# Electromagnetism: Pset #2

Due on 6th February, 2019

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(**Acknowledgements** - I would like to thank Junaid Majeed for discussions.)

## Problem 1

**Zangwill - Problem** 24.1
**Part (a)** Given,

$$\begin{aligned}
\frac{\partial F_i}{\partial \dot{r}_j} &= -\frac{\partial F_j}{\partial \dot{r}_i} \\
\implies \frac{\partial^2 F_i}{\partial \dot{r}_j \partial \dot{r}_k} &= -\frac{\partial^2 F_j}{\partial \dot{r}_i \partial \dot{r}_k} \\
\frac{\partial^2 F_i}{\partial \dot{r}_k \partial \dot{r}_j} &= -\frac{\partial^2 F_j}{\partial \dot{r}_k \partial \dot{r}_i} \\
-\frac{\partial^2 F_k}{\partial \dot{r}_i \partial \dot{r}_j} &= \frac{\partial^2 F_k}{\partial \dot{r}_j \partial \dot{r}_i} \\
\implies \frac{\partial^2 F_k}{\partial \dot{r}_j \partial \dot{r}_i} &= 0
\end{aligned}$$

From Hemholtz first relation, we know that integrating the above equation should give us an object which is antisymmetric under the exchange of indices. We can take this into account by introducing our friendly neighbourhood antisymmetric object, namely the *Levi-Civita symbol.*

$$\implies \frac{\partial F_i}{\partial \dot{r}_j} = \epsilon_{ijk} Q_k(\mathbf{r}, t) \implies F_i = \epsilon_{ijk} Q_k(\mathbf{r}, t) \dot{r}_j + P(\mathbf{r}, t)$$

Hence proved.

**Part (b)** The second Helmholtz relation says,

$$\begin{aligned}
\frac{\partial F_i}{\partial r_j} - \frac{\partial F_j}{\partial r_i} &= \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left( \frac{\partial F_i}{\partial \dot{r}_j} - \frac{\partial F_j}{\partial \dot{r}_i} \right) \\
\frac{\partial P_i}{\partial r_j} - \frac{\partial P_j}{\partial r_i} + \epsilon_{ijk} \left( \frac{\partial Q_k}{\partial r_j} + \frac{\partial Q_k}{\partial r_i} \right) &= \frac{\mathrm{d}}{\mathrm{d}t} (\epsilon_{ijk} Q_k(\mathbf{r}, t)) \\
\frac{\partial P_i}{\partial r_j} - \frac{\partial P_j}{\partial r_i} + \epsilon_{ijk} \left( \frac{\partial Q_k}{\partial r_j} + \frac{\partial Q_k}{\partial r_i} \right) &= \epsilon_{ijk} \left( \frac{\partial Q_k}{\partial t} + \frac{\partial Q_k}{\partial r_a} \dot{r}_a \right)
\end{aligned}$$

As we can see, there are no terms involving $\dot{r}_a$ on the LHS. Hence, the term involving $\dot{r}_a$ on the RHS should be zero!

$$\implies \frac{\partial Q_k}{\partial r_a} \dot{r}_a = \vec{\boldsymbol{\nabla}} Q = 0$$

Next, we multiply both sides of the equation with $\epsilon_{lji}$, and use the fact that $\epsilon_{lmk}\epsilon_{ijk} = (\delta_{li}\delta_{mj} - \delta_{lj}\delta_{mi})$,

$$\epsilon_{lji}\left(\frac{\partial P_i}{\partial r_j} - \frac{\partial P_j}{\partial r_i}\right) + \epsilon_{lji}\epsilon_{ijk}\left(\frac{\partial Q_k}{\partial r_j} + \frac{\partial Q_k}{\partial r_i}\right) = \epsilon_{lji}\epsilon_{ijk}\left(\frac{\partial Q_k}{\partial t}\right)$$

$$\epsilon_{lmk}\left(\frac{\partial P_i}{\partial r_j} - \frac{\partial P_j}{\partial r_i}\right) + \left(\frac{\partial Q_k}{\partial r_m} + \frac{\partial Q_k}{\partial r_l} - \frac{\partial Q_k}{\partial r_l} - \frac{\partial Q_k}{\partial r_m}\right) = \left(\frac{\partial Q_k}{\partial t}\right)$$

$$\therefore \epsilon_{lmk}\left(\frac{\partial P_i}{\partial r_j} - \frac{\partial P_j}{\partial r_i}\right) = \left(\frac{\partial Q_k}{\partial t}\right)$$

$$\therefore \vec{\nabla} \times P = \frac{\partial Q}{\partial t}$$

**Zangwill - Problem** 24.2
**Part (a)**

$$H(r,p) = \frac{p^2}{2m} + g(r) + p^2 f(r) = \left(\frac{1+2fm}{2m}\right)p^2 + g(r)$$

$$\implies \dot{r} = \frac{\partial H}{\partial p} = \frac{p}{m} + 2pf(r)$$

$$\implies p = \frac{m\dot{r}}{1+2fm}$$

$$\therefore L(r,\dot{r}) = \frac{m\dot{r}^2}{1+2fm} - \frac{m\dot{r}^2}{2(1+2fm)} - g(r) = \frac{m\dot{r}^2}{2(1+2fm)} - g(r)$$

