BITS Pilani
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Mathematics-I
MATH F111

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Maxima, minima and saddle points of $f(x, y)$

Critical points of f(x)

For a function of one variable, points, at which $f'(x) = 0$ or $f'(x)$ does not exist, are critical points

We use f'' to tell us if the point is a:

- max ($f'' < 0$)
- Min ($f'' > 0$)
- Or may be point of inflection ($f'' = 0$)

Critical points

- We can use the first order partial derivatives to find critical points on a surface;
- We can then use the 2nd order partial derivatives to classify the critical points
- critical points can be *maxima*, *minima*, or saddle points

Critical points

Definition:

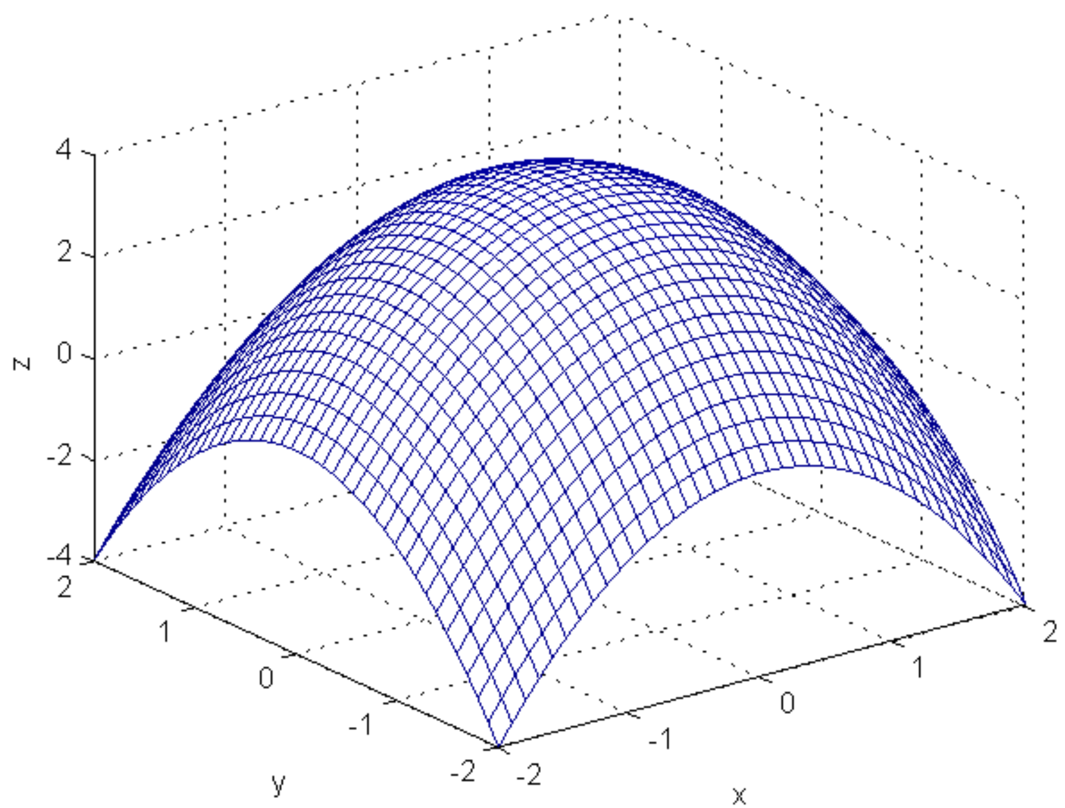
Let $f(x, y)$ be a function defined in a region R and (x_0, y_0) is an interior point of R. Then (x_0, y_0) is called critical point of f if both $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ are zero or one or both of $f_x(x_0, y_0)$ & $f_y(x_0, y_0)$ do not exist

Definitions :

Let f (x, y) be a function of two variables defined on a region R containing the point (x_0, y_0) then

1. $f(x_0, y_0)$ is a LOCAL MAXIMUM value of f if , $f(x_0, y_0) \geq f(x, y)$ for all domain points (x, y) in an open disk centered at (x_0, y_0) , point (x_0, y_0) is then called point of maxima.

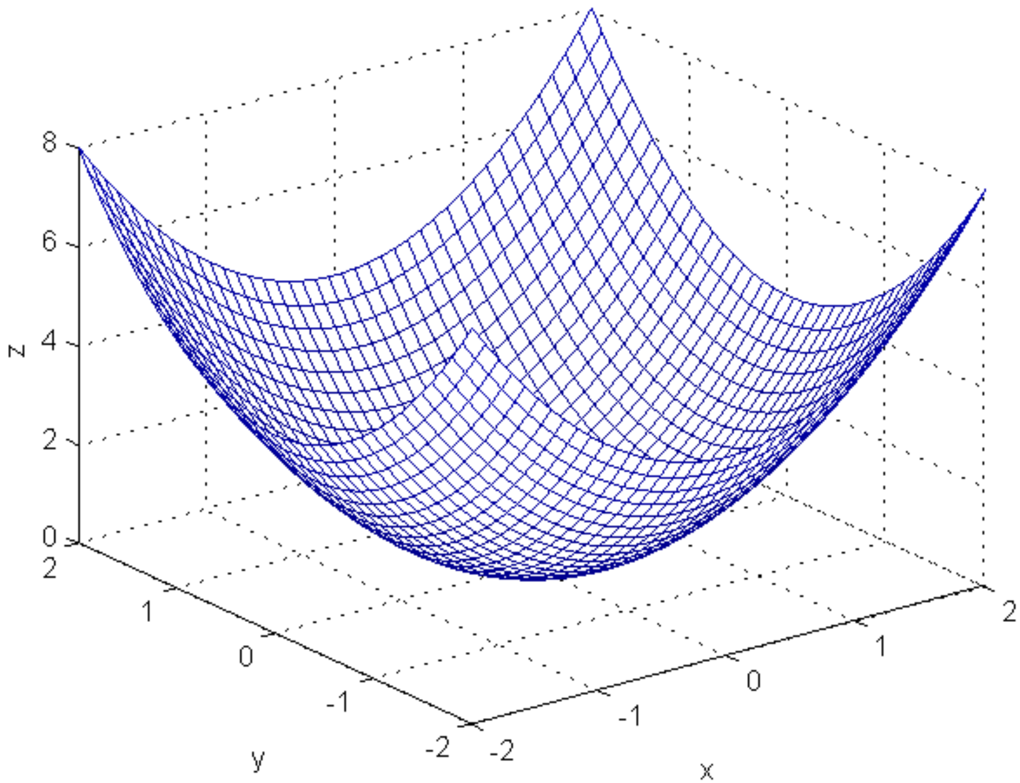
Typical maximum



Plot of $z = 4 - x^2 - y^2$ showing a maximum at (0,0)

2. $f(x_0, y_0)$ is a LOCAL MINIMUM value of f if $f(x_0, y_0) \leq f(x, y)$ for all domain points (x, y) in an open disc centered at (x_0, y_0), the point (x_0, y_0) is then called point of minima.

