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BITS Pilani
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MATH F112 (Mathematics-II)

Complex Analysis

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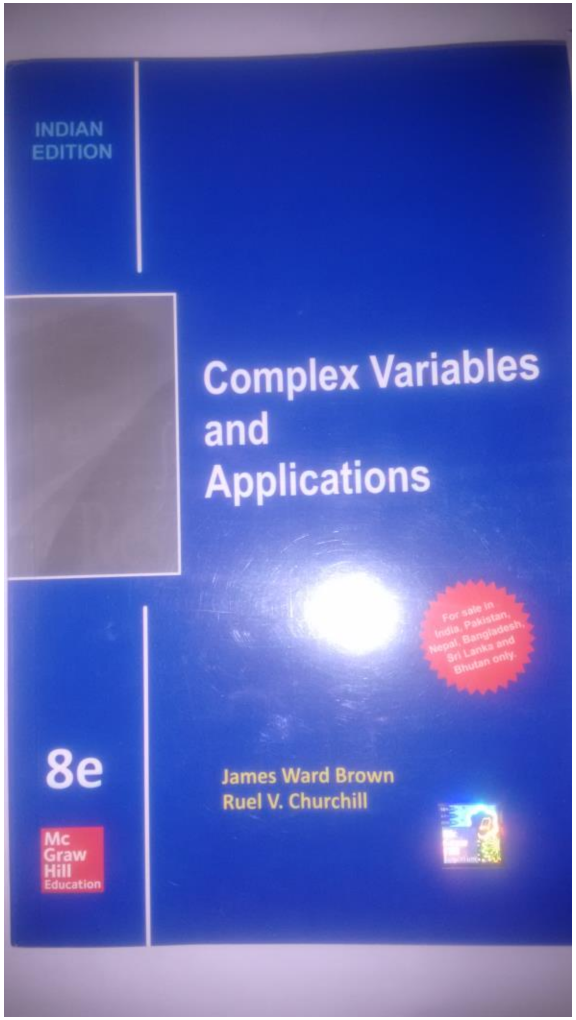
Lecture 21-22
Review of Complex Numbers

Dr. Ashish Tiwari

Text Book

- Complex Variables and Applications
- Eighth Edition, 2009.
- Authors: James Ward Brown & Ruel V. Churchill
- Publisher: McGRAW-HILL



Complex Numbers

<u>Complex Number:</u> A complex number z is an ordered pair (x, y), where x & y are real numbers i.e. $z = (x, y)$, where

x = real part of $z = \mathbf{Re}\, z$

y = imaginary part of $z = \mathbf{Im}\, z$

We usually write: $z = (x, y) = x + i\, y$

Complex Numbers

Equality of two complex numbers:

Two complex numbers $z_1 = x_1 + i\, y_1$ &

$z_2 = x_2 + i\, y_2$ are said to be equal iff

$$x_1 = \mathrm{Re}\,(z_1) = \mathrm{Re}\,(z_2) = x_2$$

&

$$y_1 = \mathrm{Im}(z_1) = \mathrm{Im}(z_2) = y_2.$$

Important Operations

1. ***Addition / Subtraction of complex numbers:***

$$z_1 \pm z_2 = (x_1 + iy_1) \pm (x_2 + iy_2)$$

$$= (x_1 \pm x_2) + i\,(y_1 \pm y_2)$$

Important Operations

2. ***Multiplication of complex numbers:***

$$z_1.z_2 = (x_1 + iy_1).(x_2 + iy_2)$$

$$= (x_1x_2 - y_1y_2) + i\,(x_1y_2 + x_2y_1)$$

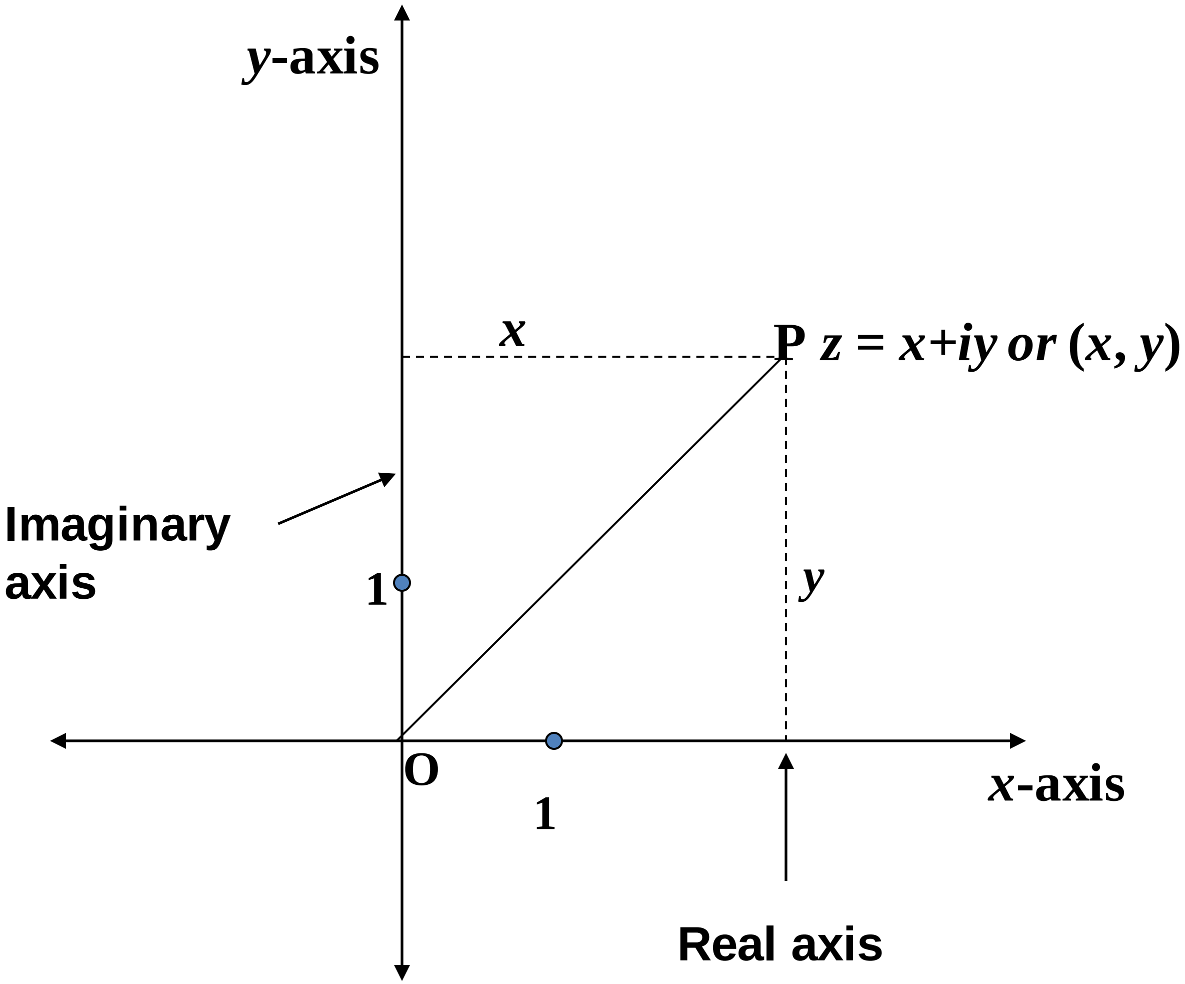
3. *Division of complex numbers:*

If $z_1 = x_1 + iy_1$ & $z_2 = x_2 + iy_2 \neq 0 + i0$, then

$$z = \frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{x_1 + iy_1}{x_2 + iy_2} \times \frac{x_2 - iy_2}{x_2 - iy_2}$$

$$= \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}$$

Complex Plane



Complex Plane

- **Choose the same unit of length on both the axis.**
- **Plot $z = (x, y) = x + iy$ as the point P with coordinates x & y.**
- **The xy-plane, in which the complex numbers are represented in this way, is called *Complex plane or Argand plane .***
- **Since complex numbers lie on a plane that's why there are no ordering relation in complex numbers**

