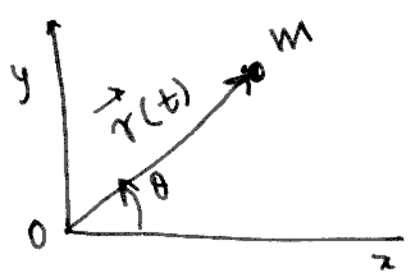
CH.6 ANGULAR MOMENTUM AND FIXED AXIS ROTATION

(Notes by: Rishikesh Vaidya).

Q. What is rotation?

→ To answer this question we must make a distinction between a point particle which is only an idealized abstraction of real bodies which have finite extension in space. With regard to rigid bodies we will as soon see that taking account of the finite size of the body leads to a difference in what constitutes rotation and what constitutes translation.

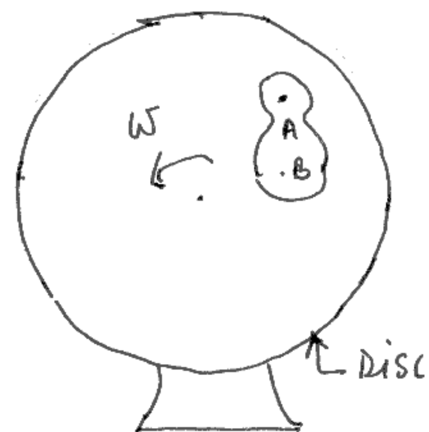
~~Point po~~ MOTION OF A POINT MASS



Consider a point mass m described by a position vector $\vec{r}(t)$. Point particle may move to a different location. The change may be pure scaling, pure rotation, or more general involving a change in magnitude as well as direction of position vector $\vec{r}(t)$. Thus pure rotation with a pure change in ~~position~~ direction of the position vector $\vec{r}$.

MOTION OF A RIGID BODY

1) Consider a rigid body hung at a frictionless pivot A, on a disc which rotates with an angular speed ω. The question is – Is the rigid body rotating or translating?



→ 2) Now consider a situation in which the rigid body is hooked to the disk by means of two frictionless nails at points A and B. Now as the disc is set spinning at angular speed ω, does the rigid body rotate, or translate, or does both? To answer this we must understand the meaning of rotation and translation for a rigid body.

RIGID BODY: An ideal rigid body is one in which its constituent atoms maintain a fixed distance throughout motion. Thus an ideal rigid body doesnot undergo deformation.

TRANSLATION: A rigid body is said to undergo translatory motion if line joining any two points inside rigid body remains parallel to itself. Thus, every rectilinear motion is translation, but all translatory motion is not necessarily rectilinear. Thus in translatory motion the displacement of all points of rigid body is identical and hence and hence all points have the same velocity and accelerations at all points in time. In case 1) when the rigid body is pivoted ~~a~~ only at A without friction it undergoes translation.

ROTATION: A rigid body is said to undergo rotation if trajectories of all the points of a rigid body are circles whose centres lie on a common straight line called axis of rotation. Thus, our rigid body in case 2) undergoes circular motion.

ROTATIONAL ANALOGUES OF PHYSICAL QUANTITIES: When we compare pure translatory motion of a rigid body with pure rotational motion, we must appreciate an important distinction. There is neither a ~~s~~ special point, nor axis, nor length scale associated with translational motion. You can refer it to any origin. ~~Whereas~~ This is not quite so with rotational motion. Whereas we are free to refer it to ~~axis~~ any

reference point, but there exist a special line called axis of rotation about which ~~the rig~~ every point of rigid body describes an arc of a circle of fixed radius. Thus there exist a special line called axis of rotation which is common for the entire rigid body, and a special length scale R - the distance from the axis of rotation ~~for~~ which is a variable for ~~a~~ every point of a rigid body.
The upshot is — for rotational motion we must expect this length scale to play an important role in defining physical quantities ~~asso~~ associated with rotation. For example,

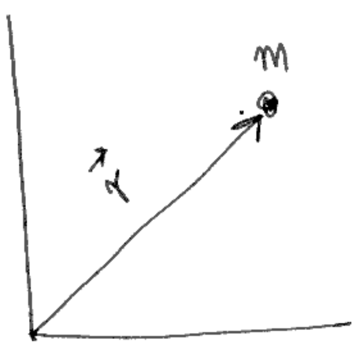
1) Displacement: $R d\theta \hat{\theta}$.
2) Velocity: $R\omega \hat{\theta}$
3) Angular momentum: $\vec{R} \times \vec{P}$ Moment of Momentum
4) Torque: $\vec{R} \times \vec{F}$ Moment of force.

ANGULAR MOMENTUM IS A FUNCTION OF THE CHOICE OF ORIGIN.

TORQUE : $\vec{\tau} = \dfrac{d\vec{L}}{dt}$.

We will try and appreciate the meaning of torque for the case of A) point particle referred to a fixed origin B) an extended body referred to a fixed origin C) an extended body referred to an accelerating origin. This will help us understand torque on an arbitrary body with respect to an arbitrary origin. Along the way, we will also appreciate that the torques due to internal forces vanish.

A) POINT MASS, FIXED ORIGIN :

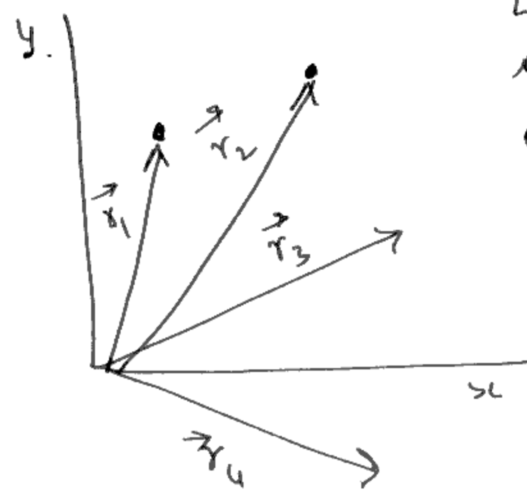


$$\vec{L} = \vec{r} \times \vec{P}$$

$$\vec{\tau} = \frac{d\vec{L}}{dt} = \underbrace{\dot{\vec{r}} \times \vec{P}}_{\to 0.} + \vec{r} \times \dot{\vec{P}}$$

$$\boxed{\vec{\tau} = \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F}}.$$

