Chapter 10 (10.1-10.8)

Infinite Sequences and Series

Note: *This module is prepared from Chapter 10 (Thomas' Calculus, 13th edition) just to help the students. The study material is expected to be useful but not exhaustive. For detailed study, the students are advised to attend the lecture/tutorial classes regularly, and consult the text book.*

Appeal: Please do not print this e-module unless it is really necessary.

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Contents

SECTION 10.1 (Sequences)

Sets of numbers

- Set of Natural Numbers is

  $\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, ...\}$.

  Natural numbers are also known as counting numbers.

- Set of Integers is

  $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, ...\}$.

  It is observed that natural numbers together with their negatives and 0 constitute the set of integers.

- Set of Rational Numbers is

  $\mathbb{Q} = \{p/q : \ p, q \in \mathbb{Z} \text{ and } q \neq 0\}$.

  It may be noted that integers are also rational numbers with denominator 1. Further, the decimal form of every rational number either terminates or repeats. For example,

  $\frac{11}{5} = 2.5$,

  $\frac{-1}{3} = -0.33333333333... = -0.\overline{3}$,

  $\frac{22}{7} = 3.142857142857... = 0.\overline{142857}$ etc.

- Set of Irrational Numbers is

  $\mathbb{I} = \{x : \ x \neq p/q \text{ for all } p, q \in \mathbb{Z} \text{ and } q \neq 0\}$.

  The decimal form of an irrational number neither terminates nor repeats. For example,

  $\sqrt{2} = 1.4142135623730 95...$,

  $\pi = 3.14159265358979323846...$,

  $e = 2.71828182845904523536...$ etc.

- Set of Real Numbers is

  $\mathbb{R} = \{x : \ x \in \mathbb{Q} \text{ or } x \in \mathbb{I}\}$.

  It may be noted that rational numbers and irrational numbers together give rise to the set of real numbers.

Algebra of infinite limits

I assume that you are familiar with the limits involving ∞ from your 12th standard. So have a look at the algebra of infinite limits.

If $\lim_{n\to\infty} f(n) = A$ and $\lim_{n\to\infty} g(n) = B$, then the following rules are true.

1. Sum rule : $\lim_{n\to\infty} [f(n) + g(n)] = \lim_{n\to\infty} f(n) + \lim_{n\to\infty} g(n) = A + B$.
2. Difference rule : $\lim_{n\to\infty} [f(n) - g(n)] = \lim_{n\to\infty} f(n) - \lim_{n\to\infty} g(n) = A - B$.
3. Constant multiple rule : $\lim_{n\to\infty} [kf(n)] = k \lim_{n\to\infty} f(n) = kA$ (any number k)
4. Product rule : $\lim_{n\to\infty} [f(n).g(n)] = \lim_{n\to\infty} f(n). \lim_{n\to\infty} g(n) = A.B$.
5. Quotient rule : $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{\lim_{n\to\infty} f(n)}{\lim_{n\to\infty} g(n)} = \frac{A}{B}$.
6. Power rule: $\lim_{n\to\infty} [f(n)]^c = \left[\lim_{n\to\infty} f(n)\right]^c = A^c$. (any real number c)
7. Exponential rule: $\lim_{n\to\infty} a^{f(n)} = a^{\left[\lim_{n\to\infty} f(n)\right]} = a^A$. (for any suitable real number a)

L'Hopital rule

Consider the limit $\lim_{n\to\infty} \frac{f(n)}{g(n)}$. If $f(n)$ and $g(n)$ both tend to ∞ or 0 as $n \to \infty$, that is, either we get ∞/∞ form or $0/0$ form, then by L'Hopital rule, we can write

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)},$$

where $'$ stands for derivative with respect to n.

Important caution: In the L'Hopital rule, ∞/∞ and $0/0$ are indeterminate forms. Note that 0^0, ∞^0, $0.\infty$, 1^∞ and $\infty - \infty$ are also indeterminate forms. If we come across any of these forms, we try to rearrange the limit function in such a way that we have either ∞/∞ form or $0/0$ form in order to apply the L'Hopital rule. Also, note that 0^∞, $\infty.\infty$, $\infty + \infty$, ∞^∞ and $\infty^{-\infty}$ are not indeterminate forms. We take $0^\infty = 0$, $\infty.\infty = \infty$, $\infty + \infty = \infty$, $\infty^\infty = \infty$ and $\infty^{-\infty} = 0$.

Sandwich theorem

Let $f(n) \leq h(n) \leq g(n)$ for all $n \geq m$, where m is some real number. If $\lim_{n\to\infty} f(n) = L = \lim_{n\to\infty} g(n)$, then $\lim_{n\to\infty} h(n) = L$.

Some illustrative examples on infinite limits

In view of the algebra of infinite limits, L'Hopital rule and Sandwich theorem, let us see the following examples pertaining to infinite limits.

Ex. $\lim\limits_{n\to\infty} \frac{1}{n} = 0.$

Sol. As n becomes larger and larger, $\frac{1}{n}$ becomes smaller and smaller, and tends to 0 as $n \to \infty$.

More generally, $\lim\limits_{n\to\infty} \frac{f(n)}{g(n)} = 0$ if $f(n)$ remains finite and $g(n)$ tends to ∞ in the limit $n \to \infty$.

