Gauge Theory in Particle Physics

Final Report

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Contents

Chapter 1

Fields Describing Free Particles

1.1 Introduction

In the realm of classical physics, we first found out by Newton's formulation that the total amount of force acting on a body is directly proportional to its acceleration. We also realized that there is an equivalent formulation given by Lagrange, which involves a system-describing quantity called the Lagrangian, and a set of equations that map the trajectory of the body. We are also aware of the principles of quantum mechanics, and the wave mechanics, matrix mechanics models. Einstein also postulated his theory of relativity, which described classical physics at very high speeds.

The special theory of relativity postulated by Einstein and quantum mechanics are unified using a quantum theory to describe fields. Any Lagrangian here would have complete information about how the wavefunction of a particular system would eveolve in time, along with its probability densities.

1.2 Vector Fields

The electromagnetic field is a vector field. JC Maxwell gave the famous laws, which describe electric and magnetic fields. Given below are Maxwell's equations in free space

$$\nabla . E = \rho, \tag{1.1a}$$

$$\nabla . B = 0 \tag{1.1b}$$

$$\frac{\partial B}{\partial t} = -\nabla \times E, \tag{1.1c}$$

$$\frac{\partial B}{\partial t} = \nabla \times B - 4\pi j, \tag{1.1d}$$

For these electric and magnetic fields, we can define potentials as follows

$$E = -\nabla E + \frac{\partial A}{\partial t} \tag{1.2a}$$

$$B = \nabla \times A \tag{1.2b}$$

If we write $A^\mu = (A^0, \overrightarrow{A})$, we could write

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu (1.3)$$

where is a tensor, called very appropriately as the field strength tensor.

Using this, the Lagrangian for the electromagnetic field can be constructed as follows

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{1.4}$$

1.3 Particles with Spin Zero

The Lagrangian for a scalar field is as follows:

$$L = \frac{1}{2}(\partial^\mu \Phi)(\partial_\mu \Phi^*) - \frac{1}{2} m \Phi \Phi^* \tag{1.5}$$

This field is called the Klein-Gordon field, and describes spin zero particles. On applying the Euler Lagrange equations using this Lagrangian, we can arrive at equations of motion for this field.

$$(\partial^\mu \partial_\mu + m^2)\Phi(x) = 0 \tag{1.6}$$

1.4 Particles with spin-$\frac{1}{2}$

British physicist Paul Dirac in 1928 derived a relativistic wave equation in 1928, which described all spin-$\frac{1}{2}$ particles. This equation, now known famously as the Dirac equation, is as follows

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0 \tag{1.7}$$

where

$$\gamma^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{1.8}$$

$$\gamma^i = \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix} \tag{1.9}$$

The Klein-Gordon equation offers only one solution. The Dirac equation on the other hand gives four solutions, which can be written in matrix form as

$$\psi_a = \begin{bmatrix} 1 \\ 0 \\ p_z/(p^0+m) \\ (p_x+ip_y)/(p^0+m) \end{bmatrix} \psi_b = \begin{bmatrix} 0 \\ 1 \\ (p_x-ip_y)/(p^0+m) \\ -p_z/(p^0+m) \end{bmatrix} \tag{1.10}$$

$$\psi_c = \begin{bmatrix} p_z/(p^0+m) \\ (p_x+ip_y)/(p^0+m) \\ 1 \\ 0 \end{bmatrix} \psi_d = \begin{bmatrix} (p_x-ip_y)/(p^0+m) \\ p_z/(p^0+m) \\ 0 \\ 1 \end{bmatrix} \tag{1.11}$$

The Dirac Field Lagrangian corresponding to the Dirac equation is given by

$$L = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \tag{1.12}$$

Chapter 2

Abelian Gauge Theory

We described global gauge invariance in the previous chapter. Global gauge invariance implies that the gauge transformation is not a function of spacetime. It sometimes might not be enough for a Lagrangian to just be globally invariant - as we know, quite a lot of transformations are spacetime dependent. We need to construct our theories keeping this spacetime dependence in mind. Such a gauge transformation, which is dependent on spacetime, is called local gauge transformation.

