Chapter 13 (13.1-13.5)

Vector-Valued Functions and Motion in Space

Note: *This module is prepared from Chapter 13 (Thomas' Calculus, 13th edition) just to help the students. The study material is expected to be useful but not exhaustive. For detailed study, the students are advised to attend the lecture/tutorial classes regularly, and consult the text book.*

Appeal: Please do not print this e-module unless it is really necessary.

Dr. Suresh Kumar, Department of Mathematics, BITS Pilani, Pilani Campus

Contents

SECTION 13.1 (Curves in space and their tangents)

Vector-valued functions

Let $x = x(t)$, $y = y(t)$ and $z = z(t)$ are continuous functions for $t \in I$. Then the points (x, y, z) trace a curve C is space. Let $\overrightarrow{OP} = \overrightarrow{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$. Then $\overrightarrow{r}(t)$ is the position vector of an arbitrary point $P(x(t), y(t), z(t))$ on the curve C, and defines a vector-valued function of the real variable t.

Ex. $x = \cos t$, $y = \sin t$, $z = 0$, $t \in [0, 2\pi]$

are the parametric equations of a circle with centre at origin and radius 1 in the XY-plane. The corresponding vector-valued function is $\overrightarrow{r}(t) = \cos t\ \hat{i} + \sin t\ \hat{j}$.

Ex. $\overrightarrow{r}(t) = \cos t\ \hat{i} + \sin t\ \hat{j} + t\ \hat{k}$, where t is any real number.

The graph of this vector-valued function is a helix. (See Figure 1)

Ex. $\overrightarrow{r}(t) = (x_0 + lt)\ \hat{i} + (y_0 + mt)\ \hat{j} + (z_0 + nt)\ \hat{k}$, where t is any real number.

Its graph is a straight line with dr's l, m, n and passing through the point (x_0, y_0, z_0).

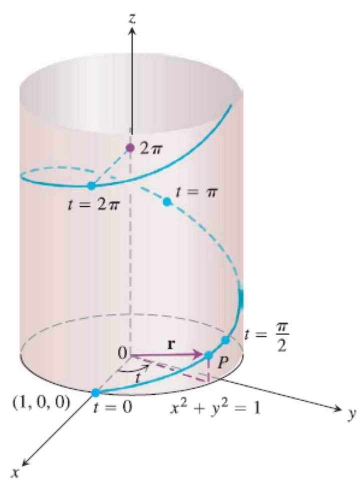


Figure 1: Blue curve is the helix $\vec{r}(t) = \cos t\ \hat{i} + \sin t\ \hat{j} + t\ \hat{k}$

Limit of a vector-valued function

The limit of a vector-valued function $\overrightarrow{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ is $\overrightarrow{l} = l_1\ \hat{i} + l_2\ \hat{j} + l_3\ \hat{k}$ as $t \to t_0$ if $\lim\limits_{t \to t_0} |\overrightarrow{r}(t) - \overrightarrow{l}| = 0$. Further,

$$\lim_{t \to t_0} \overrightarrow{r}(t) = \left(\lim_{t \to t_0} x(t)\right)\ \hat{i} + \left(\lim_{t \to t_0} y(t)\right)\ \hat{j} + \left(\lim_{t \to t_0} z(t)\right)\ \hat{k} = l_1\ \hat{i} + l_2\ \hat{j} + l_3\ \hat{k}.$$

Ex. $\lim\limits_{t \to \pi/4} (\cos t\ \hat{i} + \sin t\ \hat{j} + t\ \hat{k}) = \cos(\pi/4)\ \hat{i} + \sin(\pi/4)\ \hat{j} + (\pi/4)\ \hat{k} = (1/\sqrt{2})\ \hat{i} + (1/\sqrt{2})\ \hat{j} + (\pi/4)\ \hat{k}$.

Continuity

A vector-valued function $\overrightarrow{r}(t) = x(t)\hat{i}+y(t)\hat{j}+z(t)\hat{k}$ is said to be continuous at $t = t_0$ if $\lim\limits_{t\to t_0} \overrightarrow{r}(t) = \overrightarrow{r}(t_0)$, that is, $\lim\limits_{t\to t_0} x(t) = x(t_0)$, $\lim\limits_{t\to t_0} y(t) = y(t_0)$, $\lim\limits_{t\to t_0} z(t) = z(t_0)$. Thus, $\overrightarrow{r}(t)$ is continuous at t_0 if and only if its component functions are continuous at t_0.

Ex. The function $\overrightarrow{r}(t) = \cos t\ \hat{i} + \sin t\ \hat{j} + t\ \hat{k}$ is continuous for all real values of t.

Ex. The function $\overrightarrow{r}(t) = \cos t\ \hat{i} + \sin t\ \hat{j} + [t]\ \hat{k}$ is not continuous at any integer value of t since the greatest integer function $[t]$ is not continuous for integer values of t.

