Poisson random variable

Let k > 0 be a constant

$$f(x) = \begin{cases} \dfrac{e^{-k} k^{x}}{x!}; & \text{for } x = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Theorem : f is a density function.

Theorem : The m.g.f. of a Poisson random variable *X* with parameter *k>0* is

$$m_X(t) = e^{k(e^t - 1)}$$

***E[X]=k* and Var[*X*]=*k*.**

Poisson Process : A process occurring discretely over a continuous interval of time or length or space is called a Poisson Process.

Steps in solving a Poisson problem

1 Determine the basic unit of measurement
2 Determine the average number of occurrences of the event per unit i.e λ
3. Determine length or size of observation period i.e s
4. Random variable X the number of occurrences of the event in the interval of size s follows a Poisson distribution with parameter $k=\lambda s$

(Continuous Random Variable)

Continuous Densities:

Defn: A random variable is continuous if it can assume any value in some interval(or intervals) of real numbers and the probability that it assume any specific value is 0 (zero).

CONTINUOUS PDF (Density Function)

Definition: Let X be a continuous random variable. A function f(x):(-∞,∞)= ì R is called probability density function of X if

$$i.\ f(x) \geq 0, \forall\ x \in (-\infty, \infty)$$

$$ii.\ \int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$iii.\ P(a \leq x \leq c) = \int_{a}^{c} f(x)\,dx$$

Necessary and sufficient condition for f:(-∞,∞) ì R to be density function of continuous random variable X

$$i.\ \mathrm{f(x)} \geq 0, \forall\, \mathrm{x} \in (-\infty, \infty)$$

$$ii.\ \int_{-\infty}^{\infty} f(x)dx = 1$$

(Thus P[*a* á *X* á *c*] is area under graph of y=f(x) between *x*=*a* and *x*=*c*.)

Comment : It is a consequence of the definition that for any specified value of X, say x_o , we have P[X=x_o] = 0, since

$$P[X = x_o] = \int_{x_o}^{x_o} f(x)\,dx = 0$$

Comment : *f*(x_0) ÷ *P*[*X*=x_0]. *If f is* **probability density function** *of continuous random variable X*

Comment . X is continuous r.v. but f(x) need not be continuous function of x , f(x) is defined on $(-\infty,\infty)$.

Comment . If X assumes values in some finite interval [a,c], we simply set f(x) = 0 for all $x \notin [a,c]$.

since X is continuous r.v, we have

$$P[a \leq x \leq c]$$

$$= P[a \leq x < c]$$

$$= P[a < x \leq c]$$

$$= P[a < x < c]$$

Example : Find r if f(x) is probability density function of a random variable X

$$f(x)=\begin{cases} r(1\text{-x}) & 0\le \text{x} \le 1 \\ \text{r}(x-1) & 1< x\le 3 \\ 0 & otherwise \end{cases}$$

Conditions to be checked

$$i.\ \mathrm{f(x)} \geq 0, \forall\ \mathrm{x} \in (-\infty, \infty)$$

$$ii.\int_{-\infty}^{\infty} f(x)dx = 1$$

Let X be the continuous r.v. with density f(x). The cumulative distribution function (CDF) for X ,denoted by F(X) , is defined by

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(w)\,dw$$

Theorem : If X is continuous random variable then CDF

$F(x) : (-\infty,\infty) \to [0,1]$ is a continuous monotonic (nondecreasing) function .

Discrete r.v. X

P(a<X ≤c)

=P(X ≤c)-P(X≤a)

=F(c)-F(a)

Continuous r.v X

$$P[a \le x \le c]$$

$$= P[a \le x < c]$$

$$= P[a < x \le c]$$

$$= P[a < x < c]$$

$$= F(c) - F(a)$$

(Continuous uniform distribution)

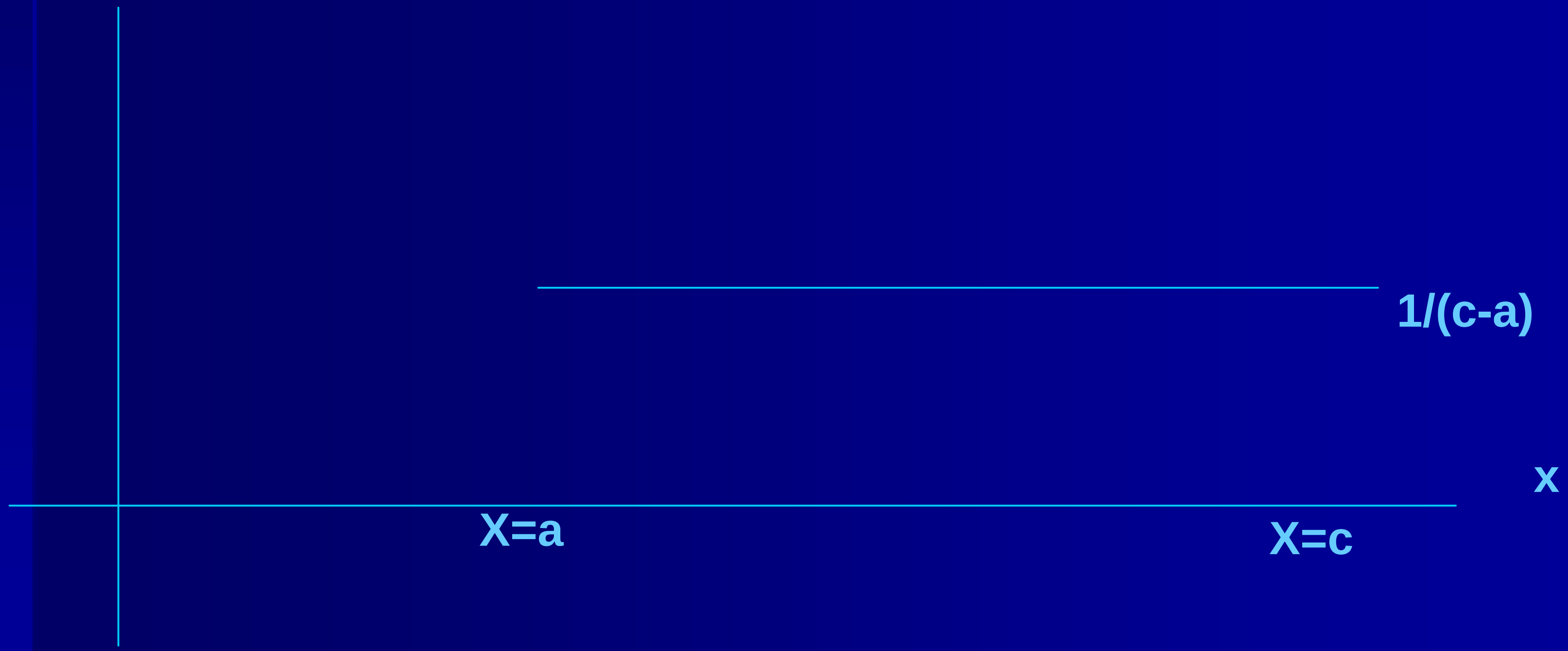
A random variable X is said to be uniformly distributed over an interval (a,c) if its density is given by

$$f(x) = \begin{cases} 0 & x \leq a \\ \dfrac{1}{c-a} & a<x<c \\ 0 & x \geq c \end{cases}$$

f(x) is a density for a continuous random variable.

$$\int_a^c \frac{1}{c-a}\,dx = 1$$

(i) graph of the uniform density.



