Chapter 5 (*A.P.French*)

Following is the list of problems of chapter 5 *of A P French*

Lecture : 9, 10, 12, 16; *Tutorial :* 7, 11, 13; *Suggested :* 2, 4, 6, 8, 14, 15, 17

5.7. *Eq.of motion of* $A: m\ddot{x}_A = -k_0 x_A - k_c x_A + k_c x_B = -k_0 x_A - k_c (x_A - x_B)$

$$\Rightarrow \ddot{x}_A + \left(\frac{k_0}{m} + \frac{k_c}{m}\right) x_A - \frac{k_c}{m} x_B = 0;\ Let: \omega_0 = \sqrt{\frac{k_0}{m}}\ and\ \omega_c = \sqrt{\frac{k_c}{m}}$$

$$So, \boxed{\ddot{x}_A + (\omega_0^2 + \omega_c^2) x_A - \omega_c^2 x_B = 0} ----(1)$$

$$Similarly: \boxed{\ddot{x}_B + (\omega_0^2 + \omega_c^2) x_B - \omega_c^2 x_A = 0} ----(2)$$

$$(1)+(2) \Rightarrow (\ddot{x}_A + \ddot{x}_B) + (\omega_0^2 + \omega_c^2)(x_A + x_B) - \omega_c^2 (x_A + x_B) \Rightarrow (\ddot{x}_A + \ddot{x}_B) + \omega_0^2 (x_A + x_B) = 0$$

$$\Rightarrow \ddot{q}_1 + \omega_0^2 q_1 = 0:\ where, q_1 = x_A + x_B ----(3)$$

$$(1)-(2) \Rightarrow \boxed{(\ddot{x}_A - \ddot{x}_B) + (\omega_0^2 + 2\omega_c^2)(x_A - x_B) = 0 \Rightarrow \ddot{q}_2 + (\omega_0^2 + 2\omega_c^2) q_2 = 0} :\ where, q_2 = x_A - x_B ----(4)$$

$$\therefore Two\ normal\ frequencies\ are: \boxed{\omega_1 = \omega_0\ and\ \omega_2 = \sqrt{\omega_0^2 + 2\omega_c^2}}$$

$$If\ B\ is\ clamped: x_B = 0;\ From\ eq.(1): Angular\ frequency\ of\ A = \omega_A = \sqrt{\omega_0^2 + \omega_c^2}$$

$$\therefore \nu_A = \frac{\omega_A}{2\pi} = 1.181 sec^{-1}\ and\ \nu_1 = \frac{\omega_1}{2\pi} = 1.14 sec^{-1} = \nu_0$$

$$\therefore \omega_0 = 2\pi\nu_0;\ Now: \omega_A = \sqrt{\omega_0^2 + \omega_c^2} \Rightarrow \omega_c = \sqrt{\omega_A^2 - \omega_0^2}$$

$$\Rightarrow \omega_c^2 = 4\pi^2 (\nu_A^2 - \nu_0^2);\ Similarly: \omega_2 = \sqrt{\omega_0^2 + 2\omega_c^2} \Rightarrow 2\pi\nu_2 = \sqrt{\omega_0^2 + 4\pi^2(\nu_A^2 - \nu_0^2)} \Rightarrow 2.29 sec^{-1}$$

$$(d)\ \omega_A = \sqrt{\omega_0^2 + \omega_c^2} \Rightarrow \omega_c^2 = \omega_A^2 - \omega_0^2 = \frac{k_c}{m}\ and\ \omega_0 = \frac{k_0}{m}$$

$$\therefore \frac{\omega_c^2}{\omega_0^2} = \frac{k_c}{k_0} = \frac{\omega_A^2 - \omega_0^2}{\omega_0^2} = \frac{\nu_A^2 - \nu_0^2}{\nu_0^2} = 1.52$$

5.8. (a) *When the force* F *is applied at a distance* $'a'$ *to hold the pendumum at an angle* θ, *the total torque acting on the pendulum must be zero. So,* $Fa\cos\theta = mgL\sin\theta \Rightarrow \tan\theta \approx \theta = \dfrac{Fa}{mgL}$.

