Chapter 2 (2.3-2.6)

Limits and Continuity

Note: *This module is prepared from Chapter 2 (Thomas' Calculus, 13th edition) just to help the students. The study material is expected to be useful but not exhaustive. For detailed study, the students are advised to attend the lecture/tutorial classes regularly, and consult the text book.*

Appeal: Please do not print this e-module unless it is really necessary.

Dr. Suresh Kumar, Department of Mathematics, BITS Pilani, Pilani Campus

Contents

SECTION 2.3 (The Precise Definition of a Limit)

Neighbourhood

Let x_0 be a point on X-axis. Then any open interval containing the point x_0 is called a neighbourhood of the point x_0. For instance, the interval $(x_0 - \delta, x_0 + \delta)$, where δ is any positive real number, is a neighbourhood of x_0. We shall denote it by $N_\delta(x_0)$ and call it δ-neighbourhood of x_0. In fact, the interval $(x_0 - \delta, x_0 + \delta)$ is neighbourhood of its every point. A neighbourhood of x_0 without x_0 is called deleted neighbourhood of x_0.

Ex. $(1, 3)$ is a neighbourhood of 2, while $(1, 3) - \{2\}$ is a deleted neighbourhood of 2.

Remark: The neighbourhood of x_0 is a continuous collection of surrounding or neighbouring points of x_0 on X-axis. That is why the word "neighbourhood" appears.

Meaning of $x \to x_0$

If a real variable $x \neq x_0$ takes infinitely many values in each neighbourhood of the point x_0, then we say that x approaches or tends to x_0, and we write $x \to x_0$. It implies that each δ-neighbourhood $(x_0 - \delta, x_0 - \delta)$ of x_0 contains infinitely many values of the variable x. We may choose positive values of δ very close to 0. Still the δ-neighbourhood will carry infinitely many values of the variable x. Thus, x takes values arbitrarily close to x_0, and it makes sense to say that x approaches x_0 or $x \to x_0$. If x approaches x_0 by taking values less than x_0, then x_0 is called as the left hand limit of x, and we write $x \to x_0^-$. If x approaches x_0 by taking values greater than x_0, then x_0 is called as the right hand limit of x, and we write $x \to x_0^+$.

Ex. Suppose x takes values 0.9, 0.99, 0.999, ... Then $x \neq 1$ and takes infinitely many values in each neighbourhood of 1. So x approaches 1. Also, all the values of x are less than 1. So $x \to 1^-$.

Ex. Suppose x takes values 1.1, 1.01, 1.001, ... Then $x \neq 1$ and takes infinitely many values in each neighbourhood of 1. So x approaches 1. Also, all the values of x are greater than 1. So $x \to 1^+$.

Ex. Suppose x takes values 1, 1/2, 1/3, ... Then $x \neq 0$ and takes infinitely many values in each neighbourhood of 0. So x approaches 0. Also, all the values of x are greater than 0. So $x \to 0^+$.

Limit of $f(x)$ as $x \to x_0$

Let f be a function defined in some neighbourhood of x_0 except possibly at x_0. Then a real number L is said to be limit of $f(x)$, symbolically written as $\lim\limits_{x \to x_0} f(x) = L$, if given any positive real number ϵ (however small), there exists $\delta > 0$ (depending on ϵ) such that

$0 < |x - x_0| < \delta \quad \implies \quad |f(x) - L| < \epsilon.$

or $x \in (x_0 - \delta, x_0 + \delta) - \{x_0\} \quad \implies \quad f(x) \in (L - \epsilon, L + \epsilon).$

Thus, $\lim_{x \to x_0} f(x) = L$ if corresponding to each ϵ-neighbourhood of L, there exists a deleted δ-neighbourhood of x_0 such that the values of $f(x)$ corresponding to the deleted δ-neighbourhood of x_0 lie in the ϵ-neighbourhood of L (see Figure 1). Geometrically, it implies that the portion of the curve $f(x)$ corresponding to the deleted δ-neighbourhood of x_0 lies inside the horizontal strip created by the lines $f(x) = L - \epsilon$ and $f(x) = L + \epsilon$ parallel to X-axis.