(iv) Find the CDF $F(x)$ for the uniform r.v. for X defined on (a, c)

F(x) must be found in every interval (-∞,a],(a,c), & [c,∞)

$$F(x) = \int_{-\infty}^{x} f(w)\,dw$$

$$\text{in } -\infty < x \le a$$

$$F(x) = 0$$

$$F(x) = \int_{-\infty}^{x} f(w)\,dw \quad x \in (a,c)$$

$$F(x) = \int_{-\infty}^{a} f(w)dw + \int_{a}^{x} f(w)dw$$

$$= F(a) + \int_{a}^{x} \frac{1}{c-a} dw = \frac{x-a}{c-a}$$

x $\in$[c,∞)

$$F(x)=\int_{-\infty}^{x} f(w)\,dw \quad x\in[c,\infty)$$

$$F(x)=\int_{-\infty}^{c} f(w)dw+\int_{c}^{x} f(w)dw$$

$$=F(c)+\int_{1}^{x} 0dw=1,\ \ x\in[c,\infty)$$

$$F(x)=\begin{cases} 0, & x \le a \\ \dfrac{x-a}{c-a}, & a < x < c \\ 1, & x \ge c \end{cases}$$

Constructed: Find CDF of random variable X if f(x) is

$$f(x)=\begin{cases}\dfrac{1}{2} & -1\le x\le 0\\ x & 0<x<1\\ 0 & \text{otherwise}\end{cases}$$

F(x) must be found in every interval $(-\infty,-1),[-1,0],(0,1)$ & $[1,\infty)$

$$F(x) = \int_{-\infty}^{x} f(w)\,dw$$

$$in\ -\infty < x < -1$$

$$F(x) = 0$$

$$F(x)=\int_{-\infty}^{x} f(w)\,dw \quad x\in[-1,0]$$

$$f(x)=\begin{cases}\frac{1}{2} & \text{-}1\le \mathrm{x}\le 0\\ x & 0\le \mathrm{x}<1\\ 0 & \text{otherwise}\end{cases}$$

$$F(x)=\int_{-\infty}^{-1} f(w)dw+\int_{-1}^{x} f(w)dw$$

$$=\int_{-\infty}^{-1} 0dw+\int_{-1}^{x}\frac{1}{2}dw=\frac{x+1}{2}$$

$x \in (0,1)$

$$F(x) = \int_{-\infty}^{x} f(w)\,dw \quad x \in (0,1)$$

$$F(x) = \int_{-\infty}^{0} f(w)dw + \int_{0}^{x} f(w)dw$$

$$= F(0) + \int_{0}^{x} w dw = \frac{1}{2} + \frac{x^2}{2}, \quad x \in (0,1)$$

$x \in [1,\infty)$

$$F(x) = \int_{-\infty}^{x} f(w)\,dw \quad x \in [1,\infty)$$

$$F(x) = \int_{-\infty}^{1} f(w)dw + \int_{1}^{x} f(w)dw$$

$$= F(1) + \int_{1}^{x} 0dw = 1, \quad x \in [1,\infty)$$

F(x)=0　　x<-1

$$F(x)=\begin{cases} \dfrac{x+1}{2} & -1\leq x\leq 0 \\ \dfrac{1}{2}+\dfrac{x^2}{2} & 0<x<1 \\ 1 & x\geq 1 \end{cases}$$

Theorem: Let F be the continuous CDF of a continuous r.v with pdf f, then

$$f(x) = \frac{d}{dx} F(x),$$

for all x at which F is differentiable

Def: Let X be a continuous random variable with pdf f. The expected value of X is defined as

$$E[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx.$$

Again E(x) exists if and only if

$$\int_{-\infty}^{\infty} |x|\, f(x)\, dx \quad is \quad finite$$

For a random variable X which is a function, say H(x), the definition takes the form:

$$E[H(x)] = \int_{-\infty}^{\infty} H(x)\; f(x)\; dx.$$

provided

$$\int_{-\infty}^{\infty} |H(x)|\, f(x)\; dx \;\; is \;\; finite$$

Moment generating function :

$$m_X(t) = E[e^{tX}]$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

where f is density of X.

X is Continuous r.v.

Thm : Prove for real numbers a& c,
$E[aX+c]=aE[X]+c$.

Def : If a Continuous random variable X has mean μ, its variance Var(X) or σ^2 is defined by

$$\mathrm{Var}(X) = E[(X-\mu)^2].$$

Properties of variance

<u>Thm</u> : $Var[X]=E[X^2]-(E[X])^2$.

Thm : Var[aX+c]=a^2 Var(X)

Proof : Let W=aX+c

$$«_W = a«_X + c$$

$$Var(W) = E[(W - «_W)^2] = E(aX + c - a«_X - c)^2$$

$$= E(aX - a«_X)^2 = E\left[a^2(X - «_X)^2\right]$$

$$= a^2 E\left[(X - «_X)^2\right] \text{(Why?)}$$

$$= a^2 Var(X)$$

mean and variance of uniform random variable on interval (a,c)

$$E(X) = \frac{a+c}{2}$$

$$Var(X) = \frac{(c-a)^2}{12}$$

$$Var(X) = \frac{1}{c-a}\int_a^c \left(x - \frac{a+c}{2}\right)^2 dx = \frac{1}{c-a}\left[\frac{\left(x - \frac{a+c}{2}\right)^3}{3}\right]_a^c$$

$$= \frac{1}{3(c-a)}\left[\left(\frac{c-a}{2}\right)^3 - \left(\frac{a-c}{2}\right)^3\right]$$

$$= \frac{1}{3(c-a)}\left[\left(\frac{c-a}{2}\right)^3 + \left(\frac{c-a}{2}\right)^3\right]$$

$$= \frac{2}{3(c-a)}\left[\left(\frac{c-a}{2}\right)^3\right] = \frac{(c-a)^2}{12}$$

Moment generating function :

$$m_X(t) = E[e^{tX}]$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

where f is density of X.

