Chapter 4:
Forced Vibrations and Resonance

Forced Vibrations and Resonance

An additional externally applied harmonic force acts on the oscillator

Without Damping

$$m\frac{d^2x}{dt^2} + kx = F_0 \cos\omega t$$

Or, $$\frac{d^2x}{dt^2} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$$

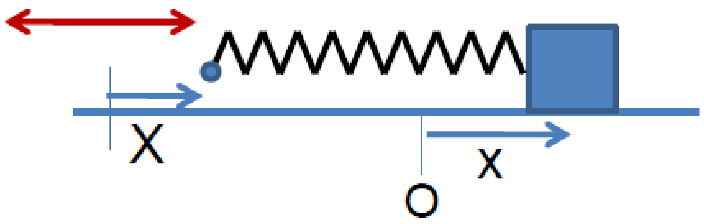
ω_0 : Natural angular frequency

ω : Angular frequency of driving force

Ex1. Spring-mass system with oscillating 'fixed' point

$$X = A\cos\omega t$$

$$F = -k(x - X)$$



Equation of motion :

$$\frac{d^2x}{dt^2} = -\,\omega_0^2\,(x - A\cos\omega t)$$

Or, $$\frac{d^2x}{dt^2} + \omega_0^2\,x = \omega_0^2\,A\cos\omega t$$

Or, $$\frac{d^2x}{dt^2} + \omega_0^2\,x = \frac{F_0}{m}\cos\omega t \qquad \left(F_0 = m\omega_0{}^2 A\right)$$

Solving the Equation of Motion

$$\frac{d^2x}{dt^2} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$$

The above is a linear, inhomogeneous differential equation.

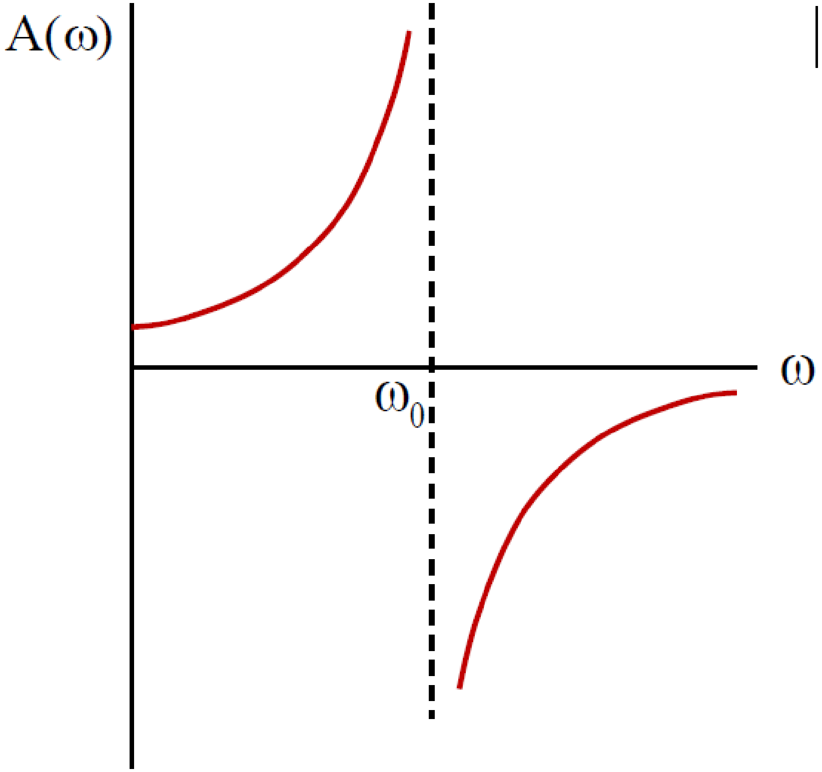
Resonance

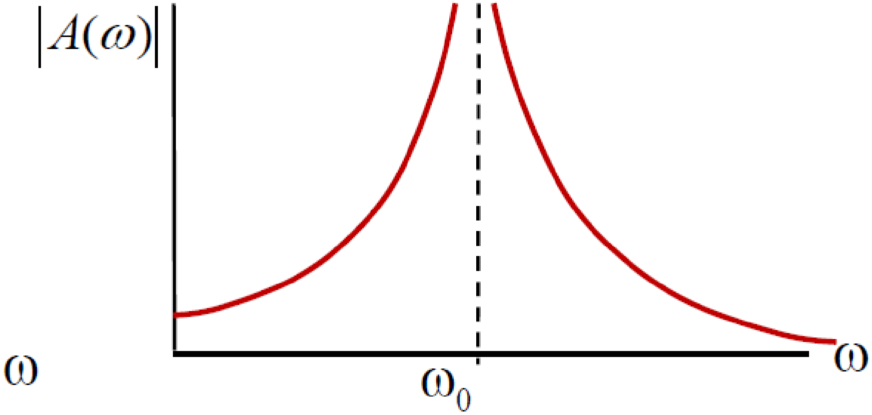
The amplitude of the oscillations :

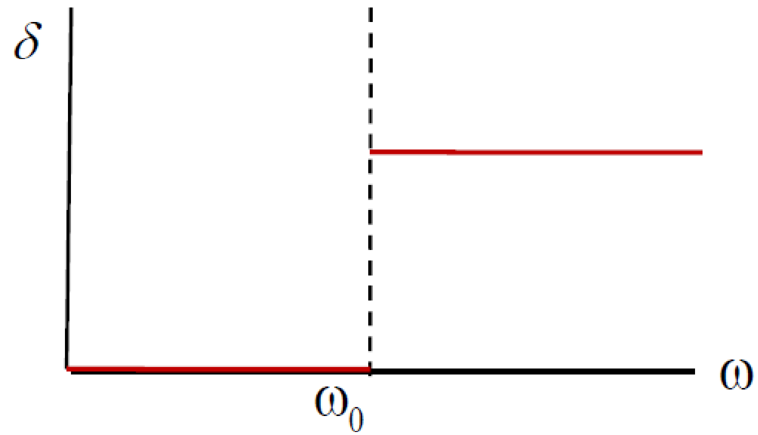
$$A(\omega) = \frac{F_0}{m(\omega_0^2 - \omega^2)}$$

varies with the driving frequency ω

$$\lim_{\omega \to \omega_{0\,\mp}} A(\omega) = \pm\,\infty$$







$$\therefore \quad x_{p.s}(t) = A(\omega)\cos(\omega t + \delta)$$

$$A(\omega) = \frac{F_0}{m\left|(\omega_0^2 - \omega^2)\right|}$$

Forced Oscillations with Damping

Equation of Motion :

