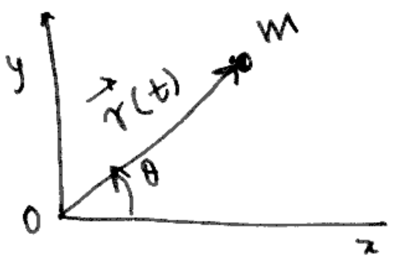
CH. 6 ANGULAR MOMENTUM AND FIXED AXIS ROTATION

(Notes by: Rishikesh Vaidya).

Q. What is rotation?

→ To answer this question we must make a distinction between a point particle which is only an idealized abstraction of real bodies which have finite extension in space. With regard to rigid bodies we will ~~us~~ soon see that taking account of the finite size of the body leads to a difference in what constitutes rotation and what constitutes translation.

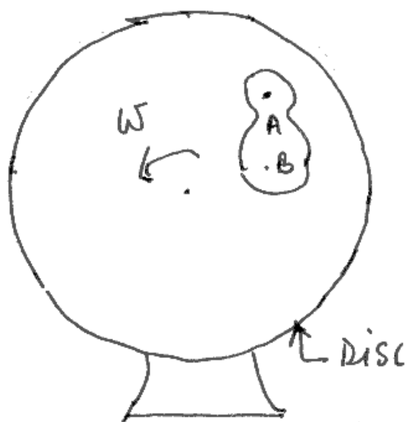
~~Point po~~ MOTION OF A POINT MASS



Consider a point mass m described by a position vector $\vec{r}(t)$. Point particle may move to a different location. The change may be pure scaling, pure rotation, or more general involving a change in magnitude as well as direction of position vector $\vec{r}(t)$. Thus pure rotation with a pure change in ~~position~~ direction of the position vector $\vec{r}$.

MOTION OF A RIGID BODY

1) Consider a rigid body hung at a frictionless pivot A, on a disc which rotates with an angular speed ω. The question is – Is the rigid body rotating or translating?



→ 2) Now consider a situation in which the rigid body is hooked to the disk by means of two frictionless nails at points A and B. Now as the disc is set spinning at angular speed ω does the rigid body rotate, or translate, or does both? To answer this we must understand the meaning of rotation and translation for a rigid body.

RIGID BODY: An ideal rigid body is one in which its constituent atoms maintain a fixed distance throughout motion. Thus an ideal rigid body doesnot undergo deformation.

TRANSLATION: A rigid body is said to undergo translatory motion if line joining any two points inside rigid body remains parallel to itself. Thus, every rectilinear motion is translation, but all translatory motion is not necessarily rectilinear. Thus in translatory motion the displacement of all points of rigid body is identical and hence and hence all points have the same velocity and accelerations at all points in time. In case 1) when the rigid body is pivoted ~~a~~ only at A without friction it undergoes translation.

ROTATION: A rigid body is said to undergo rotation if trajectories of all the points of a rigid body are circles whose centres lie on a common straight line called axis of rotation. Thus, our rigid body in case 2) undergoes circular motion.

ROTATIONAL ANALOGUES OF PHYSICAL QUANTITIES: When we compare pure translatory motion of a rigid body with pure rotational motion, we must appreciate an important distinction. There is neither a ~~s~~ special point, nor axis, nor length scale associated with translational motion. You can refer it to any origin. ~~Whereas~~ This is not quite so with rotational motion. Whereas we are free to refer it to ~~axis~~ any

reference point, but there exist a special line called axis of rotation about which ~~the rig~~ every point of rigid body describes an arc of a circle of fixed radius.
Thus there exist a special line called axis of rotation which is common for the entire rigid body, and a special length scale R – the distance from the axis of rotation ~~for~~ which is a variable for ~~a~~ every point of a rigid body.
The upshot is – for rotational motion we must expect this length scale to play an important role in defining physical quantities ~~asso~~ associated with rotation. For example,

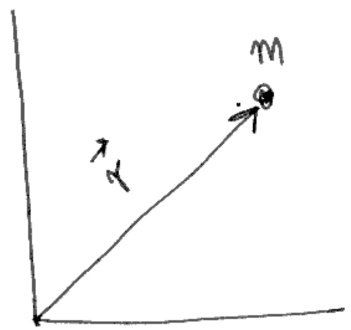
1) Displacement: $R\,d\theta\,\hat{\theta}$.
2) Velocity: $R\omega\,\hat{\theta}$.
3) Angular momentum: $\vec{R} \times \vec{P}$ Moment of Momentum
4) Torque: $\vec{R} \times \vec{F}$ Moment of force.

