Closures of Relations

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Closure of Relations

Topics:

- Reflexive closure
- Symmetric closure
- Transitive closure

Closure of Relations

- Data Centers in 6 cities (a,b,c,d,e,f)
- Direct, 1-way telephone lines
- Represent this with relation R = {(a,b), (a,c), (c,b), (b,c), (d,f)}
- Indirect link (through b) a to e
- How can we determine all such indirect links?
- Is R transitive?
- Can it be used to determine all the pairs of data centers that can be linked?
- Construct a transitive relation S containing R such that S is a subset of every transitive relation containing R

Closure of Relations

- Data Centers in 6 cities (a,b,c,d,e,f)
- Direct, 1-way telephone lines
- Represent this with relation R = {(a,b), (a,c), (c,b), (b,c), (d,f)}
- S is the smallest transitive relation containing R
- S is the *transitive closure* of R

Closure of Relations

Definition: The *closure* of a relation *R* with respect to property P is the relation obtained by adding the minimum number of ordered pairs to *R* to obtain property P.

In terms of the digraph representation of *R*

- To find the reflexive closure - add loops.
- To find the symmetric closure - add arcs in the opposite direction.
- To find the transitive closure - if there is a path from *a* to *b*, add an arc from *a* to *b*.

Note: Reflexive and symmetric closures are easy . Transitive closures can be very complicated

Reflexive Closure

- $R = \{(1,1), (1,2), (2,1), (3,2)\}$ on a set $A = \{1,2,3\}$
- R is not reflexive!
- How can we produce a reflexive relation containing R that is as small as possible?
- Add (2,2) and (3,3) to R
- To find the reflexive closure - add loops
- Reflexive closure of R!!

Diagonal Relation

Definition: Let A be a set and let $\Delta = \{\langle x, x\rangle \mid x \text{ in } A\}$. Δ is called the *diagonal relation* on *A* (sometimes called the *equality* relation *E*)

Theorem: Let *R* be a relation on A. The *reflexive closure* of *R*, denoted r(*R*), is $R \cup \Delta$.

- Add loops to all vertices on the digraph representation of *R*
- Put 1's on the diagonal of the connection matrix of *R*.

Symmetric Closure

- R = {(1,1), (1,2), (2,2), (2,3),(3,1),(3,2)} on a set A = {1,2,3}
- R is not symmetric!
- How can we produce a symmetric relation containing R that is as small as possible?
- Add (2,1) and (1,3) to R
- To find the symmetric closure - add arcs in the opposite direction
- Symmetric closure of R!!

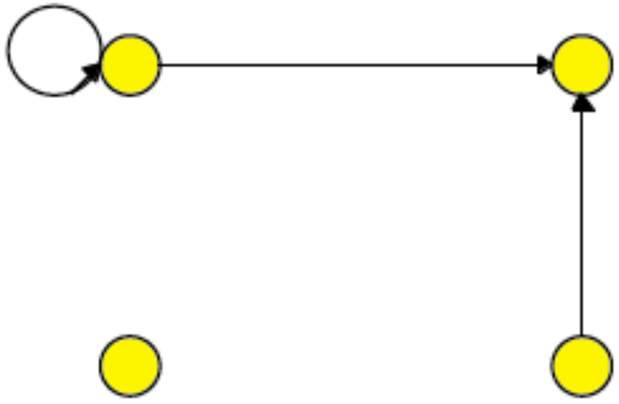
Symmetric Closure

Definition: Let R be a relation on A. Then R^{-1} or the *inverse* of R is the relation $R^{-1} = \{ < y,x > | < x,y > \in R \}$

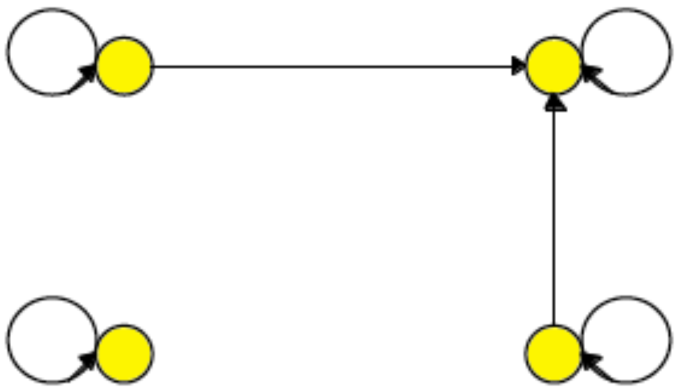
R^{-1} is sometimes denoted as R^{T} or R^{c} and called the *converse* of R

Theorem: Let R be a relation on A. The *symmetric closure* of R, denoted s(R), is the relation $R \cup R^{-1}$.

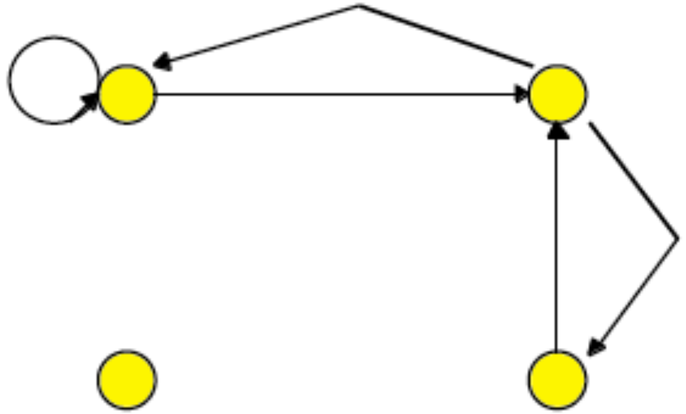
Example



R



$\mathrm{r}(R)$



$\mathrm{s}(R)$

Example

Reflexive closure of the relation, $R = \{(a,b) | a<b\}$ on the set of integers.