Derivatives and motion

Let P and Q be the positions of a moving particle along a curve C at times t and $t + \delta t$, respectively as shown in Figure 2. Let $\overrightarrow{OP} = \overrightarrow{r}(t)$ and $\overrightarrow{OQ} = \overrightarrow{r}(t + \Delta t)$ so that $\overrightarrow{PQ} = \Delta\overrightarrow{r} = \overrightarrow{r}(t + \Delta t) - \overrightarrow{r}(t)$. So displacement of the particle in time Δt is $\Delta\overrightarrow{r}$. Therefore, velocity of the particle at the point P is given by

$$\overrightarrow{v} = \lim_{\Delta t\to 0} \frac{\overrightarrow{r}(t + \Delta t) - \overrightarrow{r}(t)}{\Delta t} = \lim_{\Delta t\to 0} \frac{\Delta\overrightarrow{r}}{\Delta t} = \frac{d\overrightarrow{r}}{dt},$$

known as the derivative of $\overrightarrow{r}$ with respect to t.

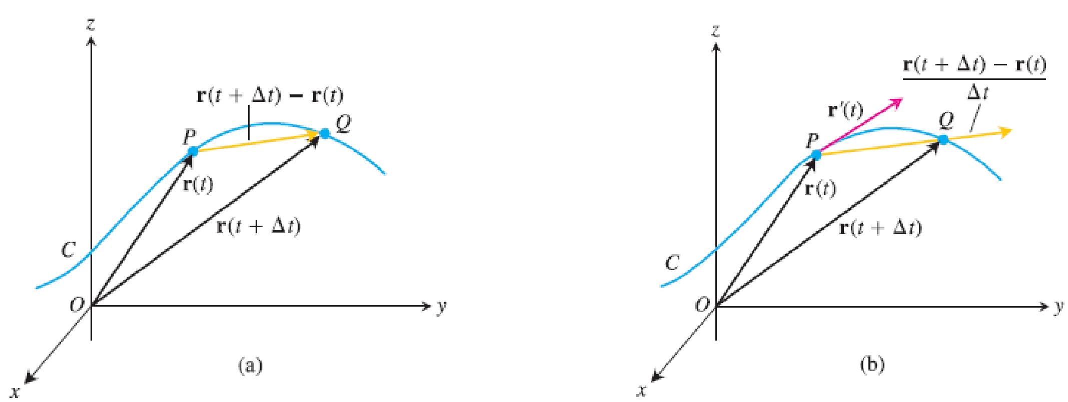


Figure 2: As $\Delta \to 0$, the point P tends to point Q along the curve C. In the limit, the vector $\vec{PQ}/\Delta t$ becomes the tangent vector $\vec{r}\,'(t)$ at P.

If $\overrightarrow{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$, then

$$\overrightarrow{v} = \frac{d\overrightarrow{r}}{dt} = \frac{dx}{dt}\,\hat{i} + \frac{dy}{dt}\,\hat{j} + \frac{dz}{dt}\,\hat{k}.$$

The curve traced by $\overrightarrow{r}(t)$ is said to be smooth if $\frac{d\overrightarrow{r}}{dt}$ is continuous and non-zero for all values of t in the interval under consideration. Further, the non-zero vector $\frac{d\overrightarrow{r}}{dt}$ is along the tangent to the curve at

P. Therefore, $\frac{d\vec{r}}{dt}$ is defined as the vector tangent to the curve at P. Thus, a smooth curve has a unique tangent at each of its points.

It would be useful to memorize the following formulas related to motion.

(1) Velocity: $\vec{v} = \dfrac{d\vec{r}}{dt}$.

(2) Speed: $|\vec{v}| = \left|\dfrac{d\vec{r}}{dt}\right|$.

(3) Acceleration: $\vec{a} = \dfrac{d\vec{v}}{dt} = \dfrac{d^2\vec{r}}{dt^2}$.

(4) Unit vector in the direction of motion: $\hat{n} = \dfrac{\vec{v}}{|\vec{v}|}$.

$\therefore \vec{v} = |\vec{v}|\left(\dfrac{\vec{v}}{|\vec{v}|}\right) = |\vec{v}|\,\hat{n} =$(Speed)(Direction)

Ex. A person on a hang glider is spiralling upward due rapidly rising air on a path $\vec{r}(t) = 3\cos t\,\hat{i} + 3\sin t\,\hat{j} + t^2\,\hat{k}$. Find the velocity vector, acceleration vector, glider's speed at time t, and the times when the glider's acceleration is orthogonal to its velocity.

Sol. We find

$\vec{v} = -3\sin t\,\hat{i} + 3\cos t\,\hat{j} + 2t\,\hat{k}$.

$\vec{a} = -3\cos t\,\hat{i} - 3\sin t\,\hat{j} + 2\,\hat{k}$.

Glider's Speed$= |\vec{v}| = \sqrt{9+4t^2}$.

When $\vec{v}$ is orthogonal to $\vec{a}$, we have $\vec{v}.\vec{a} = 0$, which gives $t = 0$.

