

Electromagnetism: Pset #2

Due on 6th February, 2019

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(Acknowledgements - I would like to thank Junaaid Majeed for discussions.)

Problem 1

Consider, with $\psi_G^R(r) = \frac{1}{(2\pi)^{d/2} R^d} \exp\left(-\frac{r^2}{2R^2}\right)$,

$$\begin{aligned}\frac{1}{R} \frac{\partial \psi_G^R}{\partial R} &= \frac{1}{R} \frac{1}{(2\pi)^{d/2} R^d} \exp\left(-\frac{r^2}{2R^2}\right) \left[\frac{-d}{R} - \frac{-2r^2}{2R^3} \right] \\ \frac{1}{R} \frac{\partial \psi_G^R}{\partial R} &= \frac{\psi_G^R(r)}{R^2} \left[-d + \frac{r^2}{R^2} \right]\end{aligned}$$

Now consider,

$$\begin{aligned}\nabla^2 \psi_G^R &= \frac{1}{r^{d-1}} \frac{\partial}{\partial r} \left(r^{d-1} \frac{\partial \psi_G^R}{\partial r} \right) \\ &= \frac{\partial^2 \psi_G^R}{\partial r^2} + \frac{d-1}{r} \frac{\partial \psi_G^R}{\partial r} \\ &= -\frac{\psi_G^R}{R^2} + \frac{r^2}{R^4} \psi_G^R + \frac{d-1}{r} \left(\frac{-r}{R^2} \right) \psi_G^R \\ \nabla^2 \psi_G^R &= \frac{\psi_G^R(r)}{R^2} \left[-d + \frac{r^2}{R^2} \right] = \frac{1}{R} \frac{\partial \psi_G^R}{\partial R}\end{aligned}$$

Consider now Green's vector field,

$$\begin{aligned}\vec{\mathbf{G}}[\psi_G^R, \phi_\lambda] &= \psi_G^R \vec{\nabla} \phi_\lambda - \phi_\lambda \vec{\nabla} \psi_G^R \\ \implies \vec{\nabla} \cdot \vec{\mathbf{G}} &= \psi_G^R \nabla^2 \phi_\lambda - \phi_\lambda \nabla^2 \psi_G^R \\ \int d^d \vec{\mathbf{r}} (\vec{\nabla} \cdot \vec{\mathbf{G}}) &= \int d^d \vec{\mathbf{r}} (\psi_G^R \nabla^2 \phi_\lambda - \phi_\lambda \nabla^2 \psi_G^R) \\ \int_{S_{d-1}} \vec{\mathbf{G}} \cdot d\vec{\mathbf{a}} &= \int d^d \vec{\mathbf{r}} (\psi_G^R \nabla^2 \phi_\lambda - \phi_\lambda \nabla^2 \psi_G^R) \\ \implies 0 &= \int d^d \vec{\mathbf{r}} (\psi_G^R \nabla^2 \phi_\lambda - \phi_\lambda \nabla^2 \psi_G^R)\end{aligned}$$

From the above manipulations, we can see that,

$$\begin{aligned}
 \int d^d \vec{r} (\psi_G^R \nabla^2 \phi_\lambda - \phi_\lambda \nabla^2 \psi_G^R) &= 0 \\
 \int d^d \vec{r} \left(\psi_G^R \lambda \phi_\lambda - \phi_\lambda \frac{1}{R} \frac{\partial \psi_G^R}{\partial R} \right) &= 0 \\
 \implies \lambda \int d^d \vec{r} (\psi_G^R \phi_\lambda) &= \frac{1}{R} \frac{\partial}{\partial R} \int d^d \vec{r} \phi_\lambda \psi_G^R \\
 \implies \int d^d \vec{r} \phi_\lambda \psi_G^R &= C \exp\left(\frac{\lambda R^2}{2}\right)
 \end{aligned}$$

Good, so now we have got an expression for the gaussian average. All that is left is to figure out the parameter C . For this we note that the Gaussian distribution approaches a Dirac delta function as $R \rightarrow 0$, and then write,

$$\begin{aligned}
 \int d^d \vec{r} \phi_\lambda \lim_{R \rightarrow 0} \psi_G^R(\vec{r} - \vec{r}_0) &= C \lim_{R \rightarrow 0} \exp\left(\frac{\lambda R^2}{2}\right) \\
 \int d^d \vec{r} \phi_\lambda \delta(\vec{r} - \vec{r}_0) &= C \\
 \implies C &= \phi_\lambda(\vec{r}_0) \\
 \implies \boxed{\int d^d \vec{r} \phi_\lambda(\vec{r}) \psi_G^R(\vec{r} - \vec{r}_0) = \phi_\lambda(\vec{r}_0) \exp\left(\frac{\lambda R^2}{2}\right)}
 \end{aligned}$$

