BITS Pilani
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MATH F112 (Mathematics-II)

Complex Analysis

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Lecture 32-35
Integrals

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Derivative of Functions $w(t)$

(1) Let $w(t)=u(t)+i\,v(t)$ be a complex valued function of a real variable t, where u and v are real-valued functions of t.

Then $\dfrac{dw}{dt}=w'(t)=u'(t)+i\,v'(t)$, provided each of the derivatives u' & v' exists at t

Derivative of Functions $w(t)$

(2) If z_0 is a complex constant, then

$$\frac{d}{dt}\left(z_0\, w(t)\right) = z_0 \frac{dw}{dt}.$$

(3)

$$\frac{d}{dt}\left(e^{z_0 t}\right) = z_0\, e^{z_0 t}.$$

(4) Mean Value Theorem for derivatives is NOT true.

Derivative of Functions $w(t)$

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Supposethat :

$(i)\ w(t) = u(t) + i\,v(t),\ a \le t \le b$ be continuous, i.e. u and v are continuous on $[a, b]$

$(ii)\ w'(t)$ exists in $a < t < b$.

Then there may NOT exist any c in (a, b) such that $w'(c) = \dfrac{w(b) - w(a)}{b - a}$

Derivative of Functions $w(t)$

Example:

Let $w(t) = e^{it},\ 0 \le t \le 2\pi$

$$\Rightarrow\ w'(t) = i\,e^{it}$$

$$\Rightarrow |w'(t)| = 1 \text{ for all } t \in [0, 2\pi]$$

$$\Rightarrow w'(t) \neq 0 \text{ for all } t \in [0, 2\pi]$$

Derivative of Functions $w(t)$

But $w(2\pi) - w(0) = e^{i2\pi} - e^{i.0}$

$$= 0$$

Þ MVT for derivative is NOT true in the complex plane.

Definite Integral of $w(t)$

Let $w(t) = u(t) + i\,v(t)$ be a complex valued function of a real-variable t.

$u(t),\ v(t)$: real-valued functions over $a \leq t \leq b$.

Then definite integral of $w(t)$ is defined as

$$\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt,$$ provided the individual integrals on the right exist.

Definite Integral of $w(t)$

$$\Rightarrow \operatorname{Re} \int_a^b w(t)\, dt = \int_a^b \operatorname{Re} w(t)\, dt,$$

$$\& \operatorname{Im} \int_a^b w(t)\, dt = \int_a^b \operatorname{Im} w(t)\, dt.$$

Definite Integral of $w(t)$

Example :

$$\int_0^1 (1 - i\,t)^2 dt = \int_0^1 \left(1 - t^2 - 2\,i\,t\right) dt$$

$$= \int_0^1 \left(1 - t^2\right) dt - i \int_0^1 2\,t\,dt$$

$$= \frac{2}{3} - i$$

Definite Integral of $w(t)$

Note: Mean Value Theorem for integral calculus also does not hold in complex plane.

i.e. for a complex valued function $w(t)$ defined on $a \le t \le b$, there need not exist a number $a < c < b$ such that

$$\int_a^b w(t)\,dt = w(c)(b-a)$$

Ex: $w(t) = e^{it} \quad 0 \le t \le 2\pi,$

Here $w(0) = w(2\pi) = 1 \Rightarrow \int_0^{2\pi} w(t)\,dt = 0$

But $\left|e^{ic}(b-a)\right| = 2\pi \neq 0,$ for any $0 < c < 2\pi$

Definite Integral of $w(t)$

Property:

Let $w(t)$ be a complex-valued function integrable on $[a,b]$. Then

$$\left|\int_a^b w(t)\,dt\right| \le \int_a^b |w(t)|\,dt$$

Contours

Definitions

(1) $Curve$: A set of points $z = (x, y)$ in the complex plane is said to be a curve C if

$$x = x(t),\ y = y(t)$$

are continuous functions of a real parameter t.

Contours

We write

$$C: x = x(t), y = y(t)$$

Or

$$C: z(t) \ = x(t) + i\, y(t).$$

Contours

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(2) *Arc* :

The portion between any two points of a curve is called an arc of the curve, i.e.

$C: \; x(t) \; + \; i\,y(t), \; a \leq t \leq b$, is an arc.

For simplicity, we shall use the single term "curve" to denote the entire curve as well as an arc of the curve.

Contours

(3) *Differentiable curve*:

The curve $C: z(t) = x(t) + i\, y(t)$ is said to be differentiable if $x'(t)$ & $y'(t)$ exist and they are continuous in $a \le t \le b$, and we write

$$z'(t) = x'(t) + i\, y'(t)$$

If $z'(t) \neq 0$, on $a < t < b$ then such a curve (arc) is said to be regular or smooth.