Differentiation rules

Let $\vec{u}$ and $\vec{v}$ be differentiable vector functions of a real variable t; $\vec{c}$ be a constant vector; h be a scalar constant and f be any differentiable scalar function of t. Then we have

(1) $\dfrac{d}{dt}(\vec{c}) = 0$, $\dfrac{d}{dt}(h\,\vec{u}) = h\dfrac{d\vec{u}}{dt}$.

(2) $\dfrac{d}{dt}(f\vec{u}) = \dfrac{df}{dt}\vec{u} + f\dfrac{d\vec{u}}{dt}$. (Scalar multiple rule)

(3) $\dfrac{d}{dt}(\vec{u} \pm \vec{v}) = \dfrac{d\vec{u}}{dt} \pm \dfrac{d\vec{v}}{dt}$. (Sum/Difference rule)

(4) $\dfrac{d}{dt}(\vec{u}.\vec{v}) = \dfrac{d\vec{u}}{dt}.\vec{v} + \vec{u}.\dfrac{d\vec{v}}{dt}$. (Dot product rule)

(5) $\dfrac{d}{dt}(\vec{u} \times \vec{v}) = \dfrac{d\vec{u}}{dt} \times \vec{v} + \vec{u} \times \dfrac{d\vec{v}}{dt}$. (Cross product rule)

(6) $\dfrac{d}{dt}[\vec{u}(f(t))] = \dfrac{d\vec{u}}{dt}\dfrac{df}{dt}$. (Chain rule)

Ex. Show that a vector function $\vec{r}(t)$ is of constant length if and only if it is orthogonal to its first derivative, that is, $|\vec{r}(t)|$ is constant if and only if $\vec{r}.\dfrac{d\vec{r}}{dt} = 0$.

Sol. We have $\vec{r}.\vec{r} = |\vec{r}|^2 = a^2$ (say) iff $\dfrac{d}{dt}(\vec{r}.\vec{r}) = 0$ iff $\vec{r}.\dfrac{d\vec{r}}{dt} = 0$.

For example, consider $\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + \hat{k}$.

Then $|\vec{r}(t)| = \sqrt{2}$. Also, $\vec{r}.\dfrac{d\vec{r}}{dt} = 0$.

SECTION 13.2 (Integrals of vector functions)

Note: Projectile motion is excluded from this section.

A differentiable function $\vec{R}(t)$ is said to be an antiderivative of a vector function $\vec{r}(t)$ on an interval I if $\dfrac{d\vec{R}}{dt} = \vec{r}(t)$ for all $t \in I$. If $\vec{c}$ is any constant vector, then $\vec{R}(t) + \vec{c}$ is also an antiderivative of $\vec{r}(t)$, which we define as indefinite integral of $\vec{r}(t)$ on I and we write

$$\int \vec{r}(t)dt = \vec{R}(t) + \vec{c}.$$

If the components of $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ are integrable over $[a, b]$, then so is $\vec{r}(t)$ and the definite integral of $\vec{r}(t)$ from a to b is Further,

$$\int_a^b \vec{r}(t)dt = \left(\int_a^b x(t)dt\right)\ \hat{i} + \left(\int_a^b y(t)dt\right)\ \hat{j} + \left(\int_a^b z(t)dt\right)\ \hat{k}.$$

For example,

$$\int_0^\pi (\cos t\ \hat{i} + \hat{j} - 2t\hat{k})dt = \left(\int_0^\pi \cos t\ dt\right)\ \hat{i} + \left(\int_0^\pi dt\right)\ \hat{j} + \left(\int_0^\pi 2t\ dt\right)\ \hat{k} = \pi\ \hat{j} - \pi^2\ \hat{k}.$$

SECTION 13.3 (Arc length in space)

Length of a smooth curve

The length of a smooth curve $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$, $a \leq t \leq b$, that is traced exactly once at t increases from a to b, is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \int_a^b \left|\frac{d\vec{r}}{dt}\right| dt = \int_a^b |\vec{v}|\, dt$$

Ex. A glider soaring upward along the helix $\vec{r}(t) = \cos t\ \hat{i} + \sin t\ \hat{j} + t\ \hat{k}$. How far does the glider travel along its path from $t = 0$ to $t = 2\pi$.

Sol. $L = \displaystyle\int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}\ dt = 2\pi\sqrt{2}$.

Arc length parameter from a fixed point

The length from a fixed point $P(t_0)$ to any point $Q(t)$ is given by

$$s(t) = \int_{t_0}^t |\vec{v}(\tau)|\, d\tau.$$

From this relation, we can obtain t in terms of the arc length s. So $\overrightarrow{r}(t) = \overrightarrow{r}(t(s))$.

Ex. Find the arc length parameter along the helix $\overrightarrow{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\,\hat{k}$ from $t_0 = 0$.

Sol. $s(t) = \displaystyle\int_0^t |\overrightarrow{v}(\tau)|\,d\tau = \int_0^t \sqrt{(-\sin\tau)^2 + (\cos\tau)^2 + 1^2}\,dt = \sqrt{2}t$.

