{M\upsilon^2}{l/2} = -\frac{2M\upsilon^2}{l}}$$

$$Now,\ from\ energy\ conservation: Mg\frac{l}{2}\sin 30^0 = \frac{1}{2}I\omega^2 \Rightarrow \omega^2 = \frac{Mgl}{2I} = \frac{Mgl}{2\left(Ml^2/3\right)} = \frac{3g}{2l}; \therefore \upsilon^2 = \frac{3gl}{8}$$

$$\therefore \boxed{F_H = -\frac{2M\upsilon^2}{l} = -\frac{2M}{l}.\frac{3gl}{8} = -\frac{3Mg}{4}}$$

6.23 A disk of mass M and radius R unwinds from a tape wrapped around it (see figure below at left). The tape passes over a frictionless pulley, and a mass m is suspended from the other end. Assume that the disk drops vertically.
a. Relate the accelerations of m and the disk, a and A, respectively, to the angular acceleration of the disk. b. Find a, A and α.



$Let, 'dx' \text{ is the distance moved down by } m \text{ and } 'dy' \text{ is the distance moved up by } M.$

$Let\ d\theta$ be the angle through which disc rotates along clock wise direction which is $-ve$.

$$\therefore -dx(\text{down ward}) = dy(\text{up ward}) - R.d\theta\ (\text{clock wise})$$

$$\therefore Rd\theta = \text{length moved by the tape} = dx + dy \therefore R\frac{d\theta}{dt} = \frac{dx}{dt} + \frac{dy}{dt} \Rightarrow \boxed{R\alpha = \dot{x} + \dot{y} = a + A}$$

$$Now,\ mg - T = ma - - - - - (1) \text{ and } Mg - T = MA - - - - (2)$$

and $TR = I\alpha = \frac{1}{2}MR^2\alpha$, where, TR is the torque on the wheel due to the rope. $\therefore T = \frac{MR\alpha}{2}$

Put $a = R\alpha - A$ and $T = \frac{MR\alpha}{2}$ we have

$$mg - \frac{MR\alpha}{2} = m(R\alpha - A) \Rightarrow mg - \left(\frac{M}{2} + m\right)R\alpha = -mA \text{ and } Mg - \frac{MR\alpha}{2} = MA$$

Solving for A and α, we have, $\boxed{a = \left(\frac{3m - M}{M + 3m}\right)g; \quad A = \left(\frac{M + m}{M + 3m}\right)g \quad \text{and} \quad \alpha = \frac{4Mg}{R(M + 3m)}}$

6.27. Torque equation : $Fb - fR = I\alpha \Rightarrow \frac{1}{2}MR^2\alpha = \frac{1}{2}MRa - - - - (1)$

Force equation : $F - f = Ma - - - - (2)$

Solving : $f = \frac{Ma\left(b + \frac{R}{2}\right)}{R - b}; F = \frac{3}{2}\frac{MaR}{R - b}; \therefore \frac{F}{f} = \left(\frac{3}{2}\frac{MaR}{R - b}\right) \Big/ \mu Mg$

Under no sliping condition : $f \le \mu Mg$

$$\therefore \frac{F}{f} = \frac{3}{2}\frac{R}{b + \frac{R}{2}} \Rightarrow F = \frac{3}{2}\frac{R\mu Mg}{b + \frac{R}{2}}$$

6.30. Bolling ball

When the ball is sliding, it has only linear velocity $= v_{CM}^{initial}$

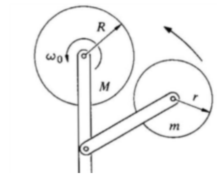
When the ball is rolling, it has linear velocity v_{CM}^{final} and also angular velocity ω.

