Mathematics-II (MATH F112)

Linear Algebra

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Section 3.4

Eigenvalues and Eigenvectors

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Example 1: For the matrix

$$A = \begin{bmatrix} -4 & 8 & -12 \\ 6 & -6 & 12 \\ 6 & -8 & 14 \end{bmatrix}.$$

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Note that $A \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$ implies $\lambda = 2$ is an **eigenvalue** of A and $X = [4, 3, 0]$ is the **eigenvector corresponding to** 2**.**

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Thus, if X is an eigenvector of A corresponding to an eigenvalue λ then, for $c \in \mathbb{R}$, cX is also an eigenvector corresponding to λ. Hence, there are infinitely many eigenvectors corresponding to an eigenvalue.

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Theorem: Let A be $n \times n$ matrix and λ be a real number. Then λ is an eigenvalue of A if and only if $|\lambda I_n - A| = 0$.

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Theorem: Let A be $n \times n$ matrix and λ be a real number. Then λ is an eigenvalue of A if and only if $|\lambda I_n - A| = 0$. The eigenvectors are the nontrivial solutions of the homogeneous system

$$(\lambda I_n - A)X = \mathbf{0}.$$

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The eigenvalues of an $n \times n$ matrix A are precisely the real roots of the characteristic polynomial $p_A(x)$.

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$$p_A(x) = |xI_3 - A| = \begin{vmatrix} x-1 & 0 & -1 \\ 0 & x-2 & 3 \\ 0 & 0 & x+5 \end{vmatrix}$$
$$= (x-1)(x-2)(x+5)$$

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In Example 3 The algebraic multiplicity of each of the eigenvalues ($\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = -5$) is 1.

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Theorem: Let A be an $n \times n$ matrix and λ be an eigenvalue of A. Then the eigenspace E_λ corresponding to λ is a subspace of $\mathbb{R}^n$.

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The geometric multiplicity of an eigenvalue λ is the dimension of its corresponding eigenspace E_λ i.e.

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$$E_1 = \{[a, 0] \mid a \in \mathbb{R}\} = \{a[1, 0] \mid a \in \mathbb{R}\}.$$

For the matrix B, the characteristic polynomial

$$p_B(x) = |xI_3 - B| = \begin{vmatrix} x-4 & 0 & 2 \\ -6 & x-2 & 6 \\ -4 & 0 & x+2 \end{vmatrix} = x(x-2)^2.$$

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$$(\lambda I_3 - B)X = 0 \text{ implies } -BX = 0.$$

The augmented matrix is

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Note that

$$E_0 = \text{span}\{[1, 3, 2]\} = \text{span}(B),$$

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$$\text{G.M. of } 0 = \dim E_0 = 1.$$

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$$[2I_3 - B|0] = \left[\begin{array}{ccc|c} -2 & 0 & 2 & 0 \\ -6 & 0 & 6 & 0 \\ -4 & 0 & 4 & 0 \end{array}\right]$$

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Then

$$E_2 = \{[c, b, c] | b, c \in \mathbb{R}\} = \{b[0, 1, 0] + c[1, 0, 1] | b, c \in \mathbb{R}\}.$$

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Now $E_2 = \text{span}(B)$, where $B = \{[0, 1, 0], [1, 0, 1]\}$. Since, B is LI ((verify it)), it is a basis for E_2.

Note that

$$\text{G.M. of } 2 = \dim E_2 = 2.$$

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Example 5: Let A be a 2×2 matrix with eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ corresponding to eigenvalues $\lambda_1 = -1, \lambda_2 = 2$.

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Solution:

$$A^{10}X = \begin{bmatrix} 2051 \\ 4093 \end{bmatrix}_{2\times 1}$$

Exercise: Consider

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- Find all the eigenvalues of A and compute their algebraic multiplicity.
- Find eigenspaces corresponding to each of the eigenvalues of A and compute their geometric multiplicity.

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Thank You