MATHEMATICS-I (MATH F111)

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Points of intersection

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Q:. How to compute the points of intersection of two curves?

Method to find the points of intersection

Q:. Find the points of intersection of $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$.

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Putting both values of r equal and solve for θ:

$$1 + \cos\theta = 1 - \cos\theta,$$

which gives

$$\cos\theta = 0 \text{ and so } \theta = \frac{\pi}{2}, \ \frac{3\pi}{2}.$$

Now substitute $\theta = \frac{\pi}{2}$ in first equation to get $r = 1$ (you can substitute in second equation also).

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Q:. Are there more points of intersection?

Now substitute $\theta = \frac{\pi}{2}$ in first equation to get $r = 1$ (you can substitute in second equation also).Thus one point of intersection is $\left(1, \frac{\pi}{2}\right)$.

Now substitute $\theta = \frac{3\pi}{2}$ in first equation to get $r = 1$. Thus the second point of intersection is $\left(1, \frac{3\pi}{2}\right)$.

Q:. Are there more points of intersection? If yes, how to find them?

Step 2 : Plot the graphs simultaneously and see the missing points of intersection in Step 1.

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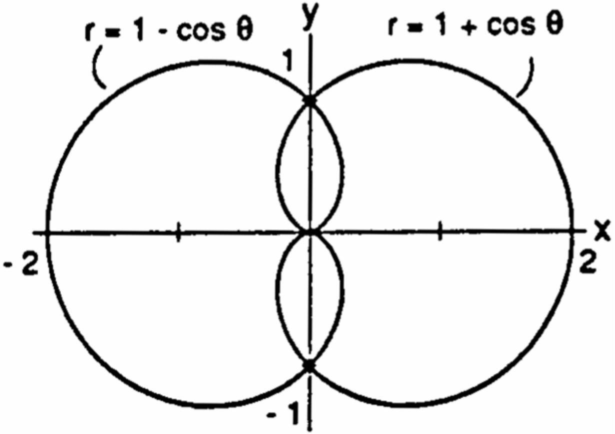
Step 1 may not give all points of intersection (because of deceptive points)



r = 1 - cos θ
y
r = 1 + cos θ
1
x
-2
2
-1

One more point of intersection $(0, 0)$ is found by graphing.

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Thus all points of intersection are $(0, 0), \left(1, \frac{\pi}{2}\right), \left(1, \frac{3\pi}{2}\right)$.

Deceptive Point

A point (r,θ) is said to be deceptive if it lies on the curve but does not satisfy the equation of the curve.

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Another Example?

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Another Example?

The point $(0,0)$ lies on the curve $r = 2\cos\theta$ but does not satisfy the equation.

Q:. How do you verify that $(0,0)$ is a deceptive point for the curve $r = 2\cos\theta$?

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Sol. Another representation $\left(0, \frac{\pi}{2}\right)$ of $(0,0)$ satisfies the equation so does lie on the curve and hence is a deceptive point.

Q:. Find the points of intersection of $r = \sin\theta$ and $r = \cos\theta$.

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Sol. **Step 1** Putting both values of r equal and solve for θ:

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Sol. **Step 1** Putting both values of r equal and solve for θ:

$$\sin\theta = \cos\theta$$
$$\Rightarrow \theta = \frac{\pi}{4}, \frac{5\pi}{4}.$$

Q:. Find the points of intersection of $r = \sin\theta$ and $r = \cos\theta$.

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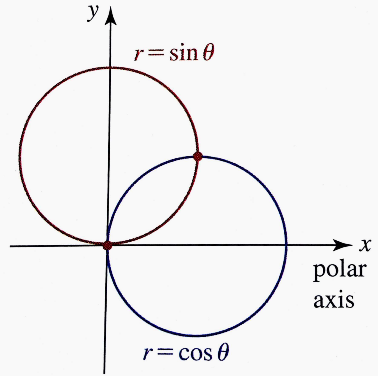
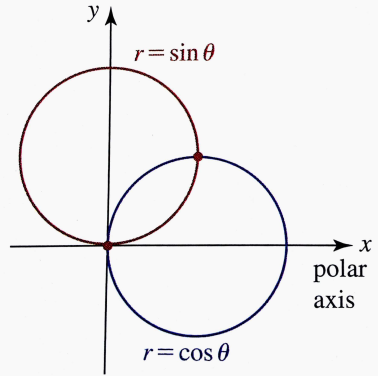
Step 2. Plot the graphs simultaneously to find the missing points of intersection in Step 1

