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BITS Pilani
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MATH F112 (Mathematics-II)

Complex Analysis

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Lecture 36
Integrals and Series Expansion

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Morera's Theorem

If a function f is continuous in a domain D and if

$$\int_C f(z)dz = 0,$$

for every closed contour C lying in D, then f is analytic in D.

Cauchy's Inequality

Theorem: If a function $f(z) = u(x, y) + i\, v(x, y)$ is analytic inside and on a positively oriented circle C_R $|z - z_0| = R$. If $|f(z)| \leq M_R \;\; \forall\, z \in C_R$, then

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!\, M_R}{R^n}, \quad n = 1, 2, \ldots.$$

This is also known as *Cauchy's inequality*.

Liouville's Theorem

A function f is entire and bounded in the complex plane then f is constant throughout the plane.

Q. Is it possible to map the entire complex plane onto a unit disc under an entire function ?

Ans. No (why ??)

Q. Prove that $\sin z$ is unbounded in $\mathbb{C}$.

Liouville's Theorem

Q1. (P-178) If a function $f(z) = u(x, y) + i\, v(x, y)$ is entire and $u(x, y) \leq u_0$ in the whole xy plane then show that $u(x, y)$ is constant throughout the plane. Also $f(z)$ is constant throughout the complex plane.

Liouville's Theorem

Sol.: If $f(z) = u(x, y) + i\, v(x, y)$ is entire then

$\phi(z) = e^{f(z)}$ is also entire and

$u(x, y) \leq u_0 \Rightarrow |\phi(z)| = e^u$ is bounded

$\Rightarrow \phi(z)$ is constant throughout the complex plane.

$\Rightarrow u(x, y)$ is constant throughout the xy plane.

$\Rightarrow v(x, y)$ is constant throughout the xy plane. (? ?)

$\Rightarrow f(z)$ is constant throughout the complex plane.

Liouville's Theorem

Q. If a function $f(z) = u(x, y) + i\, v(x, y)$ is entire and $f(z+1) = f(z+i) = f(z)$ in the entire complex plane then show that $f(z)$ is constant throughout the complex plane.

Hint: For an analytic function $f(z)$, the function $|f(z)|$ is continuous. Also, if z_1, z_2 are the periodicity of $f(z)$ then $f(z + m\, z_1 + n\, z_2) = f(z)$, for $m, n \in \mathbb{Z}$.

Fundamental Theorem of Algebra

Any polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$$

$$(a_n \neq 0) \quad \text{of degree } n \, (n \geq 1)$$

has at least one zero. That is, there exists at least one point z_0 such that $P(z_0) = 0.$

Fundamental Theorem of Algebra

Note. It follows that P can be factored into n (not necessarily distinct) linear terms:

$$P(z) = a_n(z - z_1)(z - z_2)\cdots(z - z_{n-1})(z - z_n),$$

Where the zeros of P are $z_1, z_2, \ldots, z_n$.

Problems

Ex. Evaluate $\int_C \frac{dz}{z(z^2+1)}$ for all possible choices of the contour C that does not pass through any of the points 0, $\pm\, i$.

Problems

Solution:

Case 1. Let C does not enclose 0, $\pm i$.

Then

$$I = \int_C \frac{dz}{z(z^2+1)} = 0 \text{ by C-G Theorem.}$$

Problems

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Case 2(a). Let C encloses only 0. Then

$$I = \int_C \frac{dz}{z(z^2+1)}$$

$$= \int_C \frac{f(z)dz}{z-0}, f(z) = \frac{1}{(z^2+1)}$$

$$= 2\pi i\, f(0) = 2\pi i$$

Problems

Exercise:

Case 2(b). Let C encloses only i.

Ans: $I = -\pi i$

Case 2(c). Let C encloses only $-i$.

Ans: $I = -\pi i$

Problems

Case 3 (a). Let C encloses only 0 & $-i$, then

$$I = \int_{C_0} \frac{dz}{z(z+i)(z-i)} + \int_{C_{-i}} \frac{dz}{z(z+i)(z-i)}$$

where C_0 and C_{-i} are sufficiently small circles around 0 and $-i$ resp.

