



**BITS Pilani**  
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# **MATH F112 (Mathematics-II)**

## **Complex Analysis**



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# Lecture 31-35

## Integrals

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# Derivative of Functions $w(t)$

(1) Let  $w(t) = u(t) + i v(t)$  be a complex valued function of a real variable  $t$ , where  $u$  and  $v$  are real-valued functions of  $t$ .

Then  $\frac{dw}{dt} = w'(t) = u'(t) + i v'(t)$ , provided each of the derivatives  $u'$  &  $v'$  exists at  $t$

# Derivative of Functions $w(t)$

(2) If  $z_0$  is a complex constant, then

$$\frac{d}{dt} (z_0 w(t)) = z_0 \frac{dw}{dt}.$$

$$(3) \quad \frac{d}{dt} (e^{z_0 t}) = z_0 e^{z_0 t}.$$

(4) Mean Value Theorem for  
derivatives is NOT true.

# Derivative of Functions $w(t)$

Suppose that :

(i)  $w(t) = u(t) + i v(t)$ ,  $a \leq t \leq b$  be continuous, i.e.  $u$  and  $v$  are continuous on  $[a, b]$

(ii)  $w'(t)$  exists in  $a < t < b$ .

Then there may NOT exist any  $c$  in  $(a, b)$

such that  $w'(c) = \frac{w(b) - w(a)}{b - a}$

# Derivative of Functions $w(t)$

Example:

$$\text{Let } w(t) = e^{it}, 0 \leq t \leq 2\pi$$

$$\Rightarrow w'(t) = i e^{it}$$

$$\Rightarrow |w'(t)| = 1 \text{ for all } t \in [0, 2\pi]$$

$$\Rightarrow w'(t) \neq 0 \text{ for all } t \in [0, 2\pi]$$



# Derivative of Functions $w(t)$

$$\begin{aligned}\text{But } w(2\pi) - w(0) &= e^{i2\pi} - e^{i \cdot 0} \\ &= 0\end{aligned}$$

$\Rightarrow$  MVT for derivative is NOT true in the complex plane.

# Definite Integral of $w(t)$



Let  $w(t) = u(t) + i v(t)$  be a complex valued function of a real - variable  $t$ .

$u(t), v(t)$  : real - valued functions over  $a \leq t \leq b$ .

Then definite integral of  $w(t)$  is defined as

$$\int_a^b w(t) dt = \int_a^b u(t) dt + i \int_a^b v(t) dt, \text{ where the}$$

individual integrals on the right exist.



# Definite Integral of $w(t)$



$$\Rightarrow \operatorname{Re} \int_a^b w(t) dt = \int_a^b \operatorname{Re} w(t) dt,$$

$$\& \operatorname{Im} \int_a^b w(t) dt = \int_a^b \operatorname{Im} w(t) dt.$$

# Definite Integral of $w(t)$



Example :

$$\begin{aligned}\int_0^1 (1 - it)^2 dt &= \int_0^1 (1 - t^2 - 2it) dt \\ &= \int_0^1 (1 - t^2) dt - i \int_0^1 2t dt \\ &= \frac{2}{3} - i\end{aligned}$$

# Definite Integral of $w(t)$



Property:

Let  $w(t)$  be a complex-valued function integrable on  $[a, b]$ . Then

$$\left| \int_a^b w(t) dt \right| \leq \int_a^b |w(t)| dt$$

## Definitions

(1) *Curve*: A set of points  $z = (x, y)$  in the complex plane is said to be a curve  $C$  if

$$x = x(t), y = y(t)$$

are continuous functions of a real parameter  $t$ .

# Contours



We write

$$C: x = x(t), y = y(t)$$

Or

$$C: z(t) = x(t) + i y(t).$$

(2) *Arc* :

The portion between any two points of a curve is called an arc of the curve, i.e.

$C : x(t) + i y(t), a \leq t \leq b$ , is an arc.

For simplicity, we shall use the single term "curve" to denote the entire curve as well as an arc of the curve.

## (3) *Differentiable curve*:

The curve  $C: z(t) = x(t) + i y(t)$  is said to be differentiable if  $x'(t)$  &  $y'(t)$  exist and they are continuous in  $a \leq t \leq b$ , and we write

$$z'(t) = x'(t) + i y'(t)$$

If  $z'(t) \neq 0$ , then such a curve (arc) is said to be regular or smooth.

