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MATH F112 (Mathematics-II)

Complex Analysis

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Lecture 36-38
Series Expansion & Singularities

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Taylor's Series

Let $f(z)$ is analytic throughout a disk $|z - z_0| < R_0$ centered at z_0 and with radius R_0. Then $f(z)$ has the power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad (|z - z_0| < R_0)$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!} \quad (n = 0, 1, 2.....)$$

Sir Brook Taylor 1685-1731

Maclaurin's Series

Taylor Series about the point $z_0 = 0$ is called Maclaurin series, i. e.

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n, \quad (|z| < R_0)$$

Colin Maclaurin 1698-1746

Maclaurin's Series

Examples:

1. $$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad (|z| < \infty)$$

2. $$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \quad (|z| < \infty)$$

3. $$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \quad (|z| < \infty)$$

Maclaurin's Series

4. $$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!},$$

$$(|z| < \infty)$$

5. $$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!},$$

$$(|z| < \infty)$$

Maclaurin's Series

6. $$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \qquad (|z| < 1)$$

7. $$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n, \quad (|z| < 1)$$

Laurent's Theorem

Suppose that a function $f(z)$ is analytic throughout an annular domain $R_1 < |z - z_0| < R_2$ centered at z_0 and let C denote any positively oriented simple closed contour around z_0 and lying in that domain.

Pierre Alphonse Laurent 1813-1854

Laurent's Theorem

Then, at each point in the domain, $f(z)$ has the series representation

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}$$

$$\left(R_1<|z-z_0|<R_2\right)$$

where

Laurent's Theorem

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z - z_0)^{n+1}} \quad (n = 0, 1, 2, \ldots)$$

and

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z - z_0)^{-n+1}} \quad (n = 1, 2, \ldots)$$

Laurent's Theorem

Ex. Find the Laurent series representation of $f(z) = \dfrac{z}{(z-1)(z-3)}$ when

$$(a)\quad D_1 : 0 < |z| < 1,$$

$$(b)\quad D_2 : 1 < |z| < 3,$$

$$(c)\quad D_3 : 3 < |z| < \infty,$$

Laurent's Theorem

We have

$$f(z) = \frac{z}{(z-1)(z-3)}$$

$$= -\frac{1}{2(z-1)} + \frac{3}{2(z-3)}$$

Laurent's Theorem

(a) Consider the domain

$$D_1 : 0 < |z| < 1.$$

Then $f(z)$ is analytic in D_1.

$$f(z) = -\frac{1}{2(z-1)} + \frac{3}{2(z-3)}$$

Laurent's Theorem

$$= \frac{1}{2(1-z)} - \frac{3}{2\times 3\left(1-\dfrac{z}{3}\right)}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} z^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n$$

Laurent's Theorem

$$\Rightarrow f(z) = \frac{1}{2}\sum_{n=0}^{\infty} z^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(1 - \frac{1}{3^n}\right)z^n$$

Laurent's Theorem

(b) Consider the domain

$$D_2 : 1 < |z| < 3.$$

Then $f(z)$ is analytic in D_2.

$$f(z) = -\frac{1}{2(z-1)} + \frac{3}{2(z-3)}$$

Laurent's Theorem

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$$= -\frac{1}{2z\left(1-\frac{1}{z}\right)} - \frac{3}{2\times 3\left(1-\frac{z}{3}\right)}$$

$$= -\frac{1}{2z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n$$

$$\Rightarrow f(z) = -\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{z^{n+1}} - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n$$

Laurent's Theorem

(c) Consider the domain

$$D_3 : 3 < |z| < \infty$$

Then $f(z)$ is analytic in D_3.