Contours

(4) *Piecewise Smooth curve/arc* :

The curve

$$C: z(t) = x(t) + i\, y(t), \quad a \le t \le b,$$

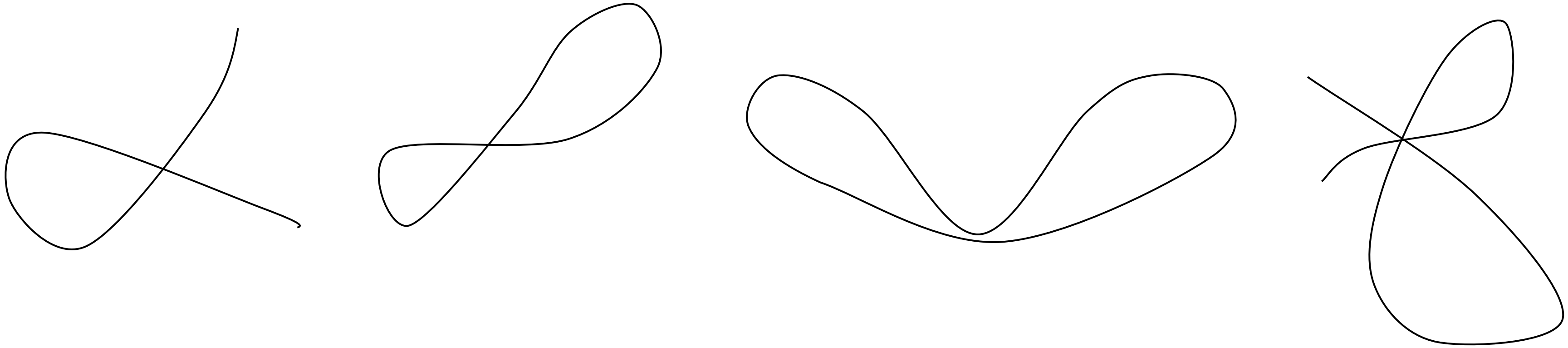
is said to be piecewise smooth if

there exists a finite no. of sub-intervals $[a, a_1], [a_1, a_2], \ldots, [a_{n-1}, b]$ of $[a, b]$, such that C is smooth on each sub-interval.

Contours

(5) $\underline{Jordan\ arc\ /\ curve\ or\ simple\ curve:}$

A curve may have points at which it intersects or touches itself. Such a point is called multiple point of the curve.



Contours

A curve having NO MULTIPLE POINTS is called a simple curve, i.e.,

a curve is said to be simple if it neither touches itself nor crosses itself,

i.e., the curve $C: z(t) = x(t) + i\, y(t)$ is said to be simple if

$z(t_1) \neq z(t_2)$ whenever $t_1 \neq t_2$.

Contours

If the curve

$$C: z(t) = x(t) + i\,y(t),\ a \le t \le b$$

is simple except for the fact that $z(a) = z(b)$, then C is said to be a simple closed curve or a Jordan curve.

Contours

(6). *Length of a differentiable curve*

Let $C: z(t) = x(t) + i\,y(t),\ a \le t \le b$

be a differeniable curve (arc).

$$\Rightarrow z'(t) = x'(t) + i\,y'(t)$$

and $|z'(t)| = \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}$

Contours

Then

$$L = \int_a^b |z'(t)|\, dt$$

is called the length of the curve C.

Contours

(7). $Contour$:

A Contour is a piecewise smooth arc, i.e. an arc consisting of finite number of smooth arcs joined end to end.

Contour Integral

Let $z = z(t),\ a \le t \le b$

denotes a contour C extending from a point $z_1 = z(a)$ to a point $z_2 = z(b)$.

Let $f(z)$ be piecewise continuous on C, i.e. $f(z(t))$ is piecewise continuous on $a \le t \le b$.

Contour Integral

Then we define the line integral or contour integral of f along C as follows:

$$\int_C f(z)\,dz = \int_a^b f\big(z(t)\big)\,z'(t)\,dt$$

Contour Integral

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Properties:

$$(1)\left|\int_C f(z)dz\right| \leq \int_a^b |f(z(t))z'(t)|dt,$$

$$C: z(t), a \leq t \leq b$$

Contour Integral

(2) If z_0 is a constant, then

$$\int_C z_0 f(z)\,dz = z_0 \int_C f(z)\,dz$$

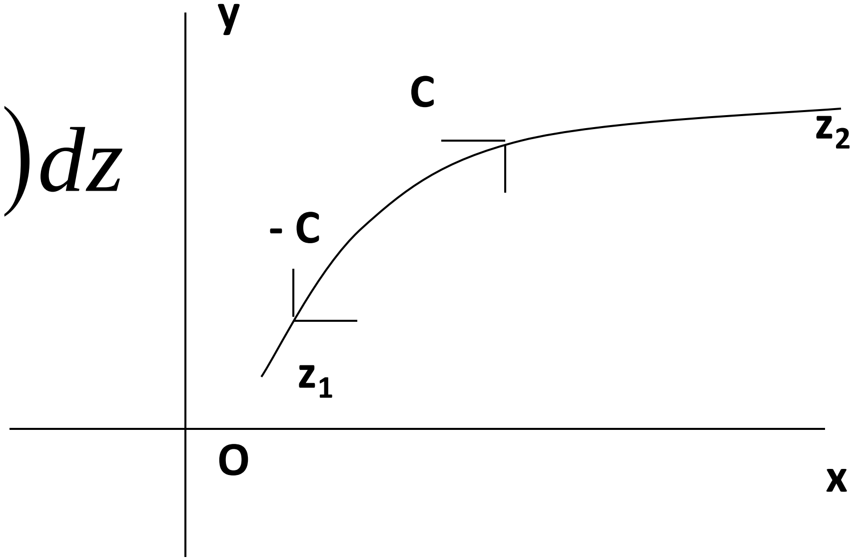
$$(3)\ \int_C \left[f(z) + g(z) \right] dz$$

$$= \int_C f(z)\,dz + \int_C g(z)\,dz$$

Contour Integral

(4) If the contour $C: z = z(t),\ a \le t \le b$ is extended from z_1 to z_2, then $-C$ is extended from z_2 to z_1 i.e. $-C : z = z(-t), -b \le t \le -a$

