Chapter 4 : A.P. French

Lec : 5, 6, 10, 17 *Tut :* 4, 8, 13 *Suggested :* 2, 3, 7, 9, 11, 12, 14, 15

Problem no. 16 *is not in the course.*

4.4. *From* (1) $kh = mg \Rightarrow \frac{k}{m} = \frac{g}{h}$ *and From* (2) $bu = mg \Rightarrow \frac{b}{m} = \frac{g}{u}$

(a) *Diff. eq. of damped harmonic motion :* $m\ddot{x} + b\dot{x} + kx = 0$

$$\Rightarrow \ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0 \Rightarrow \boxed{\ddot{x} + \frac{g}{u}\dot{x} + \frac{g}{h}x = 0}$$

$$(b)\ \gamma = \frac{b}{m} = \frac{g}{u} = \frac{g}{3\sqrt{gh}} = \frac{1}{3}\sqrt{\frac{g}{h}} \text{ and } \omega_0^2 = \frac{g}{h}; \therefore \omega^2 = \omega_0^2 - \frac{\gamma^2}{4} \Rightarrow \boxed{\omega = \sqrt{\frac{35}{36}\frac{g}{h}}}$$

$$(c)\ E = E_0 e^{-\gamma t} \Rightarrow \frac{E}{E_0} = e^{-1} = e^{-\gamma t} \Rightarrow \boxed{t = \frac{1}{\gamma} = 3\sqrt{\frac{h}{g}}}$$

$$(d)\ \boxed{Q = \frac{\omega}{\gamma} = \sqrt{\frac{35}{36}\frac{g}{h}} \times 3\sqrt{\frac{h}{g}} = \sqrt{\frac{35}{4}} \approx 3}$$

$$(e)\ x = Ae^{-\frac{\gamma}{2}t}\cos(\omega t - \delta); At, t = 0, x = 0; \therefore \cos(-\delta) = 0 \Rightarrow \boxed{\delta = \frac{\pi}{2}}$$

$$At, t = 0, \dot{x} = \upsilon_0\ (say) \Rightarrow A\left(-\frac{\gamma}{2}\right)e^{-\frac{\gamma}{2}t}\cos(\omega t - \delta) - A\omega e^{-\frac{\gamma}{2}t}\sin(\omega t - \delta) = \upsilon_0$$

$$At\ t = 0, \therefore \dot{x} = 0 - A\omega\sin(-\delta) = A\omega = \upsilon_0 \Rightarrow \boxed{\upsilon_0 = A\omega} \Rightarrow \boxed{A = \frac{\upsilon_0}{\omega}}$$

$$\therefore \boxed{x = \frac{\upsilon_0}{\omega}e^{-\frac{\gamma}{2}t}\cos(\omega t - \delta) = \frac{\upsilon_0}{\omega}e^{-\frac{\gamma}{2}t}\cos\left(\omega t - \frac{\pi}{2}\right) = \frac{\upsilon_0}{\omega}e^{-\frac{\gamma}{2}t}\sin\omega t} \text{ Do the graph.}$$

$$(f)\ F = F_0\cos\omega t \approx F_0 e^{i\omega t} = mge^{i\omega t}$$

$$\therefore m\ddot{x} + b\dot{x} + kx = mge^{i\omega t} \Rightarrow x = \frac{ge^{i\omega t}}{\left[-\omega_0^2 + \omega^2 + i\gamma\omega\right]}; x^* = \frac{ge^{-i\omega t}}{\left[-\omega_0^2 + \omega^2 - i\gamma\omega\right]} = -\frac{ge^{-i\omega t}}{\left[\omega_0^2 - \omega^2 + i\gamma\omega\right]} =$$

$$\therefore A^2(\omega) = xx^* = \frac{g^2}{\left[\left(\omega^2 - \omega_0^2\right)^2 + \gamma^2\omega^2\right]} \Rightarrow A(\omega) = \frac{g}{\sqrt{\left[\left(\omega^2 - \omega_0^2\right)^2 + \gamma^2\omega^2\right]}}$$

$$Now, \omega_0^2 = \frac{g}{h}, \omega^2 = \frac{2g}{h} \text{ and } \gamma^2\omega^2 = \left(\frac{1}{3}\sqrt{\frac{g}{h}}.\sqrt{2}\sqrt{\frac{g}{h}}\right)^2 = \frac{2}{9}\frac{g^2}{h^2}; \therefore \boxed{A(\omega) = \frac{g}{\sqrt{\frac{g^2}{h^2} + \frac{2}{9}\frac{g^2}{h^2}}} = h\frac{3}{\sqrt{11}} \approx 0.9h}$$