## (4) *Piecewise Smooth curve/arc :*

The curve

$$C: z(t) = x(t) + i y(t), \quad a \leq t \leq b,$$

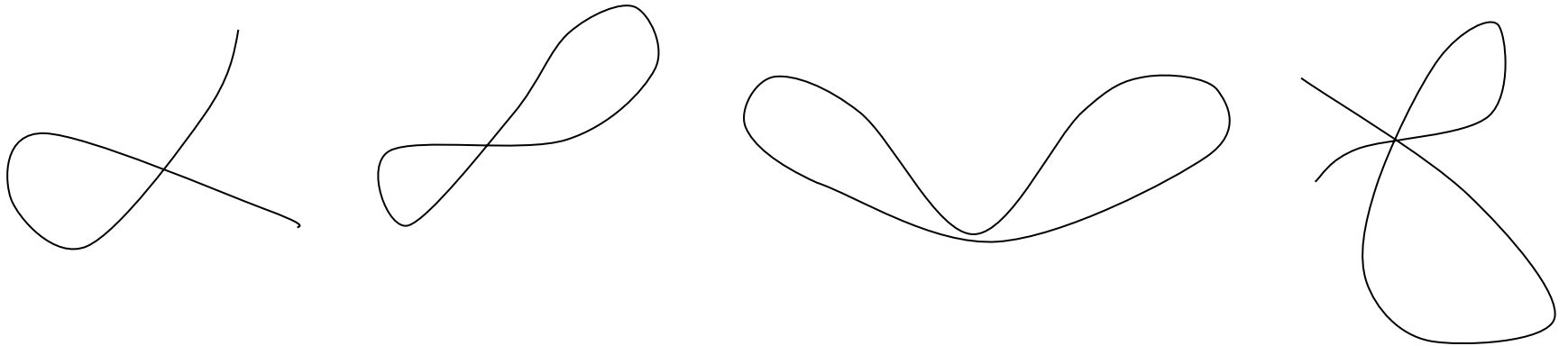
is said to be piecewise smooth if

there exists a finite no. of sub-intervals  $[a, a_1], [a_1, a_2], \dots, [a_{n-1}, b]$  of  $[a, b]$ , such that  $C$  is smooth on each sub-interval.



## (5) *Jordan arc / curve or simple curve :*

A curve may have points at which it intersects or touches itself. Such a point is called multiple point of the curve.



# Contours



A curve having NO MULTIPLE POINTS is called a simple/Jordon curve, i.e., a curve is said to be simple if it neither touches itself nor crosses itself, i.e., the curve  $C: z(t) = x(t) + i y(t)$  is said to be simple if  $z(t_1) \neq z(t_2)$  whenever  $t_1 \neq t_2$ .

# Contours



If the curve

$$C: z(t) = x(t) + i y(t), \quad a \leq t \leq b$$

is simple except for the fact that

$z(a) = z(b)$ , then  $C$  is said to be a simple closed curve or a Jordan curve.

## (6). *Length of a differentiable curve*

Let  $C: z(t) = x(t) + i y(t)$ ,  $a \leq t \leq b$   
be a differentiable curve (arc).

$$\Rightarrow z'(t) = x'(t) + i y'(t)$$

$$\text{and } |z'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2}$$

Then

$$L = \int_a^b |z'(t)| dt$$

is called the length of the curve  $C$ .

(7). *Contour* :

A Contour is a piecewise smooth arc,  
i.e. an arc consisting of finite number  
of smooth arcs joined end to end.

# Contour Integral



Let  $z = z(t)$ ,  $a \leq t \leq b$

denotes a contour  $C$  extending from a point  $z_1 = z(a)$  to a point  $z_2 = z(b)$ .

Let  $f(z)$  be piecewise continuous on  $C$ , i.e.  $f(z(t))$  is piecewise continuous on  $a \leq t \leq b$ .

# Contour Integral



Then we define the line integral or contour integral of  $f$  along  $C$  as follows:

$$\int_C f(z) dz = \int_a^b f(z(t)) z'(t) dt$$



# Contour Integral



## Properties:

$$(1) \left| \int_C f(z) dz \right| \leq \int_a^b |f(z(t)) z'(t)| dt,$$

$$C : z(t), a \leq t \leq b$$

# Contour Integral



(2) If  $z_0$  is a constant, then

$$\int_C z_0 f(z) dz = z_0 \int_C f(z) dz$$

$$(3) \int_C [f(z) + g(z)] dz$$

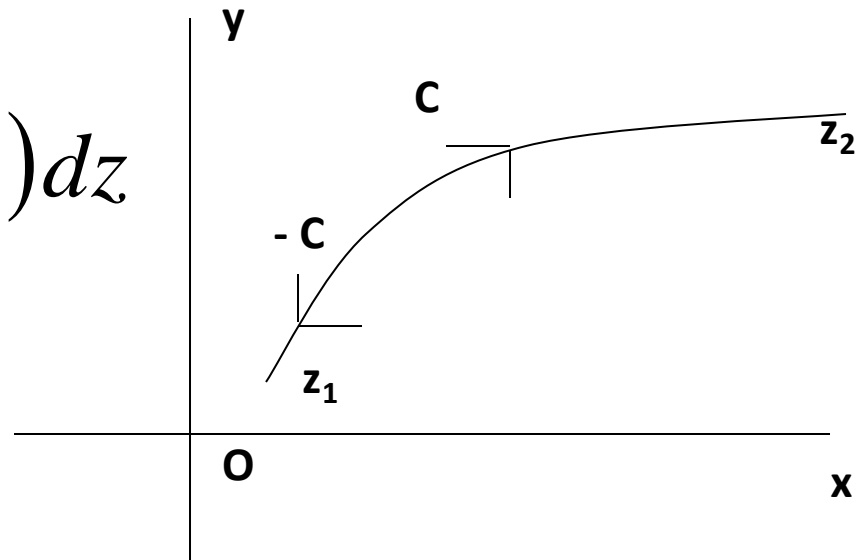
$$= \int_C f(z) dz + \int_C g(z) dz$$

# Contour Integral



(4) If the contour  $C: z = z(t)$ ,  $a \leq t \leq b$  is extended from  $z_1$  to  $z_2$ , then  $-C$  is extended from  $z_2$  to  $z_1$  i.e.  $-C: z = z(-t)$ ,  $-b \leq t \leq -a$