And $\int_{-C} f(z)dz = -\int_{C} f(z)dz$



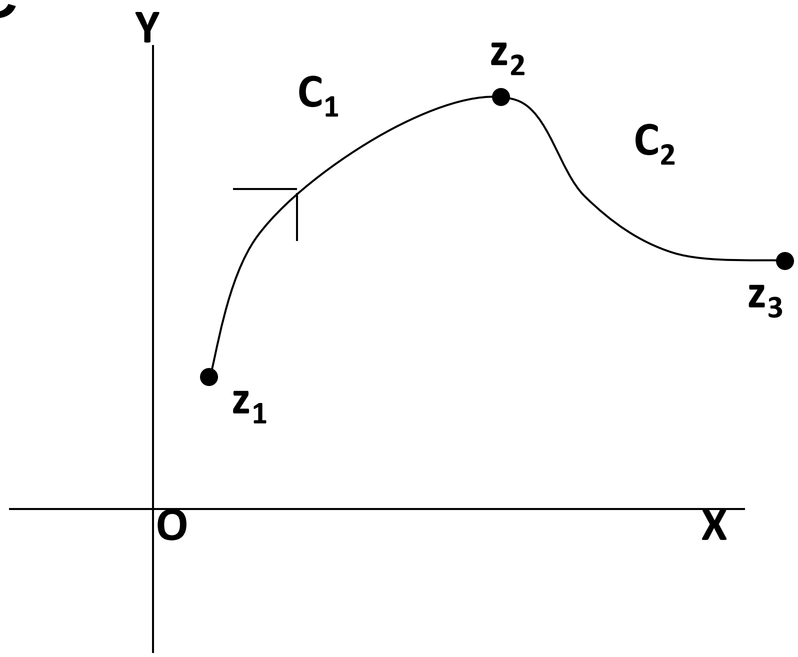
Contour Integral

(5) Let $C = C_1 \cup C_2$, where

$$C: z = z(t);\ a \le t \le b$$

$$C_1: z = z(t),\ a \le t \le c$$

$$\& \ C_2: z = z(t),\ c \le t \le b$$



Then

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

Contour Integral

Ex.1

Let $f(z) = \operatorname{Re} z$, then evaluate

$$\int_C f(z)\,dz, \quad \text{where}$$

$$C : z(t) = t + it, \quad 0 \le t \le 1$$

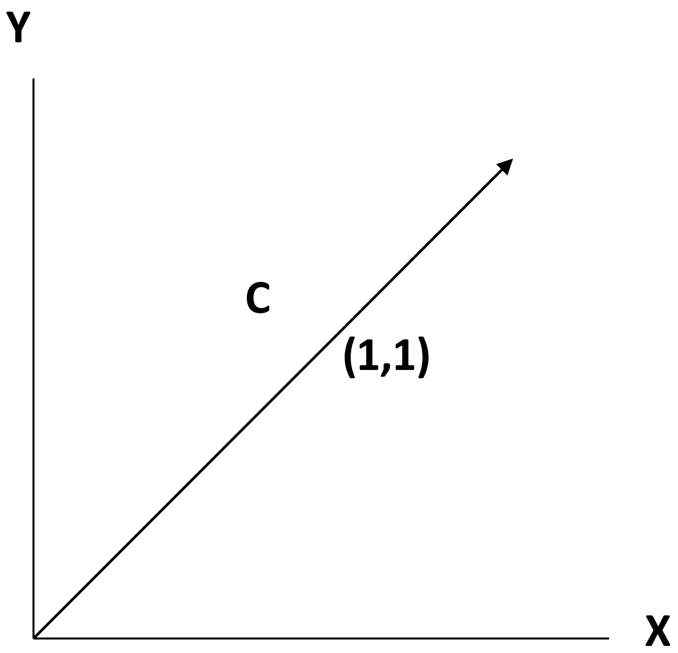
Contour Integral

$$\int_C f(z)\,dz = \int_a^b f(z(t))\dot{z}(t)\,dt$$

$$= \int_0^1 \operatorname{Re} z(t).\dot{z}(t)\,dt$$

$$= \int_0^1 t(1+i)\,dt$$

$$= \frac{1+i}{2}$$



Contour Integral

Ex.2

Let $f(z)=\dfrac{z+2}{z}$ &

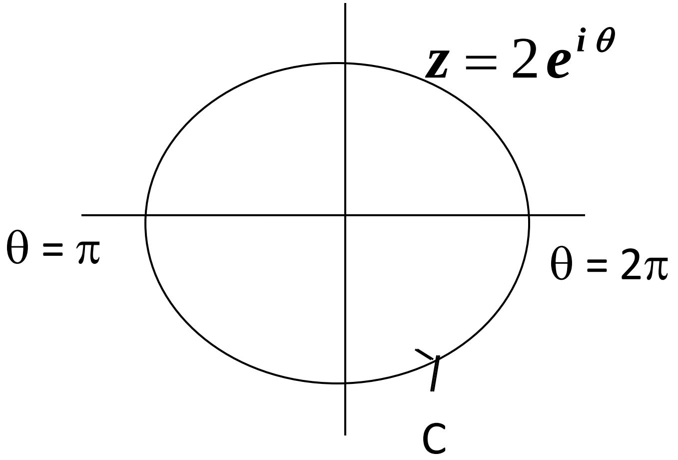
$C: z=2e^{i\theta}, \quad \pi\le\theta\le 2\pi.$