B) EXTENDED MASS, FIXED ORIGIN



Let us imagine an extended body to be a collection of N discrete particles labelled by an index i

$$\vec{L} = \sum_i \vec{r}_i \times \vec{P}_i .$$

Here $\vec{r}_i$ is the position vector of i^{th} particle and $\vec{P}_i$ its momentum. If $\vec{F}_i$ is the total force acting on it, then

$$\vec{F}_i = \vec{F}_i^{EXT} + \vec{F}_i^{INT} = \frac{d\vec{P}_i}{dt}.$$

Now,

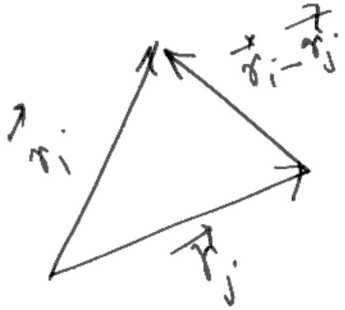
$$\vec{\tau} = \frac{d\vec{L}}{dt} = \frac{d}{dt}\sum_{i=1}^{N} \vec{r}_i \times \vec{P}_i = \sum_{i=1}^{N} \Big[\underbrace{\dot{\vec{r}}_i \times \vec{P}_i}_{\to 0.} + \vec{r}_i \times \dot{\vec{P}}_i$$

$$\vec{\tau} = \sum_{i=1}^{N} \vec{r}_i \times [\vec{F}_i^{EXT} + \vec{F}_i^{INT}].$$

$$= \tau^{EXT} + \tau^{INT}$$

We now prove that torque due to F^{INT} is zero.

Proof: Torque due to internal forces = 0.
Let $\vec{F}_{ij}^{INT}$ be the force on the i^{th} particle due to j^{th} particle, and be directed along line joining i^{th} and j^{th} particle.



$$\cancel{\vec{F}_{ij}^{INT} = \sum_j F_{ij}}$$

$$\vec{F}_i^{INT} = \sum_j \vec{F}_{ij}^{INT}$$

Total internal torque ~~due~~ ~~to~~ on all the particles relative to the chosen origin is,

$$\vec{\tau}^{INT} = \sum_{i=1}^{N} \vec{r}_i \times \vec{F}_i^{INT} = \sum_i \sum_j \vec{r}_i \times \vec{F}_{ij}^{INT} \quad (1)$$

Since indices i and j are both arbitrary and summed over we can interchange them, without affecting anything.

$$\vec{\tau}^{INT} = \sum_j \sum_i \vec{r}_j \times \vec{F}_{ji}^{INT} \quad (2)$$

Adding (1) and (2) and noting that $\vec{F}_{ij}^{INT} = -\vec{F}_{ji}$ due to Newton's third law, we have

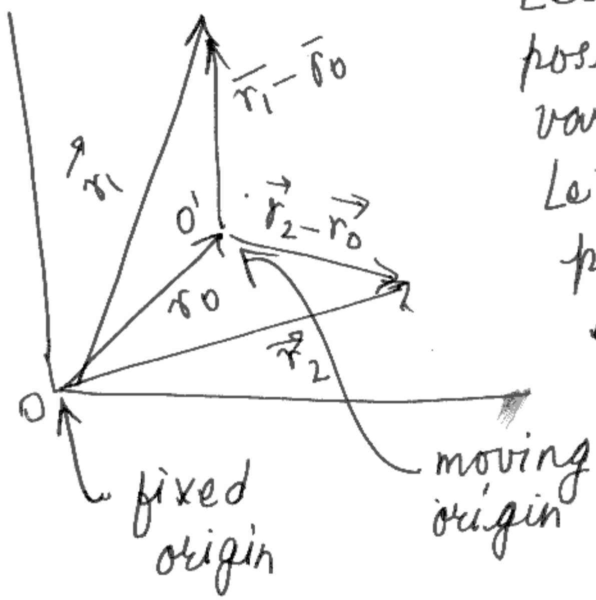
$$2\vec{\tau}^{INT} = \sum_i \sum_j (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ij}.$$

But $\vec{F}_{ij}$ is along line joining $\vec{r}_i$ and $\vec{r}_j$ and hence parallel to $\vec{r}_i - \vec{r}_j$. Thus RHS = 0 and hence total torque due to internal forces is zero. This makes perfect sense because we do not see any extended body suddenly start spinning in the absence of external torques. Thus

$$\boxed{\tau^{EXT} = \frac{d\vec{L}}{dt} = \sum_i \vec{r}_i \times \vec{F}_i^{EXT}}$$

Note that nowhere we assumed that the particles are rigidly connected to each other. Thus particles are free to move relative to one another but in that case it is hard to get a handle on $\vec{L}$ as it is no longer of $I\omega$ form.

c) EXTENDED MASS NON-FIXED (POSSIBLY ACCELERATING) ORIGIN



Let $\vec r_i$ be the position vectors of various mass points. Let $\vec r_0$ be the position vector of an accelerating origin O'. Note $\vec r_i$ and $\vec r_0$ are all measured with respect to a fixed origin O. We are interested in computing angular momentum of the extended system of ~~mass~~ N mass points relative to an accelerating origin whose position vector is $\vec r_0$ relative to fixed origin.

$$\vec L = \sum_i \underbrace{(\vec r_i - \vec r_0)}_{\substack{\text{moment arm} \\ \text{w.r.t. } O'}} \times m_i (\dot{\vec r}_i - \dot{\vec r}_0)$$

$$\vec\tau = \frac{d\vec L}{dt} = \sum_i \Big[\underbrace{(\dot{\vec r}_i - \dot{\vec r}_0) \times m_i (\dot{\vec r}_i - \dot{\vec r}_0)}_{\to 0} + (\vec r_i - \vec r_0) \times m_i (\ddot{\vec r}_i - \ddot{\vec r}_0) \Big)$$

$$\vec\tau = \sum_i (\vec r_i - \vec r_0) \times \big(\underbrace{m_i \ddot{\vec r}_i}_{F_i^{EXT} + F_i^{INT}} - m_i \ddot{\vec r}_0\big)$$

$F_i^{INT} \to 0 \; (\tau^{INT} = 0)$

$$\vec\tau = \sum_i (\vec r_i - \vec r_0) \times \vec F_i^{EXT} - \Big(\sum_i m_i \vec r_i - \sum_i m_i \ddot{\vec r}_0\Big)$$

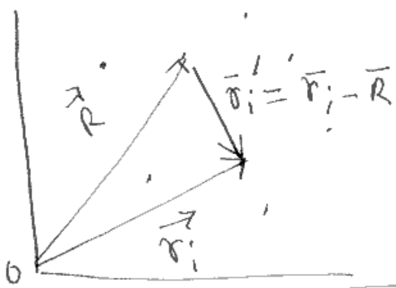
$$\vec\tau = \underbrace{\sum_i (\vec r_i - \vec r_0) \times \vec F_i^{EXT}}_{1} - \underbrace{M(\vec R_{cm} - \vec r_0) \times \ddot{\vec r}_0}_{2}$$

1 = External torque measured with respect to non-fixed origin (may be accelerating)

2 = Extra term due to non-fixed origin.
= 0 if (i) $\ddot{\vec r}_0 = 0$ or (ii) $\vec R_{CM} = \vec r_0$ or (iii) $(\vec R_{cm} - \vec r_0) \times \ddot{\vec r}_0 = 0$.

