Birla Institute of Technology & Science, Pilani

First Semester 2017-2018, MATH F111 (Mathematics I)
Mid Semester Examination (Closed Book)

Time: 90 Min. **Date: October 12, 2017 (Thursday)** **Max. Marks: 105**

1. Write solution of each question on fresh page. Answer each subpart of a question in continuation.
2. Write **END** in the answer sheet just after the final attempted solution.

1. Test the absolute and conditional convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n - \ln n}$. **[17]**

2. (a) Find the center, radius and interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n(4x-8)^n}{n}$. Also identify the values of x for which the series converges (i) absolutely (ii) conditionally. **[12]**

   (b) Find the first three non-zero terms in the Taylor series expnasion of $\sin x$ about $x = \frac{\pi}{2}$. **[5]**

3. (a) Shade the region in the first quadrant inside the circle $r = \sin\theta$ and outside the curve $r = \cos 2\theta$. Find all the intersection points and label them. Also find area of the shaded region. **[12]**

   (b) Find the length of the curve $r = \sqrt{1 + \sin 2\theta},\ 0 \le \theta \le \pi\sqrt{2}$. **[5]**

4. Find the unit tangent, normal, and binormal vectors at the point $(\sqrt{2}, \sqrt{2}, \frac{\pi}{2})$ of the curve $\mathbf{r}(t) = 2\cos(t)\mathbf{i} + 2\sin(t)\mathbf{j} + 2t\mathbf{k}$. Also find the curvature at the given point. **[18]**

5. (a) Find and sketch the domain of the function

$$f(x,y) = \frac{\sqrt{x^2+y^2-9}}{x}.$$

   Determine if the domain is an open region, a closed region, or neither. Justify your answer. **[8]**

   (b) Examine the continuity of the following function at $(0,0)$: **[10]**

$$f(x,y) = \begin{cases} \frac{\sin^{-1}(x+2y)}{\tan^{-1}(2x+4y)}, & (x,y) \neq (0,0) \\ \frac{1}{2}, & (x,y) = (0,0) \end{cases}$$

6. (a) Let $f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^4}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$. Show that $f_x(0,0)$ and $f_y(0,0)$ exist. Is f differentiable at $(0,0)$? Justify. **[9]**

   (b) Let $f(x,y) = x^2 + xy + y^2$ where $x = uv$ and $y = u/v$. Show that $uf_u + vf_v = 2xf_x$. **[9]**

END

($\cancel{1}$+10)

Sol^n-1. $a_n = (-1)^n \frac{\ln n}{n - \ln n}$

$|a_n| = \frac{\ln n}{n-\ln n}$ — (1)

$n - \ln n \le n$

$\frac{1}{n} \le \frac{1}{n-\ln n}$

$\frac{1}{n} \le \frac{\ln n}{n - \ln n} \quad \forall \ n \ge 2$. — (3)

Since $\sum \frac{1}{n}$ diverges by p-series test — (1)

$\therefore \sum \frac{\ln n}{n - \ln n}$ diverges by Direct Comparison test. — (1)

$\Rightarrow$ Given series is not absolutely convergent. — (1)

Now, $u_n > 0 \ \forall \ n \ge 1$ — (1).

Consider $f(x) = \frac{\ln x}{x - \ln x}$.

$$f'(x) = \frac{(x-\ln(x))\,1/x - \ln x\,(1 - 1/x)}{(x-\ln x)^2}$$

$$= \frac{1 - \frac{\ln x}{x} - \ln x + \frac{\ln x}{x}}{(x - \ln x)^2}$$ (3)

$$= \frac{1 - \ln x}{(x - \ln x)^2} < 0 \text{ whenever } x > e$$ — (1)

$\Rightarrow u_n \ge u_{n+1} \quad \forall \ n \ge 3$.

