Mathematics-II (MATH F112)

Linear Algebra

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Chapter: 4 (Finite Dimensional vector space)

1. Introduction to Vector Spaces
2. Subspaces
3. Span
4. Linear Independence
5. Basis and Dimension
6. Constructing Special Basis

Vector Space: A nonempty set $\mathcal{V}$ together with two operations **vector addition** (denoted as $\oplus$) and **scalar multiplication**(denoted as $\odot$) is said to be a (real) vector space if for every $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathcal{V}$ and for every $a, b \in \mathbb{R}$ the following properties hold:

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3. $(\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w} = \mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w})$ (Associativity)
4. There exists an element $0 \in \mathcal{V}$, called a **zero vector**, such that $\mathbf{u} \oplus 0 = \mathbf{u}$ (Existence of additive identity)

5. For each $\mathbf{u} \in \mathcal{V}$, there is an element $-\mathbf{u} \in \mathcal{V}$ such that $\mathbf{u} \oplus (-\mathbf{u}) = 0$ (Existence of additive inverse)

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9. $(ab) \odot \mathbf{u} = a \odot (b \odot \mathbf{u})$

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10. $1 \odot \mathbf{u} = \mathbf{u}$.

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The vector space $\mathcal{V} = \{0\}$ is called the trivial vector space.

Example 1: The set $\mathbb{R}$ of real numbers is a vector space with respect to the following operations:

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for all $a, \mathbf{u}, \mathbf{v} \in \mathbb{R}$.

Question: Does the set $\mathbb{R}^+$ of positive real numbers form a vector space under the above defined vector addition and scalar multiplication?

Example 2: The set $\mathbb{R}^+$ of a positive real numbers is a vector space with respect to the following operations:

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- $a \odot \mathbf{u} = \mathbf{u}^a$ (scalar multiplication)

for all $a \in \mathbb{R}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^+$.

Example 3: The set $\mathbb{R}^2 = \{[x_1, x_2] \mid x_1, x_2 \in \mathbb{R}\}$ is a vector space with respect to the following operations:

- $[x_1, x_2] \oplus [y_1, y_2] = [x_1 + y_1, x_2 + y_2]$ (vector addition)

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Question: Does $\mathbb{R}^2$ form a vector space under the above defined vector addition and the following scalar multiplication

$$a \odot [x_1, x_2] = [0, ax_2]$$

for all $a \in \mathbb{R}$ and $[x_1, x_2] \in \mathbb{R}^2$.

Soln. of Example 3: Let $\mathbf{u} = [x_1, x_2]$, $\mathbf{v} = [y_1, y_2]$ and $\mathbf{w} = [z_1, z_2] \in \mathbb{R}^2$ and $a, b \in \mathbb{R}$.

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$$\begin{aligned}\mathbf{u} \oplus \mathbf{v} &= [x_1 + y_1, x_2 + y_2] = [y_1 + x_1, y_2 + x_2] \\ &\qquad \text{(commutativity of } \mathbb{R} \text{ under addition)} \\ &= [y_1, y_2] \oplus [x_1, x_2] = \mathbf{v} \oplus \mathbf{u}\end{aligned}$$

3. Associative Property:
$$\begin{aligned}(\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w} &= [(x_1 + y_1) + z_1, (x_2 + y_2) + z_2] \\ &= [x_1 + (y_1 + z_1), x_2 + (y_2 + z_2)]\end{aligned}$$

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4. Existence of additive identity (zero vector):

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5. Existence of additive inverse:

$$
\begin{aligned}
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\mathbf{u} \oplus 0 = [x_1, x_2] \oplus [0, 0] &= [x_1 + 0, x_2 + 0] \\
&= [x_1, x_2] \\
&= \mathbf{u}
\end{aligned}
$$

5. Existence of additive inverse: For each $\mathbf{u} = [x_1, x_2] \in \mathbb{R}^2$ there exists $-\mathbf{u} = [-x_1, -x_2]$ in $\mathbb{R}^2$ such that

$$= [x_1, x_2] \oplus [y_1 + z_1, y_2 + z_2]$$
$$= [x_1, x_2] \oplus ([y_1, y_2] \oplus [z_1, z_2])$$
$$= \mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w})$$

4. Existence of additive identity (zero vector): For any $\mathbf{u} = [x_1, x_2] \in \mathbb{R}^2$ there exists $0 = [0, 0] \in \mathbb{R}^2$ such that
$$\mathbf{u} \oplus 0 = [x_1, x_2] \oplus [0, 0] = [x_1 + 0, x_2 + 0]$$
$$= [x_1, x_2]$$
$$= \mathbf{u}$$

