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BITS Pilani
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CS/IS F214
Logic in Computer Science

MODULE: PROPOSITIONAL LOGIC

Proof of Completeness: Theorem *Tautology-Theorem*

Theorem *Tautology-Theorem*

- **Theorem T-T**:
  - ***Every tautology (in propositional logic) is a theorem*** (i.e. it is *provable*): i.e. **if $|= \chi$ holds then $|- \chi$ holds.**
- **Proof Outline**:
  - If $|= \chi$ holds, there is a truth table for χ with all 1s in the last column
    - Using the Line Semantics Lemma
      - we construct a proof corresponding to each row of the truth table and
    - and assemble all those proofs using
      - nested application of the $\vee$e proof rule (along with LEM for each atomic proposition)

Visualization of the proof outline: Theorem T-T

Example Truth Table (for some formula ϕ)

p_0	p_1	…	p_m	ϕ
0	0	…	0	0
0	0	…	1	1
…	…	…	…	
1	1	…	**1**	

Visualization of the proof outline: Theorem T-T

Example Truth Table (for some tautology ϕ)

p_0	p_1	…	p_m	ϕ
0	0	…	0	1
0	0	…	1	1
…	…	…	…	…
1	1	…	**1**	1

Visualization of the proof outline: Theorem T-T

Example Truth Table (for some tautology ϕ)

p_1	p_2	…	p_m	ϕ
0	0	…	0	1
0	0	…	1	1
…	…	…	…	…
1	1	…	1	1

Generate proof T_1

Generate proof T_2

Generate proof T_k where $k = 2^m$

Assemble all proofs!

Lemma *Line-Semantics*

- Lemma **Line-Semantics**:
  - Let ϕ be a propositional formula on propositional atoms $\mathbf{p_1, p_2, \ldots p_m}$.
  - Let k be a line (i.e. row) number in the truth table for ϕ.
  - Define
    - $\mathbf{L_{k,i} = p_i}$ if entry for $\mathbf{p_i}$ is 1 in line ***k***
    - $\mathbf{L_{k,i} = \neg p_i}$ otherwise.
  - Then
    - $\mathbf{L_{k,1}, L_{k,2}, \ldots L_{k,m} \mid\!- \phi}$ is provable if entry for ϕ in line ***k*** is **1**
    - $\mathbf{L_{k,1}, L_{k,2}, \ldots L_{k,m} \mid\!- \neg\phi}$ is provable if entry for ϕ in line ***k*** is **0**

Lemma *Line-Semantics (proved in later slides)*

- Informally, the Line-Semantics lemma states that:
  - for each line of the truth table for ϕ there exists a proof
    - deriving ϕ if the last line of the row is 1; deriving $\neg\phi$ if it is 0.
      - with each propositional atom (*if its column is 1 in that row*) or its negation (*if it is 0*) as a premise

Visualization of the proof outline: Theorem T-T

Example Truth Table (for some tautology ϕ)

p_1	p_2	…	p_m	ϕ
0 $\neg p_1$	0 $\neg p_2$	…	0 $\neg p_m$	1 ϕ
0 $\neg p_1$	0 $\neg p_2$	…	1 p_m	1 ϕ
…	…	…	…	…
1 p_1	1 p_2	…	1 p_m	1 ϕ

Proof of the Line-Semantics Lemma

- **Lemma Line-Semantics**:
  - **L1, L2, ... Lm |-** ϕ is provable if entry for ϕ in line k is 1
  - **L1, L2, ... Lm |-** $\neg\phi$ is provable if entry for ϕ in line k is 0
- Proof Outline: (by structural induction)
  - Case ϕ *is an atomic proposition*: *see following slides*
  - Case ϕ *is of the form* $\neg\phi 1$: *see following slides*
  - Case ϕ *is of the form* $\phi 1$ **-->** $\phi 2$: *see following slides*
  - Case ϕ *is of the form* $\phi 1 \wedge \phi 2$: *similar to the case* $\phi 1$ **-->** $\phi 2$
  - Case ϕ *is of the form* $\phi 1 \vee \phi 2$: *similar to the case* $\phi 1$ **-->** $\phi 2$

