Gauge Theory and the Standard Model

PROJECT REPORT

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Abstract

Master of Science (Hons.)

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Contents

$\mu, \nu, ...$ Space-time indices of a vector or tensor.

$i, j, ...$ Spatial indices of a vector or tensor.

$g_{\mu\nu}$ (Components of) metric tensor, diag(1,-1,-1,-1).

p^{μ} Contravariant vector.

p_{μ} Covariant vector.

L Lagrangian density, also called Lagrangian.

$[A, B]$ Commutator $AB - BA$.

σ^{i} Pauli matrices.

e Electric charge of proton. Electron carries $-e$ charge.

Chapter 1

Free Particle Fields

1.1 Spin-0 Particles

Quantum Field Theory is the theory in theoretical physics that is used to study field that are bound by the laws of Quantum Mechanics and Einstein's Theory of Relativity. In Field Theory, a Lagrangian is defined that remains invariant to any coordinate transformation. The Lagrangian thus defined contains all information about the particle that is needed to be known; in classical field theory it contains the past, present, and future forms of the trajectory of the particle and in quantum field theory it contains the nature of evolution of the wavefunction, and thus the probability distribution, of the particle for all time. The Lagrangian for a scalar field (A field with just one quantifier at each point in space-time) is as follows:

$$L = \frac{1}{2}(\partial^{\mu}\Phi)(\partial_{\mu}\Phi^{*}) - \frac{1}{2}m\Phi\Phi^{*} \tag{1.1}$$

This is the Klein-Gordon Field of spin-0 particles. Equation (1.q) is called the Klein-Gordon field after physicists Oskar Klein and Walter Gordon who proposed this model to describe relativistic particles in 1926.[4] The Hamiltonian for such a particle can be derived from the Euler-Lagrange equations of motion. Alternatively, in Relativistic Mechanics, [5]

$$p^{\mu}p_{\mu} - m^{2} = 0 \tag{1.2}$$

and

$$p_{\mu} \rightarrow i\partial_{\mu} \tag{1.3}$$

in the position representation of momentum. When this relation is applied to equation (1.2) we get equation (1.4) which describes the Hamiltonian of such a particle.

$$(\partial^{\mu}\partial_{\mu} + m^{2})\Phi(x) = 0 \tag{1.4}$$

1.2 spin-$\frac{1}{2}$

1.2.1 Dirac Equation

The Dirac Hamiltonian for a spin-$\frac{1}{2}$ particle, commonly referred to as the "Dirac Equation" is [5]

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi(x) = 0 \tag{1.5}$$

where

$$\gamma^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{1.6}$$

$$\gamma^i = \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix} \tag{1.7}$$

The Dirac equation offers four interesting solutions instead of one that the Klein-Gordon field gives. These are, upon solving (1.5) [2],

$$\psi_a = \begin{bmatrix} 1 \\ 0 \\ p_z/(p^0+m) \\ (p_x+ip_y)/(p^0+m) \end{bmatrix} \psi_b = \begin{bmatrix} 0 \\ 1 \\ (p_x-ip_y)/(p^0+m) \\ -p_z/(p^0+m) \end{bmatrix} \tag{1.8}$$

$$\psi_c = \begin{bmatrix} p_z/(p^0+m) \\ (p_x+ip_y)/(p^0+m) \\ 1 \\ 0 \end{bmatrix} \psi_d = \begin{bmatrix} (p_x-ip_y)/(p^0+m) \\ p_z/(p^0+m) \\ 0 \\ 1 \end{bmatrix} \tag{1.9}$$

The first two equations describe the spin up (ψ_a) and spin down (ψ_b) particles respectively while the last two describe the spin up (ψ_c) and spin down (ψ_d) anti-particles.

1.2.2 Dirac Field

The Dirac Field for particle described by the Dirac equation is [5]

$$L = \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi \tag{1.10}$$

The Euler-Lagrangian equations of (1.10) give rise to (1.5). An interesting observation at this stage, which will be of huge significance in Chapter 3, is that the Dirac field is invariant under transformations of the kind

$$\psi \to e^{-iq\theta}\psi \tag{1.11}$$

This symmetry is an example of U(1) symmetry and the transformation elements are members of the U(1) symmetry group.

