Chapter 11 (11.3-11.5, 11.7) Parametric Equations and Polar Coordinates

Note: *This module is prepared from Chapter 11 (Thomas' Calculus, 13th edition) just to help the students. The study material is expected to be useful but not exhaustive. For detailed study, the students are advised to attend the lecture/tutorial classes regularly, and consult the text book.*

Appeal: Please do not print this e-module unless it is really necessary.

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SECTION 11.3 (Polar Coordinates)

To any point P in a given plane, we can assign coordinates by choosing difference reference frames. For instance, in cartesian system, we set up the reference frame by choosing two perpendicular lines in the given plane, usually horizontal and vertical lines called X-axis and Y-axis respectively, intersecting at a point called origin denoted by O. Then the distance x of P from Y-axis, known as abscissa, is taken as first coordinate of P while the distance y of P from X-axis, known as ordinate, is taken as second coordinate of P. The distances x and y uniquely describe the position of the point P with respect to the cartesian frame given by X-axis and Y-axis as shown in the left panel of Figure 1. The abscissa x and ordinate y are together known as coordinates of the point P and are written as an ordered pair (x, y).

In polar system, we choose a fixed point O, called pole, and a horizontal ray OX, called initial ray or polar axis, in the given plane (See right panel of Figure 1). Then the directed distance r of the point P from the pole O, known as radius vector, is taken as first polar coordinate of P. The angle θ made by OP with the polar axis OX measured in counter clockwise direction, known as vectorial angle, is taken as the second polar coordinate of P. It is obvious that the polar coordinates (r, θ) uniquely describe the position of the point P in the reference frame of polar system.

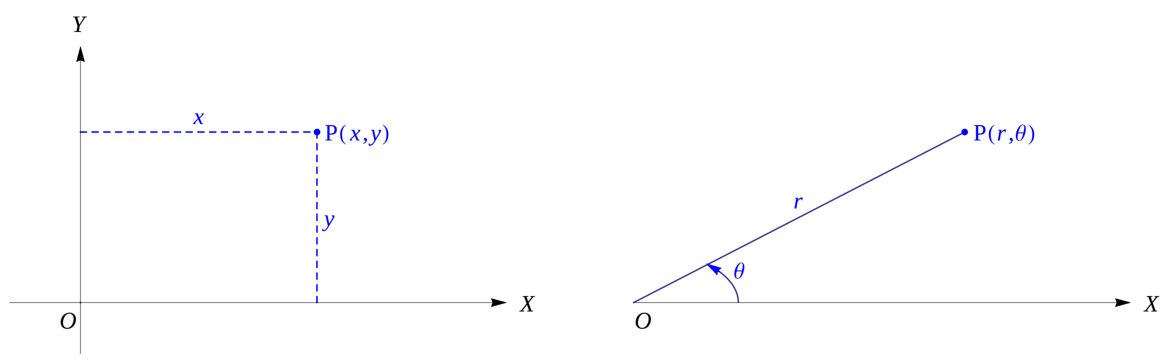

Figure 1: Left panel: System of cartesian coordinates. Right panel: System of polar coordinates

We use the cartesian or polar system as per our convenience. For instance, it is convenient to the study the motion of planets in polar system. We can switch over from cartesian to polar system and vice versa. To determine the equations of transformation, we show the two coordinate systems together in Figure 2, where the origin and pole coincide. Also, right half of X-axis coincides with the polar axis OX. It is easy to deduce that the equations of transformation from cartesian to polar system are

$$x = r\cos\theta, \quad y = r\sin\theta.$$

The equations of transformation from polar to cartesian system are

$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}(y/x).$$

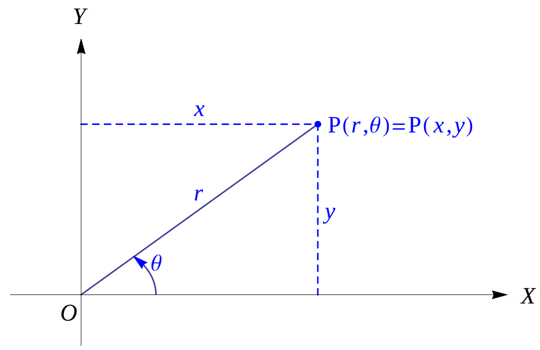

Figure 2: Cartesian and polar systems where the origin and pole coincide. Also, right half of X-axis coincides with the polar axis OX.

Remarks (i) For any integer n, we have $(r, \theta) = (r, \theta + 2n\pi)$. It implies that the second polar coordinate is not unique for a given point. However, it would be unique for each point in the plane if we allow θ to vary in an interval of length 2π such as $0 \leq \theta < 2\pi$. For example, $\left(2, \frac{\pi}{6}\right)$, $\left(2, -\frac{11\pi}{6}\right)$ both represent the same point in the polar system as shown in left panel of Figure 3.

(ii) For the pole, $r = 0$ but θ is arbitrary (not defined uniquely).

