Chapter 3: Elementary Functions

1. Exponential Functions
2. Trigonometric Functions
3. Hyperbolic Functions
4. Logarithmic Functions
5. Complex Exponents

Self Study (Sec 36, p.112-115)

6. Inverse Trigonometric Functions
7. Inverse Hyperbolic Functions

See 29: Exponential Function :

(1) Let $z = x + iy$, then

$$\exp(z) = e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots\ldots + \frac{z^n}{n!} + \ldots.$$

is called Maclaurin' series of e^z

$$e^{z_1} \cdot e^{z_2} = e^{z_1 + z_2}, \quad \frac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2}$$

(2) Let $f(z) = e^z = e^{x+iy}$

$$= e^x . e^{iy}$$

$$= e^x (\cos y + i \sin y)$$

$$\equiv u + iv$$

$$\Rightarrow u = e^x \cos y,\ v = e^x \sin y,$$

$$u = e^x \cos y,\ v = e^x \sin y$$

$$\Rightarrow u_x = e^x \cos y,\ u_y = -e^x \sin y$$

$$v_x = e^x \sin y,\ v_y = e^x \cos y$$

$$\Rightarrow u_x = v_y,\ u_y = -v_x$$

Thus CR equations are satisfied and

clearly u_x, u_y, v_x, v_y are continuous

$\Rightarrow f(z)$ *is* differentiable and

$$f'(z) = u_x + i v_x$$

$$= e^x \cos y + i e^x \sin y = e^x . e^{iy} = e^z$$

$$\Rightarrow \frac{d}{dz}\left(e^z\right) = e^z$$

$$(3)\ \mathrm{e}^{\mathrm{z}} = e^{x} . e^{iy}, \quad e^{iy} = \mathbf{cos}\, y + i\,\mathbf{sin}\, y$$

$$\Rightarrow \left|e^{iy}\right| = \sqrt{\mathbf{cos}^2 y + \mathbf{sin}^2 y} = 1$$

$$\therefore \left|e^{z}\right| = \left|e^{x}\right| = e^{x} \text{ as } \mathrm{e}^{\mathrm{x}} > 0 \ \forall\, x \in R$$

$$\Rightarrow e^{z} \neq 0 \text{ for any complex number z.}$$

We may write $\mathrm{e}^{z} = e^{x} \cdot e^{iy} = \rho e^{i\phi}$,

$when\ \rho = e^{x} = \left|e^{z}\right| > 0$ **&** $\phi = y$

$$\therefore \arg\left(\mathrm{e}^{z}\right) = y + 2n\pi,$$

$$n = 0, \pm 1, \pm 2 \ldots\ldots\ldots\ldots$$

(4) $\because \cos 2\pi = 1 \ \& \ \sin 2\pi = 0$

Hence $e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1$

$$e^{\pi i} = \cos \pi + i \sin \pi = -1$$

$$e^{-\pi i} = \cos(-\pi) + i \sin(-\pi) = -1$$

$$e^{\pi i/2} = \cos\pi/2 + i\sin\pi/2 = i$$

$$\begin{aligned} e^{-\pi i/2} &= \cos(-\pi/2) + i\sin(-\pi/2) \\ &= -i \end{aligned}$$

5. $$e^{z+2\pi i} = e^z . e^{2\pi i} = e^z$$

$\Rightarrow e^z$ is periodic with imaginary period $2\pi i$.

$$\therefore\ e^{z\pm 2n\pi i} = e^z \ \ \forall\ n = 0,1,2,3,\ldots\ldots\ldots\ldots$$

(6) $e^{x} > 0 \ \forall x \in \Re$

But e^{z} *maybe negativeif* $z \in C$

Example: Find z such that $\mathrm{e}^{\mathrm{z}} = -1$

Solution :

$$e^z = -1$$

$$\Rightarrow e^x . e^{iy} = 1.e^{\pi i}$$

$$\Rightarrow \quad e^x = 1, \quad \text{and}$$

$$y = \pi + 2n\pi, n = 0, \pm 1,. \pm 2...$$

$$\Rightarrow x = 0 \ \ \& \ \ y = \pi + 2n\pi$$

Thus, if $z = x + iy$

$$= (2n+1)\pi i,$$

$$n = 0, \pm 1, \pm 2, \ldots$$

then $e^z = -1$

Excercise :

(7) $e^{\bar{z}}$ is not analytic anywhere.

