

# CS F214 Logic in Computer Science

### MODULE: PROPOSITIONAL LOGIC

**Proof System - Natural Deduction** 

## **Propositional Logic**

- Propositional Logic is essentially Boolean Logic: i.e.
  - it is a logic of *formulas* (i.e. *propositions*)
    - that evaluate to TRUE or FALSE
  - made from *atomic propositions* and *logical operations* (AND, OR, NOT ...)
    - e.g. raining AND sunny
    - e.g. raining AND sunny OR NOT humid AND NOT hot



## **Propositional Logic - Propositions**

- An atomic proposition is a statement that is atomic (i.e. understood as is without structure):
  - It may be true or false,
    - but its truth or falsity is
      - not based on any structure
      - nor is it derived from something else
  - e.g. (from the formulas in the last slide):
    - raining, sunny, humid, hot
- In Boolean Logic <u>atomic propositions</u> are referred to as <u>variables</u>.



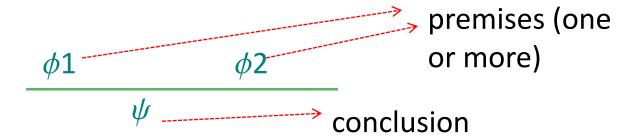
### **Proof Rules**

- Proofs in (propositional) logic can be written down in various formats:
  - one of them is (Gentzen's) *natural deduction*
- Proofs are based on rules (of inference):
  - i.e. rules that state "one can infer S<sub>1</sub> from S<sub>2</sub>"



### **Proof Rules – Generic Format**

• In natural deduction, rules are written in the form:

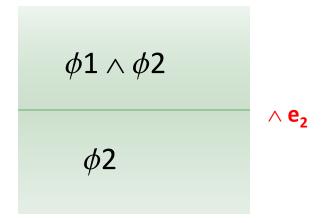


- Each premise is a (class of) propositional formula(s).
- •Similarly a *conclusion* is a (class of) propositional formula(s)



# **Proof Rules - Format - Example**

Consider this rule (labeled \( \Lambda \):



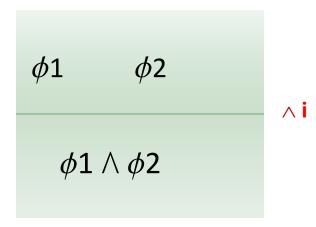
This rule is read as:

given the premise 
$$\phi 1 \land \phi 2$$
 one can conclude  $\phi 2$ 



### **Proof Rules – Different Kinds**

- In natural deduction, typically, <u>for each operation there are</u> <u>two kinds of rules</u>:
  - one kind for *elimination*
    - i.e. to prove a conclusion by <u>eliminating the operation</u>
      - e.g. see previous slide (\( \simes \) elimination)
  - one kind for *introduction* 
    - i.e. to prove a conclusion by introducing the operation
      - e.g.





# **Proof Technique: Natural Deduction: Sequents**

- In natural deduction, the statement (intended to be proved) is referred to as *a sequent*:
  - $\phi$ 1,  $\phi$ 2, ...,  $\phi$ n |--  $\psi$

### which is read as

- (the set of premises) " $\phi$ 1,  $\phi$ 2, ..., and  $\phi$ n" *entails* (the conclusion) " $\psi$ "
- e.g. (sequent):
  - $p \land q, r \mid --q \land r$
- <u>To prove the sequent</u> one has to apply (appropriate) proof rules.
  - Q: Which rule(s) is/are required to prove the example sequent given above?



