Mathematics-II (MATH F112)

Linear Algebra and Complex Analysis

Sangita Yadav

Department of Mathematics
BITS Pilani, Pilani Campus, Rajasthan

Course Structure

Instructor-Incharge: Dr. Trilok Mathur

Linear Algebra(20 Lec.): Dr. Jitender Kumar,Dr. Krishnendra Shekhawat and Dr. Sangita Yadav

Complex Variables(20 Lec.): Dr. Trilok Mathur, Prof. Balram Dubey and Dr. Ashish Tiwari

Quizzes: There will be four surprised quizzes of 20 marks each to be conducted in tutorial classes and out of 4, marks of **best** 3 quizzes will be considered.

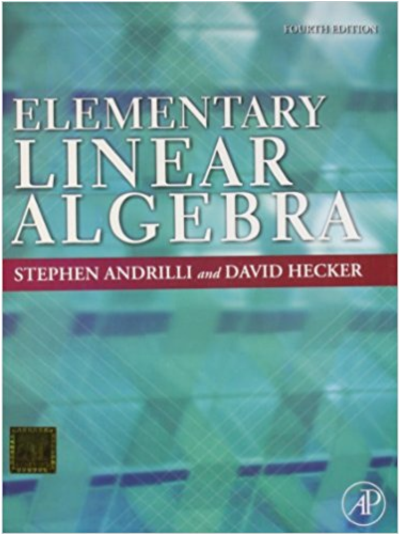
Students are requested to write all the Quizzes in their **Registered Tutorial Section Only**.

Assignments: Two assignments will be given for your practice and does not require submission. However, some of the Assignment Questions may be asked in Mid-sem/Comprehensive examination.

Notices: All course notices will be posted on **NALANDA** and **Department Notice Board**.

Chamber Consultation Hour: To be announced in your tutorial section.

Text Book: For Linear Algebra



Systems of Linear Equations

Chapter: 2

- System of Linear equations
- Row Echelon Form
- Elementary Row Operations
- Gaussian Elimination Method
- Reduced Row Echelon Form
- Gauss-Jordan Row Reduction Method
- Rank
- Inverse of a Matrix

An Example for Motivation: Solve the system of linear equations

$$\begin{aligned} x_1 - 3x_2 - x_3 &= 8 \\ x_1 - 2x_2 - 2x_3 &= 3 \\ 3x_1 - 7x_2 - 4x_3 &= 17. \end{aligned}$$

Step 1: Represent the given system of equations as follows:

$$\begin{aligned} x_1 - 3x_2 - x_3 &= 8 \\ x_1 - 2x_2 - 2x_3 &= 3 \\ 3x_1 - 7x_2 - 4x_3 &= 17 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -3 & -1 & 8 \\ 1 & -2 & -2 & 3 \\ 3 & -7 & -4 & 17 \end{array}\right]$$

Step 2: Multiply the first equation by 1 and subtract it from the 2nd equation; **Multiply the first row by** 1 **and subtract it from the 2nd row**

$$\begin{aligned} x_1 - 3x_2 - x_3 &= 8 \\ x_2 - x_3 &= -5 \\ 3x_1 - 7x_2 - 4x_3 &= 17 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -3 & -1 & 8 \\ 0 & 1 & -1 & -5 \\ 3 & -7 & -4 & 17 \end{array}\right]$$

Step 3: Multiply the first equation by 3 and subtract it from the 3rd equation; **Multiply the first row by** 3 **and subtract it from the 3rd row**

$$\begin{aligned} x_1 - 3x_2 - x_3 &= 8 \\ x_2 - x_3 &= -5 \\ 2x_2 - x_3 &= -7 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -3 & -1 & 8 \\ 0 & 1 & -1 & -5 \\ 0 & 2 & -1 & -7 \end{array}\right]$$

Step 4: Multiply the second equation by 2 and subtract it from the third equation; **Multiply the second row by 2 and subtract it from the third row**

$$\begin{aligned} x_1 - 3x_2 - x_3 &= 8 \\ x_2 - x_3 &= -5 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -3 & -1 & 8 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

By Backward substitution, we find

$$x_3 = 3, \quad x_2 = -2, \quad x_1 = 5$$

is a solution of the given system of equations.