And 
$$\int_{-C} f(z) dz = - \int_C f(z) dz$$



# Contour Integral

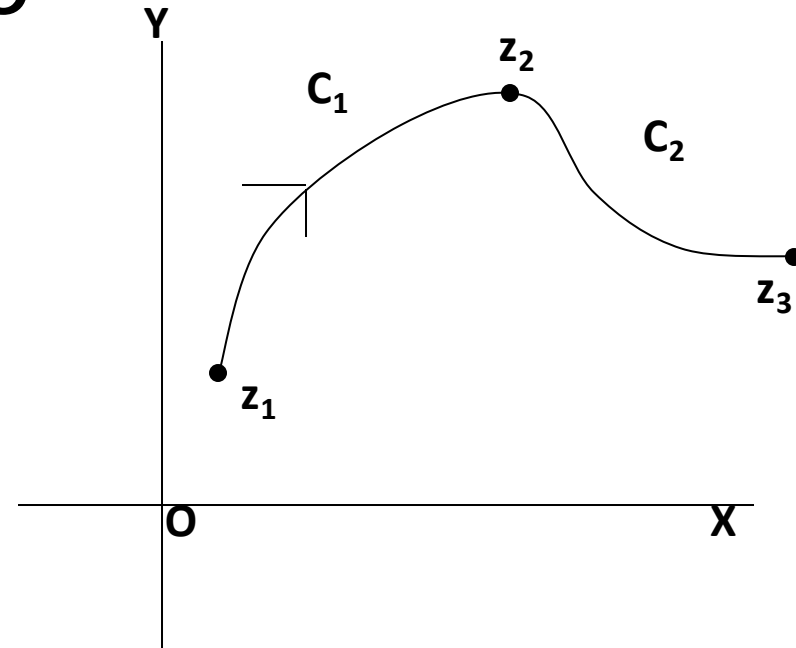


(5) Let  $C = C_1 \cup C_2$ , where

$C: z = z(t); a \leq t \leq b$

$C_1: z = z(t), a \leq t \leq c$

&  $C_2: z = z(t), c \leq t \leq b$



Then

$$\int_C f(z) dz = \int_{C_1} f(z) dz + \int_{C_2} f(z) dz$$

# Contour Integral



Ex.1

Let  $f(z) = \operatorname{Re} z$ , then evaluate

$$\int_C f(z) dz, \quad \text{where}$$

$$C : z(t) = t + it, \quad 0 \leq t \leq 1$$

# Contour Integral

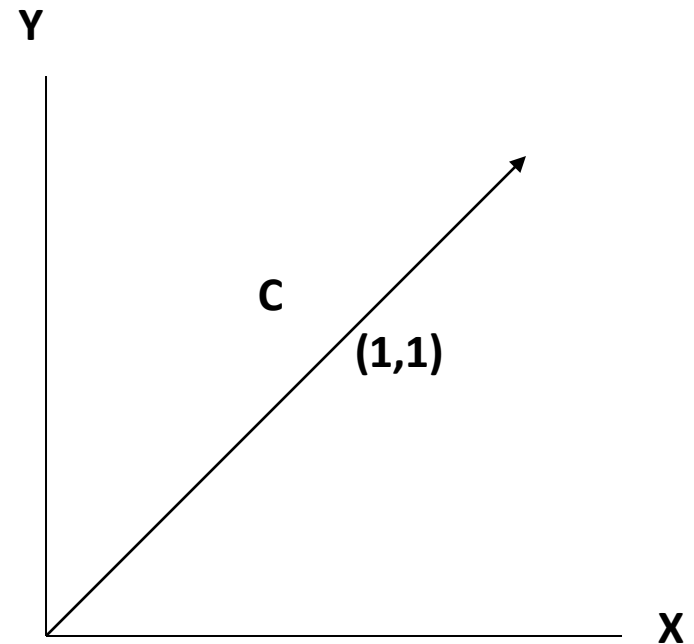


$$\int_C f(z) dz = \int_a^b f(z(t)) \dot{z}(t) dt$$

$$= \int_0^1 \operatorname{Re} z(t) \cdot \dot{z}(t) dt$$

$$= \int_0^1 t(1+i) dt$$

$$= \frac{1+i}{2}$$



# Contour Integral



Ex.2

Let  $f(z) = \frac{z+2}{z}$  &

$$C : z = 2e^{i\theta}, \quad \pi \leq \theta \leq 2\pi.$$

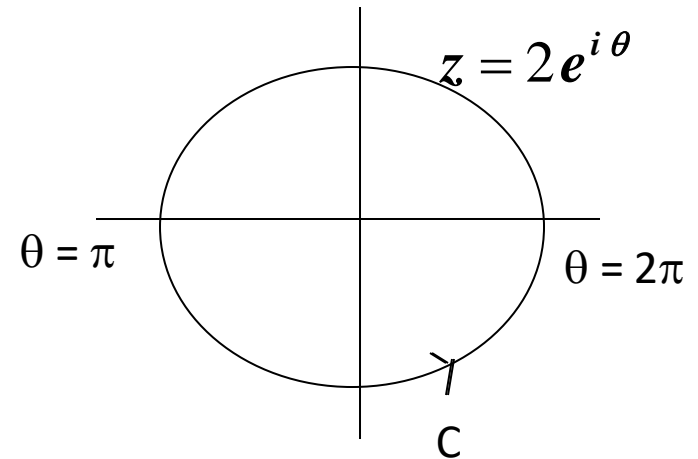
Then evaluate  $\int_C f(z) dz$

# Contour Integral



Soln:  $z = 2 e^{i \theta}$

$$\Rightarrow dz = 2 e^{i \theta} i d\theta$$





# Contour Integral



$$\begin{aligned}\therefore I &= \int_C f(z) dz \\ &= \int_{\pi}^{2\pi} \frac{2e^{i\theta} + 2}{2e^{i\theta}} \cdot 2e^{i\theta} \cdot i d\theta \\ &= 2i \int_{\pi}^{2\pi} (e^{i\theta} + 1) d\theta \\ &= 4 + 2\pi i\end{aligned}$$

# Contour Integral



Ex.3 Let  $f(z) = \begin{cases} 1, & y < 0 \\ 4y, & y > 0 \end{cases}$

&  $C$  is the arc from  $z = -1 - i$  to  $z = 1 + i$  along the curve  $y = x^3$ .

