MATHEMATICS-II (MATH F112)

Dr. Krishnendra Shekhawat

BITS PILANI
Department of Mathematics

CHAPTER 5

Linear Transformations

Topics to be covered in Chapter 5

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- Introduction to Linear Transformations

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- Introduction to Linear Transformations
- The Matrix of a Linear Transformation

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- Isomorphism

Section 5.1

Introduction to Linear Transformations

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- $L(cu) = cL(u)$ for all $c \in \mathbb{R}$ and all $u \in U$ (L preserves scalar multiplication)

Example 1

Q:. Consider the mapping $L : M_{mn} \to M_{nm}$, given by $L(A) = A^T$ for any $m \times n$ matrix A.

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- $L(A+B) = (A+B)^T = A^T + B^T = L(A) + L(B)$
- $L(cA) = (cA)^T = cA^T = cL(A)$

Hence, L is a LT.

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Q:. Consider the mapping $L : \mathbb{R}^2 \to \mathbb{R}^3$, given by $L([x, y]) = ([x, y, xy])$.

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$$\neq c([x, y, xy]),$$

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$\neq c([x,y,xy])$, i.e., $L(c([x,y])) \neq cL([x,y])$ for some $c \in \mathbb{R}$.

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Hence, L is not a LT.

Exercises

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Exercises

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- $L : \mathbb{R}^3 \to \mathbb{R}^2$, given by $L([x, y, z]) = ([x - y, y + z])$.
- $L : \mathbb{R}^2 \to \mathbb{R}^2$, given by $L([a, b]) = [a, -b]$.

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For example, the mapping $L : \mathbb{R}^3 \to \mathbb{R}^3$, given by $L([x, y, z]) = [x, y, -z]$ is a linear operator.

Theorem: Let V and W be vector spaces, and let $L : V \to W$ be a LT.

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3. For $n \geqslant 2$,
$$L(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n) = a_1L(\mathbf{v}_1) + a_2L(\mathbf{v}_2) + \cdots + a_nL(\mathbf{v}_n),$$

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for all $a_1, a_2, \ldots, a_n \in \mathbb{R}$ and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$.

Example 3

Q:. Consider the mapping $L : M_{22} \to \mathbb{R}$ given by

$$L\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a + b + c + d - 1.$$

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Sol. $L\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0+0+0+0-1 = -1 \neq 0.$

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Sol. $L\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0+0+0+0-1 = -1 \neq 0.$

Since $L(0_{M_{22}}) \neq 0_{\mathbb{R}}$, L is not a LT.

Example 4

Q:. Suppose $L : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear operator and $L([1,0,0]) = [-2,1,0], L([0,1,0]) = [3,-2,1]$, and $L([0,0,1]) = [0,-1,3]$. Find $L([-3,2,4])$.

Example 4

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Sol.

$$\begin{cases} L([-3,2,4]) = L(-3[1,0,0] + 2[0,1,0] + 4[0,0,1]) \\ \\ \\ \end{cases}$$

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Sol.

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\qquad\qquad\qquad = -3[-2,1,0]+2[3,-2,1]+4[0,-1,3] \\
\qquad\qquad\qquad = [12,-11,14]
\end{cases}
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Similarly, $L([x, y, z]) = L(x[1, 0, 0] + y[0, 1, 0] + z[0, 0, 1])$

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Similarly, $L([x,y,z]) = L(x[1,0,0] + y[0,1,0] + z[0,0,1])$

$$L([x,y,z]) = x[-2,1,0] + y[3,-2,1] + z[0,-1,3]$$

$$L([x,y,z]) = [-2x+3y, x-2y-z, y+3z]$$

Similarly, $L([x,y,z]) = L(x[1,0,0]+y[0,1,0]+z[0,0,1])$

$L([x,y,z]) = x[-2,1,0]+y[3,-2,1]+z[0,-1,3]$

$L([x,y,z]) = [-2x+3y, x-2y-z, y+3z]$

Note that

$$L\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 & 3 & 0 \\ 1 & -2 & -1 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Exercise

Q:. Suppose $L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear operator and $L([1,1]) = [3,0]$ and $L([-1,1]) = [0,1]$. Compute $L([x,y])$.