ANGULAR MOMENTUM OF A RIGID BODY THAT IS TRANSLATING AS WELL AS ROTATING: Consider a rigid body as an assembly of large number of particles each of mass m_i and have position vector $\vec{r}_i$ with respect to some fixed inertial origin.

$$\vec{L} = \sum_i \vec{r}_i \times m_i \dot{\vec{r}}_i$$



This is correct but very boring. Hardly provides any insight about the details of dynamics.

$\vec{r}_i$ = P.V. of i^{th} particle with respect O (fixed).

$\vec{R}$ = P.V of CM of rigid body

$\vec{r}_i'$ = P.V of i^{th} particle w.r.t. CM

$\vec{r}_i' = \vec{r}_i - \vec{R}$

$\vec{r}_i = \vec{r}_i' + \vec{R}$.

Note that

$\vec{r}_i = \vec{r}_i' + \vec{R}$

Such a decomposition splits the dynamics into $\vec{r}_i'$ (motion about CM) and $\vec{R}$ (motion of the CM). This looks interesting.

Upon substitution $\vec{L}$ becomes

$$L = \sum_i (\vec{r}_i' + \vec{R}) \times m_i (\dot{\vec{r}}_i' + \dot{\vec{R}}).$$

$$= \underbrace{\sum_i \vec{r}_i' \times m_i \dot{\vec{r}}_i'}_{A} + \underbrace{\sum_i \vec{r}_i' \times m_i \dot{\vec{R}}}_{B} + \underbrace{\sum_i \vec{R} \times m_i \dot{\vec{r}}_i'}_{C} + \underbrace{\sum_i \vec{R} \times m_i \dot{\vec{R}}}_{D}.$$

A $\equiv \sum_i \vec{r}_i' \times m_i \dot{\vec{r}}_i'$ This is obviously $\vec{L}$ about CM; that is L_{CM}.

B $\equiv \sum_i (\vec{r}_i') \times m_i \dot{\vec{R}} = \sum_i (\vec{r}_i - \vec{R}) m_i \times \dot{\vec{R}}$ $\boxed{M = \sum_i m_i}$

$$= \sum_i (\underbrace{m_i \vec{r}_i - M\vec{R}}_{\to\, 0 \text{ (Def}^n \text{ of CM)}}) \times \dot{\vec{R}}$$

$$= 0.$$

C $\equiv \sum_i \vec{R} \times m_i \dot{\vec{r}}_i' = 0$ (In B we proved that $\sum_i m_i \vec{r}_i' = 0$, so $\sum_i m_i \dot{\vec{r}}_i' = 0$)

D $\equiv M\vec{R} \times \dot{\vec{R}}$ $\equiv$ Ang momentum of a rigid body due to translation of CM.

Thus.

$$\vec{L} = \underbrace{\vec{L}_{CM}}_{\vec{L} \text{ in the CM frame (SPIN PART)}} + \underbrace{\vec{R} \times M\dot{\vec{R}}}_{\vec{L} \text{ due to CM motion w.r.t some fixed origin (ORBITAL PART)}}.$$

$\boxed{L^z_{CM} = (I\omega)_z \text{ for fixed axis rotation}}$

CONSERVATION OF ANGULAR MOMENTUM
CENTRAL FORCES AND KEPLER'S LAW

$\vec F = \frac{d\vec P}{dt} \Rightarrow$ if $\vec F = 0$ $\vec P$ = const.

$\vec \tau = \frac{d\vec L}{dt}$; $\vec\tau = 0 \Rightarrow \vec L$ is conserved.

$\vec\tau = \vec r \times \vec F \Rightarrow \vec F$ need not be zero for $\vec\tau = 0$.

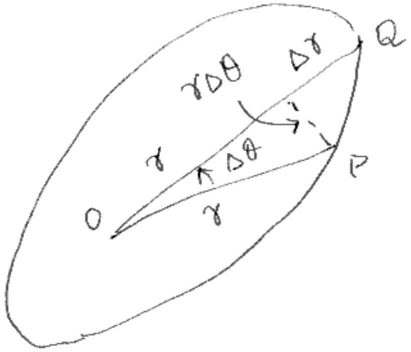
For central (radial) forces : $\vec F = f(r)\hat r$.

$\vec\tau = \vec r \times f(r)\hat r = 0 \Rightarrow \vec L$ is conserved.

If we take direction of $\vec L = |\vec L|\hat z$, conservation means it will always be $\hat z$.

Now $\vec L = \vec r \times \vec P \Rightarrow$ the motion is always in x-y plane.

MOTION OF PLANETS: Since gravity is a central force, the motion of planets is confined to the plane. Let us find Areal velocity of a planet going from P to Q.



Area $\Delta OPQ = \frac{1}{2}(r\Delta\theta)(r+\Delta r)$.

$\Delta A = \frac{1}{2}r^2\Delta\theta + \underbrace{\frac{1}{2}r\Delta\theta\Delta r}$ 2nd order in differential hence $\underset{\Delta t\to 0}{\to 0}$.

$$\frac{dA}{dt} = \lim_{\Delta t\to 0}\frac{\Delta A}{\Delta t} = \frac{1}{2}r^2\omega$$

$$\boxed{\frac{dA}{dt} = \frac{1}{2}r^2\omega}$$

$$\vec L = \vec r \times m\dot{\vec r} = r\hat r \times m(r\dot\theta\hat\theta + \dot r\hat r) = mr^2\omega\hat z$$

$\Rightarrow \boxed{\frac{d A}{dt} = \frac{L}{2m}} \Rightarrow$ CONSERVATION OF L AND AREAL VELOCITY ARE CONNECTED.

Thus, Kepler's second law of constancy of Areal velocity ~~is onl~~ holds true very generally for all central forces, because conservation of angular momentum is a generic feature of central forces as seen below:

Central force $\Rightarrow F_\theta = m a_\theta = 0$

$\vec F = f(r)\hat r$

$L = $ Const

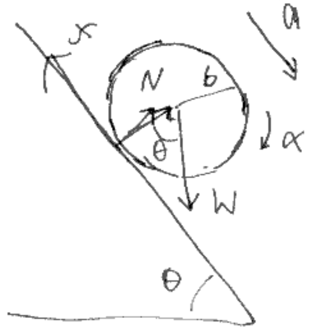
$\frac{dA}{dt} = 0 \Rightarrow A =$ const.

$$a_\theta = (r\ddot\theta + 2\dot r\dot\theta) = 0$$

$$\Rightarrow m(r^2\ddot\theta + 2\dot r r\dot\theta) = 0$$

$$\frac{d}{dt}(\underbrace{mr^2\dot\theta}_{L}) = 0$$

EXAMPLE 6.16: DRUM ROLLING DOWN PLANE



A uniform drum of radius b and mass M rolls w/o slipping down a plane inclined at an angle θ. Find the a. ($I_0 = Mb^2/2$)

sol: We will solve this problem by taking torque about three different points.