Typical minimum



Plot of $z = x^2 + y^2$ showing a minimum at (0,0)

3. If $f(x_0, y_0) \geq f(x, y)$ for ALL points (x, y) in the domain of f, then f has an ABSOLUTE MAXIMUM at (x_0, y_0)

4. If $f(x_0, y_0) \leq f(x, y)$ for ALL points (x, y) in the domain of f, then f has an ABSOLUTE MINIMUM at (x_0, y_0)

Saddle point:

Let (x_0, y_0) be a critical point of a differentiable function $f(x, y)$.

If in EVERY open disc centered at (x_0, y_0), there are domain points (x, y) where

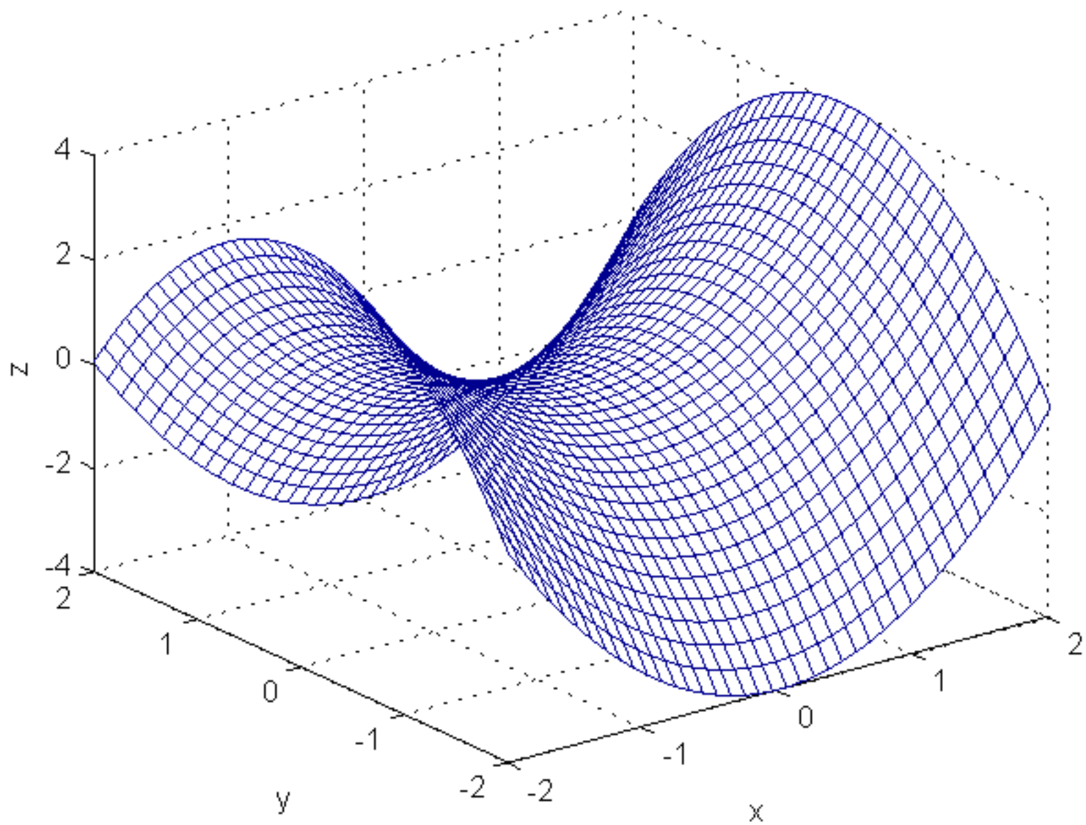
$$f(x, y) < f(x_0, y_0)$$

and domain points (x, y) where

$$f(x, y) > f(x_0, y_0),$$

then the point $(x_0, y_0, f(x_0, y_0))$ on the surface $z = f(x, y)$ is called saddle point of the surface

Typical saddle point



Plot of $z = x^2 - y^2$ showing a saddle point at (0,0)

Theorem (First derivative test):

If $f(x, y)$ has a local maximum or minimum at an *interior point* (x_0, y_0) of its domain and the first order partial derivatives of f exist there, then

$$f_x(x_0, y_0) = 0 \text{ and } f_y(x_0, y_0) = 0.$$

Proof:

Let $g(x) = f(x, y_0)$

If f has a local maximum (or minimum) at (x_0, y_0),

then g has a local maximum (or minimum) at $x = x_0$.

$\Rightarrow g'(x_0) = 0$

$\Rightarrow f_x(x_0, y_0) = 0.$

Similarly, using $h(y) = f(x_0, y)$,

We obtain $f_y(x_0, y_0) = 0.$

Second Derivative Test for Local Extreme Values:

Let $f(x, y)$ and its first & second partial derivatives are continuous throughout a disc centered at (x_0, y_0) and $f_y(x_0, y_0) = 0 = f_x(x_0, y_0)$. Then

(i) f has a local maximum at (x_0, y_0) if

$$f_{xx} < 0 \, and \, f_{xx} f_{yy} - f_{xy}^2 > 0 \text{ at } (x_0, y_0).$$

(ii) f has a local minimum at (x_0, y_0) if

$$f_{xx} > 0 \; \& \; f_{xx} f_{yy} - f_{xy}^2 > 0 \text{ at } (x_0, y_0).$$

Second Derivative Test for Local Extreme Values:

(iii) f has a saddle point at (x_0, y_0) if

$$f_{xx}f_{yy} - f_{xy}^2 < 0 \text{ at } (x_0, y_0).$$

(iv) *The* test is INCONCLUSIVE

at (x_0, y_0) if

$$f_{xx}f_{yy} - f_{xy}^2 = 0 \text{ at} (x_0, y_0)$$

Classifying critical points

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$D = f_{xx}f_{yy} - f_{xy}^2$ (Hessian or discriminant)	f_{xx} OR f_{yy}	Classification
>0	>0	Local Minimum
	<0	Local Maximum
<0		Saddle point
$=0$		Inconclusive

Example – Classifying stationary points

Find and classify all the stationary points for the function:

$$f(x,y) = x^3y - y^2 - 3x^2y$$

Find the first order partial derivatives :

$$\frac{\partial f}{\partial x} = 3x^2y - 6xy$$

$$\frac{\partial f}{\partial y} = x^3 - 2y - 3x^2$$

Example – Classifying stationary points

Set these partial derivatives equal to 0 :

$$\frac{\partial f}{\partial x} = 3x^2y - 6xy = 0 \;\Rightarrow\; 3xy(x-2) = 0$$

$$\frac{\partial f}{\partial y} = x^3 - 2y - 3x^2 = 0$$

The first equation will equal 0 if :

$$x = 0,\; x = 2 \quad \text{or} \quad y = 0$$

Example – Classifying stationary points

Put each value into the second equation and solve:

$$(i) \quad x = 0, x^3 - 2y - 3x^2 = 0$$

$$\Rightarrow -2y = 0$$

$$\Rightarrow y = 0$$

$$(ii) \quad x = 2, x^3 - 2y - 3x^2 = 0$$

$$\Rightarrow 8 - 2y - 12 = 0$$

$$\Rightarrow -2y = 4$$

$$\Rightarrow y = -2$$

Example – Classifying stationary points

Put each value into the second equation and solve:

$$(iii) \quad y = 0, x^3 - 2y - 3x^2 = 0$$

$$\Rightarrow x^3 - 3x^2 = 0$$

$$\Rightarrow x^2(x-3) = 0$$

$$\Rightarrow x = 0, 3$$

So we find 3 stationary points :

$$(x, y) = (0, 0), (2, -2), (3, 0)$$

Now we can classify them using the 2nd order partial derivatives.

