Mathematics-II (MATH F112)

Finite Dimensional Vector Spaces
Sec.$1–Sec.$5

Sangita Yadav

Department of Mathematics
BITS Pilani, Pilani Campus, Rajasthan

Chapter: 4 (Finite Dimensional vector space)

1. Introduction to Vector Spaces
2. Subspaces
3. Span
4. Linear Independence
5. Basis and Dimension

Vector Space: A nonempty set $\mathcal{V}$ together with two operations **vector addition** (denoted as $\oplus$) and **scalar multiplication**(denoted as $\odot$) is said to be a (real) vector space if for every $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathcal{V}$ and for every $a, b \in \mathbb{R}$ the following properties hold:

1. $\mathbf{u} \oplus \mathbf{v} \in \mathcal{V}$ (Closed under vector addition)
2. $\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}$ (Commutativity)
3. $(\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w} = \mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w})$ (Associativity)
4. There exists an element $0 \in \mathcal{V}$, called a **zero vector**, such that $\mathbf{u} \oplus 0 = \mathbf{u}$ (Existence of additive identity)

5. For each $\mathbf{u} \in \mathcal{V}$, there is an element $-\mathbf{u} \in \mathcal{V}$ such that $\mathbf{u} \oplus (-\mathbf{u}) = 0$ (Existence of additive inverse)
6. $a \odot \mathbf{u} \in \mathcal{V}$ (Closed under scalar multiplication)
7. $a \odot (\mathbf{u} \oplus \mathbf{v}) = (a \odot \mathbf{u}) \oplus (a \odot \mathbf{v})$ (Distributivity)
8. $(a + b) \odot \mathbf{u} = a \odot \mathbf{u} \oplus b \odot \mathbf{u}$ (Distributivity)
9. $(ab) \odot \mathbf{u} = a \odot (b \odot \mathbf{u})$
10. $1 \odot \mathbf{u} = \mathbf{u}$.

Note that the set $\mathcal{V} = \{0\}$ is a vector space with respect to

- vector addition $0 \oplus 0 = 0$
- scalar multiplication $a \odot 0 = 0$ for all $a \in \mathbb{R}$

The vector space $\mathcal{V} = \{0\}$ is called the trivial vector space.

Example 1

The set $\mathbb{R}$ of real numbers is a vector space with respect to the following operations:

- $\mathbf{u} \oplus \mathbf{v} = \mathbf{u} + \mathbf{v}$ (vector addition)
- $a \odot \mathbf{u} = a\mathbf{u}$ (scalar multiplication)

for all $a, \mathbf{u}, \mathbf{v} \in \mathbb{R}$.

Question

Does the set $\mathbb{R}^+$ of positive real numbers form a vector space under the above defined vector addition and scalar multiplication?

Example 2

The set $\mathbb{R}^+$ of a positive real numbers is a vector space with respect to the following operations:

- $\mathbf{u} \oplus \mathbf{v} = \mathbf{u}.\mathbf{v}$ (vector addition)
- $a \odot \mathbf{u} = \mathbf{u}^a$ (scalar multiplication)

for all $a \in \mathbb{R}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^+$.

Example 3

The set $\mathbb{R}^2 = \{[x_1, x_2] \mid x_1, x_2 \in \mathbb{R}\}$ is a vector space with respect to the following operations:

- $[x_1, x_2] \oplus [y_1, y_2] = [x_1 + y_1, x_2 + y_2]$ (vector addition)
- $a \odot [x_1, x_2] = [ax_1, ax_2]$ (scalar multiplication)

for all $a \in \mathbb{R}$ and $[x_1, x_2], [y_1, y_2] \in \mathbb{R}^2$.

Soln. of Example 3: Let $\mathbf{u} = [x_1, x_2]$, $\mathbf{v} = [y_1, y_2]$ and $\mathbf{w} = [z_1, z_2] \in \mathbb{R}^2$ and $a, b \in \mathbb{R}$.

1. Closure Property:
   $\mathbf{u} \oplus \mathbf{v} = [x_1, x_2] \oplus [y_1, y_2] = [x_1 + y_1, x_2 + y_2] \in \mathbb{R}^2$.
2. Commutative Property:
   $$\begin{aligned} \mathbf{u} \oplus \mathbf{v} = [x_1 + y_1, x_2 + y_2] &= [y_1 + x_1, y_2 + x_2] \\ &\text{(commutativity of } \mathbb{R} \text{ under addition)} \\ &= [y_1, y_2] \oplus [x_1, x_2] = \mathbf{v} \oplus \mathbf{u} \end{aligned}$$
3. Associative Property:
   $$\begin{aligned} (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w} &= [(x_1 + y_1) + z_1, (x_2 + y_2) + z_2] \\ &= [x_1 + (y_1 + z_1), x_2 + (y_2 + z_2)] \\ &\text{(associativity of } \mathbb{R} \text{ under addition)} \end{aligned}$$

$$
\begin{aligned}
&= [x_1, x_2] \oplus [y_1 + z_1, y_2 + z_2] \\
&= [x_1, x_2] \oplus ([y_1, y_2] \oplus [z_1, z_2]) \\
&= \mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w})
\end{aligned}
$$

