MATHEMATICS-I (MATH F111)

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CHAPTER 10

Infinite Sequences and Series

Topics to be covered in Chapter 10

Topics to be covered in Chapter 10

- Sequences (Certain Theorems on Sequences)

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- Sequences (Certain Theorems on Sequences)
- Infinite Series

Topics to be covered in Chapter 10

- Sequences (Certain Theorems on Sequences)
- Infinite Series
- Integral Test

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- Sequences (Certain Theorems on Sequences)
- Infinite Series
- Integral Test
- Comparison Tests

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- Sequences (Certain Theorems on Sequences)
- Infinite Series
- Integral Test
- Comparison Tests
- Ratio and Root Tests

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- Sequences (Certain Theorems on Sequences)
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- Integral Test
- Comparison Tests
- Ratio and Root Tests
- Alternating Series

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- Sequences (Certain Theorems on Sequences)
- Infinite Series
- Integral Test
- Comparison Tests
- Ratio and Root Tests
- Alternating Series
- Power Series

Topics to be covered in Chapter 10

- Sequences (Certain Theorems on Sequences)
- Infinite Series
- Integral Test
- Comparison Tests
- Ratio and Root Tests
- Alternating Series
- Power Series
- Taylor & Maclaurin Series

Section 10.1

Sequences [Self Study]

Sequence

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Note. Sequence is a special case for Infinite series.

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- $\left\{\frac{1}{n}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\}$
- $\left\{(-1)^{n+1}\right\} = \{1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\}$

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- $\left\{(-1)^{n+1}\right\} = \{1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\}$
- $\left\{1 - \frac{1}{n}\right\} = \left\{0, \frac{1}{2}, \frac{2}{3}, \ldots, 1 - \frac{1}{n}, \ldots\right\}$

Convergence

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A sequence $\{a_n\}$ is said to converge to a number L, if for every positive number ε, however small, we can find a positive integer N (depending on ε) such that

$$|a_n - L| < \varepsilon, \ \forall n > N.$$

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Note. A sequence can not converges to more than one limit *i.e.*, the limit of a sequence is unique.

Examples: Converge or Diverge ?

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Evaluate $\lim \frac{5n+7}{3n-5}$

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Theorem (Theorem 1.)

Let $\{a_n\}$ and $\{b_n\}$ be two convergent sequences. Then

1. $\lim ka_n = k \lim a_n$
2. $\lim(a_n \pm b_n) = \lim a_n \pm \lim b_n$
3. $\lim(a_n . b_n) = \lim a_n . \lim b_n$
4. $\lim\left(\frac{a_n}{b_n}\right) = \frac{\lim a_n}{\lim b_n}$, *if* $b_n \neq 0$ *for all* n *and* $\lim b_n \neq 0$.

Find $\lim \frac{\cos n}{n}$ (If exists).

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Theorem (Theorem 2. Sandwich Theorem or Squeeze Theorem)

Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers.

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Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $a_n \leqslant b_n \leqslant c_n$ for all $n \geqslant N$ and if $\lim a_n = \lim c_n = L$, then $\lim b_n = L$.

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Example. Find $\lim \frac{\cos n}{n}$.

Sol. $\frac{-1}{n} \leqslant \frac{\cos n}{n} \leqslant \frac{1}{n}$.

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Example. Find $\lim \frac{\cos n}{n}$.

Sol. $\frac{-1}{n} \leqslant \frac{\cos n}{n} \leqslant \frac{1}{n}$. Now, $\lim \frac{1}{n} = \lim \frac{-1}{n} = 0$, hence, using Sandwich theorem we have $\lim \frac{\cos n}{n} = 0$.

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Theorem (Theorem 3. Continuous Function Theorem)

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Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ and $f(x)$ is a function that is continuous at $x = L$ and defined at all a_n,

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Theorem (Theorem 3. Continuous Function Theorem)

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Example

Find $\lim \cos \frac{1}{n}$ (If exists).

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Find $\lim \cos \frac{1}{n}$ (If exists).

Sol. Let $a_n = \frac{1}{n}$. Then $\lim a_n = \lim \frac{1}{n} = 0$. Since $\cos x$ is continuous at $x = 0$ and defined at $a_n = \frac{1}{n}$ for all n, therefore

$$\lim \cos \frac{1}{n} = \cos(0) = 1.$$

Q:. Find $\lim \frac{n+1}{e^n}$.

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Q:. Can we use L'Hôpital's rule ?

Theorem (Theorem 4.)

Suppose $f(x)$ is a function defined for all $x \geqslant n_0$ and that $\{a_n\}$ is a sequence of real numbers s.t. $a_n = f(n)$ for $n \geqslant n_0$, then

$$\lim_{x\to\infty} f(x) = L \Rightarrow \lim_{n\to\infty} f(n) = L.$$

Q:. Find $\lim \frac{n+1}{e^n}$.

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Sol.

$$\lim \frac{x+1}{e^x} = \lim \frac{1}{e^x}, \quad \text{by L'Hôpital's rule}$$
$$= 0.$$

Q:. Find $\lim \frac{n+1}{e^n}$.

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Sol.

$$\begin{aligned}\lim \frac{x+1}{e^x} &= \lim \frac{1}{e^x}, \quad \text{by L'H}\hat{o}\text{pital's rule}\\ &= 0.\end{aligned}$$

Hence $\lim \frac{n+1}{e^n} = 0$.

Q:. Find $\lim \cos(2\pi x)$? (does not exist)

Q:. Find $\lim \frac{n+1}{e^n}$.

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Conclusion?

Q:. Find $\lim \frac{n+1}{e^n}$.

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Hence $\lim \frac{n+1}{e^n} = 0$.

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Conclusion? Converse of Theorem 4 is not true.

Bounded Sequence

A sequence $\{a_n\}$ is said to be bounded from above if there exists a number M_1 such that

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The number M_1 is called an upper bound for $\{a_n\}$.

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Similarly, a sequence $\{a_n\}$ is said to be bounded from below if there exists a number M_2 such that

$$a_n \geqslant M_2, \ \forall n.$$

The number M_2 is called a lower bound for $\{a_n\}$.

Examples: Bounded or not ?

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- The sequence $\left\{\frac{1}{n}\right\}$ is bounded below by $\mathbf{0}$ and bounded above by $\mathbf{1}$.
- The sequence $\left\{\frac{1}{2^n}\right\}$ is bounded below by

Examples: Bounded or not ?

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Bounded Sequence

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Nondecreasing/Nonincreasing/Monotonic Sequence

A sequence $\{a_n\}$ is called a nondecreasing sequence, if $a_n \leqslant a_{n+1}$ for all n. The sequence is nonincreasing, if $a_n \geqslant a_{n+1}$ for all n. The sequence is monotonic if it is either nondecreasing or nonincreasing.

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Examples: Nonincreasing or Nondecreasing ?

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Theorem (Theorem 6. Monotonic Sequence Theorem)

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Example

The sequence $\left\{\frac{1}{n}\right\}$ is monotonic (nonincreasing) and bounded (as $|a_n| \leqslant 1$), therefore it is convergent.

Section 10.2

Infinite Series

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Example: $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$ (Harmonic Series)

First method: Sequence of Partial Sums

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The n^{th} term S_n of the sequence is called n^{th} partial sum of the series.

Convergent Series

A series $\sum a_n$ is said to be convergent iff the sequence $\{S_n\}$ is convergent.

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A series $\sum a_n$ is said to be convergent iff the sequence $\{S_n\}$ is convergent. If $\{S_n\}$ converges to a real number L, then the series $\sum a_n$ also converges to L and the sum of the series is L.

A series which is not convergent is called divergent.

Convergent Series

Q:. What is the effect of addition or deletion of a finite number of terms in a series on the convergence of a series.

Convergent Series

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Sol. It does not alter the behavior (convergence or divergence) of the series. However, the sum of the series will change in the case of convergent series.

Q:. $\sum \frac{1}{(n+4)(n+5)}$

Example: Telescoping series

Q:. Find a formula for the n^{th} partial sum of $\sum \frac{1}{(n+4)(n+5)}$ and use it to determine the convergence of the series. If the series converges, find its sum.

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Sol. Here $a_n = \frac{1}{(n+4)(n+5)} = \frac{1}{n+4} - \frac{1}{n+5}$. Now,

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Example: Telescoping series

Q:. Find a formula for the n^{th} partial sum of $\sum \frac{1}{(n+4)(n+5)}$ and use it to determine the convergence of the series. If the series converges, find its sum.

