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BITS Pilani
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MATH F112 (Mathematics-II)

Complex Analysis

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Lecture 26-27
Analyticity and Harmonic Functions

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Analytic Function

A function $f(z)$ is said to be analytic at a point z_0 if

(i) $f(z)$ is differentiable at z_0, and

(ii) $f(z)$ is differentiable at every point in some neighbourhood of z_0.

Analytic Function

A function $f(z)$ is analytic in a domain D if $f(z)$ is differentiable at each point of the domain D.

Remark:

Analyticity implies differentiablity,

but differentiablity does not imply analyticity.

Analytic Function

$$\text{Ex}: f(z) = |z|^2$$

☝$f(z)$ is differentiable at origin and no where else.

☝But *f(z)* is not analytic at the origin as it is not differentiable in any neighborhood of origin.

Analytic Function

Theorem:

If $f'(z) = 0$ everywhere in a domain D,

then $f(z)$ is constant throughout in D.

Entire function:

A function $f(z)$ is said to be an Entire function

if $f(z)$ is analytic $\forall z \in \mathbb{C}$.

Analytic Function

Example: Every polynomial is an entire function.

Singular Point:

Let a function $f\,(z)$ is, not analytic at a point $z_{0,}$ but analytic at some point in every neighbourhood of z_0.

Then z_0 is called a singularity of $f(z)$.

Analytic Function

Examples

(1) $$f(z) = \frac{1}{z}$$

$\Rightarrow z = 0$ is a singularity of $f(z)$.

(2) $f(z) = |z|^2$

$\because f(z)$ is not analytic anywhere

$\Rightarrow f(z)$ has no singular point

Ex: Find the singular points of $f(z) = z|z|$ (if any).

Necessary Condition for Analyticity at $\boldsymbol{z_0}$:

Let $f(z) = u(x, y) + i\, v(x, y)$ be analytic at $z_0 = x_0 + i\, y_0$ then

- u_x, u_y, v_x, v_y exist in $N_\epsilon(z_0)$
- u_x, u_y, v_x, v_y satisfy C-R Equations in $N_\epsilon(z_0)$

Sufficient Condition for Analyticity at z_0:

Let $f(z) = u(x, y) + i\, v(x, y)$ be a function defined throughout $N_\epsilon(z_0)$ where $z_0 = x_0 + i\, y_0$ such that

- u_x, u_y, v_x, v_y exist and continuous in $N_\epsilon(z_0)$
- u_x, u_y, v_x, v_y satisfy C-R Equations in $N_\epsilon(z_0)$

then $f(z)$ is analytic at z_0.

Harmonic Functions

A real valued function $u(x, y)$ is said to be harmonic in a given domain D of x-y plane if

(i) first and second order partial derivatives of u exist & they are continuous in D,

(ii) u satisfies Laplace's eqution

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$

Example: $u(x, y) = 3x^2 y - y^3 + 2$ is harmonic in the complex plane.

Harmonic Functions

Theorem 1:

If $f(z) = u(x, y) + i\, v(x, y)$ is analytic in a domain D, then u & v are harmonic in D

Proof: Use C-R Equations for proof.

Remark: Is converse true ?

Harmonic Functions

Harmonic Conjugates :

Let u and v be two harmonic functions in a domain D and satisfies CR equations

$$u_x = v_y, \quad u_y = -v_x \quad .....(1)$$

through out in D .

Then v is said be Harmonic Conjugate of u.

Harmonic Functions

Remark1:

v is a harmonic conjugate of u

$\not\Rightarrow u$ is a harmonic conjugate of v.

For, if u is a harmonic conjugate of v,

then $v_x = u_y$ & $v_y = -u_x$, which is not same as (1)

Harmonic Functions

Remark 2 :

v is a harmonic conjugate of u

$\Rightarrow u$ is a harmonic conjugate of $-v$

$$\text{as } -v_x = u_y, -v_y = -u_x$$

$$i.e. u_x = v_y \ \& \ u_y = -v_x$$

$$\text{which is same as } (1)$$

Harmonic Functions

Theorem 2:

A function $f(z) = u(x,y) + i\,v(x,y)$

is analytic in a domain D iff

v is a harmonic conjugate of u.

$$Ex. \quad f(z) = z^2.$$

Problems

Ex. Find all the points where the function

$$f(x) = 2xy + i(x^2 - y^2) \text{ is analytic.}$$

Problems

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Let $f(z)$ be analytic in a domain D.

