Advanced Quantum Mechanics: Assignment #2

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Problem 1

Let's use the follwing convention ($|l, m\rangle$)

$$|2,2\rangle = \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} \quad \text{and} \quad |2,1\rangle = \begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix} \quad \text{and} \quad |2,0\rangle = \begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix} \quad \text{and} \quad |2,-1\rangle = \begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix} \quad \text{and} \quad |2,-2\rangle = \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}$$

We know that,

$$J_3 |l,m\rangle = m |l,m\rangle \quad \text{and} \quad J_\pm |l,m\rangle = \sqrt{(l \mp m)(l \pm m + 1)}\, |l, m \pm 1\rangle$$

$$J_3 |2,2\rangle = 2 |2,2\rangle \ , \ J_3 |2,1\rangle = |2,1\rangle \ , \ J_3 |2,0\rangle = 0 \ , \ J_3 |2,-1\rangle = - |2,-1\rangle \ , \ J_3 |2,-2\rangle = -2 |2,-2\rangle$$

Hence, we can see that,

$$J_3 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}$$

We also note that,

$$J_+ |2,2\rangle = 0 \ , \ J_+ |2,1\rangle = 2 |2,2\rangle \ , \ J_+ |2,0\rangle = \sqrt{6} |2,1\rangle \ , \ J_+ |2,-1\rangle = \sqrt{6} |2,0\rangle \ , \ J_+ |2,-2\rangle = 2 |2,-1\rangle$$

$$J_- |2,2\rangle = 2 |2,1\rangle \ , \ J_- |2,1\rangle = \sqrt{6} |2,0\rangle \ , \ J_- |2,0\rangle = \sqrt{6} |2,-1\rangle \ , \ J_- |2,-1\rangle = 2 |2,-2\rangle \ , \ J_- |2,-2\rangle = 0$$

So,

$$J_+ = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad J_- = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

Hence,

$$J_1 = \frac{J_+ + J_-}{2} = \frac{1}{2} \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & \sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & \sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix} \quad \text{and} \quad J_2 = \frac{J_+ - J_-}{2i} = \frac{i}{2} \begin{pmatrix} 0 & -2 & 0 & 0 & 0 \\ 2 & 0 & -\sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & -\sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & -2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

Problem 2

Given,

$$U = \exp\left(-i\frac{\sigma_3\alpha}{2}\right)\exp\left(-i\frac{\sigma_2\beta}{2}\right)\exp\left(-i\frac{\sigma_3\gamma}{2}\right)$$

Consider the trace of U,

$$\begin{aligned}
\operatorname{tr} U &= \langle 0|U|0\rangle + \langle 1|U|1\rangle \\
&= \langle 0|\exp\left(-i\frac{\sigma_3\alpha}{2}\right)\exp\left(-i\frac{\sigma_2\beta}{2}\right)\exp\left(-i\frac{\sigma_3\gamma}{2}\right)|0\rangle + \langle 1|\exp\left(-i\frac{\sigma_3\alpha}{2}\right)\exp\left(-i\frac{\sigma_2\beta}{2}\right)\exp\left(-i\frac{\sigma_3\gamma}{2}\right)|1\rangle \\
\operatorname{tr} U &= e^{-i\left(\frac{\alpha+\gamma}{2}\right)}\langle 0|\exp\left(-i\frac{\sigma_2\beta}{2}\right)|0\rangle + e^{i\left(\frac{\alpha+\gamma}{2}\right)}\langle 1|\exp\left(-i\frac{\sigma_2\beta}{2}\right)|1\rangle
\end{aligned}$$

We note that,

$$\begin{aligned}
\exp\left(-i\frac{\sigma_2\beta}{2}\right) &= \sum_{n=0}^{\infty}\frac{1}{2n!}\left(-i\frac{\beta}{2}\right)^{2n}\sigma_2^{2n} + \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(-i\frac{\beta}{2}\right)^{2n+1}\sigma_2^{2n+1} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{2n!}\left(\frac{\beta}{2}\right)^{2n} - \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{\beta}{2}\right)^{2n+1}\sigma_2 \impliedby \sigma_2^{2n} = 1 \\
\exp\left(-i\frac{\sigma_2\beta}{2}\right) &= \cos\frac{\beta}{2} - i\sin\frac{\beta}{2}\sigma_2 \\
\langle 0|\exp\left(-i\frac{\sigma_2\beta}{2}\right)|0\rangle &= \langle 1|\exp\left(-i\frac{\sigma_2\beta}{2}\right)|1\rangle = \cos\frac{\beta}{2}
\end{aligned}$$

