innovate achieve lead

BITS Pilani
Pilani Campus

# MATH F112 (Mathematics-II)

## Complex Analysis

# Lecture 23-25
# Complex Functions

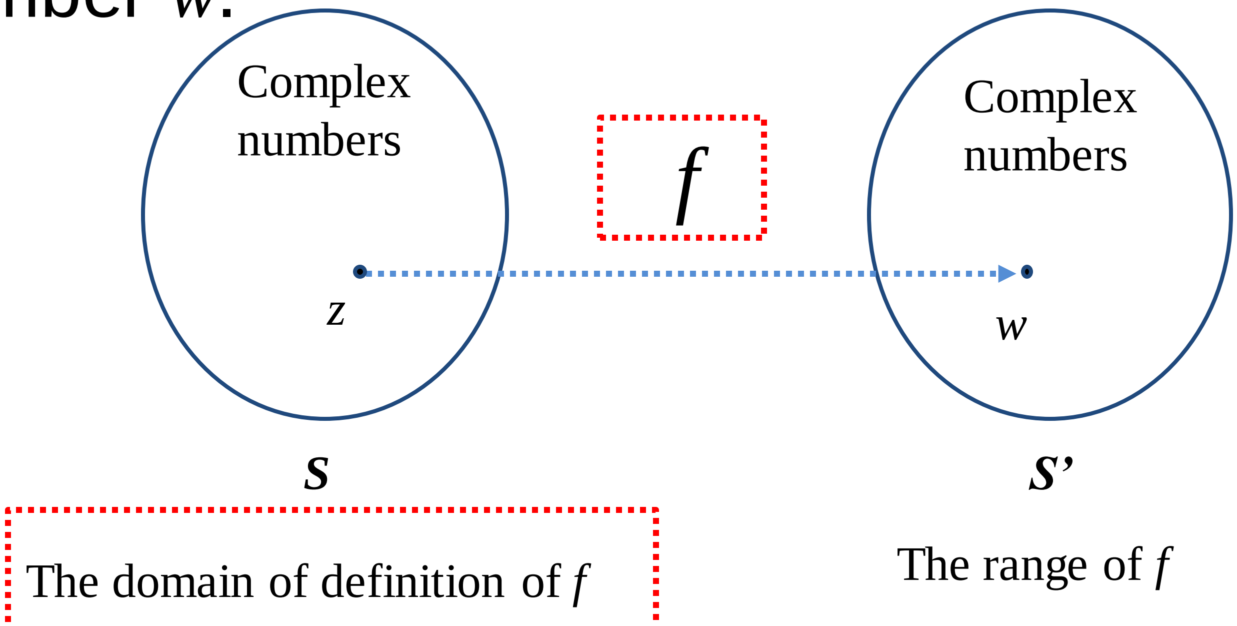
Dr. Ashish Tiwari

BITS Pilani
Pilani Campus

# Functions of Complex Variables

- Function of a complex variable

Let $S$ be a set complex numbers. A function $f$ defined on $S$ is a rule that assigns to each $z$ in $S$ a complex number $w$.



# Functions of Complex Variables

Suppose that $w = u + iv$ is the value of a function $f$ at $z = x + iy$, so that $u + iv = f(x + iy)$

Thus each of real number $u$ and $v$ depends on the real variables $x$ and $y$, meaning that

$$f(z) = u(x, y) + iv(x, y)$$

Similarly if the polar coordinates $r$ and $\theta$, instead of $x$ and $y$, are used, we get

