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BITS Pilani
Pilani Campus

CS/IS F214
Logic in Computer Science

MODULE: PREDICATE LOGIC

Proof System - Natural Deduction

- Universal Quantifier Elimination

- Existential Quantifier Introduction

Predicate Logic: Proof System: Natural Deduction

- Predicate Logic includes operations -->, $\wedge$, $\vee$, and $\neg$ with the same meanings as in Propositional Logic:
  - i.e.
    - the (introduction and elimination) rules for these operations (-->i, -->e, $\wedge$i, $\wedge$e, $\vee$i, $\vee$e, $\neg$i and $\neg$e)
  - along with
    - the rules for double negation
  - are applicable as is in Predicate Logic with the caveat:
    - the premises and the conclusion are to be treated as predicate logic formulas.

ND: Eliminating universal quantifier and Introducing existential quantifier

Recall:

$\forall X\ \phi$	$\phi[t/X]$
$\phi[t/X]$	$\exists X\ \phi$

$\forall e$

Universal Quantifier Elimination

$\exists i$

Existential Quantifier Introduction

Predicate Logic: Natural Deduction: Proofs.

- Ex. 1: Prove $\forall X\ \phi$ |-- $\exists X\ \phi$

1	$\forall X\ \phi$	Premise
2	$\phi\ [t/X]$ for some term t	$\forall$e 1
3	$\exists X\ \phi$	$\exists$i 2

Predicate Logic: Natural Deduction: Proofs.

- Ex. 2: Prove **$P(t), \forall X\ P(X) \dashrightarrow \neg Q(X)\ |\text{--}\ \exists Y\ \neg Q(Y)$**

1	P(t)	Premise
2	$\forall X(P(X) \dashrightarrow \neg Q(X))$	Premise
	$\exists Y\ \neg Q(Y)$	

Predicate Logic: Natural Deduction: Proofs.

- Ex. 2: Prove $P(t), \forall X(P(X) \rightarrow \neg Q(X)) \vdash \exists Y \neg Q(Y)$

1	$P(t)$	Premise
2	$\forall X(P(X) \rightarrow \neg Q(X))$	Premise
3	$P(t) \rightarrow \neg Q(t)$	$\forall$e 2
4	$\neg Q(t)$	$\rightarrow$e 1,3
5	$\exists Y \neg Q(Y)$	$\exists$i 4

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MODULE: PREDICATE LOGIC

Proof System - Natural Deduction
- Universal Quantifier Introduction

ND: Introducing Universal Quantifier

- What does this rule state?

introduce X0

.

.

.

ϕ[X0/X]

$\forall$X ϕ

X0 does not occur outside the bounding box

$\forall$ i
Universal Quantifier: Introduction

ND: Introducing Universal Quantifier

introduce X0
.
.
.
ϕ[X0/X]

$\forall$X ϕ

X0 does not occur outside the bounding box

$\forall$ i
Universal Quantifier: Introduction

- What does this rule state?

- **If ϕ** (*with free occurrences of variable X*) **is proved**
  - without any assumptions and without any conditions on X -
- **then $\forall$X ϕ is proved.**

- Proving ϕ without any conditions on X is achieved by
  - introducing X0 in place of X
    - where X0 is a fresh variable that does not occur elsewhere.

Predicate Logic: Natural Deduction: Proofs.

- Ex. 3: Prove **∀X(P(X) --> Q(X)), ∀X P(X) |-- ∀X Q(X)**

1	∀X P(X)	Premise
2	∀X(P(X) -->Q(X))	Premise
	∀X Q(X)	??

Predicate Logic: Natural Deduction: Proofs.

- Ex. 3: Prove **$\forall$X(P(X) --> Q(X)), $\forall$X P(X) |-- $\forall$X Q(X)**

1	$\forall$X P(X)	Premise
2	$\forall$X(P(X) -->Q(X))	Premise
3	introduce X0	Fresh variable
	Q(X0)	
	$\forall$X Q(X)	$\forall$i 3-?

Predicate Logic: Natural Deduction: Proofs.

- Ex. 3: Prove **$\forall X(P(X) \to Q(X)), \forall X\, P(X) \vdash \forall X\, Q(X)$**

1	∀X P(X)	Premise
2	∀X(P(X) -->Q(X))	Premise
3	introduce X0	Fresh variable
4	P(X0)	∀e 1
5	P(X0) --> Q(X0)	∀e 2
6	Q(X0)	-->e 4,5
7	∀X Q(X)	∀i 3-6

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MODULE: **PREDICATE LOGIC**

Proof System - Natural Deduction: Proof Rules
- Existential Quantifier Elimination

ND: Eliminating Existential Quantifier

- **What does this rule state?**

$$\dfrac{\exists X\,\phi \qquad \boxed{\begin{array}{l}\text{Introduce } X0 \\ \text{Assume } \phi[X0/X] \\ \quad\vdots \\ \quad\psi\end{array}}}{\psi}$$

