innovate achieve lead

BITS Pilani
Pilani Campus

CS/IS F214
Logic in Computer Science

MODULE: PROPOSITIONAL LOGIC

Proof of Soundness

Proving Soundness

- Theorem:
  - Let $\varphi_1, \varphi_2, \ldots, \varphi_n$ and ψ be propositional logic formulas:
    If $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$ holds
    then $\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi$ holds.

Proving Soundness

- **Theorem**:
  - Let $\varphi_1, \varphi_2, \ldots, \varphi_n$ and ψ be propositional logic formulas. If $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$ holds then $\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi$ holds.
- ***Proof Outline***:
  - If $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$ holds *then there is a proof* of ψ from the premises $\varphi_1, \varphi_2, \ldots, \varphi_n$
  - We show – *by mathematical induction on the length of this proof* – that $\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi$ must hold.
    - *Length of the proof is the number of lines (i.e. steps).*
  - To be precise, by induction on **k**, we show **M(k)** :
    - For all sequents $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$ which have a proof of length **k**,

      $\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi$ holds.

Induction on the length of a Proof

Consider the following sequent

$p \wedge q \to r \;\; |\!- \;\; p \to (q \to r)$

and the following proof of it:

1	**p∧q-->r**	**Premise**
2	**p**	**Assumption**
3	**q**	**Assumption**
4	**p∧q**	**∧i 2,3**
5	**r**	**-->e1,4**
6	**q-->r**	**-->i3-5**
7	**p-->(q-->r)**	**-->i2-6**

- Suppose we remove the last line:
  - ***we don't get a complete proof (of anything).***
    - Why?

Induction on the length of a Proof - Example

A proof of :

$p \wedge q \rightarrow r \quad \vdash \quad p \rightarrow (q \rightarrow r)$

1	p∧q-->r	Premise
2	p	Assumption
3	q	Assumption
4	p∧q	∧i 2,3
5	r	-->e1,4
6	q-->r	-->i3-5
7	p-->(q-->r)	-->i2-6

- *Removing the last line results in an incomplete proof.*

- We can fix this:
  - *by changing the assumption in line 2 into a premise*
- Then we get a (shorter) proof of

$p \wedge q \rightarrow r, p \quad \vdash \quad (q \rightarrow r)$

1	p∧q-->r	Premise
2	p	Premise
3	q	Assumption
4	p∧q	∧i 2,3
5	r	-->e1,4
6	q-->r	-->i3-5

Proving Soundness (by induction on *proof length*):

- Let M(k) be :
  - For all sequents $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$ that have a proof of length k, $\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi$ holds.
- **Base Case**: (**k=1**)
  - Then the proof must only have a premise, say ψ. Why?
  - i.e. this is a proof of $\psi \vdash \psi$.
    - But trivially, $\psi \models \psi$.
      - Why?
- **Induction Step**:
  - **Hypothesis**: Assume M(k') to be true for all k' < k.

Proving Soundness (Induction Step):

Consider the proof of the sequent $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$ of length **k**:

φ_1	Premise
…	
φ_n	Premise
…	
ψ	

What was the last rule applied?

Consider case by case:

Proving Soundness (Induction Step):

Consider the proof of the sequent $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$ of length **k**:

φ_1	Premise
...	
φ_n	Premise
...	
ψ	

What was the last rule applied?

Consider case by case:

case $\wedge$i: (*see opposite column*)

- If the last rule were $\wedge$i:
  - then ψ is of the form $\psi_1 \wedge \psi_2$
- in which case there were sub-proofs for ψ_1 and ψ_2
  - i.e. we have proofs of
    - $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi_1$ and
    - $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi_2$
  - each of length < k.
- By hypothesis,
  - $\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi_1$ and
  - $\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi_2$
- and so
  - $\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi_1 \wedge \psi_2$ (*by semantics*)

Proving Soundness (Induction Step):

Consider the proof of the sequent $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$ of length **k**:

φ_1	Premise
...	
φ_n	Premise
...	
ψ	

What was the last rule applied?

case -->i: (*see opposite column*)

- If the last rule were -->i:
  - then ψ is of the form $\psi_1 \dashrightarrow \psi_2$
- in which case there was a sub-proof for ψ_2 assuming ψ_1
  - i.e. we have a proof of
    - $\varphi_1, \varphi_2, \ldots, \varphi_n, \psi_1 \vdash \psi_2$
  - of length < k.
- By hypothesis,
  - $\varphi_1, \varphi_2, \ldots, \varphi_n , \psi_1 \models \psi_2$
- and so
  - $\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi_1 \dashrightarrow \psi_2$

  (*by truth table*)

Proving Soundness (Induction Step):

Consider the proof of the sequent $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$ of length **k**:

φ_1	Premise
...	
φ_n	Premise
...	
ψ	

Other cases are similar – do the following case:

case -->e:

- If the last rule were ->e:

... complete the proof!

Proving Soundness

- Thus we have proven that, for all **k**,
  - For all sequents $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$ which have a proof of length **k**,
    - $\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi$ holds
- by assuming that
  - for all sub-proofs deriving an intermediate ψ', which are of length less than **k**,
    - $\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi'$ holds.

Q.E.D.