Recall:

- A vector is a directed line segment that corresponds to a displacement from one point A to another point B. The vector from A to B is denoted by $\overrightarrow{AB}$.
- The point A is called its **initial point** or **tail**, and the point B is called its **terminal point** or **head**.
- The set of all ordered pair of real numbers is denoted by $\mathbb{R}^2$ i.e. $\mathbb{R}^2 = \{(a, b) \mid a, b \in \mathbb{R}\}$.
- The set $\mathbb{R}^2$ corresponds to the set of vectors whose tails are at the origin O.

- For example, the ordered pair $A = (1, 4) \in \mathbb{R}^2$ corresponds to the vector $\overrightarrow{OA}$ and we denote it as $[1, 4]$.
- For $n \in \mathbb{N}$, $\mathbb{R}^n$ is the set of all ordered n-tuples $(x_1, x_2, \ldots, x_n)$, where $x_i \in \mathbb{R}$.
- We can think the point $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ as vector and write it as $[x_1, x_2, \ldots, x_n]$ (row vector). Thus,

$$\mathbb{R}^n = \{[x_1, x_2, \ldots, x_n] \mid x_i \in \mathbb{R}\}.$$

- Sometime we will write a vector of $\mathbb{R}^n$ as a column vector

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [x_1, x_2, \ldots, x_n]^T,$$

depend on the situation.

- The vector $[0, 0, \ldots, 0]$ of $\mathbb{R}^n$, called the zero vector of $\mathbb{R}^n$ and it is denoted by the symbol $\mathbf{0}$.

System of Linear Equations

A system of **m** linear equations in **n** unknown variables $x_1, x_2, \ldots, x_n$ is given by

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad\quad \vdots \qquad\quad\quad \vdots \quad & \quad\ \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

where $a_{ij}, b_i \in \mathbb{R}$ and $1 \leq i \leq m, 1 \leq j \leq n$.

- A solution of the linear system is an n-tuple $(s_1, s_2, \ldots, s_n)$ such that each equation of the system is satisfied by substituting s_i in place of x_i.

Above linear system of equations can be written in the form $AX = B$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

- The matrix A is called the **coefficient matrix**.

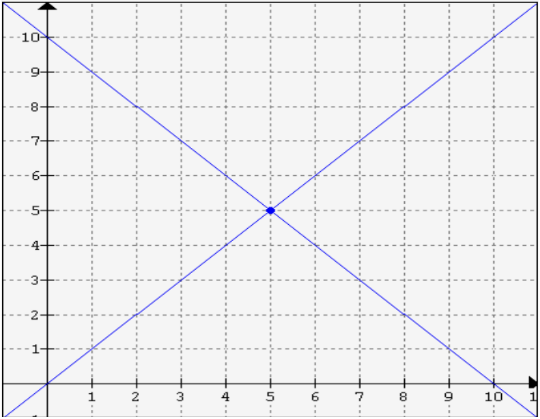
- The matrix $[A|B]$ which is formed by inserting the column of matrix B next to the column of A, is called the **augmented matrix** of the linear system $AX = B$ i.e.

$$[A|B] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right]$$

- The vertical bar is used in the augmented matrix $[A|B]$ only to distinguish the column vector B from the coefficient matrix A.

- If $B = 0 = [0, 0, \cdots, 0]^T$ i.e. $b_1 = 0 = b_2 = \cdots = b_m$, the system $AX = 0$ is called homogenous system of equations.
- If $B \neq 0$, then the system $AX = B$ is called non-homogenous system of equations.
- The solution $X = 0$ of the system $AX = 0$ is called the trivial solution and a solution other than $X = 0$ is called a non-trivial solution of the homogenous system $AX = 0$.