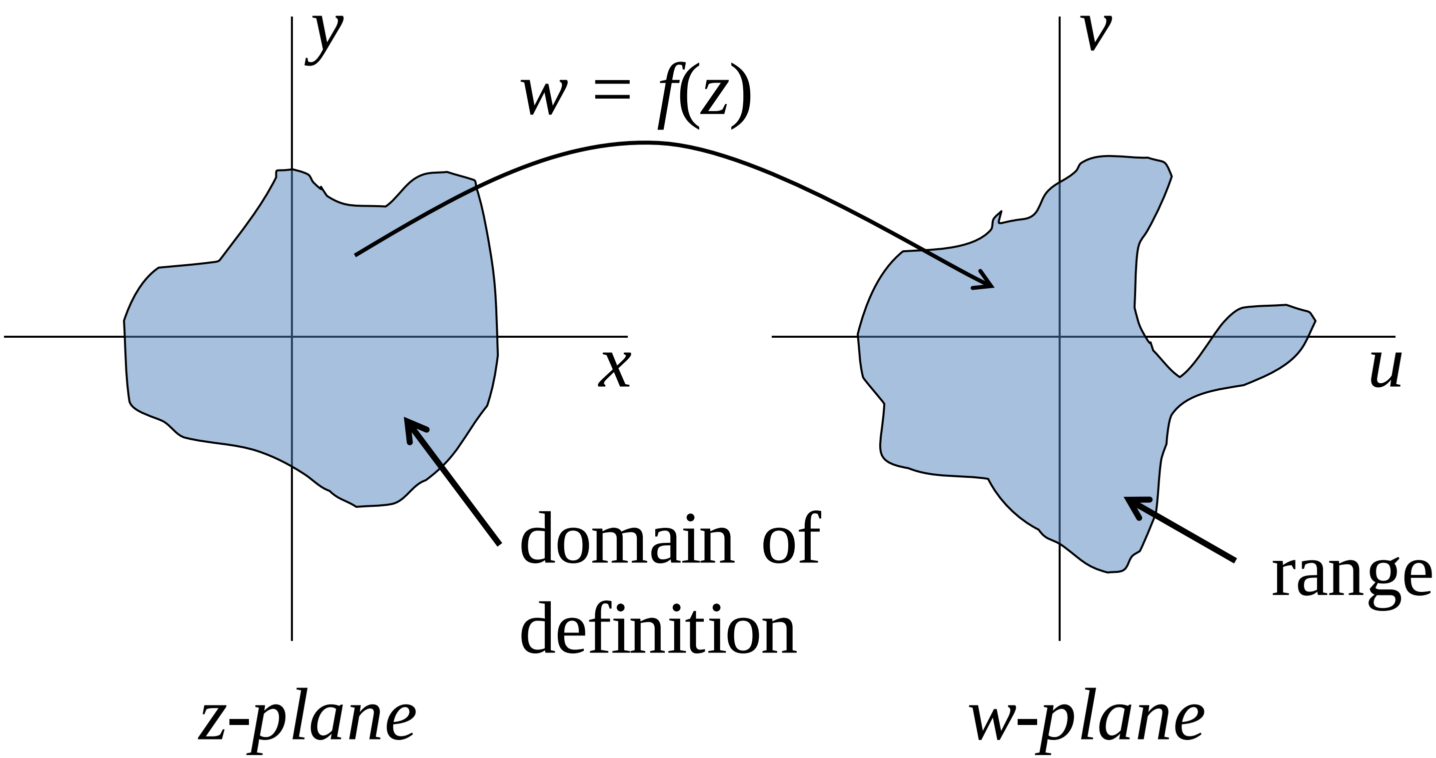
$$f(z) = u(r, \theta) + iv(r, \theta)$$

# Remark

- Properties of a real-valued function of a real variable are often exhibited by the graph of the function. But when $w = f(z)$, where $z$ and $w$ are complex, no such convenient graphical representation is available because each of the numbers $z$ and $w$ is located in a plane rather than a line.

- We can display some information about the function by indicating pairs of corresponding points $z = (x, y)$ and $w = (u, v)$. To do this, it is usually easiest to draw the $z$ and $w$ planes separately.

# Graph of Complex Functions



# Functions of Complex Variables

- Example:

  If $f(z) = z^2$, then

  case #1: $z = x + iy$

$$f(z) = (x+iy)^2 = x^2 - y^2 + i2xy$$

$$\Longrightarrow \quad u(x,y) = x^2 - y^2; v(x,y) = 2xy$$

When $v = 0$, $f$ is a real-valued function.

  case #2: $z = re^{i\theta}$

$$f(z) = (re^{i\theta})^2 = r^2 e^{i2\theta} = r^2 \cos 2\theta + ir^2 \sin 2\theta$$

$$\Longrightarrow \quad u(r,\theta) = r^2 \cos 2\theta; v(r,\theta) = r^2 \sin 2\theta$$

# Functions of Complex Variables

- Example:

A real-valued function is used to illustrate some important concepts later in this chapter is

$$f(z) = |z|^2 = x^2 + y^2 + i0$$

- Polynomial function:

$$P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n$$

where $n$ is zero or a positive integer and $a_0$, $a_1$, $\ldots a_n$ are complex constants, $a_n \neq 0$;

- The domain of definition is the entire z-plane

- Rational function:

  the quotients $P(z)/Q(z)$ of polynomials
- The domain of definition is

$$\{z \in \mathbb{C} : Q(z) \neq 0\}$$

# Limits

Suppose the function $f$ is defined in some neighborhood of $z_0$, except possibly at $z_0$ itself. Then $f$ is said to possess a limit at $z_0$, written

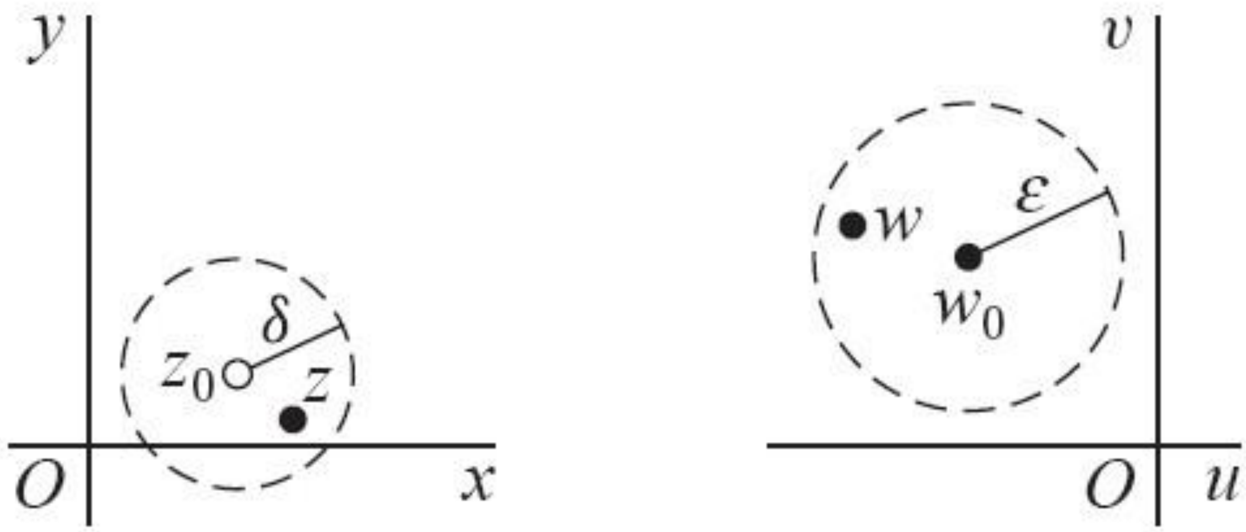
$$\lim_{z \to z_0} f(z) = w_0$$

If for a given $\epsilon > 0$, $\exists$ a $\delta > 0$ such that

$$|f(z) - w_0| < \epsilon \text{ whenever } 0 < |z - z_0| < \delta$$

# Limits

meaning the point $w = f(z)$ can be made arbitrarily close to $w_0$ if we choose the point $z$ close enough to $z_0$ but distinct from it.