$\lim_{n\to\infty} \frac{\ln n}{n - \ln n} = \lim_{n\to\infty} \frac{\ln n/n}{1 - \ln n/n} = 0.$ — (3)

Since, all the three conditions of Liebnitz test — (1).
satisfied, hence the given series is conditionally
convergent. — (1)

[Note: $\cancel{E}$ (−2) for evaluating the limit using this way.
$\lim_{n\to\infty} \frac{\ln n}{n - \ln n} = \lim_{n\to\infty} \frac{1/n}{1-1/n}$ (L'hospital rule)
$= 0.$]

Soln 2. (a). Let $a_n = \dfrac{2^n (4x-8)^n}{n}$

By using Ratio Test,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}(4x-8)^{n+1}}{n+1} \times \frac{n}{2^n(4x-8)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{2n(4x-8)}{(n+1)}\right| = |4x-8| \lim_{n\to\infty}\left|\frac{2n}{(n+1)}\right|$$

$$\left[\because \lim_{n\to\infty}\frac{n}{n+1} = 1\right]$$

$$= 2|4x-8|$$

Thus, the series converges absolutely if

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 2|4x-8| < 1$$

$$\Rightarrow |4x-8| < \frac{1}{2} \Rightarrow \frac{15}{8} < x < \frac{17}{8} \quad \text{— [3]}$$

Thus, the series converges absolutely if $x < \frac{17}{8}$ & $x > \frac{15}{8}$.

Now, we examine the end points

If $x = \frac{15}{8}$, Then

$$\sum_{n=1}^{\infty} \frac{2^n(4x-8)^n}{n} = \sum_{n=1}^{\infty} \frac{2^n}{n}\left(\frac{15}{2}-8\right)^n = \sum_{n=1}^{\infty}\frac{2^n}{n}\left(-\frac{1}{2}\right)^n$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$$

Now, we will check its Convergence

For Absolute Convergence:

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$$ which is divergent by p-test. — [1]

For <u>Conditional</u> <u>Convergence</u>.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \sum_{n=1}^{\infty} (-1)^n u_n \quad \text{where} \quad u_n = \frac{1}{n}$$

By using Alternating series test

(i) All u_n's are positive.

(ii) As $n+1 > n$

$$\frac{1}{n+1} < \frac{1}{n} \Rightarrow u_{n+1} < u_n$$

(iii) $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \frac{1}{n} = 0$

$\Rightarrow u_n \to 0$

Hence all three conditions are satisfied. So by alternating series Test $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is Conditionally Convergen — [2]

If $x = \frac{17}{8}$, Then

$$\sum_{n=1}^{\infty} \frac{2^n}{n}(4x-8)^n = \sum_{n=1}^{\infty} \frac{2^n}{n}\left(\frac{17}{2}-8\right)^n = \sum_{n=1}^{\infty} \frac{1}{n}$$

which is divergent by p- test. — [1]

So, The Interval of Convergence is

$$\left[\frac{15}{8}, \frac{17}{8}\right) \text{ — [1]}$$

Radius $= \frac{1}{8}$ — [1] & Center $= 2$ — [1]

Absolute Convergence Region $\left(\frac{15}{8}, \frac{17}{8}\right)$ — [1]

It Converges Conditionally. at $x = \frac{15}{8}$ — [1]

Soln 2. (b) $f(x) = \sin x$

$f'(x) = \cos x$ $\quad f\left(\frac{\pi}{2}\right) = 1$

$f''(x) = -\sin x$ $\quad f'\left(\frac{\pi}{2}\right) = 0$

$f'''(x) = -\cos x$ $\quad f''\left(\frac{\pi}{2}\right) = -1$

$f^{iv}(x) = \sin x$ $\quad f'''\left(\frac{\pi}{2}\right) = 0$

$f^{iv}\left(\frac{\pi}{2}\right) = 1$ —— [2]

Taylor series of $f(x)$ at $x = a$ is

$$\sum_{K=0}^{\infty} \frac{f^K(a)}{K!}(x-a)^K$$

So, taylor series of $\sin x$ at $x = \frac{\pi}{2}$ is

$$\sum_{K=0}^{\infty} \frac{f^K\left(\frac{\pi}{2}\right)}{K!}\left(x-\frac{\pi}{2}\right)^K$$ where $f(x) = \sin x$ —— [1]

First three non-zero terms are

$$1 - \frac{\left(x-\frac{\pi}{2}\right)^2}{2!} + \frac{\left(x-\frac{\pi}{2}\right)^4}{4!}$$ —— [2]

