Sec 21: Cauchy - Riemann Equations

Suppose that

$$f(z) = u(x, y) + iv(x, y)$$

and that $f'(z)$ exists at a

point $z_0 = x_0 + iy_0$

Then

(i) the first-order partial derivatives u_x, u_y, v_x and v_y must exist at (x_0, y_0),

(ii) they satisfy the CR-eqns

$$u_x = v_y,\ u_y = -v_x \quad at\ (x_0, y_0).$$

(iii)

$f'(z_0)$ can be written as

$$f'(z_0) = u_x(x_0, y_0) + i v_x(x_0, y_0).$$

Proof

Since f(z) is differentiable at z_0

$$\Rightarrow f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \quad \ldots\ldots(1)$$

Note that $z = x + iy, \;\; z_0 = x_0 + iy_0$

$$\Delta z = \Delta x + i\Delta y$$

$$f(z_0) = u(x_0, y_0) + i\,v(x_0, y_0)$$

$$\Rightarrow f(z_0 + \Delta z) = u(x_0 + \Delta x,\; y_0 + \Delta y)$$

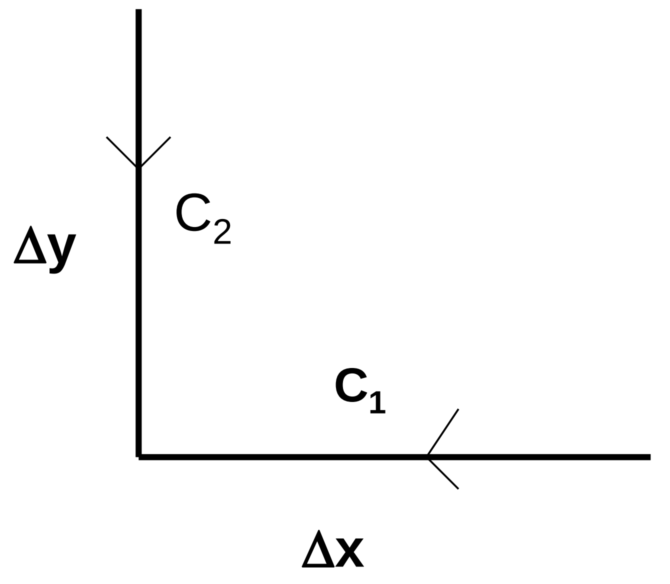
$$+ iv(x_0 + \Delta x, y_0 + \Delta y)$$

$\therefore Eq.(1)\, gives$

$$f'(z_0) =$$

$$\lim_{(\Delta x, \Delta y)\to(0,0)} \left[\frac{u(x_0+\Delta x, y_0+\Delta y) - u(x_0, y_0)}{\Delta x + i\Delta y} \right.$$

$$\left. + i\frac{v(x_0+\Delta x, y_0+\Delta y) - v(x_0, y_0)}{\Delta x + i\Delta y} \right]$$



C_2
Δy
C_1
Δx

$$f'(z_0) = \begin{cases} u_x(x_0\ y_0) + i\,v_x\,(x_0\ y_0), \\ \hfill \text{along the path } C_1 \\ -i\,u_y\,(x_0, y_0) + v_y(x_0, y_0), \\ \hfill \text{along the path } C_2 \end{cases}$$

$$\Rightarrow u_x = v_y, \quad u_y = -v_x \; at \; (x_0, y_0),$$

$$and \;\; f'(z_0) = u_x + i v_x \;\; at \; (x_0, y_0)$$

WHY ???

Sec 22. Sufficient conditions for differentiability

Let $f(z) = u(x, y) + i\, v(x, y)$ be any function defined throughout in some nbd. of the point $z_0 = x_0 + iy_0$ such that

(i) u_x, u_y, v_x, v_y exist in that nbd of z_0,

(ii) u_x, u_y, v_x, v_y are continuous at (x_0, y_0)

(iii) the first order derivatives

satisfy the CR - equations

$$u_x = v_y, \quad u_y = -v_x \ at \ (x_0, y_0).$$

Then $f'(z)$ exists at z_0.

Cauchy - Reimann Equations in Polar Form

Let $f(z) = u(r, \theta) + i\, v(r, \theta)$

be differentiable at any given point

$z_0 = r_0\, e^{i\theta_0}$.

Then the partial derivatives

$u_r,\ u_\theta, v_r, v_\theta$ exist at (r_0, θ_0)