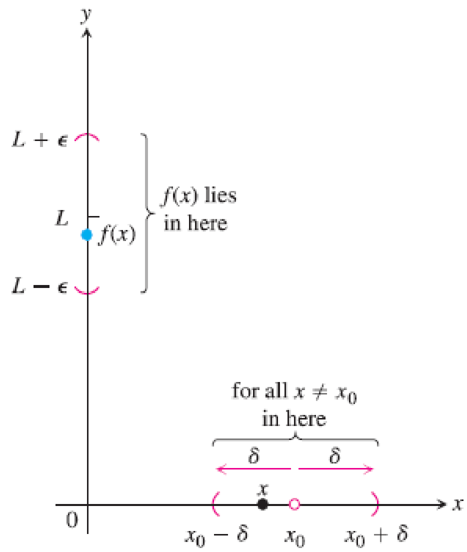


Figure 1: The relation of ϵ and δ in the definition of limit.

Ex. Show that $\lim_{x \to 1}(2x + 1) = 3$.

Sol. Let $\epsilon > 0$ be given. Then

$|2x + 1 - 3| = |2x - 2| = 2|x - 1| < \epsilon$ provided $|x - 1| < \epsilon/2$.

Choosing $\delta = \epsilon/2$, we have

$|x - 1| < \delta \quad \implies \quad |2x + 1 - 3| < \epsilon.$

Thus, $\lim_{x \to 1}(2x + 1) = 3$.

For the sake of illustration, we plot $f(x) = 2x + 1$ in Figure 2. We choose $\epsilon = 0.4$ so that $\delta = \epsilon/2 = 0.2$. The horizontal dotted blue lines are $f(x) = l - \epsilon = 3 - 0.4 = 2.6$ and $f(x) = l + \epsilon = 3 + 0.4 = 3.4$ while the vertical red lines are $x = x_0 - \delta = 1 - 0.2 = 0.8$ and $x = x_0 + \delta = 1 + 0.2 = 1.2$. We can see that the part of the curve in the vertical red strip completely lies inside the horizontal blue strip, as expected.

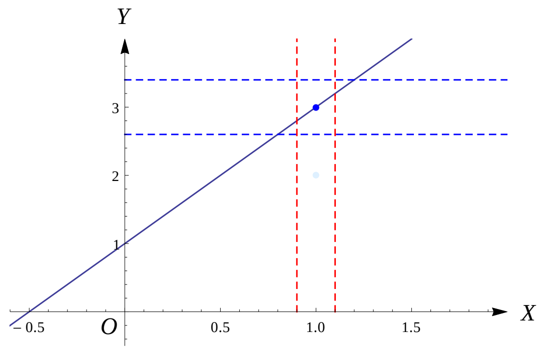


Figure 2: $f(x) = 2x + 1$.

SECTION 2.4 (One-Sided Limits)

A real number L is said to be left hand limit of $f(x)$, symbolically written as $\lim\limits_{x \to x_0^-} f(x) = L$, if given any positive real number ϵ (however small), there exists $\delta > 0$ (depending on ϵ) such that

$x \in (x_0 - \delta, x_0) \quad \Longrightarrow \quad f(x) \in (L - \epsilon, L + \epsilon)$.

Thus, $\lim\limits_{x \to x_0^-} f(x) = L$ if corresponding to each ϵ-neighbourhood of L, there exists a left δ-neighbourhood $(x_0 - \delta, x_0)$ of x_0 such that the values of $f(x)$ corresponding to the left δ-neighbourhood of x_0 lie in the ϵ-neighbourhood of L (see left panel of Figure 3).