Therefore $t = \frac{s}{\sqrt{2}}$, and the helix in terms of the arc length parameter s reads as

$$\overrightarrow{r}(s) = \cos\left(\frac{s}{\sqrt{2}}\right)\,\hat{i} + \sin\left(\frac{s}{\sqrt{2}}\right)\,\hat{j} + \left(\frac{s}{\sqrt{2}}\right)\,\hat{k}.$$

Rk. By fundamental theorem of calculus, the relation $s(t) = \displaystyle\int_{t_0}^t |\overrightarrow{v}(\tau)|\,d\tau$ gives

$$\frac{ds}{dt} = |\overrightarrow{v}(t)|,$$

which is the speed of the particle at time t. Notice that it is independent of t_0.

Unit tangent vector

Since $\overrightarrow{v} = \dfrac{d\overrightarrow{r}}{dt}$ is tangent to the curve $\overrightarrow{r}(t)$, so $\hat{T} = \dfrac{\overrightarrow{v}}{|\overrightarrow{v}|}$ is unit tangent vector to it.

Also, $\dfrac{d\overrightarrow{r}}{ds} = \dfrac{d\overrightarrow{r}}{dt}\dfrac{dt}{ds} = \dfrac{\overrightarrow{v}}{|\overrightarrow{v}|} = \hat{T}$.

So the unit tangent vector to the smooth curve $\overrightarrow{r}(t)$ is

$$\hat{T} = \frac{d\overrightarrow{r}}{ds} = \frac{\overrightarrow{v}}{|\overrightarrow{v}|}.$$

Ex. For counterclockwise motion around the circle $\overrightarrow{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$, we find $\overrightarrow{v}(t) == -\sin t\,\hat{i} + \cos t\,\hat{j}$ and $|\overrightarrow{v}(t)| = 1$. So unit tangent vector is $\hat{T} == -\sin t\,\hat{i} + \cos t\,\hat{j}$.

SECTION 13.4 (Curvature and normal vectors of a curve)

Curvature of a plane curve

The curvature of a smooth curve $\overrightarrow{r}(t)$ is defined as the rate at which the unit tangent vector $\hat{T} = \dfrac{d\overrightarrow{r}}{ds}$ turns per unit length along the curve. It is denoted by κ.

$$\therefore \quad \kappa = \left|\frac{d\hat{T}}{ds}\right| = \left|\frac{d\hat{T}}{dt}\frac{dt}{ds}\right| = \frac{1}{|\overrightarrow{v}|}\left|\frac{d\hat{T}}{dt}\right|.$$

It is the mathematical measure of bending of curve.

Ex. Show that curvature of a straight line is 0.

Sol. For a straight line, $\hat{T}$ always point in the same direction. So $\hat{T}$ has constant components and consequently $\kappa = \frac{1}{|\overrightarrow{v}|}\left|\frac{d\hat{T}}{dt}\right| = 0$, as expected, since a straight line has no bending at any point.

Ex. Find the curvature of the circle $\overrightarrow{r}(t) = a\cos t\,\hat{i} + a\sin t\,\hat{j}$.

Sol. We have
$\vec{v} = \frac{d\vec{r}}{dt} = -a\sin t\,\hat{i} + a\cos t\,\hat{j}$ and $|\vec{v}| = a$.
$\therefore\ \hat{T} = \frac{\vec{v}}{|\vec{v}|} = -\sin t\,\hat{i} + \cos t\,\hat{j}$ and $\frac{d\hat{T}}{dt} = -\cos t\,\hat{i} - \sin t\,\hat{j}$.
Finally, $\kappa = \frac{1}{|\vec{v}|}\left|\frac{d\hat{T}}{dt}\right| = \frac{1}{a}$.
We see that curvature of circle is constant, as expected, since circle has uniform bending at all the points.

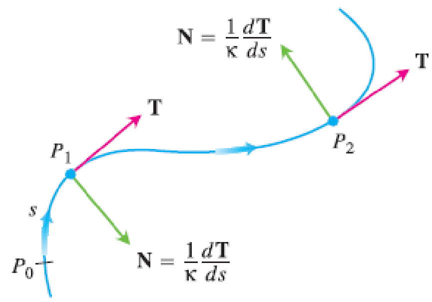


Figure 3: The vector $\frac{d\hat{T}}{ds}$ is normal to the curve, and points in the direction of turning of $\hat{T}$.