Ex. $\lim\limits_{n\to\infty} \frac{\cos n}{n^2} = 0.$

Sol. Since $-1 \leq \cos n \leq 1$ for all n and $n^2 \to \infty$ as $n \to \infty$, so $\lim\limits_{n\to\infty} \frac{\cos n}{n^2} = 0.$

Alternatively, $\frac{-1}{n^2} \leq \frac{\cos n}{n^2} \leq \frac{1}{n^2}$. Also, $\lim\limits_{n\to\infty} \frac{-1}{n^2} = 0 = \lim\limits_{n\to\infty} \frac{1}{n^2}$. So the result follows by Sandwich theorem.

Ex. $\lim\limits_{n\to\infty} \frac{n+1}{n^2+3n+1} = 0.$

Sol. $\lim\limits_{n\to\infty} \frac{n+1}{n^2+3n+1} = \lim\limits_{n\to\infty} \frac{n\left(1+\frac{1}{n}\right)}{n^2\left(1+\frac{3}{n}+\frac{1}{n^2}\right)} = \lim\limits_{n\to\infty} \frac{1}{n} \frac{\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)}{\lim\limits_{n\to\infty}\left(1+\frac{3}{n}+\frac{1}{n^2}\right)} = 0 \times \frac{1+0}{1+0+0} = 0.$

Alternatively, we can also apply the L'Hopital rule as there is ∞/∞ form in the limit. So

$$\lim_{n\to\infty} \frac{n+1}{n^2+3n+1} = \lim_{n\to\infty} \frac{1}{2n+3} = 0.$$

Definition of Sequence

Let $a : \mathbb{N} \to \mathbb{R}$ be a function. Then the expression

$$a(1), a(2), a(3), ..., a(n), ...$$

is called an infinite sequence of real numbers, where $a(n)$ is the nth or the general term of the sequence. For convenience, we denote the nth term by a_n, and the whole sequence by $\{a_n\}$. So we have

$$\{a_n\} = a_1, a_2, a_3, ..., a_n, ...$$

For example,

$$\{\sqrt{n}\} = \sqrt{1},\ \sqrt{2},\ \sqrt{3},\ ....,\ \sqrt{n}, ; ....$$

$$\left\{\frac{1}{n}\right\} = 1,\ \frac{1}{2},\ \frac{1}{3},\ ....,\ \frac{1}{n},\ ......$$

$$\left\{(-1)^{n+1}\frac{1}{n}\right\} = 1,\ -\frac{1}{2},\ \frac{1}{3},\ -\frac{1}{4},\ ....,\ (-1)^{n+1}\frac{1}{n},\ ....$$

all are real sequences. Sequences can be represented graphically as well in two ways (i) representing all the members a_1, a_2, ... of the sequence on real axis (ii) representing the points $(1, a_1)$, $(2, a_2)$,...... in the xy-plane.

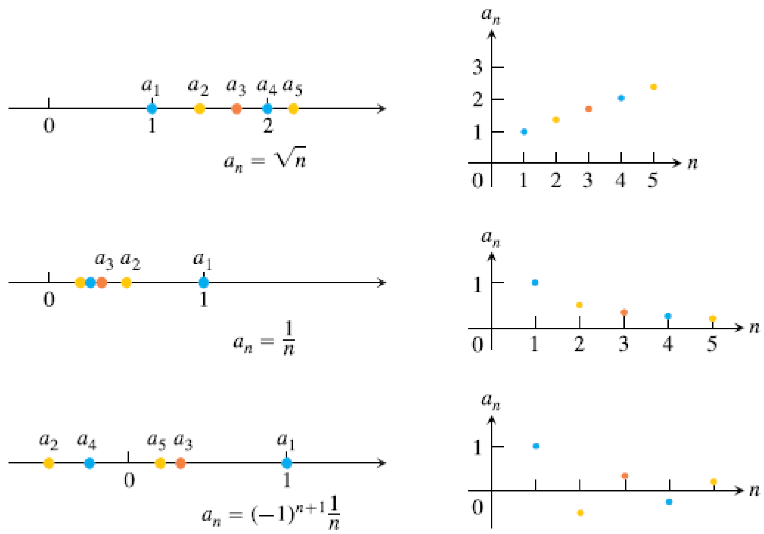

Figure 1: Graphical representation of the sequences $\{\sqrt{n}\}$, $\{\frac{1}{n}\}$ and $\{(-1)^{n+1}\frac{1}{n}\}$.

Convergent and divergent sequences

Let $\{a_n\}$ be a sequence such that $\lim_{n\to\infty} a_n = L$, a finite number. Then we say that the sequence $\{a_n\}$ converges to L (also known as limit of the sequence). Thus, a sequence with finite limit is called a convergent sequence. For example, $\{\frac{1}{n}\}$ converges to 0.

If limit of $\{a_n\}$ does not exist or is infinite, then $\{a_n\}$ is called as a divergent sequence. For example, $\{n^2\}$ is a divergent sequence since $\lim_{n\to\infty} n^2 = +\infty$.

The sequence $\{(-1)^n\}$ oscillates finitely between the limits -1 and 1 since

$$\lim_{n\to\infty}(-1)^{2n} = 1 \quad \text{and} \quad \lim_{n\to\infty}(-1)^{2n+1} = -1\ .$$

So it is divergent.

Ex. Show that $\left\{\dfrac{\ln n}{n}\right\}$ converges to 0.