From conservation of angular momentum :

$$L_i = L_f \Rightarrow MRv_{CM}^{initial} = MRv_{CM}^{final} + I\omega = MRv_{CM}^{final} + \frac{2}{5}MR^2\omega = MRv_{CM}^{final} + \frac{2}{5}MR^2\frac{v_{CM}^{final}}{R} = \left(MR + \frac{2}{5}MR\right)v_{CM}^{final}$$

$$\therefore \boxed{v_{CM}^{final} = \frac{5}{7}v_{CM}^{initial}}$$

6.32. A solid rubber wheel of radius R and mass M rotates with angular velocity ω_0 about a frictionless pivot (see sketch at left). A second rubber wheel of radius T and mass m. also mounted on a frictionless pivot. i's brought into contact with it. What is the final angular velocity of the first wheel?



6.32. *Let the frictional force acting between the two wheels is f.*

When they are in contact, the linear velocity of the wheel surfaces must be same. $\therefore \boxed{R\omega_R = r\omega_r}$ ----(1)

where, ω_R *and* ω_r *are the angular speed of the wheel with radius 'R' and 'r' respectively.*

So, $\omega_R = \omega_0 - \alpha_R t$ *where* α_R *is the −acceleration of the bigger wheel and* ω_0 *is its initial speed.*

Now, torque due to firctional force: $fR = I_R\alpha_R = \frac{MR^2}{2}\alpha_R \Rightarrow \boxed{\alpha_R = \frac{2f}{MR}}$

$\boxed{\omega_R = \omega_0 - \frac{2f}{MR}t}$ ----(2); *The other wheel starts from rest, So,* $\omega_r = \alpha_r t$

Again the torque on the smaller wheel due to frictional force: $fr = I_r\alpha_r \Rightarrow \boxed{\alpha_r = \frac{2f}{mr}}$

$\therefore \boxed{\omega_r = \alpha_r t = \frac{2ft}{mr}}$; *From equation* 1: $R\omega_R = r\omega_r \Rightarrow R\left(\pi\omega_0 - \frac{2f}{MR}t\right) = r\frac{2ft}{mr} \Rightarrow \boxed{\omega_0 = \frac{2ft}{R}\left(\frac{1}{M}+\frac{1}{m}\right)}$

$\therefore 2ft = \frac{\omega_0 RMm}{M+m}$ *and therefore,* $\boxed{\omega_R = \omega_0 - \frac{2f}{MR}t = \omega_0 - \frac{\omega_0 m}{M+m} = \frac{\omega_0 M}{M+m}}$

6.32. *Let the frictional force acting between the two wheels is* f.

When they are in contact, the linear velocity of the wheel surfaces must be same.

$\therefore \boxed{R\omega_R = r\omega_r}$ $----(1)$ *where,* ω_R *and* ω_r *are the angular speed of the wheel with radius* $'R'$ *and* $'r'$ *respectively. So,* $\omega_R = \omega_0 - \alpha_R t$ *where* α_R *is the* $-$*acceleration of the bigger wheel and* ω_0 *is its initial speed.*

Now, torque due to firctional force: $fR = I_R\alpha_R = \dfrac{MR^2}{2}\alpha_R \Rightarrow \boxed{\alpha_R = \dfrac{2f}{MR}}$

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Again the torque on the smaller wheel due to frictional force: $fr = I_r\alpha_r \Rightarrow \boxed{\alpha_r = \dfrac{2f}{mr}}$

$\therefore \boxed{\omega_r = \alpha_r t = \dfrac{2ft}{mr}}$;

From equation 1: $R\omega_R = r\omega_r \Rightarrow R\left(\pi\omega_0 - \dfrac{2f}{MR}t\right) = r\dfrac{2ft}{mr} \Rightarrow \boxed{\omega_0 = \dfrac{2ft}{R}\left(\dfrac{1}{M} + \dfrac{1}{m}\right)}$

$\therefore 2ft = \dfrac{\omega_0 RMm}{M+m}$ *and therefore,* $\boxed{\omega_R = \omega_0 - \dfrac{2f}{MR}t = \omega_0 - \dfrac{\omega_0 m}{M+m} = \dfrac{\omega_0 M}{M+m}}$

6.33.

