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BITS Pilani
Pilani Campus

Course No: MATH F113

Probability and Statistics

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Simulation

Sumanta Pasari

Office: 6067 C (NAB)
sumanta.pasari@pilani.bits-pilani.ac.in

A Few Comments

- “Monte-Carlo simulation is an extremely bad (costly) method; it should be used only when all alternative methods are worse (Sokal, 1989)”
- Nevertheless, there are times when all else is worse!
- So, we shall anyway discuss:
  - ✓ What is simulation? Its dictionary meaning?
  - ✓ How do we carry out simulation?

What is Simulation?

- Simulation is a technique by which we try to mimic or imitate real systems or process, usually via computer
- Using simulation, we basically produce *synthetic data*.
- Simulation allows us to generate many values of the random variables without physically performing the random experiment.
- Extremely useful in a wide spectrum of disciplines: Operation Research, Physics, Finance, Chemistry, Biology and Medicine, Artificial Intelligence, Catastrophe modelling.

Steps in Simulation Process

- **Step 1: Generating random digits (random numbers) –** What is the purpose? How to generate?
- **Step 2: Calculate *cdf* from *pdf*;** Why? How?
- **Step 3:** (a) For discrete RV, allocate random numbers (via generalized inverse function)

  (b) For continuous RV, allocate random numbers (via ***inverse transformation method)***

Let’s Watch…

https://www.youtube.com/watch?v=amXYVHiGTeg

Random Digits

- Let us generate one digit random number: put 10 balls with distinct levels 0,1, 2, …, 9 in an urn, and pick one ball at random; read the level on it.
- So, each digit (for each ball) has equal probability
- To generate two digit random numbers, repeat the experiment twice.
- Instead of this long process, we may use **random number tables** readily available.
- *By the way, random number generation is altogether a different topic itself, and we shall not discuss much.*

Random Number

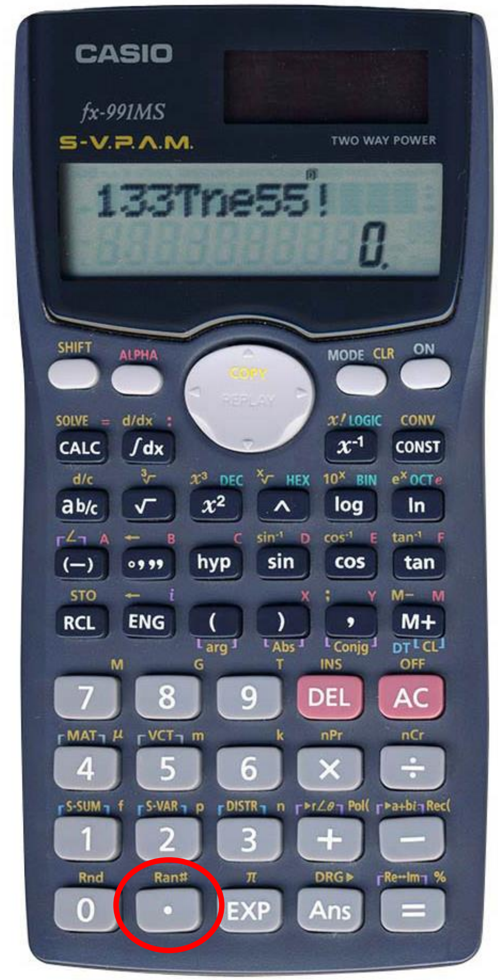
- Random numbers are numbers that occur in a sequence satisfying two conditions:
  - Values are uniformly distributed over a defined interval, usually on the interval [0,1].
  - Impossible to predict future values based on past or present values
- What are the ***pseudo random*** numbers? (can you find it using your calculator?)