**Part (b)**
We write the Euler Lagrange equations for this Lagrangian,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m\dot{r}}{1+2fm}\right) + \frac{\partial g}{\partial r} = 0$$

$$\frac{m\ddot{r}}{1+2fm} - \frac{m\dot{r}}{(1+2fm)^2}\dot{r}\frac{\mathrm{d}f}{\mathrm{d}r} + \frac{\mathrm{d}g}{\mathrm{d}r} = 0$$

$$m\ddot{r} - \frac{m\dot{r}^2}{(1+2fm)}\frac{\mathrm{d}f}{\mathrm{d}r} + (1+2fm)\frac{\mathrm{d}g}{\mathrm{d}r} = 0$$

As one can see from the above equation, the force depends on $\ddot{r}, \dot{r}, f, g$.

**Zangwill - Problem** 24.4
Let's first work with $c = 1$. We shall restore the factors of $c$ at the end.

$$S = -m\int \mathrm{d}s - g\int \mathrm{d}s\,\phi(\vec{\mathbf{r}}(s))$$

$$= -m\int \mathrm{d}t\,\frac{\mathrm{d}s}{\mathrm{d}t} - g\int \mathrm{d}t\,\frac{\mathrm{d}s}{\mathrm{d}t}\phi(\vec{\mathbf{r}}(s))$$

$$S = \int -\frac{m}{\gamma}\,\mathrm{d}t + \int \frac{-g\phi}{\gamma}\,\mathrm{d}t$$

From the above form of action $S$, the Lagrangian is evidently,

$$L = -\frac{m+g\phi}{\gamma} \quad \text{where} \quad \gamma = \frac{1}{\sqrt{1-\vec{\mathbf{v}}\cdot\vec{\mathbf{v}}}} = \frac{1}{\sqrt{1-\dot{\vec{\mathbf{r}}}\cdot\dot{\vec{\mathbf{r}}}}}$$

$$\therefore \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\vec{\mathbf{r}}}}\right) = \frac{\partial L}{\partial \vec{\mathbf{r}}}$$

$$\frac{\mathrm{d}(m+g\phi)\gamma\dot{\vec{\mathbf{r}}}}{\mathrm{d}t} = -\frac{g\vec{\nabla}\phi}{\gamma}$$

Restoring the factors of $c$,

$$\frac{\mathrm{d}(m + g\phi/c)\gamma\dot{\vec{\mathbf{r}}}}{\mathrm{d}t} = -\frac{gc\vec{\boldsymbol{\nabla}}\phi}{\gamma}$$

This differs from the electric field force equation, and has an extra term on the left (second term).

**Zangwill - Problem** 24.11

$$L_{CS} = \int \mathrm{d}^3 r[\rho\phi - \vec{\mathbf{j}} \cdot \vec{\mathbf{A}} + 1/2\{\epsilon_0(\vec{\mathbf{E}}^2 - c^2\vec{\mathbf{B}}^2) - \phi(\vec{\mathbf{d}} \cdot \vec{\mathbf{B}}/c) + \vec{\mathbf{d}} \cdot (\vec{\mathbf{A}} \times \vec{\mathbf{E}}/c) + d_0\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}\}]$$

It is given that the Lagrangian remains invariant under usual gauge tranformations $\phi \to \phi + \partial_t\lambda$ and $\vec{\mathbf{A}} \to \vec{\mathbf{A}} - \vec{\boldsymbol{\nabla}}\lambda$. Hence,

$$L'_{CS} - L_{CS} = \int \mathrm{d}^3 r[\rho\partial_t\lambda + \vec{\mathbf{j}} \cdot \vec{\boldsymbol{\nabla}}\lambda + 1/2(-\vec{\mathbf{d}} \cdot (\vec{\boldsymbol{\nabla}}\lambda \times \vec{\mathbf{E}}/c) - d_0\vec{\boldsymbol{\nabla}}\lambda \cdot \vec{\mathbf{B}} - \partial_t\lambda(\vec{\mathbf{d}} \cdot \vec{\mathbf{B}}/c))]$$

$$= \int \mathrm{d}^3 r[\partial_t(\rho\lambda) - \lambda\partial_t\rho + \vec{\boldsymbol{\nabla}} \cdot (\lambda\vec{\mathbf{j}}) - \lambda\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{j}} + 1/2(-\vec{\mathbf{d}} \cdot (\vec{\boldsymbol{\nabla}}\lambda \times \vec{\mathbf{E}}/c) - d_0\vec{\boldsymbol{\nabla}}\lambda \cdot \vec{\mathbf{B}} - \partial_t\lambda(\vec{\mathbf{d}} \cdot \vec{\mathbf{B}}/c))]$$