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m} \cos \omega t$$

To obtain the particular solution, take the complex form :

$$\frac{d^2z}{dt^2} + \gamma \frac{dz}{dt} + \omega_0^2 z = \frac{F_0}{m} e^{i\omega t}$$

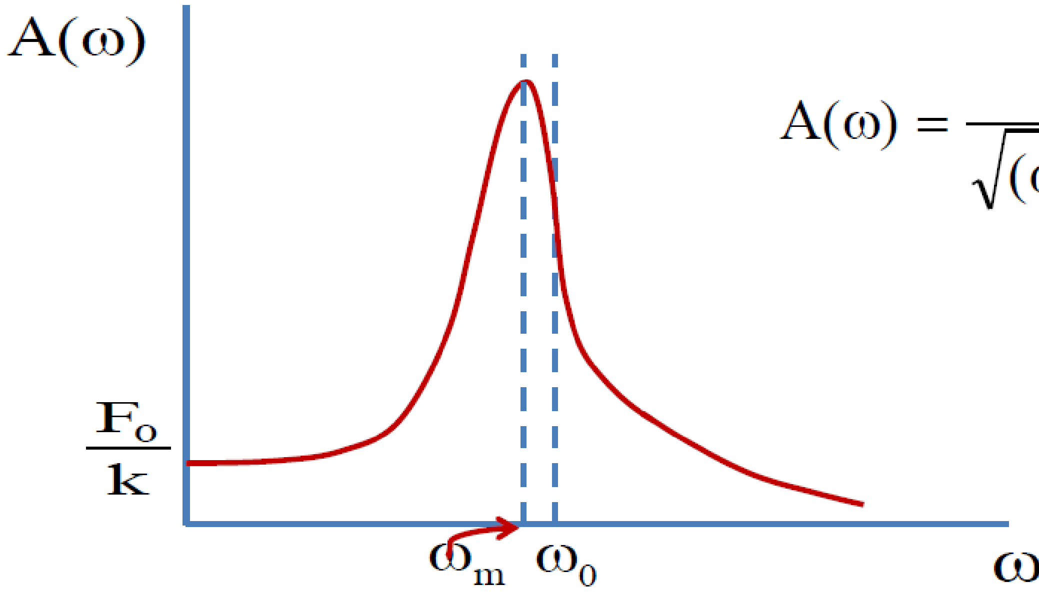
Most General Solution :

$$x(t) = Be^{-\frac{\gamma}{2}t}\cos(\omega_D t + \phi) + A(\omega)\cos(\omega t - \delta)$$

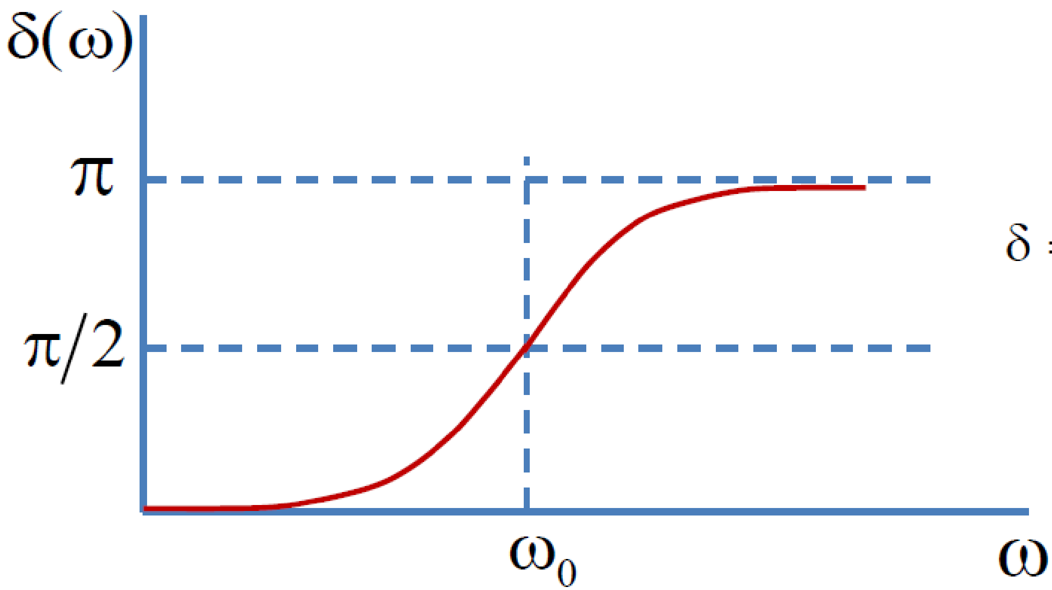
Transient

Steady State

The transient part of the solution dies out after about Q oscillations, and after that the steady state oscillations go on unabated



$$A(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}$$



$$\delta = \tan^{-1}\left(\frac{\gamma\,\omega}{\omega_0^2 - \omega^2}\right)$$

Resonance in the presence of damping

Assuming Q to be reasonably large :