Hence, we get,

$$\operatorname{tr} U = \cos\frac{\beta}{2}\left(e^{i\left(\frac{\alpha+\gamma}{2}\right)} + e^{-i\left(\frac{\alpha+\gamma}{2}\right)}\right) = \cos\frac{\beta}{2}\cos\frac{\alpha+\gamma}{2}$$

But we know that for any rotation matrix

Problem 3

We know that,

$$J_1 = \frac{J_+ + J_-}{2} \quad \text{and} \quad J_2 = \frac{J_+ - J_-}{2i} \quad \text{and} \quad J_\pm|l,m\rangle = \sqrt{(l\mp m)(l\pm m+1)}\,|l,m\pm 1\rangle$$

As successive action of the form $J_\pm^\alpha|l,m\rangle$ with integral $\alpha > 0$ takes a state to one with higher/lower m , we can see that $\langle l,m|J_\pm^\alpha|l,m\rangle = 0$. Lets consider $\langle J_1\rangle$,

$$\begin{aligned}
\langle J_1\rangle &= \langle l,m|J_1|l,m\rangle \\
&= \frac{1}{2}\left(\langle l,m|J_+|l,m\rangle + \langle l,m|J_-|l,m\rangle\right) \\
&= 0
\end{aligned}$$

Similarly for $\langle J_2\rangle$,

$$\begin{aligned}
\langle J_2\rangle &= \langle l,m|J_2|l,m\rangle \\
&= \frac{1}{2i}\left(\langle l,m|J_+|l,m\rangle - \langle l,m|J_-|l,m\rangle\right) \\
&= 0
\end{aligned}$$

Problem 3 continued on next page...

Consider $\langle J_1^2 \rangle$ and $\langle J_2^2 \rangle$,

$$\langle J_1^2 \rangle = \frac{1}{4}(\langle J_+^2 \rangle + \langle J_-^2 \rangle + \{J_+, J_-\}) = \frac{1}{4}\{J_+, J_-\} \quad \text{and}$$
$$\langle J_2^2 \rangle = \frac{1}{-4}(\langle J_+^2 \rangle + \langle J_-^2 \rangle - \{J_+, J_-\}) = \frac{1}{4}\{J_+, J_-\} \implies \langle J_1^2 \rangle = \langle J_2^2 \rangle$$

We know,

$$\langle J^2 \rangle = l(l+1)$$
$$\langle J_1^2 \rangle + \langle J_2^2 \rangle + \langle J_3^2 \rangle = l(l+1)$$
$$2\langle J_1^2 \rangle + m^2 = l(l+1)$$
$$\langle J_1^2 \rangle = \langle J_2^2 \rangle = \frac{l(l+1) - m^2}{2}$$

Problem 4

Let's use the follwing convention ($|l, m\rangle$)

$$|1,1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad |1,0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |1,-1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

We know that $J_\pm |1, m\rangle = \sqrt{(1 \mp m)(1 \pm m + 1)}\,|1, m \pm 1\rangle$, which means,

$$J_+ |1,1\rangle = 0 \quad \text{and} \quad J_+ |1,0\rangle = \sqrt{2}\,|1,1\rangle \quad \text{and} \quad J_+ |1,-1\rangle = \sqrt{2}\,|1,0\rangle$$
$$J_- |1,1\rangle = \sqrt{2}\,|1,0\rangle \quad \text{and} \quad J_- |1,0\rangle = \sqrt{2}\,|1,-1\rangle \quad \text{and} \quad J_- |1,-1\rangle = 0$$

Using the above relations, one can write J_+ and J_- as follows,

$$J_+ = \sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad J_- = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \implies J_2 = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

Thus, we have obtained J_2. Let's also note the following,

$$J_2^2 = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad J_2^3 = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = J_2$$
$$J_2^4 = J_2^3 J_2 = J_2^2$$

We can see a pattern above, which can be written in a concise form as,

$$J_2^{2n-1} = J_2 \quad \text{and} \quad J_2^{2n} = J_2^2$$

Problem 4 continued on next page...

where $n = 1, 2, 3, \ldots$. Consider $e^{-iJ_2\beta}$,

$$\begin{aligned}
e^{-iJ_2\beta} &= \sum_{n=0}^{\infty} \frac{(-i)^n \beta^n J_2^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{(-i)^{2n} \beta^{2n} J_2^{2n}}{2n!} + \sum_{n=1}^{\infty} \frac{(-i)^{2n-1} \beta^{2n-1} J_2^{2n-1}}{(2n-1)!} \\
&= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n \beta^{2n} J_2^2}{2n!} + i \sum_{n=1}^{\infty} \frac{(-1)^n \beta^{2n-1} J_2}{(2n-1)!} \\
&= 1 + (1 - \cos\beta) J_2^2 - i J_2 \sin\beta
\end{aligned}$$

Problem 5