When, the force F' *applied at the position of the bob to hold the same angle, then* $a = L$

$$\therefore \theta = \frac{F'a}{mgL} = \frac{F'L}{mgL} = \frac{F'}{mg} \therefore \frac{F'}{mg} = \frac{Fa}{mgL} \Rightarrow \boxed{F' = \frac{Fa}{L}}$$

(b) *If the angular displacements of the two pendulums are* θ_1 *and* θ_2, *the equation of motion of the 1st pendulum is :*

$I\ddot{\theta}_1 = -mgL\sin\theta_1 + (ka\sin\theta_2 - ka\sin\theta_1)a\cos(\theta_1 \sim \theta_2) \approx -mgL\theta_1 + ka(\theta_2 - \theta_1)a\cos(\theta_1 \sim \theta_2)$ *assuming* $\sin\theta_1 \approx \theta_1$ *and* $\sin\theta_2 \approx \theta_2$ *for small values of the angles.*

Here, $-mgL\sin\theta_1$ *is the torque due to gravity.* $(ka\sin\theta_2 - ka\sin\theta_1)a\cos\theta$ *is the torque due to the resultant compression* $(a(\theta_2 - \theta_2))$ *of the spring.*

$$\Rightarrow I\ddot{\theta}_1 + mgL\theta_1 - ka(\theta_2 - \theta_1)a \Rightarrow mL^2\ddot{\theta}_1 + mgl\theta_1 + ka^2(\theta_1 - \theta_2) = 0 \text{ for small displacements.}$$

$$\Rightarrow \ddot{\theta}_1 + \frac{g}{L}\theta_1 + \frac{ka^2}{mL^2}(\theta_1 - \theta_2) = 0 \Rightarrow \boxed{\ddot{\theta}_1 + \left(\frac{g}{L} + \frac{ka^2}{mL^2}\right)\theta_1 - \frac{ka^2}{mL^2}\theta_2 = 0} ----(1)$$

$$\text{similarly : } \boxed{\ddot{\theta}_2 + \left(\frac{g}{L} + \frac{ka^2}{mL^2}\right)\theta_2 - \frac{ka^2}{mL^2}\theta_1 = 0} ----(2)$$

In terms of x *:* $x_1 = a\sin\theta_1 \approx a\theta_1$ *and* $x_2 \approx a\theta_2$.

$$\therefore \text{Eq. becomes : } \boxed{\ddot{x}_1 + \left(\frac{g}{L} + \frac{ka^2}{mL^2}\right)x_1 - \frac{ka^2}{mL^2}x_2 = 0} \text{ and } \boxed{\ddot{x}_2 + \left(\frac{g}{L} + \frac{ka^2}{mL^2}\right)x_2 - \frac{ka^2}{mL^2}x_1 = 0}$$

After : $(1)+(2)$ *and* $(1)-(2)$ *we have :*

$$\boxed{\frac{d^2}{dt^2}(\theta_1 + \theta_2) + \left(\frac{g}{L} + \frac{ka^2}{mL^2}\right)(\theta_1 + \theta_2) - \frac{ka^2}{mL^2}(\theta_1 + \theta_2) = 0 \Rightarrow \ddot{q}_1 + \frac{g}{L}q_1 = 0} ----(3); \text{ where } q_1 = \theta_1 + \theta_2 \text{ or } x_1 + x_2$$

$$\text{and } \boxed{\ddot{q}_2 + \left(\frac{g}{L} + \frac{2ka^2}{mL^2}\right)q_2 = 0} ----(4); \text{ where, } q_2 = \theta_1 - \theta_2 \text{ or } x_1 - x_2$$

$$\therefore \boxed{\omega_1 = \sqrt{\frac{g}{L}} \text{ and } \omega_2 = \sqrt{\frac{g}{L} + \frac{2ka^2}{mL^2}}}$$

5.11.

$(a)\, m_1\ddot{x}_1 = -kx_1 + T\sin\theta \Rightarrow -kx_1 + m_2 g\cos\theta\sin\theta \quad where: T = m_2 g\cos\theta$

$For\ small\ angles: \cos\theta = 1; Here, \sin\theta = \frac{x_2 - x_1}{l} \approx \theta \Rightarrow m_1\ddot{x}_1 = -kx_1 + m_2 g\frac{x_2 - x_1}{l}$

$Similarly, for\ m_2: m_2\ddot{x}_2 = -T\sin\theta = -m_2 g\frac{x_2 - x_1}{l}$

$(b)\, If\ m_1 = m_2 = m,$

$m_1\ddot{x}_1 = -kx_1 + m_2 g\frac{x_2 - x_1}{l} \Rightarrow \boxed{\ddot{x}_1 + \left(\frac{k}{m} + \frac{g}{l}\right)x_1 - \frac{g}{l}x_2 = 0 - -(1)};$

$Similarly\ m_2\ddot{x}_2 = -m_2 g\frac{x_2 - x_1}{l} \Rightarrow \boxed{\ddot{x}_2 + \frac{g}{l}x_2 - \frac{g}{l}x_1 = 0 - -(2)}$

$Let\ \omega\ is\ the\ normal\ mode\ frequency\ and$

$x_1(t) = A\cos(\omega t) = Ae^{i\omega t}\ and\ x_2 = Be^{i\omega t}; \therefore \ddot{x}_1 = -\omega^2 Ae^{i\omega t}\ and\ \ddot{x}_2 = -\omega^2 Be^{i\omega t}$

