COMPLEX ANALYSIS

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TEXT BOOK:

- Complex Variable & Applications

- 8^{th} Edition

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Complex Number: A complex number z is an ordered pair (x, y), where x & y are real nos. i.e.

$$z = (x, y), \text{ where}$$

x = real part of z = Re z

y = imaginary part of z = Im z

We usually write

$$z = (x, y) = x + i\,y,$$

$$\text{where } i = \sqrt{-1} = (0, 1)$$

$$i^2 = i.\,i = (0, 1)\,.\,(0, 1) = (-1, 0)$$

- (WAIT !)

Important Operations

1. ***Addition of complex numbers:***

$$z_1 + z_2 = (x_1+iy_1) + (x_2+iy_2)$$

$$=(x_1+x_2)+i(y_1+y_2)$$

2. ***Multiplication of complex numbers:***

$$z_1\, z_2 = (x_1+iy_1)\,(x_2+iy_2)$$

$$=(x_1x_2 - y_1y_2)+i(x_1y_2+x_2y_1)$$

3. Division:

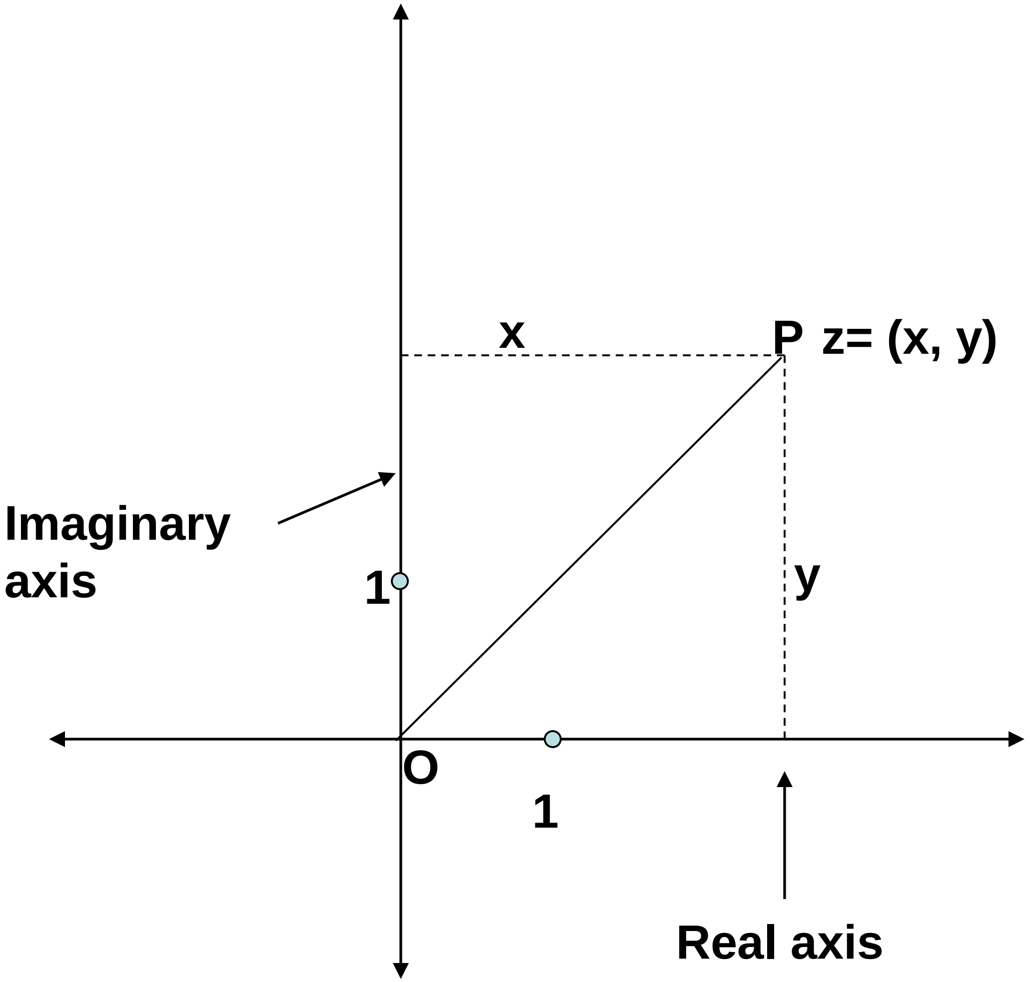
If $z_1 = x_1 + iy_1 \quad \&$

$z_2 = x_2 + iy_2 \neq 0 + i.0, then$

$$z = \frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{x_1 + iy_1}{x_2 + iy_2} \times \frac{x_2 - iy_2}{x_2 - iy_2}$$

$$= \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}$$

Complex Plane :



Complex Plane:

- **Choose the same unit of length on both the axes**

- **Plot $z = (x, y) = x + iy$ as the point P with coordinates x & y.**

- **The xy-plane, in which the complex nos. are represents in this way, is called *complex plane or Argand diagram .***