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$$\mathbf{u} \oplus (-\mathbf{u}) = [x_1, x_2] \oplus [-x_1, -x_2]$$

$$
\begin{aligned}
&= [x_1, x_2] \oplus [y_1 + z_1, y_2 + z_2] \\
&= [x_1, x_2] \oplus ([y_1, y_2] \oplus [z_1, z_2]) \\
&= \mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w})
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4. Existence of additive identity (zero vector): For any $\mathbf{u} = [x_1, x_2] \in \mathbb{R}^2$ there exists $\mathbf{0} = [0, 0] \in \mathbb{R}^2$ such that
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\begin{aligned}
\mathbf{u} \oplus (-\mathbf{u}) &= [x_1, x_2] \oplus [-x_1, -x_2] \\
&= [x_1 + (-x_1), x_2 + (-x_2)]
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\mathbf{u} \oplus (-\mathbf{u}) &= [x_1, x_2] \oplus [-x_1, -x_2] \\
&= [x_1 + (-x_1), x_2 + (-x_2)] = [0, 0]
\end{aligned}
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$$
\begin{aligned}
&= [x_1, x_2] \oplus [y_1 + z_1, y_2 + z_2] \\
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&= [x_1 + (-x_1), x_2 + (-x_2)] = [0, 0] = 0
\end{aligned}
$$

6. Closure Property of scalar multiplication:

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$a \odot \mathbf{u}$

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7. Distributivity over vector addition:

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$$\begin{aligned} a \odot (\mathbf{u} \oplus \mathbf{v}) &= a \odot ([x_1, x_2] \oplus [y_1, y_2]) \\ &= a \odot [x_1 + y_1, x_2 + y_2] \end{aligned}$$

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6. **Closure Property of scalar multiplication:** $a \odot \mathbf{u} = a \odot [x_1, x_2] = [ax_1, ax_2] \in \mathbb{R}^2$. Thus, $\mathbb{R}^2$ is closed under scalar multiplication.

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6. **Closure Property of scalar multiplication:**
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$$
\begin{aligned}
a \odot (\mathbf{u} \oplus \mathbf{v}) &= a \odot ([x_1, x_2] \oplus [y_1, y_2]) \\
&= a \odot [x_1 + y_1, x_2 + y_2] \\
&= [a(x_1 + y_1), a(x_2 + y_2)] \\
&= [ax_1 + ay_1, ax_2 + ay_2] \text{ (distributivity in } \mathbb{R}) \\
&= [ax_1, ax_2] \oplus [ay_1, ay_2] \\
&= (a \odot [x_1, x_2]) \oplus (a \odot [y_1, y_2])
\end{aligned}
$$

6. Closure Property of scalar multiplication:
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a \odot (\mathbf{u} \oplus \mathbf{v}) &= a \odot ([x_1, x_2] \oplus [y_1, y_2]) \\
&= a \odot [x_1 + y_1, x_2 + y_2] \\
&= [a(x_1 + y_1), a(x_2 + y_2)] \\
&= [ax_1 + ay_1, ax_2 + ay_2] \text{ (distributivity in } \mathbb{R}) \\
&= [ax_1, ax_2] \oplus [ay_1, ay_2] \\
&= (a \odot [x_1, x_2]) \oplus (a \odot [y_1, y_2]) \\
&= (a \odot \mathbf{u}) \oplus (a \odot \mathbf{v})
\end{aligned}
$$

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$$(a+b) \odot \mathbf{u} \quad = (a+b) \odot [x_1, x_2]$$

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$$\begin{aligned}(a+b)\odot \mathbf{u} &= (a+b)\odot [x_1, x_2]\\ &= [(a+b)x_1, (a+b)x_2]\end{aligned}$$

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$$\begin{aligned}(a+b)\odot \mathbf{u} &= (a+b)\odot[x_1,x_2]\\ &= [(a+b)x_1,(a+b)x_2]\\ &= [ax_1+bx_1, ax_2+bx_2] \text{ (distributivity in } \mathbb{R})\\ &= [ax_1,ax_2]\oplus[bx_1,bx_2]\\ &= (a\odot[x_1,x_2])\oplus(b\odot[x_1,x_2])\\ &= (a\odot \mathbf{u})\oplus(b\odot \mathbf{u})\end{aligned}$$

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9. $(ab)\odot \mathbf{u}$

8. Distributivity over scalar addition:

$$\begin{aligned}(a+b)\odot \mathbf{u} &= (a+b)\odot[x_1,x_2]\\ &= [(a+b)x_1,(a+b)x_2]\\ &= [ax_1+bx_1, ax_2+bx_2] \text{ (distributivity in } \mathbb{R})\\ &= [ax_1,ax_2]\oplus[bx_1,bx_2]\\ &= (a\odot[x_1,x_2])\oplus(b\odot[x_1,x_2])\\ &= (a\odot\mathbf{u})\oplus(b\odot\mathbf{u})\end{aligned}$$

9. $(ab)\odot \mathbf{u} = (ab)\odot[x_1,x_2]$

8. Distributivity over scalar addition:

$$\begin{aligned}(a+b)\odot \mathbf{u} &= (a+b)\odot [x_1, x_2]\\ &= [(a+b)x_1, (a+b)x_2]\\ &= [ax_1 + bx_1, ax_2 + bx_2] \text{ (distributivity in } \mathbb{R})\\ &= [ax_1, ax_2] \oplus [bx_1, bx_2]\\ &= (a \odot [x_1, x_2]) \oplus (b \odot [x_1, x_2])\\ &= (a \odot \mathbf{u}) \oplus (b \odot \mathbf{u})\end{aligned}$$

