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BITS Pilani
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MATH F112 (Mathematics-II)

Complex Analysis

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Lecture 28-31
Elementary Functions

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The Logarithmic Function

Q9(b) Show that the function

$$f(z) = \frac{Log(z+4)}{z^2+i}$$

is analytic everywhere except at the points $\pm(1-i)/\sqrt{2}$ and on the portion $x \le -4$ of the real axis.

The Logarithmic Function

Solution :

Singularities of $f(z)$ are given by

$$\operatorname{Re}(z+4)\le 0,\ \operatorname{Im}(z+4)=0\ \&\ z^2+i=0$$

$$\Rightarrow x+4\le 0,\ y=0\ \&\ z^2=-i$$

Now $z^2=-i=e^{\left(\frac{-\pi}{2}+2n\pi\right)i},\ n=0,1$

The Logarithmic Function

$$\Rightarrow z = e^{\left(\frac{-\pi}{2}+2n\pi\right)\frac{i}{2}}$$

$$\Rightarrow z = e^{\left(\frac{-\pi}{4}+n\pi\right)i}, \quad n = 0,1$$

When $n = 0$, then

$$z = e^{\frac{-\pi}{4}i} = \cos\frac{\pi}{4} - i\,\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}\left(1-i\right)$$

The Logarithmic Function

When $n = 1$, then

$$z = e^{\left(\pi - \frac{\pi}{4}\right)i} = \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)$$

$$= -\frac{1}{\sqrt{2}}(1-i)$$

Hence singularities of $f(z)$ are

$$\left\{\frac{1}{\sqrt{2}}(1-i), \frac{-1}{\sqrt{2}}(1-i)\right\} \cup \{z = x + iy : x \le -4, y = 0\}.$$

The Logarithmic Function

If z_1 & z_2 be any two non – zero complex numbers, then

$$(1)\log(z_1 z_2) = \log z_1 + \log z_2$$

$$(2)\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2$$

The Logarithmic Function

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But

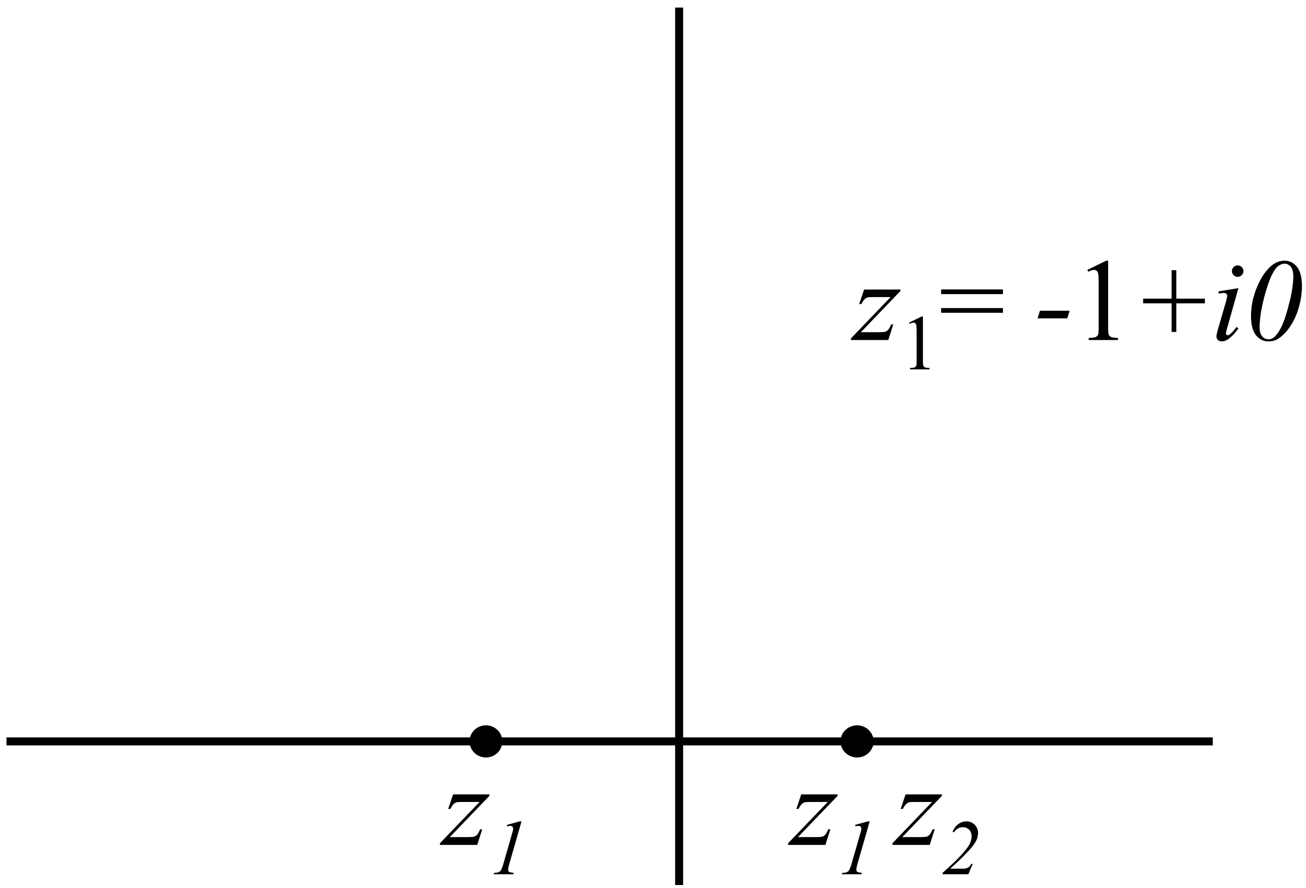
$$Log(z_1 z_2) \neq Log\, z_1 + Log\, z_2$$

