A diagnostic test has a probability 0.95 of giving a positive result when applied to a person suffering from a certain disease and a probability 0.10 of giving a positive when applied to a non-sufferer. It is estimated that 0.5% of the population are sufferers. Suppose that the test is applied to a person about whom we have no relevant information relating to the disease (apart from the fact that he/she comes from this population), determine: (1)

(i) Probability that the test result will be positive,

(ii) " " given a negative result, the person is a non-sufferer,

(iii) " " the person will be diagnosed wrongly. (8)

Soln:- Let T = test positive, A = sufferer, M = misclassified

(i) $P(T) = P(T|A)P(A) + P(T|A')P(A')$

$= 0.95 \times 0.005 + 0.10 \times 0.995 = 0.10425$ (2)

(ii) $P(A'|T') = \dfrac{P(T'|A')P(A')}{P(T')}$

$= \dfrac{0.9 \times 0.995}{1 - P(T)} = 0.9997$ (3)

(iii) $P(M) = P(T|A')P(A') + P(T'|A)P(A)$

$= (T \cap A') + P(T' \cap A)$

$= 0.10 \times 0.995 + 0.05 \times 0.005$

$= .09975$ (3)

Aes

Q(b) For a die, the probability of the face with j dots is proportional to j for j=1,2,...,6. Find the probability that a face with odd number of dots will turn up in one roll of the die. (2) (6)

Soln:- Given that

$$P(j) \propto j \Rightarrow P(j) = kj$$ for proportionality constant k — (2)

$$\Rightarrow P(1) = k, \; P(2) = 2k, \; \ldots \; P(6) = 6k$$

Since, we have

$$P(1) + P(2) + \cdots + P(6) = 1$$

$$\Rightarrow k + 2k + \cdots + 6k = 1 \Rightarrow 21k = 1$$

$$\Rightarrow k = \frac{1}{21}$$ — (2)

Hence, Prob. of the face with odd no. of dots

$$= P(1) + P(3) + P(5)$$

$$= 1k + 3k + 5k = 9k = \frac{9}{21}$$ — (2)

Solⁿ

2 (a) Sample space

$S = \{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \}$

X : Smaller value of the outcomes if they are different and the common value if they are equal.

$X \in \{1, 2, 3, 4, 5, 6\}$ — [1]

X	1	2	3	4	5	6
Pdf $f(x)$	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{3}{36}$	$\frac{1}{36}$

— [3]

$$E[X] = \sum_{x=1}^{6} x f(x)$$

$$= 1 \times \frac{11}{36} + 2 \times \frac{9}{36} + 3 \times \frac{7}{36} + 4 \times \frac{5}{36} + 5 \times \frac{3}{36} + 6 \times \frac{1}{36}$$

$$= \frac{91}{36} = 2.5278$$ — [3]

(b) Y : Absolute value of difference of the outcomes.

$Y \in \{0, 1, 2, 3, 4, 5\}$ — [1]

y	0	1	2	3	4	5
Pdf $f(y)$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$

— [3]

$$E[Y^2] = \sum_{y=0}^{5} y^2 f(y)$$

$$= 0 \times \frac{6}{36} + 1^2 \times \frac{10}{36} + 2^2 \times \frac{8}{36} + 3^2 \times \frac{6}{36} + 4^2 \times \frac{4}{36} + 5^2 \times \frac{2}{36}$$

$$= \frac{210}{36} = 5.833 \quad \text{——} \quad [3]$$

Part A

Sol^n: 3 (a) Let X denote the number of arrivals during a 6 minute period.

