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CS/IS F214
Logic in Computer Science

MODULE: PREDICATE LOGIC

Semantics – Models and Interpretation

Semantics in Propositional Logic

- Recall:
  - Semantics – i.e. the meaning of formulas – in Propositional Logic was defined by Truth Tables.
    - Atomic propositions were assigned values (TRUE or FALSE) and
    - formulas were evaluated per truth tables for operations.
- *This approach will not work in Predicate Logic!*
  - Why?
    - There are two issues:
      i. Quantification and
      ii. Context

Semantics in Predicate Logic: Issue - Context

- Predicates and Function Terms
  - e.g. Is $\forall \mathbf{X}$ older(father(mother(X)), mother(father(X))) true?

What is the context?

i.e. What assumptions are we making about predicate ***older*** and function terms ***father*** and ***mother***?
and
What is the *universal set of values* over which X is quantified?

Semantics in Predicate Logic: Issue – Quantification

- Variables and Quantification:
  - e.g. When is $\forall X\ P(X)$ true?

Potentially infinite number of possible values for **X**

Implication

One cannot write an *algorithm – that will terminate* – to evaluate this formula

Of course, if the universal set **U** is finite, then it is possible to evaluate **P(X)** for each value **X** can take in **U**.

Semantics in Predicate Logic: Example 1

- Consider the following formulas:
  1. $\forall X \ \forall Y \ \neg r(Y, X)$
  2. $\forall X \ \forall Y \ X=Y \dashrightarrow \neg \ r(Y, X)$
- What is the meaning of each of the two formulas if the *universe* is, say,
  - **N**, the set of *natural numbers* ?
  - the collection *of all sets* , and **r** is the membership relation.
  - a *finite set* **S**, and **r** is any binary relation? (or equivalently *the edge relation in a finite graph G*?)

Semantics in Predicate Logic: Example 2

- Consider formulas in predicate logic where
  - {a, b} is the set of constants,
  - { +, * } is the set of functions, and
  - { = } is the set of predicates.
- Now, consider a specific formula:
  - $\forall X \ \ {*}(X,X) = +(X,X)$
    - When is this true? When is this false?
    - For how many values of X do we have to evaluate the sub-formula?
      - For instance, is it sufficient to evaluate
        - *(a,a) = +(a,a) and
        - *(b,b) = +(b,b)

        to conclude whether the formula is true or fase?
        - Why or why not?

Semantics: Models

- Let ***F*** be a set of function symbols and ***P*** be a set of predicate symbols, each symbol with a fixed *arity*.
- A model **M** of the pair (***F***,***P***) is consists:
  1. A non-empty set **A**, the ***universe*** of concrete values
  2. for each nullary function **f** in ***F***:

     a concrete element f_M of A
  3. for each **f** in ***F*** of arity **n>0**:

     a concrete function $f_M : A^n \rightarrow A$
  4. for each **p** in ***P*** with arity **n>0**:

     a subset $p_M \subseteq A^n$

Note:

- ***arity*** denotes the number of arguments of a function term or a predicate.
- nullary function symbols are treated as **constant** symbols.

End of Note.

Semantics: Models – Example 1

- Let ***F*** = { ε, **.** } and ***P*** = { <= }
  - where **ε** is a constant, **.** and <= are binary.
- A model **M** for (***F***,***P***) is:
  - **Universe:**

    the set of all binary strings (i.e. strings of 0s and 1s)
  - **Meaning of symbols in *F*:**

    ε denotes the empty string

    . denotes concatenation of two strings
  - **Meaning of symbols in *P*:**

    <= denotes the prefix ordering of strings

    i.e. ***s1 <= s2 if s1 is a prefix of s2***

Semantics: Models and Interpretations – Example 1

- Given ***F*** = {ε, **.** } and ***P*** = { <= }
- A model M for (***F***,***P***) :
  - **Universe:** the set of all binary strings
  - **Meaning of symbols in *F***:

    ε is the empty string, **.** is concatenation of two strings
  - **Meaning of symbols in *P***:

    s1 <= s2 if s1 is a prefix of s2
- Interpret the following formulas based on this model:

1. $\forall X\ (X <= X.\ \varepsilon) \wedge (X.\ \varepsilon <= X))$
2. $\exists Y\ \forall X\ Y <= X$
3. $\forall X\ \exists Y\ Y <= X$
4. $\forall X\ \forall Y\ \forall Z\ ((X <= Y) \text{-->}\ (X.Z <= Y.Z))$
5. $\neg \forall Y\ \forall X\ ((Y <= X) \text{-->}\ (X<=Y))$

Semantics: Models: Example 2

- Let ***F*** = { **+**, **zero** } and ***P*** = { **≡** }
  - where **zero** is a constant, **+** is binary, and **≡** is binary.
- A model M for (***F***,***P***) :
  - **Universe:** the set { 0, 1, 2, 3, 4, 5, 6 }
  - **Meaning of symbols in *F***:

    \+ denotes addition modulo 7

    zero denotes 0
  - **Meaning of symbols in *P***:

    ≡ denotes the relation *congruent modulo 7*

Semantics: Models and Interpretation: Example 2

- Let ***F*** = { +, **zero** } and ***P*** = { ≡ }
- A model M for (***F***,***P***) :
  - **Universe:** the set { 0, 1, 2, 3, 4, 5, 6 }
  - **Meaning of symbols in *F***:

    zero denotes 0

    \+ denotes <u>addition modulo 7</u>
  - **Meaning of symbols in *P***:

