Chapter 15 (15.1-15.5, 15.7, 15.8)

Multiple Integrals

Note: *This module is prepared from Chapter 15 (Thomas' Calculus, 13th edition) just to help the students. The study material is expected to be useful but not exhaustive. For detailed study, the students are advised to attend the lecture/tutorial classes regularly, and consult the text book.*

Appeal: Please do not print this e-module unless it is really necessary.

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Integral of a function of single variable

Let $f(x)$ be a function defined in an interval $[a, b]$. Divide the interval $[a, b]$ into subintervals by choosing points between a and b. Let there be n subintervals of $[a, b]$ of lengths δx_i $(i = 1, 2, ..., n)$ such that $\delta x_i \to 0$ as $n \to \infty$. Let x_i be a point in the ith subinterval. Then the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\delta x_i,$$

if exists, defines the integral of $f(x)$ over the interval $[a, b]$, and is written as

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\delta x_i.$$

Geometry: Suppose the curve $y = f(x)$ lies above the x-axis from $x = a$ to $x = b$. Then $f(x_i)\delta x_i$ is the area of the rectangle of length $f(x_i)$ and breadth δx_i, erected over the ith subinterval interval of $[a, b]$. So $\sum_{i=1}^{n} f(x_i)\delta x_i$ gives the approximate area under the curve $y = f(x)$ above the x-axis from $x = a$ to $x = b$. It implies that the limit $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\delta x_i$ or $\int_a^b f(x)dx$ gives the area under the curve $y = f(x)$ above the x-axis from $x = a$ to $x = b$.

In particular, if $f(x) = 1$, then

$$\int_a^b dx = \lim_{n\to\infty} \sum_{i=1}^{n} \delta x_i = b - a,$$

gives the area of the rectangle bounded by the lines $f(x) = 1$, x-axis, $x = a$ and $x = b$. In magnitude, it is the length of the interval $[a, b]$.

Average value of $f(x)$ over $[a, b]$: It is defined as $\dfrac{\int_a^b f(x)dx}{\int_a^b dx} = \dfrac{1}{b-a}\int_a^b f(x)dx.$

Integral of a function of two variables

Let $f(x, y)$ be a function defined in a region R in the xy-plane. Divide the region R into subregions by choosing lines parallel to the coordinate axes. Let there be n complete rectangles inside R of dimensions δx_i along x-axis and δy_i along y-axis, where $i = 1, 2, ..., n$, such that $\delta x_i \to 0$ and $\delta y_i \to 0$ as $n \to \infty$. Let (x_i, y_i) be a point in the ith rectangle. Then the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i)\delta x_i \delta y_i,$$

if exists, defines the integral of $f(x, y)$ over the region R, and is written as

$$\iint_R f(x, y)dxdy = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i)\delta x_i \delta y_i.$$

Geometry: Suppose the surface $z = f(x, y)$ lies above the xy-plane with its region of projection R in the xy-plane. Then $f(x_i, y_i)\delta x_i \delta y_i$ is the volume of the cuboid of height $f(x_i, y_i)$, length δx_i and breadth δy_i, erected over the ith rectangle of the region R. So $\sum_{i=1}^{n} f(x_i, y_i)\delta x_i \delta y_i$ gives the approximate volume under the surface $z = f(x, y)$ above the region R of xy-plane. It implies that the limit $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i)\delta x_i \delta y_i$ or $\iint_R f(x, y)dxdy$ gives the volume under the surface $z = f(x, y)$ above the region R of xy-plane.

In particular, if $f(x, y) = 1$, then

$$\iint_R dxdy = \lim_{n\to\infty} \sum_{i=1}^{n} \delta x_i \delta y_i$$

gives the volume under the plane $f(x, y) = 1$ and above the region R of xy-plane. In magnitude, it is equal to the area of the region R.

Average value of $f(x, y)$ over R: It is defined as $\dfrac{\iint_R f(x, y)dxdy}{\iint_R dxdy} = \dfrac{1}{\text{Area of } R} \iint_R f(x, y)dxdy$.

