Ch. 4. WORK AND ENERGY (Lecture 7).

Carrying forward from the initial remarks in slides.

$\vec{F} = m\vec{a}$ can in principle solve everything. In practice however we are stuck because

$$\vec{F}(\vec{r}) = \frac{d\vec{v}(t)}{dt}$$

To integrate this with respect to time we need to know force as a function of time whereas very often we know it as a function of position. If we somehow manage to solve this integral, we automatically land into the notion of energy (as we will soon see) as the first integral of motion. Energy in turn derives its usefulness from its conservation and conversion. The question then is - What is the connection between energy and force, since both are capable of giving information about physical system. To understand this question, we must pose another question —

What does a force $\vec{F}$ do?

Example: We will analyze this question in terms of the effect of a constant force F on a particle of mass m moving in 1-D.

$$F = \frac{dP}{dt} = \frac{m\,dV}{dt} = ma$$

Say, the force is acting for time t. Multiply both the sides by t.

$$Ft = mat = m(V_2 - V_1).$$

If the force $F(t)$ is varying with time, then

$$F(t) = m\frac{dV}{dt}$$

$$\boxed{\int_{t_1}^{t_2} F(t)\,dt = m\int_{V_1}^{V_2} dV = m(V_2 - V_1).} \qquad (1)$$

Thus, the effect of force is in terms of its impulse (or its time integral) which results in the change in momentum. Since the change in momentum can be measured, Impulse is a good measure of the effect of force.

Now, suppose the particle covers a distance x, when the force acts on it (assume constant F). Multiplying both the sides of $F = ma$ by x, now

$$Fx = max$$
$$= ma\left(\frac{V_1 + V_2}{2}\right)t$$
$$= \frac{1}{2}m(V_2 - V_1)(V_2 + V_1)$$

$$\boxed{Fx = \frac{1}{2}mV_2^2 - \frac{1}{2}mV_1^2} \qquad (2)$$

$\Rightarrow$ Work done by $F \equiv$ Change in K.E.

If F is not constant

$$\int_{x_1}^{x_2} F\,dx = m \int_{V_1}^{V_2} \frac{dV}{dt}\,dx$$

But $\frac{dV}{dt}dx = \frac{dV}{dx}dx \cdot \frac{dx}{dt} = V\,dV$

Thus,

$$\boxed{\int_{x_1}^{x_2} F\,dx = m \int_{V_1}^{V_2} V\,dV = \frac{1}{2}m(V_2^2 - V_1^2)} \quad (2)$$

This is another way to characterize the impact of a force. ~~measure~~. Relation (1) & (2) are obtained by integrating F with respect to time and space and apriori, there is no reason to choose one over the other. However, the apparent similarity is deceptive for the following reasons.

1) Unlike t, $\vec{x}$, $\vec{P}$, and $\vec{F}$ are all vectors. This means that LHS of 1 (and hence RHS) will always be a vector.

2) Since $\vec{F}$ and $d\vec{r}$ are vectors, as you go to 2-D or 3-D, the effect of $\bar{F}$ on the mass will depend on the angle between $\vec{F}$ and $d\vec{r}$. For instance in circular motion F is constantly applied and it changes the momentum continuously but the magnitude of velocity remains unchanged.

(3) Given that $\vec{F}$ and $d\vec{r}$ are both vectors and there are two different ways to combine two vectors (dot and cross product) how do we generalize (2) as we go from 1-D to 2-D, 3-D? Ofcourse we know in hindsight that LHS is work done by $\vec{F}$ and hence we should have $\vec{F}.d\vec{r}$, but even without that, the RHS in 1-D case provides hint. Since it involves difference of two terms both quadratic in velocities it better be a scalar product (because $\vec{v}\times\vec{v}=0$). Since RHS is a scalar, LHS, better be a scalar, and $W = \int \vec{F}.d\vec{r}$.

