



BITS Pilani
Pilani Campus



CS/IS F214 Logic in Computer Science

MODULE: **PREDICATE LOGIC**

Proof System - Natural Deduction

- **Universal Quantifier Elimination**
- **Existential Quantifier Introduction**

Predicate Logic: Proof System: Natural Deduction

- Predicate Logic includes operations \rightarrow , \wedge , \vee , and \neg with the same meanings as in Propositional Logic:
 - i.e.
 - the (introduction and elimination) rules for these operations (\rightarrow i, \rightarrow e, \wedge i, \wedge e, \vee i, \vee e, \neg i and \neg e)
 - along with
 - the rules for double negation
 - are applicable as is in Predicate Logic with the caveat:
 - the premises and the conclusion are to be treated as predicate logic formulas.



ND: Eliminating universal quantifier and Introducing existential quantifier

Recall:

$\forall x \phi$
$\phi[t/x]$

$\forall e$

Universal
Quantifier
Elimination

$\phi[t/x]$
$\exists x \phi$

$\exists i$

Existential
Quantifier
Introduction



Predicate Logic: Natural Deduction: Proofs.

- Ex. 1: Prove $\forall X \phi \vdash \exists X \phi$

1	$\forall X \phi$	Premise
2	$\phi [t/X]$ for some term t	$\forall e$ 1
3	$\exists X \phi$	$\exists i$ 2



Predicate Logic: Natural Deduction: Proofs.

- Ex. 2: Prove $P(t), \forall X P(X) \rightarrow \neg Q(X) \vdash \exists Y \neg Q(Y)$

1	$P(t)$	Premise
2	$\forall X(P(X) \rightarrow \neg Q(X))$	Premise
$\exists Y \neg Q(Y)$		



Predicate Logic: Natural Deduction: Proofs.

- Ex. 2: Prove $P(t), \forall X(P(X) \rightarrow \neg Q(X)) \vdash \exists Y \neg Q(Y)$

1	$P(t)$	Premise
2	$\forall X(P(X) \rightarrow \neg Q(X))$	Premise
3	$P(t) \rightarrow \neg Q(t)$	$\forall e$ 2
4	$\neg Q(t)$	$\rightarrow e$ 1,3
5	$\exists Y \neg Q(Y)$	$\exists i$ 4





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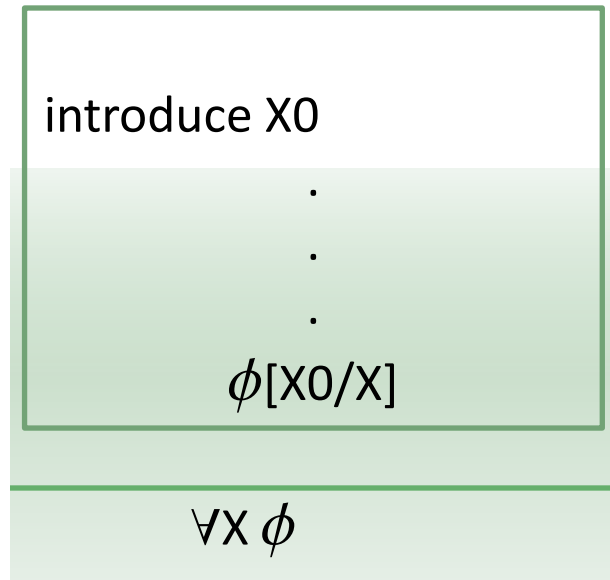
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MODULE: **PREDICATE LOGIC**

Proof System - Natural Deduction **- Universal Quantifier Introduction**

ND: Introducing Universal Quantifier

- What does this rule state?



X_0 does not occur
outside the
bounding box

$\forall i$
Universal Quantifier:
Introduction



ND: Introducing Universal Quantifier

• What does this rule state?

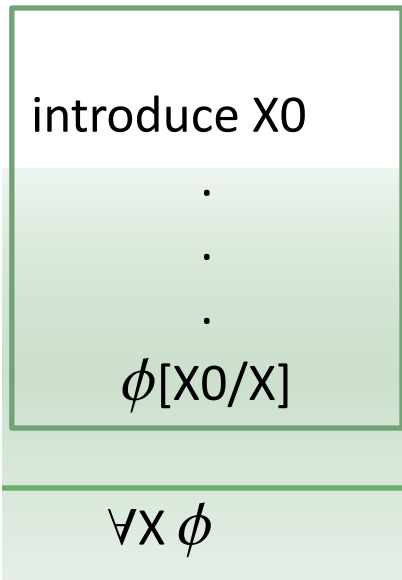
• **If ϕ (with free occurrences of variable X) is proved**

- without any assumptions
and without any conditions
on X -

• **then $\forall X \phi$ is proved.**

• Proving ϕ without any conditions on X is achieved by

- introducing $X0$ in place of X
 - where $X0$ is a fresh variable that does not occur elsewhere.



$X0$ does not occur
outside the
bounding box

$\forall i$
Universal
Quantifier:
Introduction



Predicate Logic: Natural Deduction: Proofs.