Sol. Here $a_n = \frac{1}{(n+4)(n+5)} = \frac{1}{n+4} - \frac{1}{n+5}$. Now,

$$S_n = \left(\frac{1}{5} - \frac{1}{6}\right) + \left(\frac{1}{6} - \frac{1}{7}\right) + \cdots + \left(\frac{1}{n+4} - \frac{1}{n+5}\right) = \frac{1}{5} - \frac{1}{n+5}.$$

Thus $\lim S_n = \frac{1}{5}$ and hence series converges and $\sum \frac{1}{(n+4)(n+5)} = \frac{1}{5}$.

Telescoping series

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This leads us to many more well-known series.

Second method: Comparison with well-known series

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1. The geometric series: $\sum ar^{n-1}$, $a \neq 0$ (e.g. $\sum 5\left(\frac{3}{2}\right)^{n-1}$)
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2. The p-series: $\sum \frac{1}{n^p}$, $p \in \mathbb{R}$ (e.g. harmonic series).

Convergence of Geometric series

Theorem

The geometric series $\sum ar^{n-1}$ *with* $a \neq 0$, *is*
(i) convergent if $|r| < 1$ *and* $\sum ar^{n-1} = \frac{a}{1-r}$.

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(i) convergent if $|r| < 1$ and $\sum ar^{n-1} = \frac{a}{1-r}$.
(ii) divergent if $|r| \geqslant 1$.

Proof. We know that

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}, \; r \neq 1.$$

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(i) convergent if $|r| < 1$ *and* $\sum ar^{n-1} = \frac{a}{1-r}$.
(ii) divergent if $|r| \geqslant 1$.

Proof. We know that

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}, \; r \neq 1.$$

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Therefore, GS is divergent in this case.

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$$S_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ a, & \text{if } n \text{ is odd.} \end{cases}$$

Since a is not zero, S_n does not tend to a unique limiting value and hence GS is divergent.

Example: Geometric series

Q:. $\sum_{n=0}^{\infty} \frac{\cos n\pi}{5^n}$.

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$$\implies \sum_{n=0}^{\infty} \frac{\cos n\pi}{5^n} = \frac{1}{1 - \frac{-1}{5}} = \frac{5}{6}.$$

Example: Geometric series

Q:. $\displaystyle\sum_{n=0}^{\infty} \frac{e^{n\pi}}{\pi^{ne}}.$

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Let $x_1 = \pi, x_2 = e$. Now,

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Method 3: The nth term test for divergence

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Sol. Here

$$\lim a_n = \lim \ln \frac{1}{3^n} = \lim(\ln 1 - \ln 3^n) = -\lim n \ln 3 = -\infty \neq 0,$$

therefore $\sum \ln \frac{1}{3^n}$ is divergent.

Theorem (Theorem 7. The necessary condition for a series to be convergent)

If $\sum a_n$ *converges, then* $\lim a_n = 0$.

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$$a_n = S_n - S_{n-1} \text{ for } n = 2, 3, 4, \ldots$$

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Proof. We have

$$a_1 = S_1$$
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If the given series converges then $S_n \to L$. Therefore, $\lim a_n = \lim(S_n - S_{n-1}) \to L - L = 0$. □

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If $\sum a_n$ *and* $\sum b_n$ *are two convergent series having sums* A *and* B *respectively, then*

- $\sum (a_n \pm b_n) = \sum a_n \pm \sum b_n = A \pm B.$
- $\sum k a_n = k \sum a_n = kA \quad$ *(for any real number* k*).*

For next few classes, series will be assumed to be series of nonnegative terms *i.e.*, $a_n \geqslant 0$ for all n. This means that $S_n \leqslant S_{n+1}$ for all n. That is the sequence $\{S_n\}$ is nondecreasing.

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Corollary (Corollary of Theorem 6)

A series $\sum a_n$ of nonnegative terms converges iff S_n is bounded from above.

Section 10.3

The Integral Test

Method 4: The Integral Test

Theorem (Theorem 9.)

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Let $\sum a_n$ be a series of positive terms. Let $f(x)$ be a positive, continuous, and decreasing function for all $x \geqslant N$ for some N; and let $f(n) = a_n$ for all n.

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Remark. In the case of convergence, the sum of the series and the value of integral will be different.

Q:. Check the convergence of the following series:
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Sol. Consider the function $f(x) = \frac{x}{x^2+1}$. Now,
$f'(x) = \frac{1-x^2}{(x^2+1)^2}$.

Q:. Check the convergence of the following series:
$\sum \frac{n}{n^2+1}$.
Sol. Consider the function $f(x) = \frac{x}{x^2+1}$. Now, $f'(x) = \frac{1-x^2}{(x^2+1)^2}$. Thus $f'(x) \leqslant 0$ when $x \geqslant 1 \implies f(x)$ is decreasing for all $x \geqslant 1$.

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$$\int_1^\infty \frac{x}{x^2+1}\,dx =$$

$$\frac{1}{2}\lim_{b\to\infty}[\ln(x^2+1)]_1^b \to \infty.$$

Hence by integral test the series is divergent.

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$$\int_1^\infty \frac{x}{e^{x^2}}\,dx = \frac{1}{2}\int_1^\infty \frac{1}{e^u}\,du \text{ (set } u = x^2\text{)}$$

$$= \frac{-1}{2}\lim_{b\to\infty}[e^{-u}]_1^b = \frac{1}{2e}$$

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$$= \frac{-1}{2} \lim_{b \to \infty} [e^{-u}]_1^b = \frac{1}{2e}$$

Hence by integral test the series $\sum ne^{-n^2}$ is convergent.

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Sol. The function $f(x) = \frac{1}{x \ln x}$ is positive, continuous and decreasing for all $x \geqslant 2$. Here the n^{th} term is $f(n) = \frac{1}{(n+1)\ln(n+1)}$. Now, we have

$$\int_2^{\infty} \frac{1}{(x+1)\ln(x+1)} dx \to \infty.$$

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Sol. The function $f(x) = \frac{1}{x \ln x}$ is positive, continuous and decreasing for all $x \geqslant 2$. Here the n^{th} term is $f(n) = \frac{1}{(n+1)\ln(n+1)}$. Now, we have

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(set $\ln(x+1) = t$).

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$$\int_2^{\infty} \frac{1}{(x+1)\ln(x+1)} dx \to \infty.$$

(set $\ln(x+1) = t$). Hence by integral test the series $\sum \frac{1}{n \ln n}$ is divergent.

Method 5: p-series test

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Proof. For $p \leqslant 0$, the series is trivially divergent (as $\lim a_n \neq 0$).

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Proof. For $p \leqslant 0$, the series is trivially divergent (as $\lim a_n \neq 0$).

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Proof. For $p \leqslant 0$, the series is trivially divergent (as $\lim a_n \neq 0$).

Let $p > 0$. Now for $p > 0$, the function $f(x) = \frac{1}{x^p}$ is positive, continuous and decreasing when $x \geqslant 1$ (verify yourself). Hence, we can apply integral test:

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$(b^{p-1} \to \infty$ as $b \to \infty$ because $p-1 > 0)$.

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($b^{p-1} \to \infty$ as $b \to \infty$ because $p - 1 > 0$). Thus, the series converges for $p > 1$.

- If $p > 1$,
$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^b$
$= \frac{1}{-p+1}\lim_{b\to\infty}\left(\frac{1}{b^{p-1}} - 1\right) = \frac{1}{1-p}(0-1) = \frac{1}{p-1}$
$(b^{p-1} \to \infty$ as $b \to \infty$ because $p - 1 > 0)$. Thus, the series converges for $p > 1$.
- If $0 < p < 1$, then
$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^b \to \infty.$
Hence, the series is divergent.

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- If $p > 1$,
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$(b^{p-1} \to \infty$ as $b \to \infty$ because $p-1 > 0)$. Thus, the series converges for $p > 1$.
- If $0 < p < 1$, then
$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^b \to \infty.$
Hence, the series is divergent.
- For $p = 1$, we have $\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}[\ln x]_1^b \to \infty.$
Hence, the series is divergent.

Q:. Which of the following series converge and which diverge?

- $\sum \frac{1}{n}$. (Harmonic series)

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- $\sum \frac{1}{n}$. (Harmonic series)
- $\sum \frac{1}{n^2}$.

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- $\sum \frac{1}{n}$. (Harmonic series)
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Ans. Divergent, Convergent.

Homework. For what $p > 0$, does the series $\sum_{2}^{\infty} \frac{1}{n(\ln n)^p}$ converge?

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Ans. $p > 1$.

Homework. Check the convergence of the following series:

- $\sum \frac{n^2}{e^{n/3}}$.

