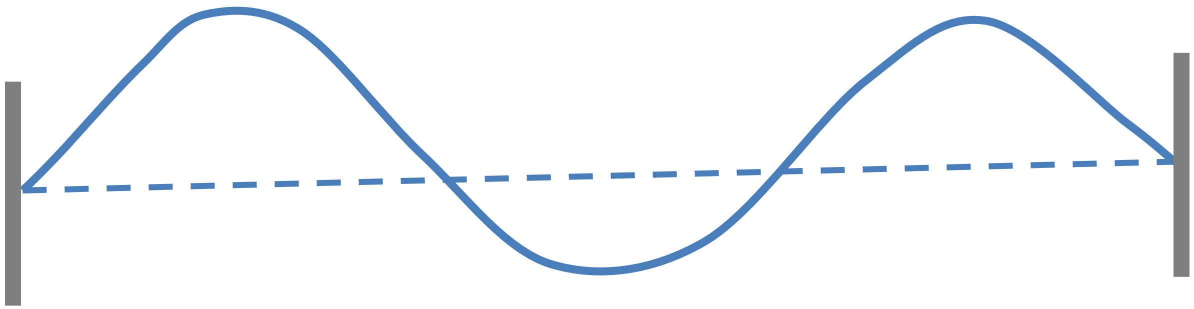
Chapter 6

Normal Modes of Continuous System

Horizontal Uniform String Stretched between Two Fixed Supports

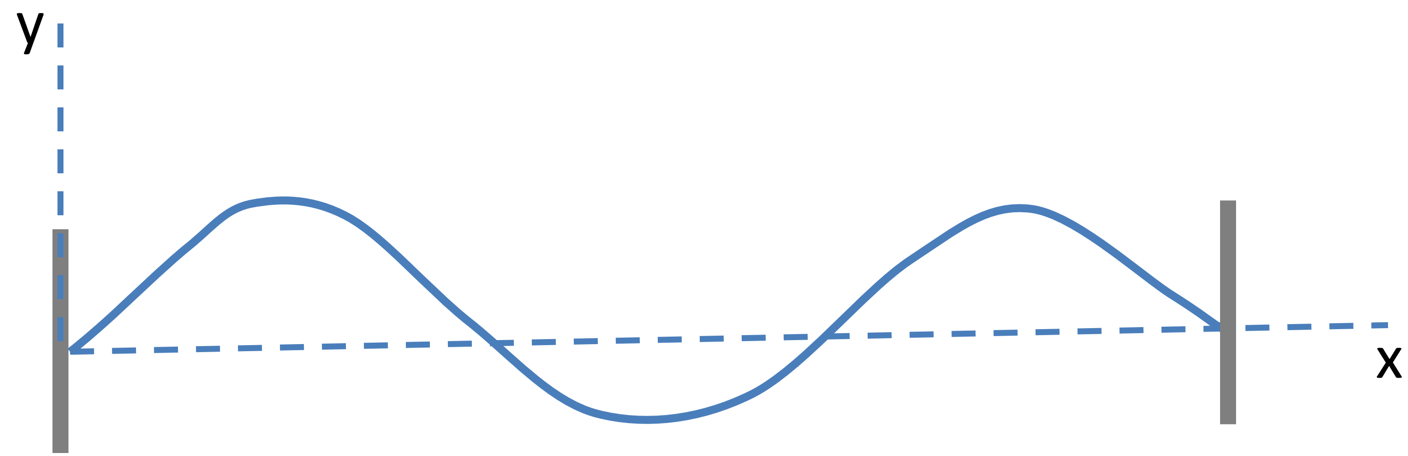


Parameters of the string :