4. Existence of additive identity (zero vector): For any $\mathbf{u} = [x_1, x_2] \in \mathbb{R}^2$ there exists $0 = [0, 0] \in \mathbb{R}^2$ such that

$$
\begin{aligned}
\mathbf{u} \oplus 0 = [x_1, x_2] \oplus [0, 0] &= [x_1 + 0, x_2 + 0] \\
&= [x_1, x_2] \\
&= \mathbf{u}
\end{aligned}
$$

5. Existence of additive inverse: For each $\mathbf{u} = [x_1, x_2] \in \mathbb{R}^2$ there exists $-\mathbf{u} = [-x_1, -x_2]$ such that

$$
\begin{aligned}
\mathbf{u} \oplus (-\mathbf{u}) &= [x_1, x_2] \oplus [-x_1, -x_2] \\
&= [x_1 + (-x_1), x_2 + (-x_2)] = [0, 0] = 0
\end{aligned}
$$

6. Closure Property of scalar multiplication: $a \odot \mathbf{u} = a \odot [x_1, x_2] = [ax_1, ax_2] \in \mathbb{R}^2$. Thus, $\mathbb{R}^2$ is closed under scalar multiplication.

7. Distributivity over vector addition:

$$
\begin{aligned}
a \odot (\mathbf{u} \oplus \mathbf{v}) &= a \odot ([x_1, x_2] \oplus [y_1, y_2]) \\
&= a \odot [x_1 + y_1, x_2 + y_2] \\
&= [a(x_1 + y_1), a(x_2 + y_2)] \\
&= [ax_1 + ay_1, ax_2 + ay_2] \ (\text{distributivity in } \mathbb{R}) \\
&= [ax_1, ax_2] \oplus [ay_1, ay_2] \\
&= (a \odot [x_1, x_2]) \oplus (a \odot [y_1, y_2]) \\
&= (a \odot \mathbf{u}) \oplus (a \odot \mathbf{v})
\end{aligned}
$$

8. Distributivity over scalar addition:

$$\begin{aligned}(a+b)\odot \mathbf{u} &= (a+b)\odot [x_1, x_2]\\ &= [(a+b)x_1, (a+b)x_2]\\ &= [ax_1+bx_1, ax_2+bx_2] \text{ (distributivity in } \mathbb{R})\\ &= [ax_1, ax_2]\oplus [bx_1, bx_2]\\ &= (a\odot [x_1, x_2])\oplus (b\odot [x_1, x_2])\\ &= (a\odot \mathbf{u})\oplus (b\odot \mathbf{u})\end{aligned}$$

9. $$\begin{aligned}(ab)\odot \mathbf{u} &= (ab)\odot [x_1, x_2]\\ &= [(ab)x_1, (ab)x_2]\\ &= [a(bx_1), a(bx_2)]\\ &\text{(associativity of } \mathbb{R} \text{ under multiplication)}\\ &= a\odot [bx_1, bx_2]\\ &= a\odot (b\odot [x_1, x_2])\\ &= a\odot (b\odot \mathbf{u})\end{aligned}$$

10. $1 \odot \mathbf{u} = 1 \odot [x_1, x_2] = [1x_1, 1x_2] = [x_1, x_2] = \mathbf{u}.$

Thus $\mathbb{R}^2$ is vector space under usual vector addition and scalar multiplication.

Question

Does $\mathbb{R}^2$ form a vector space under the above defined vector addition and the following scalar multiplication

$$a \odot [x_1, x_2] = [0, ax_2]$$

for all $a \in \mathbb{R}$ and $[x_1, x_2] \in \mathbb{R}^2$?

Answer: No as $1 \odot [x_1, x_2] = [0, x_2] \neq [x_1, x_2]$

Example 4

The set $\mathbb{R}^n = \{[x_1, x_2, \ldots, x_n] \mid x_i \in \mathbb{R}\}$ is a vector space with respect to the following operations:

- $[x_1, x_2, \ldots, x_n] \oplus [y_1, y_2, \ldots, y_n]$
$$= [x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n] \text{ (vector addition)}$$
- $a \odot [x_1, x_2, \ldots, x_n] = [ax_1, ax_2, \ldots, ax_n]$ (scalar multiplication)

for all $a \in \mathbb{R}$ and $[x_1, x_2, \ldots, x_n], [y_1, y_2, \ldots, y_n] \in \mathbb{R}^n$.