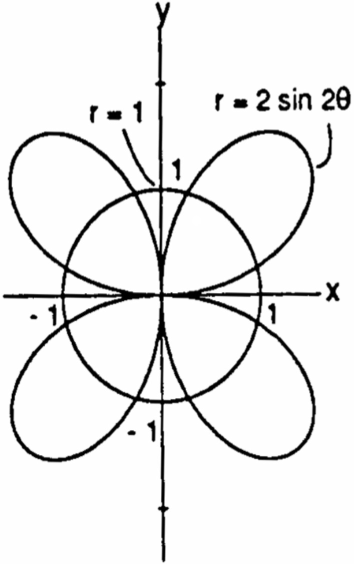

y
r = sin θ
x
polar
axis
r = cos θ



The points of intersection are

$$\left(1/\sqrt{2}, \frac{\pi}{4}\right) \equiv \left(-1/\sqrt{2}, \frac{5\pi}{4}\right), \ (0,0)$$

Q:. Find the points of intersection of $r = 1$ and $r = 2\sin 2\theta$.



Step 1: The points of intersection by solving the equations.

$$\left(1,\frac{7\pi}{12}\right),\left(1,\frac{11\pi}{12}\right),\left(1,\frac{19\pi}{12}\right),\left(1,\frac{23\pi}{12}\right),$$

Step 2: The points of intersection found by graphing and symmetry.

$$\left(1,\frac{7\pi}{12}\right),\left(1,\frac{11\pi}{12}\right),\left(1,\frac{19\pi}{12}\right),\left(1,\frac{23\pi}{12}\right).$$

Section 11.5

Areas and Lengths in Polar Coordinates

Q:. How to compute the area enclosed by a curve?

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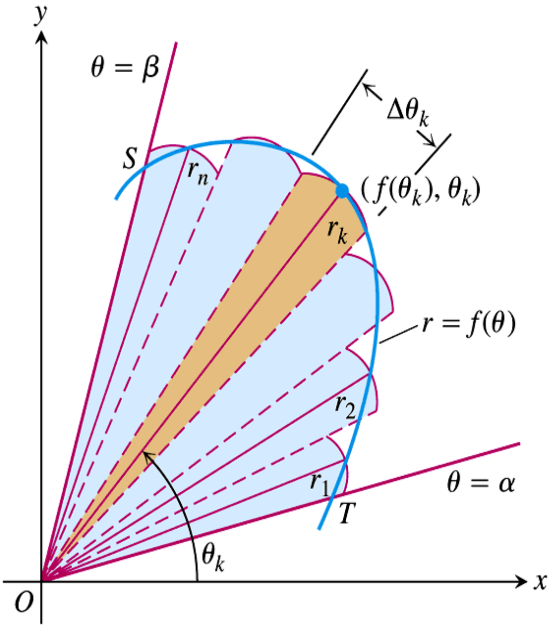
Q:. Precisely, how to compute the area enclosed by a curve $r = f(\theta)$ and by the rays $\theta = \alpha$ and $\theta = \beta$?



y
θ = β
Δθ_k
S
r_n
(f(θ_k), θ_k)
r_k
r = f(θ)
r_2
r_1
θ = α
T
θ_k
O
x

To derive a formula for the area A of region OTS, we approximate the region with n nonoverlapping fan shaped circular sectors based on a partition P of angle TOS. The typical sector has radius r_k and central angle $\Delta\theta_k$.

To derive a formula for the area A of region $\boldsymbol{OTS}$, we approximate the region with $\boldsymbol{n}$ nonoverlapping fan shaped circular sectors based on a partition P of angle $\boldsymbol{TOS}$. The typical sector has radius r_k and central angle $\Delta\theta_k$.