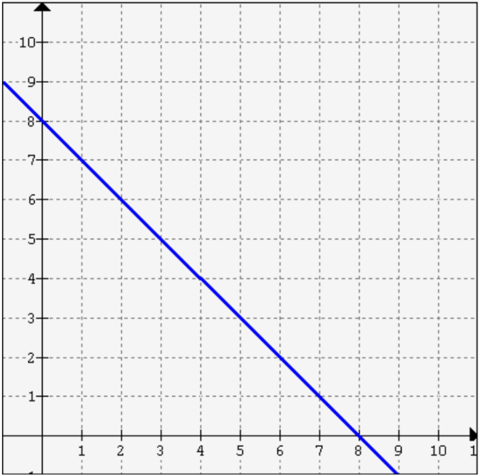
Geometrical Approach:



$$\begin{aligned} x_1 + x_2 &= 10 \\ -x_1 + x_2 &= 0 \end{aligned}$$

$x_1 = 5$, $x_2 = 5$ is the unique solution, as lines intersect at a unique point.

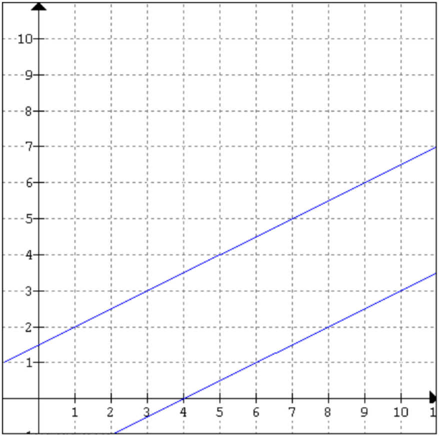
Geometrical Approach:



$$\begin{aligned} x_1 + x_2 &= 8 \\ -2x_1 - 2x_2 &= -16 \end{aligned}$$

This linear system has infinitely many solutions. Lines intersect at every point.

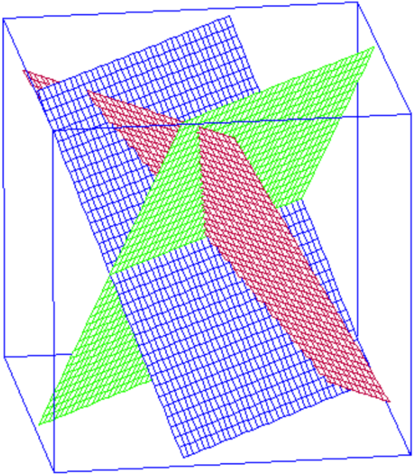
Geometrical Approach:



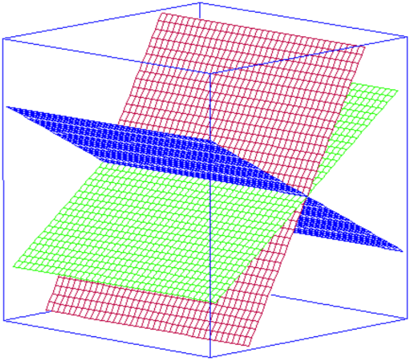
$$\begin{aligned} x_1 - 2x_2 &= \quad -3 \\ 2x_1 - 4x_2 &= \quad 8 \end{aligned}$$

This linear system has no solution. Lines do not intersect at any point.

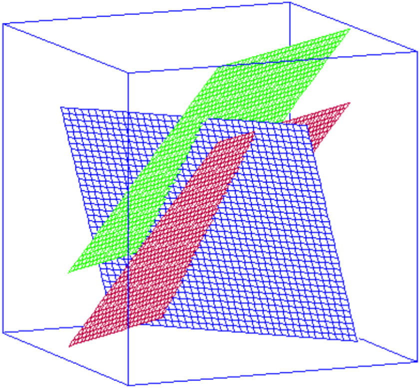
Example: Three equations in three variables. In this case each equation determines a plane in the 3D space.



This linear system has a unique solution. The planes intersect at one point.



This linear system has infinitely many solutions. The planes intersect in one line.



This linear system has no solution. There is no point in common to all three planes.

Number of Solutions to a system : There are three possibilities for the size of the solution set

- unique solution
- infinitely many solutions
- no solution

If the system $AX = B$ has atleast one solution then it is called a consistent system.
Otherwise it is called an inconsistent system.