Equality of two complex nos:

Two complex nos. z_1 & z_2 are said to be equal iff

$$Re(z_1) = Re(z_2) \;\&$$

$$Im(z_1) = Im(z_2).$$

Properties of Arithmetic operations:

(1) *Commutative Law:*

$$z_1+z_2 = z_2+z_1$$

$$z_1 z_2 = z_2 z_1$$

2. *Associative law:*

$$(z_1+z_2)+z_3=z_1+(z_2+z_3)$$
$$(z_1 z_2)\, z_3 = z_1\,(z_2 z_3)$$

3. *Distributive law*

$$z_1(z_2 + z_3)= z_1 z_2 + z_1 z_3$$
$$(z_1+z_2)z_3= z_1 z_3 + z_2 z_3$$

4. $z + (-z) = (-z) + z = 0$

5. $z.1 = z$

- **Complex conjugate number:**

  Let $z = x+iy$ be a complex number.

  Then $\bar{z} = x-iy$ is called complex conjugate of z

Properties of complex nos.:

1. $z + \bar{z} = 2x$

$$\Rightarrow x = \operatorname{Re} z = \frac{1}{2}(z + \bar{z})$$

2. $y = \operatorname{Im} z = \dfrac{1}{2i}(z - \bar{z})$

3. $$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$$

4. $$\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$$

5. $$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$$

6. $\bar{\bar{z}} = z$

7. z is real iff $z = \bar{z}$.

8. $\overline{iz} = \bar{i}\,\bar{z} = -i\,\bar{z}$

9. $\operatorname{Re}(iz) = -\operatorname{Im}(z)$, $iz = ix - y$

10. $\operatorname{Im}(iz) = \operatorname{Re}(z)$

11. $z_1 z_2 = 0 \Rightarrow z_1 = 0 \text{ or } z_2 = 0$

Polar Form of complex Numbers:

Let $z = x+iy$

Put $x = r\cos\theta$, $y = r\sin\theta$

$$\therefore z = r(\cos\theta + i\sin\theta) = r e^{i\theta}$$

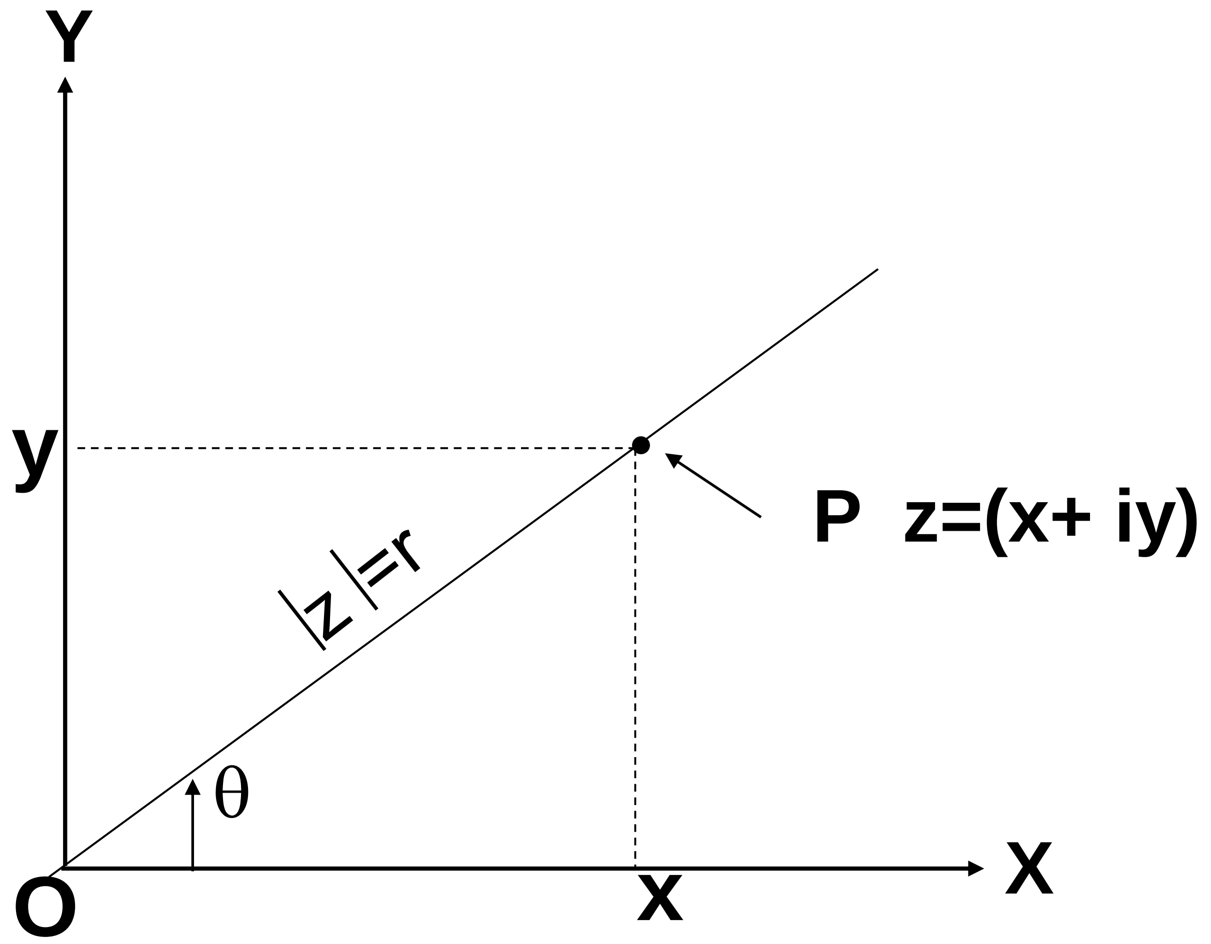
which is called polar form of

complex number.