(iii) For any point different from pole, r is always positive. However, in some situations we allow r to be negative. We denote the reflection of (r, θ) through the pole by $(-r, \theta)$, which is nothing but the point $(r, \theta + \pi)$. For example, $\left(-2, \frac{\pi}{6}\right)$ is the reflection of $\left(2, \frac{\pi}{6}\right)$ through the pole as shown in the right panel of Figure 3.

(iv) In cartesian system $x = x_0$ and $y = y_0$ are the straight lines parallel to y-axis and x-axis at distances $|x_0|$ and $|y_0|$ from the origin, respectively. In polar system, on the other hand, $r = r_0$ is a circle centred at the pole with radius $|r_0|$, and $\theta = \theta_0$ is a line through the pole at an angle θ_0 with the polar axis.

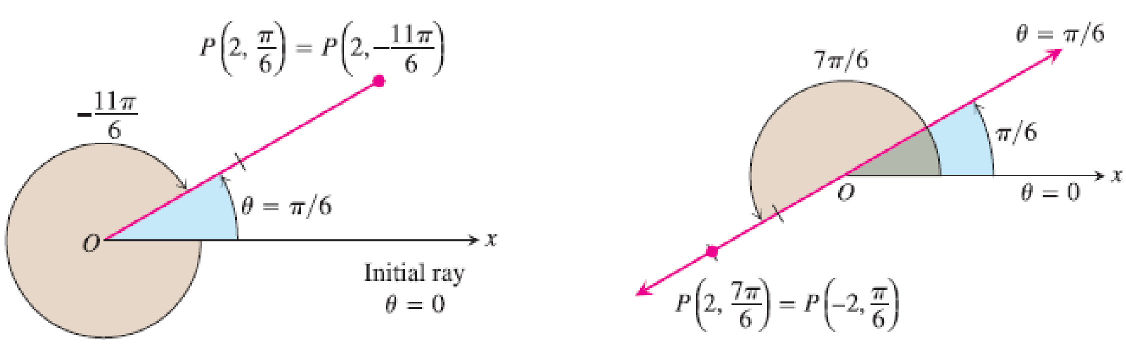

Figure 3: **Left panel:** Representation of $\left(2, \frac{\pi}{6}\right)$. **Right panel:** Reflection of $\left(2, \frac{\pi}{6}\right)$.

Ex. Graph the following:

(i) $1 \leq r \leq 2$ and $0 \leq \theta \leq \frac{\pi}{2}$

(ii) $-3 \leq r \leq 2$ and $\theta = \frac{\pi}{4}$

(iii) $\frac{2\pi}{3} \leq \theta \leq \frac{5\pi}{6}$ (no restriction on r)

Sol. The graphs are shown in Figure 4.

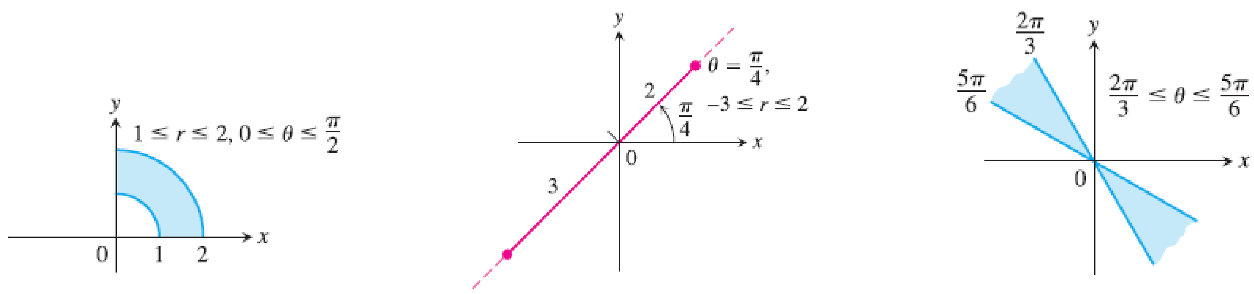

Figure 4: **Left:** (i) $1 \leq r \leq 2$ and $0 \leq \theta \leq \frac{\pi}{2}$. **Middle:** (ii) $-3 \leq r \leq 2$ and $\theta = \frac{\pi}{2}$. **Right:** (iii) $\frac{2\pi}{3} \leq \theta \leq \frac{5\pi}{6}$.

Note: The polar system, in some cases, leads to mathematical simplicity of the equations in cartesian system. For instance, the equation $x^2 + y^2 = 1$ (quadratic in x and y) transforms to $r = 1$, a simple linear equation in r.

SECTION 11.4 (Graphing in Polar Coordinates)

Tests of symmetry

Let $r = f(\theta)$ or $f(r, \theta) = 0$ be the given equation of a curve in polar system. Then the curve is symmetric about

(i) x-axis if $f(r, -\theta) = f(r, \theta)$.

(ii) y-axis if $f(r, \pi - \theta) = f(r, \theta)$.

(iii) origin if $f(-r, \theta) = f(r, \theta)$.