Properties of Arithmetic Operations

(1) ***Commutative Law:***

$$z_1 + z_2 = z_2 + z_1$$

$$z_1.z_2 = z_2.z_1$$

(2) ***Associative law:***

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$$

$$(z_1.z_2).z_3 = z_1.(z_2.z_3)$$

Properties of Arithmetic Operations

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(3) ***Distributive law:***

$$z_1.(z_2 \pm z_3) = z_1.z_2 \pm z_1.z_3$$

$$(z_1 \pm z_2).z_3 = z_1.z_3 \pm z_2.z_3$$

(4) ***Existence of Identity:***

$$z + 0 = z = z + 0$$

$$z.1 = z = 1.z$$

Properties of Arithmetic Operations

(5) *Existence of Inverse:*

$$z + (-z) = 0 = (-z) + z$$

$$z\,.(1/z) = 1 = (1/z).\,z, \text{ provided } z \neq 0$$

(6) $$z\,.0 = 0 = 0.z$$

(7) If $z_1.z_2 = 0$ then either $z_1 = 0$ *or* $z_2 = 0$ or both are zero

Complex Conjugate

- **Complex conjugate number:**

Let $z = x + iy$ **be a complex number.**

Then $\bar{z} = x - iy$ **is called complex conjugate of** z

- **Properties of complex conjugate:**

1. $z + \bar{z} = 2x$

$$\Rightarrow \quad x = \operatorname{Re} z = \frac{1}{2}(z + \bar{z})$$

2. $\boldsymbol{y = \operatorname{Im} z = \frac{1}{2i}(z - \bar{z})}$

Complex Conjugate

3. $$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$$

4. $$\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$$

5. $$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$$

Complex Conjugate

6. $\bar{\bar{z}} = z$

7. z is real $\Leftrightarrow z = \bar{z}$

8. $\overline{iz} = \bar{i}\bar{z} = -i\bar{z}$

9. $\mathrm{Re}(iz) = -\mathrm{Im}(z),\ iz = ix - y$

10. $\mathrm{Im}(iz) = \mathrm{Re}(z)$

11. $z_1 z_2 = 0 \Rightarrow z_1 = 0 \ or \ z_2 = 0$

Polar Form of Complex Number

Let $z = x + iy$

Put $x = r\cos\theta,\ \ y = r\sin\theta$

$$z = r\,(\cos\theta + i\sin\theta) = re^{i\theta}$$

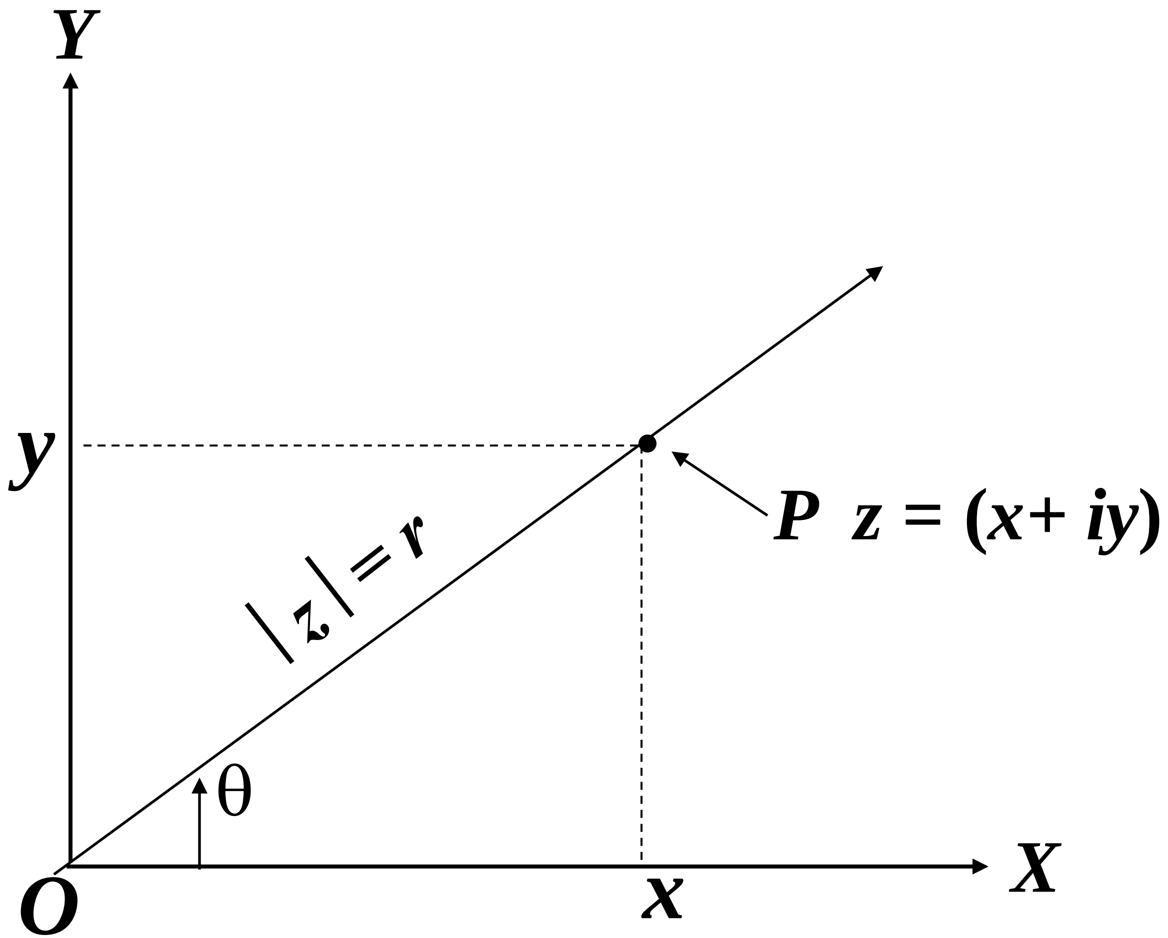
which is called polar form of complex number.

Modulus of Complex Number

$$|z| = r = \sqrt{x^2 + y^2} \geq 0$$

Geometrically, $|z|$ is the distance of the point z from the origin.

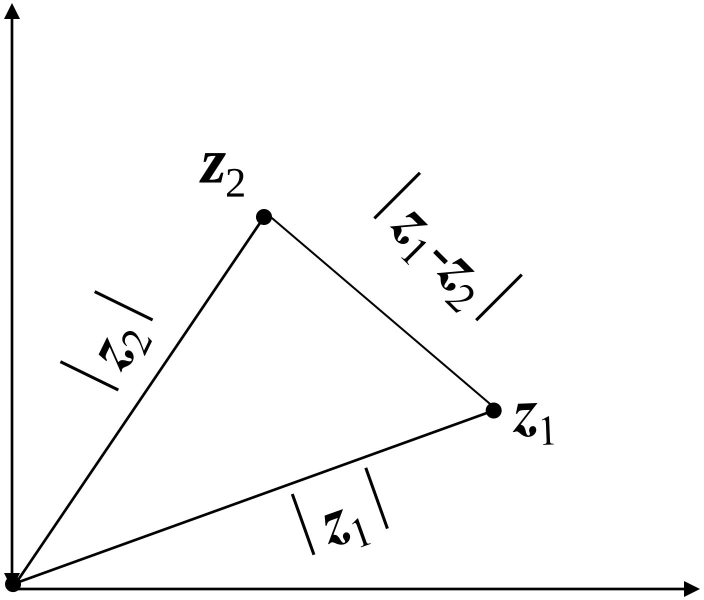
Modulus of Complex Number



Modulus of Complex Number

$|z_1| > |z_2|$ means that the point z_1 is farther from the origin than the point z_2.

