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BITS Pilani
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MATH F111 (Mathematics-I)

Lecture 17-20 (Chapter-14) Partial Derivative

BITS Pilani
Pilani Campus

Dr Trilok Mathur,
Assistant Professor,
Department of Mathematics

Topics to be Covered in Chapter-14

- Definition of functions, domain and range.
- Level curves and level surfaces.
- Types of regions in plane and space.
- Behaviour near a point : limits and continuity.
- Partial derivatives,

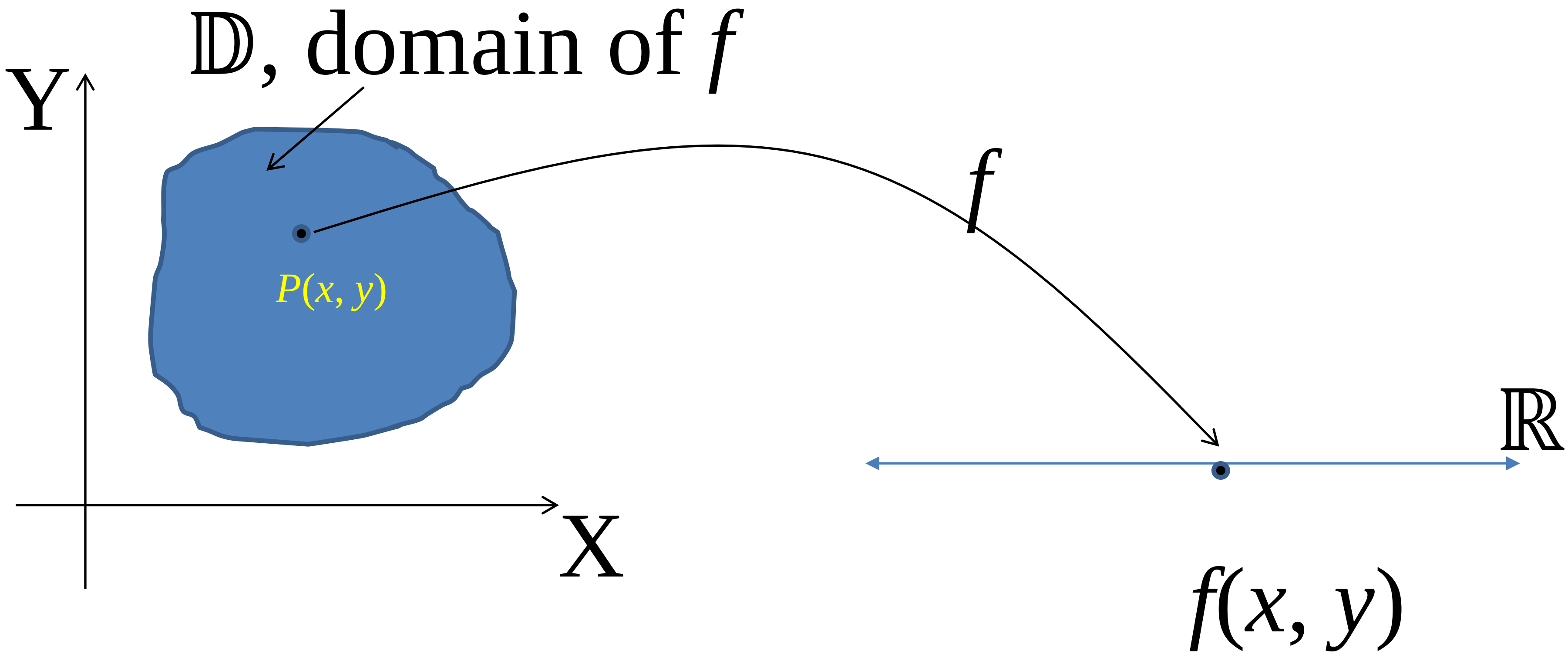
Topics to be Covered in Chapter-14

- Linearization
- Composite functions and Chain rule
- Directional derivatives
- Maxima and Minima : Relative, Absolute and with constraints
- Lagrange multipliers

Real Valued Function of one variable

Function	Domain	Range
$f(x) = 1/x$	$\mathbb{R}\sim\{0\}$	$\mathbb{R}\sim\{0\}$
$\sqrt{1-x^2}$	$[-1,1]$	$[0,1]$
$\sqrt{2-x}$	$(-\infty,2]$	$[0,\infty)$

Function of two variable



The range is the set of values that f takes on, that is $\{f(x, y) | (x, y) \in \mathbb{D}\}$

Function of two variable

We often write $z = f(x, y)$ to make explicit the value taken on by f at the general point (x, y), the variables x and y are independent variables and z is the dependent variable in $z = f(x, y)$

* Compare with $y = f(x)$

Domain of Function of two variable

Domain $\mathbb{D}$ of f:

Explicitly Mentioned In definition

Wherever defining Rule makes sense

$$\mathrm{Dom}(f) = \mathbb{D} = \{(x, y) \in \mathbb{R}^2 : f(x, y) \in \mathbb{R}\}$$

$$\mathrm{Range}(f) = \{f(x, y) \in \mathbb{R} : (x, y) \in \mathbb{D}\}$$

Domain of Function of two variable

Ex.1. $f(x, y) = 3x + y,$

$$\text{Domain} = \mathbb{R}^2$$

$$\text{Range} = \mathbb{R}$$

Domain of Function of two variable

Ex.2. $f(x,y)=\sqrt{y-x^2}$,

$$\text{Domain}=\{(x,y):y\geq x^2\},$$

$$\text{Range}=[0,\infty).$$

Domain of Function of two variable

Ex.3. Let $f(x,y)=\sqrt{x+y}, \quad if\ x>0, y>0.$

Here f has been defined for positive values of x and y, i.e. domain of f is explicitly defined,

$\text{Dom}(f)=\{(x,y): x>0, y>0\}$

$\text{Range}(f)= \mathbb{R}^{+} = (0,\infty)$

Domain of Function of two variable

Ex.4. $f(x, y) = 0$ if $x > 0, y \neq 0$

$= 1$ if $x > 0, y = 0.$

The domain of $f = \{(x, y) : x > 0\}$,

The range of $f = \{0, 1\}$.

Domain of Function of two variable

Remark : In previous Example 4, the function is defined by different rules on different parts of the domain of f. The parts should not overlap or if they overlap then rules must coincide on the overlap.

Domain of Function of two variable

Ex.5. If $f(x, y) = (x + y) / (x + 1)$, find domain and range f

Sol. : Here function is defined through a rule and domain and range is to be determined.

The domain of f is the set of points where $f(x, y)$ is well defined real number i.e.

$\text{Dom}(f) = \{(x, y) : x + 1 \neq 0\}$

Domain of Function of two variable

The range of $f = \mathbb{R}$

(To find Range of f is to find all the values of c for which the equation $f(x, y) = c$ has a solution in the domain of f)

Regions

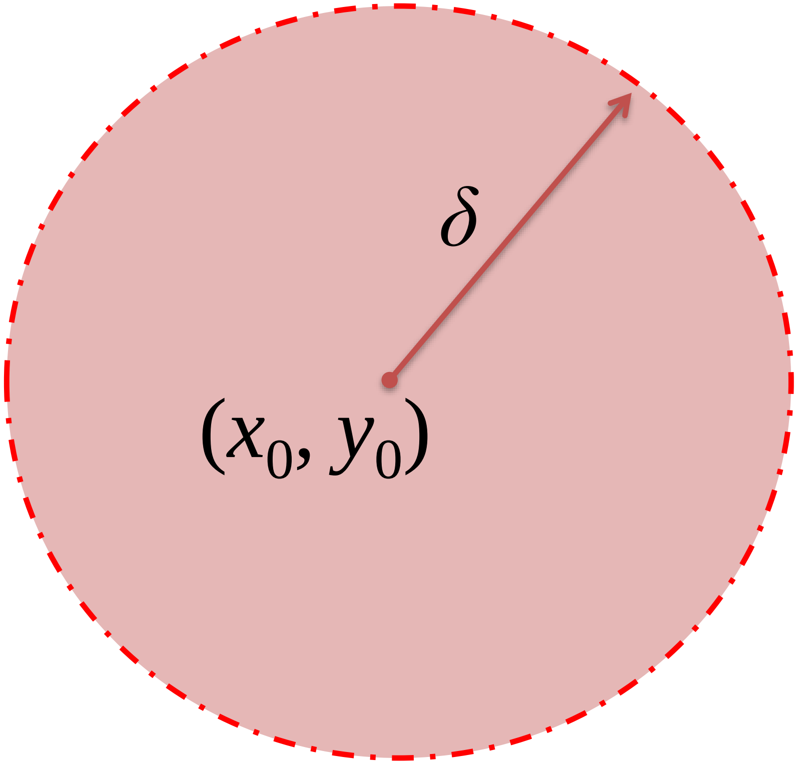
The open disk of radius $\delta > 0$ centred at (x_0, y_0) is:

$$\{(x,y) : \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta\}$$

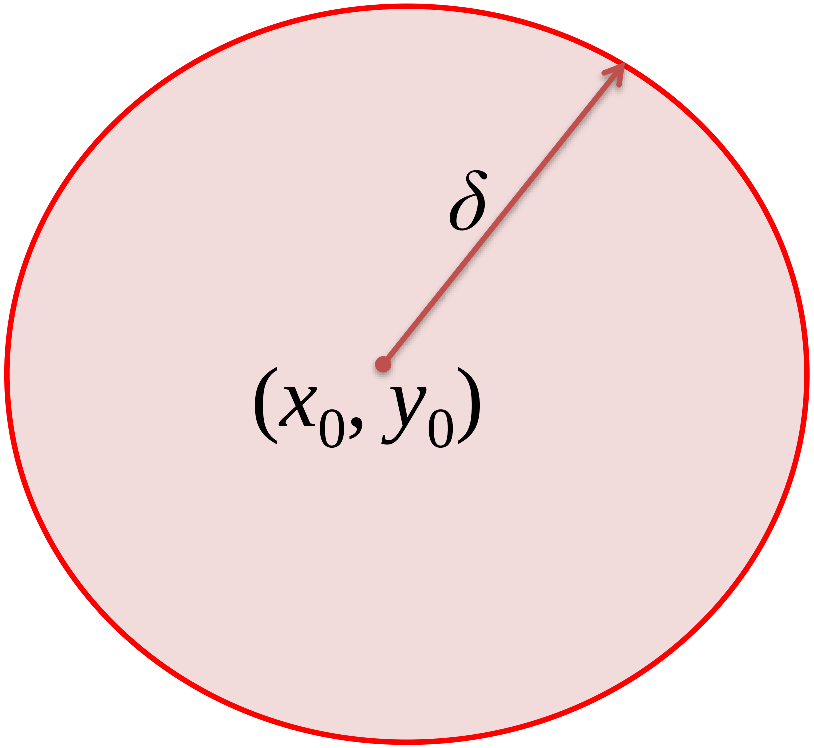
The closed disk of radius $\delta > 0$ centred at (x_0, y_0) is:

$$\{(x,y) : \sqrt{(x-x_0)^2 + (y-y_0)^2} \leq \delta\}$$

Regions



Open disk



Closed disk

Regions

The punctured disk of radius $\delta > 0$ centred at (x_0, y_0) is:

$$\{(x,y) : 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta\}$$

This is obtained by deleting the center from the open disk.