The first four terms above vanish in the integral - the first and the third because they are boundary terms and the other two because of conservation.

$$L'_{CS} - L_{CS} = \int \mathrm{d}^3 r[1/2(-\vec{\mathbf{d}} \cdot (\vec{\boldsymbol{\nabla}}\lambda \times \vec{\mathbf{E}}/c) - d_0\vec{\boldsymbol{\nabla}}\lambda \cdot \vec{\mathbf{B}} - \partial_t\lambda(\vec{\mathbf{d}} \cdot \vec{\mathbf{B}}/c))]$$

$$L'_{CS} - L_{CS} = \int \mathrm{d}^3 r[1/2(\vec{\mathbf{d}} \cdot \lambda/c\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{E}} - d_0\vec{\boldsymbol{\nabla}} \cdot \lambda\vec{\mathbf{B}} + d_0\lambda\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{B}} - \partial_t[\lambda(\vec{\mathbf{d}} \cdot \vec{\mathbf{B}}/c))] + \lambda/c(\partial_t\vec{\mathbf{d}} \cdot \vec{\mathbf{B}} + \partial_t\vec{\mathbf{B}} \cdot \vec{\mathbf{d}})]$$

$$= \int \mathrm{d}^3 r[1/2(-\vec{\mathbf{d}} \cdot \vec{\boldsymbol{\nabla}} \times (\lambda\vec{\mathbf{E}}/c) + \lambda\vec{\mathbf{d}} \cdot \vec{\boldsymbol{\nabla}} \times (\vec{\mathbf{E}}/c) + d_0\lambda\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{B}} - d_0\vec{\boldsymbol{\nabla}} \cdot -\partial_t\lambda(\vec{\mathbf{d}} \cdot \vec{\mathbf{B}}/c))]$$

**Zangwill - Problem** 24.12

$$L = \vec{\mathbf{j}} \cdot \vec{\mathbf{A}} - \rho\phi - \frac{1}{2}\epsilon_0(\vec{\mathbf{E}}^2 - c^2\vec{\mathbf{B}}^2) - \epsilon_0\vec{\mathbf{E}} \cdot (\vec{\boldsymbol{\nabla}}\phi + \dot{\vec{\mathbf{A}}}) - \epsilon_0 c^2\vec{\mathbf{B}} \cdot (\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{A}})$$

$$\frac{\partial L}{\partial\vec{\mathbf{E}}} = -\epsilon_0\vec{\mathbf{E}} - \epsilon_0(\vec{\boldsymbol{\nabla}}\phi + \dot{\vec{\mathbf{A}}}) \quad , \quad \frac{\partial L}{\partial\dot{\vec{\mathbf{E}}}} = 0 \implies \vec{\mathbf{E}} = -\vec{\boldsymbol{\nabla}}\phi - \dot{\vec{\mathbf{A}}}$$

$$\frac{\partial L}{\partial\vec{\mathbf{B}}} = \epsilon_0 c^2\vec{\mathbf{B}} - \epsilon_0 c^2\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{A}} \quad , \quad \frac{\partial L}{\partial\dot{\vec{\mathbf{B}}}} = 0 \implies \vec{\mathbf{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{A}}$$

$$\frac{\partial L}{\partial\phi} = -\rho - \epsilon_0\frac{\partial\vec{\mathbf{E}} \cdot \vec{\boldsymbol{\nabla}}\phi}{\partial\phi} = -\rho + \epsilon_0\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}} \quad , \quad \frac{\partial L}{\partial\dot{\phi}} = 0 \implies \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}} = \frac{\rho}{\epsilon_0}$$

$$\frac{\partial L}{\partial\vec{\mathbf{A}}} = \vec{\mathbf{j}} - \epsilon_0 c^2\frac{\partial\vec{\mathbf{B}} \cdot (\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{A}})}{\partial\vec{\mathbf{A}}} = \vec{\mathbf{j}} - \epsilon_0 c^2\frac{\partial(\vec{\mathbf{A}} \cdot (\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}}) + \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{A}} \times \vec{\mathbf{B}})}{\partial\vec{\mathbf{A}}} = \vec{\mathbf{j}} - \epsilon_0 c^2\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}}$$

$$\frac{\partial L}{\partial\dot{\vec{\mathbf{A}}}} = -\epsilon_0\vec{\mathbf{E}} \implies \vec{\boldsymbol{\nabla}} \times B = \frac{\vec{\mathbf{j}}}{\epsilon_0 c^2} + \frac{1}{c^2}\dot{\vec{\mathbf{E}}}$$

The first two equations can be rewritten as $\vec{\boldsymbol{\nabla}} \cdot B = 0$ and $\vec{\boldsymbol{\nabla}} \times E = -\frac{\partial\vec{\mathbf{B}}}{\partial t}$. Hence we have all the Maxwell equations.
There are 7 primary constraints (whose momenta vanish).