Evaluation of double integrals (Fubini's theorems)

- Suppose that $f(x, y)$ is a continuous function over the rectangular region

  $$R = \{(x, y) : \ a \le x \le b, \ c \le y \le d\},$$

  bounded by the four lines $x = a$, $x = b$, $y = c$ and $y = d$. Then $\iint_R f(x, y)dxdy$, $\int_a^b \left[\int_c^d f(x, y)dy\right] dx$ and $\int_c^d \left[\int_a^b f(x, y)dx\right] dy$ all exist and are equal, that is,

  $$\iint_R f(x, y)dxdy = \int_a^b \left[\int_c^d f(x, y)dy\right] dx = \int_c^d \left[\int_a^b f(x, y)dx\right] dy.$$

- If $f(x, y)$ is a continuous function over the non-rectangular region

  $$R = \{(x, y) : \ a \le x \le b, \ \phi(x) \le y \le \psi(x)\},$$

  bounded by the lines $x = a$, $x = b$ and the curves $y = \phi(x)$ and $y = \psi(x)$, then

  $$\iint_R f(x, y)dxdy = \int_a^b \left[\int_{\phi(x)}^{\psi(x)} f(x, y)dy\right] dx.$$

- If $f(x, y)$ is a continuous function over the non-rectangular region

  $$R = \{(x, y) : \ c \le y \le d, \ \phi(y) \le x \le \psi(y)\},$$

  bounded by the lines $y = c$, $y = d$ and the curves $x = \phi(y)$ and $x = \psi(y)$, then

  $$\iint_R f(x, y)dxdy = \int_c^d \left[\int_{\phi(y)}^{\psi(y)} f(x, y)dx\right] dy.$$

Ex. Solve $\int_0^1 \int_0^2 (1+xy^2)dxdy$.

Sol. We have

$$\begin{aligned}
\int_0^1 \int_0^2 (1+xy^2)dxdy &= \int_0^1 \left[\int_0^2 (1+xy^2)dx\right] dy \\
&= \int_0^1 \left[x + \frac{x^2y^2}{2}\right]_{x=0}^{x=2} dy \\
&= \int_0^1 \left(2+2y^2\right) dy \\
&= \left[2y + \frac{2y^3}{3}\right]_{y=0}^{y=1} \\
&= \frac{8}{3}
\end{aligned}$$

Ex. Solve $\int_0^1 \int_0^{x^3} x(x^2+y^2)dydx$.

Sol. Ans. $\frac{40}{231}$

Method to find limits of x and y for a given region

In some problems, the region of integration R of a function $f(x,y)$ is given without specifying the limits for x and y. In such problems, we first sketch the region R in xy-plane, and then find the limits of x and y by considering an arbitrary line either parallel to x-axis or parallel to the y-axis passing through the region R. Suppose we choose a line parallel to x-axis passing through the region R. Then for the lower limit of x, we observe the boundary curve, say $x = \phi(y)$, of the region R where the line enters the region R from the left. For the upper limit of x, we observe the boundary curve, say $x = \psi(y)$, of the region R where the line leaves the region R. The lower and upper limits of y, say $y = c$ and $y = d$, are given by the values of y at the bottommost and topmost points of the region R, respectively. Then we have

$$\iint_R f(x,y)dxdy = \int_c^d \left[\int_{\phi(y)}^{\psi(y)} f(x,y)dx\right] dy.$$

Likewise, if we choose an arbitrary line parallel to y-axis, and passing through the region R, then boundary curve of the region R through which the line enters the region R from lower side gives the lower limit of y, say $y = \phi(x)$. The boundary curve of R where the line leaves the region R gives the upper limit of y, say $y = \psi(x)$. The lower and upper limits of x, say $x = a$ and $x = b$, are given by the values of x at the leftmost and rightmost points of the region R, respectively. Then we have

$$\iint_R f(x,y)dxdy = \int_a^b \left[\int_{\phi(x)}^{\psi(x)} f(x,y)dy\right] dx.$$

Ex. Solve $\iint_R xydxdy$, where $R = \{(x, y) : x + y \leq 1,\ x \geq 0,\ y \geq 0\}$.

