Mathematics-II (MATH F112)

Linear Algebra

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Chapter: 5 (Linear Transformations)

Section 5.1: Linear Transformations

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- $L(\mathbf{u} + \mathbf{v}) = L(\mathbf{u}) + L(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in \mathcal{V}$
- $L(c\mathbf{u}) = cL(\mathbf{u})$ for all $c \in \mathbb{R}$ and all $\mathbf{u} \in \mathcal{V}$

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Example 2: For each $[x, y] \in \mathbb{R}^2$, consider a map

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Hence, L is not a LT.

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Example 3: The mapping $L : \mathbb{R}^3 \to \mathbb{R}^3$ given by $L([x, y, z]) = [x, y, -z]$ is a linear operator.

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3. For $n \geq 2$ and $a_1, a_2, \ldots, a_n \in \mathbb{R}$,

$$\text{If } \mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n, \quad \text{then}$$

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$$\begin{aligned} L(\mathbf{v}) &= L(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n) \\ &= a_1L(\mathbf{v}_1) + a_2L(\mathbf{v}_2) + \cdots + a_nL(\mathbf{v}_n). \end{aligned}$$

Example 4: Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator such that $L([1,0,0]) = [-2,1,0]$, $L([0,1,0]) = [3,-2,1]$, and $L([0,0,1]) = [0,-1,3]$.

- Find $L([-3,2,4])$.

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Solution:

$$[-3, 2, 4] = -3[1, 0, 0] + 2[0, 1, 0] + 4[0, 0, 1]$$

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$$L([-3, 2, 4]) = L(-3[1, 0, 0] + 2[0, 1, 0] + 4[0, 0, 1])$$

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[-3,2,4] &= -3[1,0,0] + 2[0,1,0] + 4[0,0,1] \\
L([-3,2,4]) &= L(-3[1,0,0] + 2[0,1,0] + 4[0,0,1]) \\
&= -3L([1,0,0]) + 2L([0,1,0]) + 4L([0,0,1])
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&= -3[-2,1,0] + 2[3,-2,1] + 4[0,-1,3] \\
&= [12,11,14]
\end{aligned}$$

Similarly,

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Note that

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -2 & 3 & 0 \\ 1 & -2 & -1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Exercise: Suppose $L : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear operator and $L([1,1]) = [3,0]$ and $L([-1,1]) = [0,1]$. Find $L([x,y])$ for all $[x,y] \in \mathbb{R}^2$.

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Answer: $L([x,y]) = \left[\frac{3x+3y}{2}, \frac{-x+y}{2}\right]$.

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Remark: Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces, and let $L : \mathcal{V} \to \mathcal{W}$ be a LT. Also, let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for $\mathcal{V}$.

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Remark: Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces, and let $L : \mathcal{V} \to \mathcal{W}$ be a LT. Also, let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for $\mathcal{V}$. If $\mathbf{v} \in \mathcal{V}$, $L(\mathbf{v})$ is completely determined by $\{L(\mathbf{v}_1), L(\mathbf{v}_2), \ldots, L(\mathbf{v}_n)\}$.

Composition of Linear transformations

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Theorem 2: Let $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{V}_3$ be vector spaces and let $L_1 : \mathcal{V}_1 \to \mathcal{V}_2$ and $L_2 : \mathcal{V}_2 \to \mathcal{V}_3$ be linear transformations. Then $L_2 \circ L_1 : \mathcal{V}_1 \to \mathcal{V}_3$ given by $(L_2 \circ L_1)(\mathbf{v}) = L_2(L_1(\mathbf{v}))$, for all $\mathbf{v} \in \mathcal{V}_1$, is a LT.

Example 5: Let $L_1 : \mathcal{P}_2 \longrightarrow \mathcal{P}_2$ and $L_2 : \mathcal{P}_2 \longrightarrow \mathcal{P}_2$ be linear operators defined as $L_1(ax^2 + bx + c) = 2ax + b$ and $L_2(ax^2 + bx + c) = 2ax^2 + bx$, respectively.

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Answer:

- $L_2 \circ L_1(ax^2 + bx + c) = 2ax$.

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Answer:

- $L_2 \circ L_1(ax^2 + bx + c) = 2ax$.
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Clearly, $L_2 \circ L_1 \neq L_1 \circ L_2$.

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$$\ker(L) = \{\mathbf{v} \in \mathcal{V} \mid L(\mathbf{v}) = \mathbf{0}_{\mathcal{W}}\}.$$

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Solution:

$$\ker(L) = \{[x, y, z] \in \mathbb{R}^3 \mid L([x, y, z]) = \mathbf{0}_{\mathbb{R}^2}\}$$

Example 6: Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ be a LT given by $L([x, y, z]) = [0, y]$. Find $\ker(L)$.

Solution:

$$\begin{aligned} \ker(L) &= \{[x, y, z] \in \mathbb{R}^3 \mid L([x, y, z]) = \mathbf{0}_{\mathbb{R}^2}\} \\ &= \{[x, y, z] \in \mathbb{R}^3 \mid [0, y] = [0, 0]\} \end{aligned}$$

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is a subspace of the vector space $\mathbb{R}^3$.

Result: If $L : \mathcal{V} \to \mathcal{W}$ is a LT, then $\ker(L)$ is a subspace of $\mathcal{V}$.

Range of a linear transformation:

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Thus a vector $\mathbf{w} \in$ range(L) implies there exists some vector $\mathbf{v} \in \mathcal{V}$ such that $L(\mathbf{v}) = \mathbf{w}$.

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Exercise: Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a LT given by

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 5 & 1 & -1 \\ -3 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

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- Is $[1, -2, 3]^T \in \ker(L)$?
- Is $[2, -1, 4]^T \in$ range(L)?

Hint: Note that $L([1,-2,3]^T) = [0,0,0]^T$

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$$\begin{aligned} 5x + y - z &= 2 \\ -3x + z &= -1 \\ x - y - z &= 4 \end{aligned}$$

is consistent or not.

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Since above system of equations is inconsistent (show it!), $[2,-1,4]^T \notin$ range(L).

Example 7: Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ be a LT given by

$$L([x, y, z]) = [0, y] \text{ for all } [x, y, z] \in \mathbb{R}^3$$

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- Find range(L).
- Find the dimension of $\ker(L)$ and range(L).

Solution:

$$\begin{aligned}\text{range}(L) &= \{L([x, y, z]) \mid [x, y, z] \in \mathbb{R}^3\} \\ &= \{[0, y] \mid y \in \mathbb{R}\}\end{aligned}$$

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$$\begin{aligned}\ker(L) &= \{[x,0,z] \mid x,z \in \mathbb{R}\} \quad (\text{see Example 6}) \\ &= \{x[1,0,0] + z[0,0,1] \mid x,z \in \mathbb{R}\}\end{aligned}$$

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Now, the set $\{[1,0,0],[0,0,1]\}$ of vectors is LI subset of $\mathbb{R}^3$ (verify!).