Bounded and Unbounded Regions

A region in plane is bounded if it is contained in some open disk, otherwise it is unbounded.

Examples of bounded regions :
Any circle, rectangle, etc.

Examples of unbounded regions :
Any ray, straight line, first quadrant, etc.

Interior and Boundary Points

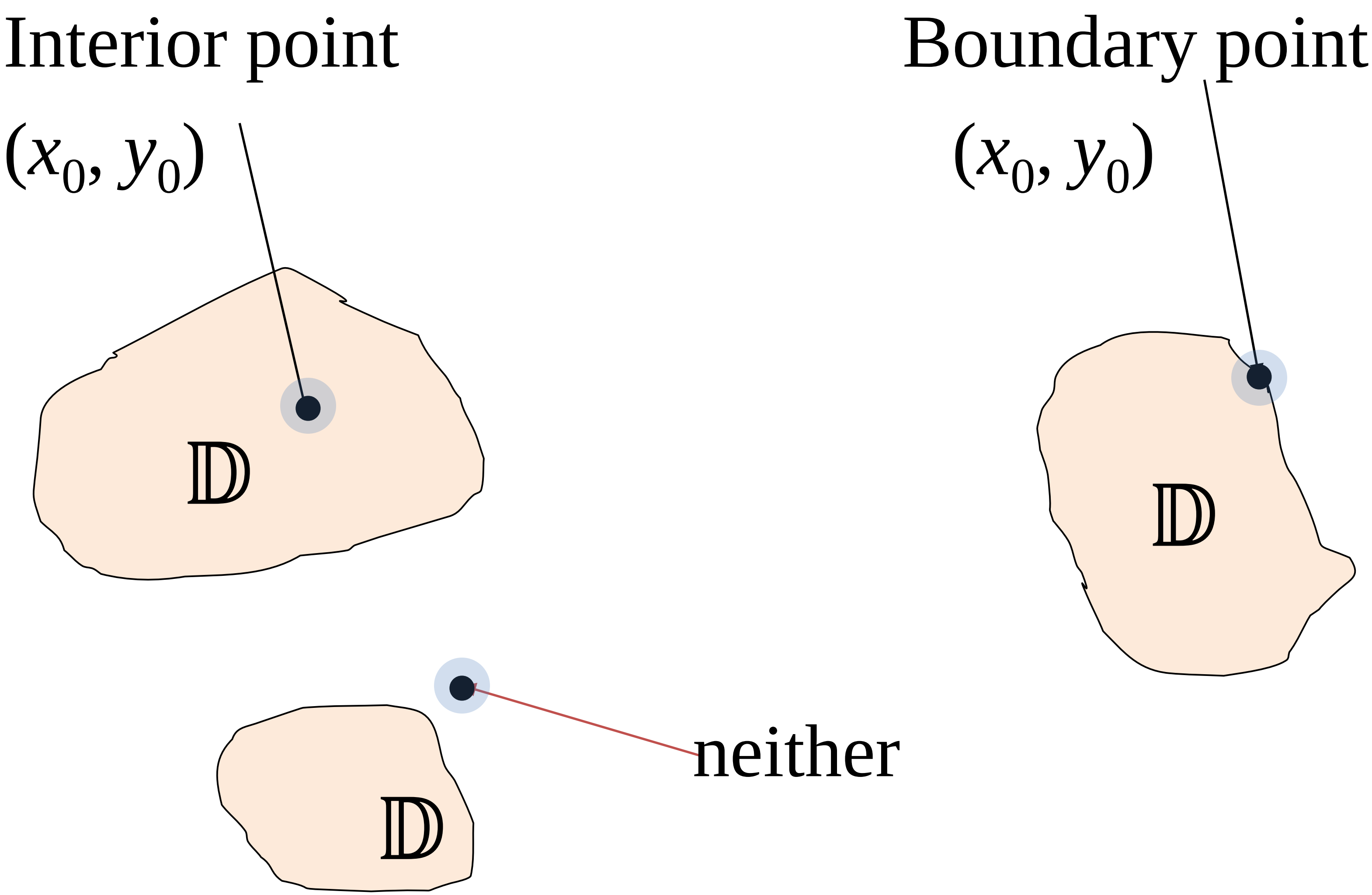
Let $\mathbb{D}$ be a subset of plane. A point (x_0, y_0) of $\mathbb{D}$ is called an interior point of $\mathbb{D}$ if *some* open disk centered at (x_0, y_0) lies completely in $\mathbb{D}$.

Interior and Boundary Points

A point (x_0, y_0) of xy plane is called a <u>boundary point</u> of $\mathbb{D}$ if *every* disk centered at (x_0, y_0) contains both points of $\mathbb{D}$ as well as points outside $\mathbb{D}$.

A boundary point of $\mathbb{D}$ may or may not belong to $\mathbb{D}$

Interior and Boundary Points



Open and Closed Sets

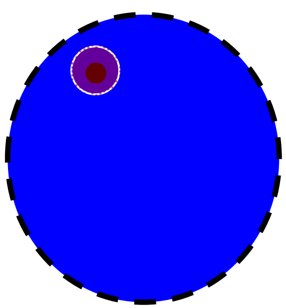
A subset $\mathbb{D}$ of plane is called **open** if **any** point of $\mathbb{D}$ is an interior point of $\mathbb{D}$.

A subset $\mathbb{D}$ of plane is called **closed** if **all** boundary points of $\mathbb{D}$ are points of $\mathbb{D}$.

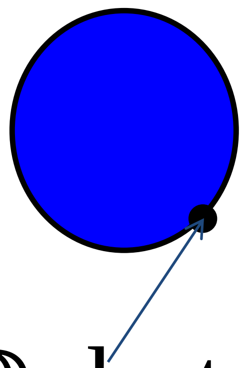
Open and Closed Sets

Examples :

Open disk $\mathbb{D}$ is open, not closed



Closed disk $\mathbb{D}$ is closed, not open



Point of $\mathbb{D}$, but not interior to $\mathbb{D}$

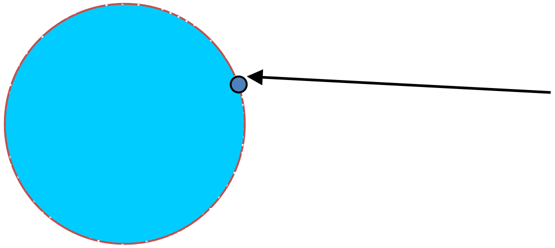
Open and Closed Sets

Remarks :

- A boundary point of $\mathbb{D}$ is ***NEVER*** an interior point of $\mathbb{D}$.
- Thus if $\mathbb{D}$ has any of its boundary points in it then it is ***NOT*** open.

Open and Closed Sets

- Empty set is both open and closed.
- A set which is neither open nor closed: Open disk with only one point on its boundary included



The point of $\mathbb{D}$ is its boundary point

Regions in 3-D Space

All the definitions of interior, boundary points and hence open and closed regions in plane continue to hold for regions in 3-D space if we replace ‘***disk***’ everywhere by ‘***ball***’.

An open ball of radius $\delta > 0$ in 3-D space with center (x_0, y_0, z_0) is $\{(x, y, z) : \sqrt{[(x - x_0)^2+(y - y_0)^2+(z - z_0)^2]} < \delta\}$.
Similarly other types of balls are defined.

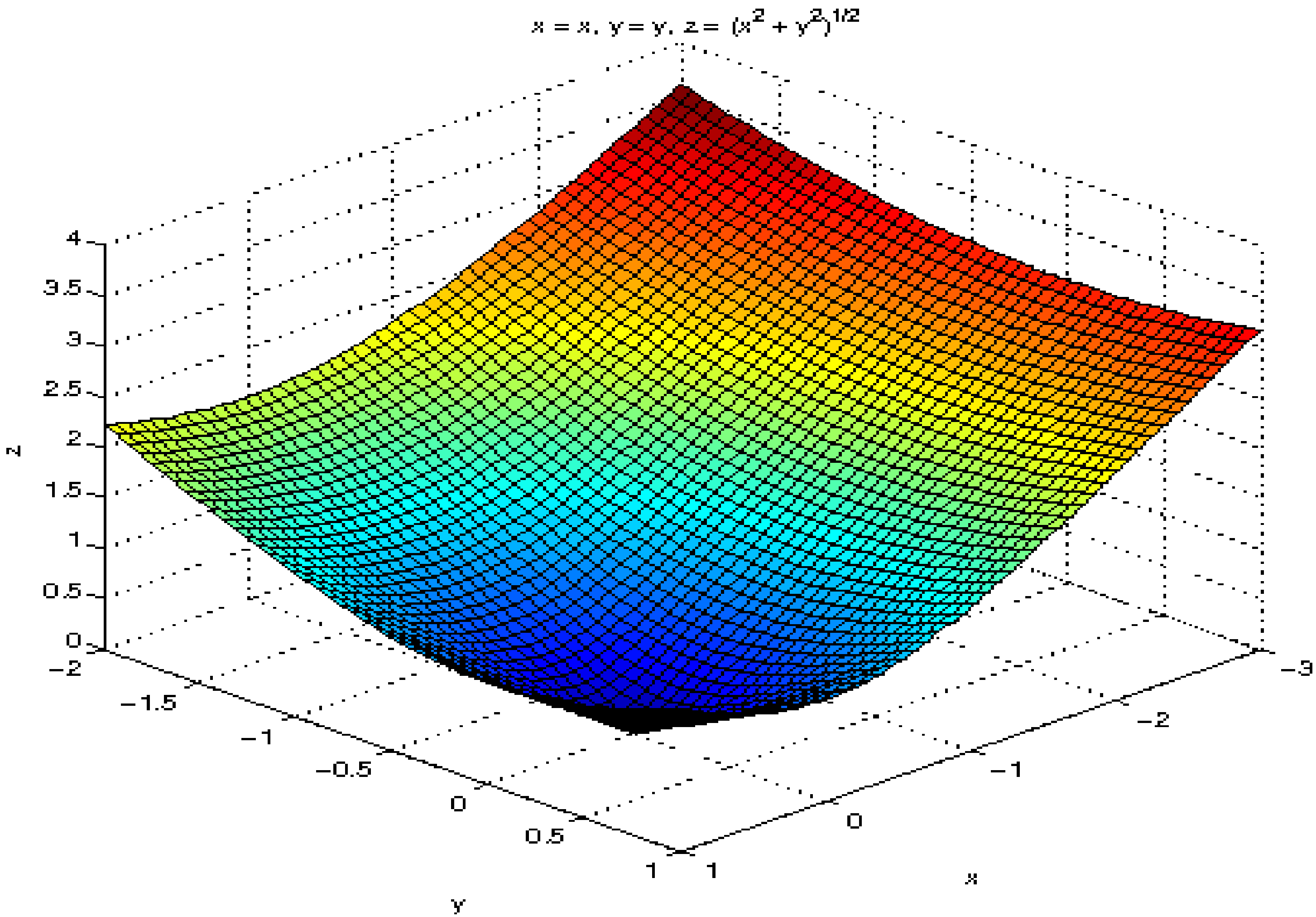
Graph of $f\,(x, y)$

The graph of $f(x, y)$ is:

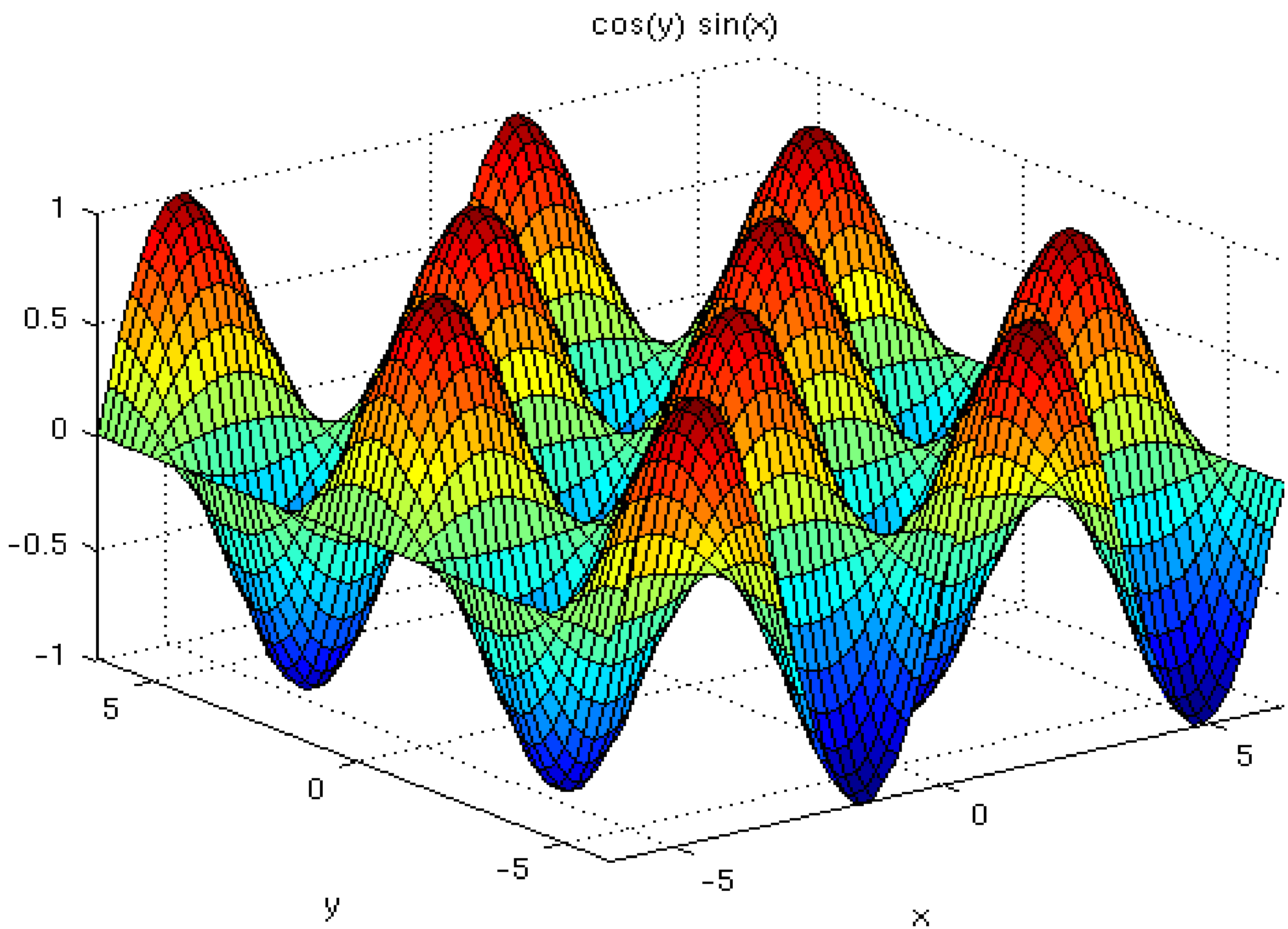
$$\{(x, y, f(x, y)) : (x, y) \in \mathbb{D}\} \subset \mathbb{R}^3$$

The graph of f is also called the surface $z = f(x, y)$.

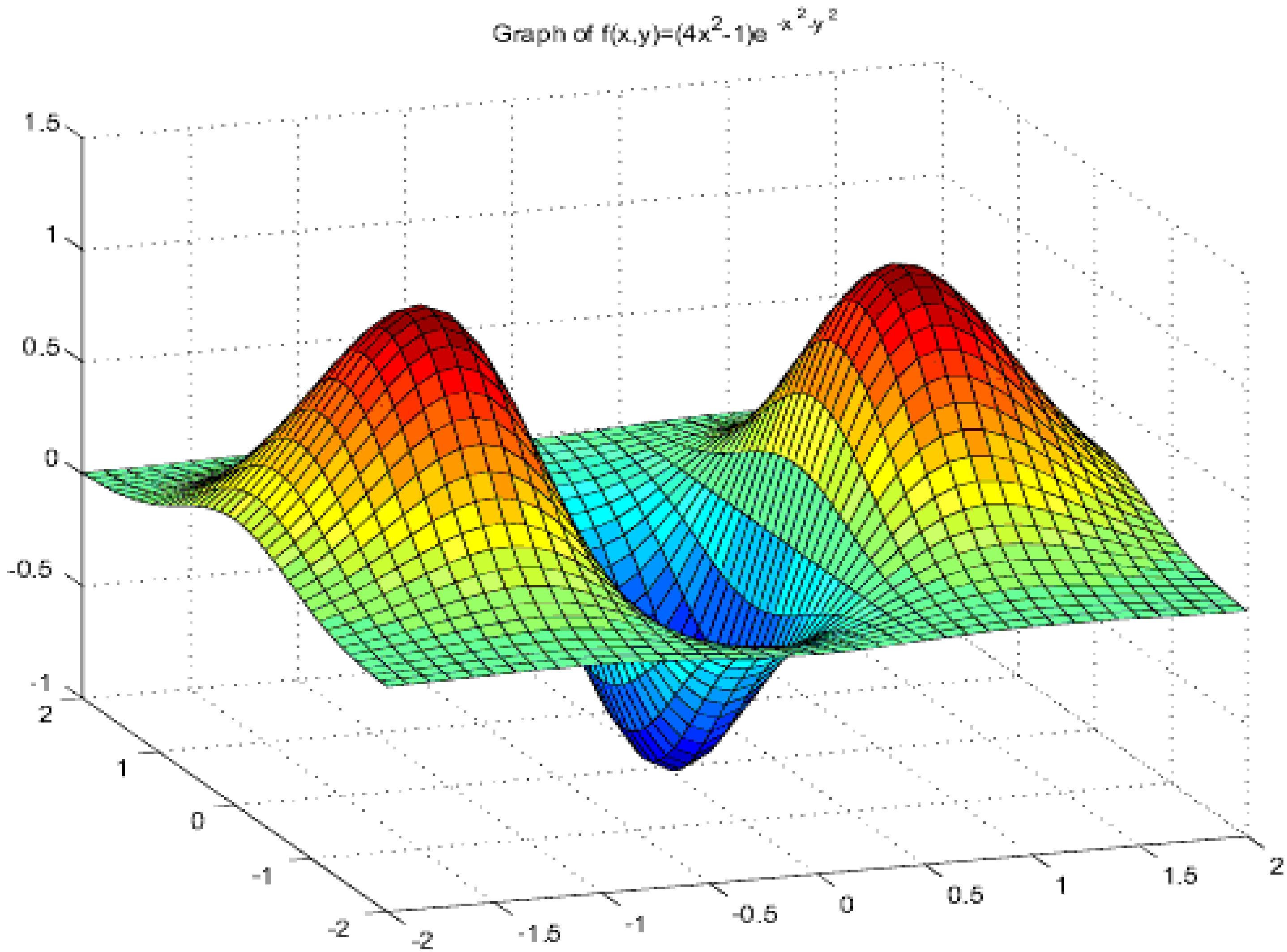
Graph of $f(x, y)$



Graph of $f(x, y)$



Graph of $f(x, y)$



Level Curve and Range

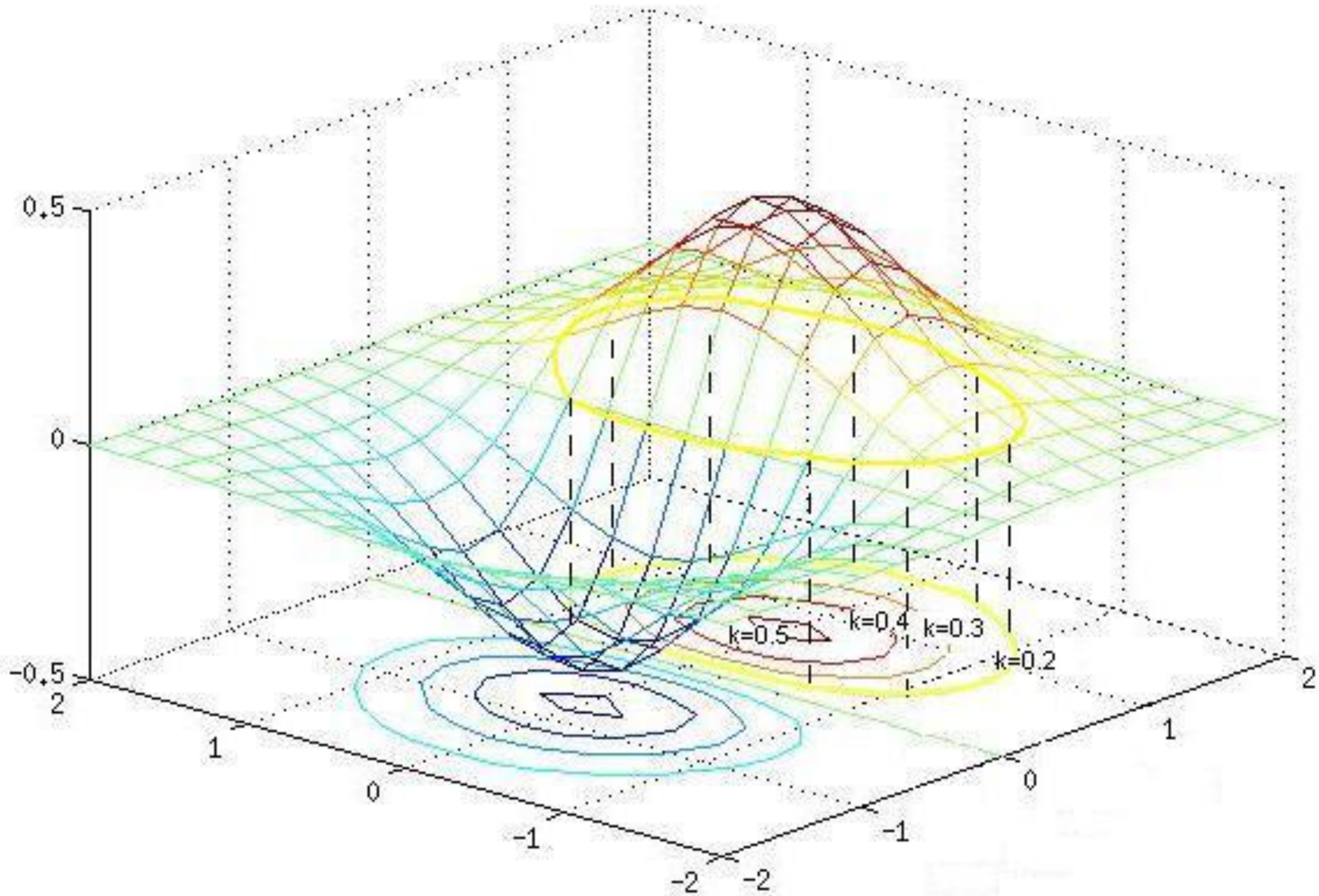
- Recall:
- Range of $f = \{f(x, y) : (x, y) \text{ in } \mathbb{D}\}$
- To find Range of f is to find all the values of c for which the equation $f(x, y) = c$ has a solution in the domain of f.