MGF of uniform distribution random variable on (a, c) :

$$E(e^{tx}) = \int_a^c e^{tx} \frac{dx}{c-a}$$

$$= \frac{1}{t}\left(\frac{e^{tc} - e^{ta}}{c-a}\right), \quad t \neq 0$$

Section 4.3:

Definition : The **Gamma function** is the function Γ defined by

$$\Gamma(\sim) = \int_0^\infty z^{\sim -1} e^{-z}\, dz, \sim > 0$$

4.3: Gamma Random variable

Gamma Function: which is an improper integral allows us to define exponential and chi- square random variables.

Gamma function:

$$\Gamma(\sim) = \int_0^{\infty} z^{\sim -1} e^{-z} \, dz, \sim > 0$$

Theorem: (Properties of Gamma function)

1. $\Gamma(1) = 1$

2. $\Gamma(\sim) = (\sim - 1)\Gamma(\sim - 1), \ \textit{for all} \sim > 1$

Proof:

by definition of Gamma function, we have

$$\Gamma(1) = \int_0^\infty z^0 \; e^{-z} \, dz = \int_0^\infty e^{-z} \; dz = 1$$

by integration by parts, we have

$$\Gamma(\sim) = \int_0^{\infty} z^{\sim -1} e^{-z}\ dz, \sim > 0$$

$$= -e^{-z}\ z^{\sim -1}\ \Big|_0^{\infty} + (\sim -1) \int_0^{\infty} e^{-z}\ z^{(\sim -1)-1}\, dz$$

$$\lim_{z\to\infty} \frac{z^{\sim -1}}{e^{z}} = 0\ \&\left[\lim_{z\to 0} \frac{z^{\sim -1}}{e^{z}} = 0 \quad \text{if } \sim > 1\right]$$

$$= (\sim -1)\,\Gamma(\sim -1)\ ,\ \sim > 1$$

$$\Gamma(n) = (n-1)!$$

$$Since, \Gamma(n) = (n-1)\ \Gamma(n-1)$$

$$= (n-1)(n-2)\Gamma(n-2)$$

$$= (n-1)(n-2)\cdots\Gamma(1) = (n-1)!$$

Thus, Gamma function is generalization of the Factorial notation

$$\Gamma(\frac{1}{2}) = \int_0^\infty z^{-1/2} e^{-z}\ dz = \sqrt{\text{®}}$$

GAMMA RANDOM VARIABLE

A random variable X with density function is said to have a Gamma Distribution with parameters α and β, for x>0, α>0, β>0.

$$f(x)=\begin{cases}\dfrac{1}{\left(\Gamma(\sim)\right)^{\sim}}x^{\sim-1}e^{-x/}\ , & x>0, \sim>0, \quad >0\\ 0, & x\le 0\end{cases}$$

To check the necessary and sufficient condition of pdf:

$$f(x) \geq 0 \quad for\ all \quad x \in (-\infty, \infty)$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{1}{\Gamma(\sim)\quad \sim} \int_{0}^{\infty} x^{\sim-1} e^{-x/} \quad dx$$

$$Let \ \frac{x}{\quad} = z \Rightarrow dx = \quad dz, \ and \ \ x = \quad z$$

$$= \frac{1}{\Gamma(\sim)\quad \sim} \int_{0}^{\infty} {}^{\sim-1} z^{\sim-1} e^{-z} \quad dz$$

$$= \frac{\sim}{\Gamma(\sim \quad)\quad \sim} \int_{0}^{\infty} z^{\sim-1} e^{-z} = 1$$

Hence f(x) is a pdf

<u>Theorem:</u> Let X be a gamma random variable with parameters α and β. Then m.g.f for X is given by:

$$i.\; m_x(t) = (1 - \quad t)^{-\sim}, t < \frac{1}{\quad}$$

$$ii.\; E[X] = \sim$$

$$iii.\; Var(x) = \sim \quad ^{2}$$

$$m_x(t) = E[e^{tx}] = \frac{1}{\Gamma(\sim)\ \ \sim} \int_0^\infty e^{tx}\ x^{\sim -1}\ e^{-x/}\ \ dx$$

$$= \frac{1}{\Gamma(\sim)\ \ \sim} \int_0^\infty x^{\sim -1} e^{-(\frac{1}{\ } - t)x}\ \ dx$$

$$z = (1- \ \ t)\frac{x}{\ } \Rightarrow x = \frac{z}{(1- \ \ t)},\ dx = \frac{dz}{(1- \ \ t)}$$

$$= \frac{1}{\Gamma(\sim)\ \ \sim} \int_0^\infty \left(\frac{z}{1- \ \ t}\right)^{\sim -1}\ e^{-z}\ \frac{dz}{(1- \ \ t)}$$

$$= \frac{1}{\Gamma(\sim) \quad \sim} \int_0^{\infty} \left(\frac{z}{1- \quad t}\right)^{\sim -1} e^{-z} \frac{dz}{(1- \quad t)}$$

$$= \frac{1}{\Gamma(\sim) \quad \sim} \frac{\sim}{(1- \quad t)^{\sim}} \int_0^{\infty} z^{\sim -1} e^{-z} \, dz$$

$$= \frac{1}{\Gamma(\sim) \quad \sim} \frac{\sim}{(1- \quad t)^{\sim}} \Gamma(\sim)$$

$$m_X(t) = (1- \quad t)^{-\sim}, \; t < \frac{1}{\quad}$$

$$m_x(t) = (1 - \quad t)^{-\sim},\ t < \frac{1}{\quad}$$

$$\frac{dm_X(t)}{dt} = -\sim(1 - \quad t)^{-\sim-1}(- \quad)$$

$$\left.\frac{dm_X(t)}{dt}\right|_{t=0} = -\sim(1 - \quad t)^{-\sim-1}(- \quad)\Big|_{t=0}$$

$$= \sim$$

$$\frac{dm_X(t)}{dt} = -\sim(1- \quad t)^{-\sim-1}(- \quad)$$

$$m''_X(t) = \frac{d}{dt}\left[-\sim(1- \quad t)^{-\sim-1}(- \quad) \right]$$

$$= -\sim \quad (-\sim -1)(1- \quad t)^{-\sim-2}$$

$$m''_X(t)\Big|_{t=0} = \sim(\sim +1) \quad ^2$$

Exponential distribution : exponential random variable is Gamma random variable with α=1.The density is

$$f(x) = \begin{cases} \dfrac{1}{} e^{-\frac{x}{}} & x > 0, \quad > 0 \\ 0 & x \leq 0 \end{cases}$$

$\beta > 0$ is the parameter of this exponential distribution.