and they satisfy

$$u_r = \frac{1}{r} v_\theta, \quad v_r = -\frac{1}{r} u_\theta$$

and

$$f'(z_0) = e^{-i\theta}\left(u_r + i v_r\right)\Big|_{(r_o,\theta_o)}$$

Example 1: For the function

$$f(z) = z^2,$$

find out the points where the function is differentiable. Also find $f'(z)$

Consider

$$\mathrm{f(z)} = \mathrm{z}^2 = x^2 - y^2 + i\,2xy \equiv u + iv$$

$$\Rightarrow u(x, y) = x^2 - y^2,\ v(x, y) = 2xy$$

$$\Rightarrow u_x = 2x,\ \ u_y = -2y,$$

$$v_x = 2y,\ \ v_y = \ 2x$$

$$\Rightarrow u_x = v_y \ \ \&\ \ u_y = -v_x$$

$\Rightarrow$ (i) CR - equations are satisfied for all x, y

(ii) u_x, u_y, v_x, and v_y are continuous for all x, y

$$\Rightarrow f(z) = z^2 \text{ is differentiable}$$

at any point z, and

$$f'(z) = u_x + iv_x = 2x + i2y = 2z$$

Example 2 : For the function

$$f(z) = |z|^2,$$

find out the points where the function is differentiable. Also find $f'(z)$

$$\text{Consider } \mathrm{f(z)} = |\mathrm{z}|^2 = x^2 + y^2$$

$$\Rightarrow u(x, y) = x^2 + y^2 \,\&\, v(x, y) = 0$$

$$\Rightarrow u_x = 2x,\ u_y = 2y,\ v_x = 0,\ v_y = 0,$$

$$\text{If CR - equations are satisfied,}$$
$$\text{then we must have}$$
$$x = 0 = y.$$

$$\Rightarrow f(z) \text{ is differentiable only at } (0,0)$$
$$\text{and no where else. Further}$$
$$f'(0) = u_x(0,0) + iv_x(0,0) = 0$$

Page 72/Q.6 Let u & v denote the real & imaginary parts of the function f defined by

$$f(z)=\begin{cases}\dfrac{(\bar{z})^2}{z}, & z\neq 0\\ 0, & z=0\end{cases}$$

Show that CR-equations are satisfied at (0,0) although

f is NOT differentiable at (0,0).

Solution :

RECALL : f is not differentiable at $(0,0)$

(already done)

$$\text{We have, when } z \neq 0,$$

$$f(z) = \frac{\left(\bar{z}\right)^2}{z} = \frac{\left(x - iy\right)^2}{x+iy} = \frac{\left(x - iy\right)^3}{(x+iy)(x-iy)}$$

$$= \frac{x^3 - 3xy^2}{x^2+y^2} - i\,\frac{3x^2y - y^3}{x^2+y^2}$$

$$\Rightarrow u(x, y) = \frac{x^3 - 3xy^2}{x^2 + y^2},$$

$$v(x, y) = \frac{y^3 - 3x^2 y}{x^2 + y^2}, (x, y) \neq (0,0)$$

When $z = 0$, then

$$u(x, y) = 0 = v(x, y)$$

Now

$$u_x(0,0) = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x}$$

$$= \lim_{x \to 0} \frac{x - 0}{x} = 1$$

$$u_y(0,0) = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y}$$

$$= \lim_{y \to 0} \frac{0 - 0}{y} = 0$$

$$v_x(0,0) = \lim_{x \to 0} \frac{v(x,0) - v(0,0)}{x}$$

$$= \lim_{x \to 0} \frac{0 - 0}{x} = 0$$

$$v_y(0,0) = \lim_{y \to 0} \frac{v(0,y) - v(0,0)}{y}$$

$$= \lim_{y \to 0} \frac{y - 0}{y} = 1$$

Thus $u_x = v_y$ & $u_y = -v_x$.

Hence, proved.

Q 2b.

Let $f(z)=e^{-z}$. Show that $f(z)$ is differentiable and find

$f'(z)$ and $f''(z)$.

Solution :

$$f(z) = e^{-z} = e^{-x-iy}$$

$$= e^{-x}(\cos y - i \sin y)$$

$$\Rightarrow u = e^{-x} \cos y \,, \quad v = -e^{-x} \sin y$$

$$\therefore \quad u_x = -e^{-x} \cos y,$$

$$u_y = -e^{-x} \sin y,$$

$$v_x = e^{-x} \sin y,$$

$$v_y = -e^{-x} \cos y.$$

Clearly

(1) $u_x = v_y \,\&\, u_y = -v_x$

(2) u_x, u_y, v_x, v_y

are continuous at any point (x, y).

$\therefore f'(z)$ exists and

$$f'(z) = u_x + iv_x$$

$$= -e^{-x}\cos y + i\,e^{-x}\sin y$$

$$= -e^{-x}.\,e^{-iy} = -e^{-z}$$

To find $f''(z)$:

Let $F(z) = f'(z)$

$$= -e^{-x}\cos y + i\,e^{-x}\sin y$$

$$\equiv U + iV \quad \text{(say)}$$

$$\therefore \quad U = -e^{-x}\cos y,$$

$$V = e^{-x}\sin y$$

$$\Rightarrow U_x = e^{-x}\cos y,$$

$$U_y = e^{-x}\sin y,$$

$$V_x = -e^{-x}\sin y,$$

$$V_y = e^{-x}\cos y.$$

Thus,

(1) $U_x = V_y \;\&\; U_y = -V_x$

(2) U_x, U_y, V_x, V_y *are* continuous at any point (x, y)

$$\therefore \mathrm{F}'(\mathrm{z}) \quad \text{exists} \quad \&$$

$$\mathrm{F}'(\mathrm{z}) = \mathrm{f}''(\mathrm{z}) = U_x + iV_x$$

$$= e^{-x}\cos y + i(-e^{-x}\sin y)$$

$$= e^{-x}.e^{-iy}$$

$$= e^{-z}$$

Ex. Page 72, Q5.

Show that when

$$f(z) = x^3 + i(1-y)^3$$

it is legitimate to write

$$f'(z) = 3x^2 \quad only \ when \quad z = i.$$