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MATH F111 (Mathematics-I)

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Lecture 12-15 (Chapter-13)
Vector Valued Functions and Motion in Space

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Notice for Remedial Classes

Dear Students,

If any of you wish for remedial classes of any subject you are studying (Including Mathematics-1), then please send an email to ***Dr. Ashish Tiwari*** latest by ***Saturday, 9th September, 2017*** on:

ashish.tiwari@pilani.bits-pilani.ac.in

Review from Senior Secondary Class

Dot Product of two vectors

Cross product of two vectors

Limit of a real valued function $f(x)$ at the point $x = a$ defined on an interval.

Continuity and Differentiability of a real valued function $f(x)$ at the point $x = a$ defined on an interval.

Vector Valued Function

A function of the form :

$$\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k},\ t \in I$$

is called vector - valued function or a vector function. Its domain is the subset of real numbers and range is in the set of vectors in three dimension.

Vector Valued Function

- The functions f, g and h are called component functions of $\vec{r}$, and are real valued functions.

- A space curve (Curve in 3D space) is traced out by points $(f(t), g(t), h(t))$. It has direction determined by giving increasing values of t in an interval I.

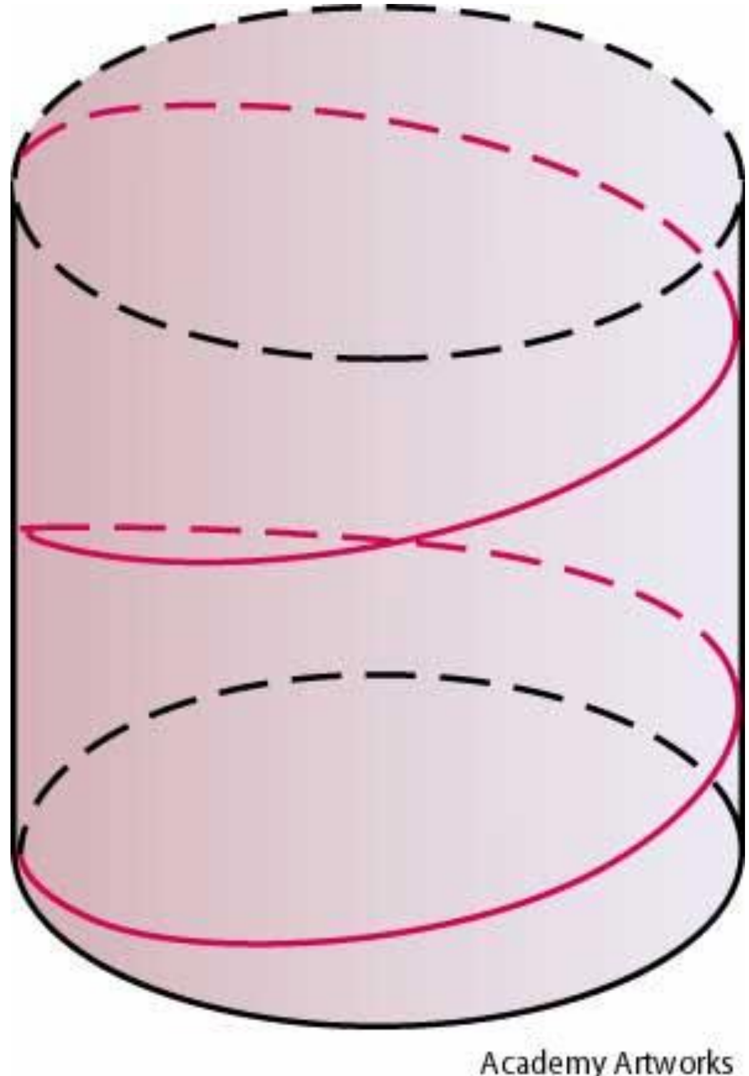
- Instead of $(f(t), g(t), h(t))$ sometimes $\vec{r}(t) = (x(t), y(t), z(t))$ is as well written

Vector Valued Function

Example: The vector function:

$$\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + t\hat{k}$$

The curve traced by $\vec{r}$ winds around the circular cylinder $x^2 + y^2 = 1$. The curve rises as the k component $z = t$ increases. The curve is called "Helix"



Limit and Continuity of Vector Valued Function

Limit : If $\vec{r}(t) = (x(t), y(t), z(t))$,

then $\lim\limits_{t \to a} \vec{r}(t) = \lim\limits_{t \to a} x(t)\hat{i} + \lim\limits_{t \to a} y(t)\,\hat{j} + \lim\limits_{t \to a} z(t)\hat{k}$,

provided all the limits of component functions exist

Continuity : A vector valued function $\vec{r}$ is continuous at a point $t = a$ if and only if

$$\lim_{t \to a} \vec{r}(t) = \vec{r}(a)$$

Thus $\vec{r}$ is continuous at a point $t = a$ if and only if each component function is continuous at $t = a$.

Differentiation of Vector Valued Function

$\vec{r}$ is continuous on an open interval I if it is continuous for all points in I.

Derivative : A vector valued function $\vec{r}$ has a derivative (is differentiable) at a point $t = a$ if and only if each component function have derivative at $t = a$. The derivative is the vector function:

$$\vec{r}\,'(t) = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

Differentiation of Vector Valued Function

Let $\vec{r}(t)$ be the position vector of a particle at time t moving along a smooth curve in space. Then

1. Velocity : $\vec{v} = \dfrac{d\vec{r}}{dt}$
2. Speed : $s = |\vec{v}|$
3. Acceleration : $\vec{a} = \dfrac{d\vec{v}}{dt} = \dfrac{d^2\vec{r}}{dt^2}$
4. Direction of motion : $\dfrac{\vec{v}}{|\vec{v}|}$

Differentiation Rules of Vector Valued Function

Smooth Curve: Curve traced by $\vec{r}(t)$ is smooth if $\dfrac{d\vec{r}(t)}{dt}$ is continuous and never 0, that is $x(t)$, $y(t)$ and $z(t)$ have continuous first derivatives that are not simultaneously zero.

Let $\vec{u} = \vec{u}(t)$ and $\vec{v} = \vec{v}(t)$ be differentiable vector functions of t, $\vec{C}$ a constant vector and a any real number and f any real valued function defined on a interval.

Differentiation Rules of Vector Valued Function

1. $\frac{d}{dt}\vec{C} = 0 \quad \vec{C}$ is a constant vector.

2. $\frac{d}{dt}(a\vec{u}(t)) = a\frac{d}{dt}(\vec{u}(t)),\ a$ constant scalar.

3. $\frac{d}{dt}(f(t)\vec{u}(t)) = f'(t)(\vec{u}(t))$

$+ f(t)\frac{d}{dt}(\vec{u}(t)),\ f(t)$ real valued function.

Differentiation Rules of Vector Valued Function

4. $$\frac{d}{dt}(\vec{u}(t).\vec{v}(t))$$

$$=\left(\frac{d}{dt}(\vec{u}(t))\right).\vec{v}(t)+\vec{u}(t).\frac{d}{dt}(\vec{v}(t))$$

5. $$\frac{d}{dt}(\vec{u}(t)\times\vec{v}(t))$$

$$=\left(\frac{d}{dt}(\vec{u}(t))\right)\times\vec{v}(t)+\vec{u}(t)\times\frac{d}{dt}(\vec{v}(t))$$

Differentiation Rules of Vector Valued Function

6. Chain Rule: $\dfrac{d\vec{r}}{ds} = \left(\dfrac{d\vec{r}}{dt}\right)\left(\dfrac{dt}{ds}\right)$

where t is a differentiable function of s.

