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Chapter 1

Introduction

It is a triumph of physics and its practitioners that we can reproduce the dynamics of extremely complex systems using mathematical equations. The problem, however, is that most of the governing equations of these complex systems are highly nonlinear, and often coupled differential equations. This makes it very difficult for us to solve the systems in question exactly,

But this was not entirely unexpected. It is easy to notice that most systems that we come across in day-to-day life are not simple systems. Something as basic as the simple pendulum does not follow $\ddot{\theta} + \omega^2\theta = 0$ if θ is large (higher order corrections to θ starts dominating). This tells us that even the most basic of the systems have to be nonlinear.

In-principle, the state of a system at present is completely determined by initial conditions. Such systems are called *deterministic system*. So for different initial conditions, we could have trajectories that diverge exponentially faster from each other in time. Such systems are said to be *chaotic systems*. Let's say δx be the separation between two trajectories. We can then give a rough mathematical definition of complexity as follows

$$\delta x(t) = \exp \lambda t \ \delta x(0)$$

The λ in the exponent is called the *Lyapunov exponent*. This quantifies the speed at which the systems moves towards chaos - a higher λ means that the system will go to chaos faster.

In this thesis, we shall explore different aspects of chaos in different physical systems. Broadly, we shall divide the systems that we'll consider into *classical* and *quantum* systems. We shall consider some specific cases in each of these, and get into the depth of it.

1.1 Outline and Description

We give a brief outline description of each of the cases that we are going to consider in this thesis :-

- **Settting up the Machinery** - we shall set up the basic analytic and numerical machinery used for calculating various quantities in chaotic systems. We shall first review a bit of Hamiltonian dynamics, and then look into mappings, Poincare sections, and also go into detail as to how one describes the stability of points and trajectories. We shall also see how we can use symbolic algebra and numerical softwares to get an idea of the dynamics of chaotic systems.
- **The Logistic Map** - The Logistic Map is a simple map of degree two, which is astoundingly effective while studying population dynamics and epidemic growth. It also holds immense interest from a mathematical point of view and will also be a useful tool when we discuss some aspects of chaos in quantum systems. We write down the logistic map and derive the stable points of the systems. We see that the system inherently gives rise to a *bifurcation diagram* in phase space, which will show clear signatures of chaotic dynamics.
- **Lorenz System of Differential Equations and the Lorenz Attractor** - The Lorenz system is probably the most well illustrated system of deterministic chaos. It was initially written down to describe the dynamics of convective atmosphere currents, but has found applications in other fields as well. We shall start with defining the system, and doing an analysis of the form of the equations, the parameters involved, and a stability analysis. We shall then evaluate the system numerically and show that it indeed is chaotic.
- **Miscellaneous Systems in Classical Chaos** - We shall define a few systems of our own, and try to study their classical chaotic dynamics using numerical methods.
- **Quantum Chaos** - We look at how chaos arises in quantum system, and investigate the link between a classically chaotic system and the quantum spectrum of the system.
- **A Bound on Chaos** - We study the conjecture by Maldacena, Stanford and Shenker [3] which gives an upper limit to how fast chaos can grown in a quantum chaotic system. We see that this bound is exactly satisfied for black holes.
- **Chaos in Quantum Channels** - We study the growth of chaos in a quantum information perspective [1], and try to define an independent information theoretic definition of chaos, referring to

Chapter 2

Setting up the Machinery

Since this chapter is mostly a review, we have borrowed from [5], [6] and [4]

2.1 Hamiltonian Dynamics

Given a Hamiltonian $H(q, p)$, one can write down Hamilton's equations of motion as follows

$$\dot{q_i} = \frac{\partial H}{\partial p_i}, \ \dot{p_i} = -\frac{\partial H}{\partial q_i}$$

Here, q_i and p_i are said to be phase space coordinates, and $x = (q, p)$ with $q = (q_1, q_2, \ldots, q_N)$ and $p = (p_1, p_2, \ldots, P_N)$ is said to be a point in phase space. We can see that Hamilton's equation indeed gives the path in time of the dynamical system in the phase space. One can write Hamilton's equation in vector form as

$$\dot{x_i} = \omega_{ij}\frac{\partial H(x)}{\partial x_j}, \ \omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

where I is the identity matrix in the applicable dimension.