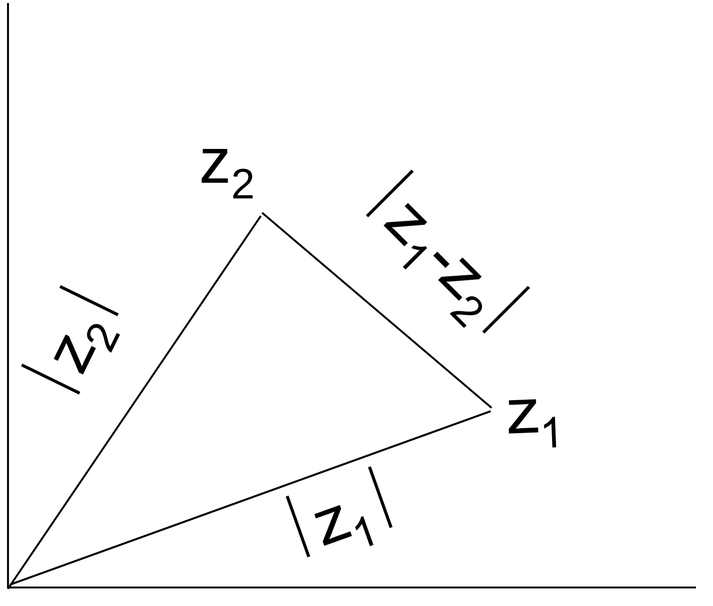
MODULUS OF COMPLEX NUMBER

$$|z| = r = \sqrt{x^2 + y^2} \geq 0$$

Geometrically, $|z|$ is the distance of the point z from the origin.

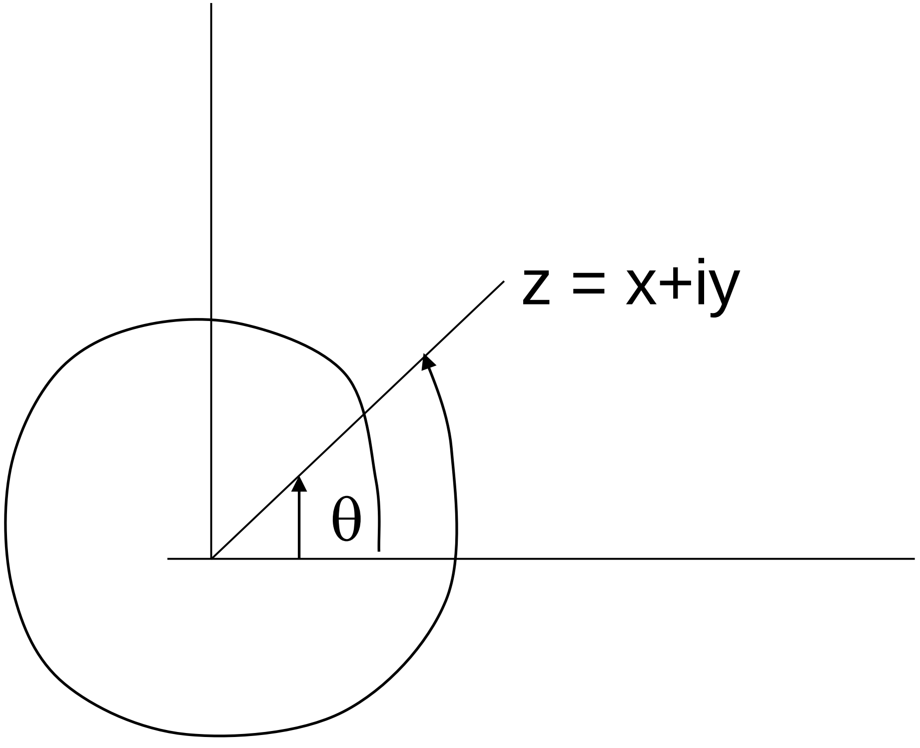

Y
y
|z|=r
P z=(x+ iy)
θ
O
x
X

- $|z_1| > |z_2|$ means that the point z_1 is farther from the origin than the point z_2.
- $|z_1 - z_2|$ = distance between z_1 & z_2



ARGUMENT OF COMPLEX NUMBER

The directed angle θ measured from the positive x-axis is called the argument of z, and we write $\theta = \arg z$.



- ***Remarks :***

1. For $z = 0$, θ is undefined.

2. θ is measured in radians, and is positive in the counterclockwise sense.