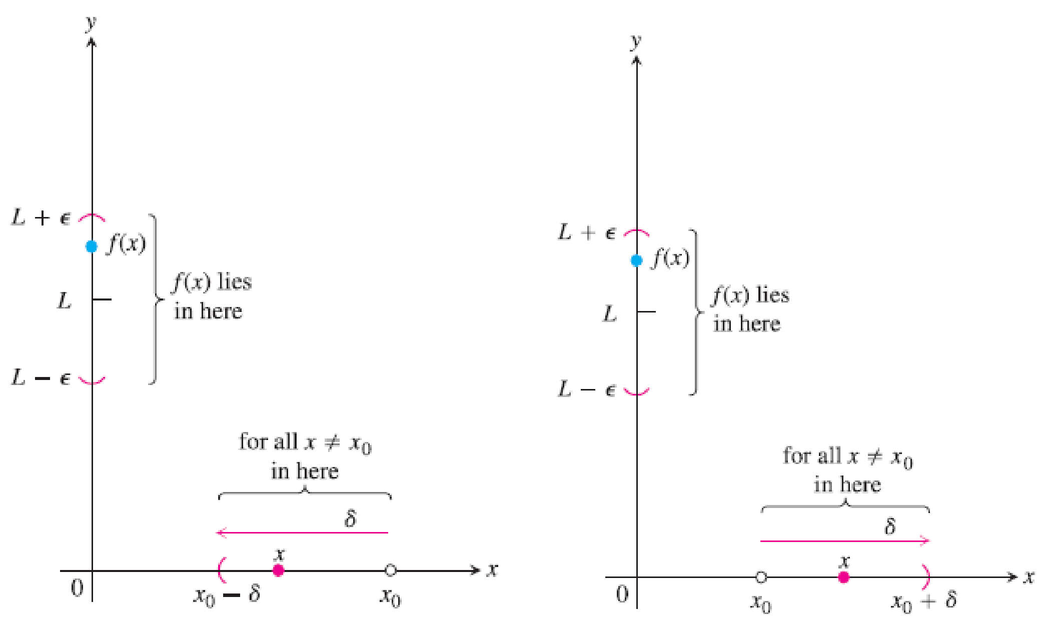


Figure 3: **Left panel:** Intervals associated with left hand limit. **Right panel:** Intervals associated with right hand limit.

Likewise, a real number L is said to be right hand limit of $f(x)$, symbolically written as $\lim\limits_{x \to x_0^+} f(x) = L$, if given any positive real number ϵ (however small), there exists $\delta > 0$ (depending on ϵ) such that

$x \in (x_0, x_0 + \delta) \quad \Longrightarrow \quad f(x) \in (L - \epsilon, L + \epsilon)$.

Thus, $\lim_{x \to x_0^+} f(x) = L$ if corresponding to each ϵ-neighbourhood of L, there exists a right δ-neighbourhood $(x_0, x_0 + \delta)$ of x_0 such that the values of $f(x)$ corresponding to the right δ-neighbourhood of x_0 lie in the ϵ-neighbourhood of L (see right panel of Figure 3).

Further, $\lim_{x \to x_0} f(x) = L$ if and only if $\lim_{x \to x_0^-} f(x) = L = \lim_{x \to x_0^+} f(x)$.

Ex. Show that $\lim_{x \to 1} f(x)$, where $f(x) = \begin{cases} x & \text{if } x \leq 1 \\ x+1 & \text{if } x > 1 \end{cases}$, does not exist.

Sol. We shall show that $\lim_{x \to 1^-} f(x) = 1$ and $\lim_{x \to 1^+} f(x) = 2$.

Let $\epsilon > 0$ be given. Then for $x < 1$, we have $|f(x) - 1| = |x - 1| < \epsilon$ provided $x \in (1 - \epsilon, 1)$.

Choosing $\delta = \epsilon$, we have

$x \in (1 - \delta, 1) \implies |f(x) - 1| < \epsilon$.

Thus, $\lim_{x \to 1^-} f(x) = 1$.

Next for $x > 1$, we have

$|f(x) - 2| = |x + 1 - 2| = |x - 1| < \epsilon$ provided $x \in (1, 1 + \epsilon)$.

Choosing $\delta = \epsilon$, we have

$x \in (1, 1 + \delta) \implies |f(x) - 2| < \epsilon$.

Thus, $\lim_{x \to 1^+} f(x) = 2$.

Since $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$, so $\lim_{x \to 1} f(x)$ does not exist.