Therefore the circular sector (with an angle $\Delta\theta_k$) is a part of circle of radius r_k. Thus the area of this sector is

$$A_k = \frac{\Delta\theta_k}{2\pi}(\pi r_k^2) = \frac{1}{2} r_k^2 \Delta\theta_k.$$

Then the area of region OTS is approximately

$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n} \frac{1}{2} r_k^2 \Delta\theta_k.$$

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$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n} \frac{1}{2} r_k^2 \Delta\theta_k.$$

Thus the area A of OTS can be given by taking $n \to \infty$. Thus

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} A_k = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2} r_k^2 \Delta\theta_k = \frac{1}{2} \int_{\alpha}^{\beta} r^2 \, d\theta.$$

Then the area A of fan shaped region OTS between the origin and the curve $r = f(\theta)$ and also between the rays $\theta = \alpha$ and $\theta = \beta$ is given by (assuming $\alpha \leqslant \beta$)

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta.$$

Example

Find the area of the region inside the lemniscate $r^2 = 2a^2 \cos 2\theta,\ a > 0$.

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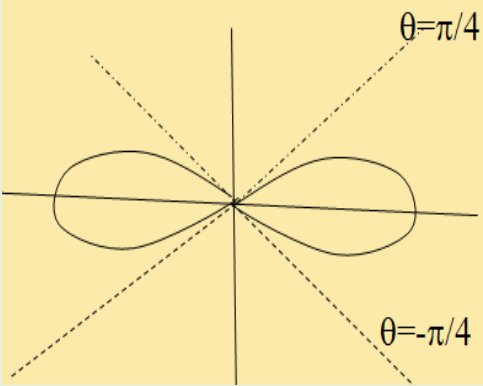
Sol.



Using symmetries, the required area is given by

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$$\begin{aligned} A &= 4 \times \text{Area in first quadrant} \\ &= 4 \times \frac{1}{2} \int_0^{\frac{\pi}{4}} r^2 \, d\theta \\ &= 2 \int_0^{\frac{\pi}{4}} 2a^2 \cos 2\theta \, d\theta \\ &= 2a^2. \end{aligned}$$

Find the area of the region inside the circle $r = 2\cos\theta$.

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Sol.

$$A = \frac{1}{2}\int_0^{2\pi} r^2 \, d\theta = 2 \times \frac{1}{2}\int_0^{\pi} 4\cos^2\theta \, d\theta$$
$$= 2\int_0^{\pi} 1 + \cos 2\theta \, d\theta$$
$$= 2\pi.$$

Find the area of the region inside the circle $r = 2\cos\theta$.

Sol.

$$
\begin{aligned}
A &= \frac{1}{2}\int_0^{2\pi} r^2 \, d\theta = 2 \times \frac{1}{2}\int_0^{\pi} 4\cos^2\theta \, d\theta \\
&= 2\int_0^{\pi} 1 + \cos 2\theta \, d\theta \\
&= 2\pi.
\end{aligned}
$$

Note: $r = 2\cos\theta$ is a circle with center at $(1,0)$ and radius 1. Hence, its actual area is π.

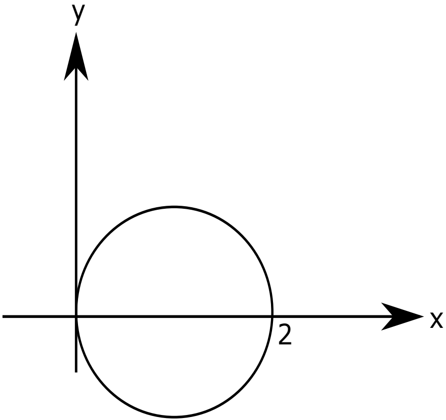

Figure: $r = 2\cos\theta$

The flaw happens because we have not traced the curve. When we trace the curve we find that the whole circle traces between 0 to π, i.e., it traces twice from 0 to 2π, which leads us to the area twice as the actual area.

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Steps to Evaluate Area

- Plot the polar graphs correctly and clearly.

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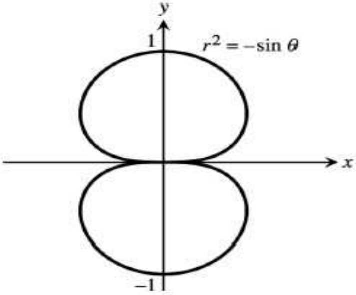
- Plot the polar graphs correctly and clearly.
- Label the relevant curves by their equations.
- Shade the area required by marking its angular spread and radial spread. You may require to find intersections or tangents at pole.
- See that area is covered exactly once by radial segment. Justify symmetries if used.

