Chapter 16

Integration in Vector Fields

Note: *This module is prepared from Chapter 16 (Thomas' Calculus, 13th edition) just to help the students. The study material is expected to be useful but not exhaustive. For detailed study, the students are advised to attend the lecture/tutorial classes regularly, and consult the text book.*

Appeal: Please do not print this e-module unless it is really necessary.

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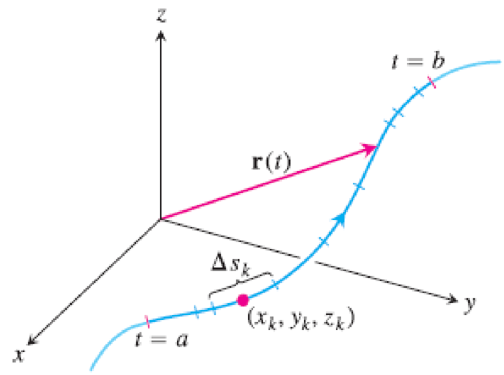
16.1 Line Integrals

Let C be a smooth curve[1] with end points A and B in the domain of a real valued function or a scalar field[2] $f(x, y, z)$.



Partition the curve C into n subarcs with lengths Δs_k $(k = 1, 2, ..., n)$ such that $\Delta s_k \to 0$ as $n \to \infty$. Let (x_k, y_k, z_k) be a point on the kth subarc. Then the limit

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta s_k,$$

if exists, defines the line integral of $f(x, y, z)$ along the curve C, and is written as

$$\int_C f(x, y, z)ds = \int_A^B f(x, y, z)ds = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta s_k,$$

where $\Delta s_k \to 0$ as $n \to \infty$.

If the curve C is parametrized by $x = x(t)$, $y = y(t)$, $z = z(t)$, and its end points A and B correspond to the parameter values $t = a$ and $t = b$, respectively, then

$$\int_C f(x, y, z)ds = \int_A^B f(x, y, z)ds = \int_a^b f(x(t), y(t), z(t))\frac{ds}{dt}dt,$$

where $\frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$.

Note that if the function f is continuous on the smooth curve C, then the line integral of f can be shown to exist along the curve C.

Further, if C is piecewise smooth curve, that is, C consists of a finite number of smooth arcs C_1, C_2, ..., C_n joined end to end, then the line integral of f along C in the sum of the line integrals of f along

[1]**Smooth Curve**: A curve C given by $\overrightarrow{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$, $a \le t \le b$ is smooth if $\frac{d\overrightarrow{r}}{dt}$ is continuous and non-zero in $[a, b]$.

[2]**Scalar Field**: A scalar field is a function $f(x, y, z)$ that yields a scalar corresponding to each point of its domain of definition. For example, the temperature function defined on the surface of earth is a scalar field.

the smooth arcs making C, that is,

$$\int_C f ds = \int_{C_1} f ds + \int_{C_2} f ds + ... + \int_{C_n} f ds.$$

Ex. Integrate $f(x, y, z) = x - 3y^2 + z$ over the line segment C joining the origin to the point $(1, 1, 1)$.

Sol. A simple parametrization of the given path C reads as

$$\overrightarrow{r}(t) = t\hat{i} + t\hat{j} + t\hat{k}, \quad 0 \leq t \leq 1.$$

$$\therefore \quad \frac{ds}{dt} = \left| \frac{d\overrightarrow{r}}{dt} \right| = \sqrt{1+1+1} = \sqrt{3}.$$

It follows that

$$\int_C f(x, y, z) ds = \int_0^1 f(t, t, t) \frac{ds}{dt} dt = \int_0^1 (t - 3t^2 + t)\sqrt{3} dt = 0.$$

Ex. Integrate $f(x, y, z) = x^2 + y + z$ along the curve C consisting of the sides of the triangle (traversed counterclockwise) with vertices $A(1, 0, 0)$, $B(0, 1, 0)$ and $C(0, 0, 1)$.

Sol. Here C is not smooth but it consists of three smooth parts which are the line segments AB, BC and CA.

$$\therefore \quad \int_C f ds = \int_{AB} f ds + \int_{BC} f ds + \int_{CA} f ds.$$

Let us first solve $\int_{AB} f ds$. The parametric equation of the line AB is

$$\frac{x-1}{1} = \frac{y-0}{-1} = \frac{z-0}{0} = t,$$

that is, $x = t + 1$, $y = -t$, $z = 0$. So $\frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} = \sqrt{2}$. Further, along AB, we find $f = (t + 1)^2 - t = t^2 + t + 1$. Also, from the relations $x = t + 1$, $y = -t$, $z = 0$, it is easy to find that the point $A(1, 0, 0)$ corresponds to $t = 0$ and the point $B(0, 1, 0)$ corresponds to $t = -1$. Thus,

$$\int_{AB} f ds = \int_0^{-1} (t^2 + t + 1)\sqrt{2}\, dt = -\frac{5}{6}\sqrt{2}.$$

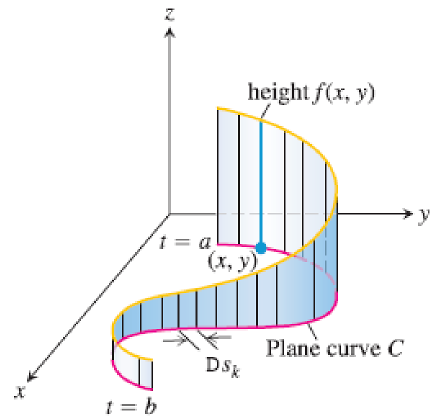
Likewise, the line integrals along the sides BC and CA can be determined.

Line integrals in plane

Here we discuss an interesting geometrical meaning of line integral. Suppose $z = f(x, y)$ is a non-negative and continuous function representing a curve in space with projection or domain curve C in the XY-plane as shown in figure below.

Then the line integral of $f(x, y)$ along C,

$$\int_C f(x, y) ds = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k) \Delta s_k,$$



where $\Delta s_k \to 0$ as $n \to \infty$, gives the area of the portion of the cylindrical surface or "wall" beneath or under the space curve $z = f(x, y) \geq 0$. It is clearly the generalization of the integral $\int_a^b f(x)dx$ (area under the plane curve $y = f(x) \geq 0$ from $x = a$ to $x = b$ above the X-axis) where the interval $[a, b]$ happens to be the projection of the plane curve $y = f(x)$ on the X-axis.

Ex. Find area of the curved surface of the cylinder $x^2 + y^2 = a^2$, $0 \leq z \leq h$.

Sol. Here $z = f(x, y) = h$, $x^2 + y^2 = a^2$ is a non-negative and continuous function representing a circular curve in space (the plane $z = h$) with projection or domain curve $C : x^2 + y^2 = a^2$ in the XY-plane. So the required area is the area beneath the circular curve $x^2 + y^2 = a^2$ lying in the plane $z = h$. The domain curve $C : x^2 + y^2 = a^2$ in the XY-plane is parametrized by $x = a\cot t$, $y = a\sin t$, $0 \leq t \leq 2\pi$.

$$\therefore \text{ Area} = \int_C f(x, y)ds = \int_0^{2\pi} h\sqrt{a^2\cos^2 t + a^2\sin^2 t}\, dt = 2\pi ah.$$

16.2 Vector Fields and Line Integrals: Work, Circulation, and Flux

A vector field is a function that assigns a vector to each point in its domain. A vector field on a three-dimensional domain in space can have a formula like

$$\overrightarrow{F} = M(x, y, z)\hat{i} + N(x, y, z)\hat{j} + P(x, y, z)\hat{k},$$

where the components M, N and P are scalar functions (scalar fields) of x, y, z.

For example, the velocity vectors in water flow in a channel constitute a vector field. See the figure below where the streamlines are shown in a contracting channel. Notice that the water speeds up as the channel narrows and the velocity vectors increase in length.

Another example of vector field is of the gravitational field of earth where each vector points towards the center of the earth as shown in the figure below.

