Q. 1 (i) The augmented matrix is given by,

$$\left[\begin{array}{ccc|c} 2 & 3 & \alpha & \beta \\ 0 & -1 & 2 & \gamma \\ 1 & 3 & -2 & 1 \end{array}\right]$$

$R_1 \leftrightarrow R_3$ $\left[\begin{array}{ccc|c} 1 & 3 & -2 & 1 \\ 0 & -1 & 2 & \gamma \\ 2 & 3 & \alpha & \beta \end{array}\right]$; $R_3 \to R_3 - 2R_1$, $R_2 \to -R_2$ $\left[\begin{array}{ccc|c} 1 & 3 & -2 & 1 \\ 0 & 1 & -2 & -\gamma \\ 0 & -3 & \alpha+4 & \beta-2 \end{array}\right]$

$R_3 \to R_3 + 3R_2$ $\left[\begin{array}{ccc|c} 1 & 3 & -2 & 1 \\ 0 & 1 & -2 & -\gamma \\ 0 & 0 & \alpha-2 & \beta-3\gamma-2 \end{array}\right]$ 3

(a) Unique solution : $\alpha - 2 \neq 0$ & $\beta, \gamma \in \mathbb{R}$ 1 * Mark deducted for not writing $\beta\gamma \in R$

ie $\alpha \neq 2$ & $\beta, \gamma \in \mathbb{R}$

Rank (A) = Ran(A|B) = no. of unknowns = 3

2

(b) No solution: $\alpha = 2$ & $\beta - 3\gamma - 2 \neq 0$

Rank(A) = 2 ; Rank(A|B) = 3, so system is inconsistent. 2

(c) Infinite Solution : Rank(A) = Rank(A|B) < no. of unknowns

$\boxed{\alpha = 2}$ & $\boxed{\beta - 3\gamma - 2 = 0}$ 2

Solution set for (c) $\left[\begin{array}{ccc|c} 1 & 3 & -2 & 1 \\ 0 & 1 & -2 & -\gamma \\ 0 & 0 & 0 & 0 \end{array}\right]$ or $\left[\begin{array}{ccc|c} 1 & 0 & 4 & 1+3\gamma \\ 0 & 1 & -2 & -\gamma \\ 0 & 0 & 0 & 0 \end{array}\right]$

let $\boxed{z = t} \in \mathbb{R}$; $y - 2z = -\gamma \Rightarrow y = -\gamma + 2z$

$\boxed{y = -\gamma + 2t}$

$x = 1 - 3y + 2z$

$x = 1 - 3(-\gamma + 2t) + 2t$

$x = 1 + 3\gamma - 6t + 2t$

$\boxed{x = 1 + 3\gamma - 4t}$

Solution set = $\{1+3\gamma-4t, -\gamma+2t, t \mid t \in \mathbb{R}\}$

or = ~~$\{3\gamma-4t, -\gamma+2t, t \mid t \in \mathbb{R}\}$~~

= $\{1+3\gamma-4t, -\gamma+2t, t \mid t \in \mathbb{R}\}$

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(ii) Given $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix}$

(a)

characteristic polynomial for A is

$$P_A(x) = |x I_{3\times3} - A| = \begin{vmatrix} x-3 & 0 & 0 \\ 0 & x-4 & -1 \\ 0 & -1 & x-4 \end{vmatrix} = (x-3)\left[(x-4)^2 - 1\right]$$

$$P_A(x) = (x-3)(x^2 - 8x + 15)$$

The eigenvalues of matrix A are solution of eqⁿ

$$\boxed{P_A(x) = 0} \Rightarrow (x-3)(x-5)(x-3) = 0$$

$(x-3)^2(x-5) = 0$ $\boxed{\lambda_1 = 3}$ Algebraic multiplicity 2

3 $\boxed{\lambda_1 = 3,3}$ $\boxed{\lambda_2 = 5}$ $\boxed{\lambda_2 = 5}$ Algebraic multiplicity 1 (1)

Now we need to find the eigenvalues of $\underline{A^{-2}}$

* 2 Marks are deducted if $|A| \neq 0$ is not mentioned

(2)

$$A^{-2}x = \lambda x$$
$$A^2 A^{-2} x = \lambda A^2 A x$$
$$x = \lambda \lambda_0^2 x$$
$$\therefore \lambda = \frac{1}{\lambda_0^2} \Rightarrow \lambda = \frac{1}{9}, \frac{1}{9}, \frac{1}{25}$$ eigenvalues for A^{-2}

as $|A| = 45 \neq 0$ & hence A^{-1} exists (1)

$\lambda = \frac{1}{9}$; Algebraic multiplicity = 2

$\lambda = \frac{1}{25}$; ——— 〃 ——— 1 (1)

(b) Now find eigenspace for lowest eigenvalue of A ie $\underline{\lambda_1 = \lambda_2 = 3}$

$\cancel{(\lambda I - x)}$ $(\lambda I - A)x = 0$ $\Rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right]$ Augmented matrix

$$\Rightarrow \left[\begin{array}{ccc|c} 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{matrix} x = a \\ y = 0 \\ z = 0 \end{matrix} \text{ or } \begin{matrix} x = 0 \\ y = b \\ z = -b \end{matrix}$$

$\therefore$ Eigenspace for $\lambda_1 = \lambda_2 = 3$ (4)

$$E_\lambda = \left\{ a[1, 0, 0] + b[0, 1, -1] \;\middle|\; a, b \in \mathbb{R} \right\}$$