Level Curve and Range

If $f(x, y)$ is a real valued function with domain $\mathbb{D}$, for a real number c, level curve of f (at level c) is $\{(x, y) \in \mathbb{D} : f(x, y) = c\}$.

As is clear from definition of level curve of $f(x, y)$, Level curves of f are in xy-plane.

Level Curve and Range



Level Curve and Range

For a real number c, c is in range of f if and only if level curve $\{(x, y) | f(x, y) = c\}$ is nonempty.

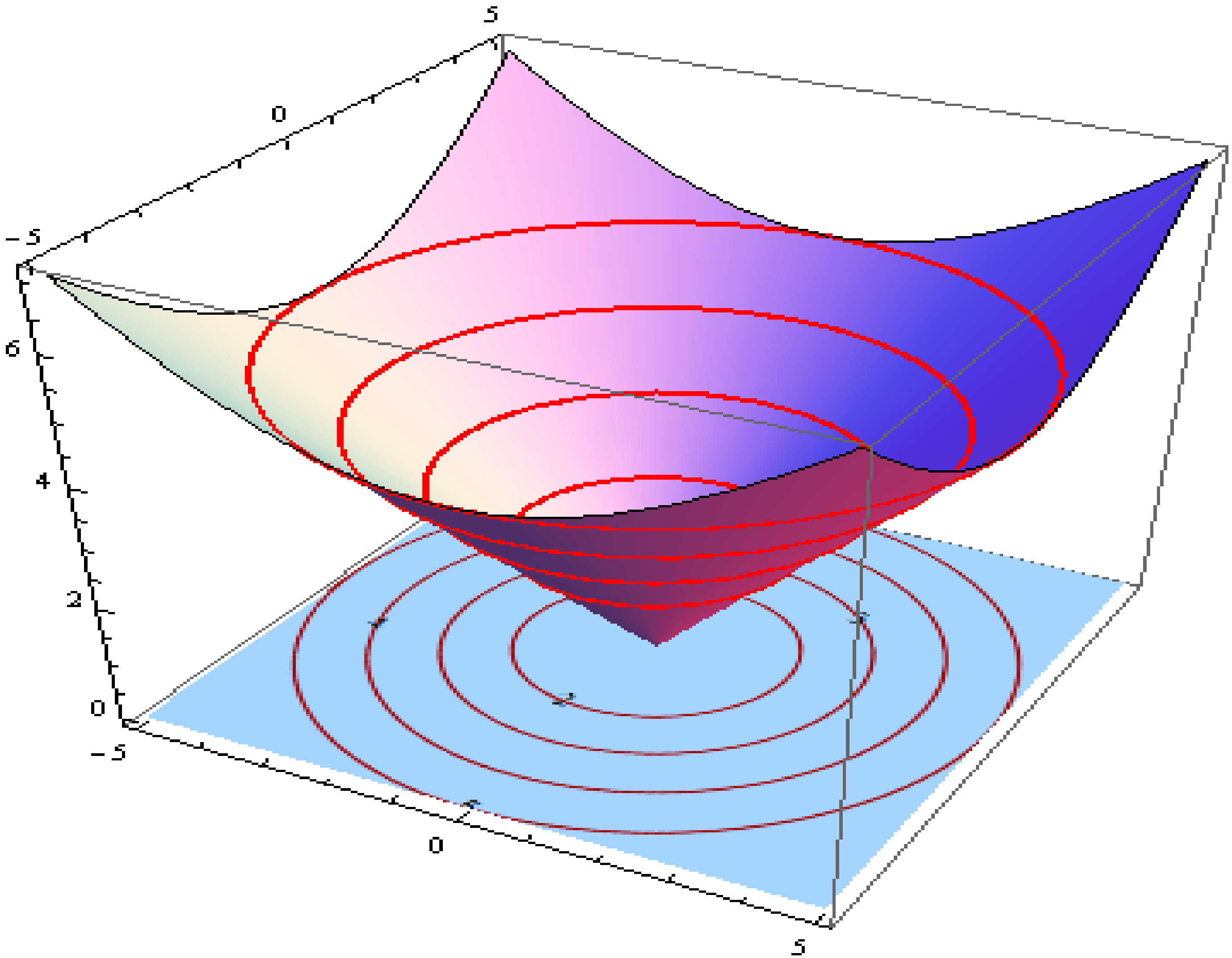
Level curve can help to determine range of f.

Contour Line

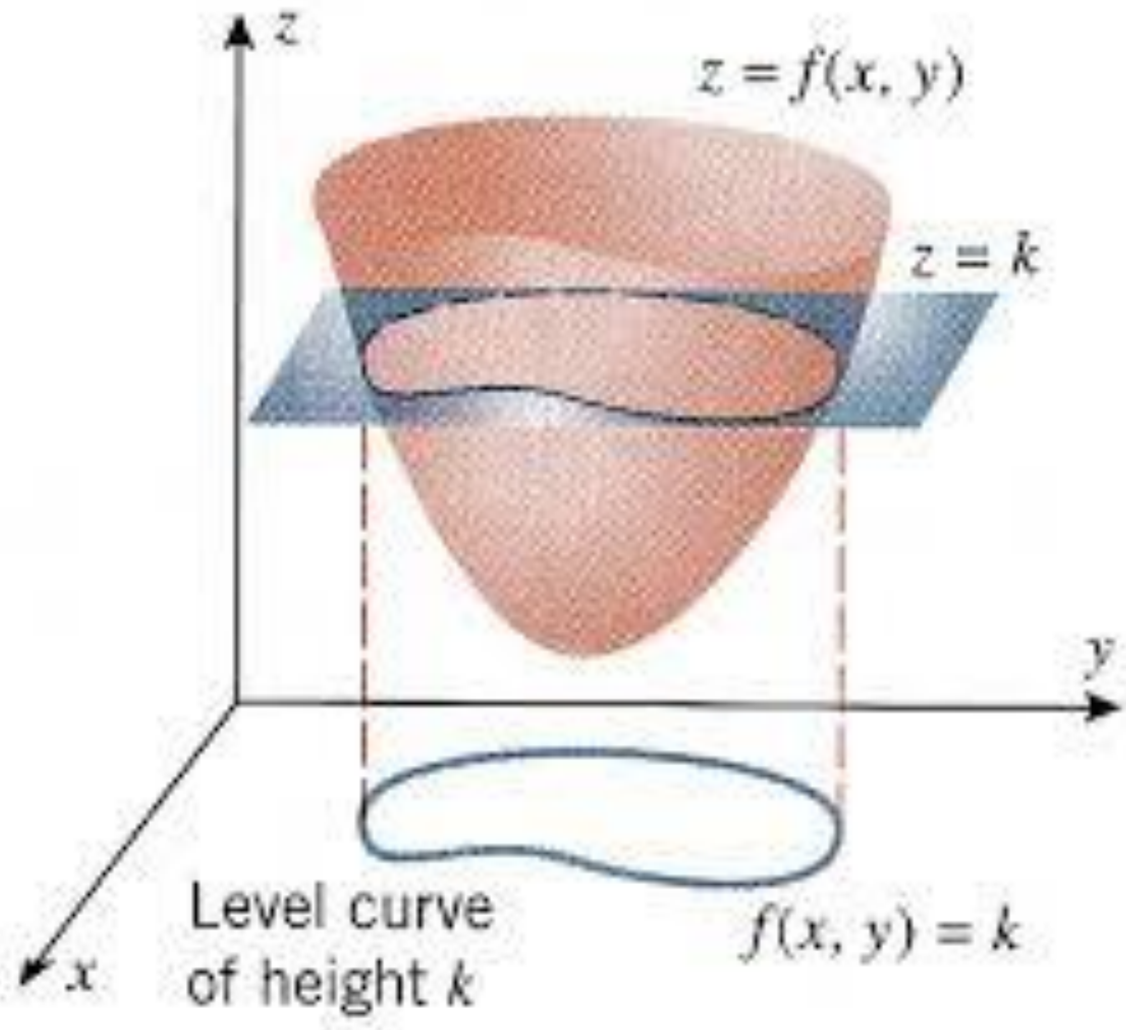
The intersection of the graph of $z = f(x, y)$ with the plane $z = c$ is called contour line $f(x, y) = c$.

