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BITS Pilani
Pilani Campus

MATH F112 (Mathematics-II)

Complex Analysis

BITS Pilani
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Lecture 29-30
Elementary Functions

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Trigonometric Function

(5).Analyticity of $\tan z$ & $\sec z$:

$$\tan z = \frac{\sin z}{\cos z}, \quad \sec z = \frac{1}{\cos z}$$

$\Rightarrow$ $\tan z$ & $\sec z$ are analytic everywhere except at the points where $\cos z = 0$

Trigonometric Function

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$$\cos z = 0$$

$$\Rightarrow \cos(x+iy) =$$

$$\cos x \cosh y - i \sin x \sinh y = 0$$

$$\Rightarrow \cos x \cosh y = 0, \ \&$$

$$\sin x \sinh y = 0$$

Trigonometric Function

$$\because \cosh y \neq 0$$

$$(\cosh y = \frac{e^{y}+e^{-y}}{2} = \frac{1}{2}\left(e^{y}+\frac{1}{e^{y}}\right)$$

$$= 0 \Rightarrow e^{2y} = -1 < 0)$$

$$\backslash \ \cos x = 0 \text{Þ} \ \ x = \left(2n+1\right)\frac{\rho}{2}, n = 0, \pm 1, \pm 2 \ldots$$

Trigonometric Function

But $\sin x \neq 0$ for $x = (2n+1)\dfrac{\pi}{2}$

$$\therefore \quad \sinh y = 0 \Rightarrow y = 0$$

$$\left\{ \sinh y = \frac{e^{y} - e^{-y}}{2} = 0 \Rightarrow e^{2y} = 1 \Rightarrow y = 0 \right\}$$

Trigonometric Function

$$\therefore\ z = x + iy = (2n+1)\frac{\pi}{2}$$

$\therefore \tan\ z\ \&\ \sec\ z$ are analytic every where exceptat

$$z = (2n+1)\frac{\pi}{2},\ \ n = 0, \pm 1 \pm 2, \ldots$$

Trigonometric Function

(6Ex.) Analyticity of $\cot z$ & $\operatorname{cosec} z$:

$$\because \quad \cot z = \frac{\cos z}{\sin z} \ \& \ \operatorname{cosec} z = \frac{1}{\sin z}$$

$\Rightarrow \cot z$ & $\operatorname{cosec} z$ are analytic everywhere except at the points where $\sin z = 0$

Trigonometric Function

$$\sin z = \sin(x + iy)$$

$$= \sin x \cosh y + i \cos x . \sinh y = 0$$

$$\Rightarrow \sin x . \cosh y = 0 \ \& \ \cos x \sinh y = 0$$

$$\because \cosh y \neq 0 \Rightarrow \sin x = 0$$

$$\Rightarrow x = n\pi, \quad n = 0, \pm 1, \pm 2, \ldots$$

Trigonometric Function

But for $x = n\pi, \quad \cos x \neq 0$

$\therefore \sinh y = 0 \Rightarrow y = 0$

$\therefore z = x + iy = n\pi$

Thus $\cot z$ & $\operatorname{cosec} z$ are analytic everywhere except at the points where

$$z = n\pi, \quad n = 0, \pm 1, \pm 2, \ldots$$

Trigonometric Function

Q 14. (Page-109) Prove that

$$\overline{\sin(iz)} = \sin(i\bar{z}) \Leftrightarrow z = n\pi i, \quad n = 0, \pm 1, \pm 2, \ldots \ldots \ldots$$

Q16. (Page-109) Show that the roots of the

equation $\cos z = 2$ are

$$z = 2n\pi + i\cosh^{-1} 2, \quad n = 0, \pm 1, \pm 2, \ldots \ldots \ldots$$

Then express the same in the form

$$z = 2n\pi \pm i\ln\left(2 + \sqrt{3}\right), \quad n = 0, \pm 1, \pm 2, \ldots \ldots \ldots$$