Sol. To test the behavior of the given sequence, we find the limit $\lim_{n\to\infty}\dfrac{\ln n}{n}$. We see that the limit function $\dfrac{\ln n}{n}$ gets ∞/∞ form in the limit $n \to \infty$. So by L'Hopital rule,
$\lim_{n\to\infty}\dfrac{\ln n}{n} = \lim_{n\to\infty}\dfrac{1/n}{1} = 0$. Hence the given sequence converges to 0.

Ex. Show that $\left\{n^{\frac{1}{n}}\right\}$ converges to 1.

Sol. Let $y = \lim_{n\to\infty} n^{\frac{1}{n}}$. Then $\ln y = \lim_{n\to\infty} \frac{1}{n} \ln n = 0$, as done in the previous example. It follows that $y = e^0 = 1$.

Ex. $\left\{x^{\frac{1}{n}}\right\}$ converges to 1 for $x > 0$ since $\lim_{n\to\infty} x^{\frac{1}{n}} = 1$.

Ex. $\{x^n\}$ converges to 0 for $|x| < 1$ since $\lim_{n\to\infty} x^n = 0$ for $|x| < 1$.

Ex. $\left\{\left(1+\frac{x}{n}\right)^n\right\}$ converges to e^x for any x since $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$ for any x. (Try to prove the limit using L'Hopital rule.)

Ex. $\left\{\frac{x^n}{n!}\right\}$ converges to 0 (why?).

The limits obtained in the above examples are quite elementary, and can be used directly in the problems. So it is useful to memorize these limits.

Bounded and Unbounded Sequences

A sequence $\{a_n\}$ is said to be

(i) **bounded above**, if there exists a real number K such that $a_n \leq K$ for all n.

(ii) **bounded below**, if there exists a real number k such that $k \leq a_n$ for all n.

(iii) **bounded**, if there exists two real numbers k and K such that $k \leq a_n \leq K$ for all n.

(iv) **unbounded**, if it is not bounded above or bounded below.

For example, the sequence $\{-n\}$ is bounded above since $-n < 0$ for all $n \in \mathbb{N}$. The sequence $\{n^2\}$ is bounded below since $n^2 > 0$ for all $n \in \mathbb{N}$. The sequence $\{\frac{1}{n}\}$ is bounded since $0 < \frac{1}{n} \leq 1$ for all $n \in \mathbb{N}$. The sequences $\{-n\}$, $\{n^2\}$, $\{n(-1)^n\}$ etc. are unbounded sequences.

Behavior of Monotonic Sequences

A sequence is said to be monotonic if either it is non-increasing or non-decreasing. More precisely, a sequence $\{a_n\}$ is said to be monotonically increasing (nondecreasing) or monotonically decreasing (non-increasing) according as $a_n \leq a_{n+1}$ or $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$.

It is useful to remember that a monotonically increasing and bounded above sequence is always convergent. A monotonically decreasing and bounded below sequence is also convergent. But a monotonically increasing and unbounded above sequence diverges to ∞ while a monotonically decreasing and unbounded below sequence diverges to $-\infty$.

For example, the monotonically decreasing and bounded below sequence $\{\frac{1}{n}\}$ converges to 0 whereas the monotonically increasing and unbounded above sequence $\{n^2\}$ diverges to ∞.

Recursively defined sequences

In a recursively defined sequence, first finite number of terms are explicitly given while the remaining terms are obtained by a recursive formula. For example, consider the sequence $\{a_n\}$ given by $a_1 = 0$, $a_{n+1} = \sqrt{8 + 2a_n}$, $n \geq 1$. Then $a_2 = \sqrt{8 + 2a_1} = \sqrt{8}$, $a_3 = \sqrt{8 + 2a_2} = \sqrt{8 + 2\sqrt{8}}$, and so on. Suppose $\lim\limits_{n\to\infty} a_n = L$. Then in the limit $n \to \infty$, the recursive formula $a_{n+1} = \sqrt{8 + 2a_n}$ gives $L = \sqrt{8 + 2L}$ or $L^2 = 8 + 2L$. So we get $L = 4, -2$. We can discard the value $L = -2$ as the given sequence is of positive terms. Hence it converges to 4.

SECTION 10.2 (Infinite Series)

Let $\{a_n\}$ be any sequence. Then the expression

$$a_1 + a_2 + a_3 + ...$$

is called as an infinite series, and is denoted by $\displaystyle\sum_{n=1}^{\infty} a_n$. So we have

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + ...$$

In what follows, we shall use the notation $\sum a_n$, where it is understood that the summation runs over the set of natural numbers. The term a_n is called the nth or the general term of the series $\sum a_n$.

If $a_n > 0$ for all n, that is, all the terms of the series are positive, then $\sum a_n$ is the series of positive terms. For example,

$$\sum n^2 = 1^2 + 2^2 + 3^2 + ...,$$

is a series of positive terms.

Behaviour of Infinite Series

Let $\sum a_n$ be an infinite series. Then the sequence $\{S_n\}$, where $S_n = a_1 + a_2 + a_3 + ... + a_n$, is called sequence of partial sums of the series $\sum a_n$.