Example – Classifying stationary points

Find the second order partial derivatives :

$$\frac{\partial f}{\partial x} = 3x^2 y - 6xy \quad \Rightarrow \frac{\partial^2 f}{\partial x^2} = 6xy - 6y$$

$$\frac{\partial f}{\partial y} = x^3 - 2y - 3x^2 \Rightarrow \frac{\partial^2 f}{\partial y^2} = -2$$

$$\Rightarrow \frac{\partial^2 f}{\partial x \partial y} = 3x^2 - 6x$$

Example – Classifying stationary points

- Now we can set up the table below:

Point	$\frac{\partial^2 f}{\partial x^2} = 6xy - 6y$	$\frac{\partial^2 f}{\partial y^2} = -2$	$\frac{\partial^2 f}{\partial x \partial y} = 3x^2 - 6x$	D	Type
(0,0)	0	-2	0	0	???
(3,0)	0	-2	9	-81	Saddle
(2,-2)	-12	-2	0	24	Max

Example – Classifying stationary points

$$Q19.\ f(x,y) = 4xy - x^4 - y^4$$

(i) *Critical* points:

$$f_x = 4y - 4x^3 = 0$$

$$f_y = 4x - 4y^3 = 0$$

$\Rightarrow$ critical points are

$$P_0(0,0),\ P_1(1,1) \text{ and } P_2(-1,-1)$$

Example – Classifying stationary points

$$(ii)\ f_{xx} = -12x^2$$

$$D = f_{xx} f_{yy} - f_{xy}^2$$

$$= 144x^2y^2 - 16$$

Example – Classifying stationary points

(iii) At the point $P_0(0,0)$, we have

$$f_{xx} = 0$$

$$D = -16 < 0$$

Conclusion:

$(0,0)$ is a saddle point of $f(x, y)$.

Example – Classifying stationary points

At the point $P_1(1,1)$:

$$f_{xx} = -12 < 0$$

$$D = 144 - 16 > 0$$

Conclusion: $f(x, y)$ has a local max at $(1,1)$ and $f(1,1) = 2$

Example – Classifying stationary points

At the point $P_2(-1,-1)$:

$$f_{xx} = -12 < 0$$

$$D = 128 > 0$$

$$\Rightarrow f(x,y) \text{ has local max at } (\text{-}1,\text{-}1)$$

$$\text{and } f(\text{-}1,\text{-}1) = 2$$

Absolute Maxima & Minima on a Closed Bounded Region R

Theorem : A continuous function on a *closed and bounded* region always has a max and min.

Remark:

The extreme values of f(x, y) can occur only at

1. Boundary points of the domain of f,
2. Critical points (interior points where $f_x=f_y=0$ or points where f_x or f_y fail to exist)

Procedure :

I. Sketch the region R

II. List the interior points of the region R

where f may have local max & min,

and evaluate f at these points.

These are the points where $f_x = f_y = 0$

OR

where one or both f_x *and* f_y fail

to exist-critical points of f.

III. List the boundary points of R where f has local max & min, and evaluate f at these points.

IV. Since the absolute max & min are also local max & min, hence absolute max & min values of f already appear somwhere in the list made in step II & III.

Thus, look through the list made in step II & III for max & min values of f. They will be absolute maximum & minimum values of f.

$Q31$. Find the absolute maxima & minima of the function

$$f(x,y) = 2x^2 - 4x + y^2 - 4y + 1$$

on a closed triangular plate bounded by the lines $x = 0, \quad y = 2, \quad y = 2x$ in the first quadrant

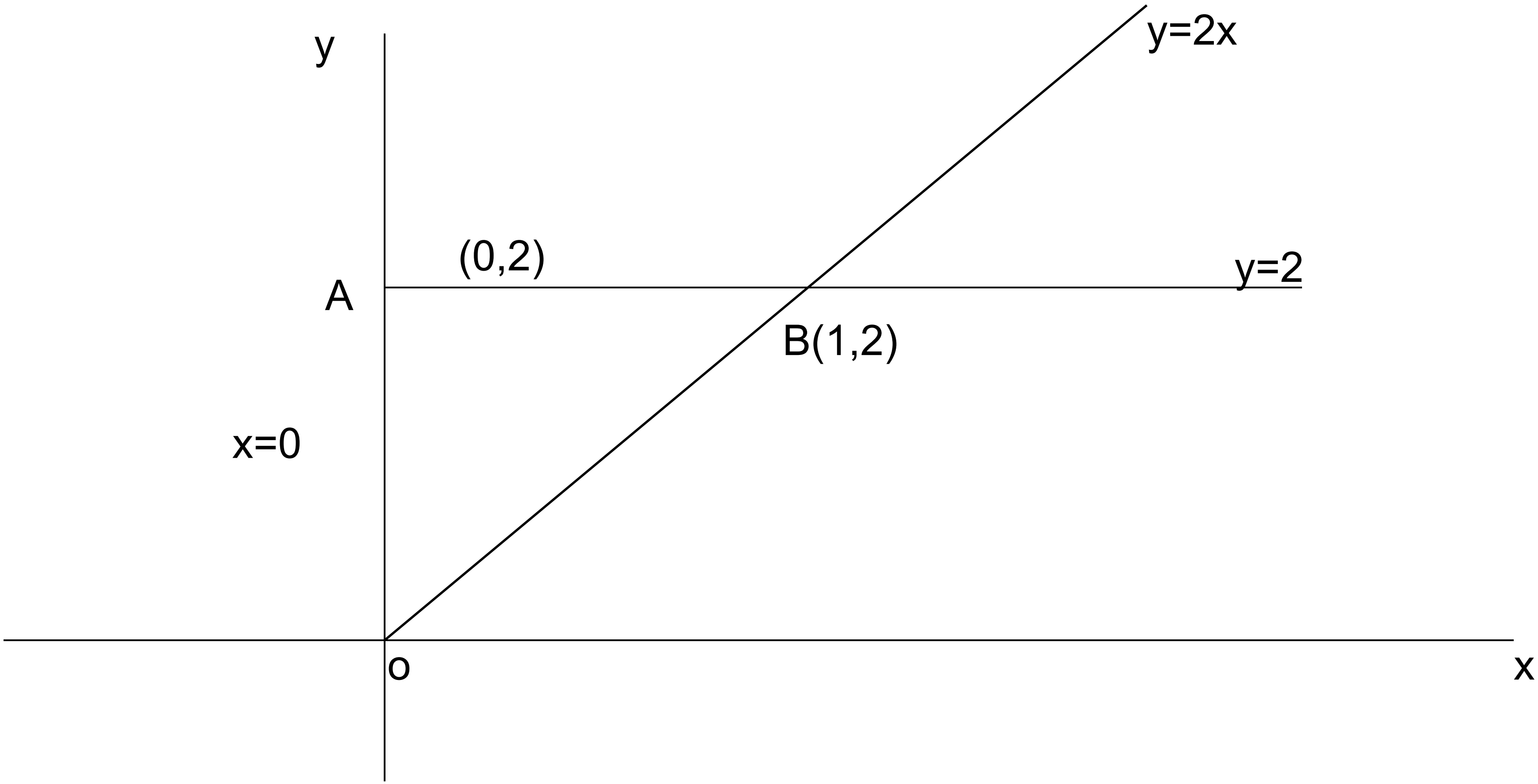
Solution: $\because f$ is differentiable