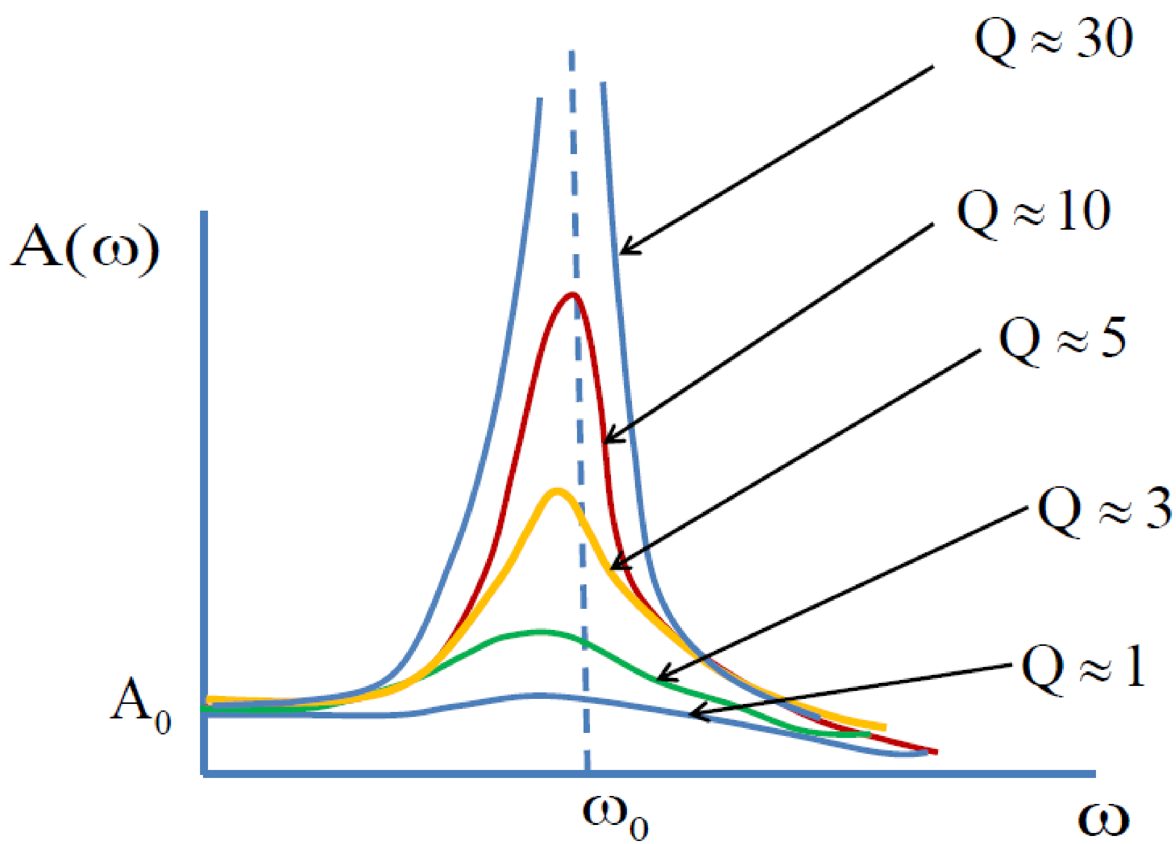
$$A(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \dfrac{(\omega\omega_0)^2}{Q^2}}}$$

$$= \frac{A_0}{\dfrac{\omega}{\omega_0}\sqrt{\left(\dfrac{\omega_0}{\omega} - \dfrac{\omega}{\omega_0}\right)^2 + \dfrac{1}{Q^2}}} \qquad (A_0 = A(0) = F/m\omega_0^2)$$

Maximizing A w.r.t. ω one gets :

$$\omega_m = \omega_0\left[1 - \frac{1}{2Q^2}\right]^{1/2} \approx \omega_0\left(1 - \frac{1}{4Q^2}\right)$$

$$A_m = A(\omega_m) = \frac{A_0 Q}{\left(1 - \frac{1}{4Q^2}\right)^{1/2}}$$



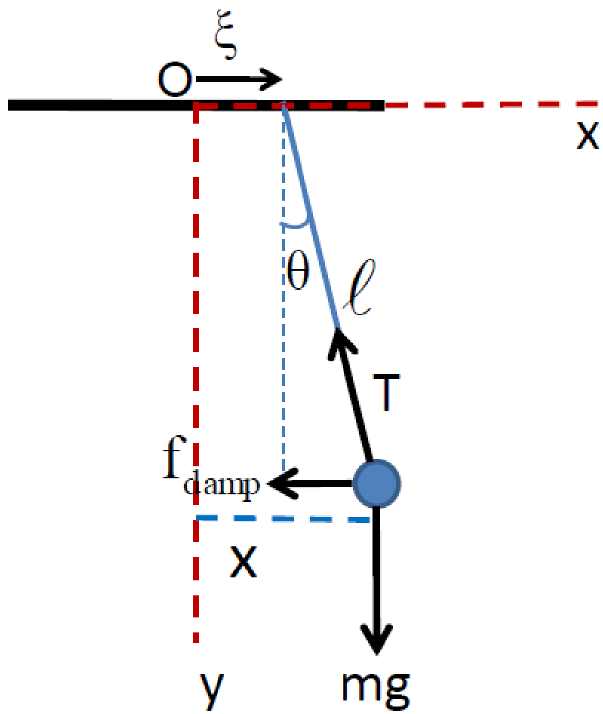
Amplitude for increasing quality

Prob. 4.5 A simple pendulum has a length of 1 m. In free vibration the amplitude falls off by a factor e in 50 swings. The pendulum is set into forced vibration by moving its point of suspension horizontally in SHM with an amplitude of 1 mm.

a) Show that if the horizontal displacement of the bob is x and the horizontal displacement of its point of suspension is ξ the equation of motion of the pendulum is :

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \frac{g}{\ell}x = \frac{g}{\ell}\xi$$

Answer :



In the x-y (inertial) frame, the Eq. of motion is :

$$m\frac{d^2x}{dt^2} = -m\frac{g}{\ell}(x - \xi) - b\frac{dx}{dt}$$

Or, $$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \frac{g}{\ell}x = \frac{g}{\ell}\xi$$

Or, $$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = \omega_0^2 \xi_0 \cos\omega t$$

Or, $$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t \qquad F_0 = m\omega_0^2\xi_0$$

b) At exact resonance, what is the amplitude of motion of the bob of the pendulum

$$A_m = A(\omega_m) = \frac{A_0 Q}{\left(1 - \frac{1}{4Q^2}\right)^{1/2}}$$

After n oscillations, the amplitude drops by a factor :