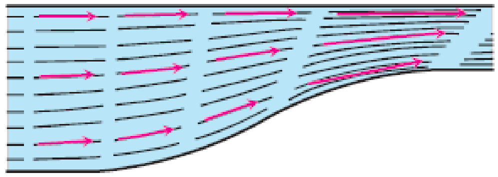


Figure 1: Water flow in a channel

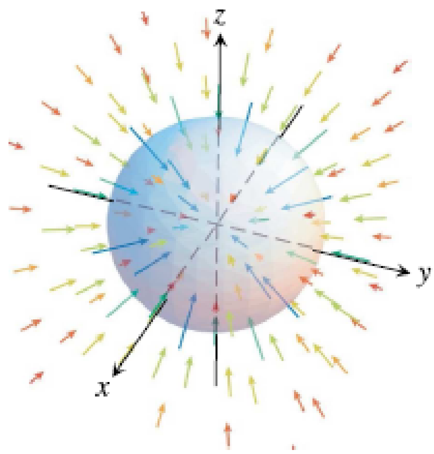


Figure 2: Gravitational field of earth

Line Integral of a Vector Field

Let C be a smooth curve with end points A and B in the domain of a vector valued function or vector field $\overrightarrow{F}(x, y, z)$. Then line integral of $\overrightarrow{F}(x, y, z)$ along the curve C is defined as the line integral of its tangential component $\overrightarrow{F}.\hat{T}$ along the curve C, that is,

$$\int_C \overrightarrow{F}.\hat{T}ds = \int_C \overrightarrow{F}.\frac{d\overrightarrow{r}}{ds}ds = \int_C \overrightarrow{F}.d\overrightarrow{r},$$

where $\overrightarrow{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is position vector of any point (x, y, z) on the curve C.

If the curve C is parametrized by $\overrightarrow{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$, and its end points A and B correspond to the parameter values t_1 and t_2, respectively, then

$$\int_C \overrightarrow{F}.\hat{T}ds = \int_C \overrightarrow{F}.d\overrightarrow{r} = \int_{t_1}^{t_2} \overrightarrow{F}.\frac{d\overrightarrow{r}}{dt}dt.$$

In the following, we mention some physical applications of line integral.

Work Done

If a particle moves along a smooth curve C from A to B under the action of force filed $\overrightarrow{F}(x,y,z)$, then the work done in moving the particle from A to B along C is given by

$$W = \int_C \overrightarrow{F}.\hat{T}ds = \int_C \overrightarrow{F}.d\overrightarrow{r} = \int_A^B \overrightarrow{F}.d\overrightarrow{r}.$$

Ex. Find the work done by the force field $\overrightarrow{F} = x\hat{i} + y\hat{j} + z\hat{k}$ in moving an object along the curve C parametrized by $\overrightarrow{r}(t) = \cos(\pi t)\hat{i} + t^2\hat{j} + \sin(\pi t)\hat{k}, \quad 0 \leq t \leq 1.$

Sol. The work done is the line integral

$$\int_0^1 \overrightarrow{F}.\frac{d\overrightarrow{r}}{dt}dt = \int_0^1 2t^3 dt = \frac{1}{2}.$$

Flow and Circulation

If $\overrightarrow{F}(x,y,z)$ represents the velocity field in a fluid flow, then the flow along a smooth curve C extending from A to B in domain of $\overrightarrow{F}(x,y,z)$ is given by

$$\text{Flow} = \int_C \overrightarrow{F}.\hat{T}ds = \int_C \overrightarrow{F}.d\overrightarrow{r} = \int_A^B \overrightarrow{F}.d\overrightarrow{r}.$$

In case, C is a closed curve, then the above integral is called circulation of $\overrightarrow{F}(x,y,z)$ around C, and is denoted by

$$\oint_C \overrightarrow{F}.\hat{T}ds = \oint_C \overrightarrow{F}.d\overrightarrow{r}.$$

Ex. Find the circulation of the field $\overrightarrow{F} = (x-y)\hat{i} + x\hat{j}$ around the circle C given by $\overrightarrow{r}(t) = \cos t\hat{i} + \sin t\hat{j}, \quad 0 \leq t \leq 2\pi.$

Sol. The circulation of the given field is

$$\int_0^{2\pi} \overrightarrow{F}.\frac{d\overrightarrow{r}}{dt}dt = \int_0^{2\pi} (1 - \sin t\cos t)dt = 2\pi.$$

Flux

Consider the planar flow of a fluid with continuous velocity field $\overrightarrow{F}(x,y)$ in the xy-plane. If a smooth and simple closed curve C lies in the domain of $\overrightarrow{F}(x,y)$, and $\hat{n}$ is a outward pointing unit normal vector on C, then the outward flux of $\overrightarrow{F}$ across C is given by line integral of normal component $\overrightarrow{F}.\hat{n}$ of $\overrightarrow{F}$ along C, that is,

$$\text{Outward Flux} = \oint_C \overrightarrow{F}.\hat{n}ds.$$

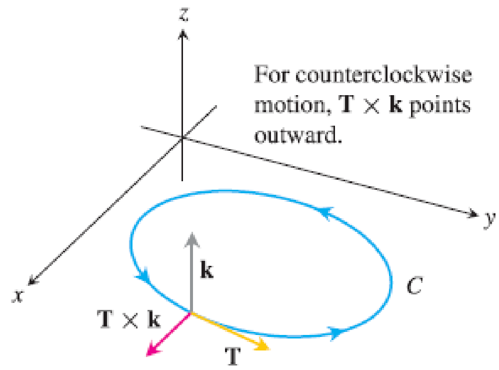


Figure 3:

If the curve C is traversed counterclockwise in the xy-plane as viewed from the tip of positive z-axis (see Figure 3), then we have

$$\hat{n} = \hat{T} \times \hat{k} = \frac{d\overrightarrow{r}}{ds} \times \hat{k} = \left(\frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j}\right) \times \hat{k} = -\frac{dx}{ds}\hat{j} + \frac{dy}{ds}\hat{i}.$$

. If $\overrightarrow{F}(x,y) = M(x,y)\hat{i} + N(x,y)\hat{j}$, then $\overrightarrow{F}.\hat{n} = M\dfrac{dy}{ds} - N\dfrac{dx}{ds}$. So

$$\text{Outward Flux} = \oint_C \overrightarrow{F}.\hat{n}ds = \oint_C Mdy - Ndx.$$

Ex. Find the flux of the field $\overrightarrow{F} = (x-y)\hat{i} + x\hat{j}$ across the circle C given by $\overrightarrow{r}(t) = \cos t\hat{i} + \sin t\hat{j}, \quad 0 \leq t \leq 2\pi$.

Sol. Here $M = x - y$ and $N = x$. On C, $x = \cos t$ and $y = \sin t$. So the flux of the given field is

$$\int_0^{2\pi} (Mdy - Ndx) = \int_0^{2\pi} \cos^2 t dt = \pi.$$

16.3 Path Independence, Conservative Fields, and Potential Functions

Consider a vector field $\overrightarrow{F}$ defined in an open region D[3]. If the line integral of $\overrightarrow{F}$ along any smooth path C extending from a point A to a point B inside D, that is, $\int_C \overrightarrow{F}.d\overrightarrow{r} = \int_A^B \overrightarrow{F}.d\overrightarrow{r}$ does not depend on the path C joining A and B rather it depends solely on the end points A and B of the path, then the line integral of $\overrightarrow{F}$ is said to be path independent in D, and $\overrightarrow{F}$ is said to be conservative field in D.