$\Rightarrow$ The max/min of f will occur only at the interior points of the domain where

$$f_x = f_y = 0,$$

or

on the boundary points of the domain


y
y=2x
(0,2)
y=2
A
B(1,2)
x=0
o
x

For the interior points, we have

$$f_x = 4x - 4 = 0$$

$$f_y = 2y - 4 = 0$$

$$\Rightarrow x = 1, y = 2 \quad \text{NOT an interior point.}$$

For the boundary points:

I. Along the line OA, $x = 0,\ 0 \le y \le 2$

$$\therefore \text{ To maximize } \phi(y) = f(0, y)$$

$$= y^2 - 4y + 1, \quad 0 \le y \le 2$$

At the end points of OA,

$$f(0,0) = 1, \quad f(0,2) = -3.$$

For the interior points of OA,

$$\phi'(y) = 2y - 4 = 0$$

$$\Rightarrow y = 2$$

But this does not lie in interval (0,2).

Hence point (0,2) not included in the list.

II. Along the line AB,

$$y = 2,\ 0 \le x \le 1.$$

$$\therefore f(x,2) = 2x^2 - 4x - 3$$

At the end points of AB,

$$f(0,2) = -3, f(1,2) = -5$$

For the interior points of AB,

$$f'(x,2) = 4x - 4 = 0$$

$$\Rightarrow x = 1$$

But (1,2) is not an int. point of AB.

III. Along the line OB,

$y = 2x,\ 0 \le x \le 1.$

$$\therefore \theta(x) = f(x, 2x)$$

$$= 6x^2 - 12x + 1,\ \ 0 \le x \le 1.$$

At the end points of OB,

$f(0,0) = 1, f(1,2) = -5$

For the interior points of OB

$$\theta'(x) = 12x - 12 = 0$$

$$\Rightarrow x = 1, \text{ not in the interval } (0,1).$$

Final Step:

List of all the values are

$$-5,\ 1,\ -3$$

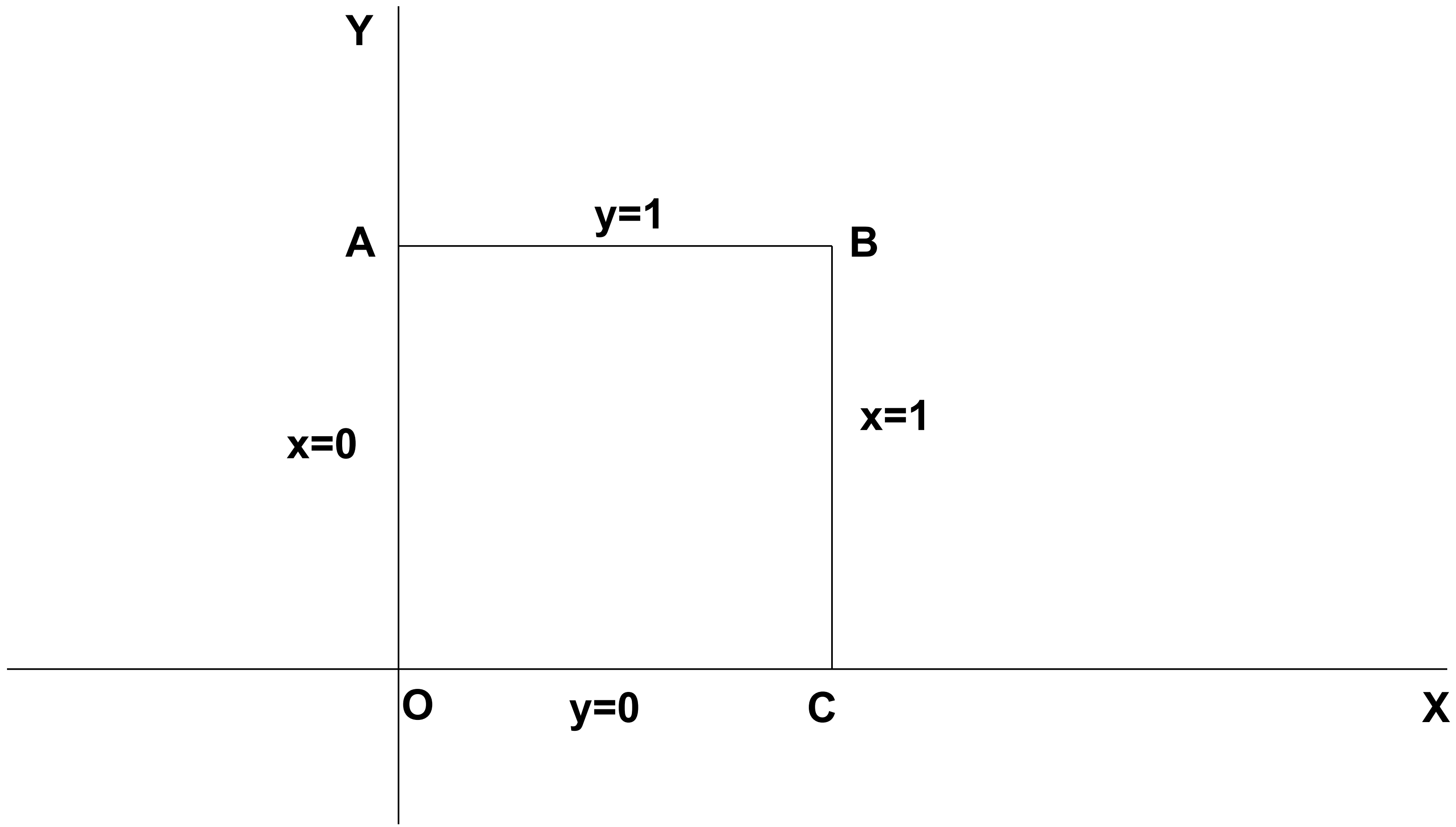
$$\Rightarrow \text{ Abs. max. of } f = 1 \text{ at } (0,0)$$

$$\text{Abs. min. of } f = -5 \text{ at } (1,2).$$

Q 36 Find the absolute maxima & minima of

$$f(x,y) = 48xy - 32x^3 - 24y^2$$

on the ractangular plate $0 \le x \le 1$, $0 \le y \le 1$.