7. If $\vec{r}(t)$ is a differentiable vector function of t of **constant length,** then

$$\vec{r}\,.\frac{d\vec{r}}{dt} = 0.$$

Differentiation Rules of Vector Valued Function

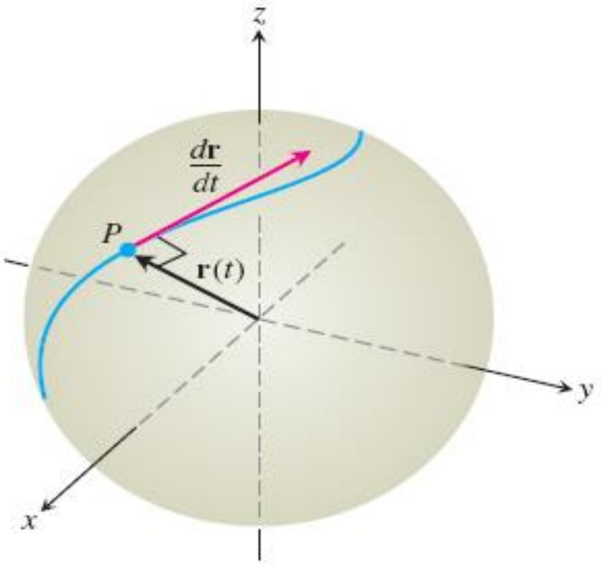
$$\text{Proof : } \left|\vec{r}(t)\right| = c\,(\text{constant})$$

$$\Rightarrow \left|\vec{r}(t)\right|^2 = \vec{r}(t).\vec{r}(t) = c^2$$

$$\Rightarrow \frac{d\left(\vec{r}(t).\vec{r}(t)\right)}{dt} = 0$$

$$\Rightarrow \vec{r}(t).\frac{d\left(\vec{r}(t)\right)}{dt} + \frac{d\left(\vec{r}(t)\right)}{dt}.\vec{r}(t) = 0$$

$$\Rightarrow \vec{r}(t).\frac{d\left(\vec{r}(t)\right)}{dt} = 0$$



Indefinite Integral

The indefinite integral of $\vec{r}$ with respect to t is the set of all antiderivatives of $\vec{r}$, written as $\int \vec{r}(t)dt$. If $\vec{R}$ is any antiderivative of $\vec{r}$, then:

$$\int \vec{r}(t)\,dt = \vec{R}(t) + \vec{C}$$

If $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$, then:

$$\int_a^c \vec{r}(t)dt = \left(\int_a^c x(t)dt\right)\hat{i} + \left(\int_a^c y(t)dt\right)\hat{j} + \left(\int_a^c z(t)dt\right)\hat{k}$$

Exercise 13.1

Q.8 The position vector of particles moving along the parabola $y = x^2 + 1$ in xy – plane is: $\vec{r}(t) = t\,\hat{i} + (t^2 + 1)\,\hat{j}$ Find the particle's velocity and acceleration vectors at the times t = -1, 0, 1 and sketch them as vectors on the curve.

Exercise 13.1

Q.14 : Given $\vec{r}(t) = e^{-t}\hat{i} + 2\cos 3t\ \hat{j} + 2\sin 3t\ \hat{k}$ is the position of a particle in space at any time t. Find the particle's speed and direction of motion at $t = 0$. Write the particle's velocity at that time as the product of its speed and direction.

Exercise 13.1

Q.19 Given that $\vec{r}(t) = (t - \sin t)\hat{i} + (1 - \cos t)\hat{j}$ is the position vector of a particle in space at time *t*. Find the time or times in the time interval $0 \leq t \leq 2\pi$ when the velocity and acceleration are orthogonal.

Exercise 13.1

Q.19 Find parametric equation for the line that is tangent to the curve:

$$\vec{r}(t) = (\sin t)\,\hat{i} + (t^2 - \cos t)\,\hat{j} + e^t\,\hat{k}$$

at $t = 0$.

Exercise 13.1

Q.25 A particle moves along the top of the parabola $y^2 = 2x$ from left to right at a constant speed of 5 units per second. Find the velocity of the particle as it moves through the point (2, 2).

Exercise 13.2

Q.4 Evaluate the integral:

$$\int_0^{\pi/3} [(\sec t \tan t)\hat{i} + (\tan t)\hat{j} + (2\sin t \cos t)\hat{k}]\, dt$$

Q.15 Solve the initial value problem for $\vec{r}$ as a vector function of t:

$$\frac{d^2\vec{r}}{dt^2} = -32\hat{k}, \text{Initial conditions}: \vec{r}(0) = 100\hat{k}$$

$$\left.\frac{d\vec{r}}{dt}\right|_{t=0} = 8\hat{i} + 8\hat{j}$$

Exercise 13.2

Q.18 A particle traveling in a straight line is located at the point (1, -1, 2) and has speed 2 at time $t = 0$, the particle moves towards the point (3, 0, 3) with constant acceleration $2i + j + k$. Find its position vector $\vec{r}(t)$ at time t.

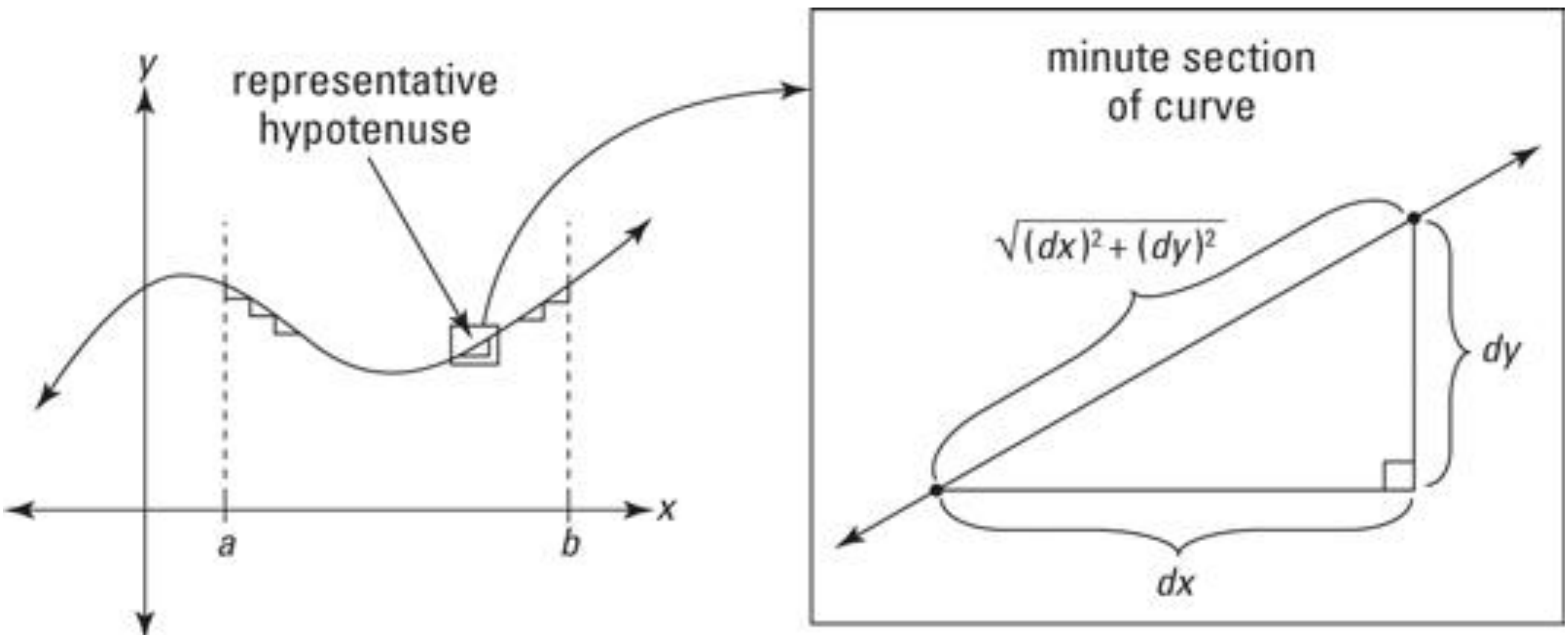
Arc Length

- One of the special features of smooth space curves is that they have a measurable length.

- That enables us to find points along these curves by giving their directed distance s along the curve from some base point.

Arc Length

Arc Length of a plane curve: Let $y = f(x)$ be a smooth function on $[a, c]$, then length of the curve $y = f(x)$, is given by



Arc Length

$$L = \int_a^c ds = \int_a^c \sqrt{(dx)^2 + (dy)^2}$$

$$= \int_a^c \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

For the smooth curve

$$x = x(t), \quad y = y(t), \quad a \le t \le c$$

$$L = \int_a^c \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Arc Length

Arc length along a curve for a vector function :

The length of a smooth curve

$$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}, \quad a \le t \le c$$

which is traced exactly once as t increases from a to c is :

$$L = \int_a^c \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$= \int_a^c \left|\vec{v}(t)\right| dt$$

Arc Length

Arc Length parameter with base point at $t = t_0$

$$s(t) = \int_{t_0}^{t} |\vec{v}(\tau)| \, d\tau$$

1. If $t > t_0$, then $s(t) > 0$.

2. If $t < t_0$, then $s(t) < 0$.

3. Every value of s determines a unique point on the curve

Arc Length

Smooth Curve

$$\frac{ds}{dt} = |\vec{v}(t)|, \quad \frac{ds}{dt} > 0 \ \text{ for a smooth curve,}$$

as $|\vec{v}|$ is never zero for a smooth curve.