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Homework. Check the convergence of the following series:

- $\sum \frac{n^2}{e^{n/3}}$.
- $\sum \frac{5^n}{4^n+3}$.

Homework. For what $p > 0$, does the series $\sum_{2}^{\infty} \frac{1}{n(\ln n)^p}$ converge?

Ans. $p > 1$.

Homework. Check the convergence of the following series:

- $\sum \frac{n^2}{e^{n/3}}$.
- $\sum \frac{5^n}{4^n+3}$.
- $\sum \frac{\ln n}{\sqrt{n}}$.

Ans. Convergent, Divergent, Divergent.

Convergence of series: Examples

Q:. $\sum \ln \sqrt{n+1} - \ln \sqrt{n}$

Convergence of series: Examples

Q:. $\sum \ln\sqrt{n+1} - \ln\sqrt{n}$

Sol. Divergent

Convergence of series: Examples

Q:. $\sum \ln \sqrt{n+1} - \ln \sqrt{n}$

Sol. Divergent (telescoping series)

Convergence of series: Examples

Q:. $\sum \ln \sqrt{n+1} - \ln \sqrt{n}$

Sol. Divergent (telescoping series)

Q:. $\sum \left(\frac{1}{2^n} + \frac{(-1)^n}{5^n}\right)$

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Sol. Convergent (sum of geometric series)

Convergence of series: Examples

Q:. $\sum \ln \sqrt{n+1} - \ln \sqrt{n}$

Sol. Divergent (telescoping series)

Q:. $\sum \left(\frac{1}{2^n} + \frac{(-1)^n}{5^n} \right)$

Sol. Convergent (sum of geometric series)

Q:. $\sum \frac{n}{n+10}$

Convergence of series: Examples

Q:. $\sum \ln\sqrt{n+1} - \ln\sqrt{n}$

Sol. Divergent (telescoping series)

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Sol. Convergent (sum of geometric series)

Q:. $\sum \frac{n}{n+10}$

Sol. Divergent

Convergence of series: Examples

Q:. $\sum \ln \sqrt{n+1} - \ln \sqrt{n}$

Sol. Divergent (telescoping series)

Q:. $\sum \left(\frac{1}{2^n} + \frac{(-1)^n}{5^n} \right)$

Sol. Convergent (sum of geometric series)

Q:. $\sum \frac{n}{n+10}$

Sol. Divergent (n^{th} term test)

Q:. $\sum \frac{n^2}{e^{n/3}}$ (use Integral test)

Q:. $\sum \frac{n^2}{e^{n/3}}$ (use Integral test)
Sol. Convergent

Q:. $\sum \frac{n^2}{e^{n/3}}$ (use Integral test)

Sol. Convergent

$f'(x) = \frac{-x(x-6)}{3e^{x/3}} < 0$ for $x > 6$.

Q:. $\sum \frac{n^2}{e^{n/3}}$ (use Integral test)

Sol. Convergent

$f'(x) = \frac{-x(x-6)}{3e^{x/3}} < 0$ for $x > 6$.

$\int_7^\infty f(x)\,dx = \frac{327}{e^{7/3}} \implies$

Q:. $\sum \frac{n^2}{e^{n/3}}$ (use Integral test)

Sol. Convergent

$f'(x) = \frac{-x(x-6)}{3e^{x/3}} < 0$ for $x > 6$.

$\int_7^{\infty} f(x)\,dx = \frac{327}{e^{7/3}} \implies \sum_{n=7}^{\infty} \frac{n^2}{e^{n/3}}$ is convergent

Q:. $\sum \frac{n^2}{e^{n/3}}$ (use Integral test)

Sol. Convergent

$f'(x) = \frac{-x(x-6)}{3e^{x/3}} < 0$ for $x > 6$.

$\int_7^\infty f(x)\,dx = \frac{327}{e^{7/3}} \implies \sum_{n=7}^\infty \frac{n^2}{e^{n/3}}$ is convergent $\implies$ $\sum_{n=1}^\infty \frac{n^2}{e^{n/3}}$ is convergent.

Section 10.4

Comparison Tests

Method 6: The Direct Comparison Test

Theorem (Theorem 10.)

Let $\sum a_n$ be a series of nonnegative terms.

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Theorem (Theorem 10.)

Let $\sum a_n$ be a series of nonnegative terms.

(a) *If $a_n \leqslant c_n$ for all $n \geqslant N$ and $\sum c_n$ is convergent, then $\sum a_n$ will be convergent.*

Method 6: The Direct Comparison Test

Theorem (Theorem 10.)

Let $\sum a_n$ be a series of nonnegative terms.

(a) *If $a_n \leqslant c_n$ for all $n \geqslant N$ and $\sum c_n$ is convergent, then $\sum a_n$ will be convergent.*

(b) *If $a_n \geqslant d_n$ for all $n \geqslant N$ and $\sum d_n$ is a divergent series of nonnegative terms, then $\sum a_n$ will be divergent.*

Q:. Investigate the convergence or divergence of $\sum \frac{1}{n^3+5n}$.

Q:. Investigate the convergence or divergence of $\sum \frac{1}{n^3+5n}$.

Sol. Here, $a_n = \frac{1}{n^3+5n}$.

Q:. Investigate the convergence or divergence of $\sum \frac{1}{n^3+5n}$.

Sol. Here, $a_n = \frac{1}{n^3+5n}$. Now, $n^3 + 5n \geqslant n^3 \implies \frac{1}{n^3+5n} \leqslant \frac{1}{n^3}$ for all n.

Q:. Investigate the convergence or divergence of $\sum \frac{1}{n^3+5n}$.

Sol. Here, $a_n = \frac{1}{n^3+5n}$. Now, $n^3 + 5n \geqslant n^3 \implies \frac{1}{n^3+5n} \leqslant \frac{1}{n^3}$ for all n. Therefore, we have,

$$a_n \leqslant \frac{1}{n^3} = c_n \text{ (say) for all } n.$$

Q:. Investigate the convergence or divergence of $\sum \frac{1}{n^3+5n}$.

Sol. Here, $a_n = \frac{1}{n^3+5n}$. Now, $n^3 + 5n \geqslant n^3 \implies \frac{1}{n^3+5n} \leqslant \frac{1}{n^3}$ for all n. Therefore, we have,

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Moreover, from p-test, $\sum c_n$ is convergent.

Q:. Investigate the convergence or divergence of $\sum \frac{1}{n^3+5n}$.

Sol. Here, $a_n = \frac{1}{n^3+5n}$. Now, $n^3 + 5n \geqslant n^3 \implies \frac{1}{n^3+5n} \leqslant \frac{1}{n^3}$ for all n. Therefore, we have,

$$a_n \leqslant \frac{1}{n^3} = c_n \text{ (say) for all } n.$$

Moreover, from p-test, $\sum c_n$ is convergent. Therefore, by DCT, $\sum a_n$ is convergent.

Q:. $\sum \frac{5^n+1}{2^n-1}$

Q:. $\sum \frac{5^n+1}{2^n-1}$

Sol. Here $a_n = \frac{5^n+1}{2^n-1} \geqslant \frac{5^n}{2^n-1} \geqslant \frac{5^n}{2^n} = \left(\frac{5}{2}\right)^n$ for all n.

Q:. $\sum \frac{5^n+1}{2^n-1}$

Sol. Here $a_n = \frac{5^n+1}{2^n-1} \geqslant \frac{5^n}{2^n-1} \geqslant \frac{5^n}{2^n} = \left(\frac{5}{2}\right)^n$ for all n.

Since the series $\sum\left(\frac{5}{2}\right)^n$ is a divergent geometric series $\left(|r| = \frac{5}{2} > 1\right)$,

Q:. $\sum \frac{5^n+1}{2^n-1}$

Sol. Here $a_n = \frac{5^n+1}{2^n-1} \geqslant \frac{5^n}{2^n-1} \geqslant \frac{5^n}{2^n} = \left(\frac{5}{2}\right)^n$ for all $\boldsymbol{n}$.

Since the series $\sum \left(\frac{5}{2}\right)^n$ is a divergent geometric series $\left(|r| = \frac{5}{2} > 1\right)$, therefore by DCT, $\sum \frac{5^n+1}{2^n-1}$ diverges.

Q:. $\sum \frac{n-1}{n^4+2}$

Q:. $\sum \frac{n-1}{n^4+2}$

Sol. $n^4 \leqslant n^4+2 \implies \frac{1}{n^4} \geqslant \frac{1}{n^4+2} \implies \frac{n}{n^4} \geqslant \frac{n}{n^4+2} \implies \frac{1}{n^3} \geqslant \frac{n}{n^4+2} \geqslant \frac{n-1}{n^4+2}$.

