Advanced Quantum Mechanics: Assignment #5

Due on 20th November, 2018

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(**Acknowledgements -** I would like to thank Chandramouli Chowdhury, Biprarshi Chakraborty and Junaid Majeed for discussions.)

Problem 1

Problem 2

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Problem 3

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Problem 4

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Problem 5

We first note for $|\psi_I(t)\rangle = \sum_n c_n(t) |\alpha_n\rangle$

$$\begin{aligned}
i\frac{\partial |\psi_I\rangle}{\partial t} &= i\frac{\partial(e^{iH_0t} |\psi_S\rangle)}{\partial t} \\
&= i\left[e^{iH_0t}\frac{\partial |\psi_S\rangle}{\partial t} + iH_0 e^{iH_0t} |\psi_S\rangle\right] \\
&= -e^{iH_0t}(H_0 + V) |\psi_S\rangle - H_0 e^{iH_0t} |\psi_S\rangle \\
&= e^{iH_0t} V |\psi_S\rangle \\
i\frac{\partial |\psi_I\rangle}{\partial t} &= V_I |\psi_I\rangle \\
i\frac{\partial \langle\alpha_n|\psi_I\rangle}{\partial t} &= \langle\alpha_n|V_I|\psi_I\rangle \\
\dot{c_n} &= -i \langle\alpha_n|V_I|\psi_I\rangle \\
\dot{c_n} &= -i \langle\alpha_n|V|\alpha_m\rangle e^{i(E_n - E_m)t} c_m
\end{aligned}$$

So for the given problem, we have

$$|\psi_I(t)\rangle = c_1(t)\,|1\rangle + c_2(t)e^{iEt}\,|2\rangle$$
$$\dot{c_1} = -iV_{11}c_1 - iV_{12}e^{-iEt}c_2 = -i\gamma e^{i(\omega-E)t}c_2 \quad \text{and} \quad \dot{c_2} = -iV_{21}e^{iEt}c_1 - iV_{22}c_2 = -i\gamma e^{i(E-\omega)t}c_1$$

To solve the above equations, we make the substitution $c_1 = b_1 e^{i\Delta t}$ and $c_2 = b_2 e^{-i\Delta t}$, where $2\Delta = \omega - E$. We then have the equations in terms of b's,

$$i\dot{b_1} = \Delta b_1 + \gamma b_2 \quad \text{and} \quad i\dot{b_2} = \gamma b_1 - \Delta b_2$$

These are coupled equations, and we can solve these by making the substitution $b_1 = Ae^{i\Omega t}$ and $b_2 = Be^{i\Omega t}$. We then have,

$$-A\Omega = \Delta A + \gamma B \quad \text{and} \quad -B\Omega = \gamma A - \Delta B$$
$$\text{For non-trivial solutions,} \quad -\frac{\gamma}{\Delta+\Omega} = \frac{\Delta-\Omega}{\gamma} \implies \Omega = \pm\sqrt{\gamma^2+\Delta^2} = \pm\Omega_0$$
$$\implies c_1 = A_1 e^{i(\Delta+\Omega_0)t} + A_2 e^{i(\Delta-\Omega_0)t} \quad \text{and} \quad c_2 = B_1 e^{i(-\Delta+\Omega_0)t} + B_2 e^{i(-\Delta-\Omega_0)t}$$

We are told that at $t = 0$, the system is in state $|1\rangle \implies c_1(0) = 0, c_2(0) = 1 \implies A_1 = -A_2, B_1 = 1 - B_2$. We also know for a fact that $|c_1(0)|^2 + |c_2(0)|^2 = 1$. Using all these facts, we can write

$$4A_1^2\cos^2\Omega t + 4(B_1^2 - B_1)\sin^2\Omega t + 1 = 1$$