Y
A
y=1
B
x=0
x=1
O
y=0
C
X

Critical points interior to domain of f :

$$f_x = 48y - 96x^2 = 0,$$

$$f_y = 48x - 48y = 0$$

$$\Rightarrow (x, y) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

(1) Along the line OA, $x = 0$.

$$\therefore \quad f(x, y) = f(0, y) = -24y^2$$

At the end points of OA,

$$f(0,0) = 0, \qquad f(0,1) = -24$$

For the interior points of OA,

$$f'(0,y) = -48y = 0$$

$$\Rightarrow y = 0$$

But (0,0) is not an interior point of OA.

(II)

Along the line AB, $y = 1$

$$\therefore \quad f(x,\ 1) = 48x - 32x^3 - 24$$

At the end points of AB,

$$f(0,1) = -24, \quad f(1,1) = -8$$

For the interior points of AB,

$$f'(x,1)=0 \Rightarrow x=\frac{1}{\sqrt{2}}, \quad y=1$$

$$or \quad x=-\frac{1}{\sqrt{2}}, \quad y=1$$

$$\therefore \; f\left(\frac{1}{\sqrt{2}},1\right)=16\sqrt{2}-24$$

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(III) Along the line BC , $x = 1$.

$$\therefore \quad f(1, y) = 48y - 32 - 24y^2$$

At the end points of BC,

$$f(1,1) = -8, \quad f(1,0) = -32$$

For the interior points of BC,

$$f'(1,y) = 48 - 48y = 0 \Rightarrow y = 1$$

$\therefore x = 1, y = 1 \rightarrow$ not an interior point

(IV) Along the line OC, $y = 0$

$$\therefore \quad f(x,0) = -32x^3$$

At the end points of OC,

$$f(0,0) = 0, \; f(1,0) = -32.$$

For the interior points of OC,

$$f'(x,0) = -96x^2 = 0 \Rightarrow x = 0 = y$$

NOT an interior point.

Final Step:

List of all the Candidates:

$$2, 0, \ -24, \ -8, 16\sqrt{2}-24, -32$$

$$\text{Abs. max of } f = 2 \text{ at } \left(\frac{1}{2}, \frac{1}{2}\right)$$

$$\text{Abs. min of } f = -32 \text{ at } (1,0)$$

The Method of Lagrange Multipliers:

Theorem: Suppose that $f(x, y, z)$ is differentiable in a region whose interior contains a smooth curve

C: $\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}.$

Let P_o is any point on C where f has a local max or minima relative to its value on C.

Then ∇f is orthogoral to the velocity vector $(d\vec{r}\,/\,dt)$ at P_o

Proof: By chain rule , we have

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$= \left(\left(\frac{\partial f}{\partial x}\right)\hat{i} + \left(\frac{\partial f}{\partial y}\right)\hat{j} + \left(\frac{\partial f}{\partial z}\right)\hat{k} \right) . \left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} \right)$$

$$\Rightarrow \frac{df}{dt} = \nabla f . \vec{v}$$

But f has a local max or min at P_0, hence

$$\frac{df}{dt} = 0 \quad at \;\; P_0$$

$$\therefore \quad \nabla f \,.\, \vec{v} = 0 \quad at \;\; P_0$$

The method of Lagrange multipliers:

Let $f(x,y,z)$ and $g(x,y,z)$ are two differentiable functions and P_0 is a point on the surface $g(x,y,z) = 0$ where f has a local max or min relative to its other values on the surface.

$\Rightarrow f$ has a local max or min relative to its values on every differentiable curve through P_0 on the surface $g(x, y, z) = 0$.

$$\Rightarrow \nabla f \text{ is orthogonal to } \frac{d\vec{r}}{dt} \text{ of}$$

every such differentiable curve through P_0

But ∇g is also orthogonal to the tangent vector at P_0.

Hence $\nabla f(P_0)$ and $\nabla g(P_0)$ must be parallel.

$\therefore$ If $\nabla g(P_0) \neq 0$, there is a number λ such that

$$\nabla f = \lambda \nabla g \quad \text{at } P_0$$

λ : a Lagrange multiplier

Procedure:

Let $f(x,y,z)$ and $g(x,y,z)$ are differentiable functions. Then to find the max. and min. values of f subject to the constraint

$$g(x,y,z) = 0$$

Step I: Find all values of x, y, z and λ such that

$$\nabla f = \lambda \nabla g \text{ and } g(x, y, z) = 0.$$

Step II: Evaluate f at all the points (x, y, z) that result from step I.

The largest of these values is the max of f
and the smallest is the min of f.

Remark:

The method says, if max or min exists then it satisfies these equations.

The condition $\vec{\nabla} f = \lambda \vec{\nabla} g$ with $g(x,\ y) = 0,\ \vec{\nabla} g(x, y) \neq 0,$ is not sufficient for existence of absolute max/min.

e.g. $f(x, y) = x + y$ with constraint $g(x, y) = xy = 16$ has neither max nor min but has two stationary points (4, 4) and (-4,-4).

$Q2$. Find the extreme values of

$$f(x,y) = xy$$

subject to the constraint

$$g(x,y) = x^2 + y^2 - 10 = 0.$$

Solution:

StepI: $\nabla f = y\hat{i} + x\hat{j}, \ \nabla g = 2x\hat{i} + 2y\hat{j}$

If λ is a Lagrange Multiplier, then

$$\nabla f = \lambda \nabla g$$

$$\Rightarrow y = 2\lambda x \quad and \quad x = 2\lambda y$$

Step II: solve

$$y = 2\lambda x, \qquad \text{(i)}$$

$$x = 2\lambda y, \qquad \text{(ii)}$$

$$x^2 + y^2 = 10 \qquad \text{(iii)}$$

$$(i)\,\&\,(ii) \Rightarrow x = 4\lambda^2 x$$

$$\Rightarrow x = 0 \; or \; \lambda = \pm\frac{1}{2}$$

$$\text{Case I: If } x = 0, \text{ then } y = 0$$

$$\text{But } (0,0) \text{ does not satisfy}$$

$$x^2 + y^2 = 10.$$

$$\therefore x \neq 0 \Rightarrow \lambda = \pm\frac{1}{2}$$

Case II: When $\lambda = \dfrac{1}{2}$, then

(i) gives $y = x$, (iii) $\Rightarrow 2x^2 = 10$

$\Rightarrow x = \pm\sqrt{5} = y.$

Thus, f assumes its extreme values

at $\left(\sqrt{5},\sqrt{5}\right)$ and $\left(-\sqrt{5},-\sqrt{5}\right)$.