Then evaluate $\displaystyle\int_C f(z)dz$

Contour Integral

Soln: $z = 2\,e^{i\,\theta}$

$$\Rightarrow \quad dz = 2\,e^{i\,\theta}\,i d\theta$$



Contour Integral

$$\therefore I = \int_C f(z)\, dz$$

$$= \int_{\pi}^{2\pi} \frac{2e^{i\theta} + 2}{2e^{i\theta}} .2e^{i\theta} .i\, d\theta$$

$$= 2i \int_{\pi}^{2\pi} \left(e^{i\theta} + 1\right) d\theta$$

$$= 4 + 2\pi i$$

Contour Integral

Ex.3 Let $f(z) = \begin{cases} 1, & y < 0 \\ 4y, & y > 0 \end{cases}$

& C is the arc from $z = -1 - i$

to $z = 1 + i$ along the curve $y = x^3$.

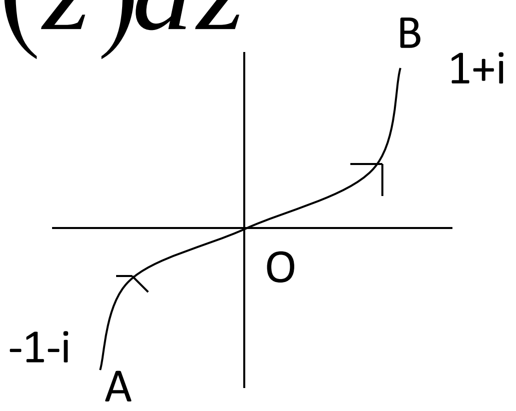
Then evaluate

$$\int_C f(z)\,dz.$$

Contour Integral

We have

$$\int_C f(z)\,dz = \int_{AO} f(z)\,dz + \int_{OB} f(z)\,dz$$



Along AO:

$$z = x + iy = x + ix^3, -1 \le x \le 0$$

Contour Integral

$$\therefore \int_{AO} f(z)dz = \int_{-1}^{0} 1(1+i3x^2)\,dx$$

$$= \left(x + i\frac{3x^3}{3} \right)_{-1}^{0}$$

$$= 1 + i$$

Contour Integral

Along OB, $z = x + i\,x^3,\quad 0 \le x \le 1$

$$\int_{OB} f(z)\,dz = \int_0^1 4y\left(1 + 3i\,x^2\right)dx$$

$$= \int_0^1 4x^3\left(1 + 3ix^2\right)dx$$

$$= 1 + 2\,i$$

$$\therefore \int_C f(z)dz = 1 + i + 1 + 2i = 2 + 3i$$

Ex. With Branch Cuts

Q7. (P-136) $f(z)$ is the principal branch

$$z^i = \exp(i\, Log\, z) \qquad (|z| > 0, -\pi < Arg\, z < \pi)$$

of this power function and C is the semicircle

$$z = e^{i\theta}\ (0 \le \theta \le \pi)$$

Sol.: $\int_C f(z)\, dz = \int_0^{\pi} f\big(z(\theta)\big) z'(\theta)\, d\theta$

$$f\big(z(\theta)\big) z'(\theta) = \exp\big(i\, Log\, e^{i\theta}\big)\, ie^{i\theta}$$

$$= \exp\big(i\, (\ln 1 + i\theta)\big)\, ie^{i\theta} = i\, e^{-\theta + i\theta}$$

Ex. With Branch Cuts

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$$\int_0^{\pi} f\big(z(\theta)\big)z'(\theta)\,d\theta = \frac{i}{-1+i}e^{-\theta+i\theta}\Big|_0^{\pi}$$

$$= \left(\frac{i}{-1+i}\right)\left(e^{(-1+i)\pi} - 1\right)$$

$$= -\frac{(1-i)}{2}\left(e^{-\pi} + 1\right)$$

ML-Inequality

Let $f(z)$ be a piecewise continuous function defined on a contour $C: z = z(t);\ a \le t \le b.$

Suppose that on the contour C, $f(z)$ satisfies $|f(z)| \le M$ for some non-negative constant M.

ML-Inequality

Then $\left|\int_C f(z)\,dz\right| = \left|\int_a^b f(z(t))z'(t)\,dt\right|$

$$\leq \int_a^b |f(z(t))||z'(t)|\,dt$$

$$\leq M\int_a^b |z'(t)|\,dt = ML,$$

where $L = \int_a^b |z'(t)|\,dt$ is the length

of C in $a \leq t \leq b$.