3(a) $r = \sin\theta$

$r = \cos 2\theta$

solving the equations simultaneously

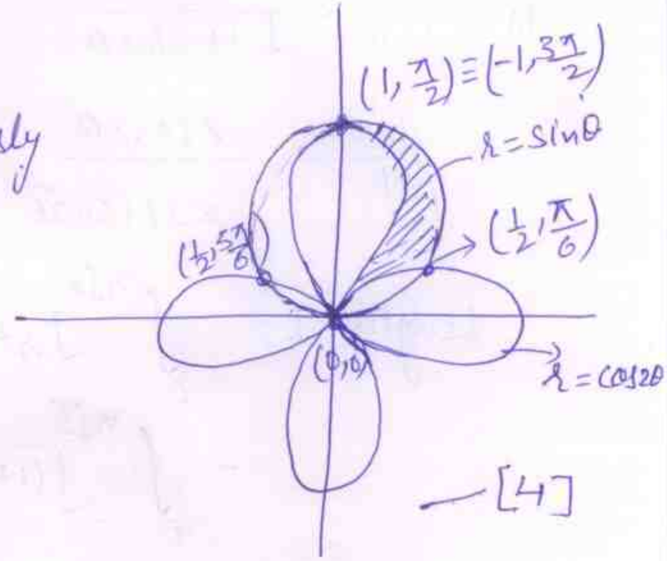
$$\sin\theta = \cos 2\theta$$

$$\Rightarrow 2\sin^2\theta + \sin\theta - 1 = 0$$

$$\Rightarrow (2\sin\theta - 1)(\sin\theta + 1) = 0$$

$$\Rightarrow \sin\theta = \tfrac{1}{2} \text{ or } -1$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$$



— [4]

So points of intersections are

$$\left(\tfrac{1}{2}, \tfrac{\pi}{6}\right), \left(\tfrac{1}{2}, \tfrac{5\pi}{6}\right), \left(-1, \tfrac{3\pi}{2}\right) \equiv \left(1, \tfrac{\pi}{2}\right)$$

From the sketch $(0,0)$ is also an intersection point } — [2]

Area of Shaded Region is

$$A = \frac{1}{2}\int_{\pi/6}^{\pi/2} (\sin^2\theta - \cos^2 2\theta)\, d\theta \qquad \text{— [2]}$$

$$= \frac{1}{2}\int_{\pi/6}^{\pi/2} \left(\frac{\cancel{1} - \cos 2\theta}{2} - \frac{\cancel{1} + \cos 4\theta}{2}\right) d\theta$$

$$= -\frac{1}{4}\int_{\pi/6}^{\pi/2} (\cos 2\theta + \cos 4\theta)\, d\theta$$

$$= -\frac{1}{4}\left[\frac{\sin 2\theta}{2} + \frac{\sin 4\theta}{4}\right]_{\pi/6}^{\pi/2} \qquad \text{— [2]}$$

$$= -\frac{1}{4}\left(-\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{8}\right) = \frac{3\sqrt{3}}{32} \qquad \text{— [2]}$$

OR $A = \frac{1}{2}\int_{\pi/6}^{\pi/2} \sin^2\theta\, d\theta - \frac{1}{2}\int_{\pi/4}^{\pi/2} \cos^2\theta\, d\theta - \frac{1}{2}\int_{0}^{\pi/6} \sin^2\theta\, d\theta - \frac{1}{2}\int_{\pi/6}^{\pi/4} \cos^2\theta\, d\theta$

$$= \frac{3\sqrt{3}}{32}$$

Noted.

3(b) $r = \sqrt{1+\sin 2\theta}$, $0 \le \theta \le \pi\sqrt{2}$

$$\frac{dr}{d\theta} = \frac{\cancel{2}\cos 2\theta}{\cancel{2}\sqrt{1+\sin 2\theta}} = \frac{\cos 2\theta}{\sqrt{1+\sin 2\theta}} \quad \text{——[1]}$$

Length $L = \int_0^{\pi\sqrt{2}} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

$$= \int_0^{\pi\sqrt{2}} \sqrt{(1+\sin 2\theta) + \left(\frac{\cos^2 2\theta}{1+\sin 2\theta}\right)}\, d\theta \quad \text{— [1]}$$

$$= \int_0^{\pi\sqrt{2}} \sqrt{\frac{(1+\sin 2\theta)^2 + \cos^2 2\theta}{1+\sin 2\theta}}\, d\theta$$

$$= \int_0^{\pi\sqrt{2}} \sqrt{2}\sqrt{\frac{1+\sin 2\theta}{1+\sin 2\theta}}\, d\theta \quad \text{——[1]}$$

$$= \int_0^{\pi\sqrt{2}} \sqrt{2}\, d\theta$$

$$= \sqrt{2}\,[\theta]_0^{\pi\sqrt{2}}$$

$$= \sqrt{2}\,[\pi\sqrt{2} - 0]$$

$$= 2\pi \quad \text{——[2]}$$

Note: ~~Those~~ The students who have taken wrong upper limit in the integral are awarded two marks less. In other words two marks are deducted for incorrect upper limit of the integral.