Case III: When $\lambda = -\frac{1}{2}$, then

Eq (i) $y = 2\lambda x$ gives $y = -x$

Now $x^2 + y^2 = 10$ gives

$$x = \pm\sqrt{5}$$

f assumes its extreme values at

$$\left(\sqrt{5},-\sqrt{5}\right) \text{ and } \left(-\sqrt{5},\sqrt{5}\right).$$

FINAL STEP:

Points for max/min are

$$\left(\sqrt{5},\sqrt{5}\right), \quad \left(-\sqrt{5},-\sqrt{5}\right),$$

$$\left(\sqrt{5},-\sqrt{5}\right), \quad \left(-\sqrt{5},\sqrt{5}\right).$$

Now, evaluate f at all these points:

$$f\left(\sqrt{5},\sqrt{5}\right)=5=f\left(-\sqrt{5},-\sqrt{5}\right)$$

$$f\left(\sqrt{5},-\sqrt{5}\right)=-5=f\left(-\sqrt{5},\sqrt{5}\right).$$

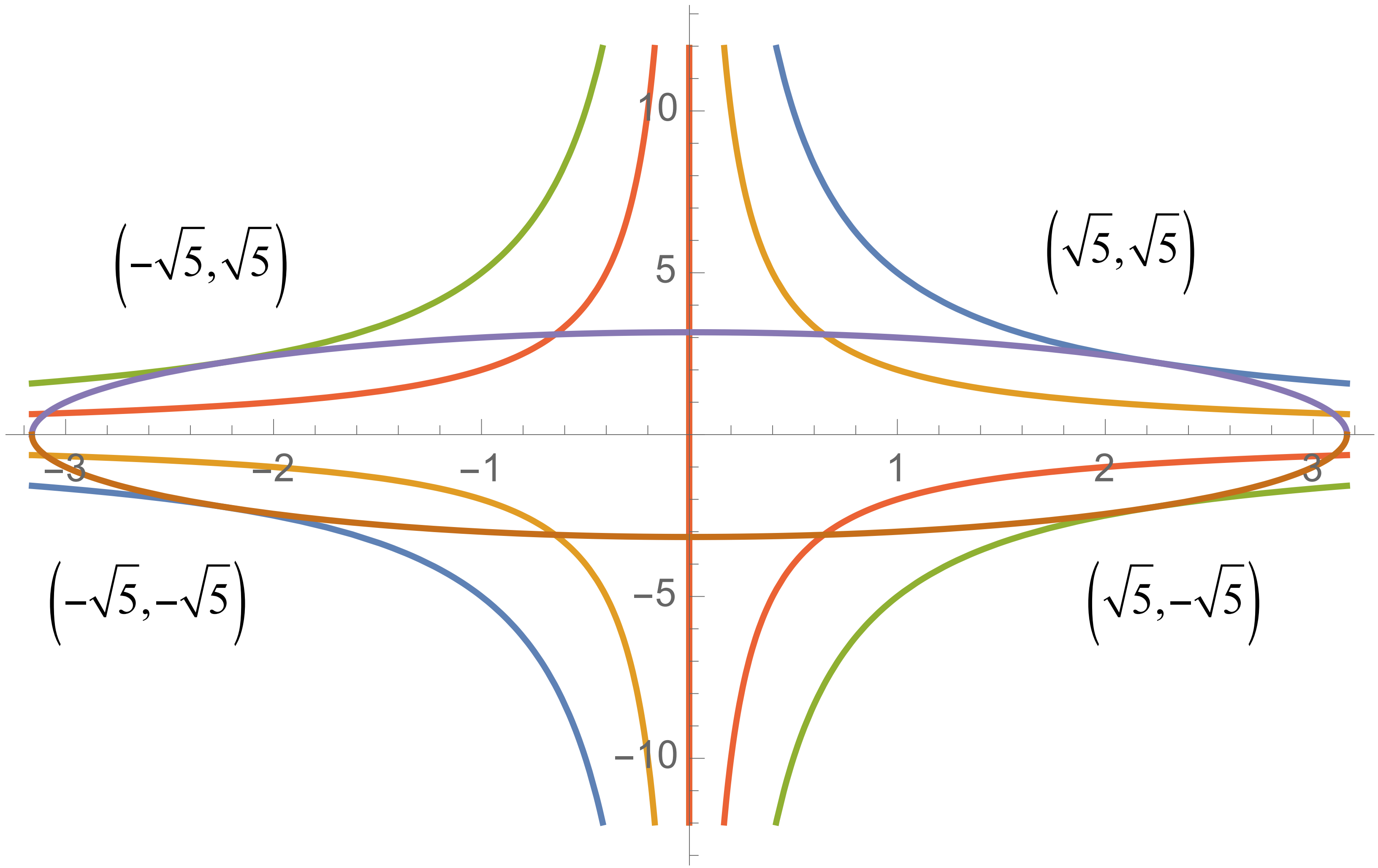
Conclusion:

$$\Rightarrow \max f = 5$$

$$\text{at } \left(\sqrt{5}, \sqrt{5}\right) \& \left(-\sqrt{5}, -\sqrt{5}\right).$$

$$\min f = -5$$

$$\text{at } \left(\sqrt{5}, -\sqrt{5}\right) \& \left(-\sqrt{5}, \sqrt{5}\right).$$



10
5
−5
−10
−3
−2
−1
1
2
3
$\left(-\sqrt{5},\sqrt{5}\right)$
$\left(\sqrt{5},\sqrt{5}\right)$
$\left(-\sqrt{5},-\sqrt{5}\right)$
$\left(\sqrt{5},-\sqrt{5}\right)$

Q. Find the points on curve

$$x^2 + xy + y^2 = 1$$

in the xy-plane that are nearest to and farthest from the origin

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Let $P(x, y)$ be any point in the given curve. Then its distance from the origin is

$$d = \sqrt{x^2 + y^2}.$$

d will be max/min whenever

$f(x,y)=x^2+y^2$ will be max/min

subject to

$$g(x,y)=x^2+xy+y^2-1=0$$

$$\nabla f = \lambda \nabla g \text{ yields}$$

$$\lambda = \frac{2x}{2x+y} = \frac{2y}{x+2y}$$

(WHY ??)

(A point satisfying all these conditions

and $2x + y = 0$,

$2x = \lambda(2x + y)$ and $g(x, y) = 0$

does not exist, similarly for $x + 2y$).

$\Rightarrow x = \pm y$

Case I: When $x = y$,

$$g(x, y) \equiv x^2 + xy + y^2 - 1 = 0$$

yields

$$3y^2 = 1 \Rightarrow y = \pm\frac{1}{\sqrt{3}}$$

$$\therefore \text{Points are}$$

$$\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right), \quad \left(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right).$$

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Case II: When $x = -y$,

$g(x, y) = 0$ gives

$$y = \pm 1.$$

$\therefore$ points are

(1,-1) and (-1,1).