Note that

$$\frac{1}{|z|} < \frac{3}{|z|} < 1.$$

Laurent's Theorem

$$f(z) = -\frac{1}{2(z-1)} + \frac{3}{2(z-3)}$$

$$= -\frac{1}{2z\left(1-\dfrac{1}{z}\right)} + \frac{3}{2\times z\left(1-\dfrac{3}{z}\right)}$$

$$= -\frac{1}{2z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^{n} + \frac{3}{2z}\sum_{n=0}^{\infty}\left(\frac{3}{z}\right)^{n}$$

Laurent's Theorem

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$$\Rightarrow f(z) = -\frac{1}{2z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n + \frac{3}{2z}\sum_{n=0}^{\infty}\left(\frac{3}{z}\right)^n$$

$$= -\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{z^{n+1}} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{3^{n+1}}{z^{n+1}}$$

$$= -\frac{1}{2}\sum_{n=0}^{\infty}\left(1-3^{n+1}\right)\frac{1}{z^{n+1}}$$

Laurent's Theorem

Exercise. Show that, when $0 < |z-1| < 2$,

the Laurent series representation

of $f(z) = \dfrac{z}{(z-1)(z-3)}$ is:

$$f(z) = -3\sum_{n=0}^{\infty} \frac{(z-1)^n}{2^{n+2}} - \frac{1}{2(z-1)}.$$

Singularity

(1) Singular Point of a function $f(z)$

(i) If a function $f(z)$ fails to be analytic at a point z_0, but it is analytic at some point in every nbd of z_0, then z_0 is called Singular Point of $f(z)$.

Singularity

(ii) Isolated Singularity

The point z_0 is called an isolated singularity of $f(z)$ if

(a) z_0 is a singular point of $f(z)$

(b) $f(z)$ is analytic in a deleted nbd

$$N: 0 < \left| z - z_0 \right| < \in .$$

Singularity

(2) (i) Let z_0 is an isolated singularity of $f(z)$

$\Rightarrow \exists\, R > 0$ such that $f(z)$ is analytic in $0 < |z - z_0| < R$.

Singularity

Hence $f(z)$ has Laurent series expansion:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n},$$

$$0 < |z - z_0| < R$$

Singularity

where $a_n = \dfrac{1}{2\pi i}\displaystyle\int_C \dfrac{f(z)dz}{(z-z_0)^{n+1}},$

$$b_n = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^{-n+1}},$$

C is any positively oriented simple closed contour around z_0 and lying in the punctured disc $0 < |z\text{-}z_0| < R.$

Singularity

$(ii)\ \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$ is called principal part (PP) of the Laurent series, i.e.

$$\text{PP} = \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$$

$$= \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \ldots$$

Singularity

If $b_k \neq 0$, for some k, say $k = m$,

and $b_n = 0 \quad \forall n > m$, then

$$\text{PP} = \frac{b_1}{(z - z_0)} + \frac{b_2}{\left(z - z_0\right)^2} + \ldots + \frac{b_m}{\left(z - z_0\right)^m}$$

Singularity

Then the singularity $z = z_0$ of $f(z)$ is called POLE OF ORDER m.

If $m = 1$, then z_0 is a pole of order 1 and is called a SIMPLE POLE.

Singularity

(iii) If an analytic function $f(z)$ has a singularity other than a pole, then this singularity is known as ESSENTIALSINGULARITY of $f(z)$, i.e.

if $b_n \neq 0$ for infinitely many n, then the singularity z_0 is called ESSENTIAL SINGULARITY of $f(z)$.

Singularity

(iv) If $b_n = 0 \quad \forall n,$

then the singularity z_0 is called REMOVABLE SINGULARITY of $f(z)$.

Singularity

(3) Consider a function $f(z)$ &

let $z = \dfrac{1}{w}$. Then

$$f(z) = f\left(\frac{1}{w}\right) = g(w)$$

Singularity

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(i) $f(z)$ is said to be analytic at infinity if $g(w)$ is analytic at $w=0.$

(ii) $f(z)$ is said to be singular at infinity if $g(w)$ is singular at $w=0.$

Singularity

(4) Zero of an analytic function :

Let $f(z)$ is analytic in a domain D.