$|z_1 - z_2| =$ distance between z_1 & z_2



Properties of Modulii

$$1.\ |z_1 z_2| = |z_1||z_2|$$

$$2.\ |z_1 / z_2| = |z_1| / |z_2|, \qquad |z_2| \text{ is non zero}$$

$$3.\ \operatorname{Re} z \pounds |z|, \qquad \operatorname{Im} z \pounds |z|$$

Properties of Modulii

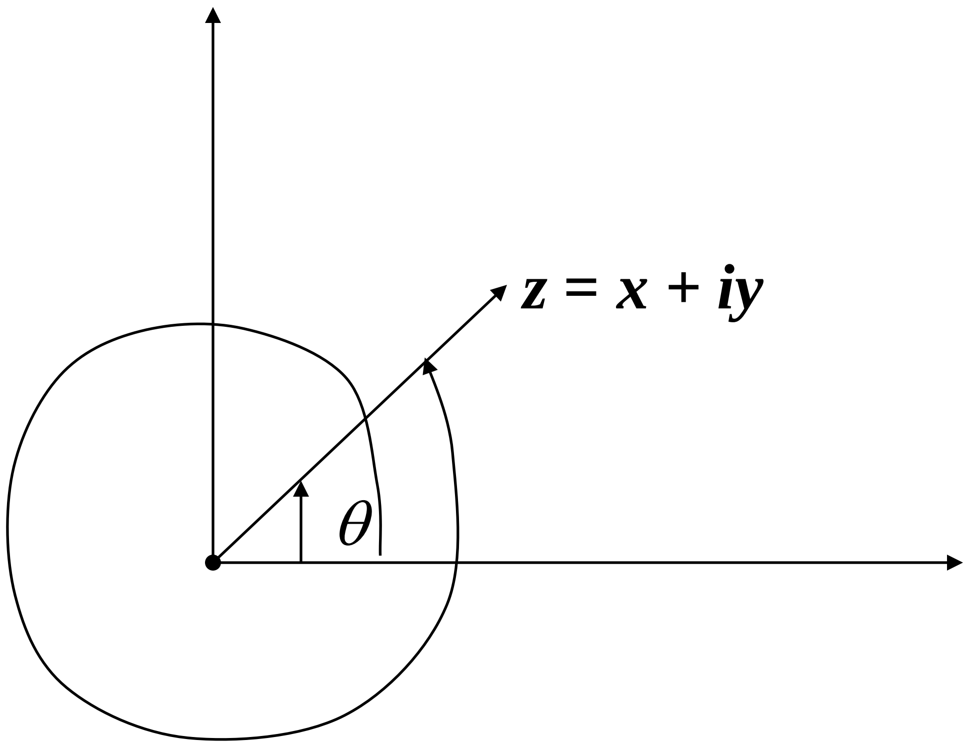
$$1.\ \left|z_1 \pm z_2\right| \text{£} \left|z_1\right| + \left|z_2\right|$$

$$2.\ \left|z_1 + z_2\right| \ 3\ \left|z_1\right| - \left|z_2\right|$$

$$3.\ \left|z_1 \pm z_2\right| \ 3\ \left\|z_1\right| - \left|z_2\right\|$$

Argument of Complex Number

The directed angle θ measured from the positive x-axis is called the argument of z, and we write $\theta = \arg z$.



Argument of Complex Number

- ***Remarks :***

1. For $z = 0$, θ is undefined.

2. θ is measured in radians, and is positive in the counterclockwise sense.

3. θ has an infinite number of possible values, that differ by integer multiples of 2π. Each value of θ is called argument of z, and is denoted by $\theta = \arg z$

Argument of Complex Number

4. When θ is such that $-\pi < \theta \leq \pi$, then such value of θ is called principal value of arg z, and is denoted by

$$\Theta = \text{Arg } z, \text{ if } -\pi < \Theta \leq \pi$$

5. $\arg z = \text{Arg } z + 2n\pi,\ n = 0, \pm 1, \pm 2, \ldots\ldots.$

6. Let $z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{i\theta_2}$.

Then $z_1 = z_2 \Leftrightarrow (i)\ r_1 = r_2$ &

$(ii)\ \theta_1 = \theta_2 + 2n\pi$

$n = 0, \pm 1, \pm 2, \ldots..$

7. $\arg(z_1 . z_2) = \arg(z_1) + \arg(z_2)$

Argument of Complex Number

Ex1. Let $z = -1 + i, \quad Argz = ?$

Sol:

We have

$$z = -1 + i = r(\cos\theta + i\sin\theta)$$

$$\Rightarrow |z| = r = \sqrt{2}$$

$$\therefore -1 + i = \sqrt{2}(\cos\theta + i\sin\theta)$$

$$\Rightarrow \sqrt{2}\cos\theta = -1, \quad \sqrt{2}\sin\theta = 1$$

$$\Rightarrow \tan\theta = -1 \Rightarrow \theta = \Theta = Argz = 3\pi/4$$

Argument of Complex Number

Hence

$$\arg z = Arg\, z + 2n\pi,\ n = 0, \pm 1, \pm 2, \ldots$$

$$= (3\pi/4) + 2n\pi,\ n = 0, \pm 1, \pm 2, \ldots$$

Argument of Complex Number

Ex2. Let $z = -2i, Argz = ?$

Sol:

We have

$$z = -2i = r(\cos\theta + i\sin\theta)$$

$$\Rightarrow |z| = r = 2$$

$$\therefore -2i = 2(\cos\theta + i\sin\theta)$$

Argument of Complex Number

$$\Rightarrow 2\cos\theta = 0, \quad 2\sin\theta = -2$$

$$\Rightarrow \theta = \Theta = \boldsymbol{Arg z} = -\pi/2$$

Hence

$$\arg z = (-\pi/2) + 2n\pi,\ n = 0, \pm 1, \pm 2, ..$$

Roots of Complex Number

For $z_0 \neq 0$, there exists n values of

z which satisfy $z^n = z_0$

Let $z = re^{i\theta} \Rightarrow z^n = r^n e^{in\theta}$

Let $z^n = z_0 = r_0 e^{i\theta_0}, n = 2, 3, \ldots\ldots$

Then $r^n e^{in\theta} = r_0 e^{i\theta_0}$

Roots of Complex Number

$$\Rightarrow \boldsymbol{r}^{\boldsymbol{n}} = \boldsymbol{r}_0,$$

$$\boldsymbol{n}\theta = \theta_0 + 2\boldsymbol{k}\pi,$$

$$Þ \quad r = (r_0)^{1/n}, q = \frac{q_0 + 2kp}{n}$$

$$\backslash \quad z = r\, e^{iq}$$

$$Þ \quad z = Z_k = (r_0)^{\frac{1}{n}} e^{i(\frac{q_0+2kp}{n})}$$

is called n^{th} roots of $z_0, k = 0,1,..,n - 1.$

Principal Root

For $k = 0,$

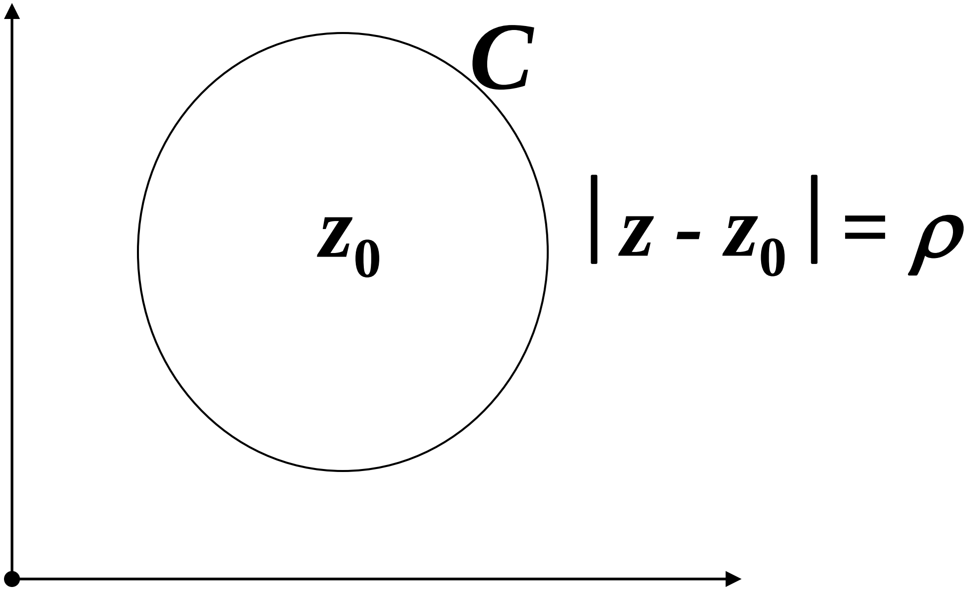
$$Z_0 = (r_0)^{1/n} \; e^{i q_0 / n}$$

is called the PRINCIPAL ROOT.