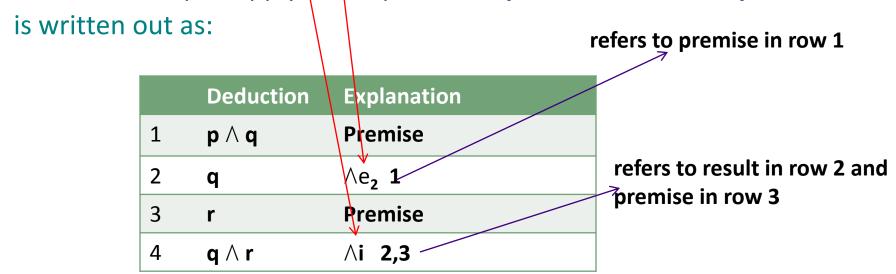
### **Proofs in Natural Deduction**

- Typically a (deduction) step in a proof involves:
  - applying a rule on one or more premises to derive a conclusion:
    - for instance, from applying the rule  $\wedge \mathbf{e_2}$  on the premise  $\mathbf{p} \wedge \mathbf{q}$  results in the conclusion, say,  $\mathbf{q}$
- Typically, then, a (natural deduction) proof is a sequence of such (deduction) steps:
  - e.g. proof for the sequent  $\mathbf{p} \wedge \mathbf{q}$ ,  $\mathbf{r} \mid --\mathbf{q} \wedge \mathbf{r}$ 
    - Step 1: apply  $\wedge \mathbf{e_2}$  on <u>premise  $\mathbf{p} \wedge \mathbf{q}$  to result in  $\mathbf{q}$ </u>
    - Step 2: apply  $\wedge$ **i** on <u>inference **q**</u> and <u>premise **r**</u> to result in **q**  $\wedge$  **r**



### **Proofs in Natural Deduction**

- Proofs are written in a particular format:
  - as a table with one column of premises and results along with corresponding explanations in the next column
- For instance, the 2-step proof from the previous slide i.e.
  - Step 1: apply  $\wedge \mathbf{e_2}$  on  $\mathbf{p} \wedge \mathbf{q}$  to result in  $\mathbf{q}$
  - Step 2: apply  $\wedge$  i on premises q and r to result in  $\mathbf{q} \wedge \mathbf{r}$



Note that the explanations may refer to the entries in the deduction column by using the row number.



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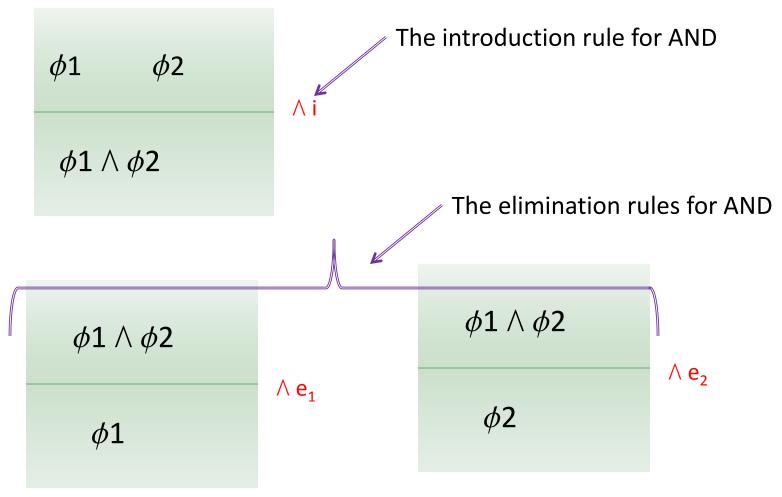
MODULE: PROPOSITIONAL LOGIC

**Natural Deduction: Rules for Conjunction** 

18-08-2018 Sundar B. CS&IS, BITS Pilani 10

# **ND: Rules for Conjunction**

• Conjunction refers to the operation "AND":





# ND: Rules for Conjunction – Example 1

- Prove the following sequent:
  - hot ∧ humid, sleepy ∧ dull |-- hot ∧ sleepy
- The proof evolves via the following steps:

	Deduction	Explanation
1	hot $\wedge$ humid	Premise
2	hot	∧e <sub>1</sub> 1

The first ∧ -elimination rule is applied on the premise in line 1 i.e. on hot ∧ humid to obtain the result hot



# ND: Rules for Conjunction – Example 1

(continued)

	Deduction	Explanation
1	$\mathbf{hot}\ \land \mathbf{humid}$	Premise
2	hot	^e <sub>1</sub> 1
3	dull $\wedge$ sleepy	Premise
4	sleepy	∧e <sub>2</sub> 3