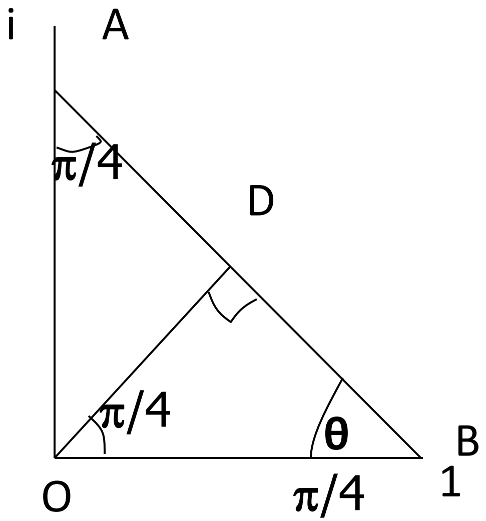
ML-Inequality

Q.2($p.140$) Let C denote the line segment from $z=i$ to $z=1$. By observing that, of all the points on that line segment, the midpoint is the closest to the origin, show that

$$\left|\int_C \frac{dz}{z^4}\right| \leq 4\sqrt{2}.$$

ML-Inequality

D is the mid point of AB



$$\sin\theta = \sin\frac{\pi}{4} = \frac{OD}{OB}$$

$$\Rightarrow OD = OB\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

ML-Inequality

If z is any point on the line AB,

then $|z| \geq \frac{1}{\sqrt{2}}$

$$\Rightarrow \frac{1}{|z|^4} \leq 4$$

ML-Inequality

$$L = \text{length of } AB = \sqrt{1^2 + 1^2}$$

$$= \sqrt{2}$$

$$\therefore \quad \left| \int_C \frac{dz}{z^4} \right| \leq ML = 4\sqrt{2}$$

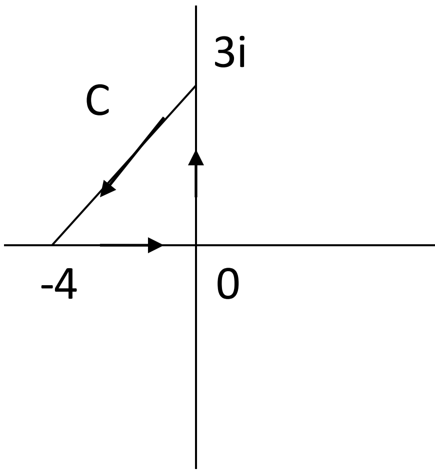
ML-Inequality

Q.3, (p.140).: Let C be the boundary of the triangle with vertices $0,\ 3i,\ -4$, oriented with counterclockwise direction.

Show that : $\left|\int_C \left(e^z - \bar{z}\right) dz\right| \leq 60.$

ML-Inequality

Let $I=\int_C f(z)\,dz,\ f(z)=e^z-\bar{z}$



Then $|I|=\left|\int_C f(z)\,dz\right|\le ML$

where $|f(z)|\le M$ on C

& $L=$ the length of the curve C

$=$ perimeter of the triangle

ML-Inequality

$L =$ Perimeter of the triangle

$$= |3| + \left|\sqrt{3^2 + 4^2}\right| + |-4|$$

$$= 3 + 5 + 4 = 12$$

We have $f(z) = e^z - \bar{z}$

$$\Rightarrow |f(z)| = \left|e^z - \bar{z}\right| \le \left|e^z\right| + |\bar{z}|$$

ML-Inequality

$$|f(z)| \le \left|e^z\right| + |z|$$

$$\le 1 + |-4| = 5$$

$$\Rightarrow M = 5$$

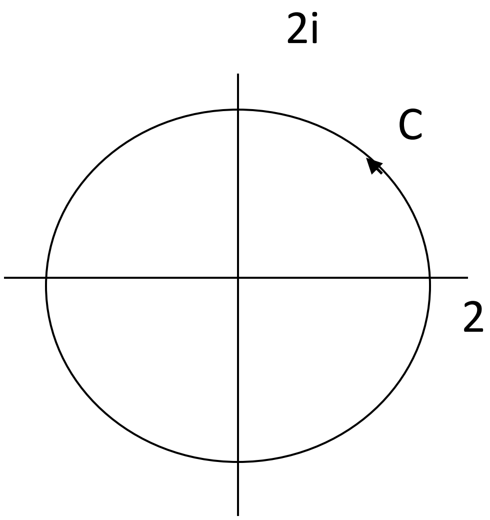
$$\therefore \quad \left|\int_C f(z)dz\right| \le ML = 60$$

ML-Inequality

Q1,(p.140): Let C be the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the first quadrant.

