MATHEMATICS-II (MATH F112)

Dr. Krishnendra Shekhawat

BITS PILANI
Department of Mathematics

Section 5.4

One-to-One and Onto Linear Transformations

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L is onto if and only if, for every $w \in W$, there is some $v \in V$ such that $L(v) = w$.

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Let $q \in P_2$. Now,there must exist $p \in P_3$ such that $L(p) = q$. Consider $p = \int q(x)dx$ with zero constant term. Because $L(p) = q$, we see that L is onto.

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Q:. Consider a LT $L : \mathbb{R}^2 \to \mathbb{R}^3$ given by $L([x,y]) = [2x, x-y, 0]$.

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Sol. Let $L([x_1, y_1]) = L([x_2, y_2]) \implies$

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L is not onto since, its range is not all of $\mathbb{R}^3$.

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Sol. Let $L([x_1, y_1]) = L([x_2, y_2]) \implies [2x_1, x_1 - y_1, 0] = [2x_2, x_2 - y_2, 0] \implies [x_1, y_1] = [x_2, y_2]$. Hence, L is one-to-one.

L is not onto since, its range is not all of $\mathbb{R}^3$. To be specific, there is no vector $[x, y] \in \mathbb{R}^2$ such that $L([x, y]) = [0, 0, 1]$.

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1. $L : \mathbb{R}^3 \to \mathbb{R}^4$ given by $L([x, y, z]) = ([y, z, -y, 0])$.

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Q:. Which of the following transformations are one-to-one? onto?

1. $L : \mathbb{R}^3 \to \mathbb{R}^4$ given by $L([x, y, z]) = ([y, z, -y, 0])$.
2. $L : M_{22} \to M_{22}$ given by $L(A) = A^T$.

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Sol. 1. neither one-to-one nor onto

2. one-to-one and onto

3. Show that the LT $L : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined by $L([x, y, z]) = ([x - y, y])$ is onto but not one-one.

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Proof. Suppose L is one-to-one and let $v \in \ker(L)$.

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Proof. Suppose L is one-to-one and let $v \in \ker(L)$. We need to show that $v = \mathbf{0}_V$. Now $L(v) = \mathbf{0}_W$ and from Theorem 5.1, $L(\mathbf{0}_V) = \mathbf{0}_W \implies L(v) = L(\mathbf{0}_V)$.

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Proof. Suppose L is one-to-one and let $v \in \mathbf{ker}(L)$. We need to show that $v = \mathbf{0}_V$. Now $L(v) = \mathbf{0}_W$ and from Theorem 5.1, $L(\mathbf{0}_V) = \mathbf{0}_W \implies L(v) = L(\mathbf{0}_V)$. Because L is one-to-one, we have $v = \mathbf{0}_V$.

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$L(v_1) - L(v_2) = \mathbf{0}_W \implies$

Conversely, let $\mathbf{ker}(L) = \{\mathbf{0}_V\}$. We need to show that L is one-to-one. Let $v_1, v_2 \in V$ with $L(v_1) = L(v_2)$. Now,
$L(v_1) - L(v_2) = \mathbf{0}_W \implies L(v_1 - v_2) = \mathbf{0}_W \implies$

Conversely, let $\mathbf{ker}(L) = \{\mathbf{0}_V\}$. We need to show that L is one-to-one. Let $v_1, v_2 \in V$ with $L(v_1) = L(v_2)$. Now, $L(v_1) - L(v_2) = \mathbf{0}_W \implies L(v_1 - v_2) = \mathbf{0}_W \implies v_1 - v_2 \in \mathbf{ker}(L)$.

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Proof. By definition, L is onto if and only if $\text{range}(L) = W$. And from Theorem 4.16, $\mathbf{dim}\, V = \mathbf{dim}\, W$ if and only if $V = W$ where W is a subspace of V.

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Proof. By definition, L is onto if and only if $\text{range}(L) = W$. And from Theorem 4.16, $\mathbf{dim}\, V = \mathbf{dim}\, W$ if and only if $V = W$ where W is a subspace of V. This concludes the proof.

Example 3

Q:. Consider a LT $L : M_{22} \to M_{23}$ given by

$$L\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a-b & 0 & c-d \\ c+d & a+b & 0 \end{bmatrix}$$

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Is L one-to-one and onto.