Suppose there exits a differentiable function ϕ such that $\overrightarrow{F} = \nabla\phi$ in D. Then ϕ is called potential of the vector field $\overrightarrow{F}$ in D. Suppose a smooth curve C parameterized by $\overrightarrow{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ lies in D, and its end points A and B correspond to the parameter values t_1 and t_2, respectively. Along the curve C, $\phi(x(t), y(t), z(t))$ is a differentiable function of t. So we have

$$\frac{d\phi}{dt} = \frac{d\phi}{dx}\frac{dx}{dt} + \frac{d\phi}{dy}\frac{dy}{dt} + \frac{d\phi}{dz}\frac{dz}{dt} = \nabla\phi.\frac{d\overrightarrow{r}}{dt} = \overrightarrow{F}.\frac{d\overrightarrow{r}}{dt}.$$

It follows that

$$\int_C \overrightarrow{F}.d\overrightarrow{r} = \int_A^B \overrightarrow{F}.d\overrightarrow{r} = \int_{t_1}^{t_2} \overrightarrow{F}.\frac{d\overrightarrow{r}}{dt}dt = \int_{t_1}^{t_2} \frac{d\phi}{dt}dt = \phi(x(t), y(t), z(t))]_{t_1}^{t_2} = \phi(B) - \phi(A).$$

This shows that line integral of $\overrightarrow{F}$ is path independent in D. The above result is known as the fundamental theorem of line integrals. We notice that the path independence of the line integral of $\overrightarrow{F}$ is ensured provided $\overrightarrow{F}$ is gradient field of some differentiable function ϕ. In this regard, we shall state some useful results. But before that let us see a practical example of conservative field from the world of Physics.

The most prominent examples of conservative forces are the gravitational force and the electric force associated to an electrostatic field. According to Newton's law of gravitation, the gravitational force $\overrightarrow{F}$ acting on a mass m due to a mass M located at distance r, obeys the equation

$$\overrightarrow{F} = -\frac{GMm}{r^2}\hat{r},$$

where G is the gravitational constant and $\hat{r}$ is a unit vector pointing from M towards m. The force of gravity is conservative because

$$\overrightarrow{F} = -\nabla\phi,$$

where

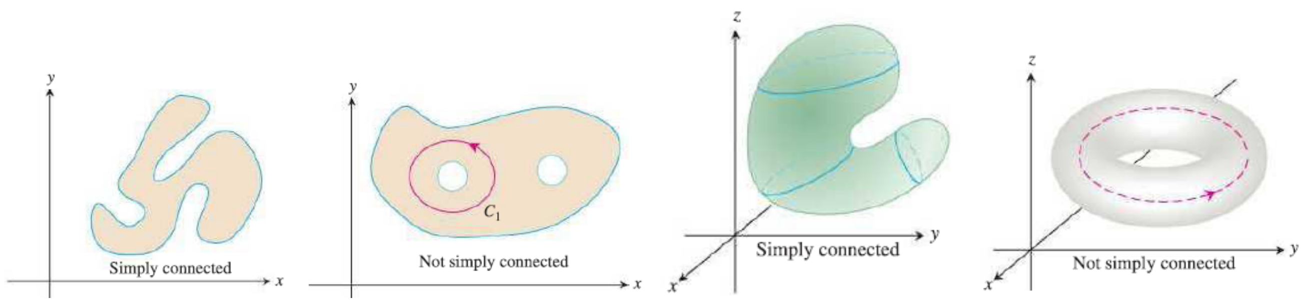
$$\phi = -\frac{GMm}{r}$$

is the gravitational potential energy.

[3]Open region is a region which contains a neighbourhood of its every point. In other words, all points of an open region are its interior points.

Simply connected regions

A region D is said to be connected if any two points of D can be joined by a polygonal path inside D. Further, a connected region D is said to be simply connected if any closed curve inside D encloses points of D only.



Conservative fields are gradient fields.

Let $\overrightarrow{F}$ be a continuous vector field in an open connected region D. Then $\overrightarrow{F}$ is conservative if and only if $\overrightarrow{F} = \nabla\phi$ for some differentiable function ϕ.

Loop property of conservative fields

Let $\overrightarrow{F}$ be a continuous vector field in an open connected region D. Then $\overrightarrow{F}$ is conservative if and only if $\oint_C \overrightarrow{F}.d\overrightarrow{r} = 0$ for every loop or closed curve C in D.

Component test for conservative fields

Let $\overrightarrow{F} = M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + P(x,y,z)\hat{k}$ be a vector field having continuous first order partial derivatives in a simply connected open region D. Then $\overrightarrow{F}$ is conservative if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},\ \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y},\ \frac{\partial P}{\partial x} = \frac{\partial M}{\partial z}.$$

By verifying the above conditions on components of $\overrightarrow{F}$, we find whether $\overrightarrow{F}$ is conserved. If $\overrightarrow{F}$ is conserved, then there exists some differentiable function ϕ such that $\overrightarrow{F} = \nabla\phi$ or

$$M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + P(x,y,z)\hat{k} = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k},$$

or

$$M = \frac{\partial\phi}{\partial x},\ N = \frac{\partial\phi}{\partial y},\ P = \frac{\partial\phi}{\partial z}.$$

To determine ϕ, we solve the above equations as illustrated in the following example.

Ex. Show that the field $\overrightarrow{F} = y\hat{i} + x\hat{j} + 4\hat{k}$ is conservative. Find its potential and hence evaluate the line integral

$$\int_{(1,1,1)}^{(2,3,-1)} ydx + xdy + 4dz$$

Sol. Here $M = y$, $N = x$ and $P = 4$. So we have

$$\frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial x}, \ \frac{\partial N}{\partial z} = 0 = \frac{\partial P}{\partial y}, \ \frac{\partial P}{\partial z} = 0 = \frac{\partial M}{\partial x}.$$

This shows that the given field is conservative.

Now, let ϕ be potential of $\overrightarrow{F}$ so that $\overrightarrow{F} = \nabla\phi$. So we have

$$\frac{\partial \phi}{\partial x} = y, \quad \frac{\partial \phi}{\partial y} = x, \quad \frac{\partial \phi}{\partial z} = 4.$$

We shall use these three equations to determine ϕ. From first equation, we get

$$\phi(x, y, z) = xy + g(y, z),$$

which gives

$$\frac{\partial \phi}{\partial y} = x + \frac{\partial g}{\partial y}.$$

In view of second equation, we have

$$\frac{\partial g}{\partial y} = 0.$$

This shows that g is a function of z alone, and therefore

$$\phi(x, y, z) = xy + h(z).$$

$$\therefore \quad \frac{\partial \phi}{\partial z} = 0 + \frac{dh}{dz} = 4, \quad \text{or} \quad h(z) = 4z + C.$$

in view of the third equation $\frac{\partial \phi}{\partial z} = 4$. Thus, finally we get

$$\phi(x, y, z) = xy + 4z + C.$$

Since the given vector field is conservative, so the required line integral is independent of the path from $(1, 1, 1,)$ to $(2, 3, -1)$, and equals

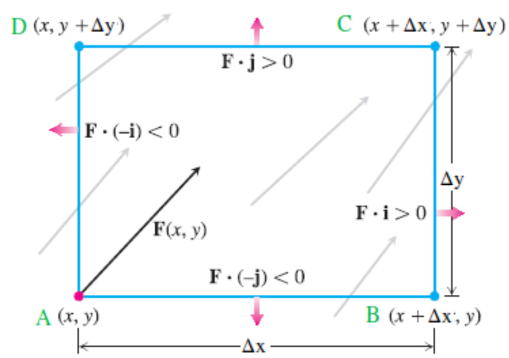
$$\phi(2, 3, -1) - \phi(1, 1, 1) = 2 + C - (5 + C) = -3.$$

16.4 Green's Theorem in the Plane

In this section we derive a method for computing a work or flux integral over a closed curve C in the plane when the field $\overrightarrow{F}$ is not conservative. This method, known as Green's Theorem, allows us to convert the line integral into a double integral over the region enclosed by C. The discussion is given in terms of velocity fields of fluid flows (a fluid is a liquid or a gas) because they are easy to visualize. However, Green's Theorem applies to any vector field, independent of any particular interpretation of the field, provided the assumptions of the theorem are satisfied. We introduce two new ideas for Green's Theorem: divergence and curl.

Divergence

Consider planar flow of a fluid in the xy-plane. Suppose $\overrightarrow{F}(x, y) = M(x, y)\hat{i} + N(x, y)\hat{j}$ is the velocity field of the fluid having continuous first order partial derivatives in a region R of the xy-plane.



