(Q1) Let S be the event that he goes to the sea.
R: he goes to the river
L: he goes to the lake
and F: he catches fish. [2]

The given information may then be summarized as,

$P(S) = 0.50$, $P(F|S) = 0.80$
$P(R) = 0.25$, $P(F|R) = 0.40$
$P(L) = 0.25$, $P(F|L) = 0.60$ [2]

(a) As the events S, R and L are mutually exclusive and collectively exhaustive, using the law of total probability, [1]

$$P(F) = P(S)P(F|S) + P(R)P(F|R) + P(L)P(F|L)$$
$$= (0.50 \times 0.80) + (0.25 \times 0.40) + (0.25 \times 0.60)$$
$$= 0.65$$ [2]

(b) From (a), we get $P(\bar{F}) = 1 - P(F) = 0.35 = \frac{7}{20}$

Now using Bayes' theorem,

$$P(S|\bar{F}) = \frac{P(S)P(\bar{F}|S)}{P(\bar{F})} = \frac{P(S)[1-P(F|S)]}{P(\bar{F})} = \frac{2}{7}$$ [3]

Similarly, $P(R|\bar{F}) = \dfrac{P(R)[1-P(F|R)]}{P(\bar{F})} = \dfrac{3}{7}$ [3]

and, $P(L|\bar{F}) = \dfrac{P(L)[1-P(F|L)]}{P(\bar{F})} = \dfrac{2}{7}$ [3]

Therefore, it is most likely that he has been to the river. [1]

—x— x — x —x—

Note: (i) marks will be deducted if you use the law of total probability or Bayes' theorem without mentioning the events to be mutually exclusive and collectively exhaustive.

(ii) for part (b), if you simply find $P(S \cap \bar{F})$, $P(R \cap \bar{F})$ and $P(L \cap \bar{F})$ and conclude from there, [4] marks will be awarded. However, a proper justification would enable getting full marks [10].

Q2(a) To find k, $\sum_{\forall x} f(x) = 1$ (1M)

i.e $k\sum_{x=3}^{5} (x-3)^2 = 1 \Rightarrow k(0+1+4) = 1 \Rightarrow k = 1/5$

$\therefore \quad f(x) = \frac{1}{5}(x-3)^2, \; x = 3,4,5$
$\qquad\qquad 0, \text{ e.w.}$ (3M)

Moment generating function

$$m_X(t) = E\left[e^{tx}\right] = \sum_{\forall x} e^{tx} f(x)$$

$$= \frac{1}{5}\sum_{x=3}^{5} e^{tx}(x-3)^2 = \frac{1}{5}\left[e^{3t}\cdot 0 + e^{4t}\cdot 1 + e^{5t}\cdot 4\right]$$

$$= \frac{e^{4t} + 4e^{5t}}{5},$$ (4M)

(b) Mean $\mu = E(X) = \frac{d}{dt} m_X(t)\Big|_{t=0}$

$$= \frac{d}{dt}\left[\frac{e^{4t}+4e^{5t}}{5}\right]_{t=0} = \frac{1}{5}\left[4e^{4t} + 20e^{5t}\right]_{t=0}$$

$$= \frac{24}{5} = \boxed{4.8}$$ (4M)

Variance $\sigma^2 = E[x^2] - \{E[x]\}^2$

$$\therefore E(x^2) = \frac{d^2}{dt^2} m_X(t)\Big|_{t=0} = \frac{1}{5}\left[16e^{4t} + 100e^{5t}\right]_{t=0}$$

$$= \frac{116}{5} = \boxed{23.2}$$ (4M)

$$\therefore \sigma^2 = 23.2 - (4.8)^2 = \frac{4}{25} = \boxed{0.16}$$ (2M)

$\boxed{\text{Sol}^{n}\text{-}3(a)}$ X : Number of operative airplane engines — 1

$X \sim B(5, p)$ for 5 - engine plane.

$X \sim B(3, p)$ for 3 engine plane. — 1

Prob. that a 3 - engine plane is operative is

$P[X \geq 2] = P[X=2] + P[X=3]$

$= {}^3C_2\, p^2 (1-p) + p^3.$ — 1

Whereas, corresponding prob. for 5 - engine plane is

$P[X \geq 3] = P[X=3] + P[X=4] + P[X=5]$

$= {}^5C_3\, p^3 (1-p)^2 + {}^5C_4\, p^4 (1-p) + p^5$ — 1

Hence the 5 - engine plane is preferable if

${}^5C_3\, p^3(1-p)^2 + {}^5C_4\, p^4 (1-p) + p^5 > {}^3C_2\, p^2 (1-p) + p^3$ — 2

$\Rightarrow 10p^3(1-p)^2 + 5p^4(1-p) + p^5 > 3p^2(1-p) + p^3$

$\Rightarrow 10p(1-p)^2 + 5p^2(1-p) + p^3 > 3(1-p) + p \quad (\because 0 < p < 1)$