Part (b)

$$\begin{aligned}
 \implies \int d^d \vec{r} \phi_\lambda(\vec{r}) \frac{1}{(2\pi)^{d/2} R^d} \exp\left(-\frac{|\vec{r} - \vec{r}_0|^2}{2R^2}\right) &= \phi_\lambda(\vec{r}_0) \exp\left(\frac{\lambda R^2}{2}\right) \\
 \implies \int d^d \vec{r} \phi_\lambda(\vec{r}) \frac{1}{(2\pi)^{d/2} R^d} \exp\left(-\frac{r^2}{2R^2}\right) \exp\left(-\frac{r_0^2}{2R^2}\right) \exp\left(\frac{\vec{r}_0 \cdot \vec{r}}{R^2}\right) &= \phi_\lambda(\vec{r}_0) \exp\left(\frac{\lambda R^2}{2}\right) \\
 \implies \int d^d \vec{r} \phi_\lambda(\vec{r}) \psi_G^R(\vec{r}) \exp\left(-\frac{r_0^2}{2R^2}\right) \exp\left(\frac{\vec{r}_0 \cdot \vec{r}}{R^2}\right) &= \phi_\lambda(\vec{r}_0) \exp\left(\frac{\lambda R^2}{2}\right)
 \end{aligned}$$

Taking averages over \vec{r}_0 and noting that $\kappa = \frac{r}{R^2}$,

$$\int d^d \vec{r} \phi_\lambda(\vec{r}) \psi_G^R(\vec{r}) \exp\left(-\frac{r_0^2}{2R^2}\right) I_0\left(d; \frac{r}{R^2} r_0\right) = \langle \phi_\lambda(\vec{r}_0) \rangle \exp\left(\frac{\lambda R^2}{2}\right)$$

Now, let's make the substitution, $r_0 = \kappa R^2$ and $\phi_\lambda(r) = J_0(d; \kappa r) \implies \lambda = -\kappa^2$

$$\begin{aligned}
 \int d^d \vec{r} J_0(d; \kappa r) \psi_G^R(\vec{r}) \exp\left(-\frac{\kappa^2 R^2}{2}\right) I_0(d; \kappa r) &= \langle J_0(d; \kappa r_0) \rangle \exp\left(-\frac{\kappa^2 R^2}{2}\right) \\
 \int d^d \vec{r} J_0(d; \kappa r) I_0(d; \kappa r) \psi_G^R(\vec{r}) &= J_0(d; \kappa R^2) \exp\left(\frac{(\kappa^2 - \kappa'^2) R^2}{2}\right)
 \end{aligned}$$

Similarly, taking $\phi_\lambda(r) = I_0(d; \kappa' r) \implies \lambda = \kappa'^2$,

$$\begin{aligned}
 \int d^d \vec{r} I_0(d; \kappa' r) \psi_G^R(\vec{r}) \exp\left(-\frac{\kappa'^2 R^2}{2}\right) I_0(d; \kappa r) &= \langle I_0(d; \kappa' r_0) \rangle \exp\left(\frac{\kappa'^2 R^2}{2}\right) \\
 \int d^d \vec{r} I_0(d; \kappa' r) I_0(d; \kappa r) \psi_G^R(\vec{r}) &= I_0(d; \kappa' \kappa R^2) \exp\left(\frac{(\kappa^2 + \kappa'^2) R^2}{2}\right)
 \end{aligned}$$

Now, let's make the substitution, $r_0 = ik_2 R^2$ and $\phi_\lambda(r) = J_0(d; k_1 r) \implies \lambda = -k_1^2$,

$$\begin{aligned} \int d^d \vec{r} J_0(d; k_1 r) \psi_G^R(\vec{r}) \exp\left(\frac{k_2^2 R^2}{2}\right) I_0(d; ik_2 r) &= \langle J_0(d; ik_1 k_2 R^2) \rangle \exp\left(-\frac{k_1^2 R^2}{2}\right) \\ \int d^d \vec{r} J_0(d; k_1 r) J_0(d; k_2 r) \psi_G^R(\vec{r}) &= I_0(d; k_1 k_2 R^2) \exp\left(-\frac{(k_1^2 + k_2^2) R^2}{2}\right) \end{aligned}$$