X0 does not occur outside the bounding box

$\exists$ e

Existential Quantifier: Elimination

ND: Eliminating Existential Quantifier

Introduce X0

$\phi[X0/X]$

$\vdots$

ψ

$\exists X\, \phi$

$$\frac{\exists X\,\phi \qquad \boxed{\begin{array}{c}\text{Introduce } X0 \\ \phi[X0/X] \\ \vdots \\ \psi\end{array}}}{\psi}\ \exists e$$

X0 does not occur outside the bounding box

$\exists$ e

Existential Quantifier: Elimination

- What does this rule state?

  - **If $\exists X\, \phi$ is true** (for formula ϕ with free occurrences of X) and
  - **if ψ can be proved assuming ϕ** (with no conditions on X)
  - **then ψ has been proved**.

- No conditions on X are being assumed in the sub-proof because we use a fresh variable X0 in place of X.

Predicate Logic: Natural Deduction: Proofs.

- Ex. 4: Prove **∀X(p(X) --> q(X)), ∃ X p(X) |-- ∃ X q(X)**

1	∃X p(X)	Premise
2	∀X(p(X) -->q(X))	Premise
	∃X q(X)	??

Predicate Logic: Natural Deduction: Proofs.

- Ex. 4: Prove **∀X(p(X) --> q(X)), ∃ X p(X) |-- ∃ X q(X)**

1	∃X p(X)	Premise
2	∀X(p(X) -->q(X))	Premise
3	introduce X0	Fresh variable
4	p(X0)	Assumption
	∃X q(X)	
	∃X q(X)	∃e 1, 3-?

Predicate Logic: Natural Deduction: Proofs.

- Ex. 4: Prove **$\forall$X(p(X) --> q(X)), $\exists$ X p(X) |-- $\exists$ X q(X)**

1	$\exists$X p(X)	Premise
2	$\forall$X(p(X) -->q(X))	Premise
3	introduce X0	Fresh variable
4	p(X0)	Assumption
5	p(X0) --> q(X0)	$\forall$e 2
6	q(X0)	-->e 4,5
7	$\exists$X q(X)	$\exists$i 6
8	$\exists$X q(X)	$\exists$e 1, 3-7

Predicate Logic: Natural Deduction: Proofs

- Exercise:
  - Prove the following sequents.
    - $\exists X\ s() \text{-->}\ p(X)\ |\text{--}\ \ s() \text{-->} \exists X\ p(X)$

ND: Comparing the rules ∀i and ∃e

$$\frac{\boxed{\begin{array}{l}\text{Introduce } X0 \\ \quad \phi[X0/X] \\ \quad \vdots \\ \quad \psi\end{array}} \qquad \exists X\, \phi}{\psi}$$

X0 does not occur outside the bounding box

∃e
Existential Quantifier: Elimination

Consider this:

Free variable X is treated as an ***existentially quantified variable*** in rule ∃e

$$\frac{\boxed{\begin{array}{l}\text{Introduce } X0 \\ \quad \vdots \\ \quad \phi[X0/X]\end{array}}}{\forall X\, \phi}$$

X0 does not occur outside the bounding box

Free variable X is treated as a ***universally quantified variable*** in rule ∀i

∀i
Universal Quantifier: Introduction

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MODULE: PREDICATE LOGIC

Predicate Logic –
Proofs - Examples

Predicate Logic: ND Proofs – Example

- Prove the following:
  - $\forall X\ (\neg p(X) \wedge q(X))\ |\text{-}\ \forall X\ (p(X) \dashrightarrow q(X))$
  - $\exists X\ (\neg p(X) \vee q(X))\ |\text{-}\ \exists X\ \neg(p(X) \wedge \neg q(X))$
  - $\forall X\ \forall Y\ (q(Y) \dashrightarrow f(X))\ |\text{-}\ (\exists Y\ q(Y)) \dashrightarrow \forall X\ f(X)$

Prove the sequent: ∀X∀Y (q(Y) ->f(X)) |- (∃Y q(Y)) -->∀X f(X)

	Step	Remark
1	∀X ∀Y (q(Y) --> f(X))	Premise
2	∃Y q(Y))	Assumption
3		fresh X0
4	∀Y (q(Y) --> f(X0))	∀e 1 [X0/X]
5	∃Y q(Y))	copy 2
6		fresh Y0
7	q(Y0)	Assumption
8	q(Y0) --> f(X0)	∀e 4 [Y0/Y]
9	f(X0)	-->e 7, 8
10	f(X0)	∃e 5, 6-9
11	∀X f(X)	∀i 3-10
12	∃Y q(Y)) --> ∀X f(X)	-->i 2-11