BIRLA INSTITUTE OF TECHNOLOGY AND SCIENCE PILANI, RAJASTHAN

Mid Semester Examination (Open Book): 2017-18, 2nd Semester

Mechanics Oscillations and Waves (MEOW): PHY F111, 7th March 2018, Duration: 90 mins., Full Marks: 75

Instruction(s): All questions are compulsory. Answer all parts of a question together. Write your final answer of each sub-part inside a box. Only text book, hand written class notes and lecture slides are allowed.

1. The trajectory of a particle of mass m is given by: $\theta(t) = \omega t;\ r(t) = c_0 t e^{-\beta t}$, where, ω, c_0 and β are positive constants. **(a)** Is $\dot{r}$ zero at any time? **(b)** At what time and under what condition is the instantaneous radial acceleration of the particle is zero? **(c)** What is the tangential acceleration (a_θ) at the instant when the radial acceleration (a_r) is zero? **(5+5+5)**

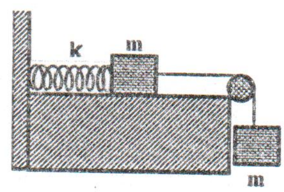
2. A uniform spherical drop of liquid of instantaneous radius r (assuming initial radius r_0) and mass M moves in one dimension through a cloud at rest such that it accretes mass into itself and still remains spherical. The rate at which it gains mass is given by kr^α, where α is a positive constant (assume there is no gravity).

(a) Show that, $\frac{1}{M}\frac{dM}{dt} = \frac{A}{t+B}$, where A and B are constants and M is the instantaneous mass of the sphere. Write down for the expressions for A and B in terms of the given constants.

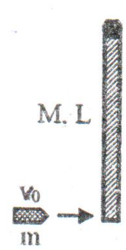
(b) If the initial velocity of the sphere at t = 0 was v_0 and if it was moving under a constant force F, write down the differential equation for v(t). Solve for v(t) when F = 0. **(5+5)**

3. Two masses m_1 and m_2 have an instantaneous separation x. Their interaction potential energy is given as: $U(x) = -\left[a + b(x-x_0)^2 + c(x-x_0)^4\right]e^{-d(x-x_0)^2}$. Here, a, b, c and d are constants of appropriate dimension and x_0 is the constant of dimension of length. **(a)** Find the equilibrium separation between m_1 and m_2. **(b)** Find the condition of stable equilibrium in terms of given constants. **(c)** Find the frequency of small oscillations (ω) about the stable equilibrium. **(4+5+4)**

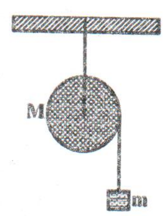
4. The figure describes a system of two equal masses m and a spring of spring constant k. The coefficient of kinetic friction between the left mass and the table is 0.25 and assume the pulley is ideal. The spring initially is in its relaxed length L with no tension in the thread (which is mass-less) and then the system of masses is released. **(a)** How far does the spring stretch before the masses come to rest? **(b)** If the thread is then cut, what is the magnitude of maximum compression of the spring during the resultant motion? **(6+6)**



5. A plank of mass M and length L is suspended from a frictionless pivot at its top end. A bullet of mass 'm' flying horizontally with a speed v_0 hits the bottom end of the plank and gets stuck. The impact displaces the plank-bullet system by an angle θ_0. **(a)** Find the initial velocity v_0 of the bullet. **(b)** What is the change in momentum of the "bullet+plank" system (ΔP) after the inelastic collision? **(c)** At what distance *y* (measured from the top end of the plank) the bullet must strike so that ΔP=0 after impact. **(d)** For m=M/6, calculate all the above three quantities obtained in **(a)**, **(b)** and **(c)**. **(5+5+5+3)**



6. A light thread with a body of mass m tied to its end is wound on a uniform solid cylinder of mass *M* and radius *R*. At time *t* = 0 the system is set in motion. Assume the friction in the axle of the cylinder to be negligible and the thread is not slipping over the cylinder. Find **(a)** the angular velocity of the cylinder as a function of time **(b)** the kinetic energy of the whole system. **(4+3)**



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Q 1 The trajectory of a particle of mass m is given by

$$\theta(t) = \omega t$$
$$r(t) = c_0\, t e^{-\beta t}$$

where ω, β and c_0 are positive constants.