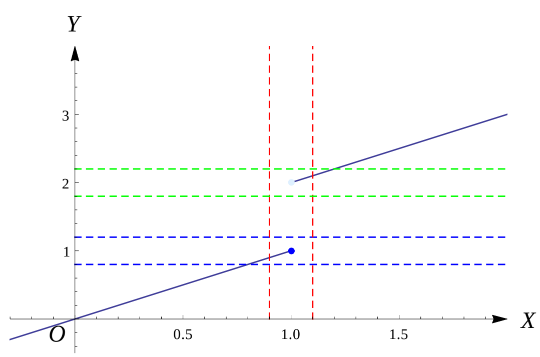


Figure 4: $f(x) = x$ for $x \leq 1$ and $f(x) = x + 1$ for $x > 1$.

For the sake of illustration, we plot $f(x) = \begin{cases} x & \text{if } x \leq 1 \\ x+1 & \text{if } x > 1 \end{cases}$ in Figure 4. We choose $\epsilon = 0.2$ so that $\delta = \epsilon = 0.2$. The horizontal dotted blue lines are $f(x) = l - \epsilon = 1 - 0.2 = 0.8$ and $f(x) = l + \epsilon = 1 + 0.2 = 1.2$ while the horizontal dotted blue lines are $f(x) = l - \epsilon = 2 - 0.2 = 1.8$ and $f(x) = 2 + \epsilon = 2 + 0.2 = 2.2$. The vertical red lines are $x = x_0 - \delta = 1 - 0.2 = 0.8$ and $x = x_0 + \delta = 1 + 0.2 = 1.2$. We can see that the part of the curve corresponding to $x \in (0.8, 1)$ in the vertical Red strip completely lies inside the horizontal blue strip while the part of the curve corresponding to $x \in (1, 1.2)$ in the vertical Red strip completely lies inside the horizontal Green strip, as expected.

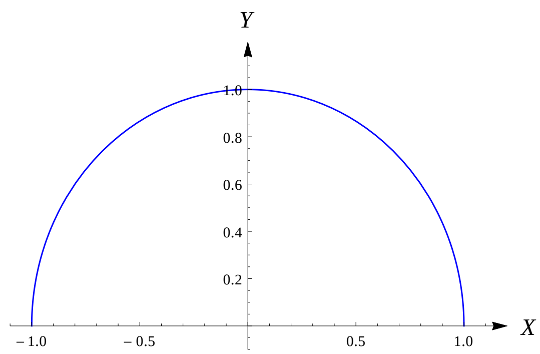


Figure 5: $f(x) = \sqrt{1-x^2}$.

Ex. Discuss the geometry of $\lim_{x\to 0} \sqrt{1-x^2} = 1$.

Sol. We know that $\sqrt{1-x^2}$ is semicircular curve with the domain $[-1,1]$ on X-axis (see Figure 5). It is easy to see that when $x \to 0$ either from left or right in the neighbourhood of 0, the corresponding part of the curve converges to the point $(0,1)$ on Y-axis. It implies that $\sqrt{1-x^2}$ tends to 1 for both the paths along which $x \to 0$. That is why, $\lim_{x\to 0} \sqrt{1-x^2} = 1$.

Remark: If $\lim_{x\to x_0} f(x)$ exists, then the graph of the function $f(x)$, geometrically, converges or strikes at the same point (whose y-coordinate is the limit of $f(x)$) from left as well as right in the neighbourhood of x_0. For instance, see Figure 2. If $\lim_{x\to x_0} f(x)$ does not exist but $\lim_{x\to x_0^-} f(x)$ and $\lim_{x\to x_0^+} f(x)$ exist finitely, then the graph of the function $f(x)$, geometrically, from the left strikes at a point different from the point where it strikes from the right in the neighbourhood of x_0. For an example, see Figure 4.

Remark: The limit of a function at a point may exist even if the function is not defined there at. For example, consider the function $f(x) = 2x+1$, $x \neq 1$, which is not defined at $x = 1$ as per its given definition. But $\lim_{x\to 1}(2x+1) = 3$.

For, let $\epsilon > 0$ be given. Then

$|2x+1-3| = |2x-2| = 2|x-1| < \epsilon$ provided $0 < |x-1| < \epsilon/2$.