If $f(z_0) = 0$ for some $z = z_0$, then $z = z_0$ is called zero of $f(z)$.

Singularity

If $f(z_0) = f'(z_0) = f''(z_0) = \ldots.$

$$= f^{(n-1)}(z_0) = 0, \text{ but}$$

$f^{(n)}(z_0) \neq 0,$ then

$z = z_0$ is called

ZERO OF ORDER n of $f(z)$.

Singularity

$i.e.\quad z = z_0$ is called zero of order n of $f(z)$ if

$$f(z) = (z - z_0)^n g(z),$$

where $g(z_0) \neq 0.$

Residue

The PP of the Laurent series is given by

$$PP = \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}, \quad \text{where}$$

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z - z_0)^{-n+1}}$$

Residue

If $n = 1$, then

$$b_1 = \frac{1}{2\pi i} \int_C f(z)\,dz$$

is called RESIDUE of $f(z)$

at $z = z_0$ and we write

Residue

$$b_1 = \operatorname*{Res}_{z=z_0} f(z)$$

$$= \frac{1}{2\pi i} \int_C f(z)\,dz$$

$$= \text{coeff of } \frac{1}{z-z_0}$$

Residue Theorem

Let C be a positively oriented simple closed contour. Suppose that $f(z)$ is analytic within and on C except for a finite number of singular points $z_k\,(k=1,2,\ldots,n)$ inside C. Then

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n}\left(\operatorname*{Res}_{z=z_k} f(z)\right)$$

Residue

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Ex1: Let $f(z) = \dfrac{\sin z}{z^4}, \quad 0 < |z| < \infty.$

Now $f(z) = \dfrac{1}{z^4}(\sin z)$

$$= \frac{1}{z^4}\left(z - \frac{z^3}{(3)!} + \frac{z^5}{(5)!} - \frac{z^7}{(7)!} + \ldots. \right)$$

Residue

$$f(z) = \frac{1}{z^3} - \frac{1}{(3)!}.\frac{1}{z} + \frac{1}{(5)!}.z - \frac{1}{(7)!}z^3 + \ldots$$

$$0 < |z| < \infty$$

$$\text{PP} = -\frac{1}{(3)!}.\frac{1}{z} - \frac{1}{z^3}$$

Note that $z = 0$ is a pole of order ???

Residue

Hence

$$\operatorname*{Res}_{z=0} f(z) = b_1 = \text{coeff of } \frac{1}{z} = -\frac{1}{6}$$

$$\therefore \int_{C:|z|=1} \frac{\sin z}{z^4} dz = 2\pi i \operatorname*{Res}_{z=0} f(z) = -\frac{\pi i}{3}$$

Residue

Ex 2. Find the residue of

$f(z) = \exp(1/z)$, and hence

evaluate

$$\int_C f(z)dz, \quad C : |z| = 1.$$

Residue

Soln:

$$f(z) = \exp\left(\frac{1}{z}\right)$$

$$= 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \ldots..$$

Residue

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Note : $z = 0$ is an essential singularity of $f(z)$.

$$\Rightarrow b_1 = \text{coeff of } \frac{1}{z} = \underset{z=0}{\mathrm{Res}}\, f(z) = 1$$

Hence $\int_C f(z)dz = 2\pi i$

Residue

Ex 3. Find the residue of

$$f(z) = \exp(1/z^2), \quad \text{and}$$

hence evaluate

$$\int_C f(z)dz, \quad C: |z| = 1.$$

Residue

1. $z = 0$ is an essential singularity of $f(z)$.
2. $b_1 = \operatorname*{Res}_{z=0} f(z) = 0.$
3. $I = 0.$

Residue

We have

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$$

Case-IA: Let $z = z_0$ is a simple pole of $f(z)$.