Principal unit normal

Since $\hat{T}$ has constant length (unity), we have $\hat{T}.\frac{d\hat{T}}{ds} = 0$. So $\frac{d\hat{T}}{ds}$ is orthogonal to $\hat{T}$ and points in the direction in which $\hat{T}$ turns as we face in the direction of increasing arc length. At a point, where $\kappa = \left|\frac{d\hat{T}}{ds}\right| \neq 0$, we define principal unit normal vector for a smooth curve as

$$\hat{N} = \frac{1}{\kappa}\frac{d\hat{T}}{ds}.$$

We can write

$$\hat{N} = \frac{\frac{d\hat{T}}{ds}}{\left|\frac{d\hat{T}}{ds}\right|} = \frac{\frac{d\hat{T}}{dt}\frac{dt}{ds}}{\left|\frac{d\hat{T}}{dt}\frac{dt}{ds}\right|} = \frac{\frac{d\hat{T}}{dt}}{\left|\frac{d\hat{T}}{dt}\right|}, \quad \text{where} \quad \hat{T} = \frac{\vec{v}}{|\vec{v}|}.$$

Ex. Find $\hat{T}$ and $\hat{N}$ for $\vec{r}(t) = t\,\hat{i} + t^2\,\hat{j}$.
Sol. $\hat{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{1+4t^2}}\,\hat{i} + \frac{2t}{\sqrt{1+4t^2}}\,\hat{j}$, $\hat{N} = \frac{\frac{d\hat{T}}{dt}}{\left|\frac{d\hat{T}}{dt}\right|} = \frac{-2t}{\sqrt{1+4t^2}}\,\hat{i} + \frac{1}{\sqrt{1+4t^2}}\,\hat{j}$

Circle of curvature for plane curves

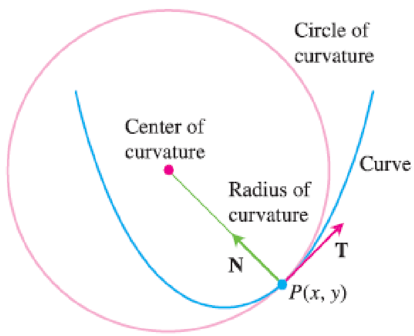
The circle of curvature or osculating circle at a point P of a plane curve where $\kappa \neq 0$ is the circle in the plane of the curve that

(i) has the same tangent line at P the curve has

(ii) has the same curvature the curve has at P

(iii) lies towards the concave or inner side of the curve.

The radius of curvature ρ of the curve at P is defined as the radius of the osculating circle at P. Then centre of the osculating circle is defined as the centre of curvature of the curve at P.



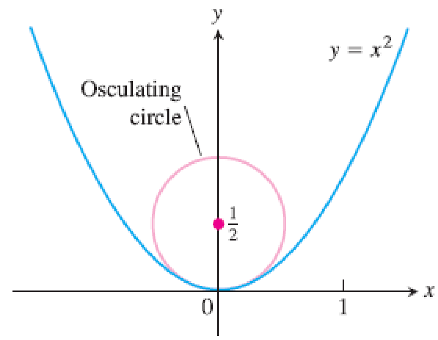


Figure 4: **Left:** The osculating circle at $P(x, y)$. **Right:** The osculating circle for the parabola $y = x^2$ at the origin.

Ex. Find the osculating circle of the parabola $y = x^2$ at the origin.

Sol. The parametric form of the given parabola is

$\vec{r}(t) = t\,\hat{i} + t^2\,\hat{j}$.

The origin corresponds to $t = 0$. We find $\kappa = \dfrac{1}{|\vec{v}|}\left|\dfrac{d\hat{T}}{dt}\right| = 2$ at $t = 0$. So radius of curvature is $\rho = 1/\kappa = 1/2$. It is easy to assess that the centre of the osculating circle is $(0, 1/2)$ that lies on Y-axis. So equation of the osculating circle is $x^2 + (y - 1/2)^2 = 1/2$.

Note: For the space curves also,

$\kappa = \dfrac{1}{|\vec{v}|}\left|\dfrac{d\hat{T}}{dt}\right|$ and $\hat{N} = \frac{1}{\kappa}\frac{d\hat{T}}{ds} = \frac{\frac{d\hat{T}}{dt}}{\left|\frac{d\hat{T}}{dt}\right|}$.

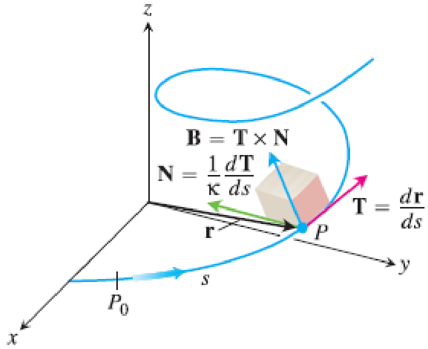
Ex. Find the curvature of the helix $\vec{r}(t) = a\cos t\,\hat{i} + a\sin t\,\hat{j} + bt\,\hat{k}$, $a, b \geq 0$, $a^2 + b^2 \neq 0$.

Sol. $\kappa = a/(a^2 + b^2)$.

SECTION 13.5 (Tangential and normal components of acceleration)

Torsion and the unit binormal vector

As we have seen $\hat{T}$ points in the direction of motion, and $\hat{N}$ points in the direction of turning of the path of motion. The tendency of the motion to "twist" out of the plane created by $\hat{T}$ and $\hat{N}$ is the direction perpendicular to this plane and is given by the vector $\hat{B} = \hat{T} \times \hat{N}$, known as the binormal vector. The moving right-handed vector frame formed by the vectors $\hat{T}$, $\hat{N}$ and $\hat{B}$, known as Frenet frame, playsan important role in calculating the paths of particles moving through space.