9. $$\begin{aligned}(ab)\odot \mathbf{u} &= (ab)\odot [x_1, x_2]\\ &= [(ab)x_1, (ab)x_2]\end{aligned}$$

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9. $$\begin{aligned}(ab)\odot \mathbf{u} &= (ab)\odot [x_1, x_2] \\ &= [(ab)x_1, (ab)x_2] \\ &= [a(bx_1), a(bx_2)]\end{aligned}$$

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(associativity of $\mathbb{R}$ under multiplication)

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$$
\begin{aligned}
(a+b)\odot \mathbf{u} &= (a+b)\odot [x_1, x_2] \\
&= [(a+b)x_1, (a+b)x_2] \\
&= [ax_1 + bx_1, ax_2 + bx_2] \text{ (distributivity in } \mathbb{R}) \\
&= [ax_1, ax_2] \oplus [bx_1, bx_2] \\
&= (a \odot [x_1, x_2]) \oplus (b \odot [x_1, x_2]) \\
&= (a \odot \mathbf{u}) \oplus (b \odot \mathbf{u})
\end{aligned}
$$

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$$
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$$= a \odot [bx_1, bx_2]$$

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$$\begin{aligned}&= a\odot[bx_1,bx_2]\\ &= a\odot(b\odot[x_1,x_2])\end{aligned}$$

8. Distributivity over scalar addition:

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(associativity of $\mathbb{R}$ under multiplication)

$$\begin{aligned}&= a\odot[bx_1, bx_2]\\ &= a\odot(b\odot[x_1,x_2])\\ &= a\odot(b\odot \mathbf{u})\end{aligned}$$

10. $1 \odot \mathbf{u} =$

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Thus $\mathbb{R}^2$ is vector space under usual vector addition and scalar multiplication.

Example 4: The set $\mathbb{R}^n = \{[x_1, x_2, \ldots, x_n] \mid x_i \in \mathbb{R}\}$ is a vector space with respect to the following operations:

- $[x_1, x_2, \ldots, x_n] \oplus [y_1, y_2, \ldots, y_n]$
$$= [x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n] \text{ (vector addition)}$$

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- $[x_1, x_2, \ldots, x_n] \oplus [y_1, y_2, \ldots, y_n]$
  $= [x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n]$ (vector addition)
- $a \odot [x_1, x_2, \ldots, x_n] = [ax_1, ax_2, \ldots, ax_n]$ (scalar multiplication)

for all $a \in \mathbb{R}$ and $[x_1, x_2, \ldots, x_n], [y_1, y_2, \ldots, y_n] \in \mathbb{R}^n$.

Example 5: The set

$$\mathcal{M}_{mn} = \{[a_{ij}]_{m\times n} \mid a_{ij} \in \mathbb{R}\}$$

of all $m \times n$ matrices with real entries is a vector space with respect to the following operations:

- $[a_{ij}]_{m\times n} \oplus [b_{ij}]_{m\times n} = [a_{ij} + b_{ij}]_{m\times n}$ (vector addition)
- $a \odot [a_{ij}]_{m\times n} = [aa_{ij}]_{m\times n}$ (scalar multiplication)

for all $a \in \mathbb{R}$ and $[a_{ij}]_{m\times n}, [b_{ij}]_{m\times n} \in \mathcal{M}_{mn}$.

Example 6: Let

$$\Phi = \{f \mid f : \mathbb{R} \to \mathbb{R}\}$$

be the set of real-valued functions defined on $\mathbb{R}$. Define

$$f \oplus g = f + g \quad (\text{vector addition}),$$

and $$a \odot f = af \quad (\text{scalar multiplication}),$$

Example 6: Let

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Define

$$f \oplus g = f + g \quad (\text{vector addition}),$$

where $(f+g)(x) = f(x) + g(x)\ \forall x \in \mathbb{R}$.

and $$a \odot f = af \quad (\text{scalar multiplication}),$$

Example 6: Let

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Example 6: Let

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and $$a \odot f = af \quad (\text{scalar multiplication}),$$

where $(af)(x) = af(x)\ \forall x \in \mathbb{R}$.

Then Φ is a vector space with respect to above defined vector addition and scalar multiplication.

Example 7: Let

$$\mathcal{P}_2 = \{a_2x^2 + a_1x + a_0 \mid a_2, a_1, a_0 \in \mathbb{R}\}$$

be the set of all polynomials of degree ≤ 2 with real coefficients. Define addition and scalar multiplication in usual way i.e. if

$$p(x) = a_2x^2 + a_1x + a_0 \text{ and } q(x) = b_2x^2 + b_1x + b_0$$

are in $\mathcal{P}_2$, then

$$p(x) \oplus q(x) = (a_2 + b_2)x^2 + (a_1 + b_1)x + (a_0 + b_0)$$

$$c \odot p(x) = ca_2x^2 + ca_1x + ca_0.$$

Show that $\mathcal{P}_2$ is a vector space.

In general, for any fixed natural number n, the set

$$\mathcal{P}_n = \{a_n x^n + \cdots + a_1 x + a_0 \mid a_n, \ldots, a_1, a_0 \in \mathbb{R}\}$$

of all polynomials of degree less than or equal to n is a vector space. under the usual addition (term by term) and scalar multiplication of polynomials.