Sol. The given region R of integration is the triangular region in the first quadrant of the xy-plane bounded by the lines $x = 0$, $y = 0$ and $x + y = 1$ as shown in Figure 1.

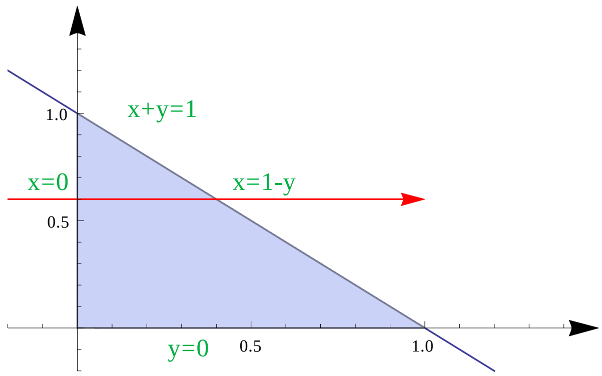


Figure 1:

Note that the line $x + y = 1$ intersects x-axis in the point $(1, 0)$, and y-axis in the point $(0, 1)$. To find the limits of x and y for the given region R, let us choose a line parallel to x-axis (the red line in Figure 1) passing through the region R. We observe that the line enters the region R from the left through the side of the triangular region on y-axis, that is, $x = 0$, and leaves the region at its side given by $x + y = 1$ or $x = 1 - y$. So lower and upper limits of x are $x = 0$ and $x = 1 - y$, respectively. Now bottommost point(s) of the region R lies on its side on x-axis, where $y = 0$, and the uppermost point is $(0, 1)$, where $y = 1$. So the given region of integration may be written as $R = \{(x, y) : 0 \leq x \leq 1 - y,\ 0 \leq y \leq 1\}$. It follows that

$$\iint_R xydxdy = \int_0^1 \left[\int_0^{1-y} xydx\right] dy = \frac{1}{24}.$$

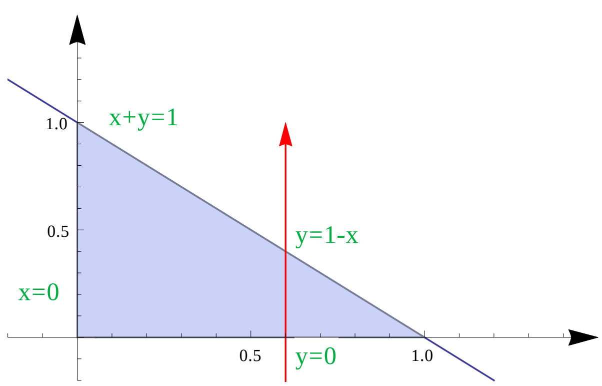


Figure 2:

We can also solve the given problem by choosing a line parallel to y-axis (the red line in Figure 2) passing through the region R. In this case, we observe that the line from lower side enters the region R

through the side of the triangular region on x-axis, that is, $y = 0$, and leaves the region at its side given by $x+y = 1$ or $y = 1-x$. So lower and upper limits of y are $y = 0$ and $y = 1-x$, respectively. Now leftmost point(s) of the region R lies on its side on y-axis, where $x = 0$, and the rightmost point is $(1, 0)$, where $x = 1$. So the given region of integration may be written as $R = \{(x, y) : 0 \leq x \leq 1,\ 0 \leq y \leq 1-x\}$. It follows that

$$\iint_R xy dx dy = \int_0^1 \left[\int_0^{1-x} xy dy\right] dx = \frac{1}{24}.$$

Note that we get the same answer in both cases. So we can choose the arbitrary line parallel to either of the axes to decide the limits of x and y.