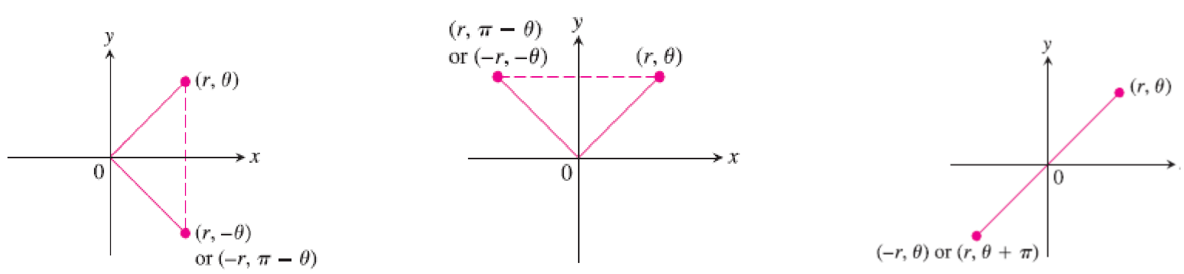

Figure 5: **Left:** (i) Symmetry about x-axis. **Middle:** (ii) Symmetry about y-axis. **Right:** (iii) Symmetry about origin.

Slope of $r = f(\theta)$

Since $x = r\cos\theta$ and $y = r\sin\theta$, the slope of the curve $r = f(\theta)$ is

$$\text{Slope} = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta},$$

where $r' = dr/d\theta$. For example, slope of the curve $r = 1+\cos\theta$ at $\theta = \pi/2$ is 1.

Note: If the curve $r = f(\theta)$ passes through the pole ($r = 0$) for $\theta = \theta_0$, then slope of the tangent at the pole is $\tan\theta_0$. It implies that $y = \tan\theta_0\, x$ or $\theta = \theta_0$ is tangent to the curve at the origin or the pole. For example, $\theta = \pi$ is tangent to the curve $r = 1+\cos\theta$ at the pole.

Steps to trace polar curves

Let $r = f(\theta)$ be the given equation of a curve in polar system. Then for tracing the curve, we follow the following steps:

(i) Check the curve for symmetries.

(ii) Check for the value of θ for which $r = 0$. If such a value exists, say θ_0, then the curve passes through the pole, and $\theta = \theta_0$ is tangent to the curve at the pole.

(iii) Find the maximum/minimum values of r, and the corresponding values of θ.

(iv) Find the range of θ for which $r'(\theta) > 0$ or $r'(\theta) < 0$, that is, r is increasing or decreasing.

(v) Find the slope of curve at some special points like end points of the range of θ where r is increasing or decreasing.

(vi) Make a table r vs θ for different values of θ. Finally plot the curve conforming to the above steps.

Ex. Trace the curve $r = 1-\cos\theta$, known as cardioid.

Sol. (i) Since $(r, -\theta)$ lies on the curve, so it is symmetric about the x-axis. So it is enough to plot the part of the curve for $0 \le \theta \le \pi$.

(ii) We see that $r = 0$ for $\theta = 0$. So the curve passes through the pole and $\theta = 0$ is tangent to the curve at the pole.

(iii) We notice that maximum value of r is 2 which occurs at $\theta = \pi$ while the minimum value of r is 0 that corresponds to $\theta = 0$.

(iv) For $0 < \theta < \pi$, we find $r'(\theta) > 0$. So r is increasing in the interval $[0, \pi]$.

(v) The slope of the curve at (r, θ) reads as

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} = \frac{\sin^2\theta + \cos\theta(1-\cos\theta)}{\sin\theta\cos\theta - \sin\theta(1-\cos\theta)}.$$

So slope of the curve at $\theta = 0$ is ∞ while at $\theta = \pi/2$ is -1.

(vi) Table of r vs θ for some selected values of θ is

θ :	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r :	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2

Conforming to the steps (i) to (vi), the graph of the given curve is shown in Figure 6.

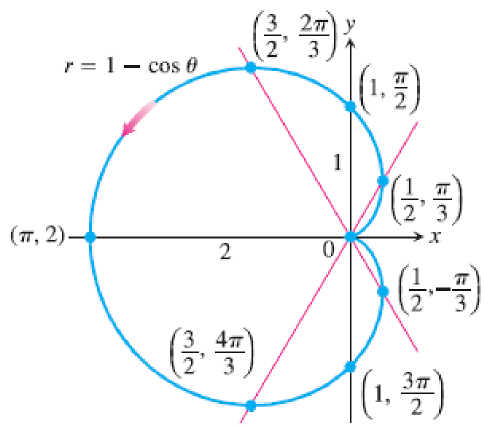

Figure 6: Plot of $r = 1 - \cos\theta$.

A technique for graphing

To plot a given curve $r = f(\theta)$, the simplest and straightforward approach is to prepare a table of (r, θ) points, and join the points of the table by free hand in the order of increasing θ. However, it requires enough collection points to reveal all the loops and dimples of the actual plot of $r = f(\theta)$. There is another technique, where we plot the curve in θr-plane treating θ along x-axis and r along y-axis, and use this plot as a guide for the polar plot as illustrated in the following example.