Prove that $f(z)$ must be constant in D

if

(a) $f(z)$ is real valued " z in D.

OR (b) $\overline{f(z)}$ is analytic in D.

OR (c) $|f(z)|$ is constant in D.

Problems

Solution:

Since $f(z)$ is analytic in a domain D.

$$\Rightarrow u_x = v_y, \quad u_y = -v_x \qquad (1)$$

and

$$f'(z) = u_x + i v_x \qquad (2)$$

(a) Given $f(z)$ is a real valued function $\forall \; z \in D$

Problems

$$\Rightarrow f(z) = u(x,y) + i\,v(x,y),$$

where

$$v(x,y) = 0 \qquad \forall\ (x,y) \in D.$$

$$\Rightarrow v_x = 0,\ v_y = 0$$

$$\because u_x = v_y,\quad u_y = -v_x$$

$$\Rightarrow u_x(x,y) = 0 = u_y(x,y)\ \forall (x,y) \in D$$

$$\therefore (2) \Rightarrow f'(z) = 0,\ \forall\ z \in D$$

$$\Rightarrow f(z) \equiv \text{constant}\ \forall\ z \in D.$$

Problems

$$(b) \because f(z) = u(x, y) + \mathrm{i}\, v(x, y)$$

$$\Rightarrow \overline{f(z)} = u(x, y) - i\, v(x, y)$$

$\because \overline{f(z)}$ is analytic in D

$\Rightarrow$ u & -v satisfy CR equations, viz.

$$u_x = -v_y, u_y = -(-v_x) = v_x \; \dots \; (3)$$

Problems

(1) and (3) $\Rightarrow u_x = v_y, u_x = -v_y$

$$\Rightarrow u_x = 0$$

Again $u_y = -v_x, u_y = v_x$

$$\Rightarrow v_x = 0$$

$$\therefore f\,'(z) = u_x + i\,v_x = 0 \;\; \forall\, z \in D$$

$\Rightarrow f(z) \equiv$ constant $\forall\, z \in D$

Problems

$(c)\ |f(z)| =$ constant in D

Let $|f(z)| = c$

If $c = 0$, then $f(z) = 0\ \forall\ z \in D$.

$\Rightarrow f(z) =$ constant $\forall\ z \in D$.

Problems

Assume $c \neq 0$. Then, $|f(z)| = c \neq 0$.

$$\Rightarrow |f(z)|^2 = c^2 \Rightarrow f(z).\overline{f(z)} = c^2$$

$$\Rightarrow \overline{f(z)} = \frac{c^2}{f(z)}$$

□ $f(z)$ is analytic in D

$\Rightarrow$ $\overline{f(z)}$ is analytic in D (as $f(z) \neq 0$ in D) (WHY)

$\therefore$ by case (b): $f(z)$ is constant in D

Problems

Ex. Consider the function

- $f(z) = u(x, y) + i\, v(x, y)$ in a domain D, where
- v is a harmonic conjugate of u and
- u is also a harmonic conjugate of v.

- Then show that $f(z)$ is constant throughout in D.

Problems

Q.10 Show that u is harmonic & find a harmonic conjugate v when

$(a)\; u(x,y) = 2x\,(1 - y)$

$$u_x = 2(1-y), u_{xx} = 0$$

$$u_y = -2x,\; u_{yy} = 0$$

$$\therefore \qquad u_{xx} + u_{yy} = 0$$

$\Rightarrow \quad u$ is harmonic.