Example 5

The set

$$\mathcal{M}_{mn} = \{[a_{ij}]_{m\times n} \mid a_{ij} \in \mathbb{R}\}$$

of all $m \times n$ matrices with real entries is a vector space with respect to the following operations:

- $[a_{ij}]_{m\times n} \oplus [b_{ij}]_{m\times n} = [a_{ij} + b_{ij}]_{m\times n}$ (vector addition)
- $\alpha \odot [a_{ij}]_{m\times n} = [\alpha a_{ij}]_{m\times n}$ (scalar multiplication)

for all $\alpha \in \mathbb{R}$ and $[a_{ij}]_{m\times n}, [b_{ij}]_{m\times n} \in \mathcal{M}_{mn}$.

Example 6

Let

$$\Phi = \{f \mid f : \mathbb{R} \to \mathbb{R}\}$$

be the set of real-valued functions defined on $\mathbb{R}$. Define

$$f \oplus g = f + g \ \ (\text{vector addition}),$$

where $(f+g)(x) = f(x) + g(x) \ \forall x \in \mathbb{R}$.

$$\text{and} \qquad a \odot f = af \ \ (\text{scalar multiplication}),$$

where $(af)(x) = af(x) \ \forall x \in \mathbb{R}$.

Then Φ is a vector space with respect to above defined vector addition and scalar multiplication.

Example 7

Let $\mathcal{P}_2$ denote the set of all polynomials of degree ≤ 2 with real coefficients. Define addition and scalar multiplication in usual way i.e. if

$$p(x) = a_0 + a_1 x + a_2 x^2 \text{ and } q(x) = b_0 + b_1 x + b_2 x^2$$

are in $\mathcal{P}_2$, then

$$p(x) \oplus q(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$$

$$c \odot p(x) = ca_0 + ca_1 x + ca_2 x^2.$$

Show that $\mathcal{P}_2$ is a vector space.

In general, for any fixed natural number n, the set $\mathcal{P}_n$ of all polynomials of degree less than or equal to n is a vector space.

Question: Does the set of all polynomials of degree 7 form a vector space under the usual operation of addition and scalar multiplication?

Answer: No.

Example 8

The set $\mathcal{P}$ of all polynomials with real coefficients is a vector space under the usual operation of polynomial (term by term) addition and scalar multiplication.

Example 9

Show that $\mathbb{R}^2$ forms a vector space with respect to the following operations:

- $[x_1, x_2] \oplus [y_1, y_2] = [x_1 + y_1 + 1, x_2 + y_2 - 2]$ (vector addition)
- $a \odot [x_1, x_2] = [ax_1 + a - 1, ax_2 - 2a + 2]$ (scalar multiplication)

for all $a \in \mathbb{R}$ and $[x_1, x_2], [y_1, y_2] \in \mathbb{R}^2$.

Solution:

- Both the operations satisfy Closure property.(1st and 6th)
- Vector addition satisfy commutativity and associativity property. (2nd and 3rd)

- Existence of Identity Element for Addition(4th): Let $[u, v]$ be the additive identity element $\mathbf{0}$.

$$\begin{aligned}[u, v] \oplus [x, y] &= [u + x + 1, v + y - 2] \\ [x, y] &= [u + x + 1, v + y - 2] \\ &\Rightarrow u = -1,\ v = 2.\end{aligned}$$

- Existence of Additive of Inverse(5th): Let the additive inverse of $[x, y]$ be $[x', y']$. Then

$$\begin{aligned}[x, y] \oplus [x', y'] &= [-1, 2] \\ [x + x' + 1, y + y, -2] &= [-1, 2] \\ \Rightarrow x' = -2 - x, y' &= 4 - y\end{aligned}$$

Hence $[x', y'] = [-2 - x, 4 - y]$.

- Distributivity for scalars over vectors:

$$
\begin{aligned}
&\alpha \odot ([x_1, y_1] \oplus [x_2, y_2]) \\
&= \alpha \odot [x_1 + x_2 + 1, y_1 + y_2 - 2] \\
&= [\alpha(x_1 + x_2 + 1) + \alpha - 1, \alpha(y_1 + y_2 - 2) - 2\alpha + 2] \\
&= [(\alpha x_1 + \alpha - 1) + (\alpha x_2 + \alpha - 1) + 1, \\
&\qquad (\alpha y_1 - 2\alpha + 2) + (\alpha y_2 - 2\alpha + 2) - 2] \\
&= [\alpha x_1 + \alpha - 1, \alpha y_1 - 2\alpha + 2] \oplus [\alpha x_2 + \alpha - 1, \alpha y_2 - 2\alpha + 2] \\
&= \alpha \odot [x_1, y_1] \oplus \alpha \odot [x_2, y_2]
\end{aligned}
$$

- Similarly Distributivity for vectors over scalars and associativity for scalar multiplication
- $1 \odot [x, y] = [x + 1 - 1, y - 2 + 2] = [x, y]$.