Q. Find all values of z such that

$$\mathrm{e}^{2\mathrm{z}-1} = 1 + i$$

Solution:

$$e^{2z-1} = 1 + i$$

$$\Rightarrow e^{2x-1} . e^{2iy} = \sqrt{2}\ e^{\frac{\pi}{4}i}$$

$$\Rightarrow e^{2x-1} = \sqrt{2},$$

$$2y = \frac{\pi}{4} + 2n\pi;$$

$$n = 0, \pm 1, \pm 2, \ldots$$

$$\Rightarrow x = \frac{1}{2}\left(1 + \ln\sqrt{2}\right), \quad y = \frac{\pi}{8} + n\pi$$

$$\therefore z = x + iy$$

$$= \frac{1}{2}\left(1 + \ln\sqrt{2}\right) + i\left(\frac{\pi}{8} + n\pi\right),$$

$$n = 0, \pm 1, \pm 2, \ldots.$$

Trigonometric Functions

(1) If x is real ,then

$$\cos x = \frac{e^{ix} + e^{-ix}}{2},$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

If z is complex, we define

$$\cos z = \frac{e^{iz} + e^{-iz}}{2},$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad ---(1)$$

$$\Rightarrow e^{iz} = \cos z + i \sin z,$$

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z},$$

$$\sec z = \frac{1}{\cos z}, \quad \cos ec\, z = \frac{1}{\sin z}$$

2. Since e^z is analytic $\forall$ z and linear combination of two analytic functions is again analytic, hence it follows that sin z and cos z are analytic functions.

3. Using (1) it is easy to prove:

i) $\sin(-z) = -\sin z$

ii) $\cos(-z) = \cos z$

iii) $\dfrac{d}{dz}(\sin z) = \cos z$

$$iv)\quad \frac{d}{dz}(\cos z) = -\sin z$$

$$v)\quad \frac{d}{dz}(\tan z) = \sec^2 z$$

$$vi)\quad \sin(z_1 \pm z_2)$$

$$= \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$$

$$vii)\quad \cos(z_1 \pm z_2)$$

$$= \cos z_1 . \cos z_2 \mp \sin z_1 \sin z_2$$

$$(4)\because \cos z=\frac{e^{iz}+e^{-iz}}{2},$$

$$\sin z=\frac{e^{iz}-e^{-iz}}{2i}.$$

Put $x=0$, then

$$\cos(iy) = \frac{e^{i(iy)} + e^{-i(iy)}}{2}$$

$$= \frac{e^{-y} + e^{y}}{2} = \cosh y$$

$$\sin(iy) = -\frac{1}{2i}(e^{y} - e^{-y})$$

$$= i\,\frac{1}{2}(e^{y} - e^{-y})$$

$$= i \sinh y$$

$$\cos z = \cos(x + iy)$$

$$= \cos x \cos(iy) - \sin x . \sin(iy)$$

$$= \cos x . \cosh y - i \sin x . \sin hy$$

$$\sin z = \sin(x + iy)$$

$$= \sin x . \cos iy + \cos x . \sin iy$$

$$= \sin x . \cosh y + i \cos x . \sin h y$$

Hence (EXCERCISE)

$$|\sin z|^2 = \sin^2 x + \sinh^2 y$$

$$|\cos z|^2 = \cos^2 x + \sin h^2 y$$

Hints : (Use)

