Derivatives: Let f(z) be a fn defined on a set S and S contains a nbd of z_0. Then derivative of f(z) at z_0, written as $f'(z_0)$, is defined by the equation

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

provided the limit on RHS exists.

The function f(z) is said to be differentiable at z_0 if its derivative at z_0 exists.

If $z - z_0 = \Delta z$, then (1) reduces to

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

Qs. Differentiablity $\Rightarrow$ Continuity

Continuity $\not\Rightarrow$ Differentiablity

Proof : Let f(z) is differentiable at z_0

$$\Rightarrow f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Now

$$\lim_{z \to z_0} \left[f(z) - f(z_0) \right]$$

$$= \lim_{z \to z_0} \left[\frac{f(z) - f(z_0)}{z - z_0} \times (z - z_0) \right]$$

$$= \lim_{z \to z_0} \left[\frac{f(z) - f(z_0)}{z - z_0} \right]$$

$$\times \left[\lim_{z \to z_0} (z - z_0) \right]$$

$$= f'(z_0) \times 0 = 0$$

$$\Rightarrow \lim_{z \to z_0} f(z) = f(z_0)$$

$$\Rightarrow f(z) \text{ is continuous at } z_0$$

Continuity $\not\Rightarrow$ ***Differentiability***

Consider the function

$$f(z) = |z|^2 = x^2+y^2$$

$$= u(x,y) + i\, v(x,y)$$

$$\Rightarrow u(x, y) = x^2 + y^2, \quad v(x, y) = 0.$$

Since u and v are continuous

everywhere, hence

f(z) is continuous everywhere

For $z \neq z_0$, *we* have

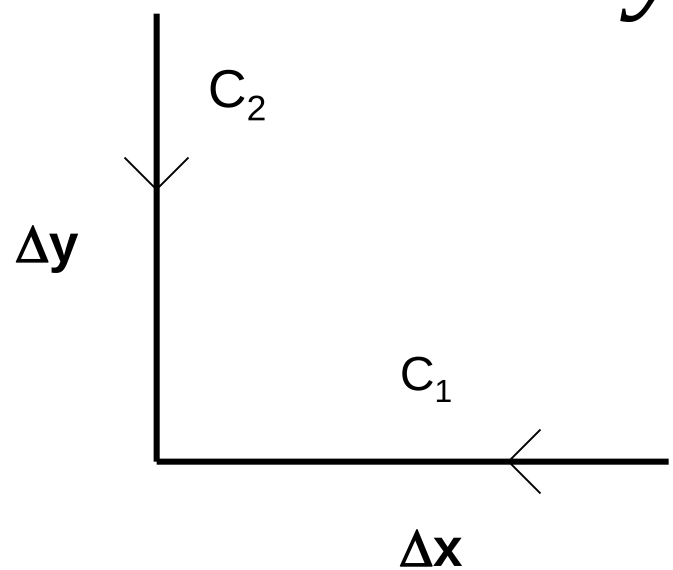
$$\frac{f(z)-f(z_0)}{z-z_0} = \frac{|z|^2-|z_0|^2}{z-z_0} = \frac{z\bar{z}-z_0\bar{z}_0}{z-z_0}$$

$$= \frac{z\,\bar{z}-\bar{z}\,z_0+\bar{z}\,z_0-z_0\,\bar{z}_0}{z-z_0}$$

$$= \frac{\bar{z}(z-z_0)+z_0(\bar{z}-\bar{z}_0)}{z-z_0}$$

$$\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}=\bar{z}+z_0\cdot\frac{\overline{\Delta z}}{\Delta z},\ z-z_0=\Delta z$$

$$=\bar{z}_0+(\Delta x-i\Delta y)+z_0\cdot\frac{\Delta x-i\Delta y}{\Delta x+i\Delta y}$$



$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$= \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$= \begin{cases} \bar{z}_0 + z_0 \; along\, the\; path C_1 \\ \bar{z}_0 - z_0 \; along\, the\; path C_2 \end{cases}$$

Thus, if $z_0 \neq (0,0)$, then

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

is not unique.

When $z_0 = (0,0)$, then

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \bar{z}_0 = 0.$$

$\Rightarrow f(z)$ *is* differentiable at the origin and no where else.