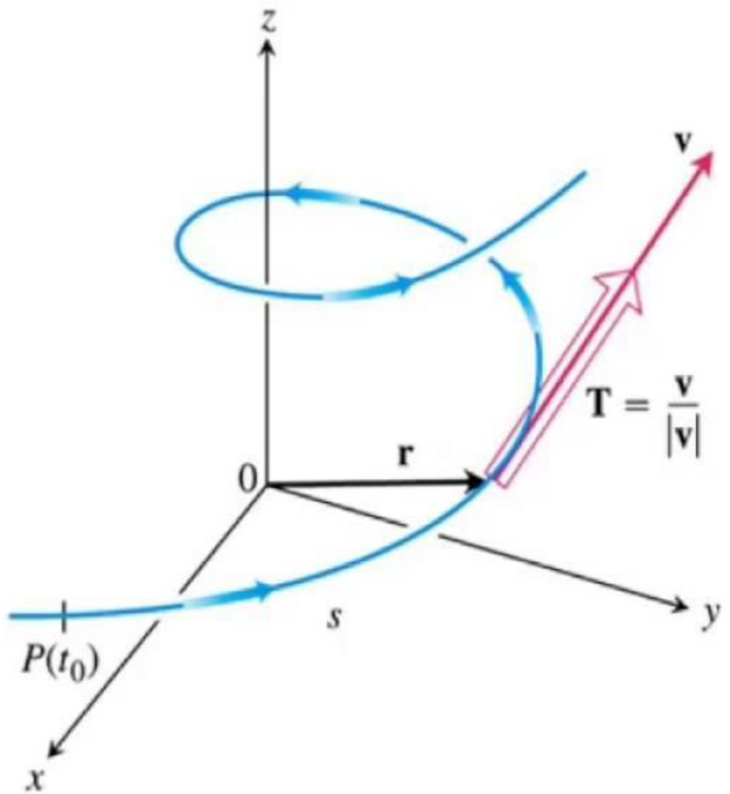
Thus $s(t)$ is a strictly increasing function of t,

hence bijection of (a, b) with $(s(a), s(b))$,

Unit Tangent Vector T

The **unit tangent vector** gets its own notation:

$$\vec{\mathbf{T}}(t) = \frac{\vec{\mathbf{r}}\,'(t)}{\left|\vec{\mathbf{r}}\,'(t)\right|} = \frac{\vec{\mathbf{v}}}{\left|\vec{\mathbf{v}}\right|}$$



Unit Tangent Vector T

We have the velocity vector $\vec{v} = \dfrac{d\vec{r}}{dt}$ is tangent to the curve and the vector, $\hat{T}(t) = \dfrac{\vec{v}(t)}{|\vec{v}(t)|}$ is therefore unit tangent vector to the curve.

For smooth curve

$$\hat{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{d\vec{r}(t)/dt}{ds/dt} = \frac{d\vec{r}}{ds}$$

Exercise 13.3

Q.8 Find the following curve's unit tangent vector and length of the indicated portion of the curve

$$\vec{r}(t) = (t\sin t + \cos t)\hat{i} + (t\cos t - \sin t)\hat{j},$$

for $\sqrt{2} \leq t \leq 2$

Exercise 13.3

Q.10 Find the point on the curve

$$\vec{r}(t) = (12\sin t)\hat{i} - (12\cos t)\hat{j} + 5t\,\hat{k}$$

at a distance 13π units along the curve from the origin (base point (0,-12,0) corresponding to $t = 0$) in the direction opposite to the direction of increasing arc length.

Exercise 13.3

Q.13 Find the arc length parameter along the curve from the point where $t = 0$.

$$\vec{r}(t) = (e^t \cos t)\hat{i} + (e^t \sin t)\hat{j} + e^t\,\hat{k}$$

Also find the length of the portion of the curve for

$$-\ln 4 \le t \le 0$$

Exercise 13.3

Q.15 Find the length of the curve

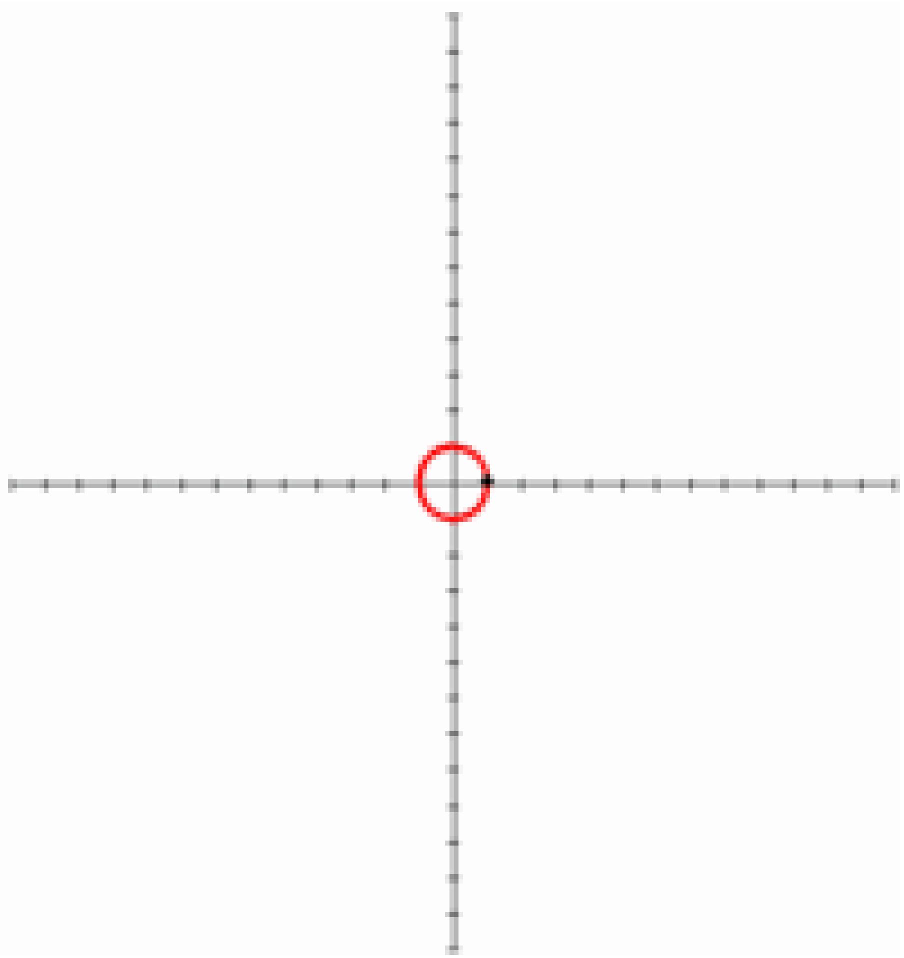
$$\vec{r}(t) = \left(\sqrt{2}t\right)\hat{i} + \left(\sqrt{2}t\right)\hat{j} + \left(1 - t^2\right)\hat{k}$$

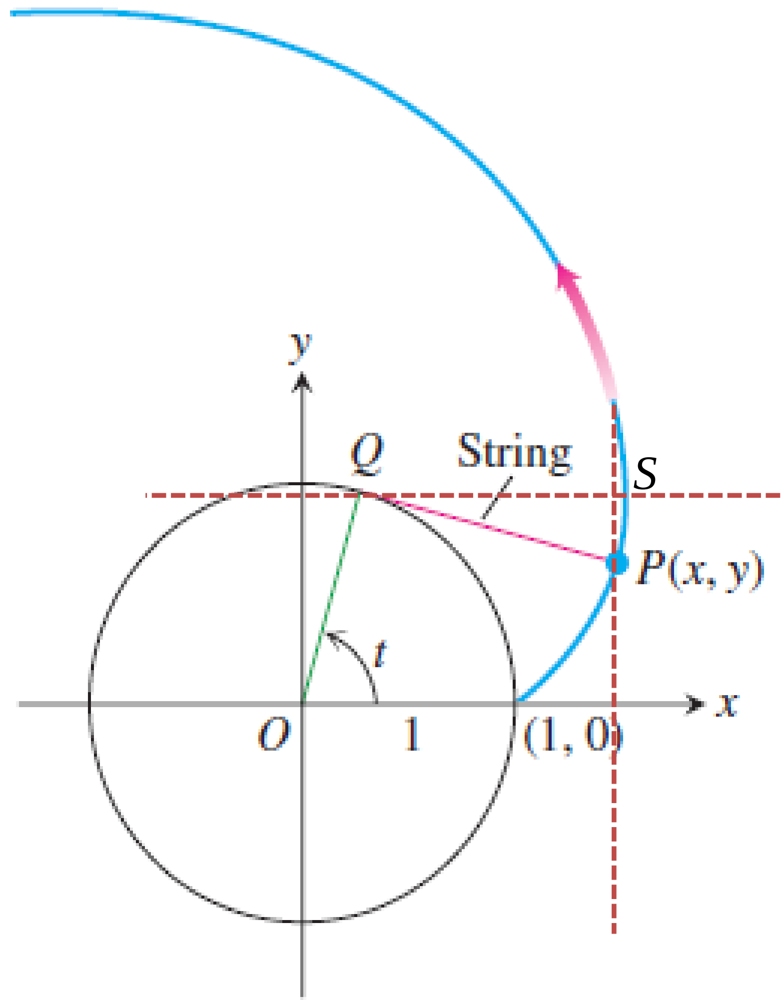
from (0,0,1) to $\left(\sqrt{2}, \sqrt{2}, 0\right)$

Exercise 13.3

Q.19 The involute of a circle If a string wound around a fixed circle is unwound while held taut in the plane of the circle, its end P traces an *involute* of the circle. In the accompanying figure, the circle in question is the circle $x^2 + y^2 = 1$and the tracing point starts at (1, 0). The unwound portion of the string is tangent to the circle at Q, and t is the radian measure of the angle from the positive x-axis to segment OQ. Derive the parametric equations of the point $P(x, y)$ for the involute.