METHOD-1: $W\sin\theta - f = ma$ Translation of CM.

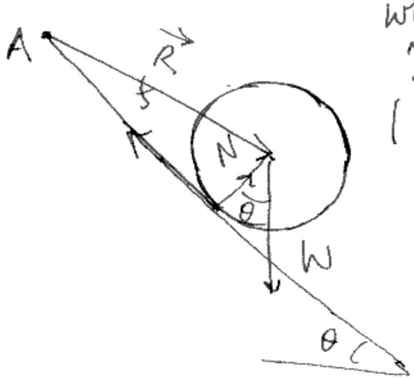
$bf = I_0\alpha$ **Torque about CM**.

$a = b\alpha$ rolling w/o slipping.

Eliminating f and using $I_0 = Mb^2/2$

$$\boxed{a = \frac{2}{3} g\sin\theta}$$

METHOD-2: Let us choose a coordinate system whose origin is A, on the plane.



Torque about A is.

$$(\vec{\tau})_{A_z} = \tau_0 + (\vec{R}\times\vec{F})_z .$$

Like L, τ also splits into two parts. Here $\vec{R}$ is position vector of CM from A. $\vec{F}$ = Net external force.

$$(\tau_A)_z = \tau_0 + (\vec{R}_\perp + \vec{R}_\parallel)\times(\vec{N} + \vec{W} + \vec{f}).$$

$$= -bf + \vec{R}_\perp\times\vec{N} + \vec{R}_\perp\times\vec{W} + \vec{R}_\perp\times\vec{f} + \vec{R}_\parallel\times\vec{N} + \vec{R}_\parallel\times\vec{W} + \vec{R}_\parallel\times\vec{f}.$$

$$= \cancel{-bf} + 0 - bW\sin\theta + \cancel{bf} + \cancel{R_\parallel N - R_\parallel W\cos\theta} + 0$$

$$(\tau_A)_z = -bW\sin\theta$$

$$(L_A)_z = L_{CM} + (\vec{R}\times M\dot{\vec{R}})_z$$

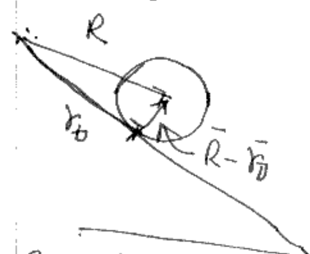
$$= -\frac{1}{2}Mb^2\omega - Mb^2\omega$$

$$= -\frac{3}{2}Mb^2\omega$$

Since $\tau_z = dL_z/dt \Rightarrow bW\sin\theta = \frac{3}{2}Mb^2\dot{\omega}$

$$\Rightarrow \dot{\omega} = \alpha = \frac{2W}{3Mb}\sin\theta \quad \text{or} \quad \boxed{a = b\alpha = \frac{2}{3}g\sin\theta}$$

METHOD 3: Origin at the point of contact. ~~The~~ Since point of contact is accelerating we must use the general formula for torque.



$$\vec{\tau} = \underbrace{\sum_i (\vec{r}_i - \vec{r}_0)\times\vec{F}^i_{ext}}_{1} - \underbrace{M(\vec{R} - \vec{r}_0)\times\ddot{\vec{r}}_0}_{2}$$

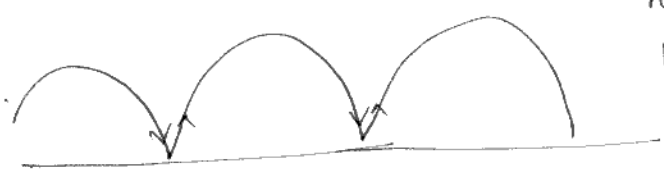
Here the 2 term vanishes because cross product vanishes. Velocity of point of contact is downwards just before it touches plane and upwards just after that. Hence $\ddot{\vec{r}}_0$ is facing down normal to incline.

1 $= -bW\sin\theta$.

So $(\bar{R}-\bar{r}_D)\times\ddot{r}_D = 0$

Pointing up normal to incline — Pointing down normal to incline

Here of course ~~o~~ the position vector of origin (point of contact) is $\bar{r}_0$. The fact that the accelaration of point of contact ~~is pointing~~ ($\ddot{r}_0$) is pointing down can be understood from the fact that trajectory of any point ~~of a~~ on a rolling circle is a cycloid.



Just when the point hits the ground its velocity is pointing dowards and immediately after it, upwards. Thus, only first term contributes

$$\tau = -bW\sin\theta = \left(\frac{Mb^2}{2}+Mb^2\right)\alpha = \frac{3}{2}Mb^2\alpha$$

$$\Rightarrow \boxed{a = \frac{2}{3}g\sin\theta} \quad \text{since } a = b\alpha.$$

The important point to realize here that in general the second term exist. You must not neglect it without knowing why it does not contribute.

METHOD-4: We will now employ energy method and find the speed of rolling drum as it descends through height h. The drum starts at rest.

Translational Work-energy theorem.

$$\int_a^b \vec{F}\cdot d\vec{r} = \frac{1}{2}MV_b^2 - \frac{1}{2}MV_a^2 = \frac{1}{2}MV^2$$

$$(W\sin\theta - f)l = \frac{1}{2}MV^2 \quad (1) \qquad \boxed{l = h/\sin\theta}$$

For the rotational motion

$$\int_{\theta_a}^{\theta_b} \tau_0\, d\theta = \frac{1}{2}I_0\omega_b^2 - \frac{1}{2}I_0\omega_a^2$$

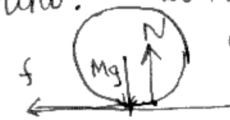
$$fb\theta = \frac{1}{2}I_0\omega^2$$ where θ is the angle through which drum rotates as it translates through l. $l = b\theta$.

$$fl = \frac{1}{2}I_0\omega^2$$

$$fl = \frac{1}{2}\frac{I_0V^2}{b^2} \quad (2)$$

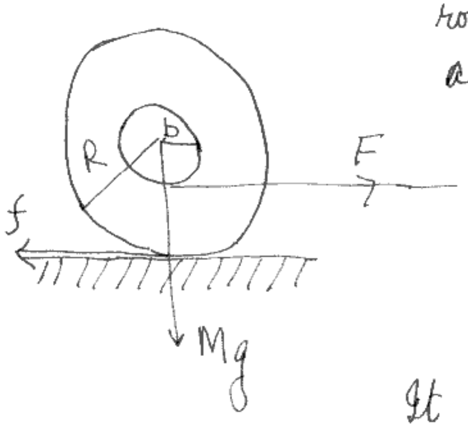
Eliminating f from (1) & (2) we get $V = \sqrt{\frac{4gh}{3}}$

Interesting thing to note here is that force of friction here is non-dissipative. It decreases translational energy by an amount fl but the torque exerted by friction increases rotational energy by same amount. It is only when a rolling wheel flattens at the bottom that the torque due to N (which doesn't pass from center) decelerates.