# Theorems on Limit

**Thm 1**

$$\text{Let } f(z) = u(x, y) + iv(x, y),$$

$$z_0 = x_0 + iy_0, \quad w_0 = u_0 + iv_0,$$

$$\text{Then } \lim_{z \to z_0} f(z) = w_0$$

$$\Leftrightarrow (i) \lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$$

$$(ii) \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$$

# Theorems on Limit



***Thm 2***

Let $\lim\limits_{z \to z_0} f(z) = w_0,$

$\lim\limits_{z \to z_0} F(z) = W_0$. Then

(i) $\lim\limits_{z \to z_0} [f(z) \pm F(z)] = w_0 \pm W_0.$

# Theorems on Limit

$$(ii)\ \lim_{z \to z_0} [f(z)F(z)] = w_0 W_0.$$

$$(iii)\ \lim_{z \to z_0} \left[\frac{f(z)}{F(z)}\right] = \frac{w_0}{W_0},\ \text{if}\ \ W_0 \neq 0.$$

# Limits

- Example

Show that $f(z) = i\bar{z}/2$ in the open disk $|z| < 1$, then

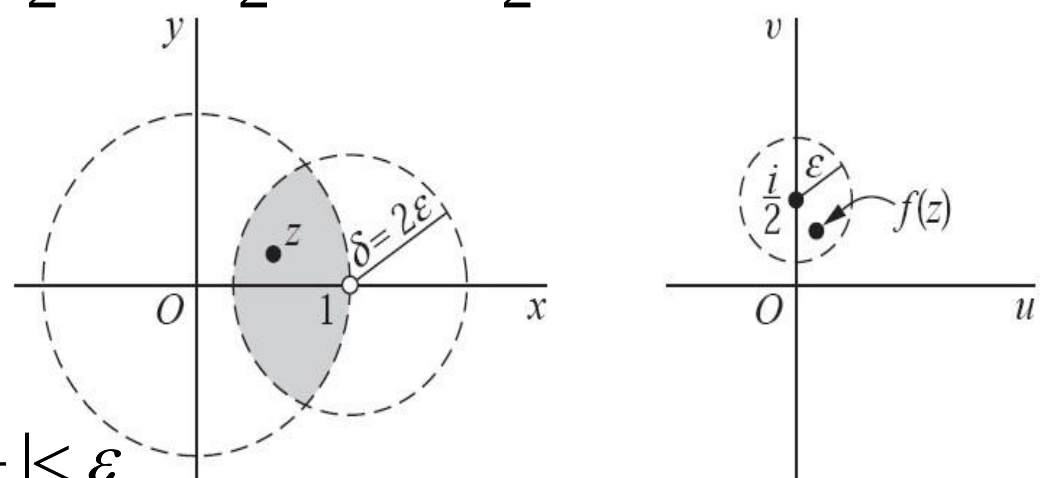
$$\lim_{z \to 1} f(z) = \frac{i}{2}$$

**Proof:**

$$|f(z) - \frac{i}{2}| = |\frac{i\bar{z}}{2} - \frac{i}{2}| = \frac{|i||\bar{z} - 1|}{2} = \frac{|z-1|}{2}$$

$$\forall \varepsilon > 0, \exists \delta = 2\varepsilon, s.t.$$

when $0 < |z-1| < \delta (= 2\varepsilon)$




$$\Rightarrow 0 < \frac{|z-1|}{2} < \varepsilon \Rightarrow |f(z) - \frac{i}{2}| < \varepsilon$$

# Limits

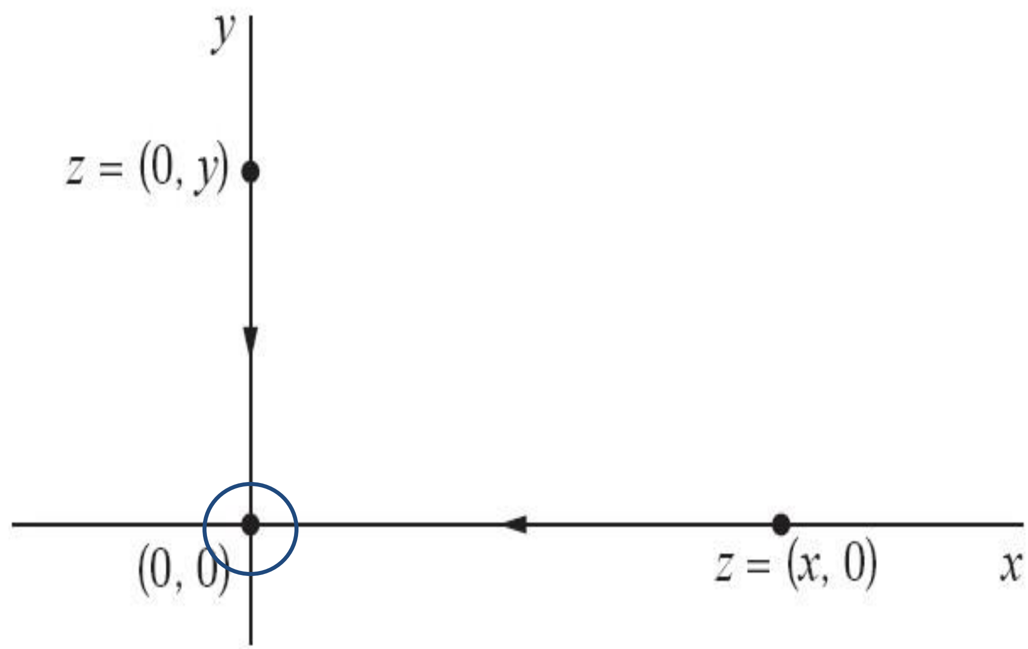
- Example

If $f(z)=\dfrac{\bar{z}}{z}$ then the limit $\lim\limits_{z\to 0} f(z)$ does not exist.