Local Transformations are given by the following

$$\psi \to e^{-ieQ\theta(x)}\psi \tag{2.1a}$$

$$A_\mu \to A'_\mu \tag{2.1b}$$

2.1 Quantum Electrodynamics

The Dirac Lagrangian is given by

$$L = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \tag{2.2}$$

If we apply local gauge transformation to this equation, we get

$$L' = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - eQ(A'_\mu - \partial_\mu \theta)\bar{\psi}\gamma^0\psi(2.3)$$

It can clearly be seen that the Lagrangian will remain invariant under the local gauge transformation if $A'_\mu \to A_\mu + \partial_\mu \theta$. This is the gauge transformation associated with electromagnetic field. Thus, adding the gauge term and the electromagnetic field Lagrangian, we get

$$L = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - eQ\bar{\psi}\gamma^0\psi A_\mu \tag{2.4}$$

This is the Lagrangian of Quantum Electrodynamics (QED).

Lets now analyze each term of the above equation. The first term $\bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi isthesameastheoneintheDiraceq$
The term $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ comes from the electromagnetic field Lagrangian, and describes a photon field. The last term $-eQ\psi\,\gamma^0\psi A_\mu$ is the term that introduces interaction between the particle field and the photon field.

The Lagrangian should be Lorentz invariant, to corroborate with relativity. This means that under a Lorentz boost $\Lambda^\mu{}_\nu$, $S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu\gamma^\nu$ holds true.

The solutions to the Hamiltonian of QED generated from the Euler-Lagrange equations are Bilinear Covariants as they transform according to $S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu\gamma^\nu$ for given S.

These bilinear covariants form the following physical quantities

$\psi\,\psi$	Scalar	Space Inversion:+
$\psi\,\gamma^\mu\psi$	Vector	Space Inversion:-
$\psi\,\sigma^{\mu\nu}$	Tensor	
$\psi\,\gamma^5\gamma^\mu\psi$	Axial Vector	Space Inversion:+
$\psi\,\gamma^5\psi$	Pseudoscalar	Space Inversion:-

where γ^5 is $i\gamma^0\gamma^1\gamma^2\gamma^3$.

Chapter 3

Non-Abelian Gauge Theory

3.1 Yang-Mills Theory of non-Abelian gauge fields

As we have already seen, a Lagrangian should remain invariant under global/local transformations. This means that the Lagrangian should have both global and local symmetries.

For spin-1 fields, like the electromagnetic field, U(1) symmetries exist. For other fields, such as the gluonic field for instance, SU(N) symmetry exists.

Let's consider one such system given by

Let $\psi^{'} = U\psi$, $U \in SU(N)$. We can then say

$$L_{\psi'} = \psi\, U^{-1}(i\gamma^{\mu}\partial_{\mu} - m)U\psi(3.1)$$

$$L_{\psi'} = L_{\psi} + \psi\, U^{-1}(i\gamma^{\mu})\partial_{\mu}(U)\psi(3.2)$$

This hence warants the introduction of a new gauge field such that $L_{\psi'} = L_{\psi}$

We introduce covariant derivative D_{μ}, such that the modified Lagrangian is given by

$$L = \psi(i\gamma^{\mu}D_{\mu} - m)\psi(3.3)$$

$D_{\mu} = \partial_{\mu} + igT_aA^a_{\mu}$, where g is coupling constant, T_a are the generators of SU(N), and A^a_{μ} is the gauge field.

It can be proved that the this Lagrangian is invariant under SU(N), and the following condition is satisfied

$$T_a A_\mu^{a\prime} = \frac{i}{g}(\partial_\mu U)U^{-1} + U T_a A_\mu^a U^{-1} \tag{3.4}$$

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