Let $A(x, y)$ be a point in R, and $ABCD$ be a small rectangle traversed counterclockwise completely lying inside R, with edges parallel to the coordinate axes having lengths Δx and Δy. To be more precise, let $AB = \Delta x$ is bottom edge; $CD = \Delta x$ is the top edge; $AD = \Delta y$ is the left edge and $BC = \Delta y$ is the right edge. We also assume that the velocity field components M and N do not change their signs throughout the small rectangle $ABCD$. Then the amount of fluid coming out of the bottom edge $AB = \Delta x$ per unit time is given by

$$\overrightarrow{F}(x, y).(-\hat{j})\Delta x = -N(x, y)\Delta x.$$

Likewise, the flow rate of the fluid across the top edge $CD = \Delta x$ of the rectangle is

$$\overrightarrow{F}(x, y + \Delta y).\hat{j}\Delta x = N(x, y + \Delta y)\Delta x.$$

The flow rate of the fluid across the left edge $AD = \Delta y$ of the rectangle is

$$\overrightarrow{F}(x, y).(-\hat{i})\Delta y = -M(x, y)\Delta y,$$

while the flow rate of the fluid across the right edge $BC = \Delta y$ of the rectangle reads as

$$\overrightarrow{F}(x+\Delta x, y).\hat{i}\Delta y = M(x+\Delta x, y)\Delta y.$$

Summing the flow rates across the opposite pairs gives

Top and Bottom: $N(x, y+\Delta y)\Delta x - N(x,y)\Delta x \approx \dfrac{\partial N}{\partial y}\Delta y \Delta x$

Right and Left: $M(x+\Delta x, y)\Delta y - M(x,y)\Delta y \approx \dfrac{\partial M}{\partial x}\Delta x \Delta y$

So the net flow rate across the entire rectangle or the flux across the rectangle $\approx \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\Delta x \Delta y$.

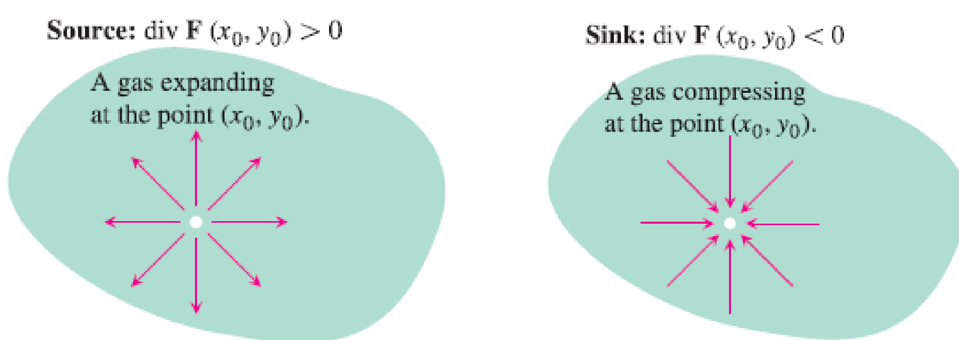
So the flux per unit area or flux density for the rectangle $\approx \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$.

In the limit $(\Delta x, \Delta y) \to (0,0)$, we get the flux density at the point $A(x,y)$, and we call it the divergence of $\overrightarrow{F}$ at the point $A(x,y)$. Formally, we have the following definition of divergence.

The divergence (flux density) of a vector field $\overrightarrow{F}(x,y) = M(x,y)\hat{i} + N(x,y)\hat{j}$ at a point (x,y) is

$$\text{div } \overrightarrow{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}.$$

If a fluid is compressible, then the divergence of its velocity field measures to what extent it is expanding or contracting at each point. Intuitively, if a fluid is expanding at a point (x,y), the lines of flow would diverge there, and since the fluid would be flowing out of a small rectangle about (x,y), the divergence of the velocity field at (x,y) would be positive. In case of compression, the divergence would be negative. The points of expansion and compression in a fluid are sometimes called as sources (something coming out) and sinks (something loosing into), respectively. Of course, there would be no sources and sinks in an incompressible fluid, and divergence of the velocity field of such a fluid would be zero at every point.



The divergence of a more general vector field $\overrightarrow{F} = M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + P(x,y,z)\hat{k}$ at a point (x,y,z) is defined as

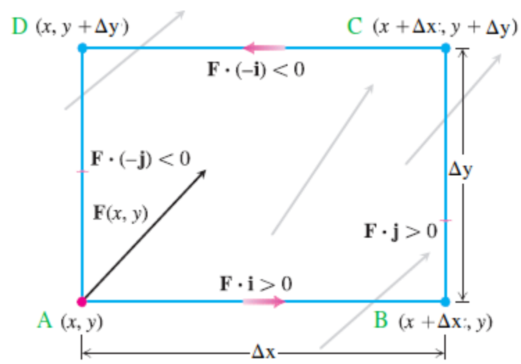
$$\text{div } \overrightarrow{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}.$$

An easy to use notation and formula for computing divergence of $\overrightarrow{F} = M(x,y,z)\hat{i}+N(x,y,z)\hat{j}+P(x,y,z)\hat{k}$ read as

$$\text{div } \overrightarrow{F} = \nabla.\overrightarrow{F} = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right).(M\hat{i}+N\hat{j}+P\hat{k}) = \frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}+\frac{\partial P}{\partial z}.$$

Spin around an axis: The k-component of curl

Consider planar flow of a fluid in the xy-plane. Suppose $\overrightarrow{F}(x,y) = M(x,y)\hat{i} + N(x,y)\hat{j}$ is the velocity field of the fluid having continuous first order partial derivatives in a region R of the xy-plane.



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$$\overrightarrow{F}(x,y).\hat{i}\Delta x = M(x,y)\Delta x.$$

Likewise, the flow rate of the fluid along the top edge $CD = \Delta x$ of the rectangle is

$$\overrightarrow{F}(x,y+\Delta y).(-\hat{i})\Delta x = -M(x,y+\Delta y)\Delta x.$$

The flow rate of the fluid along the left edge $DA = \Delta y$ of the rectangle is

$$\overrightarrow{F}(x,y).(-\hat{j})\Delta y = -N(x,y)\Delta y,$$

while the flow rate of the fluid along the right edge $BC = \Delta y$ of the rectangle reads as

$$\overrightarrow{F}(x+\Delta x,y).\hat{j}\Delta y = N(x+\Delta x,y)\Delta y.$$

Summing the flow rates across the opposite pairs gives

Top and Bottom: $-M(x, y+\Delta y)\Delta x + M(x, y)\Delta x \approx -\dfrac{\partial M}{\partial y}\Delta y \Delta x$

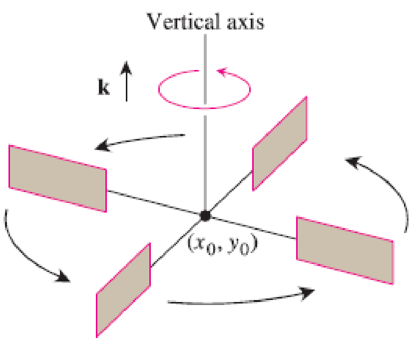
Right and Left: $N(x+\Delta x, y)\Delta y - N(x, y)\Delta y \approx \dfrac{\partial N}{\partial x}\Delta x \Delta y$

So the net flow rate or the counterclockwise circulation rate around the rectangular boundary

$$\approx \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\Delta x \Delta y.$$

So the circulation density for the rectangle $\approx \dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}$.

In the limit $(\Delta x, \Delta y) \to (0, 0)$, we get the circulation density of $\overrightarrow{F}$ at the point $A(x, y)$. Physically, it is the measure of how a floating paddle wheel with axis perpendicular to the xy-plane spins at a point (x, y) in the fluid flowing in the xy-plane.