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Note that range$(L) =$ span$\{[0,1]\}$. Since $\{[0,1]\}$ is LI subset of $\mathbb{R}^2$ (Why?). Thus, the set $B = \{[0,1]\}$ is a basis of range(L) so that $\dim(\text{range}(L)) = 1$

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Now, the set $\{[1,0,0],[0,0,1]\}$ of vectors is LI subset of $\mathbb{R}^3$ (verify!). Hence, the set $\{[1,0,0],[0,0,1]\}$ forms a basis of $\ker(L)$ and $\dim(\ker(L)) = 2$.

Example 8: Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ be a LT given by

$$L([x, y, z]) = [x - 2y, y + z].$$

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Since the set $B = \{[2,1,-1]\}$ is LI. Therefore, $B = \{[2,1,-1]\}$ is a basis of $\ker(L)$. Now

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Since the set $\{[1, 0], [0, 1]\}$ is LI. Thus,

$$\{[1, 0], [0, 1]\}$$

is a basis for range(L).

Exercise: Given a map

$$L : \mathcal{P}_3 \to \mathcal{P}_2 \quad \text{given by}$$

$$L(ax^3 + bx^2 + cx + d) = 3ax^2 + 2bx + c.$$

1. Show that L is a linear transformation.

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1. Show that L is a linear transformation.
2. Find $\ker(L)$ and range(L).

Answer:

$$\ker(L) = \left\{0x^3 + 0x^2 + 0x + d \mid d \in \mathbb{R}\right\}$$

$$\text{range}(L) = \mathcal{P}_2.$$

Exercise: Given a map

$$L : \mathbb{R}^4 \to \mathcal{P}_2 \quad \text{given by}$$

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1. Find $\ker(L)$ and $\text{range}(L)$.
2. Find a basis for $\ker(L)$ and $\text{range}(L)$.

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Answer:

$\ker(L) = \{[0, b, -b, b] \mid b \in \mathbb{R}\}$ and $B = \{[0, 1, -1, 1]\}$

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Answer:

$$\ker(L) = \{[0, b, -b, b] \mid b \in \mathbb{R}\} \text{ and } B = \{[0, 1, -1, 1]\}$$

$$\text{range}(L) = \text{span}\{1, x + x^2, x, x^2\} \text{ and } B = \{1, x, x^2\}.$$

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Example 9: Let $L : \mathbb{R}^3 \to \mathbb{R}^4$ be a LT given by

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Find a basis for $\ker(L)$ and range(L).

Answer:

- $\{[0, 1, 1]\}$ is a basis of $\ker(L)$.
- $\{[1, 0, 1, 1], [0, 1, -1, 1]\}$ is a basis for range(L).

Alternative approach for finding a basis for $\ker(L)$ (Kernel Method) Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a LT.

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$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

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Note that, Columns I and II have leading entry. Thus, the corresponding column vector of A i.e.
$\{[1, 0, 1, 1], [0, 1, -1, 1]\}$ is a basis of range (L).

The Dimension Theorem:

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Sometimes $\dim(\ker(L))$ and $\dim(\text{range}(L))$ is also known as **nullity**(L) and **rank**(L), respectively.

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$$L(a + bx + cx^2) = x(a + bx + cx^2).$$

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Answer:

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Exercise: Let $\mathcal{W}$ be the vector space of all 2×2 symmetric matrices. Define a LT $L : \mathcal{W} \to \mathcal{P}_2$ by

$$L\left(\begin{bmatrix} a & b \\ b & c \end{bmatrix}\right) = (a-b) + (b-c)x + (c-a)x^2$$

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Find $\dim(\ker(L))$ and $\dim(\text{range}(L))$.

Answer: $\dim(\ker(L)) = 1$ and $\dim(\text{range}(L)) = 2$.

Exercise: Let $\{e_1, e_2, e_3, e_4\}$ be standard basis for $\mathbb{R}^4$ and $L : \mathbb{R}^4 \to \mathbb{R}^3$ be a LT given by

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Answer: $\dim(\ker(L)) = 1$ and $\dim($range$(L)) = 3$.

Exercise: For each $\mathbf{p} \in \mathcal{P}_2$, consider $L : \mathcal{P}_2 \longrightarrow \mathcal{P}_4$ given by $L(\mathbf{p}) = x^2\mathbf{p}$.

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Answer: $\dim(\ker(L)) = 0$ and $\dim($range$(L)) = 3$.

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Section 5.4

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L is **onto** if and only if, for each $\mathbf{w} \in \mathcal{W}$, there is some $\mathbf{v} \in \mathcal{V}$ such that $L(\mathbf{v}) = \mathbf{w}$.

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Exercise: Which of the following transformations are one-to-one? onto?

1. $L : \mathbb{R}^2 \to \mathbb{R}^3$ given by $L([x, y]) = [2x, x - y, 0]$.
2. $L : \mathbb{R}^3 \to \mathbb{R}^4$ given by $L([x, y, z]) = [y, z, -y, 0]$.
3. $L : \mathcal{M}_{22} \to \mathcal{M}_{22}$ given by $L(A) = A^T$.

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3. one-to-one and onto.

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Example 12: Consider a LT $L : \mathcal{M}_{22} \to \mathcal{M}_{23}$ given by

$$L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a-b & 0 & c-d \\ c+d & a+b & 0 \end{bmatrix}$$

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$$L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a-b & 0 & c-d \\ c+d & a+b & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We have $a-b = c-d = c+d = a+b = 0$ implies $a = b = c = d = 0$.

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$$\begin{aligned}\dim(\text{range}(L)) &= \dim(\mathcal{M}_{22}) - \dim(\ker(L)) \\ &= 4 \\ &\neq \dim(\mathcal{M}_{23}).\end{aligned}$$

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Hence, L is not onto.

Try to find a basis of range(L).

Example 13: Consider a LT $L:\mathbb{R}^3 \to \mathbb{R}^3$ given by

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -7 & 4 & -2 \\ 16 & -7 & 2 \\ 4 & -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

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Solution: The RREF of matrix $A = \begin{bmatrix} -7 & 4 & -2 \\ 16 & -7 & 2 \\ 4 & -3 & 2 \end{bmatrix}$ is

$$\begin{bmatrix} 1 & 0 & -\frac{2}{5} \\ 0 & 1 & -\frac{6}{5} \\ 0 & 0 & 0 \end{bmatrix}.$$

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Example 14: Let A be a fixed $n \times n$ matrix, and consider a LT $L : \mathcal{M}_{nn} \to \mathcal{M}_{nn}$ given by $L(B) = AB - BA$.

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Solution: $L(I_n) = AI_n - I_nA = 0_{n \times n}$.

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Hence, L is not onto.

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Question: Can we apply Dimension theorem here?

Exercise: Consider a LT $L : \mathcal{M}_{23} \to \mathcal{M}_{22}$ given by

$$L\left(\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\right) = \begin{bmatrix} a+b & a+c \\ d+e & d+f \end{bmatrix}$$

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Answer: L is onto but not one-to-one.