ANGULAR MOMENTUM IS A FUNCTION OF THE CHOICE OF ORIGIN.

TORQUE : $\vec{\tau} = \dfrac{d\vec{L}}{dt}$.

We will try and appreciate the meaning of torque for the case of A) point particle referred to a fixed origin B) an extended body referred to a fixed origin C) an extended body referred to an accelerating origin. This will help us understand torque on an arbitrary body with respect to an arbitrary origin. Along the way, we will also appreciate that the torques due to internal forces vanish.

A) POINT MASS, FIXED ORIGIN :

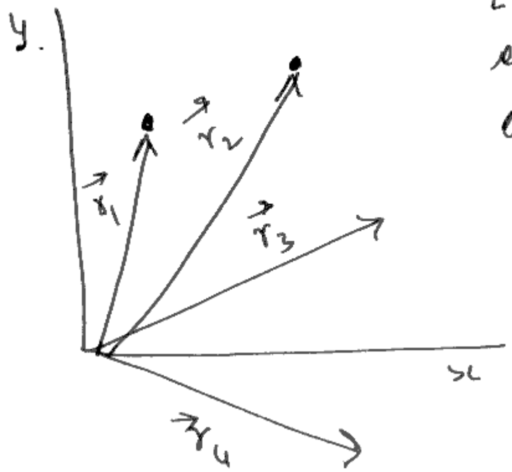


$$\vec{L} = \vec{r} \times \vec{P}$$

$$\vec{\tau} = \frac{d\vec{L}}{dt} = \underbrace{\dot{\vec{r}} \times \vec{P}}_{\to 0} + \vec{r} \times \dot{\vec{P}}$$

$$\boxed{\vec{\tau} = \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F}}.$$

B) EXTENDED MASS, FIXED ORIGIN



Let us imagine an extended body to be a collection of N discrete particles labelled by an index i

$$\vec{L} = \sum_i \vec{r}_i \times \vec{P}_i .$$

Here $\vec{r}_i$ is the position vector of i^{th} particle and $\vec{P}_i$ its momentum. If $\vec{F}_i$ is the total force acting on it, then

$$\vec{F}_i = \vec{F}_i^{EXT} + \vec{F}_i^{INT} = \frac{d\vec{P}_i}{dt}.$$

Now,

$$\vec{\tau} = \frac{d\vec{L}}{dt} = \frac{d}{dt}\sum_{i=1}^{N} \vec{r}_i \times \vec{P}_i = \sum_{i=1}^{N}\left[\underbrace{\dot{\vec{r}}_i \times \vec{P}_i}_{\to 0} + \vec{r}_i \times \dot{\vec{P}}_i\right.$$

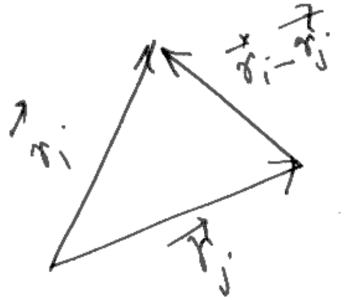
$$\vec{\tau} = \sum_{i=1}^{N} \vec{r}_i \times \left[\vec{F}_i^{EXT} + \vec{F}_i^{INT}\right].$$

$$= \tau^{EXT} + \tau^{INT}$$

We now prove that torque due to F^{INT} is zero.

Proof: Torque due to internal forces = 0.