Q:. $\sum \frac{n-1}{n^4+2}$

Sol. $n^4 \leqslant n^4+2 \implies \frac{1}{n^4} \geqslant \frac{1}{n^4+2} \implies \frac{n}{n^4} \geqslant \frac{n}{n^4+2} \implies \frac{1}{n^3} \geqslant \frac{n}{n^4+2} \geqslant \frac{n-1}{n^4+2}$. Let $c_n = \frac{1}{n^3}$.

Q:. $\sum \frac{n-1}{n^4+2}$

Sol. $n^4 \leqslant n^4+2 \implies \frac{1}{n^4} \geqslant \frac{1}{n^4+2} \implies \frac{n}{n^4} \geqslant \frac{n}{n^4+2} \implies \frac{1}{n^3} \geqslant \frac{n}{n^4+2} \geqslant \frac{n-1}{n^4+2}$. Let $c_n = \frac{1}{n^3}$.

From p-test, $\sum c_n$ is convergent $\implies \sum a_n$ is convergent.

Q:. $\sum \sqrt{\dfrac{n+4}{n^4+4}}$

Q:. $\sum \sqrt{\frac{n+4}{n^4+4}}$

Sol. We have $n^3 \leqslant n^4 \implies 4n^3 \leqslant 4n^4 \implies n^4 + 4n^3 \leqslant n^4 + 4n^4 = 5n^4 \implies n^4 + 4n^3 \leqslant 5n^4 + 20 = 5(n^4 + 4)$

Q:. $\sum \sqrt{\frac{n+4}{n^4+4}}$

Sol. We have $n^3 \leqslant n^4 \implies 4n^3 \leqslant 4n^4 \implies n^4 + 4n^3 \leqslant n^4 + 4n^4 = 5n^4 \implies n^4 + 4n^3 \leqslant 5n^4 + 20 = 5(n^4 + 4)$ $\implies \frac{n^4+4n^3}{n^4+4} \leqslant 5 \implies \frac{n+4}{n^4+4} \leqslant \frac{5}{n^3} \implies \sqrt{\frac{n+4}{n^4+4}} \leqslant \sqrt{\frac{5}{n^3}} \implies$

Q:. $\sum \sqrt{\frac{n+4}{n^4+4}}$

Sol. We have $n^3 \leqslant n^4 \implies 4n^3 \leqslant 4n^4 \implies n^4+4n^3 \leqslant n^4+4n^4 = 5n^4 \implies n^4+4n^3 \leqslant 5n^4+20 = 5(n^4+4)$ $\implies \frac{n^4+4n^3}{n^4+4} \leqslant 5 \implies \frac{n+4}{n^4+4} \leqslant \frac{5}{n^3} \implies \sqrt{\frac{n+4}{n^4+4}} \leqslant \sqrt{\frac{5}{n^3}} \implies$ $c_n = \frac{\sqrt{5}}{n^{3/2}}$.

Q:. $\sum \sqrt{\frac{n+4}{n^4+4}}$

Sol. We have $n^3 \leqslant n^4 \implies 4n^3 \leqslant 4n^4 \implies n^4 + 4n^3 \leqslant n^4 + 4n^4 = 5n^4 \implies n^4 + 4n^3 \leqslant 5n^4 + 20 = 5(n^4 + 4)$
$\implies \frac{n^4+4n^3}{n^4+4} \leqslant 5 \implies \frac{n+4}{n^4+4} \leqslant \frac{5}{n^3} \implies \sqrt{\frac{n+4}{n^4+4}} \leqslant \sqrt{\frac{5}{n^3}} \implies$
$c_n = \frac{\sqrt{5}}{n^{3/2}}$.

From p-test, $\sum c_n$ is convergent $\implies \sum a_n$ is convergent.

Q:. $\sum \frac{1}{n^3-5n}$

Q:. $\sum \frac{1}{n^3-5n}$

Sol. Here, $a_n = \frac{1}{n^3-5n}$. Now, $\frac{1}{n^3-5n} \geqslant \frac{1}{n^3} \geqslant 0$ for all $n \geqslant 3$.

But, from p-test, $\sum \frac{1}{n^3}$ is convergent.

Q:. $\sum \frac{1}{n^3-5n}$

Sol. Here, $a_n = \frac{1}{n^3-5n}$. Now, $\frac{1}{n^3-5n} \geqslant \frac{1}{n^3} \geqslant 0$ for all $n \geqslant 3$. But, from p-test, $\sum \frac{1}{n^3}$ is convergent. Therefore, by DCT, we can not conclude anything.

Q:. $\sum \frac{\sqrt{n}}{n^2+1}$
Sol. Convergent

Q:. $\sum \frac{n+2^n}{n^2 2^n}$
Sol. Convergent

Q:. $\sum \frac{1}{1+2+3+\cdots+n}$
Sol. Convergent

Method 7: The Limit Comparison Test

Theorem (Theorem 11.)

Suppose a_n *and* b_n *are positive for all* $n \geqslant N$*.*

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Theorem (Theorem 11.)

Suppose a_n and b_n are positive for all $n \geqslant N$.

(a) *If $\lim \frac{a_n}{b_n} = c$ (finite and non zero), then both $\sum a_n$ and $\sum b_n$ converge or diverge together.*

Method 7: The Limit Comparison Test

Theorem (Theorem 11.)

Suppose a_n *and* b_n *are positive for all* $n \geqslant N$.

(a) *If* $\lim \frac{a_n}{b_n} = c$ *(finite and non zero), then both* $\sum a_n$ *and* $\sum b_n$ *converge or diverge together.*

(b) *If* $\lim \frac{a_n}{b_n} = 0$, *and* $\sum b_n$ *converges, then* $\sum a_n$ *also converges.*

Method 7: The Limit Comparison Test

Theorem (Theorem 11.)

Suppose a_n and b_n are positive for all $n \geqslant N$.

- **(a)** *If* $\lim \frac{a_n}{b_n} = c$ *(finite and non zero), then both* $\sum a_n$ *and* $\sum b_n$ *converge or diverge together.*
- **(b)** *If* $\lim \frac{a_n}{b_n} = 0$, *and* $\sum b_n$ *converges, then* $\sum a_n$ *also converges.*
- **(c)** *If* $\lim \frac{a_n}{b_n} = \infty$, *and* $\sum b_n$ *diverges, then* $\sum a_n$ *also diverges.*

Q:. $\sum \frac{1}{n^3-5n}$

Q:. $\sum \frac{1}{n^3-5n}$

Sol. Here, $a_n = \frac{1}{n^3-5n}$. What could be b_n ?

Q:. $\sum \frac{1}{n^3-5n}$

Sol. Here, $a_n = \frac{1}{n^3-5n}$. What could be b_n ?

Let $b_n = \frac{1}{n^3}$.

Q:. $\sum \frac{1}{n^3-5n}$

Sol. Here, $a_n = \frac{1}{n^3-5n}$. What could be b_n ?

Let $b_n = \frac{1}{n^3}$.

Now, $\lim \frac{a_n}{b_n} = \lim \frac{n^3}{n^3-5n} = \lim \frac{1}{1-\frac{5}{n^2}} = 1 > 0.$

Q:. $\sum \frac{1}{n^3-5n}$

Sol. Here, $a_n = \frac{1}{n^3-5n}$. What could be b_n ?

Let $b_n = \frac{1}{n^3}$.

Now, $\lim \frac{a_n}{b_n} = \lim \frac{n^3}{n^3-5n} = \lim \frac{1}{1-\frac{5}{n^2}} = 1 > 0$.

From p-test, $\sum \frac{1}{n^3}$ is convergent. Therefore, by LCT, $\sum \frac{1}{n^3-5n}$ is convergent.

Q:. $\sum \frac{n+1}{n^p}$

Q:. $\sum \frac{n+1}{n^p}$

Sol. What could be b_n ?

Q:. $\sum \frac{n+1}{n^p}$

Sol. What could be b_n ?

Here, $a_n = \frac{n+1}{n^p} = \frac{1+\frac{1}{n}}{n^{p-1}}$.

Q:. $\sum \frac{n+1}{n^p}$

Sol. What could be b_n ?

Here, $a_n = \frac{n+1}{n^p} = \frac{1+\frac{1}{n}}{n^{p-1}}$. Let $b_n = \frac{1}{n^{p-1}}$.

Q:. $\sum \frac{n+1}{n^p}$

Sol. What could be b_n ?