$Sec\ 20$: Differentiation Formula

1. $$f'(z) = \frac{d}{dz} f(z)$$

2. $$\frac{d}{dz}(c) = 0,$$

3. $$\frac{d}{dz}(z) = 1,$$

4. $$\frac{d}{dz}\left(z^n\right)=nz^{n-1},$$

5. $$\frac{d}{dz}\left(cf\left(z\right)\right)=c\frac{d}{dz}f\left(z\right)$$

6. $$\frac{\mathrm{d}}{\mathrm{dz}}\left(\mathrm{f(z)} \pm \mathrm{g(z)}\right) = f'(z) \pm g'(z)$$

7. $$\frac{\mathrm{d}}{\mathrm{dz}}\left(f(z)\,g(z)\right) = f(z)\,g'(z) + f'(z)g(z)$$

8. $$\frac{\mathrm{d}}{\mathrm{dz}}\left(\frac{f(z)}{g(z)}\right) = \frac{g(z)\,f'(z) - g'(z)\,f(z)}{\left(g(z)\right)^2},$$

$$\text{if} \quad g(z) \neq 0$$

Chain Rule:

Let F(z) = g(f(z)), and assume that f(z) is differentiable at z_0 & g is differentiable at $f(z_0)$, then F(z) is differentiable at z_0 and

$$F'(z_0) = g'(f(z_0))\, f'(z_0)$$

Ex. Let $w = f(z)$ and $W = g(w)$

$\Rightarrow W = F(z)$, hence by Chain rule

$$\frac{\mathrm{dW}}{\mathrm{dz}} = \frac{\mathrm{dW}}{\mathrm{dw}}\frac{\mathrm{dw}}{\mathrm{dz}}$$

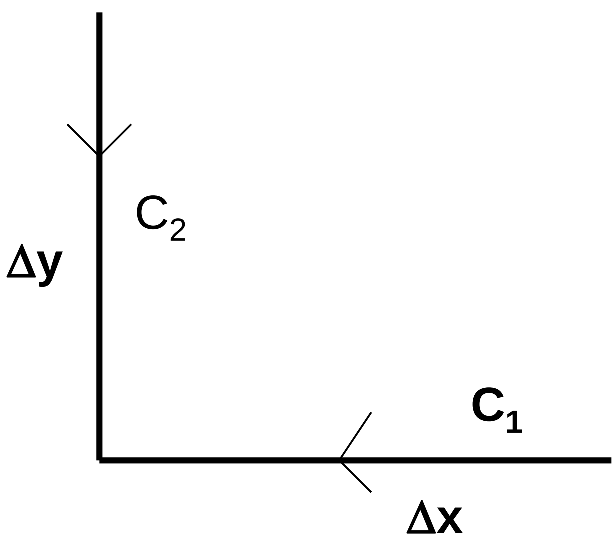
$Q. If \quad f(z) = \bar{z}$, shows that $f'(z)$ does not exist at any point z.

Solution :

Let $z \neq z_0$, then

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{\bar{z} - \bar{z}_0}{z - z_0} = \frac{\overline{z - z_0}}{z - z_0}$$

$$\Rightarrow \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z}$$



$$= \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

$$\therefore \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$= \begin{cases} 1 \; along \; C_1 \\ -1 \quad along \, C_2 \end{cases}$$

$$\Rightarrow f'(z) \, \text{does not exist any where}$$

$Q.9$ Let f be a function defined by

$$f(z) = \begin{cases} \dfrac{(\overline{z})^2}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

Show that $f'(0)$ does NOT exist.

We have ,

$$f'(0) = \lim_{\Delta z \to 0} \frac{f(0+\Delta z) - f(0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(\overline{\Delta z})^2 / \Delta z}{\Delta z}$$

$$\Rightarrow f'(0) = \lim_{\Delta z \to 0} \frac{\left(\overline{\Delta z}\right)^2}{(\Delta z)^2}$$

$$= \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\left(\Delta x - i\Delta y\right)^2}{\left(\Delta x + i\Delta y\right)^2}$$

$$\Rightarrow f'(0) = \begin{cases} 1, & \text{along real axix} \\ 1, & \text{along Im. axix} \\ -1, & \text{along line } \Delta y = \Delta x \end{cases}$$

Hence $f'(0)$ does NOT exist.