The contour line lies on the plane $z = c$ where as level curves are in xy plane

Contour Line



Contour Line



Exercise 14.1

Q.20: Identify and sketch the level curves of

$$f(x, y) = x^2 - y^2.$$

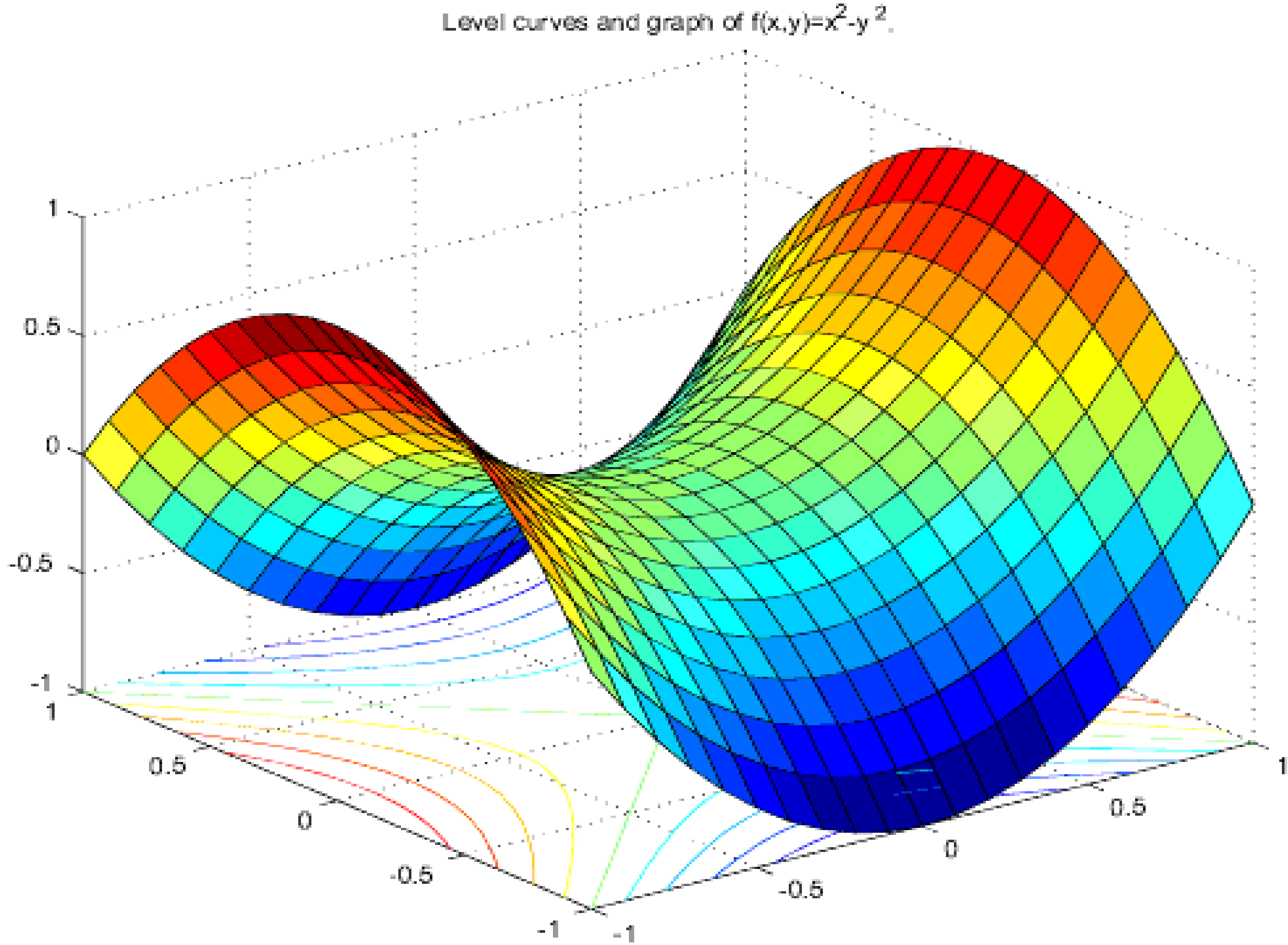
Level Curve: $x^2 - y^2 = c$

If $c = 0$, pair of lines.

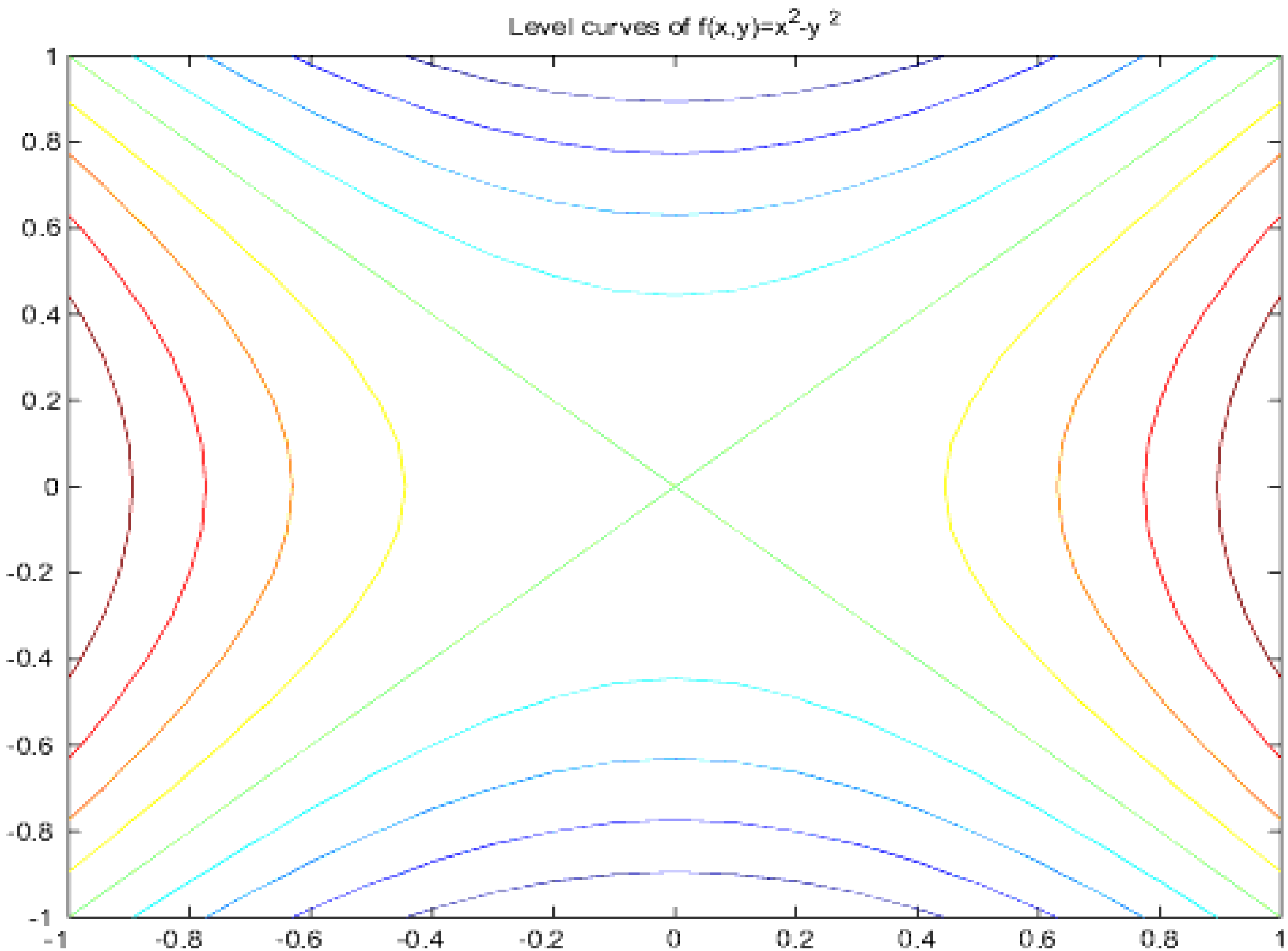
If $c > 0$, hyperbola with foci on x-axis and center at origin.

If $c < 0$, hyperbola with foci on y-axis and center at origin.

Exercise 14.1

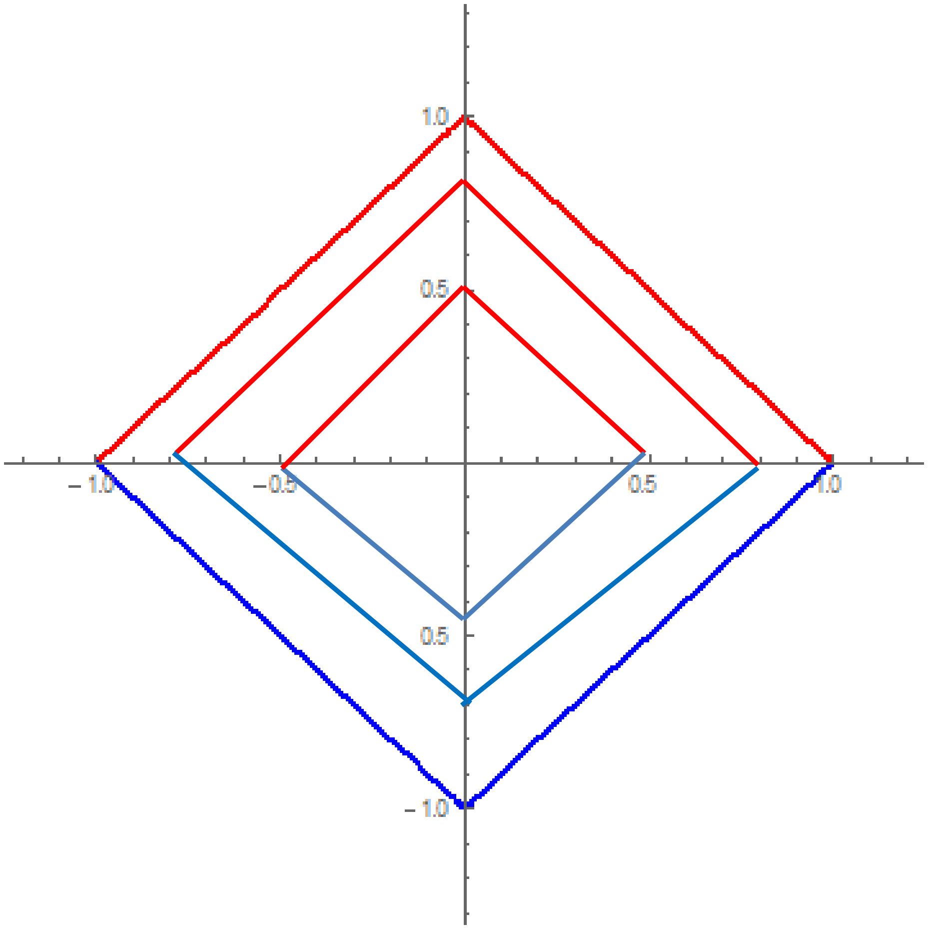


Exercise 14.1



Exercise 14.1

Q.46: Find all the level curves of f and sketch them if : $f(x, y) = 1 - |x| - |y|$.