$$\cosh^2 x - \sinh^2 x = 1$$

$$\cos^2 x + \sin^2 x = 1$$

5. Analyticity of $\tan z$ & $\sec z$:

$$\because \tan z = \frac{\sin z}{\cos z}, \quad \sec z = \frac{1}{\cos z}$$

$\Rightarrow \tan z$ & $\sec z$ are analytic everywhere except at the points where $\cos z = 0$

$$\cos z = 0$$

$$\Rightarrow \cos\left(x + iy\right) =$$

$$\cos x \ \cos hy - i\sin x \ \sin hy = 0$$

$$\Rightarrow \cos x \ \cos hy = 0, \ \&$$

$$\sin x \ \sinh \ y = 0$$

$$\because \cosh y \neq 0$$

$$(\cosh y = \frac{e^{y} + e^{-y}}{2} = \frac{1}{2}\left(e^{y} + \frac{1}{e^{y}}\right)$$

$$= 0 \Rightarrow e^{2y} = -1 < 0)$$

$$\therefore \cos x \Rightarrow 0 \Rightarrow x = (2n+1)\frac{\pi}{2}, n = 0, \pm 1, \pm 2 \ldots$$

$$But \quad \sin x \neq 0 \quad \text{for} \quad x = (2n+1)\frac{\pi}{2}$$

$$\therefore \quad \sinh y = 0 \Rightarrow y = 0$$

$$\left\{ \sinh y = \frac{e^{y} - e^{-y}}{2} = 0 \Rightarrow e^{2y} = 1 \Rightarrow y = 0 \right\}$$

$$\therefore\ z = x + iy = (2n+1)\frac{\pi}{2}$$

$\therefore$ tan z & sec z are analytic

every where except at

$$\text{z} = (2\text{n}+1)\frac{\pi}{2},\ \ n = 0, \pm 1 \pm 2, \ldots\ldots$$

$(6Ex.)$ Analyticity of $\cot z$ & $\operatorname{cosec} z$:

$$\because \cot z = \frac{\cos z}{\sin z} \ \& \ \operatorname{cosec} z = \frac{1}{\sin z}$$

$\Rightarrow \cot z \ \& \ \operatorname{cosec} z$ are analytic every where except at the points where $\sin z = 0$

CHECK??

$\cot z \;\&\; \operatorname{cosec} z$ are analytic everywhere except at the points where

$$z = n\pi, \quad n = 0, \pm 1, \pm 2, \ldots$$

Hyperbolic Functions :

Definition :

$$\sinh z = \frac{e^{z} - e^{-z}}{2},$$

$$\cosh z = \frac{e^{z} + e^{-z}}{2}.$$

$$(1) \because e^{z} \quad \& \; e^{-z} \quad \text{are} \quad \text{analytic}$$

$$\text{everywhere}$$

$$\Rightarrow \sin h\, z \,\&\, \cosh z \quad \text{are} \quad \text{analytic}$$

$$\text{everywhere}.$$

$$(2)\ \frac{d}{dz}\left[\sin h\, z\right] = \frac{d}{dz}\left[\frac{e^{z} - e^{-z}}{2}\right]$$

$$= \frac{e^{z} + e^{-z}}{2} = \cos h\, z$$

Similarly, $$\frac{d}{dz}\left[\cos hz\right] = \sin hz$$

3. $$\sinh(-z) = -\sinh z$$

$$\cosh(-z) = \cosh z$$

$$\cosh^2 z - \sinh^2 z = 1$$

4. $\mathbf{cos}\,z = \mathbf{cos}h(i\,z)$,

$$\because \quad \cos hz = \frac{e^{z} + e^{-z}}{2}$$

$$\Rightarrow \cos h(i\,z) = \frac{e^{i\,z} + e^{-i\,z}}{2} = \cos z$$

5. $\mathbf{cos}(iz) = \mathbf{cosh}\, z$

$$\because \cos z = \cosh(iz)$$

$$\Rightarrow \cos(iz) = \cosh(i^2 z)$$

$$= \cosh(-z) = \cosh z$$

6. $$\sin z = -i \sinh(iz)$$

7. $$\sin(iz) = -i \sinh(-z)$$
$$= i \sinh z$$

8. $\mathbf{sinh}\left(z_1 + z_2\right)$

$$= \mathbf{sin}\,h\,z_1 . \mathbf{cos}h\,z_2 + \mathbf{cos}h\,z_1 . \mathbf{sin}\,h\,z_2$$