# Problem 2

**Part (a)**
Given that,

$$\begin{aligned}
L_D &= \sum_a \left(\frac{1}{2}m_a\vec{\mathbf{u}}_a^2 + \frac{1}{8c^2}m_a\vec{\mathbf{u}}_a^4\right) + \sum_a\sum_{b\neq a}\left[-\frac{q_aq_b}{8\pi\epsilon_0 r_{ab}} + \frac{q_aq_b}{16\pi\epsilon_0c^2r_{ab}}(\vec{\mathbf{u}}_a\cdot\vec{\mathbf{u}}_b + (\vec{\mathbf{u}}_a\cdot\hat{\mathbf{r}}_{ab})(\vec{\mathbf{u}}_b\cdot\hat{\mathbf{r}}_{ab}))\right]\\
\implies \frac{\partial L_D}{\partial\vec{\mathbf{u}}_a} &= \left(m_a\vec{\mathbf{u}}_a + \frac{1}{2c^2}m_a\vec{\mathbf{u}}_a^2\vec{\mathbf{u}}_a\right) + \sum_{b\neq a}\left[\frac{q_aq_b}{16\pi\epsilon_0c^2r_{ab}}(2\vec{\mathbf{u}}_b + 2\hat{\mathbf{r}}_{ab}(\vec{\mathbf{u}}_b\cdot\hat{\mathbf{r}}_{ab}))\right]\\
\frac{\partial L_D}{\partial\vec{\mathbf{u}}_a} &= \underbrace{\sum_{b\neq a}\left[\frac{q_b}{8\pi\epsilon_0c^2r_{ab}}(\vec{\mathbf{u}}_b + \hat{\mathbf{r}}_{ab}(\vec{\mathbf{u}}_b\cdot\hat{\mathbf{r}}_{ab}))\right]}_{\vec{\mathbf{p}}_a^{kin}} + q_a\underbrace{\sum_{b\neq a}\left[\frac{q_b}{8\pi\epsilon_0c^2r_{ab}}(\vec{\mathbf{u}}_b + \hat{\mathbf{r}}_{ab}(\vec{\mathbf{u}}_b\cdot\hat{\mathbf{r}}_{ab}))\right]}_{\vec{\mathbf{A}}_a}
\end{aligned}$$

which is the required form.
**Part (b)**

**Part (c)**
Consider,

$$\begin{aligned}
\frac{\partial r_{ab}}{\partial\vec{\mathbf{r}}_a} &= \frac{\partial\sqrt{(\vec{\mathbf{r}}_a-\vec{\mathbf{r}}_b)\cdot(\vec{\mathbf{r}}_a-\vec{\mathbf{r}}_b)}}{\partial\vec{\mathbf{r}}_a} = \frac{2}{2r_{ab}}(\vec{\mathbf{r}}_a-\vec{\mathbf{r}}_b) = \hat{\mathbf{r}}_{ab} \quad\text{and}\\
\frac{\partial\hat{\mathbf{r}}_{ab}}{\partial\vec{\mathbf{r}}_a} &= \frac{\partial(\vec{\mathbf{r}}_{ab}/r_{ab})}{\partial\vec{\mathbf{r}}_a} = \frac{1}{r_{ab}} - \frac{\vec{\mathbf{r}}_{ab}\cdot\hat{\mathbf{r}}_{ab}}{r_{ab}^2} = 0
\end{aligned}$$