Curl **F** $(x_0, y_0) \cdot$ **k** > 0
Counterclockwise circulation

The expression $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$, in fact, is the k-component of a more general vector known as curl of $\overrightarrow{F}$. Formally, we have the following definition of curl of a vector field.

The curl of a vector field $\overrightarrow{F} = M(x, y, z)\hat{i} + N(x, y, z)\hat{j} + P(x, y, z)\hat{k}$ at a point (x, y, z) is

$$\text{curl } \overrightarrow{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\hat{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\hat{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\hat{k}.$$

An easy to use notation and formula for computing curl of $\overrightarrow{F} = M(x, y, z)\hat{i} + N(x, y, z)\hat{j} + P(x, y, z)\hat{k}$ read as

$$\text{curl } \overrightarrow{F} = \nabla \times \overrightarrow{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\hat{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\hat{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\hat{k}.$$

Note that the determinant here is not the determinant of numbers. The determinant notation is used just to facilitate the calculation of curl.

A vector field $\overrightarrow{F}$ is said to be irrotational if $\nabla \times \overrightarrow{F} = 0$.

For example, the vector field $\overrightarrow{F} = x\hat{i} + y\hat{j} + z\hat{k}$ is irrotational. Verify!

Next verify that curl of a gradient vector is zero, that is, $\nabla \times \nabla f = 0$.

Ex. The following vector fields represent the velocity of a gas flowing in the xy-plane (see the corresponding figures also).

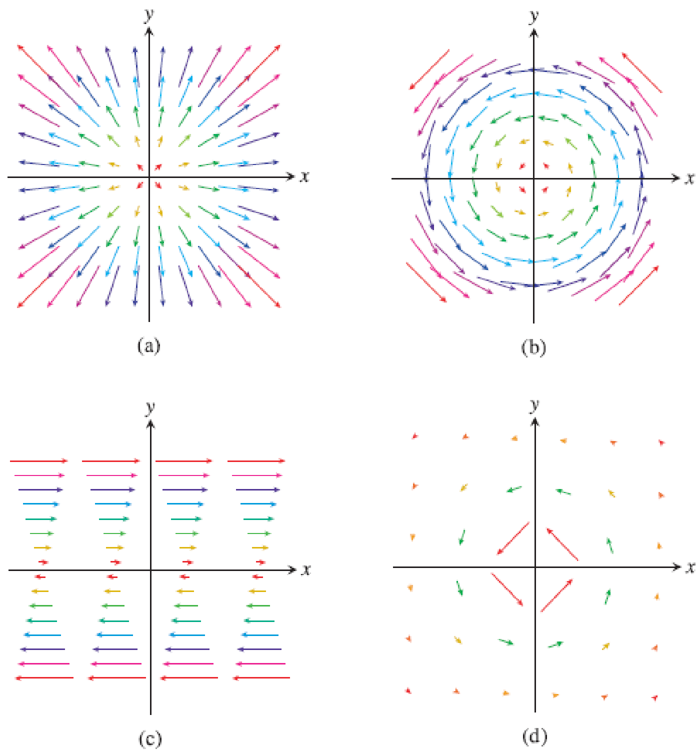
(a) **Uniform expansion or compression**: $\overrightarrow{F}(x,y) = cx\hat{i} + cy\hat{j}$, $c > 0$.

(b) **Uniform rotation**: $\overrightarrow{F}(x,y) = -cy\hat{i} + cx\hat{j}$, $c > 0$.

(c) **Shearing flow**: $\overrightarrow{F}(x,y) = y\hat{i}$.

(d) **Whirlpool effect**: $\overrightarrow{F}(x,y) = \frac{-y}{x^2+y^2}\hat{i} + \frac{x}{x^2+y^2}\hat{j}$.

Find the divergence and curl of each vector field and interpret its physical meaning.



Sol. (a) $\nabla.\overrightarrow{F} = 2c$: The gas is undergoing uniform expansion.

$\nabla \times \overrightarrow{F} = 0$: The gas is not circulating at very small scales.

(b) $\nabla.\overrightarrow{F} = 0$: The gas is neither expanding nor compressing.

$\nabla \times \overrightarrow{F} = 2c\hat{k}$: It indicates uniform counterclockwise rotation at every point.

(c) $\nabla.\overrightarrow{F} = 0$: The gas is neither expanding nor compressing.

$\nabla \times \overrightarrow{F} = -\hat{k}$: It indicates uniform clockwise rotation at every point.

(d) $\nabla.\overrightarrow{F} = 0$: The gas is neither expanding nor compressing.

$\nabla \times \overrightarrow{F} = 0$: The circulation density is 0 at every point away from the origin (where the vector field is undefined and the whirlpool effect is taking place), and the gas is not circulating at any point for which the vector field is defined.

Two forms of Green's Theorem

We have the following two forms of the Green's theorem.

Flux-Divergence or Normal Form

Let $\overrightarrow{F}(x,y) = M(x,y)\hat{i} + N(x,y)\hat{j}$ be a vector field having continuous first order partial derivatives in an open region in xy-plane containing a piecewise smooth simple closed curve C enclosing a region R. Then the outward flux of $\overrightarrow{F}$ across the curve C equals the double integral of the divergence of $\overrightarrow{F}$ over the region R, that is,

$$\oint_C \overrightarrow{F}.\hat{n}ds = \oint_C Mdy - Ndx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy.$$

Circulation-Curl or Tangential Form

Let $\overrightarrow{F}(x,y) = M(x,y)\hat{i} + N(x,y)\hat{j}$ be a vector field having continuous first order partial derivatives in an open region in xy-plane containing a piecewise smooth simple closed curve C enclosing a region R. Then the counterclockwise circulation of $\overrightarrow{F}$ around the curve C equals the double integral of curl$\overrightarrow{F}.\hat{k}$ over R, that is,

$$\oint_C \overrightarrow{F}.\hat{T}ds = \oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy.$$

Ex. Verify both forms of the Green's theorem for the vector field $\overrightarrow{F} = (x-y)\hat{i} + x\hat{j}$ and the region R enclosed by the circle C given by $\overrightarrow{r}(t) = \cos t\hat{i} + \sin t\hat{j}, \quad 0 \leq t \leq 2\pi$.

Sol. Here $M = x - y$ and $N = x$. On C, $x = \cos t$ and $y = \sin t$. So

$$\oint_C (Mdy - Ndx) = \int_0^{2\pi} \cos^2 tdt = \pi.$$

Next,

$$\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy = \iint_R (1+0)dxdy = \pi.$$

$$\therefore \quad \oint_C \overrightarrow{F}.\hat{n}ds = \oint_C Mdy - Ndx = \pi = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy$$

This verifies normal form of Green's theorem.

Similarly, it can be verified that

$$\oint_C \overrightarrow{F}.\hat{T}ds = \oint_C Mdx + Ndy = 2\pi = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy.$$

16.5 Surface and Area

Parametrization of a surface

We know that $\overrightarrow{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ represents a curve in space parametrized by the single parameter t.

For example, $\overrightarrow{r}(t) = (x_0 + lt)\hat{i} + (y_0 + mt)\hat{j} + (z_0 + nt)\hat{k}$ represents the line

$$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n},$$

with direction cosines l, m, n, and passing through the point (x_0, y_0, z_0).

Likewise, $\overrightarrow{r}(u, v) = x(u, v)\hat{i} + y(u, v)\hat{j} + z(u, v)\hat{k}$ represents a surface in space parametrized by the two parameters u and v.

For example, $\overrightarrow{r}(\phi, \theta) = a\sin\phi\cos\theta\hat{i} + a\sin\phi\sin\theta\hat{j} + a\cos\phi\hat{k}$, $0 \leq \phi \leq \pi$, $0 \leq \theta \leq 2\pi$ represents the sphere $x^2 + y^2 + z^2 = a^2$.

If we set $\phi = \pi/2$, then $\overrightarrow{r}(\pi/2, \theta) = a\cos\theta\hat{i} + a\sin\theta\hat{j}$, $0 \leq \theta \leq 2\pi$ represents the circle $x^2 + y^2 = a^2$, as expected.