Let $\vec{F}_{ij}^{INT}$ be the force on the i^{th} particle due to j^{th} particle, and be directed along line joining i^{th} and j^{th} particle.



$$\cancel{\vec{F}_{ij}^{INT} = \sum_j F_{ij}}$$

$$\vec{F}_i^{INT} = \sum_j \vec{F}_{ij}^{INT}$$

Total internal torque ~~due~~ ~~to~~ on all the particles relative to the chosen origin is,

$$\vec{\tau}^{INT} = \sum_{i=1}^{N} \vec{r}_i \times \vec{F}_i^{INT} = \sum_i \sum_j \vec{r}_i \times \vec{F}_{ij}^{INT} \quad (1)$$

Since indices i and j are both arbitrary and summed over we can interchange them, without affecting anything.

$$\vec{\tau}^{INT} = \sum_j \sum_i \vec{r}_j \times \vec{F}_{ji}^{INT} \quad (2)$$

Adding (1) and (2) and noting that $\vec{F}_{ij}^{INT} = -\vec{F}_{ji}$ due to Newton's third law, we have

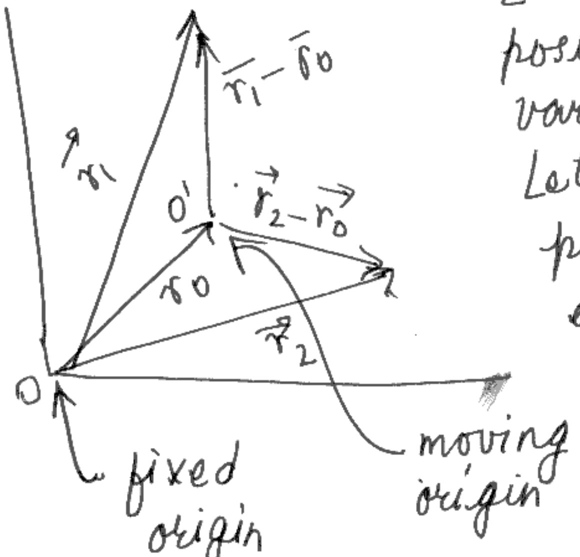
$$2\vec{\tau}^{INT} = \sum_i \sum_j (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ij}$$

But $\vec{F}_{ij}$ is along line joining $\vec{r}_i$ and $\vec{r}_j$ and hence parallel to $\vec{r}_i - \vec{r}_j$. Thus RHS = 0 and hence total torque due to internal forces is zero. This makes perfect sense because we do not see any extended body suddenly start spinning in the absence of external torques. Thus

$$\boxed{\tau^{EXT} = \frac{d\vec{L}}{dt} = \sum_i \vec{r}_i \times \vec{F}_i^{EXT}}$$

Note that nowhere we assumed that the particles are rigidly connected to each other. Thus particles are free to move relative to one another but in that case it is hard to get a handle on $\vec{L}$ as it is no longer of $I\omega$ form.

c) EXTENDED MASS NON-FIXED (POSSIBLY ACCELERATING) ORIGIN



Let $\vec{r}_i$ be the position vectors of various mass points. Let $\vec{r}_0$ be the position vector of an accelerating origin O'. Note $\vec{r}_i$ and $\vec{r}_0$ are all measured with respect to a fixed origin O. We are interested in computing angular momentum of the extended system of ~~mass~~ N mass points relative to an accelerating origin whose position vector is $\vec{r}_0$ relative to fixed origin.

$$\vec{L} = \sum_i \underbrace{(\vec{r}_i - \vec{r}_0)}_{\text{moment arm w.r.t } O'} \times m_i (\dot{\vec{r}}_i - \dot{\vec{r}}_0)$$

$$\vec{\tau} = \frac{d\vec{L}}{dt} = \sum_i \Big[\underbrace{(\dot{\vec{r}}_i - \dot{\vec{r}}_0) \times m_i (\dot{\vec{r}}_i - \dot{\vec{r}}_0)}_{\to 0} + (\vec{r}_i - \vec{r}_0) \times m_i (\ddot{\vec{r}}_i - \ddot{\vec{r}}_0)\Big]$$

$$\vec{\tau} = \sum_i (\vec{r}_i - \vec{r}_0) \times (\underbrace{m_i \ddot{\vec{r}}_i}_{F_i^{EXT} + F_i^{INT}} - m_i \ddot{\vec{r}}_0)$$