$$\begin{aligned}
\implies \frac{\partial L_D}{\partial\vec{\mathbf{r}}_a} &= 2\sum_{b\neq a}\left[-\frac{q_aq_b}{8\pi\epsilon_0r_{ab}^2}\hat{\mathbf{r}}_{ab} + \frac{q_aq_b\hat{\mathbf{r}}_{ab}}{16\pi\epsilon_0c^2r_{ab}^2}(\vec{\mathbf{u}}_a\cdot\vec{\mathbf{u}}_b + (\vec{\mathbf{u}}_a\cdot\hat{\mathbf{r}}_{ab})(\vec{\mathbf{u}}_b\cdot\hat{\mathbf{r}}_{ab}))\right]\\
-q_a\frac{\partial}{\partial\vec{\mathbf{r}}_a}(\phi_a - \vec{\mathbf{u}}_a\cdot\vec{\mathbf{A}}_a) &= \sum_{b\neq a}\left[-\frac{q_aq_b}{4\pi\epsilon_0r_{ab}^2}\hat{\mathbf{r}}_{ab} + \frac{q_aq_b\hat{\mathbf{r}}_{ab}}{8\pi\epsilon_0c^2r_{ab}^2}(\vec{\mathbf{u}}_a\cdot\vec{\mathbf{u}}_b + (\vec{\mathbf{u}}_a\cdot\hat{\mathbf{r}}_{ab})(\vec{\mathbf{u}}_b\cdot\hat{\mathbf{r}}_{ab}))\right]\\
\implies \frac{\partial L_D}{\partial\vec{\mathbf{r}}_a} &= -q_a\frac{\partial}{\partial\vec{\mathbf{r}}_a}(\phi_a - \vec{\mathbf{u}}_a\cdot\vec{\mathbf{A}}_a) = -q_a\vec{\boldsymbol{\nabla}}_a\phi_a + q_a\vec{\boldsymbol{\nabla}}_a(\vec{\mathbf{u}}_a\cdot\vec{\mathbf{A}}_a)
\end{aligned}$$

Note that, in the first expression, there is an extra factor of 2 due to the summation over $a$.
**Part (d)**
We write our equations of motion,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_D}{\partial\vec{\mathbf{u}}_a}\right) &= \frac{\partial L_D}{\partial\vec{\mathbf{r}}_a}\\
\frac{\mathrm{d}\vec{\mathbf{p}}_a^{kin}}{\mathrm{d}t} + q_a\dot{\vec{\mathbf{A}}}_a &= -q_a\vec{\boldsymbol{\nabla}}_a\phi_a + q_a\vec{\boldsymbol{\nabla}}_a(\vec{\mathbf{u}}_a\cdot\vec{\mathbf{A}}_a)
\end{aligned}$$

Using $\vec{\boldsymbol{\nabla}}(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}) = (\vec{\mathbf{a}}\cdot\vec{\boldsymbol{\nabla}})\vec{\mathbf{b}} + (\vec{\mathbf{b}}\cdot\vec{\boldsymbol{\nabla}})\vec{\mathbf{a}} + \vec{\mathbf{a}}\times(\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{b}}) + \vec{\mathbf{b}}\times(\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{a}})$,

$$\begin{aligned}
\frac{\mathrm{d}\vec{\mathbf{p}}_a^{kin}}{\mathrm{d}t} + q_a\partial_t\vec{\mathbf{A}}_a + q_a(\vec{\mathbf{u}}_a\cdot\vec{\boldsymbol{\nabla}}_a)\vec{\mathbf{A}}_a &= -q_a\vec{\boldsymbol{\nabla}}_a\phi_a + q_a((\vec{\mathbf{u}}_a\cdot\vec{\boldsymbol{\nabla}}_a)\vec{\mathbf{A}}_a + (\vec{\mathbf{A}}_a\cdot\vec{\boldsymbol{\nabla}}_a)\vec{\mathbf{u}}_a + \vec{\mathbf{u}}_a\times(\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{A}}_a) + \vec{\mathbf{A}}_a\times(\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{u}}_a))\\
\frac{\mathrm{d}\vec{\mathbf{p}}_a^{kin}}{\mathrm{d}t} + q_a\partial_t\vec{\mathbf{A}}_a &= -q_a\vec{\boldsymbol{\nabla}}_a\phi_a + q_a((\vec{\mathbf{u}}_a\cdot\vec{\boldsymbol{\nabla}}_a)\vec{\mathbf{A}}_a + (\vec{\mathbf{A}}_a\cdot\vec{\boldsymbol{\nabla}}_a)\vec{\mathbf{u}}_a + \vec{\mathbf{u}}_a\times(\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{A}}_a) + \vec{\mathbf{A}}_a\times(\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{u}}_a))
\end{aligned}$$

# Problem 3

### Part (a)

$$L_{BI} = \frac{B_0^2}{\mu_0} - \frac{B_0}{\mu_0}\sqrt{B_0^2 + \vec{\mathbf{B}}^2 - \frac{1}{c^2}\vec{\mathbf{E}}^2 - \frac{(\vec{\mathbf{E}}\cdot\vec{\mathbf{B}})^2}{(cB_0)^2}}$$