In general, for any fixed natural number n, the set

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of all polynomials of degree less than or equal to n is a vector space. under the usual addition (term by term) and scalar multiplication of polynomials.

Question: Does the set of all polynomials of degree 7 form a vector space under the usual operation of addition and scalar multiplication?

Example 8: The set $\mathcal{P}$ of all polynomials with real coefficients is a vector space under the usual operation of polynomial (term by term) addition and scalar multiplication.

Theorem: Let $\mathcal{V}$ be a vector space. Then for every $\mathbf{v} \in \mathcal{V}$ and $\alpha \in \mathbb{R}$, we have

- $\alpha \mathbf{0} = \mathbf{0}$
- $0\mathbf{v} = \mathbf{0}$
- $(-1)\mathbf{v} = -\mathbf{v}$
- If $\alpha\mathbf{v} = \mathbf{0}$, then $\alpha = 0$ or $\mathbf{v} = \mathbf{0}$.

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Note that every vector space $\mathcal{V}$ has at least two subspaces: $\{0\}$ and $\mathcal{V}$ itself. The subspace $\{0\}$ is known as trivial subspace.

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Question: Does the set

$$W = \{[x, y] \in \mathbb{R}^2 \mid x \neq y\}$$

form a subspace of $\mathbb{R}^2$?

Theorem: A **nonempty** subset W of a vector space $\mathcal{V}$ is a subspace of $\mathcal{V}$ if and only if the following conditions hold:

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- $a\mathbf{u} \in W$ for all $a \in \mathbb{R}, \mathbf{u} \in W$.

Remark: If W is a subspace of a vector space $\mathcal{V}$, then $0 \in W$.

Exercise: Examine whether the following sets are subspaces of the vector space $\mathbb{R}^3$.

- $W_1 = \{[x, y, z] \in \mathbb{R}^3 \mid x \geq 0\}$.

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Exercise: Examine whether the following sets are subspaces of the vector space $\mathcal{M}_{22}$ (see Example 5).

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- their union $W_1 \cup W_2$ **need not** be a subspace of $\mathcal{V}$.
- $W_1 \cup W_2$ is subspace of $\mathcal{V}$ if and only if either $W_1 \subset W_2$ or $W_2 \subset W_1$.
- their sum, defined as

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is a subspace of $\mathcal{V}$.

Section 4.3 (Span)

Question: Given a subset S of a vector space $\mathcal{V}$, how to construct a subspace containing S?

Linear combination: Let $\mathcal{V}$ be a vector space and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathcal{V}$. Then a vector $\mathbf{v} \in \mathcal{V}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ if

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Thus, $[3, 4]$ is a linear combination of $[1, 1]$ and $[1, 2]$ also.

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Exercise: Let $\mathcal{V} = \mathbb{R}^3$ and $S = \{[1,0,0],[0,1,0]\}$.

- Find span(S).
- Do $[3,2,0]$ and $[2,5,1]$ belong to span(S)?

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Solution:

$$\text{span}(S) = \{a[1,0,0] + b[0,1,0] \mid a,b \in \mathbb{R}\}$$

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Clearly, $[3, 2, 0] \in \text{span}(S)$ but $[2, 5, 1] \notin \text{span}(S)$.

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In this exercise **note that** span(S) is a subspace of $\mathbb{R}^3$.

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Theorem: Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a nonempty subset of a vector space $\mathcal{V}$. Then

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Convention: span($\emptyset$) $= \{0\}$.

Row space of a matrix: Let A be an $m \times n$ matrix. The row space of A, denoted by row(A), is the subspace of $\mathbb{R}^n$ spanned by the rows of A.

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Theorem: Let B be any matrix that is row equivalent to a matrix A. Then row$(B) =$ row(A).

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Theorem: Let B be any matrix that is row equivalent to a matrix A. Then row$(B) =$ row(A).

Corollary: For any matrix A, we have

$$\text{row}(A) = \text{row}\left(\text{RREF}(A)\right).$$

Exercise: Let $\mathcal{V} = \mathbb{R}^3$ and

$$S = \{[1, 3, -1], [2, 7, -3], [4, 8, -7]\}.$$

Then find span(S) in simplified form.

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$$\text{row}\left(\text{RREF}(A)\right)$$

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$$\text{row}\left(\text{RREF}(A)\right) = \text{span}(S) = \{[a,b,c] \mid a,b,c \in \mathbb{R}\}$$

$$\text{span}(S) = \mathbb{R}^3$$

Simplified Span Method: Let S be a finite subset of $\mathbb{R}^n$ containing k vectors, with $k \geq 2$.

Step 1: Construct a matrix A of order $k \times n$ by using the vectors in S as **the rows of** A.

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Step 1: Construct a matrix A of order $k \times n$ by using the vectors in S as **the rows of** A. Then span(S) $=$ row(A).

Step 2: Find RREF(A).

Step 3: Then, the set of all linear combinations of the **nonzero rows** of RREF(A) gives a simplified form for span(S).