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Exercise: Consider a LT $L : \mathbb{R}^4 \to \mathbb{R}^3$ given by

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} -5 & 3 & 1 & 18 \\ -2 & 1 & 1 & 6 \\ -7 & 3 & 4 & 19 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

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Section 5.5

Invertible linear transformation:

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Such a function M, denoted by L^{-1}, is called an inverse of L.

Isomorphism: A LT $L : \mathcal{V} \to \mathcal{W}$ that is both one-to-one and onto is called as isomorphism from $\mathcal{V}$ to $\mathcal{W}$.

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Example 16: Show that $L : \mathcal{P}_n \rightarrow \mathcal{P}_n$ given by $L(\mathbf{p}) = \mathbf{p} + \mathbf{p}'$ is an isomorphism.

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Solution: First we need to show that L is a linear operator.

$$\begin{aligned} L(\mathbf{p}+\mathbf{q}) &= (\mathbf{p}+\mathbf{q}) + (\mathbf{p}+\mathbf{q})' \\ &= \mathbf{p} + \mathbf{p}' + \mathbf{q} + \mathbf{q}' \\ &= L(\mathbf{p}) + L(\mathbf{q}) \text{ for all } \mathbf{p}, \mathbf{q} \in \mathcal{P}_n. \end{aligned}$$

Similarly, (show that) $L(c\,\mathbf{p}) = cL(\mathbf{p})$ for all real c and $\mathbf{p} \in \mathcal{P}_n$.

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By Dimension theorem,

$$\dim \mathsf{range}(L) = \dim \mathcal{P}_n = n + 1$$

so that $\mathsf{range}(L) = \mathcal{P}_n$.

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$$\dim \text{range}(L) = \dim \mathcal{P}_n = n + 1$$

so that $\text{range}(L) = \mathcal{P}_n$. Thus, L is onto. Hence, L is an isomorphism.

Example 17: Show that the linear operator

$$L : \mathbb{R}^3 \to \mathbb{R}^3 \text{ such that}$$

$$L(e_1) = e_1 + e_2, L(e_2) = e_2 + e_3, L(e_3) = e_1 + e_2 + e_3$$

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Hint: First find $L([x, y, z])$ for all $[x, y, z] \in \mathbb{R}^3$.

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and $\ker(L) = \{0_{\mathbb{R}^3}\}$. Thus, L is one-to-one. Use dimension theorem and Theorem 4 to conclude L is onto. Hence, L is an isomorphism.

Result: Let $L : \mathcal{V} \to \mathcal{W}$ be a linear transformation, where $\mathcal{V}$ and $\mathcal{W}$ be finite dimensional vector spaces such that $\dim(\mathcal{V}) = \dim(\mathcal{W})$. Then L is one-to-one if and only if L is onto.

Exercise: Show that the linear operator $L : \mathcal{P}_2 \longrightarrow \mathcal{P}_2$ given by $L(a + bx + cx^2) = (b + c) + (a + c)x + (a + b)x^2$ is an isomorphism.

Exercise: Show that the linear operator $L : \mathcal{P}_2 \to \mathcal{P}_2$ given by $L(a+bx+cx^2) = (b+c)+(a+c)x+(a+b)x^2$ is an isomorphism.

Exercise: Show that the linear transformation $L : \mathcal{M}_{mn} \to \mathcal{M}_{nm}$ given by $L(A) = A^T$ is an isomorphism.

Theorem: A LT $L : \mathcal{V} \to \mathcal{W}$ is an isomorphism if and only if L is an invertible LT.

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Example 18: Let $L : \mathbb{R}^3 \to \mathcal{P}_2$ be a LT given by

$$L([x, y, z]) = x + (x + y - z)t + (x + y + z)t^2.$$

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Is L invertible? If yes, find L^{-1}.

Solution: First show that L is both one-to-one and onto. Hence, invertible.

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$$\Rightarrow x = a, y = \frac{b + c - 2a}{2}, z = \frac{c - b}{2}.$$

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Hence, $L^{-1}(a+bx+cx^2) = \left[a, \frac{b+c-2a}{2}, \frac{c-b}{2}\right]$.

Exercise: Let $L : \mathcal{P}_2 \to \mathcal{P}_2$ be a LT given by $L(a + bx + cx^2) = (b + c) + (a + c)x + (a + b)x^2$.

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Answer:

$$L^{-1}(a+bx+cx^2) = \frac{1}{2}(b+c-a)+\frac{1}{2}(a+c-b)x+\frac{1}{2}(a+b-c)x^2.$$

Exercise: Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a LT given by $L(e_1) = e_1 + e_2, L(e_2) = e_2 + e_3, L(e_3) = e_1 + e_2 + e_3$.

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Answer: $L^{-1}([x, y, z]) = [y - z, y - x, x - y + z]$.

Isomorphic vector spaces: Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces. Then $\mathcal{V}$ is isomorphic to $\mathcal{W}$, denoted by $\mathcal{V} \cong \mathcal{W}$, if and only if there exists an isomorphism $L : \mathcal{V} \to \mathcal{W}$.

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Theorem 5: Suppose $\mathcal{V} \cong \mathcal{W}$ and $\mathcal{V}$ and $\mathcal{W}$ are finite dimensional. Then $\mathcal{V}$ is isomorphic to $\mathcal{W}$ if and only if $\dim(\mathcal{V}) = \dim(\mathcal{W})$.

Exercise: Show that $\mathbb{R}^n$ and $\mathcal{P}_n$ are not isomorphic.

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Exercise: Let W be the vector space of all symmetric 2×2 matrices. Show that W is isomorphic to $\mathbb{R}^3$.

Exercise: Show that the subspace

$$W = \{\mathbf{p} \in \mathcal{P}_3 \mid \mathbf{p}(0) = 0\}$$

is isomorphic to $\mathcal{P}_2$.

Section 4.7

Ordered Basis: An **ordered basis** for vector space $\mathcal{V}$ is an ordered n-tuple of vectors $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ such that the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for $\mathcal{V}$.

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- (e_1, e_2) and (e_2, e_1) are two ordered bases for $\mathbb{R}^2$.

Coordinatization: Let $B = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ be an ordered basis for a vector space $\mathcal{V}$.

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$$\mathbf{w} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$$

Then $[\mathbf{w}]_B$, the coordinatization or coordinates of $\mathbf{w}$ with respect to B is the n-vector $[a_1, a_2, \ldots, a_n]$.

Example 19: Let $B = ([4,2],[1,3])$ be an ordered basis for $\mathbb{R}^2$.

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Hence, $[11,13]_B = [2,3]$.

Example 20: Let

$$B = ([-4, 5, -1, 0, -1], [1, -3, 2, 2, 5], [1, -2, 1, 1, 3])$$

be an ordered basis of the subspace $\mathcal{V}$ of $\mathbb{R}^5$.