Exercise 13.3





Exercise 13.3

Q.20 The involute of a circle: Find the unit tangent vector to the involute of the circle at the point P, discussed in **Q.19.**

Curvature

The magnitude of rate at which $\hat{T}$ turns per unit of arc length along the curve is called the curvature.

$$\kappa = \left| \frac{d\hat{T}}{ds} \right| = \left| \frac{d\hat{T}/dt}{ds/dt} \right| = \frac{1}{|\vec{v}(t)|} \left| \frac{d\hat{T}}{dt} \right|$$

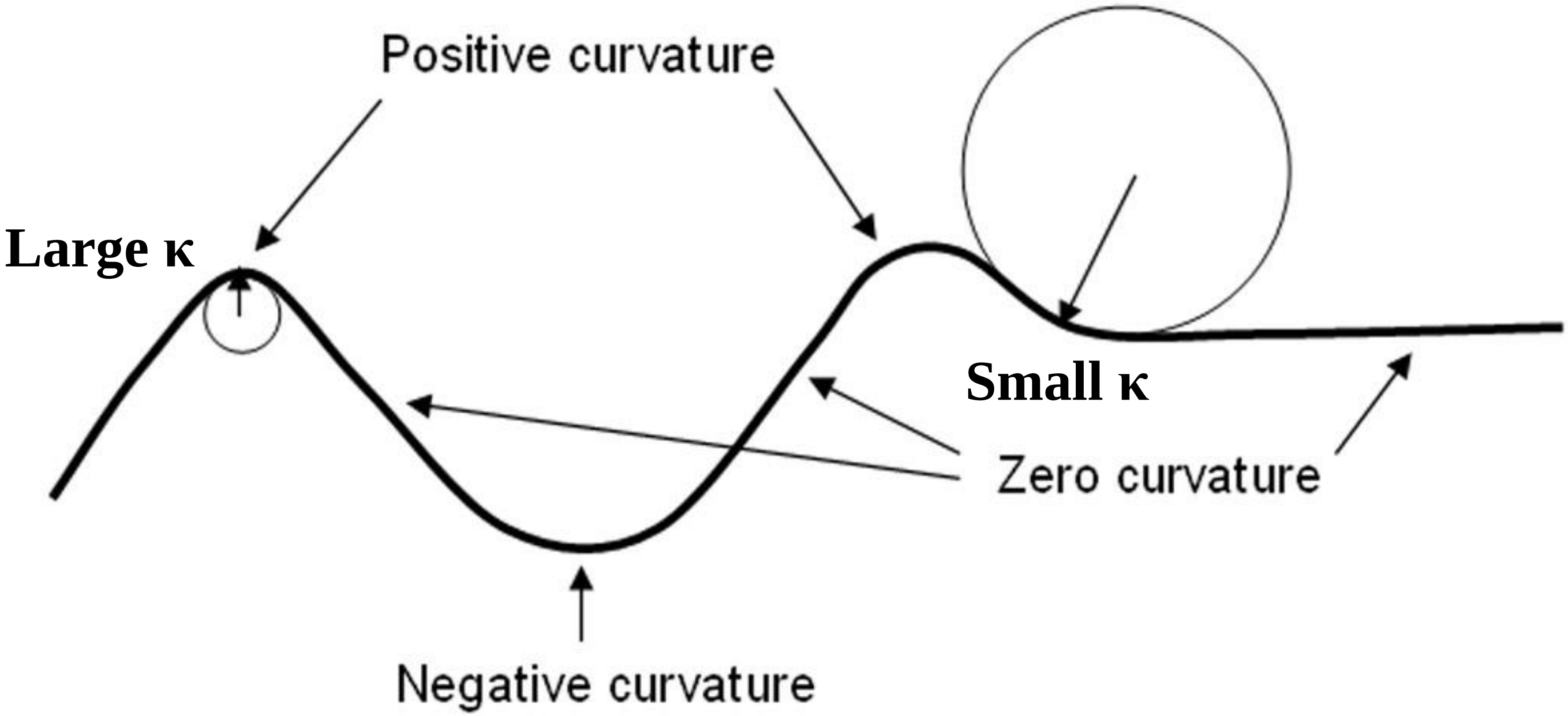
Curvature

Remark :

1. If $\left|\frac{d\hat{T}}{ds}\right|$ is small, then κ is small and $\widehat{T}$ turns slowly as the particle passes through the point.

2. If $\left|\frac{d\hat{T}}{ds}\right|$ is large, then κ is large and $\widehat{T}$ turns sharply as the particle passes through the point.

Curvature



Positive curvature
Large κ
Small κ
Zero curvature
Negative curvature

Curvature

What is curvature of straight line?

- As tangent to straight line in direction of straight line is straight line itself, rate of change of tangent line to straight line is zero with respect to arc length.

- **Hence curvature of straight line is zero.**

- One can as well prove it through parametric representation but intuitive way is simpler.

Curvature

Through parametric representation we have equation of straight line which passes through a point (x_0, y_0, z_0) and has direction ratio (α, β, γ) is:

$$x = x_0 + \alpha t, \; y = y_0 + \beta t \;, \; z = z_0 + \gamma t$$

and if we just compute curvature through formula we get that as zero.

Curvature

What is curvature of circle of radius *a* ?

If a particle moves on circle of radius a with center at origin in anti-clockwise direction in plane, vector motion of particle is represented as:

$$\vec{r}(t) = (a\cos t)\hat{i} + (a\sin t)\hat{j}, \quad 0 \le t \le 2\pi$$

$$\Rightarrow \vec{v}(t) = (-a\sin t)\hat{i} + (a\cos t)\hat{j} \Rightarrow |\vec{v}(t)| = a$$

Curvature

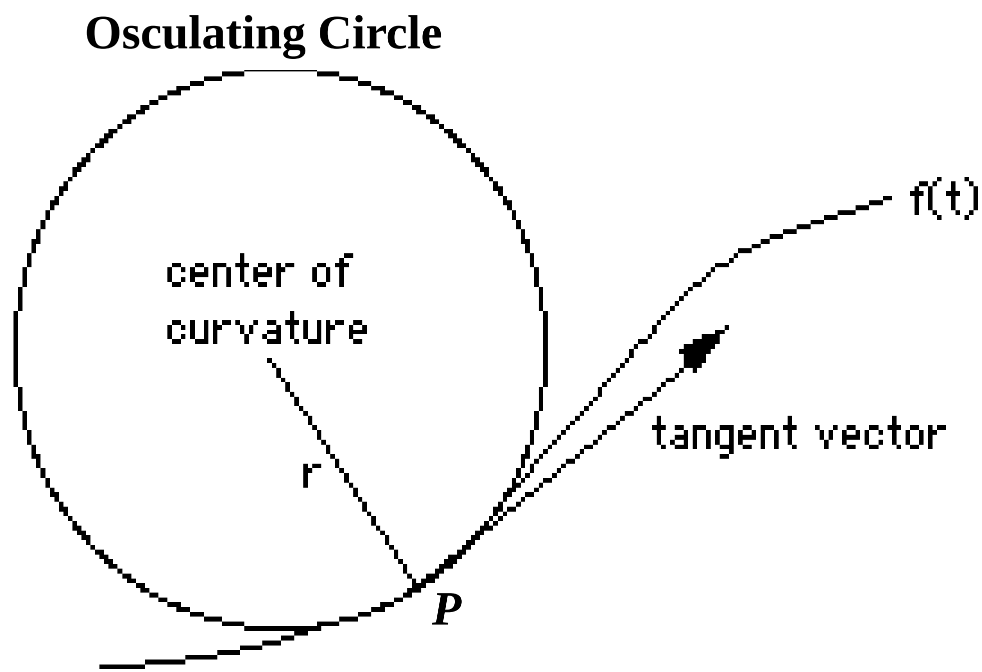
$$\text{Now } \hat{T} = \frac{\vec{v}(t)}{|\vec{v}(t)|} = (-\sin t)\hat{i} + (\cos t)\hat{j}$$

$$\Rightarrow \frac{d\hat{T}}{dt} = (-\cos t)\hat{i} + (-\sin t)\hat{j} \Rightarrow \left|\frac{d\hat{T}}{dt}\right| = 1$$

$$\text{hence curvature } \kappa = \frac{1}{|\vec{v}(t)|}\left|\frac{d\hat{T}}{dt}\right| = \frac{1}{a}$$