Average number of arrivals per minute $= \lambda = \frac{30}{60} = \frac{1}{2}$

Length of the observation period $= s = 6$

$\therefore$ X follows Poisson distribution with parameter

$$K = \lambda s = \frac{1}{2} \times 6 = 3$$

i.e. $X \sim Poi(3)$ [2]

(i) $P(X \leq 2) = \sum_{x=0}^{2} f(x)$

$$= \sum_{x=0}^{2} \frac{e^{-3} 3^x}{x!}$$

$$= e^{-3}\left[1 + 3 + \frac{9}{2}\right]$$

$$= 8.5 e^{-3}$$

$$= 0.4232$$ [2]

(ii) Let Y denote the time of the first arrival then

$Y \sim Exp(\beta)$; $\beta = \frac{1}{\lambda} = 2$. [1]

$$P(3 < Y < 5) = \int_3^5 f(y)\,dy = \int_3^5 \frac{1}{2} e^{-y/2}\,dy$$

$$= \frac{1}{2}(-2)\left(e^{-5/2} - e^{-3/2}\right)$$

$$= e^{-3/2} - e^{-5/2}$$ =0.141 [2]

(b) Let X denote the time in minutes past 7 A.M. then X is a Uniform random variable over the interval $(0,30)$

ie. $X \sim U(0,30)$

$$f(x) = \frac{1}{30} ; \quad 0 < x < 30.$$ [1]

(i) Event A : if the passengers arrives between 7:10 & 7:15 A.M.

Event B : if the passenger arrives between 7:25 & 7:30 A.M.

$\therefore$ The required probability

$= P(A \cup B)$

$= P(A) + P(B)$ $\quad$ ($\because$ A & B are mutually exclusive)

$= P(10 < X < 15) + P(25 < X < 30)$

$$= \int_{10}^{15} \frac{1}{30} dx + \int_{25}^{30} \frac{1}{30} dx$$

$$= \frac{5}{30} + \frac{5}{30} = \frac{1}{3}.$$ [3]

(ii) Event A: if the passenger arrives between 7:00 & 7:03 A.M.

Event B: if the passenger arrives between 7:15 & 7:18 A.M.

$\therefore$ The required probability $= P(A \cup B)$

$= P(A) + P(B)$ $\quad$ ($\because$ A & B are mutually exclusive)

$= P(0 < X < 3) + P(15 < X < 18)$

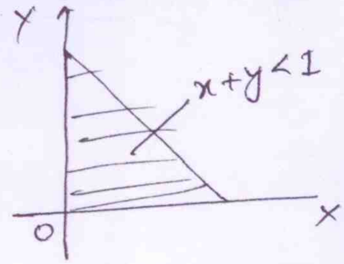
$$= \int_{0}^{3} \frac{1}{30} dx + \int_{15}^{18} \frac{1}{30} dx$$

=1/5 [3]

$= 3 + 3 = 1 .$

Sol'n 4)

$$f(x,y) = \begin{cases} kxy, & x>0,\ y>0,\ x+y<1 \\ 0, & \text{elsewhere} \end{cases}$$



i) Value of K:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$\int_0^1\int_0^{1-x} kxy\,dy\,dx = 1 \qquad [2]$$

$$\Rightarrow \frac{1}{K} = \int_0^1 x\frac{y^2}{2}\Big|_0^{1-x} dx = \int_0^1 \frac{x(1-x)^2}{2}dx$$

$$= \frac{1}{2}\int_0^1 (x + x^3 - 2x^2)\,dx \qquad [1]$$

$$= \frac{1}{2}\left[\frac{x^2}{2} + \frac{x^4}{4} - \frac{2}{3}x^3\right]_0^1$$

$$= \frac{1}{2}\left[\frac{1}{2} + \frac{1}{4} - \frac{2}{3}\right)$$

$$\frac{1}{K} = \frac{1}{24} \qquad [1]$$

$$\Rightarrow \boxed{K = 24}$$

ii) for $0<x<1$,

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_0^{1-y} 24xy\,dx$$

$$= 24y\frac{x^2}{2}\Big|_0^{1-y} \qquad [1]$$

$$= 12y(1-y)^2$$

$$f_Y(y) = \begin{cases} 12y(1-y)^2, & 0<y<1 \\ 0, & \text{elsewhere} \end{cases} \qquad [1]$$