The series $\sum a_n$ is said to be convergent if $\{S_n\}$ is convergent. For example, consider the series $\sum \frac{1}{2^n}$. In this case,

$$S_n = \frac{1}{2} + \frac{1}{2^2} + ... + \frac{1}{2^n} = \frac{\frac{1}{2}(1 - \frac{1}{2^n})}{1 - \frac{1}{2}} = 1 - \frac{1}{2^n}.$$

$$\therefore \lim_{n\to\infty} S_n = 1.$$

So $\{S_n\}$ is convergent. Hence the series $\sum \frac{1}{2^n}$ is convergent.

The series $\sum a_n$ is said to be divergent if $\{S_n\}$ is divergent. For example, the series $\sum n$ is divergent since

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (1 + 2 + ... + n) = \lim_{n\to\infty} \frac{n(n+1)}{2} = \infty.$$

The series $\sum(-1)^n$ is divergent. For, $S_{2n} = 0$ and $S_{2n+1} = -1$. So $\lim_{n\to\infty} S_n$ does not exist.

Remarks:

1. The behaviour of an infinite series $\sum_{n=1}^{\infty} a_n$ depends on the behaviour of its sequence $S_n = a_1 + a_2 + a_3 + ... + a_n$ of partial sums since $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n$. Therefore, if $\lim_{n\to\infty} S_n = l$, then $\sum_{n=1}^{\infty} a_n = l$, and we say that the series $\sum_{n=1}^{\infty} a_n$ converges to l or sum of the series $\sum_{n=1}^{\infty} a_n$ is l.
2. The sequence of partial sums of a positive term series is always monotonically increasing. Therefore, a positive term series either converges or diverges to ∞.

Ex. Test the behavior of the series

(i) $\sum \frac{1}{n(n+1)}$ (Telescoping Series)

(ii) $\sum \frac{1}{\sqrt{n}+\sqrt{n+1}}$

Sol. (i) Converges to 1. (ii) Divergent.

Theorem: Let $\sum a_n$ and $\sum b_n$ be two series converging to l and l', respectively. If c is any constant, then prove that

(i) the series $\sum(a_n + b_n)$ converges to $l + l'$.

(ii) the series $\sum(a_n - b_n)$ converges to $l - l'$.

(iii) the series $\sum(ca_n)$ converges to cl.

Proof. Let S_n, S'_n and T_n be the nth partial sums of $\sum a_n$, $\sum b_n$, $\sum(a_n + b_n)$, respectively. Therefore,

$$S_n = a_1 + a_2 + ... + a_n,$$

$$S'_n = b_1 + b_2 + ... + b_n,$$

$$T_n = (a_1 + b_1) + (a_2 + b_2) + ... + (a_n + b_n).$$

$\therefore \quad T_n = S_n + S'_n$. Since $\sum a_n$ and $\sum b_n$ converge to l and l', so $\lim_{n\to\infty} S_n = l$ and $\lim_{n\to\infty} S'_n = l'$. It follows that $\lim_{n\to\infty} T_n = l + l'$. Hence, the series $\sum(a_n + b_n)$ converges to $l + l'$.

Similarly, you can easily prove (ii) and (iii). $\square$

Remarks:

1. The above theorem suggests that if any two convergent series are added or subtracted term by term, the resulting series is also convergent. Similarly, if each term of a convergent series is multiplied by a constant, the resulting series is convergent.

2. The sum of two non-convergent series may be convergent. For example, $\sum(-1)^n$ and $\sum(-1)^{n-1}$ are non-convergent series. But the series $\sum\left[(-1)^n+(-1)^{n-1}\right]$ converges to 0.

3. The sum of a convergent series and a divergent series is always divergent.

Theorem: Addition or omission of a finite number of terms in an infinite series does not alter its behavior.

Proof. To prove the theorem it is sufficient to show that the series

$a_1+a_2+.....+a_m+a_{m+1}+a_{m+2}+.....$

and

$a_{m+1}+a_{m+2}+.....$

have same behavior.

Let S_n and T_n denote the nth partial sums of the two series. Then,

$S_n=a_1+a_2+.....+a_n$

and

$T_n=a_{m+1}+a_{m+2}+.....+a_{m+n}$.

Obviously, $T_n=S_{n+m}-S_m$. Here S_m is a constant, being sum of finite number of terms. Therefore,

$$\lim_{n\to\infty}T_n=\lim_{n\to\infty}S_{n+m}-S_m=\lim_{n\to\infty}S_n-S_m.$$

This shows that $\lim_{n\to\infty}T_n$ is finite or infinite or it does not exist according as $\lim_{n\to\infty}S_n$ is finite or infinite or it does not exist. Therefore, the sequences $\{T_n\}$ and $\{S_n\}$ converge or diverge together. This completes the proof. □

Behavior of Geometric Series

The series $\sum ar^{n-1}=a+ar+ar^2+.....$ is a geometric series with common ratio r. In this case, we have

$$S_n=a+ar+ar^2+...+ar^{n-1}=\begin{cases}\frac{a(1-r^n)}{1-r} & \text{if } r\neq 1\\ na & \text{if } r=1\end{cases}$$

$$\therefore \quad \lim_{n\to\infty} S_n = \begin{cases} \frac{a}{1-r} & \text{if } -1 < r < 1 \\ \infty & \text{if } r \geq 1 \\ 0 \;\text{ or } -a & \text{if } r = -1 \\ -\infty \;\text{ or }\; \infty & \text{if } r < -1 \end{cases}$$

This shows that the geometric series $\sum ar^{n-1}$ is convergent for $|r| < 1$, divergent for $|r| \geq 1$.