6.27 A yo-yo of mass M has an axle of radius b and a spool of radius R. The M.I = $MR^2/2$. Yo-Yo is placed upright on a table and the string is pulled with the horizontal force F. The coefficient of friction between Yo-Yo and table is μ. What is maximum value of F for which Yo-Yo will roll without slipping.

Sol: Since the Yo-Yo is supposed to roll without slipping, there is a net translational motion as well as rotational motion such that



$$\boxed{\begin{array}{l} l = R\theta \\ \text{or } a = R\alpha \end{array}}$$ rolling w/o slipping.

It is clear that Yo-Yo will translate to the right as $F > f$ (friction) for translation. Thus

$$F - f = Ma \quad \text{Here } a > 0. \quad ①$$

There are two torques bF ~~as~~ (tending to rotate the Yo-Yo counter clockwise and hence +ve) and fR (tending to rotate the Yo-Yo in clockwise direction and hence -ve). According to translational equation of motion, the Yo-Yo moves to the right. The requirement that it should roll w/o slipping means that the torque which makes it rotate to the right in the clockwise direction ~~that~~ (fR) shall dictate the sign of angular accelaration α. Thus

$$bF - fR = -\frac{MR^2}{2}\alpha = -\frac{MR^2}{2}\left(\frac{a}{R}\right) \quad ②$$

Solving ① and ② we get

$$\boxed{F = \frac{3fR}{2b+R} = \frac{3\mu MgR}{2b+R}}$$

CH. 8. NON INERTIAL FRAMES

→ We have so far formulated Newton's 2^{nd} law in inertial reference frames. All our coordinate measurements referred to a system of axis which were neither linearly accelarating nor rotating.

→ An inertial reference frame is one in which Newton's first law of inertia holds.

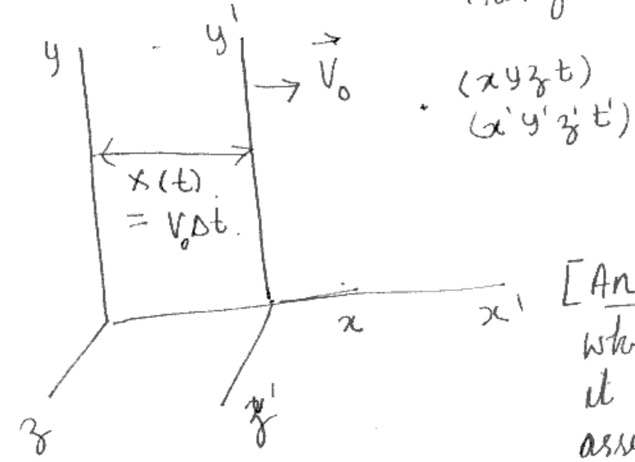
→ We have treated Earth as an inertial reference frame which is technically incorrect. Earth spins about its axis, revolves around sun as well as galactic center. All of these imply centripetal accelaration, however the corrections to g due to its revolution around sun ($a = v^2/r = .006\,m/s^2$) and its own spin ($.03\,ms^{-2}$ or $g/300$ at equator) are very small. However, depending upon the problem at hand this could be important.

→ The purpose of this chapter is to reformulate Newton's 2^{nd} law so that we can address problems of practical interest in non-inertial frames. In fact some problems become much simpler and its physics becomes more transparent when formulated in N.I.F.

GALILEAN TRANSFORMATIONS

→ These relate physical quantities in two different I.F., one of which is moving uniformly w.r.t the other frame.

Let S and S' be two I.F. with S' moving with constant velocity $\vec{v}$ along x' axis. ~~whose~~ At $t=0$, their origins as well as axis coincide. Let (x,y,z,t) and (x',y',z',t') be the coordinate and times of an event as measured in S and S'. Then, Galilean transformations are:



$$x' = x - V_0 t$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

[An Aside: Note that when we put $t'=t$, it is a tacit assumption whose validation is subject to empirical verification. Whereas above transformations had ~~emp~~ empirical support at ~~t~~ terrestrial speeds and accuracy of measurements available then. It turns out that the correct relation valid for all possible speeds is given as

$$x' = \gamma(x - vt) \; ; \; y' = y \; ; \; z' = z$$
$$t' = \gamma\left(t - \frac{vx}{c^2}\right). \quad \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

These are known as Lorentz transformations and were not discovered in trying to empirically verify Galilean transformation but in trying to fix theoretical inconsisties in the theory of light. It is a consequence of the fact that ~~spe~~ velocity of light in vacuum is a universal constant independent of the velocity of its source. this is at the heart of Einsteins special relativity].

~~Since~~ Writing in a vector form

$$\vec{r}' = \vec{r} - \vec{V}_0 t$$
$$\vec{v}' = \vec{v} - \vec{V}_0$$
$$\vec{a}' = \vec{a}$$
$$m\vec{a}' = m\vec{a}$$
$$m\vec{a}' = \vec{F}' = \vec{F} = m\vec{a}$$

This is the proof that you are free to choose any inertial frame to formulate your 2nd law.

What if S' is accelerating w.r.t. S? Then, say $\vec{A}_0$ is acceleration of S'.

$$\vec{a}' = \vec{a} - \vec{A}_0$$
$$m\vec{a}' = m\vec{a} - m\vec{A}_0$$
$$\vec{F}' = \vec{F} - m\vec{A}_0$$

Thus in the inertial frame S, the equation of motion of course takes the cannonical form

$$\vec{F} = m\vec{a}$$

where $\vec{F}$ = vector sum of all physical forces such as $m\vec{g}, \vec{N}, \vec{f}, \vec{T}$ etc.