$F_i^{INT} \to 0 \; (\tau^{INT} = 0)$

$$\vec{\tau} = \sum_i (\vec{r}_i - \vec{r}_0) \times \vec{F}_i^{EXT} - \Big(\sum_i m_i \vec{r}_i - \sum_i m_i \vec{r}_0\Big) \times \ddot{\vec{r}}_0$$

$$\vec{\tau} = \underbrace{\sum_i (\vec{r}_i - \vec{r}_0) \times \vec{F}_i^{EXT}}_{①} - \underbrace{M(\vec{R}_{cm} - \vec{r}_0) \times \ddot{\vec{r}}_0}_{②}$$

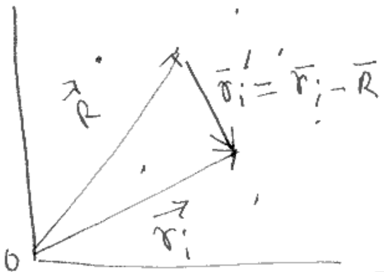
① = External torque measured with respect to non-fixed origin (may be accelerating)

② = Extra term due to non-fixed origin.
= 0 if (i) $\ddot{\vec{r}}_0 = 0$ or (ii) $\vec{R}_{cm} = \vec{r}_0$ or (iii) $(\vec{R}_{cm} - \vec{r}_0) \times \ddot{\vec{r}}_0 = 0$.

ANGULAR MOMENTUM OF A RIGID BODY THAT IS TRANSLATING AS WELL AS ROTATING:

Consider a rigid body as an assembly of large number of particles each of mass m_i and have position vector $\vec{r}_i$ with respect to some fixed inertial origin.

$$\vec{L} = \sum_i \vec{r}_i \times m_i \dot{\vec{r}}_i$$



$\vec{r}_i$ = P.V. of i^{th} particle with respect O (fixed).

$\vec{R}$ = P.V of CM of rigid body

$\vec{r}_i'$ = P.V of i^{th} particle w.r.t. CM

$\vec{r}_i' = \vec{r}_i - \vec{R}$

$\vec{r}_i = \vec{r}_i' + \vec{R}$.

This is correct but very boring. Hardly provides any insight about the details of dynamics.

Note that

$\vec{r}_i = \vec{r}_i' + \vec{R}$

Such a decomposition splits the dynamics into $\vec{r}_i'$ (motion about CM) and $\vec{R}$ (motion of the CM). This looks interesting.

Upon substitution $\vec{L}$ becomes.

$$L = \sum_i (\vec{r}_i' + \vec{R}) \times m_i (\dot{\vec{r}}_i' + \dot{\vec{R}})$$

$$= \underset{Ⓐ}{\sum_i \vec{r}_i' \times m_i \dot{\vec{r}}_i'} + \underset{Ⓑ}{\sum_i \vec{r}_i' \times m_i \dot{\vec{R}}} + \underset{Ⓒ}{\sum_i \vec{R} \times m_i \dot{\vec{r}}_i'} \cancel{+\Sigma}$$

$$+ \underset{Ⓓ}{\sum_i \vec{R} \times m_i \dot{\vec{R}}}.$$

Ⓐ $\equiv \sum_i \vec{r}_i' \times m_i \dot{\vec{r}}_i'$ This is obviously $\vec{L}$ about CM; that is L_{CM}.

Ⓑ $\equiv \sum_i (\vec{r}_i') \times m_i \dot{\vec{R}} = \sum_i (\vec{r}_i - \vec{R}) m_i \times \dot{\vec{R}}$ $\boxed{M = \sum_i m_i}$

$= \sum_i \underbrace{(m_i \vec{r}_i - M\vec{R})}_{\to\, 0 \text{ (Def}^n \text{ of CM)}} \times \dot{\vec{R}}$

$= 0$.

Ⓒ $\equiv \sum_i \vec{R} \times m_i \dot{\vec{r}}_i' = 0$ (In B we proved that $\sum_i m_i \vec{r}_i' = 0$, so $\sum_i m_i \dot{\vec{r}}_i' = 0$).

Ⓓ $\equiv M\vec{R} \times \dot{\vec{R}}$ $\equiv$ Ang momentum of a rigid body due to translation of CM.