$$Log\left(\frac{z_1}{z_2}\right) \neq Log\, z_1 - Log\, z_2$$

$$Log\, z^n \neq n\, Log\, z$$

The Logarithmic Function

Ex(1) $Let\ z_1 = -1,\ z_2 = -1$



The Logarithmic Function

$$\therefore Log\left(z_1\right) = \ln\left|z_1\right| + i\,Arg\,z_1$$

$$\Rightarrow Log\left(-1\right) = \ln(1) + i\ \ Arg\,z_1$$

$$= 0 + i\pi$$

$$\therefore Log\left(z_1\right) + Log\left(z_2\right) = 2\pi i$$

The Logarithmic Function

But $z_1.z_2 = 1$

$$\Rightarrow Log\left(z_1 z_2\right) = \ln\left|z_1 z_2\right| + i\,Arg\left(z_1\,z_2\right)$$

$$= 0 + i.0 = 0$$

Thus

$$Log\left(z_1 z_2\right) \neq Log\,z_1 + Log\,z_2$$

The Logarithmic Function

Q.3(b) p. 97 $Log(-1+i)^2 \neq 2\,Log(-1+i)$

$$\text{LHS} = Log(-1+i)^2$$
$$= Log\left[1+i^2-2i\right]$$
$$= Log(-2i)$$
$$= \ln|-2i| + iArg(-2i)$$

The Logarithmic Function

$$= \ln 2 + i\left(-\frac{\pi}{2}\right)$$

$$= \ln 2 - i\frac{\pi}{2}$$

The Logarithmic Function

$$\text{RHS} = 2\,Log\left(-1+i\right)$$

$$= 2\left[\ln\left|-1+i\right| + i\,Arg\left(-1+i\right)\right]$$

$$= 2\left[\ln\sqrt{2} + i\frac{3\pi}{4}\right]$$

The Logarithmic Function

$$= 2[\frac{1}{2}\ln 2 + i\frac{3\pi}{4}]$$

$$= \ln 2 + i\frac{3\pi}{2}$$

$$\therefore \text{LHS} \neq \text{RHS}$$

The Logarithmic Function

Q.4 p.97 : Show that

$(a) \log\left(i^2\right) = 2\log i,$ when

$$\log z = \ln r + i\theta,$$

$$r = |z| > 0, \frac{\pi}{4} < \theta < \frac{9\pi}{4}$$

The Logarithmic Function

$$(b)\log\left(i^2\right) \neq 2\log i, \quad \text{when}$$

$$\log z = \ln r + i\theta,$$

$$r = |z| > 0,$$

$$\frac{3\pi}{4} < \theta < \frac{11\pi}{4}$$

The Logarithmic Function

Soln (a):

