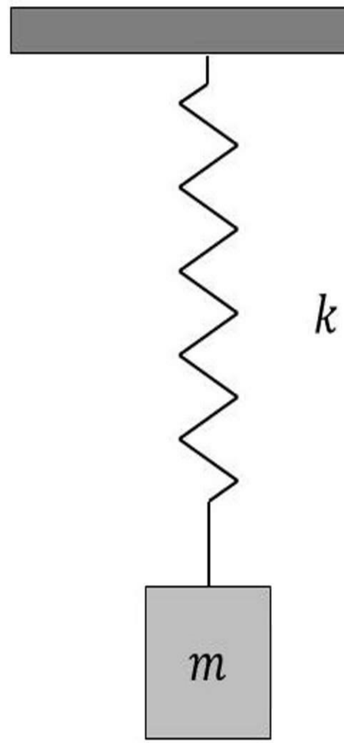
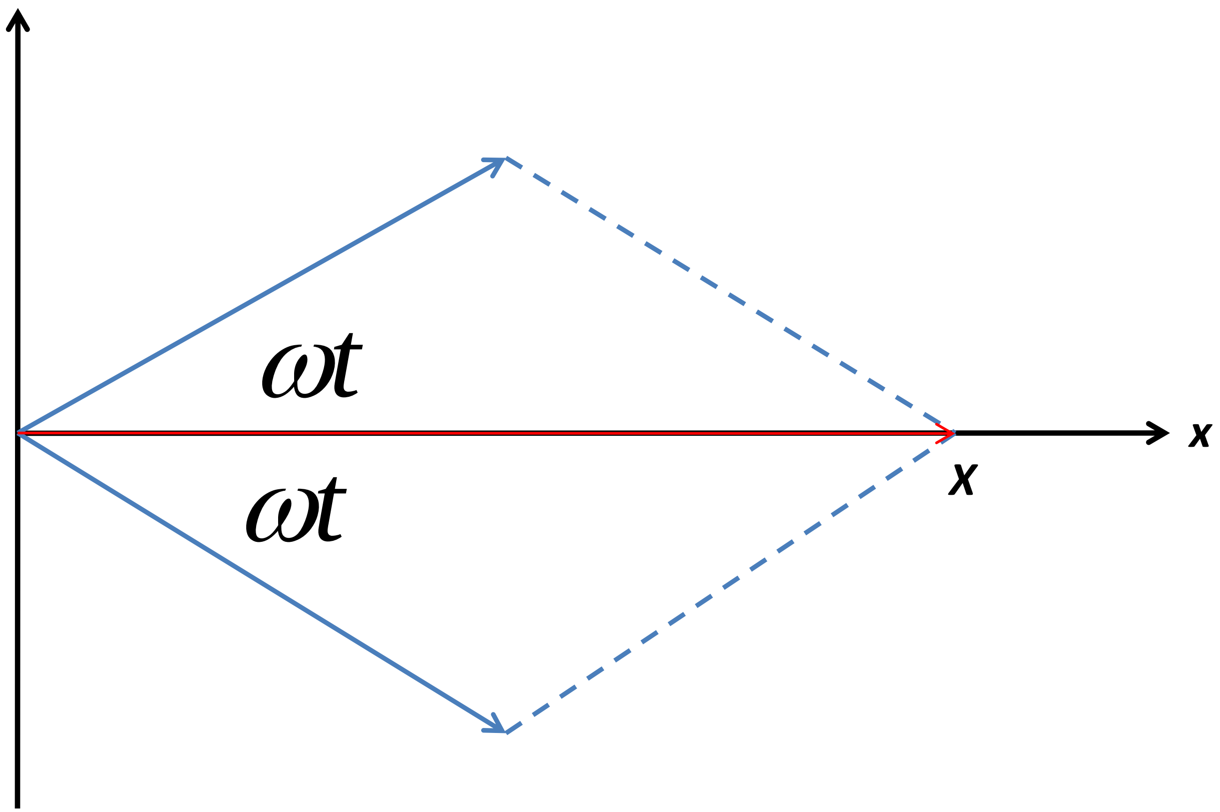
Chapter 3:
The Free Vibrations of Physical Systems