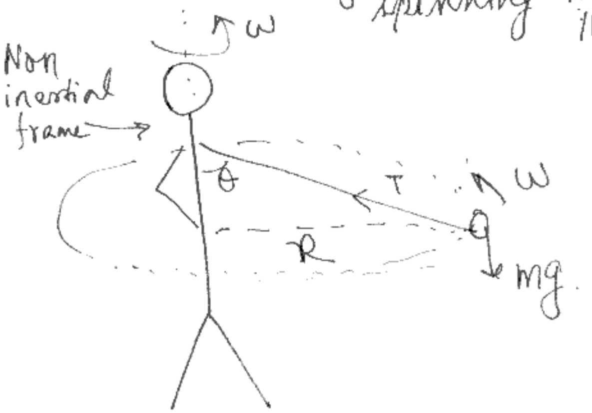
But in the accelarating and hence N.I.F.

$$\vec{F}' = m\vec{a}'$$
$$\vec{F} - m\vec{A}_0 = m\vec{a}'$$

Thus, we see that in LHS, over and above ~~sum~~ vector sum of all physical forces, there is an additional term $-m\vec{A}_0$ which does not have a physical origin but is purely a consequence of formulating the problem in a non-inertial frame. It vanishes in the limit $A_0 \to 0$.

Since it does not have a physical origin and is an artifact of accelarating frame, it is aptly called a pseudo force or fictitious force. Proportionality to mass and negative sign (points opposite to $\vec{A}_0$ the accelaration of the frame) are its tell-tale signatures. One very common example of a pseudo-force is centrifugal force (and NOT centripetal).

Example: Mass tied to an almost massless string and whirled in a circle by a spinning man.



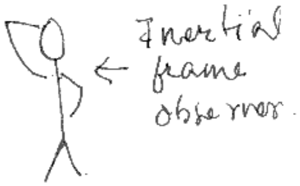
INERTIAL FRAME ~~ASPEC~~ APPROACH:

Here of course mass is going in a circle of radius R and hence needs an agent that provides for necessary centripetal force (radial component of T)

$y|$ $\quad T\cos\theta = mg$

$\hat{r}|$ $\quad -T\sin\theta = -\frac{mv^2}{R}$

$$\boxed{\tan\theta = \frac{v^2}{Rg}}$$



NON-INERTIAL FRAME APPROACH

$$* \quad \vec{F}' = m\vec{a}'$$

$$\vec{F} - m\vec{A}_0 = m\vec{a}'$$

Here $\vec{F}$ = vector sum of all physical forces

$\vec{A}_0$ = Acc. of N.I.F = $\frac{v^2}{R}(-\hat{r})$.

$\vec{a}'$ = Acc. observed in NIF.

Since NIF of spinning person has same ω as that of mass m, $\vec{a}' = 0$.

Then $\quad T\cos\theta - mg = 0$

$$\underset{\uparrow\ \text{Physical force}}{-T\sin\theta\,\hat{r}} \underbrace{- m\frac{v^2}{R}(-\hat{r})}_{\text{Centrifugal (Pseudo) force}} = 0$$

Again $\boxed{\tan\theta = \frac{v^2}{Rg}}$ Same as before

Not a surprize because if done correctly physics is truly independent of the choice of reference frame.

Centripetal force: It is a REAL force with a physical origin that is has to be provided by a physical agent (f, N, T, $m\vec{g}$, etc) to account for the observed circular motion.

Centrifugal force: It is a fictitious or pseudo force that is invoked to ~~cancel to some physical force~~ account for the observed accelaration (or lack thereof) ~~of~~ in the non-inertial frame. IT HAS NO PLACE IN ANY ANALYSIS DONE PURELY IN INERTIAL FRAME.

2.29 A car is driven on a large rotating platform which rotates with constant angular speed ω. At $t=0$, a driver leaves the origin and follows a radial line with constant speed v_0. The total weight of the car is Mg, and the coefficient of friction between car and platform is μ. a) Find the acceleration of the car as a function of time using polar coordinates. Draw a clear vector diagram showing the components of acceleration at some time $t>0$ b) Find the time t at which the car just starts to skid c) Solve the problem using rotating non-inertial frame of platform.

SOLUTION: Using Inertial reference frame

→ Car rotates with platform with ω and hence needs an agency to provide centripetal acc^n. This is radial component of frictional force f_r.

→ Since car is going with constant speed in radially outward direction $\dot{r} = V = \text{const}$. $\ddot{r} = 0$.

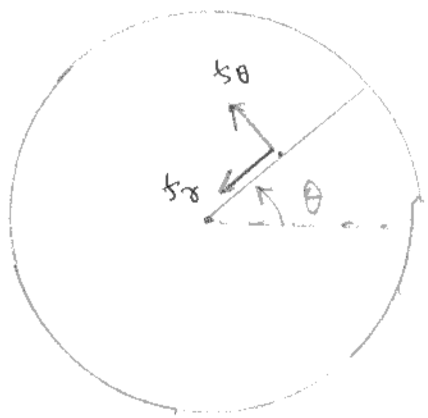
→ Since $\dot{r} \neq 0$, $\omega \neq 0$, $\therefore \dot{r}\omega \neq 0$. Thus there exist a non-zero force in tangential direction. There is nothing other than friction to provide for such a force.

Thus

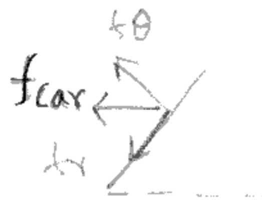
$$-f_r \hat{r} = m(\underbrace{\ddot{r}}_{\to 0} - r\dot{\theta}^2)\hat{r} \Rightarrow f_r = mr\omega^2.$$

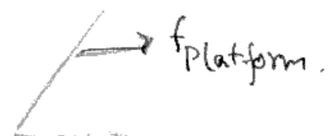
$$f_\theta \hat{\theta} = m(r\underbrace{\ddot{\theta}}_{\to 0} + 2\dot{r}\dot{\theta})\hat{\theta} \qquad \dot{r} = V$$

$$\Rightarrow f_\theta = 2mV\omega.$$



Note that since V is constant, f_θ is always constant, but $f_r = mr\omega^2$ and hence increases linearly with time ($r = Vt$) and distance from the center. The net force on the car (exerted by the platform due to friction) is given in magnitude and direction by the vector sum of f_r and f_θ. Note that the force on the car is reaction force due to the force on the platform exerted by the car, which is equal in magnitude and opposite in direction.





$$\vec{a} = -a_r\hat{r} + a_\theta\hat{\theta}$$

$$|\vec{a}| = (a_r^2 + a_\theta^2)^{1/2}$$

$$a(t) = \left[(Vt\omega^2)^2 + (2V\omega)^2\right]^{1/2}$$

car will ~~no~~ not skid until the ~~maximum~~ force f with which it pushes the platform equals the maximum frictional force $f_{max} = \mu Mg = Ma(t)$. Thus

$$(\mu Mg)^2 = M^2 a^2(t)$$

$$\mu^2 g^2 = V^2 t^2 \omega^4 + 4V^2\omega^2$$

$$t = \left[\frac{\mu^2 g^2 - 4V^2\omega^2}{V^2\omega^4}\right]^{1/2}$$

Thus if $4V^2\omega^2 > \mu^2 g^2$ the car will ~~have~~ always skid.

