MATH F113 (Probability and Statistics)

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In Lecture 14

Exercise Problem
Gamma Function

Gamma Distribution A random variable X with density function

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{\frac{-x}{\beta}} & x > 0 \quad \alpha > 0, \beta > 0 \\ 0 & \text{elsewhere} \end{cases}$$

is said to have a Gamma Distribution with parameters α, β for $\alpha > 0, \beta > 0$

To check the necessary and sufficient condition of pdf: $f(x) \geq 0$ **for all** $x > 0$
Further

$$\int_{-\infty}^{\infty} f(x)dx = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \int_{0}^{\infty} x^{\alpha-1} e^{-\frac{x}{\beta}} dx$$

Gamma Distribution (Cont...)

Let

$$\frac{x}{\beta} = t$$

$$dx = \beta dt$$

and

$$x = \beta t$$

$$= \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \int_{0}^{\infty} \beta^{\alpha-1} t^{\alpha-1} e^{-t} \beta dt$$

$$\frac{\beta^\alpha}{\Gamma(\alpha)\beta^\alpha}\int_0^\infty t^{\alpha-1}e^{-t}dt = 1$$

Hence $f(x)$ is a p.d.f

Theorem: Let X be a gamma random variable with parameter α & β, then m.g.f for X is given by

$$m_x(t) = (1 - \beta t)^{-\alpha}$$

Hence,

$$E[X] = \alpha\beta$$
$$Var(x) = \alpha\beta^2$$

$$
\begin{aligned}
m_x(t) = E[e^{tx}] &= \int_{-\infty}^{\infty} e^{tx} f(x)dx \\
&= \int_0^{\infty} e^{tx} \frac{1}{\beta^\alpha \Gamma(\alpha)} e^{-x/\beta} x^{\alpha-1} dx \\
&= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^{\infty} e^{-(1/\beta - t)x} x^{\alpha-1} dx
\end{aligned}
$$

Gamma Distribution (Cont...)

Let $z = (\frac{1}{\beta} - t)x$, **or** $z = (1-\beta t)\frac{x}{\beta} \quad \Rightarrow \quad x = \frac{z\beta}{(1-\beta t)}$ **and** $dx = \frac{\beta dz}{(1-\beta t)}$

$$m_x(t) = \int_0^\infty \frac{\beta^{\alpha-1}e^{-z}}{\beta^\alpha\Gamma(\alpha)} \frac{z^{\alpha-1}}{(1-\beta t)^{\alpha-1}} \frac{\beta dz}{(1-\beta t)}$$

$$= \frac{1}{(1-\beta t)^\alpha\Gamma(\alpha)} \int_0^\infty e^{-z} z^{\alpha-1} dz$$

$$= \frac{1}{(1-\beta t)^\alpha\Gamma(\alpha)} \Gamma(\alpha)$$

Since,

$$\int_0^\infty e^{-z} z^{\alpha-1} dz = \Gamma(\alpha)$$

Therefore,

$$m_x(t) = \frac{1}{(1-\beta t)^\alpha}, \quad t < \frac{1}{\beta}.$$

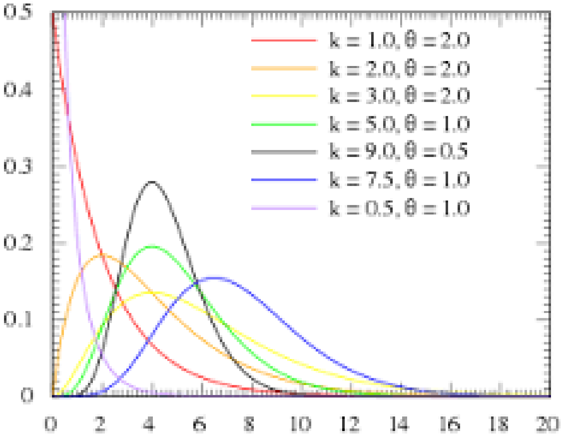
Gamma Distribution (Cont...)

$$
\begin{aligned}
E[X] &= \left[\frac{d}{dt} m_x(t)\right]_{t=0} \\
&= \left[-\alpha(1-\beta t)^{-\alpha-1}(-\beta)\right]_{t=0} = \alpha\beta
\end{aligned}
$$