Choosing $\delta = \epsilon/2$, we have

$0 < |x-1| < \delta \quad \Longrightarrow \quad |2x+1-3| < \epsilon.$

Thus, $\lim_{x\to 1}(2x+1) = 3$.

SECTION 2.5 (Continuity)

Let f be a function defined in some neighbourhood of x_0. Then f is said to be continuous at x_0 if $\lim_{x \to x_0} f(x) = f(x_0)$. Further, f is left continuous at x_0 if $\lim_{x \to x_0^-} f(x) = f(x_0)$, and right continuous if $\lim_{x \to x_0^+} f(x) = f(x_0)$. We say that f is continuous on an interval if it is continuous at each point of the interval.

Geometry: The existence of limit at x_0 ensures that the graph of $f(x)$ strikes at the same point from both sides. Also, the limit is equal to $f(x_0)$. So the point of strike is $(x_0, f(x_0))$. Therefore, continuity of f at x_0 implies that there exists at least one neighbourhood of x_0 in which the $f(x)$ has continuous graph, that is, without any break point. Consequently, continuity over an interval implies that the function has continuous graph in the interval.

Ex. The function $f(x) = \sqrt{1-x^2}$ is continuous at $x = 0$ since $\lim_{x \to 0} f(x) = 1 = f(0)$. In fact, $f(x) = \sqrt{1-x^2}$ is continuous at every point in its domain interval $[-1, 1]$. That is why, graph of $f(x) = \sqrt{1-x^2}$ is the continuous semicircular curve from $(-1, 0)$ to $(1, 0)$ as shown in the Figure 5.

Ex. The function $f(x) = \begin{cases} x & \text{if } x \leq 1 \\ x+1 & \text{if } x > 1 \end{cases}$ is not continuous at $x = 1$ since $\lim_{x \to 1} f(x)$ does not exist. The graph of $f(x)$ breaks at the point $x = 1$ as may be seen in the Figure 4. Notice that the function is continuous at every real number except $x = 1$.

Note: I move to the next section assuming that you are familiar with the algebra of continuous functions, types of discontinuity, mean value theorems etc. from your 12th class, otherwise read this section from the text book.

SECTION 2.6 (Limits Involving Infinity; Asymptotes of Graphs)

Definitions

- Limit of a function $f(x)$ is said to be L as $x \to \infty$, written as $\lim_{x \to \infty} f(x) = L$, if given $\epsilon > 0$, there exists a real number M such that $|f(x) - L| < \epsilon$ for all $x > M$.

- Limit of a function $f(x)$ is said to be L as $x \to -\infty$, written as $\lim_{x \to -\infty} f(x) = L$, if given $\epsilon > 0$, there exists a real number M such that $|f(x) - L| < \epsilon$ for all $x < M$.

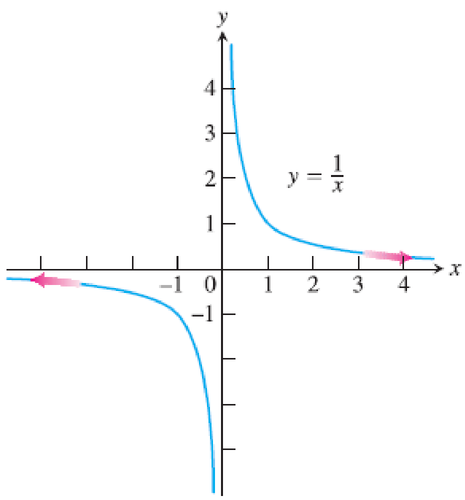
- A line $y = L$ is horizontal asymptote of $y = f(x)$ if either $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$.

- Limit of a function $f(x)$ is said to be ∞ as $x \to x_0$, written as $\lim_{x \to x_0} f(x) = \infty$, if given any real number $B > 0$ (however large), there exists a real number $\delta > 0$ such that $f(x) > B$ for all $x \in (x_0 - \delta, x_0 + \delta) - \{x_0\}$.