Q:. Find the area of the region inside $r^2 = -\sin\theta$.


y
1
$r^2 = -\sin\theta$
x
−1

Using symmetries, the required area is given by

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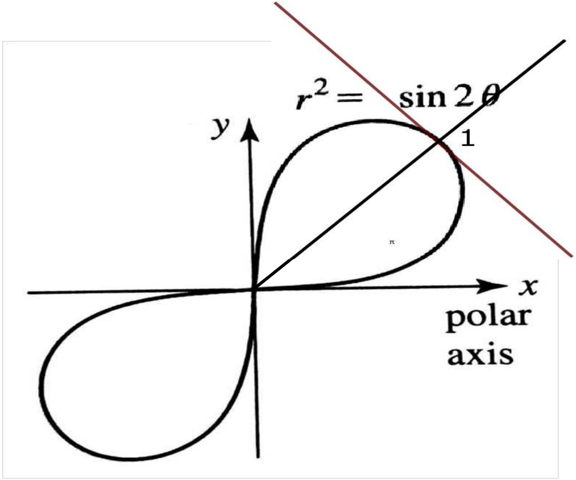
$$\begin{aligned} A &= 4 \times \text{Area lying between } \pi \text{ and } 3\pi/2 \\ &= 4 \times \frac{1}{2} \int_{\pi}^{\frac{3\pi}{2}} r^2 \, d\theta \\ &= 2 \int_{\pi}^{\frac{3\pi}{2}} -\sin\theta \, d\theta \\ &= 2. \end{aligned}$$

Q:. Find the area of the region inside $r^2 = \sin 2\theta$.



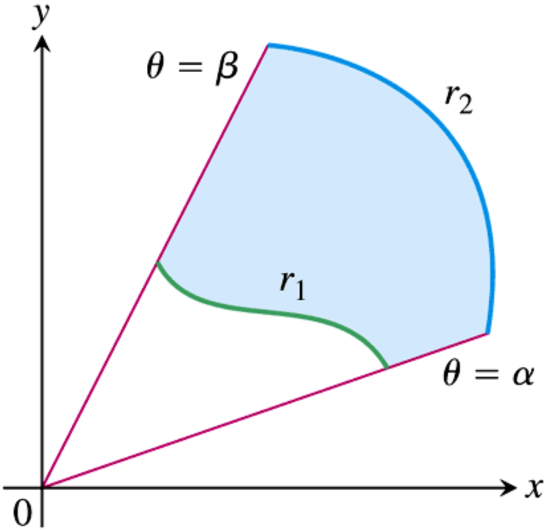
$r^2 = \sin 2\theta$
y
1
x
polar axis

Using symmetries, the required area is given by

Using symmetries, the required area is given by

$$\begin{aligned} A &= 4 \times \text{Area lying between 0 and } \pi/4 \\ &= 4 \times \frac{1}{2} \int_0^{\frac{\pi}{4}} r^2 \, d\theta \\ &= 2 \int_0^{\frac{\pi}{4}} \sin 2\theta \, d\theta \\ &= 1. \end{aligned}$$

Area Shared by Two Curves



Area Shared by Two Curves

Area of the region $0 \leqslant r_1(\theta) \leqslant r \leqslant r_2(\theta),\ \alpha \leqslant \theta \leqslant \beta$ is given by

$$A = \frac{1}{2}\int_{\alpha}^{\beta} (r_2^2 - r_1^2)\, d\theta.$$

Q:. Find the area of the region inside the circle $r = 3a\cos\theta$ and outside the cardioid $r = a(1+\cos\theta),\ a > 0.$

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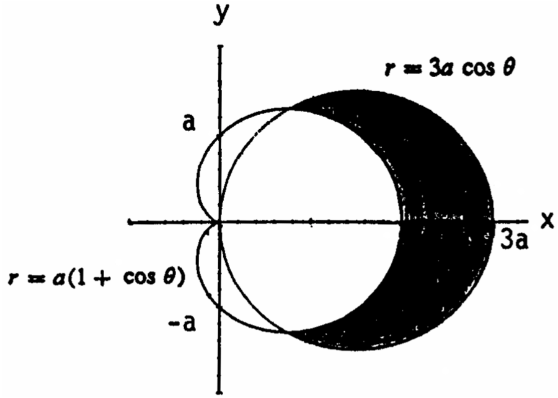

(i) The points of intersections are $\theta = -\frac{\pi}{3},\ \frac{\pi}{3}$.