Thus.

$$\vec{L} = \underbrace{\vec{L}_{CM}}_{\substack{\vec{L} \text{ in the CM frame} \\ \text{(SPIN PART)}}} + \underbrace{\vec{R} \times M\dot{\vec{R}}}_{\substack{\vec{L} \text{ due to CM motion } \cancel{\text{with}} \\ \text{w.r.t some fixed origin} \\ \text{(ORBITAL PART)}}}$$

$\boxed{L^z_{CM} = (I\omega)_z \text{ for fixed axis rotation}}$

CONSERVATION OF ANGULAR MOMENTUM
CENTRAL FORCES AND KEPLER'S LAW

$\vec{F} = \frac{d\vec{P}}{dt} \Rightarrow$ if $\vec{F} = 0$ $\vec{P}$ = const.

$\vec{\tau} = \frac{d\vec{L}}{dt}$; $\vec{\tau} = 0 \Rightarrow \vec{L}$ is conserved.

$\vec{\tau} = \vec{r} \times \vec{F} \Rightarrow \vec{F}$ need not be zero for $\vec{\tau} = 0$.

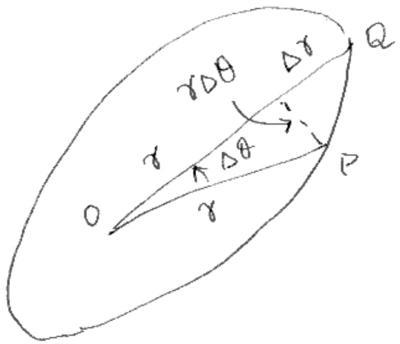
For central (radial) forces : $\vec{F} = f(r)\hat{r}$.

$\vec{\tau} = \vec{r} \times f(r)\hat{r} = 0 \Rightarrow \vec{L}$ is conserved.

If we take direction of $\vec{L} = |\vec{L}|\hat{z}$, conservation means it will always be $\hat{z}$.

Now $\vec{L} = \vec{r} \times \vec{P} \Rightarrow$ the motion is always in x-y plane.

MOTION OF PLANETS : Since gravity is a central force, the motion of planets is confined to the plane. Let us find Areal velocity of a planet going from P to Q.



Area $\Delta OPQ = \frac{1}{2}(r\Delta\theta)(r+\Delta r)$.

$$\Delta A = \frac{1}{2}r^2\Delta\theta + \underbrace{\frac{1}{2}r\Delta\theta\Delta r}$$

2^{nd} order in differential hence $\xrightarrow[\Delta t \to 0]{} 0$.

$$\frac{dA}{dt} = \lim_{\Delta t \to 0}\frac{\Delta A}{\Delta t} = \frac{1}{2}r^2\omega$$

$$\boxed{\frac{dA}{dt} = \frac{1}{2}r^2\omega}$$

$$\vec{L} = \vec{r} \times m\dot{\vec{r}} = r\hat{r} \times m(r\dot{\theta}\hat{\theta} + \dot{r}\hat{r}) = mr^2\omega\hat{z}$$

$\Rightarrow \boxed{\frac{d\vec{A}}{dt} = \frac{\vec{L}}{2m}} \Rightarrow$ CONSERVATION OF L AND AREAL VELOCITY ARE CONNECTED.

Thus, Kepler's second law of constancy of Areal velocity ~~is only~~ holds true very generally for all central forces, because conservation of angular momentum is a generic feature of central forces as seen below:

Central force $\Longrightarrow F_\theta = m a_\theta = 0$

$$a_\theta = (r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0$$

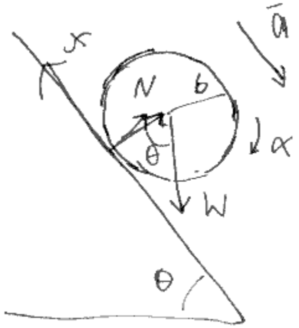
$$\Rightarrow m(r^2\ddot{\theta} + 2\dot{r}r\dot{\theta}) = 0$$

$$\frac{d}{dt}(\underbrace{mr^2\dot{\theta}}_{L}) = 0$$

$\vec{F} = f(r)\hat{r}$ $\leftrightarrow$ L = Const

$\frac{dA}{dt} = 0 \Rightarrow A$ = const.

