Russell’s Paradox and Possible Solutions

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ABSTRACT: The beginnings of set theory as a mathematical discipline can be traced back to the work of Georg Cantor. Around 1900 when the ideas of Cantor were finally being accepted, a series of logical contradictions were found to exist in the theory of sets. The most famous of these contradictions, discovered by Bertrand Russell and known as "Russell’s Paradox," caused much worry amongst mathematicians. Russell attempted to patch this logical fallacy, but the most accepted solution today is that of Zermelo and Fraenkel. A new real world example of Russell’s paradox is examined and the solution of Zermelo and Fraenkel is applied.

I. INTRODUCTION

The origins of set theory can be traced back to a Bohemian priest, Bernhard Bolzano (1781-1848), who was a professor of religion at the University of Prague. Bolzano’s paper, *Paradoxien des Unendlichen* (Paradoxes of the Infinite), is the first to introduce the term *set.* This paper discusses the relationship between the set of natural numbers and their perfect squares, an idea first considered by Galileo, but the paper also considers many other examples of infinite sets.

Georg Cantor’s paper, *Über eine Eigenshaft des Inbegriffes aller reellen algebraischen Zahlen* (On a Property of the System of all the Real Algebraic Numbers) published in 1874 is considered the first purely theoretical paper on set theory. Cantor’s ideas were quickly dismissed as being ideas for a philosopher and he was considered a mathematical heretic. There were, however, a few supporters of Cantor’s ideas including Dedekind, Weierstrass, and Hilbert. These proponents of set theory spent the next twenty years working to get their ideas accepted by mathematicians. In 1895 and 1897 Cantor published a two part journal describing all of the important results discovered in set theory from the last twenty years. Cantor defined a set as:

"By a set we are to understand any collection into a whole M of definite and distinguishable objects of our intuition or our thought. These objects are called the elements of M" (Burton, 591).

By this time, set theory had gained enough acceptance among mathematicians to be considered an independent mathematical discipline. Therefore, when Bertrand Russell presented his paradox that led Cantor’s very definition of sets to a contradiction, many mathematicians felt the foundations of mathematics had begun to erode away.

II. RUSSELL’S PARADOX

The set $S = \{A | A \text{ is a set and } A \notin A\}$. If we assume S is not in the set S, then by definition, it must belong to that set. If we assume S is in the set S, then it contradicts the definition of S. Here we have the paradox that Bertrand Russell (1872-1970) presented to Gottlob Frege (1848-1925) just as Frege's lifetime work on the logical foundations of arithmetic went to be published. In Gottlob's own words, "A scientist can hardly meet with anything more undesirable than to have the foundation give way just as the work is finished" (Burton,612)

Many ways have been found to give an example of Russell's paradox for clearer understanding. Here we present a new example of this paradox. Imagine a middle school teacher who every week passes out a list of all the materials she will pass out that week that she expects each student to have in their binders. She plans to have the list, called the "weekly list," be a table of contents for each week so the students can organize their binders. However, towards the end of the semester the teacher realizes she sometimes forgot to include the weekly list itself as one of the items she wants to be included in the binders. The teacher figures she can just create a supplementary list that will contain only the weekly lists that failed to include themselves as something needed to be in the binder. However, she finds now that she is faced with a problem. If she includes the supplementary list as one of the supplementary list's items, the list will no longer be the list of lists that do not contain themselves as an item. If she does not include the supplementary list as one of its items, then it would be considered one of the lists that failed to include itself and should be included! Thus, the teacher is faced with Russell's paradox.

III. SOLUTIONS TO THE PARADOX

Bertrand Russell devised what he called the theory of types to prevent the paradox. In this theory, a set would be defined as being of a distinct type, like type 1. The elements of type 1 sets can then only be included in a set of type 2 because sets of type 2 are defined as containing only sets of type 1. Thus, we do not need to worry about whether or not a set of type 2 can contain itself because it's defined as only containing sets of type 1. This theory creates a sort of hierarchy of sets. In the example of the teacher and her lists, she would define the additional list as containing only those lists that she had handed out weekly. Now she does not need to worry about whether or not she should include this additional list because it is not one of the weekly lists. While Russell's solution does succeed in avoiding the contradictions, mathematicians decided that the solution should be more intuitive for the foundations of mathematics.

The most accepted solution today is that of Zermelo and Fraenkel. Zermelo's axiom of specification is, "to every set A and every definite property P(x) there corresponds a set whose elements are exactly those elements x in A for which the property P(x) holds" (Burton, 616). What this axiom does is require a preexisting set A and some property P(x) to make a new set. Previously, only the property P(x) was required. This changes the set S to $S = \{A \in A | A \text{ is a set and } A \notin A\}$. Now $S \in S$ is impossible because it would have to satisfy the two conditions that $S \in A$ and $S \notin S$, which it clearly cannot. The is clear because we just state that in order for $S \in S$, it must satisfy the property that $S \notin S$. If we consider the other possibility that $S \notin S$ we see that it does satisfy the property P(x), but cannot meet our second requirement that $S \in A$. This is because if $S \in A$ then it follows, by our definition, that $S \in S$, which we already reasoned is not true. Therefore we conclude that by the law of excluded middle, which says that every proposition is either true or false (Burton, 612), $S \notin A$. Therefore, $S \notin S$ failed the second of the two requirements to be in the set S so we conclude that $S \notin S$ and have avoided the paradox.

In the example of the teacher and her lists, she has a list we will call list x that contains lists known to exist. However, in order for an item to get onto the supplementary list, the teacher makes the requirement that it must now be on list x and not contain itself. If we consider the possibility that the supplementary list is listed on the supplementary list, we find it must be on list x and must not contain itself on a list. The latter requirement clearly fails because we just said it contains itself. Now consider the other possibility that the supplementary list is not listed on itself. We can see that it does meet the first requirement of being a list that does not include itself, but it can't meet the second requirement that it's included on list x. This follows because if it were included on list x, it would imply that the supplementary list should be included on the supplementary list, but we already showed this is not allowed. Therefore, the teacher can safely avoid the paradox and not include the supplementary list as an item of itself.

IV. SUMMARY

Although the discovery of Russell's paradox came as terrible news to many mathematicians, it has also lead to many good solutions. Russell was able to give a second look at the definition of a set and found a way to avoid the contradiction by redefining what was meant by a set. It was Zermelo's elegant axiom of specification that finally provide mathematicians with a satisfactory method for avoiding the famous paradox. Mathematicians can now rest easy with the knowledge that their logical foundation still stands strong.

V. REFERENCES

Burton, David M. The History of Mathematics. New York, NY: McGraw-Hill, 1997.

Moorcroft, Francis. "Russell's Paradox." The Philosophers' Magazine. Spring 1998. <http://www.philosophers.co.uk/current/paradox2.htm>.