$E[X]=\beta$, $Var[X]= \beta^2$.

The c.d.f. of exponential distribution with parameter   is given by

$$F(x) = \int_{-\infty}^{x} f(x)dx = 0 \quad \text{if} \;\; x \le 0$$

$$\text{if} \quad x>0 \Rightarrow \mathrm{F(x)} = \int_{-\infty}^{x} f(x)dx = \int_{0}^{x} \frac{1}{\;} e^{-\frac{s}{\;}} ds$$

$$F(x)=\int_0^x \frac{1}{\quad} e^{-\frac{s}{\quad}} ds = \frac{1}{\quad} \left. \frac{e^{-\frac{s}{\quad}}}{-\frac{1}{\quad}} \right|_0^x$$

$$= -e^{-\frac{s}{\quad}} \Big|_0^x = 1 - e^{-\frac{x}{\quad}} \qquad x > 0$$

$$F(x) = \begin{cases} 0 & \text{x} \le 0 \\ 1 - e^{-x/} \quad , & x > 0, \end{cases} \quad .$$

Moment generating function, Mean and Variance of exponential distribution

Comment: Put ~=1 in the gamma distribution we get the required results.

$$m_X(t) = (1 - \quad t)^{-1}$$

$$E[X] = mean =$$

$$Var(X) = \quad ^2$$

<u>Poisson Process and Exponential dist :</u>For a Poisson process with parameter ª, the *waiting time* W is the time in the given interval before the1st success.

<u>Theorem</u> :W has an exponential distribution with parameter =1/ª.

Proof: **W is the time in the given interval before the 1st success.**

The distribution function F for W is given by

$$F(w) = P[W \le w] = 1 - P[W > w]$$

i.e., we have that the first occurrence of the event will take place after time w only if number of occurrences in the time interval [0,w] is zero

$F(w)=0$ if $w \le 0$

Let X be the number of occurrences of the event in this time interval [0,w]. $w>0$

X is a poisson random variable with parameter λw.

$$\Rightarrow P[W > w] = P[X = 0]$$

$$= \frac{e^{-aw}(aw)^0}{0!} = e^{-aw}$$

$$\Rightarrow F(w) = 1 - P[W > w] = 1 - e^{-aw} \quad w>0$$

$$=0 \quad \text{if} \quad w \leq 0$$

$$\Rightarrow F(w) = 1 - P[W > w] = 1 - e^{-a w} \qquad w>0$$

$$=0 \quad \text{if} \quad w \leq 0$$

$$f(w) = \frac{dF}{dw} = a e^{-a w} \qquad \text{if} \;\; w>0$$

$$=0 \qquad\qquad w \leq 0$$

W has an exponential distribution with parameter =1/a.

GAMMA RANDOM VARIABLE

A random variable X with density function is said to have a Gamma Distribution with parameters α and β,for x>0, α>0, β>0.

$$f(x)=\begin{cases}\dfrac{1}{\left(\Gamma(\sim)\right)\quad^{\sim}}x^{\sim-1}e^{-x/} & ,\ x>0, \sim>0, \quad >0\\[2ex] 0, & other\ wise\end{cases}$$

$$E[X]$$

$$= \int_0^\infty x \frac{1}{\Gamma(\sim)\ \ \sim} x^{\sim -1} e^{-x/}\, dx$$

$$= \frac{1}{\Gamma(\sim)\ \ \sim} \int_0^\infty x^{(\sim+1)-1} e^{-(\frac{1}{\ })x}\, dx$$

$$= \frac{1}{\Gamma(\sim)\ {}^{\sim}} \int_{0}^{\infty} x^{(\sim+1)-1} e^{-(\frac{1}{\ })x} \, dx$$

$$let \;\; z = \frac{x}{\ } \Rightarrow x = z$$

$$and \;\; dx = \;\; dz$$

$$= \frac{1}{\Gamma(\sim)\quad \sim} \int_0^\infty x^{(\sim+1)-1} e^{-(\frac{1}{\quad})x} \, dx$$

$$= \frac{1}{\Gamma(\sim)\quad \sim} \int_0^\infty (\quad z)^{\sim+1-1} e^{-z} \, dz$$

$$= \frac{\sim+1}{\Gamma(\sim)\quad \sim} \int_0^\infty z^{\sim+1-1} e^{-z} \, dz$$

$$= \frac{\sim +1}{\Gamma(\sim)} \sim \int_0^{\infty} z^{\sim +1-1} e^{-z} \, dz$$

$$= \frac{\sim +1}{\Gamma(\sim)} \sim \Gamma(\sim +1) = \sim$$

$$E[X^{2}]$$

$$= \int_{0}^{\infty} x^{2} \frac{1}{\Gamma(\sim) \quad \sim} x^{\sim -1} e^{-x/} \quad dx$$

$$= \frac{1}{\Gamma(\sim) \quad \sim} \int_{0}^{\infty} x^{(\sim +2)-1} e^{-(\frac{1}{\quad})x} \quad dx$$

$$= \frac{1}{\Gamma(\sim)\ \ \sim} \int_0^\infty x^{(\sim+2)-1} e^{-(\frac{1}{\ })x}\ dx$$

$$let\ \ z = \frac{x}{\ } \Rightarrow x = z$$

$$and\ \ dx = \ \ dz$$

$$= \frac{1}{\Gamma(\sim)\ \ \sim} \int_0^\infty (\ \ z)^{\sim+2-1} e^{-z}\ \ dz$$

$$= \frac{1}{\Gamma(\sim) \quad \sim} \int_0^\infty (\quad z)^{\sim+2-1} e^{-z} \quad dz$$

$$= \frac{\sim+2}{\Gamma(\sim) \quad \sim} \int_0^\infty z^{\sim+2-1} e^{-z} \, dz$$

$$= \frac{\sim+2}{\Gamma(\sim) \quad \sim} \Gamma(\sim+2)$$

$$=\frac{{}^{\sim+2}}{\Gamma(\sim)\quad{}^{\sim}}\Gamma(\sim+2)$$

$$=\frac{(\sim+1)\Gamma(\sim+1)\quad{}^{2}}{\Gamma(\sim)}$$

$$=\frac{(\sim+1)\sim\Gamma(\sim)\quad{}^{2}}{\Gamma(\sim)}=\sim(\sim+1)\quad{}^{2}$$