Random Digit Table

10097	32533	76520	13586	34673	54876	80959	09117	39292	74945
37542	04805	64894	74296	24805	24037	20636	10402	00822	91665
08422	68953	19645	09303	23209	02560	15953	34764	35080	33606
99019	02529	09376	70715	38311	31165	88676	74397	04436	27659
12807	99970	80157	36147	64032	36653	98951	16877	12171	76833
66065	74717	34072	76850	36697	36170	65813	39885	11199	29170
31060	10805	45571	82406	35303	42614	86799	07439	23403	09732
85269	77602	02051	65692	68665	74818	73053	85247	18623	88579
63573	32135	05325	47048	90553	57548	28468	28709	83491	25624
73796	45753	03529	64778	35808	34282	60935	20344	35273	88435
98520	17767	14905	68607	22109	40558	60970	93433	50500	73998
11805	05431	39808	27732	50725	68248	29405	24201	52775	67851
83452	99634	06288	98083	13746	70078	18475	40610	68711	77817
88685	40200	86507	58401	36766	67951	90364	76493	29609	11062
99594	67348	87517	64969	91826	08928	93785	61368	23478	34113
65481	17674	17468	50950	58047	76974	73039	57186	40218	16544
80124	35635	17727	08015	45318	22374	21115	78253	14385	53763
74350	99817	77402	77214	43236	00210	45521	64237	96286	02655
69916	26803	66252	29148	36936	87203	76621	13990	94400	56418
09893	20505	14225	68514	46427	56788	96297	78822	54382	14598

Random Digit Table
(Table 3.9; Page: 79)

Column	Random digits		
Row	**(1)**	**(2)**	**(3)**
1	10480	15011	01536
2	22368	46573	25595
3	24130	48360	22527
4	42167	93093	06243
5	37570	39975	81837
6	77921	06907	11008
7	99562	72905	56420
8	96301	91977	05463
9	89579	14342	63661
10	85485	36857	43342



Remarks

- To pick 1-digit random number, look at the 1st digit occurring on the table; alternatively, you can choose any cell and see the 1st digit.
- To produce a sequence of 10 one-digit random number, pick randomly a location; start reading either column wise or row wise; if you come to an end of a column/row, continue with the 1st entry of the next column/row. For 10 r-digits random numbers, same…
- **While producing number from table, DO NOT leave any cell in between, otherwise randomness effect will be violated.**

Simulation: Discrete RV

Ex 1: Probability that a computer software salesperson will make 0, 1, 2, 3, 4, or 5 sales on a day are 0.10, 0.30, 0.25, 0.15, 0.14, and 0.06. Use two-digit random numbers 15, 45, 23, 72, 90 to simulate his sales on 5 days. Hence, estimate his average daily sales.

Solution: Allocation of RNs

Value of RV (X=x)	Probability P(X=x)	Cumulative Prob. $P(X \le x)$	Allocation of Random Numbers
0	0.10	0.10	00-09
1	0.30	0.40	10-39
2	0.25	0.65	40-64
3	0.15	0.80	65-79
4	0.14	0.94	80-93
5	0.06	1.00	94-99

Solution: Average Sale

2-digit RNs	Value of RV
15	1
45	2
23	1
72	3
90	4

- As per the RNs given in the exercise, value of RVs (sales) correspond to outcomes 1, 2, 1, 3, 4.
- Hence, average sale = (1+2+1+3+4)/5 = 2.2

- **If you change RNs to 25, 35, 52, 36, 78, then?**
- **If you change RNs to 445, 125, 360, 787, 111, then?**
- **Can you calculate sample variance of the sale?**

Simulation: Discrete RV

Example 3.9.2. (Page 79): Suppose that at a particular airport planes arrive at an average rate of one per minute and depart at the same rate. We are interested in simulating the behavior of the random variable Z, the number of planes on the ground at a given time. We will simulate Z for five consecutive one-minute periods. Note that for each of these periods the random variables X, the number of arrivals, and Y, the number of departures, are both Poisson variables with parameters $k=1$. Assume that initially, $Z= 100$.