Neighbourhood

Let C be a circle with centre z_0 and radius ρ. Then

such a circle C can be represented by $C: |z - z_0| = \rho$



Neighbourhood

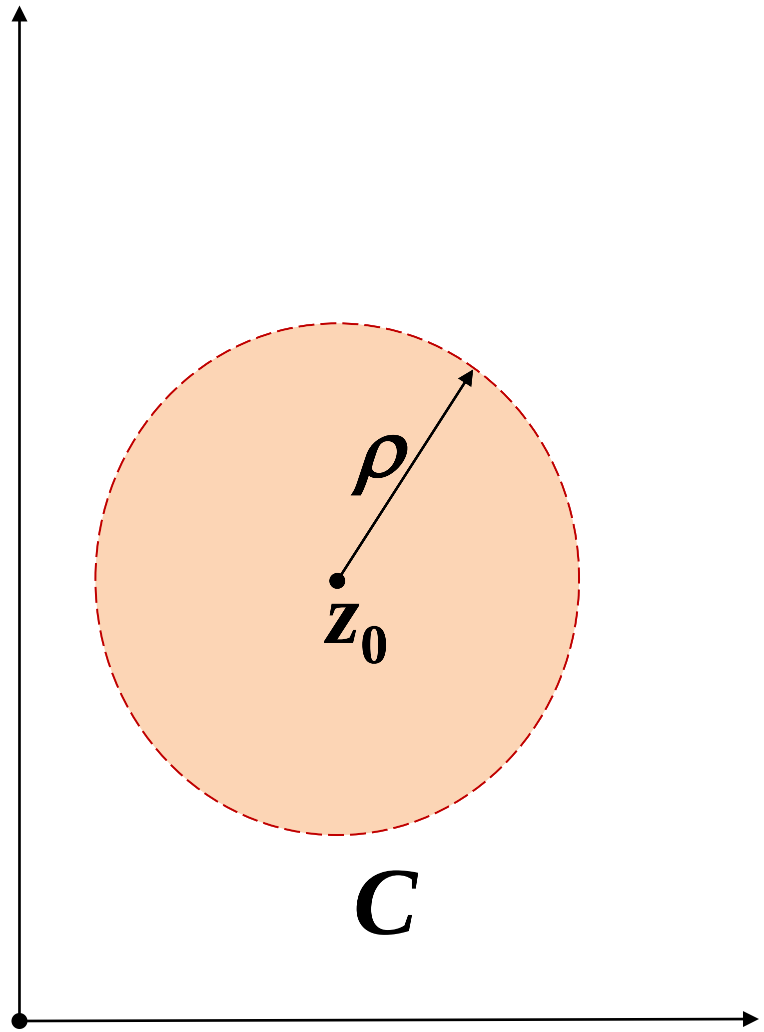
Consequently, the

inequality $|z - z_0| < \rho \qquad (1)$

holds for every z **inside** C**.**

i.e. (1) represents the

interior of C**.**



Neighbourhood

Such a region, given by (1), is called a

***ρ-neighbourhood (nbd)* of z_0 i.e. the set**

$$N_\rho(z_0) = \{z : |z - z_0| < \rho\}$$

is called a ρ-nbd. of z_0

Deleted Neighbourhood

$$N_0 = \{z : 0 < |z - z_0| < \rho\}$$

is called ***deleted nbd.*** It consists of all points z

in an ρ-nbd of z_0, except for the point z_0 itself.

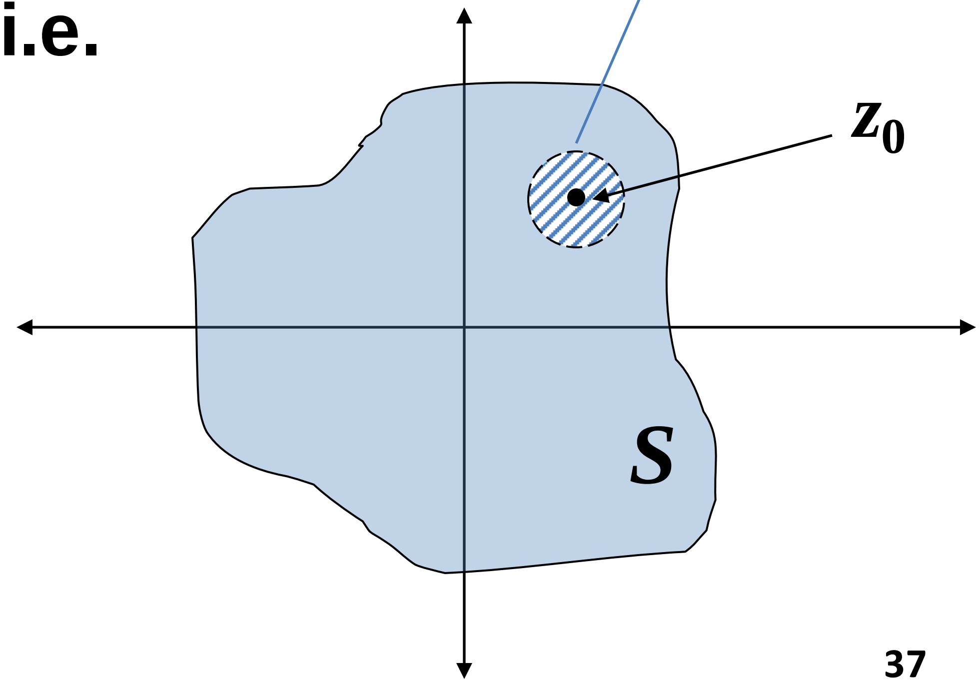
The inequality $|z - z_0| > \rho$ represents the

exterior of the circle C.

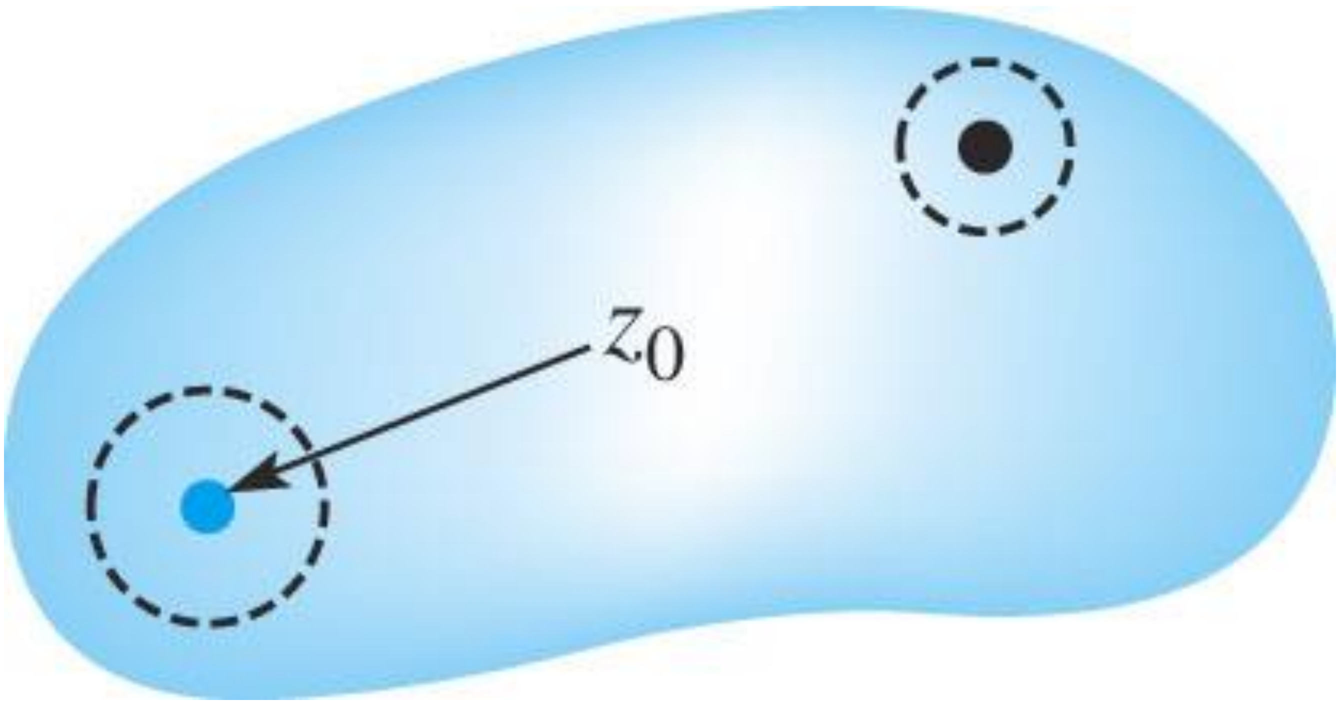
Interior Point

Interior Point: **Let S be any set. Then a point $z_0 \in S$ is called an interior point of S if $\exists$ a ρ-nbd $N_\rho(z_0)$ that contain *only points of S*, i.e.**