- Ex. 3: Prove $\forall X(P(X) \rightarrow Q(X)), \forall X P(X) \vdash \forall X Q(X)$

1	$\forall X P(X)$	Premise
2	$\forall X(P(X) \rightarrow Q(X))$	Premise
	$\forall X Q(X)$??



Predicate Logic: Natural Deduction: Proofs.

- Ex. 3: Prove $\forall X(P(X) \rightarrow Q(X)), \forall X P(X) \vdash \forall X Q(X)$

1	$\forall X P(X)$	Premise
2	$\forall X(P(X) \rightarrow Q(X))$	Premise
3	introduce X_0	Fresh variable
	$Q(X_0)$	
	$\forall X Q(X)$	$\forall i$ 3-?



Predicate Logic: Natural Deduction: Proofs.

- Ex. 3: Prove $\forall X(P(X) \rightarrow Q(X)), \forall X P(X) \vdash \forall X Q(X)$

1	$\forall X P(X)$	Premise
2	$\forall X(P(X) \rightarrow Q(X))$	Premise
3	introduce X0	Fresh variable
4	$P(X_0)$	$\forall e$ 1
5	$P(X_0) \rightarrow Q(X_0)$	$\forall e$ 2
6	$Q(X_0)$	$\rightarrow e$ 4,5
7	$\forall X Q(X)$	$\forall i$ 3-6





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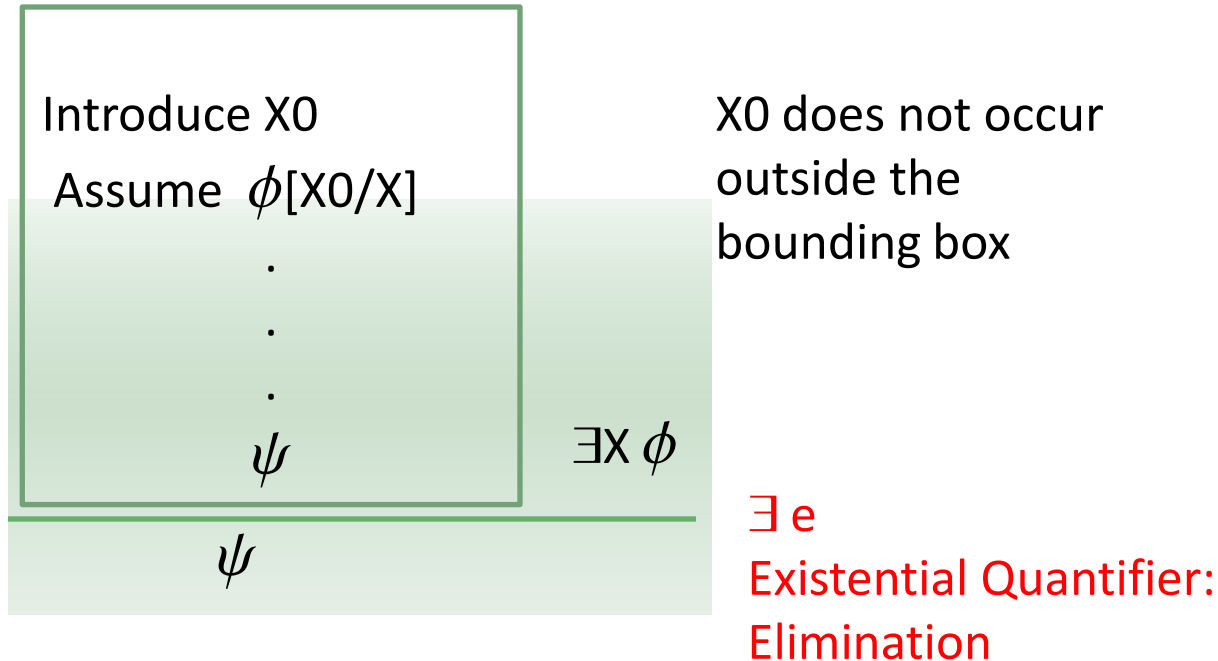
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MODULE: **PREDICATE LOGIC**

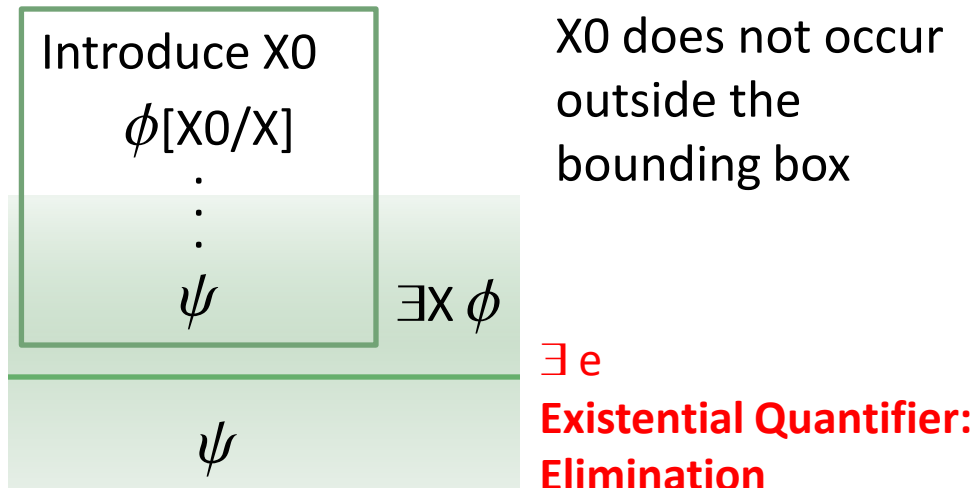
Proof System - Natural Deduction: Proof Rules **- Existential Quantifier Elimination**

