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CS/IS F214
Logic in Computer Science

MODULE: PREDICATE LOGIC

Predicate Logic – Substitution (Revisited)

RECALL: Substitution

- Definition **[Substitution]**:
  - Given a formula ϕ, a term **t**, and a variable **X**,
  - define **ϕ[t/X]** to be the formula obtained by
    - replacing each **free occurrence** of **X** in ϕ with **t**.

Substitution - Examples

$\forall Y\ (p(X) \wedge q(Y) \text{ --> } p(Y))\ \ [a/X]$

- **X** is a free variable and it occurs once:
  - *replace it with the constant* ***a***

$\forall Y\ (p(X) \wedge q(Y) \text{ --> } p(Y))\ \ [f(b)/X]$

- **X** is a free variable and it occurs once:
  - *replace it with the function term* ***f(b)***

Substitution - Examples

$\forall Y\ (p(X) \wedge q(Y) \dashrightarrow p(Y))\ \ [f(b)/Z]$

- no occurrences of **Z**:
  - *nothing to replace*

$\forall Y\ (p(X) \wedge q(Y) \dashrightarrow p(Y))\ \ [f(b)/Y]$

- **Y** is a bound variable in the formula given;
- there is no free occurrence of **Y**:
  - *nothing to replace*

Substitution - Examples

$(p(X) \wedge q(Y) \dashrightarrow p(Y))$ $[a/Y]$

- **Y** occurs free twice:
  - *replace each occurrence with the constant* **a**

$p(Y) \wedge q(Y) \dashrightarrow r(Y)$ $[f(b)/Y]$

- **Y** occurs free three times:
  - *replace each occurrence of **Y** with the function term* **f(b)**

Substitution - Examples

$(\exists Y\ p(X) \land q(Y)) \text{-->} p(Y)\ \ [a/Y]$

- **Y** is a bound variable in the antecedent of the implication
  - *nothing to be replaced*
- But **Y** is a free variable in the consequent and occurs once
  - *replace the free occurrence with the constant* **a**

$p(Y) \land (\exists Y\ q(Y)) \text{-->} r(Y)\ \ [f(b)/Y]$

- There is a bound variable **Y** and there is a free variable **Y**
- **Y** occurs free twice – once in the antecedent and once in the consequent:
  - *replace each free occurrence of* **Y** *with the function term* **f(b)**

Substitution - Examples

$\exists Y\ (p(X) \wedge q(Y) \dashrightarrow p(Y))\ [f(X)/X]$

- the replacement term (i.e. **f(X))** contains the free variable **X**
  - Is this a conflict?
    - No! (*Why not?)*

- Consider this more useful example:
  - Say, a property ϕ holds on natural numbers:
    - **∀Y natural(Y) --> ϕ**
  - A proof of this by induction would require
    - Induction Basis (e.g.):
      - **ϕ[0/Y]** to be true
    - Induction Step (e.g.):
      - **ϕ --> ϕ[succ(Y)/Y]** to be true

Substitution - Examples

$\forall Y\ p(X) \wedge q(Y) \dashrightarrow p(Y)\ [f(Y)/X]$

- the replacement term (i.e. **f(Y))** contains the bound variable **Y**
  - Is this a conflict?
    - Yes! (*Why*?)
  - What will happen if we proceed with this substitution?

- Question:
  - *Is there an analogy in programming*?

Substitution - Examples

$\forall Y\ p(X) \wedge q(Y) \text{ --> } p(Y)\ \ [f(Y)/X]$

- the replacement term i.e. **f(Y)** contains the bound variable **Y**
  - How do you handle this conflict?

- This conflict can be resolved in two steps:

1. *Rename the bound variable* (consistently!) :
   - i.e. rename the binding and all occurrences of that variable within the (sub-)formula
   - e.g. the formula **$\forall Y\ p(X) \wedge q(Y) \text{ --> } p(Y)$** would become, say,
     - **$\forall Z\ p(X) \wedge q(Z) \text{ --> } p(Z)$**
2. Substitution – after renaming – will not cause a conflict:
   - **$\forall Z\ p(X) \wedge q(Z) \text{ --> } p(Z)\ \ [f(Y) / X]$** will result in
   - **$\forall Z\ p(f(Y)) \wedge q(Z) \text{ --> } p(Z)$**

Renaming - Examples

•*Rename the variable* in the *outermost, left-most quantifier* in each of the following formulas:

- **$\forall$Y p(X) $\wedge$ q(Y) --> p(Y) $\wedge$ r(Y)**
  - Question: Can you rename **Y** to **X**?

- **($\forall$Y p(X) $\wedge$ q(Y)) --> ($\forall$Y p(Y) $\wedge$ r(Y))**
  - Question: Can you and should you rename the two bound variables **Y** to the same new variable?

Renaming

•*Rename the variable* in the *outermost, left-most quantifier* in each of the following formulas:

- **$(\forall Y\ p(X) \wedge q(Y))$ --> $(\forall X\ p(X) \wedge r(Y))$**
  - Question: Will and should the free occurrence of **Y** (in the consequent) be renamed?

- **$\forall Y\ p(X) \wedge q(Y)$ --> $(\forall X\ p(X) \wedge r(Y))$**
  - Question: Will and should the occurrence of **Y** in the consequent be renamed?

Renaming

•*Rename the variable* in the *outermost, left-most quantifier* in each of the following formulas:

- $\forall Y\ p(X) \wedge q(Y) \dashrightarrow (\forall Y\ p(X) \wedge r(Y))$
- $\forall Y\ p(X) \wedge q(Y) \dashrightarrow (\forall Y\ p(X) \wedge r(Y)) \wedge s(Y)$

Substitution and Renaming

Definition **[Substitution]**:

Given a formula ϕ, a term **t**, and a variable **X**,

define **ϕ[t/X]** to be the formula obtained

*by replacing each **free occurrence** of variable **X** in ϕ with **t**.*

- In all the examples we saw, term **t** was either a constant, or a function term:
  - but a term can be a *constant*, *variable*, or a *function term*.
- So, what does **ϕ[Y/X]** mean?

Exercises:

- Write an algorithm ***substitute***
  - that takes a formula **ϕ**, a variable **V**, and a term **t** as arguments and
  - computes **ϕ[t/V]**.
- Note that ***substitute*** would require an auxiliary procedure ***rename***
  - that takes a formula **ϕ** as argument and
  - renames the outermost, leftmost bound variable consistently to obtain a new formula **ψ** ,
    - which is exactly the same as ϕ, modulo variable names.
- (From the previous slide) recall the relation between ***renaming*** and ***substitution*** :
  - can you reuse ***substitute*** recursively, so as to avoid a separate ***rename*** procedure.