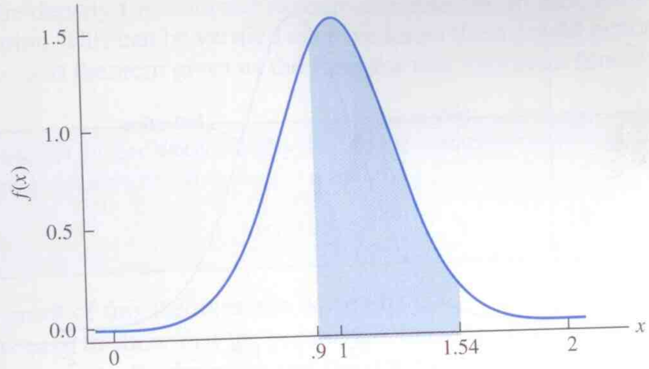

FIGURE 4.7
Shaded area = $P[.9 \le X \le 1.54]$.

The cumulative distribution function for the standard normal random variable is given in Table V of App. A. The use of the standardization theorem and this table is illustrated in the following example.

Example 4.4.2. Let X denote the number of grams of hydrocarbons emitted by an automobile per mile. Assuming that X is normal with $\mu = 1$ gram and $\sigma = .25$ gram, find the probability that a randomly selected automobile will emit between .9 and 1.54 grams of hydrocarbons per mile. The desired probability is shown in Fig. 4.7. To find $P[.9 \le X \le 1.54]$, we first standardize by subtracting the mean of 1 and dividing by the standard deviation of .25 across the inequality. That is,

$$P[.9 \le X \le 1.54] = P[(.9 - 1)/.25 \le (X - 1)/.25 \le (1.54 - 1)/.25]$$

The random variable $(X - 1)/.25$ is now Z. Therefore the problem is to find $P[-.4 \le Z \le 2.16]$ from Table V. We first express the desired probability in terms of the cumulative distribution as follows:

$$\begin{aligned} P[-.4 \le Z \le 2.16] &= P[Z \le 2.16] - P[Z < -.4] \\ &= P[Z \le 2.16] - P[Z \le -.4] \qquad (Z \text{ is continuous}) \\ &= F(2.16) - F(-.4) \end{aligned}$$

$F(2.16)$ is found by locating the first two digits (2.1) in the column headed z; since the third digit is 6, the desired probability of .9846 is found in the row labeled 2.1 and the column labeled .06. Similarly, $F(-.4)$ or .3446 is found in the row labeled -0.4 and the column labeled .00. We now see that the probability that a randomly selected automobile will emit between .9 and 1.54 grams of hydrocarbons per mile is

$$\begin{aligned} P[.9 \le X \le 1.54] &= P[-.4 \le Z \le 2.16] \\ &= F(2.16) - F(-.4) \\ &= .9846 - .3446 = .64 \end{aligned}$$

Interpreting this probability as a percentage, we can say that 64% of the automobiles in operation emit between .9 and 1.54 grams of hydrocarbons per mile driven.

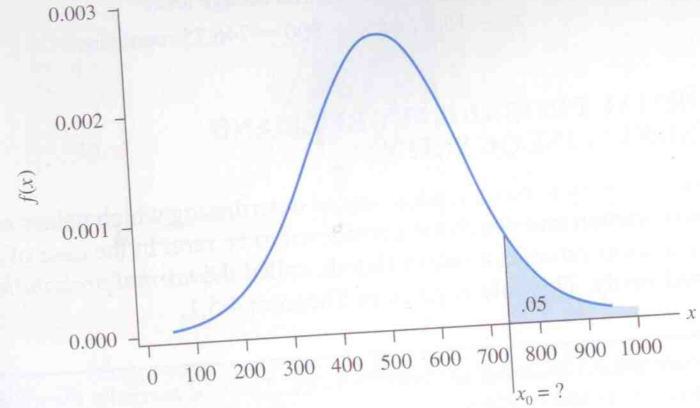

FIGURE 4.8
$P[X \ge x_0] = .05$.

We shall have occasion to read Table V in reverse. That is, given a particular probability r we shall need to find the point with r of the area to its right. This point is denoted by z_r. Thus, notationally, z_r denotes that point associated with a standard normal random variable such that

$$P[Z \ge z_r] = r$$

To see how this need arises, consider Example 4.4.3.