(4) Here is another perspective on the space integral of F. We are trying to ask - what is the effect of F applying a force in an arbitrary direction, on the motion of an object? If apply F normal to the direction of motion we only change the direction of velocity without affecting its magnitude (as in circular motion). Applying force in a parallel direction changes only magnitude of velocity without affecting direction. This is precisely what RHS is about and hence we better use $F_{\parallel}$ i.e. dot product $\vec{F}.d\vec{r}$.

Thus, starting from $\vec{F}(\vec{r}) = m\dfrac{d\vec{v}}{dt}$

we consider what happens when a particle moves a short distance $\Delta\vec{r}$ (during which $\vec{F}$ is effectively constant), and take a scalar product $\vec{F}.\Delta\vec{r}$. thus,

$$\vec{F}.\Delta\vec{r} = m\frac{d\vec{v}}{dt}.\Delta\vec{r}$$

This steps seems to assume that we know the entire trajectory and hence $\Delta\vec{r}$ before hand. Isn't this what we seek as our solution? Though an important objection, let us presume we know the trajectory and move ahead. For sufficiently short path, $\Delta\vec{r} = \vec{v}\,\Delta t$. thus,

$$m\frac{d\vec{v}}{dt}.\Delta\vec{r} = m\frac{d\vec{v}}{dt}.\vec{v}\,\Delta t$$

[An aside on a vector identity. Let $\vec{A}$ and $\vec{B}$ be two vectors: then

$$\frac{d}{dt}[\vec{A}.\vec{B}] = \frac{d\vec{A}}{dt}.\vec{B} + \vec{A}.\frac{d\vec{B}}{dt}$$

If $\vec{A} = \vec{B}$

$$\frac{d}{dt}[A^2] = 2\vec{A}.\frac{d\vec{A}}{dt}.$$

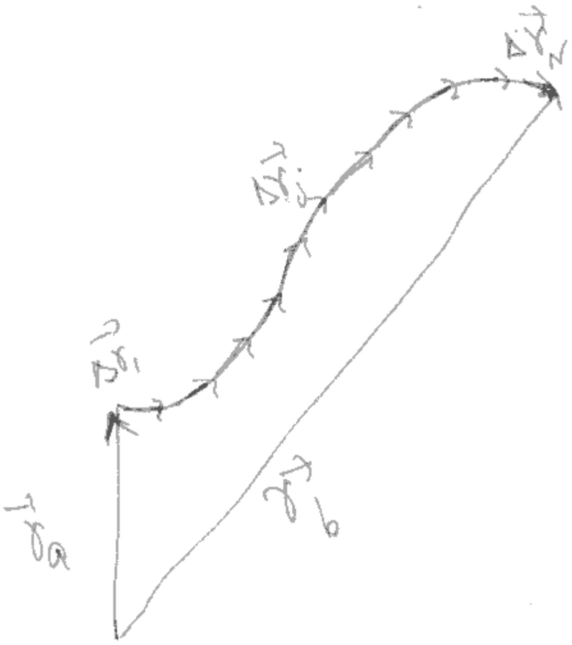
Remarks 1) If vector $\vec{A}$ stays constant in magnitude then LHS = 0. This implies $\dfrac{d\vec{A}}{dt}$ is $\perp$ $\vec{A}$. That is the only way $\vec{A}$ can change is rotate.

2) It assumes that dot product of two vectors is commutative
]

Using $\frac{d}{dt}[A^2] = 2\vec{A}\cdot\frac{d\vec{A}}{dt}$

$$\vec{V}\cdot\frac{d\vec{V}}{dt} = \frac{1}{2}\frac{d}{dt}(V^2).$$

$$\vec{F}\cdot\Delta\vec{r} = \frac{m}{2}\frac{d}{dt}(v^2)\,\Delta t.$$



If we divide the entire trajectory from the initial position r_a to final position r_b into N short segments of length Δr_j. Then

$$\vec{F}(\vec{r}_j)\cdot\Delta\vec{r}_j = \frac{m}{2}\frac{d}{dt}(V_j^2)\,\Delta t_j$$