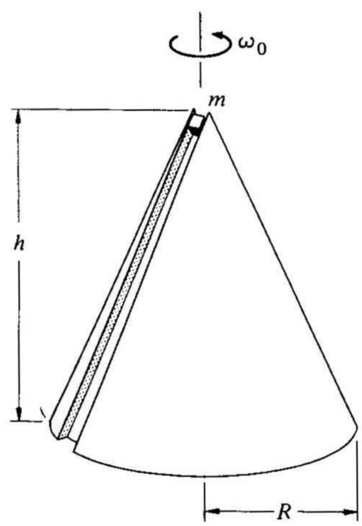
The problem is based on conservation of angular momentum. Since there is no external torque is acting on the system, so, $L_i = L_f$

Here, $L_i = I_0\omega$ *and* $L_f = I_0\omega_f + mR\omega_f \therefore \omega_f = \dfrac{I_0\omega}{I_0 + mR}$

$\Rightarrow$ *means, the mass will gain angular momentum from the cone.*

From conservation of KE:

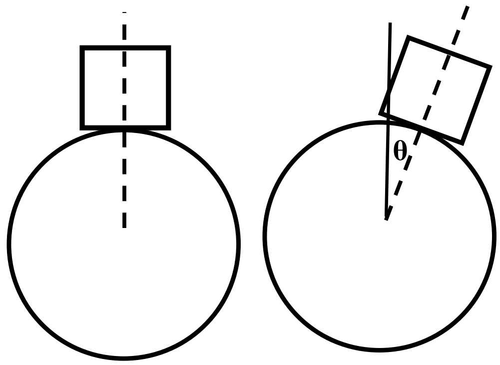
$$KE_i = \frac{1}{2}I_0\omega^2 + mgh = KE_f = \frac{1}{2}I_0\omega_f^2 + \frac{1}{2}mv_f^2 + \frac{1}{2}mR\omega_f^2 \Rightarrow v_f = ??$$



6.35. *Under equillibrium condition, CM of the Cube and the drum must will be on a vertical line. The cube always will be in contct with the drum. When the cube will rotate on the drum, the point of contact will be the junction point of the base of the cube and the radius of the drum. So, the angle created by the base line of the cube and the radius of the drum is* 90^0. *Let the angle produced at the new position of the cube contact point with the initial position and the center of the drum is* θ. *So, the weight vector of the cube also will rotate at an angle* θ.

So the horizontal shift of the CM of the cube is $\frac{L}{2}\sin\theta$ *and the shift of the contact point is* $R\sin\theta$.

As long as, $\frac{L}{2}\sin\theta < R\sin\theta$ *or* $L < 2R$, *the cube will come back to its initial position.*



6.38 A rigid massless rod of length L joins two particles each of mass m. The rod lies on a frictionless table, and is struck by a particle of mass m and velocity v_o, moving as shown. After the collision, the projectile moves straight back. Find the angular velocity of the rod about its center of mass after the collision, assuming that mechanical energy is conserved.

6.38.

Let V_{CM} is the velocity of the center of mass of the rod (AB). Since no external force is acting on the system, therefore, momentum is conserved. $\therefore \boxed{P_i = P_f}$

$$\Rightarrow mV_0 = -mV_f + 2mV_{CM} \Rightarrow \boxed{V_0 = -V_f + 2V_{CM}} ----(1)$$

Since there is no external torque, so, angular momentum about CM is conserved : $L_i = L_f$

$$\Rightarrow mV_0 \frac{L}{2}\sin 45^0 = -mV_f \frac{L}{2}\sin 45^0 + I_{rod}\omega = -mV_f \frac{L}{2}\sin 45^0 + 2m\left(\frac{L}{2}\right)^2 \omega \Rightarrow \boxed{V_f = \sqrt{2}L\omega - V_0} --(2)$$