Length : L

Tension : T

Density (Linear) : μ

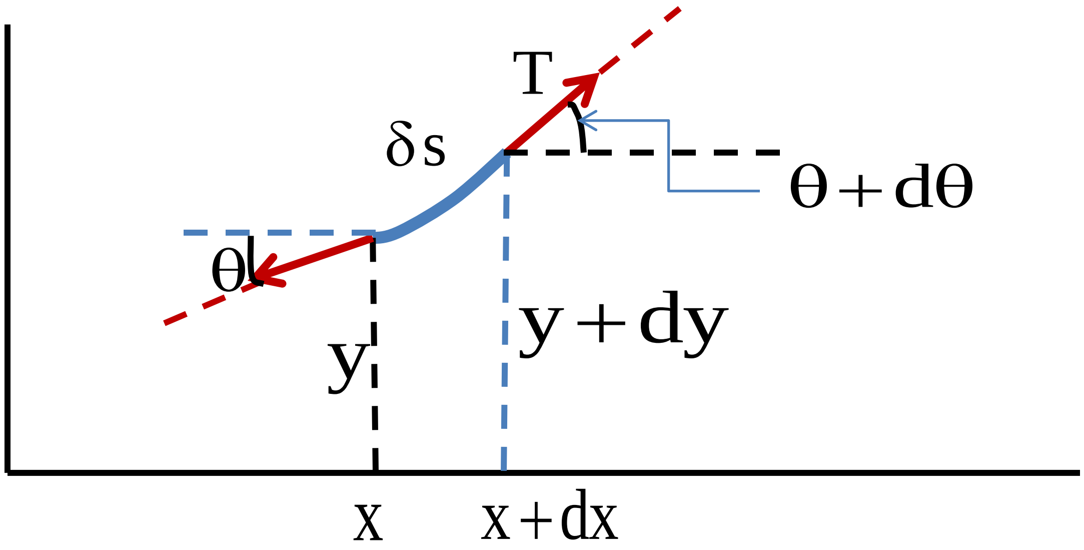


When the string vibrates, the displacement y is a function of both x and t

At a fixed time, y is a function of x

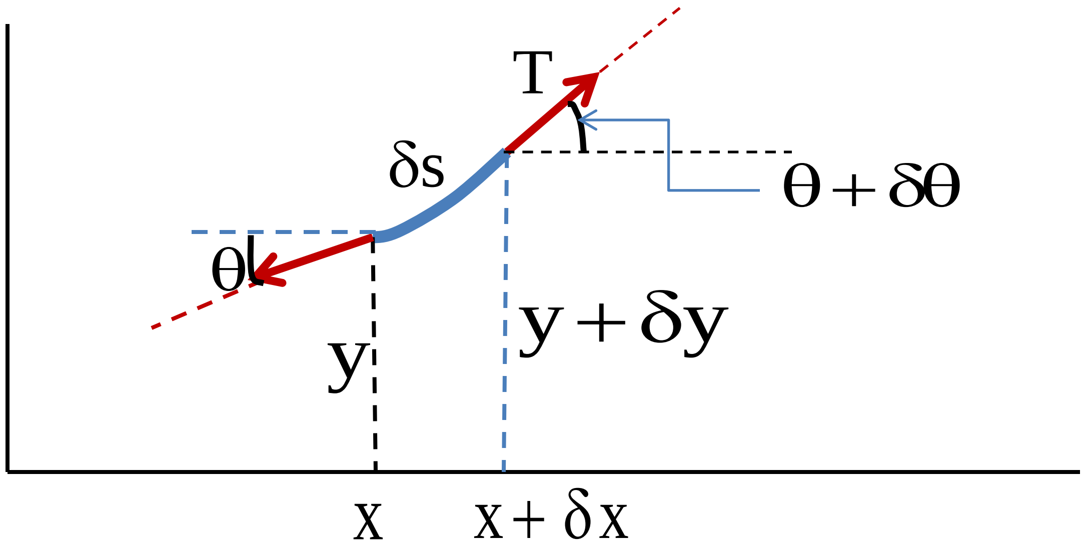
At a fixed x, y is a function of time

Equation of Motion :