Problems

$\because v$ is a harmonic conjugate of u

$\Rightarrow$ CR Equations are satisfied

i.e. $u_x = v_y, \; u_y = -v_x$

Then

$$v_y = u_x = 2(1-y)$$

$$\Rightarrow v = 2y - y^2 + \phi(x)$$

Problems

$$\Rightarrow v_x = \phi'(x) = -u_y = 2x$$

$$\Rightarrow \phi'(x) = 2x$$

$$\Rightarrow \phi(x) = x^2 + c$$

$$\therefore v = 2y - y^2 + x^2 + c$$

Problems

$$(b) \quad u(x,y) = \sinh x \, \sin y$$

$$\Rightarrow u_x = \cosh x \sin y,$$

$$u_{xx} = \sinh x \sin y,$$

$$u_y = \sinh x \cos y,$$

$$u_{yy} = -\sinh x \sin y$$

$$\therefore \quad u_{xx} + u_{yy} = 0$$

Problems

Let v be a harmonic conjugate of u

$$\therefore \quad u_x = v_y, \qquad u_y = -v_x$$

$$\therefore \; v_y = \cosh x \sin y$$

$$\Rightarrow v = -\cosh x \cos y + \varphi(x)$$

$$\Rightarrow v_x = -\sinh x \cos y + \varphi'(x)$$

Problems

$$\Rightarrow v_x = -\sinh x \cos y + \phi'(x)$$

But $v_x = -u_y = -\sinh x \cos y$

$$\Rightarrow \phi'(x) = 0$$

$$\Rightarrow \phi(x) = c$$

$$\therefore v = -\cosh x \cos y + c$$

Problems

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Show that if v and V are harmonic conjugates of u in a domain D, then $v(x, y)$ and $V(x, y)$ can differ at most by an additive constant.

Problems

Solution:

v is a harmonic conjugate of u

$$\Rightarrow u_x = v_y, \qquad u_y = -v_x \;....(1)$$

$\because V$ *is a* harmonic conjugate of u

$$\Rightarrow u_x = V_y, \; u_y = -V_x \;....(2)$$

Problems

From (1) & (2), we have

$$v_x = V_x, \qquad v_y = V_y$$

$$\Rightarrow v = V + \varphi(y), \qquad v = V + \psi(x)$$

$$\Rightarrow v_y = V_y + \varphi'(y), \quad v_x = V_x + \psi'(x)$$

Problems

$$\Rightarrow \varphi'(y) = 0, \qquad \psi'(x) = 0$$

$$\Rightarrow \varphi(y) = c_1, \qquad \psi(x) = c_2$$

$$\therefore v - V = \text{constant}$$

Q. Give example of a function, which satisfy Laplace's equation but not harmonic in a domain D.

Sol.: Consider, $f(z) = \begin{cases} Im\left(\frac{1}{\bar{z}^2}\right) & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$

For $z \neq 0$,

$$f(z) = Im\left(\frac{1}{(x - iy)^2}\right) = \frac{2xy}{(x^2 + y^2)^2} = u(x, y)$$

$$u_x = \frac{(x^2 + y^2)^2 2y - 2xy(x^2 + y^2)4x}{(x^2 + y^2)^4}$$

$$u_x = \frac{2y}{(x^2+y^2)^2} - \frac{8x^2y}{(x^2+y^2)^3}$$

$$u_{xx} = \frac{-8xy}{(x^2+y^2)^3} - \frac{16xy}{(x^2+y^2)^3} + \frac{48x^3y}{(x^2+y^2)^4}$$

$$u_y = \frac{2x}{(x^2+y^2)^2} - \frac{8xy^2}{(x^2+y^2)^3}$$

$$u_{yy} = \frac{-8xy}{(x^2+y^2)^3} - \frac{16xy}{(x^2+y^2)^3} + \frac{48y^3x}{(x^2+y^2)^4}$$

$$u_{xx} + u_{yy} = 0, \qquad \text{for} \quad z \neq 0$$

$$u_x(0,0) = \lim_{h \to 0} \frac{u(h,0) - u(0,0)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = 0$$

Using limit definition we can see that

$$u_{xx} + u_{yy} = 0 \;\; \text{for } z = 0 \text{ also}$$

$$u_{xy}(0,0) = \lim_{k \to 0} \frac{u_x(0,k) - u_x(0,0)}{k} = \lim_{k \to 0} \frac{2k^{-3} - 0}{k}$$

So, $u_{xy}(0,0)$ does not exist although $u_{xx} + u_{yy} = 0$ at $(0,0)$.