Exercise: For a given vector space $\mathcal{V}$ and a subset S of $\mathcal{V}$, find a simplified general form of span(S) using Simplified Span Method:

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3. $\mathcal{V} = \mathcal{P}_2$, $S = \left\{x^2+4x-3, 2x^2+x+5, 7x-11\right\}$.

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3. $\mathcal{V} = \mathcal{P}_2$, $S = \left\{x^2+4x-3, 2x^2+x+5, 7x-11\right\}$.
4. $\mathcal{V} = \mathcal{M}_{22}$, $S = \left\{ \begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix}, \begin{bmatrix} -2 & -5 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ -3 & 4 \end{bmatrix} \right\}$.

Section 4.4 (Linear Independence)

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Definition: A subset $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of a vector space $\mathcal{V}$ is said to be linearly dependent (LD) if there exist real numbers $a_1, a_2, \ldots, a_n$ not all zero such that

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S is linearly independent (LI) if it is not linearly dependent i.e. if

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n = \mathbf{0}$$

Then

$$a_1 = a_2 = \cdots = a_n = 0.$$

Examples

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- The subset $S = \{[1, 0, 0], [0, 1, 0], [0, 0, 1]\}$ of $\mathbb{R}^3$ is linearly independent.

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- Let S be a finite set of nonzero vectors having at least two elements. Then S is LD if and only if some vector in S can be expressed as a linear combination of the other vectors in S.

Exercise: For a given vector space $\mathcal{V}$ and a given subset S of $\mathcal{V}$, check the linear independence of S in the following:

1. $\mathcal{V} = \mathcal{P}_2$, $S = \{1 + x + x^2, 1 - x + 3x^2, 1 + 3x - x^2\}$.

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3. $\mathcal{V} = \mathcal{P}_n$, $S = \{1, x, x^2, \ldots, x^n\}$.
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5. $\mathcal{V} = \Phi$, $S = \{\sin x, \cos x\}$.
6. $\mathcal{V} = \mathcal{M}_{22}$, $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

Exercise: For a given vector space $\mathcal{V}$ and a given subset S of $\mathcal{V}$, check the linear independence of S in the following:

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7. $\mathcal{V} = \mathcal{M}_{22}$, $S = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \right\}$.

Exercise: Show that

$$S = \{[3,1,-1],[-5,-2,2],[2,2,-1]\}$$

is linearly independent subset of $\mathbb{R}^3$.

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Solution: Let $a, b, c \in \mathbb{R}$ such that

$$a[3, 1, -1] + b[-5, -2, 2] + c[2, 2, -1] = \mathsf{0}$$

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Solution: Let $a, b, c \in \mathbb{R}$ such that

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Solution: Let $a, b, c \in \mathbb{R}$ such that

$$a[3, 1, -1] + b[-5, -2, 2] + c[2, 2, -1] = 0$$

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$$[3a - 5b + 2c, a - 2b + 2c, -a + 2b - c] = [0, 0, 0]$$

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To solve above homogenous system, write augmented matrix

$$[A|\mathbf{0}] = \left[\begin{array}{rrr|r} 3 & -5 & 2 & 0 \\ 1 & -2 & 2 & 0 \\ -1 & 2 & -1 & 0 \end{array}\right]$$

reduced row echelon form of $[A|0]$ is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Thus, we have $a = 0, b = 0, c = 0$.

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Thus, we have $a = 0, b = 0, c = 0$. Hence, S is linearly independent subset of $\mathbb{R}^3$.

Independence Test Method: Let S be a finite set of vectors in $\mathbb{R}^n$. To check whether S is LI, perform the following steps:

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Step 2: Find RREF(A).

Step 3: If there is a pivot in every column of RREF(A), then S is LI. Otherwise S is LD.

Theorem: If S is any subset of $\mathbb{R}^n$ containing k distinct vectors, where $k > n$, then S is linearly dependent.

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Exercise: Examine the linear independence of a subset $S = \{[2, -5, 1], [1, 1, -1], [0, 2, -3], [2, 2, 6]\}$ of $\mathbb{R}^3$.

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Result: Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a linearly independent subset of a vector space $\mathcal{V}$. If $\mathbf{v} \in \mathcal{V}$ and $\mathbf{v} \notin \text{span}(S)$, then $S_1 = \{\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a linearly independent set.

Theorem: A nonempty finite subset S of a vector space $\mathcal{V}$ is LI iff every vector $\mathbf{v} \in \text{span}(S)$ can be expressed **uniquely** as a linear combination of the elements of S.

Definition: An **infinite** subset S of a vector space $\mathcal{V}$ is linearly dependent if there is some finite subset T of S such that T is linearly dependent.

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Example: The subset

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Section 4.5, Basis and Dimension

Basis: A subset B of a vector space $\mathcal{V}$ is said to be a basis of $\mathcal{V}$ if

1. B is LI, and
2. $\text{span}(B) = \mathcal{V}$.

Examples

- The subset $B = \{[1, 0], [0, 1]\} = \{e_1, e_2\}$ is a basis of $\mathbb{R}^2$ as B is LI and span$(B) = \mathbb{R}^2$.