The case of polar curve

If the region R is enclosed by a polar curve $r = f(\theta)$, then to decide the limits of r we choose a line through the pole, and then find the limits of r accordingly as where the line enters (starting from pole) and leaves the given region R. Limits of θ can be found according to the variation of θ for the given region R. The following example illustrates the case of polar curve.

Ex. Solve $\displaystyle\iint_R r\cos\theta dr d\theta$, where R is the region that lies inside the cardioid $r = 1+\cos\theta$ and outside the circle $r = 1$.

Sol. The given region of integration is the shaded region as shown in Figure 3.

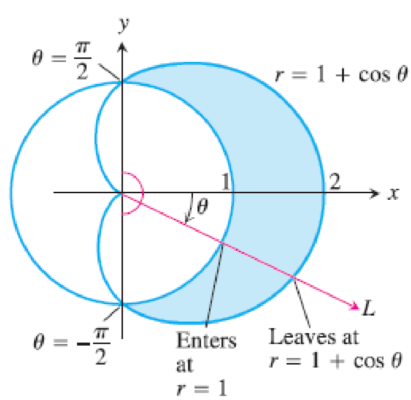


Figure 3:

Consider a line L through the pole and passing through the given region. We observe that the line enters the given region at the boundary of the circle $r = 1$ and leaves at the boundary of the cardioid $r = 1+\cos\theta$. Also, for the given region, θ varies from $\theta = -\frac{\pi}{2}$ to $\theta = \frac{\pi}{2}$. So we can write the given region as

$$R = \{(r, \theta) : 1 \leq r \leq 1+\cos\theta,\ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\}.$$

So we have

$$\iint_R r\cos\theta dr d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\int_1^{1+\cos\theta} r\sin\theta dr\right] d\theta = \frac{2}{3} + \frac{\pi}{2}.$$

Change of variables in double integrals

Let a function $f(x, y)$ be integrable over a region R in the xy-plane. Suppose R is mapped onto a region R' in uv-plane by means of the transformations

$$x = g(u, v), \quad y = h(u, v),$$

where $h(u, v)$ and $g(u, v)$ possess continuous partial derivatives in R', and the Jacobian

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \neq 0, \quad \text{in} \quad R'.$$

Then we have

$$\iint_R f(x, y) dx dy = \iint_{R'} f(g(u, v), h(u, v))|J| du dv.$$

Notice that x and y are replaced by $g(u, v)$ and $h(u, v)$, respectively while $dxdy$ is replaced by $|J|dudv$.

In particular, if we choose the polar transformations given by $x = r\cos\theta$ and $y = r\sin\theta$, then

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

Ex. Evaluate $\iint_R \sin[\pi(x^2 + y^2)] dx dy$, where $R = \{(x, y) : x^2 + y^2 \leq 1\}$.

Sol. Using $x = r\cos\theta$, $y = r\sin\theta$, the given circular region $R = \{(x, y) : x^2 + y^2 \leq 1\}$ transforms to $R' = \{(r, \theta) : 0 \leq r \leq 1,\ 0 \leq \theta \leq 2\pi\}$ while $dxdy = |J|drd\theta = rdrd\theta$. So we have

$$\iint_R \sin[\pi(x^2 + y^2)] dx dy = \iint_{R'} \sin(\pi r^2) r dr d\theta = \int_0^{2\pi} \left[\int_0^1 \sin(\pi r^2) r dr\right] d\theta = 2.$$

Note: Is change of variables essential to solve the above example? This can be realized if you try to solve it without using the change of variables. Let me know if you are able to solve it! Good luck!

Also note that the area of a closed and bounded region R in the polar coordinate plane is given by

$$A = \iint_R r dr d\theta$$

Change of order in double integrals

If the region of integration is non-rectangular, and atleast one limit of x is variable, then for changing the order of integration, choose a line parallel to y-axis through the given region, and find the limits of x

and y accordingly. In case, atleast one limit of y is variable, choose the line parallel to x-axis through the given region, and find the limits of x and y accordingly.

