Q.2. Solution

Ans: 2 (i). (a). Given the subset $S = \{2x^2 + 2x + 16, x^2 + 3, 4x^2 + x + 16\}$ of P_2.
Step-I: First, We convert the matrices in S into vectors in R^3:

$$2x^2 + 2x + 16 \longrightarrow [2,2,16]$$
$$x^2 + 3 \longrightarrow [1,0,3]$$
$$4x^2 + x + 16 \longrightarrow [4,1,16]$$

Now, we use the Simplified Span Method on these vectors.

Step-II: Construct a matrix $A = \begin{bmatrix} 2 & 2 & 16 \\ 1 & 0 & 3 \\ 4 & 1 & 16 \end{bmatrix}$ **[Marks 2]**

Step-III Find the $C = RREF(A)$

Applying $R_1 \leftarrow \frac{R_1}{2}$ then we have

$$\begin{bmatrix} 1 & 1 & 8 \\ 1 & 0 & 3 \\ 4 & 1 & 16 \end{bmatrix}$$

Applying $R_2 \leftarrow R_2 - R_1$ and $R_3 \leftarrow R_3 - 4R_1$ then we have

$$\begin{bmatrix} 1 & 1 & 8 \\ 0 & -1 & -5 \\ 0 & -3 & -16 \end{bmatrix}$$

Applying $R_2 \leftarrow (-1)R_2$ and $R_3 \leftarrow (-1)R_3$ then we have

$$\begin{bmatrix} 1 & 1 & 8 \\ 0 & 1 & 5 \\ 0 & 3 & 16 \end{bmatrix}$$

Applying $R_3 \leftarrow R_3 - 3R_2$ then we have

$$\begin{bmatrix} 1 & 1 & 8 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $R_2 \leftarrow R_2 - 5R_3$ then we have

$$\begin{bmatrix} 1 & 1 & 8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $R_1 \leftarrow R_1 - R_2$ then we have

$$\begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $R_1 \leftarrow R_1 - 8R_3$ then we have

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = C = I_3$$ **[Marks 3]**

Step-IV: We convert the non-zero rows of C to polynomial form:

$$[1,0,0] \longrightarrow x^2$$
$$[0,1,0] \longrightarrow x$$ **[Marks 1]**
$$[0,0,1] \longrightarrow 1$$

Hence, $span(S)$ is the set of linear combinations of these 3 polynomials, that is $span(S) = \{ax^2 + bx + c \vee a, b, c \in R\}$. **[Marks 1]**

Since every vector in P_2 can be obtained in the $span(S)$. Hence S spans P_2. **[Marks 1]**

Remarks:

- Please make a note that no marks are awarded if you have written $span(S) = [abc]: a, b, c \in R$ as this is subset of R^3.
- No marks are awarded if you answered Sspans P_2 without any justification.
- No marks are given if you have written basis of $span(S) = \{[1,0,0], [0,1,0], [0,0,1]\}$.

(b). The set of non-zero rows of the row reduce matrix is a basis of $span(S)$. $B = \{x^2, x, 1\}$and $dim\big(span(S)\big) = 3$. **[Marks 1+1]**

(ii). Ans: (a). Let $S = \{v_1, \cdots, v_k\}$ consider the linear combination

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = 0$$ **[Marks 1]**

Now multiplying A on the both sides

$$A(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k) = A.0$$

$$\alpha_1 A v_1 + \alpha_2 A v_2 + \cdots + \alpha_k A v_k = 0 \qquad (1)$$ **[Marks 1]**

It is given that this set $\{Av_1, Av_2, \cdots, Av_k\}$ is linearly independent. Hence, equation (1) implies

$$\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$$ **[Marks 1]**

Hence the set S is linearly independent. **[Marks 1]**

Remarks:

If the proof starts with T is linearly independent and

$$\alpha_1 A v_1 + \alpha_2 A v_2 + \cdots + \alpha_k A v_k = 0$$

$$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = 0$$ **[No Marks]**

Ans (b). The converse to the statement is: If $S = \{v_1, \cdots, v_k\}$ is a linearly independent subset ofR^m, then $T = \{Av_1, \cdots, Av_k\}$ is a linearly independent subset of R^n. Now consider $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $S = \{[1,0], [1,2]\}$. **[Marks 2+1]**

Then $T = \{[1,1], [3,3]\}$. **[Marks 1]**

Note that S is linearly independent, but the vectors in T are linearly dependent. [**Marks 1]**

(iii) Ans: By using definition of vector space, the additive identity

$$0 = 0 \odot [x, y] = [0.x + 4(0) - 4, 0.y - 5(0) + 5] = [-4, 5].$$ [**Marks 3]**

Similarly, the additive inverse of

$$[x, y] = (-1) \odot [x, y] = [(-1)x + 4(-1) - 4, (-1)y - 5(-1) + 5] = [-x - 8, -y + 10].$$

[Marks 3]