$\Rightarrow 10p(1-p)^2 + 5p^2(1-p) + p^3 - p > 3(1-p)$

$\Rightarrow 10p(1-p)^2 + 5p^2(1-p) \cancel{+} - p(1-p)(1+p) > 3(1-p)$

$\Rightarrow 10p(1-p) + 5p^2 - p(1+p) > 3 \quad (\because 1-p > 0)$

$\Rightarrow 10p - 10p^2 + 5p^2 - p - p^2 - 3 > 0$

$\Rightarrow -6p^2 + 9p - 3 > 0$

$\Rightarrow -2p^2 + 3p - 1 > 0$ — 2

$\Rightarrow -2p^2 + 2p + p - 1 > 0$

$\Rightarrow -2p(p-1) + 1(p-1) > 0$

$\Rightarrow (-2p+1)(p-1) > 0$ — 1

$\Rightarrow -2p + 1 < 0 \quad (\because p - 1 < 0)$

$\Rightarrow -2p < -1$

$\Rightarrow p > 1/2.$ — 1

$\underline{Sol^n - 3(b)}$ Let X be a poisson random variable with parameter k.

$$f(x) = \frac{e^{-k} k^x}{x!}, \quad x = 0, 1, 2, \cdots \quad k > 0 \quad — (1)$$

$$M_X(t) = E[e^{tX}] \quad — (2)$$

$$= \sum_{\text{all } x} e^{tx} f(x)$$

$$= \sum_{x=0}^{\infty} \frac{e^{tx} e^{-k} k^x}{x!} \quad — (1)$$

$$= e^{-k} \sum \frac{(e^t . k)^x}{x!} \quad — (2)$$

$$= e^{-k} \left[1 + \frac{ke^t}{1!} + \frac{(ke^t)^2}{2!} + \cdots \right] \quad — (1)$$

$$= e^{-k} (e^{ke^t}) \quad \left[\because e^x = 1 + x + \frac{x^2}{2!} + \cdots \right] — (1)$$

$$= e^{ke^t - k}$$

$$= e^{k(e^t - 1)} \quad — (1)$$

Q. 4. The diameter of an electric cable X is a continuous random variable with probability density function

$$f(x) = \begin{cases} kx(1-x) & ; \; 0 \le x < 1 \\ 0 & ; \; \text{elsewhere.} \end{cases}$$

(a) Find the value of k. Hence, **(b)** find $E[e^X]$, and **(c)** calculate $P(X \le \frac{1}{2} | \frac{1}{3} < X < \frac{2}{3})$. **[17]**

Soln. (a) We find the value of k by knowing that

$$\int_{-\infty}^{\infty} f(t)dt = \int_0^1 kt(1-t)dt = 1$$
$$\Rightarrow k\left[\frac{t^2}{2} - \frac{t^3}{3}\right] = k\left[\frac{1}{2} - \frac{1}{3}\right] = 1$$
$$\Rightarrow k = 6$$

3

(b) Next,

$$E[e^X] = \int_{-\infty}^{\infty} e^x f(x)dx$$
$$= 6\int_0^1 (x - x^2)e^x dx$$
$$= 6(3-e) = 18 - 6e$$
$$= 1.69$$

6

(c) Next, Define the events

$$A = \left\{X \le \frac{1}{2}\right\} \quad \text{and} \quad B = \left\{\frac{1}{3} < X < \frac{2}{3}\right\}$$

The required probability is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Note that

$$P(B) = \int_{\frac{1}{3}}^{\frac{2}{3}} 6t(1-t)dt = \left[3t^2 - 2t^3\right]_{\frac{1}{3}}^{\frac{2}{3}}$$
$$= \left[\left(\frac{3 \times 4}{9} - \frac{2 \times 8}{27}\right) - \left(\frac{3 \times 1}{9} - \frac{2 \times 1}{27}\right)\right]$$
$$= \left[\frac{20}{27} - \frac{7}{27}\right] = \frac{13}{27}$$

3

and

$$
\begin{aligned}
P(A \cap B) = P\left(\frac{1}{3} < X < \frac{1}{2}\right) &= \int_{\frac{1}{3}}^{\frac{1}{2}} 6t(1-t)dt \\
&= \left[3t^2 - 2t^3\right]_{\frac{1}{3}}^{\frac{1}{2}} \\
&= \left[\left(\frac{3}{4} - \frac{2}{8}\right) - \left(\frac{3}{9} - \frac{2}{27}\right)\right] \\
&= \left(\frac{1}{2} - \frac{7}{27}\right) = \frac{13}{54}
\end{aligned}
$$

3

Hence the required probability is

$$
P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{13}{54}}{\frac{13}{27}} = \frac{1}{2} = 0.50
$$

2

Q5

Let X be a gamma random variable with parameters α and β. Then find m.g.f for X , taking help of m.g.f. find mean and variance of X.

