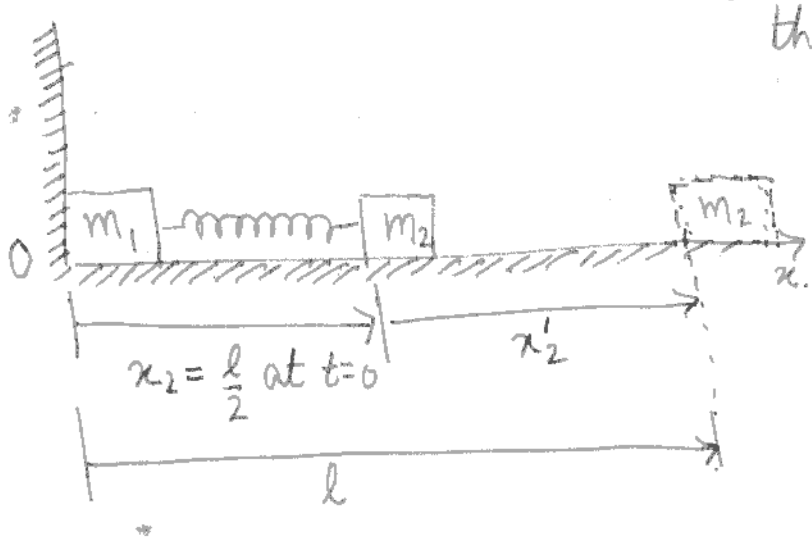
Problem 3.7 (Notes: Rishikesh Vaidya).
(K & K).

A system is composed of two blocks of masses m_1 & m_2, connected by a massless spring with spring constant k. The blocks slides on a frictionless plane. The unstretched length of the spring is l. Initially m_2 is held so that the spring is compressed to $l/2$ and m_2 is forced against a stop. m_2 is released at $t=0$. Find the motion of the center of mass of the system as a function of time.

Solution: When measured from the origin O at the wall.



At $t=0$

$x_1 = 0$

$x_2 = l/2$.

Until m_2 reaches $x_2 = l$ it moves solely under the influence of spring force, with frequency $\omega = \sqrt{k/m_2}$, $T = 2\pi/\omega$.

Hence, for $0 < t < T/4$.

m_1

$\overleftarrow{F_s}\ \overrightarrow{N}$

For m_1 $\quad N - F_s = 0$

so $x_1 = 0$.

For m_2, the equation of motion is

$$m_2 \ddot{x}_2 = k\underbrace{(l - x_2)}_{\text{compression of spring.}}$$

} +ve sign in $+k(l-x_2)$ is because the direction of spring force is same as that of increase in x_2.

Lets say $l - x_2 = x_2' \Rightarrow \ddot{x}_2 = -\ddot{x}_2'$

Thus $m_2 \ddot{x}_2' = -k x_2'$

This is an equation for a simple harmonic motion. It is a 2nd order (in derivative) linear differential equation. It is linear because every term contains first (linear) power in x. It has the property (due to linearity) that any linear combination of solution is also a solution.

Thus the most general solution is

$$x_2' = A\cos\omega t + B\sin\omega t$$

} ∵ sin & cosine both satisfy the linear d.e. above.

A and B are arbitrary constants fixed by initial conditions (values of x_2' and $\dot{x}_2'$ at $t=0$).

$$x_2'(t=0) = \frac{l}{2} = A\cos(\omega.0) + B\sin(\omega.0) = A + 0.$$

$$\Rightarrow \boxed{A = l/2}$$

$$\dot{x}_2'(t=0) = 0 = -A\omega\sin(\omega.0) + B\omega\cos(\omega.0)$$

$$\Rightarrow \boxed{B = 0}$$

Thus $\boxed{x_2' = \frac{l}{2}\cos\omega t}$ & $\boxed{\begin{aligned} x_2 &= l - x_2' \\ &= l\left(1 - \tfrac{1}{2}\cos\omega t\right)\end{aligned}}$

Thus for $0<t<T/4$

$$R_{CM} = \frac{m_2 x_2}{m_1+m_2} = \frac{m_2 l}{m_1+m_2}\left(1-\frac{1}{2}\cos\omega t\right).$$

$$\dot{R}_{CM} = \frac{m_2 \dot{x}_2}{m_1+m_2} = \frac{m_2 l\,\omega}{2(m_1+m_2)}\sin\omega t$$

Now for $t \geq T/4$.