We know that,

$$\begin{aligned}
\vec{\mathbf{D}} = \frac{\partial L_{BI}}{\partial \vec{\mathbf{E}}} &= -\frac{B_0}{2\mu_0}\frac{-\dfrac{2\vec{\mathbf{E}}}{c^2} - \dfrac{2(\vec{\mathbf{E}}\cdot\vec{\mathbf{B}})\vec{\mathbf{B}}}{c^2B_0^2}}{\sqrt{B_0^2 + \vec{\mathbf{B}}^2 - \frac{1}{c^2}\vec{\mathbf{E}}^2 - \frac{(\vec{\mathbf{E}}\cdot\vec{\mathbf{B}})^2}{(cB_0)^2}}} \\
\implies c\vec{\mathbf{D}} &= \frac{B_0}{\mu_0 c}\frac{\vec{\mathbf{E}} + \dfrac{(\vec{\mathbf{E}}\cdot\vec{\mathbf{B}})\vec{\mathbf{B}}}{B_0^2}}{\sqrt{B_0^2 + \vec{\mathbf{B}}^2 - \frac{1}{c^2}\vec{\mathbf{E}}^2 - \frac{(\vec{\mathbf{E}}\cdot\vec{\mathbf{B}})^2}{(cB_0)^2}}} \\
\implies c\vec{\mathbf{D}} &= \eta_E(\vec{\mathbf{E}} + \alpha c\vec{\mathbf{B}})
\end{aligned}$$

$$\begin{aligned}
\vec{\mathbf{H}} = -\frac{\partial L_{BI}}{\partial \vec{\mathbf{B}}} &= \frac{B_0}{2\mu_0}\frac{2\vec{\mathbf{B}} - \dfrac{2(\vec{\mathbf{E}}\cdot\vec{\mathbf{B}})\vec{\mathbf{E}}}{(cB_0)^2}}{\sqrt{B_0^2 + \vec{\mathbf{B}}^2 - \frac{1}{c^2}\vec{\mathbf{E}}^2 - \frac{(\vec{\mathbf{E}}\cdot\vec{\mathbf{B}})^2}{(cB_0)^2}}} \\
\vec{\mathbf{H}} &= \frac{B_0}{\mu_0 c}\frac{c\vec{\mathbf{B}} - \dfrac{(\vec{\mathbf{E}}\cdot\vec{\mathbf{B}})\vec{\mathbf{E}}}{cB_0^2}}{\sqrt{B_0^2 + \vec{\mathbf{B}}^2 - \frac{1}{c^2}\vec{\mathbf{E}}^2 - \frac{(\vec{\mathbf{E}}\cdot\vec{\mathbf{B}})^2}{(cB_0)^2}}} \\
\implies \vec{\mathbf{H}} &= \eta_E(c\vec{\mathbf{B}} - \alpha\vec{\mathbf{E}})
\end{aligned}$$

From the above expressions, one can write,

$$\begin{aligned}
c\vec{\mathbf{D}}\cdot\vec{\mathbf{H}} &= \eta_E^2[-\alpha\vec{\mathbf{E}}^2 + \alpha c^2\vec{\mathbf{B}}^2 + (1-\alpha^2)c\vec{\mathbf{E}}\cdot\vec{\mathbf{B}}] \\
&= \eta_E^2[-\alpha\vec{\mathbf{E}}^2 + \alpha c^2\vec{\mathbf{B}}^2 + (1-\alpha^2)c^2\alpha B_0^2] \\
&= \eta_E^2(c^2\alpha B_0^2)\left[-\frac{\vec{\mathbf{E}}^2}{c^2B_0^2} + \frac{\vec{\mathbf{B}}^2}{B_0^2} + (1-\alpha^2)\right] \\
c\vec{\mathbf{D}}\cdot\vec{\mathbf{H}} &= \eta_E^2\vec{\mathbf{E}}\cdot\vec{\mathbf{B}}c\frac{1}{\eta_E^2(\mu_0 c)^2} \\
\implies (\mu_0 c)^2\vec{\mathbf{D}}\cdot\vec{\mathbf{H}} &= \vec{\mathbf{E}}\cdot\vec{\mathbf{B}} \\
\implies \alpha &= \frac{(\mu_0 c)^2\vec{\mathbf{D}}\cdot\vec{\mathbf{H}}}{cB_0^2}
\end{aligned}$$

### Part (b)

$$\begin{aligned}
\vec{\mathbf{D}}^2 - \frac{1}{c^2}\vec{\mathbf{H}}^2 &= \frac{\eta_E^2}{c^2}(\vec{\mathbf{E}}^2 + \alpha^2c^2\vec{\mathbf{B}}^2 + 2\alpha c\vec{\mathbf{E}}\cdot\vec{\mathbf{B}} - c^2\vec{\mathbf{B}}^2 - \alpha^2\vec{\mathbf{E}}^2 + 2\alpha c\vec{\mathbf{E}}\cdot\vec{\mathbf{B}}) \\
&= \frac{\eta_E^2}{c^2}((1-\alpha^2)(\vec{\mathbf{E}}^2 - c^2\vec{\mathbf{B}}^2) + 4\alpha^2c^2B_0^2)
\end{aligned}$$

Problem 3 continued on next page... 