GAMMA RANDOM VARIABLE

A random variable X with density function is said to have a Gamma Distribution with parameters α and β,for x>0, α>0, β>0

$$f(x)=\begin{cases}\dfrac{1}{\left(\Gamma(\sim)\right)\ \ \ \sim} x^{\sim-1}e^{-x/\ } & , x>0, \sim>0, \ \ >0 \\ 0, & other\ wise\end{cases}$$

<u>Chi-square distribution</u> : If a random variable X has a gamma distribution with parameters $\beta=2$ and $\alpha=\gamma/2$, then X is said to have a chi-square (χ^2) distribution with γ degrees of Freedom and denoted by X^2_{γ}, γ is a positive Integer.

$$f(x)=\frac{1}{\Gamma(\frac{¥}{2})2^{¥/2}}x^{\frac{¥}{2}-1}e^{-x/2}, x>0$$

β=2 and α=γ/2,

i. $E[X] = \sim$

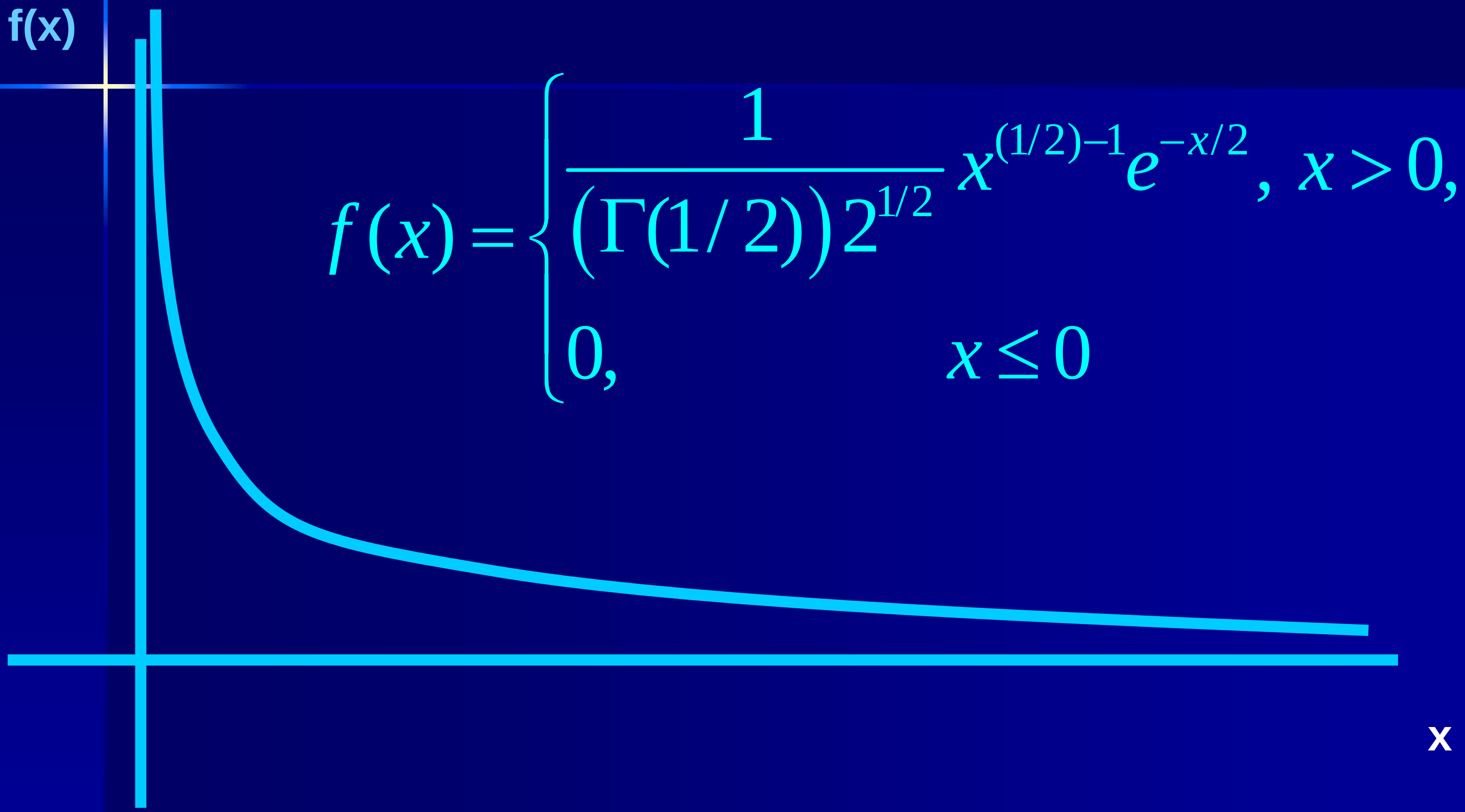
ii. $Var(x) = \sim^{2}$

$$E[X_{¥}^{2}]=¥, Var[X_{¥}^{2}]=2¥$$

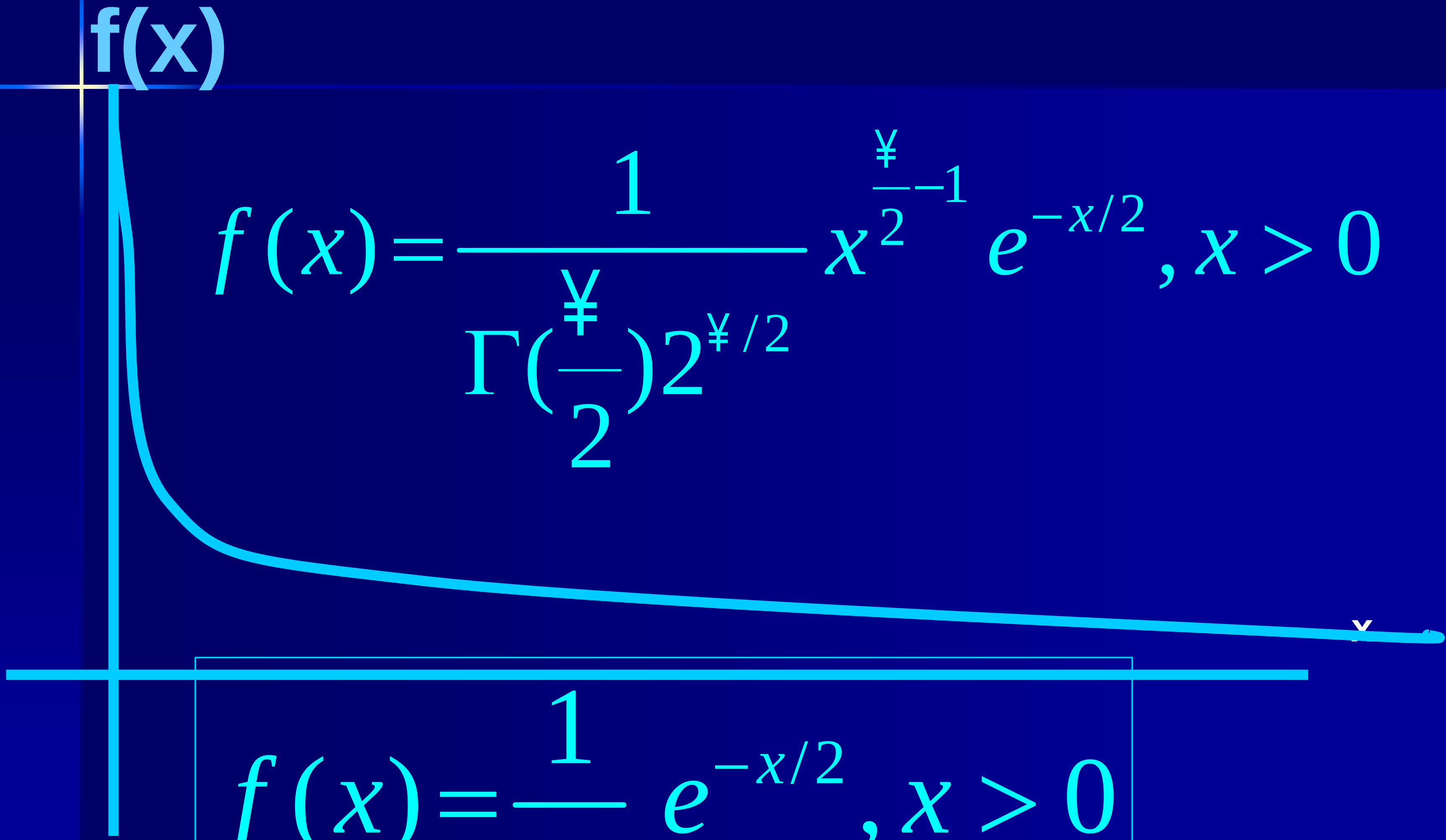
Chi square distribution

for ¥ =1

$$f(x)=\begin{cases}\dfrac{1}{\left(\Gamma(1/2)\right)2^{1/2}}x^{(1/2)-1}e^{-x/2}, & x>0,\\ 0, & x\le 0\end{cases}$$



Chi square distribution $for\ ¥ = 2$



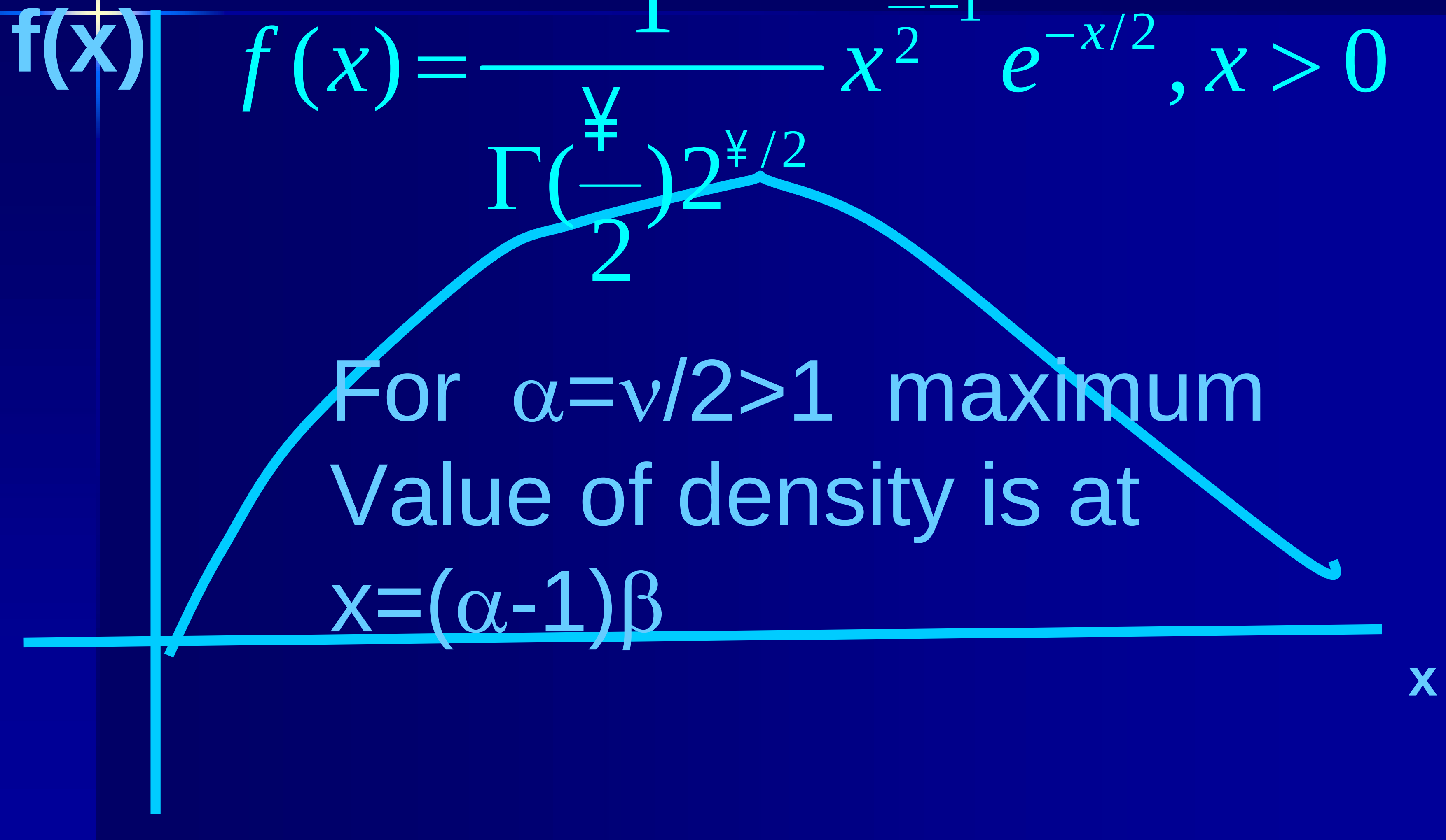
$$f(x) = \frac{1}{\Gamma(\frac{¥}{2})2^{¥/2}} x^{\frac{¥}{2}-1} e^{-x/2}, x > 0$$

$$\boxed{f(x) = \frac{1}{2} e^{-x/2}, x > 0}$$

$for\ ¥ > 2$

$$f(x) = \frac{1}{\Gamma(\frac{¥}{2})2^{¥/2}} x^{\frac{¥}{2}-1} e^{-x/2}, x > 0$$