Exercise 14.1

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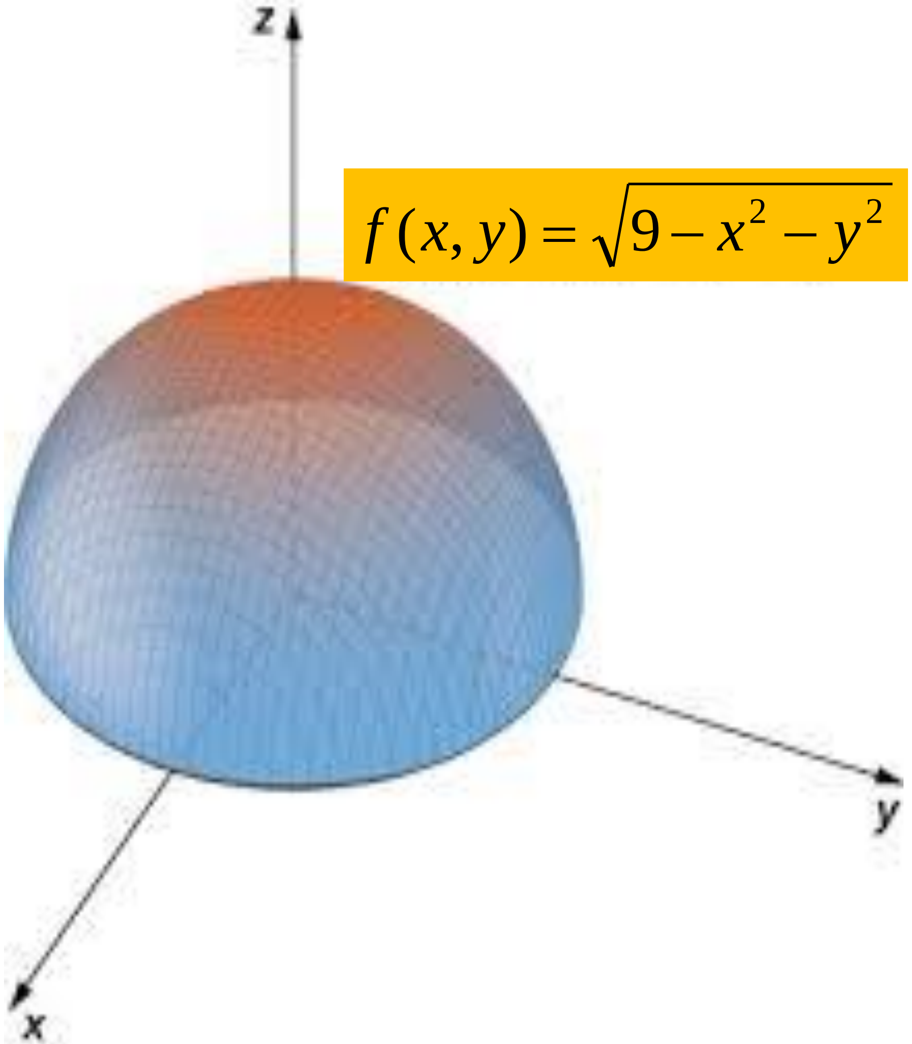
Q.24 Let $f(x,y)=\sqrt{9-x^2-y^2}$. Then

1. Domain of f is $\mathbb{D}=?$

2. Range of f is $\Omega=?$

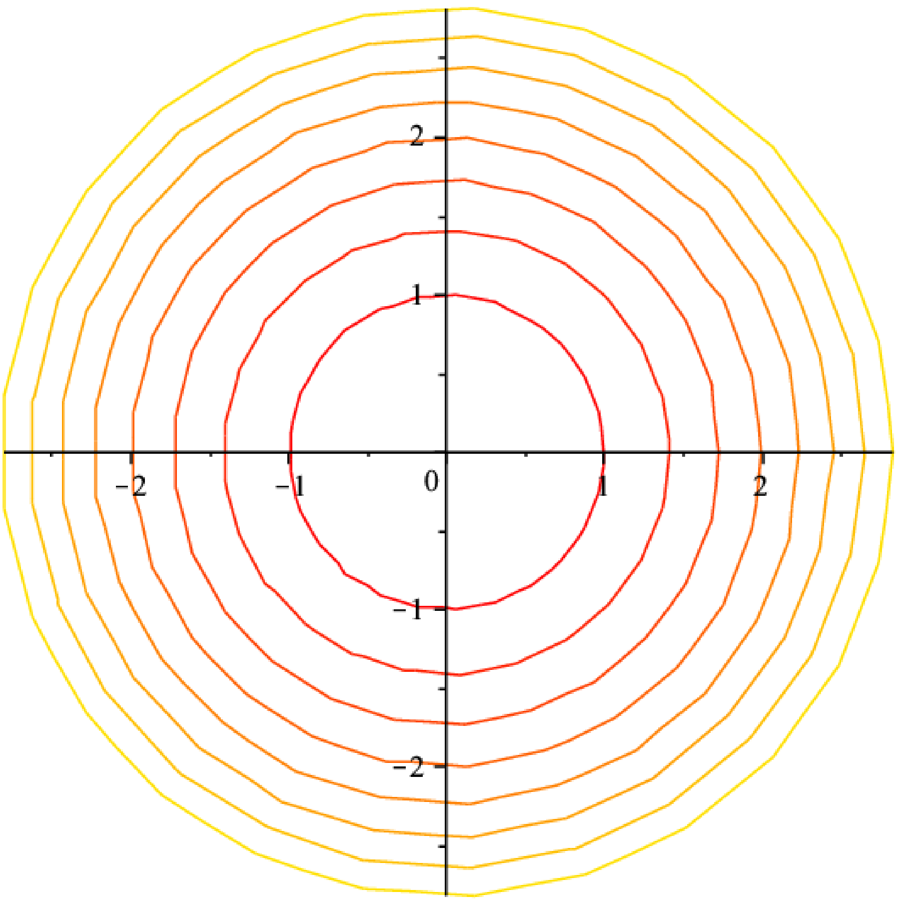
3. An equation of the level curve of f : $x^2+y^2=9-c^2$

Exercise 14.1



$$f(x,y)=\sqrt{9-x^2-y^2}$$

Level Curve



Exercise 14.1

4. Boundary of $\mathbb{D}$:

5. $\mathbb{D}$ is closed (Why?)

6. $\mathbb{D}$ is bounded (Why?)

Exercise 14.1

Example: Find an equation for the level curve of the function:

$$f(x,y)=\int_x^y \frac{t\,dt}{1+t^2} \text{ at the point } (0,0).$$

$$\text{Soln : We have } f(x,y)=\frac{1}{2}\ln\left(\frac{1+y^2}{1+x^2}\right).$$

Exercise 14.1

Equation for the level curve is

$$f(x,y) = \frac{1}{2}\ln\left(\frac{1+y^2}{1+x^2}\right) = c. \qquad (*)$$

At the point (0,0), $c = 0.$

Hence eq.(*) yields

$$y = \pm x.$$

Level Surface

Let $f(x, y, z)$ be a real valued function with domain $\mathbb{D}$. Then

$$\{(x, y, z) \in \mathbb{D} : f(x, y, z) = c\}$$

is called the level surface of f.

Example

Let $f(x,y,z) = e^{\sqrt{z-x^2-y^2}}$. Then find

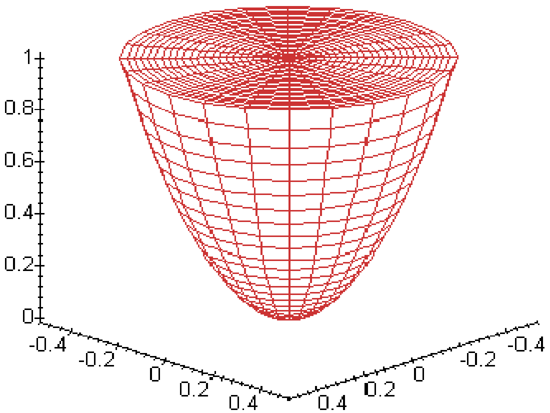
(i) the domain of f,

(ii) the range of f, and

(iii) the level surface of f

through the point (2, -1, 6).

Example



1. Domain of f is $\mathbb{D} = ?$

2. The range of f : ?

3. Eq. of the level surface is ?

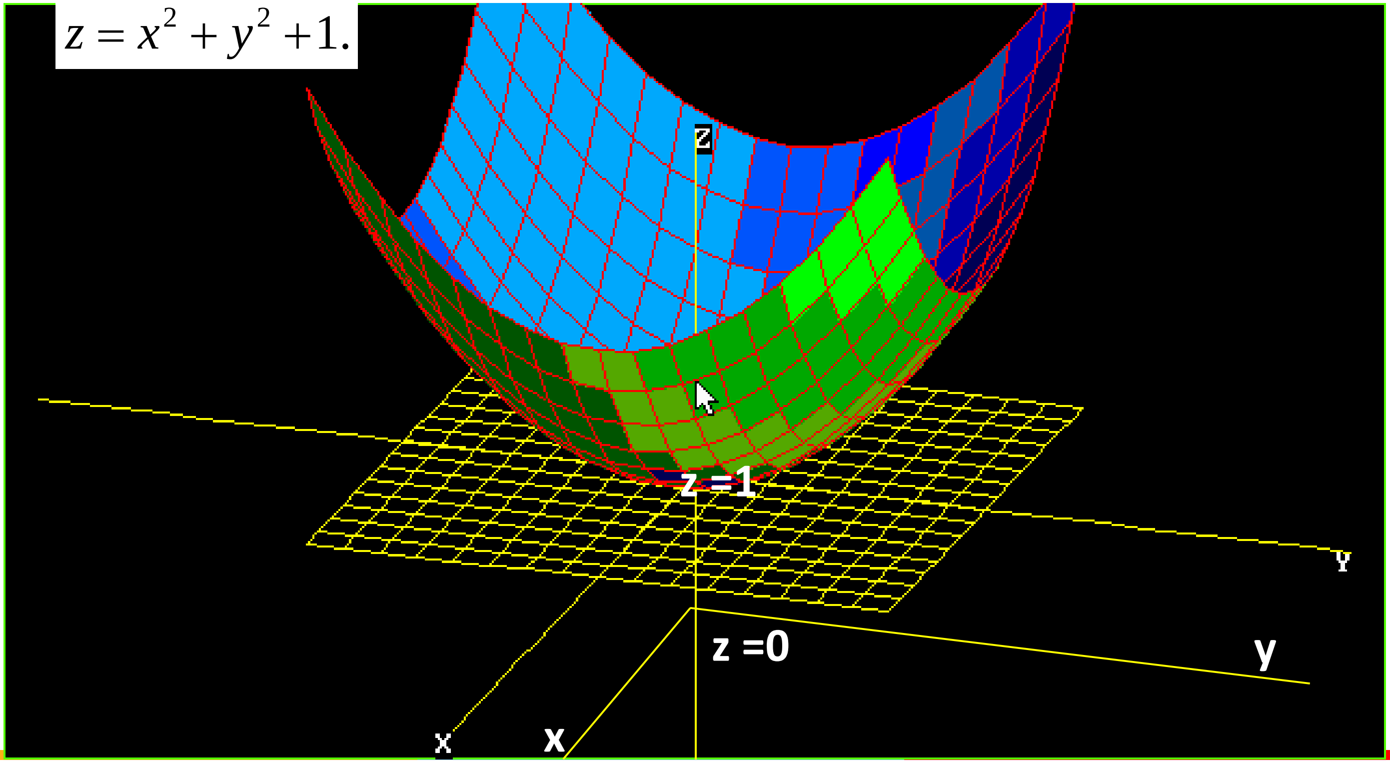
Since the surface is passing through the point (2, -1, 6), hence

$c = f(2, -1, 6) = e.$

Example

$\Rightarrow$ Thus, eq. of the level surface is

$$z = x^2 + y^2 + 1.$$



Limit

Limit : Let $f(x, y)$ be a function with domain $\mathbb{D}$ and l is a real number. Then

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = l$$

if for any $\in > 0$, their exists a corresponding $\delta > 0$ such that for all (x, y) in the domain of f

$$0 < \sqrt{(x\text{-}x_0)^2 + (y - y_0)^2} < \delta \Rightarrow \left| f(x, y) - l \right| < \in$$

Limit

OR

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = l$$

if for any $\in > 0$, there exists a corresponding

$\delta > 0$ such that for all (x, y) in the domain of f

$$0 < \left|x\text{-}x_0\right| < \delta, 0 < \left|y - y_0\right| < \delta \Rightarrow \left|f(x,y) - l\right| < \in$$

Limit

Remark

1. This condition needs to be checked for points (x,y) of the domain of f only.

2. (i) Here we fix $\in$, the target around l for the output variable first.