Show that $$\left|\int_C \frac{dz}{z^2 - 1}\right| \le \frac{\pi}{3}.$$

ML-Inequality



$$f(z) = \frac{1}{z^2 - 1}$$

$$\& \, I = \int_C f(z)\,dz$$

$$\Rightarrow |I| \le ML$$

ML-Inequality

On the curve C, we have

$$|f(z)| = \frac{1}{|z^2 - 1|} \le \frac{1}{|z^2| - 1} = \frac{1}{4-1} = \frac{1}{3}$$

$$L = \frac{2\pi \times R}{4} = \frac{2\pi \times 2}{4} = \pi$$

$$\therefore |I| \le \frac{\pi}{3}$$

ML-Inequality

Q5. (P-141) Let C_R be the circle $|z| = R \;\; (R > 1)$, described in the counterclockwise direction. Show that

$$\left| \int_{C_R} \frac{Log\, z}{z^2} dz \right| < 2\pi \left(\frac{\pi + \ln R}{R} \right).$$

Also show that the value of this integral tends to zero as $R \to \infty$. (Take principal branch of logarithm)

ML-Inequality

Sol.: $\left|\int_{C_R} f(z)\ dz\right| = \left|\int_{C_R} f\big(z(\theta)\big)z'(\theta)\ d\theta\right|$

$$\leq \int_{C_R} \left|f\big(z(\theta)\big)z'(\theta)\right| d\theta = \int_{-\pi}^{\pi} \frac{\left|Log\ Re^{i\theta}\right|}{\left|R^2 e^{2i\theta}\right|} \left|iRe^{i\theta}\right| d\theta$$

$$= \int_{-\pi}^{\pi} \frac{|\ln R + i\Theta|}{R} d\theta \quad (-\pi < \Theta < \pi)$$

$$\leq \int_{-\pi}^{\pi} \frac{|\ln R| + |i\Theta|}{R} d\theta < \int_{-\pi}^{\pi} \frac{\ln R + \pi}{R} d\theta$$

$$= 2\pi\left(\frac{\ln R + \pi}{R}\right)$$

Antiderivatives

- Let $f(z)$ be continuous function in a domain D.

- If there exists a function $F(z)$ such that

$$F'(z) = f(z) \quad \text{for all } z \text{ in } D,$$

Antiderivatives

then $F(z)$ is called an antiderivative of $f(z)$ in D.

Remark 1: An antiderivative of a given function f is an analytic function.

Remark 2: An antiderivative of a given function f is unique except for an additive complex constant.

Antiderivatives

Theorem: Suppose that a function $f(z)$ is continuous on a domain D. If any one of the following statement is true, then so are the others:

- $f(z)$ has an antiderivative $F(z)$ in D;

- The integral of $f(z)$ around closed contours lying entirely in D all have value zero.

Antiderivatives

•The integrals of $f(z)$ along contours lying entirely in D and extending from any fixed point z_1 to any fixed point z_2 all have same value;

$$\int_C f(z)\,dz = \int_{z_1}^{z_2} f(z)\,dz = F(z)\Big|_{z_1}^{z_2} = F(z_2) - F(z_1)$$

where $F(z)$ is an antiderivative of $f(z)$.

Antiderivatives

Ex.1: Use an antiderivative to evaluate $\int_{i}^{i/2} e^{\pi z}\,dz$.

Soln: Note that $f(z) = e^{\pi z}$ has an

antiderivative $F(z) = \dfrac{e^{\pi z}}{\pi}$.

$$\int_{i}^{i/2} e^{\pi z}\,dz = F(i/2) - F(i) = \frac{1}{\pi}\left[e^{i\pi/2} - e^{i\pi}\right] = \frac{1+i}{\pi}$$

Antiderivatives

Advised:

- See W.O.E. 3, p. 143
- See W.O.E. 4, p. 145
- See Q. No. 5, p. 149

Cauchy-Goursat Theorem

If a function f is analytic at all points interior to and on a simple closed contour C, then

