### MATHEMATICS-II (MATH F112)

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## Section 5.5

# Isomorphism



#### Isomorphism

A LT  $L: V \to W$  that is both one-to-one and onto is called as isomorphism from V to W.



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 for all  $p, q \in P_n$ .



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Q:. Check if the linear operator  $L: \mathbb{R}^3 \to \mathbb{R}^3$  given by  $L(e_1) = e_1 + e_2, L(e_2) = e_2 + e_3, L(e_3) = e_1 + e_2 + e_3$  is an isomorphism.

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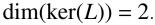
**Sol.** Basis for  $\ker(L)$  is  $\{x(x^2-1), x^2(x^2-1)\}$ .



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**Sol.** Basis for  $\ker(L)$  is  $\{x(x^2-1), x^2(x^2-1)\}$ . Hence,  $\dim(\ker(L)) = 2$ 





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**Sol.** Basis for  $\ker(L)$  is  $\{x(x^2-1), x^2(x^2-1)\}$ . Hence,

 $\dim(\ker(L)) = 2.$ 

Hence, L is not one-to-one, i.e., it is not an isomorphism.



#### Exercise

**Q:.** Show that the linear operator  $L: P_2 \to P_2$  given by  $L(a+bt+ct^2) = (b+c)+(a+c)t+(a+b)t^2$  is an isomorphism.



#### Exercise

Q:. Show that the linear operator  $L: P_2 \to P_2$  given by  $L(a+bt+ct^2) = (b+c)+(a+c)t+(a+b)t^2$  is an isomorphism.

Q:. Show that the linear operator  $L: M_{mn} \to M_{nm}$  given by  $L(A) = A^T$  is an isomorphism.



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Such a function M, denoted by  $L^{-1}$ , is called an inverse of L.



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**Theorem:** A LT  $L: V \to W$  is an isomorphism if and only if L is an invertible LT. Moreover, if L is invertible, then  $L^{-1}$  is also a LT.



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**Sol.** It can be easily verified that L is both one-to-one and onto. Hence, invertible.



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Hence, 
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 $p'(1) = 0 \implies a = -b$ . Hence,  
 $p = (x-1)(ax-a) \implies p = a(x^2-2x+1)$ . Thus  
 $\ker(L) = \{a(x^2-2x+1)|a \in \mathbb{R}\}$ .



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**Sol.** 
$$L^{-1}([x,y,z]) = [y-z,y-x,x-y+z].$$



# Isomorphism

Let V and W be vector spaces. Then V is isomorphic to W, denoted by  $V \cong W$ , if and only if there exists an isomorphism  $L: V \to W$ .





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Q:. Show that  $\mathbb{R}^n$  and  $P_n$  are not isomorphic.

**Sol.** Since,  $\dim(\mathbb{R}^n) = n \neq n+1 = \dim(P_n)$ ,  $\mathbb{R}^n$  and  $P_n$  are not isomorphic.



Q:. Let W be the vector space of all symmetric  $2 \times 2$  matrices. Show that W is isomorphic to  $\mathbb{R}^3$ .



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Q:. Check if  $P_{4n+3} \cong M_{4,n+1}$ .