Assumptions : θ **is uniformly small, so that**

$$\sin\theta \approx \theta \approx \tan\theta \qquad \cos\theta \approx 1$$



Equation of motion of the small blue piece

$$\mu\,\delta x\,\frac{\partial^2 y}{\partial t^2} = T[\sin(\theta+\delta\theta) - \sin\theta]$$

$$= T[\tan(\theta+\delta\theta) - \tan\theta]$$

$$\tan(\theta + \delta\theta) - \tan\theta = \left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_{x}$$

$$= \frac{\partial^2 y}{\partial x^2}\,\delta x$$

$$\therefore \quad \mu\,\delta x\,\frac{\partial^2 y}{\partial t^2} = T\,\frac{\partial^2 y}{\partial x^2}\,\delta x$$

$$\Rightarrow \frac{\partial^2 y}{\partial x^2} - \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2} = 0$$

Writing $\frac{T}{\mu} = v^2$, $\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = 0$

Normal Modes of a String Fixed at each End

In a normal mode, each point of the string, which is an oscillator, oscillates with the same frequency, but with different amplitude :

$$y(x,t) = f(x)\cos(\omega t)$$

Substitute the above into the equation of motion to get :

$$\frac{d^2 f}{dx^2} + \frac{\omega^2}{v^2} f = 0$$

The most general solution for f is :

$$f(x) = A\sin\left(\frac{\omega}{v}x + \phi\right)$$

The function $f(x)$ **must satisfy the boundary conditions :**

$$f(0) = 0 \quad ; \quad f(L) = 0$$

These boundary conditions require :

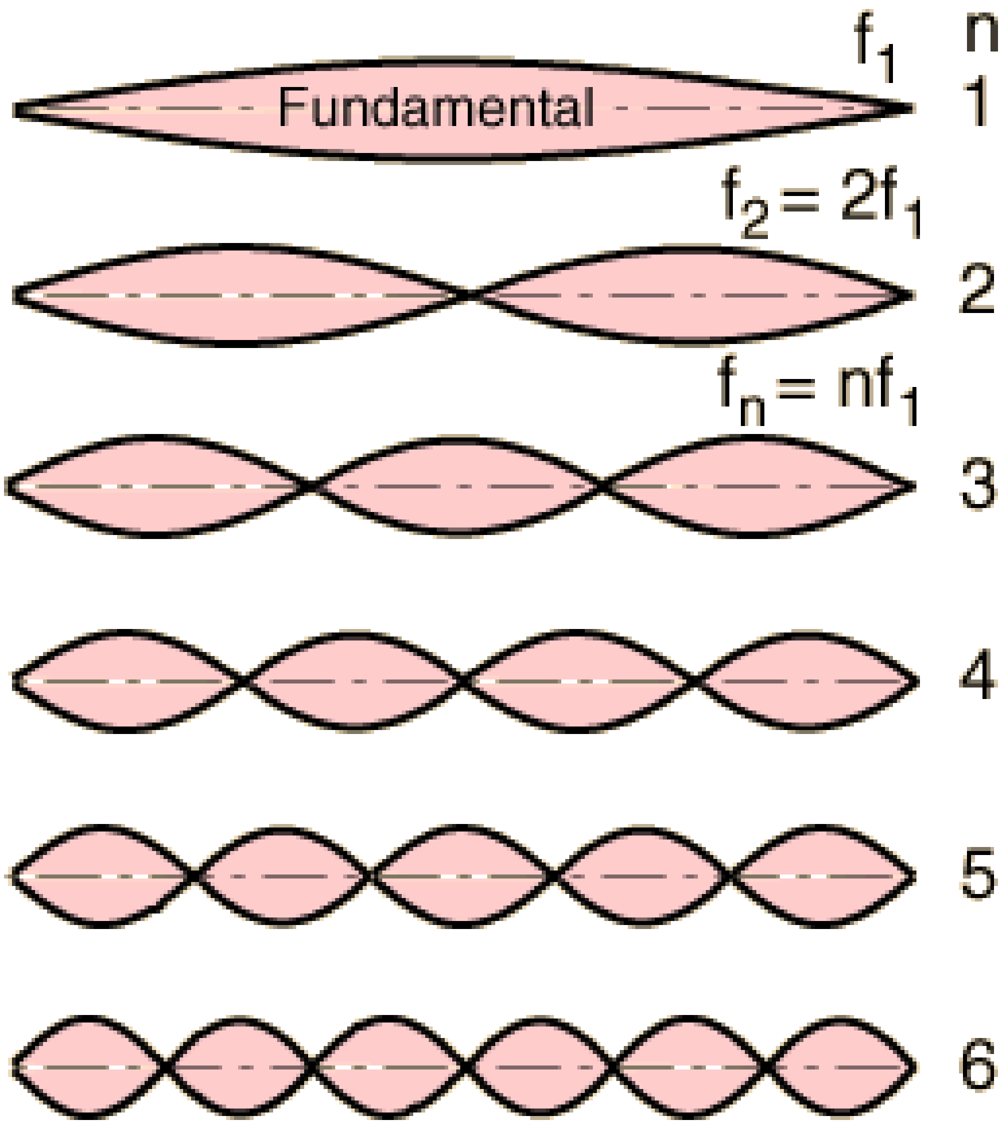
$$\phi = 0 \;\&\; \frac{\omega L}{v} = n\pi \quad (n = 1, 2, 3 \ldots\ldots\ldots)$$