Adding the equations of all segments

$$\sum_{j=1}^{N}\vec{F}(\vec{r}_j)\cdot\Delta\vec{r}_j = \sum_{j=1}^{N}\frac{m}{2}\frac{d}{dt}(V_j^2)\,\Delta t_j$$

Taking the limit $\Delta r_j \to 0$, $N \to \infty$.

$$\int_{r_a}^{r_b}\vec{F}\cdot d\vec{r} = \int_{t_a}^{t_b}\frac{m}{2}\frac{d}{dt}(v^2)\,dt = \frac{m}{2}\int_{t_a}^{t_b}\frac{d}{dt}(v^2)\,dt$$

$$\boxed{\begin{aligned}\int_{r_a}^{r_b}\vec{F}\cdot d\vec{r} &= \frac{1}{2}mV_b^2 - \frac{1}{2}mV_a^2 \\ W_{ba} &= K_b - K_a.\end{aligned}}$$

This is the famous Work-Energy Theorem. We also ~~eta~~ established the connection between energy and Force. Actually this only half the connection.

LECTURE - 9-10

We have seen that there are two broad classes of forces

a) Conservative forces (e.g. all fundamental forces)

b) Non-conservative forces (e.g. friction, viscous drag etc).

Often both kinds of forces are at work - for instance an object falling through air experiences gravity as well as viscous drag. Moreover you may be wondering as to what does non-conservation mean? After all energy cannot vanish into thin air. And what about the Work-energy theorem?

Let us write the total force acting on a body as a sum of two parts.

$$\vec{F} = \vec{F}_c + \vec{F}_{NC}$$

{ C and NC obviously ~~st~~ stand for conservative & non conservative forces.

The good part! Work energy theorem is true whether or not the force is conservative or not. The total work done by a force $\vec{F}$ as the particle moves from a to b is:

$$W_{ba}^{total} = \oint_a^b \vec{F}\cdot d\vec{r}$$

$$= \oint_a^b \vec{F}^c\cdot d\vec{r} + \oint_a^b \vec{F}^{NC}\cdot d\vec{r}$$

$$= -U_b + U_a + W_{ba}^{NC}.$$

{ Note $\oint$, here the curve on the integral stand for C for curve along which integral is carried out. A circle $\oint$ on the integral means the ~~i~~ integral is around a closed path. BOTH ARE DIFFERENT!

Here U is the P.E associated with the conservative force. For the work done by non-conservative, one cannot associate a function of position such as potential energy. Since Work-energy theorem is always true, $W_{ba}^{total} = K_b - K_a$, thus

$$-U_b + U_a + W_{ba}^{NC} = K_b - K_a .$$

or $$\boxed{K_b + U_b - (K_a + U_a) = W_{ba}^{NC}}$$

If we define total mechanical energy $E = K + U$, then E is no longer conserved, but depends on the state of the system.

$$\boxed{E_b - E_a = W_{ba}^{NC} .}$$

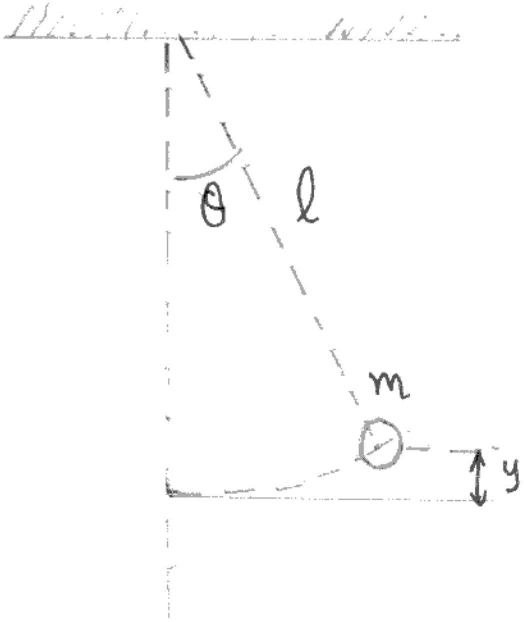
This is a generalization of the statement of conservation of total mechanical energy (word mechanical being important) to the case where non-conservative forces are present. Work done by non-conservative forces is dissipated as heat. As far as total mechanical energy is concerned it is Lost. However, if we take this loss into account, then total energy is always conserved (as it should be).