$$f(x)=\begin{cases}\dfrac{1}{(\Gamma(\alpha))\beta^{\alpha}}x^{\alpha-1}e^{-x/\beta}, & x>0,\alpha>0,\beta>0\\ 0, & \text{e.w.}\end{cases} \quad [2]$$

Comment : if any of $x,\alpha,\beta>0$ is not written (-1)

$$m_x(t)=E[e^{tx}]$$

$$=\int_0^{\infty}e^{tx}\frac{1}{\Gamma(\alpha)\beta^{\alpha}}x^{\alpha-1}e^{-x/\beta}\,dx \quad [2]$$

$$=\frac{1}{\Gamma(\alpha)\beta^{\alpha}}\int_0^{\infty}x^{\alpha-1}e^{-(\frac{1}{\beta}-t)x}\,dx$$

$$\text{let } z=(1-\beta t)\frac{x}{\beta}\Rightarrow x=\frac{z\beta}{(1-\beta t)}$$

$$\text{and } dx=\frac{\beta\,dz}{(1-\beta t)} \qquad t<1/\beta \quad [2]$$

$$x=0\Rightarrow z=0 \quad \& \quad x=\infty\Rightarrow z=\infty$$

$$=\frac{1}{\Gamma(\alpha)\beta^{\alpha}}\int_0^{\infty}\left(\frac{\beta z}{1-\beta t}\right)^{\alpha-1}e^{-z}\,\frac{\beta\,dz}{(1-\beta t)}$$

$$=\frac{1}{\Gamma(\alpha)\beta^{\alpha}}\frac{\beta^{\alpha}}{(1-\beta t)^{\alpha}}\int_0^{\infty}z^{\alpha-1}e^{-z}\,dz \quad \begin{matrix}[2]\\ [2]\end{matrix}$$

$$=\frac{1}{\Gamma(\alpha)\beta^{\alpha}}\frac{\beta^{\alpha}}{(1-\beta t)^{\alpha}}\Gamma(\alpha)$$

$$m_x(t)=(1-\beta t)^{-\alpha},\ t<\frac{1}{\beta} \quad [2]$$

Comment : If limits in integral are written as $-\infty$ to ∞ but moment generating and density Function of X are correctly written but $x,\alpha,\beta>0$ not written 2 out of 12

: If limits in integral are not written but moment generating and density Function is correct 4 out of 12

If $t<1/\beta$ is not written but all other things are correct [12-1]

$$E[X]=\left.\frac{dm_X(t)}{dt}\right|_{t=0}=-\alpha(1-\beta t)^{-\alpha-1}(-\beta)\Big|_{t=0}$$

$$=\alpha\beta \quad [1]$$

$$E[X^2]=\left.\frac{d^2m_X(t)}{dt^2}\right|_{t=0}=\alpha\beta\left.\frac{d(1-\beta t)^{-\alpha-1}}{dt}\right|_{t=0}$$

$$=-\alpha\beta\beta(-\alpha-1)(1-\beta t)^{-\alpha-2}\Big|_{t=0}=\alpha(\alpha+1)\beta^2[3]$$

$$Var[X]=E(X^2)-\left[E(X)\right]^2=\alpha(\alpha+1)\beta^2-(\alpha\beta)^2$$

$$=\alpha\beta^2.[2]$$

Q6 A company has installed 10000 electric bulbs in a metro city. Given that 10% of the bulbs are likely to fail after 744 hours of burning while 10% of the bulbs are likely to survive after 1256 hours of burning. Assuming normality, how many bulbs are expected to burn between 800 and 1200 hours.

($F(-1) = 0.1587$, $F(-1.28) = 0.1$, $F(1.42) = 0.9222$, $F(2.31) = 0.9896$)

Sol. Let X be the life of a bulb in burning hours with mean μ and S.D. σ. Then $Z = \frac{X-\mu}{\sigma}$ is a standard normal variate. [1]

By the given,

$P[X > 1256] = 0.1$ — [2]

$P[X < 744] = 0.1$,

$F(-1.28) = 0.1 \Rightarrow P[Z < -1.28] = 0.1$ — [2]

Again, by the symmetry of the normal distribution,

$P[Z > 1.28] = 0.1$. — [2]

It follows that

$$\frac{744-\mu}{\sigma} = -1.28 \quad \text{— (1)}$$

$$\frac{1256-\mu}{\sigma} = 1.28 \quad \text{— (2)} \qquad \text{— [2]}$$

Solving (1) and (2), we get

$\mu = 1000$, $\sigma = 200$ — [2]

Now $P[800 < X < 1200] = P[-1 < Z < 1]$

$= F(1) - F(-1)$

$= 1 - 2F(-1)$ [4]

$= 1 - 2(0.1587) = 0.6826$

Therefore out of 10000 bulbs, it is expected that 6826 will burn between 800 and 1200 hours. — [2]