Note that we have chosen H only to be a function of q and p. q and p are functions of time t, but in our construction, the Hamiltonian is independent of time. The explicit time-dependence of the Hamiltonian is an important check to see if the system is conservative or non-conservative. We can see this in the following way, by assuming H to be a function of t as well.

$$\frac{d}{dt}H(q, p, t) = \frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial p}\dot{p} + \frac{\partial H}{\partial t}$$

If H is not a function of t, the first two terms cancel, and hte third term goes to zero, which gives us the identity that $\frac{d}{dt}H(q,p,t) = 0$. This means that the phase space trajectories lie of curves of constant energy E. Thus, for the specific case of single degree of freedom systems, we can say that the dynamical system described by H is integrable.

2.2 Canonical Transformations

We have been used to interpreting q and p as position and momentum respectively, but it is important in chaotic dynamics to abandon this notion and think of it just as variables in the phase space. In principle, we can work with new variables which are functions of the old variables, provided that this change obeys Hamilton's equations. These new variables are called canoncical variables, and the corresponding transformations are called *canonical transformations*.

Consider,

$$\tilde{q} = f(q,p,t),\ \ \tilde{p} = g(q,p,t)$$

Corresponding to this, one could write the equations of motion as,

$$\frac{d\tilde{q}}{dt} = \frac{\partial \tilde{H}}{\partial p},\ \ \frac{d\tilde{p}}{dt} = -\frac{\partial \tilde{H}}{\partial q}$$

where $\tilde{H} = \tilde{H}(\tilde{q},\tilde{p},t)$ is the Hamiltonian of the systems under the transformation.

Chapter 3

Lorenz System of Differential Equations and the Lorenz Attractor

The Lorenz system[2] is described as follows :-

$$\dot{x} = \sigma(y - x)$$

$$\dot{y} = rx - y - xz$$

$$\dot{z} = xy - bz$$

, where x, y, z are dynamical variables and σ, r, b are constants.

These innocuous looking equations hold many secrets in them. The landmark 1963 paper by Lorenz describing these equations was written keeping in mind a convective model of atmospheric dynamics, but it has since found a wide variety of applications in lasers, circuits etc.

3.1 A Few Properties

- Though the equations look simple, we can already see that the solutions might be complicated due to the nonlinearity in the terms xy and xz. Nonlinear systems could lead to chaotic dynamics.
- One could try to solve these equations without much thought by putting it directly into a computer and getting a solution. But it is useful to note that the equation has a few symmetries. For example $(x, y, z) \to (-x, -y, z)$ keeps the solution invariant. This means that all solutions should be symmetric or have a redundant symmetric partner.

- One calls a system of equations dissipative if the phase space volume contracts as the system evolves in time. If we write the equations in a condensed form as $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$, then the volume the rate of change of volume is given by $\nabla.\boldsymbol{f}$ integrated over the volume. We can see

$$\nabla.\boldsymbol{f} = \frac{\partial}{\partial x}[\sigma(y-x)] + \frac{\partial}{\partial y}[rx - y - xz] + \frac{\partial}{\partial z}[xy - bz] = -(\sigma + 1 + b) < 0$$

  This is a negative constant. So, the phase space volume decreases in time and the Lorenz system is dissipative.