Then $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \dfrac{b_1}{z - z_0}$

Residue

$$\Rightarrow (z-z_0)\,f(z)$$

$$= b_1 + (z-z_0)\sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$$\Rightarrow \lim_{z\to z_0} (z-z_0)\ f(z) = b_1 = \operatorname*{Res}_{z=z_0}\ f(z)$$

Residue

Case-IB: Let $f(z)$ has a simple pole at $z = z_0$ and $f(z)$ is of the form

$$f(z) = \frac{p(z)}{q(z)},$$

where

(i) $p(z)$ & $q(z)$ are analytic at $z = z_0$,

Residue

$(ii)\ \ p(z_0) \neq 0,$ and

$(iii)\ q(z)$ has a simple zero at $z = z_0,$

Then

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z \to z_0} (z - z_0)\frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

Residue

Case-II: Let z_0 be a pole of order $m > 1$ for the function $f(z)$.

Then $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$

$$+\frac{b_1}{(z-z_0)}+\frac{b_2}{(z-z_0)^2}+\dots+\frac{b_m}{(z-z_0)^m}$$

Residue

$$\Rightarrow (z - z_0)^m \; f(z)$$

$$= (z - z_0)^m \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

$$+ b_1 (z - z_0)^{m-1} + b_2 (z - z_0)^{m-2}$$

$$+ \ldots\ldots + b_{m-1} (z - z_0) + b_m$$

Residue

Let $\varphi(z) = (z - z_0)^m f(z)$, then

$$\operatorname*{Res}_{z=z_0} f(z) = b_1$$

$= \text{coefficient of } (z - z_0)^{m-1}$ in the expansion of $\varphi(z)$

$$= \frac{\varphi^{(m-1)}(z_0)}{(m-1)!} \quad \text{by Taylor's Theorem}$$

Residue

Thus if z_0 is a pole of order $m>1$ of $f(z)$, then

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{\varphi^{(m-1)}(z_0)}{(m-1)!}$$

$$= \frac{1}{(m-1)!} \lim_{z\to z_0} \left[\varphi^{(m-1)}(z)\right]$$

Residue

$$\operatorname*{Res}_{z=z_0} f(z)$$

$$= \frac{1}{(m-1)!} \lim_{z \to z_0} \left[\frac{d^{m-1}}{dz^{m-1}} (z - z_0)^m f(z) \right]$$

Residue

Ex1. Find the residue of $f(z)$ at $z=0$ and $z = -1$, where

$$f(z) = \frac{1}{z+z^2}.$$

Soln: Note that $z=0$ and $z=-1$ are simple poles of $f(z)$.

Residue

$$\therefore \underset{z=0}{\operatorname{Res}} f(z) = \lim_{z\to 0} (z-0) f(z)$$

$$= \lim_{z\to 0} \left(\frac{1}{1+z} \right) = 1$$

$$\& \underset{z=-1}{\operatorname{Res}} f(z) = \lim_{z\to -1} (z+1) f(z)$$

$$= \lim_{z\to -1} \left(\frac{1}{z} \right) = -1.$$

Residue

Q.2 (a) p. 239 : Evaluate $I = \int\limits_{C:|z|=3} \frac{e^{-z}}{z^2}\, dz.$

Soln : Clearly, $z = 0$ is a pole of order 2

of $f(z) = \frac{e^{-z}}{z^2}.$

Residue

Now

$$I = \int_{C:|z|=3} f(z)dz$$

$$= 2\pi i \sum_{z=z_k} \operatorname{Res} f(z),$$

$$f(z) = \frac{e^{-z}}{z^2}$$

Residue

$$\therefore \operatorname*{Res}_{z=0} f(z) = \frac{1}{(2-1)!} \lim_{z\to 0}\left[\frac{d}{dz}\left(z^2 f(z)\right)\right]$$

$$= \lim_{z\to 0}\left[\frac{d}{dz} e^{-z}\right]$$

Residue

$$\Rightarrow \operatorname*{Res}_{z=0} f(z) = \lim_{z\to 0}\left(-e^{-z}\right) = -1$$

$$\therefore I = -2\pi i$$

Residue

Ex.2 Evaluate

$$I = \int_{C:|z-3|=1} \frac{e^{-z}}{z^2}\, dz.$$

Ans: $I = 0$ (WHY ???)