At $t=T/4$, the spring is at its natural length and hence mass m_1 ~~is~~ is free from both, the spring force as well as normal reaction of wall.

For $t \geq T/4$, there are no external forces on the system of masses and the ~~cy~~ center of mass moves with a constant speed it had $t=T/4$.

$$\dot{R}_{CM}\Big|_{t=T/4} = \frac{m_2 l\omega}{2(m_1+m_2)}\sin \omega\cdot\frac{2\pi}{\omega\cdot 4_2} = \frac{m_2 l\omega}{2(m_1+m_2)}$$

$$R_{CM}(t>T/4) = R_{CM}\Big|_{t=T/4} + \dot{R}\Big|_{t=T/4}\times t$$

$$= \frac{m_2 l}{m_1+m_2} + \frac{m_2 l\omega t}{2(m_1+m_2)}.$$

$$\boxed{R_{CM}(t>T/4) = \frac{m_2 l}{m_1+m_2}\left[1+\frac{\omega t}{2}\right]}$$

LECTURE - 6

17/08/16.

→ MASS VARYING SYSTEMS.

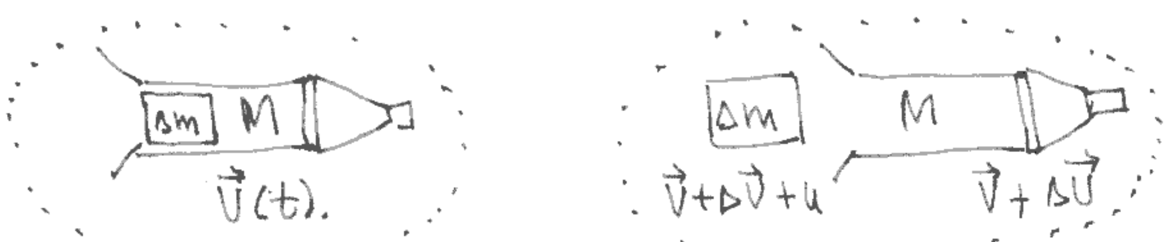
Newton's second law: $\vec{F} = m\vec{a}$.

When m changes with time, this form poses problem.

Better use $\vec{F} = \dfrac{d\vec{P}}{dt}$. How?

- Compare the momentum at two different times (infinitesimally apart), ensuring that you keep same mass at ~~the~~ both the instants, thus also taking account of momentum flowing into or out of the system.
- All measurements of velocity must be referred to inertial frames.

Example: Rocket equation



↑System at t.
Mass: $M+\Delta M$.

↑System at $t+\Delta t$.
Mass: $M+\Delta M$.

$\vec{u}$: velocity of exhaust w.r.t. rocket

$\vec{V}+\Delta\vec{V}+\vec{u}$: velocity of exhaust w.r.t. inertial frame.

Note the vector addition of velocities in $\vec{V}+\Delta\vec{V}+\vec{u}$.

Comparing momentum at t and $t+\Delta t$.

$$\vec{P}(t) = (M+\Delta m)\vec{V}(t)$$

$$\vec{P}(t+\Delta t) = M(\vec{V}+\Delta\vec{V}) + \Delta m(\vec{V}+\Delta\vec{V}+\vec{u}).$$

$$\Delta\vec{P} = \vec{P}(t+\Delta t) - \vec{P}(t).$$

$$= \cancel{M\vec{V}(t)} + M\Delta\vec{V} + \cancel{\vec{V}\Delta m} + \vec{u}\,\Delta m - \cancel{M\vec{V}(t)} - \cancel{\vec{V}(t)\Delta m}$$

(Here we have neglected the product of second order differentials such as $\Delta m\Delta\vec{V}$ which will vanish when we take $\Delta t \to 0$).

$$\Delta\vec{P} = M\Delta\vec{V} + \vec{u}\,\Delta m = \vec{F}_{ext}\,\Delta t.$$

Thus $$\vec{F}_{ext} = \lim_{\Delta t\to 0}\frac{\Delta\vec{P}}{\Delta t} = M\frac{d\vec{V}}{dt} + \vec{u}\,\frac{dm}{dt}.$$

<u>Remarks</u>: 1) Here M denotes the total mass of the rocket + exhaust at any given time t. So $M \equiv M(t)$.