Example 4.4.3. Let X denote the amount of radiation that can be absorbed by an individual before death ensues. Assume that X is normal with a mean of 500 roentgens and a standard deviation of 150 roentgens. Above what dosage level will only 5% of those exposed survive? Here we are asked to find the point x_0 shown in Fig. 4.8. In terms of probabilities, we want to find the point x_0 such that

$$P[X \ge x_0] = .05$$

Standardizing gives

$$\begin{aligned} P[X \ge x_0] &= P\left[\frac{X - 500}{150} \ge \frac{x_0 - 500}{150}\right] \\ &= P\left[Z \ge \frac{x_0 - 500}{150}\right] = .05 \end{aligned}$$

Thus $(x_0 - 500)/150$ is the point on the standard normal curve with 5% of the area under the curve to its right and 95% to its left. That is, $(x_0 - 500)/150$ is the point $z_{.05}$. From Table V the numerical value of this point is approximately 1.645 (we have interpolated). Equating these, we get

$$\frac{x_0 - 500}{150} = 1.645$$

$n = 5$, $p = .4$

x	$f(x)$
0	.0778
1	.2592
2	.3456
3	.2304
4	.0768
5	.0102

$n = 10$, $p = .4$

x	$f(x)$
0	.0060
1	.0404
2	.1209
3	.2150
4	.2508
5	.2007
6	.1114
7	.0425
8	.0106
9	.0016
10	.0001

$n = 15$, $p = .4$

x	$f(x)$	x	$f(x)$
0	.0005	8	.1181
1	.0047	9	.0612
2	.0219	10	.0245
3	.0634	11	.0074
4	.1268	12	.0016
5	.1859	13	.0003
6	.2066	14	~0
7	.1771	15	~0

$n = 20$, $p = .4$

x	$f(x)$	x	$f(x)$
0	~0	11	.0710
1	.0005	12	.0355
2	.0031	13	.0145
3	.0124	14	.0049
4	.0350	15	.0013
5	.0746	16	.0003
6	.1244	17	~0
7	.1659	18	~0
8	.1797	19	~0
9	.1597	20	~0
10	.1172		

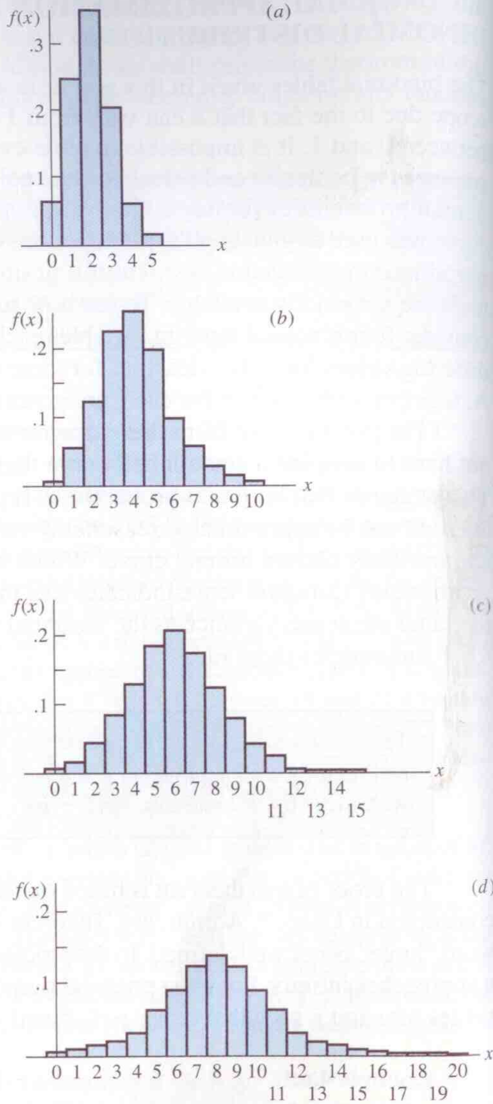

FIGURE 4.10
Density for X binomial: (*a*) $n = 5, p = .4$; (*b*) $n = 10, p = .4$; (*c*) $n = 15, p = .4$; (*d*) $n = 20, p = .4$.