Ex. Graph the lemniscate curve $r^2 = \sin 2\theta$.

Sol. The given equation yields two single valued functions given by $r = +\sqrt{\sin 2\theta}$ and $r = -\sqrt{\sin 2\theta}$. So we get real values of r for values of θ in the intervals $[0, \frac{\pi}{2}]$ and $[\pi, \frac{3\pi}{2}]$. The plots of the two functions are shown in the left panel of Figure 7. From the plot of $r = +\sqrt{\sin 2\theta}$, it is evident that as θ varies in the interval $[0, \frac{\pi}{2}]$, r starts from 0 at $\theta = 0$, then increases to its maximum value at $\theta = \frac{\pi}{4}$ and again decreases to its minimum value 0 at $\theta = \frac{\pi}{2}$. Likewise the other plots of r can easily be understood. Following these plots of r vs θ, the plot of $r^2 = \sin 2\theta$ is shown in the right panel of Figure 7. The plot of $r = +\sqrt{\sin 2\theta}$, $\theta \in [0, \frac{\pi}{2}]$ corresponds to the first quadrant loop of $r^2 = \sin 2\theta$ while the plot of $r = -\sqrt{\sin 2\theta}$, $\theta \in [0, \frac{\pi}{2}]$ corresponds to the third quadrant loop of $r^2 = \sin 2\theta$. Notice that the third quadrant loop is the reflection of the first quadrant loop through the pole. Next, the plot of $r = +\sqrt{\sin 2\theta}$, $\theta \in [\pi, \frac{3\pi}{2}]$ corresponds to the third quadrant loop of $r^2 = \sin 2\theta$ while the plot of $r = -\sqrt{\sin 2\theta}$, $\theta \in [\pi, \frac{3\pi}{2}]$ corresponds to the first quadrant loop of $r^2 = \sin 2\theta$. In this case, the first quadrant loop is the reflection of the third quadrant loop through the pole. Notice that the curve is plotted two times over itself in our new method of plotting! But it is okay. You have got the plot easily.

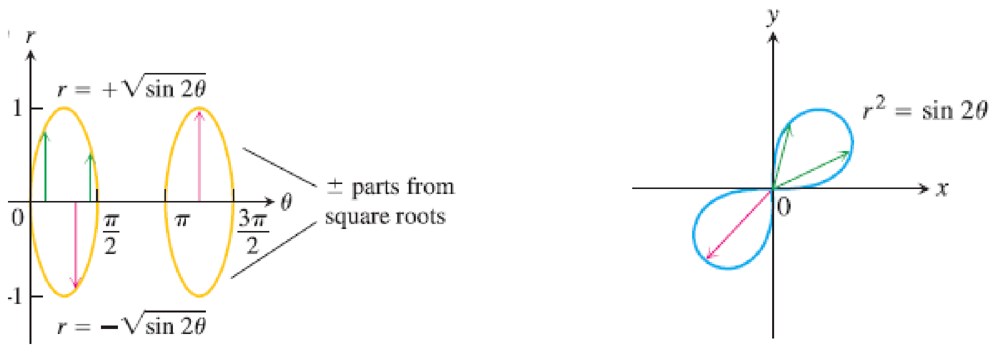

Figure 7: **Left panel:** Plots of $r = +\sqrt{\sin 2\theta}$ and $r = -\sqrt{\sin 2\theta}$ in θr-plane. **Right panel:** Plot of $r^2 = \sin 2\theta$.

Note: Some standard polar curves are given below. Try to remember these shapes logically with the equations.

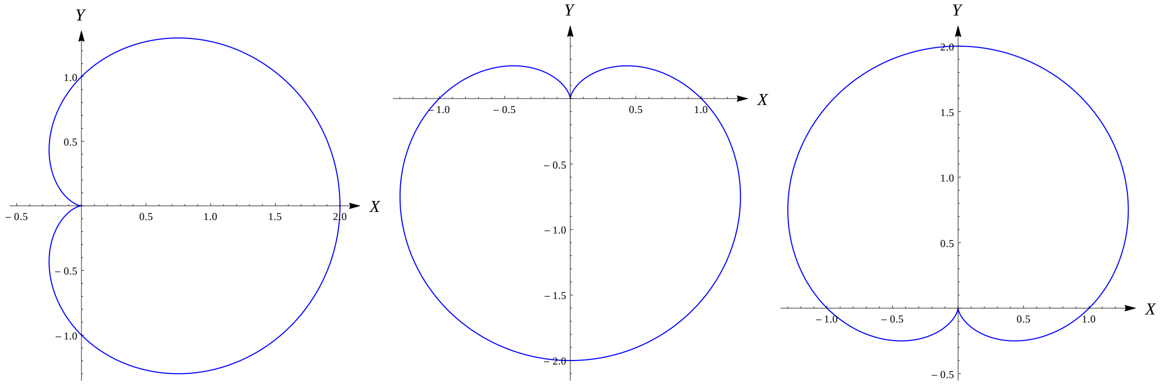

Figure 8: Some more cardioids. Left: $r = 1 + \cos\theta$. Middle: $r = 1 - \sin\theta$. Right: $r = 1 + \sin\theta$.