Proof of the Line-Semantics Lemma

- **Proof (specific cases):**
  - Case *ϕ is an atomic proposition p*:
    - Show that **p |- p** (Trivial)
    - Show that $\neg$**p |-** $\neg$**p** (Trivial)
  - Case *ϕ is of of the form $\neg\psi$*:
    - Assume that ϕ evaluates to 1 i.e. ψ evaluates to 0
      - ψ has the same atomic propositions as ϕ
      - By induction hypothesis:
        - **L1, L2, ,...,Lm |-** $\neg\psi$
          - but $\neg\psi$ is same as ϕ and so we are done.
    - Assume that ϕ evaluates to 0 i.e. ψ evaluates to 1
      - By induction hypothesis:
        - **L1, L2, ,...,Lm |-** ψ
      - By $\neg\neg$**i** rule, we get **L1, L2, ,...,Lm |-** $\neg\neg\psi$
        - but $\neg(\neg\psi)$ is same as $\neg\phi$ and so we are done.

Proof of Line Semantics Lemma

- **Proof (specific cases):**
- Case ϕ is of the form ψ **-->** χ
  - If ϕ evaluates to 0:
    - then ψ must evaluate to 1 and χ must evaluate to 0
    - We have, by induction hypotheses,
      - **Lq_1, Lq_2,…,Lq_k |-** ψ and **Lr_1, Lr_2,…,Lr_n |-** $\neg\chi$
        - where $\mathbf{q_i}$ and $\mathbf{r_j}$ are atomic propositions in ψ and χ respectively.
    - So, **Lp_1,Lp_2,…,Lp_m |-** $\psi \wedge \neg\chi$
      - where **$\{p_i | 1<=i<=m\} = \{q_j | 1<=j<=k\}$ U $\{r_j | 1<=j<=n\}$**
    - and we can prove (by ND):
      - $\psi \wedge \neg\chi$ **|-** $\neg$ **(**ψ **-->** χ**)**

Proof of Line Semantics Lemma

- **Proof (specific cases):**
- Case ϕ is of the form ψ **-->** χ (continued)
  - If ϕ evaluates to TRUE: *we have three cases*:

1) both ψ and χ evaluate to 0:

   By hypothesis: $\mathbf{Lq_1, Lq_2, \ldots, Lq_k} \mid\!- \neg\psi$ and $\mathbf{Lr_1, Lr_2, \ldots, Lr_n} \mid\!- \neg\chi$

   so $\mathbf{Lp_1, Lp_2, \ldots, Lp_m} \mid\!- \neg\psi \wedge \neg\chi$

   - and we need to show (by ND): $\neg\chi \wedge \neg\psi \mid\!- \psi \text{-->} \chi$

2) both ψ and χ evaluate to 1:

   By hypothesis: $\mathbf{Lq_1, Lq_2, \ldots, Lq_k} \mid\!- \psi$ and $\mathbf{Lr_1, Lr_2, \ldots, Lr_n} \mid\!- \chi$

   so $\mathbf{Lp_1, Lp_2, \ldots, Lp_m} \mid\!- \psi \wedge \chi$

   - and we need to show (by ND): $\chi \wedge \psi \mid\!- \psi \text{-->} \chi$

3) ψ evaluates to 0 and χ evaluates to 1:

   By hypothesis: $\mathbf{Lq_1, Lq_2, \ldots, Lq_k} \mid\!- \neg\psi$ and $\mathbf{Lr_1, Lr_2, \ldots, Lr_n} \mid\!- \chi$

   so $\mathbf{Lp_1, Lp_2, \ldots, Lp_m} \mid\!- \neg\psi \wedge \chi$

   - and we need to show (by ND): $\chi \wedge \neg\psi \mid\!- \psi \text{-->} \chi$

Theorem T-T : Using Lemma Line-Semantics

- Assuming
  - $|= \phi$
- apply Lemma ***Line-Semantics*** on the truth table for ϕ
  - Since ϕ is a tautology, it evaluates to 1 in all 2^m lines of its truth table (*where **m** is the number of atomic propositions*)
  - i.e. we will get 2^m proofs of the form
    - $L_1, L_2, \ldots L_m \mid\!- \phi$
- ***All these proofs are to be assembled into one proof (in ND) without any premises*** i.e.
  - $\mid\!- \phi$

RECALL- Example

Given the tautology

$p \wedge q \dashrightarrow p$

and its truth table:

p	q	p ∧ q	p ∧ q --> p
0	0	0	1
0	1	0	1
1	0	0	1
1	1	1	1

We generate one proof per row:

Sequents
¬p, ¬q |- p ∧ q --> p
¬p, q |- p ∧ q --> p
p, ¬q |- p ∧ q --> p
p, q |- p ∧ q --> p

Each of these sequents is valid by the **Line-Semantics Lemma**.