- Lets consider the Lorenz system without the nonlinearities. If one looks closely, this system is equivalent to linearizing the original system by considering perturbations around the origin. The equations would then read,

$$\dot{x} = \sigma(y - x) \tag{3.1}$$

$$\dot{y} = rx - y \tag{3.2}$$

$$\dot{z} = -bz, \tag{3.3}$$

  We can immediately see that the equation for z is decoupled from the other two. Also, the z solution decays exponentially as $t \to \infty$. The other two equations can be written as :-

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma \\ r & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

  The determinant $\Delta = -\sigma(1 - r)$ and trace $\tau = -(\sigma + 1)$. The system has a saddle point at $r > 1$

3.2 Numerical Solution

We have seen that a signature of chaos is that the trajectories diverge for a slight change in the initial conditions. As these equations are not really very straightforward to solve analytically, we solve it numerically using Python. We use initial conditions as $(1, 1, 1)$ and $(1.0000001, 1, 1)$, and plot the variation of x, y, z with time t.

Fig. [3.1] plots the numerically integrated equation for these slightly different initial conditions. We see that this system shows clear signs of chaos - the trajectories start diverging after $t = 30$ or so. This is clearly evident from Fig. [3.2], which plots the deviation of paths from each other with time.

Typically, if we take the two initial conditions far away from each other, we will se that the system goes to chaos faster. This is in agreement with what we had expected and conjectured.

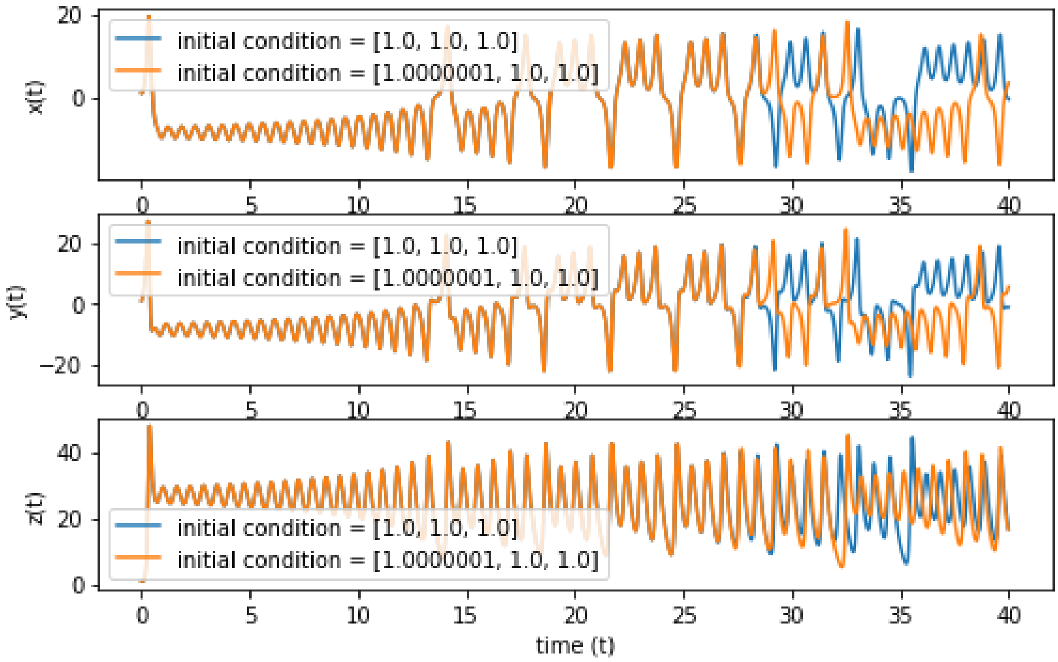

FIGURE 3.1: Numerical solutions for different initial conditions. Trajectories start to visibly diverge roughly around $t = 30$.