The Basic Mass-Spring System

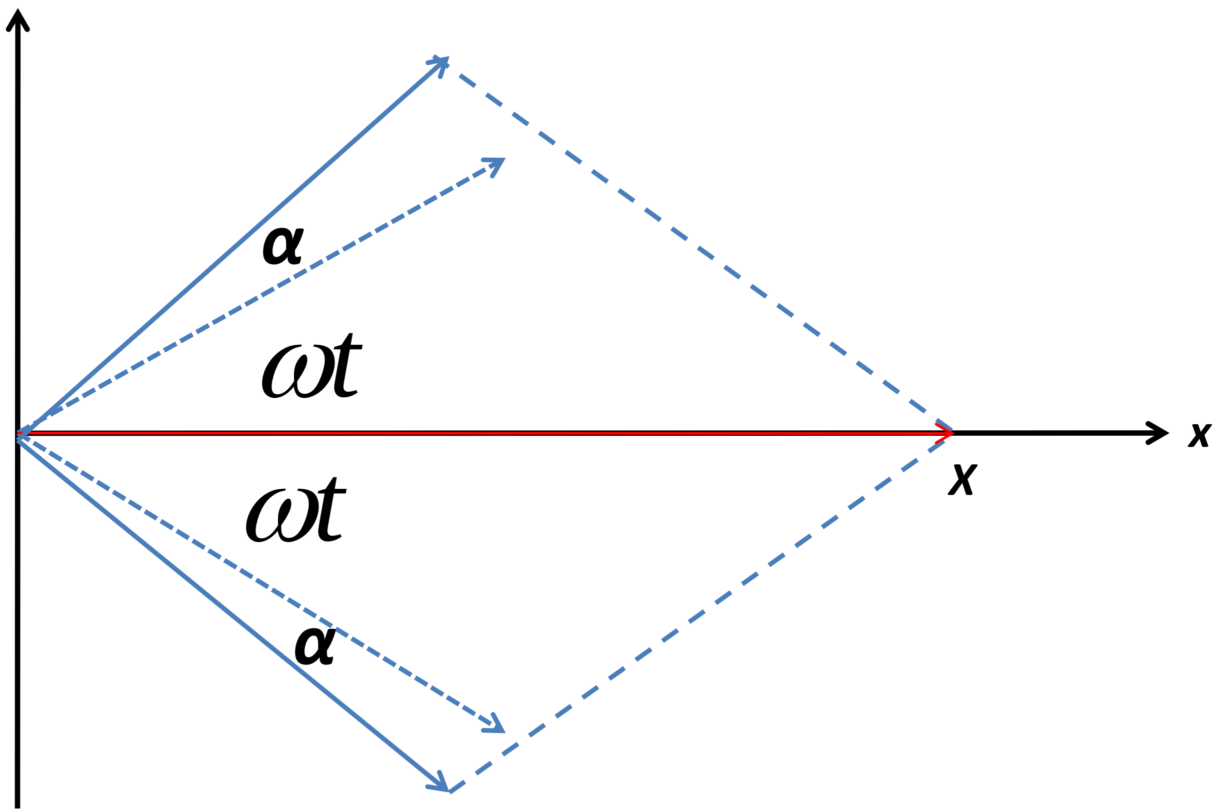


Solving the Harmonic Oscillator Equation Using Complex Exponentials



Superposition of Complex solutions

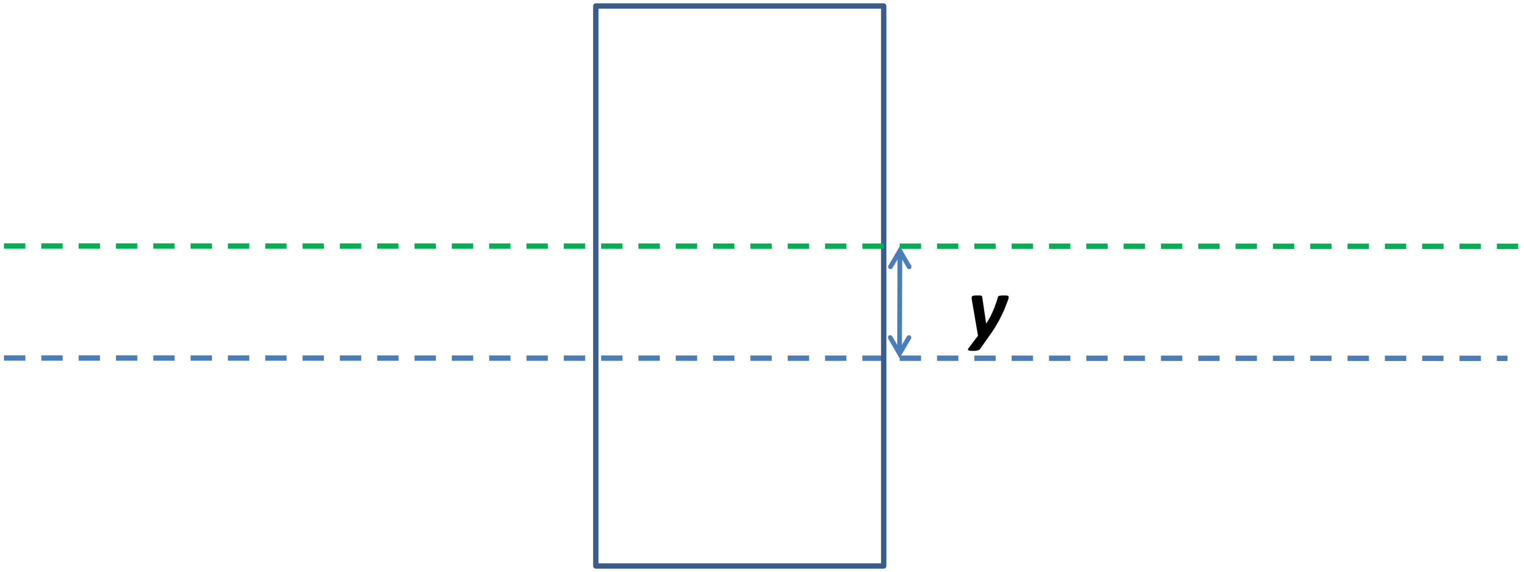
Solving the Harmonic Oscillator Equation Using Complex Exponentials



Superposition of Complex solutions with non-zero initial phase angle.