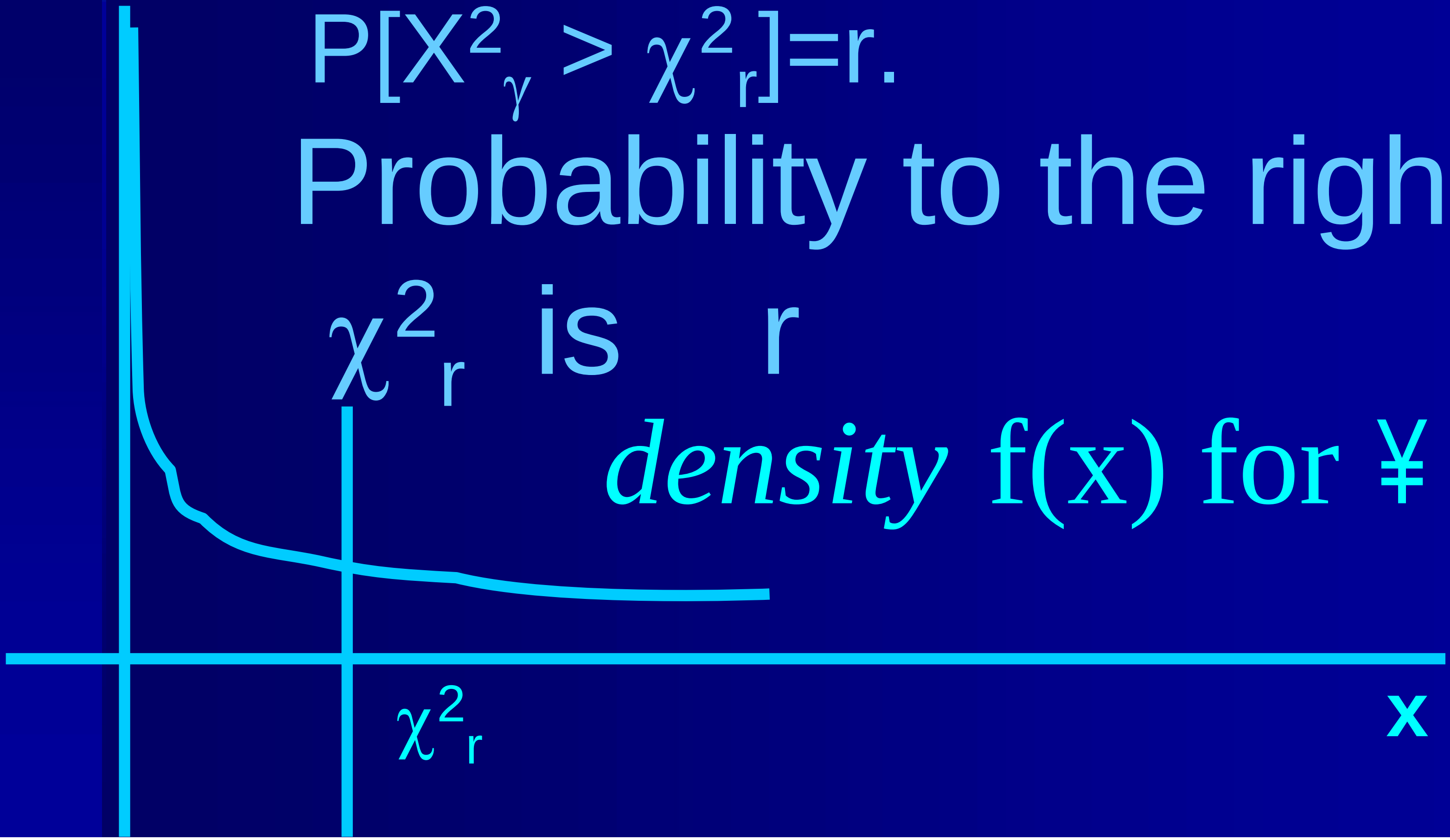
For $\alpha=\nu/2>1$ maximum Value of density is at $x=(\alpha-1)\beta$



tabulation : For $0<r<1$, we denote by χ^2_r, for a chi-square r.v. with γ degrees of freedom, a unique number such that

$$P[X^2_\gamma > \chi^2_r] = r.$$

Probability to the right of χ^2_r is r



We do not have explicit formula for CDF F of X^2_{γ}. In stead values are tabulated on
p. 695-696 as below (F occurs in margin here, and related value of r.v. inside the table):

		$P[X^2_\gamma < t]$	
γ F	0.10	0.250	0.500
5	1.61	2.67	4.35
6	2.20	3.45	5.35
7	2.83	4.25	6.35

If F is CDF for Chi square random Variable Having 5 degrees of freedom

F(1.61)= .1

F(4.35)= .5

4.4 The Normal Distribution

- A random variable X with density f(x) is said to have normal distribution with parameters μ and σ > 0, where f(x) is given by:

$$f(x)=\frac{1}{\sqrt{2\pi\sigma}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \qquad x,\mu\in(-\infty,\infty);\ \sigma>0.$$

$$(i)\ f(x) \geq 0 \quad \forall \mathrm{x} \in (-\infty,\infty) \quad (\mathrm{ii}) \int_{-\infty}^{\infty} \mathrm{f(x)dx} = 1$$

to prove

$$\frac{1}{\sqrt{2 \circledR \pm^{2}}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\text{«}}{\pm}\right)^{2}} dx = 1$$

$$\text{let}\ \left(\frac{x-\text{«}}{\pm}\right)=z\Rightarrow dx=\pm\,dz$$

$$\Rightarrow\frac{1}{\sqrt{2\text{®}\pm^{2}}}\int_{-\infty}^{\infty}e^{-\frac{1}{2}\left(\frac{x-\text{«}}{\pm}\right)^{2}}dx$$

