# *Ch. 4: Work and Energy*

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# Energy is the currency everywhere for anything and everything!

The floor is dirty? Bend and sweep – energy. Get it done by someone else – more energy. As I stand here and speak, I need energy and so do you to comprehend (even if I am incomprehensible). It takes transport of energy to see and hear things, to love and hate people. Any conceivable form of entertainment feeds heavily on energy. You don't pay for force, or momentum, or acceleration, but you sure pay for energy.It is impossible to think of a phenomena that does not consume/release energy and yet there is not a single reference to the word energy in the whole of Newton's principia.

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# Why is energy so important?

- Intimately connected to the $2^{nd}$ law and hence a useful tool to extract information about the system without having to directly confront the bull ($F = ma$).
- Its importance lies mostly in its attribute of conservation. It manages this extraordinary feat of conservation by deftly converting itself into myriad of different forms. You just can't kill this beast.
- Conservation laws in turn have deep connections with the symmetry properties of transformations in space and time. They are thus in some sense more fundamental than the Newton's laws and hence hold true even in regimes where Newton's laws fail (Relativity and Quantum Mechanics)
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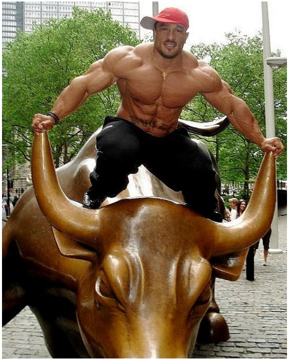
# What if I have the muscles to take the bull by the horn



After all, $\vec{F} = m\vec{a}$, I integrate twice and I am good to go!

# The problem is more serious

$$\vec{F}(\vec{r}) = m\frac{d\vec{v}(t)}{dt}$$

- Force is usually known as a function of position and not time. Cannot integrate in a straight forward manner.
- We will work around this problem using a simple trick and in the process have our first brush with difference between math and physics. What is mathematically only a matter of integration has a physical interpretation in terms of work energy theorem.

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# Total energy vs. total mechanical energy

- Of the myriad of different forms energy is capable of taking, in mechanics we shall be concerned with two fundamental forms –
  (a) the one associated with motion (kinetic energy)
  (b) the one associated with conservative forces e.g. gravitational, electrical, spring force (potential energy)
- Total mechanical energy = K.E + P.E
- When the energy transforms from these forms to chemical energy, or radiation, or random molecular or atomic motion, we call it heat. From the standpoint of mechanics it is lost. Whereas the total energy is always conserved, when the energy is lost to heat, we say that the total mechanical energy is not conserved and speak of non-conservative forces at play.

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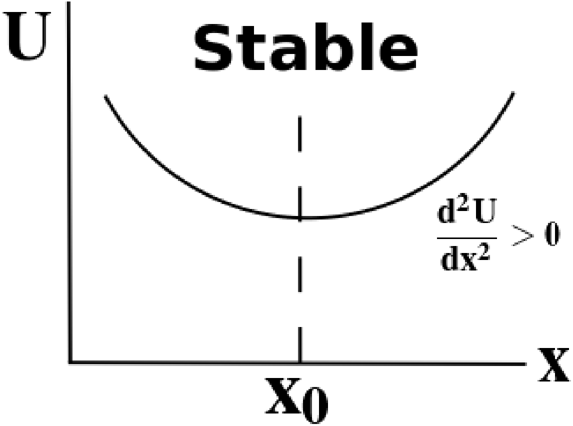
# Potential Energy determines stability of a system



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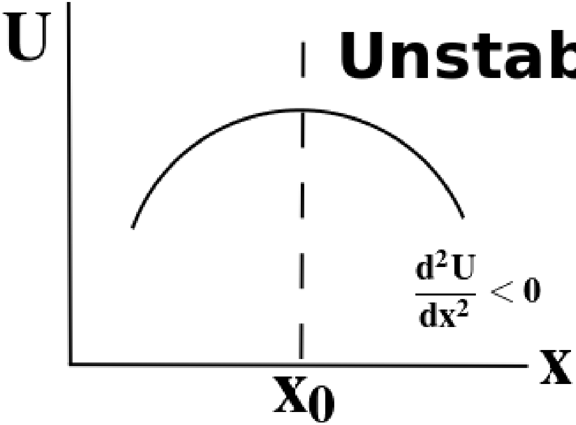