The second ∧-elimination rule is applied on premise in line 3

i.e. on dull  $\land$  sleepy to obtain the result sleepy



# ND: Rules for Conjunction – Example 1

(continued)

	Deduction	Explanation
1	hot $\land$ humid	Premise
2	hot	^e₁ 1
3	dull ∧ sleepy	Premise
4	sleepy	∧e <sub>2</sub> 3
5	hot ∧ sleepy	∧i 2,4

The  $\land$ -introduction rule is applied on <u>results in</u> <u>line 2 and 4</u> to obtain the result *hot*  $\land$  *sleepy* 



### **Natural Deduction: Proofs**

## **Observations (about proofs):**

Typically the <u>first step of a proof</u> is <u>a premise</u> from the given sequent

and the last step is the conclusion from the given sequent.

i.e. if the sequent (to be proved) is of the form

$$\phi_1, \phi_2, ... \phi_n \mid -- \psi$$

then the proof will (typically) be of the form

 $\mathbf{b}_{\mathbf{k}}$ 

**Premise** 

•••

Ψ

**Conclusion** 



## **Natural Deduction: Proofs**

## **Observations (about proofs):**

- Typically the rule (to be) applied is
  - (i) an elimination rule for an operation if

    the (desired) result should be devoid of the operation

    and one of the premises contains the operation.
  - (ii) an introduction rule for an operation if

    the (desired) result should involve the operation
    and the premises may not



# **Conjunction - Exercises**

- Prove the following sequent:
  - english\_summer \( \) green\_top \( \) fast\_bowling, indian\_batsmen\_fail|-
    - english\_summer ∧ fast\_bowling ∧ indian\_batsmen\_fail



# CS/IS F214 Logic in Computer Science

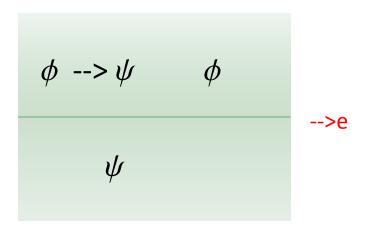
MODULE: PROPOSITIONAL LOGIC

**Proof System - Natural Deduction - Implication Rules** 

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# **Natural Deduction: Proof Rules: Implication**

Modus Ponens (Implication Elimination)



### This rule is read as:

if we know  $\phi$  implies  $\psi$  and we know  $\phi$  is true then we can conclude  $\psi$ 



Prove using "modus ponens":

rains, rains-->wet, wet-->slippery |-- slippery

The proof proceeds as follows:

	Deduction	Explanation	7	>elimination
1	rains	Premise		on premises in row 1 and 2
2	rains>wet	Premise		
3	wet	>e 1,2		
		•••	1 <sup>st</sup>	step



[continued]

Proof using "modus ponens":

rains, rains-->wet, wet-->slippery |-- slippery

	Deduction	Explanation
1	rains	Premise
2	rains>wet	Premise
3	wet	>e <b>1,2</b>
4	wet>slippery	Premise
5	slippery	>e 3,4

2<sup>nd</sup> step

-->-elimination on premises in row 3 and 4

#### Prove:

rains, rains-->wet, wet-->slippery |-- wet ∧ slippery

	Deduction	Explanation	>-elimination on premises in
1	rains	Premise	row 1 and 2
2	rains>wet	Premise	>-elimination
3	wet	>e 1,2	on premises in
4	wet>slippery	Premise	row 1 and 4
5	slippery	>e 1,4	
6	wet ∧ slippery	∧i 3,5 <u></u>	
			<ul><li>↑ -introduction</li><li>on premises in</li><li>row 3 and 5</li></ul>



## **Natural Deduction – Proofs**

- Questions:
  - How do you identify the next step in a proof?
    - How do you identify which rule is to be applied?
- Observations:
  - An ND proof is usually driven bottom up i.e.
    - one identifies the steps (deductions) of the proof by starting with the conclusion
    - The structure of the conclusion (i.e. the top level operation(s) used) leads to the identification of the appropriate rule
      - for instance the choice of &i rule vs. the choice of
         -->e rule in the two proofs in the last two slides.