$$\int_C f(z)\,dz = 0$$

Cauchy-Goursat Theorem

Example: If C is any simple closed contour, in either direction, then

$$\int_C \exp(z^3)\,dz = 0$$

because the function $f(z) = \exp(z^3)$

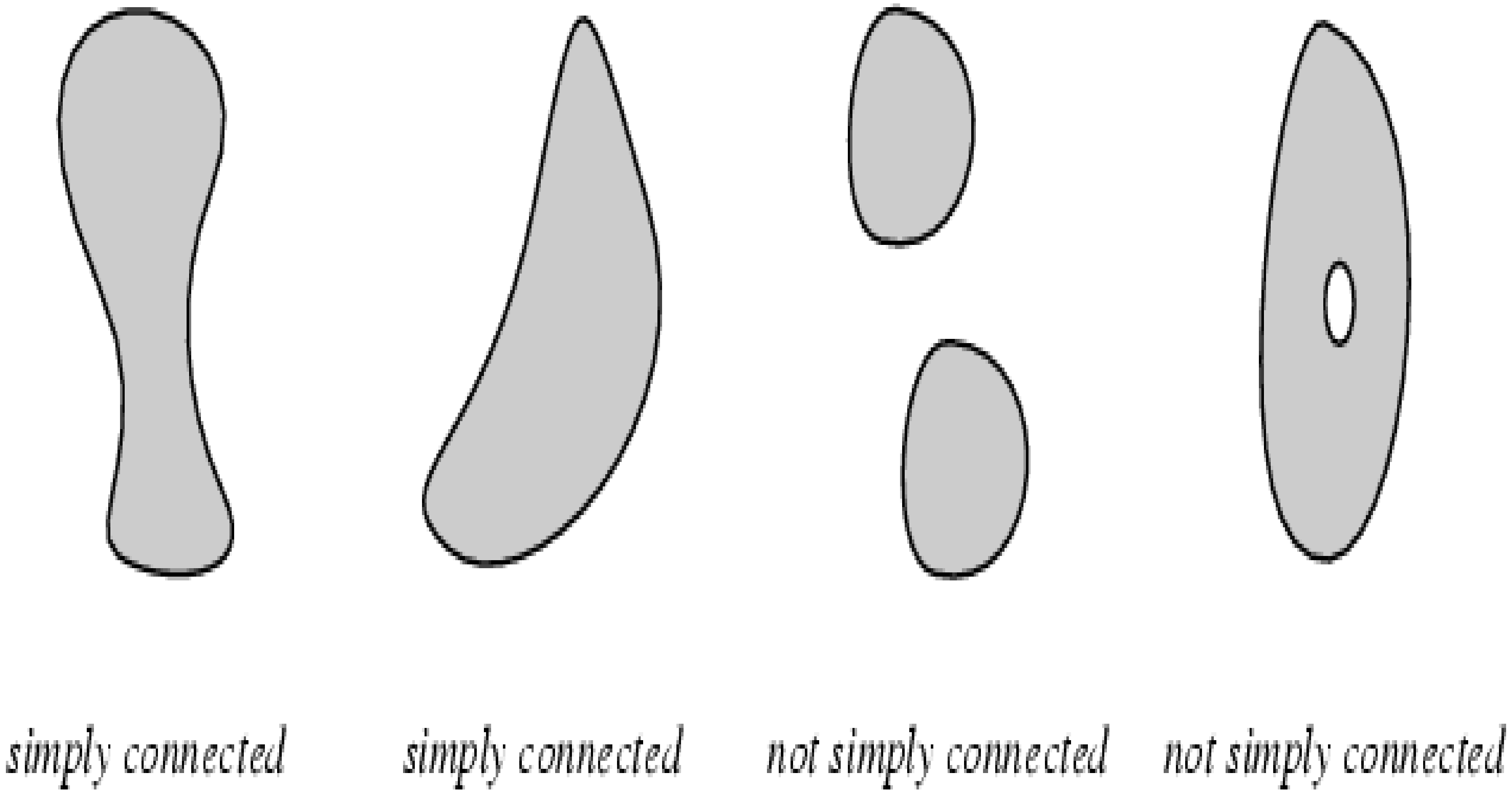
is analytic everywhere.

Cauchy-Goursat Theorem

Defn: A <u>***simply connected domain***</u> D is a domain such that every simple closed contour within it encloses only points of D.

The set of points interior to a simple closed contour is an example of simply connected domain

Cauchy-Goursat Theorem



Cauchy-Goursat Theorem

A domain that is not simply connected is said to be *multiply connected* for example, the annular domain between two concentric circles.

Cauchy-Goursat Theorem

The Cauchy – Goursat theorem for a simply connected domain D is as follows:

Theorem: If a function f is analytic throughout a simply connected domain D, then $\int_C f(z)\,dz = 0$

for every closed contour C lying in D.

Cauchy-Goursat Theorem

Theorem: Suppose that

(i) C is a simple closed contour, described in the counter-clockwise direction,

(ii) C_k $(k = 1, 2, \ldots., n)$ are finite no. of simple closed contours, all described in the clockwise direction, which are interior to C and whose interiors are disjoint.

Cauchy-Goursat Theorem

If $f(z)$ is analytic on all of these contours and throughout the multiply connected domain consisting of all points within C and exterior to each C_k, then

$$\oint_C f(z)\,dz + \sum_{k=1}^{n} \oint_{C_k} f(z)\,dz = 0.$$

Cauchy-Goursat Theorem

Corollary: Let C_1 and C_2 denote positively oriented simple closed contours, where C_2 is interior to C_1. If a function f is analytic in the closed region consisting of those contours and all points between them, then

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$

(Principle of deformation of paths)

Cauchy-Goursat Theorem

Ex.1 *Evaluate* $\int_C f(z)\,dz$

when $f(z) = ze^{-z}$

$C: \ |z| = 1$

Ans: 0 (Why??)

Cauchy-Goursat Theorem

Ex.2 *Evaluate*

$$\int_C f(z)\,dz$$

when

$$f(z) = \frac{z^2 \sin z}{z-4}, \quad C: |z| = 2.$$

Ans: 0 (Why??)

Cauchy-Goursat Theorem

<u>Qs 3/161.</u> Let C_0 denote the circle $|z - z_0| = R$, taken counter clockwise using the parametric representation

$$z = z_0 + \mathrm{Re}^{iq} \quad (- p \, £ q \, £ \, p)$$

for C_0. Then derive the following integrations:

Cauchy-Goursat Theorem

$$(a)\quad \int_{C_0} \frac{dz}{z - z_0} = 2\pi i$$

$$(b)\quad \int_{C_0} (z - z_0)^{n-1} dz = 0,\ n = \pm 1, \pm 2, \ldots$$

$$(c)\quad \int_{C_0} \left(z - z_0\right)^{a-1} dz = \frac{2iR^a}{a} \sin\left(a\pi\right),$$

where $a \neq 0$ is any real no.