Problems

$$= \int_{C_0} \frac{\frac{1}{(z+i)(z-i)}}{z} dz + \int_{C_{-i}} \frac{\frac{1}{z(z-i)}}{(z+i)} dz$$

$$= (2\pi i)\left(-\frac{1}{i^2}\right) + (2\pi i)\left(\frac{1}{-i(-2i)}\right)$$

$$= \pi i$$

Problems

Case 3 (b). Let C encloses only 0 & i, then

$$I = \int_{C_0} \frac{dz}{z(z+i)(z-i)} + \int_{C_i} \frac{dz}{z(z+i)(z-i)}$$

where C_0 and C_i are sufficiently small circles around 0 and i resp.

Problems

$$I = \int_{C_0} \frac{\frac{1}{(z+i)(z-i)}}{z} dz + \int_{C_i} \frac{\frac{1}{z(z+i)}}{(z-i)} dz$$

$$= 2\pi i + (2\pi i)\left(\frac{1}{i.2i}\right) = \pi i$$

Problems

Case 3 (c). Let C encloses only -i & +i. Then

$$I = \int_{C_i} \frac{\frac{1}{z(z+i)}}{z-i} dz + \int_{C_{-i}} \frac{\frac{1}{z(z-i)}}{(z+i)} dz$$

$$= (2\pi i)\left(\frac{1}{i.2i}\right) + (2\pi i)\left(\frac{1}{-i.(-2i)}\right) = -2\pi i$$

Problems

Case 3 (d). Let C encloses all of the points 0, -i, +i. Then

$$I = \int_{C_0} \frac{\frac{1}{z^2+1}}{z} dz + \int_{C_i} \frac{\frac{1}{z(z+i)}}{z-i} dz$$

$$+ \int_{C_{-i}} \frac{\frac{1}{z(z-i)}}{(z+i)} dz$$

$$= 2\pi i - \pi i - \pi i = 0$$

Taylor's Series

Let $f(z)$ be analytic throughout a disc $|z - z_0| < R_0$ centered at z_0 with radius R_0 then $f(z)$ has the power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad (|z - z_0| < R_0)$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!} \quad (n = 0, 1, 2 \ldots..)$$

Maclaurin's Series

Taylor Series about the point $z_0 = 0$ is called Maclaurin series, i. e.

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n, \quad (|z| < R_0)$$

Maclaurin's Series

Examples:

1. $$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad (|z| < \infty)$$

2. $$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \quad (|z| < \infty)$$

3. $$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \quad (|z| < \infty)$$

Maclaurin's Series

4. $$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!},$$

$$(|z| < \infty)$$

5. $$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!},$$

$$(|z| < \infty)$$

Maclaurin's Series

6. $$\frac{1}{1-z}=\sum_{n=0}^{\infty} z^n, \qquad (|z|<1)$$

7. $$\frac{1}{1+z}=\sum_{n=0}^{\infty} (-1)^n z^n, \quad (|z|<1)$$

Laurent's Theorem

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Suppose that a function $f(z)$ is analytic throughout an annular domain

$$R_1 < |z - z_0| < R_2,$$

centered at z_0 and let C denote any positively oriented simple closed contour around z_0 and lying in that domain.

Laurent's Theorem

Then, at each point in the domain,

$f(z)$ has the series representation

$$f(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

$$\left(R_1 < |z-z_0| < R_2\right)$$

where

Laurent's Theorem

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z - z_0)^{n+1}} \quad (n = 0, 1, 2, \ldots)$$

and

$$b_n = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z - z_0)^{-n+1}} \quad (n = 1, 2, \ldots)$$

Laurent's Theorem

Ex. Find the Laurent series representation

of $f(z) = \dfrac{z}{(z-1)(z-3)}$ when

$(a)\ \ D_1 : 0 < |z| < 1,$

$(b)\ \ D_2 : 1 < |z| < 3,$

$(c)\ \ D_3 : 3 < |z| < \infty,$

Laurent's Theorem

We have

$$f(z) = \frac{z}{(z-1)(z-3)}$$

$$= -\frac{1}{2(z-1)} + \frac{3}{2(z-3)}$$

Laurent's Theorem

(a) Consider the domain

$$D_1 : 0 < |z| < 1.$$

Then $f(z)$ is analytic in D_1.

$$f(z) = -\frac{1}{2(z-1)} + \frac{3}{2(z-3)}$$

Laurent's Theorem

$$= \frac{1}{2(1-z)} - \frac{3}{2\times 3\left(1-\frac{z}{3}\right)}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} z^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n$$

Laurent's Theorem

$$\Rightarrow f(z) = \frac{1}{2}\sum_{n=0}^{\infty} z^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(1 - \frac{1}{3^n}\right) z^n$$

THANK YOU