Residue

Q.2(b) p. 239: Evaluate $I = \int\limits_{c:|z|=3} \dfrac{e^{-z}}{(z-1)^2}.$

Soln: $z=1$ is pole of order 2 of

$$f(z) = \frac{e^{-z}}{(z-1)^2}.$$

Residue

$$\therefore \operatorname*{Res}_{z=1} f(z) = \frac{d}{dz}\left(e^{-z}\right)\Big|_{z=1}$$

$$= -e^{-z}\Big|_{z=1} = -\frac{1}{e}$$

$$\therefore I = -\frac{2\pi i}{e}$$

Residue

Q2(c)p.239: $I = \int_{|z|=3} z^2 . e^{\frac{1}{z}} \, dz$

Let $f(z) = z^2 e^{\frac{1}{z}}$

$\Rightarrow z = 0$ is an essential singularity of $f(z)$

Residue

$$f(z) = z^2\left(1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \frac{1}{4!z^4} + \ldots\right)$$

$$= z^2 + z + \frac{1}{2!} + \frac{1}{3!z} + \frac{1}{4!z^2} + \ldots.$$

Residue

$$\therefore \underset{z=0}{\operatorname{Res}} f(z) = \text{coeff. of } \frac{1}{z} = \frac{1}{6}$$

$$\therefore I = 2\pi i \times \frac{1}{6} = \frac{\pi i}{3}$$

Residue

Q.2(d)p.239: $I = \int_{|z|=3} \frac{z+1}{z^2 - 2z}\, dz$

Let $f(z) = \frac{z+1}{z^2 - 2z} = \frac{z+1}{z(z-2)}$

$\Rightarrow z = 0 \;\; \& \, z = 2$ are simple poles

Residue

$$\operatorname*{Res}_{z=0} f(z) = \lim_{z\to 0} z \; f(z)$$

$$= \lim_{z\to 0} \frac{z+1}{z-2} = -\frac{1}{2}$$

$$\operatorname*{Res}_{z=2} f(z) = \lim_{z\to 2} (z-2) f(z) = \frac{3}{2}$$

Residue

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$$\therefore I = 2\pi i \sum \operatorname{Res} f(z)$$

$$= 2\pi i\left(-\frac{1}{2}+\frac{3}{2}\right) = 2\pi i.$$

Residue

Q.3,p.243: Let $f(z)$ be analytic at z_0,

and consider $g(z) = \dfrac{f(z)}{z - z_0}$.

Then Show that :

Residue

(a) If $f(z_0) \neq 0$,

then z_0 is a simple pole

of $g(z)$ and

$$\operatorname*{Res}_{z=z_0} g(z) = f(z_0)$$

Residue

(b) If $f(z_0) = 0$,

then z_0 is a removable singularity of $g(z)$

and $\operatorname*{Res}_{z=z_0} g(z) = 0.$

Residue

Sol: $\because f(z)$ is analytic at z_0

$\Rightarrow f(z)$ has Taylor's series expansion about z_0, &

Residue

$$f(z) = f(z_0) + (z - z_0) f'(z_0)$$
$$+ (z - z_0)^2 \frac{f''(z_0)}{2!}$$
$$+ (z - z_0)^3 \frac{f'''(z_0)}{3!} + \ldots$$