Ex. Examine the convergence of the series

(i) $\sum \left(\frac{3}{4}\right)^{n-1}$

(ii) $\frac{3}{5} + \frac{4}{5^2} + \frac{3}{5^3} + \frac{4}{5^4} + ...$

Sol. (i) Convergent. (ii) Convergent to 19/24.

Necessary Condition for Convergence and the nth term test of divergence

If a positive term series $\sum a_n$ is convergent, then $\lim\limits_{n\to\infty} a_n = 0$.

Proof. Let S_n denote the nth partial sum of the series $\sum a_n$. Then, we have $S_n = a_1 + a_2 + ... + a_n$, $S_{n+1} = a_1 + a_2 + ... + a_n + a_{n+1}$.

$\therefore \quad a_n = S_{n+1} - S_n$.

Suppose the series $\sum a_n$ converges to l. Then, $\lim_{n\to\infty} S_n = l$. Therefore,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n+1} = l - l = 0.$$

□

Remark: If $\lim\limits_{n\to\infty} a_n = 0$,, then the series $\sum a_n$ need not be convergent. For example, consider the series $\sum \frac{1}{n}$. In this case, $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$. But the series $\sum \frac{1}{n}$ is divergent since it is the series $\sum \frac{1}{n^p}$ with $p = 1$.

On the other hand, if $\lim\limits_{n\to\infty} a_n \neq 0$, then the series $\sum a_n$ will be divergent. It is called the test of divergence. For example, the series $\sum n^2$ is divergent since $\lim\limits_{n\to\infty} n^2 \neq 0$.

SECTION 10.3 (The Integral Test)

Let f be a continuous, non-negative and a decreasing function of x, for all $x \geq 1$. If $a_n = f(n)$ for all $n \geq 1$, then series $\sum a_n$ and the sequence $\{I_n\}$, where $I_n = \int_1^n f(x)dx$, both converge or diverge together.

Proof. For a natural number n, we can choose a real number x such that

$$n + 1 \geq x > n.$$

Since f is a decreasing function of x, so

$$f(n+1) \leq f(x) < f(n).$$

$$\Rightarrow \quad \int_n^{n+1} f(n+1)dx \leq \int_n^{n+1} f(x)dx < \int_n^{n+1} f(n)dx.$$

$$\Rightarrow \quad f(n+1) \leq \int_n^{n+1} f(x)dx < f(n).$$

$$\Rightarrow \quad a_{n+1} \leq \int_n^{n+1} f(x)dx < a_n.$$

Substituting $n = 1, 2, 3, ..., n-1$ and adding the resulting inequalities, we get

$$a_2 + a_3 + ... + a_n \leq \int_1^n f(x)dx < a_1 + a_2 + ... + a_{n-1}.$$

$\Rightarrow \quad S_n - a_1 \leq I_n < S_n - a_n$, where $S_n = a_1 + a_2 + ... + a_n$.

Since $a_n \geq 0$, so $S_n - a_n \leq S_n$. Therefore,

$$S_n - a_1 \leq I_n < S_n. \tag{1}$$

If $\sum a_n$ is convergent, then limit of S_n is finite. So second part of the inequality (1) suggests that limit of I_n is finite. So $\{I_n\}$ is also convergent.

If $\{I_n\}$ is also convergent, that is, limit of I_n is finite, then from the first part of the inequality (1) we see that limit of S_n is finite, which in turn implies that $\sum a_n$ is convergent.

Thus, $\sum a_n$ and $\{I_n\}$ converge together. Similarly, again using the inequality (1), we can that $\sum a_n$ and $\{I_n\}$ diverge together. □

Note: In the integral test, the condition $x \geq 1$ can be replaced by $x \geq N$, where N is some positive integer, since addition or deletion of a finite number of terms in an infinite series does not change its behavior.

Ex. 0.0.1. Using integral test, discuss the behavior of the series

(i) $\sum \frac{1}{n^p} \quad (p > 0)$

(ii) $\sum \frac{1}{n(\log n)^p} \quad (p > 0)$

Sol. (i) Here $a_n = \frac{1}{n^p} = f(n)$ (say), where $p > 0$. Clearly $f(x) = \frac{1}{x^p}$ is a continuous, non-negative and a decreasing function of x, for all $x \geq 1$. So Cauchy Integral test is applicable. We have

$$I_n = \int_1^n f(x)dx = \int_1^n \frac{1}{x^p}dx = \left[\frac{x^{1-p}}{1-p}\right]_1^n = \frac{n^{1-p}}{1-p} - \frac{1}{1-p} \quad \text{for} \quad p \neq 1.$$

For $p = 1$, we get $I_n = \displaystyle\int_1^n f(x)dx = \int_1^n \frac{1}{x}dx = \ln n$.

We see that $I_n \to \frac{1}{p-1}$ for $p > 1$ while $I_n \to \infty$ for $p \leq 1$ as $n \to \infty$. Therefore, $\{I_n\}$ and hence the given series $\sum \frac{1}{n^p}$ converges for $p > 1$ and diverges for $p \leq 1$.

(ii) Converges for $p > 1$ and diverges for $p \leq 1$. (Please try yourself.)