Examples of SHM

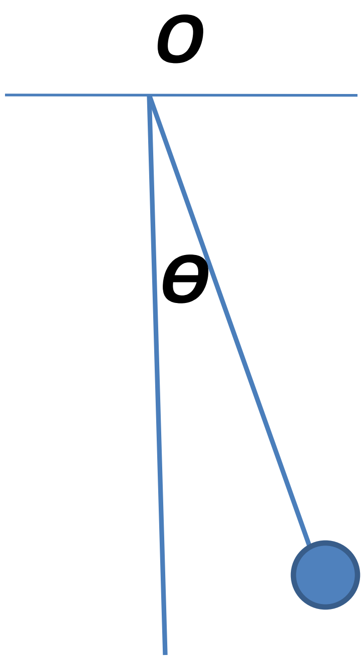
Floating Objects



Examples of SHM

Pendulums

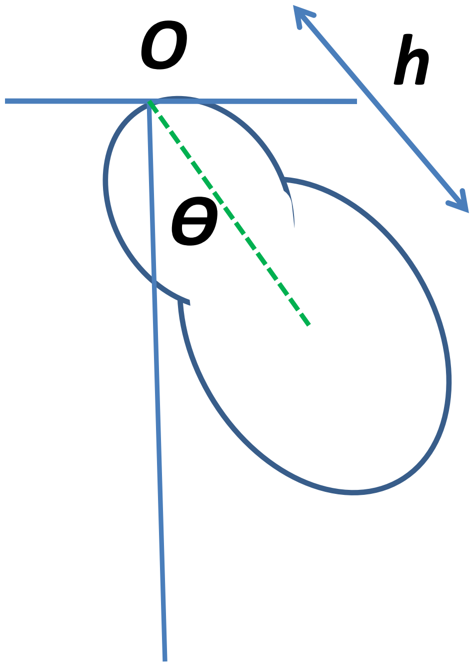
a) Simple Pendulum



Examples of SHM

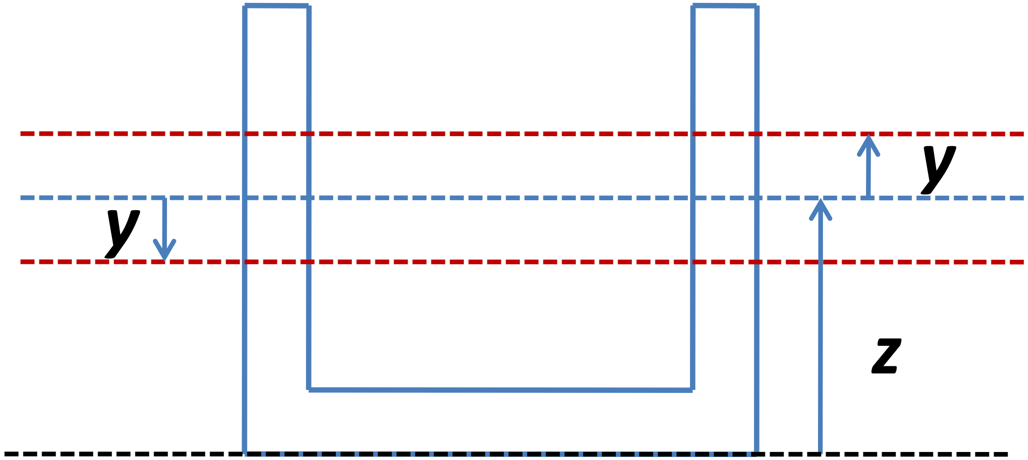
Pendulums

b) Compound Pendulum



Examples of SHM

c) Water in a U-tube



Decay of Free Vibrations

$$m\frac{d^2X}{dt^2}+b\frac{dX}{dt}+kX=0$$

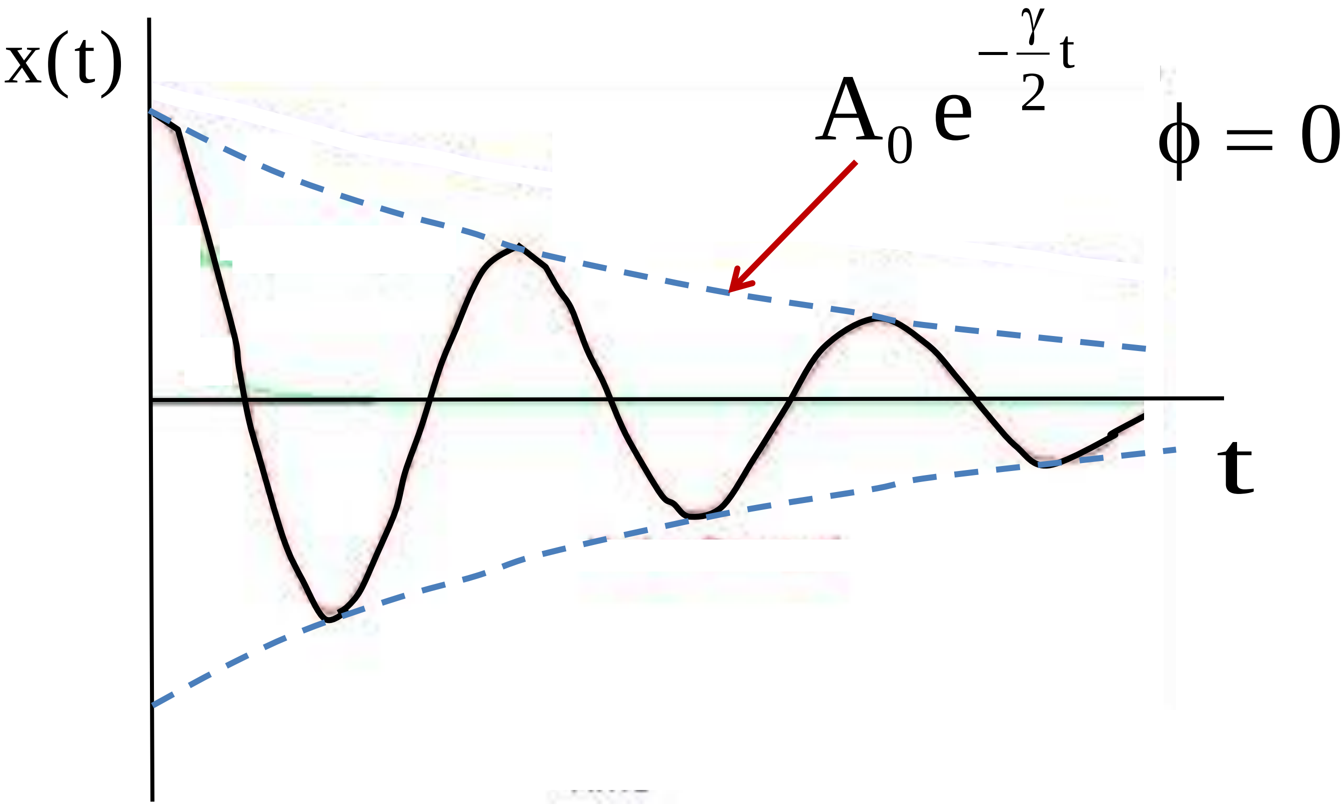
$$\frac{d^2X}{dt^2}+\gamma\frac{dX}{dt}+\omega_0^2X=0$$

