# Gauge Theory and the Standard Model

PROJECT REPORT

*Submitted in partial fulfillment of the requirements of*
*PHY F266 Study Oriented Project*

*By*

Vishnu SHARMA
ID No. 2013B5A3467P

*Under the supervision of:*

Dr. Biswanath LAYEK

BIRLA INSTITUTE OF TECHNOLOGY AND SCIENCE PILANI, PILANI CAMPUS

November 2016

*"If anybody says he can think about quantum physics without getting giddy, that only shows he has not understood the first thing about them."*

Niels Bohr

BIRLA INSTITUTE OF TECHNOLOGY AND SCIENCE PILANI, PILANI CAMPUS

# *Abstract*

Master of Science (Hons.)

## Gauge Theory and the Standard Model

by Vishnu SHARMA

The Project is an attempt to understand the physics of partilces and their interactions through the systemic study of the fields involved. It comprises of a study of the Klein-Gordon Field, the Dirac Field, and the gauge theories of Quantum Electrodynamics and Quantum Chromodynamics.

# *Acknowledgements*

# Contents

$\mu, \nu, ...$ Space-time indices of a vector or tensor.

$i, j, ...$ Spatial indices of a vector or tensor.

$g_{\mu\nu}$ (Components of) metric tensor, diag(1,-1,-1,-1).

$p^\mu$ Contravariant vector.

$p_\mu$ Covariant vector.

$L$ Lagrangian density, also called Lagrangian.

$[A, B]$ Commutator $AB - BA$.

$\sigma^i$ Pauli matrices.

$e$ Electric charge of proton. Electron carries $-e$ charge.

# Chapter 1

# Free Particle Fields

## 1.1 Spin-0 Particles

Quantum Field Theory is the theory in theoretical physics that is used to study field that are bound by the laws of Quantum Mechanics and Einstein's Theory of Relativity. In Field Theory, a Lagrangian is defined that remains invariant to any coordinate transformation. The Lagrangian thus defined contains all information about the particle that is needed to be known; in classical field theory it contains the past, present, and future forms of the trajectory of the particle and in quantum field theory it contains the nature of evolution of the wavefunction, and thus the probability distribution, of the particle for all time. The Lagrangian for a scalar field (A field with just one quantifier at each point in space-time) is as follows:

$$L = \frac{1}{2}(\partial^\mu \Phi)(\partial_\mu \Phi^*) - \frac{1}{2}m\Phi\Phi^* \tag{1.1}$$

This is the Klein-Gordon Field of spin-0 particles. Equation (1.q) is called the Klein-Gordon field after physicists Oskar Klein and Walter Gordon who proposed this model to describe relativistic particles in 1926.[**KGE**] The Hamiltonian for such a particle can be derived from the Euler-Lagrange equations of motion. Alternatively, in Relativistic Mechanics, [**lahiri04**]

$$p^\mu p_\mu - m^2 = 0 \tag{1.2}$$

and

$$p_\mu \rightarrow i\partial_\mu \tag{1.3}$$

in the position representation of momentum. When this relation is applied to equation (1.2) we get equation (1.4) which describes the Hamiltonian of such a particle.

$$(\partial^\mu \partial_\mu + m^2)\Phi(x) = 0 \tag{1.4}$$

1

## 1.2   spin-$\frac{1}{2}$

### 1.2.1   Dirac Equation

The Dirac Hamiltonian for a spin-$\frac{1}{2}$ particle, commonly referred to as the "Dirac Equation" is [**lahiri04**]

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0 \tag{1.5}$$

where

$$\gamma^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{1.6}$$

$$\gamma^i = \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix} \tag{1.7}$$

The Dirac equation offers four interesting solutions instead of one that the Klein-Gordon field gives. These are, upon solving (1.5) [**griffiths08**],

$$\psi_a = \begin{bmatrix} 1 \\ 0 \\ p_z/(p^0+m) \\ (p_x+ip_y)/(p^0+m) \end{bmatrix} \quad \psi_b = \begin{bmatrix} 0 \\ 1 \\ (p_x-ip_y)/(p^0+m) \\ -p_z/(p^0+m) \end{bmatrix} \tag{1.8}$$

$$\psi_c = \begin{bmatrix} p_z/(p^0+m) \\ (p_x+ip_y)/(p^0+m) \\ 1 \\ 0 \end{bmatrix} \quad \psi_d = \begin{bmatrix} (p_x-ip_y)/(p^0+m) \\ p_z/(p^0+m) \\ 0 \\ 1 \end{bmatrix} \tag{1.9}$$

The first two equations describe the spin up ($\psi_a$) and spin down ($\psi_b$) particles respectively while the last two describe the spin up ($\psi_c$) and spin down ($\psi_d$) anti-particles.