3. θ has an infinite number of possible values, that differ by integer multiples of 2π. Each value of θ is called argument of z, and is denoted by $\theta = \arg z$

4. When θ is such that $-\pi < \theta \leq \pi$, then such value of θ is called principal value of arg z, and is denoted by

$$\Theta = \text{Arg } z, \text{ if } -\pi < \Theta \leq \pi$$

5. $\arg z = \text{Arg } z + 2n\pi$, $n = 0, \pm 1, \pm 2, \ldots\ldots\ldots$

6. Let $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$.

Then $z_1 = z_2 \Leftrightarrow (i)\ r_1 = r_2$ &

$(ii)\ \theta_1 = \theta_2 + 2n\pi$

$n = 0, \pm 1, \pm 2, \ldots..$

7. $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

How to find argz / Argz ?

Ex1. Let $z=-1+i, \quad Argz=?$

Sol:

We have

$$z=-1+i=r(\cos\theta+i\sin\theta)$$

$$\Rightarrow |z|=r=\sqrt{2}$$

$$\therefore -1+i=\sqrt{2}(\cos\theta+i\sin\theta)$$

$$\Rightarrow \sqrt{2}\cos\theta = -1, \quad \sqrt{2}\sin\theta = 1$$

$$\Rightarrow \tan\theta = -1$$

$$\Rightarrow \theta = \Theta = Argz = 3\pi/4$$

Hence

$$\arg z = Argz + 2n\pi,\ n = 0, \pm 1, \pm 2, ..$$

$$= (3\pi/4) + 2n\pi,\ n = 0, \pm 1, \pm 2, ..$$

Ex2. Let $z=-2i, Arg z = ?$

Sol:

We have

$$z=-2i=r(\cos\theta+i\sin\theta)$$

$$\Rightarrow |z|=r=2$$

$$\therefore -2i=2(\cos\theta+i\sin\theta)$$

$$\Rightarrow 2\cos\theta = 0, \quad 2\sin\theta = -2$$

$$\Rightarrow \theta = \Theta = Argz = -\pi / 2$$

Hence

$$\arg z = (-\pi / 2) + 2n\pi,\ n = 0, \pm 1, \pm 2, ..$$

Roots of Complex Numbers:

For $z_0 \neq 0$, there exists n values of

z which satisfy $z^n = z_0$

$$\text{Let} \quad z = re^{i\theta} \Rightarrow z^n = r^n e^{in\theta}$$

$$Let \quad z^n = z_0 = r_0 e^{i\theta_0}, n = 2, 3, \ldots\ldots$$

$$Then \quad r^n e^{in\theta} = r_0 e^{i\theta_0}$$

$$\Rightarrow r^n = r_0,$$

$$n\theta = \theta_0 + 2k\pi,$$

$$\Rightarrow r = (r_0)^{1/n}, \theta = \frac{\theta_0 + 2k\pi}{n}$$

$$\therefore z = \mathrm{r\,e}^{\,\mathrm{i}\theta}$$

$$\Rightarrow z = z_k = (r_0)^{\frac{1}{n}} e^{i(\frac{\theta_0 + 2k\pi}{n})}$$

is called nth roots of $z_0, k = 0,1,..,n-1.$

Principal Root.

For $\mathrm{k} = 0,$

$$z_0 = (r_0)^{1/n} \; e^{i\theta_0 / n}$$

is called the PRINCIPAL ROOT.

Triangular inequality:

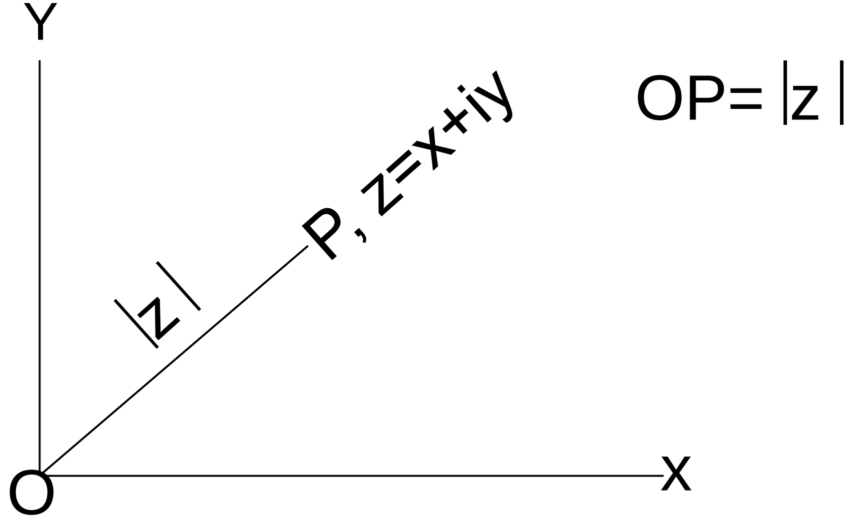
1. $|z_1 + z_2| \le |z_1| + |z_2|$

2. $|z_1 - z_2| \le |z_1| + |z_2|$

3. $|z_1 + z_2| \ge |z_1| - |z_2|$

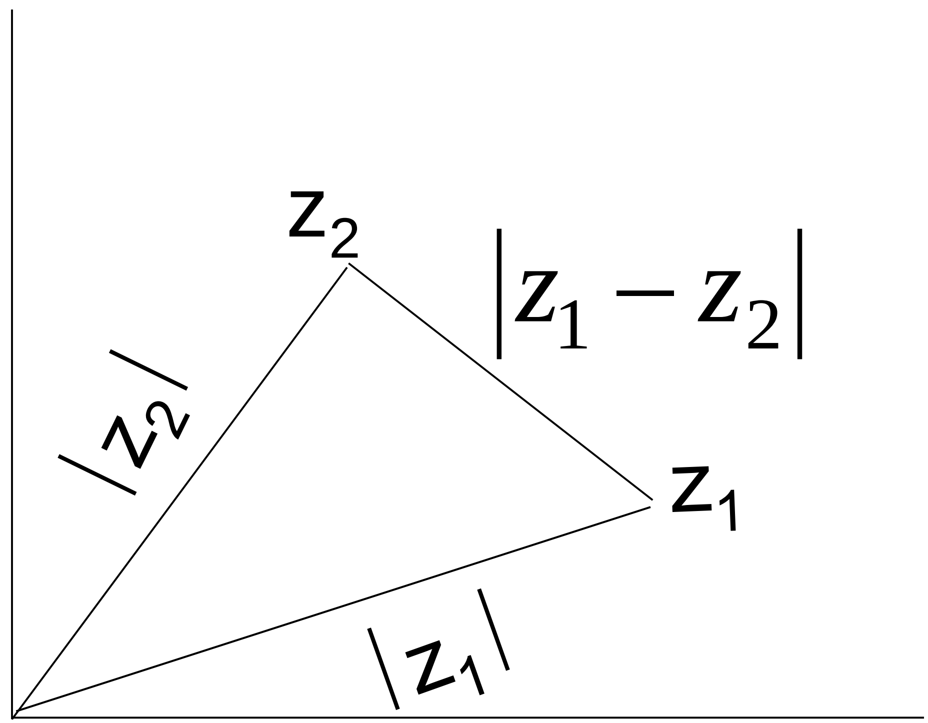
4. $|z_1 - z_2| \ge \big||z_1| - |z_2|\big|$