$$e^{-n\pi/Q}$$

$$\therefore \quad 50\pi/Q = 1 \Rightarrow Q = 50\pi$$

$$A_0 = \frac{F_0}{m\omega_0^2} = \xi_0 = 1 \text{ mm}$$

$$\Rightarrow A_m \approx A_0 Q = 50\pi \text{ mm} = 15.7 \text{ cm}$$

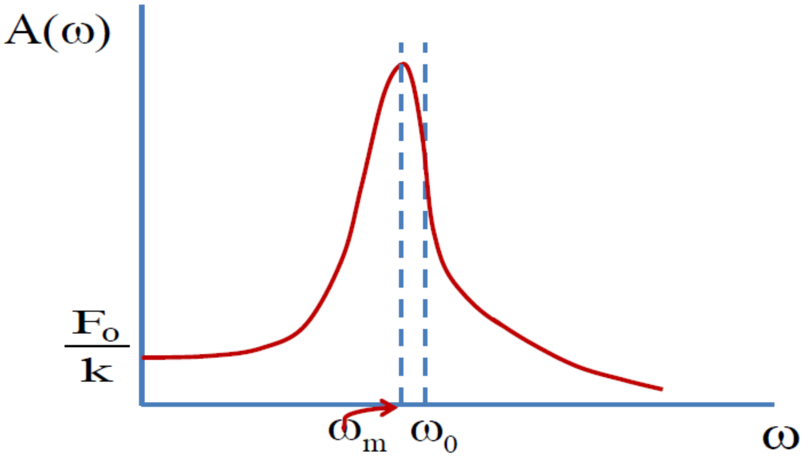
c) At what angular frequency, is the amplitude half its resonance value?

$$A(\omega) = \frac{A_0}{\frac{\omega}{\omega_0}\sqrt{\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}}} = \frac{A_0 Q}{2}$$

Putting $\dfrac{\omega}{\omega_0} = x,$ the equation to be solved :

$$\frac{4}{Q^2} = x^2\left[\left(\frac{1}{x} - x\right)^2 + \frac{1}{Q^2}\right] = (1 - x^2)^2 + \frac{x^2}{Q^2}$$

Since x is expected to be extremely close to 1, put : $x = 1 + \alpha$



$$\therefore\ 1 - x^2 \approx -2\alpha$$

$$\therefore\ \frac{4}{Q^2} = 4\alpha^2 + \frac{1+2\alpha}{Q^2}$$

Or, $\alpha = \pm\frac{\sqrt{3}}{2Q} = \pm\frac{\sqrt{3}}{100\pi} = \pm 5.5\times 10^{-3}$

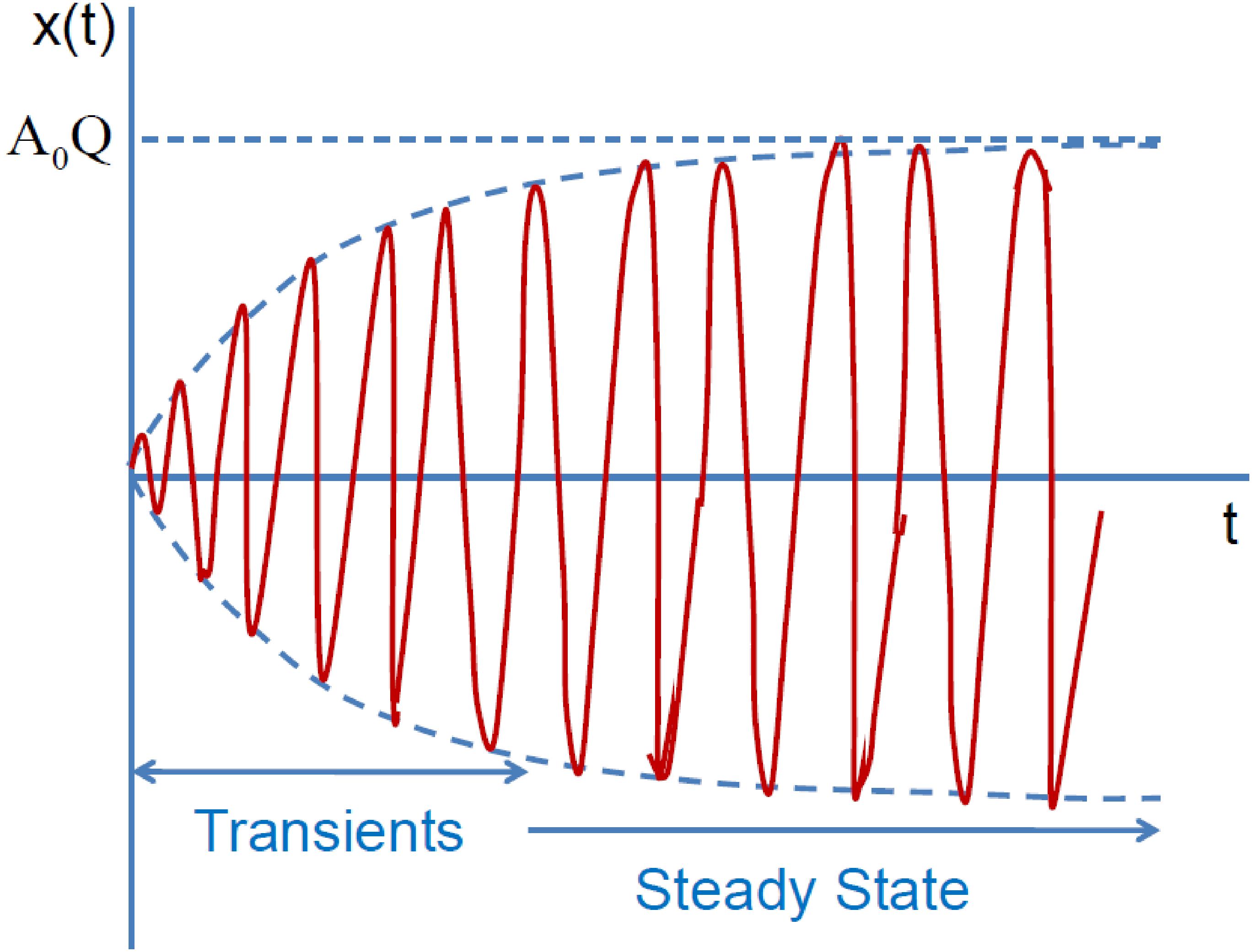
$$\therefore\ \omega = \omega_0\left(1 \pm 0.0055\right) \quad ; \quad \omega_0 = \sqrt{10} = 3.16\ s^{-1}$$

Transient Phenomena

In a driven oscillator, the motion in the beginning is not quite simple harmonic. This part of the motion is called the transients. Afterwards, the motion settles to a SHM of a frequency, that is equal to the driving frequency.

