

Advanced Quantum Mechanics: Assignment #2

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Problem 1

Let's use the following convention ($|l, m\rangle$)

$$|2, 2\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad |2, 1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad |2, 0\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad |2, -1\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |2, -2\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

We know that,

$$J_3 |l, m\rangle = m |l, m\rangle \quad \text{and} \quad J_{\pm} |l, m\rangle = \sqrt{(l \mp m)(l \pm m + 1)} |l, m \pm 1\rangle$$

$$J_3 |2, 2\rangle = 2 |2, 2\rangle, \quad J_3 |2, 1\rangle = |2, 1\rangle, \quad J_3 |2, 0\rangle = 0, \quad J_3 |2, -1\rangle = -|2, -1\rangle, \quad J_3 |2, -2\rangle = -2 |2, -2\rangle$$

Hence, we can see that,

$$J_3 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}$$

We also note that,

$$J_+ |2, 2\rangle = 0, \quad J_+ |2, 1\rangle = 2 |2, 2\rangle, \quad J_+ |2, 0\rangle = \sqrt{6} |2, 1\rangle, \quad J_+ |2, -1\rangle = \sqrt{6} |2, 0\rangle, \quad J_+ |2, -2\rangle = 2 |2, -1\rangle$$

$$J_- |2, 2\rangle = 2 |2, 1\rangle, \quad J_- |2, 1\rangle = \sqrt{6} |2, 0\rangle, \quad J_- |2, 0\rangle = \sqrt{6} |2, -1\rangle, \quad J_- |2, -1\rangle = 2 |2, -2\rangle, \quad J_- |2, -2\rangle = 0$$

So,

$$J_+ = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad J_- = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

Hence,

$$J_1 = \frac{J_+ + J_-}{2} = \frac{1}{2} \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & \sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & \sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix} \quad \text{and} \quad J_2 = \frac{J_+ - J_-}{2i} = \frac{i}{2} \begin{pmatrix} 0 & -2 & 0 & 0 & 0 \\ 2 & 0 & -\sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & -\sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & -2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

Problem 2

Problem 3

We know that,

$$J_1 = \frac{J_+ + J_-}{2} \quad \text{and} \quad J_2 = \frac{J_+ - J_-}{2i} \quad \text{and} \quad J_{\pm} |l, m\rangle = \sqrt{(l \mp m)(l \pm m + 1)} |l, m \pm 1\rangle$$

As successive action of the form $J_{\pm}^{\alpha} |l, m\rangle$ with integral $\alpha > 0$ takes a state to one with higher/lower m , we can see that $\langle l, m | J_{\pm}^{\alpha} |l, m\rangle = 0$. Lets consider $\langle J_1 \rangle$,

$$\begin{aligned} \langle J_1 \rangle &= \langle l, m | J_1 |l, m\rangle \\ &= \frac{1}{2} (\langle l, m | J_+ |l, m\rangle + \langle l, m | J_- |l, m\rangle) \\ &= 0 \end{aligned}$$

Similarly for $\langle J_2 \rangle$,

$$\begin{aligned} \langle J_2 \rangle &= \langle l, m | J_2 |l, m\rangle \\ &= \frac{1}{2i} (\langle l, m | J_+ |l, m\rangle - \langle l, m | J_- |l, m\rangle) \\ &= 0 \end{aligned}$$

Consider $\langle J_1^2 \rangle$ and $\langle J_2^2 \rangle$,

$$\begin{aligned} \langle J_1^2 \rangle &= \frac{1}{4} (\langle J_+^2 \rangle + \langle J_-^2 \rangle + \{J_+, J_-\}) = \frac{1}{4} \{J_+, J_-\} \quad \text{and} \\ \langle J_2^2 \rangle &= \frac{1}{-4} (\langle J_+^2 \rangle + \langle J_-^2 \rangle - \{J_+, J_-\}) = \frac{1}{4} \{J_+, J_-\} \implies \langle J_1^2 \rangle = \langle J_2^2 \rangle \end{aligned}$$

We know,

$$\begin{aligned} \langle J^2 \rangle &= l(l+1) \\ \langle J_1^2 \rangle + \langle J_2^2 \rangle + \langle J_3^2 \rangle &= l(l+1) \\ 2\langle J_1^2 \rangle + m^2 &= l(l+1) \\ \langle J_1^2 \rangle = \langle J_2^2 \rangle &= \frac{l(l+1) - m^2}{2} \end{aligned}$$

Problem 4

Let's use the following convention ($|l, m\rangle$)

$$|1, 1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad |1, 0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |1, -1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

We know that $J_{\pm} |1, m\rangle = \sqrt{(1 \mp m)(1 \pm m + 1)} |1, m \pm 1\rangle$, which means,

$$\begin{aligned} J_+ |1, 1\rangle &= 0 \quad \text{and} \quad J_+ |1, 0\rangle = \sqrt{2} |1, 1\rangle \quad \text{and} \quad J_+ |1, -1\rangle = \sqrt{2} |1, 0\rangle \\ J_- |1, 1\rangle &= \sqrt{2} |1, 0\rangle \quad \text{and} \quad J_- |1, 0\rangle = \sqrt{2} |1, -1\rangle \quad \text{and} \quad J_- |1, -1\rangle = 0 \end{aligned}$$

Using the above relations, one can write J_+ and J_- as follows,

$$J_+ = \sqrt{2} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad J_- = \sqrt{2} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \implies J_2 = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

Thus, we have obtained J_2 . Let's also note the following,

$$J_2^2 = \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad J_2^3 = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = J_2$$

$$J_2^4 = J_2^3 J_2 = J_2^2$$

We can see a pattern above, which can be written in a concise form as,

$$J_2^{2n-1} = J_2 \quad \text{and} \quad J_2^{2n} = J_2^2$$

where $n = 1, 2, 3, \dots$. Consider $e^{iJ_2\beta}$,

$$\begin{aligned} e^{-iJ_2\beta} &= \sum_{n=0}^{\infty} \frac{(-i)^n \beta^n J_2^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(-i)^{2n} \beta^{2n} J_2^{2n}}{2n!} + \sum_{n=1}^{\infty} \frac{(-i)^{2n-1} \beta^{2n-1} J_2^{2n-1}}{(2n-1)!} \\ &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n \beta^{2n} J_2^2}{2n!} + i \sum_{n=1}^{\infty} \frac{(-1)^n \beta^{2n-1} J_2}{(2n-1)!} \\ &= 1 + (1 - \cos \beta) J_2^2 - i J_2 \sin \beta \end{aligned}$$

Problem 5