Residue

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$$\Rightarrow g(z) = \frac{f(z)}{z - z_0} = \frac{f(z_0)}{z - z_0} + f'(z_0)$$

$$+ (z - z_0)\frac{f''(z_0)}{2!}$$

$$+ (z - z_0)^2 \frac{f'''(z_0)}{3!} + \ldots$$

Residue

(a) Clearly if $f(z_0) \neq 0$, Then principal part (P.P) of

$$g(z) \text{ is } = \frac{f(z_0)}{z - z_0}$$

Residue

$\therefore z_0$ is a simple pole of $g(z)$ and

$$\operatorname*{Res}_{z=z_0} g(z) = b_1 = \text{coeff of } \frac{1}{z-z_0}$$

$$= f(z_0)$$

Residue

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(b)If $f(z_0) = 0$, then

PP of $g(z) \quad = 0$

$\Rightarrow b_n = 0 \; \forall \, n$

$\Rightarrow z = z_0$ is a removable

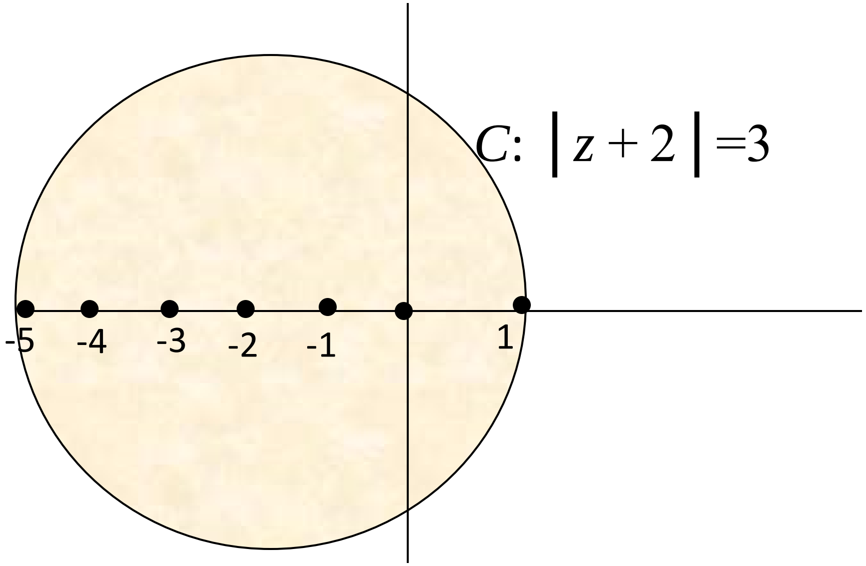
singularity of $g(z)$, and

$$\operatorname*{Res}_{z=z_0} g(z) = 0$$

Residue

Q.4(b),p.248: $I=\oint_{c}\frac{dz}{z^{3}(z+4)}, c:|z+2|=3$

Let $f(z)=\frac{1}{z^{3}(z+4)}$



Residue

$\Rightarrow z = 0$ is a pole of order 3

and $z = -4$ is a simple pole

& both lie inside C.

$$\therefore \operatorname*{Res}_{z=0} f(z) = \frac{1}{2} \cdot \frac{d^2}{dz^2}\left[\frac{1}{z+4}\right]_{z=0} = \frac{1}{4^3}$$

Residue

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$$\operatorname*{Res}_{z=-4} f(z) = \frac{1}{z^3}\bigg|_{z=-4} = -\frac{1}{4^3}$$