SOLUTION IN NON-INERTIAL FRAM

$$\vec{F}_{rot} = \vec{F}_{IN} - m\vec{\omega}\times(\vec{\omega}\times\vec{r}) - 2m\vec{\omega}\times\vec{V}_{rot}.$$

Some observations:

1) In the rotating frame, the car does not rotate. Since it is going with constant speed in radially outward direction $\dot{r} = V_{rot} = V$ and $\ddot{r} = 0$. This there is <u>no radial</u> acceleration in rotating frame ($a_r^{rot} = 0$).

2) Since ω is zero in rotating frame, there is no tangential acceleration in rotating frame ($a_\theta^{rot} = 0$).

Thus LHS of the above equation is:

$$\vec{F}_{rot} = m(\underbrace{a_r^{rot}}_{\to 0}\hat{r} + \underbrace{a_\theta^{rot}}_{\to 0}\hat{\theta}) = 0$$

3) Thus, the three terms on RHS must also "conspire" to give zero. Let us look at them individually and then add components

$$\vec{F}_{IN} = \vec{F}_r^{IN} + \vec{F}_\theta^{IN} = m(\vec{a}_r^{IN} + \vec{a}_\theta^{IN}) \quad (a)$$

$$= f_r\hat{r} + f_\theta\hat{\theta}$$

$$m\vec{\omega}\times(\vec{\omega}\times\vec{r}) = mr\omega^2\hat{r} \quad (b)$$

$$2m\vec{\omega}\times\vec{V}_{rot} = 2m\omega V\hat{\theta}. \quad (c)$$

Thus, adding (a) (b) and (c) and equating LHS = RHS

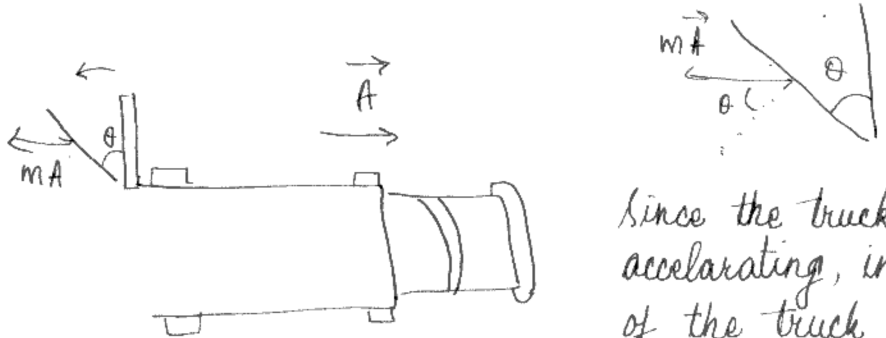
$$0\hat{r} + 0\hat{\theta} = f_r\hat{r} + f_\theta\hat{\theta} - mr\omega^2\hat{r} - 2m\omega V\hat{\theta}.$$

$$0\hat{r} + 0\hat{\theta} = (f_r - mr\omega^2)\hat{r} + (f_\theta - 2m\omega V)\hat{\theta}.$$

$$\Rightarrow f_r = mr\omega^2 \;;\; f_\theta = 2m\omega V.$$

This is precisely the result we obtained in the inertial frame.

8.2



Since the truck is accelarating, in the frame of the truck there is a torque due to pseudo force whose normal component $mA\cos\theta$ brings about a change in the angular momentum of the door. Equations of motion are:

$$mA\cos\theta \frac{l}{2} = I\omega .$$

The door starts at rest and hence initial $\omega = 0$.

$$-N + mA\sin\theta = -m\frac{l}{2}\omega^2 .$$

Work-energy theorem: $\int \tau\, d\theta = \frac{1}{2} I \omega^2$
(pure rotation about pivot)

$$\int_0^{90} mA\cos\theta \frac{l}{2}\, d\theta = \frac{1}{2} I\omega^2 .$$

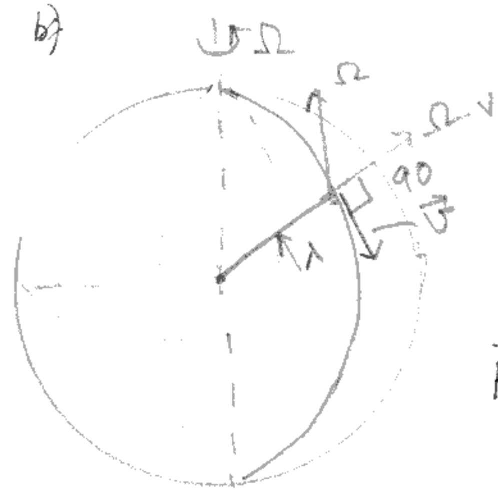
$$\omega^2 = \frac{mAl}{I} .$$

Horizontal component of force when it has swung through 90° : $F_H = m\frac{l}{2}\omega^2 = \frac{m^2 l^2 A}{2I}$.

8.9

A 400 ton train runs south at a speed 60 miles/hr at a latitude of 60° north.
a) What is the horizontal component of force on the track b) What is the direction of force?

b)



Sol: When you are asked horizontal comp what is meant is the horizontal component of Coriolis force.

$$\vec{F}_{COR} = -2m\vec{\Omega}\times\vec{v}$$

$$= -2m(\vec{\Omega}_V + \vec{\Omega}_H)\times\vec{v}$$

$$= -2m(\vec{\Omega}_V\times\vec{v} + \vec{\Omega}_H\times\vec{v})$$

Now whatever moves on the surface of the earth has its velocity on the plane of the earth. $\vec{\Omega}$ of the earth can be resolved into $\vec{\Omega}_V$ which points in radially outward direction and $\vec{\Omega}_H$ which is on the "plane" of earth. Thus $\vec{\Omega}_H\times\vec{v}$ points in vertical (radial) direction and hence we are no not interested in it. $\vec{\Omega}_V\times\vec{v}$ is what leads to horizontal component of Coriolis force.