Then evaluate

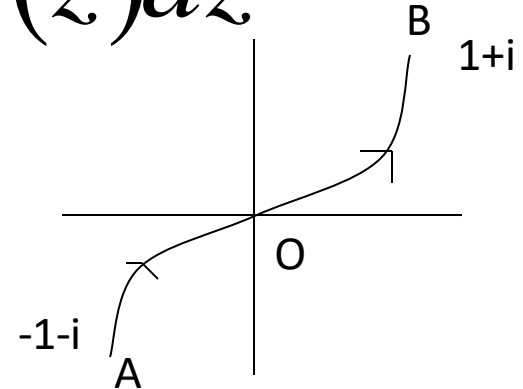
$$\int_C f(z) dz.$$

# Contour Integral



We have

$$\int_C f(z) dz = \int_{AO} f(z) dz + \int_{OB} f(z) dz$$



Along  $AO$  :

$$z = x + iy = x + ix^3, -1 \leq x \leq 0$$

# Contour Integral



$$\therefore \int_{AO} f(z) dz = \int_{-1}^0 1(1 + i 3 x^2) dx$$

$$= \left( x + i \frac{3x^3}{3} \right)_{-1}^0$$

$$= 1 + i$$

# Contour Integral



Along  $OB$ ,  $z = x + i x^3$ ,  $0 \leq x \leq 1$

$$\int_{OB} f(z) dz = \int_0^1 4y(1 + 3i x^2) dx$$

$$= \int_0^1 4x^3(1 + 3ix^2) dx$$

$$= 1 + 2i$$

$$\therefore \int_C f(z) dz = 1 + i + 1 + 2i = 2 + 3i$$

# ML-Inequality



Let  $f(z)$  be a piecewise continuous function defined on a contour  $C: z = z(t); a \leq t \leq b$ .

Suppose that on the contour  $C$ ,  $f(z)$  satisfies  $|f(z)| \leq M$  for some non-negative constant  $M$ .

# ML-Inequality



$$\text{Then } \left| \int_C f(z) dz \right| = \left| \int_a^b f(z(t)) z'(t) dt \right|$$

$$\leq \int_a^b |f(z(t))| |z'(t)| dt$$

$$\leq M \int_a^b |z'(t)| dt = ML,$$

where  $L = \int_a^b |z'(t)| dt$  is the length of  $C$  in  $a \leq t \leq b$ .

# ML-Inequality



Q .2(*p*.140) Let  $C$  denote the line segment from  $z = i$  to  $z = 1$ . By observing that, of all the points on that line segment, the midpoint is the closest to the origin, show that

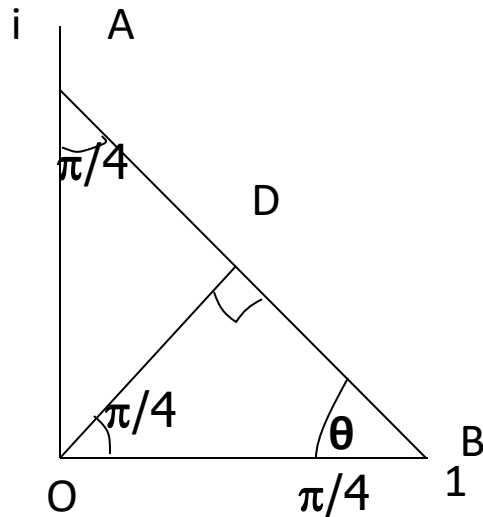
$$\left| \int_C \frac{dz}{z^4} \right| \leq 4\sqrt{2}.$$



# ML-Inequality



$D$  is the mid point of  $AB$



$$\sin \theta = \sin \frac{\pi}{4} = \frac{OD}{OB}$$

$$\Rightarrow OD = OB \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

# ML-Inequality



If  $z$  is any point on the line  $AB$ ,

$$\text{then } |z| \geq \frac{1}{\sqrt{2}}$$

$$\Rightarrow \frac{1}{|z|^4} \leq 4$$

# ML-Inequality



$$L = \text{length of } AB = \sqrt{1^2 + 1^2} \\ = \sqrt{2}$$

$$\therefore \left| \int_C \frac{dz}{z^4} \right| \leq ML = 4\sqrt{2}$$

# ML-Inequality



Q.3, (p. 140).: Let  $C$  be the boundary of the triangle with vertices  $0$ ,  $3i$ ,  $-4$ , oriented with counterclockwise direction.

Show that :  $\left| \int_C (e^z - \bar{z}) dz \right| \leq 60$ .