# ND – Structure of Proofs - Example

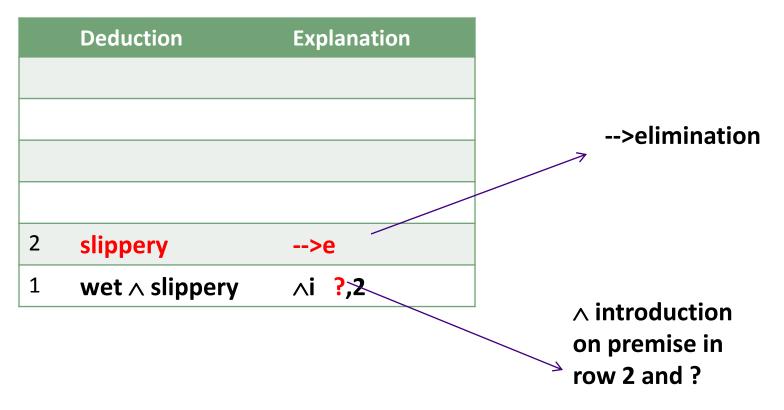
Prove: rains, rains-->wet, wet-->slippery |-- wet ∧ slippery

Deduction	Explanation	
wet ∧ slippery	^i	



#### Prove:

rains, rains-->wet, wet-->slippery |-- wet ∧ slippery





#### Prove:

rains, rains-->wet, wet-->slippery |-- wet ∧ slippery

	Deduction	Explanation	
4	wet	?	>-elimination on Premise in
3	wet> slippery	Premise	row 3 and resu
2	slippery	>e 3,4	in row 4
1	wet ∧ slippery	∧i <b>?,2</b>	△ -introduction
			on premise in row 2 and ?

#### Prove:

rains, rains-->wet, wet-->slippery |-- wet ∧ slippery

	Deduction	Explanation
6	rains	Premise
5	rains>wet	Premise
4	wet	>e 5,6
3	wet> slippery	Premise
2	slippery	>e 3,4
1	wet ∧ slippery	∧i <b>?,2</b>

-->-elimination on Premises in rows 5 and 6



#### Prove:

rains, rains-->wet, wet-->slippery |-- wet ∧ slippery

	Deduction	Explanation
6	rains	Premise
5	rains>wet	Premise
4	wet	>e 5,6
3	wet> slippery	Premise
2	slippery	>e 3,4
1	wet ∧ slippery	∧i 4,2

-->-elimination on Premise in row 4 and result in row 3

> ∧ -introduction on premise in row 2 and result in row 4

# **ND: Proof Rules: Implication Introduction**

A "Proof" introduces implication

Assume  $\phi$ 1

.

 $\phi 1 --> \phi 2$ 

i.e. if we can assume  $\phi 1$  and prove  $\phi 2$ , then  $\phi 1 --> \phi 2$  can be inferred

### Question:

How is this rule different from the ones seen before?

[Hint: *Note that there is a sub-proof*. End of Hint.]



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# **ND: Proofs using Implication Introduction**

- Example:
  - Prove the sequent  $|--p \land q --> q$
  - Proof:

	Deduction	Explanation
1	p∧q	Assumption
2	q	^e <sub>2</sub> - 1
3	p ∧ q> q	>i - 1-2

These two rows constitute a proof of the sequent  $p \land q \mid --q$ 

#### **Observation:**

Proof of a sequent of the form  $\phi \mid -- \psi$  can be treated as the proof of a sequent of the form  $\mid -- \phi \rightarrow \psi$ 



# **ND: Proofs using Implication Introduction**

### Observation:

- Proof of a sequent of the form  $\phi \mid -- \psi$  can be treated as the proof of a sequent of the form  $\mid -- \phi --> \psi$
- Inductively, proof of  $\phi 1$ ,  $\phi 2$ , ...  $\phi n \mid --\psi$  is proof of  $\mid --\phi 1 --> (\phi 2 --> (... --> (\phi n --> \psi)...))$

#### Exercise:

Prove the following sequent:

and thereby prove the following sequent, using the observation mentioned above:

