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BITS Pilani
Pilani Campus

CS/IS F214
Logic in Computer Science

MODULE: **PROPOSITIONAL LOGIC**

Natural Deduction: Rules for Negation

ND: Negation–Elimination Rule

$$\frac{\phi \qquad \neg\phi}{\bot}\ \neg e$$

This symbol – *read **bottom*** – denotes a contradiction.

This rule is also referred to as *<u>contradiction introduction</u>* (_|_ introduction)

ND: Contradiction-Elimination Rule

$$\frac{\bot}{\phi}\ \bot e$$

You can infer anything from a contradiction!

ND: Negation–Introduction Rule

$$\dfrac{\begin{matrix}\text{Assume } \phi \\ . \\ . \\ . \\ \bot\end{matrix}}{\neg \phi}\ \neg i$$

This rule is read as:

if you can derive a contradiction assuming φ is true then φ must be false (i.e. ¬ φ must be true)

Exercise: *Prove :*

i_am_god --> happy, i_am_god --> ¬ happy |-- ¬ i_am_god

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MODULE: **PROPOSITIONAL LOGIC**

Natural Deduction: Rules for Double Negation

ND – Rules for Double negation

$$\frac{\neg\neg\phi}{\phi}\ \neg\neg e \qquad \frac{\phi}{\neg\neg\phi}\ \neg\neg i$$

Double Negation – Example

Exercise:

Prove the following sequent: **$p, \neg\neg(q \wedge r) \vdash \neg\neg p \wedge r$**

	Deduction	Explanation
1	**p**	**Premise**
2	**$\neg\neg(q \wedge r)$**	**Premise**
3	**$q \wedge r$**	**$\neg\neg$e 2**
4	**r**	**$\wedge$e2 3**
5	**$\neg\neg p$**	**$\neg\neg$i 1**
6	**$\neg\neg p \wedge r$**	**$\wedge$i 3,4**

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MODULE: **PROPOSITIONAL LOGIC**

Natural Deduction: Rules for Disjunction

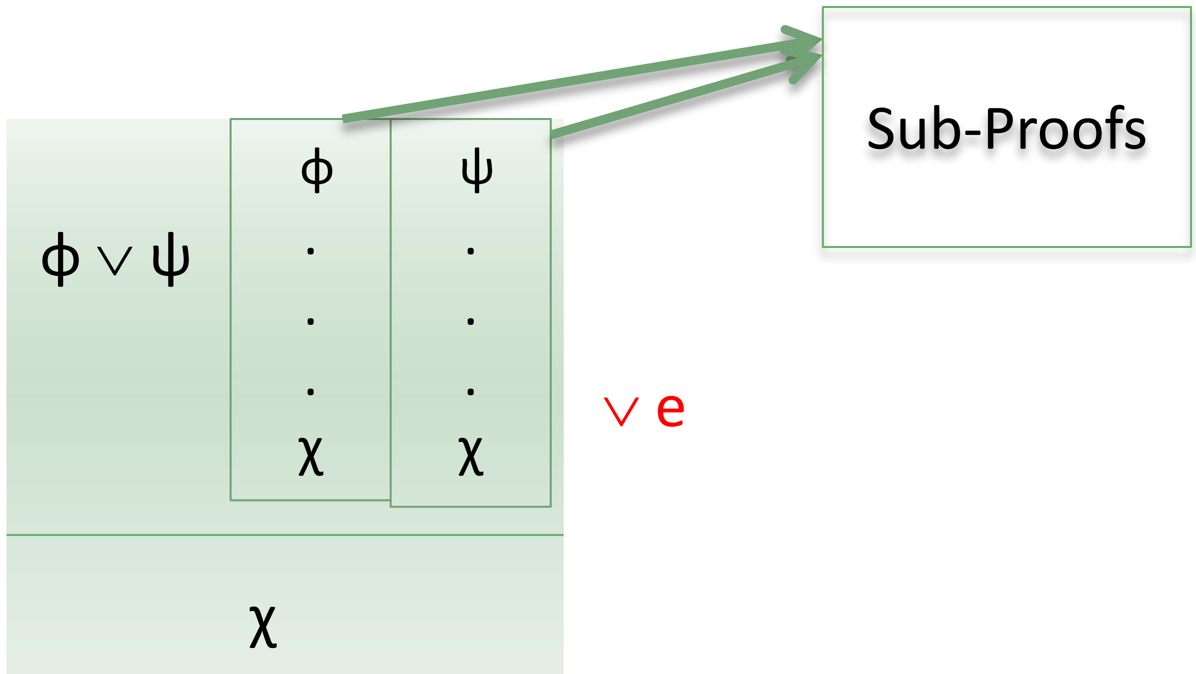
ND: Proof Rules: OR-introduction

Rules for Disjunction Introduction

$$\frac{\phi}{\phi \vee \psi} \ \vee i_1 \qquad\qquad \frac{\psi}{\phi \vee \psi} \ \vee i_2$$