Complete motion :

$$x(t) = B e^{-\frac{\gamma}{2}t} \cos(\omega_D t + \phi) + A(\omega) \cos(\omega t - \delta)$$

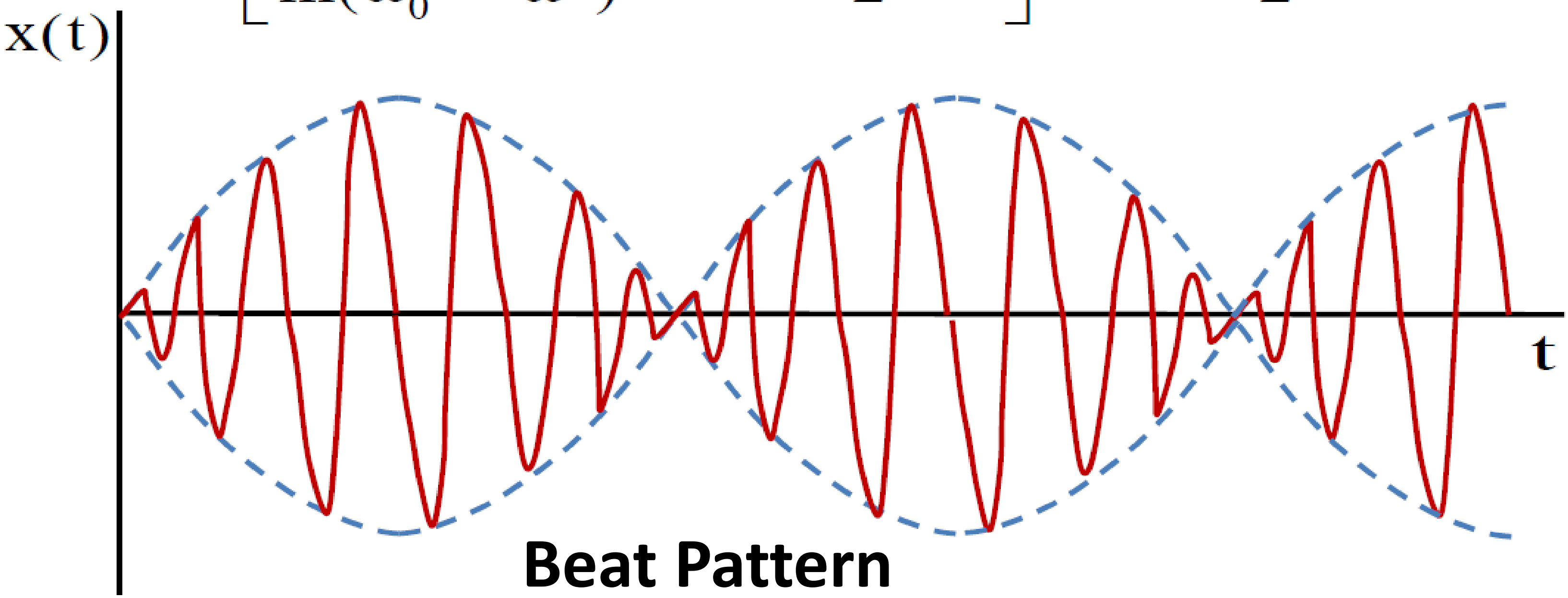

x(t)
A_0Q
t
Transients
Steady State

Most general solution for the driven oscillator :

$$x(t) = A\cos(\omega_0 t + \phi) + \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos\omega t$$

With the initial conditions : $x(0) = \dot{x}(0) = 0$

$$x(t) = \left[\frac{2F_0}{m(\omega_0^2 - \omega^2)}\sin\frac{(\omega_0 - \omega)t}{2}\right]\sin\frac{(\omega + \omega_0)t}{2}$$



Beat Pattern

Power Input to a Driven Oscillator in the Steady State

Instantaneous power input to the oscillator by the driving force :

$$P = F\,v$$

$$F = F_0 \cos \omega t$$

$$v = \frac{dx}{dt} = -\,A(\omega)\,\omega\,\sin(\omega t - \delta)$$

$$= -\,v_0 \sin(\omega t - \delta)$$

$$v_0 = \omega A(\omega) = \frac{A_0\omega_0}{\sqrt{\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}}}$$

Resonance for velocity amplitude occurs exactly at the natural frequency

$$P(t) = -F_0\, v_0 \cos\omega t \sin(\omega t - \delta)$$

$$= -F_0\, v_0 (\cos\delta \cos\omega t \sin\omega t - \sin\delta \cos^2 \omega t)$$

$$\overline{P}(\omega) = \frac{1}{T}\int_0^T P(t)\, dt = \frac{1}{2} F_0\, v_0 \sin\delta$$

Prob. 4.10 The power required to maintain forced vibration must be equal to the power loss due to damping.

a) Find the instantaneous rate of doing work against the damping force.

b) Find the mean rate of doing work against damping

c) Show that the above answer is the negative of the mean power delivered by the driving force.