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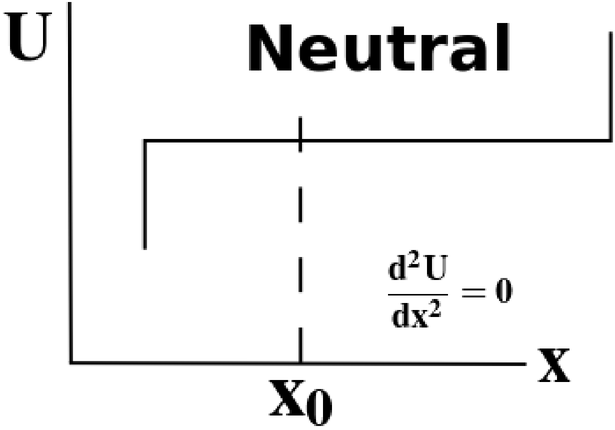
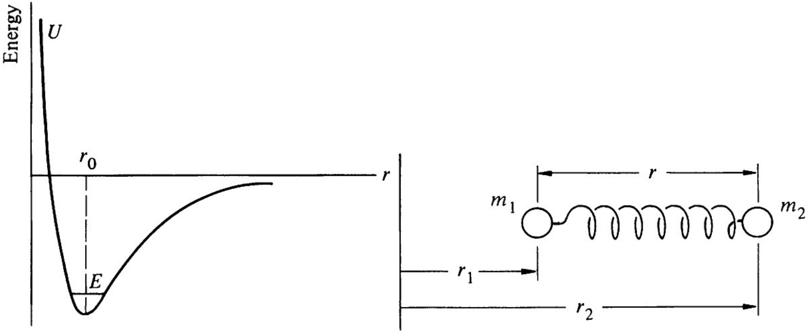


# Why do atoms of a molecule vibrate?



# Reducing a two body problem to one body problem

Referring to the previous slide, with $r_0$ being the equilibrium length of 'spring', $r$ being their instantaneous separation, equations of motion for $m_1$ and $m_2$ are:

$$m_1 \ddot{r}_1 = k(r - r_0)$$
$$m_2 \ddot{r}_2 = -k(r - r_0)$$

Dividing $1^{st}$ eq. by $m_1$ and $2^{nd}$ eq. by $m_2$ and subtracting, we get

$$\ddot{r}_2 - \ddot{r}_1 = \ddot{r} = -k\left(\frac{1}{m_1} + \frac{1}{m_2}\right)(r - r_0) \quad (1)$$

or

$$\mu \ddot{r} = -k(r - r_0) \quad (2)$$

where $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is the reduced mass.

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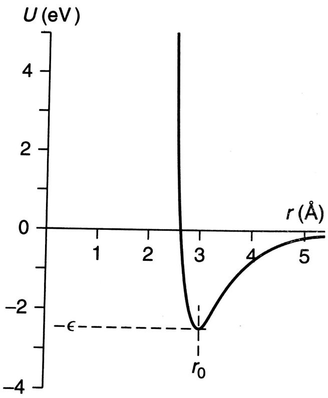
# Prob. 4.13. Lennard-Jones 6-12 potential




Commonly used function to describe the interaction between two atoms is Lennard-Jones 6-12 potential.

$$U = \epsilon \left[ \left( \frac{r_0}{r} \right)^{12} - 2 \left( \frac{r_0}{r} \right)^{6} \right]$$

a. Show that the radius at the potential minimum is $r_0$ and that the depth is $\epsilon$.

b. Find the frequency of small oscillations for two identical atoms.

# Solution. 4.13. Lennard Jones Potential

Engineering the right potential: The term $(r_0/r)^{12}$ rises steeply for $r < r_0$, and hence models the strong "hard sphere" repulsion between two atoms at close separation. The term $(r_0/r)^6$ decreases slowly for $r > r_0$ to model long attractive tail between two atoms at large separation.

To find minimum:

$$\frac{dU}{dr} = \left(\frac{\epsilon}{r_0}\right)\left[-12\left(\frac{r_0}{r}\right)^{13} + 12\left(\frac{r_0}{r}\right)^{7}\right]$$

Clearly $\frac{dU}{dr} = 0$ at $r = r_0$ and $U(r_0) = -\epsilon$.