But we know,

$$\eta_E = \frac{1}{\mu_0 c}\frac{1}{\sqrt{1-\alpha^2+\frac{1}{B_0^2}\left(\vec{\mathbf{B}}^2-\frac{1}{c^2}\vec{\mathbf{E}}^2\right)}} \implies B_0^2\left(\frac{1}{(\mu_0 c)^2\eta_E^2}+\alpha^2-1\right) = \left(\vec{\mathbf{B}}^2-\frac{1}{c^2}\vec{\mathbf{E}}^2\right)$$

$$\eta_H = \frac{\mu_0 c}{\sqrt{1-\alpha^2+\left(\frac{\mu_0 c}{B_0}\right)^2\left(\vec{\mathbf{D}}^2-\frac{1}{c^2}\vec{\mathbf{H}}^2\right)}} \implies \left[\left(\frac{\mu_0 c}{\eta_H}\right)^2+\alpha^2-1\right]\left(\frac{B_0}{\mu_0 c}\right)^2 = \vec{\mathbf{D}}^2-\frac{1}{c^2}\vec{\mathbf{H}}^2$$

Hence, we can rewrite the expression as,

$$\begin{aligned}
\left[\left(\frac{\mu_0 c}{\eta_H}\right)^2+\alpha^2-1\right]\left(\frac{B_0}{\mu_0 c}\right)^2 &= -\eta_E^2\left((1-\alpha^2)\left[B_0^2\left(\frac{1}{(\mu_0 c)^2\eta_E^2}+\alpha^2-1\right)\right]+4\alpha^2c^2B_0^2\right)\\
\left(\frac{\mu_0 c}{\eta_H}\right)^2+\alpha^2-1 &= (\alpha^2-1)\left[1+\eta_E^2(\mu_0 c)^2(\alpha^2-1)\right]-4\alpha^2c^2\eta_E^2(\mu_0 c)^2\\
\left(\frac{\mu_0 c}{\eta_H}\right)^2 &= (\mu_0 c)^2(\alpha^2-1)\eta_E^2(\alpha^2-1)-4\alpha^2c^2\eta_E^2\\
\left(\frac{1}{\eta_H}\right)^2 &= \eta_E^2\left[(\alpha^2-1)^2-4\alpha^2c^2\right] = \eta_E^2(\alpha^2+1)^2
\end{aligned}$$

$$\text{Hence,}\quad \eta_E\eta_H(1+\alpha^2)=1$$

We know,

$$\vec{\mathbf{H}} = \eta_E(c\vec{\mathbf{B}}-\alpha\vec{\mathbf{E}}) \quad\text{and}\quad c\vec{\mathbf{D}} = \eta_E(\vec{\mathbf{E}}+\alpha c\vec{\mathbf{B}})$$

Hence,

$$\begin{aligned}
\alpha c\vec{\mathbf{D}}+\vec{\mathbf{H}} = \eta_E c(1+\alpha^2)\vec{\mathbf{B}} &\implies \vec{\mathbf{B}} = \eta_H\left(\alpha\vec{\mathbf{D}}+\frac{\vec{\mathbf{H}}}{c}\right)\\
c\vec{\mathbf{D}}-\alpha\vec{\mathbf{H}} = \eta_E(1+\alpha^2)\vec{\mathbf{E}} &\implies \vec{\mathbf{E}} = \eta_H(c\vec{\mathbf{D}}-\alpha\vec{\mathbf{H}})
\end{aligned}$$

Applying $\dfrac{\vec{\mathbf{E}}}{\mu_0 c}\leftrightarrow\vec{\mathbf{H}}$ and $\dfrac{\vec{\mathbf{B}}}{\mu_0 c}\leftrightarrow\vec{\mathbf{D}}$, we have,

$$\eta_H = \frac{\mu_0 c}{\sqrt{1-\alpha^2+\left(\frac{1}{B_0}\right)^2\left(\vec{\mathbf{B}}^2-\frac{1}{c^2}\vec{\mathbf{E}}^2\right)}} = (\mu_0 c)^2\eta_E$$

$$\begin{aligned}
\frac{\vec{\mathbf{E}}}{\mu_0 c} = \eta_E(\mu_0 c^2\vec{\mathbf{D}}-\alpha\mu_0 c\vec{\mathbf{H}}) &\implies \vec{\mathbf{E}} = (\mu_0 c)^2\eta_E(c\vec{\mathbf{D}}-\alpha\vec{\mathbf{H}}) = \eta_H(c\vec{\mathbf{D}}-\alpha\vec{\mathbf{H}})\\
c\frac{\vec{\mathbf{B}}}{\mu_0 c} = \eta_E(\mu_0 c)(\vec{\mathbf{H}}+\alpha c\vec{\mathbf{D}}) &\implies \vec{\mathbf{B}} = \eta_H\left(\alpha\vec{\mathbf{D}}+\frac{\vec{\mathbf{H}}}{c}\right)\\
\mu_0 c\vec{\mathbf{D}} = \eta_H\frac{1}{\mu_0 c}\left(\alpha\vec{\mathbf{B}}+\frac{\vec{\mathbf{E}}}{c}\right) &\implies c\vec{\mathbf{D}} = \eta_E(\vec{\mathbf{E}}+\alpha c\vec{\mathbf{B}})\\
\mu_0 c\vec{\mathbf{H}} = \frac{\eta_H}{\mu_0 c}(c\vec{\mathbf{B}}-\alpha\vec{\mathbf{E}}) &\implies \vec{\mathbf{H}} = \eta_E(c\vec{\mathbf{B}}-\alpha\vec{\mathbf{E}})
\end{aligned}$$