For $0<y<1$,

Conditional: $f_{X|Y}(x) = \dfrac{f(x,y)}{f_Y(y)} = \dfrac{24xy}{12y(1-y)^2}$ [2]

$$f_{X|Y}(x) = \begin{cases} \dfrac{2x}{(1-y)^2}, & 0<x<1-y \\ 0, & \text{elsewhere} \end{cases}$$ [1]

iii) We know that

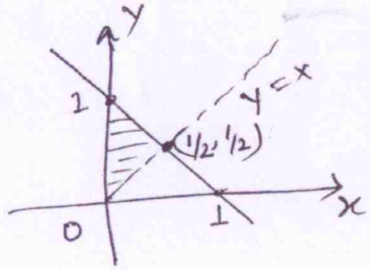
$$\mu_{X|Y} = \int_x x\, f_{X|Y} \cdot dx$$

so for $0<y<1$,

$$\mu_{X|Y} = \int_0^{1-y} x \cdot \frac{2x}{(1-y)^2}\,dx$$

$$= \frac{2}{3}(1-y)$$ —— [2]

(iv) $P(Y \geq X) = \int_0^{1/2}\int_x^{1-x} f(x,y)\,dy\,dx$ [1]



$$= 24\int_0^{1/2} x\frac{y^2}{2}\Big|_x^{1-x}\,dx$$

$$= 12\int_0^{1/2} [x(1-x)^2 - x^3]\,dx$$

$$= 12\int_0^{1/2} (x + \cancel{x^3} - 2x^2 - \cancel{x^3})\,dx$$ [1]

$$= 12\left[\frac{x^2}{2} - \frac{2}{3}x^3\right]_0^{1/2}$$

$$= 12\left[\frac{1}{8} - \frac{2}{3}\cdot\frac{1}{8}\right]$$

$$= 12 \cdot \frac{1}{24}$$ [1]

$P(Y \geq X) = 1/2$

Alternatively, $P(X>Y) + P(Y \geq X) = 1$

$\Rightarrow P(Y \geq X) = 1 - P(X>Y)$

$= \cancel{1 - \text{find } P(X>Y)}$

find $P(X>Y)$

Part A

5(a)

	x	y	x^2	xy
	0	6	0	0
	4	7	16	28
	10	7	100	70
	15	8	225	120
	20	8	400	160
Sum	49	36	741	378

$n = 5$

$\bar{y} = \frac{36}{5} = 7.2$

$\bar{x} = \frac{49}{5} = 9.8$

2 pts

The line of regression is given by $y = b_0 + b_1 x$
where the coefficients b_0 and b_1 are given by
the Normal Equations

$$n b_0 + b_1 \Sigma x = \Sigma y \quad \text{——— (I)}$$

$$b_0 \Sigma x + b_1 \Sigma x^2 = \Sigma xy$$

$\boxed{\text{OR}}$ we can directly write ——— 2 pts

$$b_1 = \frac{n \Sigma xy - \Sigma x \Sigma y}{n \Sigma x_i^2 - (\Sigma x_i)^2} \quad \text{——— (II)}$$

$$b_0 = \bar{y} - b_1 \bar{x}$$

Substituting the values from Table in Eq (I)
or Eq (II), we get

$b_0 = \frac{4077}{652} = 6.2592$ 2 pts

$b_1 = \frac{126}{\ldots} = 0.0966$

Hence estimated linear regression Equation of Y on x is

$$\hat{\mu}_{Y|x} = 6.2592 + 0.0966\, x$$

when $x = 30$, $\hat{\mu}_{Y|x} = (6.2592) + (0.0966) \times (30)$
$= 9.15$ — 2 pts

(b)

$$\rho = \frac{Cov(X,Y)}{\sqrt{Var\,X}\sqrt{Var\,Y}} = \frac{E(XY) - E(X)(E(Y))}{\sqrt{E(X^2) - (E(X))^2}\sqrt{E(Y^2) - (E(Y))^2}}$$

$$= \frac{n\sum xy - (\sum x)(\sum y)}{\sqrt{n\sum x^2 - (\sum x)^2}\sqrt{n\sum y^2 - (\sum y)^2}}$$ — 2 pts

Now substituting the given values

$$= \frac{12(452) - (84)(56)}{\sqrt{(12)(672) - (84)^2}\sqrt{(12)(308) - (56)^2}}$$ — 2 pts

$$= \frac{720}{\sqrt{1008}\sqrt{560}} = \frac{720}{751.32} \approx 0.95$$

$\because$ $\rho > 0$, Yes I will support the guess that small values of Y tend to be associated to small values of X.
2 pts