Mathematics-II (MATH F112)

Linear Algebra and Complex Variables

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Announcements

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Instructor-Incharge: Dr. Trilok Mathur

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For more details about the course structure, please go through the course (MATH F112) handout available on ID website.

Text Book: For Linear Algebra

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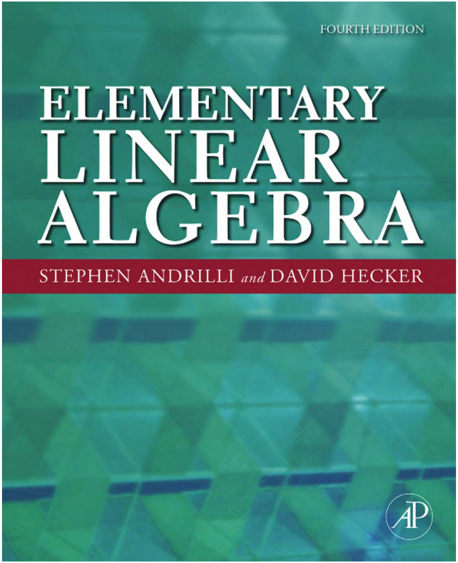

Chapter: 2

- System of Linear equations
- Row Echelon Form
- Elementary Row Operations
- Gaussian Elimination Method
- Reduced Row Echelon Form
- Gauss-Jordan Row Reduction Method
- Rank
- Inverse of a Matrix

An Example for Motivation: Solve the system of linear equations

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Step 1: Represent the given system of equations as follows:

$$\begin{aligned} x - y - z &= 2 \\ 3x - 3y + 2z &= 16 \\ 2x - y + z &= 9 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 3 & -3 & 2 & 16 \\ 2 & -1 & 1 & 9 \end{array}\right]$$

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$$\begin{aligned} x - y - z &= 2 \\ 5z &= 10 \\ 2x - y + z &= 9 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & 0 & 5 & 10 \\ 2 & -1 & 1 & 9 \end{array}\right]$$

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Step 3: Multiply the first equation by 2 and subtract it from the 3rd equation; **Multiply the first row by 2 and subtract it from the 3rd row**

$$\begin{aligned} x - y - z &= 2 \\ 5z &= 10 \\ y + 3z &= 5 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & 0 & 5 & 10 \\ 0 & 1 & 3 & 5 \end{array}\right]$$

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By Backward substitution we find

$$z = 2, \;\; y = -1, \;\; x = 3$$

is a solution of the given system of equations.

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2. In each nonzero row, the first nonzero entry (called the leading entry or pivot) is 1 and it is in a column to the left (strictly) of any leading entry below it. (The column containing a pivot element is called a pivot column).

- The following matrices are in row echelon form:

$$\begin{bmatrix} \boxed{1} & 2 \\ 0 & \boxed{1} \end{bmatrix}, \begin{bmatrix} \boxed{1} & -1 & -1 \\ 0 & \boxed{1} & 3 \\ 0 & 0 & \boxed{1} \end{bmatrix}, \begin{bmatrix} 0 & \boxed{1} & -1 & 2 & 1 \\ 0 & 0 & \boxed{1} & 4 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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- If a matrix A is in row echelon form, then in each column of A containing a leading entry, the entries below that leading entry are zero.

- The following matrices are **not** in row echelon form:

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix}, \begin{bmatrix} 0 & -1 & -1 & 2 & 1 \\ 1 & 0 & 5 & 10 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Elementary Row Operations: The following row operations are called **elementary row operations** of a matrix:

- Interchange of two rows ($R_i \leftrightarrow R_j$)
- Multiply a row R_i by a nonzero constant c ($R_i \to cR_i$)
- Add a multiple of a row R_j to another row R_i ($R_i \to R_i + cR_j$)

Example: Transform the following matrix into row echelon form

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 2 & 5 \\ 3 & 4 & 4 & 12 \end{bmatrix}$$

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Answer:

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Answer:

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- Row echelon form of a matrix is not unique.

