Calculus: Homework #2

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Aditya Vijaykumar

Problem 1

Show that commutators in quantum mechanics and Poisson brackets in classical mechanics both obey the Jacobi identity.

$$[A, [B, C]] + [C, [A, B]] + [B, [C, A]] = 0.$$

$$\{A, \{B, C\}_{PB}\}_{PB} + \{C, \{A, B\}_{PB}\}_{PB} + \{B, \{C, A\}_{PB}\}_{PB} = 0$$

Solution
We solve each part separately.

Part One - Commutators
We expand out each term as follows

$$[A, [B, C]] = ABC - ACB - BCA + CBA$$

$$[C, [A, B]] = CAB - CBA - ABC + BAC$$

$$[B, [C, A]] = BCA - BAC - CAB + ACB$$

Adding the three expressions above, we arrive at the expression

$$[A, [B, C]] + [C, [A, B]] + [B, [C, A]] = 0.$$

Hence Proved.
Part One - TO BE DONE

Problem 1

Show that

$$[AB, CD] = -AC\{D, B\} + A\{C, B\}D - C\{D, A\}B + \{C, A\}DB$$

Solution

$$\begin{aligned}
[AB, CD] &= A[B, CD] + [A, CD]B \\
&= A[B, C]D + AC[B, D] + C[A, D]B + [A, C]DB \\
&= A(\{B, C\} - 2CB)D + AC(2BD - \{B, D\}) + C(2AD - \{A, D\}) + (\{A, C\} - 2CA)DB \\
&= A\{B, C\}D - 2ACBD + 2ACBD - AC\{B, D\} + 2CADB - C\{A, D\}B + \{A, C\}DB - 2CADB \\
&= -AC\{B, D\} + A\{B, C\}D - C\{A, D\}B + \{A, C\}DB \\
&= -AC\{D, B\} + A\{C, B\}D - C\{D, A\}B + \{C, A\}DB
\end{aligned}$$

Hence Proved.

Problem 3

Let $\vec{\mathbf{n}} = n_x\hat{\mathbf{x}} + n_y\hat{\mathbf{y}} + n_z\hat{\mathbf{z}}$. $\vec{\sigma} \cdot \vec{\mathbf{n}} = n_x\sigma_x + n_y\sigma_y + n_z\sigma_z$. Find its eigenvalues and eigenvectors.

Problem 4

Let A be an observable whose spectral decomposition is$A = \sum_i \lambda_i P_i$. What is the significance of

$$\prod_{i \neq j} \frac{A - \lambda_i}{\lambda_j - \lambda_i}$$

where note that the product runs only over i and j is held fixed

Problem 5

Show that commutators in quantum mechanics and Poisson brackets in classical mechanics both obey the Jacobi identity.

$$[A, [B, C]] + [C, [A, B]] + [B, [C, A]] = 0.$$

$$\{A, \{B, C\}_{PB}\}_{PB} + \{C, \{A, B\}_{PB}\}_{PB} + \{B, \{C, A\}_{PB}\}_{PB} = 0$$

1. $f(n) = n^2 + n + 1$, $g(n) = 2n^3$
2. $f(n) = n\sqrt{n} + n^2$, $g(n) = n^2$
3. $f(n) = n^2 - n + 1$, $g(n) = n^2/2$

Solution
We solve each solution algebraically to determine a possible constant c.

Part One

$$\begin{aligned} n^2 + n + 1 &= \\ &\leq n^2 + n^2 + n^2 \\ &= 3n^2 \\ &\leq c \cdot 2n^3 \end{aligned}$$

Thus a valid c could be when $c = 2$.

Part Two

$$\begin{aligned} n^2 + n\sqrt{n} &= \\ &= n^2 + n^{3/2} \\ &\leq n^2 + n^{4/2} \\ &= n^2 + n^2 \\ &= 2n^2 \\ &\leq c \cdot n^2 \end{aligned}$$

Thus a valid c is $c = 2$.

Problem 5 continued on next page. . .

Part Three

$$
\begin{aligned}
n^2 - n + 1 &= \\
&\leq n^2 \\
&\leq c \cdot n^2/2
\end{aligned}
$$

Thus a valid c is $c = 2$.