For this to be a minimum $\frac{d^2U}{dr^2} > 0\ at\ r = r_0$

$$\frac{d^2U}{dr^2} = \left(\frac{\epsilon}{r_0^2}\right)\left[(12)(13)\left(\frac{r_0}{r}\right)^{14} - (12)(7)\left(\frac{r_0}{r}\right)^{8}\right]$$

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# Frequency of small oscillations

Since spring constant $k$ is given as:

$$k = \left.\frac{d^2U}{dr^2}\right|_{r=r_0} = \frac{72\epsilon}{r_0^2}$$

$$\omega = \sqrt{k/\mu} = 12\sqrt{\epsilon/r_0^2 m}$$

**For Chlorine molecule ($Cl_2$):**
$m = 5.89 \times 10^{-26}\ Kg$ and calculated value $r_0 = 2.98 \times 10^{-10} m$ and $\epsilon = 3.97 \times 10^{-19} J$. This gives $\omega = 1.05 \times 10^{14} rad/s$ which is in excellent agreement with experimentally observed frequency $1.05 \times 10^{14} rad/s$.

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# Ubiquitous Oscillatory Systems

But how do we 'smell' them? Look for quadratic energy forms.

In many problems, energy is naturally written in terms of variables other then linear displacement. For instance, $q$ and $\dot{q}$ where $q$ is a variable other than displacement.

$$U = \frac{1}{2} A q^2$$
$$K = \frac{1}{2} B \dot{q}^2$$

$U$ is a measure of the energy storing potential owing to elastic attribute ($A$) whenever $q \neq 0$. $K$ is a measure of energy storing inertial attributes owing to ($B$) whenever $\dot{q} \neq 0$. Thus, $\omega = \sqrt{\frac{A}{B}}$

mass-spring system: $q = x, \quad A = k, \quad , B = m,$
LC-circuit: $q$ is charge on capacitor, $A = 1/C$, $B = L$.

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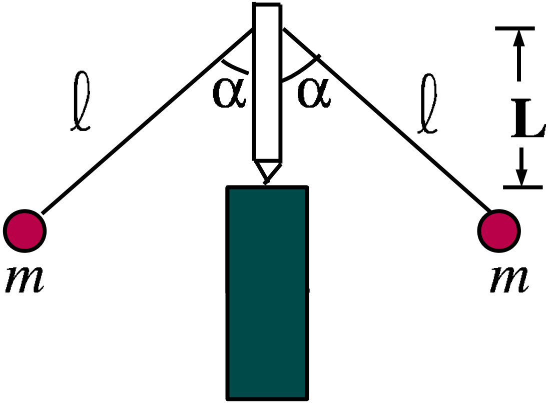
# Rock me, spin me, but topple I don't: Amazingly stable teeter-toy



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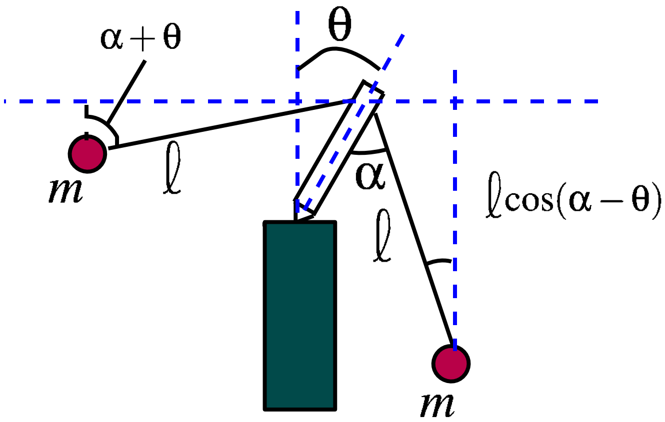

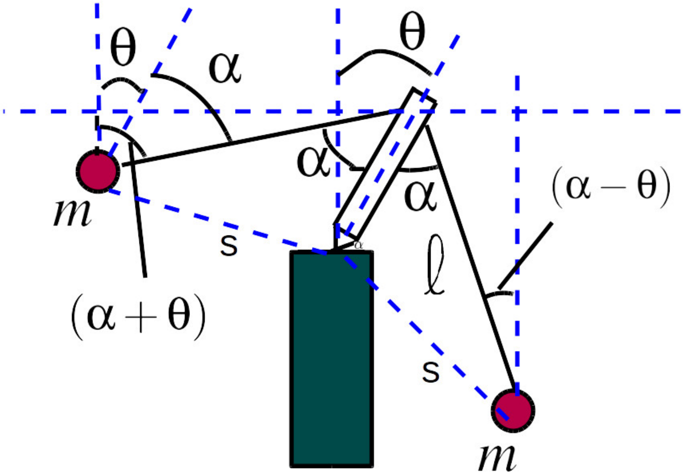


$$U(\theta) = mg\,[L\cos\theta - l\cos(\alpha+\theta)] - mg\,[l\cos(\alpha-\theta) - L\cos\theta] \quad (3)$$

# P.E. for Teeter-toy

Using $\cos(\alpha \pm \theta) = \cos\alpha\cos\theta \mp \sin\alpha\sin\theta$

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\begin{aligned}
U(\theta) &= 2mg\cos\theta(L - l\cos\alpha) \\
&= -A\cos\theta \quad \textit{where } A = 2mg(l\cos\alpha - L) \textit{ a constant} \\
&= -A\left(1 - \frac{\theta^2}{2} + \ldots\right) \quad \textit{Taylor expansion for } \cos\theta \\
&= -A + \frac{1}{2}A\theta^2 \quad \textit{cannonical oscillator P.E. form}
\end{aligned}
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If $s$ is the distance of each mass from the pivot, and the toy rocks with angular speed $\dot{\theta}$, then the speed of each mass is $s\dot{\theta}$. Thus,

$$K = \frac{1}{2}(2m)s^2\dot{\theta}^2 = \frac{1}{2}B\dot{\theta}^2$$

$$\omega = \sqrt{\frac{A}{B}} = \sqrt{\frac{g(l\cos\alpha - L)}{s^2}}$$

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Using $\cos(\alpha \pm \theta) = \cos\alpha\cos\theta \mp \sin\alpha\sin\theta$

$$\begin{aligned} U(\theta) &= 2mg\cos\theta(L - l\cos\alpha) \\ &= -A\cos\theta \quad \textit{where } A = 2mg(l\cos\alpha - L) \textit{ a constant} \\ &= -A\left(1 - \frac{\theta^2}{2} + \ldots\right) \quad \textit{Taylor expansion for } \cos\theta \\ &= -A + \frac{1}{2}A\theta^2 \quad \textit{cannonical oscillator P.E. form} \end{aligned}$$

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# Stability Analysis of Teeter-toy

$$U(\theta) = 2mg\cos\theta(L - l\cos\alpha)$$

Equilibrium occurs when

$$\frac{dU}{d\theta} = -2mg\sin\theta(L - l\cos\alpha) = 0$$

This implies $\theta = 0$ (we rule out $\theta = \pi$ as unphysical). To investigate stability we must find second derivative.

$$\begin{aligned} \frac{d^2U}{d\theta^2} &= -2mg\cos\theta(L - l\cos\alpha) \\ \left[\frac{d^2U}{d\theta^2}\right]_{\theta=0} &= -2mg(L - l\cos\alpha) \end{aligned}$$

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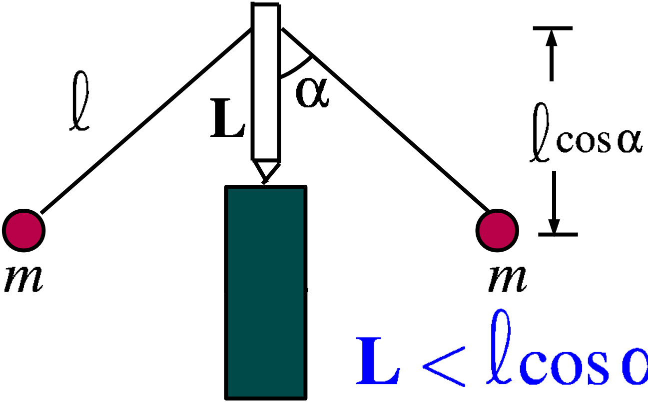



$$L < \ell\cos\alpha$$

Stability requires low center of mass and hence the peculiar design of a sports car.

# A word of Caution!

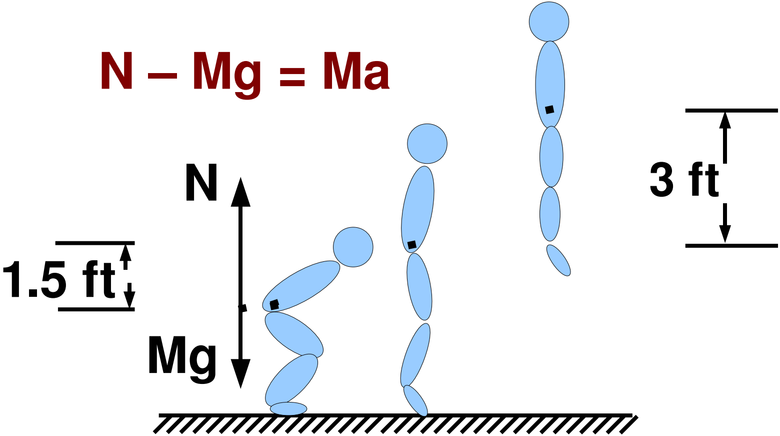
You are
neither a teeter-toy
nor a formula one car
low CG = big instability
better rev up.

