MATH F113 (Probability and Statistics)

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In Lecture 15

Gamma Distribution
Exponential Distribution

Exercise 34: A particular nuclear plant releases a detectable amount of radioactive gases twice a month on average. Find the probability that at least 3 months will elapse before the release of first detectable emission. What is the average time one must wait to observe the first emission?

Solution: Let X time elapsed before the release of the first detectable emission and X is an exponential distribution with $\beta = \frac{1}{2}$. Therefore,

$$f(x) = 2e^{-2x} \qquad x > 0$$

Required Probability,

$$P(x > 3) = 1 - F(3), \textbf{since } F(x) = 1 - e^{-\frac{x}{\beta}}$$

$$P(x > 3) = 1 - (1 - e^{-6}) = e^{-6}$$

$$E(x) = \beta = \frac{1}{2} \textbf{ months}$$

Exercise 37: California is hit every year by approximately 500 earthquakes that are large enough to be felt. However, those of destructive magnitude occur, on the average, once a year. Find the probability that at least 3 months elapse before the first earthquake of destructive magnitude occur.

Solution: X **be the time elapsed before first earthquake of destructive magnitude occur**

$$X \sim \textbf{Exp. Distribution}\left(\beta = \frac{1}{\frac{1}{12}} = 12\right)$$

$$P[X \geq 3] = 1 - F[3] = e^{-\frac{1}{4}} = 0.7788$$

Exercise 35: The average number of lightning strikes on transformers during the severe thunderstorm season in a given area is 2 per week. Assume that a Poisson process is in operation, and find the probability that during the next storm season one must wait at most 1 week in order to see the first transformer strike.

Solution: Let X be the time elapsed until the first transformer strike. Hence, X follows an exponential distribution with $\beta = 1/2$.

Required Probability

$$P[X \leq 1] = F(1) = 1 - e^{-2} = 0.8647$$

Exercise 86: A computer center maintains a telephone consulting service to troubleshoot for its users. The service is available from 9 a.m to 5 p.m. each working day. Past experience shows that the variable X, the number of calls received per day, follows a Poisson distribution with $\lambda = 50$.

Exercise 86: For a given day, find the probability that the first call of the day will be received by 9.15 am; after 3 pm; between 9.30 am and 10 am.

Solution: λ = **50**= **average number of calls per 8 hours**

Hence, λ_s= $50/8 = 6.25 =$ **average number of calls per hour. Let** W **be the time of occurrence of the first call and hence** W **is an exponential distribution with** $\beta = \frac{1}{\lambda_s}$

Prob. that the first call will be recorded by 9.15 am

$$P\left[W \leq \frac{1}{4}\right] = F\left(\frac{1}{4}\right) = 1 - e^{-6.25\frac{1}{4}} = 0.7904$$

Prob. that the first call will be recorded after 3 pm

$$P\left[W > 6\right] = 1 - F(6) = e^{-37.5}$$

Prob. that the first call will be recorded between 9:30 am and 10 am

$$P\left[\frac{1}{2} < W < 1\right] = F(1) - F\left(\frac{1}{2}\right) = 0.042$$

Chi-Squared Distribution: If a random variable X has a gamma distribution with parameters $\beta = 2$ and $\alpha = \frac{\gamma}{2}$, then X is said to have a chi-squared χ^2 distribution with γ degrees of freedom and denoted by χ^2_γ, γ is a positive integer.

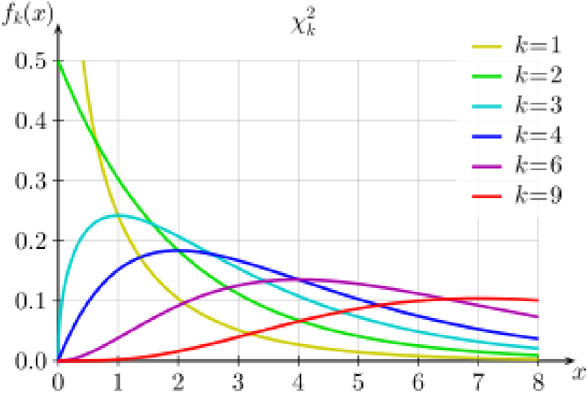
$$f(x) = \frac{1}{\Gamma\left(\frac{\gamma}{2}\right) 2^{\frac{\gamma}{2}}} x^{\frac{\gamma}{2}-1} e^{-\frac{x}{2}}, \qquad x > 0.$$

$$E[\chi^2_\gamma] = \gamma, \quad Var[\chi^2_\gamma] = 2\gamma$$

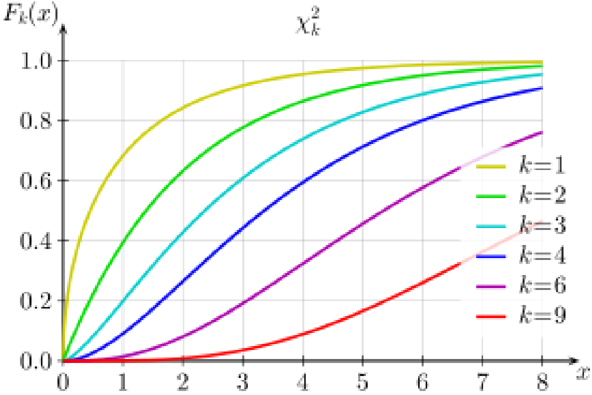
Applications

- Used extensively in applied statistics
- Provides the basis for making inferences about the variance of a population based on a sample.