Part (c)

Part (d)

Consider,

$$I_1 = \int_0^\infty \frac{dR}{R^{2\Delta+1}} \exp\left(-\frac{r^2}{2R^2}\right)$$

$$\text{Substitute } R^2 = \frac{r^2}{2u} \implies du = -\frac{r^2}{R^3} dR \implies dR = -\frac{r du}{(2u)^{3/2}},$$

$$\begin{aligned} I_1 &= \int_0^\infty \frac{r du}{(2u)^{3/2}} \left(\frac{2u}{r^2}\right)^{\Delta+1/2} \exp(-u) \\ &= \frac{2^{\Delta-1}}{r^{2\Delta}} \int_0^\infty du u^{\Delta-1} \exp(-u) \\ I_1 &= \frac{2^{\Delta-1}}{r^{2\Delta}} \Gamma(\Delta) \quad ; \quad \text{for } \Delta > 0 \end{aligned}$$

Now consider,

$$I_2 = \int_0^\infty \frac{dR}{R^{2\Delta+1}} (2\pi)^{d/2} R^d \exp\left(-\frac{k^2 R^2}{2}\right)$$

$$\text{Substitute } \frac{k^2 R^2}{2} = u \implies du = k^2 R dR, R = \left(\frac{2u}{k^2}\right)^{1/2},$$

$$\begin{aligned} I_2 &= \int_0^\infty \frac{du}{k^2} \frac{1}{R^{2\Delta+2}} (2\pi)^{d/2} \left(\frac{2u}{k^2}\right)^{d/2} \exp(-u) \\ &= \int_0^\infty \frac{du}{k^2} \left(\frac{k^2}{2u}\right)^{\Delta+1} (2\pi)^{d/2} \left(\frac{2u}{k^2}\right)^{d/2} \exp(-u) \\ &= \frac{2^{d-\Delta-1} \pi^{d/2}}{k^{2\Delta-d}} \int_0^\infty du u^{d/2-\Delta-1} \exp(-u) \\ I_2 &= \frac{2^{d-\Delta-1} \pi^{d/2}}{k^{2\Delta-d}} \Gamma(d/2 - \Delta) \quad ; \quad \text{for } \frac{d}{2} > \Delta \end{aligned}$$

From Gaussian averaging, we know that,

$$\begin{aligned} \int d^d \vec{r} \phi_{-k^2}(\vec{r}) \psi_G^R(\vec{r} - \vec{r}_0) &= \phi_{-k^2}(\vec{r}_0) \exp\left(-\frac{k^2 R^2}{2}\right) \\ \int d^d \vec{r} \phi_{-k^2}(\vec{r}) \int_0^\infty \frac{dR}{R^{2\Delta+1}} (2\pi)^{d/2} R^d \psi_G^R(\vec{r} - \vec{r}_0) &= \phi_{-k^2}(\vec{r}_0) \int_0^\infty \frac{dR}{R^{2\Delta+1}} (2\pi)^{d/2} R^d \exp\left(-\frac{k^2 R^2}{2}\right) \\ \int d^d \vec{r} \phi_{-k^2}(\vec{r}) \frac{2^{\Delta-1}}{|\vec{r} - \vec{r}_0|^{2\Delta}} \Gamma(\Delta) &= \phi_{-k^2}(\vec{r}_0) \frac{2^{d-\Delta-1} \pi^{d/2}}{k^{2\Delta-d}} \Gamma(d/2 - \Delta) \\ \implies \int d^d \vec{r} \phi_{-k^2}(\vec{r}) \frac{\Gamma(\Delta)}{|\vec{r} - \vec{r}_0|^{2\Delta}} &= \phi_{-k^2}(\vec{r}_0) \frac{2^{d-2\Delta} \pi^{d/2}}{k^{2\Delta-d}} \Gamma(d/2 - \Delta) \end{aligned}$$

Problem 2

Part (a)

