MATHEMATICS-II (MATH F112)

Dr. Krishnendra Shekhawat

BITS PILANI
Department of Mathematics

Section 3.4

Eigenvalues and Eigenvectors

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Also, any nonzero vector X for which $AX = \lambda X$, is an eigenvector corresponding to the eigenvalue λ.

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We can see that $A\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} \implies \lambda = 2$ is an eigenvalue for A and $X = [4, 3, 0]$ is the corresponding eigenvector.

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Theorem: Let A be $n \times n$ matrix and λ be a real number. Then λ is an eigenvalue of A if and only if $|\lambda I_n - A| = \mathbf{0}$.

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Theorem: Let A be $n \times n$ matrix and λ be a real number. Then λ is an eigenvalue of A if and only if $|\lambda I_n - A| = \mathbf{0}$. The eigenvectors are the nontrivial solutions of the homogeneous system $(\lambda I_n - A)X = \mathbf{0}$.

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The eigenvalues of an $n \times n$ matrix A are precisely the real roots of the characteristic polynomial $p_A(x)$.

Example 2

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$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -3 \\ 0 & 0 & -5 \end{bmatrix}$$

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Now, eigenvalues of A are the roots of $p_A(x)$. Hence, eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = -5$.

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Corresponding to Example 2, the algebraic multiplicities of $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = -5$ are 1, 1, 1 respectively.

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Sol. There are no eigenvalues of A.

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Theorem: Let A be an $n \times n$ matrix and λ be an eigenvalue for A having eigenspace E_λ. Then E_λ is a subspace of $\mathbb{R}^n$.

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$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

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To compute eigenspace E_1 for $\lambda = 1$, we need to solve the homogeneous system

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To compute eigenspace E_1 for $\lambda = 1$, we need to solve the homogeneous system $\lambda I_2 - AX = 0$, i.e., $I_2 - AX = 0$.

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Then $E_1 = \{[a, 0] | a \in \mathbb{R}\} = \{a[1, 0] | a \in \mathbb{R}\}$.

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$$\text{G.M. of } \lambda = \dim(E_\lambda).$$

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- Find all the eigenvalue of A and compute their algebraic multiplicity.
- Find eigenspaces corresponding to each of the eigenvalues of A and compute their geometric multiplicity.

For $A = \begin{bmatrix} 4 & 0 & -2 \\ 6 & 2 & -6 \\ 4 & 0 & -2 \end{bmatrix}$, characteristic polynomial

$$p_A(x) = |xI_3 - A| = \begin{vmatrix} x-4 & 0 & 2 \\ -6 & x-2 & 6 \\ -4 & 0 & x+2 \end{vmatrix} = x(x-2)^2.$$

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To compute eigenspace E_0 for $\lambda = 0$, we need to solve the homogeneous system

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To compute eigenspace E_0 for $\lambda = 0$, we need to solve the homogeneous system $\lambda I_3 - AX = 0$, i.e., $-AX = 0$.

The augmented matrix is $[-A|0]$, i.e., $\left[\begin{array}{rrr|r} -4 & 0 & 2 & 0 \\ -6 & -2 & 6 & 0 \\ -4 & 0 & 2 & 0 \end{array}\right]$,

which reduces to

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which reduces to $\left[\begin{array}{ccc|c} 1 & 0 & -1/2 & 0 \\ 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

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Then $E_0 = \{[c, 3c, 2c] | c \in \mathbb{R}\} = \{c[1, 3, 2] | c \in \mathbb{R}\}$.

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Now $E_0 = \text{span}\{[1, 3, 2]\} = \text{span}(B)$. Since, B is LI, it is a basis for E_0. Note that

$$\text{G.M. of eigenvalue } 0 = \dim(E_0) = 1.$$

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The augmented matrix is $[-A|0]$, i.e., $\left[\begin{array}{ccc|c} -2 & 0 & 2 & 0 \\ -6 & 0 & 6 & 0 \\ -4 & 0 & 4 & 0 \end{array}\right]$,

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Now $E_2 = \text{span}\{[0, 1, 0], [1, 0, 1]\} = \text{span}(B)$. Since, B is LI, it is a basis for E_2. Note that

$$\text{G.M. of eigenvalue } 2 = \dim(E_2) = 2.$$

Exercise: Consider

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

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Sol. $\lambda = -2, 3, 6$ where $E_{-2} = \{[-z, 0, z] | z \in \mathbb{R}\}$, $E_3 = \{[z, -z, z] | z \in \mathbb{R}\}$ and $E_6 = \{[z, 2z, z] | z \in \mathbb{R}\}$.

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Sol. $\lambda = -1, -1, 3$ where $E_{-1} = \{[x, 2x - z, z] | x, z \in \mathbb{R}\}$ and $E_3 = \{[z/2, z/2, z] | z \in \mathbb{R}\}$.

Result: Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be n eigenvalues of a $n \times n$ matrix A.

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Theorem: Let A be a square matrix with eigenvalue λ and corresponding eigenvector X.

- If λ is an eigenvalue of a matrix A, then for any positive integer n, λ^n is an eigenvalue of A^n with corresponding eigenvector X.

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Theorem: Let A be a square matrix with eigenvalue λ and corresponding eigenvector X.

- If λ is an eigenvalue of a matrix A, then for any positive integer n, λ^n is an eigenvalue of A^n with corresponding eigenvector X.
- If A is nonsingular, then for any integer n, λ^n is an eigenvalue of A^n with corresponding eigenvector X.

Example 8

Q:. Let A be a 2×2 matrix with eigenvectors $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ corresponding to eigenvalues $\lambda_1 = 1, \lambda_2 = 4$.

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Sol. $\begin{pmatrix} -1021 \\ -515 \end{pmatrix}$

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Sol. If λ is not an eigenvalue for A, then, by the definition of an eigenvalue, there are no nonzero vectors X such that $AX = \lambda X$. Thus, no nonzero vector can be in S. However, $A0 = 0 = \lambda 0$, hence $0 \in S$. $S = \{0\}$ is a trivial subspace of $\mathbb{R}^n$.