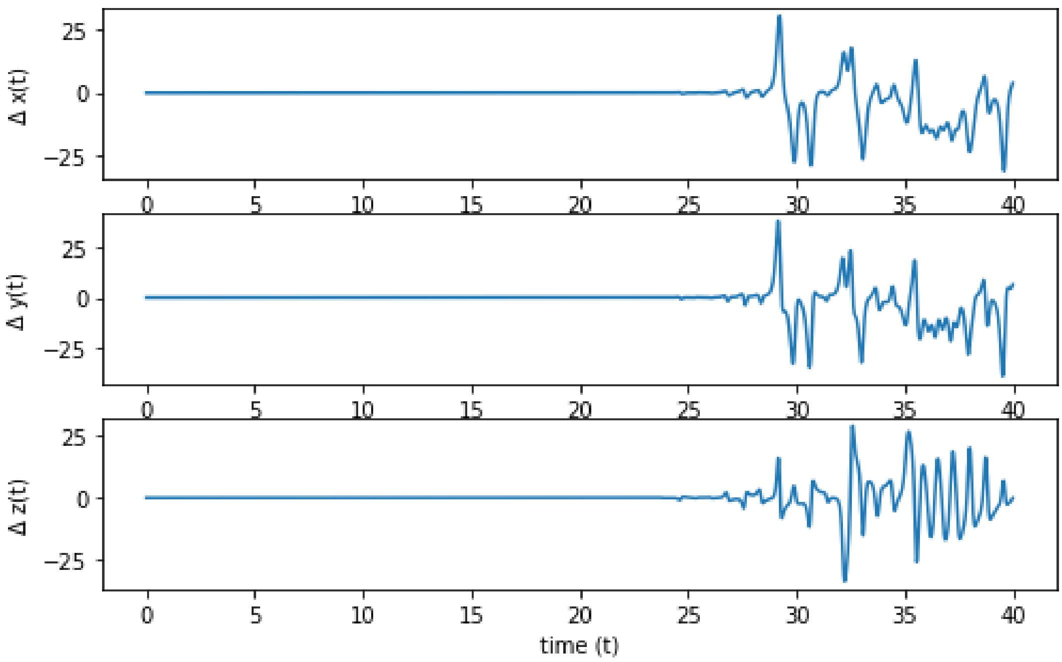

FIGURE 3.2: The deviation of paths with start out very close to each other at tie $t = 0$.

Fig. [3.3] shows a beautiful 3-D plot of the numerically integrated solution, which looks like the wings of a butterfly. Probably because of this, chaos is also sometimes referred to as the *butterfly effect*.

Lorenz's model is so powerful that it continues to be studied till today. We shall investigate this further in the thesis, and hope to make some more comments in the final report.

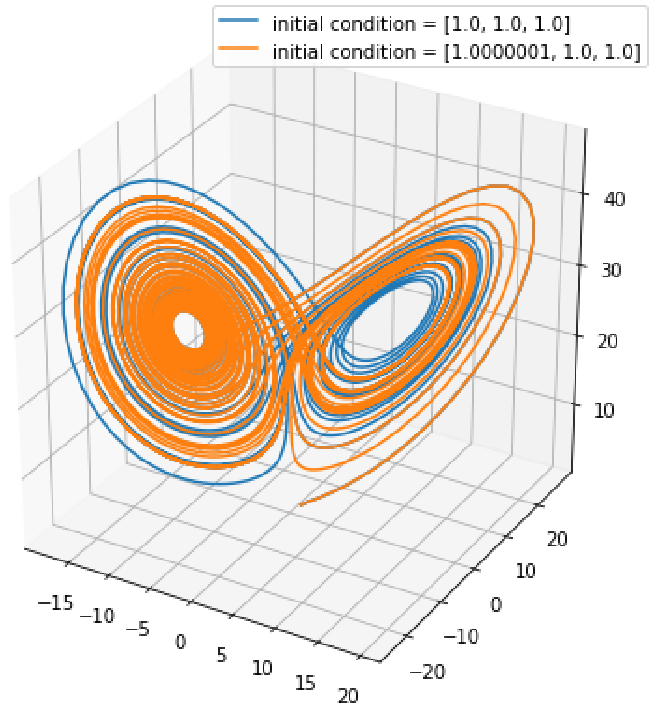

FIGURE 3.3: A 3-D plot of x, y, z. Chaos is evident here too.