The exact probability of .0565 is given by the sum of the areas of the blocks centered at 12, 13, 14, 15, 16, 17, 18, 19, and 20, as shown in Fig. 4.11. The approximate probability is given by the area under the normal curve shown above 11.5. That is,

$$P[X \geq 12] \doteq P[Y \geq 11.5]$$

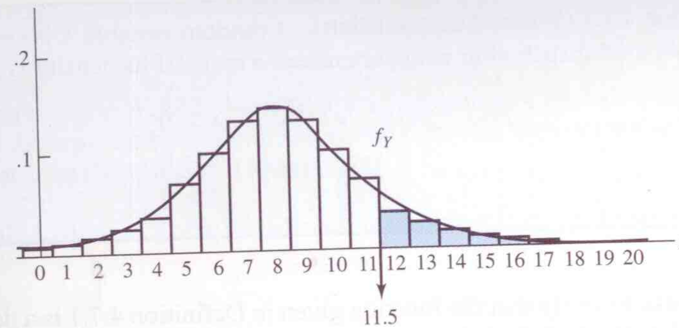

FIGURE 4.11
$P[X \geq 12]$ = area of shaded blocks $\doteq$ area under curve beyond 11.5.

The number .5 is called the *half-unit correction* for continuity. It is subtracted from 12 in the approximation because otherwise half the area of the block centered at 12 will be inadvertently ignored, leading to an unnecessary error in the calculation. From this point on the calculation is routine:

$$\begin{aligned} P[X \geq 12] &\doteq P[Y \geq 11.5] \\ &= P\left[\frac{Y-8}{\sqrt{4.8}} \geq \frac{11.5-8}{\sqrt{4.8}}\right] \\ &= P[Z \geq 1.59] \\ &= 1 - .9441 = .0559 \end{aligned}$$

Note that even with n as small as 20, the approximated value of .0559 compares quite favorably with the exact value of .0565. In practice, of course, one would not approximate a probability that could be found directly from a binomial table. This was done here only for comparative purposes.

4.7 WEIBULL DISTRIBUTION AND RELIABILITY

In 1951 W. Weibull introduced a distribution that has been found to be useful in a variety of physical applications. It arises quite naturally in the study of reliability as we shall show. The most general form for the Weibull density is given by

$$f(x) = \alpha\beta(x-\gamma)^{\beta-1}e^{-\alpha(x-\gamma)^{\beta}} \qquad \begin{aligned} x &> \gamma \\ \alpha &> 0 \\ \beta &> 0 \end{aligned}$$

The implication of this definition of the density is that there is some minimum or "threshold" value γ below which the random variable X cannot fall. In most physical applications this value is 0. For this reason, we shall define the Weibull density with this fact in mind. Be careful when reading scientific literature to note the form of the Weibull density being used.