$$z_0 \in N_\rho(z_0) \subseteq S$$



Interior Point



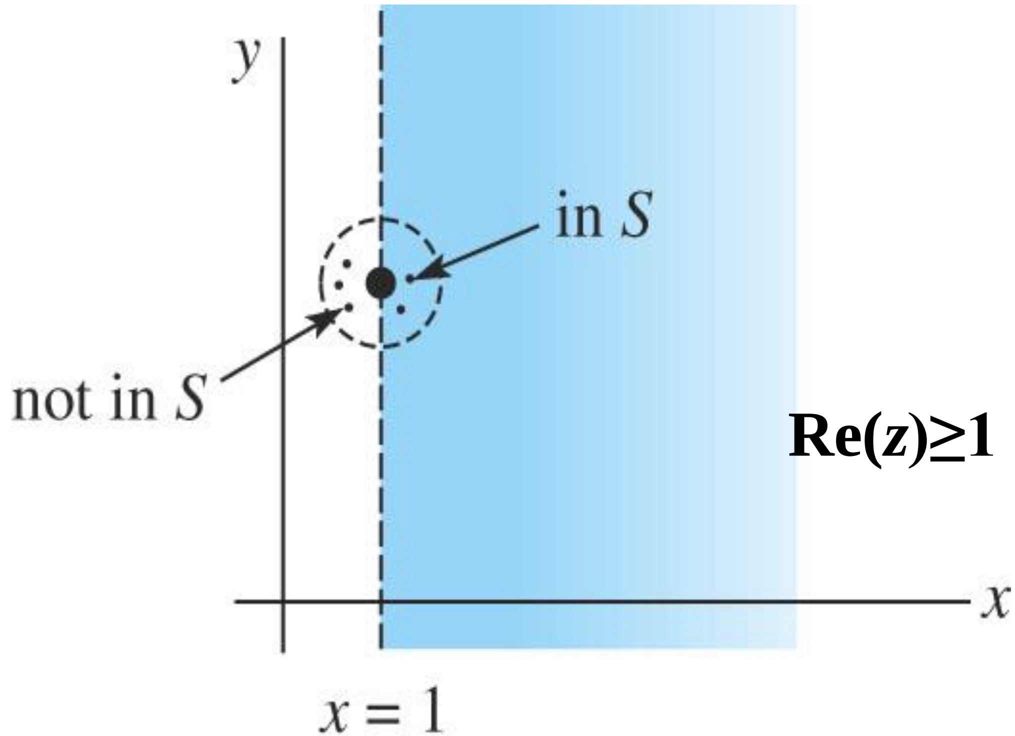
Exterior Point

Exterior Point: A point z_0 is called an exterior point of the set S if $\exists$ a ρ-nbd $N_\rho(z_0)$ of z_0 that contains ***no points of S***.

z_0 is an exterior point of $S \Leftrightarrow z_0$ is an interior point of S^c.

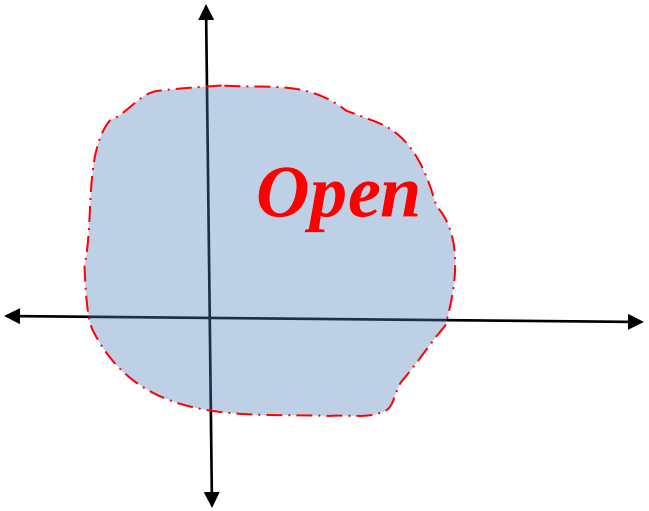
Boundary Point

***Boundary Point*:** A point z_0 is called boundary point for the set S if it is neither **interior point** nor **exterior point** of S.



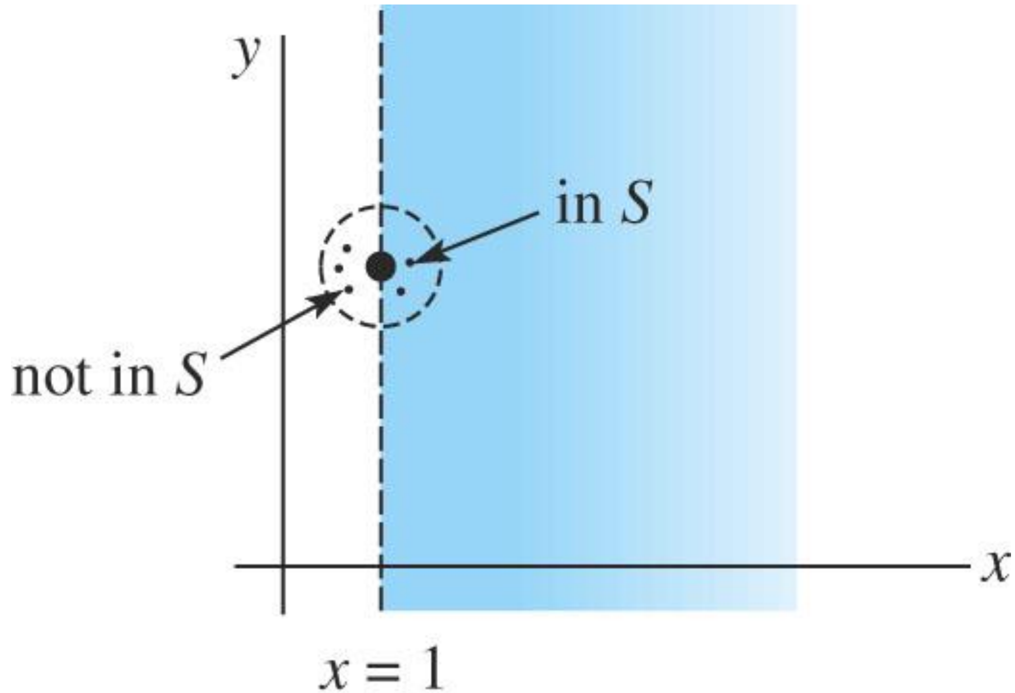
Open Set

Open Set: **A set S is said to be open if every point of S is an interior point of S. i.e. S is open if it contains none of its boundary points.**



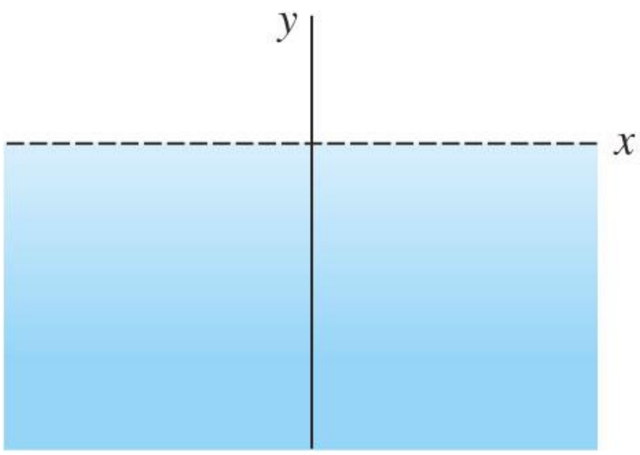
Open Set

- The graph of $\mathrm{Re}(z) \geq 1$ is shown in Following figure. It is not an open set.

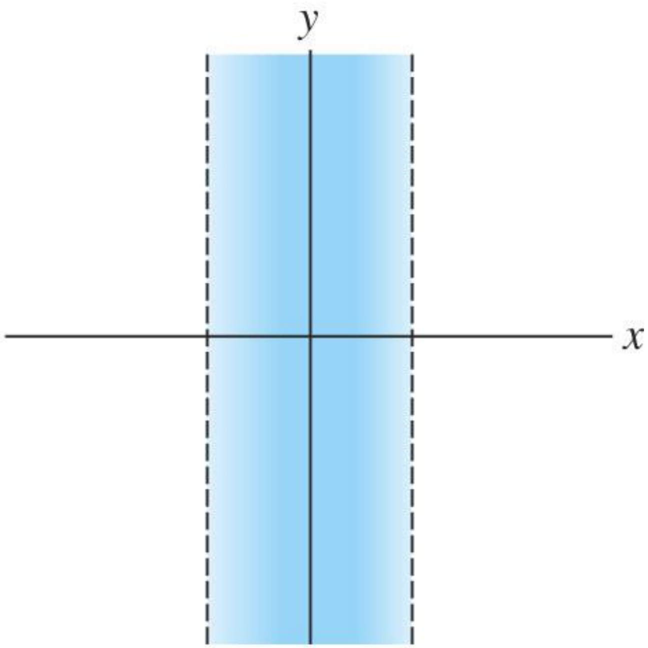


Open Set

- Some examples of open sets.