$$=\frac{1}{\sqrt{2\text{®}}}\int_{-\infty}^{\infty}e^{-\frac{1}{2}z^{2}}dz$$

$$\int_0^\infty e^{-\frac{1}{2}z^2}\,dz = I$$

$$I \times I = \left(\int_0^\infty e^{-\frac{1}{2}x^2}\,dx\right)\left(\int_0^\infty e^{-\frac{1}{2}y^2}\,dy\right)$$

$$I \times I = \left(\int_0^\infty e^{-\frac{1}{2}x^2} dx \right) \left(\int_0^\infty e^{-\frac{1}{2}y^2} dy \right)$$

$$= \left(\int_0^\infty \int_0^\infty e^{-\frac{1}{2}(x^2+y^2)} dxdy \right)$$

$$\left(\int_0^{\infty} \int_0^{\infty} e^{-\frac{1}{2}(x^2+y^2)} dxdy \right)$$

$$= \lim_{R\to\infty} \left(\int_0^{\text{®}/2} \int_0^{R} e^{-\frac{1}{2}(r^2)} rdrd\text{¯} \right)$$

$$= \lim_{R\to\infty} \left(\int_0^{\text{®}/2} \int_0^{R} e^{-\frac{1}{2}(r^2)}\, r\,dr\,d\text{¯} \right)$$

$$= \lim_{R\to\infty} \left(\int_0^{\text{®}/2} \int_0^{R^2/2} e^{-w}\, dw\, d\text{¯} \right)$$

$$= \frac{\text{®}}{2}$$

$$i.e.\ \ \mathrm{I} = \sqrt{\frac{\circledR}{2}}$$

$$\frac{1}{\sqrt{2\circledR}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\, dz = 2\frac{1}{\sqrt{2\circledR}} \int_{0}^{\infty} e^{-\frac{1}{2}z^2}\, dz$$

$$= \frac{\sqrt{2}}{\sqrt{\circledR}} \times \frac{\sqrt{\circledR}}{\sqrt{2}} = 1$$

Standard Normal distribution

A random variable Z with density f(z) is said to be standard normal random variable if f(z) is

$$f(z)=\frac{1}{\left(\sqrt{2\circledR}\right)}e^{-\frac{z^2}{2}},-\infty<z<\infty$$

Moment Generating Function

Let Z be normally distributed with parameters μ=0 and σ=1 , then the moment generating function for Z is given by

$$m_Z(t) = e^{t^2/2}$$

$$m_Z(t) = E(e^{tz}) = \frac{1}{\sqrt{2\text{®}}}\int_{-\infty}^{\infty} e^{tz} e^{-\frac{1}{2}(z)^2}\, dz$$

$$= \frac{1}{\sqrt{2\text{®}}}\int_{-\infty}^{\infty} e^{tz-\frac{1}{2}(z)^2}\, dx$$

$$tz - \frac{1}{2}(z)^2 = -\left(\frac{z^2 - 2tz + t^2}{2}\right) + \frac{t^2}{2}$$

$$= \frac{t^2}{2} - \left(\frac{(z-t)^2}{2}\right)$$

$$m_Z(t) = \left(e^{t^2/2}\right)\frac{1}{\sqrt{2\text{®}}}\int_{-\infty}^{\infty} e^{-\frac{(z-t)^2}{2}}\,dz$$

$$\text{let }\ (\text{z-t}) = \text{w} \Rightarrow dz = dw$$

$$m_Z(t) = \left(e^{t^2/2}\right)\frac{1}{\sqrt{2\text{®}}}\int_{-\infty}^{\infty} e^{-\frac{w^2}{2}}\,dw$$

$$= \left(e^{t^2/2}\right) \qquad \left(as\,\frac{1}{\sqrt{2\text{®}}}\int_{-\infty}^{\infty} e^{-\frac{w^2}{2}}\,dw = 1\right)$$

$$m_Z(t) = \left(e^{t^2/2}\right)\frac{1}{\sqrt{2\circledR}}\int_{-\infty}^{\infty} e^{-\frac{(z-t)^2}{2}}\,dz$$

$$\text{let } (\text{z-t}) = \text{w} \Rightarrow dz = dw$$

$$m_Z(t) = \left(e^{t^2/2}\right)\frac{1}{\sqrt{2\circledR}}\int_{-\infty}^{\infty} e^{-\frac{w^2}{2}}\,dw$$

$$= \left(e^{t^2/2}\right) \qquad \left(as\ \frac{1}{\sqrt{2\circledR}}\int_{-\infty}^{\infty} e^{-\frac{w^2}{2}}\,dw = 1\right)$$

Theorem: Let X be normal with parameters μ & σ. Then variable

$$W = \frac{X - «}{\pm}$$

is standard normal.

$$F_W(a) = P(W \le a) = P\left(\frac{X - \text{«}}{\pm} \le a\right)$$

$$= P(X \le \pm a + \text{«}) = \frac{1}{\sqrt{2\text{®}\,\pm}} \int_{-\infty}^{\pm a + \text{«}} e^{-\frac{(s-\text{«})^2}{2\pm^2}}\, ds,$$

$$\frac{s - \text{«}}{\pm} = z, \quad \Rightarrow \mathrm{s} = \text{«} + \pm z, \; \mathrm{ds} = \pm dz$$

$$F_W(a) = P(X \le \pm a + «) = \frac{1}{\sqrt{2® \pm}} \int_{-\infty}^{\pm a + «} e^{-\frac{(s-«)^2}{2\pm^2}} \, ds,$$

$$\frac{s - «}{\pm} = z, \quad \Rightarrow s = « + \pm z, \ \mathrm{ds} = \pm \, dz$$

$$F_W(a) = P(X \le \pm a + «) = \frac{1}{\sqrt{2®}} \int_{-\infty}^{a} e^{-z^2/2} \, dz,$$

$$= \mathrm{F_Z(a)}$$

Mean and Standard deviation for Normal distribution

Theorem : Let X be a normal random variable with parameters μ and σ. Then μ is the mean of X and σ is its standard deviation.

Moment Generating Function

Let X be normally distributed with parameters μ and σ, then the moment generating function for X is

$$m_X(t) = e^{\text{«}t + \frac{\pm^2 t^2}{2}}$$

$$m_X(t) = E(e^{tx}) = E(e^{t(\pm z + \text{«})})$$

$$= E(e^{t\text{«}} e^{t\pm z})$$

$$= e^{t\text{«}} E(e^{(t\pm)z})$$

$$= e^{t\text{«}} e^{t^2 \pm^2/2}$$

Mean and Standard deviation for Normal random variable

Theorem : Let X be a normal random variable with parameters μ and σ. Then μ is the mean of X and σ is its standard deviation.