MATH F113 (Probability and Statistics)

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In Lecture 13

Expectation
Mean, Variance and Moment Generating Function
Uniform Distribution

Exercise 17/4.2/pp. 141

Let X denote the length in minutes of a long distance telephone conversation. The density for X is given by

$$f(x) = \frac{1}{10}e^{-\frac{x}{10}}; \qquad x > 0$$

(a) Find the moment generating function $m_x(t)$.

Exercise (Cont...)

$$m_x(t) = E\left[e^{tx}\right] = \int_0^\infty e^{tx}\frac{1}{10}e^{-\frac{x}{10}}dx$$

$$= \int_0^\infty \frac{1}{10}e^{-\left(\frac{1}{10}-t\right)x}dx = (1-10t)^{-1}, \quad t < \frac{1}{10}$$

(b) Use $m_x(t)$ to find the average length of such a call

$$E(X) = \left[\frac{d}{dx}(m_x(t))\right]_{t=0} = 10 \quad minutes$$

(c) Find the variance and standard deviation of X.

$$E(X^2) = \left[\frac{d^2}{dx^2}(m_x(t))\right]_{t=0} = 200$$

Hence,

$$\sigma^2 = 200 - 10^2 = 100, \qquad \sigma = 10 \quad minutes$$

Exercise Let X be a uniformly distributed over $(0, 1)$. Calculate $E\left[X^3\right]$

Solution Let $Y = X^3$ we calculate the distribution Y as follows. For $0 \leq a \leq 1$

$F_Y(a) = P\,[Y \leq a] = P\left[X^3 \leq a\right]$

$$= P\left[X \leq a^{\frac{1}{3}}\right] = a^{\frac{1}{3}}$$

Since X is a uniformly distributed over $(0, 1)$

Now, differentiating $F_Y(a)$, we shall get density of Y.

$$f_y(a) = \frac{1}{3}a^{-\frac{2}{3}} \quad 0 \leq a \leq 1$$

Hence

$$E\left[X^3\right] = E\left[Y\right] = \int\limits_{-\infty}^{\infty} a\frac{1}{3}a^{-\frac{2}{3}}da = \frac{1}{4}$$

Exercise: **For the uniform random variable X on the interval $(1, 2)$ find the probability that $0 < X < 3/2$ given that $5/4 < X < 9/4$.**

Solution:

$$P[0 < X < 3/2|5/4 < X < 9/4]$$

$$= \frac{P[x \in (0, 3/2) \cap (5/4, 9/4)]}{P[x \in (5/4, 9/4)]}$$

$$= \frac{P[5/4 < X < 3/2]}{P[5/4 < X < 9/4]}$$

$$= \frac{P[5/4 < X < 3/2]}{P[5/4 < X < 2]}$$

$$= \frac{(1/4)}{(3/4)} = 1/3$$

Exercise 22 **A random variable X with density**

$$f(x) = \frac{1}{\pi}\frac{a}{a^2 + (x-b)^2};$$

$$-\infty < x < \infty \qquad -\infty < b < \infty \qquad a > 0$$

A random variable X with density is said to have a Cauchy distribution with parameters a and b. This distribution is Interestingly in that it provides an example of a continuous random variable whose mean does not exist. Let $a = 1$, $b = 0$ to obtain a special Case of the Cauchy distribution with density

$$f(x) = \frac{1}{\pi}\frac{1}{1+x^2} \qquad -\infty < x < \infty$$

Show that $\int\limits_{-\infty}^{\infty} |x| f(x) dx$ **does not exist**

Solution

$$\int\limits_{-\infty}^{\infty} |x| f(x) dx = \int\limits_{-\infty}^{\infty} |x| \frac{1}{\pi}\frac{1}{1+x^2} dx$$

$$\int_{-\infty}^{\infty} |x| f(x) dx = \int_{-\infty}^{0} \frac{1}{\pi} \frac{-x}{1+x^2} dx + \int_{0}^{\infty} \frac{1}{\pi} \frac{x}{1+x^2} dx$$

Multiply and divide by 2, we get

$$= -\frac{1}{2\pi} \ln|1+x^2|\}_{-\infty}^{0} + \frac{1}{2\pi} \ln|1+x^2|\}_{0}^{\infty}$$

which does not exist, as $\ln(\infty) \to \infty$

Exercise 24 Assume that the increase in demand for electric power in millions of kilowatt hours over the next 2 years in particular area is a random variable whose density is given by

$$f(x) = \begin{cases} \frac{1}{64}x^3 & 0 < x < 4 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Verify that this is a valid density
(b) Find the expression for the cumulative distribution function F for X, and use it to find the probability that the demand will be at most 2 million kilowatt hours

(c) If the area only has the capacity to generate an additional 3 million kilowatt hours, what is the probability that demand will exceed supply ?

(d) Find the average increase in demand

Solution

(a)(i)

$$f(x) \geq 0 \quad \textbf{for all} \quad x > 0$$

(a)(ii)

$$\int_0^4 f(x)dx = 1$$

(b)

$$F(x) = \int_0^x \frac{1}{64}x^3 dx = \frac{x^4}{256} \quad 0 < x < 4$$

Therefore, $P\left[x \leq 2\right] = F(2) = \frac{16}{256}$

(c)

$$P\left[X \geq 3\right] = 1 - P\left[X \leq 3\right]$$

$$1 - F(3) = 0.6836$$

(d)

$$E\left[X\right] = \int_{0}^{4} \frac{1}{64} x^4 dx = 3.2$$

Gamma function

$$\Gamma(\alpha) = \int_0^\infty z^{\alpha-1} e^{-z} dz \quad \alpha > 0$$

Theorem: Properties of Gamma function

$$\Gamma(1) = 1$$

$$\Gamma(\alpha) = (\alpha - 1)\ \Gamma(\alpha - 1) \qquad \textbf{for all } \alpha > 1$$

By definition of Gamma function, we have

$$\Gamma(1) = \int_0^\infty z^{1-1} e^{-z} dz \qquad = 1$$

By integration by parts, we have

$$\Gamma(\alpha) = \int_0^\infty z^{\alpha-1} e^{-z} dz \qquad \alpha > 0$$

Gamma Function (Cont...)

$$= -e^{-z} z^{\alpha-1}\Big|_0^{\alpha} + (\alpha-1)\int_0^{\infty} z^{(\alpha-1)-1} e^{-z}\,dz$$

$$= (\alpha-1)\Gamma(\alpha-1)$$

Hint: by repeated use of L hospital rule, we shall have

$$\lim_{z\to\infty} \frac{-z^{\alpha-1}}{e^z} = \lim_{z\to\infty} \frac{-(\alpha-1)z^{\alpha-2}}{e^z}$$

Case 1: If α is integer

$$= -(\alpha - 1)! \lim_{z\to\infty} \frac{1}{e^z} \to 0$$

Case 2: If α is not integer

$$= \lim_{z\to\infty} \frac{-(\alpha-1)(\alpha-2)...(\alpha-k+1)}{z^{\alpha-k}e^z}$$

$$; 0 < (\alpha - k) < 1$$

$$= 0$$

Further $\Gamma\alpha = (\alpha - 1)!$

$$\Gamma\alpha = (\alpha - 1)\Gamma(\alpha - 1)$$

$$\Gamma\alpha = (\alpha - 1).(\alpha - 2)...3.2.1.\Gamma 1$$

Thus, Gamma function is a generalization of the Factorial notation

$$\Gamma(\frac{1}{2}) = \int\limits_0^\infty z^{-1/2} e^{-z} dz = \sqrt{\pi}$$