# MATHEMATICS-II (MATH F112)


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# Section 5.2

# *The Matrix of a Linear Transformation*

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Let $V$ and $W$ be two finite dimensional real vector spaces such that $\mathbf{dim}(V) = n$ and $\mathbf{dim}(W) = m$. Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $C = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ be an ordered basis of $V$ and $W$, respectively.

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Let $L : V \to W$ be any linear transformation. For $\mathbf{v}_j \in V$, $L(\mathbf{v}_j) \in W$, for each $j, 1 \leqslant j \leqslant n$. Since $C$ is a basis of $W$, for $a_{ij} \in \mathbb{R}$, we can write

$$L(\mathbf{v}_j) = a_{1j}\mathbf{w}_1 + a_{2j}\mathbf{w}_2 + \cdots + a_{mj}\mathbf{w}_m$$

Thus, we have

$$
\begin{aligned}
L(\mathbf{v}_1) &= a_{11}\mathbf{w}_1 + a_{21}\mathbf{w}_2 + \cdots + a_{m1}\mathbf{w}_m \\
L(\mathbf{v}_2) &= a_{12}\mathbf{w}_1 + a_{22}\mathbf{w}_2 + \cdots + a_{m2}\mathbf{w}_m \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
L(\mathbf{v}_n) &= a_{1n}\mathbf{w}_1 + a_{2n}\mathbf{w}_2 + \cdots + a_{mn}\mathbf{w}_m
\end{aligned}
$$

Then the matrix

$$A_{BC} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

is called the matrix of linear transformation $L$ w.r.t. the bases $B$ and $C$.

## Example 1

**Q:.** Consider the LT $L : \mathbb{R}^3 \to \mathbb{R}^2$, given by

$$L([x, y, z]) = [-2x + 3z, x + 2y - z]$$

with ordered bases
$B = ([1, -3, 2], [-4, 13, -3], [2, -3, 20])$ and
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$$[C|L[B]] = \left[\begin{array}{cc|ccc} -2 & 5 & 4 & -1 & 56 \\ -1 & 3 & -7 & 25 & -24 \end{array}\right]$$

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Using row reduction, we obtain

$$[C|L[B]] = \left[\begin{array}{cc|ccc} 1 & 0 & -47 & 128 & -288 \\ 0 & 1 & -18 & 51 & -104 \end{array}\right]$$

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$[L(2,-3,20)]_C = [-288,-104]$.

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Clearly, $[58, 19]_C = -79[-2, -1] - 20[5, 3]$.

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- Use row reduction on $[w_1, \ldots, w_m | L(v_1), \ldots, L(v_n)]$ to produce $[I_m | A_{BC}]$.

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Let $L : \mathbb{R}^4 \to \mathbb{R}^3$, given by

$L([0,4,0,1]) = [3,1,2]$, $L([-2,5,0,2]) = [2,-1,1]$,
$L([-3,5,1,1]) = [-4,3,0]$, $L([-1,2,0,1]) = [6,1,-1]$.

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$[3k_1 + 2k_2 - 4k_3 + 6k_4, k_1 - k_2 + 3k_3 + k_4, 2k_1 + k_2 - k_4]$

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Clearly, if $A_{BC} = \begin{bmatrix} 3 & 2 & -4 & 6 \\ 1 & -1 & 3 & 1 \\ 2 & 1 & 0 & -1 \end{bmatrix}$

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$$A_{BC} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 3k_1 + 2k_2 - 4k_3 + 6k_4 \\ k_1 - k_2 + 3k_3 + k_4 \\ 2k_1 + k_2 - k_4 \end{bmatrix}$$

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Hence, $A_{BC}[v]_B = [L(v)]_C$.

**Theorem:** Let $V$ and $W$ be non-trivial vector spaces, with $\mathbf{dim}(V) = n$ and $\mathbf{dim}(W) = m$. Let $B = (v_1, \ldots, v_n)$ and $C = (w_1, \ldots, w_m)$ be ordered bases for $V$ and $W$, respectively. Let $L : V \to W$ be a LT.

