MATH F113 (Probability and Statistics)

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In Lecture 18

Problems on Normal Distribution
Log Normal Distribution

Normal Probability Rule and Chebyshev's inequality

It is sometimes useful to have a quick way of determining which values of a random variables are common and which are considered to be rare.

Normal Probability rule and Chebyshev's inequality (Cont...)

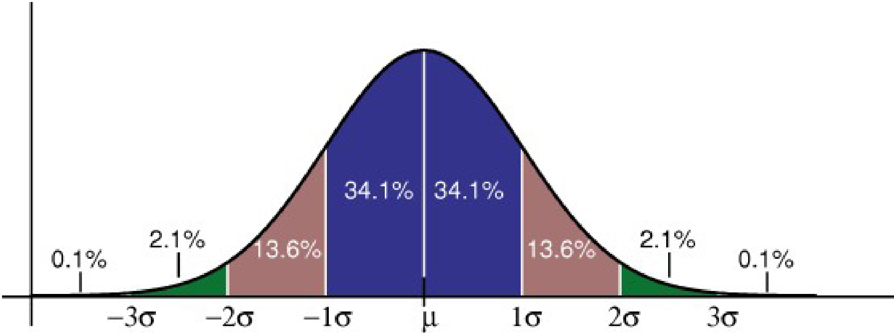
Let X be normally distributed with parameters mean μ and standard deviation σ then

$$P\left[-\sigma < X - \mu < \sigma\right] = 0.68$$

$$P\left[-2\sigma < X - \mu < 2\sigma\right] = 0.95$$

$$P\left[-3\sigma < X - \mu < 3\sigma\right] = 0.997$$

Normal Probability rule and Chebyshev's inequality (Cont...)



- The rule will be used later to obtain a quick estimate of the standard deviation of a normally distributed random variable.
- A second rule of thumb that can be used to gauge the rarity of observed values of a random variable is Chebyshev's inequality.

Chebyshev's Inequality: **Let X be a random variable with finite variance, then for any positive number k,**

$$P\left[|X-\mu| \geq k\sigma\right] \leq \frac{1}{k^2}$$

$$P\left[|X-\mu| < k\sigma\right] \geq 1-\frac{1}{k^2}$$

where μ the mean and σ standard deviation.

Normal Probability rule and Chebyshev's inequality (Cont...)

Put $k\sigma = \epsilon$ **then**

$$P\left[|X - \mu| \geq \epsilon\right] \leq \frac{\sigma^2}{\epsilon^2}$$

Problem: Show, by Chebysheff's inequality, that in 2000 throws with a coin, the probability that the number of heads lies between 900 and 1100 is at least 19/20.

Solution: **Let** X **be the number of heads (success) in throwing of a coin, then** $X \sim Bin(n, p)$, **where** $n = 2000$, $p = \frac{1}{2}$ $E(X) = np = 1000$, $Var(X) = npq = 500$. **Using, Chebysheff's inequality we get,**

Normal Probability rule and Chebyshev's inequality (Cont...)

$$P\left[|X-\mu|<k\sigma\right]\geq 1-\frac{1}{k^2}$$

$$P\left[|X-1000|<\sqrt{500}k\right]\geq 1-\frac{1}{k^2}$$

$$P\left[900<X<1100\right]\geq \frac{19}{20}$$

Problem: If X is the number scored in a throw of a fair die, show that the Chebychev's inequality gives

$$P\left[|X-\mu|>2.5\right]<0.47$$

where μ is the mean of X, while the actual probability is zero.

Solution: **Let** X **be the random variable which takes the values 1, 2, 3, 4, 5, 6, each with probability** $1/6$**. Then** $E(X) = 7/2$ **and** $Var(X) = 35/12 = 2.9167$. **Now,**

$$P\left(|X - \mu| > \epsilon\right) < \frac{\sigma^2}{\epsilon^2} = \frac{2.9167}{(2.5)^2}$$

Normal Probability rule and Chebyshev's inequality (Cont...)

$$P\left(|X-3.5|>2.5\right)$$

$=P(X$ **lies outside the limit** $3.5-2.5, 3.5+2.5) = 0$

Exercise 50/4.5/pp.147: For a normal distribution

$$P\left[|X-\mu|<3\sigma\right]=0.997$$

What value is associated to this probability via Chebyshev's inequality? Are the results consistent? Which rule gives a stronger statement in case of a normal variable?