4 Sol^n: For $t = \frac{\pi}{4}$,

$$\vec{r}(t) = \sqrt{2}\,\hat{i} + \sqrt{2}\,\hat{j} + \frac{\pi}{2}\hat{k}.$$

$$\vec{r}\,'(t) = -2\sin t\,\hat{i} + 2\cos t\,\hat{j} + 2\hat{k}. \quad \text{— (1)}$$

$$|\vec{r}\,'(t)| = 2\sqrt{2}.$$

$$\vec{T}(t) = -\frac{\sin t}{\sqrt{2}}\hat{i} + \frac{\cos t\,\hat{j}}{\sqrt{2}} + \frac{\hat{k}}{\sqrt{2}}. \quad \text{— (2)}$$

$$\vec{T}\left(\frac{\pi}{4}\right) = \frac{-\hat{i}}{2} + \frac{\hat{j}}{2} + \frac{\hat{k}}{\sqrt{2}}. \quad \text{— (1)}$$

$$\vec{T}\,'(t) = \frac{1}{\sqrt{2}}(-\cos t\,\hat{i} - \sin t\,\hat{j} + 0\cdot\hat{k}).$$

$$|\vec{T}\,'(t)| = \frac{1}{\sqrt{2}}.$$

$$\therefore \vec{N}(t) = -\cos t\,\hat{i} - \sin t\,\hat{j} + 0\cdot\hat{k}. \quad \text{— (2)}$$

$$\therefore \vec{N}\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}. \quad \text{— (1)}$$

$$\vec{B}\left(\frac{\pi}{4}\right) = \vec{T}\left(\frac{\pi}{4}\right) \times \vec{N}\left(\frac{\pi}{4}\right). \quad \text{— (2)}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{vmatrix}. \quad \text{— (1)}$$

$$= \frac{1}{2}\hat{i} - \frac{1}{2}\hat{j} + \frac{1}{2}\hat{k}. \quad \text{— (3)}$$

$$\vec{T}\,'(t) = \frac{-\cos t\,\hat{i} - \sin t\,\hat{j}}{\sqrt{2}}. \quad \text{— (4)}$$

$$\kappa = \frac{1}{|v|}\left|\frac{d\vec{T}}{dt}\right| = \frac{1}{2\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{4} \quad \text{— (5)}$$

K-S

5(e) Discuss the continuity of the follow function at $(0,0)$.

$$f(x,y)=\begin{cases}\dfrac{\sin^{-1}(x+2y)}{\tan^{-1}(2x+4y)} & (x,y)\neq(0,0)\\ 1/2 & (x,y)=(0,0).\end{cases}$$

Let $x+2y=t$ 1

Now $(x,y)\to(0,0)$ then $t\to 0$ 1

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{t\to 0}\frac{\sin^{-1}t}{\tan^{-1}(2t)} \quad 1$$

$$= \lim_{t\to 0}\frac{(\sin^{-1}t)/t}{\tan^{-1}(2t)/2t}\times\frac{t}{2t} \quad 2$$

But $\lim_{t\to 0}(\sin^{-1}t)/t = 1$ 1 & $\lim_{t\to 0}\frac{\tan^{-1}(2t)}{2t} = 1$ 1

(Using L'Hôpital Rule) 1

$$\Rightarrow \lim_{t\to 0}\frac{1}{2} = \frac{1}{2} \quad 1$$

$\Rightarrow$ function is continuous at $(0,0)$.