We know,

$$\begin{aligned}
 I_0(d; x) &= \sum_{m=0}^{\infty} \frac{\Gamma(d/2)}{m! \Gamma(d/2 + m)} \frac{x^{2m}}{2^{2m}} \\
 \frac{x^2}{d^2} I_0(d+2; x) &= \sum_{m=0}^{\infty} \frac{\Gamma(d/2 + 1)}{m! \Gamma(d/2 + m + 1)} \frac{x^{2m+2}}{2^{2m}} \\
 &= \sum_{m=0}^{\infty} \frac{\Gamma(d/2) \cdot d/2}{m! \Gamma(d/2 + m + 1)} \frac{x^{2m+2}}{2^{2m+2}} \frac{2^2}{d^2} \\
 \frac{x^2}{d^2} I_0(d+2; x) &= \frac{2}{d} \sum_{m=0}^{\infty} \frac{\Gamma(d/2)}{m! \Gamma(d/2 + m + 1)} \frac{x^{2m+2}}{2^{2m+2}}
 \end{aligned}$$

Let's proceed and take the derivative,

$$\begin{aligned}
 \frac{x}{d} \frac{d}{dx} I_0(d; x) &= \frac{x}{d} \sum_{m=0}^{\infty} \frac{\Gamma(d/2)}{m! \Gamma(d/2 + m)} \frac{d}{dx} \left(\frac{x^{2m}}{2^{2m}} \right) \\
 &= \frac{1}{d} \sum_{m=1}^{\infty} \frac{\Gamma(d/2)}{m! \Gamma(d/2 + m)} \frac{(2m)x^{2m}}{2^{2m}} \\
 &= \frac{1}{d} \sum_{m=0}^{\infty} \frac{\Gamma(d/2)}{(m+1)! \Gamma(d/2 + m + 1)} \frac{2(m+1)x^{2m+2}}{2^{2m+2}} \iff (m \rightarrow m+1) \\
 \frac{x}{d} \frac{d}{dx} I_0(d; x) &= \frac{2}{d} \sum_{m=0}^{\infty} \frac{\Gamma(d/2)}{m! \Gamma(d/2 + m + 1)} \frac{x^{2m+2}}{2^{2m+2}}
 \end{aligned}$$

Let's now consider the third part,

$$\begin{aligned}
 I_0(d-2; x) - I_0(d; x) &= \sum_{m=0}^{\infty} \left[\frac{\Gamma(d/2 - 1)}{\Gamma(d/2 + m - 1)} - \frac{\Gamma(d/2)}{\Gamma(d/2 + m)} \right] \frac{x^{2m}}{m! 2^{2m}} \\
 &= \sum_{m=1}^{\infty} \left[\frac{\Gamma(d/2 - 1)}{\Gamma(d/2 + m - 1)} - \frac{\Gamma(d/2)}{\Gamma(d/2 + m)} \right] \frac{x^{2m}}{m! 2^{2m}} \\
 &= \sum_{m=0}^{\infty} \left[\frac{\Gamma(d/2 - 1)}{\Gamma(d/2 + m)} - \frac{\Gamma(d/2)}{\Gamma(d/2 + m + 1)} \right] \frac{x^{2m+2}}{(m+1)! 2^{2m}} \\
 &= \sum_{m=0}^{\infty} \left[\frac{\Gamma(d/2)}{\Gamma(d/2 + m + 1)} \frac{(d/2 + m)}{(d/2 - 1)} - \frac{\Gamma(d/2)}{\Gamma(d/2 + m + 1)} \right] \frac{x^{2m+2}}{(m+1)! 2^{2m}} \\
 &= \sum_{m=0}^{\infty} \left[\frac{2(m+1)}{d-2} \right] \frac{\Gamma(d/2)}{\Gamma(d/2 + m + 1)} \frac{x^{2m+2}}{(m+1)! 2^{2m}} \\
 &= \frac{2}{d-2} \sum_{m=0}^{\infty} \frac{\Gamma(d/2)}{\Gamma(d/2 + m + 1)} \frac{x^{2m+2}}{m! 2^{2m}} \\
 \frac{d-2}{d} (I_0(d-2; x) - I_0(d; x)) &= \frac{2}{d} \sum_{m=0}^{\infty} \frac{\Gamma(d/2)}{m! \Gamma(d/2 + m + 1)} \frac{x^{2m+2}}{2^{2m+2}}
 \end{aligned}$$

$$\implies \boxed{\frac{x}{d} \frac{d}{dx} I_0(d; x) = \frac{x^2}{d^2} I_0(d+2; x) = \frac{d-2}{d} [I_0(d-2; x) - I_0(d; x)]}$$

Part (b)