Solution: Allocation of RNs

Value of RV (number of arrivals, departures)	Probability	Cumulative Prob.	Allocation of Random Numbers
0	0.368	0.368	000-367
1	0.368	0.736	368-735
2	0.184	0.920	736-919
3	0.061	0.981	920-980
4	0.015	0.996	981-995
5	0.003	0.999	996-998
6	0.001	1.000	999

Random Number Table
(Table 3.9; Page: 79)

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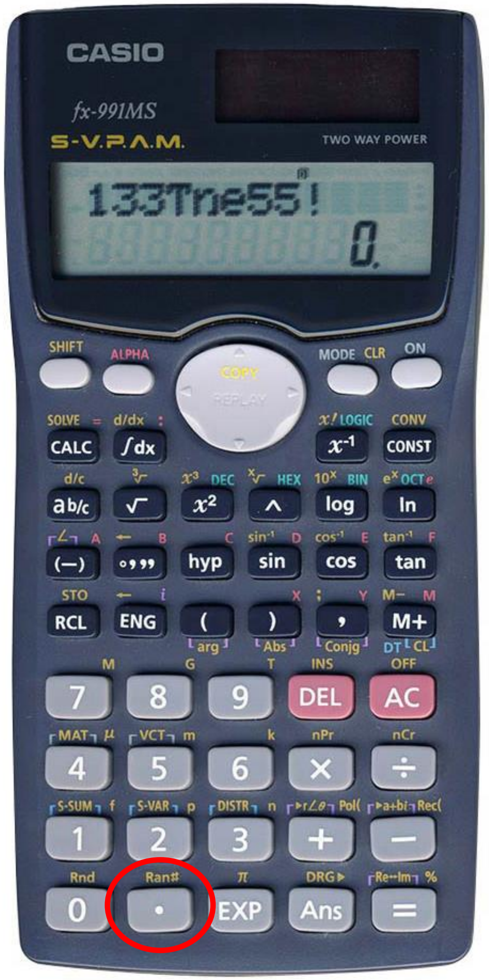

Number of Planes on Ground

Time span (min)	3-digit RNs (from table 3.9)	No of arrivals (x)	No of departures (y)	Number on ground at end of time period (z)
1	015	**0**		
	255		**0**	**100**
2	225	**0**		
	062		**0**	**100**
3	818	**2**		
	110		**0**	**102**
4	564	**1**		
	054		**0**	**103**
5	636	**1**		
	433		**1**	**103**

- On the average, how many planes are on the ground?
- How much variability is there in the number of planes on the ground?

Homework

HW 1: Let X be a discrete random variable with the following distribution :

X	: 1	2	3	4
Probability	: 0.2	0.1	0.4	0.3

(a) Simulate five values of X based on the random numbers 101, 001, 205, 989, 871

(b) Simulate five values of X based on the random numbers 201, 111, 255, 999, 181

(c) Simulate five values of X based on the random numbers 21, 11, 55, 99, 81

(d) Compare.

Rationale: Simulation of Discrete RV

X: discrete RV; $F(x)$ is a step function

We know, $u = F(x) \in [0,1]$

To simulate the values x of X, we find

$$\boxed{x = F^{-1}(u) = \inf\left\{x : F(x) > u\right\}; u \in [0,1)}$$

Here u is a random number $\in [0,1)$, and

values of u are generally provided to you innovate achieve lead

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Probability and Statistics

Simulation: Continuous RV

It is based on the following fact:

If X is a continuous RV with cdf $F(x)$,

then $U = F(X)$ has uniform dist. on $[0,1]$.

Thus, given a value $u \in [0,1)$ of the uniform RV

U on $[0,1]$, a value x of X can be generated

as follows $(\text{Inverse Transformation Method})$:

$$\boxed{u = F(x) \quad \forall x \Rightarrow x = F^{-1}(u)}$$

Example: Exponential (β)

Let X has the exponential distribution with parameter β. How to generate exponential random variate?

HW 2:

If $X \sim \exp\left(\beta = 5\right)$, simulate five values of X using the RNs 97, 23, 09, 40, and 99.