Trigonometric Function

$$\cos z = \cos x \cosh y - i \sin x \sinh y = 2$$

$$\Rightarrow \cos x \cosh y = 2, \qquad \sin x \sinh y = 0$$

$$\sin x \sinh y = 0 \Rightarrow \sin x = 0 \text{ or } \sinh y = 0$$

$$\text{For } \sinh y = 0, \text{we have } \quad y = 0$$

$$\text{Then } \cos x \cosh y = 2 \Rightarrow \cos x = 2 \text{ (not possible)}$$

So, we must have

$$\sin x = 0 \Rightarrow x = n\pi, \qquad n = 0, \pm 1, \pm 2, \dots \dots$$

Trigonometric Function

$$\cos x \cosh y = 2, \Rightarrow (-1)^n \cosh y = 2$$

$$\Rightarrow \cosh y = 2(-1)^n \Rightarrow n \text{ must be even}$$

$$\text{since} \quad \cosh y > 0$$

$$\Rightarrow y = \cosh^{-1} 2 \Rightarrow z = 2n\pi + i \cosh^{-1} 2$$

$$\text{where} \;\; n = 0, \pm 1, \pm 2, \dots \dots$$

$$\text{Let} \quad t = \cosh^{-1} 2 \Rightarrow \cosh t = 2 \Rightarrow \frac{e^t + e^{-t}}{2} = 2$$

Trigonometric Function

$$e^{2t} - 4e^{t} + 1 = 0$$

$$\Rightarrow e^{t} = 2 \pm \sqrt{3} \Rightarrow t = \ln\left(2 \pm \sqrt{3}\right)$$

But

$$\ln\left(2 - \sqrt{3}\right) = \ln\left(\frac{\left(2-\sqrt{3}\right)}{\left(2+\sqrt{3}\right)}\left(2 + \sqrt{3}\right)\right) = \ln\left(\frac{1}{\left(2+\sqrt{3}\right)}\right)$$

$$t = \ln\left(2 \pm \sqrt{3}\right) \Rightarrow t = \pm \ln\left(2 + \sqrt{3}\right)$$

$$z = 2n\pi \pm \ln\left(2 + \sqrt{3}\right) i, \qquad n = 0, \pm 1, \pm 2, \dots \dots$$

Hyperbolic Function

Definition:

$$\sinh z = \frac{e^{z} - e^{-z}}{2},$$

$$\cosh z = \frac{e^{z} + e^{-z}}{2}.$$

Hyperbolic Function

(1). $\because e^{z}$ & e^{-z} are analytic everywhere

$\Rightarrow \sinh z$ & $\cosh z$ are analytic everywhere.

Hyperbolic Function

$$(2).\ \frac{d}{dz}[\sinh z] = \frac{d}{dz}\left[\frac{e^{z}-e^{-z}}{2}\right]$$

$$= \frac{e^{z}+e^{-z}}{2} = \cosh z$$

Similarly, $\dfrac{d}{dz}[\cosh z] = \sinh z$

Hyperbolic Function

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$$(3).\ \sinh(-z) = -\sinh z$$

$$\cosh(-z) = \cosh z$$

$$\cosh^2 z - \sinh^2 z = 1$$

Hyperbolic Function

$$(4).\cos z = \cosh(iz),$$

$$\because \quad \cosh z = \frac{e^{z} + e^{-z}}{2}$$

$$\Rightarrow \cosh(iz) = \frac{e^{iz} + e^{-iz}}{2} = \cos z$$

Hyperbolic Function

$$(5). \quad \cos(iz) = \cosh z$$

□ $\cos z = \cosh(iz)$

Þ $\cos(iz) = \cosh(i^2 z)$

$$= \cosh(-z) = \cosh z$$

Hyperbolic Function

(6). $$\sin z = -i \sinh(iz)$$

(7). $$\sin(iz) = -i \sinh(-z)$$

$$= i \sinh z$$

Hyperbolic Function

(8). $\sinh(z_1 + z_2)$

$$= \sinh z_1 . \cosh z_2 + \cosh z_1 . \sinh z_2$$

(9). $\cosh(z_1 + z_2)$

$$= \cosh z_1 . \cosh z_2 + \sinh z_1 . \sinh z_2$$

Hyperbolic Function

$$(10).\sinh z$$

$$=\sinh x.\cos y+i\cosh x.\sin y$$

Soln:

$$\because \quad \sin(iz)=i\sinh z$$

Hyperbolic Function

$$\Rightarrow \quad \sinh z = -i\sin(iz)$$

$$= -i\sin(ix - y)$$

$$= -i[\sin(ix)\cos y$$

$$-\cos(ix)\,\sin y]$$

Hyperbolic Function

$$= -i[i\sinh x\cos y - \cosh x\sin y]$$

$$\Rightarrow \sinh z = \sinh x\cos y + i\cosh x\sin y$$

Hyperbolic Function

Excercise :

$$|\sinh z|^2 = \sinh^2 x + \sin^2 y$$

Hyperbolic Function

Similarly

$$(a) \quad \cosh z = \cosh x . \cos y$$

$$+ i \sinh x . \sin y$$

Use $\cosh z = \cos(i z) = \cos(ix - y)$

$$(b) \quad |\cosh z|^2 = \sinh^2 x + \cos^2 y$$

Hyperbolic Function

(11). Analyticity of $\tanh z$ & $\sec hz$:

$$\because \quad \tanh z = \frac{\sinh z}{\cosh z},$$

$$\sec hz = \frac{1}{\cosh z}.$$

Hyperbolic Function

$\Rightarrow \tanh z$ & $\operatorname{sec}hz$ are analytic everywhere except at the points where

$$\cosh z = 0.$$

Hyperbolic Function

Now $\cosh z = 0$

$$\Rightarrow \cos(i\,z) = \cos(ix - y) = 0$$

$$\Rightarrow \cos\,(i\,x).\cos(y) + \sin\,(ix).\sin(y) = 0$$

$$\Rightarrow \cosh x\,.\cos y + i\sinh x.\sin y = 0$$

Hyperbolic Function

$$\Rightarrow \cosh x.\cos y = 0,$$

and

$$\sinh x.\sin y = 0.$$

$$\because \cosh x \neq 0 \Rightarrow \cos y = 0$$

$$\Rightarrow y = (2n+1)\frac{\pi}{2}, n = 0, \pm 1, \pm 2, \ldots$$

Hyperbolic Function

For $y=(2n+1)\dfrac{\pi}{2}, \sin y \neq 0$

$$\therefore \sinh x = 0 \Rightarrow x = 0$$

$$\therefore z = x + iy$$

$$= (2n+1)\frac{i\pi}{2},$$

$$n = 0, \pm 1, \pm 2, \ldots$$

Hyperbolic Function

$\Rightarrow \tanh z$ & $\operatorname{sec} hz$ are analytic everywhere except at

$$z = (2n+1)\frac{i\pi}{2},\ n = 0, \pm 1, \pm 2, \ldots.$$

Hyperbolic Function

Exercise:

$\coth z$ and $\operatorname{cosech} z$ are analytic everywhere except at $z = n\pi i$,

$$n = 0, \pm 1, \pm 2, \ldots\ldots$$

Hyperbolic Function

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Q. Show that :

$$(i)\ |\sinh(\operatorname{Im} z)| \le |\sin z| \le \cosh(\operatorname{Im} z)$$

$$(ii)\ |\sinh(\operatorname{Im} z)| \le |\cos z| \le \cosh(\operatorname{Im} z)$$

Sol: $(i) \because \sin z = \sin x.\cosh y$

$$+ i \cos x.\sin h\, y$$

Hyperbolic Function

$$\Rightarrow |\sin z|^2 = \sin^2 x.\cosh^2 y$$

$$+\cos^2 x.\sinh^2 y$$

$$= \sin^2 x\left(1+\sinh^2 y\right)$$

$$+\left(1-\sin^2 x\right).\sinh^2 y$$

$$= \sin^2 x+\sinh^2 y$$

Hyperbolic Function

$$\Rightarrow \sinh^2 y \le |\sin z|^2 = \sin^2 x + \sinh^2 y$$

$$\le 1 + \sinh^2 y$$

$$= \cosh^2 y$$

$$\Rightarrow |\sinh y| \le |\sin z| \le \cosh y$$

Hyperbolic Function

$$(ii)\cos z = \cos x.\cosh y$$

$$- i\sin x.\sinh y$$

$$\Rightarrow |\cos z|^2 = \cos^2 x.\cosh^2 y$$

$$+ \sin^2 x.\sinh^2 y$$

Hyperbolic Function

$$\Rightarrow |\cos z|^2 = \cos^2 x\left(1+\sinh^2 y\right) + \left(1-\cos^2 x\right).\sinh^2 y$$