Schaffli's contour integral is given by,

$$I_0(d; x) = \oint_{\mathcal{C}} \frac{dz}{2\pi i} \frac{\Gamma(d/2)}{z^{d/2}} \exp\left(z + \frac{x^2}{4z}\right)$$

where the contour $\mathcal{C} = C[-\infty - i0, 0+, -\infty + i0]$.

$$\begin{aligned} \frac{x}{d} \frac{d}{dx} I_0(d; x) &= \oint_{\mathcal{C}} \frac{x}{d} \frac{dz}{2\pi i} \frac{\Gamma(d/2)}{z^{d/2}} \frac{x}{2z} \exp\left(z + \frac{x^2}{4z}\right) \\ &= \frac{x^2}{d^2} \oint_{\mathcal{C}} \frac{dz}{2\pi i} \frac{(d/2) \cdot \Gamma(d/2)}{z^{(d+2)/2}} \exp\left(z + \frac{x^2}{4z}\right) \\ &= \frac{x^2}{d^2} \oint_{\mathcal{C}} \frac{dz}{2\pi i} \frac{\Gamma((d+2)/2)}{z^{(d+2)/2}} \exp\left(z + \frac{x^2}{4z}\right) \\ \Rightarrow \frac{x}{d} \frac{d}{dx} I_0(d; x) &= \frac{x^2}{d^2} I_0(d+2; x) \end{aligned}$$

$$\begin{aligned} I_0(d-2; x) - I_0(d; x) &= \oint_{\mathcal{C}} \frac{dz}{2\pi i} \left[\frac{\Gamma(d/2-1)}{z^{d/2-1}} - \frac{\Gamma(d/2)}{z^{d/2}} \right] \exp\left(z + \frac{x^2}{4z}\right) \\ &= \end{aligned}$$

Part (c)

We are given,

$$\begin{aligned} I_0(d; x) &\approx \frac{e^x}{|S^{d-1}|} \left(\frac{2\pi}{x} \right)^{\frac{d-1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\frac{d-1}{2} + n)}{(2x)^n n! \Gamma(\frac{d-1}{2} - n)} \\ \Rightarrow \frac{x}{d} \frac{d}{dx} I_0(d; x) &\approx \frac{x}{d} \left[\frac{e^x}{|S^{d-1}|} \left(\frac{2\pi}{x} \right)^{\frac{d-1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\frac{d-1}{2} + n)}{(2x)^n n! \Gamma(\frac{d-1}{2} - n)} \right] \end{aligned}$$

Problem 3**Part (a)**

Given,

$$\begin{aligned} V(z, \rho) &= \int_0^{2\pi} \frac{d\alpha}{2\pi} g(z - i\rho \cos \alpha) \\ \nabla^2 V(z, \rho) &= \int_0^{2\pi} \frac{d\alpha}{2\pi} \nabla^2 g(z - i\rho \cos \alpha) \\ &= \int_0^{2\pi} \frac{d\alpha}{2\pi} \left[\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial (g(z - i\rho \cos \alpha))}{\partial \rho} \right) + \frac{\partial^2 g(z - i\rho \cos \alpha)}{\partial z^2} \right] \\ &= \int_0^{2\pi} \frac{d\alpha}{2\pi} \left[\frac{1}{\rho} \frac{\partial \{ \rho g'(t) (-i \cos \alpha) \}}{\partial \rho} + g''(t) \right] \quad ; \quad t = z - i\rho \cos \alpha \\ &= \int_0^{2\pi} \frac{d\alpha}{2\pi} \left[-\frac{ig'(t) \cos \alpha}{\rho} - g''(t) \cos^2 \alpha + g''(t) \right] \\ &= \int_0^{2\pi} \frac{d\alpha}{2\pi} \left[-\frac{ig'(t) \cos \alpha}{\rho} + g''(t) \sin^2 \alpha \right] \end{aligned}$$

Now we proceed to integrate the first term by parts,

$$\begin{aligned}\nabla^2 V(z, \rho) &= - \left. \frac{ig'(t) \sin \alpha}{\rho} \right|_0^{2\pi} + \int_0^{2\pi} \frac{d\alpha}{2\pi} [ig''(t)i \sin \alpha \sin \alpha + g''(t) \sin^2 \alpha] \\ \Rightarrow \nabla^2 V(z, \rho) &= 0\end{aligned}$$

Hence, we have proved that V solves Laplace equation.