Behavior of p-series

The series $\sum \frac{1}{n^p}$, where p is any real number, is known as p-series. For $p = 1$, it is called harmonic series. This p-series converges for $p > 1$ and diverges for $p \leq 1$.

We have already seen the behavior of p-series by the Cauchy Integral test that it converges for $p > 1$ and diverges for $0 < p < 1$. For $p \leq 0$, $\frac{1}{n^p}$ does not tend to 0 as $n \to \infty$. So by test of divergence, the series $\sum \frac{1}{n^p}$ diverges for $p \leq 0$.

SECTION 10.4 (Comparison Tests)

(1) If $\sum a_n$ and $\sum b_n$ are two positive term series $a_n \leq b_n$ for all $n > m$ (m being some positive integer), then convergence of $\sum b_n$ implies the convergence of $\sum a_n$.

(2) If $\sum a_n$ and $\sum b_n$ are two positive term series $a_n \geq b_n$ for all $n > m$ (m being some positive integer), then divergence of $\sum b_n$ implies the divergence of $\sum a_n$.

(3) Let $\sum a_n$ and $\sum b_n$ be two positive term series such that $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = l$.

(i) If $l > 0$, then $\sum a_n$ and $\sum b_n$ converge or diverge together.

(ii) If $l = 0$ and $\sum b_n$ converges, then $\sum a_n$ also converges.

(iii) If $l = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ also diverges.

Ex. Examine the convergence of the series

(i) $\sum \frac{1}{n!}$

(ii) $\sum \frac{1}{\log n}$

(iii) $\sum \frac{n+1}{n(2n-1)}$

(iv) $\sum(\sqrt{n^4+1} - \sqrt{n^4-1})$

(v) $\frac{1}{2^2+1} + \frac{\sqrt{2}}{3^2+1} + \frac{\sqrt{3}}{4^2+1} + ...$

(vi) $\sum \frac{1}{n} \sin\left(\frac{1}{n}\right)$

(vii) $\sum \frac{x^{n-1}}{1+x^n}, \quad (x > 0)$

Sol. (i) Convergent $(n! > 2^{n-1})$

(ii) Divergent $(\log n < n \ \ \forall\, n \geq 2)$

(iii) Convergent $(b_n = \frac{1}{n})$ (iv) Convergent $(b_n = \frac{1}{n^2})$

(v) Convergent $(a_n = \frac{\sqrt{n}}{(n+1)^2+1},\ b_n = \frac{1}{n^{3/2}})$

(vi) Convergent $(b_n = \frac{1}{n^2})$ (vi) Convergent for $0 < x < 1$, $(b_n = x^n)$ and diverges for $x \geq 1$ ($a_n = 1/2$ and $1/x \nrightarrow 0$).

SECTION 10.5 (The Ratio and Root Tests)

Ratio Test

A positive term series $\sum a_n$ is convergent if $\lim\limits_{n\to\infty} \dfrac{a_n}{a_{n+1}} > 1$ and divergent if $\lim\limits_{n\to\infty} \dfrac{a_n}{a_{n+1}} < 1$.

Note that the ratio test fails in the case $\lim\limits_{n\to\infty} \dfrac{a_n}{a_{n+1}} = 1$. For example, consider the series $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$. For the first series,

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{n+1}{n} = 1.$$

For the second series

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{(n+1)^2}{n^2} = 1.$$

But the series $\sum \frac{1}{n}$ being a p-series with $p = 1$, is divergent, whereas the series $\sum \frac{1}{n^2}$ being a p-series with $p = 2$, is divergent.

Ex. Examine the convergence of the series

(i) $\sum \frac{n!}{n^n}$

(ii) $\frac{1}{2} + \frac{1.3}{2.5} + \frac{1.3.5}{2.5.8} + ...$

(iii) $1 + \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + ...$

Sol. (i) Convergent ($e > 1$)

(ii) Convergent ($a_n = \frac{1.3.5...(2n-1)}{2.5.8...(3n-1)}$, $3/2 > 1$)

(iii) Convergent for $x^2 < 1$ and div. for $x^2 \geq 1$ ($a_n = \frac{x^{2n}}{2n}$)

Root Test

If a positive term series $\sum a_n$ is convergent if $\lim\limits_{n\to\infty} a_n^{\frac{1}{n}} < 1$ and divergent if $\lim\limits_{n\to\infty} a_n^{\frac{1}{n}} > 1$.

Note that the root test is inconclusive in the case $\lim_{n\to\infty} a_n^{\frac{1}{n}} = 1$. For example, consider the series $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$. For the first series,

$$\lim_{n\to\infty} a_n^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{n^{\frac{1}{n}}} = 1,$$

and for the second one

$$\lim_{n\to\infty} a_n^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{1}{n^{\frac{1}{n}}}\right)^2 = 1.$$

But the series $\sum \frac{1}{n^2}$ is convergent, whereas the series $\sum \frac{1}{n}$ is divergent.

Ex. Examine the convergence of the series

(i) $\sum \frac{n^{n^2}}{(n+1)^{n^2}}$

(ii) $\sum \left(1+\frac{1}{n}\right)^n x^n$

Sol. (i) Convergent $(1/e < 1)$

(ii) Convergent for $x < 1$ and divergent for $x \geq 1$

SECTION 10.6 (Alternating Series, Absolute and Conditional Convergence)

Alternating Series

A series of the form

$$\sum_{n=1}^{\infty}(-1)^{n-1}a_n = a_1 - a_2 + a_3 - a_4 + ... ,$$

where $a_n > 0$ for all n, is called an alternating series. Therefore, in an alternating series, positive and negative terms appear alternatively.