Here, $a_n = \frac{n+1}{n^p} = \frac{1+\frac{1}{n}}{n^{p-1}}$. Let $b_n = \frac{1}{n^{p-1}}$.

Now, $\lim \frac{a_n}{b_n} = \lim 1 + \frac{1}{n} = 1 > 0$.

Q:. $\sum \frac{n+1}{n^p}$

Sol. What could be b_n ?

Here, $a_n = \frac{n+1}{n^p} = \frac{1+\frac{1}{n}}{n^{p-1}}$. Let $b_n = \frac{1}{n^{p-1}}$.

Now, $\lim \frac{a_n}{b_n} = \lim 1 + \frac{1}{n} = 1 > 0$.

From p-test, $\sum b_n$ is convergent if $p - 1 > 1$ and divergent if $p - 1 \leqslant 1$.

Q:. $\sum(\sqrt{n^4+1}-\sqrt{n^4-1})$

Q:. $\sum(\sqrt{n^4+1}-\sqrt{n^4-1})$

Sol. Here

$$a_n = \sqrt{n^4+1}-\sqrt{n^4-1}$$

$$= \sqrt{n^4+1}-\sqrt{n^4-1}.\frac{\sqrt{n^4+1}+\sqrt{n^4-1}}{\sqrt{n^4+1}+\sqrt{n^4-1}}$$

$$= \frac{2}{\sqrt{n^4+1}+\sqrt{n^4-1}}.$$

Q:. $\sum(\sqrt{n^4+1}-\sqrt{n^4-1})$

Sol. Here

$$\begin{aligned}
a_n &= \sqrt{n^4+1}-\sqrt{n^4-1} \\
&= \sqrt{n^4+1}-\sqrt{n^4-1}.\frac{\sqrt{n^4+1}+\sqrt{n^4-1}}{\sqrt{n^4+1}+\sqrt{n^4-1}} \\
&= \frac{2}{\sqrt{n^4+1}+\sqrt{n^4-1}}.
\end{aligned}$$

b_n ?

Consider $b_n = \frac{1}{n^2}$.

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Consider $b_n = \frac{1}{n^2}$. Note that $\sum b_n$ is convergent. Now

$$\lim \frac{a_n}{b_n} = \lim \frac{\frac{2}{\sqrt{n^4+1}+\sqrt{n^4-1}}}{\frac{1}{n^2}} = 1.$$

Consider $b_n = \frac{1}{n^2}$. Note that $\sum b_n$ is convergent. Now

$$\lim \frac{a_n}{b_n} = \lim \frac{\frac{2}{\sqrt{n^4+1}+\sqrt{n^4-1}}}{\frac{1}{n^2}} = 1.$$

Hence $\sum a_n$ is convergent, by LCT (a).

Q:. $\sum \tan \frac{1}{n}$

Q:. $\sum \tan \frac{1}{n}$

Sol. Let $b_n = \frac{1}{n} \implies \lim \frac{\tan \frac{1}{n}}{\frac{1}{n}} = \lim \frac{1}{\cos \frac{1}{n}} . \frac{\sin \frac{1}{n}}{\frac{1}{n}} =$

Q:. $\sum \tan \frac{1}{n}$

Sol. Let $b_n = \frac{1}{n} \implies \lim \frac{\tan \frac{1}{n}}{\frac{1}{n}} = \lim \frac{1}{\cos \frac{1}{n}} . \frac{\sin \frac{1}{n}}{\frac{1}{n}} =$ $\lim_{x \to 0} \frac{1}{\cos x} . \frac{\sin x}{x} = 1.1 = 1.$

Q:. $\sum \tan \frac{1}{n}$

Sol. Let $b_n = \frac{1}{n} \implies \lim \frac{\tan \frac{1}{n}}{\frac{1}{n}} = \lim \frac{1}{\cos \frac{1}{n}} . \frac{\sin \frac{1}{n}}{\frac{1}{n}} =$
$\lim_{x \to 0} \frac{1}{\cos x} . \frac{\sin x}{x} = 1.1 = 1.$
Now, $\sum b_n$ is a divergent p-series, hence, $\sum a_n$ is divergent, by LCT.

Q:. $\sum \frac{(1+n\ln n)}{n^2+5}$.

Q:. $\sum \frac{(1+n\ln n)}{n^2+5}$.

Sol. Here $a_n = \frac{(1+n\ln n)}{n^2+5}$.

Q:. $\sum \frac{(1+n\ln n)}{n^2+5}$.

Sol. Here $a_n = \frac{(1+n\ln n)}{n^2+5}$. For large n, a_n would behave like $\frac{n\ln n}{n^2} = \frac{\ln n}{n}$. Consider $b_n = \frac{1}{n}$.

Q:. $\sum \frac{(1+n\ln n)}{n^2+5}$.

Sol. Here $a_n = \frac{(1+n\ln n)}{n^2+5}$. For large n, a_n would behave like $\frac{n\ln n}{n^2} = \frac{\ln n}{n}$. Consider $b_n = \frac{1}{n}$. Note that $\sum b_n$ is divergent. Now

$$\lim \frac{a_n}{b_n} = \lim \frac{n^2 \ln n + n}{n^2+5} = \infty$$

Hence $\sum a_n$ is divergent, by LCT.

Q:. $\sum \frac{1}{\sqrt{n^3+2}}$.

Sol. Convergent.

Q:. $\sum \frac{2^n-n}{n2^n}$

Sol. Divergent

Q:. $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\ln n}$.

Sol. Divergent

Section 10.5

Ratio and Root Tests

Method 8: Ratio Test

Theorem (Theorem 12.)

Let $\sum a_n$ *be a series of positive terms. Suppose* $\lim \frac{a_{n+1}}{a_n} = r$.

Method 8: Ratio Test

Theorem (Theorem 12.)

Let $\sum a_n$ be a series of positive terms. Suppose $\lim \frac{a_{n+1}}{a_n} = r$.

(a) *If $r < 1$, the series $\sum a_n$ is convergent;*

Method 8: Ratio Test

Theorem (Theorem 12.)

Let $\sum a_n$ *be a series of positive terms. Suppose* $\lim \frac{a_{n+1}}{a_n} = r$.

(a) *If* $r < 1$, *the series* $\sum a_n$ *is convergent;*

(b) *If* $r > 1$, *the series* $\sum a_n$ *is divergent;*

Method 8: Ratio Test

Theorem (Theorem 12.)

Let $\sum a_n$ *be a series of positive terms. Suppose* $\lim \frac{a_{n+1}}{a_n} = r$.

(a) *If* $r < 1$, *the series* $\sum a_n$ *is convergent;*

(b) *If* $r > 1$, *the series* $\sum a_n$ *is divergent;*

(c) *If* $r = 1$, *the test is inconclusive.*

Q:. $\sum \frac{n}{2^n}$.

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Sol. Here $a_n = \frac{n}{2^n}$. So

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$$\frac{a_{n+1}}{a_n} = \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \frac{n+1}{2^{n+1}} . \frac{2^n}{n}$$

$$\lim \frac{a_{n+1}}{a_n} = \lim \frac{1}{2} . \frac{n+1}{n} = \frac{1}{2} < 1.$$

Q:. $\sum \frac{n}{2^n}$.

Sol. Here $a_n = \frac{n}{2^n}$. So

$$\frac{a_{n+1}}{a_n} = \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \frac{n+1}{2^{n+1}}.\frac{2^n}{n}$$

$$\lim \frac{a_{n+1}}{a_n} = \lim \frac{1}{2}.\frac{n+1}{n} = \frac{1}{2} < 1.$$

Therefore by ratio test, the series converges.

Q:. $\sum e^{-n} n^3$.

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Sol. Here $a_n = e^{-n} n^3$. So

Q:. $\sum e^{-n} n^3$.

Sol. Here $a_n = e^{-n} n^3$. So

$$\frac{a_{n+1}}{a_n} = \frac{e^{-(n+1)}(n+1)^3}{e^{-n} n^3} = e^{-1}\left(1 + \frac{1}{n}\right)^3.$$

Thus $\lim \frac{a_{n+1}}{a_n} = e^{-1} < 1$. Therefore by ratio test, the series converges.

Q:. $\sum \frac{1}{n^2}$.

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Q:. $\sum \frac{1}{n^2}$.

Sol. Here $a_n = \frac{1}{n^2}$. So

$$\frac{a_{n+1}}{a_n} = \frac{n^2}{(n+1)^2}$$

$$\lim \frac{a_{n+1}}{a_n} = 1.$$

Q:. $\sum \frac{1}{n^2}$.