- Limit of a function $f(x)$ is said to be $-\infty$ as $x \to x_0$, written as $\lim\limits_{x\to x_0} f(x) = -\infty$, if given any negative real number $-B$ (however small), there exists a real number $\delta > 0$ such that $f(x) < -B$ for all $x \in (x_0 - \delta, x_0 + \delta) - \{x_0\}$.

- A line $x = x_0$ is vertical asymptote of $y = f(x)$ if either $\lim\limits_{x\to x_0^-} f(x) = \pm\infty$ or $\lim\limits_{x\to x_0^+} f(x) = \pm\infty$.

Ex. Show that $\lim\limits_{x\to\infty} \dfrac{1}{x} = 0$ and $\lim\limits_{x\to-\infty} \dfrac{1}{x} = 0$.
Sol. Let $\epsilon > 0$ be given. Then we have $\left|\frac{1}{x} - 0\right| < \epsilon$ if $x > 1/\epsilon$ or $x < -1/\epsilon$. Choosing $M = 1/\epsilon$, we have $\left|\frac{1}{x} - 0\right| < \epsilon$ for all $x > M$. Therefore, $\lim\limits_{x\to\infty} \dfrac{1}{x} = 0$.
Likewise, $\left|\frac{1}{x} - 0\right| < \epsilon$ for all $x < -M$. Therefore, $\lim\limits_{x\to-\infty} \dfrac{1}{x} = 0$.
For geometrical description, see Figure 6.



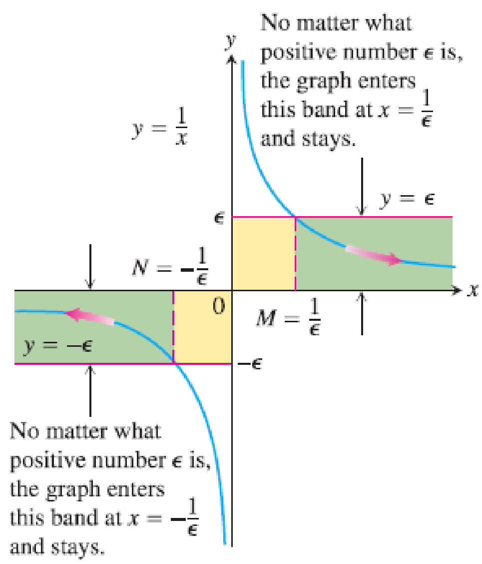


Figure 6: Plots of $y = \frac{1}{x}$ displaying $\lim\limits_{x\to\infty} \dfrac{1}{x} = 0$ and $\lim\limits_{x\to-\infty} \dfrac{1}{x} = 0$.

Ex. Show that $\lim\limits_{x\to 0^+} \dfrac{1}{x} = \infty$ and $\lim\limits_{x\to 0^-} \dfrac{1}{x} = -\infty$.
Sol. Let $B > 0$ be given. Then $\frac{1}{x} > B$ if $0 < x < 1/B$. Choosing $\delta = 1/B$, we get $\frac{1}{x} > B$ for all $0 < x < \delta$. This shows that $\lim\limits_{x\to 0^+} \dfrac{1}{x} = \infty$.
Likewise, $\lim\limits_{x\to 0^-} \dfrac{1}{x} = -\infty$. For geometrical description, see Left panel of Figure 7.
Ex. Find horizontal and vertical asymptotes of $y = \frac{1}{x}$.
Sol. The horizontal asymptote is $y = 0$ since $\lim\limits_{x\to\infty} \dfrac{1}{x} = 0$ and $\lim\limits_{x\to-\infty} \dfrac{1}{x} = 0$. The vertical asymptote is $x = 0$ since $\lim\limits_{x\to 0^-} \dfrac{1}{x} = -\infty$ and $\lim\limits_{x\to 0^+} \dfrac{1}{x} = \infty$. See Right panel of Figure 7.