# ML-Inequality

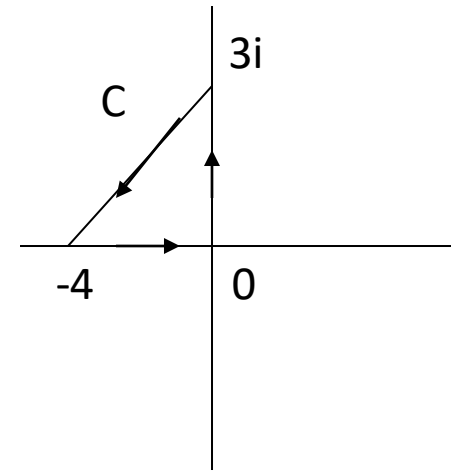


$$\text{Let } I = \int_C f(z) dz, f(z) = e^z - \bar{z}$$

$$\text{Then } |I| = \left| \int_C f(z) dz \right| \leq ML$$

where  $|f(z)| \leq M$  on  $C$

&  $L$  = the length of the curve  $C$   
= perimeter of the triangle



# ML-Inequality



$L$  = Perimeter of the triangle

$$= |3| + \left| \sqrt{3^2 + 4^2} \right| + |-4|$$

$$= 3 + 5 + 4 = 12$$

We have  $f(z) = e^z - \bar{z}$

$$\Rightarrow |f(z)| = |e^z - \bar{z}| \leq |e^z| + |\bar{z}|$$

# ML-Inequality



$$|f(z)| \leq |e^z| + |z|$$

$$\leq 1 + |-4| = 5$$

$$\Rightarrow M = 5$$

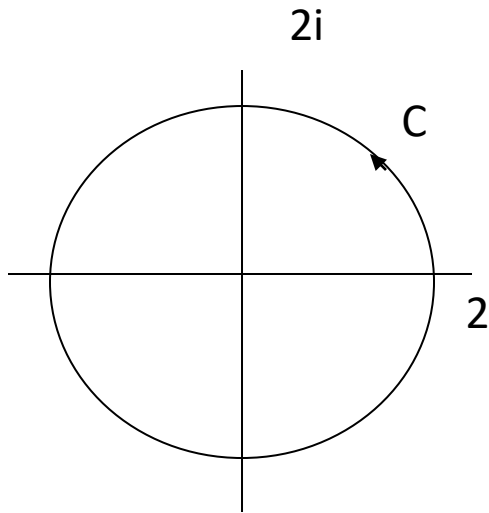
$$\therefore \left| \int_C f(z) dz \right| \leq ML = 60$$

Q1, (p.140): Let  $C$  be the arc of the circle  $|z| = 2$  from  $z = 2$  to  $z = 2i$  that lies in the first quadrant.

Show that 
$$\left| \int_C \frac{dz}{z^2 - 1} \right| \leq \frac{\pi}{3}.$$



# ML-Inequality



$$f(z) = \frac{1}{z^2 - 1}$$

$$\& I = \int_c f(z) dz$$

$$\Rightarrow |I| \leq ML$$

# ML-Inequality



On the curve  $C$ , we have

$$|f(z)| = \frac{1}{|z^2 - 1|} \leq \frac{1}{|z^2| - 1} = \frac{1}{4 - 1} = \frac{1}{3}$$

$$L = \frac{2\pi \times R}{4} = \frac{2\pi \times 2}{4} = \pi$$

$$\therefore |I| \leq \frac{\pi}{3}$$

# Antiderivatives



- Let  $f(z)$  be continuous function in a domain  $D$ .
- If there exists a function  $F(z)$  such that

$$F'(z) = f(z) \quad \text{for all } z \text{ in } D,$$

# Antiderivatives



then  $F(z)$  is called an antiderivative of  $f(z)$  in  $D$ .

Remark 1: An antiderivative of a given function  $f$  is an analytic function.

Remark 2: An antiderivative of a given function  $f$  is unique except for an additive complex constant.

# Antiderivatives



Theorem: Suppose that a function  $f(z)$  is continuous on a domain  $D$ . If any one of the following statement is true, then so are the others:

- $f(z)$  has an antiderivative  $F(z)$  in  $D$ ;
- The integrals of  $f(z)$  along contours lying entirely in  $D$  and extending from any fixed point  $z_1$  to any fixed point  $z_2$  all have same value.

# Antiderivatives



- The integral of  $f(z)$  around closed contours lying entirely in  $D$  all have value zero.

Corollary: Let  $f(z)$  is continuous in a domain  $D$  and  $F(z)$  is an antiderivative of  $f(z)$  in  $D$ .  
 $\Rightarrow F'(z) = f(z)$  for all  $z$  in  $D$ .

# Antiderivatives



Let  $z_1$  and  $z_2$  be any two points in  $D$  and  $C$  is any contour joining  $z_1$  &  $z_2$ , and lying ENTIRELY in  $D$ . Then

$$\int_C f(z) dz = F(z_2) - F(z_1).$$

# Antiderivatives



Ex.1: Use an antiderivative to evaluate  $\int_i^{i/2} e^{\pi z} dz$ .

Soln: Note that  $f(z) = e^{\pi z}$  has an antiderivative  $F(z) = \frac{e^{\pi z}}{\pi}$ .

$$\int_i^{i/2} e^{\pi z} dz = F(i/2) - F(i) = \frac{1}{\pi} [e^{i\pi/2} - e^{i\pi}] = \frac{1+i}{\pi}$$



# Antiderivatives



## Advised:

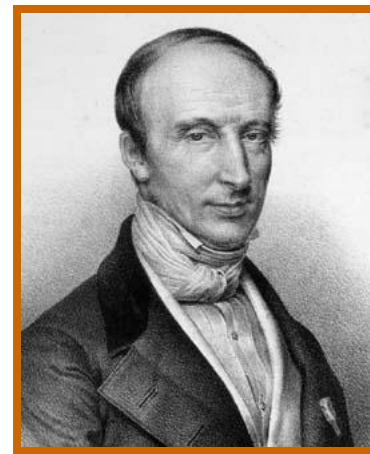
- See W.O.E. 3, p. 143
- See W.O.E. 4, p. 145
- See Q. No. 5, p. 149

# Cauchy-Goursat Theorem



If a function  $f$  is analytic at all points interior to and on a simple closed contour  $C$ , then

$$\int_C f(z) dz = 0$$



Augustin Cauchy  
(1789-1857)



Edourd Goursat  
(1858-1936)

# Cauchy-Goursat Theorem



**Example:** If  $C$  is any simple closed contour, in either direction, then

$$\int_C \exp(z^3) dz = 0$$

because the function  $f(z) = \exp(z^3)$

is analytic everywhere.