Cauchy-Goursat Theorem

Sol. We have $\left|z - z_0\right| = R$

$$\Rightarrow \quad z - z_0 = Re^{i\theta}$$

$$\Rightarrow \quad dz = Re^{i\theta}\,.\,i\,d\theta$$

$$I = \oint_{C_0} \frac{dz}{z - z_0}$$

Cauchy-Goursat Theorem

$$= \int_{-\pi}^{\pi} \frac{\mathrm{R}e^{i\theta}\,.id\theta}{\mathrm{R}e^{i\theta}}$$

$$= i\left(\pi - \left(-\pi\right)\right) = 2\pi i$$

Cauchy-Goursat Theorem

b)

$$I = \int_{C_0} (z - z_0)^{n-1} dz$$

$$= \int_{-\pi}^{\pi} R^{n-1} e^{i(n-1)\theta} . \mathrm{Re}^{i\theta} i d\theta$$

$$= 0 \quad \text{(after simplification)}$$

Cauchy-Goursat Theorem

c)

$$I = \int_{C_0} \left(z - z_0\right)^{a-1} dz$$

$$= \int_{-\pi}^{\pi} R^{a-1} e^{i(a-1)\theta} . \mathrm{Re}^{i\theta} i d\theta$$

$$= \frac{2i\, R^{a}}{a} \sin\left(a\pi\right)$$

Cauchy-Goursat Theorem

Exercise:

Does Cauchy–Goursat Theorem hold separately for the real or imaginary part of an analytic function $f(z)$? Justify your answer.

Cauchy-Integral Formula

Let f be analytic everywhere inside and on a simple closed contour C, taken in the positive sense, then:

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{z - z_0}.$$

Cauchy-Integral Formula

Suppose that a function f is analytic everywhere inside and on a simple closed contour C, taken in the positive sense. If z_0 is any point interior to C, then:

Cauchy-Integral Formula

$$(a) \quad f'(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^2},$$

$$(b) \quad f''(z_0) = \frac{(2)!}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^3},$$

$$(c) \; f^{(n)}(z_0) = \frac{(n)!}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^{n+1}}.$$

Cauchy-Integral Formula

Theorem:

If $f(z)$ is analytic at z_0, then its derivatives of all orders exist at z_0 and are themselves analytic at z_0.

Cauchy-Integral Formula

Corollary: If a function $f(z) = u(x, y) + i\, v(x, y)$ is analytic at a point $z = (x, y)$, then the component functions u, v have continuous partial derivatives of all orders at that point.

Cauchy-Integral Formula

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Qs.1(b)/170: Let C denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate the following integral: $\oint_C \frac{\cos z \, dz}{z(z^2+8)}$.

Ans : pi/4.

Cauchy-Integral Formula

Qs. 2(b)/170: Find the value of the integral of $g(z)$ around the circle $|z - i| = 2$ in the positive sense when

$$g(z) = \frac{1}{(z^2+4)^2}.$$

Cauchy-Integral Formula

$$Sol: \int_C \frac{dz}{\left(z^2+4\right)^2} = \int_C \frac{dz}{\left(z+2i\right)^2\left(z-2i\right)^2}$$

$$= 2\pi i \frac{d}{dz}\left(\frac{1}{\left(z+2i\right)^2}\right)_{z=2i} = \frac{\pi}{16}$$

Cauchy-Integral Formula

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<u>Qs.4/170:</u> Let C be any simple closed contour, described in the positive sense in the z-plane and write

$$g(w) = \oint_C \frac{z^3 + 2z}{(z - w)^3}\, dz$$

Cauchy-Integral Formula

Show that:

$$g(w) = 6\mathrm{p}\, iw$$

when w is inside C and that

$$g(w) = 0$$

when w is outside C.

Cauchy-Integral Formula

Case I: Let w be inside C.

Let $f(z) = z^3 + 2z.$ Then

$$g(w) = \int_C \frac{f(z)}{(z-w)^3} dz,$$

$$= \frac{2\pi i}{2} f''(w)$$

Cauchy-Integral Formula

$$f(z) = z^3 + 2z$$

$$\Rightarrow f'(z) = 3z^2 + z$$

$$\Rightarrow f''(z) = 6z$$

$$\Rightarrow f''(w) = 6w$$

$$\therefore I = g(w) = 6\pi i w$$

Cauchy-Integral Formula

Case 2: When w is outside C, then by Cauchy-Goursat Theorem $g(w)=0$.

Cauchy-Integral Formula

Qs. 5/170: Show that if f is analytic within and on a simple closed contour C and z_0 is not on C, then

$$\int_C \frac{f'(z)}{(z-z_0)}dz = \int_C \frac{f(z)}{(z-z_0)^2}dz$$

Cauchy-Integral Formula

Sol. Let

$$I_1 = \int_C \frac{f'(z)}{(z - z_0)} dz \text{ and}$$

$$I_2 = \int_C \frac{f(z)}{(z - z_0)^2} dz$$

Cauchy-Integral Formula

Case I: Let z_0 be inside C, then:

$$I_1 = \int_C \frac{f'(z)}{(z - z_0)} dz = 2\pi i\, f'(z)\Big|_{z=z_0}$$

$$= 2\pi i\, f'(z_0)$$

Cauchy-Integral Formula

and

$$I_2 = \int_C \frac{f(z)}{(z - z_0)^2} dz$$

$$= 2\pi i \, f'(z_0)$$

$$\therefore I_1 = I_2 \, .$$

Cauchy-Integral Formula

Case II: Let z_0 be outside C

Then $I_1 = I_2 = 0.$

(WHY ???)