Obviously, the partial derivative $\overrightarrow{r_\theta}(\pi/2, \theta) = -a\sin\theta\hat{i} + a\cos\theta\hat{j}$, $0 \leq \theta \leq 2\pi$ is tangent vector to the circle $x^2 + y^2 = a^2$.

In general, the vectors $\overrightarrow{r_u}$ and $\overrightarrow{r_v}$ evaluated at some point (u_0, v_0) of the plane of parameters u and v lie in the tangent plane to the surface $\overrightarrow{r}(u, v) = x(u, v)\hat{i} + y(u, v)\hat{j} + z(u, v)\hat{k}$ at the point $(x(u_0, v_0), y(u_0, v_0), z(u_0, v_0))$ of the surface provided $\overrightarrow{r_u}$ and $\overrightarrow{r_v}$ are non-zero at (u_0, v_0). It implies that $\overrightarrow{r_u} \times \overrightarrow{r_v}$ is normal vector to the surface $\overrightarrow{r}(u, v)$.

Smooth surface

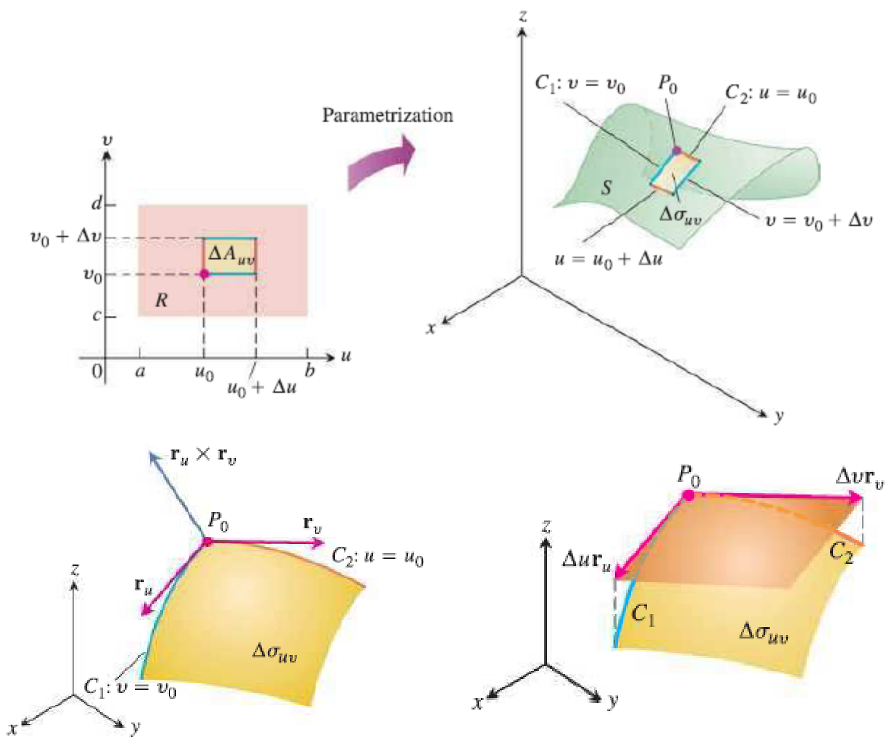
The parametrized surface $\overrightarrow{r}(u, v) = x(u, v)\hat{i} + y(u, v)\hat{j} + z(u, v)\hat{k}$ is said to be smooth if $\overrightarrow{r_u}$ and $\overrightarrow{r_v}$ are continuous with $\overrightarrow{r_u} \times \overrightarrow{r_v} \neq \overrightarrow{0}$ at every point of the parameters domain under consideration.

The condition that $\overrightarrow{r_u} \times \overrightarrow{r_v} \neq \overrightarrow{0}$ in the definition of smoothness implies that the vectors $\overrightarrow{r_u}$ and $\overrightarrow{r_v}$ are non-zero and are not parallel, so these always determine a plane tangent to the surface. We relax this condition on the boundary of the domain, but this does not affect the area computations.

Area of a smooth surface

Consider a smooth surface S parametrized by $\overrightarrow{r}(u, v) = x(u, v)\hat{i} + y(u, v)\hat{j} + z(u, v)\hat{k}$, where u and v belong to some region R in the uv-plane. Divide the region R into meshes by choosing lines parallel to $u-$ and v-axes. Let there be n complete rectangles with dimensions Δu_i and Δv_i $(i = 1, 2, ..., n)$. Let $\Delta\sigma_i$

be the area of the patch on the surface S corresponding to the ith rectangle. We approximate the area $\Delta\sigma_i$ by the area of the parallelogram described by the vectors $\Delta u_i \overrightarrow{r_u}$ and $\Delta v_i \overrightarrow{r_v}$, that lies in the tangent plane to the surface.



So we have

$$\Delta\sigma_i \approx |\Delta u_i \overrightarrow{r_u} \times \Delta v_i \overrightarrow{r_v}| = |\overrightarrow{r_u} \times \overrightarrow{r_v}| \Delta u_i \Delta v_i.$$

So the approximate area of the surface $S = \sum_{i=1}^{n} |\overrightarrow{r_u} \times \overrightarrow{r_v}| \Delta u_i \Delta v_i$. In the limit $n \to \infty$ such that $(\Delta u_i, \Delta v_i) \to (0,0)$, the sum $\sum_{i=1}^{n} |\overrightarrow{r_u} \times \overrightarrow{r_v}| \Delta u_i \Delta v_i$ defines the double integral $\iint_R |\overrightarrow{r_u} \times \overrightarrow{r_v}| dudv$, and gives the area of the surface S, that is,

$$\text{Area of } S = \iint_S d\sigma = \iint_R |\overrightarrow{r_u} \times \overrightarrow{r_v}| dudv.$$

Ex. Find the surface area of the sphere $x^2 + y^2 + z^2 = a^2$.

Sol. We use the following well known parametrization:

$\overrightarrow{r}(\phi, \theta) = a \sin\phi \cos\theta \hat{i} + a \sin\phi \sin\theta \hat{j} + a \cos\phi \hat{k}$, $0 \leq \phi \leq \pi$, $0 \leq \theta \leq 2\pi$.

Then we find

$$|\overrightarrow{r_\phi} \times \overrightarrow{r_\theta}| = a^2 \sin\phi.$$

$$\therefore \quad \text{Area of the sphere} = \int_0^{2\pi} \int_0^{\pi} a^2 \sin\phi \; d\phi d\theta = 4\pi a^2.$$

Ex. Find the area of the curved surface of the cylinder $x^2 + y^2 = 16$, $z = 0$, $z = 3$.

Sol. The given cylindrical surface is parametrized by $\vec{r}(\theta, z) = 4\cos\theta\hat{i} + 4\sin\theta\hat{j} + z\hat{k}$, $0 \leq \theta \leq 2\pi$, $0 \leq z \leq 3$. So $x = 4\cos\theta$, $y = 4\sin\theta$, $|\vec{r_\theta} \times \vec{r_z}| = |4\cos\theta\hat{i} + 4\sin\theta\hat{j}| = 4$, and hence

$$\therefore \quad \text{Area} = \int_0^{2\pi}\int_0^3 4\, d\theta dz = 24\pi.$$

Area of an explicitly defined smooth surface

Suppose a smooth surface S is explicitly defined, say, $z = f(x, y)$. Then its vector equation reads as

$$\vec{r}(x, y) = x\hat{i} + y\hat{j} + f(x, y)\hat{k},$$

which is clearly a parametrization of the surface in terms of two parameters x and y. Obviously, the ranges of these parameters are given by the domain or the projected region R of the surface in the XY-plane.

