PART B

Q1 X & Y are independent RV s.t.

IES 17

$$P[2X+4Y \le 10] + P[3X+Y \le 9] = 1$$

$$P[2X-4Y \le 6] + P[Y-3X \ge 1] = 1.$$

Assume $X \sim N(\mu, \sigma_1^2)$ & $Y \sim N(\mu, \sigma_2^2)$. Compute μ.

If X & Y are independent normal variates with mean μ_1 & μ_2 & variance σ_1^2 & σ_2^2 respectively,

then
$$\alpha X + \beta Y \sim N(\alpha + \beta,\ \alpha^2\sigma_1^2 + \beta^2\sigma_2^2) \quad (1)$$

Hence
$$2X + 4Y \sim N(2\mu + 4\mu,\ 4\sigma_1^2 + 16\sigma_2^2)$$
$$3X + Y \sim N(3\mu + \mu,\ 9\sigma_1^2 + \sigma_2^2)$$
$$2X - 4Y \sim N(2\mu - 4\mu,\ 4\sigma_1^2 + 16\sigma_2^2)$$
$$Y - 3X \sim N(\mu - 3\mu,\ 9\sigma_1^2 + \sigma_2^2) \quad (2)$$

let $4\sigma_1^2 + 16\sigma_2^2 = \alpha^2$, $9\sigma_1^2 + \sigma_2^2 = \beta^2$

Now $P[2X+4Y \le 10] + P[3X+Y \le 9] = 1$

$$\Rightarrow P\left[Z \le \frac{10-6\mu}{\alpha}\right] + P\left[Z \le \frac{9-4\mu}{\beta}\right] = 1$$

$$\Rightarrow P\left[Z \le \frac{10-6\mu}{\alpha}\right] = 1 - P\left[Z \le \frac{9-4\mu}{\beta}\right]$$
$$= P\left[Z \ge \frac{9-4\mu}{\beta}\right]$$

$$\Rightarrow \frac{10-6\mu}{\alpha} = -\left(\frac{9-4\mu}{\beta}\right) = \frac{4\mu-9}{\beta}$$

/4 mu -9 [3]

$$\Rightarrow \frac{\alpha}{\beta} = \frac{10-6\mu}{} \quad (3)$$

Similarly

$$P[2X-4Y \le 6] + P[Y-3X \ge 1] = 1$$

$$\Rightarrow P\left[Z \le \frac{6+2\mu}{\alpha}\right] + P\left[Z \ge \frac{1+2\mu}{\beta}\right] = 1$$

$$\Rightarrow P\left[Z \le \frac{6+2\mu}{\alpha}\right] = 1 - P\left[Z \ge \frac{1+2\mu}{\beta}\right]$$

$$= P\left[Z \le \frac{1+2\mu}{\beta}\right]$$

$$\Rightarrow \frac{6+2\mu}{\alpha} = \frac{1+2\mu}{\beta}$$

$$\Rightarrow \frac{\alpha}{\beta} = \frac{6+2\mu}{1+2\mu} \quad 3$$

From Eqn 1 & 2

$$\frac{10-6\mu}{4\mu-9} = \frac{6+2\mu}{1+2\mu} \quad 1+1$$

$$\Rightarrow 5\mu^2 - 2\mu - 16 = 0 \quad \cancel{2}\ 1$$

$$\Rightarrow \mu = 2 \quad \text{or} \quad \mu = -1.6 \quad 1$$

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Q:2 A continuous random variable X has the following probability density function

$$f(x) = \frac{1}{2\sigma} \exp\left(-\frac{|x|}{\sigma}\right), \quad -\infty < x < \infty, \quad \sigma > 0$$

Based on a random sample of size n, find an estimator of σ by the method of (i) moments and (ii) MLE.

Solution: Given $f(x) = \frac{1}{2\sigma} \exp\left(-\frac{|x|}{\sigma}\right)$

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{\infty} x \frac{1}{2\sigma} \exp\left(-\frac{|x|}{\sigma}\right) dx = 0 \quad \text{— 2M}$$

(∵ the integrand is an odd f^n)

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_{-\infty}^{\infty} x^2 \frac{1}{2\sigma} \exp\left(-\frac{|x|}{\sigma}\right) dx$$

$$= 2\int_{0}^{\infty} x^2 \frac{1}{2\sigma} \exp\left(-\frac{|x|}{\sigma}\right) dx \quad (\because \text{even integrand})$$

$$= \int_{0}^{\infty} x^2 \frac{1}{\sigma} \exp\left(-\frac{x}{\sigma}\right) dx$$

$$= E(Y^2) \quad \text{where } Y \sim Exp\left(\frac{1}{\sigma}\right)$$

3M

$$E(X^2) = 2\sigma^2$$

by method of moments substitute M_2 for $E(X^2)$ we get

$$M_2 = 2\hat{\sigma}^2 \Rightarrow \hat{\sigma} = \sqrt{\frac{M_2}{2}}$$

$$\boxed{\hat{\sigma} = \sqrt{\frac{M_2}{2}}} \quad \text{where } M_2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 \quad \text{2M}$$

