Multipole Expansion of Gravitational Waves

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1 Gravitational Waves

Main Reference : Gravitation - Foundations and Frontiers by Thanu Padmanabhan

1.1 Facts about the Field Equations

The field equations are given by,

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R = 8\pi T_{ab}$$

where,

$$R_{ijkl} = \frac{1}{2}(\partial_k \partial_l g_{im} + \partial_i \partial_m g_{kl} - \partial_k \partial_m g_{il} - \partial_i \partial_l g_{km}) + g_{np}(\Gamma^n_{kl}\Gamma^p_{im} - \Gamma^p_{il}\Gamma^n_{km})$$
$$R_{ab} = R_{iajb} g^{ij}$$
$$R = R_{ab} g^{ab}$$

The equations are symmetric under the exchange of indices, and hence there are really **ten** equations from the face of it. But we also note that the Bianchi identity tells us that $\nabla_a G^a_b = 0$, Where does this come from? Read. giving us **four** constraint equations. So really, we have **six** independent functions of spacetime coordinates to solve for. This suggests naively that though we have ten independent components in g_{ab} only six of them play a part in dynamics, and the other four are just fixed by the evolution of these six components. Let's now observe a few other things :-

- Second derivatives of time are contained only in $R_{0\alpha 0\beta}$, where α, β run only over the space coordinates. *The only terms which have double time derivatives*, hence, are $\ddot{g}_{\alpha\beta}$.
- $\nabla_a G^a_b = 0 \implies \nabla_0 G^0_b = -\nabla_\alpha G^\alpha_b$. The RHS has terms with double time derivatives, which means that G^0_b should have *only one time derivative.*
- Further, the space-time and the time-time components have first time derivatives which are propotional to $\dot{g}_{\alpha\beta}$. The time derivatives of the type $\dot{g}_{0a}$ *do not appear in Einstein's equations.*

This now gives us an alternate way to understand the evolution equations. To evolve the equations, we need to provide the initial values of $g_{\alpha\beta}$ and $\dot{g}_{\alpha\beta}$ on a time slice. The space-time and the time-time Einstein's equations fix the value of the other four components. Add Harmonic Gauge details here.

1.2 Weak field limit of gravity

Consider $g_{ab} = \eta_{ab} + \epsilon h_{ab}$, with $\eta = \text{diag}(-1, 1, 1, 1)$, and ϵ being a small number. We now write down the field equations with terms only up to $\mathcal{O}(\epsilon)$. After the dust settles, we get,

$$\partial_n \partial_m h + \Box h_{mn} - \partial_n \partial_r h^r_m - \partial_m \partial_r h^r_n - \eta_{mn}(\Box h - \partial_r \partial_s h^{sr}) = -16\pi\kappa T_{mn}$$

We effect a change of variables as $\bar{h}_{mn} = h_{mn} - \dfrac{\eta_{mn}}{2} h$. The equation now becomes,

$$\Box \bar{h}_{mn} + \eta_{mn} \partial_r \partial_s \bar{h}^{rs} - \partial_n \partial_r \bar{h}^r_m - \partial_m \partial_r \bar{h}^r_n = -16\pi\kappa T_{mn}$$

Noticing that the above equation is invariant under gauge transformations of the form

$$\bar{h}'^{mr} = \bar{h}^{mr} - \partial^m \xi^r - \partial^r \xi^m + \eta^{mr} \partial_s \xi^s$$

and imposing the harmonic gauge constraint $\Box \xi^m = 0$, review this calculation we get,

$$\Box \bar{h}^{mn} = -16\pi\kappa T_{mn}$$

This shows us that some sort of gravitational waves exist, with or without a source.

1.3 Gravitational waves in a flat background

By symmetry considerations of the Riemann tensor and upto first order in perturbation, we can write,

$$\Box R_{bcmn} = 8\pi[\partial_b(\partial_m \bar{T}_{nc} - \partial_n \bar{T}_{mc}) - \partial_c(\partial_m \bar{T}_{nb} - \partial_n \bar{T}_{mb})] = 8\pi \bar{T}_{bcmn}$$

where $\bar{T}_{ij} = T_{ij} - \dfrac{g_{ij} T}{2}$. The equation in invariant under infinitesimal coordinate transformations. Let's first consider the vacuum case $\Box R_{bcmn} = 0 \implies R_{abmn} = C_{abmn} e^{ik_a x^a}, k^a k_a = 0$. The Bianchi identity gives, Read Up

$$C_{bcmn} k_a + C_{camn} k_b + C_{abmn} k_c = 0$$

We now choose the wavevector to be oriented aong the z-axis *ie* $k_a = (-\omega, 0, 0, \omega)$. Setting index $c = 0$, we get

$$C_{abmn} = \frac{1}{\omega}(C_{b0mn} k_a + C_{0amn} k_b) = \frac{1}{\omega}(C_{b0mn} k_a - C_{a0mn} k_b)$$
$$n \to 0 \implies C_{a0bm} = \frac{1}{\omega}(C_{b0m0} k_a - C_{a0m0} k_b)$$

Substituting the second equation into the first, we can figure that C_{abmn} can be specified completely in terms of the form C_{i0j0}. Furthermore, substituting $a = 0$ in the second equation gives,

$$C_{00bm} = \frac{1}{\omega}(C_{b0m0}(-\omega) - C_{00m0} k_b) \implies C_{00m0} = 0$$

This means that i, j cannot be zero, and hence only the terms of the form $C_{\alpha 0 \beta 0}$ survive this ordeal.

2 Symmetric Trace-free Tensors