(a) Is $\ddot{r}$ zero at any time?

$$\dot{r} = c_0 e^{-\beta t} - \beta c_0 t e^{-\beta t}$$
$$\ddot{r} = -c_0 \beta e^{-\beta t} - \beta c_0 e^{-\beta t} + \beta^2 c_0 t e^{-\beta t}$$
$$\ddot{r} = 0 \implies t = 2/\beta$$

[5]

(b) At what time and under what condition is the instantaneous radial acceleration of the particle zero ?

$$a_r = \ddot{r} - r\dot{\theta}^2$$
$$\dot{\theta} = \omega$$
$$\left(\beta^2 c_0 t - 2c_0\beta\right) e^{-\beta t} - c_0 t \omega^2 e^{-\beta t} = 0$$
$$a_r = 0 \implies t = \frac{2\beta}{\beta^2 - \omega^2} [4]$$

Condition $\beta > \omega$ [1]

(c) What is the angular acceleration at the instant when the radial acceleration is zero?

$$a_\theta = 2\dot{r}\dot{\theta}$$
$$a_\theta = 2c_0\left(1 - \beta t\right) e^{-\beta t}$$
$$= 2c_0 \left(1 - \frac{2\beta^2}{\beta^2 - \omega^2}\right) e^{-2\beta^2/(\beta^2 - \omega^2)} [5]$$

Q 2

An uniform spherical drop of liquid of instantaneous radius r and mass M moves in one dimension through a cloud at rest such that it acretes mass into itself and still remains spherical. The rate at which it gains mass is given by kr^α

(a) Show that

$$\frac{1}{M}\frac{dM}{dt} = \frac{A}{t+B}$$

where A and B are constants and M is the instantaneous mass of the sphere. Find A and B.

$$\frac{dM}{dt} = \frac{d}{dt}\frac{4}{3}\pi r^3 \rho = 4\pi r^2 \rho \frac{dr}{dt} = kr^\alpha$$
$$r^{2-\alpha} dr = \frac{k}{4\pi\rho} dt = a dt$$

where $a = k/4\pi\rho$. Integrating

$$r^{3-\alpha} = b + (3-\alpha) a t$$

where $b = r_0^{3-\alpha}$

$$\frac{1}{M}\frac{dM}{dt} = \frac{1}{\frac{4}{3}\pi r^3 \rho} 4\pi r^2 \rho \frac{dr}{dt} = \frac{3}{r}\frac{dr}{dt}$$
$$\frac{dr}{dt} = \frac{a}{r^{2-\alpha}} \implies \frac{1}{M}\frac{dM}{dt} = \frac{3a}{b + (3-\alpha) a t} = \frac{A}{t+B}$$

$$A = \frac{3}{3-\alpha}$$

$$B = \frac{3r_0^{-\alpha} M(t=0)}{(3-\alpha)k}$$

[2.5 + 2.5]

(b) If the initial velocity of the sphere at $t = 0$ was v_0 and if it was moving under a constant force F write down the differential equation for $v(t)$. Solve for $v(t)$ when $F = 0$.

$$M\frac{dv}{dt} + v\frac{dM}{dt} = 0$$

Or

$$\frac{1}{v}\frac{dv}{dt} = -\frac{A}{t+B}[1]$$

$$\ln\frac{v}{v_0} = -A\ln(t+B) + A\ln B$$

$$v = v_0\left(\frac{B}{t+B}\right)^A \quad [4]$$

Model Answers to Q.3 & Q.4

Ans 3 (a)

$$U(x) = -[a + b(x-x_0)^2 + c(x-x_0)^4]e^{-d(x-x_0)^2}$$

For equilibrium: $\frac{dU}{dx} = 0$; $Let\ x - x_0 = x';\ \therefore\ U(x') = -[a + bx'^2 + cx'^4]e^{-dx'^2}$

$$\Rightarrow \frac{dU}{dx'} = 2dx'e^{-dx'^2}(a + bx'^2 + cx'^4) - e^{-dx'^2}(2bx' + 4cx'^3)$$

$$\Rightarrow e^{-dx'^2}[x'(2ad - 2b) + x'^3(2db - 4c) + x'^5(2dc)] = 0 \cdots\cdots\cdots [2]$$

$$\because e^{-dx'^2} \neq 0 \Rightarrow x' = 0 \Rightarrow x - x_0 = 0;\ \boxed{\Rightarrow x = x_0} \text{............}[2]$$

Ans.3(b)

The condition of stable equilibrium: $\frac{d^2U}{dx^2} > 0\ at\ x = x_0 \ \Rightarrow \frac{d^2U}{dx'^2} > 0$ at $x' = 0$

$$\frac{d^2U}{dx'^2} = -2dx'e^{-dx'^2}[x'(2ad-2b) + x'^3(2db-4c) + x'^5(2dc)]$$

$$+e^{-dx'^2}[(2ad-2b) + 3x'^2(2db-4c) + x'^4(10dc)] \text{ ....}[3]$$

$$\frac{d^2U}{dx'^2}\mid x' = 0 \quad \boxed{= 2(ad-b) > 0 \Rightarrow ad > b} \quad \text{...................}[2]$$

Ans.3(c) $k = \frac{d^2U}{dx'^2}\mid x' = 0 \Rightarrow k = \frac{d^2U}{dx^2}\mid x = x_0 \ = 2(ad-b) \ldots\ldots\ldots\ldots\ldots [1]$

$$\omega = \sqrt{\frac{k}{\mu}} \quad where\ \mu = \frac{m_1 m_2}{m_1 + m_2} \quad \text{......................}[1]$$

$$\boldsymbol{\omega = \sqrt{\frac{2(ad-b)(m_1+m_2)}{m_1 m_2}}} \text{ ....}[2]$$

Ans. 4 (a) Let x_0 be the maximum stretching in the spring. Work done by gravity on m = mgx_0.