Using MLE The likelihood f^n is given by

$$L(\sigma) = \prod_{i=1}^{n} \frac{1}{2\sigma} \exp\left(-\frac{|x_i|}{\sigma}\right) \quad 1M$$

Taking ln on both side we get

$$\ln L(\sigma) = \ln\left[\left(\frac{1}{2\sigma}\right)^n \exp\left(-\frac{\sum |x_i|}{\sigma}\right)\right]$$

$$\ln L(\sigma) = \ln\left(\frac{1}{2\sigma}\right)^n - \frac{\sum_{i=1}^{n}|x_i|}{\sigma}$$

$$= n \ln \frac{1}{2\sigma} - \frac{\sum_{i=1}^{n}|x_i|}{\sigma}$$

$$= -n \ln 2 - n \ln \sigma - \frac{\sum_{i=1}^{n}|x_i|}{\sigma} \quad 2M$$

$$\frac{d \ln L(\sigma)}{d\sigma} = \frac{-n}{\sigma} + \frac{\sum_{i=1}^{n}|x_i|}{\sigma^2}$$

$$\frac{d \ln L(\sigma)}{d\sigma} = 0 \Rightarrow \frac{-n}{\sigma} = -\frac{\sum_{i=1}^{n}|x_i|}{\sigma^2} \Rightarrow \sigma = \frac{1}{n}\sum_{i=1}^{n}|x_i|$$

$$\frac{d^2 \ln L(\sigma)}{d\sigma^2} < 0 \quad \text{at } \sigma = \frac{1}{n}\sum_{i=1}^{n}|x_i|$$

Therefore, $\boxed{\hat{\sigma} = \frac{1}{n}\sum_{i=1}^{n}|x_i|}$ 3M

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Part-B

Q.3 a)

x	2	3	4	5	6	7	8	9	10	11	12
$f(x)$	$1/36$	$2/36$	$3/36$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$$E(X) = \sum_{x=2}^{12} x f(x) = \frac{1}{36}\left[2+6+12+20+30+42+40+36+30+22+12\right]$$

$$= \frac{252}{36} = 7 \qquad (3m)$$

$$E(X^2) = \sum_{x=2}^{12} x^2 f(x) = \frac{1}{36}\left[4+18+48+100+180+294+320+324+300+242+144\right]$$

$$= 1974/36 = 54.833$$

$$\sigma^2 = Var(X) = E(X^2) - (E(X))^2 = 54.833 - 49 = 5.833 \Rightarrow \sigma = 2.415 \qquad (2m)$$

$n = 75$, $\bar{x} = 6.5$; for large sample, CLT ~~can~~ ~~be~~ used

$\Rightarrow \bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ so 98% CI for $\mu \Rightarrow \bar{X} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$

$\boxed{z_{\alpha/2} = 2.33}$ (1m)

$$\Rightarrow 6.5 \pm 2.33 * \frac{2.415}{8.66} = \left[6.5 \pm 0.6498\right] = \left[5.850, 7.15\right] \qquad (2m)$$

b) 95% CI for σ is $\left[\sqrt{\frac{30s^2}{\chi^2_{1-\alpha/2,30}}}, \sqrt{\frac{30s^2}{\chi^2_{\alpha/2,30}}}\right]$ (1m)

$n = 31$,

$$= \left[\sqrt{\frac{30s^2}{47}}, \sqrt{\frac{30s^2}{16.8}}\right] \qquad (1m)$$

given that $\sqrt{\frac{30s^2}{47}} = 15 \Rightarrow s^2 = \frac{225 * 47}{30} = 352.5 \Rightarrow s = 18.775$ (2m)

The right end pt. of CI for $\sigma = \sqrt{\frac{30 * 352.5}{16.8}} = \sqrt{629.464}$

$= 25.089$ (1m)

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Solution:

(a) X = number in millions of successful foreign investors in a year
$H_0 : \mu = 1$ Null hypothesis
$H_1 : \mu < 1$ Alternative Hypothesis — 3Marks