Compearing equation (1) and (2), we get, $\boxed{V_{CM} = \frac{L\omega}{\sqrt{2}} \text{ and } V_f = \sqrt{2}L\omega - V_0}$. *From energy conservtion*:

$$\frac{1}{2}mV_0^2 = \frac{1}{2}mV_f^2 + \frac{1}{2}(2m)V_{CM}^2 + \frac{1}{2}I_{rod}\omega^2 = \frac{1}{2}m\left(\sqrt{2}L\omega - V_0\right)^2 + \frac{1}{2}(2m)\left(\frac{L\omega}{\sqrt{2}}\right)^2 + \frac{1}{2}\left(2m \times \frac{L^2}{4}\right)\omega^2$$

$$\Rightarrow \cancel{V_0^2} = 2L^2\omega^2 + \cancel{V_0^2} - 2\sqrt{2}L\omega V_0 + L^2\omega^2 + \frac{1}{2}L^2\omega^2 \Rightarrow \frac{7}{2}L^2\omega\cancel{^2} = 2\sqrt{2}L\cancel{\omega}V_0 \Rightarrow \boxed{\omega = \frac{4\sqrt{2}V_0}{7L}}$$

6.39. (a) *Let* $M = m$ *for simplicity. Position of center of mass of the rod* : $\frac{L}{2}$ *above the ground.*

$\therefore$ *Position of center of mass of the rod + boy* : $\frac{L}{4}$ *above the ground.*

$\therefore$ *Velocity of the center of mass of the system* : $m\upsilon_0 = (M+m)V_{cm} \Rightarrow \boxed{V_{cm} = \frac{m\upsilon_0}{M+m}}$

the combined system will rotate in anti – closkwise direction.

If new CM position is at a height z from ground, then, $z = \frac{ML}{2(M+m)}$

So, from conservation of angular momentum about the new CM is : $m\upsilon_0 z = I'_{rod}\omega$,

$I'_{rod} = MI$ *of the rod + boy system* $= I_{rod} + I_{boy}$ *about new CM* $= \left[\frac{ML^2}{12} + M\left(\frac{L}{2} - z\right)^2\right] + \left[mz^2\right] = \frac{ML^2}{12}\left(\frac{M+4m}{M+m}\right)$

$\therefore \boxed{\omega = \frac{6m\upsilon_0}{(M+4m)L}}$

(b) *Let the point P is at rest due to the resultant effect of the rotational and translational motion of the rod + boy system. Let the point is 'x' above the new CM.*

So, $\omega x = V_{CM} = \frac{m\upsilon_0}{M+m} \Rightarrow x = \frac{m\upsilon_0}{(M+m)\omega} = \frac{m\upsilon_0}{(M+m)} \times \frac{(M+4m)L}{6m\upsilon_0} = \frac{M+4m}{M+m}.\frac{L}{6}$

$$\boxed{\text{So, distance from the boy is}: z + x = \frac{ML}{2(M+m)} + \frac{M+4m}{M+m}.\frac{L}{6} = \frac{2}{3}L}$$

6.40.

(a) *Just before and after collition:*

$L_i = MR\upsilon_i \left(No\,rotation\right);\; KE_i = E_i = \frac{1}{2}Mv_i^2 = \frac{L_i^2}{2MR^2}$

$And: L_f = I_f\omega_f \left(With\,rotation\right); I_f = I \text{ about the contact point} = I_0 + MR^2 = 2MR^2$

$\therefore L_f = 2MR^2\omega_f \text{ and } E_f = \frac{1}{2}I\omega_f^2 = \frac{L_f^2}{2MR^2}; \text{Since}, L_f = L_i, \therefore E_f = \frac{1}{2}E_i$

Let, b' is the compression of the spring.

$\therefore \frac{1}{2}kb'^2 = E_f = \frac{1}{2}E_i = \frac{1}{2}.\frac{1}{2}kb^2 \Rightarrow \boxed{b' = \frac{b}{\sqrt{2}}}$

$So, \text{distance from the wall}: l - b' = l - \frac{b}{\sqrt{2}}$

(b) *When it will back, at the point of 1st collision, torque and the linear velocity will be just reverse.*

If b'' is the extension of the spring, then,

$\frac{1}{2}kb''^2 = E_f = \frac{1}{2}E_i = \frac{1}{4}kb^2 \Rightarrow b'' = \frac{b}{\sqrt{2}} \Rightarrow \left(Pl.\,check\right)$

(c)