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(ii) Both the curves are symmetric about x-axis.

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(iii) The graph also gives the point of intersection $(0,0)$.

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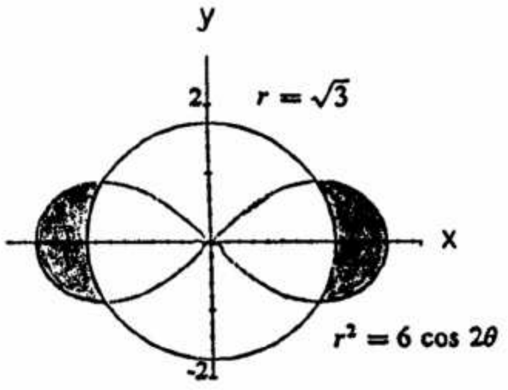
(iii) The graph also gives the point of intersection $(0,0)$.

(iv) Required area is the shaded area.

$$
\begin{aligned}
A &= 2 \times \text{Area in first quadrant} \\
&= 2 \times \frac{1}{2} \int_0^{\frac{\pi}{3}} [(3a\cos\theta)^2 - a^2(1+\cos\theta)^2] \, d\theta \\
&= \pi a^2.
\end{aligned}
$$

Q:. Find the area of the region inside the lemniscate $r^2 = 6\cos 2\theta$ and outside the circle $r = \sqrt{3}$.



y
2
$r = \sqrt{3}$
X
$r^2 = 6\cos 2\theta$
-2

(i) The point of intersection is $\theta = \frac{\pi}{6}$ (in the first quadrant).

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(iii) Required area is the shaded area.

Here $A = 4\times$ area in first quadrant, i.e.,

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$$A = 4 \times \frac{1}{2} \int_0^{\frac{\pi}{6}} ((6\cos 2\theta) - 3)\, d\theta$$

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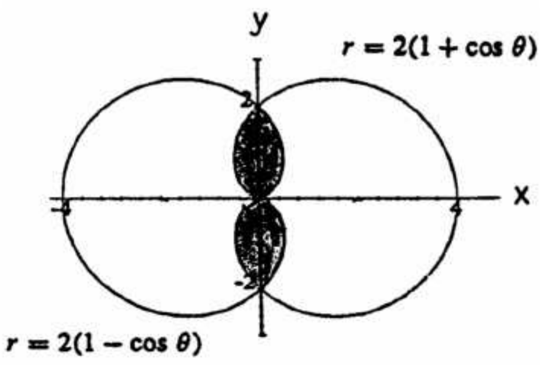
$$A = 4 \times \frac{1}{2} \int_0^{\frac{\pi}{6}} ((6\cos 2\theta) - 3)\, d\theta$$

$$= 3\sqrt{3} - \pi.$$

Q:. Find the area of the region shared by the cardioids $r = 2(1 + \cos\theta)$ and $r = 2(1 - \cos\theta)$.



(i) The point of intersections are $\theta = 0, \frac{\pi}{2}, \frac{3\pi}{2}$.

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Method 1

Method 1

$$A = \left[2 \times \frac{1}{2} \int_0^{\frac{\pi}{2}} (2(1-\cos\theta))^2 \, d\theta\right] + \left[2 \times \frac{1}{2} \int_{\frac{\pi}{2}}^{\pi} (2(1+\cos\theta))^2 \, d\theta\right]$$

Method 1

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$$= 6\pi - 16.$$

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$$= 6\pi - 16.$$

Method 2

Method 1

$$A = \left[2 \times \frac{1}{2} \int_0^{\frac{\pi}{2}} (2(1-\cos\theta))^2 \, d\theta\right] + \left[2 \times \frac{1}{2} \int_{\frac{\pi}{2}}^{\pi} (2(1+\cos\theta))^2 \, d\theta\right]$$

$$= 6\pi - 16.$$

Method 2

$$A = 4 \times \left[\frac{1}{2} \int_0^{\frac{\pi}{2}} (2(1-\cos\theta))^2 \, d\theta\right]$$

Note:

To apply Method 2, it is required to show that $r = 2(1 + \cos\theta)$ and $r = 2(1 - \cos\theta)$ are mirror images of each other.