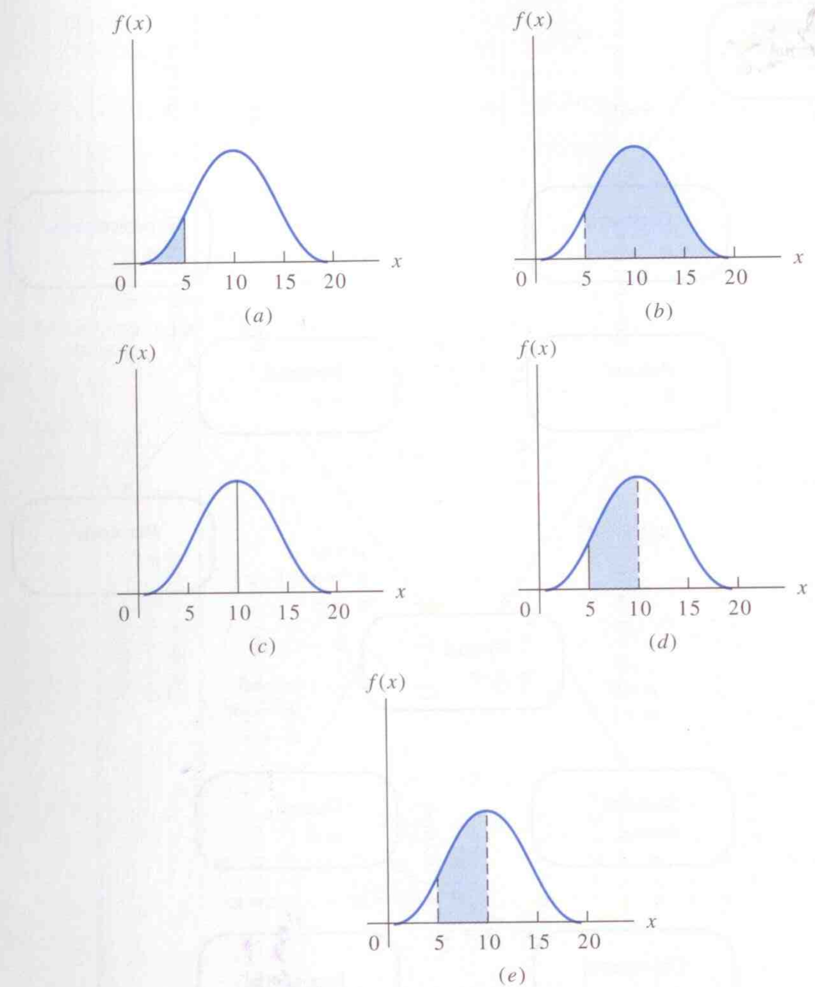

FIGURE 14.17

EXERCISES

Section 4.1

1. Consider the function

$$f(x) = kx \qquad 2 \le x \le 4$$

(*a*) Find the value of k that makes this a density for a continuous random variable.
(*b*) Find $P[2.5 \le X \le 3]$.
(*c*) Find $P[X = 2.5]$.

2. Consider the areas shown in Fig. 4.17. In each case, state what probability is being depicted. What is the relationship between the areas depicted in Figs. 4.17(*a*) and (*b*)? Between those in Figs. 4.17(*d*) and (*e*)?

3. Let X denote the length in minutes of a long-distance telephone conversation. Assume that the density for X is given by

$$f(x) = (1/10)e^{-x/10} \qquad x > 0$$

(*a*) Verify that f is a density for a continuous random variable.
(*b*) Assuming that f adequately describes the behavior of the random variable X, find the probability that a randomly selected call will last at most 7 minutes; at least 7 minutes; exactly 7 minutes.
(*c*) Would it be unusual for a call to last between 1 and 2 minutes? Explain, based on the probability of this occurring.
(*d*) Sketch the graph of f and indicate in the sketch the area corresponding to each of the probabilities found in part (*b*).

k13 4. Some plastics in scrapped cars can be stripped out and broken down to recover the chemical components. The greatest success has been in processing the flexible polyurethane cushioning found in these cars. Let X denote the amount of this material, in pounds, found per car. Assume that the density for X is given by

$$f(x) = \frac{1}{\ln 2}\frac{1}{x} \qquad 25 \le x \le 50$$

(*a*) Verify that f is a density for a continuous random variable.
(*b*) Use f to find the probability that a randomly selected auto will contain between 30 and 40 pounds of polyurethane cushioning.
(*c*) Sketch the graph of f, and indicate in the sketch the area corresponding to the probability found in part (*b*).

5. (*Continuous uniform distribution.*) A random variable X is said to be uniformly distributed over an interval (a, b) if its density is given by

$$f(x) = \frac{1}{b - a} \qquad a < x < b$$

(*a*) Show that this is a density for a continuous random variable.
(*b*) Sketch the graph of the uniform density.
(*c*) Shade the area in the graph of part (*b*) that represents $P[X \le (a + b)/2]$.
(*d*) Find the probability pictured in part (*c*).
(*e*) Let (c, d) and (e, f) be subintervals of (a, b) of equal length. What is the relationship between $P[c \le X \le d]$ and $P[e \le X \le f]$? Generalize the idea suggested by this example, thus justifying the name "uniform" distribution.

6. If a pair of coils were placed around a homing pigeon and a magnetic field was applied that reverses the earth's field, it is thought that the bird would become disoriented. Under these circumstances it is just as likely to fly in one direction as in any other. Let θ denote the direction in radians of the bird's initial flight. See Fig. 4.18. θ is uniformly distributed over the interval $[0, 2\pi]$.
(*a*) Find the density for θ.