Chapter 3 A.P. French

Lect: 4, 12, 19

Tut. 13, 15, 17

Suggest List. 1,2,3,5,6, 14, 16

Not in course. 7,8,9,10,11 and 18.

3.13.

$Given,\ x = Ae^{-\alpha t}\cos\omega t = Ae^{-\alpha t + i\omega t}$

$\therefore \dfrac{dx}{dt} = A(-\alpha + i\omega)e^{-\alpha t + i\omega t} = (-\alpha + i\omega)x;\ \dfrac{d^2x}{dt^2} = (-\alpha + i\omega)\dfrac{dx}{dt} = (-\alpha + i\omega)^2 x$

$\therefore \dfrac{d^2x}{dt^2} + \gamma\dfrac{dx}{dt} + \omega_0^2 x = (-\alpha + i\omega)^2 x + \gamma(-\alpha + i\omega)x + \omega_0^2 x = 0$

$\Rightarrow \alpha^2 - \omega^2 + \omega_0^2 - \alpha\gamma + 2i\alpha\omega + i\omega\gamma = 0 \therefore \alpha^2 - \omega^2 + \omega_0^2 - \alpha\gamma = 0\ and\ 2i\alpha\omega + i\omega\gamma = 0$

$From,\ 2i\alpha\omega + i\omega\gamma = 0 \Rightarrow \boxed{\alpha = \dfrac{\gamma}{2}}$

$From,\ \alpha^2 - \omega^2 + \omega_0^2 - \alpha\gamma = 0 \Rightarrow \dfrac{\gamma^2}{4} - \omega^2 + \omega_0^2 - \dfrac{\gamma^2}{2} = 0 \Rightarrow \boxed{\omega = \sqrt{\omega_0^2 - \dfrac{\gamma^2}{4}}}$

3.17

$Let\ y\ and\ y'\ be\ the\ shift\ of\ CM\ in\ the\ 1st\ and\ the\ 2nd\ column.$

$\therefore \pi\rho r^2 y = \pi\rho(2r)^2 y' \Rightarrow \boxed{y' = \dfrac{y}{4}}$

$Change\ in\ PE\ in\ 1st\ column: \pi\rho r^2.(L+y)g.\dfrac{L+y}{2} - \pi\rho r^2 gL.\dfrac{L}{2} = \pi\rho r^2 g\left(Ly + \dfrac{y^2}{2}\right)$

$Change\ in\ PE\ in\ 2nd\ column: \pi\rho(2r)^2 g.(L-y').\dfrac{L-y'}{2} - \pi\rho(2r)^2 g.L.\dfrac{L}{2} = \pi\rho(2r)^2 g\left(-Ly' + \dfrac{y'^2}{2}\right)$

$Total\ change\ in\ P.E. = \left|\pi\rho r^2\left(Ly + \dfrac{y^2}{2}\right) + \pi\rho(2r)^2\left(-Ly' + \dfrac{y'^2}{2}\right)\right| = \left|\pi\rho r^2\left(Ly + \dfrac{y^2}{2}\right) + \pi\rho r^2\left(-Ly + \dfrac{y^2}{8}\right)\right|$

$\Rightarrow \pi\rho r^2\left(\dfrac{y^2}{2} + \dfrac{y^2}{8}\right) = \dfrac{5}{8}\pi\rho r^2 y^2 \Rightarrow \boxed{Change\ in\ PE = \dfrac{5}{8}\pi\rho r^2 y^2}$

$Not,\ \pi\rho r^2\left(\dfrac{y^2}{2} - \dfrac{y^2}{8}\right) = \dfrac{3}{8}\pi\rho r^2 y^2$

Kinetic energy of the left and right arms at any time instant t:

$$K_L = \frac{1}{2}\pi r^2 \rho h \left(\frac{dy}{dt}\right)^2 \text{ and } K_R = \frac{1}{2}\pi (2r)^2 \rho h \left(\frac{dy'}{dt}\right)^2 = \frac{1}{8}\pi r^2 \rho h \left(\frac{dy}{dt}\right)^2$$

under small oscillation condition.

Kinetic energy of the horizontal arm:

Radius of the horizontal arm at $x = 0$ *is* r_0. *So, radius at* $x : r(x) = r + k.x$

$$\therefore r(l) = r_0 + k.l = 2r_0 \Rightarrow k = \frac{r_0}{l}; \therefore r(x) = r_0 + k.x = r_0\left(1 + \frac{x}{l}\right)$$

$$\therefore dK_H = \text{Kinetic energy of a small section at } x \text{ with thickness } dx = \frac{1}{2}\pi r^2(x)\rho \upsilon^2(x)\,dx$$

$$\therefore K_H = \int_{x=0}^{l} \frac{1}{2}\pi r^2(x)\rho\upsilon^2(x)\,dx = \int_{x=0}^{l} \frac{1}{2}\pi r_0^2 \rho\left(1+\frac{x}{l}\right)^2 \upsilon^2(x)\,dx; \upsilon(x) \text{ is the velocity at } x = x.$$