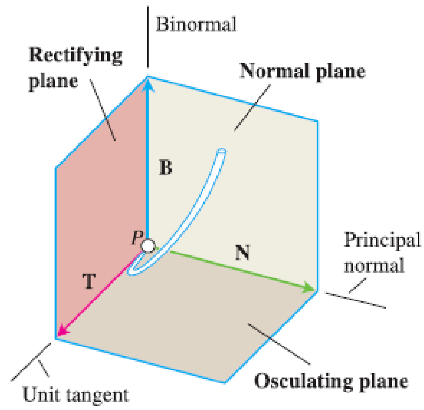


Figure 5: **Left:** The TNB frame of mutually orthogonal unit vectors travelling along a curve in space. **Right:** The three planes in TNB frame.

Since $\hat{B} = \hat{T} \times \hat{N}$, so we have

$$\frac{d\hat{B}}{ds} = \frac{d\hat{T}}{ds} \times \hat{N} + \hat{T} \times \frac{d\hat{N}}{ds} = \hat{T} \times \frac{d\hat{N}}{ds}. \quad (\because \frac{d\hat{T}}{ds} \text{ and } \hat{N} \text{ have same direction.})$$

It implies that $\frac{d\hat{B}}{ds}$ is orthogonal to $\hat{T}$. Also, $\hat{B}$ is of constant length, so $\frac{d\hat{B}}{ds}$ is orthogonal to $\hat{B}$. Hence $\hat{B}$ is orthogonal to $\hat{T}$ and $\hat{B}$ and therefore parallel to $\hat{N}$.

$$\therefore \frac{d\hat{B}}{ds} = -\tau\hat{N},$$

where minus sign is conventional. The scalar τ is called torsion along the curve. Also, we notice that

$$\frac{d\hat{B}}{ds}.\hat{N} = -\tau\hat{N}.\hat{N} = -\tau.$$

$$\therefore \quad \boxed{\tau = -\frac{d\hat{B}}{ds}.\hat{N}}$$

Since $\hat{T}$, $\hat{N}$ and $\hat{B}$ constitute a right-handed frame, there are three planes determined by $\hat{T}$, $\hat{N}$ and $\hat{B}$. The plane of $\hat{T}$ and $\hat{N}$ is called osculating plane; the plane of $\hat{N}$ and $\hat{B}$ is called normal plane, and the plane of $\hat{B}$ and $\hat{T}$ is called rectifying plane. The curvature $\kappa = \left|\frac{d\hat{T}}{ds}\right|$ can be thought of as the rate at which the normal plane turns as the point P moves along its path. Similarly, the torsion $\tau = -\frac{d\hat{B}}{ds}.\hat{N}$ is the rate of turning of osculating plane about $\hat{T}$ as P moves along its path. Torsion measures the twist of the curve.

If we think of the curve as the path of a moving body, then the curvature of the curve (object path) $\left|\frac{d\hat{T}}{ds}\right|$ tells us how much the path turns to the left or right as the object moves along. The torsion $-\frac{d\hat{B}}{ds}.\hat{N}$ tells how much the body's path rotates or twists out of its plane of motion as the body moves along.

Every moving body travels with a TNB frame that characterizes the geometry of its path of motion. (See Figure 6)

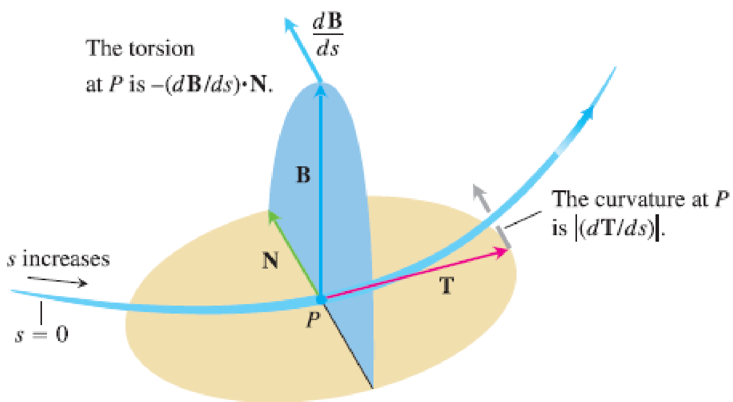


Figure 6:

Derivation of the tangential and normal components of acceleration

We have $\overrightarrow{v} = \dfrac{d\overrightarrow{r}}{dt} = \dfrac{d\overrightarrow{r}}{ds}\dfrac{ds}{dt} = \dfrac{ds}{dt}\hat{T}$.

$$\therefore\ \overrightarrow{a} = \frac{d\overrightarrow{v}}{dt} = \frac{d^2s}{dt^2}\hat{T} + \frac{ds}{dt}\frac{d\hat{T}}{dt} = \frac{d^2s}{dt^2}\hat{T} + \frac{ds}{dt}\frac{d\hat{T}}{ds}\frac{ds}{dt} = \frac{d^2s}{dt^2}\hat{T} + \kappa\left(\frac{ds}{dt}\right)^2\hat{N},$$

where $\dfrac{d\hat{T}}{ds} = \kappa\hat{N}$.