$$\text{LHS} = \log(i^2) = \log(-1)$$

$$= \ln|-1| + i\theta,\ \theta = \arg(-1)$$

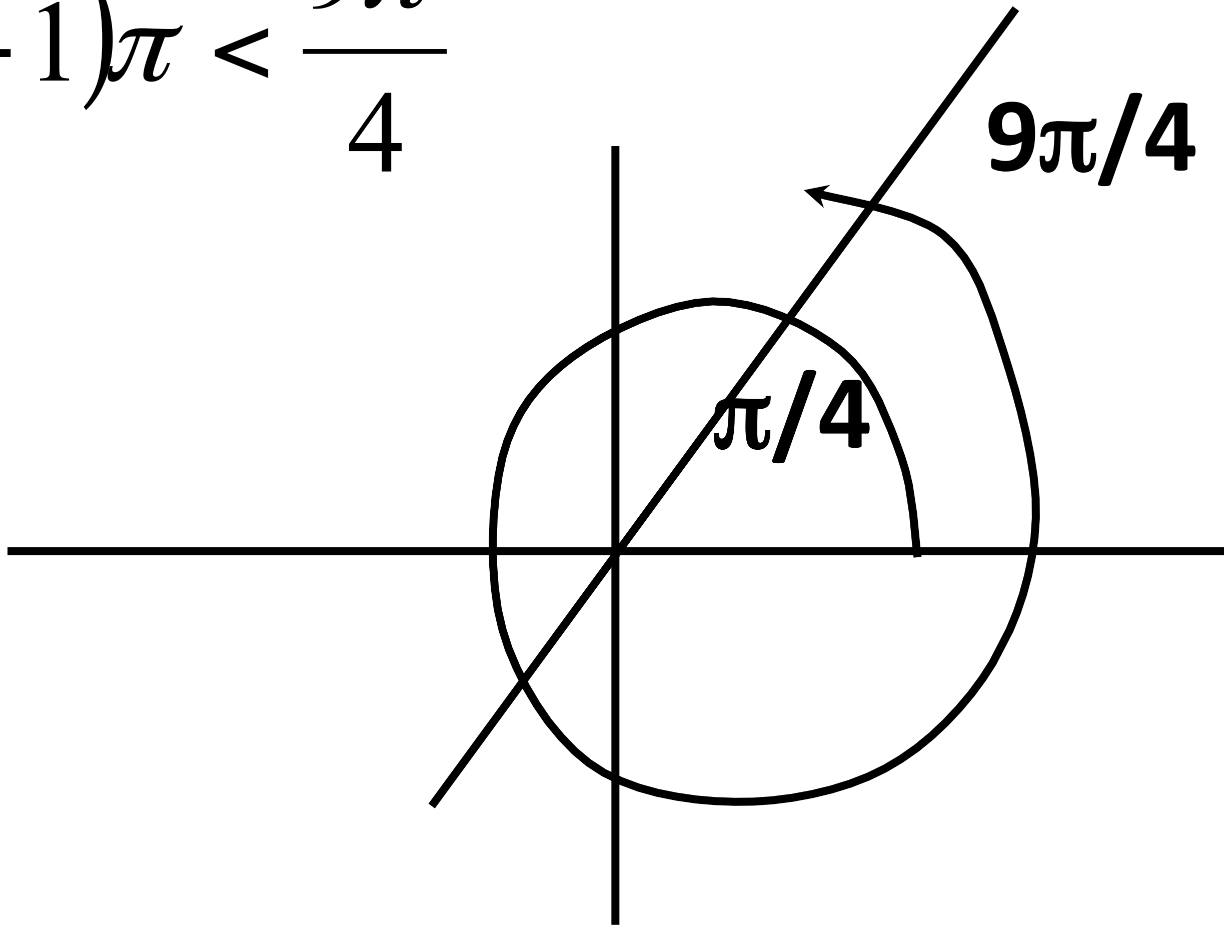
$$= 0 + (\pi + 2n\pi)i$$

The Logarithmic Function

$$\frac{\pi}{4} < \theta = (2n+1)\pi < \frac{9\pi}{4}$$

$$\Rightarrow n = 0$$

$$\Rightarrow LHS = \pi i$$



The Logarithmic Function

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But when

$$\frac{\pi}{4} < \theta = \left(2n + \frac{1}{2}\right)\pi < \frac{9\pi}{4}$$

$$\Rightarrow n = 0 \ \text{ \& hence } \ \theta = \frac{\pi}{2}$$

The Logarithmic Function

$$\text{RHS} = 2\log i = 2i\frac{\pi}{2} = \pi i$$

$$\therefore \text{LHS} = \text{RHS}$$

$$\text{i.e.} \log\left(i^2\right) = 2\log i$$

$$\text{if } \frac{\pi}{4} < \Theta < \frac{9\pi}{4}$$

The Logarithmic Function

(b) NOTE :