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Is L one-to-one and onto. Also find a basis for range(L).

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$\dim(\ker(L)) = 0$

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Hence, $\mathbf{ker}(L)$ contains only the zero matrix, i.e., $\mathbf{dim}(\mathbf{ker}(L))=0$ implies L is one-to-one.

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$a - b = c - d = c + d = a + b = 0$ implies $a = b = c = d = 0$.

Hence, $\mathbf{ker}(L)$ contains only the zero matrix, i.e.,

$\mathbf{dim}(\mathbf{ker}(L)) = 0$ implies L is one-to-one.

From Dimension Theorem, we can directly conclude that

$\mathbf{dim}(\text{range}(L)) = \mathbf{dim}(M_{22}) - \mathbf{dim}(\mathbf{ker}(L)) = 4$.

Sol. If $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \ker(L)$ then $a-b=c-d=c+d=a+b=0$ implies $a=b=c=d=0$.

Hence, $\ker(L)$ contains only the zero matrix, i.e., $\dim(\ker(L))=0$ implies L is one-to-one.

From Dimension Theorem, we can directly conclude that $\dim(\text{range}(L)) = \dim(M_{22}) - \dim(\ker(L)) = 4$. Hence, $\dim(\text{range}(L)) = 4 \neq \dim(M_{23}) \implies$

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$a - b = c - d = c + d = a + b = 0$ implies $a = b = c = d = 0$.

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$\mathbf{dim}(\text{range}(L)) = \mathbf{dim}(M_{22}) - \mathbf{dim}(\mathbf{ker}(L)) = 4$. Hence,
$\mathbf{dim}(\text{range}(L)) = 4 \neq \mathbf{dim}(M_{23}) \implies L$ is not onto.

$$\text{range}(L) = \left\{ \begin{bmatrix} a-b & 0 & c-d \\ c+d & a+b & 0 \end{bmatrix} \middle| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{22} \right\}$$

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Now, $\begin{bmatrix} a-b & 0 & c-d \\ c+d & a+b & 0 \end{bmatrix} =$

$$a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + b\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \implies$$

$$\text{range}(L) = \left\{ \begin{bmatrix} a-b & 0 & c-d \\ c+d & a+b & 0 \end{bmatrix} \middle| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{22} \right\}$$

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$\text{range}(L) =$

$$\text{span}\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \right\} =$$

$\text{span}(B)$.

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$\text{span}(B)$.

Since B is LI, it forms a basis for range(L).

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Q:. Consider a LT $L : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -7 & 4 & -2 \\ 16 & -7 & 2 \\ 4 & -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

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Sol. The matrix $\begin{bmatrix} -7 & 4 & -2 \\ 16 & -7 & 2 \\ 4 & -3 & 2 \end{bmatrix}$ row reduces to

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$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -7 & 4 & -2 \\ 16 & -7 & 2 \\ 4 & -3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Is L one-to-one and onto.

Sol. The matrix $\begin{bmatrix} -7 & 4 & -2 \\ 16 & -7 & 2 \\ 4 & -3 & 2 \end{bmatrix}$ row reduces to

$$\begin{bmatrix} 1 & 0 & -2/5 \\ 0 & 1 & -6/5 \\ 0 & 0 & 0 \end{bmatrix}.$$

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Result: Let V and W be finite dimensional vector spaces with $\mathbf{dim}(V) = \mathbf{dim}(W)$. Let $L : V \to W$ be a LT. Then L is one-to-one if and only if L is onto.

Example 5

Q:. Let A be a fixed $n \times n$ matrix, and consider a LT $L : M_{nn} \to M_{nn}$ given by $L(B) = AB - BA$.

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Sol. $L(I_n) = AI_n - I_nA = 0_n$. Hence, $I_n \in \ker(L)$ and so, L is not one-to-one. Now from Dimension Theorem, we can conclude L is not onto.

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Exercises

Q:. Consider a LT $L: M_{23} \to M_{22}$ given by

$$L\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} a+b & a+c \\ d+e & d+f \end{bmatrix}$$

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Q:. Consider a LT $L : \mathbb{R}^4 \to \mathbb{R}^3$ given by

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} -5 & 3 & 1 & 18 \\ -2 & 1 & 1 & 6 \\ -7 & 3 & 4 & 19 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

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