Fluid Mechanics: Assignment #3

Due on 23rd October, 2018

Aditya Vijaykumar

Acknowledgements - I thank Saumav Kapoor for discussions.

Problem 1

Part (a)

The unsteady state Bernoulli equation tells us that,

$$\frac{\partial \phi}{\partial t} + \frac{P_{atm}}{\rho} + \frac{v^2}{2} + gz = constant$$
$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial y} + g = 0$$

where we have differentiated with y going from the first to the second line.
From the geometry of the cone, we have,

$$r_1 = r_0 + y \tan\alpha$$
$$\pi r_1^2 = \pi r_0^2 + \pi y^2 \tan^2\alpha + 2\pi r_0 y \tan\alpha$$
$$A_1 = \pi r_0^2 + \pi y^2 \tan^2\alpha + 2\sqrt{A\pi} y \tan\alpha$$

From this and the continuity equation, we get,

$$v = \frac{r_0^2}{\beta^2} v_0 \quad , \quad \beta^2 = r_0^2 + y^2 \tan^2\alpha + 2 r_0 y \tan\alpha$$

Substituting back into earlier equation,

$$\frac{r_0^2}{\beta^2}\frac{\partial v_0}{\partial t} - \frac{r_0^2}{\beta^2} v_0^2 \frac{r_0^2}{\beta^4}(2y\tan\alpha + 2r_0\tan\alpha) + g = 0$$

The above equation holds for all y and specifically $y = 0$. Let's put $y = 0$ and $\beta^2 = r_0^2$,

$$\frac{\partial v_0}{\partial t} - \frac{2}{r_0} v_0^2 \tan\alpha + g = 0$$

Solving this gives,

$$v_0 = \sqrt{\frac{g r_0}{2\tan\alpha}} \coth\left(\sqrt{\frac{2g\tan\alpha}{r_0}} t + C\right)$$

All that is left is to evaluate the constant C.
We go back to the continuity equation, which says,

$$\pi(y+y_0)^2 v(y,t)\tan^2\alpha = K(t) \implies v(y,t) = \frac{K(t)}{\pi(y+y_0)^2\tan^2\alpha} \implies \phi(y,t) = -\frac{K(t)}{\pi(y+y_0)\tan^2\alpha}$$

Writing Bernoulli between points $y = h$ and $y = r_0 \tan\alpha$,

$$-\frac{K'}{\pi(h+y_0)\tan^2\alpha} + \frac{K^2}{2\pi^2\tan^4\alpha h^4} + gh = -\frac{K'}{\pi(y_0)\tan^2\alpha} + \frac{K^2}{2\pi^2\tan^4\alpha r_0^4}$$

Substituting $K(t) = v_0\pi r_0^2\tan^2\alpha$, we can get an expression for the constant C in terms of height h.

Part (b)

We know that,

$$\frac{\mathrm{d}V}{\mathrm{d}t} = Q \implies t = \int \frac{dV}{Q}$$

$$\therefore t_1 = \int \frac{dV_1}{Q_1} \qquad t_2 = \int \frac{dV_2}{Q_2}$$

$$t_1 = \int \frac{\pi h^2 dh}{A_1\sqrt{2gh}} \qquad t_2 = \int \frac{\pi h^2 dh}{A_2\sqrt{2gh}}$$

$$t_1 - t_2 = \left(\frac{1}{A_1} - \frac{1}{A_2}\right)\int \frac{\pi h^2 dh}{\sqrt{2gh}} \implies \quad \text{tank with larger base area will drain faster}$$

Problem 2

We first write the Bernoulli equation for between the point where water leaves the tap ($z_1 = 0$) and a point distance h below ($z_2 = -h$),

$$\frac{P_0}{\rho} + \frac{v_1^2}{2} = \frac{P_0}{\rho} + \frac{v_2^2}{2} - gh \implies \frac{v_2^2}{v_1^2} = 1 + \frac{2gh}{v_1^2}$$

The continuity equation gives,

$$\pi r_1^2 v_1 = \pi r_2^2 v_2 \implies \frac{v_2}{v_1} = \frac{r_1^2}{r_2^2}$$

Using the above two equations, we get,

$$\frac{r_1^4}{r_2^4} = 1 + \frac{2gh}{v_1^2} \implies \boxed{\frac{R_0^4}{r^4} = 1 + \frac{2gH}{v_0^2}}$$

where r is the cross-sectional radius at height H below the tap, and R_0 and v_0 and the cross-sectional radius and velocity of the water the moment it leaves the tap.

Problem 3

We work in cylindrical coordinates. The assumption of laminar flow $\implies u_r = u_\phi = 0$. The assumption of axisymmetry $\implies u_z = u_z(r,z)$ The continuity condition $\boldsymbol{\nabla}\cdot\vec{\mathbf{u}} = 0$ gives,

$$\frac{\partial u_z}{\partial z} = 0 \implies u_z = u_z(r)$$

We now proceed and write the Navier-Stokes equation in cylindrical coordinates component-wise,

$$0 = -\frac{1}{\rho}\frac{\partial P}{\partial r}$$

$$0 = -\frac{1}{\rho r}\frac{\partial P}{\partial \phi}$$

$$0 = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)$$

We can see from the first two equations that $P = P(z)$. In the third equation, since the first term on the RHS depends only on z and the second term depends only on r, we say that each of the terms should be constants. We get,