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Example: Matrices

$$A = \begin{bmatrix} 3 & 2 & 7 \\ -4 & 1 & 6 \\ 2 & 5 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 2 & 7 \\ -2 & 6 & 10 \\ 2 & 5 & 4 \end{bmatrix}$$

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Result: Matrices A and B are row equivalent if and only if they can be reduced to same row echelon form.

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- The set $\mathbb{R}^2$ corresponds to the set of vectors whose tails are at the origin O.

- For example, the ordered pair $A = (3, 2) \in \mathbb{R}^2$ corresponds to the vector $\overrightarrow{OA}$ and we denote it as square bracket $[3, 2]$.

For $n \in \mathbb{N}$, $\mathbb{R}^n$ is the set of all ordered n-tuples $(x_1, x_2, \ldots, x_n)$, where $x_i \in \mathbb{R}$.

- We can think the point $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ as vector and write it as $[x_1, x_2, \ldots, x_n]$ (row vector). Thus,

$$\mathbb{R}^n = \{[x_1, x_2, \ldots, x_n] \mid x_i \in \mathbb{R}\}.$$

- Sometime we will write a vector of $\mathbb{R}^n$ as a column vector

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [x_1, x_2, \ldots, x_n]^T,$$

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- The vector $[0, 0, \ldots, 0]$ of $\mathbb{R}^n$, called the zero vector of $\mathbb{R}^n$ and it is denoted by the symbol $\mathbf{0}$.

Let **u** $= [u_1, u_2, \ldots, u_n]$ and **v** $= [v_1, v_2, \ldots, v_n] \in \mathbb{R}^n$ and $k \in \mathbb{R}$.

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- $\mathbf{u} + \mathbf{v} = [u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n]$ (Vector addition)
- $k\mathbf{u} = [ku_1, ku_2, \ldots, ku_n]$ (Scalar Multiplication)

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- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutativity).
- $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ (associativity).
- $\mathbf{u} + 0 = \mathbf{u}$.
- $\mathbf{u} + (-\mathbf{u}) = 0$, where $-\mathbf{u} = [-u_1, -u_2, \ldots, -u_n]$.
- $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ (distributivity over vector addition).
- $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ (distributivity over scalar addition).
- $c(d\mathbf{u}) = (cd)\mathbf{u}$.
- $1\mathbf{u} = \mathbf{u}$.
- $0\mathbf{u} = 0$.

System of Linear Equations

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A system of **m** linear equations in **n** unknown variables $x_1, x_2, \ldots, x_n$ is given by

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad & \quad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

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where $a_{ij}, b_i \in \mathbb{R}$ and $1 \leq i \leq m, 1 \leq j \leq n$.

- A solution of the linear system is an n-tuple $(s_1, s_2, \ldots, s_n)$ such that each equation of the system is satisfied by substituting s_i in place of x_i.

Above linear system of equations can be written in the form $AX = B$, where

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- The matrix A is called the **coefficient matrix**.

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$$[A|B] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right]$$

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- The vertical bar is used in the augmented matrix $[A|B]$ only to distinguish the column vector B from the coefficient matrix A.

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- If $B \neq 0$, then the system $AX = B$ is called non-homogenous system of equations.
- The solution $X = 0$ of the system $AX = 0$ is called the trivial solution and a solution other than $X = 0$ is called a non-trivial solution of the homogenous system $AX = 0$.

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are equivalent (Why?).

Theorem: Let $AX = B$ be a system of linear equations. If $[C|D]$ is row equivalent to $[A|B]$, then the system $CX = D$ is equivalent to $AX = B$.

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- Use back substitution to solve the equivalent system that corresponds to row echelon form.

Exercise: Solve the linear system of equations by Gaussian elimination method

$$\begin{aligned} x + y + z &= 3 \\ 2x + 3z &= 5 \\ y + z &= 2 \end{aligned}$$

Exercise: Solve the linear system of equations

$$x + y + z = 3, \quad x + 2y + 2z = 5, \quad 3x + 4y + 4z = 12$$

by Gaussian elimination method.

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- The variables which are not dependent are called independent (free) variables.

Exercise: Solve the system of linear equations

$$x + y + z = 3, \quad x + 2y + 2z = 5, \quad 3x + 4y + 4z = 11$$

by Gaussian elimination method.