Circle of Curvature

The circle of curvature or **Osculating circle** at a point P on a plane curve where $\kappa \neq 0$ is the circle in the plane of the curve that:



1. is tangent to the curve at P
2. has the same curvature the curve has at P
3. has center that lies towards the concave or inner side of the curve

Radius of Curvature

The **radius of curvature** of the curve at P is the radius of the circle of curvature, i.e. $\rho = 1/\kappa$

The **center of curvature** of the curve at P is the center of circle of curvature.

Principle Unit Normal Vector N

1. $\hat{T}$ has constant length implies $\dfrac{d\hat{T}}{ds}$ is orthogonal to $\hat{T}$

(how?)

2. Therefore if we divide $\dfrac{d\hat{T}}{ds}$ by its length κ we obtain

a unit vector $\hat{N}$ orthogonal to $\hat{T}$ called principal Unit

Normal vector $\hat{N} \Rightarrow \hat{N} = \dfrac{1}{\kappa}\dfrac{d\hat{T}}{ds} = \dfrac{d\hat{T}/ds}{\left|d\hat{T}/ds\right|} = \dfrac{d\hat{T}/dt}{\left|d\hat{T}/dt\right|}$

Principle Unit Normal Vector N

- The vector $d\hat{T}/ds$ always points in the direction in which $\hat{T}$ is turning.
- If we face in the direction of increasing arc length, then $d\hat{T}/ds$ points:

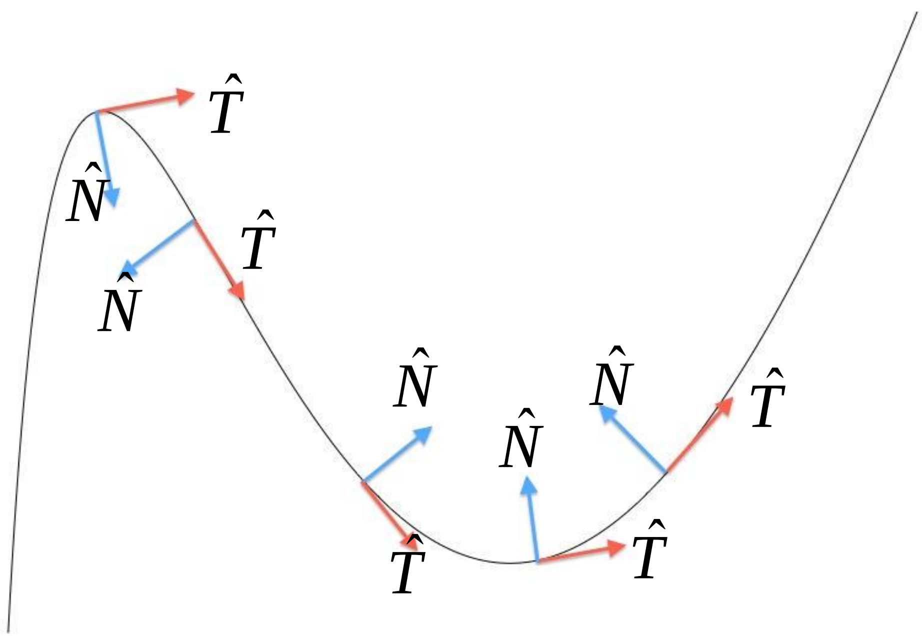
(i) towards the right if $\hat{T}$ turns clockwise,

Principle Unit Normal Vector N

(ii) towards the left if $\hat{T}$ turns anti-clockwise.

Thus, $\hat{N}$ will point towards the concave side of the curve.

$$\hat{N} \times \frac{d\hat{T}}{ds} = 0$$



Principle Unit Normal Vector N

$$\text{Problem}: \vec{a}(t) = \frac{d^2 s}{dt^2}\hat{T}(t) + \left(\frac{ds}{dt}\right)^2 \kappa(t)\hat{N}(t)$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt} = \left(\frac{d\vec{r}/dt}{ds/dt}\right)\frac{ds}{dt} = \hat{T}\frac{ds}{dt}$$

$$\text{as } |\vec{v}(t)| = \frac{ds}{dt}, \quad \hat{T} = \frac{\vec{v}}{|\vec{v}|}$$

Principle Unit Normal Vector N

$$\Rightarrow \vec{a}(t) = \frac{d}{dt}\left(\hat{T}\frac{ds}{dt}\right) = \hat{T}\frac{d^2 s}{dt^2} + \frac{ds}{dt}\frac{d}{dt}\left(\hat{T}(t)\right)$$

$$\text{But } \hat{N}(t) = \frac{1}{\kappa}\frac{d\hat{T}}{ds} = \frac{1}{\kappa}\frac{\frac{d\hat{T}}{dt}}{\frac{ds}{dt}}, \text{ hence } \frac{d\hat{T}}{dt} = \kappa\,|\vec{v}|\,\hat{N}$$

$$\vec{a}(t) = \frac{d^2 s}{dt^2}\hat{T} + \kappa\left|\vec{v}\right|^2 \hat{N}.$$

Principle Unit Normal Vector N

We write $\vec{a}(t) = a_T \hat{\mathrm{T}} + a_N \hat{N}$,

where

$$a_T = \frac{d^2 s}{dt^2} = \frac{d}{dt}\left|\vec{v}\right|, \quad a_N = \kappa \left|\vec{v}\right|^2 \geq 0$$

are called tangential and normal components of acceleration.

Principle Unit Normal Vector N

$$\text{Problem: Prove } \kappa(t) = \left|\vec{v} \times \vec{a}\right| / \left|\vec{v}\right|^3$$

$$\vec{v}(t) = \frac{ds}{dt}\hat{T} \text{ and } \vec{a}(t) = \frac{d^2 s}{dt^2}\hat{T}(t) + \left(\frac{ds}{dt}\right)^2 \kappa(t)\hat{N}(t)$$

$$\vec{v} \times \vec{a} = \frac{ds}{dt}\hat{T} \times \left(\frac{d^2 s}{dt^2}\hat{T} + \left(\frac{ds}{dt}\right)^2 \kappa(t)\hat{N}\right)$$

$$\Rightarrow \kappa(t) = \frac{\left|\vec{v} \times \vec{a}\right|}{\left|\vec{v}\right|^3}$$

Exercise 13.4

Q.4 Find $\hat{T}$, $\hat{N}$, κ for the plane curve

$$\vec{r}(t) = (\cos t + t\sin t)\hat{i} + (\sin t - t\cos t)\hat{j}, \;\; t > 0$$

.