1.2.3 Vector Fields

It is enriching to explore Vector Fields, especially the Electromagnetic Field, at this stage. From, Classical Physics, Maxwell's equations are of the form [1] Maxwell's equations in free space:

$$\nabla . E = \rho, \tag{1.12a}$$

$$\nabla . B = 0 \tag{1.12b}$$

$$\frac{\partial B}{\partial t} = -\nabla \times E, \tag{1.12c}$$

$$\frac{\partial B}{\partial t} = \nabla \times B - 4\pi j, \tag{1.12d}$$

Potential can be defined for these fields such that

$$E = -\nabla E + \frac{\partial A}{\partial t} \tag{1.13a}$$

$$B = \nabla \times A \tag{1.13b}$$

Defining a tensor $F_{\mu\nu}$, called the field strength tensor while writing $A^\mu = (A^0, \overrightarrow{A})$ such that, [5]

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu (1.14)$$

A Lagrangian can be written for the electromagnetic field as follows,

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{1.15}$$

The Electomagnetic Field is known to be invariant under gauge transformations of the sort

$$A_\mu^{'} \to A_\mu + \partial_\mu \theta (1.16)$$

Chapter 2

Abelian Gauge Theory

2.1 Local Gauge Invariance

The previous chapter concluded with a gauge transformation of the Dirac Field that kept it invariant. The gauge transformation was a global one due to the fact that the actual nature of the transformation didn't depend on the position of the particle in space-time. We, however, can't expect all situations that appear in nature to be so. There could be transformations that depend on where the particle is, and the Lagrangian has to remain invariant to that transformation as well. Such transformations are called local transformations and are given by,

$$\psi \to e^{-ieQ\theta(x)}\psi \tag{2.1a}$$

$$A_\mu \to A'_\mu \tag{2.1b}$$

[5] The transformed Lagrangian will be [5]

$$L' = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - eQ(A'_\mu - \partial_\mu\theta)\bar{\psi}\gamma^0\psi(2.2)$$

The Lagrangian will remain invariant if $A'_\mu \to A_\mu + \partial_\mu\theta$. This is the familiar gauge transformation of the Electromagnetic field that was introduced in the previous chapter. Thus, upon adding a gauge field term and the Lagrangian of the Electromagnetic Field for logical consistency, we get

$$L = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - eQ\bar{\psi}\gamma^0\psi A_\mu \tag{2.3}$$

This is the Lagrangian of Quantum Electrdynamics or QED.

2.2 Quantum Electrodynamics

As was shown in the previous section, the Lagrangian of QED is

$$L = \psi(i\gamma^{\mu}\partial_{\mu} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - eQ\psi\gamma^{0}\psi A_{\mu}(2.4)$$

While $\psi(i\gamma^{\mu}\partial_{\mu} - m)\psi$ describes the free spin $\frac{1}{2}$ particle and $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ describes the photon field, $-eQ\psi\gamma^{0}\psi A_{\mu}$ describes the interaction between the two. An observation to make is that the quantum of electric charge, together with the net total number of charged particles, that enable the interaction as they act as the coupling parameter. This reinforces the fact that only charged particles interact with the Electromagnetic field. The Electromangetic field is also called the bosonic field as they represent photons that act as spin-1 carriers of the field when interacting with an electron current $eQ\psi\gamma^{0}\psi$.
The form of the Lagrangian is Lorentz invariant. Under a Lorentz boost $\Lambda^{m}u_{\nu}$, the wavefunction of the particle transforms as follows to preserve invariance [3].

$$S^{-1}\gamma^{\mu}S = \Lambda^{\mu}{}_{\nu}\gamma^{\nu} \tag{2.5}$$

This equation demonstrated that while the representation of γ^{μ} changes from frame to frame, the underlying physics remains the same.
The solutions to the Hamiltonian of QED that is generated from the Euler-Lagrange equations are Bilinear covariants as they transform according to (2.5). They however form a wide variety of physical quantities as follows [3]

$\psi\psi$	Scalar	Space Inversion:+
$\psi\gamma^{\mu}\psi$	Vector	Space Inversion:-
$\psi\sigma^{\mu\nu}$	Tensor	
$\psi\gamma^{5}\gamma^{\mu}\psi$	Axial Vector	Space Inversion:+
$\psi\gamma^{5}\psi$	Pseudoscalar	Space Inversion:-

where γ^{5} is $i\gamma^{0}\gamma^{1}\gamma^{2}\gamma^{3}$

Bibliography

[1] David Griffiths. *Introduction to Electrodynamics*. 2013.

[2] David Griffiths. *Introduction to Elementary Particles*. 2008.

[3] Francis Halzen and Alan D. Martin Martin. *Quarks and Leptons, An Introductory Course in Modern Partice Physics*. 1984.

[4] *Klein-Gordon equation*. URL: `https://en.wikipedia.org/wiki/Klein%E2%80%93Gordon_equation`.

[5] Amitabha Lahiri and Palash B. Pal. *A First Book of Quantum Field Theory*. 2004.