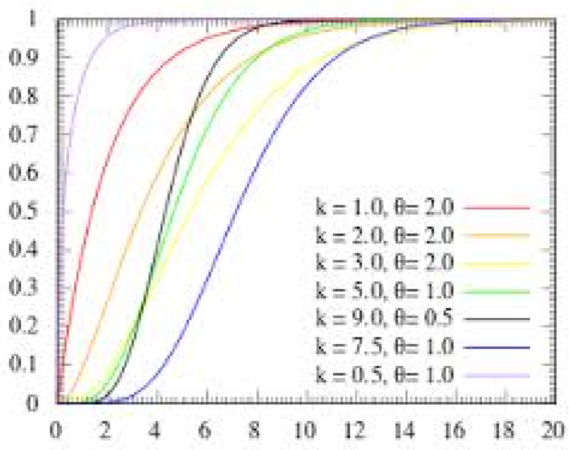
$$
\begin{aligned}
E[X^2] &= \left[\frac{d^2}{dt^2}(m_x(t))\right]_{t=0} \\
&= \left[\frac{d}{dt}(-\alpha(1-\beta t)^{-\alpha-1}(-\beta))\right]_{t=0} \\
&= \alpha\beta(-\alpha-1)(-\beta)
\end{aligned}
$$

$$
Var(X) = \alpha\beta(-\alpha-1)(-\beta) - \alpha^2\beta^2 = \alpha\beta^2
$$

Probability Density Function ($\alpha = k, \beta = \theta$)



Cumulative Distribution Function ($\alpha = k, \beta = \theta$)



- α and β both play a role in determining the mean and the variance of the random variable
- Curves are not symmetric and are located entirely to the right of the vertical axis
- For $\alpha > 1$, the maximum value of the density occurs at the point $x = (\alpha - 1)\beta$

Example 4.3/pp.143 Let X be a gamma random variable with parameters $\alpha = 3$ and $\beta = 4$

(a) What is the probability density function?

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{\frac{-x}{\beta}} & ; \; x > 0 \\ 0 & ; \; \text{elsewhere} \end{cases}$$

Hence,

$$f(x) = \begin{cases} \frac{1}{128} x^2 e^{-\frac{x}{4}} & ; \; x > 0 \\ 0 & ; \; \textbf{elsewhere} \end{cases}$$

(b) What is the moment generating function for X

$$m_x(t) = \frac{1}{(1-\beta t)^{\alpha}} = (1-4t)^{-3}, \quad t < \frac{1}{4}.$$

(c) Find mean, variance and standard deviation

$$\mu = \alpha\beta = 12$$
$$\sigma^2 = \alpha\beta^2 = 48$$
$$\sigma = \sqrt{48} = 6.9282$$

Exponential Distribution

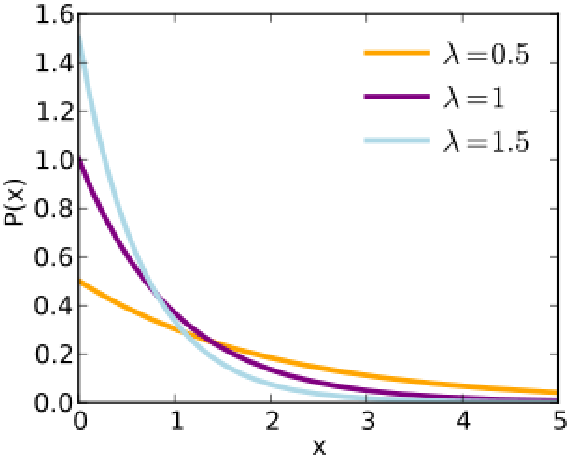
In Gamma Distribution, Put $\alpha = 1$, we get

$$f(x) = \begin{cases} \frac{1}{\beta} e^{-\frac{x}{\beta}} & x > 0, \beta > 0 \\ 0 & \textbf{elsewhere} \end{cases}$$

or if $\lambda = 1/\beta$**,**

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0, \lambda > 0 \\ 0 & \textbf{elsewhere} \end{cases}$$