We know that the solution of the system of linear equations **does not change** if we

1. Multiply any equation by a non-zero scalar
2. Replace an equation by the sum of itself and a scalar multiple of another equation.
3. Interchange any two equations

Elementary Row Operations: The following row operations are called **elementary row operations** of a matrix:

- Multiply a row R_i by a nonzero constant c ($R_i \to cR_i$)
- Add a multiple of a row R_j to another row R_i ($R_i \to R_i + cR_j$)
- Interchange of two rows ($R_i \leftrightarrow R_j$)

Row Equivalent Matrices: Matrices A and B are said to be row equivalent if there is a finite sequence of elementary row operations that converts A into B or B into A.

Example: Matrices

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & 3 \\ 3 & 1 & 4 \\ 1 & 2 & 2 \end{bmatrix}$$

are row equivalent.

Note: If a matrix A is row equivalent to a matrix B, then B is row equivalent to A (Why ?).

Row Echelon Form (REF): A matrix A is said to be in row echelon form if it satisfies the following properties:

1. The first nonzero entry (called the leading entry or pivot) in each row is 1.
2. For each nonzero row, leading entry or pivot comes to the right and below any leading entry of previous rows.(The column containing a pivot element is called a pivot column).
3. All zero rows are at the bottom.

- The following matrices are in row echelon form:

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 & 2 & 1 \\ 0 & 0 & 1 & 4 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- If a matrix A is in row echelon form, then in each column of A containing a leading entry, the entries below that leading entry are zero.

- The following matrices are **not** in row echelon form:

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix}, \begin{bmatrix} 0 & -1 & -1 & 2 & 1 \\ 1 & 0 & 5 & 10 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot Position: in a matrix is a location where a leading 1 (a **pivot**) appears in the row echlon form of the matrix.

Pivot: in a matrix is a nonzero number which is changed into a leading 1 used to create zeros below the pivot.

Pivot row/column: in a matrix is a row/column that contains a pivot position.

Example: Find Row Echelon form of the matrix

$$A = \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

- **Step 1: Selecting Pivot column :** Begin with the leftmost nonzero column.

$$A = \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

- **Step 2: Selecting Pivot Position:** Select a nonzero entry in the pivot column as a pivot. If necessary interchange rows to move this entry into the pivot position

Step 3: Use elementary row operations to create zeros in all positions below the pivot.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \xrightarrow[R_3 \to R_3 + 2R_1]{R_2 \to R_2 + R_1} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Step 4: Ignore the row containing the pivot position and cover all rows, if any, above it.
Apply steps 1-3 to the remaining sub- matrix. Repeat the process until there are no more nonzero rows to modify.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Make this pivot element to 1 by applying $R_2 \to \frac{1}{2}R_2$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \xrightarrow[R_4 \to R_4 + 3R_2]{R_3 \to R_3 - 5R_2} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

Apply $R_3 \leftrightarrow R_4$ and make this pivot -5 to 1.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 \to -\frac{1}{5} R_3} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This is a row echelon form.

Remark:

- Row echelon form of a matrix may not be unique.

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

- Every matrix is row equivalent to its row echelon form.

Equivalent Systems: Two system of m linear equations in n variables are equivalent if and only if they have exactly the same solution set.

Example: The systems

$$\begin{aligned} 2x_1 - x_2 &= 1 \\ 3x_1 + x_2 &= 9 \end{aligned} \quad \text{and} \quad \begin{aligned} x_1 + 4x_2 &= 14 \\ 5x_1 - 2x_2 &= 4 \end{aligned}$$

are equivalent. (Why?)

Theorem

Let $AX = B$ be a system of linear equations. If $[C|D]$ is row equivalent to $[A|B]$, then the system $CX = D$ is equivalent to $AX = B$.

Gaussian Elimination Method: Use the following steps to solve a system of equations $AX = B$

- Write the augmented matrix $[A \mid B]$.
- Find a row echelon form of the matrix $[A \mid B]$.
- Use back substitution to solve the equivalent system that corresponds to row echelon form.