Theorem:

Let $\mathcal{V}$ be a vector space. Then for every $\mathbf{v} \in \mathcal{V}$ and $\alpha \in \mathbb{R}$, we have

1. $\alpha \odot \mathbf{0} = \mathbf{0}$
2. $0 \odot \mathbf{v} = \mathbf{0}$
3. $(-1) \odot \mathbf{v} = -\mathbf{v}$
4. If $\alpha \odot \mathbf{v} = \mathbf{0}$, then $\alpha = 0$ or $\mathbf{v} = \mathbf{0}$.

Exercises:

- Does set of all *nxn* matrices $\mathcal{M}_{nn}$ forms a vector space with matrix addition and scalar multiplication?
  Yes (Verify)
- What if above set is replaced by matrices of order *nxn* in RREF?
  No (Closure property does not hold)
- What if above set is replaced by non-singular matrices of order *nxn*?
  No (Closure property does not hold)
- What if above set is replaced by singular matrices of order *nxn*?
  No (Closure property does not hold)

Section 4.2 (Subspaces)

Subspace: A nonempty subset $\mathcal{W}$ of a vector space $\mathcal{V}$ is said to be a subspace of $\mathcal{V}$ if $\mathcal{W}$ is itself a vector space with respect to the same operations (vector addition and scalar multiplication) of $\mathcal{V}$.

Note that every vector space $\mathcal{V}$ has at least two subspaces: $\{0\}$ and $\mathcal{V}$ itself. The subspace $\{0\}$ is known as trivial subspace.

Example: The set

$$\mathcal{W} = \{[x, y] \in \mathbb{R}^2 \mid y = 0\}$$

forms a vector space with respect to usual vector addition and scalar multiplication in $\mathbb{R}^2$. Thus, $\mathcal{W}$ is a subspace of $\mathbb{R}^2$.

Question: Does the set

$$\mathcal{W} = \{[x, y] \in \mathbb{R}^2 \mid x \neq y\}$$

form a subspace of $\mathbb{R}^2$?

Theorem

A **nonempty** subset $\mathcal{W}$ of a vector space $\mathcal{V}$ is a subspace of $\mathcal{V}$ if and only if the following conditions hold:

- $\mathbf{u} \oplus \mathbf{v} \in W$ for all $\mathbf{u}, \mathbf{v} \in \mathcal{W}$.
- $a \odot \mathbf{u} \in \mathcal{W}$ for all $a \in \mathbb{R}, \mathbf{u} \in \mathcal{W}$.

Remark: If $\mathcal{W}$ is a subspace of a vector space $\mathcal{V}$, then $0 \in \mathcal{W}$.

Exercise: Examine whether the following sets are subspaces of $\mathbb{R}^3$.

- $W_1 = \{[x, y, z] \in \mathbb{R}^3 \mid x \geq 0\}$.
  No (Closure property for scalar multiplication).
- $W_2 = \{[x, y, z] \in \mathbb{R}^3 \mid x + y + z = 0\}$.
  Yes (Verify).
- $W_3 = \{[x, y, z] \in \mathbb{R}^3 \mid x = y^2\}$.
  No (Closure property for vector adition).
- $W_4 = \{[x, y, z] \in \mathbb{R}^3 \mid x + y + z = 2\}$.
  No (Closure property for scalar multiplication).
- $W_5 = \{[x, y, z] \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$.
  No (Closure property for scalar multiplication).

Exercise: Examine whether the following sets are subspaces of $\mathcal{M}_{22}$ under usual operations.

- $W_1 = \{A \in \mathcal{M}_{22} \mid A \text{ is singular}\}$.
  No (Closure property for vector adition fails).
- $W_2 = \{A \in \mathcal{M}_{22} \mid A \text{ is nonsingular}\}$.
  No (Closure property for vector adition fails).
- $W_3 = \{A \in \mathcal{M}_{22} \mid A \text{ is in RREF}\}$.
  No (Closure property for scalar multiplication fails).
- $W_4 = \{A \in \mathcal{M}_{22} \mid A \text{ is symmetric}\}$.
  Yes (Verify).
- $W_5 = \{A \in \mathcal{M}_{22} \mid A^2 = A\}$.
  No (Closure property for vector adition fails).

Exercise: Examine whether the following sets are subspaces of Φ (see Example 6).

- $W_1 = \{f \in \Phi \mid \ f(-x) = f(x) \text{ for all } x \in \mathbb{R}\}$.
  Yes.
- $W_2 = \{f \in \Phi \mid \ f(-x) = -f(x) \text{ for all } x \in \mathbb{R}\}$.
  Yes.
- $W_3 = \{f \in \Phi \mid \ f(1) = 0\}$.
  Yes (Verify).
- $W_4 = \left\{f \in \Phi \mid \ f\left(\frac{1}{2}\right) = f(1)\right\}$.
  Yes.
- $W_5 = \left\{f \in \Phi \mid \ f(1) = \frac{1}{2}\right\}$.
  No (closure property for vector addition fails).