Example 20: Let

$$B = ([-4, 5, -1, 0, -1], [1, -3, 2, 2, 5], [1, -2, 1, 1, 3])$$

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Compute $[-23, 30, -7, -1, -7]_B, [1, 2, 3, 4, 5]_B$.

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be an ordered basis of the subspace $\mathcal{V}$ of $\mathbb{R}^5$.

Compute $[-23, 30, -7, -1, -7]_B$, $[1, 2, 3, 4, 5]_B$.

Solution: To find $[-23, 30, -7, -1, -7]_B$, we need to solve the following equation
$[-23, 30, -7, -1, -7] =$
$a[-4, 5, -1, 0, -1] + b[1, -3, 2, 2, 5] + c[1, -2, 1, 1, 3]$

or equivalently

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$$
\begin{aligned}
-4a + b + c &= -23 \\
5a - 3b - 2c &= 30 \\
-a + 2b + c &= -7 \\
2b + c &= -1 \\
-a + 5b + 3c &= -7
\end{aligned}
$$

or equivalently

$$\begin{aligned} -4a + b + c &= -23 \\ 5a - 3b - 2c &= 30 \\ -a + 2b + c &= -7 \\ 2b + c &= -1 \\ -a + 5b + 3c &= -7 \end{aligned}$$

To solve this system, note that the RREF of the augmented matrix

$$\left[\begin{array}{rrr|r} -4 & 1 & 1 & -23 \\ 5 & -3 & -2 & 30 \\ -1 & 2 & 1 & -7 \\ 0 & 2 & 1 & -1 \\ -1 & 5 & 3 & -7 \end{array}\right]$$

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Hence, the unique solution for the system is

$$a = 6, b = -2, c = 3$$

implies

$$\left[\begin{array}{ccc|c} -4 & 1 & 1 & -23 \\ 5 & -3 & -2 & 30 \\ -1 & 2 & 1 & -7 \\ 0 & 2 & 1 & -1 \\ -1 & 5 & 3 & -7 \end{array}\right] \text{ is } \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

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This system has no solution,

$$\left[\begin{array}{ccc|c} -4 & 1 & 1 & 1 \\ 5 & -3 & -2 & 2 \\ -1 & 2 & 1 & 3 \\ 0 & 2 & 1 & 4 \\ -1 & 5 & 3 & 5 \end{array}\right] \text{ is } \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This system has no solution, implies that the vector $[1, 2, 3, 4, 5]$ is not in span$(B) = \mathcal{V}$.

Coordinatization Method:

Let $\mathcal{V}$ be a nontrivial subspace of $\mathbb{R}^n$, let $B = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ be an ordered basis for $\mathcal{V}$, and let $\mathbf{v} \in \mathbb{R}^n$.

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- Form an augmented matrix $[A|\mathbf{v}]$ by using the vectors in B as the columns of A, in order, and using $\mathbf{v}$ as a column on the right.

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- Find RREF($[A|\mathbf{v}]$), say $[C|\mathbf{w}]$ = RREF($[A|\mathbf{v}]$).

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- Find RREF($[A|\mathbf{v}]$), say $[C|\mathbf{w}]$ = RREF($[A|\mathbf{v}]$).
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- If there is a row of $[C|\mathbf{w}]$ that contains all zeros on the left and has a nonzero entry on the right, then $\mathbf{v} \notin \text{span}(B) = \mathcal{V}$, i.e., coordinatization is not possible. Otherwise, $\mathbf{v} \in \text{span}(B) = \mathcal{V}$.

- Eliminate all rows consisting entirely of zeros in $[C|\mathbf{w}]$ to obtain $[I_k|\mathbf{y}]$.

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Example 21: Let

$$B = \left(\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix} \right)$$

be an ordered basis of the subspace $\mathcal{W}$ of $\mathcal{M}_{22}$.

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Compute $[\mathbf{v}]_B$ if exists, where $\mathbf{v} = \begin{bmatrix} -3 & -2 \\ 0 & 3 \end{bmatrix}$.

Solution: Consider

$$[A|\mathbf{v}] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & -3 \\ -2 & -1 & -1 & -2 \\ 0 & 1 & 3 & 0 \\ 1 & 0 & 1 & 3 \end{array}\right]$$

Solution: Consider

$$[A|\mathbf{v}] = \left[\begin{array}{rrr|r} 1 & 2 & 1 & -3 \\ -2 & -1 & -1 & -2 \\ 0 & 1 & 3 & 0 \\ 1 & 0 & 1 & 3 \end{array}\right]$$

Note that the row reduced echelon form is

$$\text{RREF}[A|\mathbf{v}] = \left[\begin{array}{rrr|r} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The row reduced matrix contains no rows with all zero entries on the left and a nonzero entry on the right, so $[\mathbf{v}]_B$ exists, The row reduced matrix contains no rows with all zero entries on the left and a nonzero entry on the right, so $[\mathbf{v}]_B$ exists, and

$$[\mathbf{v}]_B = [2, -3, 1].$$

Fundamental properties of Coordinatization: Let $B = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ be an ordered basis for a vector space $\mathcal{V}$. Suppose $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k \in \mathcal{V}$ and $a_1, a_2, \ldots, a_k$ are scalars. Then

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- $[\mathbf{w}_1 + \mathbf{w}_2]_B = [\mathbf{w}_1]_B + [\mathbf{w}_2]_B$

Fundamental properties of Coordinatization: Let $B = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ be an ordered basis for a vector space $\mathcal{V}$. Suppose $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k \in \mathcal{V}$ and $a_1, a_2, \ldots, a_k$ are scalars. Then

- $[\mathbf{w}_1 + \mathbf{w}_2]_B = [\mathbf{w}_1]_B + [\mathbf{w}_2]_B$
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- $[a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + \ldots a_k\mathbf{w}_k]_B$
  $= a_1[\mathbf{w}_1]_B + a_2[\mathbf{w}_2]_B + \cdots + a_k[\mathbf{w}_k]_B$.

Exercise: Let

$$B = (3x^2 - x + 2, x^2 + 2x - 3, 2x^2 + 3x - 1)$$

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Exercise: Let

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be an ordered basis of the subspace $\mathcal{W}$ of $\mathcal{P}_2$. Compute $[\mathbf{v}]_B$ if exists, where $\mathbf{v} = 13x^2 - 5x + 20$.

Answer: $[\mathbf{v}]_B = [4, -5, 3]$.

Exercise: Let

$$B = ([-4, 5, -1, 0, -1], [1, -3, 2, 2, 5], [1, -2, 1, 1, 3])$$

be an ordered basis of the subspace $\mathcal{W}$ of $\mathbb{R}^5$.