Exercise: Solve the linear system of equations by Gaussian elimination method

$$\begin{aligned} x_1 + x_2 + x_3 &= 3, \\ 2x_1 + 3x_3 &= 5, \\ x_2 + x_3 &= 2. \end{aligned}$$

Hint: The augmented matrix of the given system of equations $AX = B$ is

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 3 \\ 2 & 0 & 3 & : & 5 \\ 0 & 1 & 1 & : & 2 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 & : & 3 \\ 0 & -2 & 1 & : & -1 \\ 0 & 1 & 1 & : & 2 \end{bmatrix}$$

$$\xrightarrow{R_2 \to -\frac{1}{2} R_2} \begin{bmatrix} 1 & 1 & 1 & \vdots & 3 \\ 0 & 1 & -\frac{1}{2} & \vdots & \frac{1}{2} \\ 0 & 1 & 1 & \vdots & 2 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 1 & 1 & \vdots & 3 \\ 0 & 1 & -\frac{1}{2} & \vdots & \frac{1}{2} \\ 0 & 0 & \frac{3}{2} & \vdots & \frac{3}{2} \end{bmatrix}$$

$$\xrightarrow{R_3 \to \frac{3}{2} R_3} \begin{bmatrix} 1 & 1 & 1 & \vdots & 3 \\ 0 & 1 & -\frac{1}{2} & \vdots & \frac{1}{2} \\ 0 & 0 & 1 & \vdots & 1 \end{bmatrix}$$

- Row echelon form of $[A : B]$ is

$$\begin{bmatrix} 1 & 1 & 1 & \vdots & 3 \\ 0 & 1 & -\frac{1}{2} & \vdots & \frac{1}{2} \\ 0 & 0 & 1 & \vdots & 1 \end{bmatrix}$$

- Corresponding system of equations

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_2 - \frac{1}{2}x_3 &= \frac{1}{2} \\ x_3 &= 1 \end{aligned}$$

- By backward substitution, we find

$$x_3 = 1, \quad x_2 = 1, \quad x_1 = 1$$

is a solution of the given system of equations.

Exercise: Solve the linear system of equations

$$x_1+x_2+x_3=3, \quad x_1+2x_2+2x_3=5, \quad 3x_1+4x_2+4x_3=12$$

by Gaussian elimination method.

Solution: The corresponding augmented matrix is

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 2 & 5 \\ 3 & 4 & 4 & 12 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- The REF of the augmented matrix is

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Corresponding system of equations

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_2 + x_3 &= 2 \; . \\ 0x_3 &= 1 \end{aligned}$$

- The given system of equations is inconsistent.

Exercise: Solve the system of linear equations

$$x_1+x_2+x_3=3,\quad x_1+2x_2+2x_3=5,\quad 3x_1+4x_2+4x_3=11$$

by Gaussian elimination method.

Solution: The corresponding Augmented matrix is

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 2 & 5 \\ 3 & 4 & 4 & 11 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

$$R_3 \to R_3 - R_2 \Big\downarrow$$

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- The REF of the augmented matrix is
$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
- Corresponding system of equations
$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_2 + x_3 &= 2 \\ 0x_3 &= 0 \end{aligned} .$$
- By backward substitution, we find

$$x_3 = a, \quad x_2 = 2 - a, \quad x_1 = 1$$

is a solution of the given system of equations.

Independent and dependent variables:

- Consider the linear system $AX = B$ in n variables and m equations.
- Let $[C \mid D]$ be a row echelon form of the augmented matrix $[A \mid B]$.
- The variables corresponding to the pivot columns in the first n columns of $[C \mid D]$ are called the dependent (or basic) variables.
- The variables which are not dependent are called independent (free) variables.

Reduced Row Echelon Form (RREF): A matrix A is said to be in reduced row echelon form if it satisfies the following properties:

- A is in row echelon form.
- If a column contains a leading entry (or pivot) then all other entries in that column must be zero.

Example: The following matrix are in reduced row echelon form

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Result:

- Every matrix has a unique reduced row echelon form.

- Matrices A and B are row equivalent if and only if they have same reduced row echelon form.

Gauss-Jordan Row Reduction Method: Use the following steps to solve a system of equations $AX = B$

- Write the augmented matrix $[A \mid B]$.
- Find the reduced row echelon form of the matrix $[A \mid B]$.
- Use back substitution to solve the equivalent system that corresponds to row echelon form.