Exercise 13.4

Q6. Show that curvature of a smooth curve

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$$

defined by a twice differentiable

function $x = x(t), \quad y = y(t) \quad is$

$$\kappa = \frac{|\dot{x}\,\ddot{y} - \ddot{x}\,\dot{y}|}{\left(\dot{x}^2 + \dot{y}^2\right)^{3/2}}.$$

Exercise 13.4

Solution : $\vec{v} = \dot{x}\,\hat{i} + \dot{y}\,\hat{j},$

$\vec{a} = \ddot{x}\,\hat{i} + \ddot{y}\,\hat{j},$

$$\vec{v} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dot{x} & \dot{y} & 0 \\ \ddot{x} & \ddot{y} & 0 \end{vmatrix}$$

$$= (\dot{x}\,\ddot{y} - \ddot{x}\,\dot{y})\hat{k}$$

Exercise 13.4

$$\Rightarrow \left|\vec{v} \times \vec{a}\right| = \left|\dot{x}\,\ddot{y} - \ddot{x}\,\dot{y}\right|$$

$$\left|\vec{v}\right| = \sqrt{\dot{x}^2 + \dot{y}^2}$$

$$\therefore \kappa = \frac{\left|\vec{v} \times \vec{a}\right|}{\left|\vec{v}\right|^3} = \frac{\left|\dot{x}\,\ddot{y} - \ddot{x}\,\dot{y}\right|}{\left(\dot{x}^2 + \dot{y}^2\right)^{3/2}}$$

Exercise 13.4

Q.7 For the plane curve $y = f(x)$, show that its curvature at the point $(x, f(x))$ is:

$$\kappa = \frac{|f''(x)|}{\left(1 + (f'(x))^2\right)^{3/2}}$$

Exercise 13.4

Solution : $\vec{r}(t) = x(t)\,\hat{i} + f\,(x(t))\,\hat{j},$

$$\Rightarrow \vec{v}(t) = \dot{x}\,\hat{i} + \dot{x}f\,'(x)\,\hat{j},$$

$$\Rightarrow \vec{a}(t) = \ddot{x}\hat{i} + \left((\dot{x})^2\, f\,''(x) + \ddot{x}f\,'(x)\right)$$

$$\vec{v} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dot{x} & \dot{x}f\,'(x) & 0 \\ \ddot{x} & \left((\dot{x})^2\, f\,''(x) + \ddot{x}f\,'(x)\right) & 0 \end{vmatrix}$$

$$= (\dot{x})^3\, f\,''(x)\hat{k}$$

Exercise 13.4

$$\Rightarrow \left|\vec{v}\times\vec{a}\right| = \left|(\dot{x})^3 f''(x)\right|$$

$$\left|\vec{v}\right| = \sqrt{\dot{x}^2 + \dot{x}^2 (f'(x))^2}$$

$$\therefore \kappa = \frac{\left|\vec{v}\times\vec{a}\right|}{\left|\vec{v}\right|^3} = \frac{\left|f''(x)\right|}{\left(1+(f'(x))^2\right)^{3/2}}$$

Exercise 13.4

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Q.17 Show that the parabola has its largest curvature at its vertex.

We have $y = ax^2 = f(x), \; a > 0$

$$\Rightarrow f'(x) = 2ax \text{ and } f''(x) = 2a$$

$$\therefore \kappa = \frac{|f''(x)|}{\left(1 + (f'(x))^2\right)^{3/2}} = \frac{|2a|}{\left(1 + 4a^2x^2\right)^{3/2}}$$

$$\Rightarrow \frac{d\kappa}{dx} = \frac{-24|a|a^2x}{\left(1 + 4a^2x^2\right)^{5/2}} = 0 \Rightarrow x = 0$$

Exercise 13.4

$$\left.\frac{d^2\kappa}{dx^2}\right|_{x=0} = -24|a|a^2 < 0$$

$$\Rightarrow \kappa \text{ is maximum at } x = 0, \text{i.e. at vertex } (0, 0)$$

$$\kappa_{\max} = 2|a|$$

Exercise 13.5

Prob. Write $\vec{a}$ in the form $\vec{a} = a_T \hat{T} + a_N \hat{N}$ without finding $\hat{T}$ and $\hat{N}$ for

$$\vec{r}(t) = (2t+3)\hat{i} + (t^2-1)\hat{j}.$$

Exercise 13.5

Solution $a_T = \dfrac{d}{dt}\left|\vec{v}\right| = \dfrac{2t}{\sqrt{1+t^2}}$

$$\vec{a} = 2\,\hat{j} \Rightarrow \left|\vec{a}\right| = 2$$

$$\Rightarrow \vec{a}_N = \sqrt{\left|\vec{a}\right|^2 - a_T^2} = \frac{2}{\sqrt{1+t^2}}$$

$$\vec{a} = \frac{2t}{\sqrt{1+t^2}}\hat{T} + \frac{2}{\sqrt{1+t^2}}\hat{N}$$

Binormal Vector

The binormal vector of a curve in space is

$$\hat{B} = \hat{T} \times \hat{N}$$

which is a unit vector orthogonal to both $\hat{T}$ and $\hat{N}$.

Serret-Frenet Frame

The vectors $\hat{T}, \hat{N}, \hat{B}$ form a right-handed frame naturally (or intrinsically) associated to the curve which moves from point to point. This exists at a point P when the *curvature* at that point is nonzero.

Co-ordinate planes in Serret-Frenet frame:

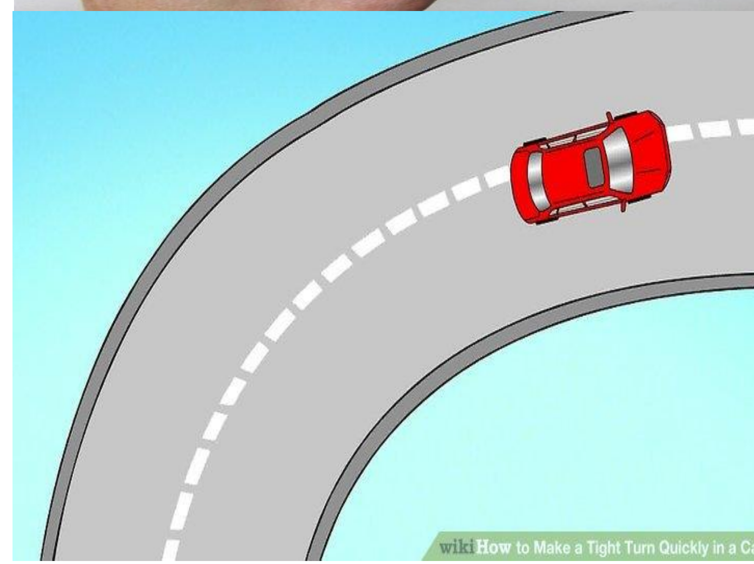
Osculating plane at *P* : plane through P spanned by $\hat{T}$ and $\hat{N}$ (normal to vector $\hat{B}$)

Normal plane : spanned by $\hat{N}$ and $\hat{B}$

Rectifying plane : spanned by $\hat{T}$ and $\hat{B}$.

Torsion

Torsion : We define torque as the capability of rotating objects around a fixed axis. Or how much a vehicle's path rotates or twists out of its plane of motion as the vehicle moves along it.



Torsion

Geometric significance of τ :

$$\hat{B} = \hat{T} \times \hat{N}$$

$$\Rightarrow \frac{d\hat{B}}{ds} = \frac{d\hat{T}}{ds} \times \hat{N} + \hat{T} \times \frac{d\hat{N}}{ds}$$

$$\Rightarrow \frac{d\hat{B}}{ds} = \hat{T} \times \frac{d\hat{N}}{ds}$$

Torsion

$$\Rightarrow \frac{d\hat{B}}{ds} \text{ is orthogonal to } \hat{T}$$

$$\because \frac{d\hat{B}}{ds} \text{ is orthogonal to } \hat{B} \text{ (WHY ?)}$$

$$\text{Hence } \frac{d\hat{B}}{ds} \text{ is orthogonal to the plane}$$

$$\text{of } \hat{B} \;\&\; \hat{T} \Rightarrow \frac{d\hat{B}}{ds} \text{ is parallel to } \hat{N}$$

Torsion

$$\Rightarrow \frac{d\hat{B}}{ds} = -\tau\,\hat{N}, \quad \text{where the}$$

scalar τ is called Torsion along the curve.

(minus sign is a convention which we have to follow)

$$\boxed{\therefore \frac{d\hat{B}}{ds}\cdot\hat{N} = -\tau\,\hat{N}\cdot\hat{N} = -\tau.}$$

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A Computational formula for Torsion

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{\left|\vec{v} \times \vec{a}\right|^2}, \quad \left|\vec{v} \times \vec{a}\right| \neq 0.$$

Exercise 13.5

Q. Determine $\hat{T}$, $\hat{N}$, $\hat{B}$, κ and τ

for the curve given by

$$\vec{r}(t) = (\sin t)\hat{i} + (\sqrt{2}\cos t)\hat{j} + (\sin t)\hat{k}$$

at the point $t = \pi/2$.