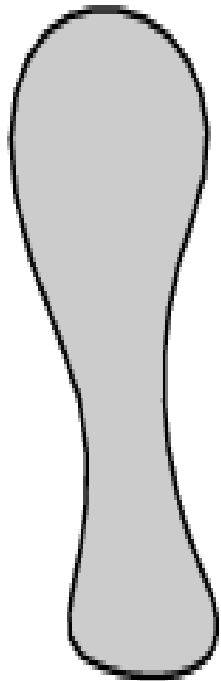
# Cauchy-Goursat Theorem



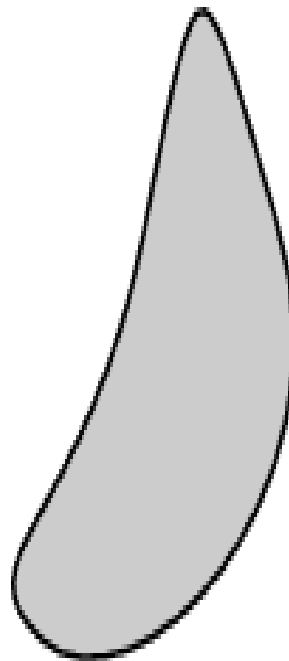
Defn: A **simply connected domain**  $D$  is a domain such that every simple closed contour within it encloses only points of  $D$ .

The set of points interior to a simply closed contour is an example of simply connected domain

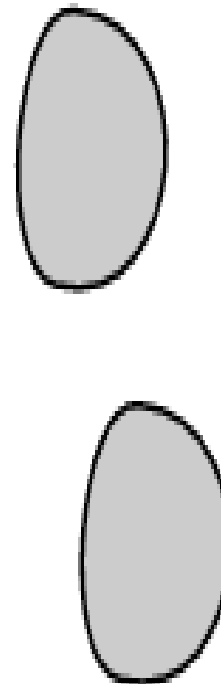
# Cauchy-Goursat Theorem



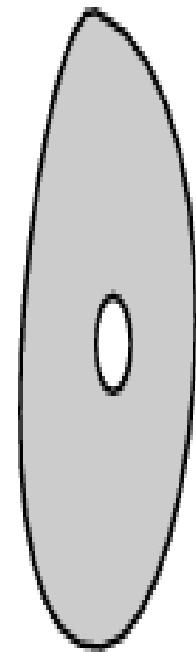
*simply connected*



*simply connected*



*not simply connected*



*not simply connected*

# Cauchy-Goursat Theorem

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A domain that is not simply connected is said to be *multiply connected* for example, the annular domain between two concentric circles.

# Cauchy-Goursat Theorem



The Cauchy – Goursat theorem for a simply connected domain  $D$  is as follows:

**Theorem:** If a function  $f$  is analytic throughout a simply connected domain  $D$ , then  $\int_C f(z) dz = 0$  for every closed contour  $C$  lying in  $D$ .

# Cauchy-Goursat Theorem



**Result:** Let  $C_1$  and  $C_2$  denote positively oriented simple closed contours, where  $C_2$  is interior to  $C_1$ . If a function  $f$  is analytic in the closed region consisting of those contours and all points between them, then

$$\int_{C_1} f(z) dz = \int_{C_2} f(z) dz$$



# Cauchy-Goursat Theorem



Ex. 1 Evaluate  $\int_C f(z) dz$

when  $f(z) = ze^{-z}$

$C: |z|=1$

Ans: 0 (Why??)

# Cauchy-Goursat Theorem



Ex.2 Evaluate

$$\int_C f(z) dz$$

when

$$f(z) = \frac{z^2 \sin z}{z - 4}, \quad C : |z| = 2.$$

Ans: 0 (Why??)

# Cauchy-Goursat Theorem



## Exercise:

Does Cauchy–Goursat Theorem hold separately for the **real or imaginary part** of an analytic function  $f(z)$  ? Justify your answer.

# Cauchy-Goursat Theorem



**Q3/161.** Let  $C_0$  denote the circle  $|z - z_0| = R$ , taken counter clockwise using the parametric representation 
$$z = z_0 + R e^{i\theta} \quad (-\pi \leq \theta \leq \pi)$$
 for  $C_0$ . Then derive the following integrations:

# Cauchy-Goursat Theorem



$$(a) \int_{C_0} \frac{dz}{z - z_0} = 2\pi i$$

$$(b) \int_{C_0} (z - z_0)^{n-1} dz = 0, n = \pm 1, \pm 2, \dots$$

$$(c) \int_{C_0} (z - z_0)^{a-1} dz = \frac{2iR^a}{a} \sin(a\pi),$$

where  $a \neq 0$  is any real no.