Result: Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be two subspaces of vector space $\mathcal{V}$. Then

- their intersection i.e. $\mathcal{W}_1 \cap \mathcal{W}_2$ is a subspace of $\mathcal{V}$.
- their union $\mathcal{W}_1 \cup \mathcal{W}_2$ **need not** be a subspace of $\mathcal{V}$.
- $\mathcal{W}_1 \cup \mathcal{W}_2$ is subspace of $\mathcal{V}$ if and only if either $\mathcal{W}_1 \subset \mathcal{W}_2$ or $\mathcal{W}_2 \subset \mathcal{W}_1$.
- their sum, defined as

$$\mathcal{W}_1 + \mathcal{W}_2 = \{w_1 + w_2 \mid w_1 \in \mathcal{W}_1, w_2 \in \mathcal{W}_2\},$$

is a subspace of $\mathcal{V}$.

Section 4.3 (Span)

Question: Given a subset S of a vector space $\mathcal{V}$, how to construct a subspace containing S?

Linear combination: Let $\mathcal{V}$ be a vector space and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathcal{V}$. Then a vector $\mathbf{v} \in \mathcal{V}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ if

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k; \quad a_i(1 \leq i \leq k) \in \mathbb{R}$$

Example: The vector $[4, 3]$ is a linear combination of $[1, 0]$ and $[0, 1]$ in $\mathbb{R}^2$.
Note that

$$[4, 3] = 2[1, 1] + [2, 1].$$

Thus, $[4, 3]$ is a linear combination of $[1, 1]$ and $[1, 2]$ also.

Span of a set: Let S be a nonempty subset of a vector space $\mathcal{V}$. Then the span of S is the set of all possible (finite) linear combinations of the vectors in S and it is denoted by span(S) i.e.

$$\text{span}(S) = \{a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k \mid \mathbf{v}_i \in S, a_i \in \mathbb{R}, 1 \leq i \leq k\}$$

- For a subset $S = \{[1,0],[0,1]\}$ of $\mathbb{R}^2$, we have span$(S) = \mathbb{R}^2$.
- For a subset $S = \{[1,0,0],[0,1,0],[0,0,1]\}$ of $\mathbb{R}^3$, we have span$(S) = \mathbb{R}^3$.

Exercise: Let $\mathcal{V} = \mathbb{R}^3$ and $S = \{[1,1,0],[0,1,1]\}$.

- Find span(S).
- Do $[3,4,1]$ and $[2,5,1]$ belong to span(S)?

Solution:

$$\begin{aligned} \text{span}(S) &= \{a[1,1,0] + b[0,1,1] \mid a,b \in \mathbb{R}\} \\ &= \{[a, a+b, b] \mid a,b \in \mathbb{R}\} \end{aligned}$$

Clearly, $[3,4,1] \in \text{span}(S)$ but $[2,5,1] \notin \text{span}(S)$.

In this exercise **note that** span(S) is a subspace of $\mathbb{R}^3$.

Theorem:

Let S be a nonempty subset of a vector space $\mathcal{V}$. Then span(S) is the smallest subspace of $\mathcal{V}$ containing S.

- Convention: $\text{span}(\emptyset) = \{0\}$.

Remark:

Let S_1, S_2 be two subsets of a vector space $\mathcal{V}$. If $S_1 \subset S_2$ then span(S_1) is a subset of span(S_2).

Row space of a matrix: Let A be an $m \times n$ matrix. The row space of A, denoted by $\text{row}(A)$, is the subspace of $\mathbb{R}^n$ spanned by the rows of A.

Theorem: Let B be any matrix that is row equivalent to a matrix A. Then $\text{row}(B) = \text{row}(A)$.

Corollary: For any matrix A, we have

$$\text{row}(A) = \text{row}\left(\text{RREF}(A)\right).$$

Exercise: Let $\mathcal{V} = \mathbb{R}^3$ and

$$S = \{[1, 3, -1], [2, 7, -3], [4, 8, -7]\}.$$

Then find span(S) in simplified form.

Solution: To determine span(S) in simplified form consider

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 7 & -3 \\ 4 & 8 & -7 \end{bmatrix}$$

Note that span(S) = row(A) = row$\,(\text{RREF}(A))$.

Note that

$$\text{RREF}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

row (RREF(A))

$$= \{a[1,0,0] + b[0,1,0] + c[0,0,1] \mid a, b, c \in \mathbb{R}\}$$

$$\text{row}\,(\text{RREF}(A)) = \text{span}(S) = \{[a,b,c] \mid a, b, c \in \mathbb{R}\}$$

$$\text{span}(S) = \mathbb{R}^3$$

Simplified Span Method: Let S be a finite subset of $\mathbb{R}^n$ containing k vectors, with $k \geq 2$.

Step 1: Construct a matrix A of order $k \times n$ by using the vectors in S as the rows of A. Then $\text{span}(S) = \text{row}(A)$.