Thus, $F_H^{COR} = -2m(\vec{\Omega}_V \times \vec{V})$

$|F_H^{COR}| = +2m\Omega\sin\lambda\, V$

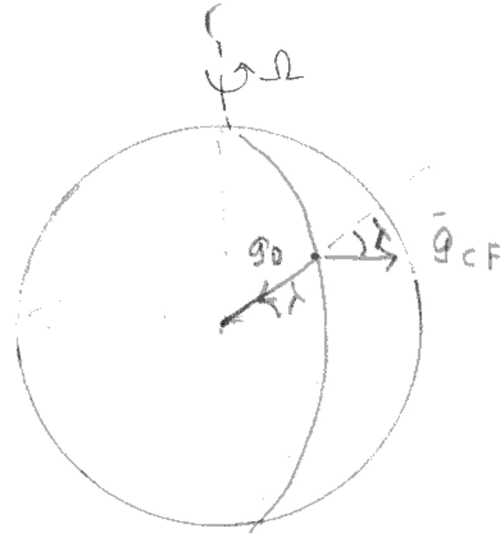
$= \dfrac{2m\Omega V\sqrt{3}}{\cancel{\sqrt{3}}\cdot 2}$ $\quad M = 4\times10^5$ kg

$\Omega = \dfrac{2\pi}{24\times60\times60}$ rad/s

$v = 60$ miles/hr.

The force on the train is in westward direction and hence the force on the track is ~~eastward~~ also westward.

8.10 The acceleration due to gravity measured in a earth bound system is denoted by $\vec{g}$. However due to earth's rotation, g differs from the true acceleration due to gravity g_0. Assuming that the earth is perfectly round, with radius R_e and angular velocity Ω_0, find g as a function of latitude λ. (Assuming that earth is perfectly round is not justified here – the contribution due to polar flattening is comparable to the effect calculated here).



Here,
$\vec{g}_0 = |\vec{g}_0|(-\hat{r})$
$\vec{g}_0$ is the value if earth was not rotating.
$\vec{g}_{CF} = \vec{F}_{CF}/m$
that is correction to $\vec{g}_0$ due to earth's rotation.

Thus $\vec{g}$ as measured on ~~a~~ rotating earth:

$\vec{g} = \vec{g}_0 + \vec{g}_{CF}$ $\qquad |\vec{g}_{CF}| = \Omega^2 R_e \cos\lambda$

Comparing with $(\vec{a}_{rot} = \vec{a}_{in} - \vec{\Omega}\times(\vec{\Omega}\times\vec{r}))$ if you are puzzled with a + sign in front of $\vec{g}_{CF}$ then you must note that we have already taken care of direction of $\vec{g}_{CF}$ as it is pointing axially outwards. Thus,

$$|\vec{g}| = [\vec{g}_0\cdot\vec{g}_0 + \vec{g}_{CF}\cdot\vec{g}_{CF} + 2\vec{g}_0\cdot\vec{g}_{CF}]^{1/2}$$

$$= g_0^2\left[1 + \frac{(\Omega^2 R_e\cos\lambda)^2}{g_0^2} \cancel{+} - \frac{2\Omega^2 R_e\cos^2\lambda}{g_0}\right]^{1/2}$$

$$= g_0^2\left[1 + x^2\cos^2\lambda - 2x\cos^2\lambda\right]^{1/2} \qquad x = \frac{\Omega^2 R_e}{g_0}.$$

Example 6.11 Analysis of Coriolis force on to calculate the motion of Foucalt pendulum demonstrates rotation of earth beyond doubt.

Consider a pendulum of mass m and frequency $\beta = \sqrt{g/l}$. If we describe the motion of pendulum's bob in a horizontal plane by coordinates r, θ then

$$r = r_0 \sin \beta t$$

r_0 = amplitude of oscillation

In the absence of Coriolis force there are no tangential forces and θ is constant. Since the bob is moving in a rotating frame, $\vec{F}_{COR} \neq 0$.

$$\vec{F}_{COR} = -2m\Omega \sin\lambda \, \dot{r} \, \hat{\theta}.$$

Hence tangential eqn of motion is $F^{\theta}_{COR} = m a_\theta$

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = -2m\Omega \sin\lambda \, \dot{r}$$

The simplest solution is found by taking $\ddot{\theta} = 0$

$$\dot{\theta} = \text{const.} \Rightarrow \dot{\theta} = -\Omega \sin\lambda.$$

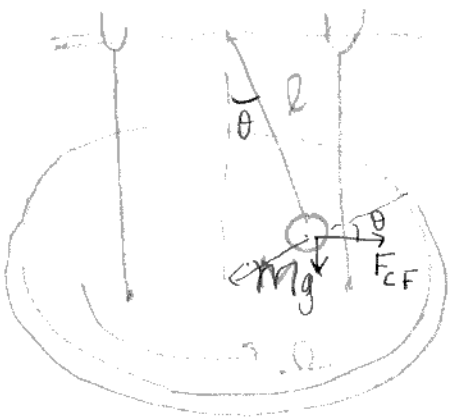
The pendulum precesses uniformly in clockwise due. The time for the plane of oscillation to rotate once is

$$T = \frac{2\pi}{\dot{\theta}} = \frac{2\pi}{\Omega \sin\lambda} = \frac{24h}{\sin\lambda}.$$

at $\lambda = 45^\circ$ $T = 34h$.

The rotation of the plane of oscillation of the pendulum demonstrates the rotation of earth. From an inertial frame one can actually see that plane of oscillation of pendulum remains fixed, but it is the earth beneath which is rotating.

[8.12] A pendulum is rigidly fixed to an axle by two ~~of~~ supports so that it can swing only in a plane perpendicular to the axle. The pendulum consists of a mass m attached to a rod of length l. The supports are mounted on a platform which rotates with constant angular velocity Ω. Find the pendulum's frequency assuming that the amplitude is small.



Sol: Note that Coriolis force would tend to move the pendulum out of the plane of oscillation and in clockwise direction. However it cannot succeed because the pendulum is rigidly supported at the pivot. Thus the pendulum is only subject to gravity and centrifugal force due to rotation of platform. Thus equation of motion is:

$$I\ddot\theta = \vec{F}_g + \vec{F}_{CF}.$$

$$\vec{F}_g = -mg\sin\theta\,\hat\theta \qquad |\vec{F}_{CF}| = m\Omega^2 l\sin\theta$$

direction is shown in fig.

EOM is (small θ) $\sin\theta \approx \theta$ $\cos\theta \approx 1$.

$$I\ddot\theta = -mgl\theta + m\Omega^2 l^2\theta\underbrace{\cos\theta}_{\approx 1}$$

$$\cancel{m}l^{\cancel{2}}\ddot\theta = -(\cancel{m}g\cancel{l} - \cancel{m}\Omega^2 l\cancel{l})\theta$$

$$\boxed{\text{Frequency} = \left(\frac{g}{l} - \Omega^2\right)^{1/2}}$$