$\therefore$ Tangential component of $\overrightarrow{a}$ $= a_T = \dfrac{d^2s}{dt^2} = \dfrac{d}{dt}(|\overrightarrow{v}|)$.

Normal component of $\overrightarrow{a}$ $= a_N = \kappa\left(\dfrac{ds}{dt}\right)^2 = \kappa|\overrightarrow{v}|^2$.

Obviously, $\overrightarrow{a} = a_T\hat{T} + a_N\hat{N}$ and $|\overrightarrow{a}|^2 = |a_T|^2 + |a_N|^2$.

Ex. Without solving $\hat{T}$ and $\hat{N}$, find acceleration vector of the motion $\overrightarrow{r}(t) = (\sin t - t\cos t)\,\hat{i} + (\cos t + t\sin t)\,\hat{j}$.

Sol. We have $\overrightarrow{v} = \frac{d\overrightarrow{r}}{dt} = t\sin t\,\hat{i} + t\cos t\,\hat{j}$ and $\frac{d\overrightarrow{v}}{dt} = (\sin t + t\cos t)\,\hat{i} + (\cos t - t\sin t)\,\hat{j}$.

$\therefore\ |\overrightarrow{v}| = t$ and $a_T = \frac{d}{dt}(|\overrightarrow{v}|) = 1$.

Also, $|\overrightarrow{a}| = \left|\frac{d\overrightarrow{v}}{dt}\right| = t^2 +$. So $a_N = \sqrt{|\overrightarrow{a}|^2 - a_T^2} = t$.

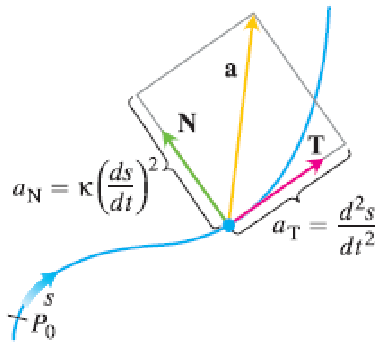


Figure 7: The tangential and normal components of acceleration.

Hence, $\vec{a} = a_T\hat{T} + a_N\hat{N} = \hat{T} + t\hat{N}$.

Useful formulas for curvature and torsion

It can be proved that (please try yourself or see from the text book)

$$\kappa = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}, \quad |\vec{v}| \neq 0.$$

and

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\vec{v} \times \vec{a}|^2}, \quad |\vec{v} \times \vec{a}| \neq 0.$$

Here an overdot denotes the derivative with respect to t.

Ex. Find the curvature and torsion of the helix $\vec{r}(t) = a\cos t\ \hat{i} + a\sin t\ \hat{j} + bt\ \hat{k}$, $a, b \geq 0$, $a^2 + b^2 \neq 0$.
Sol. $\kappa = a/(a^2 + b^2)$, $\tau = b/(a^2 + b^2)$.

SECTION 13.6 (Velocity and Acceleration in Polar Coordinates)

Consider a particle moving along a curve in the polar coordinate plane with position at $P(r, \theta)$ at time t. We shall express its velocity, and acceleration in terms of the moving unit vectors

$$\hat{u}_r = \cos\theta\hat{i} + \sin\theta\hat{j}, \quad \hat{u}_\theta = -\sin\theta\hat{i} + \cos\theta\hat{j}$$

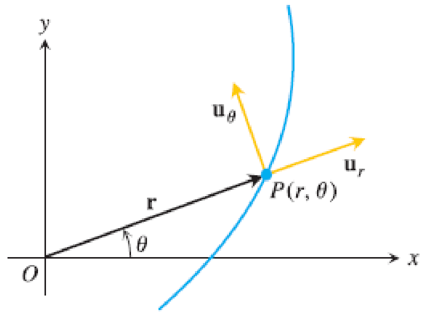


Figure 8:

shown in Figure 8. The vector $\hat{u}_r$ points along the position vector $\overrightarrow{OP}$ so that $\overrightarrow{r} = r\hat{u}_r$. The vector $\hat{u}_\theta$, orthogonal to $\hat{u}_r$ points in the direction of increasing θ. We find that

$$\frac{d\hat{u}_r}{dt} = (-\sin\theta\hat{i} + \cos\theta\hat{j})\dot{\theta} = \dot{\theta}\hat{u}_\theta,$$

$$\frac{d\hat{u}_\theta}{dt} = (-\cos\theta\hat{i} - \sin\theta\hat{j})\dot{\theta} = -\dot{\theta}\hat{u}_r.$$