EXAMPLE 6.16: DRUM ROLLING DOWN PLANE



A uniform drum of radius b and mass M rolls w/o slipping down a plane inclined at an angle θ. Find the a. ($I_0 = Mb^2/2$)

sol: We will solve this problem by taking torque about three different points.

METHOD-1:

$W\sin\theta - f = Ma$ Translation of CM.

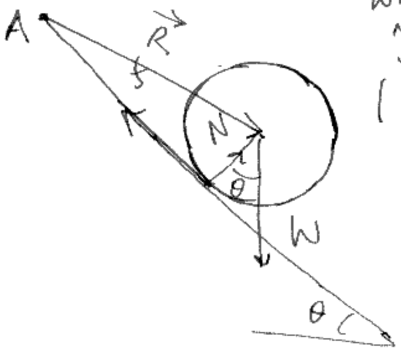
$bf = I_0\alpha$ $\boxed{\text{Torque about CM}}$.

$a = b\alpha$ rolling w/o slipping.

Eliminating f and using $I_0 = Mb^2/2$

$$\boxed{a = \frac{2}{3} g\sin\theta}$$

METHOD-2: Let us choose a coordinate system whose origin is A, on the plane.



Torque about A is.

$$(\vec{\tau}_{A})_z = \tau_0 + (\vec{R}\times\vec{F})_z.$$

Like L, τ also splits into two parts. Here $\vec{R}$ is position vector of CM from A. $\vec{F}$ = Net external force.

$$(\tau_A)_z = \tau_0 + (\vec{R}_\perp + \vec{R}_{\parallel})\times(\vec{N}+\vec{W}+\vec{f})$$

$$= -bf + \vec{R}_\perp\times\vec{N} + \vec{R}_\perp\times\vec{W} + \vec{R}_\perp\times\vec{f}$$
$$+ \vec{R}_{\parallel}\times\vec{N} + \vec{R}_{\parallel}\times\vec{W} + \vec{R}_{\parallel}\times\vec{f}.$$

$$= \cancel{-bf} + 0 - bW\sin\theta + \cancel{bf} + \cancel{R_\parallel N} - R_\parallel W\cos\theta + 0$$

$$(\tau_A)_z = -bW\sin\theta$$

$$(L_A)_z = L_{CM} + (\vec{R}\times M\dot{\vec{R}})_z$$

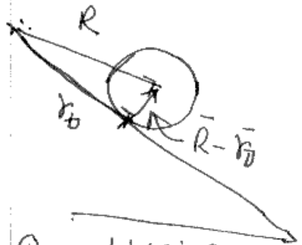
$$= -\frac{1}{2}Mb^2\omega - Mb^2\omega$$

$$= -\frac{3}{2}Mb^2\omega$$

since $\tau_z = dL_z/dt \Rightarrow bW\sin\theta = \frac{3}{2}Mb^2\dot{\omega}$

$$\Rightarrow \dot{\omega} = \alpha = \frac{2W}{3Mb}\sin\theta \quad \text{or} \quad \boxed{a = b\alpha = \frac{2}{3}g\sin\theta}$$

METHOD 3: Origin at the point of contact. ~~The~~ since point of contact is accelarating we must use the general formula for torque.



$$\tau = \underbrace{\sum_i (r_i - r_0)\times F^i_{ext}}_{①} - \underbrace{M(R - r_0)\times\ddot{r}_0}_{②}$$

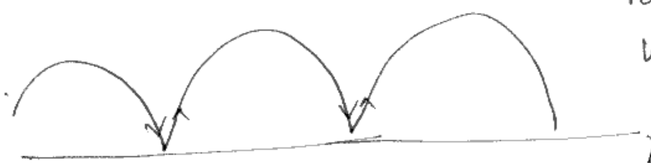
Here the ② term vanishes because cross product vanishes. Velocity of point of contact is downwards just before it touches plane and upwards just after that. Hence $\ddot{r}_0$ is facing down normal to incline.

① = $-bW\sin\theta$.