Leibniz's test, as given below, is used for testing the convergence of an alternating series.

Leibniz's Test

An alternating series $\sum(-1)^{n-1}a_n$ converges if $\{a_n\}$ monotonically decreases to 0 as $n \to \infty$.

Ex. Determine the behavior of the series

(i) $\sum \frac{(-1)^{n-1}n}{10n-1}$

(ii) $\frac{\log 2}{2^2} - \frac{\log 3}{3^2} + \frac{\log 4}{4^2} - ....$

Sol. (i) Divergent (ii) Convergent ($f(x) = \log x/x^2$ decreases if $x > e^{1/2}$.)

Remark. Consider the alternating series

$$\frac{1}{3} - \frac{1}{2} + \frac{1}{3^2} - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{2^3} + ... + \frac{1}{3^n} - \frac{1}{2^n} + ...$$

Comparing it with $\sum(-1)^{n-1}a_n$, we have

$$\{a_n\} = \frac{1}{3}, \frac{1}{2}, \frac{1}{3^2}, \frac{1}{2^2}, \frac{1}{3^3}, \frac{1}{2^3}, ...$$

Clearly, $\{a_n\}$ does not decrease monotonically, though $a_n \to 0$ as $n \to \infty$. So we can not apply Leibniz's test. However, the given series is convergent being the difference of the two convergent series $\sum \frac{1}{3^n}$ and $\sum \frac{1}{2^n}$. Therefore, the monotonically decreasing condition in the Leibniz's test is sufficient but not necessary.

Absolute and Conditional Convergence

A series $\sum a_n$ is said to be absolutely convergent if $\sum |a_n|$ is convergent.

For example, the series $\sum(-1)^{n-1}\frac{1}{n^2}$ is absolutely convergent since $\sum \left|(-1)^{n-1}\frac{1}{n^2}\right| = \sum \frac{1}{n^2}$ is convergent.

If $\sum a_n$ is convergent but $\sum |a_n|$ is not convergent, then the series $\sum a_n$ is said to be conditionally convergent or non-absolutely convergent or semi-convergent.

For example, the series $\sum(-1)^{n-1}\frac{1}{n}$ is conditional convergent since $\sum \left|(-1)^{n-1}\frac{1}{n}\right| = \sum \frac{1}{n}$ is divergent but $\sum(-1)^{n-1}\frac{1}{n}$ is convergent.

Theorem: (The absolute convergence test) Every absolutely convergent series is convergent.

Ex. Discuss the absolute convergence of the series

(i) $1 + \frac{x}{1!} + \frac{x^2}{2!} + ...$

(ii) $x - \frac{x^3}{3} + \frac{x^5}{5} - ....$

(iii) $\sum(-1)^{n-1}\sin\frac{1}{n}$

(iv) $\sum \frac{(-1)^n(x+1)^n}{2^n n^2}$

Sol. (i) Absolutely convergent (ii) Convergent if $-1 \leq x \leq 1$ and absolutely convergent if $-1 < x < 1$ (iii) Conditionally convergent (iv) Absolutely convergent if $-3 \leq x \leq 1$.

Note: If a series is absolutely convergent, then the rearrangement of its terms does not change its behavior.

Tips for applying different tests

Here, we give some useful tips for applying different tests of convergence.

- If nth term of a series of positive terms does not tend to 0, then the series is divergent.
- If nth term of a series of positive terms tends to 0, apply ratio test to test its behavior. If nth term contains exponents involving n, the root test would be suitable. In case, ratio or root test do not work, try comparison tests or something else.
- To test the convergence of an alternating series, first test the absolute convergence because absolute convergence implies convergence. In case it is not absolutely convergent, the Leibniz's test is suggested for testing the convergence.

SECTION 10.7 (Power Series)

An infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + ... \tag{2}$$

is called a power series in $x - x_0$.

The power series (2) is said to converge at a point x if $\lim_{m\to\infty} \sum_{n=0}^{m} a_n(x-x_0)^n$ exists finitely, and the sum of the series is defined as the value of the limit. Obviously the power series (2) converges at $x = x_0$, and in this case its sum is a_0. The following theorem gives a criterion for the convergence and divergence of the power series.

Theorem: If power series (2) converges at a point $x = c \neq x_0$, then it converges absolutely for all x with $|x - x_0| < |c - x_0|$. If the power series (2) diverges at a point $x = d \neq x_0$, then it diverges for all x with $|x - x_0| > |d - x_0|$.

From the above theorem, it is obvious that there exists some largest positive real number R such that the power series (2) converges for all x with $|x - x_0| < R$. We call R as the **radius of convergence** of the power series, and $(x_0 - R, x_0 + R)$ is called the **interval of convergence**. If the power series converges only for $x = x_0$, then $R = 0$. If the power series converges for every real value of x, then $R = \infty$.

We can derive a formula for R by using ratio test. For, by ratio test the power series (2) converges if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| |x - x_0| < 1$, that is, if $|x - x_0| < R$ where $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$.

