Advanced Quantum Mechanics: Assignment #1

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Problem 1

Solution
We solve each part separately.

Part 1 - Commutators
We expand out each term as follows,

$$[A,[B,C]] = ABC - ACB - BCA + CBA$$
$$[C,[A,B]] = CAB - CBA - ABC + BAC$$
$$[B,[C,A]] = BCA - BAC - CAB + ACB$$

Adding the three expressions above, we arrive at the expression

$$[A,[B,C]] + [C,[A,B]] + [B,[C,A]] = 0.$$

Hence Proved.

Part 2 - Poisson Brackets
$\{.,.\}$ denotes Poisson Bracket, $X_y = \frac{\partial X}{\partial y}$ and $X_{yz} = \frac{\partial^2 X}{\partial y \partial z}$. We expand each term as follows,'

$$\begin{aligned}
\{A,\{B,C\}\} &= \{A, B_q C_p - B_p C_q\} \\
&= A_q B_{pq} C_p + A_q B_q C_{pp} + A_p B_{pq} C_p + A_p B_q C_{pq} - A_q B_{pp} C_q - A_q B_p C_{pq} - A_p B_{pq} C_q - A_p B_p C_{qq} \\
\{C,\{A,B\}\} &= C_q A_{pq} B_p + C_q A_q B_{pp} + C_p A_{pq} B_p + C_p A_q B_{pq} - C_q A_{pp} B_q - C_q A_p B_{pq} - C_p A_{pq} B_q - C_p A_p B_{qq} \\
\{B,\{C,A\}\} &= B_q C_{pq} A_p + B_q C_q A_{pp} + B_p C_{pq} A_p + B_p C_q A_{pq} - B_q C_{pp} A_q - B_q C_p A_{pq} - B_p C_{pq} A_q - B_p C_p A_{qq}
\end{aligned}$$

Adding the three expressions above, we arrive at the expression

$$\{A,\{B,C\}\} + \{C,\{A,B\}\} + \{B,\{C,A\}\} = 0$$

Hence Proved.

Problem 2

Solution

$$\begin{aligned}
[AB,CD] &= A[B,CD] + [A,CD]B \\
&= A[B,C]D + AC[B,D] + C[A,D]B + [A,C]DB \\
&= A(\{B,C\} - 2CB)D + AC(2BD - \{B,D\}) + C(2AD - \{A,D\}) + (\{A,C\} - 2CA)DB \\
&= A\{B,C\}D - 2ACBD + 2ACBD - AC\{B,D\} + 2CADB - C\{A,D\}B + \{A,C\}DB - 2CADB \\
&= -AC\{B,D\} + A\{B,C\}D - C\{A,D\}B + \{A,C\}DB \\
&= -AC\{D,B\} + A\{C,B\}D - C\{D,A\}B + \{C,A\}DB
\end{aligned}$$

Hence Proved.

Problem 3

Solution

$$\vec{\sigma} \cdot \vec{\mathbf{n}} = \begin{pmatrix} n_z & n_x - in_y \\ n_x + in_y & -n_z \end{pmatrix}$$

The corresponding equation for eigenvalues of this matrix is,

$$\lambda^2 - n_x^2 - n_y^2 - n_z^2 = 0$$

which gives as eigenvalues $\lambda = \pm\sqrt{n_x^2 + n_y^2 + n_z^2}$. Substituting these values in the eigenvalue equation $(\vec{\sigma} \cdot \vec{\mathbf{n}})X = \lambda X$, we get the following eigenvectors

$$\begin{pmatrix} \frac{-\sqrt{n_x^2+n_y^2+n_z^2}+n_z}{n_x+in_y} \\ 1 \end{pmatrix} \text{and} \begin{pmatrix} \frac{\sqrt{n_x^2+n_y^2+n_z^2}+n_z}{n_x+in_y} \\ 1 \end{pmatrix}$$

Problem 4

Solution

Let $|\beta\rangle$ be an arbitrary state, and $|\lambda_i\rangle$ be the eigenstates such that $\sum_k |\lambda_k\rangle \langle\lambda_k| = 1$. Consider the following,

$$\begin{aligned}
\prod_{i \neq j} \left(\frac{A - \lambda_i}{\lambda_j - \lambda_i} \right) |\beta\rangle &= \prod_{i \neq j} \left(\frac{A - \lambda_i}{\lambda_j - \lambda_i} \right) \sum_k |\lambda_k\rangle \langle\lambda_k|\beta\rangle \\
&= \sum_k \prod_{i \neq j} \left(\frac{\lambda_k - \lambda_i}{\lambda_j - \lambda_i} \right) |\lambda_k\rangle \langle\lambda_k|\beta\rangle
\end{aligned}$$