Exercise 13.5

$$\text{Solution}: \vec{v}(t) = \cos t\,\hat{i} - \sqrt{2}\sin t\,\hat{j} + \cos t\,\hat{k}$$

$$\left|\vec{v}(t)\right| = \sqrt{2},\ \hat{T}(t) = \frac{\vec{v}(t)}{\left|\vec{v}(t)\right|}$$

$$\Rightarrow \hat{T}(t) = \frac{\left[\cos t\,\hat{i} - \sqrt{2}\sin t\,\hat{j} + \cos t\,\hat{k}\right]}{\sqrt{2}}$$

$$\hat{T}(\pi/2) = \frac{1}{\sqrt{2}}(-\sqrt{2}\,\hat{j}) = -\hat{j}$$

Exercise 13.5

$$\frac{d\hat{T}(t)}{dt} = \frac{\left(-\sin t\,\hat{i} - \sqrt{2}\cos t\,\hat{j} - \sin t\,\hat{k}\right)}{\sqrt{2}}$$

$$\left|\frac{d\hat{T}(t)}{dt}\right| = 1 \Rightarrow \kappa = \frac{1}{|\vec{v}(t)|}\left|\frac{d\hat{T}(t)}{dt}\right| = \frac{1}{\sqrt{2}}\,(\text{Constant})$$

$$\hat{N}(t) = \frac{1}{\kappa|\vec{v}(t)|}\frac{d\hat{T}(t)}{dt} = \frac{\left(-\sin t\,\hat{i} - \sqrt{2}\cos t\,\hat{j} - \sin t\,\hat{k}\right)}{\sqrt{2}}$$

$$\hat{N}(\pi/2) = \frac{1}{\sqrt{2}}(-\hat{i} - \hat{k})$$

Exercise 13.5

$$\hat{B}(t) = \hat{T}(t) \times \hat{N}(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\cos t}{\sqrt{2}} & -\sin t & \dfrac{\cos t}{\sqrt{2}} \\ \dfrac{-\sin t}{\sqrt{2}} & -\cos t & \dfrac{-\sin t}{\sqrt{2}} \end{vmatrix}$$

$$= \frac{1}{\sqrt{2}}(\hat{i} - \hat{k})$$

$$\hat{B}(\pi/2) = \frac{1}{\sqrt{2}}(\hat{i} - \hat{k})$$

Exercise 13.5

$$\tau = -\frac{d\hat{B}}{ds}.\hat{N} = \frac{-1}{\left|\vec{v}(t)\right|}\frac{d\hat{B}}{dt}.\hat{N} = 0$$

Since $\hat{B}$ is a constant function.

Hence τ is zero throughout.

What does it mean for torsion to be zero for all time? Study Q.25

Exercise 13.5

Plane through point (x_0, y_0, z_0)

for which vector $A\hat{i} + B\hat{j} + C\hat{k}$

is normal vector is

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

Exercise 13.5

Parametric equation of line through point (x_0, y_0, z_0) that is parallel to vector $A\hat{i} + B\hat{j} + C\hat{k}$ is

$$x = x_0 + tA,$$

$$y = y_0 + tB,$$

$$z = z_0 + tC \;, \quad -\infty < t < \infty$$

Exercise 13.5

Q.7 Find $\hat{T}$, $\hat{N}$ and $\hat{B}$ at $t = \pi/4$ for

$$\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} - \hat{k}$$

Then find equations for osculating, normal and rectifying planes at $t = \pi/4$

Exercise 13.5

$$\text{Solution}: \vec{v}(t) = (-\sin t)\hat{i} + (\cos t)\hat{j}$$

$$\Rightarrow \hat{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = (-\sin t)\hat{i} + (\cos t)\hat{j} \Rightarrow \hat{T}\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}(-\hat{i} + \hat{j})$$

$$\Rightarrow \hat{N}(t) = \frac{\dfrac{d\hat{T}}{dt}}{\left|\dfrac{d\hat{T}}{dt}\right|} = (-\cos t)\hat{i} - (\sin t)\hat{j} \Rightarrow \hat{N}\left(\frac{\pi}{4}\right) = \frac{-1}{\sqrt{2}}(\hat{i} + \hat{j})$$

$$\Rightarrow \hat{B}(t) = \hat{T}(t) \times \hat{N}(t) = \hat{k} \Rightarrow \hat{B}\left(\frac{\pi}{4}\right) = \hat{k};$$

Exercise 13.5

$$\vec{r}\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}\hat{i}+\frac{1}{\sqrt{2}}\hat{j}-\hat{k}\Rightarrow P\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},-1\right)$$

lies on the osculating plane and $\hat{B}=\hat{k}$ is perpendicular to osculating plane.

Therefore equation of osculating plane

$$\Rightarrow 0\left(x-\frac{1}{\sqrt{2}}\right)+0\left(y-\frac{1}{\sqrt{2}}\right)+\left(z-(-1)\right)=0$$

$$\Rightarrow z+1=0$$

Exercise 13.5

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Equation of Normal plane :

$$\hat{T} = \frac{1}{\sqrt{2}}(-\hat{i} + \hat{j}) \text{ is perpendicular to Normal plane.}$$

Therefore equation of Normal plane

$$\Rightarrow \frac{-1}{\sqrt{2}}\left(x - \frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\left(y - \frac{1}{\sqrt{2}}\right) + 0.\left(z - (-1)\right) = 0$$

$$\Rightarrow x - y = 0$$

Exercise 13.5

Equation of Rectifying plane :

$\hat{N} = \frac{-1}{\sqrt{2}}(\hat{i} + \hat{j})$ is perpendicular to Rectifying plane.

Therefore equation of Rectifying plane

$$\Rightarrow \frac{-1}{\sqrt{2}}\left(x - \frac{1}{\sqrt{2}}\right) + \frac{-1}{\sqrt{2}}\left(y - \frac{1}{\sqrt{2}}\right) + 0.\left(z - (-1)\right) = 0$$

$$\Rightarrow x + y = \sqrt{2}$$

Exercise 13.5

Q. Find the parametric equation of the tangent line & equation of normal plane to the curve given as

$$\vec{r}\left(t\right)=(t+\sin t)\hat{i}+(1-\cos t)\hat{j}+\frac{\sin t}{\sqrt{2}}\hat{k}$$

$$\text{at}\, t=\pi/2$$

Motion in Polar Coordinates

When a particle moves along a curve in the polar coordinate plane, we express its position, velocity and acceleration in the terms of unit vectors $\hat{u}_r$ and $\hat{u}_\theta$.

$\hat{u}_r$: A unit vector that points along the position vector (radial direction), $\overrightarrow{OP}$ hence

$$\hat{u}_r = \frac{\vec{r}}{|\vec{r}|} = \frac{\vec{r}}{r}$$

Motion in Polar Coordinates

$\hat{u}_\theta$: A unit vector, normal to $\hat{u}_r$ (obtained by rotating $\hat{u}_r$ anticlockwise $\pi/2$) that points in the direction of increasing θ

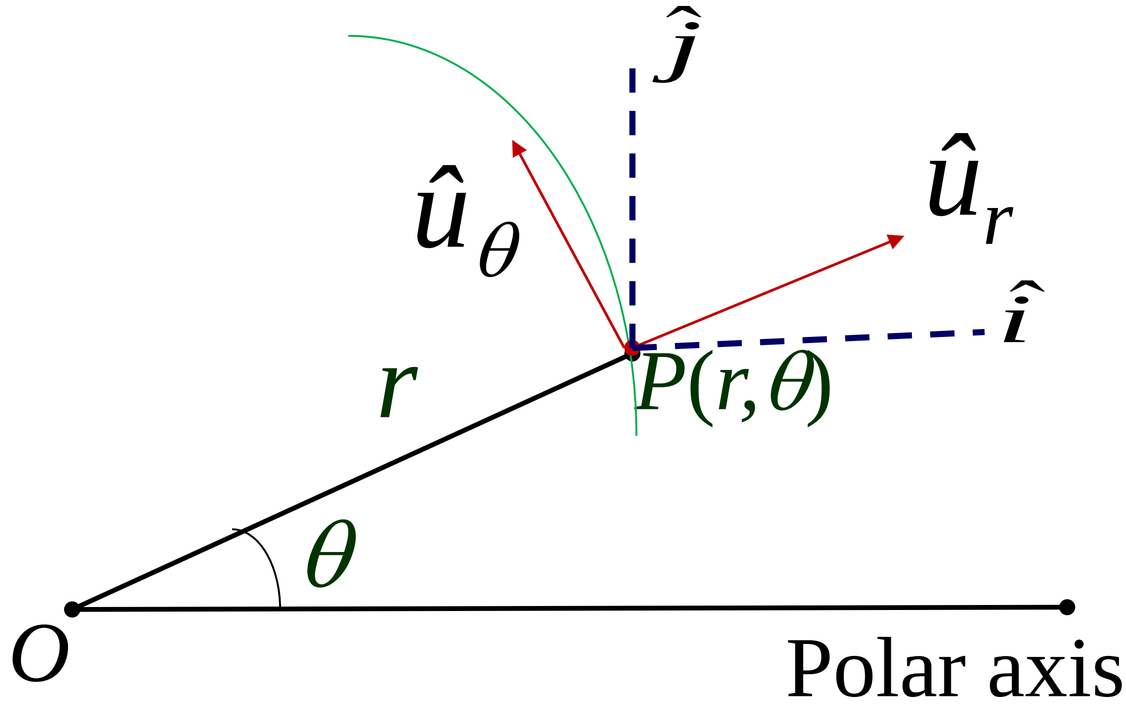
r : length of $\vec{r}$ which is the positive polar coordinate r of the point $P(r,\theta)$.