Exercise: Solve the system of linear equations equations

$$x_1 + x_2 + x_3 = 5, \;\; 2x_1 + 3x_2 + 5x_3 = 8, \;\; 4x_1 + 5x_3 = 2$$

by Gauss-Jordan method.

Solution: The corresponding augmented matrix is

$$\begin{bmatrix} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & -4 & 1 & -18 \end{bmatrix}$$

$$\xrightarrow[R_1 \to R_1 - R_2]{R_3 \to R_3 + 4R_2} \begin{bmatrix} 1 & 0 & -1 & 7 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 13 & -26 \end{bmatrix}$$

$$\xrightarrow{R_3 = \frac{1}{13} R_3} \begin{bmatrix} 1 & 0 & -2 & 7 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$$\xrightarrow[R_2 \to R_2 - 3R_3]{R_1 \to R_1 + 2R_3} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

- The RREF of the augmented matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

- Corresponding system of equations is

$$\begin{aligned} x_1 &= 3 \\ x_2 &= 4 \\ x_3 &= -2 \end{aligned}$$

- The solution is $x_1 = 3, x_2 = 4$ and $x_3 = -2$.

Exercise: Solve the system of linear equations

$$4x_2 + x_3 = 2,\ 2x_1 + 6x_2 - 2x_3 = 3,\ 4x_1 + 8x_2 - 5x_3 = 4$$

by Gauss-Jordan method.

Answer: Infinitely many solutions and solution set is

$$\left\{ \left(\frac{7}{4}d,\ \frac{1}{2} - \frac{1}{4}d,\ d \right) \mid d \in \mathbb{R} \right\}.$$

Exercise: Solve the system of linear equations

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 2, \\ 6x_1 + 3x_2 - 9x_3 &= 6, \\ 7x_1 + 14x_2 - 21x_3 &= 13 \end{aligned}$$

by Gauss-Jordan method.

Answer: No solution.

Question: Whether there are conditions under which the linear system $AX = B$ is consistent?

Rank: The rank of a matrix A is the number of non-zero rows in its row echelon form. It is denoted by rank(A).

Remark: The number of non-zero rows in either the row echelon form or the reduced row echelon form of a matrix are same.

Exercise: Determine the rank of $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

Solution: REF of A is

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

No. of nonzero rows is 2.

$$rank(A) = 2.$$

Theorem: Let $AX = B$ be a system of equations with n variables.

1. If $\text{rank}(A) = \text{rank}([A \mid B]) = n$ then the system $AX = B$ has a unique solution.
2. if $\text{rank}(A) = \text{rank}([A \mid B]) < n$ then the system $AX = B$ has a infinitely many solutions.
3. If $\text{rank}(A) \neq \text{rank}([A \mid B])$ then the system $AX = B$ is inconsistent.

Theorem: Let $AX = 0$ be a homogenous system of equations with n variables.

1. If $\text{rank}(A) = n$ then the system has a unique solution (trivial solution).
2. If $\text{rank}(A) < n$ then the system $AX = B$ has infinitely many solutions.

Exercise: Test the consistency of the given system of equations

$$\begin{aligned} 3x_1 + x_2 + x_4 &= -9 \\ -2x_2 + 12x_3 - 8x_4 &= -6 \\ 2x_1 - 3x_2 + 22x_3 - 14x_4 &= -17. \end{aligned}$$

Find all the solutions, if it is consistent.

Solution: The Augmented matrix and its REF is given by

$$\left[\begin{array}{cccc|c} 3 & 1 & 0 & 1 & -9 \\ 0 & -2 & 12 & -8 & -6 \\ 2 & -3 & 22 & -14 & -17 \end{array}\right] \xrightarrow{REF} \left[\begin{array}{cccc|c} 1 & \frac{1}{3} & 0 & \frac{1}{3} & -3 \\ 0 & 1 & -6 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Solution set$=\{(-4-2a+b, 3+6a-4b, a, b), a, b \in \mathbb{R}\}$

Exercise: For what value of $k \in \mathbb{R}$, the following system of equations is inconsistent

$$kx_1 + x_2 = 0,$$
$$x_1 + kx_2 = 1.$$

Answer: $k = \pm 1$

Exercise: For what value of $k \in \mathbb{R}$, the following system of equation has (i) a unique solution (ii) infinitely many solutions and (iii) no solution

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 0 \\ -x_1 + x_2 - x_3 &= 0 \\ x_1 + kx_2 + x_3 &= 0. \end{aligned}$$

Also find the solutions, whenever they exist.