Now

$$f\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)=\frac{2}{3},$$

$$f\left(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)=\frac{2}{3},$$

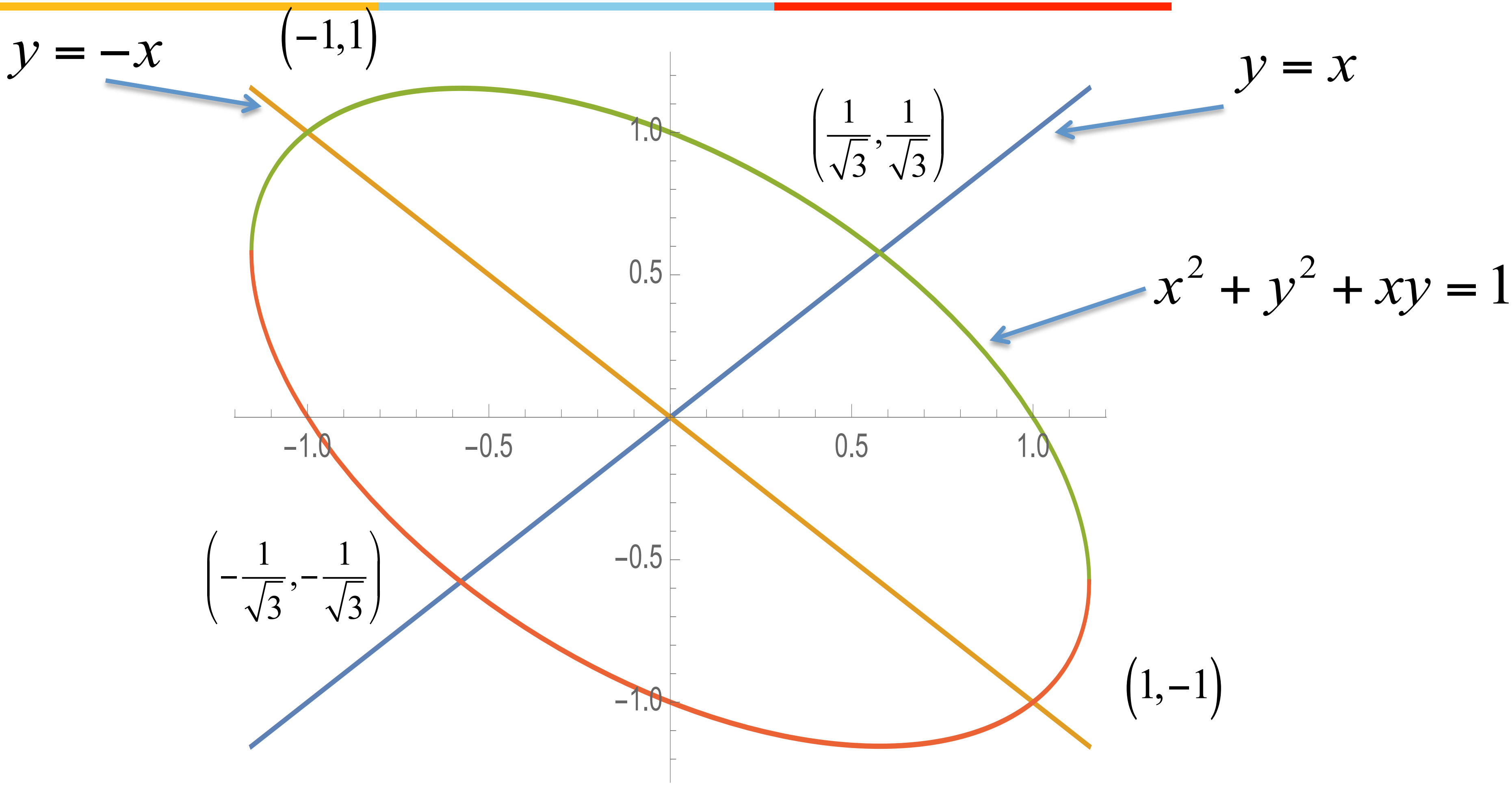
$$f(1,-1)=2,\quad f(-1,1)=2.$$

$$\Rightarrow (1,-1) \,\&\, (-1,1):$$

farthest from the origin

$$\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right) \& \left(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right):$$

nearest from the origin


$y=-x$
$(-1,1)$
$y=x$
$\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$
$x^2+y^2+xy=1$
$\left(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)$
$(1,-1)$
1.0
0.5
−0.5
−1.0
−1.0
−0.5
0.5
1.0

Q. A space probe in the shape of the ellipsoid

$$4x^2 + y^2 + 4z^2 = 16$$

enters the earth atmosphere and its surface begins to hot. After one hour, the temperature at the point (x, y, z) on the probe's surface is

$$T(x,y,z) = 8x^2 + 4yz - 16z + 600.$$

Find the hottest point on the probe's surface.

Soln: We wish to find

$\max\,(T(x,y,z))$ subject to the

constraint

$$g(x,y,z) \equiv 4x^2 + y^2 + 4z^2 - 16 = 0.$$

We have

$$\nabla T = 16x\hat{i} + 4z\hat{j} + (4y - 16)\hat{k},$$

$$\nabla g = 8x\hat{i} + 2y\hat{j} + 8z\hat{k}.$$

If λ is the Lagrange multiplier, then

$$\nabla T = \lambda \nabla g.$$

We wish to solve:

$$16x = 8\lambda x, \qquad (1)$$

$$4z = 2\lambda y, \qquad (2)$$

$$4y - 16 = 8\lambda z, \qquad (3)$$

$$4x^2 + y^2 + 4z^2 - 16 = 0. \qquad (4)$$

$$\text{Eq.}(1) \Rightarrow x = 0 \quad or \quad \lambda = 2.$$

$\lambda = 2$ gives

$$x = \pm\frac{4}{3}, \quad y = -\frac{4}{3}, \quad z = -\frac{4}{3}.$$

$x = 0$ gives $y \neq 0$, thus $\lambda = 2z / y$.

$\therefore 4y - 16 = 16z^2 / y$ or $4z^2 = y^2 - 4y$.

Substituting in $g(x, y, z) = 0$,

$y = -2, \quad 4.$

$y = -2$ gives $z = \pm\sqrt{3}$.

$y = 4$ gives $z = 0$.

Points of max/min are among :

$$\left(\pm\frac{4}{3},\ -\frac{4}{3},\ -\frac{4}{3}\right),$$

$$\left(0,\quad -2,\quad \pm\sqrt{3}\right),$$

$$\left(0,\quad 4,\quad 0\right).$$

$$\because T(x,y,z) = 8x^2 + 4yz - 16z + 600$$

$$\Rightarrow T(\pm 4/3, -4/3, -4/3) = 642.667 \; units$$

$$T(0, -2, \sqrt{3}) = 558.431 \quad units$$

$$T(0, -2, -\sqrt{3}) = 641.569 \quad units$$

$$T(0, 4, 0) = 600 \quad units$$

$\Rightarrow (\pm 4/3, -4/3, -4/3)$ are the hottest points on the space probe.

Method of Lagrange Multipliers with two constraint

Lagrange multipliers with two constraints:

How to find extreme values of $f(x,y,z)$ subject to

$$g_1(x,y,z)=0 \;\;\&\;\; g_2(x,y,z)=0.$$

Method of Lagrange Multipliers with two constraint

Let C: be a curve of intersection of

$g_1 = 0$ & $g_2 = 0$, P_0 : be a point on C

(which lies on both the surfaces $g_1 = 0$

& $g_2 = 0$) where f has local max./min.