Step 2: Find $\text{RREF}(A)$.

Step 3: Then, the set of all linear combinations of the **nonzero rows** of $\text{RREF}(A)$ gives a simplified form for $\text{span}(S)$.

Exercise: For a given vector space $\mathcal{V}$ and a subset S of $\mathcal{V}$, find a simplified general form of span(S) using Simplified Span Method:

1. $\mathcal{V} = \mathbb{R}^3$, $S = \{[1,1,1],[2,1,1],[1,1,2]\}$.
2. $\mathcal{V} = \mathcal{P}_2$, $S = \left\{x^2+x+1, x+1, 1\right\}$.
3. $\mathcal{V} = \mathcal{P}_2$, $S = \left\{x^2+4x-3, 2x^2+x+5, 7x-11\right\}$.
4. $\mathcal{V} = \mathcal{M}_{22}$, $S = \left\{\begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix}, \begin{bmatrix} -2 & -5 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ -3 & 4 \end{bmatrix}\right\}$.

Section 4.4 (Linear Independence)

Definition: A subset $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of a vector space $\mathcal{V}$ is said to be linearly dependent (LD) if there exist real numbers $a_1, a_2, \ldots a_n$ not all zero such that

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n = 0.$$

S is linearly independent (LI) if it not linearly dependent i.e. if

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n = 0$$

Then

$$a_1 = a_2 = \ldots = a_n = 0.$$

Examples

- The subset $S = \{[1,0],[0,1]\}$ of $\mathbb{R}^2$ is linearly independent.
- The subset $S = \{[1,2],[5,10]\}$ of $\mathbb{R}^2$ is linearly dependent.
- The subset $S = \{[1,0,0],[0,1,0],[0,0,1]\}$ of $\mathbb{R}^3$ is linearly independent.

- The singleton set containing $0 \in \mathcal{V}$ i.e. $\{0\}$ is LD.
- For $\mathbf{v} \neq 0$ of $\mathcal{V}$, the set $\{\mathbf{v}\}$ is LI.
- Any set containing zero vector is linearly dependent.
- Let $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ be a set of nonzero vectors of $\mathcal{V}$. Then S is linearly dependent iff one of a vector is scalar multiple of other.
- Let S be a finite set of nonzero vectors having at least two elements. Then S is LD if and only if some vector in S can be expressed as a linear combination of the other vector in S.

Exercise: For a given vector space $\mathcal{V}$ and a given subset S of $\mathcal{V}$, check the linear independence of S in the following:

1. $\mathcal{V} = \mathcal{P}_2,\ S = \{1 + x + x^2, 1 - x + 3x^2, 1 + 3x - x^2\}$.
2. $\mathcal{V} = \mathcal{P}_2,\ S = \{1 + x, x + x^2, 1 + x^2\}$.
3. $\mathcal{V} = \mathcal{P}_n,\ S = \{1, x, x^2, \ldots, x^n\}$.
4. $\mathcal{V} = \Phi,\ S = \{\sin^2 x, \cos^2 x, \cos 2x\}$.
5. $\mathcal{V} = \Phi,\ S = \{\sin x, \cos x\}$.
6. $\mathcal{V} = \mathcal{M}_{22},\ S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.
7. $\mathcal{V} = \mathcal{M}_{22},\ S = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \right\}$.

Exercise: Show that

$$S = \{[3, 1, -1], [-5, -2, 2], [2, 2, -1]\}$$

is linearly independent subset of $\mathbb{R}^3$.

Solution: Let $a, b, c \in \mathbb{R}$ such that

$$a[3, 1, -1] + b[-5, -2, 2] + c[2, 2, -1] = 0$$

$$[3a, a, -a] + [-5b, -2b, 2b] + [2c, 2c, -c] = [0, 0, 0]$$
$$[3a - 5b + 2c, a - 2b + 2c, -a + 2b - c] = [0, 0, 0]$$

To find $a, b, c \in \mathbb{R}$, we need to solve the following homogenous system:

$$\begin{aligned} 3a - 5b + 2c &= 0 \\ a - 2b + 2c &= 0 \\ -a + 2b - c &= 0 \end{aligned}$$

To solve above homogenous system, write augmented matrix

$$[A|0] = \left[\begin{array}{rrr|r} 3 & -5 & 2 & 0 \\ 1 & -2 & 2 & 0 \\ -1 & 2 & -1 & 0 \end{array}\right]$$

reduced row echelon form of $[A|0]$ is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Thus, we have $a = 0, b = 0, c = 0$. Hence, S is linearly independent subset of $\mathbb{R}^3$.

Independence Test Method: Let S be a finite set of vectors in $\mathbb{R}^n$. To check whether S is LI, perform the following steps:

Step 1: Form a matrix A whose columns are the vectors in S.

Step 2: Find RREF(A).