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Sol. We can write $r = 1-\cos\theta = 1+\cos(\theta-\pi)$.

Note:

To apply Method 2, it is required to show that $r = 2(1 + \cos\theta)$ and $r = 2(1 - \cos\theta)$ are mirror images of each other.

Sol. We can write $r = 1 - \cos\theta = 1 + \cos(\theta - \pi)$.
Thus the curve $r = 1 - \cos\theta$ is obtained from $r = 1 + \cos\theta$ by replacing θ by $\theta + \pi$. Therefore, to obtain the curve of $r = 1 - \cos\theta$, we just need to rotate the curve of $r = 1 + \cos\theta$ by an angle π.

Q:. Find the area of the region inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = \cos\theta$.

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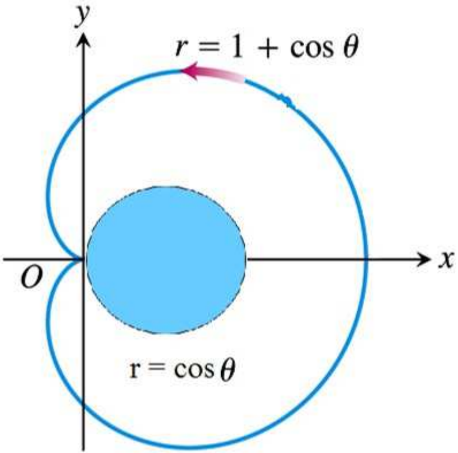
Sol.



$$A = 2 \times \frac{1}{2} \int_0^{\pi} [(1 + \cos\theta)^2 - (\cos\theta)^2] \, d\theta$$

$$A = 2 \times \frac{1}{2} \int_0^{\pi} [(1+\cos\theta)^2 - (\cos\theta)^2] \, d\theta$$

Q:. Is the above expression correct?

$$A = 2 \times \frac{1}{2} \int_0^{\pi} [(1+\cos\theta)^2 - (\cos\theta)^2] \, d\theta$$

Q:. Is the above expression correct? No?

$$A = 2 \times \frac{1}{2} \int_0^{\pi} [(1+\cos\theta)^2 - (\cos\theta)^2] \, d\theta$$

Q:. Is the above expression correct? No? Why?

$$A = 2 \times \frac{1}{2} \int_0^{\pi} [(1+\cos\theta)^2 - (\cos\theta)^2] \, d\theta$$

Q:. Is the above expression correct? No? Why?

Sol. If we consider θ from 0 to 2π, then the circle $r = \cos\theta$ would be traced twice.

Method 1

Method 1

Let A_1 and A_2 are the areas in first and second quadrant respectively. Then

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$$A_1 = \frac{1}{2}\int_0^{\pi/2} [(1+\cos\theta)^2 - (\cos\theta)^2]\; d\theta$$
$$= \frac{1}{2}\left(\frac{\pi}{2}+2\right).$$

$$A_2 = \frac{1}{2}\int_{\pi/2}^{\pi} (1+\cos\theta)^2\; d\theta$$
$$= \frac{1}{2}\left(\frac{3\pi}{4}-2\right).$$

Thus the required area

$$\begin{aligned} A &= 2(A_1 + A_2) \\ &= \left(\frac{\pi}{2} + 2\right) + \left(\frac{3\pi}{4} - 2\right) \\ &= \frac{5\pi}{4}. \end{aligned}$$

Method 2

Method 2

$$A_1 = 2 \times \frac{1}{2}\int_0^{\pi}[(1+\cos\theta)^2]\,d\theta = \frac{3\pi}{2}$$

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$$A_1 = 2 \times \frac{1}{2}\int_0^{\pi}[(1+\cos\theta)^2]\, d\theta = \frac{3\pi}{2}$$

$$A_2 = \text{ Area of circle } r = \cos\theta = \frac{\pi}{4}$$

Method 2

$A_1 = 2 \times \frac{1}{2} \int_0^{\pi} [(1+\cos\theta)^2] \, d\theta = \frac{3\pi}{2}$

$A_2 =$ Area of circle $r = \cos\theta = \frac{\pi}{4}$

Thus the required area$=A_1 - A_2 = \frac{5\pi}{4}$

Length of a Polar Curve

Let $r = f(\theta)$ has a continuous first derivative for $\alpha \leqslant \theta \leqslant \beta$ and the point $P(r, \theta)$ traces the curve exactly once as θ varies from α to β. Then the length of the curve from $\theta = \alpha$ to $\theta = \beta$ is given by

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Q:. Find the length of the spiral $r = \theta^2,\ 0 \leqslant \theta \leqslant \sqrt{5}$.