$R \cup \Delta = \{(a,b) | a<b\} \cup \{(a,a) | a \in \boldsymbol{Z}\} = \{(a,b) | a \leq b\}$

Example

Symmetric closure of the relation, $R = \{(a,b) | a>b\}$ on the set of positive integers.

$R \cup R^{-1} = \{(a,b) | a<b\} \cup \{(b,a) | a>b\} = \{(a,b) | a \neq b\}$

Paths & Cycles in a Digraph

Definition: A *path* of *length n* in a digraph G is a

sequence of edges $<x_0, x_1><x_1, x_2> . . . <x_{n-1}, x_n>$.

The terminal vertex of the previous arc matches with the initial vertex of the following arc.

If $x_0 = x_n$ the path is called a *cycle* or *circuit*.

Similarly for relations.

Paths & Cycles in a Relation

The term path also applies to relations. Carrying over the definition from directed graphs to relations, there is a path from a to b in R if there is a sequence of elements

$a, x_1, x_2, \ldots, x_{n-1}, b$

with $(a, x_1) \in R$, $(x_1, x_2) \in R$, $\ldots$, and $(x_{n-1}, b) \in R$.

Paths & Cycles in a Digraph

Programming problem #1:

Input: R

Input: 2 nodes a & b in R

Output – all paths in R between a and b along with their lengths

Paths & Cycles in a Digraph

Programming problem #2:

Input: R

Input: a node in R

Output – all cycles involving the node along with their lengths

Paths & Cycles

Theorem: Let R be a relation on A. There is a path of length *n* from *a* to *b* iff $<a,b> \in R^n$.

Proof: (by induction)

• *Basis*: An arc from *a* to *b* is a path of length 1 which is in $R^1 = R$. Hence the assertion is true for $n = 1$.

• *Induction Hypothesis*: Assume the assertion is true for *n*.

Show it must be true for $n+1$.

There is a path of length n+1 from a to b iff there is an x in A such that there is a path of length 1 from *a* to *x* and a path of length *n* from *x* to *b.*

Paths & Cycles

From the Induction Hypothesis,

$<a, x> \in R$ and since <x , b> is a path of length n, $<x, b> \in R^n$.

If $<a, x> \in R$ and $<x, b> \in R^n$, then

$<a,b> \in R^n \circ R = R^{n+1}$

by the inductive definition of the powers of R.

Q. E. D.

Transitive Closures

Finding the TC of R is equivalent to determining which pair of vertices in the associated DG are connected by a path

Defn. – Let R be a relation on a set A, The connectivity relation R* consists of the pairs (a,b) such that there is a path of at least 1 from a to b in R.

$$R^* = \bigcup_{n=1}^{\infty} R^n$$

Transitive Closures

Theorem: The transitive closure of R equals the connectivity relation R*

Lemma 1: Let A be a set with n elements and R be a relation on A. If there is a path of length at least 1 from a to b, then there is such a path with length not exceeding n. Moreover, when a !=b, if there is a path of length at least 1 from a to b, then there is such a path with length not exceeding n-1.

$R^* = R \cup R2 \cup R3 \cup \ldots \cup Rn$

Transitive Closures

LetMR be the zero–one matrix of the relation R on a set with n elements. Then the zero–one matrix of the transitive closure R∗ is

$$M_{R*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} \vee \ldots \vee M_R^{[n]}$$

Transitive Closures

Theorem: The transitive closure of R equals the connectivity relation R^*

Outline of the proof:

1. Show that $R^* \supseteq R$ (by defn.of R^*)
2. Show that R^* is transitive
3. Show that R^* is the smallest set having properties 1 & 2.

Lemma 1: Let A be a set with n elements and R be a relation on A. If there is a path of length at least 1 from a to b, then there is such a path with length not exceeding n. Moreover, when a !=b, if there is a path of length at least 1 from a to b, then there is such a path with length not exceeding n-1.

Outline of the proof:

Using pigeon-hole principle!!

Both proofs will be done in class next week!!

Transitive Closures

Algorithm for finding the transitive closure of R.

procedure *transitive_closure* ($\mathbf{M}_R$: zero-one $n \times n$ matrix)

$\mathbf{A} := \mathbf{M}_R$

$\mathbf{B} := \mathbf{A}$

for $i := 2$ **to** n

begin

$\quad \mathbf{A} := \mathbf{A} \odot \mathbf{M}_R$

$\quad \mathbf{B} := \mathbf{B} \vee \mathbf{A}$

end { $\mathbf{B}$ is the zero-one matrix for R^* }

Transitive Closures: Warshall's Algorithm

Algorithm for finding the transitive closure of R.

Procedure *Warshall* ($\mathbf{M}_R$: rank-n 0-1 matrix)

W := $\mathbf{M}_R$

for k := 1 **to** n

  for i := 1 **to** n

    for j := 1 **to** n

      $w_{ij} := w_{ij} \vee (w_{ik} \wedge w_{kj})$

return W {this represents R^*}

$w_{ij} = 1$ means there is a path from i to j going only through nodes $\leq k$.
Indices i and j may have index higher than k.

Transitive Closures: Warshall's Algorithm

Working of the algorithm!

Example: The matrix below is the matrix representation for a relation . Find the matrix representation of , the transitive closure of .

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Transitive Closures: Warshall's Algorithm

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We know that . To compute W1, we notice that in the first column of W0, there are "1"s in rows 1 and 4. Thus, we replace rows 1 and 4 with the OR of themselves and row 1. We obtain:

Transitive Closures: Warshall’s Algorithm

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$$W_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$