$$\therefore I = 2\pi i\left(\frac{1}{4^3} - \frac{1}{4^3}\right) = 0$$

Residue

Q.3(a), p.248 : $C : |z-2| = 2$

$$I = \int_C \frac{3z^3+2}{(z-1)(z^2+9)}\,dz,$$

Let $f(z) = \dfrac{3z^3+2}{(z-1)(z^2+9)}$

Then $1,\ 3i, -3i$ are simple poles of $f(z)$

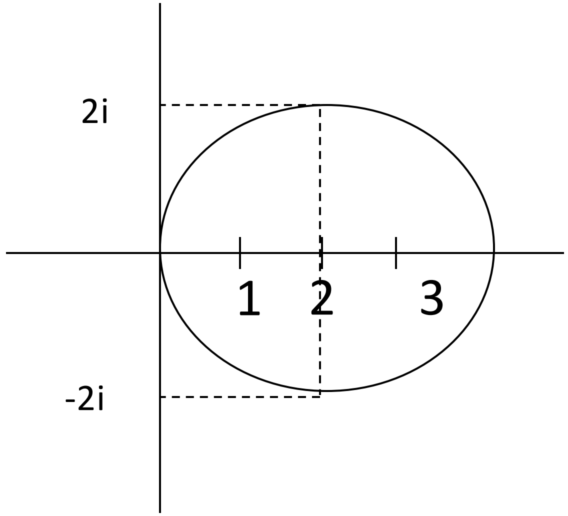
Residue

Note : $z=1$ is only inside C

$$\therefore \operatorname*{Res}_{z=1} f(z) = \left[\frac{3z^3+2}{z^2+9}\right]_{z=1}$$

$$=\frac{5}{10}=\frac{1}{2}$$

$$\therefore I = 2\pi i \times \operatorname*{Res}_{z=1} f\left(z\right) = \pi i$$

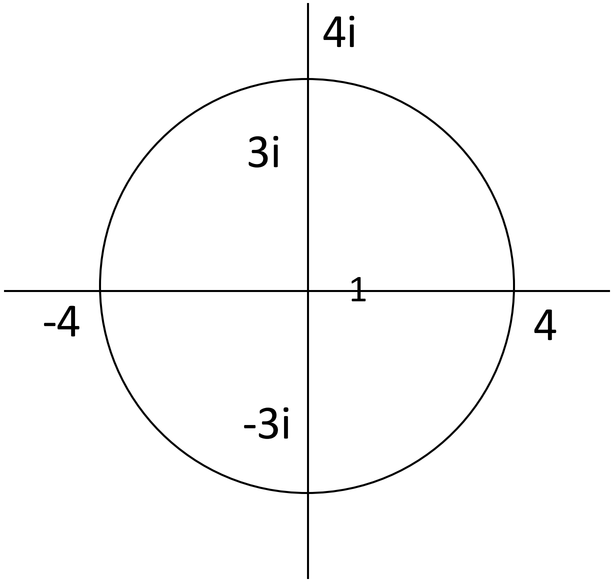


Residue

(b) $C: |z| = 4$

Then $1, 3i, -3i$ are all inside C

$$\therefore \operatorname*{Res}_{z=1} f(z) = \frac{1}{2}$$



Residue

$$\operatorname*{Res}_{z=3i} f(z) = \frac{3z^3+2}{(z-1)(z+3i)}\Bigg|_{z=3i}$$

$$= \frac{-81i+2}{(3i-1)(6i)} = \frac{2-81i}{-18-6i}$$

Residue

$$\operatorname*{Res}_{z=-3i} f(z) = \frac{3z^3+2}{(z-1)(z-3i)}\Bigg|_{z=-3i}$$

$$= \frac{+81i+2}{(-3i-1)(-6i)} = \frac{2+81i}{-18+6i}$$

Residue

$$\therefore \sum \operatorname{Res} f(z)$$

$$= \frac{1}{2} + \frac{2+81i}{6i-18} - \frac{2-81i}{6i+18}$$

$$= 3$$

$$\therefore I = 2\pi i \sum \operatorname{Res} f(z) = 6\pi i$$

Residue

Theorem: If a function f is analytic in the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C, then

$$\int_C f(z)dz = 2i\pi \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$$

Residue

In previous Q.3(b) $C: |z| = 4$

Then $1, 3i, -3i$ all singularities are inside C

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{\left(3+2z^2\right)}{z^2(1-z)\left(1+9z^2\right)}, \text{has double pole}$$

at $z = 0$

$$I = 2i\pi \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] = 2i\pi\left[\frac{d}{dz}\left(\frac{3+2z^2}{(1-z)\left(1+9z^2\right)}\right)\right]_{z=0}$$

$$= 2i\pi \times 3 = 6i\pi$$

THANK YOU
FOR YOUR PATIENCE !!!