Limit

(ii) It should then be possible to choose δ for a punctured disk around (x_0, y_0) such that the points of domain lying in the punctured disk are mapped inside the given target.

3. The radius of the punctured disk δ may depend on the size of $\in$.

Limit

Let $f(x,y)$ and $g(x,y)$ have the same domain $\mathbb{D}$ in the plane and suppose

$$\lim_{(x,y)\to(x_o,y_o)} f(x,y) = l,$$

$$\lim_{(x,y)\to(x_o,y_o)} g(x,y) = m$$

exist for real numbers l, m.

For sum/difference/product/division/roots of functions, rules similar to one variable can be used

Continuity

Continuity : A function $f(x, y)$ is said to be continuous at (x_0, y_0) if

(i) $f(x, y)$ is defined at (x_0, y_0),

(ii) $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ exists

(iii) $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$

Continuity

* $f(x, y) = x, \quad g(x, y) = y$ and $h(x, y) = c$, a constant function, are continuous at all points of domain.

* Sum and product of continuous functions is also continuous.

* Thus, polynomials in x and y are continuous every where on domain.

* Rational functions are continuous at all points where the denominator is non zero

Continuity

Composition Rule: Let f: $\mathbb{D} \to \mathbb{R}$ and g: $\mathbb{R} \to \mathbb{R}$ for a subset $\mathbb{D}$ of plane

Let $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = l$ exist and $g(z)$ be **continuous** at $z = l$.

If $w(x, y) = g(f(x, y))$ is composite function from $\mathbb{D}$ to $\mathbb{R}$,

then $\lim\limits_{(x,y)\to(x_0,y_0)} w(x,y)$ exists and equals $g(l)$.

Continuity

In other words, If $z = f(x, y)$ is a real valued continuous function of x and y and $w = g(z)$ is a continuous function of z, then composite function $w = g(f(x, y))$ is continuous function.

Ex. (1) $\sin(xy)$ is continuous at every (x,y).

(2) $\sin\left(\tan(xy)\right)$ is continuous at all points where

$$xy \neq \frac{(2n+1)\pi}{2}; n \in \mathbb{Z}$$

Continuity

IMPORTANT REMARKS

1. Let $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = l$ exists.

Then along any path in the domain of f, limit of $f(x,y)$ as $(x,y) \to (x_0, y_0)$ must exist and equal to l.

Continuity

The path must pass through (x_0, y_0) and lie in the domain of f. If path is given by parameterisation $(x(t), y(t))$ and $(x_0, y_0) = \lim_{t \to t_0}(x(t), y(t))$, then limit along path is $\lim_{t \to t_0} f(x(t), y(t))$.

Continuity

2. TWO PATH TEST :

If $f(x, y)$ has different limits along two different paths in the domain of f approaching (x_0, y_0), then $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y)$.

DOES NOT exist.

3. **Two path test** can only be used to show that limit of $f(x, y)$ **does not exist**.

Exercise 14.2

Q.44 Show that $f(x, y) = \dfrac{xy}{|xy|}$ has no limit as $(x, y) \to (0,0)$.

Q.9 Find $\displaystyle\lim_{(x, y)\to(0, 0)} \frac{e^y \sin x}{x}$, if it exists.

Q.18 Find $\displaystyle\lim_{(x,y)\to(2,2)} \frac{x+y-4}{\sqrt{x+y}-2}$, if it exists.

The Sandwich Theorem

Let $g(x,y) \le f(x,y) \le h(x,y)$
for all $(x,y) \ne (x_0, y_0)$ of Domain of f
in an open disk centred at (x_0, y_0).

If $\lim\limits_{(x,y)\to(x_0,y_0)} g(x,y) = \lim\limits_{(x,y)\to(x_0,y_0)} h(x,y) = l$,
l a real number, then $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = l$

The Sandwich Theorem

Q.57 Does $\lim_{(x,y)\to(0,0)} y \sin\frac{1}{x}$ exist?

Find if it does.

Note that the domain of $f(x,y) = y\sin\frac{1}{x}$ is

$D = \{(x,y) : x \neq 0\}$ and $-|y| \leq y \sin\frac{1}{x} \leq |y|$

for all $(x,y) \neq (0,0)$ in a disk centred at $(0,0)$ which are in D

The Sandwich Theorem

$$\Rightarrow \lim_{(x,y)\to(0,0)} y \sin\frac{1}{x} = 0$$

$$\text{(By the Sandwich Theorem)}$$

Exercise 14.2

Q.68 If $f(x,y)=\dfrac{3x^2y}{x^2+y^2}$, find

$\lim\limits_{(x,y)\to(0,0)} f(x,y)$, if it exists.

By Sandwich Theorem,

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}=0.$$

Exercise 14.2

Q.54 If $f(x_0, y_0) = 3$, what can you say about

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)$$

(a) if f is continuous at (x_0, y_0)?

(b) if f is not continuous at (x_0, y_0)?

Exercise 14.2

If $f(x,y)=\dfrac{xy}{x^2+y^2}$, find

$$\lim_{(x,y)\to(0,0)} f(x,y), \text{ if it exists.}$$

$\text{Soln: If } y=0, \text{ then } f(x,0)=0.$

$\Rightarrow f(x,y)\to 0 \quad \text{as} \quad (x,y)\to(0,0)$

along the x-axis.

Exercise 14.2

If $x = 0$, then $f(0, y) = 0$.

$\Rightarrow f(x, y) \to 0$ as $(x, y) \to (0,0)$

along the y-axis.

Conclusion : INCONLUSIVE

Let's now approach $(0,0)$ along the line $y = x$. Then

Exercise 14.2

$$f(x,x) = \frac{x^2}{x^2 + x^2} = \frac{1}{2} \text{ for } x \neq 0.$$

CONCLUSION:

limit does NOT exist.

Alternative Method

See the limits as (x, y) approaches origin along line $y = mx$.

Examples

If $f(x,y) = \dfrac{xy^2}{x^2 + y^4}$, find

$\lim\limits_{(x,y)\to(0,0)} f(x,y)$, if it exists.

Soln : See what happens along the line $y = mx$?

Exercise 14.2

Along the curve $x = my^2$, for $(x,y) \neq (0,0)$ we have

$$f(x,y) = f(my^2, y) = \frac{m}{m^2+1},$$

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } x=my^2}} f(x,y) = \lim_{y\to 0} \frac{m}{m^2+1} = \frac{m}{m^2+1}$$

Thus, limit does NOT exist.

Exercise 14.2

Q.66 Where is the function

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2} \quad \text{continuous?}$$

Soln : Note that

(i) f is discontinuous at $(0,0)$, since it is not defined there.

Exercise 14.2

(ii) f is a rational function, it is continuous on its domain $\mathrm{D} = \{(x, y) : (x, y) \neq (0,0)\}$.

Exercise 14.2

$$\text{Define } f(x,y)=\begin{cases}\dfrac{x^2-y^2}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

Is f continuous at $(0,0)$?

Ans: NO (WHY?)

Exercise 14.2

$$\text{Define } f(x,y) = \begin{cases} \dfrac{3x^2 y}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Ans : Note that

(i) f is continuous at $(0,0)$, in FACT,

(ii) f is continuous in whole plane.

Limits in Polar Coordinates

If for any $\in > 0$, there exists a $\delta > 0$ such that for all (r, θ) in the domain of f,

$$0 < |r| < \delta \Rightarrow | f(r, \theta) - l| < \in,$$

then we say that

$$\lim_{(x,y)\to(0,0)} g(x, y) = l, \text{ where}$$

$$g\left(r\cos\theta, r\sin\theta\right) = g(x, y) = f(r, \theta)$$

and l is a real number

Limits in Polar Coordinates

In particular, if there exists a function $\phi(r)$ such that $\left| f(r,\theta) - l \right| < \phi(r)$ for all points (r,θ) in the domain of f in a punctured disk around origin and if $\lim_{r \to 0} \phi(r) = 0$, then

$$\lim_{(x,y)\to(0,0)} g(x,y) = l,$$

where $g(r\cos\theta, r\sin\theta) = f(r,\theta)$.

Exercise 14.2

Q.62 Find $\lim_{(x,y)\to(0,0)} \cos\left(\frac{x^3-y^3}{x^2+y^2}\right)$.

Solution :

First consider

$$\lim_{(x,y)\to(0,0)} \left(\frac{x^3-y^3}{x^2+y^2}\right)$$

Exercise 14.2

$$\text{here } g(x, y) = \frac{x^3 - y^3}{x^2 + y^2}$$

$$\Rightarrow g(r\cos\theta, r\sin\theta)$$

$$= \frac{r^3(\cos^3\theta - \sin^3\theta)}{r^2}$$

$$= f(r, \theta)$$

Exercise 14.2

$$\text{on } 0 < |r| < \delta$$

$$|f(r,\theta) - 0| = |r|\,\left|\cos^3\theta - \sin^3\theta\right|$$

$$\leq |r|\left(\left|\cos^3\theta\,| + |\sin^3\theta\right|\right) \leq 2|r| = \phi(r)$$

$$\lim_{r\to 0} \phi(r) = 0$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} g(x,y) = 0$$

Exercise 14.2

Now by composition rule,

$$\lim_{(x,y)\to(0,0)} \cos\left(\frac{x^3-y^3}{x^2+y^2}\right)$$

$$=\cos\left(\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}\right)$$

$$=1$$

Summary

To show a function has a limit and to find it use :

- Theorems on limits of sum, product, quotient, powers, if applicable
- Simplification on its domain, if possible
- Sandwich theorem, if applicable
- Method of polar coordinates, if possible.

Summary

You may, meanwhile, keep yourself open to

Possibility of nonexistence of limits.

To show a function f does not have limit, need to use two path test as follows:

- Search for a path in domain of f through given point along which limit does not exist.

- Search for two paths or family of paths as above along which limits are different.