Chapter 4

The Logistic Map

4.1 The Map

The logistic map is really simple. But as we have seen earlier with the Lorenz system earlier, simple maps can have very complicated dynamics. The logistic map is a one-dimensional map, and is defined by :-

$$x_{n+1} = rx_n(1 - x_n) \tag{4.1}$$

where r is a parameter. Note that this is the discrete version of the map, and a continuous description can also be defined similarly. This model is usually used to study population dynamics, so r is referred to as the *growth parameter*.

4.2 Preliminary Numerics

For all intents and purposes we shall restrict ourselves in the regime of $0 < x \leq 1$ and $0 < r \leq 4$. We shall see that this is the only interesting regime of parameters for this model. It is useful to think of x as population to gain some physical intuition.

All the plots were made with initial condition $x = 0.2$, and varying r.

First, let's look at what happens when $r \leq 1$. Fig. [4.1] plots the evolution for $r = 0.1, 0.5, 1.0, 1.1$. We see that for $r = 0.1, 0.5$, the population goes to zero pretty fast. When $r = 1.0$, the population tends to go to zero as $n \to \infty$. On the other hand if r is just above 1.0, we see that there is a stable value to which the population converges to.

What does this tell us? It gives us the gloomy result that for low growth rates, the population of a certain species will go to zero at some point in time, ie. the species will become extinct.

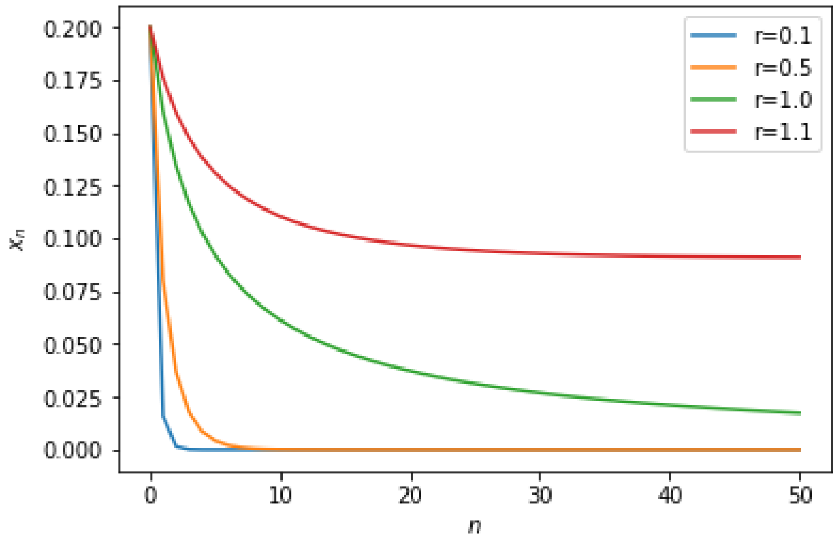

FIGURE 4.1: x_n vs n. We see that for $r < 1$, x_n goes to zero pretty fast.

Let's take a look at how the behaviour is for values of $r > 1$ (Fig. [4.2]). For $r = 2.8$, we see that the population oscillates a bit at the start and then settles down to a steady state value. But on the other hand, for $r = 3.3$, the population just keeps oscillating periodically around the steady state value forever. This is where the logistic map becomes an important topic of study. One should note that the period of oscillations in rough $n = 2$.