JUST HOW USEFUL IS ENERGY PERSPECTIVE

Newton's laws of motion and energy methods offer two different approaches to solve problems of dynamical systems. From the standpoint of mechanics, the two approaches are equivalent. However, as we discussed in the class, conservation laws follow directly from the symmetry properties of the transformations in space-time (details are beyond the scope of this course) and hence, in some sense more fundamental than Newton's laws which break down at high speeds (Well Newton's second still holds good}, it is the Newton's conception of absolute and independent notion of space-time that is challenged) and atomic scales. In both these regimes conservation laws hold true.

ENERGY SOLUTION TO A DYNAMICAL PROBLEM

In class we had solved the problem of simple harmonic motion of a mass spring system (problem 3.7) using Newton's 2^{nd} law. To illustrate the power of energy method, we will now solve the problem of simple pendulum using energy methods. As you will learn, this method offers far more insight than you can glean from Newton's 2^{nd} law.



The work done by gravitational force $-m\vec{g}$ on mass m, ~~is~~ $\cancel{m\vec{g}}$ as it moves from $y=0$ to $y=y$ is $U(y)-U(0) = mgy$. The total energy of pendulum at any θ is

$$E = K + U$$
$$= \frac{1}{2} m l \dot{\theta}^2 + \cancel{\tfrac{1}{2}}\, mgy .$$

Here l = length of pendulum
$$y = l(1-\cos\theta).$$

At the end of the swing, say $\theta = \theta_0$ and $\dot{\theta} = 0$. Since there are no non-conservative forces, total energy is conserved. So

$$\frac{1}{2} m l^2 \dot{\theta}^2 + mgl(1-\cos\theta) = mgl(1-\cos\theta_0).$$

$$\Rightarrow \quad \frac{d\theta}{dt} = \sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_0)} \quad \text{—— } ★$$

which can rearranged to give

$$\int \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} = \sqrt{\frac{2g}{l}} \int dt$$

Let us look at the solution for the case of small amplitude, so that, $\cos\theta \simeq 1 - \theta^2/2$

$$\int \frac{d\theta}{\sqrt{\frac{1}{2}}\sqrt{\theta_0^2 - \theta^2}} = \sqrt{\frac{2g}{l}} \int dt$$

Introducing $\omega = \sqrt{g/l}$, we can rewrite

$$\int \frac{d\theta/\theta_0}{\sqrt{1-(\theta/\theta_0)^2}} = \omega \int dt \qquad \left\{ \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x \right.$$
Put $x = \theta/\theta_0$

Taking the lower limits of integration ($\theta=0, t=0$) and upper limits to be (θ, t).

$$\sin^{-1}\theta/\theta_0 - 0 = \sqrt{\frac{g}{l}}(t-0)$$

$$\boxed{\theta = \theta_0 \sin\omega t}$$

This is what we would obtain by solving $\vec{F} = m\vec{a}$. More importantly, the starred eq^n (*) on the previous page is a general equation that is not limited to the small-angle approximation. It has mathematically exact solution in terms of functions called elliptic integrals. Even without going into that complexity we can use equation (*) to find an important result: that is, correction to the pendulum period due its finite amplitude. Such a correction would be very difficult to extract starting with the Newtonian equation of motion.

TWO IMPORTANT QUESTIONS

1) What distinct advantage energy method had to offer the bonus advertised above?

Ans: Did you notice that as opposed to solving Newton's second order differential equation, we integrated only once here. This is because when you start with energy equation, half of the problem is already solved for you, as you exploit work energy theorem. When you ~~we~~ further write work-done by a conservative force in terms of a potential energy function, you are exploiting energy as first ~~int~~ integral of motion. With one more integral and you have solved it. In starting with Newton's 2^{nd} order d.e, you do not avail the benefits of energy conservation.