**Theorem:** Let $V$ and $W$ be non-trivial vector spaces, with $\mathbf{dim}(V) = n$ and $\mathbf{dim}(W) = m$. Let $B = (v_1, \ldots, v_n)$ and $C = (w_1, \ldots, w_m)$ be ordered bases for $V$ and $W$, respectively. Let $L : V \to W$ be a LT. Then there is a unique $m \times n$ matrix $A_{BC}$ such that $A_{BC}[v]_B = [L(v)]_C$, for all $v \in V$.

**Theorem:** Let $V$ and $W$ be non-trivial vector spaces, with $\mathbf{dim}(V) = n$ and $\mathbf{dim}(W) = m$. Let $B = (v_1, \ldots, v_n)$ and $C = (w_1, \ldots, w_m)$ be ordered bases for $V$ and $W$, respectively. Let $L : V \to W$ be a LT. Then there is a unique $m \times n$ matrix $A_{BC}$ such that $A_{BC}[v]_B = [L(v)]_C$, for all $v \in V$. Furthermore, for $1 \leqslant i \leqslant n$, the $i^{th}$ column of $A_{BC} = [L(v_i)]_C$.

## Example 2

**Q:.** Consider the LT $L:\mathbb{R}^2 \to P_2$, given by

$$L([a,b]) = (-a+5b)x^2 + (3a-b)x + 2b$$

with ordered bases $B = ([5,3],[3,2])$ and $C = (3x^2-2x, -2x^2+2x-1, x^2-x+1)$ of $\mathbb{R}^2$ and $P_2$ respectively.

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**Sol.** $L[5,3] = 10x^2 + 12x + 6$

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$$[C|L[B]] = \left[\begin{array}{rrr|rr} 3 & -2 & 1 & 10 & 7 \\ -2 & 2 & -1 & 12 & 7 \\ 0 & -1 & 1 & 6 & 4 \end{array}\right]$$

**Sol.** $L[5,3] = 10x^2 + 12x + 6$ and $L[3,2] = 7x^2 + 7x + 4$.

$$[C|L[B]] = \left[\begin{array}{rrr|rr} 3 & -2 & 1 & 10 & 7 \\ -2 & 2 & -1 & 12 & 7 \\ 0 & -1 & 1 & 6 & 4 \end{array}\right]$$

Using row reduction, we obtain

$$[C|L[B]] = \left[\begin{array}{rrr|rr} 1 & 0 & 0 & 22 & 14 \\ 0 & 1 & 0 & 62 & 39 \\ 0 & 0 & 1 & 68 & 43 \end{array}\right] \implies A_{BC} = \begin{bmatrix} 22 & 14 \\ 62 & 39 \\ 68 & 43 \end{bmatrix}$$

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**Q:.** Consider the LT $L : P_3 \to P_2$, given by $L(p) = p'$.

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**Sol.** Standard basis of $P_3$ is $\{x^3, x^2, x, 1\}$.

Now $L(x^3) = 3x^2$, $L(x^2) = 2x$, $L(x) = 1$, $L(1) = 0 \implies$

$$[C|L[B]] = \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right] \implies A_{BC} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

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Now $[L(4x^3 - 5x^2 + 6x - 7)]_C = A_{BC}[(4x^3 - 5x^2 + 6x - 7)]_B$

$$[C|L[B]] = \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right] \implies A_{BC} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Now $[L(4x^3 - 5x^2 + 6x - 7)]_C = A_{BC}[(4x^3 - 5x^2 + 6x - 7)]_B$

$$= A_{BC} \begin{bmatrix} 4 \\ -5 \\ 6 \\ -7 \end{bmatrix} = \begin{bmatrix} 12 \\ -10 \\ 6 \end{bmatrix}$$

$$[C|L[B]] = \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right] \implies A_{BC} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

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**Note: Here**

$[L(4x^3 - 5x^2 + 6x - 7)]_C = L(4x^3 - 5x^2 + 6x - 7) \implies$

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**Note: Here**

$[L(4x^3-5x^2+6x-7)]_C = L(4x^3-5x^2+6x-7) \implies$
$[L(4x^3-5x^2+6x-7)]_C = (4x^3-5x^2+6x-7)' =$
$12x^2-10x+6.$

# Exercises

**Q:.** Consider the LT $L : \mathbb{R}^3 \to \mathbb{R}^2$, given by $L([x,y,z]) = [x+y, y-z]$.

## Exercises

**Q:.** Consider the LT $L : \mathbb{R}^3 \to \mathbb{R}^2$, given by $L([x,y,z]) = [x+y, y-z]$. Compute $A_{BC}$ with respect to bases $B = ([1,0,1],[0,1,1],[1,1,1])$ and $C = ([1,2],[-1,1])$.