Lets look closer at the sum above. For $k \neq j$, the coefficient $\frac{\lambda_k - \lambda_i}{\lambda_j - \lambda_i}$ in the sum will vanish for some i, rendering the whole product to be zero. So all that remains in the summation is the term corresponding to $k = j$. Hence,

$$\begin{aligned}
\prod_{i \neq j} \left(\frac{A - \lambda_i}{\lambda_j - \lambda_i} \right) |\beta\rangle &= \prod_{i \neq j} \left(\frac{\lambda_j - \lambda_i}{\lambda_j - \lambda_i} \right) |\lambda_j\rangle \langle\lambda_j|\beta\rangle \\
&= \prod_{i \neq j} (1) |\lambda_j\rangle \langle\lambda_j|\beta\rangle \\
&= |\lambda_j\rangle \langle\lambda_j|\beta\rangle \\
&= P_j |\beta\rangle
\end{aligned}$$

Hence, $\prod_{i \neq j} \left(\frac{A - \lambda_i}{\lambda_j - \lambda_i} \right) = P_j$, the Projection operator onto $|\lambda_j\rangle$.

Problem 5

$F(\hat{x})$ and $G(\hat{p})$ have regular series expansions. So, for some constants α_i and β_i,

$$F(\hat{x}) = \alpha_0 + \alpha_1 \hat{x} + \alpha_2 \hat{x}^2 + \ldots$$

$$G(\hat{p}) = \beta_0 + \beta_1 \hat{p} + \beta_2 \hat{p}^2 + \dots$$

Consider $[\hat{p}, \hat{x}^n]$,

$$\begin{aligned}[\hat{p}, \hat{x}^n] &= [\hat{p}, \hat{x}]\hat{x}^{n-1} + \hat{x}[\hat{p}, \hat{x}]\hat{x}^{n-2} + \hat{x}^2[\hat{p}, \hat{x}]\hat{x}^{n-3} + \dots \text{n terms} \\ &= -in\hat{x}^{n-1}\end{aligned}$$

Consider $[\hat{p}, F(\hat{x})]$.

$$\begin{aligned}[\hat{p}, F(\hat{x})] &= \left[\hat{p}, \alpha_0 + \alpha_1 \hat{x} + \alpha_2 \hat{x}^2 + \dots\right] \\ &= [\hat{p}, \alpha_0] + \alpha_1 [\hat{p}, \hat{x}] + \alpha_2 \left[\hat{p}, \hat{x}^2\right] + \dots \\ &= \sum_{j=0}^{\infty} \alpha_j \left[\hat{p}, \hat{x}^j\right] \\ &= -i \sum_{j=1}^{\infty} \alpha_j (j\hat{x}^{j-1}) \\ &= -iF'(\hat{x})\end{aligned}$$

Similarly, $[\hat{x}, \hat{p}^n] = in\hat{p}^{n-1}$, and,

$$\begin{aligned}[\hat{x}, G(\hat{p})] &= \sum_{j=0}^{\infty} \beta_j \left[\hat{x}, \hat{p}^j\right] \\ &= i \sum_{j=1}^{\infty} \beta_j (j\hat{p}^{j-1}) \\ &= iG'(\hat{p})\end{aligned}$$

Hence Proved

$$\begin{aligned}\left[\hat{x}^2, \hat{p}^2\right] &= \hat{x}\left[\hat{x}, \hat{p}^2\right] + \left[\hat{x}, \hat{p}^2\right]\hat{x} \\ &= 2i\{x, p\}\end{aligned}$$

Problem 6

Solution

Part (a)

The normalized coherent states are given by,

$$\begin{aligned}|z\rangle &= \frac{1}{\sqrt[4]{\pi} \exp\left(\frac{\alpha^2}{2}\right)} \exp\left(-\frac{x^2}{2} + \alpha x + i\beta x\right) \\ &= \frac{1}{\sqrt[4]{\pi}} \exp\left(-\frac{(x-\alpha)^2}{2} + i\beta x\right)\end{aligned}$$