$Putting\ these\ values\ in\ eq.(1)\ and\ (2)\ we\ have:$

$$-\omega^2 Ae^{i\omega t} + \left(\frac{k}{m} + \frac{g}{l}\right)Ae^{i\omega t} - \frac{g}{l}Be^{i\omega t} = 0 \Rightarrow \left(\frac{k}{m} + \frac{g}{l} - \omega^2\right)A - \frac{g}{l}B = 0 - - - -(3)$$

$$And: -\omega^2 Be^{i\omega t} + \frac{g}{l}Be^{i\omega t} - \frac{g}{l}Ae^{i\omega t} = 0 \Rightarrow -\frac{g}{l}A + \left(\frac{g}{l} - \omega^2\right)B = 0 - - - -(4)$$

These equations only will have a non-trivial solution if :

$$\begin{vmatrix} \frac{k}{m} + \frac{g}{l} - \omega^2 & -\frac{g}{l} \\ -\frac{g}{l} & \frac{g}{l} - \omega^2 \end{vmatrix} = 0 \Rightarrow \omega^4 - \left(\frac{k}{m} + \frac{2g}{l}\right)\omega^2 + \frac{gk}{ml} = 0$$

$$\therefore \boxed{\omega^2 = \frac{\left(\frac{2g}{l} + \frac{k}{m}\right) \pm \sqrt{\left(\frac{2g}{l} + \frac{k}{m}\right)^2 - \frac{4gk}{ml}}}{2} = \frac{\left(\frac{2g}{l} + \frac{k}{m}\right) \pm \sqrt{\frac{4g^2}{l^2} + \frac{k^2}{m^2}}}{2}}$$

$$(c)\, \frac{g}{l} >> \frac{k}{m} \Rightarrow \omega^2 = \frac{\left(\frac{2g}{l} + \frac{k}{m}\right) \pm \sqrt{\frac{4g^2}{l^2} + \frac{k^2}{m^2}}}{2} \approx \frac{\left(\frac{2g}{l} + \frac{k}{m}\right) \pm \frac{2g}{l}}{2} = \left(\frac{2g}{l} + \frac{k}{2m}\right)\ and\ \frac{k}{2m}$$

$\omega = \pm\sqrt{\left(\frac{2g}{l} + \frac{k}{2m}\right)}\ and\ \pm\sqrt{\frac{k}{2m}}; '-'\ frequency\ has\ no\ physical\ meaning.$

$Therefore\ acceptable\ frequencies\ are: \boxed{\sqrt{\left(\frac{2g}{l} + \frac{k}{2m}\right)}\ and\ \sqrt{\frac{k}{2m}}}$

5.13.*This tyoes of problems are known as beaded string. For one and two beaded string the solutions are as follows:*

(a) *Single beaded string: Let the horizontal displacement is x. So, eq.of motion of the mass :*

$$m\ddot{x} = -T\left(\sin\alpha_1 + \sin\alpha_2\right) \approx -T\left(\frac{x}{l} + \frac{x}{2l}\right) = -\frac{3T}{2l}x$$

$$\Rightarrow \ddot{x} + \frac{3T}{2ml}x = 0 \Rightarrow \omega = \sqrt{\frac{3T}{2ml}}$$

(b) *Two beaded string : Eq.of motion of the 1st mass with displacement* $x_1 : m\ddot{x}_1 = -T\left(\sin\alpha_1 + \sin\alpha_2\right);$

$$\sin\alpha_1 = \frac{x_1}{l} \text{ and } \sin\alpha_2 = \frac{x_1 - x_2}{l}$$

$$\therefore m\ddot{x}_1 = -T\left(\frac{2x_1 + x_2}{l}\right) \Rightarrow \ddot{x}_1 + \frac{T}{ml}\left(2x_1 + x_2\right) = 0 ----(1)$$

Similarly, for 2nd mass the if the displacement is x_2*, the the equation of motion is:*

$$\ddot{x}_2 + \frac{T}{ml}\left(2x_2 + x_1\right) = 0 ----(2) \text{ [Change the suffix only]}$$

$$(1) + (2) \Rightarrow \left(\ddot{x}_1 + \ddot{x}_2\right) + \frac{T}{ml}\left(x_1 + x_2\right) = 0 \Rightarrow \omega_1 = \sqrt{\frac{T}{ml}}$$

and

$$(1) - (2) \Rightarrow \left(\ddot{x}_1 - \ddot{x}_2\right) + \frac{3T}{ml}\left(x_1 - x_2\right) = 0 \Rightarrow \omega_2 = \sqrt{\frac{3T}{ml}} \text{ Higher mode}$$