Decay of Free Vibrations

Complex Exponential Method:

$$Z(t) = A\exp\left(jpt + j\alpha\right)$$

Decay of Free Vibrations



Under damped Simple Harmonic motion

The Quality Factor (Quality of the Oscillatory System)

The Damped Oscillator is characterized by two parameters:

ω_0 and γ

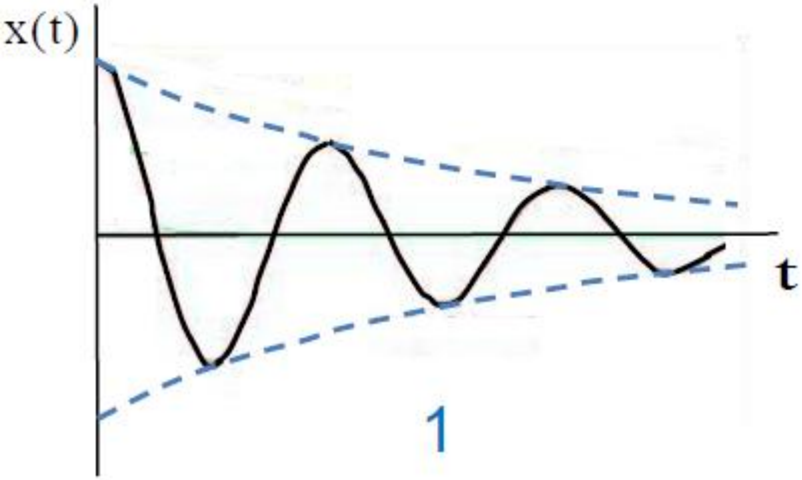
The Quality Factor

$$Q = \frac{\omega_0}{\gamma}$$

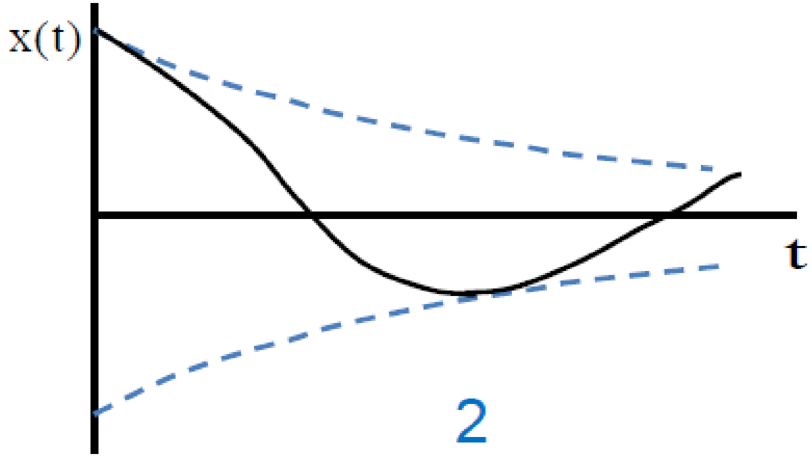
The Quality Factor (Quality of the Oscillatory System)

Quality Factor (Q) : the number of oscillations after which the amplitude of a damped oscillator drops to $e^{-\pi}$ of its initial value.

The Quality Factor



1



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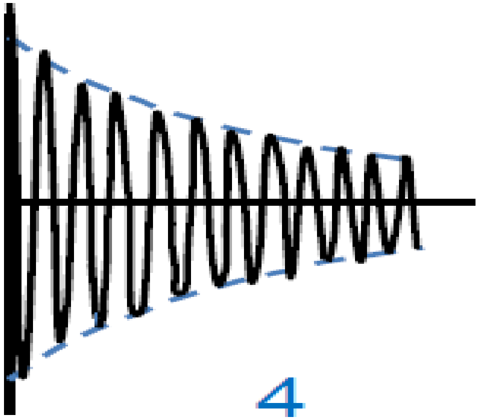


4

The Quality Factor

Time required to complete Q oscillations :

$$t = QT = \frac{2\pi Q}{\omega}$$

$$\therefore\ e^{-\frac{\pi Q \gamma}{\omega}} = e^{-\pi}$$

$$\therefore\ Q = \frac{\omega}{\gamma}$$

The Quality Factor

$$\omega^2 = \omega_0^2 - \frac{\gamma^2}{4} = \omega_0^2 - \frac{\omega^2}{4Q^2}$$

$$\therefore\ \omega = \omega_0 \left[1 + \frac{1}{4Q^2}\right]^{-\frac{1}{2}}$$

For large Q (Q > 5),

$$\omega = \omega_0 \left[1 - \frac{1}{8Q^2}\right] \approx \omega_0$$

$$\therefore\ Q \approx \frac{\omega_0}{\gamma} \quad (\mathrm{Large\ Q})$$

The Effects of Very Large Damping:

If :

$$\frac{b^2}{4m^2} > \frac{k}{m}$$

Then:

$$x(t) = A_1 \exp\left(\frac{-\gamma t}{2} - \beta t\right) + A_2 \exp\left(\frac{-\gamma t}{2} + \beta t\right)$$

Overdamped

The Effects of Very Large Damping:

If : $$\frac{b^2}{4m^2} = \frac{k}{m}$$

Then:

$$x(t) = (A + Bt)\exp\left(\frac{-\gamma t}{2}\right)$$

Critically Damped

Prob. 3.12 . The motion of a linear oscillator may be represented by means of a graph in which x is shown as abscissa and $\dot{x}$ is shown as ordinate. The history of the oscillator is thus a curve.

a) Show that for an undamped oscillator, this curve is an ellipse.

b) Show that if a damping term is introduced, one gets a curve spiraling into the origin

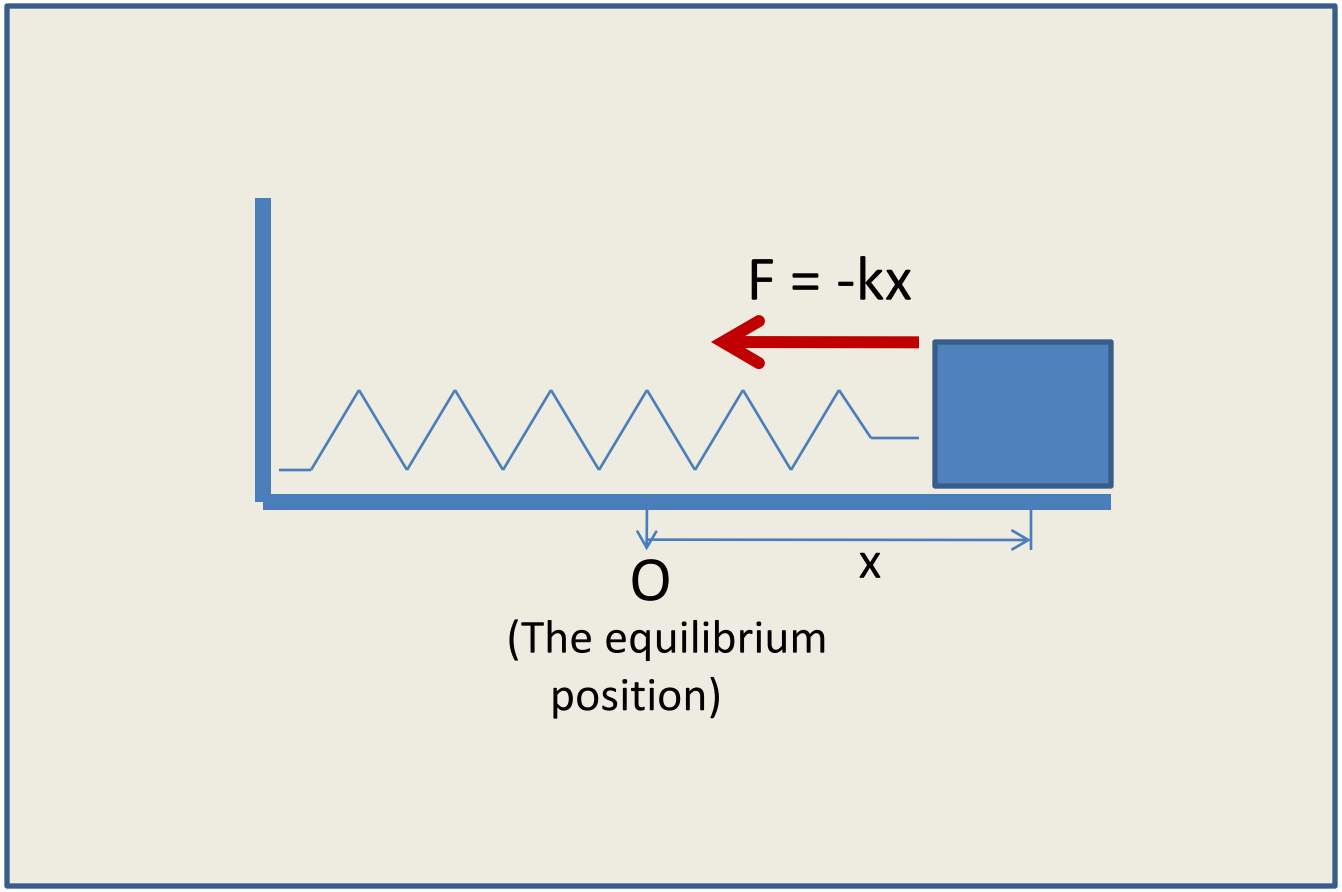
Prob. 3.14. Object of mass 0.2 kg hung from a spring of spring constant 80 N/m. Resistive force $-\mathrm{bv}$ acting on the object.