$$z=(x,0) \quad \lim_{x\to 0}\frac{x+i0}{x-i0}=1$$

$$\neq$$

$$z=(0,y) \quad \lim_{y\to 0}\frac{0+iy}{0-iy}=-1$$

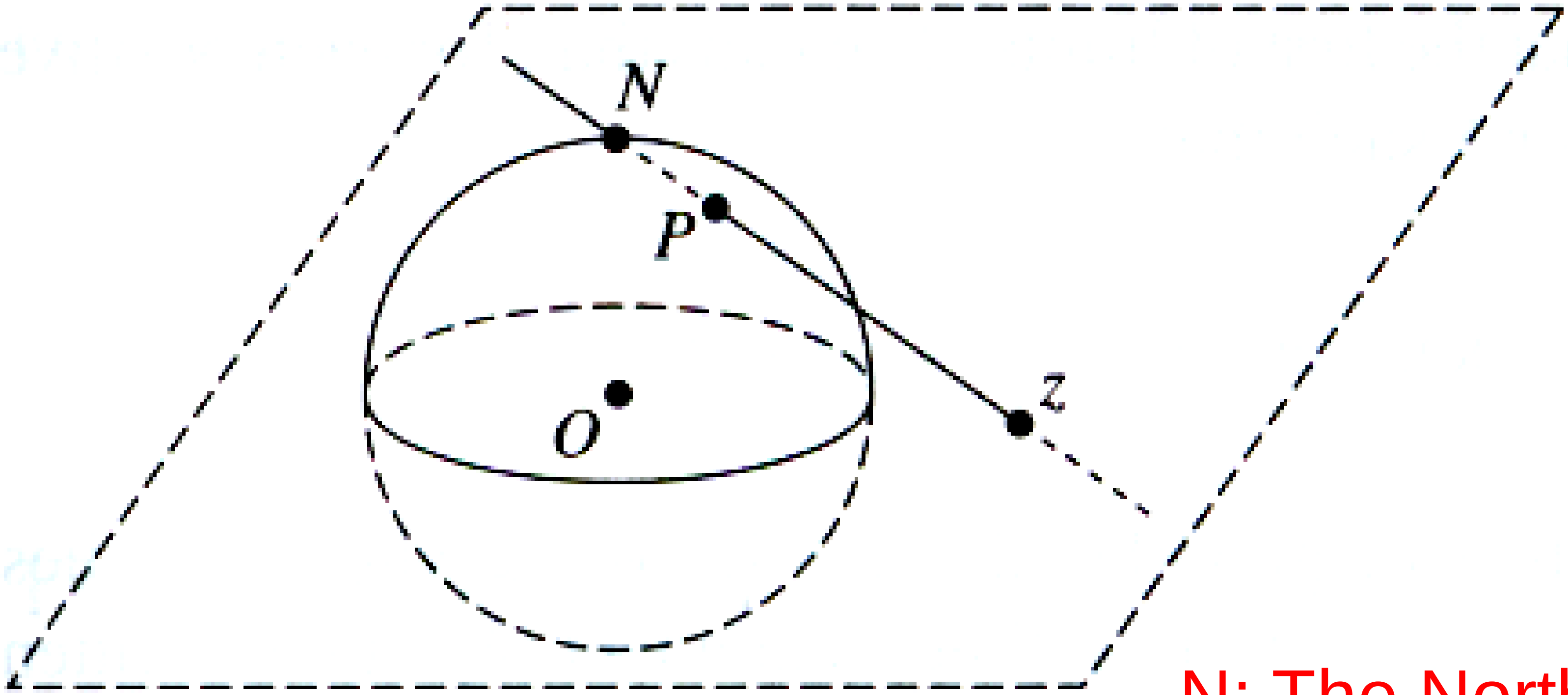


# The point at Infinity

The point at infinity is denoted by $\infty$, and the complex plane together with the point at infinity is called the ***<u>Extended complex Plane</u>***.

# The point at Infinity

## Riemann Sphere & Stereographic Projection




N: The North pole

# The point at Infinity

- The ε-Neighborhood of Infinity

When the radius $R$ is large enough

***i.e.*** for each small positive number $R=1/\varepsilon$

The region of $|z| > R = 1/\varepsilon$ is called the ε -Neighborhood of Infinity (∞)

# Theorems

1. $$\lim_{z \to z_0} f(z) = \infty \Leftrightarrow \lim_{z \to z_0} \frac{1}{f(z)} = 0.$$

2. $$\lim_{z \to \infty} f(z) = w_0 \Leftrightarrow \lim_{z \to 0} f\left(\frac{1}{z}\right) = w_0$$

3. $$\lim_{z \to \infty} f(z) = \infty \Leftrightarrow \lim_{z \to 0} \frac{1}{f\left(\frac{1}{z}\right)} = 0$$

1.

$$\lim_{z \to z_0} f(z) = \infty \implies \text{for each } \epsilon > 0, \exists\, \delta > 0 \text{ such that}$$

$$|f(z)| > \frac{1}{\epsilon} \text{ whenever } 0 < |z - z_0| < \delta$$

$$i.e. \left|\frac{1}{f(z)} - 0\right| < \epsilon \text{ whenever } 0 < |z - z_0| < \delta$$

Similarly, converse part can also be done.

# Theorems

Thus,

$$\lim_{z \to z_0} f(z) = \infty \Leftrightarrow \lim_{z \to z_0} \frac{1}{f(z)} = 0.$$

2.

$\lim_{z\to\infty} f(z) = w_0 \implies$ for each $\epsilon > 0, \exists\, \delta > 0$ such that

$$|f(z) - w_0| < \epsilon \text{ whenever } |z| > \frac{1}{\delta}$$

Replacing $z$ by $1/z$,

$$i.e. \left|f\left(\frac{1}{z}\right) - w_0\right| < \epsilon \text{ whenever } 0 < |z| < \delta$$

Similarly, converse part can also be done.