1

4 $f(x,y) = \begin{cases} \frac{1}{1+e^{1/x}} + y^2 & (x,y) \neq (0,0) \\ & (x,y) = (0,0) \end{cases}$

Now for $x > 0$ 1

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim \frac{1}{1+e^{1/x}} + y^2$$

$$= \lim \frac{1}{\infty} + 0 = 0 \quad 3$$

For $x < 0$ 1

$$\lim f(x,y) = \lim \frac{1}{1+e^{1/x}} + y^2$$

$$= \frac{1}{1+0} + 0 = 1 \quad 3$$

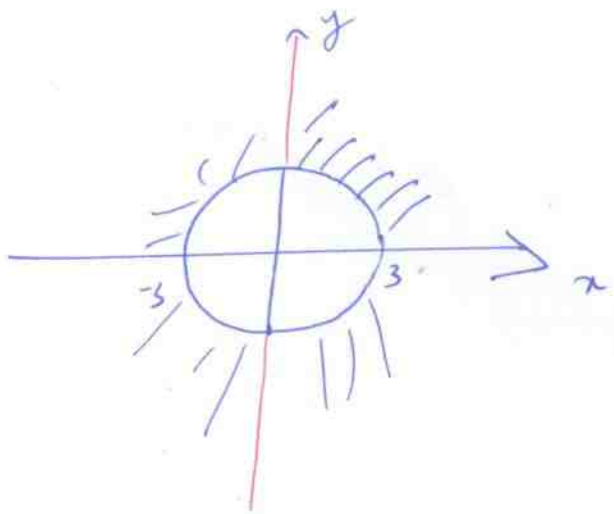
$\Rightarrow f(x,y)$ is not continuous at $(0,0)$.

1

$$f(x,y) = \frac{\sqrt{x^2+y^2-9}}{x}$$

~~$f(x,y) =$~~

$D = \{ (x,y) \in \mathbb{R}^2 \ / \ x^2+y^2 \geq 9 \text{ and } x \neq 0 \}$ 2



2

D is not open because any point lying on the boundary is not an interior point. 2

D is not closed because it doesnot contain many boundary points lying on y-axis. 2

Solution (Q.6)

(a) $f_x(0,0) = \lim_{h\to 0} \dfrac{f(0+h,0) - f(0,0)}{h}$

$= \lim_{h\to 0} \dfrac{\frac{h(0)^2}{h^2+0^4} - 0}{h} = 0$ (exist) ——— [2M]

$f_y(0,0) = \lim_{h\to 0} \dfrac{f(0,0+h) - f(0,0)}{h} = \lim_{h\to 0} \dfrac{\frac{0(0+h)^2}{0^2+(0+h)^4} - 0}{h} = 0$. (exist) ⟶ [2M]

Now checking the existance of the limit of $f(x,y)$ as $(x,y)\to(0,0)$

taking path $x = my^2$ ——— [1M]

$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{y\to 0} \dfrac{my^4}{m^2y^4+y^4} = \dfrac{m}{m^2+1}$ ——— [2M]

$\Rightarrow \lim_{(x,y)\to(0,0)} f$ depends on values of m.

$\Rightarrow \lim_{(x,y)\to(0,0)} f$ does not exist $\Rightarrow f(x,y)$ is not continuous at $(0,0)$ ——— [1M]

$\Rightarrow f(x,y)$ is not differentiable at $(0,0)$ ——— [1M]

x ——— x ——— x ——— x ——— x ——— x ——— x

(b) $f_u = \dfrac{\partial f}{\partial x}\cdot\dfrac{\partial x}{\partial u} + \dfrac{\partial f}{\partial y}\cdot\dfrac{\partial y}{\partial u} = v\cdot f_x + \dfrac{1}{v} f_y$. ——— [2M]

$f_v = \dfrac{\partial f}{\partial x}\cdot\dfrac{\partial x}{\partial v} + \dfrac{\partial f}{\partial y}\cdot\dfrac{\partial y}{\partial v} = u\cdot f_x + \left(-\dfrac{u}{v^2}\right) f_y$ ——— [2M]

$f_x = (2x+y) \Rightarrow 2x f_x = 2x(2x+y)$ ——— [2M]

Now $u f_u + v f_v = \left(uv\cdot f_x + \dfrac{u}{v} f_y\right) + \left(uv\cdot f_x - \dfrac{u}{v} f_y\right)$

$= 2uv f_x$

$= 2x f_x$ $[\because uv = x]$ ——— [3M].

——— x ——— x ———

Note 1. [1 marks] is deducted in (a) if continuity of $f(x,y)$ at $(0,0)$ is not discussed.

2. [1 mark] is deducted in (b) if $\dfrac{d}{dx}$ or $\dfrac{d}{du}$ is used for partial