Analytic function

A function f(z) is said to be analytic at a point z_0 if

(i) f(z) is differentiable at z_0, *and*

(ii) f(z) is differentiable in some neighbourhood of z_0.

A function f(z) is analytic in an open set S if f is differentiable at each point of the set S.

Remark :

Differentiablity *d*oes not imply analyticity.

$$\text{Ex}: f(z) = |z|^2$$

☝ f(z) is differentiable at origin and no where else.

☝ But f(z) is not analytic at the origin as it is not differentiable in any neighborhood of origin.

Theorem :

If $f'(z) = 0$ everywhere in a domain D,

then $f(z)$ is constant throughout in D.

Entire function: **A function f(z) is said to be an Entire function if f(z) is analytic at each point in the entire finite plane.**

Example: Every polynomial is an entire function.

Singular Point:

Let a function f(z) is, ***not analytic at a point z_{0}, but analytic at some point in every neighbourhood of z_{0}.***

Then z_0 is called a singularity of f(z).

Examples

$$(1) \qquad f(z) = \frac{1}{z}$$

$$\Rightarrow z = 0 \text{ is a singularity of } f(z).$$

$$(2) \qquad f(z) = |z|^2$$

$\because f(z)$ is not analytic anywhere

$\Rightarrow f(z)$ has no singular point

Section 25
Harmonic Function :

A real valued function $u(x, y)$ is said to be harmonic in a given domain D if

(i) u_x, u_{xx}, u_y & u_{yy} exist & they are continuous in D,

(ii) u satisfies Laplace eqution

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$

Example:

$$u(x, y) = 3x^2 y - y^3 + 2$$

is harmonic in the complex plane.

$$\text{Theorem 1:}$$

$$\text{If } f(z) = u(x, y) + i\,v(x, y) \text{ is}$$

$$\text{analytic in a domain } D,$$

$$\text{then } u \,\&\, v \text{ are harmonic in } D$$

$$\text{Remark: Is converse true?}$$

$Defn$ **:** Harmonic Conjugate :

Assume:

(i) u and v be two harmonic functions in a domain D and

(ii) the first partial derivatives of u and v satisfy CR equations:

$$u_x = v_y, \quad u_y = -v_x \quad .....(1)$$

through out in D ...

Then v is said be

Harmonic Conjugate of u.

$\text{Re}\,m\text{ark}\,1:$

v is a harmonic conjugate of u

$\not\Rightarrow u$ is a harmonic conjugate of v.

For, if u is a harmonic conjugate of v, then

$$v_x = u_y \;\&\; v_y = -u_x$$

which is not same as (1)

Remark 2 :

v is a harmonic conjugate of u

$\Rightarrow u$ is a harmonic conjugate of $-v$

$$\text{as } -v_x = u_y, -v_y = -u_x$$

$$i.e. u_x = v_y \ \& \ u_y = -v_x$$

which is same as (1)

Theorem 2 :

A function $f(z) = u(x, y) + i\, v(x, y)$

is analytic in a domain D iff

v is a harmonic conjugate of u.

$$Ex. \quad f(z) = z^2.$$

Ex. Find all the points where the function

$$f(x) = 2xy + i(x^2 - y^2)$$

is analytic.

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Let f(z) be analytic in a domain D. Prove that f(z) must be constant in D if

(a) $f(z)$ is real valued $\forall$ z in D.

(b) $\overline{f(z)}$ is analytic in D.

(c) $|f(z)|$ is constant in D.

Ex. Consider the function

- $f(z) = u(x, y) + i\, v(x, y)$ in a domain D, where
- v is a harmonic conjugate of u and
- u is also a harmonic conjugate of v.

- Then show that $f(z)$ is constant throughout in D.

Q.10 Show that u is harmonic & find a harmonic conjugate v when

(a) $u(x, y) = 2x(1 - y)$

$$u_x = 2(1-y), u_{xx} = 0$$

$$u_y = -2x, \; u_{yy} = 0$$

$$\therefore \qquad u_{xx} + u_{yy} = 0$$

$$\Rightarrow \quad u \text{ is harmonic.}$$

$\because v$ is a harmonic conjugate of u

$\Rightarrow$ CR Equations are satisfied

i.e. $u_x = v_y$, $u_y = -v_x$

Then

$$v_y = u_x = 2(1-y)$$

$$\Rightarrow v = 2y - y^2 + \phi(x)$$

$$\Rightarrow v_x = \phi'(x) = -u_y = 2x$$

$$\Rightarrow \phi'(x) = 2x$$

$$\Rightarrow \phi(x) = x^2 + c$$

$$\therefore v = 2y - y^2 + x^2 + c$$

Problem:

Show that if v and V are harmonic conjugates of u in a domain D, then $v(x,y)$ and $V(x,y)$ can differ at most by an additive constant.

Solution :

v is a harmonic conjugate of u

$$\Rightarrow u_x = v_y, \qquad u_y = -v_x \text{ ....(1)}$$

$\because V$ *is a* harmonic conjugate of u

$$\Rightarrow u_x = V_y, \; u_y = -V_x \text{ .... (2)}$$

From (1) & (2), we have

$$v_x = V_x, \qquad v_y = V_y$$

$$\Rightarrow v = V + \varphi(y), \qquad v = V + \psi(x)$$

$$\Rightarrow v_y = V_y + \varphi'(y), \qquad v_x = V_x + \psi'(x)$$

$$\Rightarrow \varphi'(y) = 0, \qquad \psi'(x) = 0$$

$$\Rightarrow \varphi(y) = c_1, \qquad \psi(x) = c_2$$

$$\therefore v - V = \text{constant}$$

Q. If $u(x, y) = \dfrac{x}{x^2 + y^2}$, find a harmonic conjugate v of u.

Soln : Observe the following :

(*i*) If $f(z) = \dfrac{1}{z}$, then $u = \mathbf{Re}\, f(z)$.

(*ii*) $f(z)$ is analytic in a domain

$$D = C - \{(0,0)\}.$$

(iii) $\text{Im}\, f(z) = v = -\dfrac{y}{x^2 + y^2}$.

Conclude that v is a H.C. of u.