$$\begin{aligned}
\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u_z}{\mathrm{d}r}\right) &= \frac{1}{\mu}\frac{\mathrm{d}P}{\mathrm{d}z} = constant \\
\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u_z}{\mathrm{d}r}\right) &= \frac{r}{\mu}\frac{\mathrm{d}P}{\mathrm{d}z} \\
\implies r\frac{\mathrm{d}u_z}{\mathrm{d}r} &= \frac{r^2}{2\mu}\frac{\mathrm{d}P}{\mathrm{d}z} + A \\
\implies \frac{\mathrm{d}u_z}{\mathrm{d}r} &= \frac{r}{2\mu}\frac{\mathrm{d}P}{\mathrm{d}z} + \frac{A}{r} \\
\implies u_z &= \frac{r^2}{4\mu}\frac{\mathrm{d}P}{\mathrm{d}z} + A\ln r + B
\end{aligned}$$

We need the flow to be well-defined at $r = 0$. As it stands, for non-zero A, the flow will not be well-defined for $r = 0$, which is undesirable. Hence, $A = 0$.
If R is the radius of the pipe, and the pipe is not moving, we get $u_z(R) = 0$, which means,

$$0 = \frac{R^2}{4\mu}\frac{\mathrm{d}P}{\mathrm{d}z} + B \implies B = -\frac{R^2}{4\mu}\frac{\mathrm{d}P}{\mathrm{d}z}$$

So the final answer is,

$$u_z = \frac{1}{4\mu}\frac{\mathrm{d}P}{\mathrm{d}z}(r^2 - R^2)$$

Problem 4

We solve the problem for a two-dimensional jet. The orthogonal directions are taken to be x and y. We assume that steady state.
The Continuity equation gives us,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \implies u\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial y} = 0$$

The x-component of the Navier-Stokes gives us,

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \tag{1}$$

Adding up the two equations, one has,

$$\begin{aligned}
2u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + u\frac{\partial v}{\partial y} &= \nu\frac{\partial^2 u}{\partial y^2} \\
\frac{\partial(u^2)}{\partial x} + \frac{\partial(uv)}{\partial y} &= \nu\frac{\partial^2 u}{\partial y^2}
\end{aligned}$$

Integrating both sides with respect to y,

$$\frac{\partial}{\partial x}\int_{-\infty}^{\infty} u^2 dy + uv\Big|_{-\infty}^{\infty} = \nu\left.\frac{\partial u}{\partial y}\right|_{-\infty}^{\infty}$$

We now would like to impose boundary conditions. The velocity is purely along the x-axis at $y = 0$. As $y \to \pm\infty$, both $u \to 0$ and $v \to 0$, and so do their derivatives.
We then have,

$$\frac{\partial}{\partial x}\int_{-\infty}^{\infty} u^2 dy = 0 \implies \int_{-\infty}^{\infty} u^2 dy = constant = M \tag{2}$$

We now try to guess the form of the similarity solution for this problem. Let's assume,

$$x \to \lambda^a x' \qquad y \to \lambda^b y' \qquad \psi \to \lambda^c \psi' \tag{3}$$

Using the fact that $u = \psi_y$ and $v = -\psi_x$, one can write (1) as,

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} = \nu \psi_{yyy} \tag{4}$$

and (2) as,

$$\int_{-\infty}^{\infty} \psi_y^2 dy = M \tag{5}$$

Substituting (3) into (4) and (5), we get,

$$\begin{aligned} 2c - 2b - a = c - 3b &\implies a = b + c \quad \text{and} \\ 2(c-b) + b = 0 &\implies b = 2c \end{aligned}$$

Solving which we get,

$$b = \frac{2a}{3} \qquad c = \frac{a}{3}$$

and the final form being,

$$x \to \lambda^a x' \qquad y \to \lambda^{2a/3} y' \qquad \psi \to \lambda^{a/3} \psi'$$

This suggests that,

$$\frac{\psi}{x^{1/3}} \sim f\left(\frac{y}{x^{2/3}}\right) \implies \psi = A x^{1/3} f(\eta)$$

where $\eta = \dfrac{y}{x^{2/3}}$. We note the following,

$$\begin{aligned} \psi_y &= A x^{1/3} f'(\eta) \frac{\mathrm{d}\eta}{\mathrm{d}y} \\ &= A x^{-1/3} f' \\ \psi_x &= \frac{A}{3} x^{-2/3} f(\eta) + A x^{1/3} f'(\eta) \frac{\mathrm{d}\eta}{\mathrm{d}x} \\ &= \frac{A}{3} x^{-2/3} f(\eta) - \frac{2A}{3} x^{-2/3} f'(\eta)\eta \\ &= \frac{A x^{-2/3}}{3}(-2\eta f' + f) \\ \psi_{xy} &= \frac{A x^{-4/3}}{3}(-2\eta f'' - f') \\ \psi_{yy} &= A x^{-1} f'' \\ \psi_{yyy} &= A x^{-5/3} f''' \end{aligned}$$

Putting all this into (4), we get,

$$-f'(-2\eta f'' + f') - (-2\eta f' + f) f'' = \frac{3\nu}{A} f''' \implies \frac{3\nu}{A} f''' + f'^2 + f'' f = 0$$

If we set $A = \nu$ (we can always do that since it is an arbitrary constant), we get our final answer,

$$3 f''' + f'^2 + f'' f = 0$$