Exponential Distribution PDF



The c.d.f of exponential distribution with parameter β is given by

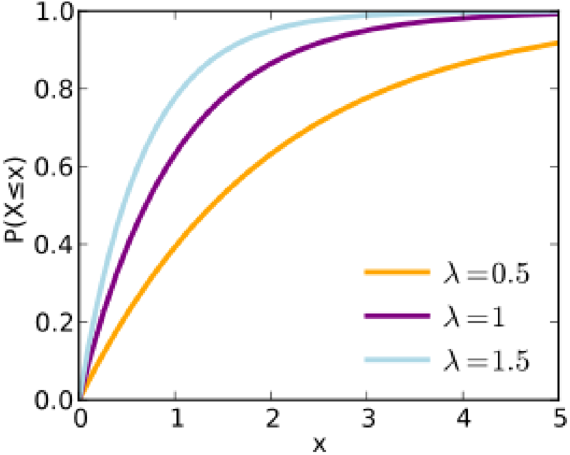
$$= \int_0^x f(t)dt = \int_0^x \frac{1}{\beta} e^{-\frac{t}{\beta}} dt$$

$$= \frac{1}{\beta}\left[\frac{1}{(-1/\beta)} e^{-\frac{t}{\beta}}\right]_0^x = 1 - e^{-\frac{x}{\beta}}$$

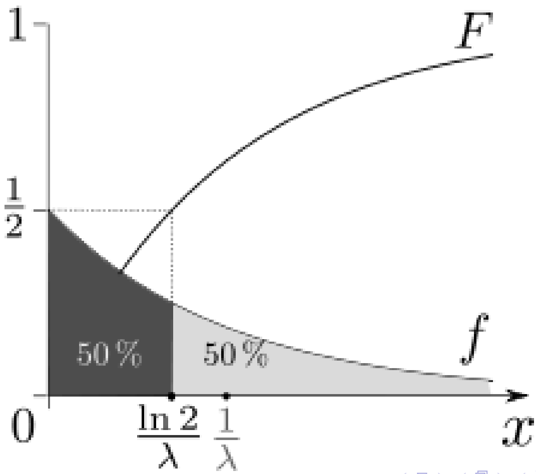
Thus,

$$F(x) = \begin{cases} 1 - e^{-\frac{x}{\beta}} & ; \; x > 0 \\ 0 & ; \; \textbf{otherwise} \end{cases}$$

Exponential Distribution CDF



$\exp(\lambda)$



Moment generating function, Mean and Variance of exponential distribution

Note: Put $\alpha = 1$ in the gamma distribution, we get the required results.

$$m_x(t) = (1 - \beta t)^{-1} \quad t < \frac{1}{\beta}$$

$$E[X] = \beta$$

$$Var(X) = \beta^2$$

The distribution arises in practice in conjunction with the study of Poisson processes, where we have discrete events are being observed in continuous time interval. If we let W denote the time of the occurrence of the first event, then W is a continuous random variable

Theorem

Consider a Poisson process with parameter λ. Let W denote the time of the occurrence of the first event. W has an Exponential distribution with

$$\beta = \frac{1}{\lambda}$$

Proof This theorem is distribution of waiting time. The distribution function F for W is given by

$$F(w) = P[W \leq w] = 1 - P[W > w]$$

Here, we note that, the first occurrence of the event will take place after time w only if no occurrence of the event are recorded in the time interval $[0, w)$

Let X denote the number of occurrences of the event in this time interval $[0, w)$.
X is a Poisson random variable with parameter λw. Thus,

$$\begin{aligned} P[W > w] &= P[X = 0] \\ &= \frac{e^{-\lambda w}(\lambda w)^0}{0!} = e^{-\lambda w} \end{aligned}$$

By substitution, we get

$$F(w) = 1 - P[W > w] = 1 - e^{-\lambda w}$$

Since, in the continuous case, the derivative of the cumulative distribution function is the density

$$F'(w) = f(w) = \lambda e^{-\lambda w}$$

This is exactly density for an exponential random variable with $\beta = \frac{1}{\lambda}$