Ex. Change the order of integration in $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dydx$, and hence evaluate it.

Sol.

$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dydx = \int_0^\infty \int_0^y \frac{e^{-y}}{y} dxdy = 1.$$

Note: Is change of order of integration essential to solve the above example? This can be realized if you try to solve it without using the change of order. Let me know if you are able to solve it! Good luck!

Ex. Change the order of integration in $\int_0^1 \int_{x^2}^{2-x} xy \, dydx$, and hence evaluate it.

Sol.

$$\int_0^1 \int_{x^2}^{2-x} xy \, dydx = \int_0^1 \int_0^{\sqrt{y}} xy \, dxdy + \int_1^2 \int_0^{2-y} xy \, dxdy = \frac{1}{6} + \frac{5}{24} = \frac{3}{8}.$$

Integral of a function of three variables

Let $f(x, y, z)$ be a function defined in a volumetric region V in the xyz-space. Divide the region V into subregions by choosing planes parallel to the coordinate planes. Let there be n complete cuboids inside V of dimensions δx_i along x-axis, δy_i along y-axis and δz_i along z-axis, where $i = 1, 2, ..., n$, such that $\delta x_i \to 0$, $\delta y_i \to 0$ and $\delta z_i \to 0$ as $n \to \infty$. Let (x_i, y_i, z_i) be a point in the ith cuboid. Then the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i, z_i)\delta x_i \delta y_i \delta z_i,$$

if exists, defines the integral of $f(x, y, z)$ over the region V, and is written as

$$\iiint_V f(x, y, z)dxdydz = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i, z_i)\delta x_i \delta y_i \delta z_i.$$

Geometry: Since we can not plot the graph of $f(x, y, z)$ geometrically, so we can not interpret the geometry of $\iiint_V f(x, y, z)dxdydz$, in general. However, in the particular case $f(x, y, z) = 1$, the expression

$$\iiint_V dxdydz = \lim_{n\to\infty} \sum_{i=1}^{n} \delta x_i \delta y_i \delta z_i,$$

gives the volume of the region V, in magnitude.

Average value of $f(x, y, z)$ over V: It is defined as

$$\frac{\iiint_V f(x, y, z)dxdydz}{\iint_V dxdydz} = \frac{1}{\text{Volume of } V} \iiint_V f(x, y, z)dxdydz.$$

Evaluation of triple integrals

- Suppose that $f(x, y, z)$ is a continuous function in the cuboidal region

  $$V = \{(x, y, z) : \ a \le x \le b, \ c \le y \le d, \ \ e \le z \le f\},$$

  bounded by the six planes $x = a$, $x = b$, $y = c$, $y = d$, $z = e$ and $z = f$. Then

  $$\iiint_V f(x, y, z)dxdydz = \int_e^f \left[\int_c^d \left[\int_a^b f(x, y, z)dx\right] dy\right] dz.$$

- If $f(x, y, z)$ is a continuous function in the non-cuboidal region

  $$V = \{(x, y, z) : \ a \le x \le b, \ \phi(x) \le y \le \psi(x), \ g(x, y) \le z \le h(x, y)\},$$

  bounded by the planes $x = a$, $x = b$, the curves $y = \phi(x)$ and $y = \psi(x)$, and the surfaces $z = g(x, y)$, $z = h(x, y)$, then

  $$\iiint_V f(x, y, z)dxdydz = \int_a^b \left[\int_{\phi(x)}^{\psi(x)} \left[\int_{g(x,y)}^{h(x,y)} f(x, y, z)dz\right] dy\right] dx.$$

Ex. Solve $\int_0^1 \int_0^x \int_0^{x+y} e^{x+y+z} dz dy dx.$

Sol. We find

$$\int_0^1 \int_0^x \int_0^{x+y} e^{x+y+z} dz dy dx = \int_0^1 \left[\int_0^x \left[\int_0^{x+y} e^{x+y+z} dz \right] dy \right] dx = \frac{1}{8}e^4 - \frac{3}{4}e^2 + e.$$