{"a year" is important for r.v.,
{$\bar{X}$, θ, p etc are wrong

(b) $\beta = P\left[H_0 \text{ is accepted} \middle| H_1 \text{is true}\right]$
$\beta = P\left[H_0 \text{ is accepted} \middle| \mu = 0.95, \sigma^2 = 0.01\right]$
$\beta = P\left[\bar{X} > 0.98\right]_{\mu=0.95,\sigma^2=0.01}$
Since X is normal with $\mu_X = 0.95$, $\sigma_X = 0.1$, $\bar{X}$ is normal with $\mu_{\bar{X}} = 0.95$, $\sigma_{\bar{X}} = \frac{0.1}{\sqrt{5}}$
$\beta = P\left[\bar{X} > 0.98\right] = P\left[Z \geq \frac{0.98-0.95}{\frac{0.1}{\sqrt{5}}}\right] = P\left[Z \geq 0.67\right] = 1 - F(0.67)$ — 1Mark

power $= 1 - \beta = F(0.67) = 0.7486$ — 1Mark

(c) $\bar{x} = \frac{\sum x}{n} = \frac{0.8+1.2+0.7+0.95+1.02}{5} = \frac{4.67}{5} = 0.934$
$\sum x^2 = 0.8^2 + 1.2^2 + 0.7^2 + 0.95^2 + 1.02^2 = 4.5129$
$s = \sqrt{\frac{n\sum x^2 - (\sum x)^2}{n(n-1)}} = \sqrt{\frac{5\times4.5129-(4.67)^2}{5\times4}} = 0.1944$ — 3Marks

"σ" is wrong.

$t_{obs} = \frac{\bar{x}-\mu}{\frac{s}{\sqrt{n}}} = \frac{0.934-1}{\frac{0.1944}{\sqrt{5}}} = -0.7592$ — 2Marks, "Z" is wrong

Alternative 1 Testing of Hypothesis.
$t_\alpha = t_{0.1} = 1.533$ at 4 degree of freedom — 1Mark, "Z_α" is wrong

Since $t_{obs} > -t_{0.1}$, null hypothesis can not be rejected i.e. finance ministry claim can not be rejected at 10% level of significance. — 2Marks

Alternative 2 Significance Testing.
P Value$=P[t_4 \leq -0.7592]$
Since $P[t_4 \leq -1.533] = 0.1$ and $P[t_4 \leq -0.741] = 0.25$, P value $\in [0.1, 0.25]$ therefore, P value $> \alpha$, null hypothesis can not be rejected i.e. finance ministry claim can not be rejected at 10% level of significance. — 3Marks

"Z_α" is wrong

Q.5

T_1 follows continous uniform distribution in $(6, 14)$, therefore, the c.d.f. is:

$$F(t) = \begin{cases} 0; & t < 6 \\ \frac{t-6}{8}, & 6 < t < 14 \\ 1, & t > 14 \end{cases} \qquad \text{———— [1]}$$

setting $u = F(t)$, we get $t = 6 + 8u$, $0 < u < 1$ ———— [1]

Table for T_1:

Random No.	953	139	512
T_1	13.62	7.11	10.10

———— [2]

T_2 has density function

$$f(t) = \begin{cases} \frac{1}{25}t, & 0 \le t \le 5 \\ \frac{2}{5} - \frac{1}{25}t, & 5 < t \le 10 \\ 0, & \text{otherwise} \end{cases}$$

$$F(t) = \int_{-\infty}^{t} f(s)\,ds$$

If $t < 0$, $F(t) = 0$

$0 \le t \le 5$, $F(t) = \int_0^t \frac{s}{25}\,ds = t^2/50$

$5 < t \le 10$, $F(t) = \int_0^5 \frac{s}{25}\,ds + \int_5^t \left(\frac{2}{5} - \frac{1}{25}s\right)ds$

$-t^2 \quad 2t - 1$

$$F(t) = \begin{cases} t^2/50, & 0 \le t \le 5 \\ -t^2/50 + \frac{2t}{5} - 1, & 5 < t \le 10 \\ 1, & t > 10 \end{cases}$$ —— [3]

Next, we find F^{-1}: $u = F(t)$

When $0 \le t \le 5$, $F(t) = t^2/50 \Rightarrow 0 \le u \le 1/2$

$u = t^2/50 \Rightarrow t = \sqrt{50u}, \quad 0 \le u \le 1/2$

$5 < t \le 10$, $F(t) = -t^2/50 + 2t/5 - 1 \Rightarrow \frac{1}{2} < u \le 1$

$u = -t^2/50 + 2t/5 - 1$

$t^2 - 20t + 50 + 50u = 0$

$$t = \frac{20 \pm \sqrt{400 - 200 - 200u}}{2} = 10 \pm 5\sqrt{2-2u}$$

$$t = F^{-1}(u) = \begin{cases} \sqrt{50u}, & 0 \le u \le 1/2 \\ 10 - 5\sqrt{2-2u}, & \frac{1}{2} < u \le 1 \end{cases}$$ —— [2]

Table for T_2:

Random No.	25	12	96
T_2	3.54	2.45	8.59

—— [2]

Average time $= \frac{1}{3}\left[(13.62 + 3.54) + (7.11 + 2.45) + (10.10 + 8.59)\right]$

$= 15.13$ —— [1]

Probability of meeting deadline $= \frac{1}{3}$. —— [1]