Power Resonance Curve

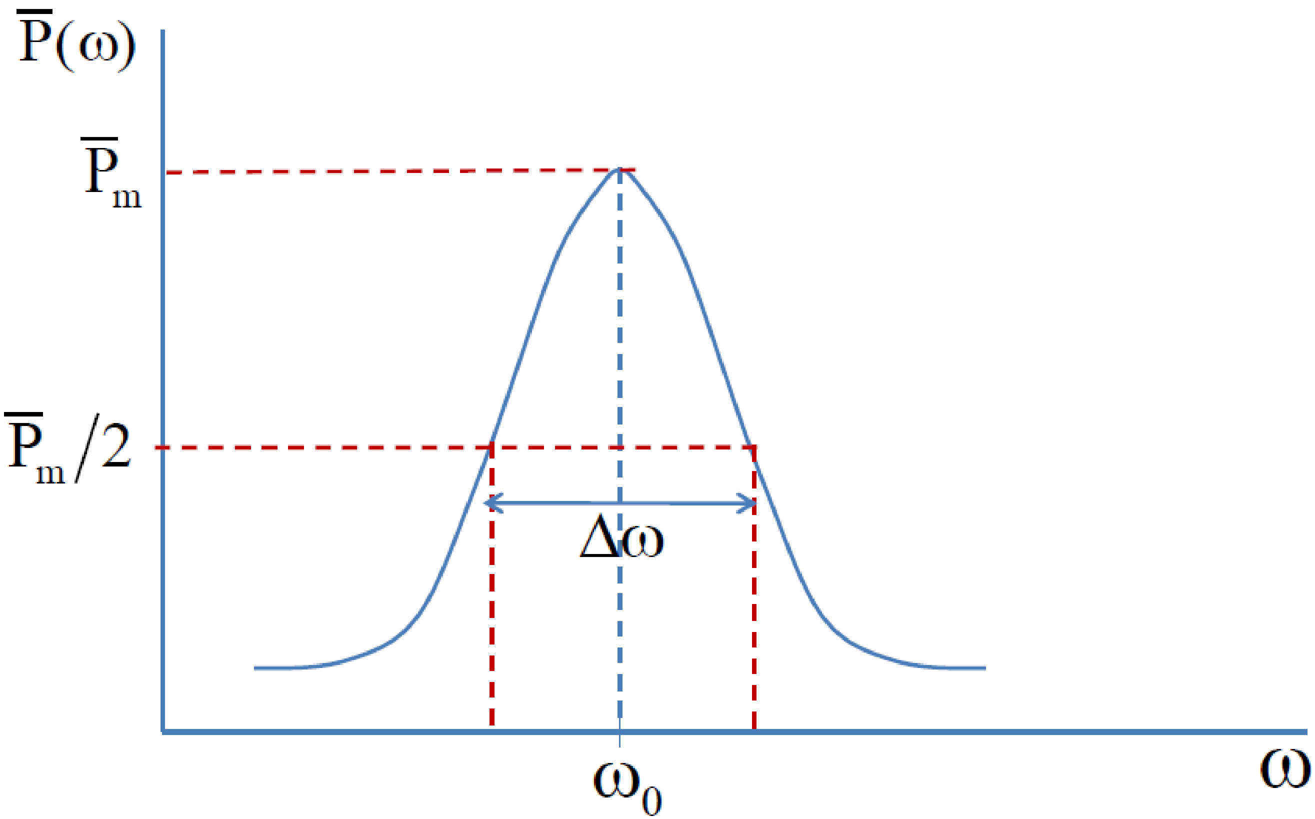
$$\overline{P}(\omega) = \frac{1}{2} b\,\omega^2 A^2 = \frac{bF_0^2 \big/ (2m^2\omega_0^2)}{\left[\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}\right]}$$

$\overline{P}$ is maximum at $\omega = \omega_0$

$$\boxed{\Rightarrow\ \overline{P}_m = \frac{bF_0^2\,Q^2}{2m^2{\omega_0}^2} = \frac{F_0^2}{2b}}$$

$$\therefore\ \overline{P}(\omega) = \frac{\overline{P}_m}{Q^2} \frac{1}{\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}}$$

Width of Power Resonance Curve (Full Width at Half Maximum (FWHM))



$\Delta\omega$: FWHM

Finding FWHM

Equating $\overline{P}(\omega)$ to $\dfrac{\overline{P}_m}{2}$

$$\frac{1}{2} = \frac{1}{Q^2} \frac{1}{\left(\dfrac{\omega_0}{\omega} - \dfrac{\omega}{\omega_0}\right)^2 + \dfrac{1}{Q^2}}$$

$$\Rightarrow \quad Q^2 \left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 = 1$$

$$\Rightarrow \quad \left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right) = \pm \frac{1}{Q}$$

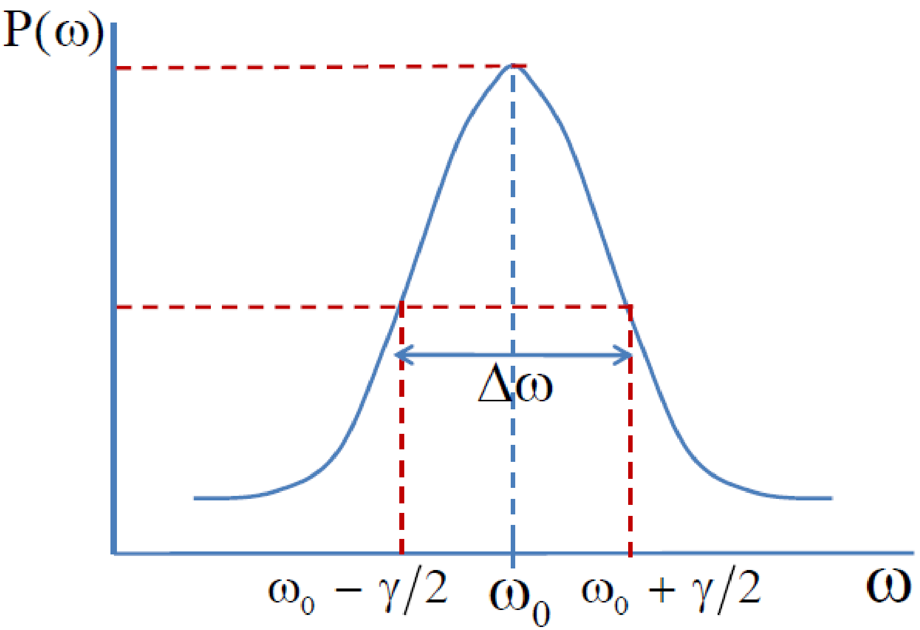
Putting $\dfrac{\omega}{\omega_0} = 1 + \alpha$, where $\alpha << 1$