2) $\frac{dm}{dt} \equiv$ rate at which the rocket is losing the mass. Thus expressed in terms of the total instantaneous mass of the rocket, we have $\boxed{\frac{dM}{dt} = -\frac{dm}{dt}}$

3) Note that $\vec{u}$ denotes velocity of exhaust relative to the rocket. May be we can write it as: $\vec{u} = \vec{V}_{rel}$. Thus

$$\boxed{F_{ext} = M(t)\frac{d\vec{V}}{dt} - \vec{V}_{rel}\frac{dM(t)}{dt}}$$

This is Newton's 2nd law applied to any mass varying system.

Example: Rocket moving in a free space.

$$F_{ext} = M(t)\frac{d\vec{V}}{dt} - \vec{V}_{rel}\frac{dM(t)}{dt}$$

0 (free space) $\Rightarrow$

$$M(t)\frac{d\vec{V}}{dt} = \vec{V}_{rel}\frac{dM(t)}{dt}$$

$$\Rightarrow \int_{V_0}^{V_f} dV = \vec{V}_{rel}\int_{M_0}^{M_f}\frac{dM}{M}$$

$$\vec{V}_f - \vec{V}_0 = \vec{V}_{rel}\ln\left(\frac{M_f}{M_0}\right)$$

If $V_0 = 0$ and $\vec{V}_{rel}$ is always opposite to $\vec{V}$.

$$\boxed{V_f = V_{rel}\ln\left(\frac{M_0}{M_f}\right)}$$

The final velocity is thus independent of how the mass is released - rapidly or slowly. It only depends on the exhaust speed and the ratio of initial to final mass.

Rocket in constant gravitational speed ($\vec{F}_{ext} = M\vec{g}$).

$$\vec{F}_{ext} = M(t)\frac{d\vec{V}}{dt} - \vec{V}_{rel}\frac{dM}{dt}$$

$$-\cancel{M}g = \cancel{M}\frac{dV}{dt} + \frac{V_{rel}}{M}\frac{dM}{dt}.$$

$$\int_{V_0}^{V_f} dV = -V_{rel}\int_{M_0}^{M_f}\frac{dM}{M} - g\int_{t_0}^{t_f} dt$$

$$V_f = V_{rel}\ln\left(\frac{M_0}{M_f}\right) - g(t_f).$$

Thus faster you burn your fuel higher is your V_f

3.20 A rocket ascends from rest in a uniform gravitational field by ejecting exhaust with constant speed u. Assume that the rate at which fuel is expelled is $dM/dt = -|r|M$, where M is the instantaneous mass of rocket and $|r|$ is a constant, and that the rocket is retarded by air resistance with a force Mbv, where b is a constant. Find the velocity of rocket as a function of time.

sol.
$$\vec{F}_{ext} = M(t)\frac{dV(t)}{dt} - \vec{V}_{rel}\frac{dM(t)}{dt}.$$

$$-\cancel{M(t)}g - b\cancel{M}v = \cancel{M}\frac{dv}{dt} - (+u)\cdot(+)|r|\cancel{M}.$$

$$-g - bv = \frac{dv}{dt} - ru.$$

$$\frac{dv}{dt} = \underbrace{\underbrace{ru - g}_{\alpha} - bv}_{v'}$$

$\alpha = ru - g$

$v' = \alpha - bv$

$dv' = -b\,dv.$

$v: 0 \to v_f$

$v': \alpha \to \alpha - bv_f.$

$$-\frac{1}{b}\int_{\alpha}^{\alpha - bv_f}\frac{dv'}{v'} = \int dt.$$

$$\ln v'\Big|_{\alpha}^{\alpha - bv_f} = -bt.$$

$$\frac{\alpha - bv_f}{\alpha} = e^{-bt}.$$

$$\frac{\alpha}{b}[1 - e^{-bt}] = v_f.$$

$$\Rightarrow \boxed{\frac{ru - g}{b}\left[1 - e^{-bt}\right] = v_f}.$$