Sec 12: Function of a complex variable:

Let S be a set of complex numbers. Then function f defined on S is a rule that assigns to each $z \in S$ a ~~unique~~ complex number w, and we write

$$f(z) = w$$

The set S is called domain of definition of f.

Let $z = x + i\,y$ &

$$w = u\,(x,y) + i\,v(x,y)$$

Then $f(z) = w = u\,(x, y) + iv\,(x, y)$

$\operatorname{Re} f(z) = u(x,y)$ & $\operatorname{Im} f(z) = v(x,y)$

- In polar coordinates,

$$z = x+iy = re^{i\theta} ,$$

$$f(z) = u(r, \theta) + i\, v(r, \theta).$$

Limit:

- Let f be a function defined at ALL POINTS of z in some deleted nbd of z_0.
  Then

$$\lim_{z\to z_0} f(z) = w_0 \Rightarrow \text{given } \varepsilon > 0,$$

$$\exists \, a \quad \delta > 0 \text{ such that}$$

$$|f(z)\text{-}w_0| < \varepsilon$$

$$\text{whenever } \; 0 < |z - z_0| < \delta$$

Theorems on limits:

Thm 1 Let $f(z) = u(x,y) + iv(x,y)$,

$$z_0 = x_0 + iy_0, \quad w_0 = u_0 + iv_0,$$

Then $\lim\limits_{z \to z_0} f(z) = w_0$

$$\Leftrightarrow (i) \lim_{(x,y)\to(x_0,y_0)} \mathrm{u(x, y)} = u_0$$

$$(ii) \lim_{(x,y)\to(x_0,y_0)} v(\mathrm{x, y}) = v_0$$

Thm 2 Let $\lim_{z \to z_0} f(z) = w_0$,

$\lim_{z \to z_0} g(z) = W_0$. Then

(i) $\lim_{z \to z_0} [f(z) \pm g(z)] = w_0 \pm \mathrm{W}_0$.

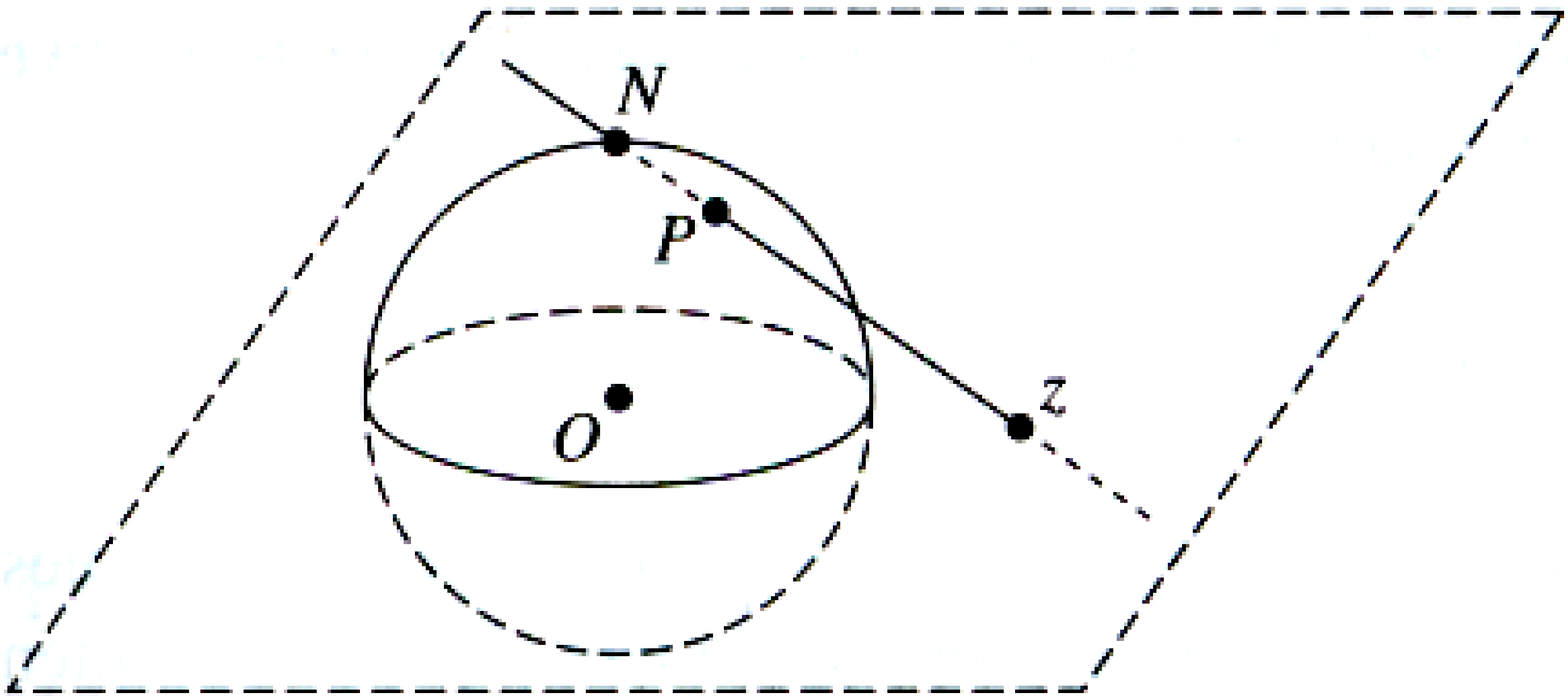
$$(ii)\ \lim_{z\to z_0} [f(z)g(z)] = w_0 W_0.$$

$$(iii)\ \lim_{z\to z_0} \left[\frac{f(z)}{g(z)}\right] = \frac{w_0}{W_0},\ \text{if}\quad W_0 \neq 0.$$