# Problem 4.18

A 160 *lb* man leaps into the air from a crouching position. His center of gravity rises 1.5 *ft* before he leaves the ground, and it then rises 3 *ft* to the top of his leap. What power does he develop assuming that he pushes the ground with constant force?

# Problem 4.18

$$N - Mg = Ma$$



# Solution to 4.18

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\begin{aligned}
P &= W/T \quad (W = \textit{work done by } N) \\
W &= N \cdot 1.5 \quad (c.g.\ \textit{rises by } 1.5 ft) \\
N &= mg + ma \quad or \quad N = 160 + \frac{160}{32}a \\
a &= \frac{v^2}{2s} = 64 \quad \left[ v = \sqrt{2gs'} = \sqrt{2 \cdot 32 \cdot 3} = 8\sqrt{3} \right] \\
N &= 480\ lb \quad W = 720\ lb.ft \quad T = v/a = \sqrt{3}/8
\end{aligned}
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$$P = W/T = 3325 lb.ft/s \approx 6hp$$

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# Prob. 4.19 leaping man (now with varying force $F(t)$)

In the preceeding problem take $F(t) = F_0 \cos \omega t$ where $F_0$ is the peak force, and the contact with ground ends at $\omega t = \pi/2$. Find the peak power that the man develops during the jump.

$$\begin{aligned}
P(t) &= N(t)v(t) \quad [N(t) = -F(t)] \\
N(t) - mg &= ma(t) \\
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x(t) &= \frac{F_0}{m\omega^2}(1 - \cos \omega t) - \frac{1}{2}gt^2
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Two equations, two unknowns $F_0, \omega$. We can solve for these using boundary conditions $x(t = \pi/2\omega)$, $v(t = \pi/2\omega)$.

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# Except for $F(t) = F_0 \cos\omega t$ nothing has changed from previous problem

$$\begin{aligned} v(t = \pi/2\omega) &= 8\sqrt{3} = \frac{F_0}{m\omega} - \frac{g\pi}{2\omega} \\ x(t = \pi/2\omega) &= 1.5\ ft = \frac{F_0}{m\omega^2} - \frac{g\pi^2}{8\omega^2} \end{aligned}$$

$$\omega = 9.96s^{-1} \qquad F_0 = 832\ lb \qquad t = \frac{\pi}{2\omega} = 0.16s$$

$$\begin{aligned} P(t) &= F(t)v(t) \\ & \frac{F_0}{2m\omega}\left[\underbrace{F_0 \sin 2\omega t}_{1} - \underbrace{2mg\omega t \cos\omega t}_{2}\right] \end{aligned}$$

A reasonable approximation: $F_0 >> mg$ then $1^{st} >> 2^{nd}$

$$P(t) \approx \frac{F_0^2}{2m\omega} \sin 2\omega t$$

$$\text{For} \quad P_{max.} : \frac{dP}{dt} = 0$$

$$\frac{dP}{dt} = \frac{F_0^2}{m} \cos 2\omega t = 0$$

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Check:

$$\left.\frac{d^2P}{dt^2}\right|_{t=\frac{\pi}{4\omega}} = \left.\frac{-F_0^2}{m} 2\omega \sin 2\omega t\right|_{t=\frac{\pi}{4\omega}} < 0$$

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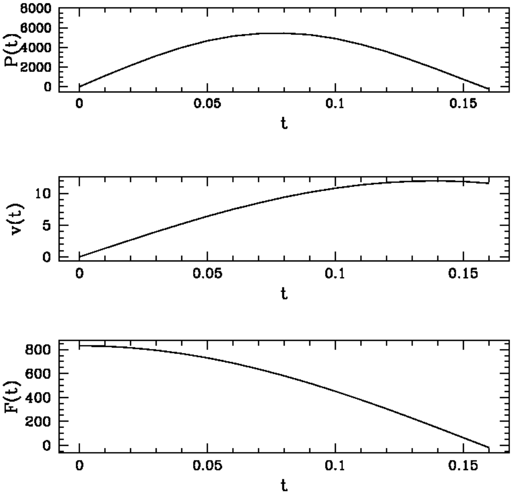
# Graphically

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