$\Rightarrow \nabla f \perp$ C at P_0 & $\nabla g_1, \nabla g_2 \perp$ C at P_0

∇f lies on the plane containing ∇g_1 and ∇g_2

Method of Lagrange Multipliers with two constraint

innovate achieve lead

Method: Find x, y, z, λ and μ that satisfy

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2,$$

$$g_1(x, y, z) = 0,$$

$$g_2(x, y, z) = 0.$$

Method of Lagrange Multipliers with two constraint

Q 41. Find the extreme values of

$$f(x,y,z)=x^2yz+1$$

on the intersection of the plane $z = 1$ with the sphere $x^2+y^2+z^2=10.$

Method of Lagrange Multipliers with two constraint

innovate achieve lead

Solution : We wish to find extrema $f(x, y, z)$ sublect to

$$\left.\begin{aligned} g_1(x,y,z) &\equiv z-1=0 \\ g_2(x,y,z) &\equiv x^2+y^2+z^2-10=0 \end{aligned}\right\}$$

Method of Lagrange Multipliers with two constraint

$Let\ \ \lambda \,\&\, \mu$ be Lagrange Multipliers, then

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$$

$$\Rightarrow 2xyz\,\hat{i} + x^2 z\,\hat{j} + x^2 y\,\hat{k}$$

$$= \lambda(0.\hat{i} + 0.\hat{j} + \hat{k}) + \mu(2x\,\hat{i} + 2y\,\hat{j} + 2z\,\hat{k})$$

Method of Lagrange Multipliers with two constraint

innovate achieve lead

$$xyz = \mu x \qquad (1)$$

$$x^2 z = 2\mu y \qquad (2)$$

$$x^2 y = \lambda + 2\mu z \qquad (3)$$

$$\mathrm{z} = 1 \qquad (4)$$

$$x^2 + y^2 + z^2 = 10 \qquad (5)$$

Method of Lagrange Multipliers with two constraint

Eq. (1) gives $\quad x = 0 \quad or \;\; \mu = yz.$

When $\quad x = 0,$

then eq. (5) yields $\; y = \pm 3$ as $z = 1.$

Thus the points are

$(0,3,1) \quad \text{and} \quad (0,-3,1).$

Method of Lagrange Multipliers with two constraint

When $\mu = yz,$ then eq.(2) gives

$$z(x^2 - 2y^2) = 0$$

$$\Rightarrow z = 0 \quad \text{or} \quad x^2 - 2y^2 = 0.$$

$z = 0$ NOT POSSIBLE (??)

Hence $x = \pm\sqrt{2}\, y$

Method of Lagrange Multipliers with two constraint

When $x = +\sqrt{2}\, y,$ then

eq. (4) & eq. (5) give

$$y = \pm\sqrt{3}\,.$$

$$\therefore x = \pm\sqrt{6}$$

Method of Lagrange Multipliers with two constraint

innovate achieve lead

Thus, points for max/min are

$$(\sqrt{6}, \sqrt{3}, 1), \quad (-\sqrt{6}, -\sqrt{3}, 1),$$

Method of Lagrange Multipliers with two constraint

When $x = -\sqrt{2}\, y,$ then

eq. (4) &eq. (5) give

$$y = \pm\sqrt{3}\,.$$

$$\therefore x = \pm\sqrt{6}$$

Method of Lagrange Multipliers with two constraint

innovate achieve lead

Thus, points for max/min are

$$(-\sqrt{6}, \sqrt{3}, 1), \quad (\sqrt{6}, -\sqrt{3}, 1).$$

Method of Lagrange Multipliers with two constraint

Since $f(x,y,z) = x^2yz + 1$, we have

$$f(0,3,1) = 1,\ f(0,\text{-}3,1) = 1,$$

$$f(\sqrt{6},\sqrt{3},1) = 6\sqrt{3}+1,$$

$$f(\sqrt{6},-\sqrt{3},1) = -6\sqrt{3}+1,$$

$$f(-\sqrt{6},\sqrt{3},1) = 6\sqrt{3}+1,$$

$$f(-\sqrt{6},-\sqrt{3},1) = -6\sqrt{3}+1,$$

$$Q\,17.\ \text{Find the point on the plane}$$

$$x + 2y + 3z = 13$$

$$\text{closest to the point } (1,1,1)$$

$$\text{Solution: we have}$$

$$\min f(x, y, z) = (x-1)^2 + (y-1)^2 + (z-1)^2$$

$$\text{subject to}$$

$$g(x, y) = x + 2y + 3z - 13 = 0$$

$\nabla f = \lambda \nabla g$ gives

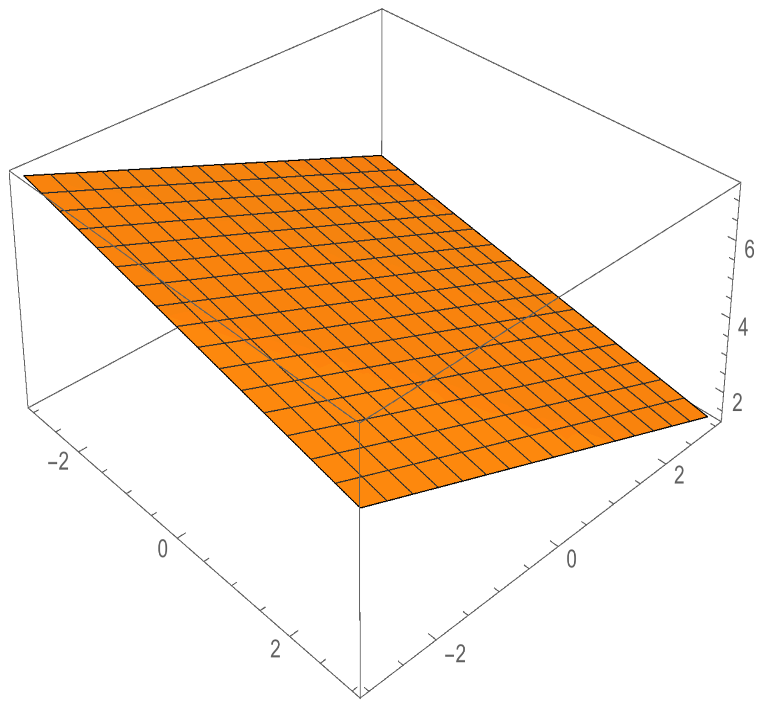
$\lambda = 2(x-1), \;\; \lambda = y-1,$

$\lambda = \dfrac{2}{3}(z-1).$

and $x + 2y + 3z - 13 = 0$

solve the above equation to get

$$x = \frac{3}{2}, y = 2, z = \frac{5}{2}$$



Closest point is being identified by using the geometrical interpretation that the line joining points (1,1,1) and (3/2, 2, 5/2) is perpendicular to the plane, making the above point on plane closest to (1,1,1). Justification is needed as the region is not closed and bounded.

END