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- The subset $B = \{[1,0,0],[0,1,0],[0,0,1]\}$, also denoted by $\{e_1, e_2, e_3\}$, is a basis of $\mathbb{R}^3$ as it is LI and span$(B) = \mathbb{R}^3$.

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Think about some more basis of $\mathbb{R}^2$ and $\mathbb{R}^3$.

- The subset $B = \{1, x, x^2, \ldots, x^n\}$ is a basis of $\mathcal{P}_n$ as B is LI (verify!) and span$(B) = \mathcal{P}_n$ (verify!).

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- The subset

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

  is a basis of $\mathcal{M}_{22}$.

- The subset $B = \{1, x, x^2, \ldots, x^n\}$ is a basis of $\mathcal{P}_n$ as B is LI (verify!) and span$(B) = \mathcal{P}_n$ (verify!).
- The subset

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is a basis of $\mathcal{M}_{22}$.

Verify that B is LI and span$(B) = \mathcal{M}_{22}$.

Theorem: Every vector space has a basis.

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Theorem: If a vector space V has a finite basis, then all bases for V are finite and have the same number of elements.

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Examples

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- $\dim(\mathbb{R}^n) = n$.
- $\dim(\mathcal{P}_n) = n + 1$.
- $\dim(\mathcal{M}_{mn}) = mn$.
- Since $\{1, x, x^2, x^3, \ldots\}$ is a basis of $\mathcal{P}$ (the vector space of all polynomials with real coefficients), thus the vector space $\mathcal{P}$ is infinite dimensional.

Exercise: Find a basis and the dimension of a subspace W of $\mathbb{R}^3$, where

$$W = \{[x, y, z] \in \mathbb{R}^3 \mid x + 2z = 0\}.$$

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Solution: The general solution of the equation $x + 2z = 0$ is given by $\{[-2s, t, s] \mid t, s \in \mathbb{R}\}$. Thus

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$$W = \{s[-2, 0, 1] + t[0, 1, 0] \mid t, s \in \mathbb{R}\}$$

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$$\begin{aligned} W &= \{[-2s, t, s] \mid t, s \in \mathbb{R}\} \\ W &= \{s[-2, 0, 1] + t[0, 1, 0] \mid t, s \in \mathbb{R}\} \\ W &= \text{span}\left(\{[-2, 0, 1], [0, 1, 0]\}\right). \end{aligned}$$

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Solution: The general solution of the equation $x + 2z = 0$ is given by $\{[-2s, t, s] \mid t, s \in \mathbb{R}\}$. Thus

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$$W = \{s[-2, 0, 1] + t[0, 1, 0] \mid t, s \in \mathbb{R}\}$$
$$W = \text{span}\left(\{[-2, 0, 1], [0, 1, 0]\}\right).$$

Note that the set $\{[-2, 0, 1], [0, 1, 0]\}$ is linearly independent (show it).

Hence, the subset $\{[-2, 0, 1], [0, 1, 0]\}$ is a basis of W and $\dim(W) = 2$.

Hence, the subset $\{[-2, 0, 1], [0, 1, 0]\}$ is a basis of W and dim$(W) = 2$.

Exercise: Find a basis and the dimension of a subspace W of $\mathcal{P}_3$, where

$$W = \{\mathbf{p} \in \mathcal{P}_3 \mid \mathbf{p}(2) = 0\}.$$

Theorem: Let W be a subspace of a finite dimensional vector space $\mathcal{V}$. Then

- W is also finite dimensional and $\dim W \leq \dim \mathcal{V}$.
- $\dim W = \dim \mathcal{V}$ if and only if $W = \mathcal{V}$.

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Exercise: For a given vector space $\mathcal{V}$ and a given subset B of $\mathcal{V}$, determine which of following B form a basis of the respective vector space $\mathcal{V}$:

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4. $\mathcal{V} = \mathcal{P}_2$, $B = \{1 - x, x - x^2, 1 - x^2\}$.
5. $\mathcal{V} = \mathcal{M}_{22}$,

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right\}.$$

Exercise: Let $S = \{[1,2],[3,0],[0,2]\} \subseteq \mathbb{R}^2$.

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- span(S) = span(B);
- B is LI.

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Exercise: Let $W = \{X \in \mathbb{R}^5 : AX = 0\}$,

Exercise: Let $W = \{X \in \mathbb{R}^5 : AX = 0\}$, where

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 & -1 \\ 2 & -1 & 0 & 1 & 3 \\ 1 & -3 & -1 & 1 & 4 \\ 2 & 9 & 4 & -1 & -7 \end{bmatrix}.$$

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Find a basis for W.

Answer:

$$B = \left\{ \begin{bmatrix} -\frac{1}{5} \\ -\frac{2}{5} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{2}{5} \\ \frac{1}{5} \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Exercise: Let $S = \{[4, 2, 1], [2, 6, -5], [1, -2, 3]\}$ be a subset of vector space $\mathbb{R}^3$.

- Examine the linear independence of S.

Exercise: Let $S = \{[4, 2, 1], [2, 6, -5], [1, -2, 3]\}$ be a subset of vector space $\mathbb{R}^3$.

