MATHEMATICS-II (MATH F112)

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Section 4.6

Constructing Special Bases

Basis

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The subset $B = \{1, x, x^2, \ldots, x^n\}$ is a basis of P_n as B is LI (why) and $\operatorname{span}(B) = P_n$ (why).Here, $\mathbf{dim}(P_n) = n+1$.

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$W = \{p \in P_5 | p(2) = p(3) = 0\}$.

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Hence, B is a basis for W and $\mathbf{dim}(B) = 4$.

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Sol.

$\{[-1/5, -2/5, 1, 0, 0]^T, [-2/5, 1/5, 0, 1, 0]^T, [-1, 1, 0, 0, 1]^T\}$.

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Since, B is LI, B forms a basis for span(S).

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Q:. Does there always exists a basis for span(S).

Recall

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Theorem: Let V be a finite dimensional vector space such that $\dim(V) = n$.

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Theorem: Let V be a finite dimensional vector space such that $\dim(V) = n$. Suppose S is a finite subset of V that spans V. Then $|S| \geqslant n$. Moreover, $|S| = n$ if and only if S is a basis of V.

Simplified Span Method to find a Basis for span(S)

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Let $S \subseteq \mathbb{R}^n$ containing k vectors.

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- Compute $C = \text{RREF}(A)$.

Simplified Span Method to find a Basis for span(S)

Let $S \subseteq \mathbb{R}^n$ containing k vectors.

- Construct a matrix A of order $k \times n$ by using vectors of S as rows of A.
- Compute $C = \text{RREF}(A)$.
- Non-zero rows of C forms a basis for span(S).

Example

Q:. Let $S = \{[1, 2, 3, -1, 0], [3, 6, 8, -2, 0], [-1, -1, -3, 1, 1], [-2, -3, -5, 1, 1]\}$ be a subset of $\mathbb{R}^5$.

Example

Q:. Let $S = \{[1,2,3,-1,0],[3,6,8,-2,0],$ $[-1,-1,-3,1,1],[-2,-3,-5,1,1]\}$ be a subset of $\mathbb{R}^5$. Find a basis for span(S).

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Q:. Let $S = \{[1,2,3,-1,0],[3,6,8,-2,0],$ $[-1,-1,-3,1,1],[-2,-3,-5,1,1]\}$ be a subset of $\mathbb{R}^5$. Find a basis for span(S).

Sol. Step 1.

Example

Q:. Let $S = \{[1,2,3,-1,0],[3,6,8,-2,0],$
$[-1,-1,-3,1,1],[-2,-3,-5,1,1]\}$ be a subset of $\mathbb{R}^5$.
Find a basis for span(S).

Sol. Step 1.

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 & 0 \\ 3 & 6 & 8 & -2 & 0 \\ -1 & -1 & -3 & 1 & 1 \\ -2 & -3 & -5 & 1 & 1 \end{bmatrix}$$

Step 2.

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$$C = \mathrm{RREF}(A) = \begin{bmatrix} \boxed{1} & 0 & 0 & 2 & -2 \\ 0 & \boxed{1} & 0 & 0 & 1 \\ 0 & 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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Corresponding to non-zero rows,

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Corresponding to non-zero rows,
$B = \{[1, 0, 0, 2, -2], [0, 1, 0, 0, 1], [0, 0, 1, -1, 0]\}$ is a basis for span(S).

Exercise

Q:. Let $S = \{x^3 - 3x^2 + 2, 2x^3 - 7x^2 + x - 3, 4x^3 - 13x^2 + x + 5\}$ be a subset of P_3. Use Simplified Span Method to find a basis for span(S).

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Sol. $B = \{x^3 - 3x, x^2 - x, 1\}$

Next is to reduce a spanning set to a basis

Next is to reduce a spanning set to a basis by eliminating redundant vectors

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Next is to reduce a spanning set to a basis by eliminating redundant vectors without changing the form of the original vectors. How?