$$\hat{u}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

$$\hat{u}_\theta = \cos(\pi/2+\theta)\,\hat{i} + \sin(\pi/2+\theta)\,\hat{j}$$

$$\hat{u}_\theta = \sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

Motion in Polar Coordinates



Motion in Polar Coordinates

$$\frac{d\hat{u}_r}{d\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} = \hat{u}_\theta$$

$$\frac{d\hat{u}_\theta}{d\theta} = -\cos\theta\,\hat{i} - \sin\theta\,\hat{j} = -\hat{u}_r$$

distinguish between r and $\vec{r}$

Velocity for Motion in Polar Coordinates

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{u}_r) = \frac{dr}{dt}\hat{u}_r + r\frac{d\hat{u}_r}{dt}$$

$$= \frac{dr}{dt}\hat{u}_r + r\frac{d\hat{u}_r}{d\theta}\frac{d\theta}{dt}$$

$$= \frac{dr}{dt}\hat{u}_r + r\left(\frac{d\theta}{dt}\right)\hat{u}_\theta$$

$$= \dot{r}\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta$$

Acceleration for Motion in Polar Coordinates

$$\vec{a}(t) = \frac{d}{dt}\vec{v}(t) = \frac{d}{dt}\frac{d\vec{r}}{dt}$$

$$= \frac{d}{dt}\left(\frac{dr}{dt}\hat{u}_r\right) + \frac{d}{dt}\left[r\left(\frac{d\theta}{dt}\right)\hat{u}_\theta\right]$$

$$\Rightarrow \vec{a}(t) = \frac{d^2r}{dt^2}\hat{u}_r + \frac{dr}{dt}\frac{d\hat{u}_r}{dt}$$

$$+ \frac{dr}{dt}\frac{d\theta}{dt}\hat{u}_\theta + r\frac{d}{dt}\left(\frac{d\theta}{dt}\right)\hat{u}_\theta + r\left(\frac{d\theta}{dt}\right)\left(\frac{d\hat{u}_\theta}{dt}\right)$$

Acceleration for Motion in Polar Coordinates

$$\Rightarrow \vec{a}(t) = \frac{d^2r}{dt^2}\hat{u}_r + \frac{dr}{dt}\hat{u}_\theta\frac{d\theta}{dt}$$

$$+\frac{dr}{dt}\frac{d\theta}{dt}\hat{u}_\theta + r\frac{d^2\theta}{dt^2}\hat{u}_\theta$$

$$+r\left(\frac{d\theta}{dt}\right)\left(-\hat{u}_r\right)\left(\frac{d\theta}{dt}\right)$$

Acceleration for Motion in Polar Coordinates

$$\vec{a}(t) = \left[\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{u}_r + \left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\hat{u}_\theta$$

$$= (\ddot{r} - r\dot{\theta}^2)\hat{u}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{u}_\theta$$

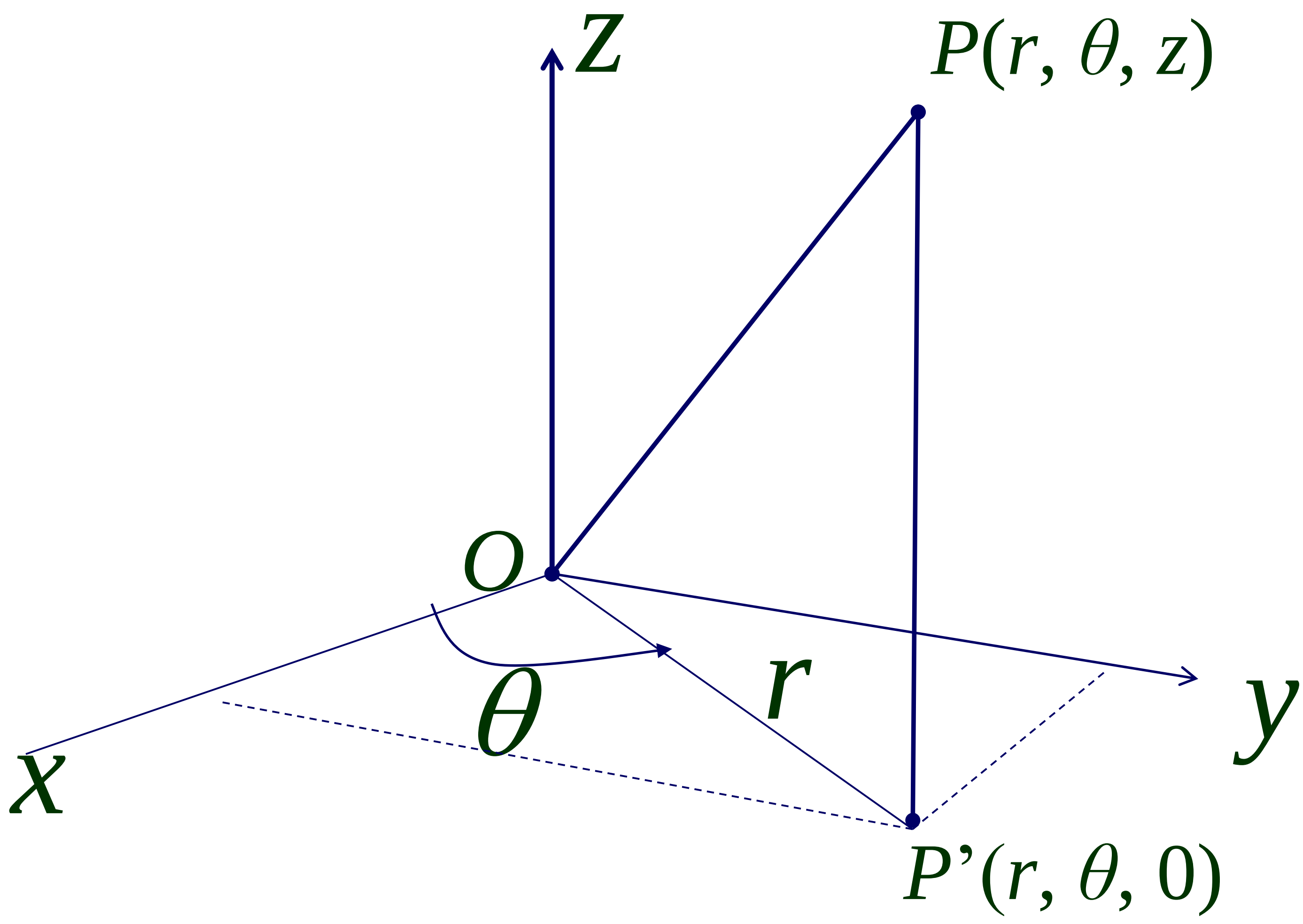
Cylindrical Coordinates

We obtain cylindrical coordinates of space by combining polar coordinates in $r\theta$ – plane with the usual z – axis, in (r, θ, z).

r and θ are polar co-ordinates of the projection of P onto the $r\theta$ – plane.

z : the directed distance from the $r\theta$ – plane to the point P.

Cylindrical Coordinates



Cylindrical Coordinates

Equations relating Rectangular (x, y, z) and Cylindrical (r, θ, z) coordinates

$$x = r\cos\theta,\ y = r\sin\theta,\ \ z = z$$

$$r^2 = x^2 + y^2, \quad \tan\theta = y/x$$

Motion in Cylindrical Coordinates

$$\hat{u}_r \times \hat{u}_\theta = \hat{k},$$

$$\hat{u}_\theta \times \hat{k} = \hat{u}_r,$$

$$\hat{k} \times \hat{u}_r = \hat{u}_\theta,$$

$$\vec{r} = r\hat{u}_r + z\hat{k}$$

$\Rightarrow (\hat{u}_r, \hat{u}_\theta, \hat{k})$ is right handed orthogonal frame of unit vectors.

Motion in Cylindrical Coordinates

$$\vec{v}(t) = \dot{r}\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta + \dot{z}\hat{k}$$

$$\vec{a}(t) = (\ddot{r} - r\dot{\theta}^2)\hat{u}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{u}_\theta + \ddot{z}\hat{k}$$

Motion in Cylindrical Coordinates

Q. Express velocity and acceleration in polar coordinate if

$$r = a(1-\cos\theta), \quad \frac{d\theta}{dt} = 3.$$

THANK YOU
FOR YOUR PATIENCE !!!