a) Set up the differential eq. of motion

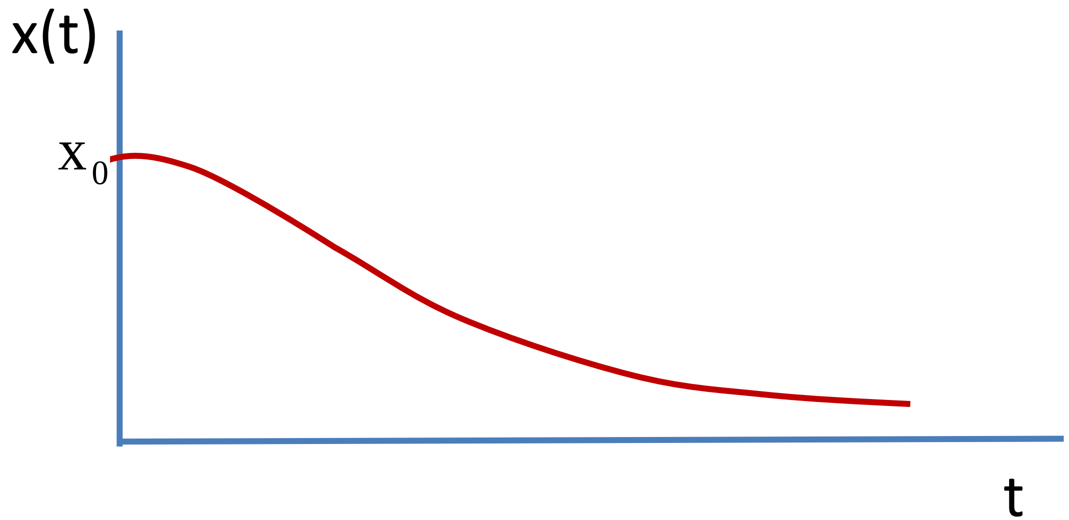
b) If the damped frequency is $\frac{\sqrt{3}}{2}$ times the undamped frequency, what is the value of b?

c) What is the Q of the oscillator and by what factor is the amplitude reduced after 10 complete cycle?

- Overdamped

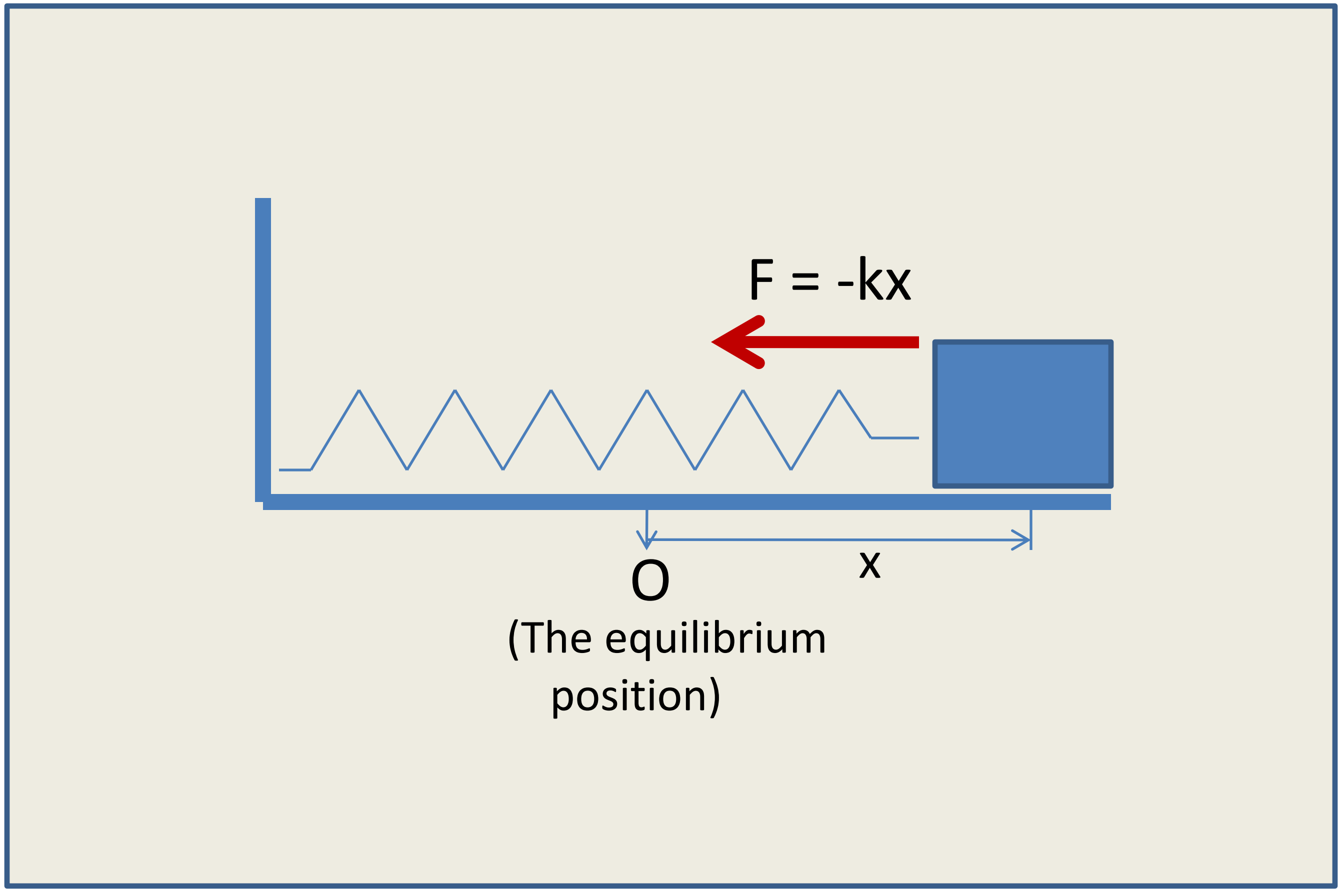


With the initial conditions : $x(0) = x_0$; $\dot{x}(0) = 0$



$$x(t) = \frac{x_0}{2\beta}\left[\left(\frac{\gamma}{2} + \beta\right) e^{-\left(\frac{\gamma}{2} - \beta\right)t} - \left(\frac{\gamma}{2} - \beta\right) e^{-\left(\frac{\gamma}{2} + \beta\right)t}\right]$$