- Examine the linear independence of S.
- Find $\dim(\text{span}(S))$.

Exercise: Let $S = \{[4,2,1],[2,6,-5],[1,-2,3]\}$ be a subset of vector space $\mathbb{R}^3$.

- Examine the linear independence of S.
- Find dim(span(S)).

Hint:

- Let

$$a_1[4,2,1] + a_2[2,6,-5] + a_3[1,-2,3] = \mathbf{0} = [0,0,0]$$

On solving above system of equations, we get

$$a_1 = -1, a_2 = 1, a_3 = 2$$

implies S is not LI.

- Note that

$$[2,6,-5] = [4,2,1] - 2[1,-2,3]$$

implies span$(S) =$ span(B), where

$$B = \{[4,2,1],[1,-2,3]\}.$$

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Now, note that B is LI (Show it). Thus B (a set of two elements) is a basis of span(S) and

$$\text{dim}(\text{span}(S)) = 2$$

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Simplified Span Method to find a Basis for span(S)

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Let $S \subseteq \mathbb{R}^n$ containing k vectors.

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- Compute $C = \text{RREF}(A)$.

Simplified Span Method to find a Basis for span(S)

Let $S \subseteq \mathbb{R}^n$ containing k vectors.

- Construct a matrix A of order $k \times n$ by using vectors of S **as rows of** A.
- Compute $C = \text{RREF}(A)$.
- **Nonzero rows** of C forms a basis for span(S).

Example: Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, where

$$\mathbf{v}_1 = [1, 2, 3, -1, 0],\ \mathbf{v}_2 = [3, 6, 8, -2, 0]$$

$$\mathbf{v}_3 = [-1, -1, -3, 1, 1],\ \mathbf{v}_4 = [-2, -3, -5, 1, 1]$$

be a subset of $\mathbb{R}^5$.

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be a subset of $\mathbb{R}^5$. Find a basis for span(S).

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be a subset of $\mathbb{R}^5$. Find a basis for span(S).

Solution:

Step 1:

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 & 0 \\ 3 & 6 & 8 & -2 & 0 \\ -1 & -1 & -3 & 1 & 1 \\ -2 & -3 & -5 & 1 & 1 \end{bmatrix}$$

Step 2:

$$C = \text{RREF}(A) = \begin{bmatrix} 1 & 0 & 0 & 2 & -2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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Step 3:

$$B = \{[1,0,0,2,-2],[0,1,0,0,1],[0,0,1,-1,0]\}$$

is a basis for span(S).

Exercise: Let

$$S = \{x^3 - 3x^2 + 2, 2x^3 - 7x^2 + x - 3, 4x^3 - 13x^2 + x + 5\}$$

be a subset of $\mathcal{P}_3$. Find a basis for span(S).

Exercise: Let

$$S = \{x^3 - 3x^2 + 2, 2x^3 - 7x^2 + x - 3, 4x^3 - 13x^2 + x + 5\}$$

be a subset of $\mathcal{P}_3$. Find a basis for span(S).

Answer: $B = \{x^3 - 3x, x^2 - x, 1\}$.

Next is to reduce a spanning set to a basis

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Theorem: If S is a spanning set for a finite dimensional vector space $\mathcal{V}$, then there is a set $B \subseteq S$ that is a basis for $\mathcal{V}$.

Independence Test Method to find a Basis for $\mathcal{V} =$ span(S)

Let $S \subseteq \mathbb{R}^n$ containing k vectors.

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Independence Test Method to find a Basis for $\mathcal{V} = \textbf{span}(S)$

Let $S \subseteq \mathbb{R}^n$ containing k vectors.

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Independence Test Method to find a Basis for $\mathcal{V} = \textbf{span}(S)$

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Independence Test Method to find a Basis for $\mathcal{V} = \textbf{span}(S)$

Let $S \subseteq \mathbb{R}^n$ containing k vectors.

- Construct a matrix A of order $n \times k$ by using vectors of S **as columns of** A.
- Compute $C = \text{RREF}(A)$.
- Column vectors in A corresponding to **pivot columns** of C forms a basis for $\text{span}(S)$.

Example: Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$, where

$$\mathbf{v}_1 = [1, 2, -2, 1], \;\; \mathbf{v}_2 = [-3, 0, -4, 3]$$

$$\mathbf{v}_3 = [2, 1, 1, -1], \;\; \mathbf{v}_4 = [-3, 3, -9, 6]$$

$$\text{and } \; \mathbf{v}_5 = [9, 3, 7, -6]$$

be a subset of $\mathbb{R}^4$.

Example: Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$, where

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be a subset of $\mathbb{R}^4$. Find a basis for span(S).

Example: Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$, where

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$$\text{and } \mathbf{v}_5 = [9, 3, 7, -6]$$

be a subset of $\mathbb{R}^4$. Find a basis for span(S).

