MATH F113
(Probability and Statistics)

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What have you covered?

In Lecture 12

Continuous Random Variable

Probability Density Function

Cumulative Distribution Function

Definition

Let X be a continuous random variable with pdf f. The expected value of X is defined as

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

Again $E(X)$ exists if and only if $\int_{-\infty}^{\infty} |x| f(x) dx$ is finite.

Expectation (Cont...)

Definition

For a random variable X and function, say $H(x)$, the definition takes the form

$$E\left[H(x)\right] = \int_{-\infty}^{\infty} H(x) f(x) dx$$

provided $\int_{-\infty}^{\infty} |H(x)| f(x) dx$ is finite

Variance of X

$$Var(X) = E\left[x^2\right] - [E(x)]^2 = \sigma^2$$

where,

$$[E(x^2)] = \int_{-\infty}^{\infty} x^2 f(x)dx$$

Moment Generating Function (mgf)

$$E\left[e^{tx}\right] = m_x(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

$$E\left[x\right] = \left[\frac{d}{dt}(m_x(t))\right]_{t=0} = \mu_x$$

μ_x **is a location parameter since it indicates the position of the center of the density along the** x **axis.**

$$E\left[x^2\right] = \left[\frac{d^2}{dt^2}(m_x(t))\right]_{t=0}$$

Variance is shape parameter in the sense that a random variable with small variance will have a compact density; one with a large variance will have a density that is rather spread out or flat.

Exercise 5/4.1/139

Definition

A random variable X is said to be uniformly distributed over an interval (a, b) if its density is given by

$$f(x) = \frac{1}{b-a} \quad a < x < b$$

(a) Show that this is a density for a continuous random variable.

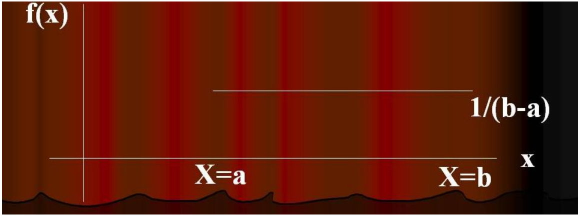
Since

$$f(x) = \frac{1}{b-a} > 0 \quad \textbf{for} \quad b > a$$

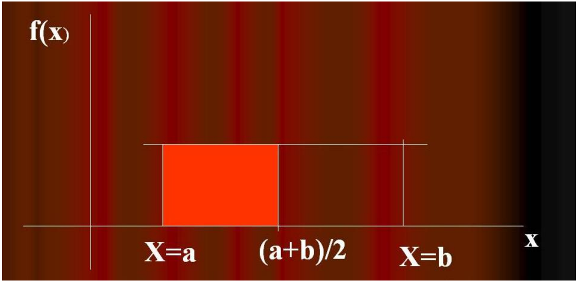
Secondly,

$$\int_a^b \frac{1}{b-a} dx = 1$$

(b) Sketch the graph of the uniform density.



(c) Shade the area in the graph of part (b) that represents $P\left[X \leq (a+b)/2\right]$.



(d) Find the probability pictured in part (c)

$$P\left[X \le \frac{a+b}{2}\right] = \int_{a}^{\frac{a+b}{2}} \frac{1}{b-a}\,dt = 0.5$$

(e) Let (c, d) **and** (e, f) **be sub intervals of** (a, b) **of equal length. What is the relationship between** $P\left[c \leq X \leq d\right]$ **and** $P\left[e \leq f \leq d\right]$**.**

Probabilities are constant over equal length of interval

Exercise 10/4.1/pp.139

Find the general expression for the cumulative uniform distribution for a random variable X over (a, b)

By definition of cdf, we have

$$F(x) = P\left[X \leq x\right] = \int_{-\infty}^{x} \frac{1}{b-a} dt$$

$$= \frac{x-a}{b-a} \quad a < x < b$$

Therefore, we have the following cdf:

$$F(x) = \begin{cases} 0 & \mathbf{x < a} \\ \frac{x-a}{b-a} & \mathbf{a \leq x < b} \\ 1 & \mathbf{x \geq b} \end{cases}$$

A random variable is said to be uniformly distributed over $(0,1)$ if its pdf is given by

$$f(x) = \begin{cases} 1 & \mathbf{0} < x < \mathbf{1} \\ 0 & \textbf{otherwise} \end{cases}$$

Since $f(x) > 0$ only when $x \in (0,1)$, it means X must assume a value in $(0,1)$.

Hence, cumulative distribution function is

$$F(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

MGF of uniform distribution on (a, b):
Density is $f(x) = \frac{1}{b-a}, \quad a < x < b$

$$m_x(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx = \int_a^b \frac{e^{tx}}{b-a}dx$$

$$= \frac{(e^{tb} - e^{ta})}{t(b-a)}; \quad t \neq 0$$

$$m_x(0) = \int_a^b \frac{e^{0x}}{b-a}dx = 1$$

Either by using mgf or directly, mean and variance can be found.

$$E[X] = \frac{a+b}{2}$$

$$Var[X] = \frac{(b-a)^2}{12}$$

Proof: By definition

$$E\left[X\right] = \int_a^b \frac{x}{b-a}dx = \left[\frac{x^2}{2(b-a)}\right]_a^b = \frac{a+b}{2}$$

$$E\left[X^2\right] = \int_a^b \frac{x^2}{b-a}dx = \frac{b^2+a^2+ab}{3}$$

Proof

$$var\left(X\right) = \frac{b^2 + a^2 + ab}{3} - \frac{(b+a)^2}{4}$$

$$= \frac{(b-a)^2}{12}$$