Q. Let $f(z) = u(x, y) + i\, v(x, y)$ be analytic in a domain D. Prove that

$arg\big(f(z)\big) = \text{constant}\ \forall \text{z} \in D \Rightarrow f(z) = \text{constant}\ \forall \text{z} \in\ D$

Sol.: If $arg\big(f(z)\big) = \tan^{-1}\frac{v}{u} = 0, \forall \text{z} \in D$

then $v = 0 \Rightarrow u = const. \forall \text{z} \in D$

(By C-R Equations) and hence $f(z) = \text{constant},\ \forall \text{z} \in D$

If $arg\left(f(z)\right) = \tan^{-1}\frac{v}{u} = c \neq 0, \forall \mathrm{z} \in D$

then $v = ku \ \forall \mathrm{z} \in D$, where $k = \tan c$

$$\Rightarrow f(z) = (1 + ik)u \text{ analytic,} \qquad \forall \mathrm{z} \in D$$

and hence $\overline{f(z)} = (1 - ik)u$, is analytic $\forall \mathrm{z} \in D$

$$\Rightarrow f(z) = constant.$$

Q. Let $f(z) = u(x, y) + i\, v(x, y)$ be analytic in a domain D. Prove that

$$Re\big(f(z)\big) = \text{constant}\ \forall \mathrm{z} \in D \Rightarrow f(z) = \text{constant}\ \forall \mathrm{z} \in D$$

Sol.: If $u(x, y) = const.\ \forall \mathrm{z} \in D \Rightarrow u_x = u_y = 0\ \ \forall \mathrm{z} \in D$

$$\Rightarrow v_x = v_y = 0\ \ \forall \mathrm{z} \in D \Rightarrow f'(z) = 0 \quad \forall \mathrm{z} \in D$$

(By C-R Equations)

$$\Rightarrow f(z) = const. \quad \forall \mathrm{z} \in D$$

Q. Show that u_x, u_y, v_x, v_y exist and satisfy C-R equations at the origin but $f(z)$ is not differentiable at the origin for the following functions:

(i) $$f(z) = \begin{cases} \dfrac{Im(z^2)}{|z|^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

(ii) $$f(z) = \begin{cases} \dfrac{z^5}{|z|^4} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

Q. Let $f(z) = u(x,y) + i\, v(x,y)$ be analytic in a domain D such that $au + bv = const.$, where a, b are real constant (not vanishing simultaneously), then

$$f(z) = \text{constant} \quad \forall z \in D$$

Sol.: $\quad au_x + bv_x = 0 \;\; \forall z \in D$

$$au_y + bv_y = 0 \;\; \forall z \in D \Rightarrow a(-v_x) + bu_x = 0 \;\; \forall z \in D$$

For non-trivial solution of system (as a, b not vanishing simultaneously)

$$a u_x + b v_x = 0$$
$$a(-v_x) + b u_x = 0$$

$$\Rightarrow u_x^2 + v_x^2 = 0 \;\; \forall z \in D \Rightarrow |f'(z)|^2 \; = 0 \;\; \forall z \in D$$

$$\Rightarrow f'(z) \; = 0 \;\; \forall z \in D \Rightarrow f(z) \; = const. \;\; \forall z \in D$$

Q. Let $f(z)$ be analytic function in a domain D then prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$$

Q. Let $f(z) = u(x, y) + i\, v(x, y)$ be analytic in a domain D and $f(z) \neq 0 \quad \forall z \in D$, then check whether $\phi(x, y) = \ln|f(z)|$ is harmonic in D or not?

Q. Is product of two harmonic functions harmonic ?

Ans: No, $u = x^2 - y^2,\ v = -u$ are harmonic but uv is not harmonic.

Note: In above question, if v is harmonic conjugate of u then uv will be harmonic.

Q. Let $f(z) = u(x, y) + i\, v(x, y)$ be analytic in a domain D and

$$u - v = e^{-x}\big((x - y)\sin y - (x + y)\cos y\big)$$

then find $f(z)$.

Q. If $u(x, y), v(x, y)$ are harmonic functions in a domain D then check whether the function

$$f(z) = \left(u_y - v_x\right) + i\left(u_x + v_y\right)$$

is analytic in D or not ?

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THANK YOU