9. $\cos h\left(z_1 + z_2\right)$

$$= \cos h\,z_1 . \cos h\,z_2 + \sin h\,z_1 . \sin h\,z_2$$

$$(10)\sinh z$$

$$= \sinh x.\cos y + i \cosh x.\sin y$$

Soln :

$$\because \quad \sin(iz) = i\sinh z$$

$$\Rightarrow \quad \sinh z = -i\sin(iz)$$

$$= -i\sin(ix - y)$$

$$\Rightarrow \sin h\, z = -i \sin(ix - y)$$

$$= -i[\sin(ix)\cos y$$

$$- \cos(ix)\ \sin y]$$

$$= -i[i \sinh x\ \cos y$$

$$- \cosh x \sin y]$$

$$\Rightarrow \sin h\, z$$

$$= \sinh x \cos y + i \cosh x \sin y$$

Excercise :

$$|\sin h\, z|^2 = \sin h^2 x + \sin^2 y$$

Similarly

$$a) \quad \cosh z = \cosh x \cos y$$

$$+ i \sinh x . \sin y$$

Use $\cosh z = \cos(i z) = \cos(ix - y)$

$$b) \quad |\cosh z|^2 = \sinh^2 x + \cos^2 y$$

(11) <u>Analyticity of tanhz & sechz:</u>

$$\because \qquad \tanh z = \frac{\sinh z}{\cosh z},$$

$$\operatorname{sech} z = \frac{1}{\cosh z}.$$

$\Rightarrow \tanh z$ & $\sec hz$ are analytic everywhere except at the points where

$$\cos h z = 0.$$

Now $\cosh z = 0$

$$\Rightarrow \cos(\mathrm{i}\,\mathrm{z}) = \cos(ix - y) = 0$$

$$\Rightarrow \cos(i\,x).\cos(y) + \sin(ix).\sin(y) = 0$$

$$\Rightarrow \cosh x\ .\cos y + i\sinh x.\sin y = 0$$

$$\Rightarrow \cosh x . \cos y = 0,$$

and

$$\sinh x . \sin y = 0.$$

$$\because \cosh x \neq 0 \Rightarrow \cos y = 0$$

$$\Rightarrow y = (2n+1)\frac{\pi}{2}, n = 0, \pm 1, \pm 2, \ldots$$

$$For\ y = (2n+1)\frac{\pi}{2}, \sin y \neq 0$$

$$\therefore \sinh x = 0 \Rightarrow x = 0$$

$$\therefore z = x + iy$$

$$= (2n+1)\frac{i\pi}{2},$$

$$n = 0, \pm 1, \pm 2, \ldots$$

$\Rightarrow \tan hz$ & $\sec hz$ are analytic everywhere except at

$$z = (2n+1)\frac{i\pi}{2},\ n = 0, \pm 1, \pm 2, \ldots.$$

Exercise:

coth z and cosech z are analytic everywhere except at z = nπi,

$$\boldsymbol{n} = 0, \pm 1, \pm 2, \ldots\ldots$$

$Q. Show that$

$$(i)\ \left|\sinh(\operatorname{Im} z)\right| \le |\sin z| \le \cosh(\operatorname{Im} z)$$

$$(ii)\ \left|\sinh(\operatorname{Im} z)\right| \le |\cos z| \le \cosh(\operatorname{Im} z)$$