Step 3: If there is a pivot in every column of RREF(A), then S is LI. Otherwise S is LD.

Exercise:
Consider a subset of $\mathcal{P}_3$

$$S = \{2x^3 - x + 3, 3x^3 + 2x - 2, x^3 - 4x + 8, 4x^3 + 5x - 7\}.$$

(a) Show that S is linearly dependent.
(b) Show that every three-element subset of S is linearly dependent.
(c) Explain why every subset of S containing exactly two vectors is linearly independent. (Note:There are six possible two-element subsets.)

Theorem: If S is any subset of $\mathbb{R}^n$ containing k distinct vectors, where $k > n$, then S is linearly dependent.

Exercise: Examine the linear independence of a subset $S = \{[2, -5, 1], [1, 1, -1], [0, 2, -3], [2, 2, 6]\}$ of $\mathbb{R}^3$.

Result: Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a linearly independent subset of a vector space $\mathcal{V}$. If $\mathbf{v} \in \mathcal{V}$ and $\mathbf{v} \notin \text{span}(S)$, then $S_1 = \{\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a linearly independent set.

Theorem: A nonempty finite subset S of a vector space $\mathcal{V}$ is LI iff every vector $\mathbf{v} \in \text{span}(S)$ can be expressed **uniquely** as a linear combination of the elements of S.

Definition: An **infinite** subset S of a vector space $\mathcal{V}$ is linearly dependent if there is some finite subset T of S such that T is linearly dependent.

Example: The subset

$$S = \{A \in \mathcal{M}_{22} \mid A \text{ is nonsingular}\}$$

of vector space $\mathcal{M}_{22}$ is linearly dependent.

Solution: Note that the finite subset $T = \{I_2, 2I_2\}$ of S is linearly dependent as $2I_2$ is scalar multiple of I_2. Hence, S is linearly dependent.

Definition: An **infinite** subset S of a vector space $\mathcal{V}$ is linearly independent if every finite subset of S is linearly independent.

Result: An infinite subset S of a vector space $\mathcal{V}$ is linearly independent if and only if no vector in S is a finite linear combination of other vector in S.

Example: The subset $S = \{1, x, x^2, x^3, x^4, \ldots\}$ of vector space $\mathcal{P}$ is linearly independent.

Section 4.5, Basis and Dimension

Basis: A subset B of a vector space $\mathcal{V}$ is said to be a basis of $\mathcal{V}$ if

1. B is LI, and
2. $\text{span}(B) = \mathcal{V}$.

Examples

- The subset $B = \{[1,0],[0,1]\}$ is a basis of $\mathbb{R}^2$ as B is LI and $\text{span}(B) = \mathbb{R}^2$.The subset B is called the **standard basis** of $\mathbb{R}^2$.
- The subset $B = \{[1,2],[3,4]\}$ is a basis of $\mathbb{R}^2$ as B is LI (verify!) and $\text{span}(B) = \mathbb{R}^2$ (verify!).
- The subset $B = \{[1,0,0],[0,1,0],[0,0,1]\}$ is a basis of $\mathbb{R}^3$ as it is LI and $\text{span}(B) = \mathbb{R}^3$. The subset B is called the **standard basis** of $\mathbb{R}^3$.

Think about some more basis of $\mathbb{R}^2$ and $\mathbb{R}^3$.

- The subset $B = \{1, x, x^2, \ldots, x^n\}$ is a basis of $\mathcal{P}_n$ as B is LI (verify!) and span$(B) = \mathcal{P}_n$ (verify!).
- The subset

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

is a basis of $\mathcal{M}_{22}$.

Verify that B is LI and span$(B) = \mathcal{M}_{22}$.

Example: Let $S = \{t+1, t-1, t^2+t\}$. Show that S is a basis of $\mathcal{P}_2$.

Solution: For $span(S) = \mathcal{P}_2$:
Can we find α, β and γ such that

$$at^2 + bt + c = \alpha(t+1) + \beta(t-1) + \gamma(t^2+t)$$

for any a, b, c??
It gives the system of linear equations in α, β and γ as

$$\begin{aligned} \gamma &= a, \\ \alpha + \beta + \gamma &= b, \\ \alpha - \beta &= c. \end{aligned}$$

The augmented matrix is

$$\begin{bmatrix} 0 & 0 & 1 & a \\ 1 & 1 & 1 & b \\ 1 & -1 & 0 & c \end{bmatrix}$$

The RREF form is

$$\begin{bmatrix} 1 & 0 & 0 & \frac{b-a+c}{2} \\ 0 & 1 & 0 & \frac{b-a-c}{2} \\ 0 & 0 & 1 & a \end{bmatrix}$$

The system is consistent for any values of a, b, c.
Therefore

$$\text{span}(S) = \mathcal{P}_2.$$

For S is LI:
Consider

$$\alpha(t+1)+\beta(t-1)+\gamma(t^2+t)=0$$

which gives

$$\begin{aligned} \alpha-\beta &= 0, \\ \alpha+\beta+\gamma &= 0, \\ \gamma &= 0. \end{aligned}$$

The coefficient matrix is $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

and its RREF is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

This means system will have only trivial solution

$\alpha = \beta = \gamma = 0$.