Method to find limits of x, y and z for a given region

In some problems, the region of integration V of a function $f(x, y, z)$ is given without specifying the limits for x, y and z. In such problems, we first sketch the region V in xyz-space. To find the limits of z, we choose a line parallel to z-axis through the given region V. Then the lower and upper limits of z are the values of z corresponding to the parts of surface of the region V where the line enters and and leaves (along the direction of positive z-axis) the region V. Let R be the region of projection of the three dimensional region V on the xy-plane. We find the limits of x and y by considering the region R using the method applied in double integrals.

Ex. Solve $\iiint_V \frac{dxdydz}{(x+y+z+1)^3}$ over a tetrahedron bounded by the coordinate planes and the plane $x + y + z = 1$.

Sol. We find

$$\iiint_V \frac{dxdydz}{(x+y+z+1)^3} = \int_0^1 \left[\int_0^{1-x} \left[\int_0^{1-x-y} \frac{1}{(x+y+z+1)^3} dz \right] dy \right] dx = \frac{1}{2}\ln 2 - \frac{5}{16}.$$

Change of variables in triple integrals

Let a function $f(x, y, z)$ be integrable over a region V in the xyz-space. Suppose V is mapped onto a region V' in uvw-space by means of the transformations

$$x = g(u, v, w), \quad y = h(u, v, w), \quad y = k(u, v, w)$$

where $g(u, v, w)$, $h(u, v, w)$ and $k(u, v, w)$ possess continuous partial derivatives in V', and the Jacobian

$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \neq 0, \quad \text{in} \quad V'.$$

Then we have

$$\iiint_V f(x, y)dxdy = \iiint_{V'} f(g(u, v, w), h(u, v, w), k(u, v, w))|J|dudvdw.$$

Notice that x, y and z are replaced by $g(u, v, w)$, $h(u, v, w)$ and $k(u, v, w)$, respectively while $dxdydz$ is replaced by $|J|dudvdw$.

Change to cylindrical polar coordinates

Cylindrical coordinates represent a point P in space by ordered triples (r, θ, z) in which r and θ are polar coordinates for the vertical projection of P on the xy-plane, and z is the rectangular vertical coordinate. The equations of transformation from cartesian coordinates (x, y, z) to cylindrical polar coordinates (r, θ, z) are

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z,$$

where $r \geq 0$, $0 \leq \theta \leq 2\pi$.

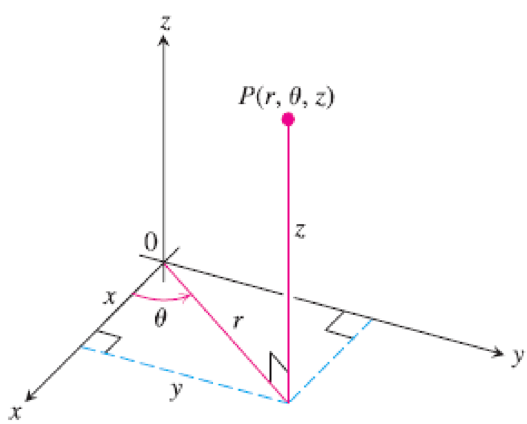


Figure 4: Cylindrical coordinates (r, θ, z)

The Jacobian of the cylindrical polar transformations is obtained as

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = r.$$

Ex. Evaluate $\iiint_V z(x^2+y^2)dxdydz$, where $V = \{(x,y,z) : x^2+y^2 \leq 1,\ 2 \leq z \leq 3\}$.

