Advanced Quantum Mechanics: Assignment #4

Due on 8th November, 2018

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Problem 1

Problem 2

Part (a)
From the form of the Hamiltonian, we can see that the energy will have the form,

$$E_{n_x,n_y} = (n_x + 0.5 + n_y + 0.5)\omega = (n_x + n_y + 1)\omega$$

The three lowest lying states are,

$$\begin{aligned} n_x = 0 \quad &, \quad n_y = 0 \implies E_{00}^{(0)} = \omega \\ n_x = 1 \quad &, \quad n_y = 0 \implies E_{10}^{(0)} = 2\omega \\ n_x = 0 \quad &, \quad n_y = 1 \implies E_{01}^{(0)} = 2\omega \end{aligned}$$

Part (b)
Let's denote states by $|n_x n_y\rangle$. x and y can be written in terms of corresponding creation and annhilation operators as follows,

$$x = \frac{1}{\sqrt{2m\omega}}(a_x + a_x^\dagger) \quad \text{and} \quad y = \frac{1}{\sqrt{2m\omega}}(a_y + a_y^\dagger)$$

The perturbation is $V = \lambda m\omega^2 xy$. Consider $\langle q_x q_y|V|n_x n_y\rangle$

$$\begin{aligned} \langle q_x q_y|V|n_x n_y\rangle &= \lambda m\omega^2 \big(\langle q_x q_y|a_x a_y|n_x n_y\rangle + \langle q_x q_y|a_x a_y^\dagger|n_x n_y\rangle + \langle q_x q_y|a_x^\dagger a_y^\dagger|n_x n_y\rangle + \langle q_x q_y|a_x^\dagger a_y|n_x n_y\rangle \big) \\ &= \lambda m\omega^2 (\sqrt{n_x n_y}\delta_{q_x,n_x-1}\delta_{q_y,n_y-1} + \sqrt{n_x(n_y+1)}\delta_{q_x,n_x-1}\delta_{q_y,n_y+1} \\ &\quad + \sqrt{(n_x+1)(n_y+1)}\delta_{q_x,n_x+1}\delta_{q_y,n_y+1} + \sqrt{(n_x+1)(n_y)}\delta_{q_x,n_x+1}\delta_{q_y,n_y-1}) \end{aligned}$$

$$\implies \langle n_x n_y|V|n_x n_y\rangle = 0 \implies E_{n_x n_y}^{(1)} = 0$$

This means that there will be no energy shift at the first order in λ for any state under consideration. We now proceed to calculate $\left|n_x n_y^{(1)}\right\rangle$,

$$\begin{aligned} \left|00^{(1)}\right\rangle &= \sum_{(q_x,q_y)\neq(0,0)} \frac{\langle q_x q_y|V|00\rangle}{E_{00}^{(0)} - E_{q_x q_y}^{(0)}} |q_x q_y\rangle \\ &= \lambda m\omega^2 \sum_{(q_x,q_y)\neq(0,0)} \frac{\delta_{q_x,1}\delta_{q_y,1}}{E_{00}^{(0)} - E_{q_x q_y}^{(0)}} |q_x q_y\rangle \\ \left|00^{(1)}\right\rangle &= -\frac{\lambda m\omega}{2} |11\rangle \end{aligned}$$

$$
\begin{aligned}
\left|10^{(1)}\right\rangle &= \sum_{(q_x,q_y)\neq(1,0)} \frac{\langle q_x q_y|V|10\rangle}{E^{(0)}_{10} - E^{(0)}_{q_x q_y}} |q_x q_y\rangle \\
&= \lambda m\omega^2 \sum_{(q_x,q_y)\neq(0,0)} \frac{\delta_{q_x,1}\delta_{q_y,1}}{E^{(0)}_{00} - E^{(0)}_{q_x q_y}} |q_x q_y\rangle \\
\left|10^{(1)}\right\rangle &= -\frac{\lambda m\omega}{2} |11\rangle
\end{aligned}
$$