Similarly, by Cauchy's root test the power series (2) converges if $\lim_{n\to\infty} |a_n|^{1/n} |x - x_0| < 1$, that is, if $|x - x_0| < R$ where $R = \lim_{n\to\infty} |a_n|^{-1/n}$.

Ex. $\sum_{n=0}^{\infty} x^n$ ($R = 1$. So the power series converges for $-1 < x < 1$.

Ex. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ ($R = \infty$. So the power series converges for all x.)

Ex. $\sum_{n=0}^{\infty} n!x^n$ ($R = 0$. So the power series converges only for $x = 0$.)

Now suppose that the power series (2) converges to $f(x)$ for $|x - x_0| < R$, that is,

$$f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + ... \tag{3}$$

Then $f(x)$ is continuous and has derivatives of all orders for $|x - x_0| < R$. Also the series can be differentiated termwise in the sense that

$$f'(x) = \sum_{n=1}^{\infty} na_n(x-x_0)^{n-1} = a_1 + 2a_2(x-x_0) + 3a_3(x-x_0)^2 + ...,$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2} = 2a_2 + 3.2a_3(x-x_0) + ...,$$

and so on, and each of the resulting series converges for $|x - x_0| < R$.

Next, the power series (3) can be integrated termwise provided the limits of integration lie inside the interval of convergence.

If we have another power series $\sum_{n=0}^{\infty} b_n(x-x_0)^n$ converging to $g(x)$ for $|x - x_0| < R$, that is,

$$g(x) = \sum_{n=0}^{\infty} b_n(x-x_0)^n = b_0 + b_1(x-x_0) + b_2(x-x_0)^2 + b_3(x-x_0)^3 + ..., \quad (4)$$

then (3) and (4) can be added or subtracted termwise, that is,

$$f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n)(x-x_0)^n = (a_0 \pm b_0) + (a_1 \pm b_1)(x-x_0) + (a_2 \pm b_2)(x-x_0)^2 + ...$$

The two series can be multiplied also in the sense that

$$\begin{aligned} f(x)g(x) &= \sum_{n=0}^{\infty} (a_0 b_n + a_1 b_{n-1} + ... + a_n b_0)(x-x_0)^n \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0)(x-x_0) + (a_0 b_2 + a_1 b_1 + a_2 b_0)(x-x_0)^2 + ... \end{aligned}$$

Ex. Find the power series of $f'(x)$, where $f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + ..., (-1 < x < 1)$.

Sol. We see that $-1 < x < 1$ is the interval of convergence of the given power series. So it can be differentiated termwise to obtain

$$f'(x) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + ... \quad (-1 < x < 1)$$

SECTION 10.8 (Taylor and Maclaurin Series)

Suppose the power series

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + ...$$

converges to $f(x)$ for $|x-x_0| < R$, that is,

$$f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + ...$$

Then $f(x)$ is continuous and has derivatives of all orders for $|x-x_0| < R$. Also the series can be differentiated termwise in the sense that

$$f'(x) = \sum_{n=1}^{\infty} na_n(x-x_0)^{n-1} = a_1 + 2a_2(x-x_0) + 3a_3(x-x_0)^2 + ...,$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2} = 2a_2 + 3.2a_3(x-x_0) + ...,$$

and so on, and each of the resulting series converges for $|x-x_0| < R$. The successive differentiated series suggest that $a_n = f^{(n)}(x_0)/n!$.

The power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + ...$$

is called **Taylor series** generated by f at x_0. For $x_0 = 0$, we call it **Maclaurin series**.

The expression

$$P_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + ... + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n.$$

is defined as **Taylor polynomial** of order n generated by f at $x = a$.

Ex. Find the Taylor series and Taylor polynomials generated by e^x at $x = 0$.

Sol. Let $f(x) = e^x$. Then $f(0) = 1$, $f'(0) = 1$, $f''(0) = 1$, ...

So Taylor series generated by e^x at $x = 0$ is given by

$$1 + x + \frac{x^2}{2!} + ...$$

The Taylor polynomial of order n generated by e^x at $x = 0$ is

$$P_n(x) = 1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!}.$$

The information in the following note is not a part of your Mathematics-I course but is very useful for the further understanding of power series. So I suggest you to go through it.

Note: If $f(x)$ possesses derivatives up to $(n+1)th$ order in $|x-x_0|<R$, then we can prove the Taylor's formula given by

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+........+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+R_n,$$

where $R_n=\dfrac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$, ξ is some number between x_0 and x. Obviously the power series $\sum_{n=0}^{\infty}\dfrac{f^{(n)}(x_0)}{n!}(x-x_0)^n$ converges to $f(x)$ for those values of $x\in(x_0-R,x_0+R)$ for which $R_n\to 0$ as $n\to\infty$. Thus for a given function $f(x)$, the Taylor's formula enables us to find the power series that converges to $f(x)$. On the other hand, if a convergent power series is given, then it is not always possible to find/recognize its sum function. In fact, very few power series have sums that are elementary functions.

If the power series $\sum_{n=0}^{\infty}\dfrac{f^{(n)}(x_0)}{n!}(x-x_0)^n$ converges to $f(x)$ for all values of x in some neighbourhood of x_0 (open interval containing x_0), then $f(x)$ is said to be analytic at x_0 and the power series is, of course, the Taylor series of $f(x)$ about x_0.