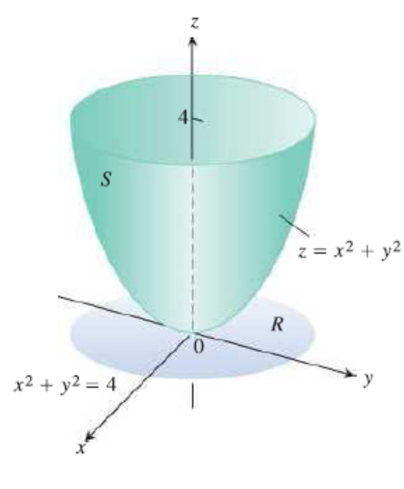
$$\text{Area of } S = \iint_S d\sigma = \iint_R |\vec{r_x} \times \vec{r_y}| dxdy = \iint_R \sqrt{f_x^2 + f_y^2 + 1}\, dxdy.$$

Remark: In case the surface S lies in XY-plane, we have $z = 0$ and $\vec{r}(x, y) = x\hat{i} + y\hat{j}$. Therefore,

$$\text{Area of } S = \iint_S d\sigma = \iint_R |\vec{r_x} \times \vec{r_y}| dxdy = \iint_R |\hat{i} \times \hat{j}|\, dxdy = \iint_R |\hat{k}|\, dxdy = \iint_R dxdy,$$

as expected.

Ex. Find the surface area of paraboloid $z = x^2 + y^2$, $0 \leq z \leq 4$.



Sol. Here $f(x, y) = x^2 + y^2$, and the projected region R in the xy-plane of the given surface is the circle $x^2 + y^2 = 4$. So the required surface area equals to

$$\iint_R \sqrt{f_x^2 + f_y^2 + 1}\, dxdy = \iint_R \sqrt{4x^2 + 4y^2 + 1}\, dxdy = \int_0^{2\pi}\int_0^2 \sqrt{4r^2 + 1}\, r dr d\theta = \frac{\pi}{6}(17\sqrt{17} - 1).$$

16.6 Surface Integrals

Integral over a smooth surface

Let $G(x, y, z)$ be a function defined in an open region containing a smooth surface S. Divide the surface S into n patches with areas $\Delta\sigma_i$ $(i = 1, 2, ..., n)$ such that $\Delta\sigma_i \to 0$ in the limit $n \to \infty$. Choose a point (x_i, y_i, z_i) on the ith patch of S and construct the sum $\sum_{i=1}^{n} G(x_i, y_i, z_i)\Delta\sigma_i$, which defines the integral of $G(x, y, z)$ over the surface S in the limit $n \to \infty$, that is,

$$\iint_S G(x,y,z)d\sigma = \lim_{n\to\infty} \sum_{i=1}^{n} G(x_i, y_i, z_i)\Delta\sigma_i,$$

provided the limit exists finitely.

If the surface S is parametrized by $\overrightarrow{r}(u, v) = x(u, v)\hat{i} + y(u, v)\hat{j} + z(u, v)\hat{k}$, where u and v belong to some region R in the uv-plane, then

$$\iint_S G(x,y,z)d\sigma = \iint_R G(x(u,v), y(u,v), z(u,v))|\overrightarrow{r_u} \times \overrightarrow{r_v}|dudv.$$

If the surface S is explicitly defined by $z = f(x, y)$ and R is its projected region in the xy-plane, then $d\sigma = |\overrightarrow{r_x} \times \overrightarrow{r_y}|dxdy = \sqrt{f_x^2 + f_y^2 + 1}dxdy$, and therefore

$$\iint_S G(x,y,z)d\sigma = \iint_R G(x,y,f(x,y))\sqrt{f_x^2 + f_y^2 + 1}\, dxdy.$$

Ex. Integrate $\sqrt{4x^2 + 4y^2 + 1}$ over the surface of the paraboloid $z = x^2 + y^2$, $0 \leq z \leq 2$.

Sol. The projected region R in the xy-plane of the given surface is the circle $x^2 + y^2 = 2$. So the integral of $\sqrt{4x^2 + 4y^2 + 1}$ equals to

$$\iint_R \sqrt{4x^2 + 4y^2 + 1}\sqrt{f_x^2 + f_y^2 + 1}\, dxdy = \iint_R \sqrt{4x^2 + 4y^2 + 1}\sqrt{4x^2 + 4y^2 + 1}\, dxdy$$

$$= \int_0^{2\pi}\int_0^{\sqrt{2}} (4r^2 + 1)\, rdrd\theta = 2\pi(4 + 1) = 10\pi.$$

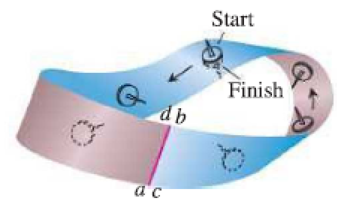
Remark: In case a surface S consists of a number of smooth surfaces say S_1, S_2, ..., S_n, then

$$\iint_S G(x,y,z)d\sigma = \iint_{S_1} G(x,y,z)d\sigma + \iint_{S_2} G(x,y,z)d\sigma + ... + \iint_{S_n} G(x,y,z)d\sigma.$$

For example, a cuboid consists of six smooth surfaces which are the faces of the cuboid.

Orientable surface

A smooth surface S is said to be orientable, if it possesses continuous unique normal vector at each point. For example, the surface of a sphere is orientable. A Mobius band is not orientable surface. Take a strip



of paper, give half twist and join the two ends of the strip to get a Mobius band. By moving continuously along the surface, we can reach to the opposite of starting point. It means there exist two different normal vectors at each point of the surface. So it is not orientable.

If the surface is orientable, conventionally we consider unit normal vector $\hat{n}$ drawn outwards to the surface, and say that the surface is oriented in the direction of $\hat{n}$.

Surface integral of a vector field

Let $\overrightarrow{F}$ be a continuous vector field in an open region containing a smooth surface S oriented in the direction of $\hat{n}$. Then the integral of $\overrightarrow{F}$ over the surface S is defined by

$$\iint_S \overrightarrow{F}.\hat{n}d\sigma.$$

If the surface S is parametrized by $\overrightarrow{r}(u,v) = x(u,v)\hat{i} + y(u,v)\hat{j} + z(u,v)\hat{k}$, where u and v belong to some region R in the uv-plane, then we have

$$\iint_S \overrightarrow{F}.\hat{n}d\sigma = \iint_R \overrightarrow{F}.(\overrightarrow{r_u} \times \overrightarrow{r_v})dudv$$

since $\hat{n} = (\overrightarrow{r_u} \times \overrightarrow{r_v})/|\overrightarrow{r_u} \times \overrightarrow{r_v}|$ and $d\sigma = |\overrightarrow{r_u} \times \overrightarrow{r_v}|dudv$. Note that it is very useful formula for finding surface integral of the vector field over the surfaces, which can be parametrized easily, for example the spherical and cylindrical surfaces.

Flux

Let $\overrightarrow{F}$ be a continuous vector field in an open region containing a smooth surface S oriented in the direction of $\hat{n}$. Then the the flux of $\overrightarrow{F}$ across S is defined as

$$\text{Flux} = \iint_S \overrightarrow{F}.\hat{n}d\sigma = \iint_R \overrightarrow{F}.(\overrightarrow{r_u} \times \overrightarrow{r_v})dudv.$$

Ex. Find the flux of $\overrightarrow{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ across the curved surface of the cylinder $x^2 + y^2 = 16$, $z = 0$, $z = 3$.

