# MATH F113 (Probability and Statistics)

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# In Lecture 11

Application of Discrete Distribution
Binomial Approximation
Poisson Approximation

# Chapter 4
# Continuous Distribution

# Continuous Densities

- Consider the random variable $T$, the time of the peak demand for electricity at a particular power plant.
- Here, we cannot limit the set of possible values for $T$ to some finite or countable infinite collection of times.

## Continuous Densities

- **Time $T$ can conceivably assume any value in the time interval $[0, 24)$, where 0 denotes mid night of one day and 24 denotes 12 mid night of next day.**

# Continuous Densities

- Further, we could pose the question, what is the probability that the peak demand will occur exactly $12.013278\ldots$?
- It is virtually impossible for the peak load to occur at this split. Hence the answer is zero.

# Continuous Densities

- Suppose the range of $X$ is made up of large finite number of values say, $X$ in $0 \leq x \leq 1$ of the form 0, 0.01, 0.02,. . . ,0.99, 1.0. Each of the values is associated with a non-negative number whose sum is 1.
- $X$ can assume all possible values $0 \leq x \leq 1$.

# Continuous Densities

- **Since possible values of $X$ are non countable in $0 \leq x \leq 1$, then what happens to point probabilities? We can't really speak of $i^{th}$ value of $X$ for all $X$, $P[X = x_i]$ becomes meaningless**

Definition

**Continuous Random Variable**

**A random variable is continuous if it can assume any value in some interval (or intervals) of real numbers and the probability that it assume any specific value is 0 (zero).**

### Definition

## Continuous Density

**Let $X$ be a continuous random variable. A function $f(x)$ is called continuous density (probability density function i.e pdf ) if**

**(i)** $f(x) \geq 0$

**(ii)** $\int_{-\infty}^{+\infty} f(x)dx = 1$

**(iii) For any $a, b$ (real) with $-\infty < x < +\infty$ we have**

$$P[a \leq x \leq b] = \int_a^b f(x)dx$$

**Remark 1: If $f(x)$ is not pdf (or density) of $X$ if**

$$\int_a^b f(x)dx = k,$$

$k$ **is not one, then** $\frac{f(x)}{k}$ **is the pdf of** $X$.

**Remark 2:** $X$ **is continuous r.v,** $X$ **assumes all values in** $(a, b)$**, where** $a, b$ **may be replaced by** $-\infty$ **and** $+\infty$ **respectively. We are considering the idealized description of** $X$**.**

**Remark 3:** It is a consequence of the probabilistic description that for any specified value of $X$, say $x_0$, we have $P[X = x_0] = 0$, since

$$P[X = x_0] = \int_{x_0}^{x_0} f(x)dx = 0$$

**Remark 4: However, if you allow $X$ to assume all values in some interval, then probability zero is not equivalent with impossibility. Hence for continuous case $P(A) = 0$ does not imply $A = \Phi$(empty set)**

**Remark 5:** **If** $X$ **assumes values in some finite interval** $[a, b]$**, we simply set** $f(x) = 0$ **for all** $x \notin [a, b]$**.**

**Remark 6:** Consider the line segment

$$\{X \,|\, 0 \leq x \leq 2\}$$

Every conceivable point on the line segment could be the outcome of the experiment.

**Since $X$ is continuous $r.v$, we have**

$$
\begin{aligned}
&P\left[a \le x \le b\right] \\
&= P\left[a \le x \;\; < b\right] \\
&= P\left[a < x \le b\right] \\
&= P\left[a < x < b\right]
\end{aligned}
$$

**Example:** Suppose that the r.v $X$ is continuous, Let the pdf $f$ is given by:

$$f(x) = \begin{cases} 2X & \mathbf{0 < x < 1} \\ 0, & \textbf{elsewhere} \end{cases}$$

Clearly, $f(x) \geq 0$ and also

$$\int_{-\infty}^{\infty} f(x)dx = \int_{0}^{1} 2xdx = 1$$

**To compute**

$$P[X \leq 1/2] = \int_{0}^{1/2} 2x dx = \frac{1}{4}$$

**Conditional Probability: For example,**

$$P[X \leq 1/2 | 1/3 \leq X \leq 2/3],$$

$$= \frac{P[1/3 \leq X \leq 1/2]}{P[1/3 \leq X \leq 2/3]}$$

$$= \frac{\int_{1/3}^{1/2} 2x dx}{\int_{1/3}^{2/3} 2x dx} = \frac{5}{12}$$

## Definition

Let $X$ be the continuous r.v. with density $f(x)$. The **cumulative distribution function (cdf) for $X$, denoted by $F(X)$, is defined by**

$$F(X) = P(X \leq x) \quad \forall \quad x$$

$$= \int_{-\infty}^{x} f(s)ds$$

**PROBABILITY by using cdf $F(x)$**

$$P(a \leq X \leq b) = F(b) - F(a).$$

**For example, in the previous case, we have**

$$f(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & elsewhere \end{cases}$$