Let $z = x+iy$, Then $|z|$ is the distance of the point P (x,y) from the origin

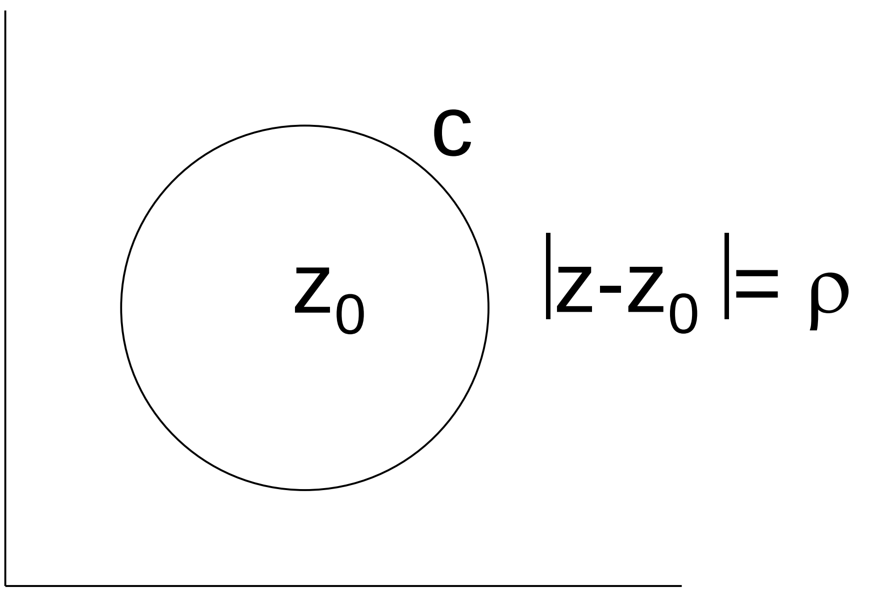


If $z_1 = x_1 + iy_1 \quad and\ z_2 = x_2 + iy_2,$

then $|z_1 - z_2| = \text{distance between } z_1 \ \& \ z_2.$



Let C be a circle with centre z_0 and radius ρ. Then such a circle C can be represented by C: $|z-z_0| = \rho$.



Consequently, the inequality

$$|z-z_0| < \rho \quad \text{-----------(1)}$$

holds for every z inside C.

i.e. (1) represents the interior of C.

Such a region, given by (1), is called a ***neighbourhood (nbd) of z_0,*** i.e. the set

$$N(z_0) = \{z : |z - z_0| < \rho\}$$

is called a nbd. of z_0

Deleted neighborhood:

$N_0 = \{z: 0 < |z-z_0| < \rho\}$ is called ***deleted nbd.***

It consists of all points z in an ρ -nbd of z_0, except for the point z_0 itself.

- The inequality $|z-z_0| > \rho$

  represents the exterior of the

circle C.

Interior Point:

Let S be any set. Then a point $z_0 \in S$ is called an interior point of S if $\exists$ a nbd $N(z_0)$ that contain ***only points of S***, i.e.

$$z_0 \in N(z_0) \subseteq S$$

Exterior Point: A point z_0 is called an exterior point of the set S if $\exists$ a nbd N of z_0 that contains ***no points of S***.

z_0 is an ext. pt. of S $\Leftrightarrow$ z_0 is an int. pt of S^c.

Boundary point:

A point z_0 is called boundary point for the set S if it is neither interior point nor exterior point of S.

Open Set:

A set S is said to be open if every point of S is an interior point of S, i.e. S is open iff it contains none of its boundary points.

Closed set:

A set S is said to be closed if its complement S^c is open, i.e. S is closed iff it contains all of its boundary points.

Closure of a set:

- Closure of a set S is the closed set consisting of all points in S together with the boundary of S.

$$Ex1. \quad Let \quad S = \{z : |z| < 1\}.$$

$$Then \quad Cl(S) = \{z : |z| \leq 1\}.$$

$$Ex2. \quad Let \quad S = \{z : |z| \leq 1\}.$$

$$Then \quad Cl(S) = \{z : |z| \leq 1\}.$$

Bounded set:

A set S is called bounded if all of its points lie within a circle of sufficiently large radius, otherwise it is unbounded.

Connected Set:

An open set S is said to be connected if any of its two points can be joined by a broken line of finitely many line segments, all of whose points belong to S.

- Q. Is the set

$$S=\{z:|z|<1\}\cup\{z:|z-2|<1\}$$

connected ?

Domain:

An open connected set is called a domain.