    ≡ denotes the relation *<u>congruent modulo 7</u>*
- Interpret the following formulas based on this model:

1. $\forall X\ \forall Y\ \exists Z\ X + Y \equiv Z$

2. $\forall X\ \mathbf{zero} + X \equiv X$

3. $\forall X\ \exists Y\ \ X + Y \equiv \mathbf{zero}$

Semantics: Models: Example 3

- Let ***F*** = { Φ} and ***P*** = { $\in$, $\subseteq$ }
- A model M for (***F***,***P***) :
  - **Universe:** the set of all finite sets
  - **Meaning of symbols in *F***:

    Φ denotes the empty set
  - **Meaning of symbols in *P***:

    $\in$ denotes the membership relation

    $\subseteq$ denotes the subset relation

Semantics: Models and Interpretation: Example 3

Let ***F*** = { Φ} and ***P*** = {$\in$, $\subseteq$}

A model M for (***F***,***P***) :

Universe: *the set of all finite sets*

Meaning of symbols in *F*:

Φ denotes the empty set

Meaning of symbols in *P*:

$\in$ denotes the membership relation

$\subseteq$ denotes the subset relation

Interpret the following formulas based on this model:

1. $\forall X\ \Phi \subseteq X$
2. $\forall X\ \neg(X \in \Phi)$
3. $\forall X\ X \subseteq X$
4. $\forall X\ (X \subseteq \Phi) \text{ --> } (X=\Phi)$
5. $\forall X\ \forall Y\ (X \subseteq Y) \wedge (Y \subseteq X) \text{ --> } (X=Y)$
6. $\forall X\ \forall Y\ \forall Z\ (X\subseteq Y) \wedge (Y\subseteq Z) \text{ -->} (X\subseteq Z)$

Semantics: Models: Example 4

- Let ***F*** = { } and ***P*** = { →, ⇒ }
- A model M for (***F***,***P***) :
  - **Universe:** the vertices of a given graph **G**
  - **Meaning of symbols in *P***:

    → denotes the (*undirected*) ***edge*** relation on vertices

    ⇒ denotes the ***path*** relation on vertices

Semantics: Models: Example 4

Let ***F*** = { } and ***P*** = { →, ⇒ }

A model M for (***F***,***P***) :

Universe: the vertices of a given graph **G**

Meaning of symbols in *P*:

→ denotes the (*undirected*) ***edge*** relation on vertices

⇒ denotes the ***path*** relation on vertices

Note:

In general, **path** relation captures "reachability" i.e. a path from vertex **u** to vertex **v** in captures the notion **v** is reachable from **u** (and *vice versa* in an undirected graph).

End of Note.

⇒ is

- *reflexive*:

  i.e. an empty path makes vertex **v** reachable from itself

- *symmetric*:

  because edges are undirected and

- *transitive*:

  concatenating a path from u to v and a path from v to w yields a path from u to w.

Semantics: Models and Interpretation: Example 4

Let ***F*** = { } and ***P*** = { $\rightarrow$, $\Rightarrow$ }

A model M for (***F***,***P***) :

Universe: the vertices of a given graph **G**

Meaning of symbols in *P*:

$\rightarrow$ denotes the (*undirected*) ***edge*** relation on vertices of G

$\Rightarrow$ denotes the path relation on vertices of G

- Which of the formulas 1 through 6 are true for all graphs G?
- Are there graphs for which formula 5 is true but not formula 6?

Interpret the following formulas based on this model:

1. $\forall X\ \forall Y\ (X \rightarrow Y) \dashrightarrow (Y \rightarrow X)$
2. $\forall X\ \forall Y\ (X \rightarrow Y) \dashrightarrow (X \Rightarrow Y)$
3. $\forall X\ (X \Rightarrow X)$
4. $\forall X\ \forall Y\ \forall Z\ (X \Rightarrow Y) \wedge (Y \Rightarrow Z) \dashrightarrow (X \Rightarrow Z)$
5. $\forall X\ \exists Z\ (X \Rightarrow Z) \wedge (Z \Rightarrow X)$
6. $\forall X\ \exists Z\ \neg(X=Z) \dashrightarrow ((X \Rightarrow Z) \wedge (Z \Rightarrow X))$

Predicate Logic: Models and Interpretations

- A model – in essence – provides
  - a **context** in which the values and symbols (i.e. constants, functions, and predicates) can be assigned a meaning
  - and thereby enables us to "***evaluate***" a formula
- For instance,
  - the formula **$\forall$X succ(pred(X)) = pred(succ(X))**
  - can be evaluated
    - (only) if we know what **pred** and **succ** stand for
- Of course the result of the evaluation will be dependent on the model and may differ for different models (for even closely related ones!).

Predicate Logic: Models and Interpretations – Example 5

- Evaluate the formula **∀X succ(pred(X)) = pred(succ(X))**
  - using the model of numbers, where:
    - **succ(X)** means **X+1**
    - **pred(X)** means **X-1**
  - Will the formula result in different valuations for
    - the set of *integers* vs. the set of *natural numbers*?

Predicate Logic: Models and Interpretations – Example 5

- Evaluate the formula **∀X succ(pred(X)) = pred(succ(X))**
  - using the model of linked lists, where:
    - **succ(X)** means the node pointed to by **X**
    - **pred(X)** means the node pointing to **X**
  - Will the formula result in different valuations for
    - the set of nodes in a *linear linked list* vs. the set of nodes in a *circular linked list* ?