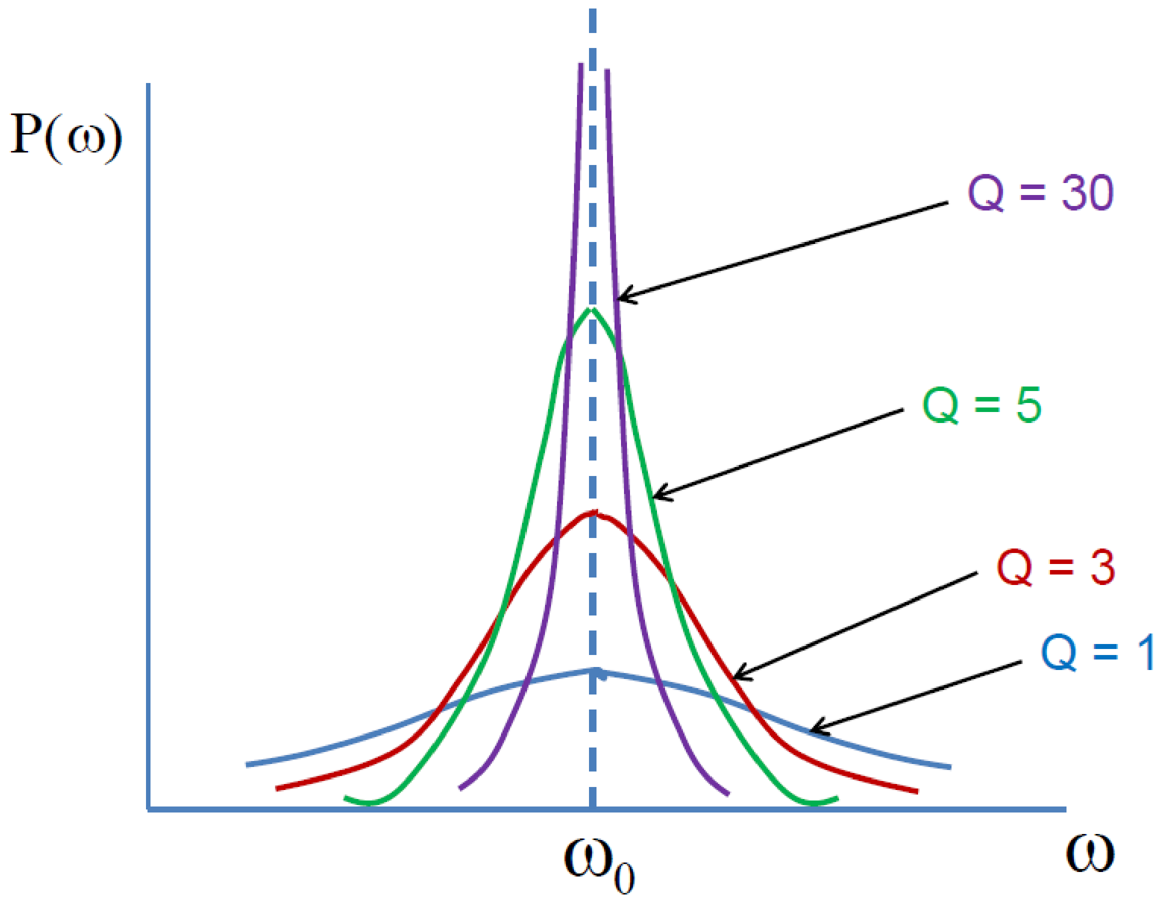
$$\frac{1}{1+\alpha} - (1+\alpha) = \pm \frac{1}{Q} \quad \Rightarrow \quad \alpha = \pm \frac{1}{2Q}$$

$$\frac{\omega}{\omega_0} = 1 \pm \frac{1}{2Q}$$

$$\therefore \quad \Delta\omega = \frac{\omega_0}{Q} = \gamma$$

$$\boxed{\therefore \quad Q = \frac{\omega_0}{\Delta\omega}}$$




P(ω)
Q = 30
Q = 5
Q = 3
Q = 1
ω0
ω

Prob. 4.12 A mass of 2 kg is hung from a spring that is extended by 2.5 cm. The top end of the spring is oscillated up and down with an amplitude 1 mm. The Q of the system is 15.

a) What is ω_0 for this system?

b) What is the amplitude of the oscillations at $\omega = \omega_0$?

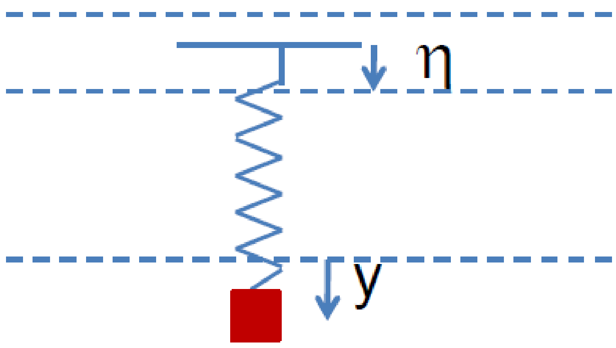
c) What is the mean power input to maintain an oscillation at 2% higher than ω_0?

a) What is ω_0 for this system?

$$k = 800\ \mathrm{N/m}\ ;\ \omega_0 = 20\ \mathrm{s}^{-1}$$

b) What is the amplitude of the oscillations at $\omega = \omega_0$?

$$\eta = \eta_0 \cos \omega t \qquad \eta_0 = 1 \text{ mm}$$



Equation of motion is

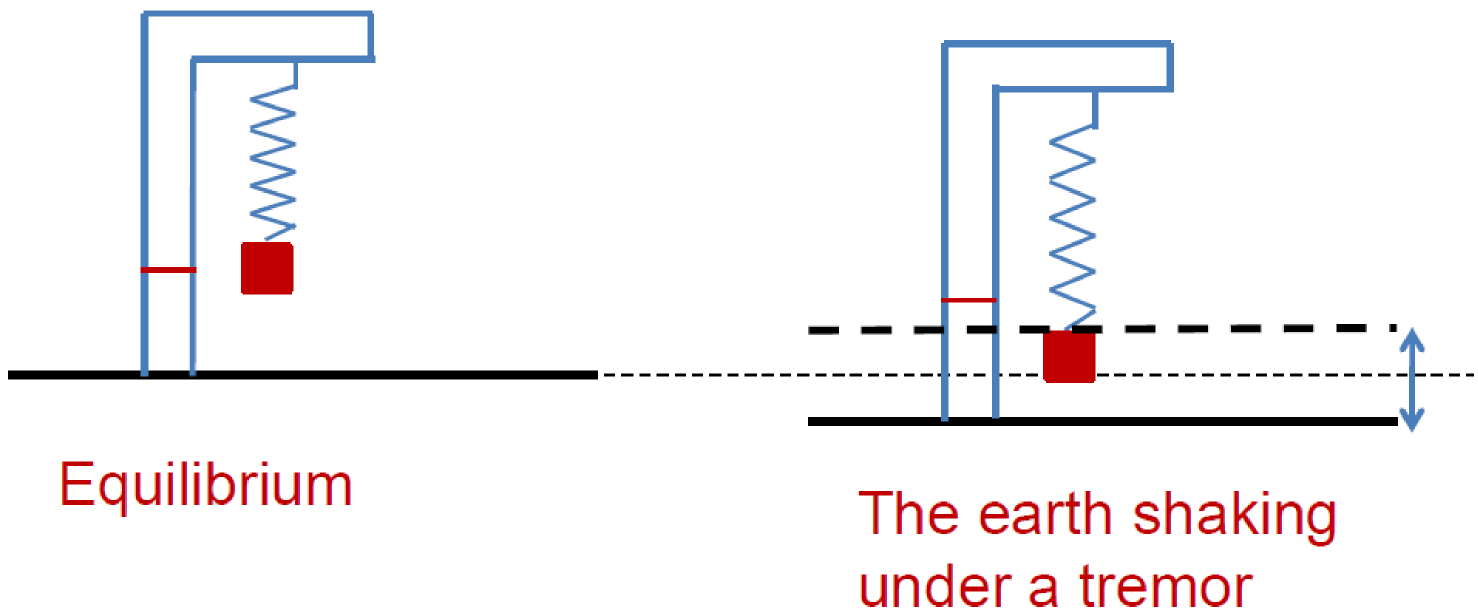
$$\frac{d^2y}{dt^2} + \gamma \frac{dy}{dt} + \omega_0^2 y = \eta_0 \omega^2 \cos \omega t \qquad F_0 = m\eta_0\omega^2$$