- Critically Damped



Critical Damping

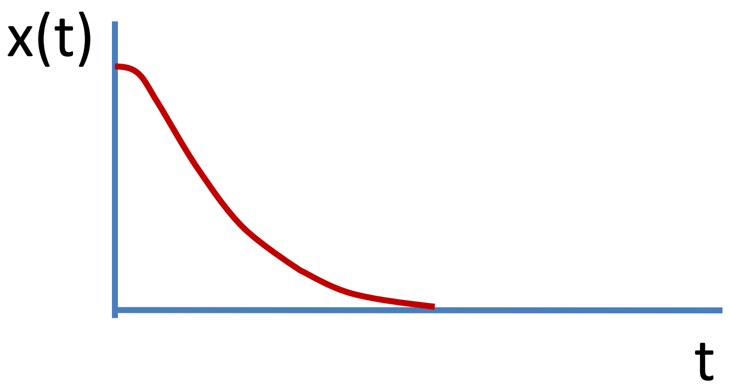
$$\beta = 0$$

The most general solution :

$$x(t) = (A + Bt)e^{-\frac{\gamma}{2}t}$$

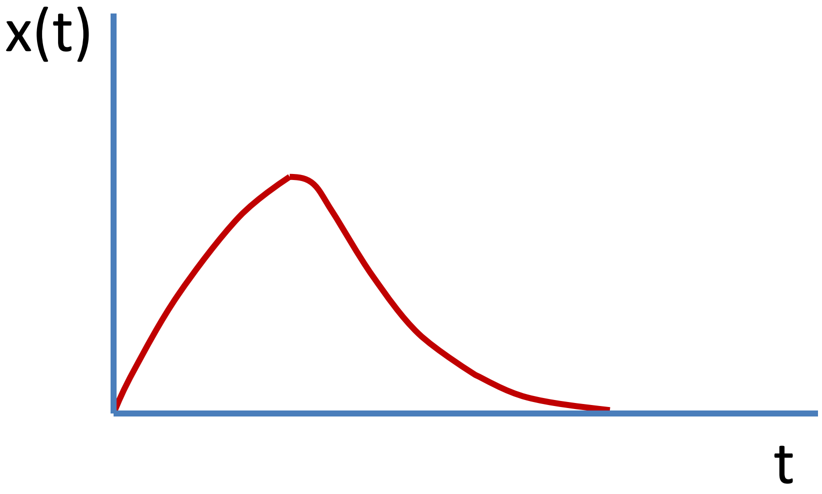
i) With the initial conditions : $x(0) = x_0 \; ; \; \dot{x}(0) = 0$

$$x(t) = x_0\left(1 + \frac{\gamma}{2}t\right)e^{-\frac{\gamma}{2}t}$$



ii) With the initial conditions : $x(0) = 0 \; ; \; \dot{x}(0) = v_0$

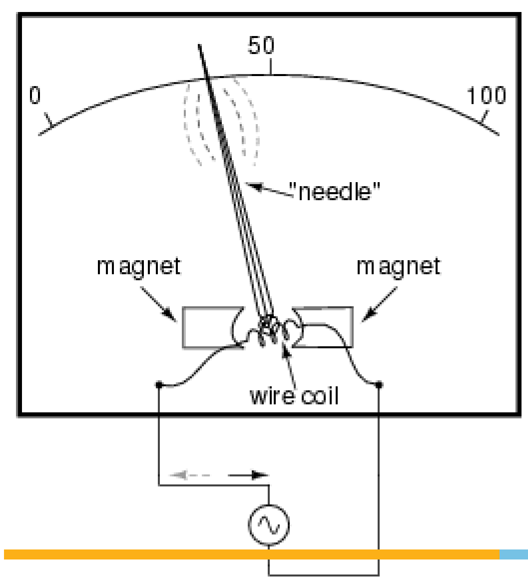
$$x(t) = v_0 \, t \, e^{-\frac{\gamma}{2} t}$$



Applications of Critical Damping

In many systems, quick damping is desirable to bring the system to a quick stop.

