innovate achieve lead

BITS Pilani
Pilani Campus

CS/IS F214
Logic in Computer Science

MODULE: PROPOSITIONAL LOGIC

Completeness of Propositional Logic

Completeness

- Definition:
  - A proof system for a logic is said to be ***complete*** if all ***true statements*** in the logic are ***provable*** using the proof system.
  - A logic is said to be ***complete*** if it admits a complete proof system.

Completeness of Propositional Logic

- Theorem (***Completeness of Propositional Logic***):
  - Let $\square_1, \square_2, \ldots \square_n$ and $\square$ be propositional logic formulas.

    If $\square_1, \square_2, \ldots \square_n \models \square$ holds then $\square_1, \square_2, \ldots \square_n \vdash \square$ is valid. i.e.
    - if a propositional logic formula is true given a set of premises
    - then the formula is provable from those premises (using **Natural Deduction**)

Outline of the Proof

- Outline of the proof of completeness of propositional logic:
  - Assume $\square_1, \square_2, \ldots \square_n \models \square$.
  - The proof proceeds in three steps:
    1. Show that $\models \square_1$ --> $(\square_2$ --> $(\square_3$ --> $(\ldots \square n$ --> $\square)\ldots)))$ holds
    2. Show that $\vdash \square_1$ --> $(\square_2$ --> $(\square_3$ --> $(\ldots \square n$ --> $\square)\ldots)))$ is valid (*assuming 1*).
    3. Show that $\square_1, \square_2, \ldots \square_n \vdash \square$ is valid *assuming 2*

Proof of Completeness – Step 1

- Given

  $\square_1, \square_2, \ldots\ \square_n \models \square$

- show that

  $\models \square_1 \text{ --> } (\square_2 \text{ --> } (\square_3 \text{ --> } (\ldots\ \square_n \text{ --> } \square)\ldots)))$

Proof of Completeness – Proof of Step 1

- Given

  $\square_1, \square_2, \ldots \square_n \mid= \square$

- show that

  $\mid= \square_1 \dashrightarrow (\square_2 \dashrightarrow (\square_3 \dashrightarrow (\ldots \square_n \dashrightarrow \square)\ldots)))$

- Proof:
  - If $\square_1 \mid= \square_2$ then when $\square_1$ is true $\square_2$ is also true.

    i.e. $\square_1 \dashrightarrow \square_2$ is true (by truth table).
  - By repeated application of the above step:
    - if $\square_1, \square_2, \ldots \square_n \mid= \square$
    - then $\mid= \square_1 \dashrightarrow (\square_2 \dashrightarrow (\square_3 \dashrightarrow (\ldots \square_n \dashrightarrow \square)\ldots)))$
      - i.e. $\square_1 \dashrightarrow (\square_2 \dashrightarrow (\square_3 \dashrightarrow (\ldots \square_n \dashrightarrow \square)\ldots)))$

        is a ***tautology***, which means it is true under all evaluations.

Proof of Completeness – Step 2

- Theorem T-T:
  - ***Every tautology (in propositional logic) is a theorem*** (i.e. *it is provable*):
    - i.e. if **|=** $\square$ holds then **|-** $\square$ holds.

- Applying Theorem T-T on the tautology from Step 1:
  - Since |= $\square_1$ **--> (**$\square_2$ **--> (**$\square_3$ **--> (…** $\square_n$ **-->** $\square$**)…)))** holds

    |- $\square_1$ **--> (**$\square_2$ **--> (**$\square_3$ **--> (…** $\square$**n -->** $\square$**)…)))** is valid

[Caveat: We have to prove Theorem T-T. End of Caveat]

Proof of Completeness – Step 3

- Given
  - **$\vert$- $\square_1$ --> ($\square_2$ --> ($\square_3$ --> (… $\square$n --> $\square$)…)))** is valid
- show that
  - $\square_1$, $\square_2$, … $\square_n$ **$\vert$-** $\square$ is valid.

Proof of Completeness – Proof of Step 3

- Given
  - **|- □$_1$ --> (□$_2$ --> (□$_3$ --> (... □n --> □)...)))** is valid
- show that
  - **□$_1$, □$_2$, ... □$_n$ |- □** is valid.
- Proof:
  - Consider the proof for the sequent: |- **□1 --> □2**
    - Prefix the proof with **□1** as a premise.
      - Apply --> e to conclude **□2.**
    - This is a proof of the sequent **□1 |- □2**
  - Repeated application of the previous step on the proof of
    |- **□1 --> (□2 --> (□3 --> (... □n --> □)...)))**
  - will yield a proof of
    □1, □2, ... □n |- □