# Theorems

Thus,

$$\lim_{z\to\infty} f(z) = w_0 \Leftrightarrow \lim_{z\to 0} f\left(\frac{1}{z}\right) = w_0$$

3.

$$\lim_{z\to\infty} f(z) = \infty \implies \text{for each } \epsilon > 0, \exists\ \delta > 0 \text{ such that}$$

$$|f(z)| > \frac{1}{\epsilon} \ \text{ whenever } \ |z| > \frac{1}{\delta}$$

Replacing $z$ by $1/z$,

$$i.e. \left|\frac{1}{f\left(\frac{1}{z}\right)} - 0\right| < \epsilon \text{ whenever } \ 0 < |z - 0| < \delta$$

Similarly, converse part can also be done.

# Theorems



Thus,

$$\lim_{z\to\infty} f(z) = \infty \Leftrightarrow \lim_{z\to 0} \frac{1}{f\left(\frac{1}{z}\right)} = 0$$

# Continuity

A function $f$ is continuous at a point $z_0$ if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

meaning that

1. f is defined at $z_0$.
2. the limit of function $f$ exist at point $z_0$ and
3. the limit is equal to the value of $f(z_0)$

For a given positive number $\varepsilon$, there exists a positive number $\delta$, s.t.

When $|z - z_0| < \delta \quad \Rightarrow |f(z) - f(z_0)| < \varepsilon$

# Continuity

The function $f(z)$ is said to be continuous in a region $R$ if it is continuous at all points of the region $R$.

# Continuity

**Theorem 1.**

A composition of two continuous functions is itself continuous.

**Theorem 2.**

If $f(z) = u(x, y) + iv(x, y)$, then $f(z)$ is continuous iff

$\mathrm{Re}(f(z)) = u(x, y)$ and $\mathrm{Im}(f(z)) = v(x, y)$

are continuous functions.

# Continuity

Theorem3. If $f(z)$ and $g(z)$ are continuous, then

(a) $f(z) \pm g(z)$

(b) $f(z)g(z)$

(c) $\dfrac{f(z)}{g(z)}, \quad g(z) \neq 0$

are all continuous.

# Continuity

## Theorem 4.

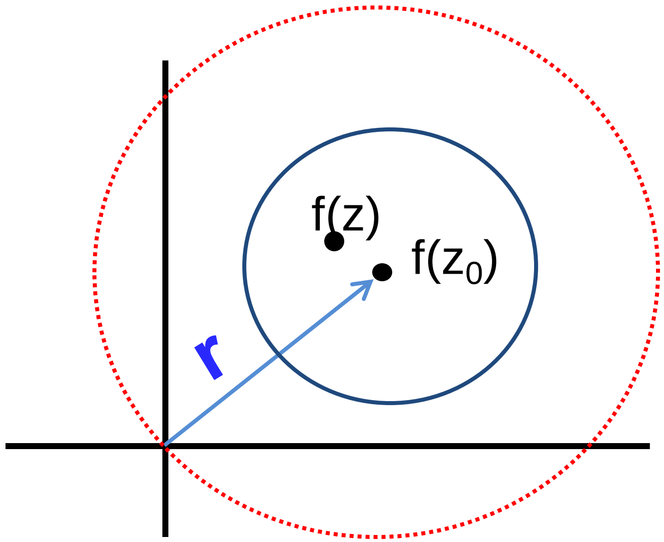
If a function $f\,(z)$ is continuous and nonzero at a point $z_0$, then $f\,(z) \neq 0$ throughout some neighborhood of that point.

**Proof** $\lim_{z \to z_0} f(z) = f(z_0) \neq 0$ **Why?**

For $e = \dfrac{|f(z_0)|}{2} > 0, \$ d > 0, s.t.$

When $|z - z_0| < \delta$

$$|f(z) - f(z_0)| < \varepsilon = \frac{|f(z_0)|}{2}$$




$\forall \varepsilon \leq |f(z_0)|$

If $f\,(z)$ = 0, then $|f(z_0)| < \dfrac{|f(z_0)|}{2}$

**Contradiction!**

# Continuity

**Theorem 5.**

Every continuous function $f$ in a closed and bounded region $R$, is bounded i.e. there exists a nonnegative real number $M$ such that

$$|f(z)| \leq M \qquad \text{for all points } z \text{ in } R$$

where equality holds for at least one such $z$.

# Continuity

***Example:*** Discuss the continuity of $f(z)$ at $z = 0$ if

$$(i)\ f(z) = \frac{\text{Re}\, z}{1+|z|}$$

$$(ii)\ f(z) = z^{-1}\text{Re}\, z$$

# Continuity



***Sol. (i)***

$$f(z) = \frac{\operatorname{Re} z}{1+|z|}$$

$$= \frac{x}{1+\sqrt{x^2+y^2}}$$

# Continuity

$$\therefore \lim_{z \to 0} f(z) = \lim_{(x,y)\to(0,0)} \frac{x}{1+\sqrt{x^2+y^2}}$$

$$= 0 = f(0)$$

$$\Rightarrow f(z) \text{ is continuous at } z = 0$$

# Continuity

(ii) $f(z) = \dfrac{\operatorname{Re} z}{z} = \dfrac{x}{x+iy}$

We have

$$\lim_{z\to 0} f(z) = \lim_{(x,y)\to(0,0)} \frac{x}{x+iy}$$

# Continuity

$$\Rightarrow \lim_{z\to 0} f(z) = \lim_{(x,y)\to(0,0)} \frac{x}{x+imx},$$

(along $y = mx$)

$$= \frac{1}{1+im}$$

which is not unique

$\Rightarrow f(z)$ is not continuous at $z = 0$

# Derivative of Complex Function

***Derivatives:*** Let $f(z)$ be a function defined on a set $S$ and $S$ contains $N_\rho(z_0)$. Then derivative of $f(z)$ at $z_0$, written as $f'(z_0)$, is defined as the limit:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}, \qquad (1)$$

provided the limit on RHS exists.

