Fluid Mechanics: Assignment #4

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Acknowledgements -

Problem 1

In ideal 2D flow, $\nabla \cdot \vec{\mathbf{u}} = 0$. This means,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \implies u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}$$

Hence, $\nabla \psi = \frac{\partial \psi}{\partial x}\hat{\mathbf{x}} + \frac{\partial \psi}{\partial y}\hat{\mathbf{y}} = -v\hat{\mathbf{x}} + u\hat{\mathbf{y}}$. By definition $\vec{\mathbf{u}} = \nabla \phi = u\hat{\mathbf{x}} + v\hat{\mathbf{y}}$. Now we can do the calculations required in the problem,

- $\nabla \psi \cdot \nabla \phi = (-v\hat{\mathbf{x}} + u\hat{\mathbf{y}}) \cdot (u\hat{\mathbf{x}} + v\hat{\mathbf{y}}) = -vu + uv = 0.$
- $-\nabla \psi \times \nabla \phi = -(-v\hat{\mathbf{x}} + u\hat{\mathbf{y}}) \times (u\hat{\mathbf{x}} + v\hat{\mathbf{y}}) = -(-v^2 - u^2)\hat{\mathbf{z}} = |\vec{\mathbf{u}}|^2\hat{\mathbf{z}}$
- $|\nabla \psi|^2 = u^2 + v^2 \quad \text{and} \quad |\nabla \phi|^2 = u^2 + v^2 \implies |\nabla \psi|^2 = |\nabla \phi|^2$
- $-\hat{\mathbf{z}} \times \nabla \psi = -\hat{\mathbf{z}} \times (-v\hat{\mathbf{x}} + u\hat{\mathbf{y}}) = u\hat{\mathbf{x}} + v\hat{\mathbf{y}} = \nabla \phi$

Problem 2

- For point source, $\vec{\mathbf{u}} = \dfrac{q_s}{2\pi r}\hat{\mathbf{r}}$, where q_s is the source strength. In spherical polar coordinates, $\vec{\mathbf{u}} = \frac{\partial \phi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\hat{\theta}$. This means

$$\phi = \int \frac{\partial \phi}{\partial r} dr + \int \frac{1}{r}\frac{\partial \phi}{\partial \theta} d\theta = \frac{q_s}{2\pi} \ln r + constant$$

For lines of constant ϕ,

$$\frac{q_s}{2\pi} \ln r = C$$
$$\implies \frac{q_s}{2\pi r}\frac{\mathrm{d}r}{\mathrm{d}x} = 0$$
$$\implies \frac{q_s}{2\pi r^2}\left(2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x}\right) = 0 \implies \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y} = m$$

As velocity is radial, the streamlines are also radial straight lines passing through the origin. The slope of such straight lines is $\dfrac{y}{x} = -\dfrac{1}{m}$. Hence the streamlines and lines of constant ψ are perpendicular.

- For point vortex, $\vec{\mathbf{u}} = \frac{\Gamma}{2\pi r}\hat{\theta} = \frac{-\Gamma y}{2\pi(x^2+y^2)}\hat{\mathbf{x}} + \frac{\Gamma x}{2\pi(x^2+y^2)}\hat{\mathbf{y}}$. This means,

$$\phi = \frac{\Gamma}{2\pi}\tan^{-1}\frac{y}{x} \quad \text{and} \quad \psi = -\frac{\Gamma}{2\pi}\ln r$$

For lines of constant ϕ,

$$\begin{aligned}\frac{\Gamma}{2\pi}\tan^{-1}\frac{y}{x} &= C_1\\ \implies \frac{y}{x} &= constant = m_1\\ \implies \frac{\mathrm{d}y}{\mathrm{d}x} &= m_1\end{aligned}$$

For lines of constant ψ,

$$\begin{aligned}\frac{\Gamma}{2\pi}\ln r &= C_2\\ \implies r &= constant\\ \implies x^2+y^2 &= constant\\ \implies \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{x}{y} = -\frac{1}{m_1}\end{aligned}$$

Hence proved.

Problem 3

Given $A = \begin{bmatrix}-1 & p\\ 0 & -2\end{bmatrix}$. The eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = -2$ with corresponding (normalized) eigenvectors are $v_1 = \begin{bmatrix}1 & 0\end{bmatrix}^T$ and $v_2 = \begin{bmatrix}\frac{-p}{\sqrt{1+p^2}} & \frac{1}{\sqrt{1+p^2}}\end{bmatrix}^T$. So, the resultant vector is,

$$\begin{aligned}v &= v_1 e^{\lambda_1 t} + v_2 e^{\lambda_2 t}\\ &= \begin{bmatrix}1\\0\end{bmatrix}e^{-t} + \frac{1}{\sqrt{1+p^2}}\begin{bmatrix}-p\\1\end{bmatrix}e^{-2t}\\ v &= \begin{bmatrix}e^{-t} - \frac{p}{\sqrt{1+p^2}}e^{-2t}\\ \frac{1}{\sqrt{1+p^2}}e^{-2t}\end{bmatrix}\\ |v|^2 &= \left(e^{-t} - \frac{p}{\sqrt{1+p^2}}e^{-2t}\right)^2 + \left(\frac{1}{\sqrt{1+p^2}}e^{-2t}\right)^2\\ &= e^{-2t} + e^{-4t} - \frac{2p}{\sqrt{1+p^2}}e^{-3t}\\ \frac{\mathrm{d}|v|^2}{\mathrm{d}t} &= -2e^{-2t} - 4e^{-4t} + \frac{6p}{\sqrt{1+p^2}}e^{-3t}\end{aligned}$$

For the resultant to grow, $\frac{\mathrm{d}|v|^2}{\mathrm{d}t} > 0$.

