Dynamical Systems: Homework #3

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Problem 1

Given $\overline{\lim_{t\to\infty}} f(t) = a \neq \pm\infty$, we can think of two broad cases :-

Case I - $\lim_{t\to\infty} f(t)$ exists $\implies \lim_{t\to\infty} f(t) \leq a$.

(a) $\lim_{t\to\infty} [f(t) - a - \epsilon] \leq -\epsilon$

(b) $\overline{\lim_{t\to\infty}} [f(t) - a - \epsilon] = -\epsilon$

(c) $\lim_{t\to\infty} [f(t) - a + \epsilon] \leq \epsilon$

(d) $\overline{\lim_{t\to\infty}} [f(t) - a + \epsilon] = \epsilon$

Case II - $\lim_{t\to\infty} f(t)$ does not exist $\implies f(t) \leq a$ as $t \to \infty$.

(a) $\lim_{t\to\infty} [f(t) - a - \epsilon]$ does not exist.

(b) $\overline{\lim_{t\to\infty}} [f(t) - a - \epsilon] = -\epsilon$

(c) $\lim_{t\to\infty} [f(t) - a + \epsilon]$ does not exist

(d) $\overline{\lim_{t\to\infty}} [f(t) - a + \epsilon] = \epsilon$

For the next part, we note,

$$\begin{aligned}
\chi[x(t)] &= \ln\left[\left\|\exp\left(\int A(\tau)\,\mathrm{d}\tau\right)x_0\right\|\right] \\
&\leq \ln\left[\left\|\exp\left(\int A(\tau)\,\mathrm{d}\tau\right)\right\|\|x_0\|\right] \\
&\leq \ln\left[\exp\left(\int \|A(\tau)\|\,\mathrm{d}\tau\right)\|x_0\|\right] \\
\chi[x(t)] &\leq \int \|A(\tau)\|\,\mathrm{d}\tau + \ln\|x_0\|
\end{aligned}$$

Hence, if $\chi[x] = \pm\infty$ is not bounded, then $\|A(\tau)\|$ should also go to ∞ at some point. Hence, $\chi[x] = \pm\infty \implies \|A(\tau)\|$ is unbounded.
For the converse statement, we give a counterexample. Consider $x(t) = \exp(t\sin t) \implies A(t) = \sin t + t\cos t$. $\chi[x] = 1$. So even though $A(t)$ is unbounded, the Lyapunov exponent is finite.

Problem 2

The eigenvalues are given to be as $u_k + iv_k$. Therefore the set $\{u_k\}$ is the set of Lyapunov characteristic exponents. A general non-trivial linear combination of the eigensolutions will have Lyapunov exponent $\max u_k$.

Problem 3

The most general quadratic time-independent Hamiltonian for a system with n degrees of freedom is,

$$H = \frac{1}{2} \sum_{i=1,j=1}^{n} A_{ij} q_i q_j + B_{ij} p_i p_j + C_{ij} q_i p_j$$

where $A_{ij} = A_{ji}, B_{ij} = B_{ji}, C_{ij} = C_{ji}$, and q_i and p_i are the generalized positions and momenta respectively. Using Hamilton's equations of motion, one can write,

$$\dot{q}_i = \sum_{j=1}^{n} B_{ij} p_j + C_{ij} q_j (2 - \delta_{ij}) \quad \text{and} \quad \dot{p}_i = -\sum_{j=1}^{n} A_{ij} q_j - C_{ij} p_j (2 - \delta_{ij})$$

One can really write this above set of equations in matrix form $\dot{x} = Ax$, with $x = \begin{bmatrix} q_1 & \dots q_N & p_1 & \dots & p_N \end{bmatrix}^T$, and hence, as per the previous question, the Lyapunov exponent would be $\max[u_k]$, for eigenvalues $u_k + iv_k$. For action angle variables Q_k, P_k, the equations of motion would be linear in t, and the Lyapunov exponent would be zero. The Hamiltonian can be written as,

$$H = \sum_{i=1,j=1}^{n} P_i P_j$$

Canonical transformations keep the equations of motion invariant, hence the Lyapunov exponents stay invariant too.