# Derivative of Complex Function

The function $f(z)$ is said to be differentiable at $z_0$ if its derivative at $z_0$ exists.

If $z\text{-}z_0 = \Delta z$, then (1) reduces to

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

# Derivative of Complex Function

Problem: Differentiablity $\Rightarrow$ Continuity

Continuity $\not\Rightarrow$ Differentiablity

Proof : Let $f(z)$ is differentiable at $z_0$

$$\Rightarrow f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Now

$$\lim_{z \to z_0} \left[ f(z) - f(z_0) \right] = \lim_{z \to z_0} \left[ \frac{f(z) - f(z_0)}{z - z_0} \times (z - z_0) \right]$$

# Derivative of Complex Function



$$= \lim_{z \to z_0} \left[ \frac{f(z) - f(z_0)}{z - z_0} \right] \times \left[ \lim_{z \to z_0} (z - z_0) \right]$$

$$= f'(z_0) \times 0 = 0$$

$$\Rightarrow \lim_{z \to z_0} f(z) = f(z_0)$$

$$\Rightarrow f(z) \text{ is continuous at } z_0$$

# Derivative of Complex Function

***Continuity $\not\Rightarrow$ Differentiability***

To show this consider the function

$$f(z) = |z|^2 = x^2 + y^2 = u(x, y) + i\, v(x, y)$$

$$\Rightarrow u(x, y) = x^2 + y^2, \quad v(x, y) = 0.$$

Since $u$ and $v$ are continuous everywhere,

hence $f(z)$ is continuous everywhere

# Derivative of Complex Function

For $z \neq z_0$, we have

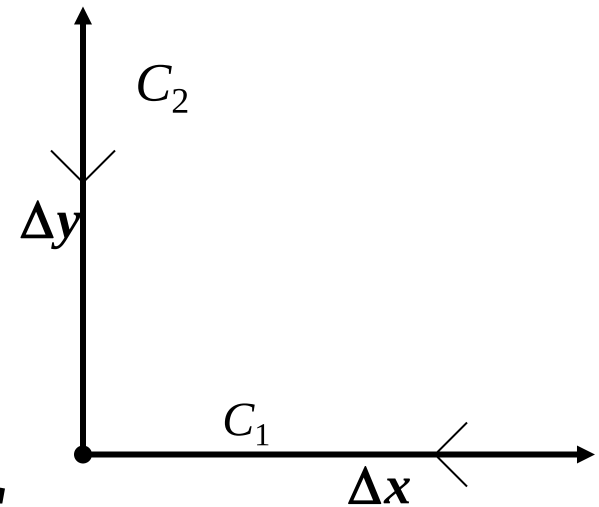
$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{|z|^2 - |z_0|^2}{z - z_0} = \frac{z\,\bar{z} - z_0 \bar{z}_0}{z - z_0}$$

$$= \frac{z\,\bar{z} - \bar{z}\,z_0 + \bar{z}\,z_0 - z_0\,\bar{z}_0}{z - z_0} = \frac{\bar{z}(z - z_0) + z_0(\bar{z} - \bar{z}_0)}{z - z_0}$$

# Derivative of Complex Function

$$= \bar{z} + z_0 . \frac{\overline{\Delta z}}{\Delta z}, \qquad z - z_0 = \Delta z$$

$$= \bar{z} + z_0 . \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$




$$= \begin{cases} \bar{z} + z_0, \text{along the path } C_1 \\ \bar{z} - z_0, \text{along the path } C_2 \end{cases}$$

# Derivative of Complex Function

Thus, if $z_0 \neq (0,0)$, then

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$ is not unique.

When $z_0 = (0,0)$, then

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \bar{z}_0 = 0.$$

$\Rightarrow f(z)$ is differentiable at the origin and no where else.

# Differentiation Formulas



1. $$f'(z) = \frac{d}{dz} f(z)$$

2. $$\frac{d}{dz}(c) = 0,$$

3. $$\frac{d}{dz}(z) = 1,$$

# Differentiation Formulas

4. $$\frac{d}{dz}\left(z^{n}\right)=nz^{n-1},$$

5. $$\frac{d}{dz}\left(cf\left(z\right)\right)=c\frac{d}{dz}f\left(z\right)$$

# Differentiation Formulas

innovate achieve lead

6. $$\frac{d}{dz}\left(f(z)\pm g(z)\right)=f'(z)\pm g'(z)$$

7. $$\frac{d}{dz}\left(f(z)g(z)\right)=f(z)g'(z)+f'(z)g(z)$$

# Differentiation Formulas

innovate achieve lead

8. $$\frac{d}{dz}\left(\frac{f(z)}{g(z)}\right)=\frac{g(z)\,f'(z)-g'(z)\,f(z)}{\left(g(z)\right)^2},$$

if $g(z)\neq 0$

# Chain Rule

Let $F(z) = g(f(z))$, and assume that $f(z)$ is differentiable at $z_0$ & $g$ is differentiable at $f(z_0)$, then $F(z)$ is differentiable at $z_0$ and

$$F'(z_0) = g'(f(z_0))f'(z_0)$$

# Problems

Ex. Let $w = f(z)$ and $W = g(w)$

$\Rightarrow W = F(z)$, hence by Chain rule $\dfrac{dW}{dz} = \dfrac{dW}{dw}\dfrac{dw}{dz}$

Q8:(a) $f(z) = \bar{z}$, show that $f'(z)$ does not exist at any point $z$.