Exercise: Let

$$B = ([-4, 5, -1, 0, -1], [1, -3, 2, 2, 5], [1, -2, 1, 1, 3])$$

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Exercise: Let

$$B = ([-4, 5, -1, 0, -1], [1, -3, 2, 2, 5], [1, -2, 1, 1, 3])$$

be an ordered basis of the subspace $\mathcal{W}$ of $\mathbb{R}^5$. Consider $x = [1, 0, -1, 0, 4], y = [0, 1, -1, 0, 3]$ and $z = [0, 0, 0, 1, 5]$. Compute $[2x - 7y + 3z]_B$.

Exercise: Let

$$B = ([-4, 5, -1, 0, -1], [1, -3, 2, 2, 5], [1, -2, 1, 1, 3])$$

be an ordered basis of the subspace $\mathcal{W}$ of $\mathbb{R}^5$. Consider $x = [1, 0, -1, 0, 4], y = [0, 1, -1, 0, 3]$ and $z = [0, 0, 0, 1, 5]$. Compute $[2x - 7y + 3z]_B$.

Answer: $[2x - 7y + 3z]_B = [-2, 9, -15]$.

Example 22: Let $C = ([-4, 5, -1, 0, -1], [1, -3, 2, 2, 5], [1, -2, 1, 1, 3])$ be an ordered basis of the subspace $\mathcal{W}$ of $\mathbb{R}^5$.

Example 22: Let $C = ([-4, 5, -1, 0, -1], [1, -3, 2, 2, 5], [1, -2, 1, 1, 3])$ be an ordered basis of the subspace $\mathcal{W}$ of $\mathbb{R}^5$. Using simplified span method on C, compute an ordered basis $B = (x, y, z)$ for $\mathcal{W}$.

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Example 22: Let $C = ([-4, 5, -1, 0, -1], [1, -3, 2, 2, 5], [1, -2, 1, 1, 3])$ be an ordered basis of the subspace $\mathcal{W}$ of $\mathbb{R}^5$. Using simplified span method on C, compute an ordered basis $B = (x, y, z)$ for $\mathcal{W}$. Also, compute $[x]_C, [y]_C, [z]_C$.

Solution: We have the following augmented matrix

$$\left[\begin{array}{c|ccc} A & x & y & z \end{array}\right] = \left[\begin{array}{ccc|ccc} -4 & 1 & 1 & 1 & 0 & 0 \\ 5 & -3 & -2 & 0 & 1 & 0 \\ -1 & 2 & 1 & -1 & -1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \\ -1 & 5 & 3 & 4 & 3 & 5 \end{array}\right]$$

Row reduce echelon form of the above matrix is

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$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & -5 & -4 & -3 \\ 0 & 0 & 1 & 10 & 8 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

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Clearly, $[x]_C = [1, -5, 10]$, $[y]_C = [1, -4, 8]$ and $[z]_C = [1, -3, 7]$.

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Clearly, $[x]_C = [1, -5, 10]$, $[y]_C = [1, -4, 8]$ and $[z]_C = [1, -3, 7]$. Here, the matrix

$$P = \begin{bmatrix} 1 & 1 & 1 \\ -5 & -4 & -3 \\ 10 & 8 & 7 \end{bmatrix}$$

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Clearly, $[x]_C = [1, -5, 10]$, $[y]_C = [1, -4, 8]$ and $[z]_C = [1, -3, 7]$. Here, the matrix $P = \begin{bmatrix} 1 & 1 & 1 \\ -5 & -4 & -3 \\ 10 & 8 & 7 \end{bmatrix}$ is called the transition matrix from B-coordinates to C-coordinates.

Transition Matrix:

Transition Matrix: Suppose that $\mathcal{V}$ is a nontrivial n-dimensional vector space with ordered bases B and C.

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$$\left[\begin{array}{cccc|cccc} 1^{st} & 2^{nd} & & k^{th} & 1^{st} & 2^{nd} & & k^{th} \\ \text{vector} & \text{vector} & \cdot\cdot & \text{vector} & \text{vector} & \text{vector} & \cdot\cdot & \text{vector} \\ \text{in} & \text{in} & & \text{in} & \text{in} & \text{in} & & \text{in} \\ C & C & & C & B & B & & B \end{array}\right]$$

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to produce

$$\left[\begin{array}{c|c} I_k & P \\ \hline \text{rows of} & \text{zeroes} \end{array}\right]$$

Example 23: For the ordered bases

$$B = \left(\begin{bmatrix} 7 & 3 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \right)$$

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$$B = \left(\begin{bmatrix} 7 & 3 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \right)$$

and

$$C = \left(\begin{bmatrix} 22 & 7 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 12 & 4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 33 & 12 \\ 0 & 2 \end{bmatrix} \right)$$

of $\mathcal{U}_2$ (the set of 2×2 upper triangular matrices),

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Solution: Apply row reduction on

$$\left[\begin{array}{ccc|ccc} 22 & 12 & 33 & 7 & 1 & 1 \\ 7 & 4 & 12 & 3 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 2 & 0 & -1 & 1 \end{array}\right]$$

we get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & -4 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

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Example 24: Let $C = (a, b, c) = ([1, 0, 1], [1, 1, 0], [0, 0, 1])$ and $B = (x, y, z)$ be ordered bases of $\mathbb{R}^3$.

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Solution:

$$x = 1 \cdot a + 2 \cdot b - 1 \cdot c = [3, 2, 0]$$

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Solution:

$$x = 1 \cdot a + 2 \cdot b - 1 \cdot c = [3, 2, 0]$$
$$y = 1 \cdot a + 1 \cdot b - 1 \cdot c = [2, 1, 0]$$
$$z = 2 \cdot a + 1 \cdot b + 1 \cdot c = [3, 1, 3].$$

Hence,

$$B = ([3, 2, 0], [2, 1, 0], [3, 1, 3]).$$

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Change of Coordinates Using the Transition Matrix

Theorem: Suppose that $\mathcal{V}$ is a nontrivial n-dimensional vector space with ordered bases B and C.

Hence,

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Change of Coordinates Using the Transition Matrix

Theorem: Suppose that $\mathcal{V}$ is a nontrivial n-dimensional vector space with ordered bases B and C. Let P be an $n \times n$ matrix.

Hence,

$$B = ([3, 2, 0], [2, 1, 0], [3, 1, 3]).$$

Change of Coordinates Using the Transition Matrix

Theorem: Suppose that $\mathcal{V}$ is a nontrivial n-dimensional vector space with ordered bases B and C. Let P be an $n \times n$ matrix. Then P is the transition matrix from B to C if and only if for every $\mathbf{v} \in \mathcal{V}$, $P[\mathbf{v}]_B = [\mathbf{v}]_C$.

Example 25: For the ordered bases

$$B = \left(\begin{bmatrix} 7 & 3 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \right)$$

Example 25: For the ordered bases

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of $\mathcal{U}_2$ (set of 2×2 upper triangular matrices). Find $[\mathbf{v}]_B$ and $[\mathbf{v}]_C$, where $\mathbf{v} = \begin{bmatrix} 25 & 24 \\ 0 & -9 \end{bmatrix}$.