Chi-squared PDF ($k = \gamma$)



Chi-squared CDF ($k = \gamma$)



There is no explicit formula for cdf F of χ^2_γ. Instead value are tabulated on p. 695-696 as below

	$P(\chi^2_\gamma < t)$			
	F	**0.100**	**0.250**	**0.500**
	5	**1.61**	**2.67**	**4.35**
γ	**6**	**2.20**	**3.45**	**5.35**
	7	**2.83**	**4.25**	**6.35**

(F occurs in margin here, and corresponding value of r.v. inside the table)

Another notation: For $0 < r < 1$, we denote by χ^2_r for a chi-squared r.v. with γ degrees of freedom, a unique number such that

$$P\left[\chi^2_\gamma \geq \chi^2_r\right] = r$$

χ^2_r technically $\chi^2_{r,\gamma}$ is the point such that the area to its right is r

Exercise 38: Consider a chi – squared random variable with 15 degrees of freedom.

(a) What is the mean of χ^2_{15}

Solution: Mean = γ=15 and σ^2 =2γ= 30

(b) What is the expression for the density for χ^2_{15}

Solution: χ^2_γ is a random variable with $\beta = 2$ $\alpha = \frac{\gamma}{2}$ Hence

$$f(x) = \frac{1}{\gamma\left(\frac{15}{2}2^{\frac{15}{2}}\right)} x^{\frac{15}{2}-1} e^{\frac{-x}{2}} \quad x > 0$$

(c) What is the expression for the moment generating function for χ^2_{15}

Solution:

$$m_x(t) = (1-\beta t)^{-\alpha} = (1-2t)^{-\frac{15}{2}}, \quad t < \frac{15}{2}$$

(d)

$$P[\chi^2_\gamma \leq 5.23] = 0.010$$

$$P[\chi^2_\gamma \geq 22.3] = 1 - P[\chi^2_\gamma \leq 22.3]$$
$$= 1 - 0.900 = 0.10$$

$$P[6.26 \leq \chi^2_\gamma \leq 27.5] = F(27.5) - F(6.26)$$
$$= 0.975 - 0.025 = 0.95$$

Chi-squared Distribution (Cont...)

$$\chi^2_{0.05} = 25.0 \quad i.e. \quad \textbf{for} \quad \gamma \ \textbf{degrees of freedon}$$

$$\chi^2_{0.01} = 30.6 \quad (\textbf{Hint: } 1 - 0.01 = 0.99)$$

$$\chi^2_{0.95} = 7.26 \quad (\textbf{Hint: } 1 - 0.95 = 0.05)$$

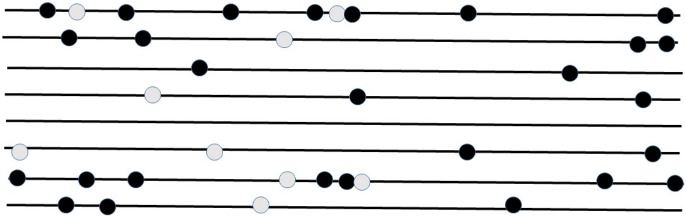
Discrete Distribution

- Bernoulli
- Binomial
- Geometric
- Pascal
- Poisson
- Hypergeometric

Continuous Distribution

- Uniform
- Exponential
- Gamma
- Chi-squared
- Normal
- Lognormal

Discrete events in Continuous time



$X_{Bernoulli(p)}$ = **0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 1, 1, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 1, 0**

$X_{Binomial(5,p)}$ = **1, 1, 1, 1, 2, 1**

$X_{Geometric(p)}$ = **2, 4, 5, 4, 2, 1, 5, 3, 4**

$X_{Pascal(r=3,p)}$ = **12, 9, 14**

$X_{Poisson(k)}$ = **2, 1, 0, 1, 0, 2, 2, 1**

$X_{Hypergeometric(N=36,r=9,n=6)}$ = **2, 1, 1, 2, 1, 1**

Assume that independent Bernoulli trials are being performed

- R.V: Number of occurrences of event A in a fixed number of trials. Distribution: Binomial
- R.V: Number of Bernoulli trials required to obtain first occurrence of event A Distribution: Geometric

- **R.V: Number of Bernoulli trials required to obtain r^th occurrence of event A Distribution: Pascal**

Assume a Poisson Process

- R.V: Number of occurrences of event A during a fixed time interval Distribution: Poisson
- R.V: Time required until first occurrence of event A Distribution: Exponential
- R.V: Time required until r^th occurrence of event A . Distribution: Gamma