# Cauchy-Goursat Theorem



Sol. We have  $|z - z_0| = R$

$$\Rightarrow z - z_0 = R e^{i\theta}$$

$$\Rightarrow dz = R e^{i\theta} . i d\theta$$

a)

$$I = \int_{C_0} \frac{dz}{z - z_0}$$

# Cauchy-Goursat Theorem



$$\begin{aligned} &= \int_{-\pi}^{\pi} \frac{R e^{i\theta} \cdot i d\theta}{R e^{i\theta}} \\ &= i(\pi - (-\pi)) = 2\pi i \end{aligned}$$

# Cauchy-Goursat Theorem



b)

$$I = \int_{C_0} (z - z_0)^{n-1} dz$$

$$= \int_{-\pi}^{\pi} R^{n-1} e^{i(n-1)\theta} \cdot R e^{i\theta} i d\theta$$

$$= 0 \quad (\text{after simplification})$$



# Cauchy-Goursat Theorem



c)

$$I = \int_{C_0} (z - z_0)^{a-1} dz$$

$$= \int_{-\pi}^{\pi} R^{a-1} e^{i(a-1)\theta} \cdot R e^{i\theta} i d\theta$$

$$= \frac{2i R^a}{a} \sin(a\pi)$$

# Cauchy-Goursat Theorem



Theorem: Suppose that

- (i)  $C$  is a simple closed contour, described in the counter-clockwise direction,
- (ii)  $C_k$  ( $k = 1, 2, \dots, n$ ) are finite no. of simple closed contours, all described in the clockwise direction, which are interior to  $C$  and whose interiors are disjoint.

# Cauchy-Goursat Theorem



If  $f(z)$  is analytic throughout the closed region consisting of all points within and on  $C$  except for the points interior to  $C_k$ , then

$$\int_C f(z) dz + \sum_{k=1}^n \int_{C_k} f(z) dz = 0.$$

# Cauchy-Integral Formula



Let  $f$  be analytic everywhere inside and on a simple closed contour  $C$ , taken in the positive sense, then :

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{z - z_0}.$$

# Cauchy-Integral Formula



Suppose that a function  $f$  is analytic everywhere inside and on a simple closed contour  $C$ , taken in the positive sense. If  $z_0$  is any point interior to  $C$ , then :

# Cauchy-Integral Formula



$$(a) \quad f'(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z - z_0)^2},$$

$$(b) \quad f''(z_0) = \frac{(2)!}{2\pi i} \int_C \frac{f(z)dz}{(z - z_0)^3},$$

$$(c) \quad f^{(n)}(z_0) = \frac{(n)!}{2\pi i} \int_C \frac{f(z)dz}{(z - z_0)^{n+1}}.$$

# Cauchy-Integral Formula



## Theorem:

If  $f(z)$  is analytic at  $z_0$ , then its derivatives of all orders exist at  $z_0$  and are themselves analytic at  $z_0$ .

# Cauchy-Integral Formula



Qs.1(b)/170: Let  $C$  denote the positively oriented boundary of the square whose sides lie along the lines  $x = \pm 2$  and  $y = \pm 2$ . Evaluate the

following integral:  $\int_C \frac{\cos z \, dz}{z(z^2 + 8)}$ .

**Ans :  $\pi i/4$ .**



# Cauchy-Integral Formula



Qs. 2(b)/170: Find the value of the integral of  $g(z)$  around the circle  $|z - i| = 2$  in the positive sense when

$$g(z) = \frac{1}{(z^2 + 4)^2}.$$

# Cauchy-Integral Formula



$$\text{Sol : } \int_c \frac{dz}{(z^2 + 4)^2} = \int_c \frac{dz}{(z + 2i)^2 (z - 2i)^2}$$

$$= 2\pi i \frac{d}{dz} \left( \frac{1}{(z + 2i)^2} \right)_{z=2i} = \frac{\pi}{16}$$

# Cauchy-Integral Formula



**Q.4/170:** Let  $C$  be any simple closed contour, described the positive sense in the  $z$ -plane and write

$$g(w) = \int_C \frac{z^3 + 2z}{(z - w)^3} dz$$

# Cauchy-Integral Formula



Show that:

$$g(w) = 6\pi i w$$

when  $w$  is inside  $C$  and that

$$g(w) = 0$$

when  $w$  is outside  $C$ .

# Cauchy-Integral Formula



Case I: Let  $w$  is inside  $C$ .

Let  $f(z) = z^3 + 2z$ . Then

$$g(w) = \int_C \frac{f(z)}{(z-w)^3} dz,$$

$$= \frac{2\pi i}{2} f''(w)$$

# Cauchy-Integral Formula



$$f(z) = z^3 + 2z$$

$$\Rightarrow f'(z) = 3z^2 + 2$$

$$\Rightarrow f''(z) = 6z$$

$$\Rightarrow f''(w) = 6w$$

$$\therefore I = g(w) = 6\pi i w$$

# Cauchy-Integral Formula



Case 2: When  $w$  is outside  $C$ ,  
then by Cauchy-Goursat

Theorem  $g(w) = 0$ .

# Cauchy-Integral Formula



Q.5/170: Show that if  $f$  is analytic within and on a simple closed contour  $C$  and  $z_0$  is not on  $C$ , then

$$\int_C \frac{f'(z)}{(z - z_0)} dz = \int_C \frac{f(z)}{(z - z_0)^2} dz$$



# Cauchy-Integral Formula



Sol. Let

$$I_1 = \int_C \frac{f'(z)}{(z - z_0)} dz \text{ and}$$

$$I_2 = \int_C \frac{f(z)}{(z - z_0)^2} dz$$

# Cauchy-Integral Formula



Case I: Let  $z_0$  is inside  $C$ , then:

$$\begin{aligned} I_1 &= \int_C \frac{f'(z)}{(z - z_0)} dz = 2\pi i f'(z) \Big|_{z=z_0} \\ &= 2\pi i f'(z_0) \end{aligned}$$

# Cauchy-Integral Formula



and

$$\begin{aligned} I_2 &= \int_C \frac{f(z)}{(z - z_0)^2} dz \\ &= 2\pi i f'(z_0) \end{aligned}$$

$$\therefore I_1 = I_2 .$$

# Cauchy-Integral Formula



Case II: Let  $z_0$  is outside  $C$

Then  $I_1 = I_2 = 0$ .