ND: Proof Rules: OR Elimination

Rule for Disjunction Elimination



Proofs using rules for Disjunction – Example 1

- Recall from Boolean algebra, the distribution rule
  - $p \wedge (q \vee r)$ "is equivalent to" $(p \wedge q) \vee (p \wedge r)$
- *To prove that the two formulas are equivalent*
  - we must *prove that one of them can be derived from the other* and *vice versa*
- For this example:
  - We will prove
    - $\mathbf{p \wedge (q \vee r) \;|\text{--}\; (p \wedge q) \vee (p \wedge r)}$
  - and this will be an exercise for you:
    - $\mathbf{(p \wedge q) \vee (p \wedge r) \;|\text{--}\; p \wedge (q \vee r)}$

Proofs using rules for Disjunction – Example 1

- Prove:
  - $p \wedge (q \vee r) \quad |\text{--} \quad (p \wedge q) \vee (p \wedge r)$

	Deduction	Explanation	
	p ∧ (q ∨ r)	**Premise**	
?	**q** ∨ **r**		
	q	**Assumption**	Sub-proof
	…		
?	**(p**∧q**)** ∨ **(p**∧**r)**		
	r	**Assumption**	Sub-proof
	…		
2	**(p**∧**q)** ∨ **(p**∧**r)**		
1	**(p** ∧ **q)** ∨ **(p** ∧ **r)**	∨**e ?-2, ?-?,**	

Proofs using rules for Disjunction – Example 1

- Prove:
  - $p \wedge (q \vee r)$ |-- $(p \wedge q) \vee (p \wedge r)$

	Deduction	Explanation	
	$p \wedge (q \vee r)$	Premise	
	p		
?	$q \vee r$		
	q	Assumption	
	…		
5	$(p \wedge q) \vee (p \wedge r)$		
4	r	Assumption	
3	$p \wedge r$	$\wedge i$?, 4	Sub-proof
2	$(p \wedge q) \vee (p \wedge r)$	$\vee i_2$ 3	
1	$(p \wedge q) \vee (p \wedge r)$	$\vee e$ 4-2, ?-5,	

Proofs using rules for Disjunction – Example 1

- Prove:
  - $p \wedge (q \vee r) \quad |\text{--} \quad (p \wedge q) \vee (p \wedge r)$

	Deduction	Explanation	
10	$p \wedge (q \vee r)$	Premise	
9	p	$\wedge e_1$ 10	
8	q|r	$\wedge e_2$ 10	
7	q	Assumption	Sub-proof
6	$p \wedge q$	$\wedge i$ 9,7	
5	$(p \wedge q) \vee (p \wedge r)$	$\vee i_1$ 6	
4	r	Assumption	
3	$p \wedge r$	$\wedge i$ 9, 4	
2	$(p \wedge q) \vee (p \wedge r)$	$\vee i_2$ 3	
1	$(p \wedge q) \vee (p \wedge r)$	$\vee e$ 4-2, 7-5,	

Proofs using rules for Disjunction - Example

- Consider the following program fragment in C:
  - if (x>y) { m = x; }
  - else /* x <= y */ { m = y; }
- *Prove the post-condition* (i.e. condition after execution)
  - m *holds the maximum of the two values* x *and* y
- How would the proof proceed?

1. ϕ is ____________
2. ψ is _____________
3. $\phi \vee \psi$ is true.
4. χ is ______________
5. Now, apply disjunction elimination.

- **Exercise:** *Prove*:
  - **¬rains|wet_road |-- rains --> wet_road**

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MODULE: **PROPOSITIONAL LOGIC**

Natural Deduction: Derived Rules

Modus Tollens

$$\frac{\phi \dashrightarrow \psi \qquad \neg\psi}{\neg\phi}$$

MT (modus tollens)

What is the relation between this and modus ponens?

How do you derive it?

Proof by Contradiction

$$\dfrac{\begin{array}{c}\neg\phi \\ \vdots \\ \bot\end{array}}{\phi}\ \text{PBC}$$

- One can infer anything from a contradiction.
- But in this case the contradiction resulted from an assumption i.e. $\neg \phi$.
  - Therefore it is meaningful to infer ϕ that the assumption led to the contradiction
    - Implicit meta-assumption: that the proof is sound!
- In fact one must infer ϕ to eliminate the assumption $\neg \phi$.
  - Why?