$$A_0 = \frac{F_0}{m\omega_0^{\;2}} = \eta_0 = 1\text{mm} \quad \text{for} \quad \omega = \omega_0$$

$$\boxed{A(\omega_0) = A_0 Q = 15 \text{ mm}}$$

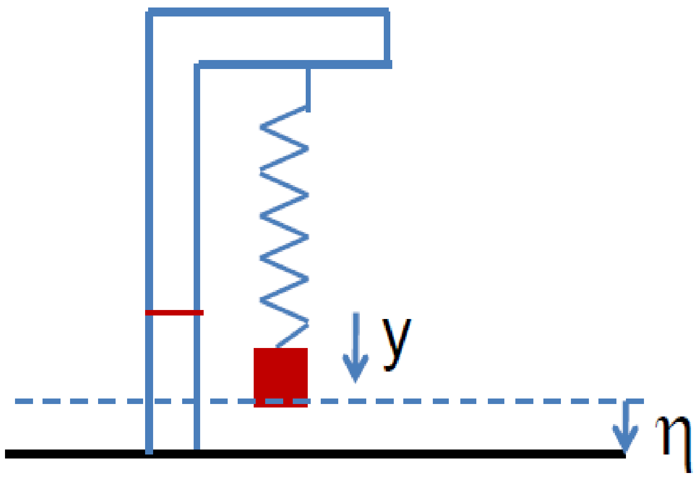
Prob. 4.6. Simple Seismograph as in figure below.

It consists of a mass m hung from a spring on a rigid framework attached to the earth. The spring force and damping force depend on displacement and velocity relative to the earth's surface, but the dynamically significant acceleration is acceleration of M relative to the fixed stars.



a) Show that the equation of motion is :

$$\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + \omega_0^2 y = -\frac{d^2\eta}{dt^2}$$



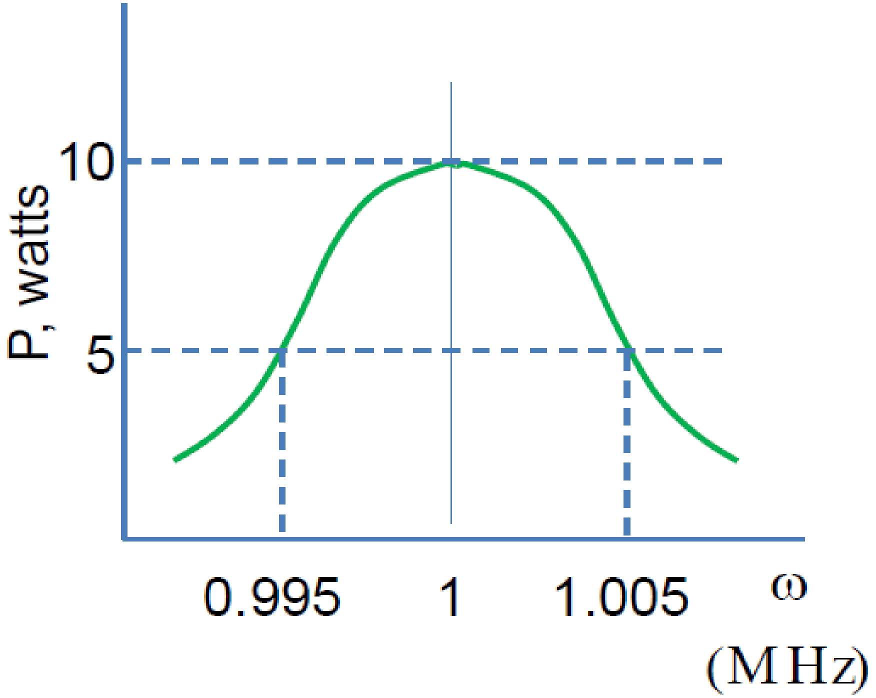
y is displacement of m relative to earth and η is displacement of earth's surface itself.

b) Solve for y if $\eta = C \cos \omega t$

c) Plot a graph of amplitude versus driving frequency.

d) A typical long period seismometer has a period of about 30 sec. and quality of 2. As a result of earthquake the earth's surface may oscillate with a period of 20 min. and with an amplitude such that the maximum acceleration is about $10^{-9}\,\mathrm{m-s^{-2}}$ sec. How small a value of the displacement of the block must be observable, if the quake is to be detected.

Prob. 4.17 The graph shows the mean power absorbed by an oscillator when driven by a force of constant magnitude but variable frequency.



a) At exact resonance, how much work per cycle is being done against the resistive force?

b) At exact resonance, what is the total mechanical energy E_0 of the oscillator?

c) If the driving force is turned off, how long does it take for the energy of the oscillator to drop to $E_0 e^{-1}$?

Problem 4.7 Consider a system with a damping force undergoing forced oscillations at an angular frequency ω.

(a) What is the instantaneous kinetic energy of the system?

(b) What is the instantaneous potential energy of the system?

(c) What is the ratio of the average kinetic energy to the average potential energy?

(d) For what value(s) of ω are the average kinetic energy and average potential energy equal? What is the total energy of the system under these conditions?

(e) How does the total energy of the system vary with time for an arbitrary value of ω? For what value(s) of ω is the total energy constant in time?