Part (b)

Consider,

$$\begin{aligned}g(z - i\rho \cos \alpha) &= \sum_{n=0}^{\infty} g^{(n)}(z) (-i)^n \cos^n \alpha \frac{\rho^n}{n!} \\ \int_0^{2\pi} \frac{d\alpha}{2\pi} g(z - i\rho \cos \alpha) &= \sum_{n=0}^{\infty} g^{(n)}(z) (-i)^n \frac{\rho^n}{n!} \int_0^{2\pi} \frac{d\alpha}{2\pi} \cos^n \alpha \\ \int_0^{2\pi} \frac{d\alpha}{2\pi} \cos^n \alpha &= 0 \quad \text{for } n = 1, 3, 5, \dots \quad ; \quad \int_0^{2\pi} \frac{d\alpha}{2\pi} \cos^n \alpha = \frac{1}{2^n} {}^nC_{n/2} \quad \text{for } n = 0, 2, 4, \dots \\ \Rightarrow V(g, z) &= \sum_{m=0}^{\infty} g^{(2m)}(z) (-1)^m \frac{\rho^{2m}}{(2m)!} \frac{1}{2^{2m}} {}^{2m}C_m\end{aligned}$$

But, $g(z) = V(z, 0) \Rightarrow g^{(n)}(z) = V^{(n)}(z, 0)$. Hence, we write,

$$V(g, z) = V(z, 0) - \frac{\rho^2}{4} V^{(2)}(z, 0) + \frac{\rho^4}{64} V^{(4)}(z, 0) + \dots$$

Part (c)

$$\begin{aligned}V(g, z) &= \sum_{m=0}^{\infty} (-1)^m \frac{\rho^{2m}}{(2m)!} \frac{1}{2^{2m}} {}^{2m}C_m \frac{\partial^{2m}}{\partial z^{2m}} V(z, 0) \\ &= \sum_{m=0}^{\infty} (-1)^m \frac{1}{(m!)^2} \left(\frac{1}{2} \rho \frac{\partial}{\partial z} \right)^{2m} V(z, 0) \\ V(g, z) &= J_0 \left(d = 2; \rho \frac{\partial}{\partial z} \right) V(z, 0)\end{aligned}$$

Consider,

$$\begin{aligned}E_z(z, \rho) &= - \frac{\partial V(z, \rho)}{\partial z} = - \sum_{m=0}^{\infty} (-1)^m \frac{1}{(m!)^2} \left(\frac{1}{2} \rho \frac{\partial}{\partial z} \right)^{2m} \frac{\partial V(z, 0)}{\partial z} = J_0 \left(d = 2; \rho \frac{\partial}{\partial z} \right) E_z(z, 0) \\ E_\rho(z, \rho) &= - \frac{\partial V(z, \rho)}{\partial \rho} = - \rho \frac{1}{\rho} \frac{\partial}{\partial \rho} J_0 \left(d = 2; \rho \frac{\partial}{\partial z} \right) V(z, 0) = - \frac{\rho}{2} J_0 \left(d = 4; \rho \frac{\partial}{\partial z} \right) \frac{\partial^2 V}{\partial z^2} = \frac{\rho}{2} J_0 \left(d = 4; \rho \frac{\partial}{\partial z} \right) \frac{\partial E_z}{\partial z}\end{aligned}$$

Part (d)

$$\begin{aligned}
V_l(z, \rho) &= J_0 \left(\rho \frac{\partial}{\partial z} \right) \\
&= \sum_{m=0}^{\infty} (-1)^m \frac{1}{(m!)^2} \left(\frac{1}{2} \rho \frac{\partial}{\partial z} \right)^{2m} \frac{z^l}{l!} \\
&= \sum_{m=0}^{\infty} (-1)^m \frac{1}{(m!)^2} \frac{\rho^{2m}}{2^{2m}} \frac{l!}{(l-2m)!} \frac{z^{l-2m}}{l!} \quad \text{if } l \geq 2m \quad \text{else} \quad = 0 \\
V_l(z, \rho) &= \sum_{m=0}^{\infty} (-1)^m \frac{1}{(m!)^2} \frac{\rho^{2m}}{2^{2m}} \frac{z^{l-2m}}{(l-2m)!}
\end{aligned}$$

$$V_1(z, \rho) = z \quad \text{and} \quad V_2(\rho, z) = \frac{z^2}{2} - \frac{\rho^2}{4}$$