Sol. Here $a_n = \frac{1}{n^2}$. So

$$\frac{a_{n+1}}{a_n} = \frac{n^2}{(n+1)^2}$$

$$\lim \frac{a_{n+1}}{a_n} = 1.$$

Therefore the ratio test yields no conclusion.

Q:. $\sum \frac{3^n}{n^3 2^n}$.

Q:. $\sum \frac{3^n}{n^3 2^n}$.

Sol. Divergent.

Q:. $\sum \frac{(2n+3)(2^n+3)}{3^n+2}$.

Sol. Convergent.

Method 9: Root Test or n^{th} Root Test

Theorem (Theorem 13.)

Let $\sum a_n$ be a series of non-negative terms, and suppose that $\lim(a_n)^{\frac{1}{n}} = r$.

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Theorem (Theorem 13.)

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- **(a)** *If* $r < 1$, *the series* $\sum a_n$ *converges;*
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Let $\sum a_n$ *be a series of non-negative terms, and suppose that* $\lim(a_n)^{\frac{1}{n}} = r$.

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(c) *If* $r = 1$, *the test is inconclusive.*

Q:. $\sum \left(\frac{2n+4}{5n-1}\right)^n$.

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Sol. Here

$$a_n = \left(\frac{2n+4}{5n-1}\right)^n$$
$$\Rightarrow (a_n)^{1/n} = \frac{2n+4}{5n-1}$$
$$\Rightarrow \lim (a_n)^{1/n} = \frac{2}{5} < 1.$$

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Therefore the series converges by root test.

Q:. $\sum \frac{(n!)^n}{(n^n)^2}$.

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Sol. Here

$$a_n = \frac{(n!)^n}{(n^n)^2}$$
$$(a_n)^{1/n} = \frac{n!}{n^2}$$
$$\lim (a_n)^{1/n} = \lim \frac{n!}{n^2} = \infty.$$

Therefore the series diverges by root test.

Section 10.6

Alternating Series, Absolute and Conditional Convergence

Alternating Series

A series whose terms are alternately positive and negative is called an alternating series.

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An alternating series is one of the form $\sum(-1)^{n+1}u_n$ or $\sum(-1)^n u_n$, where $u_n > 0$ for all n.

Examples: Alternating series

- $\sum(-1)^{n+1}$

Examples: Alternating series

- $\sum(-1)^{n+1}$
- $\sum(-1)^n \frac{1}{n \ln n}$

Examples: Alternating series

- $\sum(-1)^{n+1}$
- $\sum(-1)^n \frac{1}{n \ln n}$
- $\sum(-1)^n \ln\left(1 + \frac{1}{n}\right)$

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- $\sum(-1)^{n+1}$
- $\sum(-1)^{n}\frac{1}{n\ln n}$
- $\sum(-1)^{n}\ln\left(1+\frac{1}{n}\right)$
- $\sum\frac{\cos n\pi}{n^2+1}$

Theorem (Theorem 14. Leibniz's Theorem for Alternating Series)

The alternating series $\sum(-1)^{n+1}u_n$ *converges if*

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$$\textbf{\textit{(i)}}\ u_n \geqslant u_{n+1} \text{ for all } n \geqslant N \text{ for some } N; \text{ and}$$

Theorem (Theorem 14. Leibniz's Theorem for Alternating Series)

The alternating series $\sum(-1)^{n+1}u_n$ *converges if*

(i) $u_n \geqslant u_{n+1}$ *for all* $n \geqslant N$ *for some* N*; and*

(ii) $\lim u_n = 0$.

Q:. $\sum(-1)^{n+1}\frac{1}{(2n-1)!}$

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Sol. **(i)** Here $u_n = \frac{1}{(2n-1)!}$. It is easy to verify that $u_n \geqslant u_{n+1}$ for all $n \geqslant 1$.

Q:. $\sum(-1)^{n+1}\frac{1}{(2n-1)!}$

Sol. **(i)** Here $u_n = \frac{1}{(2n-1)!}$. It is easy to verify that $u_n \geqslant u_{n+1}$ for all $n \geqslant 1$.

(ii) Also, we have $\lim u_n = 0$.

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Hence by Leibniz's test the series is convergent.

Proof. To prove Leibniz Theorem, we will show that the sequence S_n is convergent, which is proved using the following result:

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Ex. 133 on p. 572: For a sequence $\{S_n\}$, if $\{S_{2n}\}$ and $\{S_{2n+1}\}$ converge to the same number L, then $S_n \to L$.

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First consider the sequence $\{S_{2n}\}$.

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Ex. 133 on p. 572: For a sequence $\{S_n\}$, if $\{S_{2n}\}$ and $\{S_{2n+1}\}$ converge to the same number L, then $S_n \to L$.

First consider the sequence $\{S_{2n}\}$. We'll show that S_{2n} converges (by showing that S_{2n} is non-decreasing and bounded from above). Why ?

S_{2n} is non-decreasing. We have

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$$S_{2n+2} = S_{2n} + (u_{2n+1} - u_{2n+2}).$$

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Therefore, using condition (ii):

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Thus $S_n \to L$ and hence $\sum(-1)^{n+1}u_n$ converges to L.

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If second condition of Leibniz's test fails, i.e., $\lim u_n \neq 0$, then the series would be divergent.

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Sol. Here $u_n = \frac{2^n}{n^2}$ and $\lim u_n = \infty \implies$ the series is divergent.

Remark 2

The Leibniz's test is sufficient but not necessary for the convergence of an alternating series. There are examples of alternating series for which condition (i) fails but the series converges. Thus, if the first condition of Leibniz's test fails, the result would be inconclusive. In this case, we can not say that the series is divergent as it may be convergent.

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Sol. Clearly, the series is not monotonically decreasing but still it is convergent. how? (difference of geometric series)

Q:. $\sum(-1)^{n+1}\dfrac{\sqrt{n}}{n+1}$.

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Sol. **(i)** Here $u_n = \frac{\sqrt{n}}{n+1}$. Is $u_n \geqslant u_{n+1}$?

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Consider

$$f(x) = \frac{\sqrt{x}}{x+1} \Rightarrow f'(x) = -\frac{x-1}{2\sqrt{x}(x+1)^2} < 0 \forall x > 1.$$

Hence, $f(x)$ is non-increasing for all $x > 1 \implies u_n \geqslant u_{n+1},\ \forall n > 1.$

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Hence by Leibniz's test the series is convergent.

Q:. $\sum(-1)^n \ln\left(1+\frac{1}{n}\right)$.

Sol. Convergent

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A series $\sum a_n$ is said to be

- absolutely convergent if the series $\sum |a_n|$ is convergent.

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A series $\sum a_n$ is said to be

- absolutely convergent if the series $\sum |a_n|$ is convergent.
- conditionally convergent if the series $\sum |a_n|$ is divergent but $\sum a_n$ is convergent.

Example: Absolute and Conditional Convergence

- $\sum(-1)^n \frac{1}{2^n}$

Example: Absolute and Conditional Convergence

- $\sum(-1)^n \frac{1}{2^n}$ (absolutely convergent)

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- $\sum(-1)^{n+1} \frac{3+n}{5+n}$ (divergent)

Theorem (Theorem 16. The Absolute Convergence Test)

If $\sum |a_n|$ converges, then $\sum a_n$ also converges. That is "absolute convergence implies convergence".

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Now the series $\sum 2|a_n|$ converges as the series $\sum |a_n|$ converges

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Therefore by Direct Comparison Test, the non-negative terms series $\sum(|a_n| + a_n)$ converges.

Now we can write

$$\sum a_n = \sum(|a_n| + a_n - |a_n|) = \sum(|a_n| + a_n) - \sum |a_n|.$$

Therefore $\sum a_n$, being a difference of two convergent series, converges. $\square$

Steps to check a series for absolute and conditional convergence

- First check the convergence of the series $\sum |a_n|$, i.e., if $\sum |a_n|$ is convergent then $\sum a_n$ is absolutely convergent.

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- First check the convergence of the series $\sum |a_n|$, i.e., if $\sum |a_n|$ is convergent then $\sum a_n$ is absolutely convergent.
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Steps to check a series for absolute and conditional convergence

- First check the convergence of the series $\sum |a_n|$, i.e., if $\sum |a_n|$ is convergent then $\sum a_n$ is absolutely convergent.
- If $\sum |a_n|$ is divergent but $\sum a_n$ is convergent then $\sum a_n$ is conditionally convergent.
- Otherwise $\sum a_n$ is divergent.

Q:. $\sum(-1)^{n+1}\frac{\sin nx}{n^3}$.