Thus, we have shown that BI theory has a duality transformation.

Problem 3 continued on next page...

**Part (c)**

$$
\begin{aligned}
\delta L_{BI} =& -\frac{B_0}{2\mu_0}\frac{1}{\sqrt{B_0^2+\vec{\mathbf{B}}^2-\frac{1}{c^2}\vec{\mathbf{E}}^2-\frac{(\vec{\mathbf{E}}\cdot\vec{\mathbf{B}})^2}{(cB_0)^2}}}\left(2\vec{\mathbf{B}}\cdot\delta\vec{\mathbf{B}}-\frac{2\vec{\mathbf{E}}\cdot\delta\vec{\mathbf{E}}}{c^2}-\frac{2\vec{\mathbf{E}}\cdot\vec{\mathbf{B}}}{(cB_0)^2}(\vec{\mathbf{B}}\cdot\delta\vec{\mathbf{E}}+\vec{\mathbf{E}}\cdot\delta\vec{\mathbf{B}})\right)\\
&= c\eta_E(\vec{\mathbf{E}}\cdot\delta\vec{\mathbf{E}}) = -\vec{\boldsymbol{\nabla}}\cdot\left(\eta_E\frac{\vec{\mathbf{E}}}{c}\delta\phi\right)-\partial_t\left(\eta_E\frac{\vec{\mathbf{E}}}{c}\cdot\delta\vec{\mathbf{A}}\right)+\vec{\boldsymbol{\nabla}}\cdot\left(\eta_E\frac{\vec{\mathbf{E}}}{c}\right)+\partial_t\left\{\eta_E\frac{\vec{\mathbf{E}}}{c}\right\}\cdot\delta\vec{\mathbf{A}}
\end{aligned}
$$

$$
\begin{aligned}
c\eta_E(\vec{\mathbf{E}}\cdot\delta\vec{\mathbf{E}}) &= -\vec{\boldsymbol{\nabla}}\cdot\left(\eta_E\frac{\vec{\mathbf{E}}}{c}\delta\phi\right)-\partial_t\left(\eta_E\frac{\vec{\mathbf{E}}}{c}\cdot\delta\vec{\mathbf{A}}\right)+\vec{\boldsymbol{\nabla}}\cdot\left(\eta_E\frac{\vec{\mathbf{E}}}{c}\right)+\partial_t\left\{\eta_E\frac{\vec{\mathbf{E}}}{c}\right\}\cdot\delta\vec{\mathbf{A}}\\
-c\eta_E(\vec{\mathbf{B}}\cdot\delta\vec{\mathbf{B}}) &= \vec{\boldsymbol{\nabla}}\cdot(c\eta_E\vec{\mathbf{B}}\times\delta\vec{\mathbf{A}}) = \delta\vec{\mathbf{A}}\cdot\vec{\boldsymbol{\nabla}}\times c\eta_E\vec{\mathbf{B}}\\
\eta_E\alpha\vec{\mathbf{E}}\cdot\delta\vec{\mathbf{B}} &= \eta_E\alpha\vec{\mathbf{E}}\cdot\vec{\boldsymbol{\nabla}}\times\delta\vec{\mathbf{A}}-\delta\vec{\mathbf{A}}\cdot\vec{\boldsymbol{\nabla}}\times(c\eta_E\vec{\mathbf{B}})\\
\eta_E\alpha\vec{\mathbf{B}}\cdot\delta\vec{\mathbf{E}} &= -\vec{\boldsymbol{\nabla}}\cdot(\eta_E\alpha\vec{\mathbf{E}}\times\delta\vec{\mathbf{A}})
\end{aligned}
$$

**Part (d)**

Including the point charge,

$$
L = \frac{B_0^2}{\mu_0}-\frac{B_0}{2\mu_0}\sqrt{B_0^2+\vec{\mathbf{B}}^2-\frac{1}{c^2}\vec{\mathbf{E}}^2-\frac{(\vec{\mathbf{E}}\cdot\vec{\mathbf{B}})^2}{(cB_0)^2}}+\vec{\mathbf{J}}\cdot\vec{\mathbf{A}}-\rho\phi
$$