$$= \cos^2 x + \sinh^2 y$$

Hyperbolic Function

$$\Rightarrow \sin h^2 y \le |\cos z|^2 = \cos^2 x + \sin h^2 y$$

$$\le 1 + \sin h^2 y$$

$$= \cosh^2 y$$

$$\Rightarrow |\sin h\, y| \le |\cos z| \le \cosh y$$

Hyperbolic Function

Q 14. (Page-112) Why is the function $\sinh(e^z)$ entire? Write its real and imaginary parts and discuss why they are harmonic functions everywhere?

Q16. (Page-112) Prove that

$$\sinh 2z = 2\sinh z \cosh z$$

Hyperbolic Function

Q. Find all values of z such that

$$\sinh z = \frac{1}{2} + i\frac{\sqrt{3}}{2}.$$

$$\sinh x \cos y = \frac{1}{2} \quad \& \quad \cosh x \sin y = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \sinh x = \frac{1}{2\cos y} \quad \& \quad \cosh x = \frac{\sqrt{3}}{2\sin y}$$

$$\Rightarrow \cosh^2 x - \sinh^2 x = 1 \Rightarrow \frac{3}{4\sin^2 y} - \frac{1}{4\cos^2 y} = 1$$

Hyperbolic Function

$$\Rightarrow 3\cos^2 y - \sin^2 y = 4\sin^2 y\cos^2 y$$

$$\Rightarrow 4\sin^4 y - 8\sin^2 y + 3 = 0$$

$$\Rightarrow \sin y = \pm\frac{1}{\sqrt{2}}, \pm\sqrt{\frac{3}{2}} \text{ but } \sin y \neq \pm\sqrt{\frac{3}{2}}$$

$$\sin y = -\frac{1}{\sqrt{2}} \Rightarrow \cosh x = -\sqrt{\frac{3}{2}} \text{ (not possible)}$$

$$\sin y = \frac{1}{\sqrt{2}} \Rightarrow y = n\pi + (-1)^n\frac{\pi}{4}, n = 0, \pm 1, \pm 2, \ldots..$$

Hyperbolic Function

When n is even,

$$\sin y = \frac{1}{\sqrt{2}} = \cos y \Rightarrow \cosh x = \sqrt{\frac{3}{2}}, \sinh x = \frac{1}{\sqrt{2}}$$

$$\Rightarrow e^x = \cosh x + \sinh x = \frac{\sqrt{3}+1}{\sqrt{2}}$$

$$\Rightarrow x = \ln\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)$$

Hyperbolic Function

When n is odd,

$$\sin y = \frac{1}{\sqrt{2}}, \cos y = -\frac{1}{\sqrt{2}} \Rightarrow \cosh x = \sqrt{\frac{3}{2}}, \sinh x = -\frac{1}{\sqrt{2}}$$

$$\Rightarrow e^x = \cosh x + \sinh x = \frac{\sqrt{3} - 1}{\sqrt{2}}$$

$$\Rightarrow x = \ln\left(\frac{\sqrt{3} - 1}{\sqrt{2}}\right)$$

Hyperbolic Function

$$\Rightarrow z = \begin{cases} \ln\left(\dfrac{\sqrt{3}+1}{\sqrt{2}}\right) + \left(n\pi + \dfrac{\pi}{4}\right) i & \text{when } n \text{ is even} \\ \ln\left(\dfrac{\sqrt{3}-1}{\sqrt{2}}\right) + \left(n\pi - \dfrac{\pi}{4}\right) i & \text{when } n \text{ is odd} \end{cases}$$