Partial Derivative

Let (x_0, y_0) be a point in the domain of $f(x, y)$. The partial derivative of $f(x, y)$ with respect to x at the point (x_0, y_0) is

$$\left(\frac{\partial f}{\partial x}\right)\Bigg|_{(x_0, y_0)} = \left(\frac{d}{dx} f(x, y_0)\right)\Bigg|_{x=x_0}$$

Partial Derivative

$$= \lim_{h \to 0} \frac{f\left(x_0 + h, y_0\right) - f\left(x_0, y_0\right)}{h},$$

Provided the limit exist.

Partial Derivative

The partial derivative of $f(x, y)$ with respect to y at the point (x_0, y_0) is

$$\left(\frac{\partial f}{\partial y}\right)\Bigg|_{(x_0, y_0)} = \left(\frac{d}{dy} f(x_0, y)\right)\Bigg|_{y=y_0}$$

Partial Derivative

$$= \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h},$$

provided the limit exists.

Geometric Interpretation of Partial Derivative

Recall:

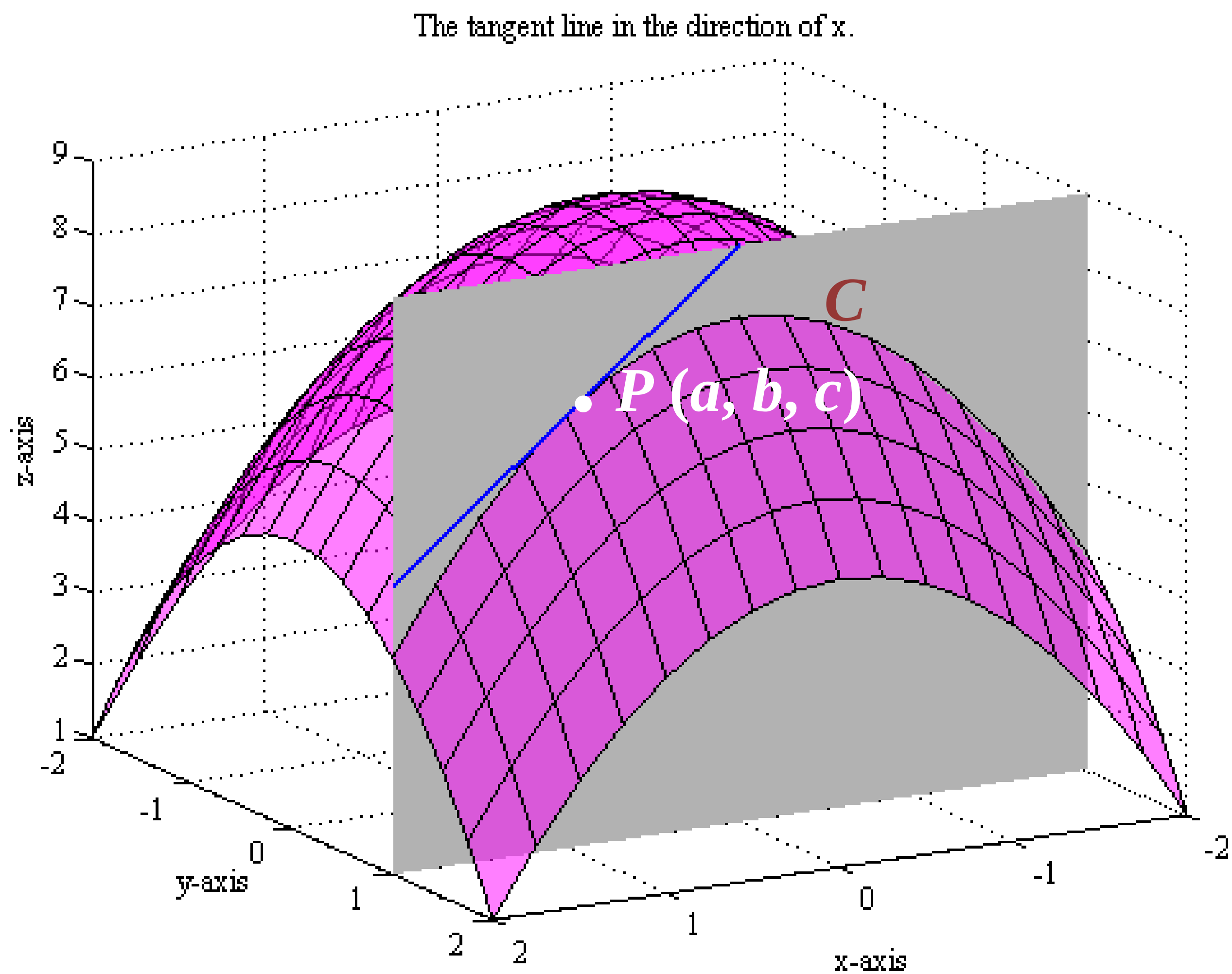
1. $z = f(x, y)$ represents a surface S (graph of f)

2. If $f(a, b) = c$, then the point $P(a, b, c)$ lies on S

3. Fix $y = b$, then the vertical plane $y = b$ intersects S in a curve C, i.e.
$C: \; z = f(x, b)$ trace of S in the plane $y = b$.

$$\Rightarrow \left.\frac{\partial f}{\partial x}\right|_{(a,b)} = \text{slope of tangent to } C \text{ at } P(a,b,c)$$

Geometric Interpretation of Partial Derivative



Symbols of Partial Derivative

$$\frac{\partial f}{\partial x} = f_x, \qquad \frac{\partial f}{\partial y} = f_y,$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = (f_x)_x = f_{xx},$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy},$$

Symbols of Partial Derivative

innovate achieve lead

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \left(f_y\right)_x = f_{yx},$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \left(f_x\right)_y = f_{xy},$$

Examples

Find f_x and f_y at the point $(2,1)$

if $f(x,y) = x^2 + xy.$

$$\Rightarrow \left(\frac{\partial f}{\partial x}\right)\Bigg|_{(x_0,y_0)} = \lim_{h\to 0} \frac{(x_0+h)^2 + (x_0+h)y_0 - x_0^2 - x_0 y_0}{h}$$

$$\Rightarrow \left(\frac{\partial f}{\partial x}\right)\Bigg|_{(2,1)} = \lim_{h\to 0} \frac{h^2+5h}{h} = 5$$

Examples

$$\left(\frac{\partial f}{\partial y}\right)\Bigg|_{(x_0, y_0)} = \lim_{h \to 0} \frac{x_0^2 + x_0(y_0 + h) - x_0^2 - x_0 y_0}{h}$$

$$= \lim_{h \to 0} \frac{x_0 h}{h} = x_0$$

$$\Rightarrow \left(\frac{\partial f}{\partial y}\right)\Bigg|_{(2,1)} = 2$$

Alternative Method

$$\frac{\partial f}{\partial x}(a,b) = \frac{dg}{dx}(a) \text{ where}$$

$$g(x) = f(x,b).$$

$$\text{Thus } \frac{\partial f}{\partial x}(2,1) = \left.\frac{d(x^2 + x.1)}{dx}\right|_{x=2} = (2)(2) + 1 = 5.$$

In general, $\frac{\partial f}{\partial x}(x,y)$ is obtained by differentiating f treating y as constant.

Mixed Derivative Theorem

If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy} and f_{yx} are defined through out an open region containing a point (x_0, y_0) and are all **continuous** at (x_0, y_0), then

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$$

Functions of Three Variable

$$f_x(x_0, y_0, z_0) = \lim_{h\to 0} \frac{f(x_0 + h, y_0, z_0) - f(x_0, y_0, z_0)}{h}$$

if the limit exists.

Exercise 14.3

Q.2 Find $f_x, f_y, f_{xx}, f_{yy}, f_{xy}, f_{yx}$

if $f(x, y) = x^2 - xy + y^2$

Q.80 If $f(x, y, z) = e^{3x+4y}\cos 5z$, show that

$$f_{xx} + f_{yy} + f_{zz} = 0$$

Examples

Implicit diffferentiation :

Find $\frac{\partial z}{\partial x}$ if the equation

$xz - \ln z = xy$ defines z as a function of two independent variables x and y and partial derivaties exist.

Exercise 14.3

Q.72 Let

$$f(x,y)=\begin{cases}\dfrac{xy\left(x^2-y^2\right)}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

Show that $f_{xy}(0,0) \neq f_{yx}(0,0).$

Exercise 14.3

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$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = 0$$

$$f_y(0,0) = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} = 0$$

Exercise 14.3

For $(x, y) \neq (0,0)$, Calculate f_x :

$$f_x(x,y) = \frac{x^4 y + 4x^2 y^3 - y^5}{(x^2 + y^2)^2}$$

$$\therefore\ f_{xy}(0,0) = (f_x)_y\Big|_{(x,y)=(0,0)}$$

Exercise 14.3

$$= \lim_{h \to 0} \frac{f_x(0,h) - f_x(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{-h-0}{h} = -1$$

Exercise 14.3

For $(x, y) \neq (0,0)$, Calculate f_y :

$$f_y(x,y) = -\frac{y^4x + 4y^2x^3 - x^5}{(x^2+y^2)^2}$$

$$\therefore\ f_{yx}(0,0) = (f_y)_x\Big|_{(x,y)=(0,0)}$$

Exercise 14.3

$$= \lim_{h \to 0} \frac{f_y(h,0) - f_y(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{h-0}{h} = 1$$

To find Partial Derivative

- If a specific rule works in an open set containing the given point, one can differentiate that rule directly while treating other variables constant. If two rules are required in every open set containing that point, use limit definition.

- For 2nd order partials, 1st order partial which are differentiated must be obtained on relevant nearby points.

Increment Theorem for $z = f(x, y)$

Assumptions:

1. Let the first partial derivatives f_x and f_y of $f(x, y)$ be defined throughout an open region R containing the point (x_0, y_0).
2. f_x and f_y are continuous at (x_0, y_0).

Increment Theorem for $z = f(x, y)$

Then the change

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$

in the value of f that results in moving from (x_0, y_0) to another point $(x_0 + \Delta x, y_0 + \Delta y)$ in open region R satisfies an equation of the form

Increment Theorem for $z = f(x, y)$

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \in_1 \Delta x + \in_2 \Delta y, \qquad \text{........}(1)$$

$$\text{where } \in_1, \in_2 \to 0 \text{ as } \Delta x, \Delta y \to 0.$$

Differentiability

A function $f(x, y)$ is **differentiable** at (x_0, y_0) if f_x and f_y exists at (x_0, y_0) and equation (1) holds for f at (x_0, y_0).

We say that f is differentiable if it is differentiable at every point in its domain

Differentiability

Remark 1: Continuity of partial derivatives at (x_0, y_0) implies differentiability at (x_0, y_0)

<u>Remark 2:</u> If f_x and f_y of $f(x, y)$ are continuous throughout an open region R, then f is differentiable at every point of R.

Differentiability

Theorem:

If a function $f(x, y)$ is differentiable at (x_0, y_0), then f is continuous at (x_0, y_0)

THANK YOU
FOR YOUR PATIENCE !!!