So $(\bar{R}-\bar{r}_D)\times\ddot{r}_D = 0$

Pointing up normal to incline — Pointing down normal to incline

Here of course ~~o~~ the position vector of origin (point of contact) is $\bar{r}_0$. The fact that the accelaration of point of contact ~~is pointing~~ ($\ddot{r}_0$) is pointing down can be understood from the fact that trajectory of any point of ~~a~~ on a rolling circle is a cycloid.



Just when the point hits the ground its velocity is pointing dowards and immediately after it, upwards. Thus, only first term contributes

$$\tau = -bW\sin\theta = \left(\frac{Mb^2}{2} + Mb^2\right)\alpha = \frac{3}{2}Mb^2\alpha$$

$$\Rightarrow \boxed{a = \frac{2}{3}g\sin\theta} \quad \text{since } a = b\alpha.$$

The important point to realize here that in general the second term exist. You must not neglect it without knowing why it does not contribute.

METHOD-4: We will now employ energy method and find the speed of rolling drum as it descends through height h. The drum starts at rest.

Translational Work-energy theorem.

$$\int_a^b \vec{F}\cdot d\vec{r} = \frac{1}{2}MV_b^2 - \frac{1}{2}MV_a^2 = \frac{1}{2}MV^2$$

$$(W\sin\theta - f)l = \frac{1}{2}MV^2 \quad (1) \qquad \boxed{l = h/\sin\theta}$$

For the rotational motion

$$\int_{\theta_a}^{\theta_b} \tau_0\, d\theta = \frac{1}{2}I_0\omega_b^2 - \frac{1}{2}I_0\omega_a^2$$

$$fb\theta = \frac{1}{2}I_0\omega^2$$

where θ is the angle through which drum rotates as it translates through l. $l = b\theta$.

$$fl = \frac{1}{2}I_0\omega^2$$

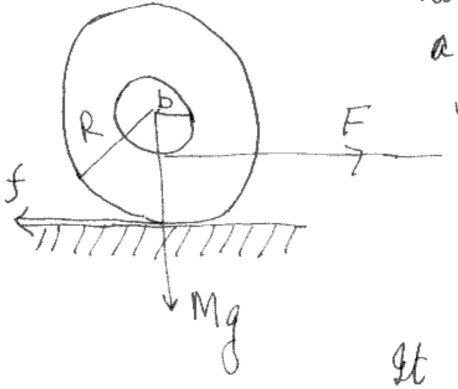
$$fl = \frac{1}{2}\frac{I_0V^2}{b^2} \quad (2)$$

Eliminating f from (1) & (2) we get $V = \sqrt{\frac{4gh}{3}}$

Interesting thing to note here is that force of friction here is non-dissipative. It decreases translational energy by an amount fl but the torque exerted by friction increases rotational energy by same amount. It is only when a rolling wheel flattens at the bottom that the torque due to N (which doesn't pass from center) decelerates.

$\boxed{6.27}$ A yo-yo of mass M has an axle of radius b and a spool of radius R. The M.I $= MR^2/2$. Yo-Yo is placed upright on a table and the string is pulled with the horizontal force F. The coefficient of friction between Yo-Yo and table is μ. What is maximum value of F for which Yo-Yo will roll without slipping.

Sol:



Since the Yo-Yo is supposed to roll without slipping, there is a net translational motion as well as rotational motion such that

$$\boxed{\begin{array}{l} l = R\theta \\ \text{or } a = R\alpha \end{array}} \text{ rolling w/o slipping.}$$

It is clear that Yo-Yo will translate to the right as $F > f$ (friction) for translation. Then

$$F - f = Ma \quad \text{Here } a > 0. \quad ①$$

There are two torques bF ~~as~~ (tending to rotate the Yo-Yo counter clockwise and hence +ve) and fR (tending to rotate the Yo-Yo in clockwise direction and hence −ve). According to translational equation of motion, the Yo-Yo moves to the right. The requirement that it should roll w/o slipping means that the torque which makes it rotate to the right in the clockwise direction ~~that~~ (fR) shall dictate the sign of angular accelaration α. Thus

$$bF - fR = -\frac{MR^2}{2}\alpha = -\frac{MR^2}{2}\left(\frac{a}{R}\right) \quad ②$$

Solving ① and ② we get

$$\boxed{F = \frac{3fR}{2b+R} = \frac{3\mu MgR}{2b+R}}$$