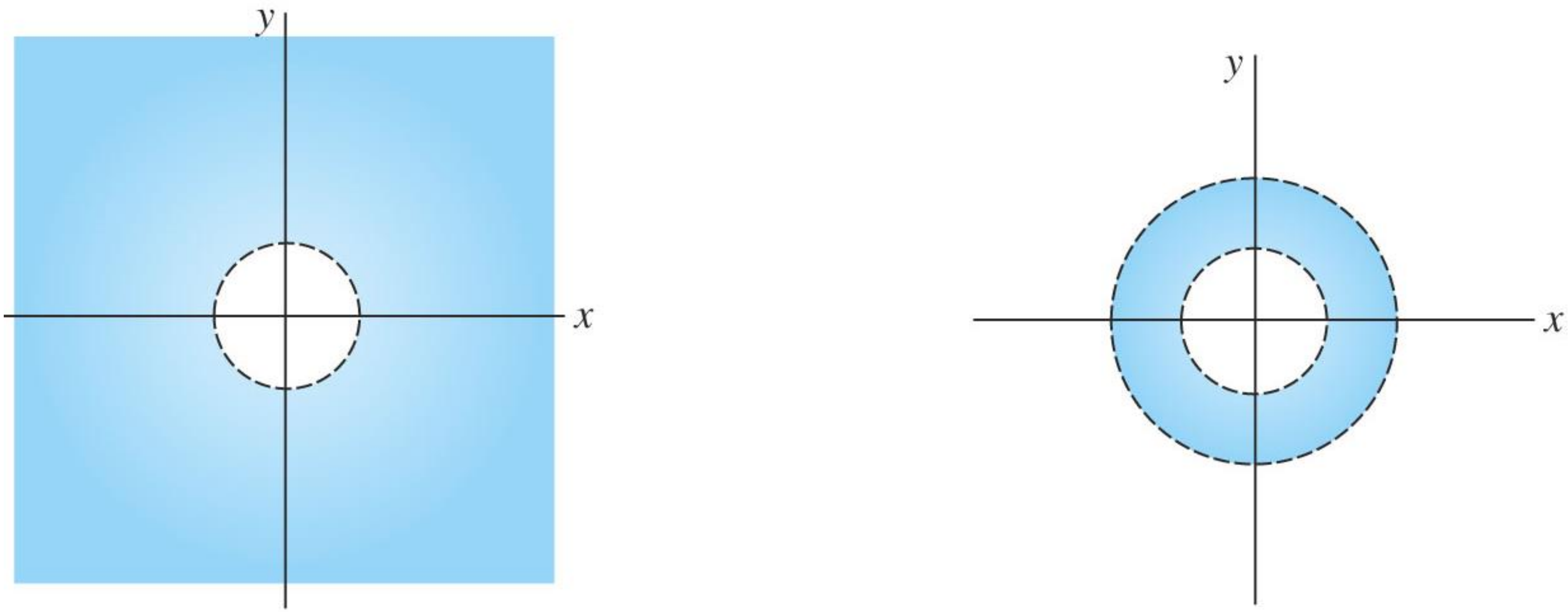


$\mathrm{Im}(z) < 0$
lower half-plane
(a)



$-1 < \mathrm{Re}(z) < 1$
infinite strip
(b)

Open Set

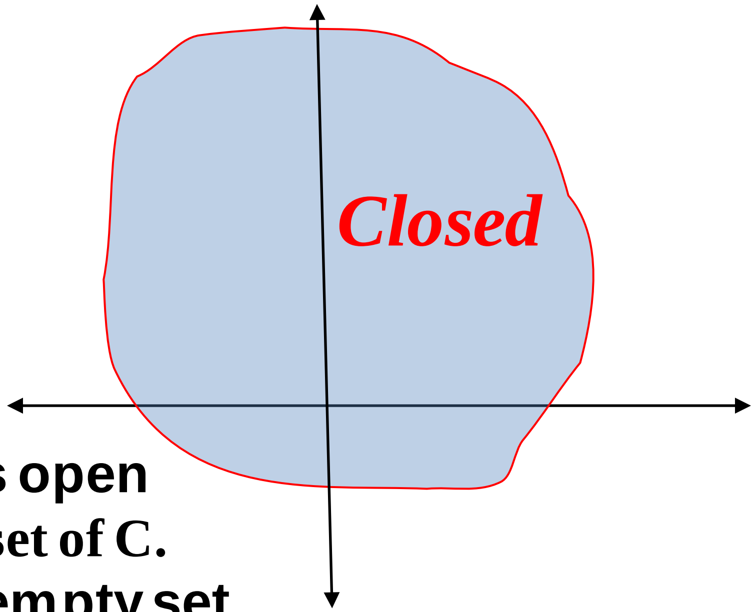


$|z| > 1$
exterior of unit circle
(c)

$1 < |z| < 2$
circular ring
(d)

Closed Set

Closed Set: **A set S is said to be closed, iff it contains all of its boundary points.**



Note: S is closed iff its complement S^c is open provided both S and S^c are non empty subset of C.
i.e. S is neither entire complex plane nor empty set.

Closure of a Set

Closure of a set: **Let S be any subset of $\mathbb{C}$. Smallest closed set F which contains S is called closure of S**

Closure of a set S is the closed set consisting of all points in S together with the boundary of S.

Ex 1. Let $S = \{z : |z| < 1\}.$

Then $Cl(S) = \overline{S} = \{z : |z| \leq 1\}.$

Ex 2. Let $S = \{z : |z| \leq 1\}.$

Then $Cl(S) = \overline{S} = \{z : |z| \leq 1\}.$

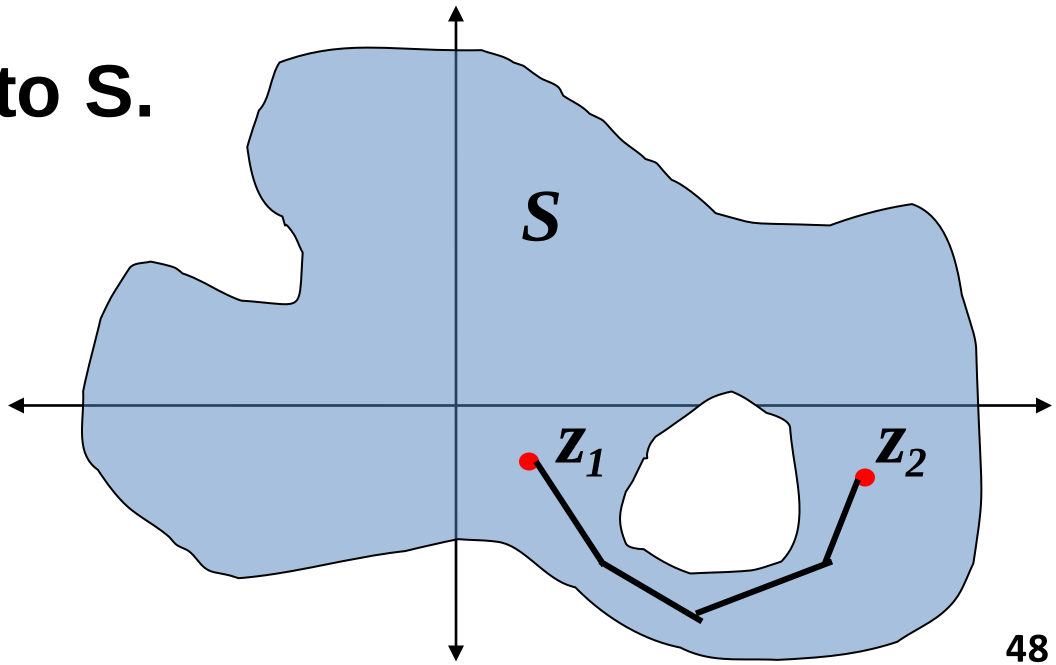
Bounded Set

Bounded Set: A set S is called bounded if all of its points lie within a circle of finite radius, otherwise it is unbounded.

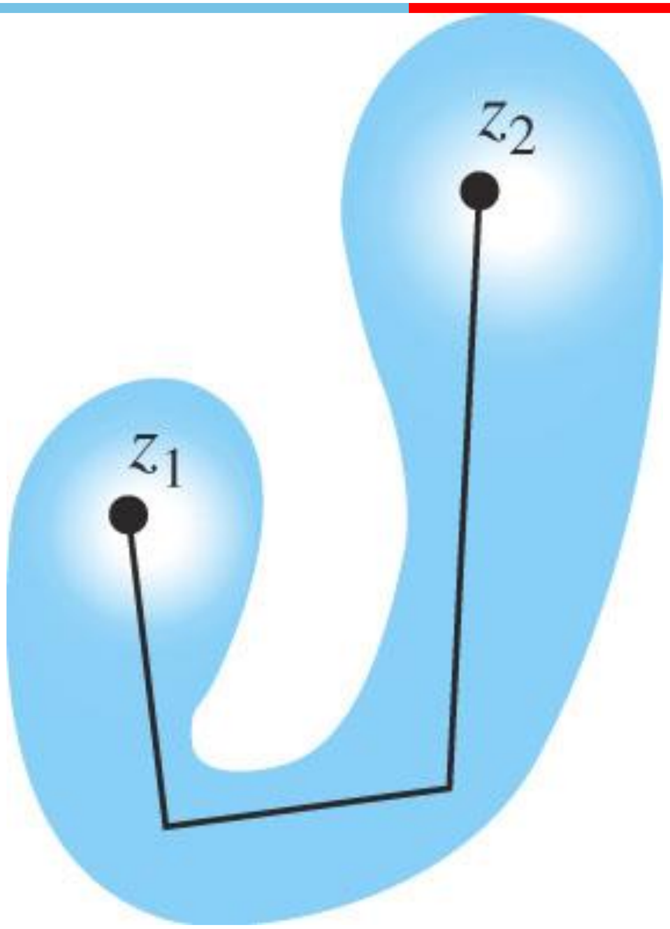
Connected Set

Connected Set: **An open set S is said to be connected if any of its two points can be joined by a broken line of finitely many line segments, all of whose points belong to S.**

Roughly speaking, this means that S consists of a "single piece", although it may contain holes.