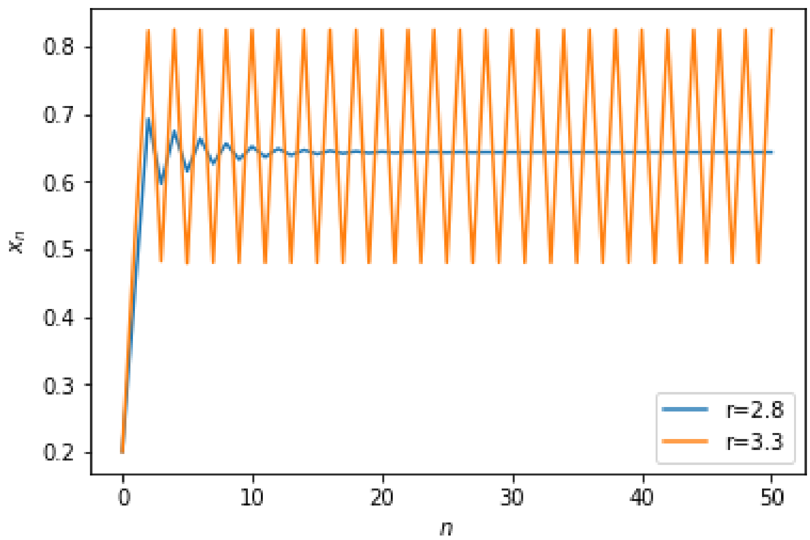

FIGURE 4.2: x_n vs n. For $r = 2.8$, x_n settles to a steady state value. For $r = 3.3$, x_n oscillates about the steady state.

Things really start getting interesting after $r = 3.3$. For example, take a look at Fig. [4.3], which plots the x_n for $r = 3.8$. We see that like $r = 3.3$, x_n oscillates around the steady state value, but now the period is not $n = 2$! The period has now changed to a higher value. We could conjecture that for some value of the growth parameter, the period would really tend to infinity. This is a clear sign of chaos!

This is a really deep feature of the logistic map, which we shall hope to investigate further in this thesis, among other things.

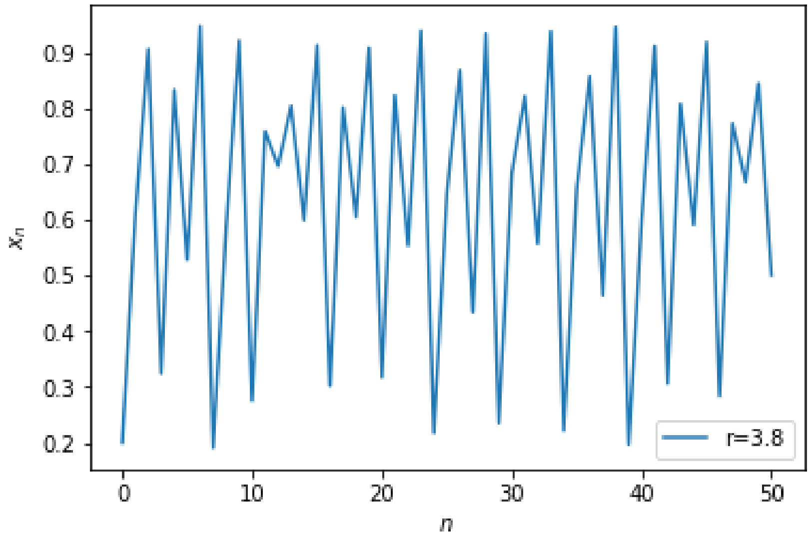

FIGURE 4.3: Numerical solutions for different initial conditions. Trajectories start to visibly diverge roughly around $t = 30$.

Chapter 5

A Bound on Chaos

Recall how we diagnose chaos in classical systems. For a dynamical variable q, we say that the system is chaotic if :-

$$\frac{\partial q}{\partial q_0} = \{q(t), p_0\} \sim e^{\lambda t},$$

where q_0 and p_0 are the initial conditions.

But in quantum mechanics, our dynamical variable is the wavefunction $\psi(x)4$, which respects unitary evolution. Due to this the inner product of the wavefunction will always have the same norm.

On thinking a bit more, one can easily see what's wrong with this analysis. We should ideally compare our wavefunction with not with the individual dynamical variables, but the phase space distribution $\rho(q, p)$.

Let us work for the moment with a spin system with spin centers $S_1, \ldots S_N$. At $t = 0$, we have $[S_i, S_j] = 0$. The system can be time-evolved using a Hamiltonian H, which we consider to be k-local and built out of these spin operators.

In quantum mechanics, we can use the following object for our investigations of chaos

$$[W(t), V(0)]^2,$$

where

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