# Exercises

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**Q:.** Consider the LT $L : P_3 \to M_{22}$, given by

$$L(ax^3 + bx^2 + cx + d) = \begin{bmatrix} -3a-2c & -b+4d \\ 4b-c+3d & -6a-b+2d \end{bmatrix}.$$

Compute $A_{BC}$ with respect to standard bases for $P_3$ and $M_{22}$.

**Sol.** $\begin{bmatrix} -3 & 0 & -2 & 0 \\ 0 & -1 & 0 & 4 \\ 0 & 4 & -1 & 3 \\ -6 & -1 & 0 & 2 \end{bmatrix}$

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**Sol.** $[15,2]$.

**Q:.** Let

$$B = ([1,1,0,0],[0,1,1,0],[0,0,1,1],[0,0,0,1]) \text{ and}$$

$$C = ([1,1,1],[1,2,3],[1,0,0])$$

be ordered bases for $\mathbb{R}^4$ and $\mathbb{R}^3$, respectively. If $L : \mathbb{R}^4 \to \mathbb{R}^3$ be a LT such that $A_{BC} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$. Find $L$?

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**Sol.**

$$L([x_1, x_2, x_3, x_4]) = [-2x_1 + 3x_2 + x_4, x_2 + 2x_3, x_2 + 3x_3].$$

**Theorem:** Suppose $V$ and $W$ are nontrivial finite dimensional vector spaces with ordered bases $B$ and $C$, respectively, and let $L : V \to W$ be a LT. Then $L$ is an isomorphism if and only if the matrix representation $A_{BC}$ for $L$ with respect to $B$ and $C$ is nonsingular.

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**Remark:** Let $D_{CB}$ be the matrix for $L^{-1}$ with respect to $C$ and $B$. Also let $\mathbf{dim}(V) = \mathbf{dim}(W)$. Then $A_{BC}^{-1} = D_{CB}$ provided that $A_{BC}$ is nonsingular.

## Example 4

**Q:.** Let $L_1$ and $L_2$ be linear operators. Also, let

$$A = \begin{bmatrix} 0 & -2 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & 1 \\ 0 & -3 & 0 \end{bmatrix}$$ be matrices for $L_1$ and $L_2$ respectively, with respect to standard basis.

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Hence, $L_1$ and $L_2$ are isomorphisms.

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$$A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & -2 & 0 \end{bmatrix} \text{ and } B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1/3 \\ 2 & 1 & 0 \end{bmatrix}.$$

# Matrix for the composition of Linear Transformations

**Theorem:** Let $V_1, V_2$ and $V_3$ be nontrivial finite dimensional vector spaces with ordered bases $\boldsymbol{B}, \boldsymbol{C}$ and $\boldsymbol{D}$, respectively.

# Matrix for the composition of Linear Transformations

**Theorem:** Let $V_1, V_2$ and $V_3$ be nontrivial finite dimensional vector spaces with ordered bases $B, C$ and $D$, respectively. Let $L_1 : V_1 \to V_2$ be a linear transformation with matrix $A_{BC}$ and let $L_2 : V_2 \to V_3$ be a linear transformation with matrix $A_{CD}$. Then matrix

$$A_{BD} = A_{CD} A_{BC}$$

is the matrix of linear transformation $L_2 \circ L_1 : V_1 \to V_3$ with respect to the bases $B$ and $D$.

**Q:.** Let $L_1 : \mathbb{R}^2 \to \mathbb{R}^2$ and $L_2 : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$L_1([x,y]) = [y,x]$$

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- Find the matrix of $L_1$ and $L_2$ with respect to the standard basis in each case.

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- Find the matrix of $L_1$ and $L_2$ with respect to the standard basis in each case.
- Find the matrix of $L_2 \circ L_1$ with respect to standard basis of $\mathbb{R}^2$ and $\mathbb{R}^3$.