Solution:

(i) $k \neq -1$, $x_1 = x_2 = x_3 = 0$

(ii) $k = -1$, The solution set is $\{(a, a, 0) : \ a \in \mathbb{R}\}$

(iii) No value of k

Exercise: For what value of $\lambda \in \mathbb{R}$, the following system of equation has (i) a unique solution (ii) infinitely many solutions and (iii) no solution

$$\begin{aligned}(5-\lambda)x_1 + 4x_2 + 2x_3 &= 4\\ 4x_1 + (5-\lambda)x_2 + 2x_3 &= 4\\ 2x_1 + 2x_2 + (2-\lambda)x_3 &= 2.\end{aligned}$$

Also find the solutions, whenever they exist.

Solution: The augmented matris has the REF as for $\lambda \neq 1, 10$

$$\begin{bmatrix} 1 & 1 & \frac{2-\lambda}{2} & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & \frac{2}{10-\lambda} \end{bmatrix}$$

(i) Unique solution for $\lambda \neq 1, 10$ and the solution is

$$x_1 = \frac{4}{10-\lambda},\ x_2 = \frac{4}{10-\lambda},\ x_3 = \frac{2}{10-\lambda}$$

(ii) Infinitely many solutions for $\lambda = 1$ and the solution set is

$$\{(1 - a - \frac{b}{2}, a, b) :\ a, b \in \mathbb{R}\}$$

(iii) No solution for $\lambda = 10$.

Definition: Let A be an $n \times n$ matrix. Then an $n \times n$ matrix B is said to be a (multiplicative) inverse of A if and only if

$$AB = BA = I_n,$$

where I_n is the $n \times n$ identity matrix.

- If such a matrix B exists, then A is called nonsingular.
- If there exists no such matrix B, then A is called singular.

Example: Show that the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix}$ is nonsingular.

Solution: For $B = \begin{bmatrix} 9 & -4 \\ -2 & 1 \end{bmatrix}$, we have $AB = BA = I_2$.

Theorem

Let A and B be $n \times n$ matrices.

- *If $AB = I_n$ then $BA = I_n$.*
- *If $BA = I_n$ then $AB = I_n$.*

Theorem

Inverse of a matrix is unique if it exists.

As the inverse of a matrix A is unique, we denote it by A^{-1}. That is, $AA^{-1} = A^{-1}A = I$.

Theorem

Let A and B be an $n \times n$ nonsingular matrices. Then

- $(A^{-1})^{-1} = A$.
- $(AB)^{-1} = B^{-1}A^{-1}$.
- $(A^T)^{-1} = (A^{-1})^T$.

Question:

- How can we know when a matrix has an inverse?
- If a matrix does have an inverse, how can we find it?

Method for finding the Inverse of a matrix (if it exists): Let A be a given $n \times n$ matrix.

Step 1: Write the augmented matrix $[A \mid I_n]$.

Step 2: Transform the augmented matrix $[A \mid I_n]$ to the matrix $[C \mid D]$ in reduced row echelon form via elementary row operations.

Step 3: If

- $C = I_n$ then $D = A^{-1}$.
- $C \neq I_n$ then A is singular and A^{-1} does not exist.

Exercise: Using row reduction method, find the inverse of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$, if it exists.

Hint: Note that reduced row echelon form of the matrix $[A|I_3]$ is

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right]$$

Thus, $A^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 2 \\ -1 & 1 & -1 \end{bmatrix}$

Theorem

Let A be an $n \times n$ matrix. The following statements are equivalent:

- *A is nonsingular.*
- *$AX = B$ has a unique solution for every $B \in \mathbb{R}^n$.*
- *$AX = 0$ has only the trivial solution.*
- *The reduced row echelon form of A is I_n.*
- *$rank(A) = n$.*

Thank You