Solution: Clearly,

$$\begin{bmatrix} 25 & 24 \\ 0 & -9 \end{bmatrix} = 4\begin{bmatrix} 7 & 3 \\ 0 & 0 \end{bmatrix} + 3\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} - 6\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

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Hence, $[\mathbf{v}]_B = [4, 3, -6]^T$.

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Hence, $[\mathbf{v}]_B = [4, 3, -6]^T$. Now, since $[\mathbf{v}]_C = P[\mathbf{v}]_B$ and

$$P = \begin{bmatrix} 1 & -2 & 1 \\ -4 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$ (see Example 23)

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implies $[\mathbf{v}]_C = [-8, -19, 13]^T$.

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$$\begin{bmatrix} 25 & 24 \\ 0 & -9 \end{bmatrix} = -8 \begin{bmatrix} 22 & 7 \\ 0 & 2 \end{bmatrix} - 19 \begin{bmatrix} 12 & 4 \\ 0 & 1 \end{bmatrix} + 13 \begin{bmatrix} 33 & 12 \\ 0 & 2 \end{bmatrix}$$

Theorem: Let B and C be be ordered bases for a nontrivial finite dimensional vector space $\mathcal{V}$, and let P be the transition matrix from B to C. Then P is nonsingular, and P^{-1} is the transition matrix from C to B.

Example 26: For an ordered basis $B = ([1, -4, 1, 2, 1], [6, -24, 5, 8, 3], [3, -12, 3, 6, 2])$ of a subspace $\mathcal{V}$ of $\mathbb{R}^5$.

Example 26: For an ordered basis $B = ([1, -4, 1, 2, 1], [6, -24, 5, 8, 3], [3, -12, 3, 6, 2])$ of a subspace $\mathcal{V}$ of $\mathbb{R}^5$.

- Use the Simplified Span Method to find a second ordered basis C.

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$$B = \begin{bmatrix} 1 & -4 & 1 & 2 & 1 \\ 6 & -24 & 5 & 8 & 3 \\ 3 & -12 & 3 & 6 & 2 \end{bmatrix}$$

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- Use the Simplified Span Method to find a second ordered basis C.

Solution: Consider

$$B = \begin{bmatrix} 1 & -4 & 1 & 2 & 1 \\ 6 & -24 & 5 & 8 & 3 \\ 3 & -12 & 3 & 6 & 2 \end{bmatrix}$$

Note that

$$\text{RREF}(B) = \begin{bmatrix} 1 & -4 & 0 & -2 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C = ([1, -4, 0, -2, 0], [0, 0, 1, 4, 0], [0, 0, 0, 0, 1]).$$

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$$C = ([1, -4, 0, -2, 0], [0, 0, 1, 4, 0], [0, 0, 0, 0, 1]) .$$

- Find the transition matrix P from B to C.

Answer:

$$P = \begin{bmatrix} 1 & 6 & 3 \\ 1 & 5 & 3 \\ 1 & 3 & 2 \end{bmatrix}$$

- Find the transition matrix Q from C to B.

- Find the transition matrix Q from C to B.

Answer:

$$Q = P^{-1} = \begin{bmatrix} 1 & -3 & 3 \\ 1 & -1 & 0 \\ -2 & 3 & -1 \end{bmatrix}$$

- For the given vector $\mathbf{v} = [2, -8, -2, -12, 3] \in \mathcal{V}$, calculate $[\mathbf{v}]_B$ and $[\mathbf{v}]_C$.

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$$[B|\mathbf{v}] = \left[\begin{array}{ccc|c} 1 & 6 & 3 & 2 \\ -4 & -24 & -12 & -8 \\ 1 & 5 & 3 & -2 \\ 2 & 8 & 6 & -12 \\ 1 & 3 & 2 & 3 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 17 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -13 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

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Thus,

$$[\mathbf{v}]_B = [17, 4, -13]$$

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Thus,

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Exercise: For the ordered bases

$$B = \left(2x^2 + 3x - 1, 8x^2 + x + 1, x^2 + 6\right)$$

and

$$C = \left(x^2 + 3x + 1, 3x^2 + 4x + 1, 10x^2 + 17x + 5\right)$$

of $\mathcal{P}_2$, find the transition matrix P from B to C.

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of $\mathcal{P}_2$, find the transition matrix P from B to C.

Answer: $P = \begin{bmatrix} 20 & -30 & -69 \\ 24 & -24 & -80 \\ -9 & 11 & 31 \end{bmatrix}$

Exercise: Let $P = \begin{bmatrix} 2 & 2 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$ be the transition matrix from B to C. If $C = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$, find the basis B.

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Answer: $B = \left\{ \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 2 \end{bmatrix} \right\}$

Exercise: For an ordered basis

$$B = ([3,-1,4,6],[6,7,-3,-2],[-4,-3,3,4],[-2,0,1,2])$$

of a subspace $\mathcal{W}$ of $\mathbb{R}^4$, perform the following steps:

1. Use the Simplified Span Method to find a second ordered basis C.
2. Find the transition matrix P from B to C.
3. Find the transition matrix Q from C to B.
4. For the given vector $\mathbf{v} = [10, 14, 3, 12]$, calculate $[\mathbf{v}]_B$ and $[\mathbf{v}]_C$.

The Matrix of a linear transformation:

Section 5.2

The Matrix of a linear transformation: Let $\mathcal{V}$ and $\mathcal{W}$ be two finite dimensional real vector spaces such that $\dim(\mathcal{V}) = n$ and $\dim(\mathcal{W}) = m$. Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $C = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ be an ordered basis of $\mathcal{V}$ and $\mathcal{W}$, respectively.

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Let $L : \mathcal{V} \to \mathcal{W}$ be any linear transformation. For $\mathbf{v}_j \in \mathcal{V}$, $L(\mathbf{v}_j) \in \mathcal{W}$. For each $j, 1 \leq j \leq n$. Since C is a basis of $\mathcal{W}$, for $a_{ij} \in \mathbb{R}$, we can write

$$L(\mathbf{v}_j) = a_{1j}\mathbf{w}_1 + a_{2j}\mathbf{w}_2 + \cdots + a_{mj}\mathbf{w}_m$$

Thus, we have

$$L(\mathbf{v}_1) = a_{11}\mathbf{w}_1 + a_{21}\mathbf{w}_2 + \cdots + a_{m1}\mathbf{w}_m$$
$$L(\mathbf{v}_2) = a_{12}\mathbf{w}_1 + a_{22}\mathbf{w}_2 + \cdots + a_{m2}\mathbf{w}_m$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$L(\mathbf{v}_n) = a_{1n}\mathbf{w}_1 + a_{2n}\mathbf{w}_2 + \cdots + a_{mn}\mathbf{w}_m$$

Define

$$A_{BC} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}.$$

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The matrix A_{BC} is called the matrix of linear transformation L w.r.t. the bases B and C.

Remark: i^{th} column of the matrix A_{BC} is $[L(\mathbf{v}_i)]_C$.