Theorem: If S is a spanning set for a finite dimensional vector space V, then there is a set $B \subseteq S$ that is a basis for V.

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Independence Test Method to find a Basis for span(S)

Let $S \subseteq \mathbb{R}^n$ containing k vectors.

- Construct a matrix A of order $n \times k$ by using vectors of S as columns of A.
- Compute $C = \text{RREF}(A)$.
- Vectors corresponding to pivot columns of C forms a basis for span(S).

Example

Q:. Let $S = \{[1, 2, -2, 1], [-3, 0, -4, 3], [2, 1, 1, -1], [-3, 3, -9, 6], [9, 3, 7, -6]\}$ be a subset of $\mathbb{R}^4$.

Example

Q:. Let $S = \{[1,2,-2,1],[-3,0,-4,3],[2,1,1,-1],$ $[-3,3,-9,6],[9,3,7,-6]\}$ be a subset of $\mathbb{R}^4$. Find a basis for span(S).

Example

Q:. Let $S = \{[1,2,-2,1],[-3,0,-4,3],[2,1,1,-1],$ $[-3,3,-9,6],[9,3,7,-6]\}$ be a subset of $\mathbb{R}^4$. Find a basis for span(S).

Sol. **Step 1.**

Example

Q:. Let $S = \{[1,2,-2,1],[-3,0,-4,3],[2,1,1,-1],$ $[-3,3,-9,6],[9,3,7,-6]\}$ be a subset of $\mathbb{R}^4$. Find a basis for span(S).

Sol. Step 1.

$$A = \begin{bmatrix} 1 & -3 & 2 & -3 & 9 \\ 2 & 0 & 1 & 3 & 3 \\ -2 & -4 & 1 & -9 & 7 \\ 1 & 3 & -1 & 6 & -6 \end{bmatrix}$$

Step 2.

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$$C = \mathrm{RREF}(A) = \begin{bmatrix} \boxed{1} & 0 & 1/2 & 3/2 & 3/2 \\ 0 & \boxed{1} & -1/2 & 3/2 & -5/2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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$B = \{[1,2,-2,1],[-3,0,-4,3]\}$ forms a basis for span(S).

Exercise

Q:. Let $S = \{x^3 - 3x^2 + 1, 2x^2 + x, 2x^3 + 3x + 2, 4x - 5\}$ be a subset of P_3. Use Independence Test Method to find a basis for span(S).

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Sol. $B = \{x^3 - 3x^2 + 1, 2x^2 + x, 4x - 5\}$

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Q:. Let $S = \{x^3 - 3x^2 + 1, 2x^2 + x, 2x^3 + 3x + 2, 4x - 5\}$ be a subset of P_3. Use Independence Test Method to find a basis for span(S).

Sol. $B = \{x^3 - 3x^2 + 1, 2x^2 + x, 4x - 5\}$

Q:. Does Simplified Span Method give a basis B for span(S) such that $B \subset S$.

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Sol. $B = \{x^3 - 3x^2 + 1, 2x^2 + x, 4x - 5\}$

Q:. Does Simplified Span Method give a basis $\boldsymbol{B}$ for span(S) such that $B \subset S$.

Sol. No. Counterexample

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Exercise

Q:. Let $S = \{x^3 - 3x^2 + 1, 2x^2 + x, 2x^3 + 3x + 2, 4x - 5\}$ be a subset of P_3. Use Independence Test Method to find a basis for span(S).

Sol. $B = \{x^3 - 3x^2 + 1, 2x^2 + x, 4x - 5\}$

Q:. Does Simplified Span Method give a basis B for span(S) such that $B \subset S$.

Sol. No. Counterexample

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

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Q:. Let $T = \{x^3 - x^2, x^4 - 3x^3 + 5x^2 - x\}$ be a LI subset of P_4.

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Exercise

Q:. Let $T = \left\{ \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} \right\}$ be a LI subset of M_{32}.

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Sol.

$$B = \left\{ \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$