(WHY ???)

# Morera's Theorem



If a function  $f$  is continuous in a domain  $D$  and if

$$\int_C f(z) dz = 0,$$

for every closed contour  $C$  lying in  $D$ , then  $f$  is analytic in  $D$ .

**Giacinto Morera (1856-1909)**



# Liouville's Theorem



If  $f$  is a function analytic in the entire complex plane ( $f$  is called an *entire* function) which is bounded in modulus, then  $f$  is a constant function.



**Joseph Liouville (1809-1882)**

# Fundamental Theorem of Algebra



Any polynomial

$$P(z) = a_0 + a_1z + a_2z^2 + \dots + a_nz^n$$

( $a_n \neq 0$ ) of degree  $n$  ( $n \geq 1$ )

has at least one zero.

That is, there exists at least one point  $z_0$  such that  $P(z_0) = 0$ .



**Carl Friedrich Gauss (1777-1855)**

# Fundamental Theorem of Algebra



**Note.** It follows that  $P$  can be factored into  $n$  (not necessarily distinct) linear terms:

$$P(z) = a_n (z - z_1)(z - z_2) \cdots (z - z_{n-1})(z - z_n),$$

Where the zeros of  $P$  are  $z_1, z_2, \dots, z_n$ .



# Problems



Ex. Evaluate  $\int_C \frac{dz}{z(z^2 + 1)}$  for all possible choices of the contour  $C$  that does not pass through any of the points  $0, \pm i$ .

# Problems



Solution:

Case 1. Let  $C$  does not enclose  $0, \pm i$ .

Then

$$I = \int_C \frac{dz}{z(z^2 + 1)} = 0 \text{ by C-G Theorem.}$$

# Problems



Case 2(a). Let  $C$  encloses only 0. Then

$$\begin{aligned} I &= \int_C \frac{dz}{z(z^2 + 1)} \\ &= \int_C \frac{f(z)dz}{z - 0}, \quad f(z) = \frac{1}{(z^2 + 1)} \\ &= 2\pi i f(0) = 2\pi i \end{aligned}$$

# Problems



Exercise:

Case 2(b). Let  $C$  encloses only  $i$ .

Ans:  $I = -\pi i$

Case 2(c). Let  $C$  encloses only  $-i$ .

Ans:  $I = -\pi i$

# Problems



Case 3 (a). Let  $C$  encloses only  $0$  &  $-i$ , then

$$I = \int_{C_0} \frac{dz}{z(z+i)(z-i)} + \int_{C_{-i}} \frac{dz}{z(z+i)(z-i)}$$

where  $C_0$  and  $C_{-i}$  are sufficiently small circles around  $0$  and  $-i$  resp.

# Problems



$$= \int_{C_0} \frac{1}{\frac{(z+i)(z-i)}{z}} dz + \int_{C_{-i}} \frac{1}{\frac{z(z-i)}{(z+i)}} dz$$

$$= (2\pi i) \left( -\frac{1}{i^2} \right) + (2\pi i) \left( \frac{1}{-i(-2i)} \right)$$

$$= \pi i$$

# Problems



Case 3 (b). Let  $C$  encloses only  $0$  &  $i$ , then

$$I = \int_{C_0} \frac{dz}{z(z+i)(z-i)} + \int_{C_i} \frac{dz}{z(z+i)(z-i)}$$

where  $C_0$  and  $C_i$  are sufficiently small circles around  $0$  and  $i$  resp.

# Problems



$$I = \int_{C_0} \frac{1}{\frac{(z+i)(z-i)}{z}} dz + \int_{C_i} \frac{1}{\frac{z(z+i)}{(z-i)}} dz$$
$$= 2\pi i + (2\pi i) \left( \frac{1}{i \cdot 2i} \right) = \pi i$$



# Problems



Case 3 (c). Let  $C$  encloses only  $-i$  &  $+i$ . Then

$$\begin{aligned} I &= \int_{C_i} \frac{1}{\frac{z(z+i)}{z-i}} dz + \int_{C_{-i}} \frac{1}{\frac{z(z-i)}{z+i}} dz \\ &= (2\pi i) \left( \frac{1}{i \cdot 2i} \right) + (2\pi i) \left( \frac{1}{-i \cdot (-2i)} \right) = -2\pi i \end{aligned}$$

# Problems



Case 3 (d). Let  $C$  encloses all of the points  $0, -i, +i$ . Then

$$\begin{aligned} I &= \int_{C_0} \frac{1}{\frac{z^2 + 1}{z}} dz + \int_{C_i} \frac{1}{\frac{z(z + i)}{z - i}} dz \\ &\quad + \int_{C_{-i}} \frac{1}{\frac{z(z - i)}{(z + i)}} dz \\ &= 2\pi i - \pi i - \pi i = 0 \end{aligned}$$

# Problems



Q.1 (p.178): Suppose  $f(z)$  is entire and that the harmonic function  $u(x, y) = \operatorname{Re}[f(z)]$  has an upper bound  $u_0$ . Show that  $u(x, y)$  must be a constant throughout the plane.

Prob : Suppose  $f(z)$  is entire and that the harmonic function  $v(x, y) = \operatorname{Im}[f(z)] > 0$ . Show that  $f(z)$  must be a constant throughout the plane.

Prob : Show that the function  $f(z) = \int_0^1 \frac{dt}{1-tz}$  is analytic

in the unit disk  $|z| < 1$ .