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Sol. Here $|a_n| = \frac{|\sin nx|}{n^3}$.

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Sol. Here $|a_n| = \frac{|\sin nx|}{n^3}$. Let $b_n = \frac{1}{n^3}$.

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Sol. Here $|a_n| = \dfrac{|\sin nx|}{n^3}$. Let $b_n = \dfrac{1}{n^3}$. Clearly, $\dfrac{|\sin nx|}{n^3} \leqslant \dfrac{1}{n^3}$ and $\sum \dfrac{1}{n^3}$ is convergent

Q:. $\sum(-1)^{n+1}\frac{\sin nx}{n^3}$.

Sol. Here $|a_n| = \frac{|\sin nx|}{n^3}$. Let $b_n = \frac{1}{n^3}$. Clearly, $\frac{|\sin nx|}{n^3} \leqslant \frac{1}{n^3}$ and $\sum \frac{1}{n^3}$ is convergent (from p-series test). Hence $\sum \frac{|\sin nx|}{n^3}$ is convergent.

Q:. $\sum(-1)^{n+1}\frac{\sin nx}{n^3}$.

Sol. Here $|a_n| = \frac{|\sin nx|}{n^3}$. Let $b_n = \frac{1}{n^3}$. Clearly, $\frac{|\sin nx|}{n^3} \leqslant \frac{1}{n^3}$ and $\sum\frac{1}{n^3}$ is convergent (from p-series test). Hence $\sum\frac{|\sin nx|}{n^3}$ is convergent. This implies that $\sum(-1)^{n+1}\frac{\sin nx}{n^3}$ is absolutely convergent and hence, convergent.

Q:. $\sum(-1)^{n+1}\frac{n}{n^2+1}$.

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Sol.

- First we check the behavior of $\sum|a_n| = \sum\frac{n}{n^2+1}$.

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Sol.

- First we check the behavior of $\sum|a_n| = \sum\frac{n}{n^2+1}$. Let $b_n = \frac{1}{n}$.
  Now $\lim\frac{|a_n|}{b_n} = \lim\frac{n^2}{n^2+1} = 1$.

Q:. $\sum(-1)^{n+1}\frac{n}{n^2+1}$.

Sol.

- First we check the behavior of $\sum|a_n| = \sum\frac{n}{n^2+1}$. Let $b_n = \frac{1}{n}$.
  Now $\lim\frac{|a_n|}{b_n} = \lim\frac{n^2}{n^2+1} = 1$.
  Since $\sum\frac{1}{n}$ is divergent so $\sum\frac{n}{n^2+1}$ is also divergent (by LCT).

- Now we check the behavior of $\sum a_n = \sum(-1)^{n+1}\frac{n}{n^2+1}$ by using Leibniz's test.

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Therefore by Leibniz's test $\sum(-1)^{n+1}\frac{n}{n^2+1}$ is convergent.

Hence $\sum(-1)^{n+1}\frac{n}{n^2+1}$ is conditionally convergent.

Q:. $\sum(-1)^{n+1}\frac{(2n)!}{2^n n! n}$

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Sol. $\lim \frac{(2n)!}{2^n n! n} = \lim \frac{(n+1)(n+2)\ldots(2n)}{2^n n} = \lim \frac{(n+1)(n+2)\ldots(n+(n-1))}{2^{n-1}}$

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By n^{th} term test, $\sum(-1)^{n+1}\frac{(2n)!}{2^n n! n}$ is divergent.

Q:. $\sum(-1)^n \frac{10^n}{(n+1)!}$

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Now, $\lim |\frac{a_{n+1}}{a_n}| = \lim \frac{10}{n+2} = 0 < 1$.

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Now, $\lim |\frac{a_{n+1}}{a_n}| = \lim \frac{10}{n+2} = 0 < 1$. By ratio test, $\sum(-1)^n \frac{10^n}{(n+1)!}$ is absolutely convergent and hence, convergent.

Q:. $\sum(-1)^n(\sqrt{n+1}-\sqrt{n})$

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Sol. Now,

$$\sqrt{n+1}-\sqrt{n}=(\sqrt{n+1}-\sqrt{n}).\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}=\frac{1}{\sqrt{n+1}+\sqrt{n}}.$$

Q:. $\sum(-1)^n(\sqrt{n+1}-\sqrt{n})$

Sol. Now,

$$\sqrt{n+1}-\sqrt{n}=(\sqrt{n+1}-\sqrt{n}).\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}=\frac{1}{\sqrt{n+1}+\sqrt{n}}.$$

Here $\sum|a_n|=\sum\frac{1}{\sqrt{n+1}+\sqrt{n}}$.

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Here $\sum|a_n|=\sum\frac{1}{\sqrt{n+1}+\sqrt{n}}$. Let $b_n=\frac{1}{\sqrt{n}}$.

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Here $\sum|a_n|=\sum\frac{1}{\sqrt{n+1}+\sqrt{n}}$. Let $b_n=\frac{1}{\sqrt{n}}$.

Now, $\lim\frac{|a_n|}{b_n}=\lim\frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}=\lim\frac{1}{\sqrt{1+\frac{1}{n}}+1}=\frac{1}{2}\neq 0.$

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Here, $\sum b_n$ is a divergent p-series.

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Here, $\sum b_n$ is a divergent p-series. Hence, $\sum|a_n|$ is divergent implies $\sum(-1)^n(\sqrt{n+1}-\sqrt{n})$ is not absolutely convergent.

Again, $\{\frac{1}{\sqrt{n+1}+\sqrt{n}}\}$ is a decreasing sequence of positive terms, i.e., $u_n \geqslant u_{n+1} \forall n$.

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Also, $\lim u_n \to 0$.

Hence, using Leibniz's test, $\sum(-1)^n(\sqrt{n+1}-\sqrt{n})$ is conditionally convergent.

Q:. $\sum \frac{\cos n\pi}{n\sqrt{n}}$.
Sol. Convergent.

Q:. $\sum (-1)^n \frac{\ln n}{n - \ln n}$.
Sol. Conditionally convergent.

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- If $\lim a_n \nrightarrow 0$, the series diverges.

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- For series with some negative terms (not necessarily alternating series) see whether $\sum |a_n|$ converges. If yes, so does $\sum a_n$.
- For alternating series apply Leibniz's test.

Section 10.7

Power Series

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Examples

- $\sum_{n=0}^{\infty} x^n$ (geometric series).

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- $\sum (-1)^{n+1} \frac{x^n}{n}$ (logarithmic series).

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For what values of x a power series converge.

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$\sum(x-2)^n$ it is a geometric series which converges whenever $|r| = |x-2| < 1$, i.e., for each x, $1 < x < 3$, the series converges to $\frac{1}{3-x}$.

Ratio Test (in general form)

Let $\sum a_n$ be any series and $\lim \left|\frac{a_{n+1}}{a_n}\right| = r$.

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(a) If $r < 1$, the series $\sum a_n$ is absolutely convergent;

(b) If $r > 1$, the series $\sum a_n$ is divergent;

(c) If $r = 1$, the test is inconclusive.

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Let $\sum a_n$ be any series and suppose that $\lim(|a_n|)^{\frac{1}{n}} = r$.

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Step-1. Testing for absolute convergence: Use Ratio (or Root) test to find an open interval where the series converges absolutely as follows:

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$$\begin{aligned}\lim\left|\frac{u_{n+1}}{u_n}\right| &= \lim\left|\frac{a_{n+1}}{a_n}(x-a)\right| \\ &= |x-a|\lim\left|\frac{a_{n+1}}{a_n}\right| \\ &= \frac{|x-a|}{R}.\end{aligned}$$

Testing a Power Series for Convergence

Now for convergence, we must have (ratio test)

$$\lim \left| \frac{u_{n+1}}{u_n} \right| < 1$$

$$\frac{|x-a|}{R} < 1$$

$$|x-a| < R$$

$$x \in (a-R, a+R).$$

Testing a Power Series for Convergence

Step-2. Testing at the end points: If R is finite, then test for absolute (or conditional) convergence at both end points $x=a-R$ and $x=a+R$ (by substituting $x=a-R$ and $x=a+R$ respectively in the given power series).

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Step-3. Make a conclusion based on Step 1 and Step 2.

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Remark.

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- If series converges for all x then we say $R = \infty$.

Formula for Radius of Convergence

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or

$$\frac{1}{R} = \lim |a_n|^{1/n}.$$

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For a power series $\sum_{n=0}^{\infty} a_n(x-a)^n$, exactly one of the following is true:

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- The series converges absolutely for all real numbers x
- The series converges absolutely over a finite interval, $|x-a| < R$, i.e., $(a-R, a+R)$ such that $R > 0$. The series may converge absolutely, converge conditionally or diverge, at the end points $x = a-R$ and $x = a+R$.

Q:. $\sum_{n=0}^{\infty} \frac{x^n}{2^n(n+1)^2}$.

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Step-1. Testing for absolute convergence (Ratio Test):

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Step-1. Testing for absolute convergence

(Ratio Test): $\lim \left|\frac{u_{n+1}}{u_n}\right| = \lim \left|\frac{(x)^{n+1}}{2^{n+1}(n+2)^2} . \frac{2^n(n+1)^2}{x^n}\right| = \lim \left(\frac{n+1}{n+2}\right)^2 \frac{|x|}{2} = \frac{|x|}{2}$.

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For the absolute convergence, we must have $\mathbf{lim}\left|\frac{u_{n+1}}{u_n}\right| < 1$ *i.e.*, $|x| < 2$ or $x \in (-2, 2)$. Thus the given series is absolutely convergent for all $x \in (-2, 2)$.

Step-2. Testing at the end points $x = -2$ and $x = 2$:

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- At $x = 2$, the given series becomes $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$

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Step-3. Conclusion: For the given series

(i) $R = 2$.

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Step-1. Testing for absolute convergence (Ratio Test): $\lim \left|\frac{u_{n+1}}{u_n}\right| = \lim \left|\frac{(x)^{n+1}(n+1)}{n+3} . \frac{n+2}{nx^n}\right| = |x| \lim \frac{(n+1)(n+2)}{n(n+3)} = |x|.$

For the absolute convergence, we must have $\mathbf{lim}\left|\frac{u_{n+1}}{u_n}\right| < 1$ *i.e.*, $|x| < 1$ or $x \in (-1,1)$. Thus the given series is absolutely convergent for all $x \in (-1,1)$.

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- At $x = 1$, the given series becomes $\sum_{n=0}^{\infty} \frac{n}{n+2}$ which is divergent (n^{th} term test).

- At $x = -1$, the given series becomes $\sum_{n=0}^{\infty} (-1)^n \frac{n}{n+2}$ which is again divergent.

Step-3. Conclusion: For the given series

(i) $R = 1$.

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Step-1. Testing for absolute convergence

(Ratio Test): $\lim \left|\frac{u_{n+1}}{u_n}\right| = \lim \left|\frac{(3x+1)^{n+2}}{2n+4} . \frac{2n+2}{(3x+1)^{n+1}}\right| = |3x+1| \lim \frac{(2n+2)}{(2n+4)} = |3x+1|.$

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For the absolute convergence, we must have $\mathbf{lim}\left|\frac{u_{n+1}}{u_n}\right| < 1$ $i.e.,|3x+1| < 1$ or $x \in (-\frac{2}{3}, 0)$.

Q:. $\sum_{n=1}^{\infty} \frac{(3x+1)^{n+1}}{2n+2}$

Step-1. Testing for absolute convergence

(Ratio Test): $\lim \left|\frac{u_{n+1}}{u_n}\right| = \lim \left|\frac{(3x+1)^{n+2}}{2n+4} . \frac{2n+2}{(3x+1)^{n+1}}\right| = |3x+1| \lim \frac{(2n+2)}{(2n+4)} = |3x+1|.$

For the absolute convergence, we must have $\lim \left|\frac{u_{n+1}}{u_n}\right| < 1$ *i.e.*, $|3x+1| < 1$ or $x \in (-\frac{2}{3}, 0)$. Thus the given series is absolutely convergent for all $x \in (-\frac{2}{3}, 0)$.

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Step-3. Conclusion: For the given series

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(ii) For $-\frac{2}{3} < x < 0$, it is absolutely convergent.

(iii) It converges conditionally for $x = -\frac{2}{3}$

Q:. $\sum_{n=1}^{\infty} n!(x+10)^n.$

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Sol. Here

$$\lim\left|\frac{u_{n+1}}{u_n}\right| = \lim\left|\frac{(n+1)!(x+10)^{n+1}}{n!(x+10)^n}\right| = \lim(n+1)|x+10|.$$

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Now $\lim(n+1)|x+10| < 1$ only if $|x+10| = 0$, i.e., when $x = -10$. Thus, $R = 0$ and the series converges only at $x = -10$.

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Sol. $\lim |u_n|^{\frac{1}{n}} = \lim\left(\left(\frac{n}{n+1}\right)^{n^2} |x|^n\right)^{\frac{1}{n}} = |x| \lim\left(\frac{n}{n+1}\right)^n$

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For the absolute convergence, we must have

$\lim |u_n|^{\frac{1}{n}} < 1 \implies |x| < e \implies R = e.$

Questions

Discuss the nature of convergence of the following power series:

1. $\sum_{n=0}^{\infty} \frac{(x-3)^{2n}}{3^n}$ **2.** $\sum \frac{x^n}{n!}$.

Ans 1. (i) $R = \sqrt{3}$.
(ii) For $3 - \sqrt{3} < x < 3 + \sqrt{3} \rightarrow$ absolutely convergent.
(iii) For $3 - \sqrt{3} < x < 3 - \sqrt{3} \rightarrow$ convergent.
(iv) No point of conditional convergence.

Ans 2. Absolutely convergent for all values of x.

Section 10.8

Taylor and Maclaurin Series

Q:. Can we expand an infinitely differentiable function (such as $f(x) = \sin x$ or $f(x) = e^x$) into a power series $\sum a_n(x-a)^n$ that converge to the correct function value $f(x)$ for all x in some open interval $(a-R, a+R)$, where $R > 0$ or $R = \infty$.

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Let $f(x)$ be a function with derivatives of all orders in an interval containing a as an interior point.

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That is the Taylor series is

$$f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\cdots$$

Maclaurin Series

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$$\sum_{n=0}^{\infty} a_n x^n, \text{ where } a_n = \frac{f^{(n)}(0)}{n!}.$$

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The Maclaurin series is the Taylor series generated by f at $x = 0$.

Q:. Determine the Maclaurin series of $\cos x$.

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Sol. Here

$$\begin{aligned}
f(x) &= \cos x & \Rightarrow f(0) &= 1 \\
f'(x) &= -\sin x & \Rightarrow f'(0) &= 0 \\
f''(x) &= -\cos x & \Rightarrow f''(0) &= -1 \\
f'''(x) &= \sin x & \Rightarrow f'''(0) &= 0 \\
\vdots \quad \vdots & \quad \vdots \quad \vdots & \vdots \quad & \vdots
\end{aligned}$$

Maclaurin series is given as

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Thus, the required series is

$$1-\frac{1}{2!}x^2+\frac{1}{4!}x^4+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$$

Q:30. Determine the Taylor series of 2^x at $x = 1$.

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Sol. Here

$$\begin{aligned}
f(x) &= 2^x & &\Rightarrow f(1) = 2\\
f'(x) &= 2^x \ln 2 & &\Rightarrow f'(1) = 2\ln 2\\
f''(x) &= 2^x (\ln 2)^2 & &\Rightarrow f''(1) = 2(\ln 2)^2\\
f'''(x) &= 2^x (\ln 2)^3 & &\Rightarrow f'''(1) = 2(\ln 2)^3\\
f^{iv}(x) &= 2^x (\ln 2)^4 & &\Rightarrow f^{iv}(1) = 2(\ln 2)^4\\
\vdots\quad & \vdots \quad \vdots & & \vdots \quad \vdots\\
f^{(n)}(x) &= 2^x (\ln 2)^n & &\Rightarrow f^{(n)}(1) = 2(\ln 2)^n\\
\vdots\quad & \vdots \quad \vdots & & \vdots \quad \vdots
\end{aligned}$$

The Taylor series is

$$f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\cdots$$

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Thus, the required series is

$$2+2\ln 2(x-1)+\frac{2(\ln 2)^2}{2!}(x-1)^2+\cdots+\frac{2(\ln 2)^n}{n!}(x-1)^n+\cdots$$

Taylor Polynomial

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$$P_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

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Q:. Find the Taylor polynomial of order 3 generated by $f(x) = \frac{1}{x}$ at $a = 2$.

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Sol.

$$f(x) = x^{-1} \qquad \Rightarrow f(2) = 1/2$$
$$f'(x) = -x^{-2} \qquad \Rightarrow f'(2) = -1/4$$
$$f''(x) = 2x^{-3} \qquad \Rightarrow f''(2) = 1/4$$
$$f'''(x) = -6x^{-4} \qquad \Rightarrow f'''(2) = -3/8$$

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$$P_3(x) = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3$$