Hyperbolic Function

Q. Find all values of z, such that

$$\sqrt{2}\sin z = \cosh\beta + i\sinh\beta\,, \qquad \beta \in \mathbb{R}.$$

Q. Show that $\tan z = \dfrac{\sin 2x + i\sinh 2y}{\cos 2x + \cosh 2y}$.

The Logarithmic Function

The natural logarithm of $z = x + iy$

is denoted by $\log z$,

i.e. $w = \log z$,

and $\log z$ is defined for $z \neq 0$

by the relation

$$e^{w} = z \quad \ldots\ldots\ldots\ldots\ldots(i)$$

The Logarithmic Function

i.e. if $e^w = z$, then we write

$$w = \log z$$

Let $w = u + iv$,

$$z = x + iy = r\cos\Theta + i\,r\sin\Theta$$

$= r\,e^{i\Theta}$, where

$$-\pi < \Theta \le \pi,\ \Theta = Arg\ z$$

The Logarithmic Function

$$\text{Then}(i) \Rightarrow e^{u+iv} = r\, e^{i\Theta}$$

$$\Rightarrow e^{u}.e^{iv} = r\, e^{i\Theta}$$

$$\Rightarrow e^{u} = r = |z|,$$

$$v = \Theta + 2n\pi,$$

$$n = 0, \pm 1, \pm 2, \ldots\ldots$$

The Logarithmic Function

$$\Rightarrow u = \ln r = \ln|z|,$$

$$v = \Theta + 2n\pi$$

$$\therefore w = \log z = u + iv$$

$$= \ln|z| + i(\Theta + 2n\pi)$$

The Logarithmic Function

Since $Arg\, z = \Theta, -\pi < \Theta \leq \pi$

and $\quad \arg z = \Theta + 2n\pi,$

n is any integer

$$\therefore \log z = \ln|z| + i \arg z, \qquad z \neq 0$$

The Logarithmic Function

When $n = 0$, then $\arg\ z\ = Arg\ z$

When $n = 0$, then the value of $\log\ z$ is called the principal value of $\log\ z$ and is denoted by $Log\ z$, i.e.

$$Log\ z\ =\ \ln|z| + i\ Arg\ z,\ z \neq 0.$$

The Logarithmic Function

$$\therefore \log z = \ln|z| + i \arg z$$

$$= \ln|z| + i(\Theta + 2n\pi)$$

$$= (\ln|z| + i\Theta) + i\,2n\pi$$

$$\Rightarrow \log z = Log\, z + i\,2n\pi,$$

$$n = 0, \pm 1, \pm 2, \ldots$$

The Logarithmic Function

Remark1:

Since $\log\ z = \ln|z| + i \arg z$

$$= \ln|z| + i(\Theta + 2n\pi),$$

$$n = 0, \pm 1, \pm 2, \ldots\ldots$$

Þ $\log z$ is a multivalued function.

The Logarithmic Function

Remark 2:

Since $Log\, z = \ln|z| + i\,\Theta,$

$$\Theta = Arg\, z$$

$\Rightarrow Log\, z$ is a single-valued function.

The Logarithmic Function

Remark 3 :

$$\ln|z| = \frac{1}{2}\ln\left(x^2 + y^2\right)$$

is continuous everywhere except at $(0,0)$.