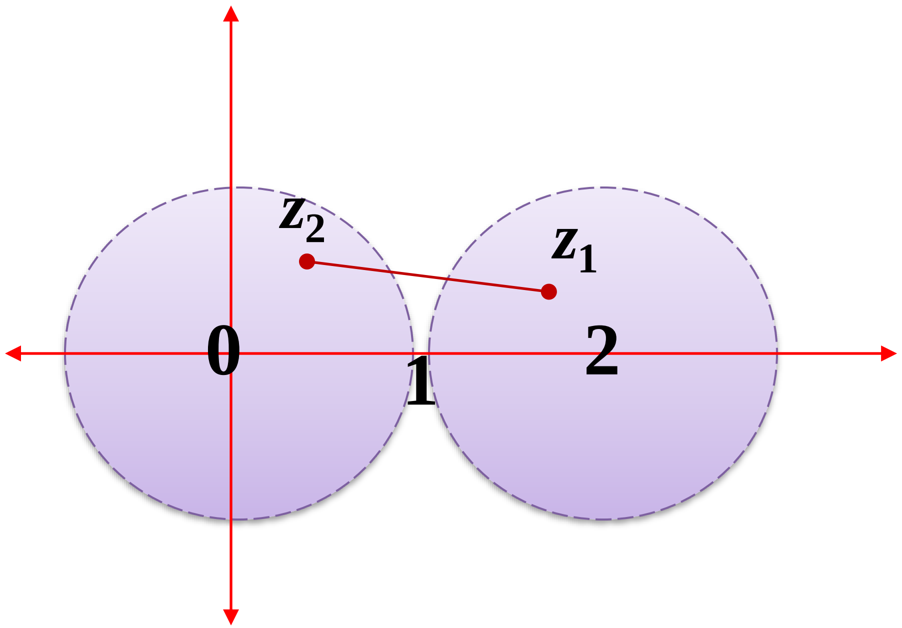


Connected Set



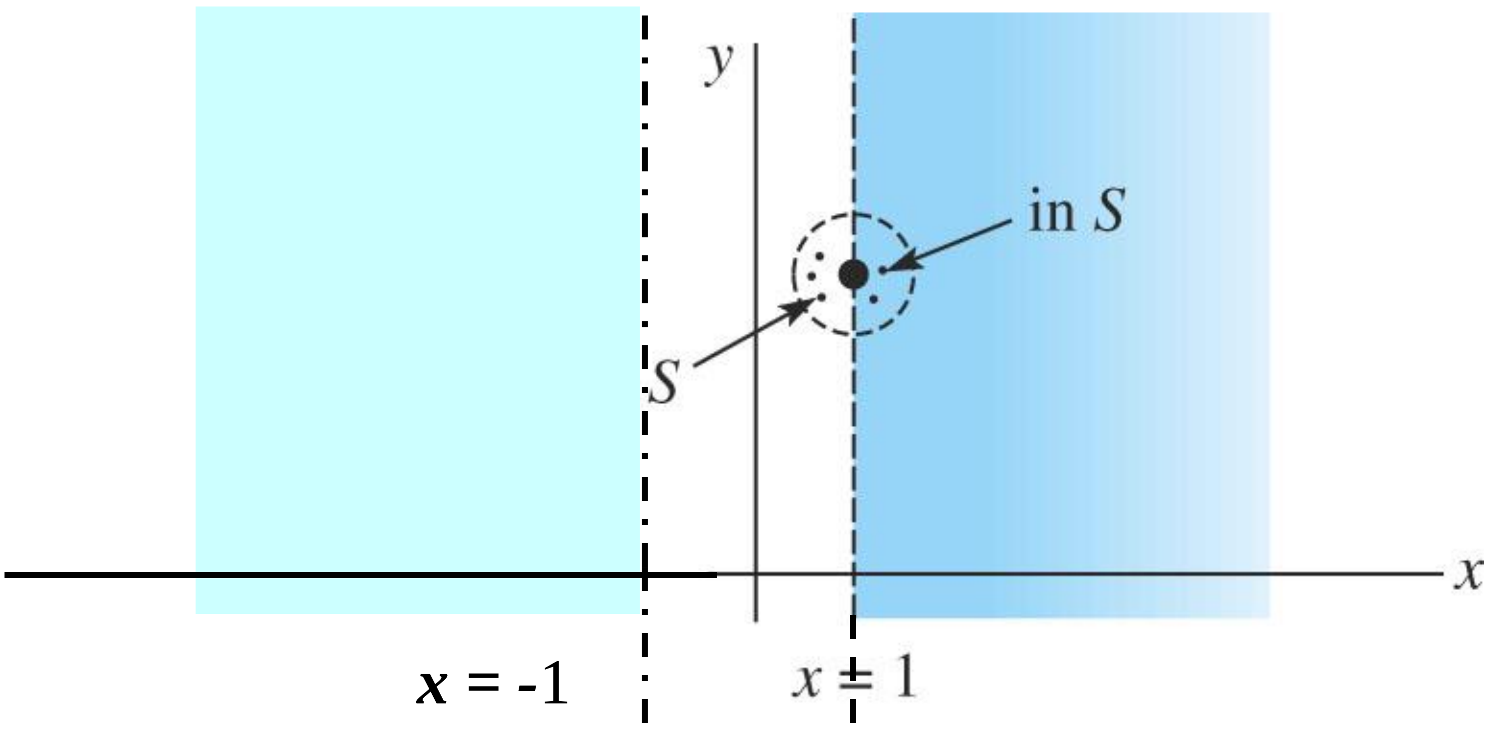
Connected Set

Is the set $S = \{z : |z| < 1\} \cup \{z : |z-2| < 1\}$ connected?



Connected Set

- The graph of $|\text{Re}(z)| \geq 1$ is shown in following figure. It is not an open set.



Is it connected?

Domain

Domain: **An open connected set is called a domain.**

Ex1: **Sketch & determine which are domains**

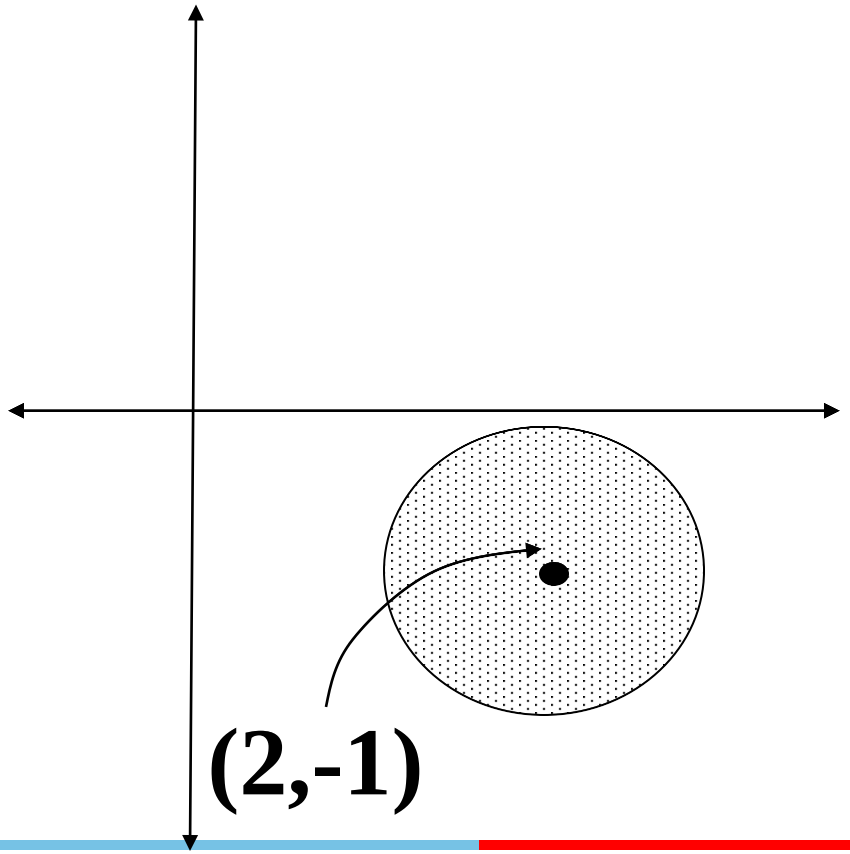
(a) $S = \{z: |z - 2 + i| \leq 1\}$

We have $|z - 2 + i| \leq 1$

$\Rightarrow |x + iy - 2 + i| \leq 1$

$\Rightarrow |(x - 2) + i(y + 1)| \leq 1$

$\Rightarrow (x - 2)^2 + (y + 1)^2 \leq 1$



Domain

$\Rightarrow$ S contains the interior & boundary points of a circle with centre $(2, -1)$ & radius 1.