Solution:

$$A = \begin{bmatrix} 1 & -3 & 2 & -3 & 9 \\ 2 & 0 & 1 & 3 & 3 \\ -2 & -4 & 1 & -9 & 7 \\ 1 & 3 & -1 & 6 & -6 \end{bmatrix}$$

$$C = \text{RREF}(A) = \begin{bmatrix} 1 & 0 & 1/2 & 3/2 & 3/2 \\ 0 & 1 & -1/2 & 3/2 & -5/2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

implies

$$C = \text{RREF}(A) = \begin{bmatrix} 1 & 0 & 1/2 & 3/2 & 3/2 \\ 0 & 1 & -1/2 & 3/2 & -5/2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

implies the set of vectors corresponding to pivot columns i.e.

$$B = \{[1, 2, -2, 1], [-3, 0, -4, 3]\}$$

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forms a basis for the subspace span(S).

Exercise: Let

$$S = \{x^3 - 3x^2 + 1, 2x^2 + x, 2x^3 + 3x + 2, 4x - 5\}$$

be a subset of $\mathcal{P}_3$. Find a basis for span(S).

Exercise: Let

$$S = \{x^3 - 3x^2 + 1, 2x^2 + x, 2x^3 + 3x + 2, 4x - 5\}$$

be a subset of $\mathcal{P}_3$. Find a basis for span(S).

Answer: $B = \{x^3 - 3x^2 + 1, 2x^2 + x, 4x - 5\}$.

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Note that the set S is **NOT** a basis for $\mathbb{R}^4$ (Why?).

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$$B_1 = \{[1,0,1,0], [-1,1,-1,0], e_1, e_2, e_3, e_4\}.$$

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Now B' spans $\mathbb{R}^4$ implies span$(B_1) = \mathbb{R}^4$. Consider

$$A = \begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C = \text{RREF}(A) = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C = \text{RREF}(A) = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Using Independence Test Method, we have

$$C = \text{RREF}(A) = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Using Independence Test Method, we have

$$B = \{[1, 0, 1, 0], [-1, 1, -1, 0], e_1, e_4\}$$

is a basis of $\mathbb{R}^4$ containing S.

Theorem: Every LI subset of a finite dimensional vector space $\mathcal{V}$ can be extended to form a basis of $\mathcal{V}$.

Method for finding a Basis by Enlarging a LI Subset

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Let $T = \{t_1, \ldots, t_k\}$ be a LI subset of a finite dimensional vector space $\mathcal{V}$.

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Let $T = \{t_1, \ldots, t_k\}$ be a LI subset of a finite dimensional vector space $\mathcal{V}$.

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Method for finding a Basis by Enlarging a LI Subset

Let $T = \{t_1, \ldots, t_k\}$ be a LI subset of a finite dimensional vector space $\mathcal{V}$.

- Find a spanning set $A = \{a_1, \ldots, a_n\}$ for $\mathcal{V}$.
- Form the ordered spanning set

$$S = \{t_1, \ldots, t_k, a_1, \ldots, a_n\}$$

for $\mathcal{V}$.

Method for finding a Basis by Enlarging a LI Subset

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$$S = \{t_1, \ldots, t_k, a_1, \ldots, a_n\}$$

  for $\mathcal{V}$.
- Use Independence Test Method to produce a subset B of S.

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- Find a spanning set $A = \{a_1, \ldots, a_n\}$ for $\mathcal{V}$.
- Form the ordered spanning set

  $$S = \{t_1, \ldots, t_k, a_1, \ldots, a_n\}$$

  for $\mathcal{V}$.
- Use Independence Test Method to produce a subset B of S. Then B is a basis for $\mathcal{V}$ containing T.

Example: Let

$$T = \{x^3 - x^2, x^4 - 3x^3 + 5x^2 - x\}$$

be a LI subset of $\mathcal{P}_4$.

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Solution: By considering the polynomials in T as vectors in $\mathbb{R}^5$, note that

$$T = \{\mathbf{v}_1, \mathbf{v}_2\},$$

where $\mathbf{v}_1 = [0, 1, -1, 0, 0]$ and $\mathbf{v}_2 = [1, -3, 5, -1, 0]$.

Step 1: We know that $A = \{e_1, e_2, e_3, e_4, e_5\}$ is a standard basis for $\mathbb{R}^5$.

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of $\mathbb{R}^5$.

Step 3:

$$C = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -3 & 0 & 1 & 0 & 0 & 0 \\ -1 & 5 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{RREF}(C) = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & -5 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

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Since columns I, II, III, IV and VII are pivot columns, so by independent test method, the set

$$B = \{\mathbf{v}_1, \mathbf{v}_2, e_1, e_2, e_5\}$$

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Since columns I, II, III, IV and VII are pivot columns, so by independent test method, the set

$$B = \{\mathbf{v}_1, \mathbf{v}_2, e_1, e_2, e_5\}$$

is a basis of span(S), i.e.

$$B = \{x^3 - x^2, x^4 - 3x^3 + 5x^2 - x, x^4, x^3, 1\}$$

is a basis of $\mathcal{P}_4$ containing T.

Exercise: Let

$$T = \left\{ \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} \right\}$$

be a LI subset of $\mathcal{M}_{32}$.

Exercise: Let

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be a LI subset of $\mathcal{M}_{32}$. Extend T to form a basis for $\mathcal{M}_{32}$.

Answer:

$$B = \left\{ \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

Thank You