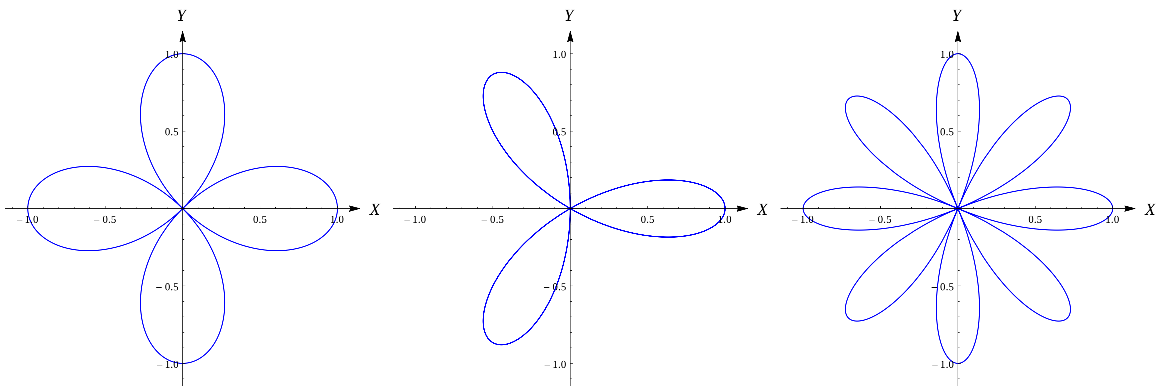

Figure 9: Roses. Left: $r = \cos 2\theta$ (Four leaved). Middle: $r = \cos 3\theta$ (Three leaved). Right: $r = \cos 4\theta$ (Eight leaved).

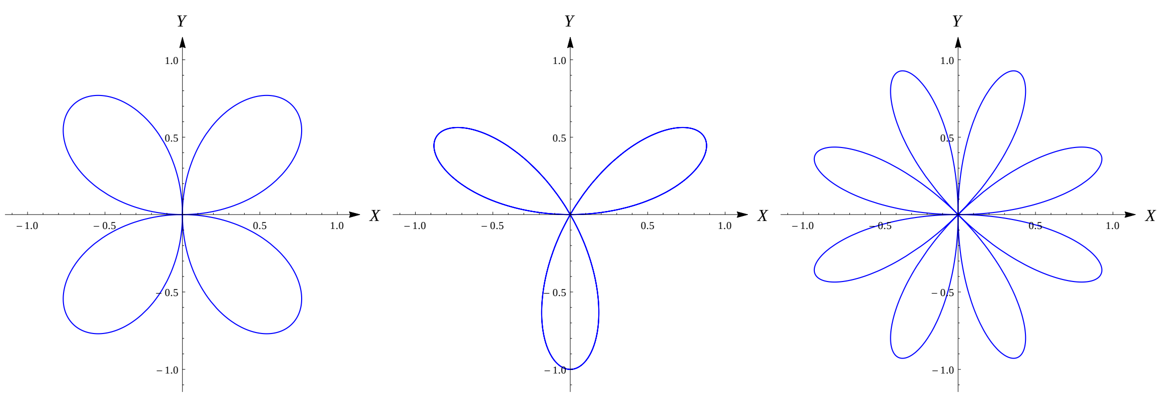

Figure 10: Roses. Left: $r = \sin 2\theta$ (Four leaved). Middle: $r = \sin 3\theta$ (Three leaved). Right: $r = \sin 4\theta$ (Eight leaved).

Ex. Find the points of intersection of the curves $r = \cos 2\theta$ and $r = 1 - \cos\theta$.

Sol. From the given equations, we have $\cos 2\theta = 1 - \cos\theta$ or $2\cos^2\theta + \cos\theta - 2 = 0$. This leads to one acceptable solution (the other solution gives $\cos\theta = \frac{-1-\sqrt{17}}{4} < -1$)

$$\theta = \cos^{-1}\left(\frac{-1+\sqrt{17}}{4}\right), \quad r = \frac{5-\sqrt{17}}{4}.$$

So the point of intersection is $\left(\frac{5-\sqrt{17}}{4}, \cos^{-1}\left(\frac{-1+\sqrt{17}}{4}\right)\right)$. Since the curves are symmetric about the X-axis, one more point of intersection is $\left(\frac{5-\sqrt{17}}{4}, -\cos^{-1}\left(\frac{-1+\sqrt{17}}{4}\right)\right)$. Thus, we are able to find two points of intersection by the direct calculation from the given curves. However from Figure 11, we observe that there are seven points of intersection of the two curves. So we are yet to obtain the remaining five points of intersection.