Sol. Using $x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$, the given cylindrical region $V = \{(x,y,z) : x^2+y^2 \leq 1,\ 2 \leq z \leq 3\}$ transforms to $V' = \{(r,\theta,z) : 0 \leq r \leq 1,\ 0 \leq \theta \leq 2\pi,\ 2 \leq z \leq 3\}$ while $dxdydz$ transforms to $|J|drd\theta d\phi = rdrd\theta dz$. So we have

$$\iiint_V z(x^2+y^2)dxdydz = \iiint_{V'} z(r^2)rdrd\theta d\phi = \int_2^3 \left[\int_0^{2\pi} \left[\int_0^1 r^3 z dr\right] d\theta\right] dz = \frac{5\pi}{4}.$$

Change to spherical polar coordinates

Spherical coordinates represent a point P in space by ordered triples (ρ, ϕ, θ) in which ρ is the distance from P to the origin; ϕ is the angle made by $\overrightarrow{OP}$ with the positive z-axis, and θ is the angle from cylindrical coordinates

The equations of transformation from cartesian coordinates (x, y, z) to spherical polar coordinates (ρ, ϕ, θ) are

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi,$$

where $\rho \geq 0$, $0 \leq \phi \leq \pi$ and $0 \leq \theta \leq 2\pi$.

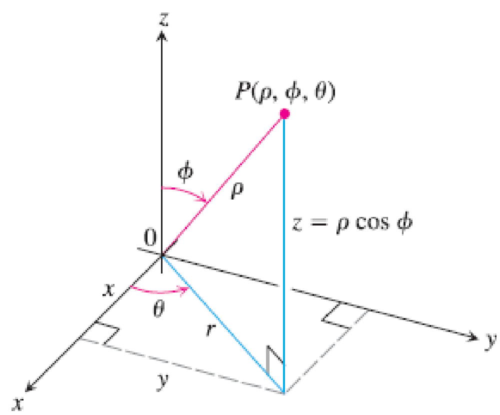


Figure 5: Spherical coordinates (ρ, ϕ, θ)

The Jacobian of the spherical polar transformations is obtained as

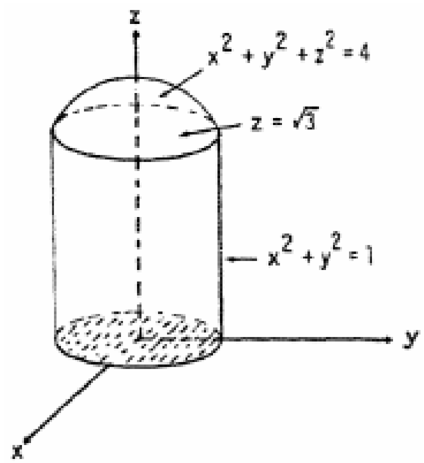
$$J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \rho^2 \sin\phi.$$

Ex. Evaluate $\iiint_V (x^2+y^2+z^2)^3 dxdydz$, where V is the region enclosed by the unit sphere $x^2+y^2+z^2 = 1$.
Sol. Using $x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = r \cos\phi$, the given spherical region $V = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$ transforms to $V' = \{(\rho, \phi, \theta) : 0 \leq \rho \leq 1, \ 0 \leq \phi \leq \pi, \ 0 \leq \theta \leq 2\pi\}$ while $dxdydz$ transforms to $|J|d\rho d\theta d\phi = \rho^2 \sin\phi d\rho d\theta d\phi$. So we have

$$\iiint_V (x^2 + y^2 + z^2)^3 dxdydz = \iiint_{V'} (\rho^2)^3 \rho^2 \sin\phi d\rho d\theta d\phi = \int_0^{2\pi} \left[\int_0^{\pi} \left[\int_0^1 \rho^8 \sin\phi d\rho\right] d\phi\right] d\theta = \frac{4\pi}{9}.$$

Ex. Find volume of the cylinder $x^2 + y^2 = 1$, $z \geq 0$ cut-off by the sphere $x^2 + y^2 + z^2 = 4$ using (i) spherical polar coordinates (ii) cylindrical polar coordinates.

Sol.



(i) In spherical polar coordinates, we have

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi.$$

So $x^2 + y^2 = 1 \implies \rho^2 \sin^2\phi \implies \rho = \csc\phi$.