Ex1: Sketch & determine which are domains

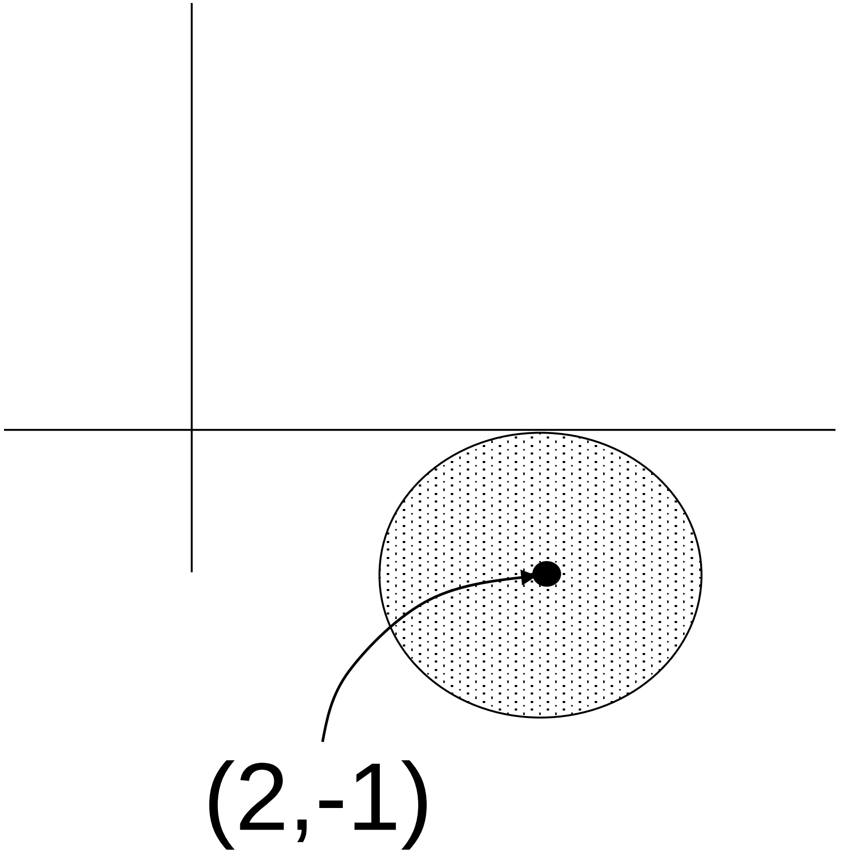
(a) $\mathbf{S = \{z : |z-2+i| \leq 1\}}$

We have $|z-2+i| \leq 1$

$\Rightarrow |x+iy-2+i| \leq 1$

$\Rightarrow |(x-2)+i(y+1)| \leq 1$

$\Rightarrow (x-2)^2 + (y+1)^2 \leq 1$



$\Rightarrow$ S contains the interior & boundary pts. of a circle with centre (2, -1) & radius 1.

$\Rightarrow$ (i) S is not a domain

(ii) S is bounded.

Ex2. **S = { z: $|2z+3|>4$ }**

We have $|2z+3|>4$

$\Rightarrow |2x+3+ i\,2y| > 4$

$\Rightarrow (2x+3)^2 + 4y^2 > 16$

$\Rightarrow (x+3/2)^2 + y^2 > 4$

- Clearly S contents the exterior pts of a circle with centre $(-\frac{3}{2},0)$ & radius 2.
- S is a domain and it is unbounded

Ex. 3 $S = \left\{ z : \left| \frac{z+1}{z-1} \right| < 1 \right\}$

Sol. Note that : $|z+1| < |z-1|$

$$\Rightarrow |z+1|^2 < |z-1|^2$$

$$\Rightarrow (z+1)(\bar{z}+1) < (z-1)(\bar{z}-1)$$

$$\Rightarrow x < 0.$$

S is a domain and it is unbounded.

END