ICTS graduate course : Classical Electrodynamics

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Assignment 1 : **Vectors and Differential Forms**

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Good reference for differential forms are

- Introduction to differential forms by Donu Arapura
  https://www.math.purdue.edu/ ˜dvb/preprints/diffforms.pdf
- Vector Calculus, Linear Algebra, and Differential Forms by John H. Hubbard and Barbara Burke Hubbard (See Chapter 6)
  https://archive.org/details/HubbardJ.H.HubbardB.B.VectorCalculusLinearAlgebraAndDiffer
- A Geometric Approach to Differential Forms by David Bachman
  (free access via ICTS)
  https://link.springer.com/book/10.1007%2F978-0-8176-8304-7
- Differential Forms. A complement to vector calculus by Steven H. Weintraub
- Advanced Calculus: A Differential Forms Approach by Harold M. Edwards
  (free access via ICTS)
  https://link.springer.com/book/10.1007%2F978-0-8176-8412-9
- Differential Forms, A Heuristic Introduction by M. Schreiber
  (free access via ICTS)
  https://link.springer.com/book/10.1007%2F978-1-4612-9940-0
- Differential Forms with Applications to the Physical Sciences by Harley Flanders

from which many of the problems below are taken from. Many modern texts on General relativity cover differential forms.

1. **Index notation and Levi-Civita tensor :**

   (a) Use Einstein summation convention to evaluate the following expressions

   $$\delta^i_i,\ \delta^{ij}\epsilon_{ijk},\ \epsilon^{ijk}\epsilon_{mjk},\ \epsilon^{ijk}\epsilon_{ijk}\ .$$

   (b) Given $B^i = \epsilon^{ijk}\partial_j A_k$, evaluate $\epsilon_{ijk}B^k$.

   (c) Prove the following using index notation

   $$\begin{aligned}\vec{\nabla}\times(\vec{A}\times\vec{B}) &= \vec{A}(\vec{\nabla}\cdot\vec{B}) - \vec{B}(\vec{\nabla}\cdot\vec{A}) - (\vec{A}\cdot\vec{\nabla})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{A} \\ \vec{\nabla}(\vec{A}\cdot\vec{B}) &= \vec{A}\times(\vec{\nabla}\times\vec{B}) + \vec{B}\times(\vec{\nabla}\times\vec{A}) + (\vec{A}\cdot\vec{\nabla})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{A} \\ \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) &= \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}^2\vec{A}\ .\end{aligned}$$

2. **Exercises in Wedge product :**

   (a) By writing out the components explicitly, show the following identities for vectors/forms in $\mathbb{R}^3$:

   $$\begin{gathered}(\vec{A}\cdot d\vec{\ell})\wedge(\vec{B}\cdot d\vec{\ell}) = (\vec{A}\times\vec{B})\cdot d\vec{a} \\ (\vec{A}\cdot d\vec{\ell})\wedge(\vec{B}\cdot d\vec{\ell})\wedge(\vec{C}\cdot d\vec{\ell}) = (\vec{A}\times\vec{B})\cdot\vec{C}\ d\forall\end{gathered}$$

   Use these relations along with

   $$d\vec{\ell} = ds\ \hat{e}_s + sd\phi\ \hat{e}_\phi + dz\ \hat{e}_z = dr\ \hat{e}_r + rd\theta\ \hat{e}_\theta + r\sin\theta d\phi\ \hat{e}_\phi$$

   to compute $d\vec{a}$ and $d\forall$ in cylindrical and spherical co-ordinates.

   (b) Compute the following wedge products

   $$\begin{aligned}&(dx + dy - dz)\wedge(dx + dy + dz), \\ &\quad[(x-y)dx + (x+y)dy + zdz]\wedge[(x-y)dx + (x+y)dy]\end{aligned} \tag{1}$$

   (c) If ω is a 2-form on $\mathbb{R}^4$ such that $\omega\wedge\omega = 0$, then ω can be written as the wedge product of two 1-forms.

3. **Exercises in exterior derivative :**

   (a) Consider a p-form given by

   $$\Omega \equiv \frac{1}{p!}\Omega_{i_1 i_2 \ldots i_p} dx^{i_1}\wedge dx^{i_2}\wedge\ldots\wedge dx^{i_p}$$

   where without loss of generality, one can take $\Omega_{i_1 i_2 \ldots i_p}$ to be completely anti-symmetric in its indices. Then show that its exterior derivative is given by

   $$d\Omega \equiv \frac{1}{(p+1)!}(d\Omega)_{i_1 i_2 \ldots i_p i_{p+1}}\ dx^{i_1}\wedge dx^{i_2}\wedge\ldots\wedge dx^{i_p}\wedge dx^{i_{p+1}}$$

where

$$(d\Omega)_{i_1 i_2 \ldots i_p i_{p+1}} \equiv \partial_{i_1}\Omega_{i_2 \ldots i_{p+1}} + (-1)^p \partial_{i_2}\Omega_{i_3 \ldots i_{p+1} i_1} + (-1)^{2p}\partial_{i_3}\Omega_{i_4 \ldots i_{p+1} i_1 i_2} \\ + \ldots + (-1)^{(p-1)p}\partial_{i_p}\Omega_{i_{p+1} i_1 i_2 \ldots i_{p-1}} + (-1)^{p^2}\partial_{i_{p+1}}\Omega_{i_1 i_2 \ldots i_p} \tag{2}$$

(b) Prove that $(d\Omega)_{i_1 i_2 \ldots i_p i_{p+1}}$ is completely anti-symmetric under the exchange of indices.

4. **Generalised Stokes' theorem :**

(a) Calculate

$$\int_C x^3 dx + \Big(\frac{x^3}{3} + xy^2\Big) dy$$

where C is the anti-clockwise circle of radius 2, centered about the origin.

(b) Let S be the can-shaped surface in $\mathbb{R}^3$ whose side is the cylinder of radius 1 (centered on the z-axis), and whose top and bottom are in the planes $z = 1$ and $z = 0$, respectively. Use the generalised Stokes' Theorem to calculate $\int_S z^2 dx \wedge dy$.

(c) Use the generalised Stokes' theorem relations $d(\vec{A}\cdot d\vec{\ell}) = (\vec{\nabla}\times\vec{A})\cdot d\vec{a}$ as well as $d(\vec{B}\cdot d\vec{a}) = (\vec{\nabla}\cdot\vec{B})d\forall$ to compute the formula for curl and divergence in spherical and cylindrical co-ordinates.

Assignment feedback : Please take time give your feedback on this assignment

1. Time taken to finish the assignment :
2. Make up an exam question on the topic of the assignment :
3. How many other students you collaborated with ?
4. How much was the class useful for solving the problems of the assignment ?
5. Was assignment useful to understand topics covered in the class ?
6. How much useful were the tutorials/tutor for the last assignment ?