Example: Exponential (β)

$$f(x)=\begin{cases}\frac{1}{\beta}e^{-\frac{x}{\beta}}; x\geq 0, \beta>0\\ 0\ ; \text{o.w}\end{cases} \qquad F(x)=\begin{cases}1-e^{-\frac{x}{\beta}}; x\geq 0, \beta>0\\ 0; \text{o.w}\end{cases}$$

Let $u\in[0,1)$ be a value of U, then setting $u=F(x)$,

$$1-e^{-\frac{x}{\beta}}=u \Rightarrow x=-\beta\ln(1-u), 0\leq u<1$$

$$\boxed{\Rightarrow x=-\beta\ln(u), 0<u\leq 1}$$

$$(\because U\sim unif(0,1)\Rightarrow(1-U)\sim unif(0,1))$$

Example: Uniform (a,b)

Let X has the uniform distribution on $[a,b]$.

How to generate uniform random variate?

HW 3:

If $X \sim U(-3,5)$, simulate five values of X

using the RNs 97, 23, 09, 40, and 99.

Example: Uniform (a,b)

$$f(x)=\begin{cases}\dfrac{1}{b-a};a\le x\le b\\ 0\ ;\text{o.w}\end{cases}\qquad F(x)=\begin{cases}0;x<a\\ \dfrac{x-a}{b-a};a\le x\le b\\ 1;x>b\end{cases}$$

Let $u\in[0,1)$ be a value of U, then setting $u=F(x)$,

$$\frac{x-a}{b-a}=u\ \Rightarrow \boxed{x=a+(b-a)u,0\le u<1}$$

Example: Uniform (a,b)

HW 4: If $X \sim U(4,7)$, simulate three values of X using 3-digit RNs 235, 789, and 178.

Hint:

$$F(x) = \begin{cases} 0; x < 4 \\ \dfrac{x-4}{3}; 4 \le x \le 7 \\ 1; x > 7 \end{cases}$$

3-digit RNs	***u***	***x=F*** $^{-1}$ ***(u)***
235	0.235	4+3(0.235)
789	0.789	4+3(0.789)
178	0.178	4+3(0.178)

Example

HW 5: A continuous random variable *X* has the following pdf:

$$f(x) = \begin{cases} -x^3, & -1 < x < 0 \\ \frac{1}{2}x, & 0 < x < 1 \\ \frac{1}{2}, & 1 < x < 2 \\ 0, & \text{elsewhere} \end{cases}$$

Using the inverse transform method, find a formula for generating values of *X*. Use two digit RNs 37, 13, 83 to generate three values of *X*.

Example

$$F(x)=\begin{cases} 0 & ,\ x<-1 \\ \frac{1}{4}\left(1-x^4\right) & ,\ -1\le x<0 \\ \frac{1}{4}\left(1+x^2\right) & ,\ 0\le x<1 \\ \frac{x}{2} & ,\ 1\le x<2 \\ 1 & ,\ x\ge 2 \end{cases}$$

- **Also, F(0) = 0.25, F(1) = 0.50, F(2) = 1**

Example

Let $u \in [0,1)$ be a value of U, then setting $u = F(x)$,

If $0 \le u < 0.25$, $\quad x = -(1-4u)^{1/4}$

If $0.25 \le u < 0.50$, $\quad x = (4u-1)^{1/2}$

If $0.5 \le u < 1$, $\quad x = 2u$

For $u_i = 0.37,\ 0.13,\ 0.83,$ calculate x_i.

Example

HW 6: A continuous random variable *X* has the following dist:

(a)

$$f(x)=\begin{cases} x; \ 0\le x\le 1 \\ 2-x; \ 1<x\le 2 \\ 0; \ \text{e.w} \end{cases}$$

$\text{RNs}: \ 25, 84, 49, 76, 08$

(b)

$$F(x)=\begin{cases} 0 & , x<1 \\ \dfrac{6x-x^2-5}{5} & , 1\le x<3 \\ \dfrac{x^2-6x+13}{5} & , 3\le x<4 \\ 1 & , x\ge 4 \end{cases}$$

RNs: 152, 928, 041

Using the inverse transform method, generate values of *X*.

Example

HW 7: Simulate one value of Binomial RV with $n=3$, $p=0.40$, using two-digit RNs 34, 45, 98.