4.8. (a) *If the spring force is absent, the diff. eq.*

$m\ddot{x}+b\dot{x}=0$; *The trial solution given* : $x=C-\dfrac{\upsilon_0}{\gamma}e^{-\gamma t}$. $\therefore\ \dot{x}=\upsilon_0 e^{-\gamma t}$ *and* $\ddot{x}=-\upsilon_0\gamma e^{-\gamma t}$

$$\Rightarrow m\ddot{x}+b\dot{x}=-m\upsilon_0\gamma e^{-\gamma t}+b\upsilon_0 e^{-\gamma t}=\upsilon_0\left(-m\gamma+b\right)e^{-\gamma t}=\upsilon_0\left(-m\frac{b}{m}+b\right)e^{-\gamma t}=0.$$

So, the trial solution satisfies the diff. eq. and hence is a possible form of solution.

(b) $x=A\cos(\omega t-\delta)$; *At*, $t=0$, $x=0$ *and* $\dot{x}=\upsilon_0$ *(say)* $\therefore \cos(-\delta)=0$ *and* $\delta=\dfrac{\pi}{2}$

In absence of spring force, the differential equation : $m\ddot{x}+b\dot{x}=F_0\cos\omega t\approx$ *real part of* $F_0e^{i\omega t}$

$\therefore\ \ddot{x}+\gamma\dot{x}=\dfrac{F_0}{m}e^{i\omega t}\Rightarrow$ *Following the mathematical method :*

$$x=\frac{\frac{F_0}{m}e^{i\omega t}}{-\omega^2+i\gamma\omega}=-\frac{\frac{F_0}{m}e^{i\omega t}}{\omega^2-i\gamma\omega};\ \therefore\ xx^*=\left(-\frac{\frac{F_0}{m}e^{i\omega t}}{\omega^2-i\gamma\omega}\right)\left(-\frac{\frac{F_0}{m}e^{i\omega t}}{\omega^2-i\gamma\omega}\right)^*$$

$$=\frac{\left(\frac{F_0}{m}\right)^2}{\left(\omega^2-i\gamma\omega\right)\left(\omega^2+i\gamma\omega\right)}=\frac{\left(\frac{F_0}{m}\right)^2}{\left(\omega^4+\gamma^2\omega^2\right)};$$

$$\therefore\ \boxed{Amplitude=\sqrt{xx^*}=\frac{F_0}{m\sqrt{\omega^4+\gamma^2\omega^2}}\ \text{and}\ \tan\delta=\frac{\gamma\omega}{\omega^2}=\frac{\gamma}{\omega}}$$

$\therefore$ *Complete solution* : $x=C-\dfrac{\upsilon_0}{\gamma}e^{-\gamma t}+A\cos(\omega t-\delta)$

At, $t=0, x=0;\Rightarrow C-\dfrac{\upsilon_0}{\gamma}+A\cos\delta=0\Rightarrow\boxed{C=\dfrac{\upsilon_0}{\gamma}-A\cos\delta}$

At, $t=0,\dot{x}=0;\Rightarrow \upsilon_0-A\omega\sin(-\delta)=\upsilon_0+A\omega\sin\delta=0\Rightarrow\boxed{\upsilon_0=-A\omega\sin\delta}$

$$\therefore\ \boxed{C=\frac{\upsilon_0}{\gamma}-A\cos\delta=-A\left(\frac{\omega}{\gamma}\sin\delta+\cos\delta\right)}$$

4.13.

$\left(a\right) w.r.t.\ the\ \ fig.$

$\omega_0 = 40\,s^{-1}\ \ and\ \ \Delta\omega = 2s^{-1} \Rightarrow \boxed{Q = \dfrac{\omega_0}{\Delta\omega} = 20}$

$\left(b\right) \gamma = \dfrac{\omega_0}{Q} = \Delta\omega = 2s^{-1}$

$\therefore E\left(t\right) = E_0 e^{-\gamma t} = E_0 .e^{-5} \Rightarrow \gamma t = 5 \Rightarrow t = \dfrac{5}{\gamma} = \dfrac{5}{2} = 2.5\,s^{-1}.$

$Time\ \ period\ \ of\ \ oscillation: \dfrac{2\pi}{\omega_0} = \dfrac{2\pi}{40} = \dfrac{\pi}{20} = 0.05\pi\ \ s$

$\therefore 0.05\pi\ \ sec \equiv 1 cycle \Rightarrow 0.05\pi\ \ sec = \dfrac{2.5}{0.05\pi} = \dfrac{50}{3.14} = 15.92 \approx 16\ \ cycle.$