$\Rightarrow$ (i) S is not a domain

(ii) S is bounded.

*A domain together with some, none or all of its boundary points ie referred to as a ***region***

Domain

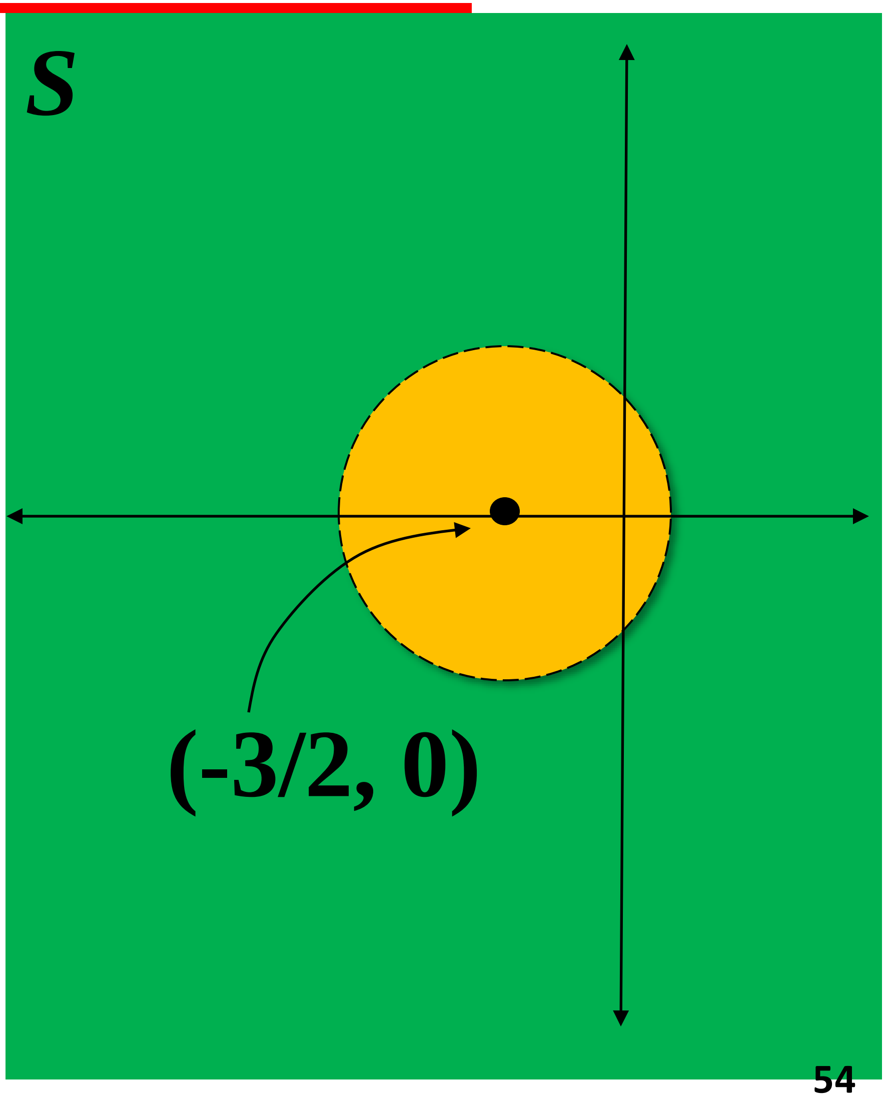
Ex2. $S = \{z : |2z + 3| > 4\}$

We have $|2z + 3| > 4$

$\Rightarrow |2x + 3 + i\,2y| > 4$

$\Rightarrow (2x + 3)^2 + 4y^2 > 16$

$\Rightarrow (x+3/2)^2 + y^2 > 4$

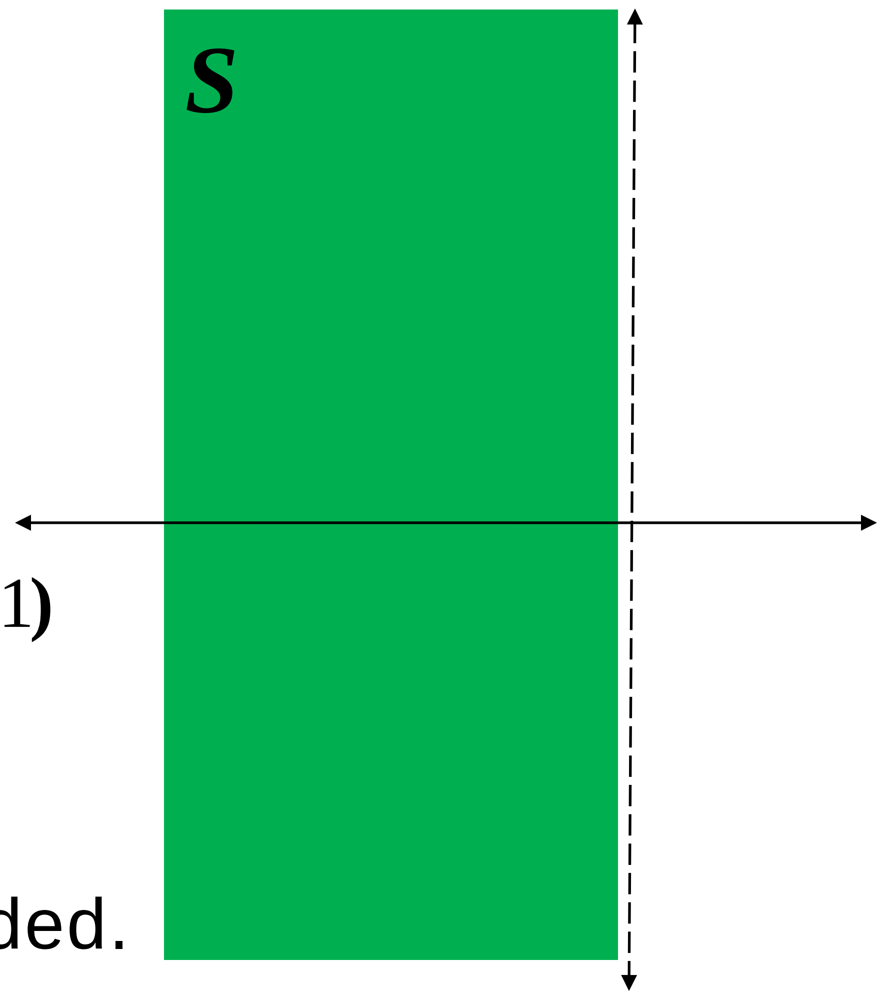


Domain

- Clearly S contains the exterior points of a circle with centre $(-3/2, 0)$ & radius 2.
- S is a domain and it is unbounded

Domain

Ex. 3 $S = \left\{ z : \left| \dfrac{z+1}{z-1} \right| < 1 \right\}$



Sol. Note that : $|z+1| < |z-1|$

$$\Rightarrow |z+1|^2 < |z-1|^2$$

$$\Rightarrow (z+1).(\bar{z}+1) < (z-1).(\bar{z}-1)$$

$$\Rightarrow x < 0.$$

S is left half open plane

S is a domain and it is unbounded.

Accumulation Point

A point z_0 is said to be an accumulation point of a set S if every ρ-neighbourhood $N_\rho(z_0)$ of z_0 contains at least one point of S other than z_0, i.e.

if $S \cap \{N_\rho(z_0) \backslash \{z_0\}\} \neq \phi$, then z_0 is called accumulation point of S.

Remark: z_0 may be or may not be a point of S.

THANK YOU