Therefor S is LI.

Hence S is a basis for $\mathcal{P}_2$.

Theorem: Every vector space has a basis.

Theorem: If a vector space V has a finite basis, then all bases for V are finite and have the same number of elements.

Definition: The dimension of a vector space $\mathcal{V}$ is the number of vectors in a basis of $\mathcal{V}$ and it is denoted by $\dim(\mathcal{V})$.

The dimension of the trivial vector space $\{0\}$ is defined to be zero.

Definition: A vector space $\mathcal{V}$ is said to be finite dimensional if it has a basis containing finite number of elements. If a vector space $\mathcal{V}$ has no finite basis then $\mathcal{V}$ is called infinite dimensional.

Examples

- $\dim(\mathbb{R}^2) = 2$.
- $\dim(\mathbb{R}^3) = 3$.
- $\dim(\mathbb{R}^n) = n$.
- $\dim(\mathcal{P}_n) = n + 1$.
- $\dim(\mathcal{M}_{mn}) = mn$.
- Since $\{1, x, x^2, x^3, \ldots\}$ is a basis of $\mathcal{P}$ (the vector space of all polynomials with real coefficients), thus the vector space $\mathcal{P}$ is infinite dimensional.

Exercise: Find a basis and the dimension of a subspace W of $\mathbb{R}^3$, where

$$W = \{[x, y, z] \in \mathbb{R}^3 \mid x - 3y + 4z = 0\}.$$

Solution: The general solution of the equation $x - 3y + 4z = 0$ is given by $\{[3t - 4s, t, s] \mid t, s \in \mathbb{R}\}$. Thus

$$\begin{aligned} W &= \{[3t - 4s, t, s] \mid t, s \in \mathbb{R}\} \\ W &= \{s[-4, 0, 1] + t[3, 1, 0] \mid t, s \in \mathbb{R}\} \\ W &= \operatorname{span}\left(\{[-4, 0, 1], [3, 1, 0]\}\right). \end{aligned}$$

Note that the set $\{[-4, 0, 1], [3, 1, 0]\}$ is linearly independent (show it).

Hence, the subset $\{[-4,0,1],[3,1,0]\}$ is a basis of W and $\dim(W) = 2$.

Exercise: Find a basis and the dimension of a subspace W of $\mathcal{P}_4$, where

$$W = \{\mathbf{p} \in \mathcal{P}_4 \mid \mathbf{p}(2) = 0\}.$$

Exercise: Find the dimension of a subspace W of $\mathcal{P}_2$ consisting of all vectors of the form $ax^2 + bx + c$, where $a = b + c$.

Answer: dim(W)=2.

Theorem: Let W be a subspace of a finite dimensional vector space $\mathcal{V}$. Then

- W is also finite dimensional and $\dim W \leq \dim \mathcal{V}$.
- $\dim W = \dim \mathcal{V}$ if and only if $W = \mathcal{V}$.

Notation: $|A|$-The number of elements in A.

Theorem: Let $\mathcal{V}$ be a finite dimensional vector space such that $\dim(\mathcal{V}) = n$.

- Suppose S is a finite subset of $\mathcal{V}$ that spans $\mathcal{V}$. Then $|S| \geq n$. Moreover, $|S| = n$ if and only if S is a basis of $\mathcal{V}$.
- Suppose T is a linearly independent subset of $\mathcal{V}$. Then T is finite and $|T| \leq n$. Moreover, $|T| = n$ if and only if T is a basis for $\mathcal{V}$.

Exercise: For a given vector space $\mathcal{V}$ and a given subset B of $\mathcal{V}$, determine which of following B form a basis of the respective vector space $\mathcal{V}$:

1. $\mathcal{V} = \mathbb{R}^3$, $B = \{[3, 1, -1], [-5, -2, 2], [2, 2, -1]\}$.
2. $\mathcal{V} = \mathbb{R}^4$, $B = \{[7, 1, 2, 0], [8, 0, 1, -1], [1, 0, 0, -2]\}$.
3. $\mathcal{V} = \mathcal{P}_2$, $B = \{1 + x, x + x^2, 1 + x^2\}$.
4. $\mathcal{V} = \mathcal{P}_2$, $B = \{1 - x, x - x^2, 1 - x^2\}$.
5. $\mathcal{V} = \mathcal{M}_{22}$,

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right\}.$$

Exercise: Let $S = \{[4, 2, 1], [2, 6, -5], [1, -2, 3]\}$ be a subset of vector space $\mathbb{R}^3$.

- Examine the linear independence of S.
- Find $\dim(\text{span}(S))$.

Thank You