$$3\pi/4 < \theta < 11\pi/4,$$

$$\theta = \arg(-1) = \pi + 2n\pi$$

and hence $n = 0.$

Hence, $\mathrm{LHS} = \pi i$

The Logarithmic Function

We have

$$\log\ i = \ln|i| + i \arg i$$

$$= \ln|1| + i\left(\frac{\pi}{2} + 2n\pi\right), \quad \text{where}$$

n is an integer

$$= i\pi\left(2n + \frac{1}{2}\right)$$

The Logarithmic Function

$$\because \frac{3\pi}{4} < \theta = \left(2n + \frac{1}{2}\right)\pi < \frac{11\pi}{4}$$

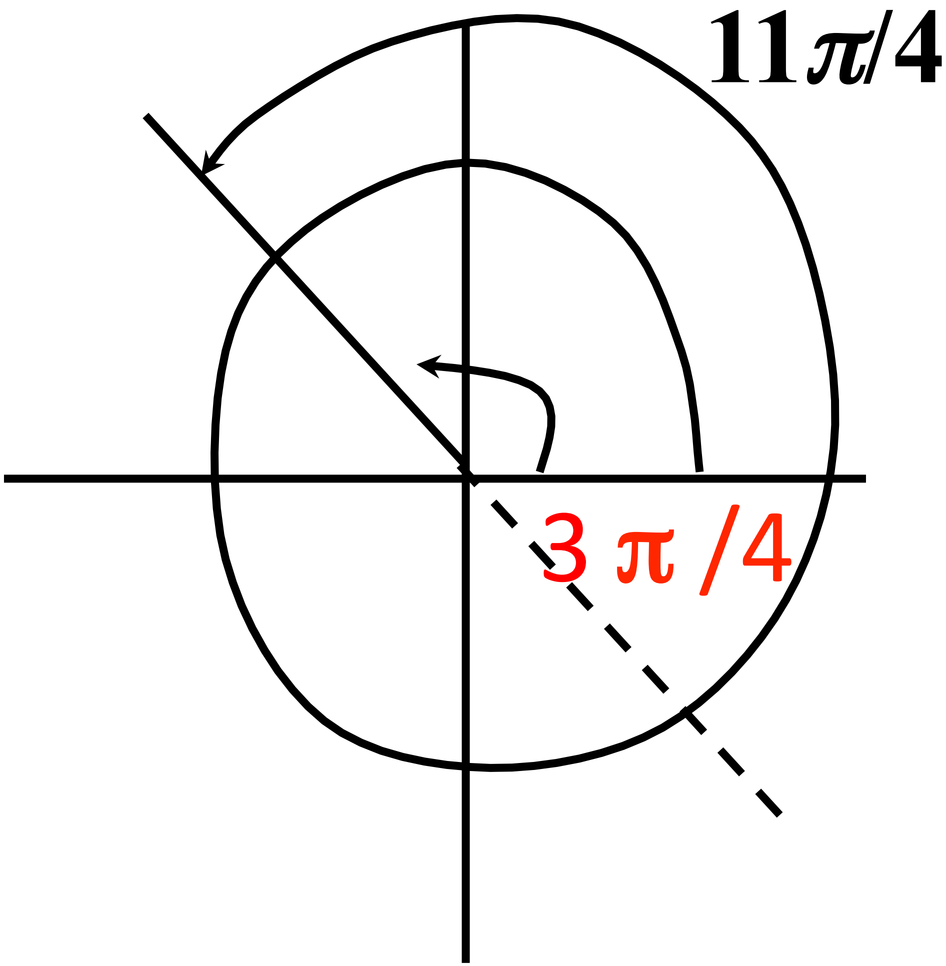
$\Rightarrow n = 1$ & hence

$$\theta = \frac{5\pi}{2}$$

The Logarithmic Function

$$\therefore \text{RHS} = 2\log i$$

$$= 2.i\frac{5\pi}{2} = 5\pi i$$



$$\text{LHS} \neq \text{RHS}$$

The Logarithmic Function

Q1.(P-100) Show that if $Re\ z_1 > 0$ and $Re\ z_2 > 0$ then $Log\ (z_1 z_2) = Log\ z_1 + Log\ z_2$