Now, volume of the water crossing any cross-section in the left vertical column in one second must will flow through any section in the horizontal tube in one second because of no accumulation of water in the tube. So,

$$\pi r_0^2 \frac{dy}{dt} = \pi r^2(x)\upsilon(x) = \pi r_0^2\left(1+\frac{x}{l}\right)^2 \upsilon(x) \Rightarrow \upsilon(x) = \frac{dy/dt}{\left(1+\frac{x}{l}\right)^2}$$

$$\therefore K_H = \int_{x=0}^{l} \frac{1}{2}\pi r_0^2\rho\left(1+\frac{x}{l}\right)^2 \upsilon^2(x)\,dx = \int_{x=0}^{l} \frac{1}{2}\pi r_0^2 \rho\left(1+\frac{x}{l}\right)^2 \frac{(dy/dt)^2}{\left(1+\frac{x}{l}\right)^4}dx = \frac{1}{2}\pi r_0^2\rho\left(\frac{dy}{dt}\right)^2 \int_{x=0}^{l}\left(1+\frac{x}{l}\right)^2 \frac{1}{\left(1+\frac{x}{l}\right)^4}dx$$

$$\Rightarrow K_H = \frac{1}{2}\pi r_0^2\rho\left(\frac{dy}{dt}\right)^2 \int_{x=0}^{l} \frac{1}{\left(1+\frac{x}{l}\right)^2}dx = \frac{1}{2}\pi r_0^2 \rho\left(\frac{dy}{dt}\right)^2 . \frac{l}{2} = \frac{l}{4}\pi r_0^2\rho\left(\frac{dy}{dt}\right)^2$$

$$\boxed{KE_{Total} = \frac{1}{2}\pi r_0^2 \rho h\left(\frac{dy}{dt}\right)^2 + \frac{1}{8}\pi r_0^2\rho h\left(\frac{dy}{dt}\right)^2 + \frac{1}{2}\pi r_0^2\left(\frac{dy}{dt}\right)^2 . \frac{l}{2} = \frac{1}{4}\pi r_0^2 \rho\left(l + \frac{5h}{2}\right)\left(\frac{dy}{dt}\right)^2}$$

$$\therefore \text{Total energy} = KE + PE = \frac{1}{4}\pi r_0^2\rho\left(l + \frac{5h}{2}\right)\left(\frac{dy}{dt}\right)^2 + \frac{5}{8}\pi\rho r_0^2 y^2 = E = \text{constant}.$$

$$\text{Differentiating wrt } t: \frac{1}{2}\pi r_0^2\rho\left(l+\frac{5h}{2}\right).\frac{dy}{dt}.\frac{d^2y}{dt^2} + \frac{5}{4}\pi\rho r_0^2 y\frac{dy}{dt} = 0$$

$$\Rightarrow \frac{1}{2}\left[\pi r_0^2\rho\left(\frac{5h}{2}+\frac{5h}{2}\right)\right].\frac{d^2y}{dt^2} + \frac{1}{2}\left(\frac{5}{2}\pi\rho r_0^2\right)y = 0 \Rightarrow h\frac{d^2y}{dt^2} + \frac{1}{2}y = 0 \Rightarrow \frac{d^2y}{dt^2} + \frac{1}{2h}y = 0 \Rightarrow \frac{d^2y}{dt^2} + \omega_0^2 y = 0$$

$$\therefore \boxed{\omega_0 = \sqrt{\frac{1}{2h}}}$$

$$3.15.(a)\,E = E_0 e^{-\gamma t} \Rightarrow \frac{E_0}{2} = E_0 e^{-\gamma t} \Rightarrow \gamma = \ln 2$$

$$So, Q = \frac{\omega_0}{\gamma} = \frac{2\pi f}{\gamma} = \frac{2\pi \times 256}{\ln 2} = \frac{512\pi}{\ln 2} \Rightarrow \boxed{Q = \frac{512\pi}{\ln 2} = Q_0\,(say)}$$

$$(b)\,If\;\; \omega_0 = 512 \times 2\pi \Rightarrow \boxed{Q = 2Q_0}$$

$$(c)\,E = E_0 e^{-\gamma t} \Rightarrow \frac{E}{E_0} = \frac{1}{e} = e^{-\gamma.4} \Rightarrow \boxed{\gamma = \frac{1}{4}}\, Now, \omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{0.9}{0.1}} = 3 \Rightarrow \boxed{Q = \frac{\omega_0}{\gamma} = 3 \times 4 = 12}$$

$$Now,\; \gamma = \frac{b}{m} = \frac{b}{0.1} = \frac{1}{4} \Rightarrow \boxed{b = \frac{1}{40} = 0.025 kg / s}$$