where $z = \alpha + i\beta$, $\hat{a}\,|z\rangle = z\,|z\rangle$. Hence,

$$\begin{aligned}\langle z'|z\rangle &= \int_{-\infty}^{\infty} \frac{1}{\sqrt[4]{\pi}\exp\left(\frac{\alpha^2}{2}\right)} \frac{1}{\sqrt[4]{\pi}\exp\left(\frac{\alpha'^2}{2}\right)} \exp\left(-x^2 + (\alpha+\alpha')x + i(\beta-\beta')x\right) \\ &= \exp\left(-\frac{\alpha^2}{4} - \frac{1}{2}i\alpha'\beta' + \frac{1}{2}i\beta\alpha' + \frac{\alpha\alpha'}{2} - \frac{(\alpha')^2}{4} - \frac{1}{2}i\alpha\beta' + \frac{i\alpha\beta}{2} - \frac{\beta^2}{4} + \frac{\beta\beta'}{2} - \frac{(\beta')^2}{4}\right) \\ &= \exp\left(-\frac{(\alpha-\alpha')^2}{4} - \frac{(\beta-\beta')^2}{4} + \frac{\alpha\alpha'(\beta-\beta')}{2}i\right)\end{aligned}$$

Here, $z = \alpha + i\beta$ and $z' = \alpha' + i\beta'$

Part (b)

Consider $\langle x'|x\rangle = \delta(x-x')$. The completeness relation is of the form $\int d^2z f(z)\,|z\rangle\langle z| = 1$. Using this identity, we insert the complete states in $\langle x|x\rangle$ as follows,

$$\int d^2z f(z)\,\langle x'|z\rangle\,\langle z|x\rangle = \delta(x-x')$$

$$\int d\alpha\; d\beta f(\alpha,\beta)\frac{1}{\sqrt[4]{\pi}}\exp\left(-\frac{(x'-\alpha)^2}{2} + i\beta x'\right)\frac{1}{\sqrt[4]{\pi}}\exp\left(-\frac{(x-\alpha)^2}{2} - i\beta x\right) = \int d\beta \exp(i\beta(x-x'))$$

This relation should hold for all x', specifically for $x = x'$. Hence,

$$\int d\alpha\; d\beta f(\alpha,\beta)\frac{1}{\sqrt{\pi}}\exp\{-(x-\alpha)^2\} = \int d\beta$$

$$\int d\beta\left\{\int d\alpha f(\alpha,\beta)\frac{1}{\sqrt{\pi}}\exp\{-(x-\alpha)^2\} - 1\right\} = 0$$

For this to hold for all x, the term in curly brackets should be zero.

$$\int d\alpha f(\alpha,\beta)\frac{1}{\sqrt{\pi}}\exp\{-(x-\alpha)^2\} = 1 \tag{1}$$

At this point, we note that,

$$\int d\alpha\frac{1}{\sqrt{\pi}}\exp\{-(x-\alpha)^2\} = 1$$

By comparing preceding two equations, we can claim that $f(\alpha,\beta) = 1$ is **one** possibility and the corresponding completeness relation is

$$\int d^2z\,|z\rangle\langle z| = 1$$

Note that this is **a** completeness relation and not **the** completeness relation. In principle, any $f(\alpha,\beta)$ that satisfies (1) can be included in the completeness relation.

Problem 7

Solution

Part (a)

We know that $\langle z|\hat{a}|z\rangle = z$ and $\langle z|\hat{a^\dagger}|z\rangle = z^*$. Adding these two up, we get,

$$\sqrt{2}\,\langle z|\hat{x}|z\rangle = z + z^* = 2\,\mathrm{Re}\{z\}$$

Problem 7 continued on next page...

$$\langle z|\hat{x}|z\rangle = \sqrt{2}\,\mathrm{Re}\{z\}$$

Similarly, subtracting the two, we get,

$$\sqrt{2}i\,\langle z|\hat{p}|z\rangle = z - z^* = 2i\,\mathrm{Im}\{z\}$$
$$\langle z|\hat{p}|z\rangle = \sqrt{2}\,\mathrm{Im}\{z\}$$

Part (b)

$$\hat{a}^2 = \frac{1}{2}(\hat{x}^2 - \hat{p}^2 + i\{x,p\}) \tag{2}$$
$$\hat{a^\dagger}^2 = \frac{1}{2}(\hat{x}^2 - \hat{p}^2 - i\{x,p\}) \tag{3}$$
$$\hat{a}\hat{a^\dagger} = \frac{1}{2}(\hat{x}^2 + \hat{p}^2 - i[x,p]) \tag{4}$$
$$\hat{a^\dagger}\hat{a} = \frac{1}{2}(\hat{x}^2 + \hat{p}^2 + i[x,p]) \tag{5}$$