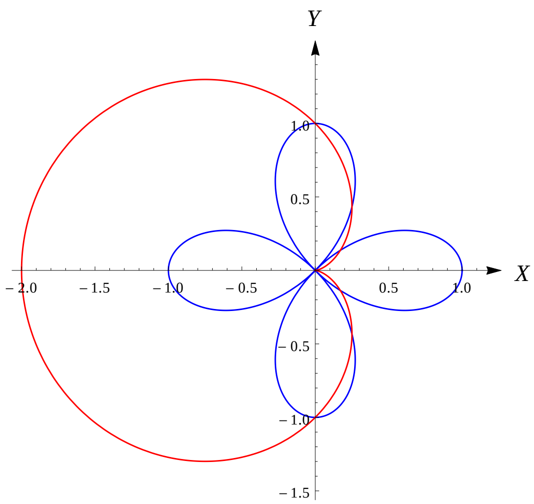

Figure 11: $r = \cos 2\theta$ (Blue color) and $r = 1 - \cos\theta$ (Red color).

The above problem poses a natural question "Why the direct calculation from the polar curves does not give all the points of intersection?" The answer is that the coordinates of points in polar system are not unique. For instance, (r, θ),$(-r, \theta + \pi)$ and $(r, \theta + 2\pi)$ all represent the same point. Likewise, $(0, \theta)$ are the coordinates of pole for any θ.

In the above problem, we see that $(-r, \theta + \pi)$ lies on the curve $r = \cos 2\theta$, and therefore $-r = \cos 2\theta$ is simply another representation of the curve $r = \cos 2\theta$. Solving $-r = \cos 2\theta$ with $r = 1 - \cos\theta$, we get $\cos 2\theta = 1 - \cos\theta$ or $1 - 2\cos^2\theta = 1 - \cos\theta$ or $\cos\theta(2\cos\theta - 1) = 0$. This gives $\cos\theta = 0, 1/2$ and hence $\theta = \pi/3, \pi/2$. So using symmetry, the points of intersection are $(-1/2, \pm\pi/3)$ and $(-1, \pm\pi/2)$.

Finally, we find that $(0, \pi/4)$ lies on $r = \cos 2\theta$ and $(0, 0)$ lies on the curve $r = 1 - \cos\theta$. This shows that both the curves pass through the pole. Hence the pole is also a point of intersection of the given curves.

A plausible strategy for finding all the points of intersection

The graphs for two polar functions $r = f(\theta)$ and $r = g(\theta)$ have possible intersections in 3 cases:

1. In the origin if the equations $f(\theta) = 0$ and $g(\theta) = 0$ have at least one solution each.
2. All the points $(g(\theta_i), \theta_i)$ where θ_i are the solutions to the equation $f(\theta) = g(\theta)$.
3. All the points $(g(\theta_i), \theta_i)$ where θ_i are the solutions to the equation $f(\theta + (2k+1)\pi) = -g(\theta)$.

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SECTION 11.5 (Areas and Lengths in Polar Coordinates)

Area in polar coordinates

Area of the region bounded by $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is given by

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta.$$

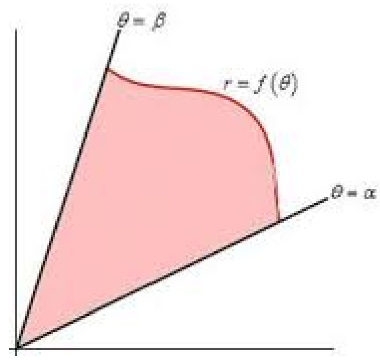

Figure 12: The shaded region is the area bounded by $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$.

Ex. Find the area of the lemniscate $r^2 = \cos 2\theta$.

Sol. See left panel of Figure 12 for the graph of the given curve. Because of symmetry, the area enclosed by the two loops of the curve is 4 times the area in the first quadrant. Also, θ varies from 0 to $\pi/4$ for the part of loop in the first quadrant.

$$\therefore \text{Req. Area} = 4.\frac{1}{2}\int_0^{\pi/4} r^2 d\theta = 2\int_0^{\pi/4} \cos 2\theta d\theta = 1.$$

Ex. Find the area inside the circle $r = 3\cos\theta$ but outside the cardioid $r = 1 + \cos\theta$.