Theorem: Let $\mathcal{V}$ and $\mathcal{W}$ be non-trivial vector spaces, with $\dim(\mathcal{V}) = n$ and $\dim(\mathcal{W}) = m$. Let $B = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ and $C = (\mathbf{w}_1, \ldots, \mathbf{w}_m)$ be ordered bases for $\mathcal{V}$ and $\mathcal{W}$, respectively. Let $L : \mathcal{V} \to \mathcal{W}$ be a LT.

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Example: Consider the LT $L : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$, given by

$$L(\mathbf{p}(x)) = x\mathbf{p}(x)$$

with ordered bases $B = (x, 1)$ and $C = (x^2, x - 1, x + 1)$ of $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively.

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$$[L(x)]_C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

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Hence,

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Method for computing A_{BC}: Let $B = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ and $C = (\mathbf{w}_1, \ldots, \mathbf{w}_m)$ be ordered bases for $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Also, let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a LT.

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- Compute $L(\mathbf{v}_i)$ for all $i = 1, 2, \ldots, n$.
- Form the augmented matrix

$$[\mathbf{w}_1 \ \mathbf{w}_2 \ \ldots \mathbf{w}_m \mid L(\mathbf{v}_1) | \ L(\mathbf{v}_2) | \ \ldots | L(\mathbf{v}_n)]$$

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- Compute $L(\mathbf{v}_i)$ for all $i = 1, 2, \ldots, n$.
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to produce $[I_m \mid A_{BC}]$.

Example: Consider the LT $L : \mathbb{R}^2 \to \mathcal{P}_2$, given by

$$L([a, b]) = (-a + 5b)x^2 + (3a - b)x + 2b$$

with ordered bases $B = ([5, 3], [3, 2])$ and

$$C = (3x^2 - 2x, -2x^2 + 2x - 1, x^2 - x + 1)$$

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Solution: Since $L[5,3] = 10x^2 + 12x + 6$ and $L[3,2] = 7x^2 + 7x + 4$. Consider

$$\left[\begin{array}{ccc|cc} 3 & -2 & 1 & 10 & 7 \\ -2 & 2 & -1 & 12 & 7 \\ 0 & -1 & 1 & 6 & 4 \end{array}\right]$$

RREF of the above matrix is

$$\left[\begin{array}{ccc|cc} 1 & 0 & 0 & 22 & 14 \\ 0 & 1 & 0 & 62 & 39 \\ 0 & 0 & 1 & 68 & 43 \end{array}\right]$$

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Example: Consider the LT $L : \mathcal{P}_3 \to \mathcal{P}_2$, given by $L(\mathbf{p}) = \mathbf{p}'$. Compute A_{BC} with respect to standard bases $B = \{x^3, x^2, x, 1\}$ of $\mathcal{P}_3$ and $C = \{x^2, x, 1\}$ of $\mathcal{P}_2$.

Example: Consider the LT $L : \mathcal{P}_3 \to \mathcal{P}_2$, given by $L(\mathbf{p}) = \mathbf{p}'$. Compute A_{BC} with respect to standard bases $B = \{x^3, x^2, x, 1\}$ of $\mathcal{P}_3$ and $C = \{x^2, x, 1\}$ of $\mathcal{P}_2$. Using A_{BC}, find $L(4x^3 - 5x^2 + 6x - 7)$.

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Solution: Standard basis of $\mathcal{P}_3$ is $\{x^3, x^2, x, 1\}$. Since

$L(x^3) = 3x^2,\ L(x^2) = 2x,\ L(x) = 1,\ L(1) = 0,$ we have

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Solution: Standard basis of $\mathcal{P}_3$ is $\{x^3, x^2, x, 1\}$. Since

$L(x^3) = 3x^2,\ L(x^2) = 2x,\ L(x) = 1,\ L(1) = 0,$ we have

$$A_{BC} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Since

$$[L(4x^3 - 5x^2 + 6x - 7)]_C = A_{BC}[(4x^3 - 5x^2 + 6x - 7)]_B$$

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$$= A_{BC}\begin{bmatrix} 4 \\ -5 \\ 6 \\ -7 \end{bmatrix} = \begin{bmatrix} 12 \\ -10 \\ 6 \end{bmatrix}$$

Thus, $L(4x^3 - 5x^2 + 6x - 7) = 12x^2 - 10x + 6$

Since

$$[L(4x^3-5x^2+6x-7)]_C = A_{BC}[(4x^3-5x^2+6x-7)]_B$$

$$= A_{BC}\begin{bmatrix}4\\-5\\6\\-7\end{bmatrix} = \begin{bmatrix}12\\-10\\6\end{bmatrix}$$

Thus, $L(4x^3-5x^2+6x-7) = 12x^2-10x+6$

Also, note that

$$L(4x^3-5x^2+6x-7) = (4x^3-5x^2+6x-7)' = 12x^2-10x+6.$$

Example: Let the matrix of LT $L : \mathcal{P}_1 \to \mathcal{P}_1$ with respect to basis $B = (x+1, x-1)$ be $\begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$.

Example: Let the matrix of LT $L : \mathcal{P}_1 \rightarrow \mathcal{P}_1$ with respect to basis $B = (x+1, x-1)$ be $\begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$. Find the matrix of L with respect to basis $C = (x, 1)$.

Example: Let the matrix of LT $L : \mathcal{P}_1 \to \mathcal{P}_1$ with respect to basis $B = (x+1, x-1)$ be $\begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$. Find the matrix of L with respect to basis $C = (x, 1)$.

Solution: Since $A_{BB} = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$, we have

$$L(x+1) = 2(x+1) - 1(x-1) = x+3$$
$$L(x-1) = 3(x+1) - 2(x-1) = x+5$$

$$L(ax+b) = L\left(\frac{a+b}{2}(x+1) + \frac{a-b}{2}(x-1)\right)$$

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so that $L(x) = x+4$ and $L(1) = -1$.

$$L(ax+b) = L\left(\frac{a+b}{2}(x+1) + \frac{a-b}{2}(x-1)\right)$$
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so that $L(x) = x + 4$ and $L(1) = -1$.
Hence,

$$A_{CC} = \begin{bmatrix} 1 & 0 \\ 4 & -1 \end{bmatrix}.$$

Exercise: Consider the LT $L : \mathbb{R}^3 \to \mathbb{R}^2$, given by $L([x, y, z]) = [x + y, y - z]$.

Exercise: Consider the LT $L : \mathbb{R}^3 \to \mathbb{R}^2$, given by $L([x, y, z]) = [x + y, y - z]$. Compute A_{BC} with respect to bases $B = ([1, 0, 1], [0, 1, 1], [1, 1, 1])$ and $C = ([1, 2], [-1, 1])$.

Exercise: Consider the LT $L : \mathbb{R}^3 \to \mathbb{R}^2$, given by $L([x, y, z]) = [x + y, y - z]$. Compute A_{BC} with respect to bases $B = ([1, 0, 1], [0, 1, 1], [1, 1, 1])$ and $C = ([1, 2], [-1, 1])$.