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- If a column contains a leading entry (or pivot) then all other entries in that column must be zero.

Example: The following matrices are in reduced row echelon form

$$\begin{bmatrix} \boxed{1} & 0 \\ 0 & \boxed{1} \end{bmatrix}, \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \boxed{1} & 0 & 2 & 1 \\ 0 & 0 & \boxed{1} & 4 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Result: Every matrix has a unique reduced row echelon form.

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- Write the augmented matrix $[A \mid B]$.
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- Use back substitution to solve the equivalent system that corresponds to the reduced row echelon form.

Exercise: Solve the system of linear equations equations

$$x + y + z = 5, \;\; 2x + 3y + 5z = 8, \;\; 4x + 5z = 2$$

by Gauss-Jordan method.

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$$x + y + z = 5, \;\; 2x + 3y + 5z = 8, \;\; 4x + 5z = 2$$

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Answer: $x = 3, y = 4$ and $z = -2$.

Exercise: Solve the system of linear equations

$$4y + z = 2, \; 2x + 6y - 2z = 3, \; 4x + 8y - 5z = 4$$

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Answer: Infinitely many solutions and the solution set is

$$\left\{ \left(\frac{7}{4}d,\ \frac{1}{2} - \frac{1}{4}d,\ d \right) \mid d \in \mathbb{R} \right\}.$$

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Answer: No solution.

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- existence of a unique solution
- existence of an infinite number of solutions, and
- no solution.

- If the system $AX = B$ has some solution then it is called a consistent system. Otherwise it is called an inconsistent system.

Question: Whether there are conditions under which the linear system $AX = B$ is consistent?

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Remark: The number of nonzero rows in either the row echelon form or the reduced row echelon form of a matrix are same.

Exercise: Determine the rank of $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

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3. If rank(A) $\neq$ rank($[A \mid B]$) then the system $AX = B$ is inconsistent.

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Exercise: Test the consistency of the given system of equations

$$\begin{aligned} 3x + y + w &= -9 \\ -2y + 12z - 8w &= -6 \\ 2x - 3y + 22z - 14w &= -17. \end{aligned}$$

Find all the solutions, if it is consistent.

Exercise: For what value of $\lambda \in \mathbb{R}$, the following system of equations has (i) a unique solution (ii) infinitely many solutions and (iii) no solution

$$\begin{aligned}(5-\lambda)x + 4y + 2z &= 4\\ 4x + (5-\lambda)y + 2z &= 4\\ 2x + 2y + (2-\lambda)z &= 2.\end{aligned}$$

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Also find the solutions, whenever they exist.

Definition: Let A be an $n \times n$ matrix. Then an $n \times n$ matrix B is a (multiplicative) inverse of A if and only if

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- If such a matrix B exists then A is called nonsingular. Otherwise it is called singular.

Example: Show that the matrix $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ is nonsingular.

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Theorem: Inverse of a matrix is unique if it exists.

As the inverse of a matrix A is unique, we denote it by A^{-1}. That is, $AA^{-1} = A^{-1}A = I$.

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Theorem: Let A and B be an $n \times n$ nonsingular matrices. Then

- $(A^{-1})^{-1} = A$.
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- $(A^T)^{-1} = (A^{-1})^T$.

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- If a matrix does have an inverse, how can we find it?

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Step 3: If

- $C = I_n$ then $D = A^{-1}$.
- $C \neq I_n$ then A is singular and A^{-1} does not exist.

Exercise: Using row reduction method, find the inverse of $A = \begin{bmatrix} 1 & -1 & -2 \\ 2 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$, if it exists.

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Hint: Note that reduced row echelon form of the matrix $[A|I_3]$ is

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 5 & -3 & -1 \\ 0 & 0 & 1 & -3 & 2 & 1 \end{array}\right]$$

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Thus, $A^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 5 & -3 & -1 \\ -3 & 2 & 1 \end{bmatrix}$

Theorem: Let A be an $n \times n$ matrix. The following statements are equivalent:

- A is nonsingular.
- The homogenous system $AX = 0$ has only the trivial solution.
- rank$(A) = n$.
- The reduced row echelon form of A is I_n.

Thank You