Each value of n corresponds to a normal mode, with a frequency :

$$\omega_n = \frac{n\pi v}{L}$$

Thus, the normal mode vibrations of the string are given as :

$$y_n(x,t) = A \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t)$$



f_1
n
Fundamental
1
$f_2 = 2f_1$
2
$f_n = nf_1$
3
4
5
6

The most general solution of the vibrating string is a superposition of normal modes with different amplitudes and phase angles :

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t + \phi_n)$$

The constants A_n & ϕ_n, are determined from the displacement and velocity profile of the string at time t = 0, by using Fourier series analysis.

6.1 A uniform string of length 2.5 m and mass 0.01 kg is placed under a tension 10 N.

a) What is the frequency of its fundamental mode?
b) If the string is plucked transversely and is then touched at a point 0.5 m from one end, what frequencies persist?

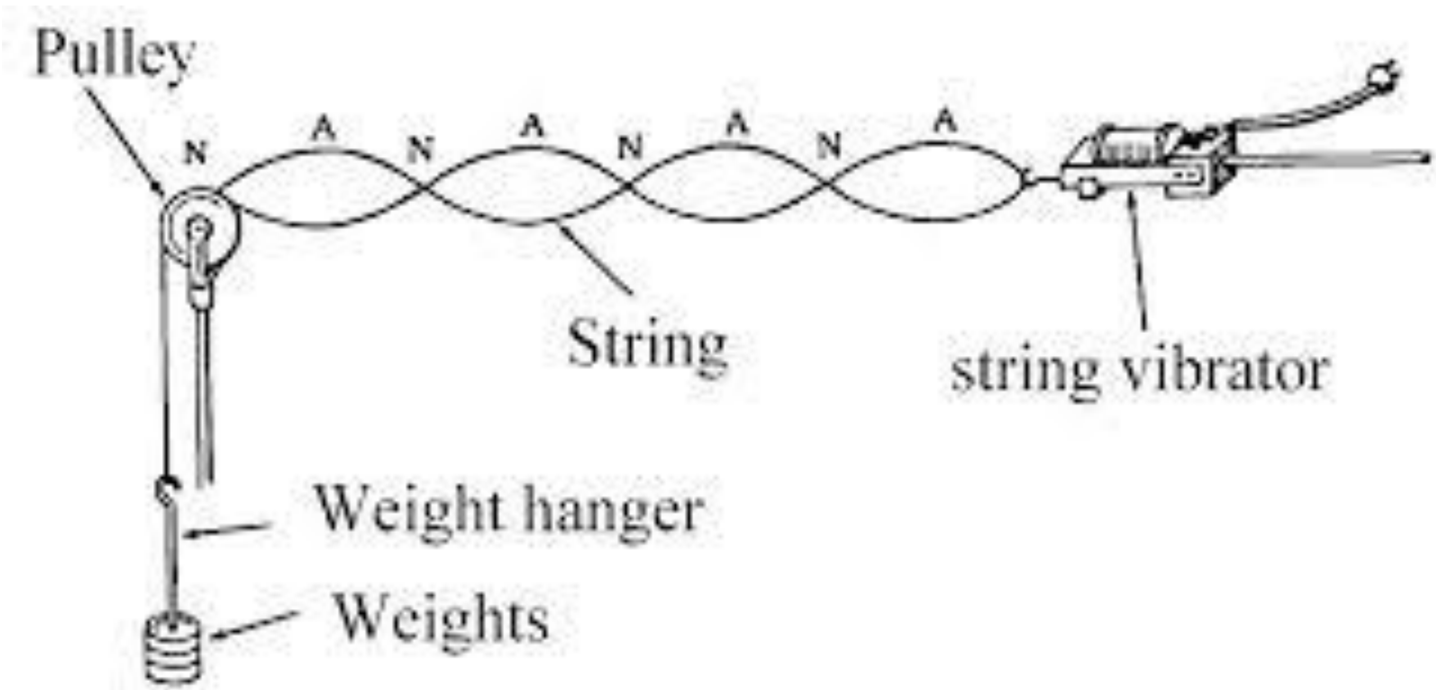
Ans :