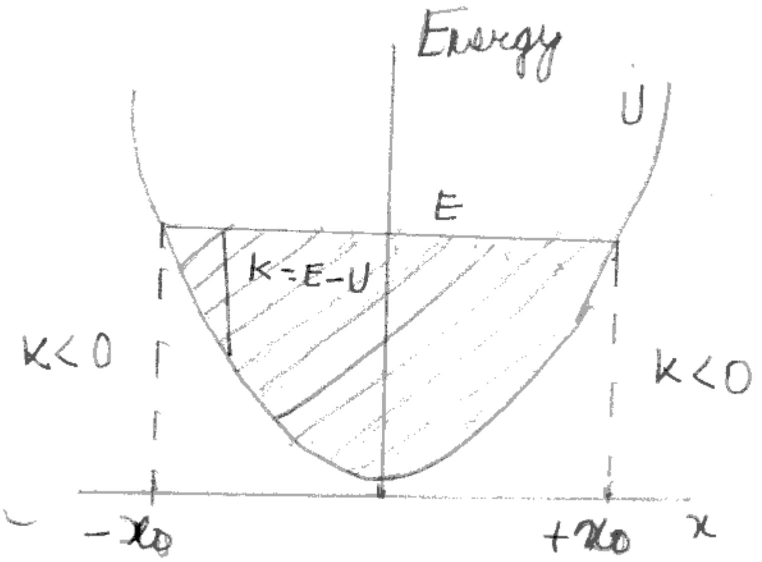
2) ~~Th~~ When was this done in the class?

Ans. Short answer: it was not done for the want of time. However, this being such an important and powerful aspect that compares favourably to the much venerated dynamical approach that I could not resist including in my notes.

ENERGY DIAGRAMS

An energy diagram is a plot of total energy and potential energy U as a function of position. Such a simple plot can reveal many key features of the problem without having to solving it. We will look at three situations a) Potentials that lead to bound states b) Only unbound states c) possibility of bound as well as unbounded states.

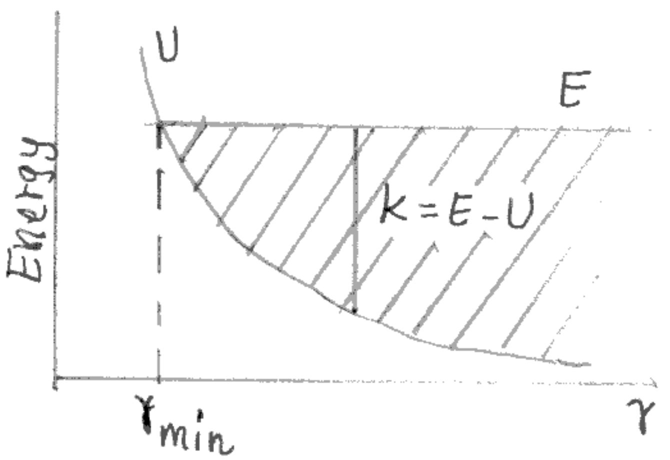
a) HARMONIC OSCILLATOR POTENTIAL: $\boxed{U = \frac{1}{2} k x^2}$



Here is the energy diagram for harmonic oscillator. P.E is a parabola centered at the origin. Because total energy E is a constant for a conservative system E is shown as a horizontal line. Motion is limited to the shaded region where $E \geq U$. This determines the limits of the motion $x = \pm x_0$ which are called turning points. Since $K = E - U$, beyond these turning points, the K.E of the particle would be negative and hence the particle is permanantly confined or bounded within $x = \pm x_0$.

Note that these turning points are determined by the value of the total energy E. Also note that these turning points exist because the potential energy grows indefinitely with distance. It also shows that k.E is zero at the turning points and maximum at the origin and hence the particle accelerating back and forth. Greater the E, further away are the turning points.

b) REPULSIVE INVERSE SQUARE LAW POTENTIAL



$\vec{F} = \frac{A}{r^2}\hat{r}$ (A is a positive constant)

$U = \frac{A}{r}$ $U(\infty) = 0$.