Problem 3

Problem 4

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Problem 5

Let $L^2 = L_x^2 + L_y^2 + L_z^2$. We work in the basis of states $|l,m\rangle$ such that $L^2|l,m\rangle = l(l+1)|l,m\rangle$ and $L_z|l,m\rangle = m|l,m\rangle$. The Hamiltonian then is,

$$H = H_0 + \lambda V = AL^2 + BL_z + \lambda C L_y$$

The eigenstates of H_0 are,

$$H_0|l,m\rangle = (Al(l+1) + Bm)|l,m\rangle = E_{lm}|l,m\rangle$$

The first order energy shift is given by,

$$
\begin{aligned}
\Delta^{(1)} &= \langle l,m|V|l,m\rangle \\
&= C\langle l,m|L_y|l,m\rangle \\
&= \frac{C}{2i}\langle l,m|L_+ - L_-|l,m\rangle \\
\Delta^{(1)} &= 0
\end{aligned}
$$

So we need to find out the higher order energy shifts.

$$
\begin{aligned}
\left|\psi^{(1)}_{lm}\right\rangle &= \frac{C}{2i}\sum_{l'm'\neq lm} \frac{\langle l'm'|L_+ - L_-|lm\rangle}{E_{lm} - E_{l'm'}} |l'm'\rangle \\
&= \frac{C}{2i}\sum_{l'm'\neq lm} \frac{\sqrt{(l-m)(l+m+1)}\delta_{l,l'}\delta_{m+1,m'}|l'm'\rangle - \sqrt{(l+m)(l-m+1)}\delta_{l,l'}\delta_{m-1,m'}|l'm'\rangle}{E_{lm} - E_{l'm'}} \\
\left|\psi^{(1)}_{lm}\right\rangle &= \frac{Ci}{2B}\left[\sqrt{(l-m)(l+m+1)}|l,m+1\rangle + \sqrt{(l+m)(l-m+1)}|l,m-1\rangle\right]
\end{aligned}
$$

$$
\begin{aligned}
\Delta^{(2)} &= \frac{C}{2i}\left\langle l,m\middle|L_+ - L_-\middle|\psi^{(1)}_{lm}\right\rangle \\
&= -\frac{C^2}{4B}\left[\sqrt{(l-m)(l+m+1)(l+m-1)(l-m+2)} + \sqrt{(l+m)(l-m+1)(l-m+1)(l+m)}\right]
\end{aligned}
$$

Problem 6

The Hamiltonian to deal with is,

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2$$

and the given trial wavefunction is,

$$\psi_\beta(x) = Ne^{-\beta|x|}$$

where N is some normalization. Let's calculate $\langle\psi_\beta|H|\psi_\beta\rangle$,

$$\begin{aligned}
\langle\psi_\beta|H|\psi_\beta\rangle &= N^2 \int_{-\infty}^{\infty} e^{-2\beta|x|}\left(-\frac{1}{2m}\beta^2 + \frac{m\omega^2}{2}x^2\right)dx \\
&= 2N^2 \int_0^{\infty} e^{-2\beta x}\left(-\frac{1}{2m}\beta^2 + \frac{m\omega^2}{2}x^2\right)dx \\
&= 2N^2\left[\int_0^{\infty} e^{-2\beta x}\left(-\frac{1}{2m}\right)\beta^2 dx + \int_0^{\infty}\frac{m\omega^2}{2}x^2 dx\right] \\
&= 2N^2\left[-\frac{\beta}{4m} + \frac{m\omega^2}{8\beta^3}\right]
\end{aligned}$$

$$\begin{aligned}
N^2 \int_{-\infty}^{\infty} e^{-2\beta|x|} &= 1 \\
N^2 &= \beta
\end{aligned}$$

$$\langle\psi_\beta|H|\psi_\beta\rangle = -\frac{\beta^2}{2m} + \frac{m\omega^2}{4\beta^2}$$