a) $$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{10}{0.004}} \qquad \Rightarrow \omega_1 =$$

b) Initially, the vibration was a superposition of all modes, in general.

After it was touched at the point $x = 0.5$ m, only those modes will remain, that have a node at $x = 0.5$ m. These modes are multiples of 5, i.e., $5n$, $n = 1,2,3,\ldots\ldots$

Forced Harmonic Vibrations of a String



Left end is fixed and right end is driven in a SHM :

$$y(L, t) = \eta_0 \cos(\omega t)$$

The right end, x = L, is driven with a frequency ω and amplitude A.

The Equation of Motion is :

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = 0$$

In the steady state, the entire string vibrates with the driving frequency ω :

$$y(x,t) = f(x)\cos(\omega t)$$

Substituting this into the Eq. of Motions :

$$\frac{d^2 f}{dx^2} + \frac{\omega^2}{v^2} f = 0$$

The most general solution for $f(x)$:

$$f(x) = A \sin\left(\frac{\omega}{v} x + \phi\right)$$

Boundary Conditions on $f(x)$:

$$f(0) = 0 \;;\; f(L) = \eta_0$$

From the first BC, $\phi = p\pi, \quad p = 0, \pm 1, \pm 2, \ldots\ldots.$

From the second BC,

$$A = \frac{\eta_0}{\sin\left(\frac{\omega L}{v} + p\pi\right)}$$

Steady State Solution :

$$y(x,t) = \frac{\eta_0}{\sin\left(\frac{\omega L}{v} + p\pi\right)} \sin\left(\frac{\omega x}{v} + p\pi\right)\cos(\omega t)$$

Clearly, the amplitude of oscillations blows up when :

$$\omega = \frac{n\pi v}{L}\ ,\ \ n = 1, 2, 3, \ldots..$$

which is one of the normal-mode frequencies.

6.5. **A stretched string of mass m, length L, and tension T is driven by two sources, one at each end. The sources both have the same frequency ω and amplitude A, but are exactly 180^{o} out of phase w.r.t. one another. What is the smallest possible value of ω consistent with stationary vibrations of the string?**

Stationary Solution :

$$y(x,t) = f(x)\cos\omega t$$

$$f(x) = B\sin\left(\frac{\omega}{v}x + \phi\right)$$

$$y(0,t) = -A\cos(\omega t)\ ;\ y(L,t) = A\cos(\omega t)$$

$$\Rightarrow \phi = -\frac{\pi}{2}$$

$$\Rightarrow \frac{\omega_{\min} L}{v} - \frac{\pi}{2} = \frac{\pi}{2}$$

$$\therefore\ \omega_{\min} = \frac{\pi v}{L}$$