Sol. See right panel of Figure 12 for the graphs of the given curves. Because of symmetry, the required area is 2 times the area in the first quadrant. Also the two curves intersect at $\theta = \pi/3$ in the first quadrant. Let $r_1 = 3\cos\theta$ and $r_2 = 1 + \cos\theta$. Then we have

$$\text{Req. Area} = 2\left[\frac{1}{2}\int_0^{\pi/3} r_1^2 d\theta - \frac{1}{2}\int_0^{\pi/3} r_2^2 d\theta\right] = \pi + 1 - \sqrt{3}.$$

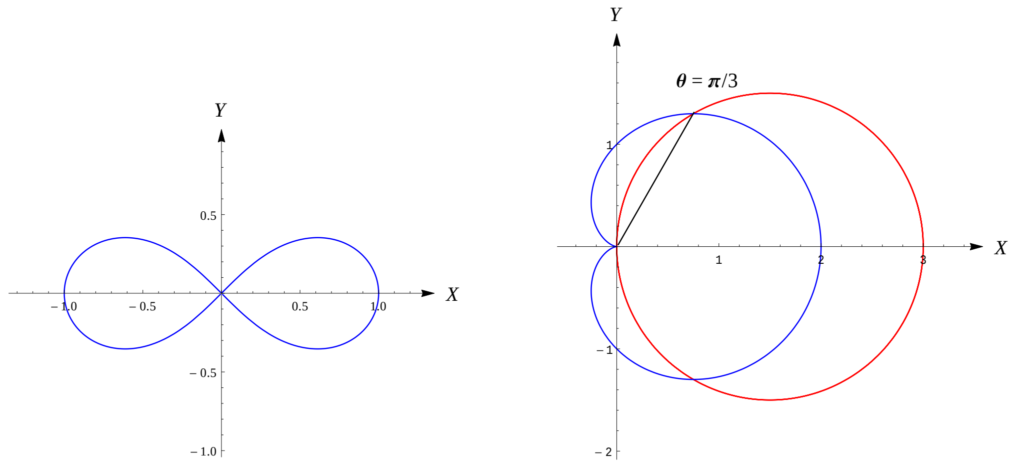

Figure 13: Left: $r^2 = \cos 2\theta$. Right: $r = 3\cos\theta$ (Red color), $r = 1 + \cos\theta$ (Blue color).

Length of polar curve

Length of the curve $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is given by

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Ex. Find the length of the circle $r = a$.

Sol. It is given by

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_{0}^{2\pi} \sqrt{a^2 + 0}\, d\theta = 2\pi a.$$

SECTION 11.7 (Conics in Polar Coordinates)

Conic sections

Locus of points whose distances from a fixed point (known as focus) and fixed line (known as directrix) are in constant ratio, defines a conic section. Let F be focus of a conic section and P be any point on the conic section at distance PD from the directrix of the conic section. Then the ratio PF/PD is defined as the eccentricity of the conic section and is denoted by e. So $PF = e.PD$. The conic section is a parabola for $e = 1$, ellipse for $e < 1$ and hyperbola for $e > 1$.

If we place one focus at the origin and the corresponding directrix to the right of the origin along the line $x = k$ $(k > 0)$ as shown in Figure 13, then $PF = r$ and $PD = k - FB = k - r\cos\theta$. So the relation $PF = e.PD$ gives $r = e(k - r\cos\theta)$ or

$$r = \frac{ke}{1 + e\cos\theta},$$

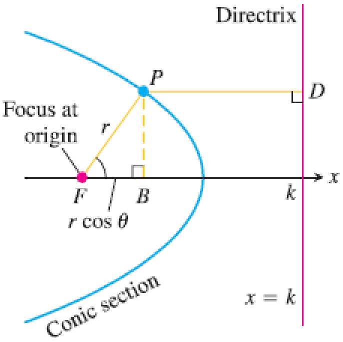

Figure 14:

the equation of the conic section.

Other forms of the conic sections are

$r = \dfrac{ke}{1 - e\cos\theta}$. (Focus at origin, Directrix $x = -k$)

$r = \dfrac{ke}{1 + e\sin\theta}$. (Focus at origin, Directrix $y = k$)

$r = \dfrac{ke}{1 - e\sin\theta}$. (Focus at origin, Directrix $y = -k$)

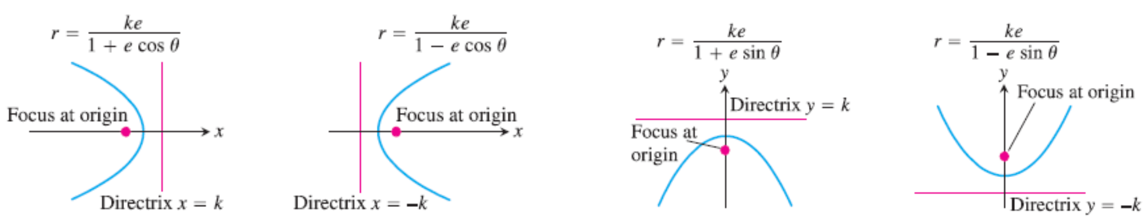

Figure 15: Various conics.

For an ellipse with semi-major axis a and eccentricity e, we have $k = \frac{a}{e} - ae$. So $r = \dfrac{a(1-e^2)}{1 + e\cos\theta}$.

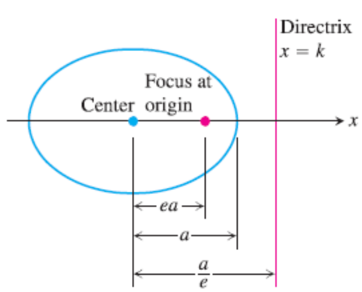

Figure 16: Ellipse with focus at the origin.