Clearly for $r < r_{min}$

$K < 0$

Repulsive inverse square law radial force compels a particle to move along a radial line since for $r < r_{min}$, $K < 0$ there is a distance of closest approach determined by total energy E. Since for $r > r_{min}$ $E > U$ particle accelerates (K.E is increasing) all the way to infinity and hence motion is unbounded (not confined to any region like in harmonic oscillator potential). So if throw the particle towards origin, it rebounds at r_{min} (determined by energy with which we threw) ~~it rebounds at r_{min}~~ and goes back to infinity. Its speed at each point being same during in-bound and out bound journey.

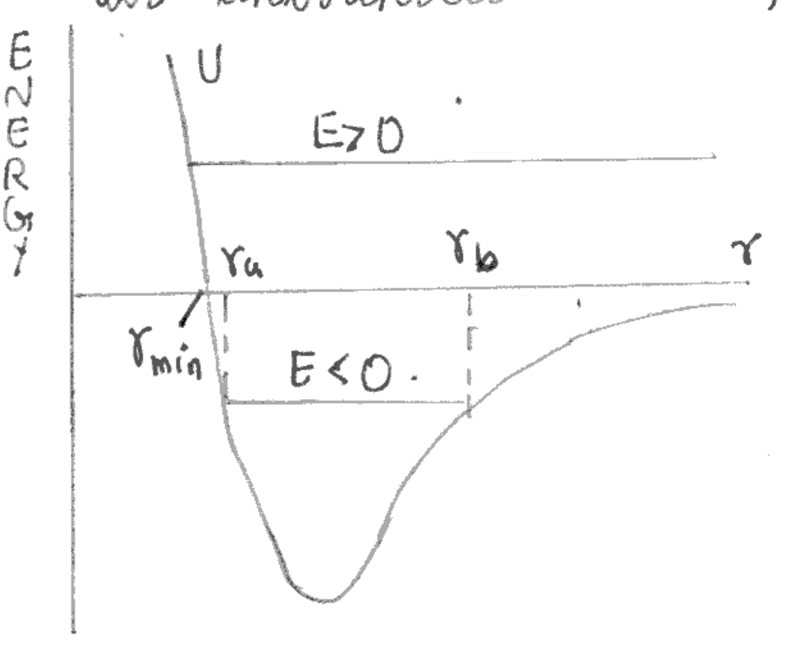
c) POTENTIALS WHICH ALLOW BOUNDED AS WELL AS UNBOUNDED MOTION DEPENDING UPON TOTAL ENERGY

By now it should be pretty clear as to what does it take to have a bounded motion and unbounded motion.

Bounded motion: Potential should have a minimum so that $\frac{dU}{dx} = 0$.

Unbounded motion Potential should monotonically fall to zero.

To have a possibility of bounded as well as unbounded motion, both these ~~possi~~ features are present. Van der Waals potential shown here is a case in point. For $E > 0$, though there is a fixed distance of closest approach, for $r > r_{MIN}$ K.E is always positive



and the motion is unbounded. However, for $E<0$, K.E is <0 for ~~r~~ $r<r_0$ and $r>r_0$. The motion is clearly bounded. This tells us that when two atoms approach ~~each~~ each other with $E>0$ (such as collision of two hydrogen atoms in a gaseous state) they will reflect after reaching the distance of closest approach (r_{MIN}) and will never form a molecule. However, if there is some means to loose excess energy to make E negative, then they may form a bound state. The means could be the presence of third atom or a surface. For instance if we insert a piece of platinum in the hydrogen gas, then the hydrogen atoms tightly adhere to the surface of the platinum and if a collision occurs between two atoms at the surface, the excess energy is released to the surface, and the molecule which is not strongly attracted to the surface, leaves. In fact, so much energy is delivered to the platinum that it glows brightly. A third atom can also ~~a~~ take away the excess energy, but that is a rare event at low pressures, but it become important at high pressure.