**Then, we can compute** $F(x)$**.**

$$F(x) = \begin{cases} 0 & \textbf{if} \quad x < 0 \\ \int_0^x 2s ds = x^2 & \textbf{if} \quad 0 \leq x < 1 \\ 1 & \textbf{if} \quad x \geq 1 \end{cases}$$

**Theorem**

*Let $F$ be the continuous cdf of a continuous r.v with pdf $f$, then*

$$f(x) = \frac{d}{dx} F(X)$$

*for all $x$ at which $F$ is differential*

# Exercise 1/4.1/pp 138

**Consider the function**

$$f(x) = k\,x \quad 2 \leq x \leq 4$$

**(a) Find the value of $k$ that makes this a density for a continuous random variable.**

$$\int_2^4 kx dx = 1 \implies k = \frac{1}{6}$$

**(b) Find** $P[2.5 \leq X \leq 3]$

$$P\left[2.5 \leq x \leq 3\right] = \int_{2.5}^{3} \frac{1}{6} x dx = 0.2292$$

**(c) Find** $P[X = 2.5]$

$$P\left[x = 2.5\right] = \int_{2.5}^{2.5} \frac{1}{6} x dx = 0$$

**(d) Find** $P[2.5 < X \leq 3]$

$$P\left[2.5 < x \leq 3\right] = P\left[2.5 \leq x \leq 3\right] = 0.2292$$

## Exercise 3/4.1/pp. 139

Let $X$ denote the length in minutes of a long distance telephone conversation. Assume that density for $X$ is given by

$$f(x) = \frac{1}{10}e^{-\frac{x}{10}}, \quad x > 0$$

# Exercise 3/4.1/pp. 139

**(a) Verify that $f$ is a density for a continuous random variable.**

$f(x) > 0$, **since for all** $x$, $e^{-\frac{x}{10}} > 0$ **and**

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx = 1$$

**Hence,** $f(x)$ **is a pdf**

**(b) Assume that $f$ adequately describes the behavior of the random variable $X$, find the probability that a randomly selected call will last at most 7 minutes; at least 7 minutes; exactly 7 minutes.**

**At most 7 minutes, we have**

$$P[X \le 7] = \int_{-\infty}^{0} f(x)dx + \int_{0}^{7} f(x)dx$$

$$= 0 + \int_{0}^{7} \frac{1}{10} e^{\frac{-x}{10}} dx = 0.5034$$

**Exactly 7 minutes, we have**

$$P\left[X = 7\right] = 0$$

**Probability of at least 7 minutes, we have**

$$P\left[x \geq 7\right] = 1 - P\left[x \leq 7\right] = 1 - 0.5034 = 0.4966$$

**(c) Would it be unusual for a call to last between 1 and 2 minutes? Explain, based on the probability of this occurring.**

**We have**

$$P\left[1 < x < 2\right] = \int_{1}^{2} \frac{1}{10} e^{-\frac{x}{10}} dx = 0.09861$$

**Which is relatively small value**

**(d) What is c.d.f for the above p.d.f? Given that**

$$f(x) = \frac{1}{10} e^{-\frac{x}{10}}, x > 0$$

$$F(x) = \int_{0}^{x} \frac{1}{10} e^{-\frac{t}{10}} dt$$

$$= 1 - e^{-\frac{x}{10}}, x > 0$$

**Hence**

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\frac{x}{10}} & x \geq 0 \end{cases}$$

**Exercise:** **Find the CDF of the following density function**

**(i)**

$$f(x) = \begin{cases} 1/3 & 0 \leq x \leq 1 \\ 2/3 & 1 < x \leq 2 \\ 0 & \textbf{elsewhere} \end{cases}$$

**(ii)**

$$f(x) = \begin{cases} |x| & -1 < x < 1 \\ 0 & \textbf{elsewhere} \end{cases}$$