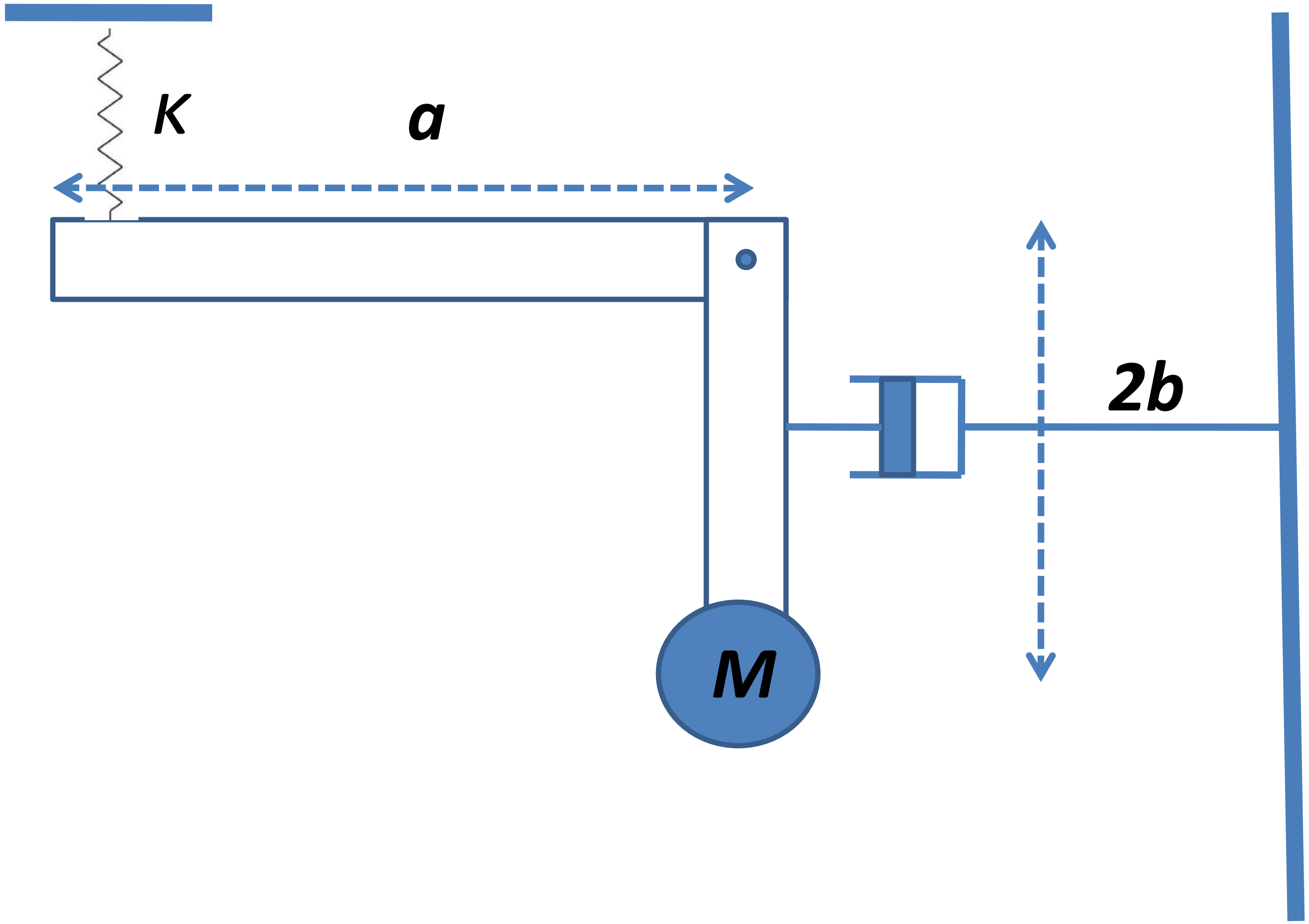
i) Needle in meters such as ammeter, voltmeter etc.



i) Door closers :

Out of the two non-oscillatory damping – over damping and critical damping – it is the latter that brings the system back to equilibrium quicker

Problem: Damped Oscillator



Problem: 3.16 According to classical electromagnetic theory an accelerated electron radiates energy at the rate Ke^2a^2/c^3, where $K = 6*10^9 N\text{-}m^2/C^2$.

(a) If an electron were oscillating along a straight line with frequency ν (Hz) and amplitude A, how much energy it would radiate away during one cycle. Assume that the motion is described adequately by $x = A\sin 2\pi\nu t$ during any cycle.

Problem: 3.16 According to classical electromagnetic theory an accelerated electron radiates energy at the rate Ke^2a^2/c^3, where $K = 6*10^9 N\text{-}m^2/C^2$.

(b) What is the Q of this Oscillator.

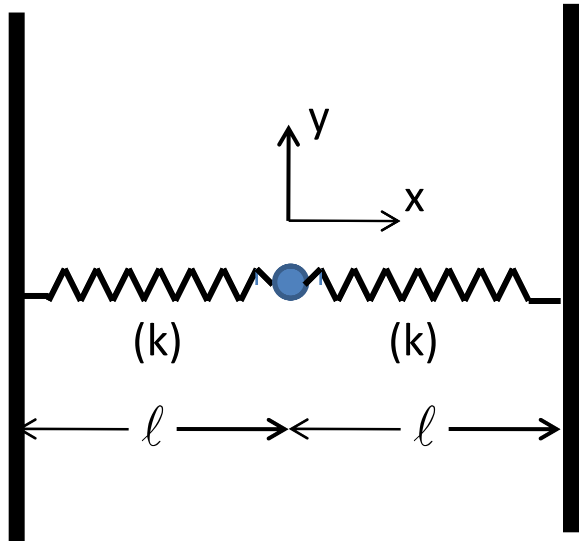
(c) How many periods of oscillations would elapse before the energy of the motion was down to half the initial value.

(d) Putting for ν a typical optical frequency estimate numerically the approximate Q and half-life of the radiating system.

Superposition of SHMs at right angles

Prob. 3.19

Mass m connected to two springs on frictionless horizontal table. Unstretched lengths of springs ℓ_0



a) Eq. of motion along x

b) Eq. of motion along y

c) Ratio of periods along x & y

d) x & y as functions of time if x(0) = y(0) = A and mass starts from rest

e) Sketch path of mass for $\ell = \frac{9}{5}\ell_0$

e) $\omega_x = \frac{3}{2}\omega_y$