### 1.2.2   Dirac Field

The Dirac Field for particle described by the Dirac equation is [**lahiri04**]

$$L = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \tag{1.10}$$

The Euler-Lagrangian equations of (1.10) give rise to (1.5). An interesting observation at this stage, which will be of huge significance in Chapter 3, is that the Dirac field is invariant under transformations of the kind

$$\psi \rightarrow e^{-iq\theta}\psi \tag{1.11}$$

This symmetry is an example of U(1) symmetry and the transformation elements are members of the U(1) symmetry group.

### 1.2.3   Vector Fields

It is enriching to explore Vector Fields, especially the Electromagnetic Field, at this stage. From, Classical Physics, Maxwell's equations are of the form [**griffiths13**] Maxwell's equations in free space:

$$\nabla.E = \rho, \tag{1.12a}$$

$$\nabla.B = 0 \tag{1.12b}$$

$$\frac{\partial B}{\partial t} = -\nabla \times E, \tag{1.12c}$$

$$\frac{\partial B}{\partial t} = \nabla \times B - 4\pi j, \tag{1.12d}$$

Potential can be defined for these fields such that

$$E = -\nabla E + \frac{\partial A}{\partial t} \tag{1.13a}$$

$$B = \nabla \times A \tag{1.13b}$$

Defining a tensor $F_{\mu\nu}$, called the field strength tensor while writing $A^\mu = (A^0, \vec{A})$ such that, [**lahiri04**]

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{1.14}$$

A Lagrangian can be written for the electromagnetic field as follows,

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{1.15}$$

The Electomagnetic Field is known to be invariant under gauge transformations of the sort

$$A'_\mu \rightarrow A_\mu + \partial_\mu\theta \tag{1.16}$$

# Chapter 2

# Abelian Gauge Theory

## 2.1   Local Gauge Invariance

The previous chapter concluded with a gauge transformation of the Dirac Field that kept it invariant. The gauge transformation was a global one due to the fact that the actual nature of the transformation didn't depend on the position of the particle in space-time. We, however, can't expect all situations that appear in nature to be so. There could be transformations that depend on where the particle is, and the Lagrangian has to remain invariant to that transformation as well. Such transformations are called local transformations and are given by,

$$\psi \rightarrow e^{-ieQ\theta(x)}\psi \qquad (2.1a)$$

$$A_\mu \rightarrow A_\mu^{'} \qquad (2.1b)$$

[**lahiri04**] The transformed Lagrangian will be [**lahiri04**]

$$L^{'} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - eQ(A_\mu^{'} - \partial_\mu\theta)\bar{\psi}\gamma^0\psi (2.2)$$

The Lagrangian will remain invariant if $A_\mu^{'} \rightarrow A_\mu + \partial_\mu\theta$. This is the familiar gauge transformation of the Electromagnetic field that was introduced in the previous chapter. Thus, upon adding a gauge field term and the Lagrangian of the Electromagnetic Field for logical consistency, we get

$$L = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - eQ\bar{\psi}\gamma^0\psi A_\mu \qquad (2.3)$$

This is the Lagrangian of Quantum Electrdynamics or QED.

## 2.2   Quantum Electrodynamics

As was shown in the previous section, the Lagrangian of QED is

$$L = \psi(i\gamma^\mu \partial_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - eQ\psi\,\gamma^0\psi A_\mu \quad (2.4)$$

While $\psi(i\gamma^\mu \partial_\mu - m)\psi$ describes the free spin $\frac{1}{2}$ particle and $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ describes the photon field, $-eQ\psi\,\gamma^0\psi A_\mu$ describes the interaction between the two. An observation to make is that the quantum of electric charge, together with the net total number of charged particles, that enable the interaction as they act as the coupling parameter. This reinforces the fact that only charged particles interact with the Electromagnetic field. The Electromangetic field is also called the bosonic field as they represent photons that act as spin-1 carriers of the field when interacting with an electron current $eQ\psi\,\gamma^0\psi$.

The form of the Lagrangian is Lorentz invariant. Under a Lorentz boost $\Lambda^m u_\nu$, the wavefunction of the particle transforms as follows to preserve invariance [**halzen84**].

$$S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu \gamma^\nu \quad (2.5)$$

This equation demonstrated that while the representation of $\gamma^\mu$ changes from frame to frame, the underlying physics remains the same.

The solutions to the Hamiltonian of QED that is generated from the Euler-Lagrange equations are Bilinear covariants as they transform according to (2.5). They however form a wide variety of physical quantities as follows [**halzen84**]

| | | |
|---|---|---|
| $\psi\,\psi$ | Scalar | Space Inversion:+ |
| $\psi\,\gamma^\mu\psi$ | Vector | Space Inversion:- |
| $\psi\,\sigma^{\mu\nu}$ | Tensor | |
| $\psi\,\gamma^5\gamma^\mu\psi$ | Axial Vector | Space Inversion:+ |
| $\psi\,\gamma^5\psi$ | Pseudoscalar | Space Inversion:- |

where $\gamma^5$ is $i\gamma^0\gamma^1\gamma^2\gamma^3$