Proof: Let $z_1 = r_1 e^{i\theta_1},\ z_2 = r_2 e^{i\theta_2}$ then

$$z_1 z_2 = r_1 r_2 e^{i(\Theta_1 + \Theta_2)}$$

$$Re\ z_1 > 0, Re\ z_2 > 0 \Rightarrow -\frac{\pi}{2} < \Theta_1 < \frac{\pi}{2}, -\frac{\pi}{2} < \Theta_2 < \frac{\pi}{2}$$

$$\Rightarrow -\pi < \Theta_1 + \Theta_2 < \pi$$

The Logarithmic Function

$$Log\,(z_1 z_2) = \ln(r_1 r_2) \ + i(\Theta_1 + \Theta_2)$$

$$= \ln(r_1) + \ln(r_2) \ + i(\Theta_1 + \Theta_2)$$

$$= \ln(r_1) + i\Theta_1 + \ln(r_2) \ + i\Theta_2 = Log\,z_1 + Log\,z_2$$

Complex Exponent

(1) Let $z \neq 0$ be a complex no., and c is any complex no. Then z^c *is* defined as $z^c = e^{c \log z}$

If $\log z$ is replaced by $Log\, z,$ then

$$z^c = e^{c\, Log z}$$

is called the principal value of z^c.

Complex Exponent

Q.2(a) p. 104 : Show that i^i is real and find its principal value.

Soln : $i^i = e^{i \log i}$

$$\log(i) = \ln|i| + i\arg(i)$$

$$= 0 + i\left(\frac{\pi}{2} + 2n\pi\right) = \left(2n + \frac{1}{2}\right)\pi i$$

Complex Exponent

$$\therefore\ i^i = e^{-\left(2n+\frac{1}{2}\right)\pi}$$, which is real,

Principal value of i^i is

$$e^{-\frac{\pi}{2}} \quad (n=0).$$

Complex Exponent

(b) Find P.V. of i^{-i}.

Solution :

$$i^{-i} = e^{-i\log i} = e^{-i\left(2n+\frac{1}{2}\right)\pi i}$$

$$= e^{\left(2n+\frac{1}{2}\right)\pi}, \; n = 0, \pm 1, \pm 2, ..$$

Principal value of $i^{-i} = e^{\pi/2}$

Complex Exponent

(c) Write $\log(Log\ i)$ in terms of $a+ib$

We have $Log\, i = \frac{\pi}{2} i$

$$\Rightarrow \log(Log\, i) = \log\left(\frac{\pi}{2} i\right) = \ln\left|\frac{\pi}{2} i\right| + i \arg\left(\frac{\pi}{2} i\right)$$