Solution: Let $z \neq z_0$, then

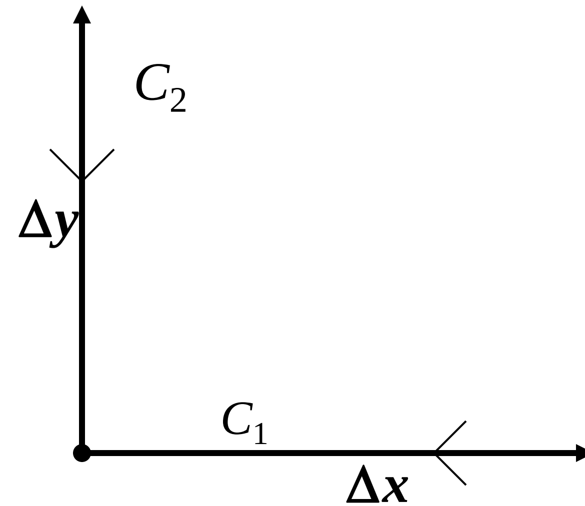
$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{\bar{z} - \bar{z}_0}{z - z_0} = \frac{\overline{z - z_0}}{z - z_0}$$

# Problems

$$\Rightarrow \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}=\frac{\overline{\Delta z}}{\Delta z}=\frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

$$\therefore \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$$

$$=\begin{cases} 1 \; along \; C_1 \\ -1 \quad along \, C_2 \end{cases}$$




$$\Rightarrow f'(z) \text{ does not exist any where}$$

# Problems

Q.9 Let $f$ be a function defined by

$$f(z) = \begin{cases} \dfrac{(\bar{z})^2}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

Show that $f'(0)$ does NOT exist.

# Problems

We have ,

$$f'(0) = \lim_{\Delta z \to 0} \frac{f(0+\Delta z) - f(0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\left(\overline{\Delta z}\right)^2 / \Delta z}{\Delta z}$$

# Problems

$$\Rightarrow f'(0) = \lim_{\Delta z \to 0} \frac{(\overline{\Delta z})^2}{(\Delta z)^2}$$

$$= \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{(\Delta x - i\Delta y)^2}{(\Delta x + i\Delta y)^2}$$

# Problems

$$\Rightarrow f'(0) = \begin{cases} 1, & \text{along real axis} \\ 1, & \text{along imaginary axis} \\ -1, & \text{along line } \Delta y = \Delta x \end{cases}$$

Hence $f'(0)$ does NOT exist.

# Cauchy-Riemann Equations

Suppose that $f(z) = u(x, y) + iv(x, y)$

and that $f'(z)$ exists at a point

$z_0 = x_0 + iy_0$

Then the first-order partial derivatives

$u_x, u_y, v_x$ and $v_y$ must exist at $(x_0, y_0)$

and they satisfy the CR-equations

i.e. $u_x = v_y$ and $u_y = -v_x$ at $(x_0, y_0)$

# Cauchy-Riemann Equations

Also,

$f'(z_0)$ can be written as

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0).$$

***Proof***

Since $f(z)$ is differentiable at $z_0$

$$\Rightarrow f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \quad \ldots\ldots (1)$$

# Cauchy-Riemann Equations

Note that $z = x + iy, \;\; z_0 = x_0 + iy_0$

$$\Delta z = \Delta x + i\Delta y$$

$$f(z_0) = u(x_0, y_0) + i\,v(x_0, y_0)$$

$$\Rightarrow f(z_0 + \Delta z) = u(x_0 + \Delta x,\; y_0 + \Delta y)$$

$$+ iv(x_0 + \Delta x, y_0 + \Delta y)$$

# Cauchy-Riemann Equations

$\therefore$ Eq.(1) gives

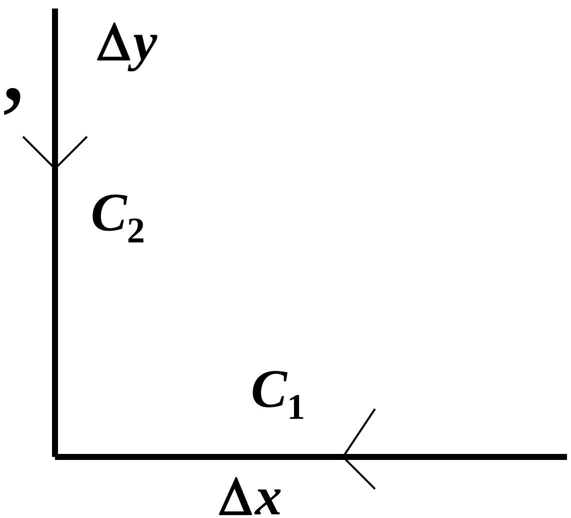
$$f'(z_0) =$$

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \left[ \frac{u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)}{\Delta x + i\Delta y} \right.$$

$$\left. + i\, \frac{v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)}{\Delta x + i\Delta y} \right]$$

# Cauchy-Riemann Equations

$$f'(z_0) = \begin{cases} u_x(x_0\ y_0) + i\,v_x(x_0\ y_0), \\ \qquad\qquad\qquad \text{along } C_1 \\ -i\,u_y(x_0, y_0) + v_y(x_0, y_0), \\ \qquad\qquad\qquad \text{along } C_2 \end{cases}$$



# Cauchy-Riemann Equations

$$\Rightarrow u_x = v_y, \quad u_y = -v_x \; at \; (x_0, y_0),$$

$$and \;\; f'(z_0) = u_x + i v_x \;\; at \; (x_0, y_0)$$

WHY ???

# Sufficient Condition for Differentiability

Let $f(z) = u(x,y) + i\,v(x, y)$ be any function defined throughout in some neighbourhood of the point $z_0 = x_0 + iy_0$ such that :

$(i)$ $u_x, u_y, v_x, v_y$ exist in that nbd. of $z_0$,

$(ii)$ $u_x, u_y, v_x, v_y$ are continuous at $(x_0, y_0)$

# Sufficient Condition for Differentiability

$(iii)$ the first order partial derivatives satisfy the CR-equations, $u_x = v_y, \ \ u_y = -v_x$ at $(x_0, y_0)$.

Then $f'(z)$ exists at $z_0$.