How do we combine them?

Theorem T-T: Proof-assembly - Example

- How do you assemble proofs?
  - Example: $p \wedge q \dashrightarrow p$

1	$p \vee \neg p$			LEM
2	p	Assumption	$\neg p$	Assumption
3				
	$p \wedge q \dashrightarrow p$		$p \wedge q \dashrightarrow p$	
	$p \wedge q \dashrightarrow p$			$\vee e$

- Suppose you prove (i) $p \mid\!- p \wedge q \dashrightarrow p$ and (ii) $\neg p \mid\!- p \wedge q \dashrightarrow p$
- Then you have proved $\mid\!- p \wedge q \dashrightarrow p$
- i.e. $p \wedge q \dashrightarrow p$ is true independent of truth (or falsity) of p.

Theorem T-T: Proof-assembly - Example (contd.)

- How do you assemble proofs?
  - Example: $p \wedge q \dashrightarrow p$

1	$p \vee \neg p$						LEM	
2	p	Assumption			$\neg p$		Assumption	
3	$q \vee \neg q$	LEM			$q \vee \neg q$		LEM	
4	q	Assum	$\neg q$	Assum	q	Assum	$\neg q$	Assum
5	..		..		..		..	
6	..		..		..		..	
	..		..		..		..	
7	$p \wedge q \dashrightarrow p$		$p \wedge q \dashrightarrow p$		$p \wedge q \dashrightarrow p$		$p \wedge q \dashrightarrow p$	
8	$p \wedge q \dashrightarrow p$			$\vee e$	$p \wedge q \dashrightarrow p$			$\vee e$
9	$p \wedge q \dashrightarrow p$							$\vee e$

$p \wedge q \dashrightarrow p$ is true independent of the truth (or the falsity) of p or that of q.

Theorem T-T: Proof

- How do you assemble proofs?

Applying this proof-assembly technique on the 2^m proofs of the form **L1, L2, ... Lm |- ϕ** yields a proof of **|- ϕ i.e.**

i. starting from **|=** ϕ,
ii. we draw a truth table for ϕ,
iii. write proofs for each line (based on Line-Semantics),
iv. and assemble them all to get **|-** ϕ

- i.e. two proofs of the form:

 L1, L2, ... Lm-1, p |- ϕ and **L1, L2, ... Lm-1, ¬p |- ϕ**

can be combined to yield:

 L1, L2, ... Lm-1 |- ϕ

- *By induction on* **m** (the number of propositional atoms):
  - all propositional atoms (and their negations) can be eliminated as premises, yielding:

   |- ϕ

RECAP: Proof of completeness – Steps: I

- We proved **Lemma Line-Semantics**

  by structural induction on formulas in propositional logic

- We proved **Theorem T-T**
  - **i.e. If |= ψ then |- ψ** in two steps:

    1. We showed by Lemma Line-Semantics

       there is a proof corresponding to every row of truth table for ψ

    2. We showed by induction on **n**

       that we can assemble all 2^n proofs into a single proof of **|- ψ**

RECAP: Proof of completeness – Steps: II

- Assume $\phi_1, \phi_2, \ldots \phi_n \models \psi$.
  1. We showed by induction on **n** that

     $\models \phi_1 \dashrightarrow (\phi_2 \dashrightarrow (\phi_3 \dashrightarrow (\ldots \phi_n \dashrightarrow \psi)\ldots)))$
  2. We showed <u>by Theorem T-T</u>

     $\vdash \phi_1 \dashrightarrow (\phi_2 \dashrightarrow (\phi_3 \dashrightarrow (\ldots \phi_n \dashrightarrow \psi)\ldots)))$ is valid
  3. We showed by induction on **n**

     $\phi_1, \phi_2, \ldots \phi_n \vdash \psi$ is valid