Therefore velocity vector $\overrightarrow{v}$ can be expressed as

$$\overrightarrow{v} = \frac{d\overrightarrow{r}}{dt} = \frac{d(r\hat{u}_r)}{dt} = \dot{r}\hat{u}_r + r\frac{d\hat{u}_r}{dt} = \dot{r}\hat{u}_r + r\dot{\theta}\hat{u}_\theta$$

as shown in Figure 9. The acceleration vector $\overrightarrow{a}$ can be expressed as

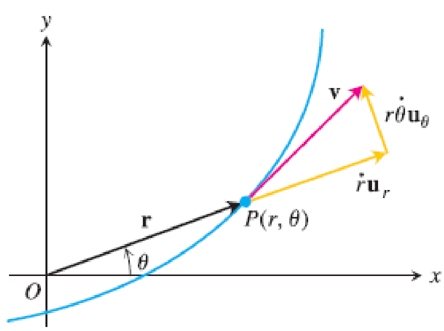


Figure 9:

$$\overrightarrow{a} = \frac{d\overrightarrow{v}}{dt} = \frac{d}{dt}(\dot{r}\hat{u}_r + r\dot{\theta}\hat{u}_\theta) = \ddot{r}\hat{u}_r + \dot{r}\frac{d\hat{u}_r}{dt} + \dot{r}\dot{\theta}\hat{u}_\theta + r\ddot{\theta}\hat{u}_\theta + r\dot{\theta}\frac{d\hat{u}_\theta}{dt}$$

$$= \ddot{r}\hat{u}_r + \dot{r}\dot{\theta}\hat{u}_\theta + \dot{r}\dot{\theta}\hat{u}_\theta + r\ddot{\theta}\hat{u}_\theta + r\dot{\theta}(-\dot{\theta}\hat{u}_r) = (\ddot{r} - r\dot{\theta}^2)\hat{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{u}_r.$$

To extend these equations of motion to space, we add $z\hat{k}$ to the right-hand side of the equation $\overrightarrow{r} = r\hat{u}_r$. Then, in these cylindrical coordinates, we have

$$\overrightarrow{r} = r\hat{u}_r + z\hat{k},$$

$$\vec{v} = \dot{r}\hat{u}_r + r\dot{\theta}\hat{u}_\theta + \dot{z}\hat{k}$$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{u}_r + \ddot{z}\hat{k}.$$

The vectors $\hat{u}_r$, $\hat{u}_\theta$ and $\hat{k}$ make a right-handed frame (see Figure 10) in which

$$\hat{u}_r \times \hat{u}_\theta = \hat{k}, \quad \hat{u}_\theta \times \hat{k} = \hat{u}_r, \quad \hat{k} \times \hat{u}_r = \hat{u}_\theta.$$

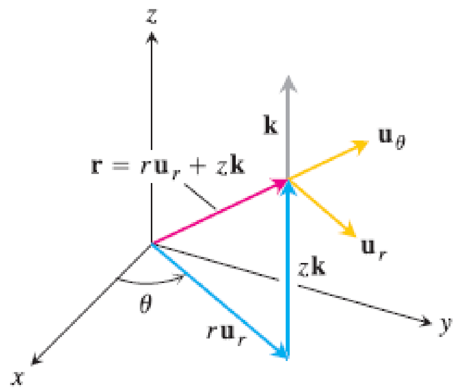


Figure 10:

8. Arc length in cylindrical coordinates

a. Show that when you express $ds^2 = dx^2 + dy^2 + dz^2$ in terms of cylindrical coordinates, you get $ds^2 = dr^2 + r^2\,d\theta^2 + dz^2$.

b. Interpret this result geometrically in terms of the edges and a diagonal of a box. Sketch the box.

c. Use the result in part (a) to find the length of the curve $r = e^\theta, z = e^\theta, 0 \le \theta \le \theta \ln 8$.

8. (a) $x = r\cos\theta \Rightarrow dx = \cos\theta\,dr - r\sin\theta\,d\theta$; $y = r\sin\theta \Rightarrow dy = \sin\theta\,dr + r\cos\theta\,d\theta$; thus
$dx^2 = \cos^2\theta\,dr^2 - 2r\sin\theta\cos\theta\,dr\,d\theta + r^2\sin^2\theta\,d\theta^2$ and
$dy^2 = \sin^2\theta\,dr^2 + 2r\sin\theta\cos\theta\,dr\,d\theta + r^2\cos^2\theta\,d\theta^2 \Rightarrow ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2\,d\theta^2 + dz^2$

(c) $r = e^\theta \Rightarrow dr = e^\theta\,d\theta$

$$\Rightarrow L = \int_0^{\ln 8} \sqrt{dr^2 + r^2\,d\theta^2 + dz^2}$$
$$= \int_0^{\ln 8} \sqrt{e^{2\theta} + e^{2\theta} + e^{2\theta}}\,d\theta$$
$$= \int_0^{\ln 8} \sqrt{3}e^\theta\,d\theta = \left[\sqrt{3}\,e^\theta\right]_0^{\ln 8}$$
$$= 8\sqrt{3} - \sqrt{3} = 7\sqrt{3}$$

(b)