ND: Eliminating Existential Quantifier

- What does this rule state?



ND: Eliminating Existential Quantifier



- What does this rule state?
 - **If $\exists X \phi$ is true** (for formula ϕ with free occurrences of X) **and**
 - **if ψ can be proved assuming ϕ** (with no conditions on X)
 - **then ψ has been proved.**
- No conditions on X are being assumed in the sub-proof because we use a fresh variable X_0 in place of X .



Predicate Logic: Natural Deduction: Proofs.

- Ex. 4: Prove $\forall X(p(X) \rightarrow q(X)), \exists X p(X) \vdash \exists X q(X)$

1	$\exists X p(X)$	Premise
2	$\forall X(p(X) \rightarrow q(X))$	Premise
	$\exists X q(X)$??



Predicate Logic: Natural Deduction: Proofs.

- Ex. 4: Prove $\forall X(p(X) \rightarrow q(X)), \exists X p(X) \vdash \exists X q(X)$

1	$\exists X p(X)$	Premise
2	$\forall X(p(X) \rightarrow q(X))$	Premise
3	introduce X_0	Fresh variable
4	$p(X_0)$	Assumption
	$\exists X q(X)$	
	$\exists X p(X)$	$\exists e$ 1, 3-?



Predicate Logic: Natural Deduction: Proofs.

- Ex. 4: Prove $\forall X(p(X) \rightarrow q(X)), \exists X p(X) \vdash \exists X q(X)$

1	$\exists X p(X)$	Premise
2	$\forall X(p(X) \rightarrow q(X))$	Premise
3	introduce X_0	Fresh variable
4	$p(X_0)$	Assumption
5	$p(X_0) \rightarrow q(X_0)$	$\forall e$ 2
6	$q(X_0)$	$\rightarrow e$ 4,5
7	$\exists X q(X)$	$\exists i$ 6
8	$\exists X q(X)$	$\exists e$ 1, 3-7



Predicate Logic: Natural Deduction: Proofs

- Exercise:
 - Prove the following sequents.
 - $\exists X s() \rightarrow p(X) \vdash s() \rightarrow \exists X p(X)$



ND: Comparing the rules $\forall i$ and $\exists e$

Introduce $X0$

$\phi[X0/X]$

\vdots

ψ

$X0$ does not occur
outside the
bounding box

$\exists X \phi$

ψ

$\exists e$

Existential Quantifier:
Elimination

Consider this:

Free variable X is treated as an
existentially quantified variable in
rule $\exists e$

Free variable X is treated as a
universally quantified variable in
rule $\forall i$

Introduce $X0$

\vdots

$\phi[X0/X]$

$X0$ does not occur
outside the
bounding box

$\forall X \phi$

$\forall i$

Universal Quantifier:
Introduction





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MODULE: PREDICATE LOGIC

Predicate Logic – Proofs - Examples

Predicate Logic: ND Proofs – Example

- Prove the following:
 - $\forall X (\neg p(X) \wedge q(X)) \mid - \forall X (p(X) \rightarrow q(X))$
 - $\exists X (\neg p(X) \vee q(X)) \mid - \exists X \neg(p(X) \wedge \neg q(X))$
 - $\forall X \forall Y (q(Y) \rightarrow f(X)) \mid - (\exists Y q(Y)) \rightarrow \forall X f(X)$



Prove the sequent: $\forall X \forall Y (q(Y) \rightarrow f(X)) \vdash (\exists Y q(Y)) \rightarrow \forall X f(X)$

Step		Remark
1	$\forall X \forall Y (q(Y) \rightarrow f(X))$	Premise
2	$\exists Y q(Y)$	Assumption
3		fresh X0
4	$\forall Y (q(Y) \rightarrow f(X0))$	$\forall e\ 1\ [X0/X]$
5	$\exists Y q(Y)$	copy 2
6		fresh Y0
7	$q(Y0)$	Assumption
8	$q(Y0) \rightarrow f(X0)$	$\forall e\ 4\ [Y0/Y]$
9	$f(X0)$	$\rightarrow e\ 7, 8$
10	$f(X0)$	$\exists e\ 5, 6-9$
11	$\forall X f(X)$	$\forall i\ 3-10$
12	$\exists Y q(Y) \rightarrow \forall X f(X)$	$\rightarrow i\ 2-11$