Sol. $L([x,y]) = \left[\frac{3x+3y}{2}, \frac{-x+y}{2}\right]$

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Remark

Let V and W be vector spaces, and let $L : V \to W$ be a LT. Also, let $\{v_1, v_2, \ldots, v_n\}$ be a basis for V.

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Remark

Let V and W be vector spaces, and let $L : V \to W$ be a LT. Also, let $\{v_1, v_2, \ldots, v_n\}$ be a basis for V. If $v \in V$, $L(v)$ is completely determined by $\{L(v_1), L(v_2), \ldots, L(v_n)\}$.

Composition of Linear transformations

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Theorem: Let V_1, V_2, and V_3 be vector spaces. Let $L_1 : V_1 \to V_2$ and $L_2 : V_2 \to V_3$ be linear transformations. Then $L_2 \circ L_1 : V_1 \to V_3$ given by
$(L_2 \circ L_1)(\mathbf{v}) = L_2(L_1(\mathbf{v}))$, for all $\mathbf{v} \in V_1$, is a LT.

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Note: $(L_2 \circ L_1)(v)$ is called composite of L_2 with L_1.

Example 5

Q:. Let $L_1 : P_2 \rightarrow P_2$ and $L_2 : P_2 \rightarrow P_2$ be linear operators.

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Q:. Let $L_1 : P_2 \longrightarrow P_2$ and $L_2 : P_2 \longrightarrow P_2$ be linear operators. Also, let $L_1(at^2 + bt + c) = 2at + b$ and $L_2(at^2 + bt + c) = 2at^2 + bt$.

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Sol. $L_2 \circ L_1(at^2 + bt + c) = L_2(L_1(at^2 + bt + c)) =$
$L_2(2at + b) = 2at$.

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Sol. $L_2 \circ L_1(at^2 + bt + c) = L_2(L_1(at^2 + bt + c)) = L_2(2at + b) = 2at$.

Similarly, $L_1 \circ L_2(at^2 + bt + c) = 4at + b$.

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Sol. $L_2 \circ L_1(at^2 + bt + c) = L_2(L_1(at^2 + bt + c)) = L_2(2at + b) = 2at$.

Similarly, $L_1 \circ L_2(at^2 + bt + c) = 4at + b$.

Clearly, $L_2 \circ L_1 \neq L_1 \circ L_2$.

Example 6

Q:. Find two linear operators $L_1 : \mathbb{R}^2 \to \mathbb{R}^2$ and $L_2 : \mathbb{R}^2 \to \mathbb{R}^2$ such that $L_2 \circ L_1([a,b]) = [0,0]$ and $L_1 \circ L_2([a,b]) \neq [0,0]$.

Example 6

Q:. Find two linear operators $L_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and $L_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $L_2 \circ L_1([a,b]) = [0,0]$ and $L_1 \circ L_2([a,b]) \neq [0,0]$.

Sol. Let $L_2([a,b]) = [a,0]$ and $L_1([a,b]) = [0,a]$.

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Sol. Let $L_2([a,b]) = [a,0]$ and $L_1([a,b]) = [0,a]$.

$$L_2 \circ L_1([a,b]) = L_2(L_1([a,b])) = L_2([0,a]) = [0,0].$$

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Sol. Let $L_2([a,b]) = [a,0]$ and $L_1([a,b]) = [0,a]$.

$L_2 \circ L_1([a,b]) = L_2(L_1([a,b])) = L_2([0,a]) = [0,0]$.

Similarly,

$L_1 \circ L_2([a,b]) = L_1(L_2([a,b])) = L_1([a,0]) = [0,a]$.