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CS/IS F214
Logic in Computer Science

MODULE: **PREDICATE LOGIC**

Semantics – Undecidability: Proof by Diagonalization

Undecidability of Validity in Predicate Logic

- Theorem:
  - ***Validity of formulas in Predicate Logic is undecidable***.
- Proof :
  - By diagonalization.
- Note:
  - The complete proof is tedious.
  - A sketch of the proof adapted from Enderton's *Mathematical Introduction to Logic* follows.

Size of the set of well-formed formulas

- **Lemma**: The set of all well-formed-formulas in Predicate Logic is *countably infinite*.
- **Proof:**
  - Use an encoding of formulas into numbers the way we encoded C programs.

Undecidability of Validity in Predicate Logic

Proof Sketch:

- Let each well-formed-formula be assigned a unique natural number, say j,
  - and we can index that formula using j i.e. we can refer to ϕ_j
- Define a *binary relation* **p** on natural numbers:
  - **(m,n)** in **p** iff
    - ϕ_m is a formula with a single free variable **V1**, and
    - $\models^{N}_{[V1 \mapsto n]} \phi_m$
      - i.e. ϕ_m is true under
        - the model ***N***, the set of natural numbers, and
        - the lookup table which maps V1 to n

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- Proof Sketch (*continued*):
  - Then for any natural number **i** in ***N***, define the set $\mathbf{S_i}$:
    - **S_i = { j | (i,j) in p }**
      - i.e. the set of all numbers **j** such that ϕ_i evaluates to **true** under ***N*** when **V1** is mapped to ***j***
    - *i.e. we have one set **S_i** defined for each **i***

      *and any subset of **N** corresponds to some such **S_i***

      - *Why?*

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- Proof Sketch (*continued*):
  - Now we diagonalize:
    - **D = { j | (j,j) not in p }**
      - i.e. **j** $\in$ **D** iff ϕ_j evaluates to false when V1 is mapped to j
  - What is **k**, such that $\mathbf{S_k}$ **= D** ?
    - No such **k** exists because:
      - **k** $\in$ **D** <==> **(k,k) not in p** (by definition of **D**)
        <==> **k** $\notin$ $\mathbf{S_k}$ (by definition of S_k)

    [Exercise:

    Draw the (infinite) matrix and identify the diagonal to visualize the proof.]

Undecidability of Validity in Predicate Logic

- Proof Sketch (*continued*):
  - *i.e. there exist formulas for which the set of numbers evaluating to true cannot be computed. (even with one variable under one model).*
  - *i.e. validity is not decidable.*

- Note:
  - What we actually proved is this:
    - ***set of values (assigned to a free variable) for which a formula evaluates to true, under the model of natural numbers, is not computable.***

Relation between Countability and Computability

- Theorem:
  - If a decision problem π is decidable,
  - then $L_\pi = \{ x \mid \pi(x) \text{ is true} \}$ is countable.
    - Why?

- Corollary?