Answer: $A_{BC} = \begin{bmatrix} 0 & 1/3 & 2/3 \\ -1 & -2/3 & -4/3 \end{bmatrix}$

Exercise: Consider the LT $L : \mathcal{P}_3 \to \mathcal{M}_{22}$, given by

$$L(ax^3 + bx^2 + cx + d) = \begin{bmatrix} -3a - 2c & -b + 4d \\ 4b - c + 3d & -6a - b + 2d \end{bmatrix}.$$

Compute A_{BC} with respect to standard bases for $\mathcal{P}_3$ and $\mathcal{M}_{22}$.

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Compute A_{BC} with respect to standard bases for $\mathcal{P}_3$ and $\mathcal{M}_{22}$.

Answer:

$$A_{BC} = \begin{bmatrix} -3 & 0 & -2 & 0 \\ 0 & -1 & 0 & 4 \\ 0 & 4 & -1 & 3 \\ -6 & -1 & 0 & 2 \end{bmatrix}$$

Exercise: Consider the LT $L : \mathbb{R}^2 \to \mathcal{P}_2$, given by

$$L([a, b]) = (-a + 5b)x^2 + (3a - b)x + 2b.$$

Exercise: Consider the LT $L : \mathbb{R}^2 \longrightarrow \mathcal{P}_2$, given by

$$L([a, b]) = (-a + 5b)x^2 + (3a - b)x + 2b.$$

Compute A_{BC} with respect to bases $B = ([5, 3], [3, 2])$ and $C = (3x^2 - 2x, -2x^2 + 2x - 1, x^2 - x + 1)$.

Exercise: Consider the LT $L : \mathbb{R}^2 \to \mathcal{P}_2$, given by

$$L([a,b]) = (-a+5b)x^2 + (3a-b)x + 2b.$$

Compute A_{BC} with respect to bases $B = ([5,3],[3,2])$ and $C = (3x^2-2x, -2x^2+2x-1, x^2-x+1)$.

Answer:

$$A_{BC} = \begin{bmatrix} 22 & 14 \\ 62 & 39 \\ 68 & 43 \end{bmatrix}$$

Exercise: Let $B = ([1,2],[2,-1])$ and $C = ([1,0],[0,1])$ be ordered bases for $\mathbb{R}^2$. If $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a LT such that $A_{BC} = \begin{bmatrix} 4 & 3 \\ 2 & -4 \end{bmatrix}$. Find $L([5,5])$, Also, find $L([x,y])$ for all $[x,y] \in \mathbb{R}^2$.

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Answer: $L([5,5]) = [15,2]$.

Exercise: Let

$$B = ([1,1,0,0],[0,1,1,0],[0,0,1,1],[0,0,0,1]) \text{ and}$$

$$C = ([1,1,1],[1,2,3],[1,0,0])$$

be ordered bases for $\mathbb{R}^4$ and $\mathbb{R}^3$, respectively. If

$L : \mathbb{R}^4 \to \mathbb{R}^3$ be a LT such that $A_{BC} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$.

Find L?

Exercise: Let

$$B = ([1,1,0,0],[0,1,1,0],[0,0,1,1],[0,0,0,1]) \text{ and}$$

$$C = ([1,1,1],[1,2,3],[1,0,0])$$

be ordered bases for $\mathbb{R}^4$ and $\mathbb{R}^3$, respectively. If $L : \mathbb{R}^4 \to \mathbb{R}^3$ be a LT such that $A_{BC} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$.

Find L?

Answer:

$$L([x_1,x_2,x_3,x_4]) = [-2x_1+3x_2+x_4, x_2+2x_3, x_2+3x_3].$$

Matrix for the composition of Linear Transformations:

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Theorem: Let $\mathcal{V}_1, \mathcal{V}_2$ and $\mathcal{V}_3$ be nontrivial finite dimensional vector spaces with ordered bases B, C and D, respectively. Let $L_1 : \mathcal{V}_1 \to \mathcal{V}_2$ be a linear transformation with matrix A_{BC} and let $L_2 : \mathcal{V}_2 \to \mathcal{V}_3$ be a linear transformation with matrix A_{CD}. Then matrix

$$A_{BD} = A_{CD}A_{BC}$$

is the matrix of linear transformation $L_2 \circ L_1 : \mathcal{V}_1 \to \mathcal{V}_3$ with respect to the bases B and D.

Example: Let $L_1 : \mathbb{R}^2 \to \mathbb{R}^2$ and $L_2 : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$L_1([x, y]) = [y, x]$$

$$L_2([x, y]) = [x + y, x - y, y]$$

- Find the matrix of L_1 and L_2 with respect to the standard basis in each case.

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- Find the matrix of L_1 and L_2 with respect to the standard basis in each case.
- Find the matrix of $L_2 \circ L_1$ with respect to standard basis of $\mathbb{R}^2$ and $\mathbb{R}^3$.

Answer: The matrix of L_1 w.r. to $B = \{[1,0],[0,1]\}$ is

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The matrix of L_2 w. r. to the bases $C = [1,0],[0,1]$ and $D = \{[1,0,0],[0,1,0],[0,0,1]\}$ is

$$A_{CD} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$$

Thus, the matrix A_{BD} of the linear transformation $L_2 \circ L_1 : \mathbb{R}^2 \to \mathbb{R}^3$ w.r. to the bases B and D is

$$A_{BD} = A_{CD}A_{BB} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 0 \end{bmatrix}$$

Theorem: Let $L : \mathcal{V} \to \mathcal{W}$ be a linear transformation between n-dimensional vector spaces $\mathcal{V}$ and $\mathcal{W}$ and let B and C are ordered bases for $\mathcal{V}$ and $\mathcal{W}$, respectively. Then L is an isomorphism (or invertible) if and only if the matrix representation A_{BC} for L with respect to B and C is nonsingular.

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In this case If D_{CB} is the matrix for L^{-1} with respect to C and B then $A_{BC}^{-1} = D_{CB}$.

Example: Let L_1 and L_2 be linear operators on $\mathbb{R}^3$. Let

$$A = \begin{bmatrix} 0 & -2 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & 1 \\ 0 & -3 & 0 \end{bmatrix}$$

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be matrices for L_1 and L_2 respectively, with respect to standard basis.

- Show that L_1 and L_2 are isomorphisms.

Answer: Since rank(A) $= 3$ and rank(B) $= 3$, the matrices A and B are nonsingular. Hence, L_1 and L_2 are isomorphisms.

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Answer: Since $L_1^{-1}(\mathbf{v}) = A^{-1}(\mathbf{v})$. Using row reduction (see Chapter 3), we have

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Similarly, $L_2^{-1}(\mathbf{v}) = B^{-1}(\mathbf{v})$, where

$$B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1/3 \\ 2 & 1 & 0 \end{bmatrix}.$$

Thank You