$$= \ln(\pi/2) + i\left(\frac{\pi}{2} + 2n\pi\right)$$

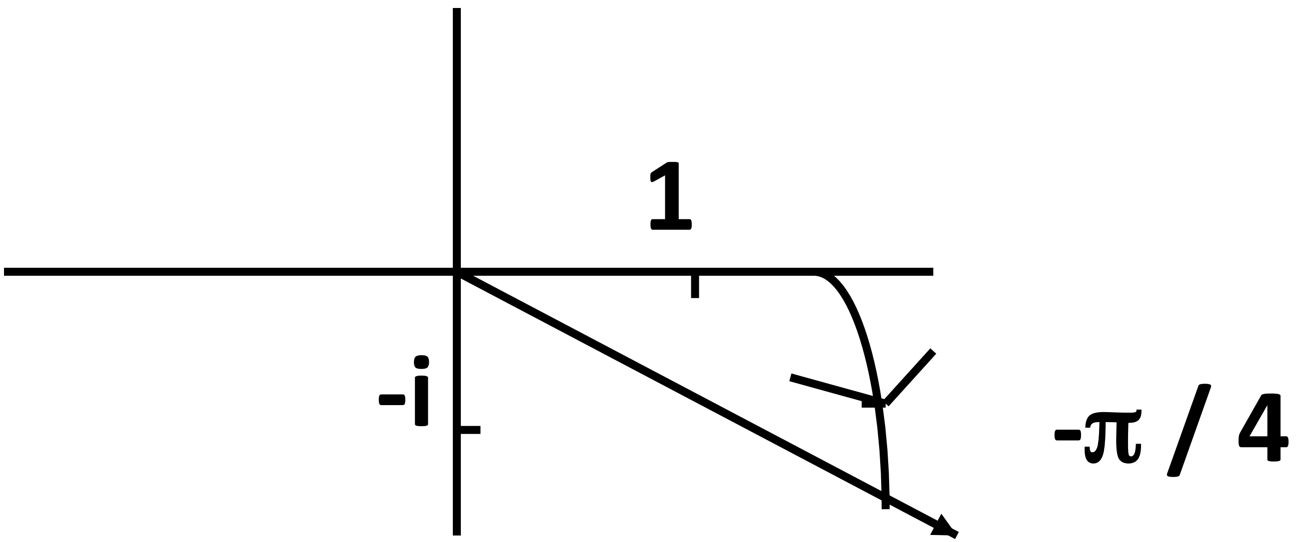
Principal value of

$Log\,(Log\, i)$ is $\ln(\pi/2) + i\frac{\pi}{2}$

Complex Exponent

Q.Find the principal value of $(1-i)^{1+i}$

Solution : $(1-i)^{1+i} = e^{(1+i)\log(1-i)}$



Complex Exponent

Now,

$$\log\,(1-i) = \ln|1-i| + i\arg(1-i)$$

$$= \ln\sqrt{2} + i\left(-\frac{\pi}{4} + 2n\pi\right)$$

Complex Exponent

$$\therefore \quad (1-i)^{1+i} = e^{\log(1-i)+i\log(1-i)}$$

$$= e^{\log(1-i)} \cdot e^{i\log(1-i)}$$

$$= (1-i) \cdot e^{i\ln\sqrt{2} - \left(2n - \frac{1}{4}\right)\pi}$$

Complex Exponent

$$= (1-i)e^{i\ln\sqrt{2}} \cdot e^{-\left(2n-\frac{1}{4}\right)\pi}$$

Principal value of

$(1-i)^{1+i}$ is $(1-i)e^{i\ln\sqrt{2}}\ e^{\frac{\pi}{4}}$

$$= e^{(\ln\sqrt{2}+\frac{\pi}{4})}\left(\cos(\ln\sqrt{2}-\frac{\pi}{4})+i\sin(\ln\sqrt{2}-\frac{\pi}{4})\right)$$

Complex Exponent

Q3. (P-104) Show that $\left(-1+\sqrt{3}i\right)^{\frac{3}{2}} = \pm 2\sqrt{2}$

Soln.: $\left(-1+\sqrt{3}\mathrm{i}\right)^{\frac{3}{2}} = \mathrm{e}^{\frac{3}{2}\log\left(-1+\sqrt{3}\mathrm{i}\right)}$

$$= \mathrm{e}^{\frac{3}{2}\left[\log\left|-1+\sqrt{3}\mathrm{i}\right| + \mathrm{i}\arg\left(-1+\sqrt{3}\mathrm{i}\right)\right]}$$

$$= \mathrm{e}^{\frac{3}{2}\left[\log 2 + \mathrm{i}\left(\frac{2\pi}{3} + 2n\pi\right)\right]}$$

$$= \mathrm{e}^{\frac{3}{2}[\log 2] + (3n+1)\pi \mathrm{i}} = \pm 2\sqrt{2}$$

Complex Exponent

Ex. (P-102) For $z_1 = 1 + i, z_2 = 1 - i, z_3 = -1 - i$

Show that $(z_2 z_3)^i \neq z_2^i z_3^i$ (when principal values are taken)

Soln.: $(z_2 z_3)^i = (-2)^i = e^{i\,Log(-2)} = e^{-\pi} e^{i \ln 2}$

$$z_2^i = e^{\frac{\pi}{4}} e^{\frac{i \ln 2}{2}}$$

$$z_3^i = e^{\frac{3\pi}{4}} e^{\frac{i \ln 2}{2}}$$

$$(z_2 z_3)^i = z_2^i z_3^i e^{-2\pi}$$

THANK YOU