Adding the above equations, we see,

$$\begin{aligned}
2\,\langle z|\hat{x}^2|z\rangle &= \langle z|\hat{a}^2 + \hat{a^\dagger}^2 + \left\{a, a^\dagger\right\}|z\rangle \\
&= \langle z|\hat{a}^2|z\rangle + \langle z|\hat{a^\dagger}^2|z\rangle + \langle z|\left[a, a^\dagger\right]|z\rangle + 2\,\langle z|a^\dagger a|z\rangle \\
&= z^2 + {z^*}^2 + 1 + 2zz^* \\
&= 1 + 4\,\mathrm{Re}(z)^2 \\
\langle z|\hat{x}^2|z\rangle &= \frac{1}{2} + 2\,\mathrm{Re}(z)^2
\end{aligned}$$

(1) + (2) - (3) - (4) gives,

$$\begin{aligned}
-2\,\langle z|\hat{p}^2|z\rangle &= \langle z|\hat{a}^2 + \hat{a^\dagger}^2 - \left\{a, a^\dagger\right\}|z\rangle \\
&= -4\,\mathrm{Im}\{z\}^2 - 1 \\
\langle z|\hat{p}^2|z\rangle &= \frac{1}{2} + 2\,\mathrm{Im}\{z\}^2
\end{aligned}$$

Substituting all required values above, we get,

$$\Delta x = \sqrt{\frac{1}{2}}\ ;\ \Delta p = \sqrt{\frac{1}{2}}\ ;\ \Delta x \Delta p = \frac{1}{2}$$

As is evident, this saturates the uncertainty relation $\Delta x \Delta p \geq \frac{1}{2}$.

Problem 8

Solution

Part (a)

$$|\psi\rangle = a_{00}\,|00\rangle + a_{01}\,|01\rangle + a_{10}\,|10\rangle + a_{11}\,|11\rangle$$

The density matrix of system 1 is obtained by tracing over the degrees of freedom of system 2. $\rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|)$

$$\begin{aligned}
\rho_A &= \langle 0_B|\psi\rangle\,\langle\psi|0_B\rangle + \langle 1_B|\psi\rangle\,\langle\psi|1_B\rangle \\
&= (a_{00}\,|0\rangle + a_{10}\,|1\rangle)(a_{00}^*\,\langle 0| + a_{10}^*\,\langle 1|) + (a_{11}\,|1\rangle + a_{01}\,|0\rangle)(a_{11}^*\,\langle 1| + a_{01}^*\,\langle 0|) \\
&= (|a_{00}|^2 + |a_{01}^2|)\,|0\rangle\langle 0| + (a_{11}^* a_{01} + a_{10}^* a_{00})\,|0\rangle\langle 1| + (a_{11} a_{01}^* + a_{10} a_{00}^*)\,|1\rangle\langle 0| + (|a_{11}|^2 + |a_{10}^2|)\,|1\rangle\langle 1| \\
&= \begin{pmatrix} |a_{00}|^2 + |a_{01}^2| & a_{11}^* a_{01} + a_{10}^* a_{00} \\ a_{11} a_{01}^* + a_{10} a_{00}^* & |a_{11}|^2 + |a_{10}^2| \end{pmatrix}
\end{aligned}$$

Similarly, one can find ρ_B

$$\begin{aligned}
\rho_B &= \langle 0_A|\psi\rangle \langle\psi|0_A\rangle + \langle 1_A|\psi\rangle \langle\psi|1_A\rangle \\
&= (a_{00}\,|0\rangle + a_{01}\,|1\rangle)(a_{00}^*\,\langle 0| + a_{01}^*\,\langle 1|) + (a_{11}\,|1\rangle + a_{10}\,|0\rangle)(a_{11}^*\,\langle 1| + a_{10}^*\,\langle 0|) \\
&= (|a_{00}|^2 + |a_{10}^2|)\,|0\rangle\langle 0| + (a_{11}^* a_{10} + a_{01}^* a_{00})\,|0\rangle\langle 1| + (a_{11} a_{10}^* + a_{01} a_{00}^*)\,|1\rangle\langle 0| + (|a_{11}|^2 + |a_{01}^2|)\,|1\rangle\langle 1| \\
&= \begin{pmatrix} |a_{00}|^2 + |a_{10}^2| & a_{11}^* a_{10} + a_{01}^* a_{00} \\ a_{11} a_{10}^* + a_{01} a_{00}^* & |a_{11}|^2 + |a_{01}^2| \end{pmatrix}
\end{aligned}$$