Now to find ρ-limits, consider a ray from the origin passing through the given region. We see that the ray will come out of either the surface of the sphere or that of the cylinder. Let us call the corresponding parts of the given 3-D region to be Region I (cone shaped region cut-off by the sphere (the ice-cream shape)) and Region II (the remaining region).

In Region I, for any values of θ and ϕ, we see that ρ varies from $\rho = 0$ to $\rho = 2$ (for the points on the surface of the sphere $x^2 + y^2 + z^2 = 4$). Further for any value of θ, we see that ϕ varies from $\phi = 0$ to $\phi = \pi/6$. Also, θ varies from $\theta = 0$ to $\theta = 2\pi$.

In Region II, for any values of θ and ϕ, we see that ρ varies from $\rho = 0$ to $\rho = \csc\phi$ (for the points on the curved surface of the cylinder $x^2 + y^2 = 1$). Further for any value of θ, we see ϕ varies from $\phi = \pi/6$ to $\phi = \pi/2$. Also, θ varies from $\theta = 0$ to $\theta = 2\pi$.

$$\therefore \text{ Required Volume} = \int_0^{2\pi}\int_0^{\pi/6}\int_0^{2} \rho^2 \sin\phi \, d\rho d\phi d\theta + \int_0^{2\pi}\int_{\pi/2}^{\pi/6}\int_0^{\csc\phi} \rho^2 \sin\phi \, d\rho d\phi d\theta$$

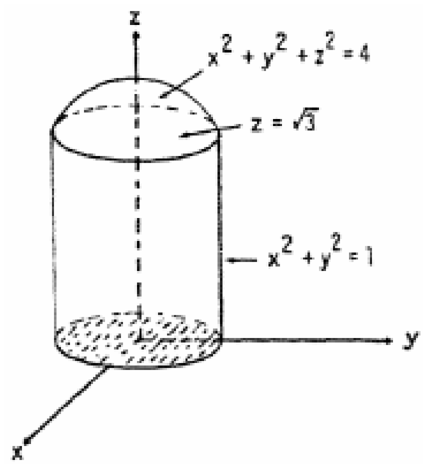
$$= \frac{8\pi}{3}(2-\sqrt{3}) + \frac{2\pi\sqrt{3}}{3} = \frac{2\pi}{3}(8 - 3\sqrt{3})$$

(ii) In cylindrical polar coordinates, we have

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

To find z-limits, consider a line parallel to z-axis through the given region. For any values of r and θ, we see that z varies from $z = 0$ to $z = \sqrt{4 - r^2}$ (for points on the surface of the sphere $x^2 + y^2 + z^2 = 4$). For any value of θ, we see that r varies from $r = 0$ to $r = 1$. Also, θ varies from $\theta = 0$ to $\theta = 2\pi$.

$$\therefore \text{ Required Volume} = \int_0^{2\pi}\int_0^1\int_0^{\sqrt{4-r^2}} r\, dzdrd\theta = \frac{2\pi}{3}(8 - 3\sqrt{3}).$$



Remark: In part (i), if the limits are found in the order ϕ, ρ and θ. Then

$$\text{Required Volume} = \int_0^{2\pi}\int_0^2\int_0^{\pi/6} \rho^2 \sin\phi\, d\phi d\rho d\theta + \int_0^{2\pi}\int_1^2\int_{\pi/6}^{\sin^{-1}(1/\rho)} \rho^2 \sin\phi\, d\phi d\rho d\theta.$$

In part (ii), if the limits are found in the order r, z and θ. Then

$$\text{Required Volume} = \int_0^{2\pi}\int_0^{\sqrt{3}}\int_0^1 r\, drdzd\theta + \int_0^{2\pi}\int_{\sqrt{3}}^2\int_0^{\sqrt{4-z^2}} r\, drdzd\theta$$

In part (ii), if the limits are found in the order θ, z and r. Then

$$\text{Required Volume} = \int_0^1\int_0^{\sqrt{4-r^2}}\int_0^{2\pi} r\, d\theta dzdr.$$