# C-R Equations in Polar Form

Let $f(z) = u(r,\theta)+i\,v(r,\theta)$ be differentiable at any given point $z_0 = r_0\,e^{i\theta_0}$. Then the partial derivatives $u_r,\ u_\theta, v_r, v_\theta$ exist at $(r_0,\theta_0)$ and they satisfy

$$u_r = \frac{1}{r} v_\theta, \quad v_r = -\frac{1}{r} u_\theta$$

and

$$f'(z_0) = e^{-i\theta}\left(u_r + i\,v_r\right)\Big|_{(r_o,\theta_o)}$$

Note: For a complex variable function

$$f(z) = u(x, y) + i\, v(x, y),$$

Differentiability $\nRightarrow$ continuity of first order partial derivatives of $u$ and $v$.

Ex: $f(z) = \begin{cases} (x^2 + y^2)\sin\dfrac{1}{\sqrt{(x^2+y^2)}} \; ; & (x, y) \neq (0,0) \\ 0 \; ; & (x, y) = (0,0) \end{cases}$

Here, $f(z)$ is differentiable at $(0,0)$ but $u_x, u_y$ are not continuous at $(0,0)$.

$$u_x(0,0) = \lim_{h\to 0} \frac{u(h,0) - u(0,0)}{h}$$

$$\lim_{h\to 0} \frac{h^2 \sin \frac{1}{|h|}}{h} = 0$$

similarly, $u_y(0,0) = 0$.

Away from origin

$$u_x(x,y) = 2x \sin\left(\frac{1}{\sqrt{(x^2+y^2)}}\right) - \frac{x}{\sqrt{(x^2+y^2)}} \cos\left(\frac{1}{\sqrt{(x^2+y^2)}}\right)$$

$$u_y(x,y) = 2y \sin\left(\frac{1}{\sqrt{(x^2+y^2)}}\right) - \frac{y}{\sqrt{(x^2+y^2)}} \cos\left(\frac{1}{\sqrt{(x^2+y^2)}}\right)$$

$u_x(x,y), u_y(x,y)$ oscillate wildly near origin and hence are discontinuous at origin.

For example

$$u_x(x,0) = 2x\sin\left(\frac{1}{|x|}\right) - sign(x)\cos\left(\frac{1}{|x|}\right), \quad x \neq 0$$

Where $sign(x) = \pm 1$

Second term of $u_x(x,0)$ oscillates wildly between 1 and -1 as $x \to 0$ ensuring the non-existence of limit and hence discontinuity of $u_x(x,y)$ at $(0,0)$.

Similarly, $u_y(x,y)$ is discontinuous at origin.

# Problems

Example 1: For the function

$$f(z) = z^2,$$

find out the points where the function is differentiable. Also find $f'(z)$

Solution: Consider

$$f(z) = z^2 = x^2 - y^2 + i\,2xy \equiv u + iv$$

$$\Rightarrow u(x,y) = x^2 - y^2,\; v(x,y) = 2xy$$

# Problems

$$\Rightarrow u_x = 2x,\ \ u_y = -2y,$$

$$v_x = 2y,\ \ v_y = \ 2x$$

$$\Rightarrow u_x = v_y \ \ \& \ \ u_y = -v_x$$

$\Rightarrow$ (i) CR - equations are satisfied for all $x, y$

(ii) $u_x$, $u_y$, $v_x$, and $v_y$ are continuous for all $x, y$

# Problems

$\Rightarrow f(z)=z^2$ is differentiable at any point $z$,

and

$$f'(z)=u_x+iv_x=2x+i2y=2z$$

# Problems

Example 2 : For the function

$$f(z) = |z|^2,$$

find out the points where the function is differentiable. Also find $f'(z)$

Consider $f(z) = |z|^2 = x^2 + y^2$

$$\Rightarrow u(x, y) = x^2 + y^2 \; \& \; v(x, y) = 0$$

# Problems

Þ $u_x = 2x, u_y = 2y, v_x = 0, v_y = 0,$

If CR -equations are satisfied,

then we must have $x = 0 = y.$

Also $u_x, u_y, v_x, v_y$ are continuous at (0,0)

$\Rightarrow f(z)$ is differentiable only at (0,0)

and no where else. Further

$$f'(0) = u_x(0,0) + iv_x(0,0) = 0$$

# Problems

Page 72/Q.6 : Let $u$ & $v$ denote the real & imaginary parts of the function $f$ defined by

$$f(z)=\begin{cases}\dfrac{\left(\bar{z}\right)^2}{z}, & z\neq 0\\ 0, & z=0\end{cases}$$

Show that CR-equations are satisfied at $(0,0)$ although $f$ is NOT differentiable at $(0,0)$.

# Problems

Solution :

RECALL: $f$ is not differentiable at $(0,0)$

(already done)

We have, when $z \neq 0,$

$$f(z) = \frac{(\bar{z})^2}{z} = \frac{(x\text{-}iy)^2}{x+iy} = \frac{(x\text{-}iy)^3}{(x+iy)(x-iy)}$$

$$= \frac{x^3 - 3xy^2}{x^2 + y^2} - i\,\frac{3x^2 y - y^3}{x^2 + y^2}$$

# Problems

$$\Rightarrow u(x,y)=\frac{x^3-3xy^2}{x^2+y^2},$$

$$v(x,y)=\frac{y^3-3x^2y}{x^2+y^2},(x,y)\neq(0,0)$$

When $z=0$, then

$$u(x,y)=0=v(x,y)$$

# Problems

$$u_x(0,0) = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0} \frac{x-0}{x} = 1$$

$$v_x(0,0) = \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0$$

$$u_y(0,0) = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0$$

$$v_y(0,0) = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y} = \lim_{y\to 0} \frac{y-0}{y} = 1$$

Thus $u_x = v_y$ & $u_y = -v_x$. Hence, proved.

# THANK YOU