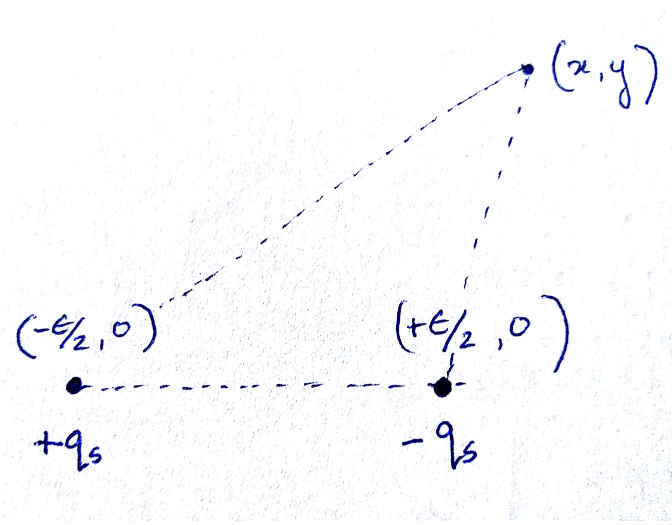
$$\begin{aligned}\implies -2e^{-2t} - 4e^{-4t} + \frac{6p}{\sqrt{1+p^2}}e^{-3t} > 0\\ \implies \frac{p}{\sqrt{1+p^2}} > \frac{e^t + 2e^{-t}}{3}\end{aligned}$$

The function $\frac{e^t + 2e^{-t}}{3}$ has a minimum value of $\frac{2\sqrt{2}}{3}$. Hence, for the resultant to grow for some finite time,

$$\begin{aligned}
\frac{p}{\sqrt{1+p^2}} &> \frac{2\sqrt{2}}{3} \\
\implies \frac{p^2}{1+p^2} &> \frac{8}{9} \\
\implies p^2 &> 8 \\
\implies p &> 2\sqrt{2}
\end{aligned}$$

Hence, the resultant will grow for some finite time if $p > 2\sqrt{2}$.

Problem 4



Let's first find the potential. As described in Problem 1, $\phi = \frac{q_s}{2\pi}\ln r$. So for this problem,

$$\begin{aligned}
\phi &= -\frac{q_s}{4\pi}\ln[(x-\epsilon/2)^2 + y^2] + \frac{q_s}{4\pi}\ln[(x+\epsilon/2)^2 + y^2] \\
&= \frac{q_s}{4\pi}\ln\frac{(x+\epsilon/2)^2 + y^2}{(x-\epsilon/2)^2 + y^2} \\
&= \frac{q_s}{4\pi}\ln\frac{1+\epsilon x/r^2}{1-\epsilon x/r^2} \\
&= \frac{q_s}{4\pi}\ln\left(1 + 2\epsilon x/r^2\right) \\
\phi &= \frac{q_s \epsilon\cos\theta}{2\pi r}
\end{aligned}$$

Hence, the velocity profile is,

$$\vec{\mathbf{u}} = -\frac{q_s\epsilon\cos\theta}{2\pi r^2}\hat{\mathbf{r}} - \frac{q_s\epsilon\sin\theta}{2\pi r^2}\hat{\theta}$$

For the streamlines, we note that for a single point source at origin,

$$u = \frac{q_s x}{2\pi(x^2+y^2)} \quad \text{and} \quad v = \frac{q_s y}{2\pi(x^2+y^2)}$$

$$\therefore \psi = \frac{q_s}{2\pi}\tan^{-1}\frac{y}{x}$$

Problem 4 continued on next page. . .

Hence, for this problem,

$$\begin{aligned}
\psi &= -\frac{q_s}{2\pi}\tan^{-1}\frac{y}{x-\epsilon/2} + \frac{q_s}{2\pi}\tan^{-1}\frac{y}{x+\epsilon/2} \\
&= \frac{q_s}{2\pi}\left(\tan^{-1}\frac{y}{x+\epsilon/2} - \tan^{-1}\frac{y}{x-\epsilon/2}\right) \\
&= \frac{q_s}{2\pi}\tan^{-1}\frac{y(x-\epsilon/2-x-\epsilon/2)}{x^2+y^2} \\
&= \frac{q_s}{2\pi}\tan^{-1}\frac{-y\epsilon}{r^2} \\
\psi &= -\frac{q_s}{2\pi}\frac{\sin\theta}{r}
\end{aligned}$$

$$\psi = -\frac{q_s}{2\pi}\frac{\sin\theta}{r} = constant \implies \frac{\sin\theta}{r} = \frac{y}{x^2+y^2} = \frac{1}{2C} \implies x^2 + (y-C)^2 = C^2$$

which is the equation of a circle with centre at $(0, C)$ and radius C.