The Logarithmic Function

Remark 4 : Let α be any real number,

and consider

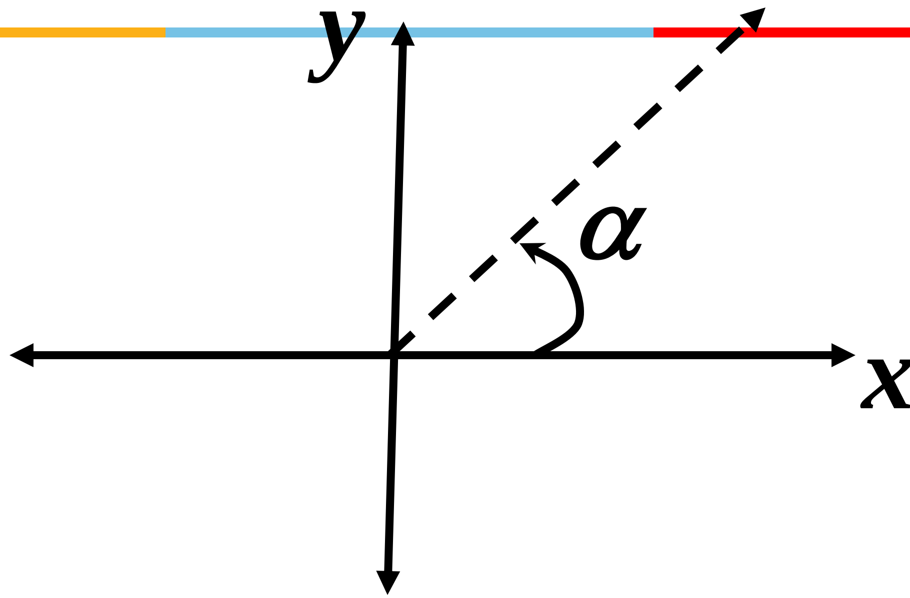
$$f(z) = \log z = \ln|z| + i\theta$$

$$= \ln r + i\theta,$$

$$(r > 0, \alpha < \theta < \alpha + 2\pi)$$

$$\Rightarrow u(r,\theta) = \ln r, \ \ v(r,\theta) = \theta$$

The Logarithmic Function



Then $\log z$ is single - valued and continuous in the domain

$$D = \{z : |z| > 0, \alpha < \theta < \alpha + 2\pi\}$$

The Logarithmic Function

Remark 5: The function $\log\ z$ is NOT continuous on the ray $\theta = \alpha$ as $\arg z$ is NOT continuous on the ray $\theta = \alpha$.

For if z is a point on the ray $\theta = \alpha$ then there are points arbitrary close to z at which the values of v are nearer to α, and also there are points such that the values of v are nearer to $\alpha + 2\pi$.

$$\Rightarrow \lim_{z \to \alpha} \arg z \text{ does not exist.}$$

The Logarithmic Function

Remark 6 :

$(i)\ \log z = \ln r + i\theta$ is analytic in domain

$$D_1 = \{z : |z| = r > 0, \alpha < \theta (= \arg z) < \alpha + 2\pi\}$$

$(ii)\ Log\ z = \ln r + i\Theta$ is analytic in the domain

$$D_2 = \{z : |z| = r > 0, -\pi < \Theta (= Arg\ z) < \pi\}$$

The Logarithmic Function

$$\text{As}, u(r,\theta) = \ln r, \;\; v(r,\theta) = \theta$$

$$\Rightarrow u_r = \frac{1}{r}, u_\theta = 0$$

$$v_r = 0, \; v_\theta = 1$$

$\Rightarrow$ CR-equations in polar form

$$r\,u_r = v_\theta, \;\; u_\theta = -r\,v_r$$

are satisfied and first-order partial derivatives are continuous.

The Logarithmic Function

$$\Rightarrow f'(z) = \frac{d}{dz}(\log z) = e^{-i\theta}(u_r + i v_r)$$

$$= \frac{1}{r e^{i\theta}} = \frac{1}{z} \text{ in } \quad D_1$$

In particular, when $\alpha = -\pi$

$$\frac{d}{dz}(Log\, z) = \frac{1}{z} \text{ in } \; D_2.$$

The Logarithmic Function

Remark : 7

$Log\ z$ is analytic on the whole complex plane except at $(0,0)$ and on the ray $\theta = -\pi$, i.e. on negative real axis.

i.e. singularties of $Log\ z$ are given by

$$\operatorname{Re} z \leq 0 \text{ and } \operatorname{Im} z = 0.$$

The Logarithmic Function

Definition:

A branch of a multiple-valued function $f(z)$ defined on a set S is any single valued function $F(z)$ that is analytic in some domain $D \subseteq S$ such that for all $z \in D$, $F(z)$ is one of the values of $f(z)$.

The Logarithmic Function

Ex. For each fixed α,

$$\log z = \ln|z| + i\theta,$$

$$\left(|z| > 0, \alpha < \theta < \alpha + 2\pi\right)$$

is a branch of

$$\log z = \ln|z| + i \arg z$$

The Logarithmic Function

$$Log\, z = \ln|z| + i\,\Theta,$$

$$\left(|z| > 0, -\pi < \Theta < \pi\right)$$

is called the principal branch.