Normal Probability rule and Chebyshev's inequality (Cont...)

Exercise A symmetric die is thrown 600 times. Find the lower bound for the probability of getting 80 to 120 sixes.

Exercise: For geometric distribution

$$f(x) = 2^{-x}; \qquad x = 1, 2, 3...$$

prove that Chebyshev's inequality gives

$$P\left[|x-2| \leq 2\right] > \frac{1}{2}$$

while the actual probability is $\frac{15}{16}$

Exercise: For the discrete variate with density

x:	-1	0	1
$f(x)$:	1/8	6/8	1/8

Evaluate

$$P\left[|X-\mu_x| \leq 2\sigma_x\right]$$

Compare this result with that obtained on using Chebyshev's inequality

Exercise: How many times should we toss a balanced coin so that we can assert with probability at least 0.99 that the proportion of heads occurs between 0.45 and 0.55?

Solution: **Let n be the number of trials and X be number of heads in n trials**

$$X \sim BD\left(n, p = \frac{1}{2}\right)$$

Hence, $\mu = \frac{n}{2}$ **and** $\sigma^2 = \frac{n}{4}$
Proportion of heads $\frac{X}{n}$ **is between 0.45 and 0.55**
if and only if $0.45n < X < 0.55n$

Normal Probability rule and Chebyshev's inequality (Cont...)

$$0.45n < X < 0.55n = |X - \mu| < 0.05n$$

By Chebyshev's inequality

$$P\left(|X - \mu| < k\sigma\right) \geq 1 - \frac{1}{k^2}$$

So, $k\sigma = k\frac{\sqrt{n}}{2} = 0.05n$. **Hence,** $k = 0.1\sqrt{n}$

Thus if we choose n **such that**

$$1 - \frac{1}{k^2} = 1 - \frac{100}{n} > 0.99$$

Hence, $n > 10000$

This justifies the relative frequency approach to the probability. The proportion of successes (i.e. relative frequency) can be brought as close to the actual probability by increasing the number of trials.

Exercise: If X is Gamma random variable with $\alpha = 0.05$ & $\beta = 100$, find an upper bound on

$$P([(X-4)(X-6)] \geq 999)$$

Exercise: From a usual pack of 52 cards, cards are drawn randomly with replacement till the red card appears. If X denotes the number of card drawn, using Chebyshev's inequality, find a lower bound for

$$P[|X-2|<2]$$

Normal Probability rule and Chebyshev's inequality (Cont...)

Exercise: In one out of 6 cases, material for bulletproof vests fails to meet puncture standards. If 405 specimens are tested, what does Chebyshev's theorem tell us about the probability of getting at most 30 or more than 105 cases that do not meet puncture standards?

Solution: Let X be number of bulletproof vests out of 405 which fail to meet puncture standards. Hence,

$$X \sim BD\left(n = 405, p = \frac{1}{6}\right)$$

So, $\mu = np = 67.5$ and $\sigma = \sqrt{npq} = 7.5$

Take $k\sigma = 37.5$ or $k = 5$

Required Probability

$$P\left(X \leq 30 \,\textbf{or}\, X > 105\right)$$

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$$P\left(X \leq 30 \,\mathbf{or}\, X > 105\right)$$

$$\leq P\left(X \leq 30 \,\mathbf{or}\, X \geq 105\right)$$

$$= P\left(|X - \mu| \geq k\sigma\right) \leq \frac{1}{k^2} = 0.04$$

Exercise 44/4.4/pp.146 Assuming that during seasons of normal rainfall the water level in feet at a particular lake follows a normal distribution with mean of 1876 feet and standard deviation of 6 inches.

(a) During such a season, would it be unusual to observe a water level of at most 1875 feet.

(b) Suppose that the water will crest the spillway if the water level exceeds 1878 feet. What is the probability that this will occur during a season of normal rainfall?

Exercise: A computer firm introduces a new home computer. Past experience shows that the random variable X, the time of peak demand measured in months after its introduction, follows a gamma distribution with variance 36

(a) **If the expected value of X is 18 months, find α and β**

(b) **Find** $P\left[X \leq 7.01\right]$, $P\left[X \geq 26\right]$, $P\left[13.7 \leq X \leq 31.5\right]$