The point at infinity:

The point at infinity is denoted by ∞, and the complex plane together with the point at infinity is called the ***Extended complex Plane.***

Riemann Sphere & Stereographic Projection



Theorem

1. $$\lim_{z\to z_0} f(z)=\infty \Leftrightarrow \lim_{z\to z_0}\frac{1}{f(z)}=0.$$

2. $$\lim_{z\to\infty} f(z)=w_0 \Leftrightarrow \lim_{z\to 0} f\left(\frac{1}{z}\right)=w_0$$

3. $$\lim_{z\to\infty} f(z)=\infty \Leftrightarrow \lim_{z\to 0}\frac{1}{f\left(\frac{1}{z}\right)}=0$$

Sec 18. Continuity

1. A function f(z) is said to be continuous at a point z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

$i.e.$ *for* each $\in > 0$, $\exists\ \delta > 0$ such that

$$\left| f(z) - f(z_0) \right| < \in$$

whenever $\left| z - z_0 \right| < \delta$.

- The function $f(z)$ is said to be continuous in a region R if it is continuous at all points of the region R.

2.

If $f(z) = u(x, y) + iv(x, y)$, *then*

$f(z)$ is continuous iff

$$\operatorname{Re} f(z) = u(x, y) \quad and$$

$$\operatorname{Im} f(z) = v(x, y)$$

are continuous.

3. If f(z) and g(z) are continuous, then

(a) $f(z) \pm g(z)$

(b) $f(z)g(z)$

(c) $\dfrac{f(z)}{g(z)}, \quad g(z) \neq 0$

are all continuous.

4. Composition of two continuous map is continuous

Qs. Let f (z) is continuous at z_0 and $f(z_0) \neq 0$. Then show that $f(z) \neq 0$ throughout in some nbd of z_0.

- ***Solution:*** f (z) is continuous at z_0

$$\Rightarrow \lim_{z \to z_0} f(z) = f(z_0)$$

$\Rightarrow$ For each $\in>0$, $\exists$ a $\delta>0$ s.t.

$|f(z) - f(z_0)| < \in$

whenever $|z-z_0| < \delta$. (1)

Note that $f(z_0) \neq 0$ & (1) is valid for each $\in > 0$.

$$\text{Let } \in = \frac{1}{2}\left| f(z_0) \right| > 0.$$

If possible, let

$$\exists\, z = \bar{z} \in N(z_0, \delta) : |z - z_0| < \delta$$

such that $f(\bar{z}) = 0$

Then (1) gives

$$\left|f(\bar{z}) - f(z_0)\right| < \frac{1}{2}\left|f(z_0)\right|,$$

whenever $\left|\bar{z} - z_0\right| < \delta$

$$\Rightarrow \left| f(z_0) \right| < \frac{1}{2} \left| f(z_0) \right|$$

whenever $\left| \bar{z} - z_0 \right| < \delta$

a contradiction

$$\therefore \; f(z) \neq 0 \; \forall \, z \in N(z_0, \delta)$$

Result: Every continuous function in a closed & bounded region is bounded.

Let f (z) is continuous in a closed & bounded region R

$\Rightarrow \exists\ M > 0$ s. t $|f(z)| \leq M\ \forall\ z \in R$.

Ex1. If $f(z) = \dfrac{z}{\bar{z}}$, then

$$\lim_{z \to 0} f(z) \text{ does NOT exist.}$$

Soln: Use two path test.

Ex2. If $f(z) = \left(\dfrac{z}{\bar{z}}\right)^2$, then

$$\lim_{z \to 0} f(z) \text{ does NOT exist.}$$

Soln : Use two path test.

Ex 3. Discuss the continuity of f(z) at z = 0 if

$$(i)\ f(z) = \frac{\operatorname{Re} z}{1+|z|}$$

$$(ii)\ f(z) = z^{-1}\operatorname{Re} z$$

Sol. (i) $$f(z) = \frac{\operatorname{Re} z}{1+|z|}$$

$$= \frac{x}{1+\sqrt{x^2+y^2}}$$

$$\therefore \lim_{z\to 0} f(z) = \lim_{(x,y)\to(0,0)} \frac{x}{1+\sqrt{x^2+y^2}}$$

$$= 0 = f(0)$$

$$\Rightarrow f(z) \text{ is continuous at } z = 0$$

(ii) $f(z) = \dfrac{\operatorname{Re} z}{z} = \dfrac{x}{x+iy}$

We have

$$\lim_{z\to 0} f(z) = \lim_{(x,y)\to(0,0)} \frac{x}{x+iy}$$

$$\Rightarrow \lim_{z\to 0} f(z) = \lim_{(x,y)\to(0,0)} \frac{x}{x+imx},$$

$$(\text{along } y = mx)$$

$$= \frac{1}{1+im}$$

which is not unique

$\Rightarrow$ f(z) is not continuous at $z = 0$