Spring energy = $\frac{1}{2}kx_0^2$. Work done against frictional force $\mu mgx_0 = \frac{1}{4}mgx_0$.

$$\frac{1}{2}kx_0^2 + \frac{1}{4}mgx_0 = mgx_0 \ \Rightarrow \frac{1}{2}kx_0^2 = \frac{3}{4}mgx_0 \ \Rightarrow \boldsymbol{x_0 = \frac{3mg}{2k}} \text{ ....}[3+3]$$

Ans.4 (b) Let the maximum compression of the spring about its normal length L be d.

Distance travelled by m on table against friction = $d + x_0$. Work done against friction = ¼ $mg\ (d + x_0)$

Spring energy stored (in length d) = $\frac{1}{2}kd^2$.

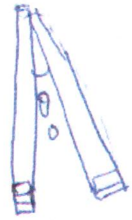
$$\frac{1}{2}kx_0^2 = \tfrac{1}{4}mg\ (d + x_0) + \frac{1}{2}kd^2 \quad \text{..............................}[3]$$

$$\Rightarrow kx_0^2 = \frac{1}{2}mg\ (d + x_0) + kd^2 \ \Rightarrow 2k(x_0^2 - d^2) = mg\ (d + x_0)$$

$$\Rightarrow 2k\ (x_0 + d)(x_0 - d) = mg(d + x_0) \ \Rightarrow 2k(x_0 - d) = mg$$

$Substituting\ x_0\ from\ (a),\ \ 2k.\frac{3mg}{2k} - mg = 2kd \Rightarrow\ 2mg = 2kd\ \Rightarrow \boldsymbol{d = mg/k}$ ....[3]

(5)



(a) $m v_0 l = \left(m l^2 + \frac{M L^2}{3}\right)\omega_0 \Rightarrow \omega_0 = \frac{3 m v_0}{(M+3m)L}$

$\frac{1}{2} I \omega^2 = Mg\frac{L}{2}(1-\cos\theta_0) + mgl(1-\cos\theta_0)$ (2M)

$v_0 = \frac{1}{m}\sqrt{\frac{2gL}{3}(M+3m)(M+2m)}\ \sin\frac{\theta_0}{2}$

$\omega_0 = \sqrt{\frac{6g}{L}\left(\frac{M+2m}{M+3m}\right)}\ \sin\frac{\theta_0}{2}$ (3M)

(b)

$\Delta P = \left(M\omega_0\frac{L}{2} + m\omega_0 L\right) - m v_0$ --- (1M)

$= \frac{M m v_0}{2[M+3m]}$ --- 4M

$= \left(\sin\frac{\theta_0}{2}\right) M \sqrt{\frac{gL}{6}\left(\frac{M+2m}{M+3m}\right)}$

(c) $m v y = \left[\frac{ML^2}{3} + m y^2\right]\omega'$ --- (1M)

$P_f = m y \omega' + M\omega'\frac{L}{2}$; $P_i = m v$ --- (1M)

$\Delta P = P_f - P_i = \frac{3my^2}{ML^2+3my^2} + \frac{3MyL}{2(ML^2+3My^2)} = 1$

$6my^2 + 3MyL - 2ML^2 - 6My^2 = 0$

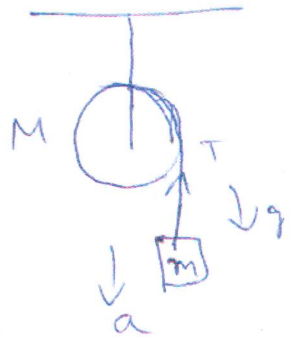
$\Rightarrow 3y = 2L$ or $\boxed{y = \frac{2}{3}L}$ --- (3M)

(d) For $m = M/6$ (i) $v_0 = \sqrt{48gL}\ \sin\frac{\theta_0}{2} = \sqrt{24gL(1-\cos\theta_0)}$

(ii) $\Delta P = \frac{M v_0}{18} = \frac{2M^2}{3}\sqrt{\frac{gL}{3}}\ \sin\theta_0/2$; $y = \frac{2L}{3}$ (3M)

~~(iii) y~~

(6)



$$mg - T = ma \quad \text{--- } 1M$$

$$TR = \frac{MR^2}{2}\alpha \quad \text{--- } 1M$$

$$mg - T = mR\alpha$$

$$\alpha = \frac{2mg}{(M+2m)R} \quad \text{--- } 1M$$

$$\omega = \frac{2mgt}{(M+2m)R} \quad \text{--- } 1M$$

(b) $$K.E = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

$$= \frac{1}{2}mR^2\omega^2 + \frac{1}{2}\frac{MR^2}{2}\omega^2$$

$$= \frac{R^2\omega^2}{4}(2m+M) = \frac{m^2g^2t^2}{(M+2m)} \quad 3M$$