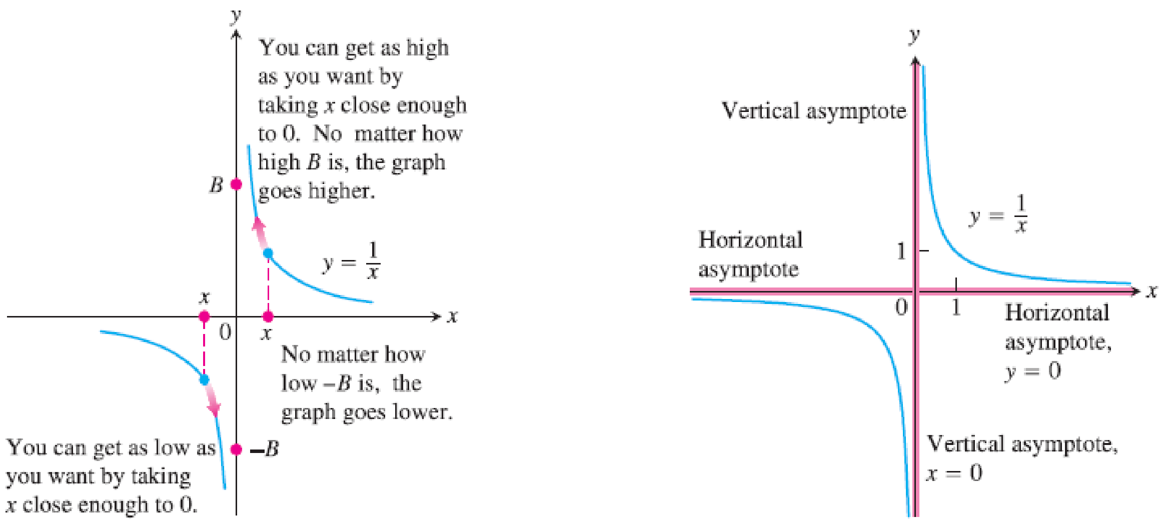


Figure 7: **Left panel:** Plots of $y = \frac{1}{x}$ displaying $\lim\limits_{x\to 0^-} \frac{1}{x} = -\infty$ and $\lim\limits_{x\to 0^+} \frac{1}{x} = \infty$. **Right panel:** Asymptotes of $y = \frac{1}{x}$

Oblique Asymptotes

If the degree of the numerator $n(x)$ of a rational function $y = f(x) = \frac{n(x)}{d(x)}$ is 1 greater than the degree of the denominator $d(x)$, then it has an oblique or slant line asymptote given by $y = ax + b$, where $ax + b$ is the quotient when $n(x)$ is divided by $d(x)$.

Ex. Find asymptotes of $y = \frac{x^2-3}{2x-4}$.

Sol. We have

$$y = \frac{x^2-3}{2x-4} = \frac{x}{2} + 1 + \frac{1}{2x-4}.$$

Therefore $y = \frac{x}{2} + 1$ is an oblique asymptote of the given curve as shown in Figure 8. Notice that $x = 2$ is also an asymptote (vertical asymptote) since $\lim\limits_{x\to 2^-} \frac{x^2-3}{2x-4} = -\infty$ and $\lim\limits_{x\to 2^+} \frac{x^2-3}{2x-4} = \infty$.

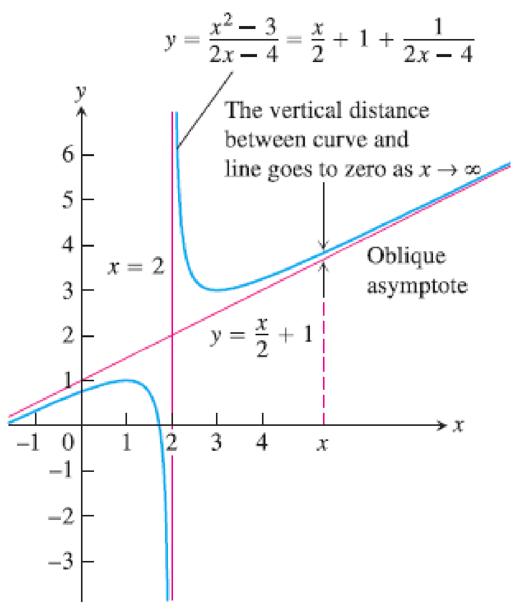


Figure 8: Plot of $y = \frac{x^2-3}{2x-4}$ with the asymptote $y = \frac{x}{2} + 1$