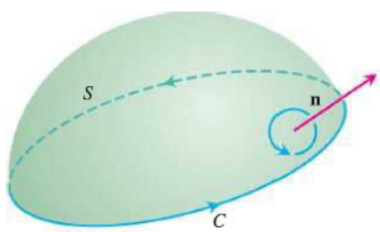
Sol. The given cylindrical surface is parametrized by $\overrightarrow{r}(\theta, z) = 4\cos\theta\hat{i} + 4\sin\theta\hat{j} + z\hat{k}$, $0 \leq \theta \leq 2\pi$, $0 \leq z \leq 3$. So $x = 4\cos\theta$, $y = 4\sin\theta$, $\overrightarrow{r_\theta} \times \overrightarrow{r_z} = 4\cos\theta\hat{i} + 4\sin\theta\hat{j}$, and hence

$$\text{Flux} = \iint_R \overrightarrow{F}.(\overrightarrow{r_\theta} \times \overrightarrow{r_z})d\theta dz = \int_0^3 \int_0^{2\pi} (16\cos\theta\hat{i} - 32\sin^2\theta\hat{j}).(4\cos\theta\hat{i} + 4\sin\theta\hat{j})d\theta dz = 192\pi.$$

16.7 Stokes Theorem

Let $\overrightarrow{F} = M\hat{i} + N\hat{j} + P\hat{k}$ be a vector field having continuous first order partial derivatives in an open region containing a piecewise smooth oriented surface S with piecewise smooth boundary C. Then the circulation of $\overrightarrow{F}$ around C in the direction counterclockwise with respect to the unit normal vector $\hat{n}$ on S equals the integral of $\nabla \times \overrightarrow{F}.\hat{n}$ over S, that is,

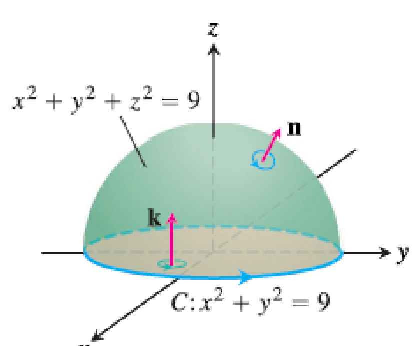
$$\oint_C \overrightarrow{F}.d\overrightarrow{r} = \iint_S \nabla \times \overrightarrow{F}.\hat{n}d\sigma.$$



Notice that if $\overrightarrow{F} = M\hat{i}+N\hat{j}$ and S is the region enclosed by C in the xy-plane, then the above theorem reduces to the tangential form of Green's theorem.

If $\overrightarrow{F} = \nabla\phi$, then it is easy to verify that $\nabla \times \overrightarrow{F} = 0$. Therefore, curl of a conservative vector field always vanishes. On the other hand, Stokes theorem can be used to show that if $\nabla \times \overrightarrow{F} = 0$ in a simply connected open region D, then $\oint_C \overrightarrow{F}.d\overrightarrow{r} = 0$ for any piecewise smooth closed path C inside the region D, which in turn implies that $\overrightarrow{F}$ is conservative in D.

Ex. Verify Stoke's theorem for the vector field $\overrightarrow{F} = y\hat{i} - x\hat{j}$ over the surface of $S : x^2+y^2+z^2 = 9,\ z \geq 0$ with bounding circle $x^2 + y^2 = 9$ in the XY-plane.



Sol. First we calculate the counterclockwise circulation around C (as viewed from above) using the parametrization $\overrightarrow{r}(t) = 3\cos t\,\hat{i} + 3\sin t\,\hat{j}$, $0 \leq t \leq 2\pi$:

$$\oint_C \overrightarrow{F}.d\overrightarrow{r} = \int_0^{2\pi} (3\sin t\hat{i} - 3\cos t\hat{j}).(-3\sin t\,\hat{i} + 3\cos t\,\hat{j})dt = -18\pi.$$

To determine $\iint_S \nabla \times \overrightarrow{F}.\hat{n}d\sigma$, we consider the projection of S over XY-plane. Clearly the projected region is $R : x^2 + y^2 \leq 9$. Here $z = f(x, y) = \sqrt{1 - x^2 - y^2}$. So we have

$$\sqrt{f_x^2 + f_y^2 + 1} = \sqrt{\frac{x^2}{9 - x^2 - y^2} + \frac{y^2}{9 - x^2 - y^2} + 1} = \frac{3}{\sqrt{9 - x^2 - y^2}} = \frac{3}{z}.$$

Next we find $\nabla \times \overrightarrow{F} = -2\hat{k}$ and $\hat{n} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{3}$. So we have

$$\iint_S \nabla \times \overrightarrow{F}.\hat{n}d\sigma = \iint_R \nabla \times \overrightarrow{F}.\hat{n}\sqrt{f_x^2 + f_y^2 + 1}dxdy = \iint_{x^2+y^2\leq 9} \frac{-2z}{3}\frac{3}{z}dxdy = -2(9\pi) = -18\pi.$$

Hence the stoke's theorem is verified.

Attempt the following problems.

1. Verify Stoke's theorem for the vector field
   $\overrightarrow{F} = (2x - y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$ over the upper half surface of $x^2 + y^2 + z^2 = 1$, bounded by its projection on the XY-plane.

   Ans. π.

2. Using Stoke's theorem, evaluate $\oint_C \overrightarrow{F}.d\overrightarrow{r}$, where
   $\overrightarrow{F} = y^2\hat{i} + x^2\hat{j} - (x + z)\hat{k}$ and C is boundary of the triangle with vertices $(0, 0, 0)$, $(1, 0, 0)$ and $(1, 1, 0)$.

   Ans. $1/3$.

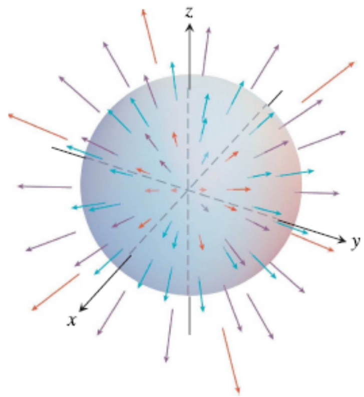
16.8 Divergence Theorem

Let $\overrightarrow{F} = M\hat{i} + N\hat{j} + P\hat{k}$ be a vector field having continuous first order partial derivatives in an open region containing a piecewise smooth oriented closed surface S. Then the flux of $\overrightarrow{F}$ across S in the direction of the outward drawn unit normal vector $\hat{n}$ on S equals the integral of $\nabla.\overrightarrow{F}$ over the region D enclosed by S, that is,

$$\iint_S \overrightarrow{F}.\hat{n}d\sigma = \iiint_D \nabla.\overrightarrow{F}dV.$$

Notice that if $\overrightarrow{F} = M\hat{i} + N\hat{j}$ and D is the region enclosed by S in the xy-plane, then the above theorem reduces to the normal form of Green's theorem.

Ex. Verify Divergence theorem for the expanding vector field $\overrightarrow{F} = x\hat{i} + y\hat{j} + z\hat{k}$ over the surface S of the sphere $x^2 + y^2 + z^2 = a^2$.



Sol. The outer unit normal vector to S is

$$\hat{n} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{a^2}} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{a}.$$

So we have

$$\iint_S \overrightarrow{F}.\hat{n}d\sigma = \iint_S \frac{x^2+y^2+z^2}{a}d\sigma = \iint_S \frac{a^2}{a}d\sigma = a\iint_S d\sigma = a.4\pi a^2 = 4\pi a^3.$$

Next we find $\nabla.\overrightarrow{F} = \nabla.(x\hat{i} + y\hat{j} + z\hat{k}) = 3$. So

$$\iiint_D \nabla.\overrightarrow{F}dV = \iiint_{x^2+y^2+z^2\leq a^2} 3dV = 3\iiint_{x^2+y^2+z^2\leq a^2} dV = 3.\frac{4}{3}\pi a^3 = 4\pi a^3.$$

Hence the divergence theorem is verified.

Attempt the following problems.

1. Verify divergence theorem for $\overrightarrow{F} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$ over the cuboidal region given by $0 \leq x \leq 1$, $0 \leq y \leq 2$, $0 \leq z \leq 3$.

   Ans. 36.

2. Verify divergence theorem for $\overrightarrow{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ over the region bounded by the cylinder $x^2 + y^2 = 16$, $z = 0$, $z = 3$.

   Ans. $0 + 192\pi + 144\pi = 336\pi$. Here 0, 192π and 144π are integrals over the lower circular face, curved face and the upper circular face of the given cylinder, respectively.

3. Using divergence theorem, integrate $\overrightarrow{F} = (2x + 3z)\hat{i} - (xz + y)\hat{j} + (y^2 + 2z)\hat{k}$ over the surface of the sphere having center at origin and radius 3.

   Ans. 108π.