Ex. Assuming focus at the origin, find the vertices and centre of the conic section $r = \dfrac{400}{16 + 8\sin\theta}$.

Sol. Graph of the given curve is shown in the left panel of Figure 16. Comparing the given conic equation with $r = \dfrac{ke}{1 + e\sin\theta}$, we get $e = 1/2$ and $k = 50$. Using $ke = a(1 - e^2)$, we obtain $ae = 50/3$. Center is $(50/3, 3\pi/2)$ while the vertices are $(50/3, \pi/2)$ and $(50, 3\pi/2)$.

Ex. Find vertex of the parabola $r = 4/(1 + \sin\theta)$.

Sol. The graph of the given curve is shown in the right panel of Figure 16. For the given curve $e = 1$, $k = 4$. Vertex is $(2, \pi/2)$.

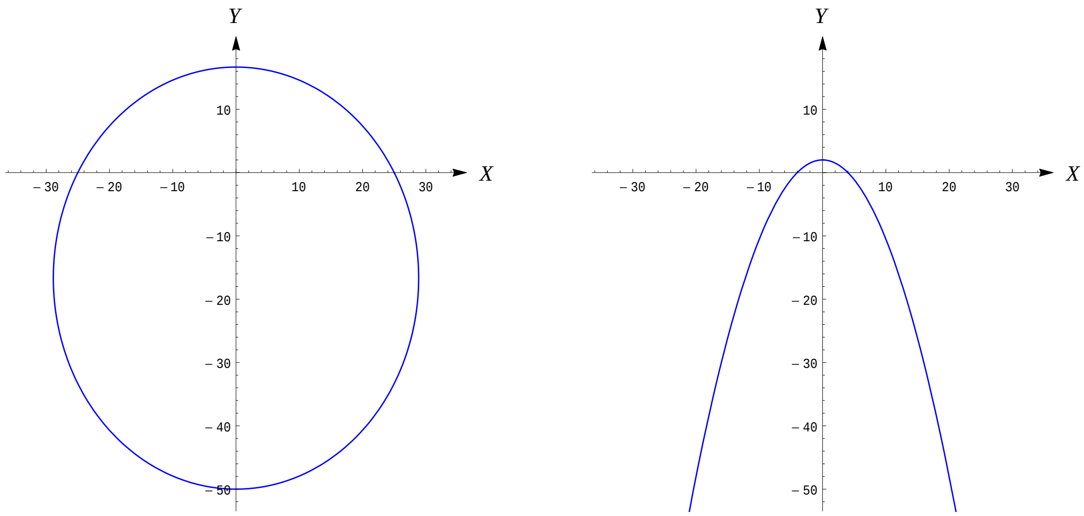

Figure 17: Left: $r = \dfrac{400}{16 + 8\sin\theta}$. Right: $r = 4/(1 + \sin\theta)$.

Polar equation of a line

If $P_0(r_0, \theta_0)$ is foot of perpendicular from pole onto a line L and $P(r, \theta)$ is any point of the line L, then the equation of line L is $r\cos(\theta - \theta_0) = r_0$.

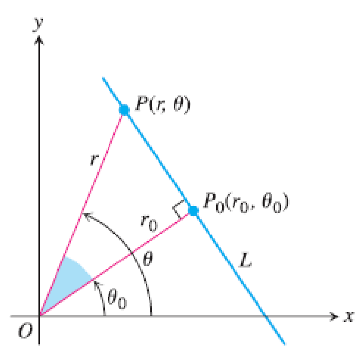

Figure 18: Line.

Ex. Find polar equation of $\sqrt{3}x - y = 1$ in the form $r\cos(\theta - \theta_0) = r_0$.

Sol. Using $x = r\cos\theta$, $y = r\sin\theta$ and rearranging, we get $r\cos(\theta + \pi/6) = 1/2$.

Polar equation of a circle

Consider a circle of radius a and centred at $P_0(r_0, \theta_0)$. If $P(r, \theta)$ is any point on the circle, then we have
$\cos(\theta - \theta_0) = \dfrac{r^2 + r_0^2 - a^2}{2rr_0}$, or $r^2 + r_0^2 - 2rr_0\cos(\theta - \theta_0) = a^2$,
the equation of the circle.

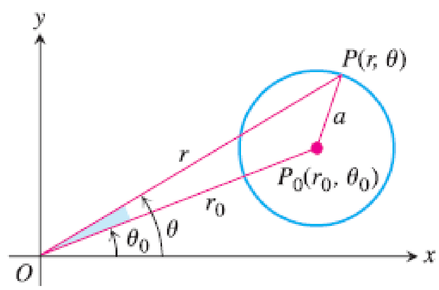

Figure 19: Circle.

If the circle passes through pole, then $r_0 = a$ and therefore
$r = 2a\cos(\theta - \theta_0)$.
Further, if the centre of the circle lies on x-axis, then $\theta_0 = 0$. If the centre lies on y-axis, $\theta_0 = \pi/2$.