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Sol. The required length is

$$\begin{aligned} L &= \int_0^{\sqrt{5}} \sqrt{(\theta^2)^2 + (2\theta)^2}\ d\theta \\ &= \int_0^{\sqrt{5}} \sqrt{\theta^4 + 4\theta^2}\ d\theta \\ &= \int_0^{\sqrt{5}} |\theta|\ \sqrt{\theta^2 + 4}\ d\theta \\ &= \int_0^{\sqrt{5}} \theta\sqrt{\theta^2 + 4}\ d\theta \end{aligned}$$

Substitute $u = \theta^2 + 4$

$$L = \frac{1}{2}\int_4^9 \sqrt{u}\, du$$
$$= \frac{19}{3}.$$

Q:. Find the length of cardioid

$$r = 1 + \cos\theta.$$

Q:. Find the length of cardioid

$$r = 1 + \cos\theta.$$

Sol.

$$L = \int_0^{2\pi} \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2}\, d\theta$$

$$= 2\int_0^{\pi} \sqrt{2 + 2\cos\theta}\, d\theta$$

$$= 4\int_0^{\pi} \sqrt{(1+\cos\theta)/2}\, d\theta$$

$$= 8$$

Q:. Find the length of the curve

$$r = a\sin^2\frac{\theta}{2},\ 0 \leqslant \theta \leqslant \pi,\ a > 0.$$

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$$r = a\sin^2\frac{\theta}{2},\ 0 \leqslant \theta \leqslant \pi,\ a > 0.$$

Sol. Note that the curve is a cardioid. The required length is

$$L = \int_0^{\pi} \sqrt{\left(a\sin^2\frac{\theta}{2}\right)^2 + \left(a\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^2}\, d\theta$$

$$= \int_0^{\pi} \left|a\sin\frac{\theta}{2}\right| d\theta$$

$$
\begin{aligned}
L &= \int_0^{\pi} a \sin\frac{\theta}{2}\, d\theta \\
&= -2a\left(\cos\frac{\theta}{2}\right)_0^{\pi} \\
&= 2a.
\end{aligned}
$$

Q:. Find the length of the parabolic segment

$$r = \frac{6}{1+\cos\theta}, \; 0 \leqslant \theta \leqslant \frac{\pi}{2}.$$

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Sol. Better to convert $r = \frac{6}{1+\cos\theta} = 3\sec^2\frac{\theta}{2}$. Thus the required length is

Q:. Find the length of the parabolic segment

$$r = \frac{6}{1+\cos\theta}, \ 0 \leqslant \theta \leqslant \frac{\pi}{2}.$$

Sol. Better to convert $r = \frac{6}{1+\cos\theta} = 3\sec^2\frac{\theta}{2}$. Thus the required length is

$$L = \int_0^{\frac{\pi}{2}} \sqrt{\left(3\sec^2\frac{\theta}{2}\right)^2 + \left(3\sec^2\frac{\theta}{2}\tan\frac{\theta}{2}\right)^2}\, d\theta$$

$$= 3\int_0^{\frac{\pi}{2}} \sec^2\frac{\theta}{2}\sqrt{1+\tan^2\frac{\theta}{2}}\, d\theta$$

Put $\tan\frac{\theta}{2} = t$ so that $\sec^2\frac{\theta}{2}\,d\theta = 2dt$ and so

$$
\begin{aligned}
L &= 6\int_0^1 \sqrt{1+t^2}\,dt \\
&= 6\left[\frac{t}{2}\sqrt{1+t^2} + \frac{1}{2}\ln(t+\sqrt{1+t^2})\right]_0^1 \\
&= 3[\sqrt{2} + \ln(1+\sqrt{2})].
\end{aligned}
$$