The Logarithmic Function

Q.9(a) p. 97 : Show that the function $Log\left(z-i\right)$ is analytic everywhere except on the half line $y=1\ \left(x\le 0\right)$.

Solution : We have $f\left(z\right)=Log\left(z-i\right)$

singularity of $f\left(z\right)$ is given by

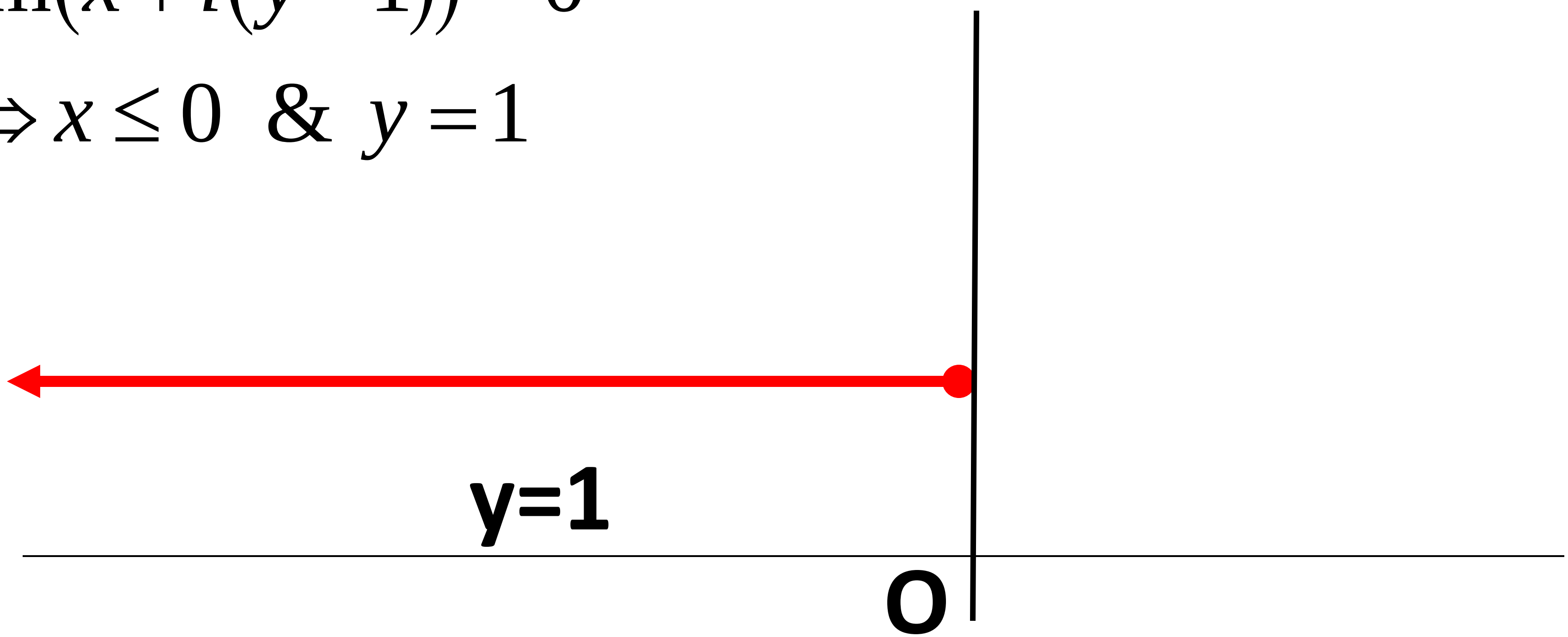
The Logarithmic Function

$$\operatorname{Re}(z-i)\le 0 \;\&\; \operatorname{Im}(z-i)=0$$

$$\Rightarrow \operatorname{Re}(x+i(y-1))\le 0\,\&$$

$$\operatorname{Im}(x+i(y-1))=0$$

$$\Rightarrow x\le 0 \;\;\&\;\; y=1$$



THANK YOU