Classical Mechanics: Assignment #6

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Problem 1

Liouville Theorem states that in a Hamiltonian system, the phase space density is constant in time.
Let our system consist of N points (q_k, p_k) in a $2N$ dimensional phase space.

Problem 2

Transformations of coordinates $(q, p, t) \rightarrow (Q, P, t)$ which preserves the form of Hamilton's equations are called canonical transformations. So, by definition,

$$\dot{p} = \frac{\partial H}{\partial q} \quad , \quad \dot{q} = -\frac{\partial H}{\partial p} \quad \text{and} \quad \dot{P} = \frac{\partial K}{\partial Q} \quad , \quad \dot{Q} = -\frac{\partial K}{\partial P}$$

The definition also implies that,

$$\delta(p\dot{q} - H) = 0 \quad \text{and} \quad \delta(P\dot{Q} - K) = 0$$
$$\lambda(p\dot{q} - H) = P\dot{Q} - K + \frac{\mathrm{d}F}{\mathrm{d}t}$$

We deal with the $\lambda = 1$ case. The $\frac{\mathrm{d}F}{\mathrm{d}t}$ term comes from the fact that Lagrangians are not unique and we can always add a total time derivative term without changing the equations of motion. If the above condition is satisfied, the transformation $(q, p, t) \rightarrow (Q, P, t)$ is guaranteed to be canonical, and the function F is called a generating function. We deal with four classes of generating functions case-by-case,

- $F = F_1(q, Q, t)$,

$$p\dot{q} - H = P\dot{Q} - K + \frac{\mathrm{d}F_1}{\mathrm{d}t} = P\dot{Q} - K + \frac{\partial F_1}{\partial q}\dot{q} + \frac{\partial F_1}{\partial Q}\dot{Q} + \frac{\partial F_1}{\partial t}$$

  As q and Q are independent, the coefficients should vanishh independently, such that $K = H + \frac{\partial F_1}{\partial t}$. This implies,

$$\frac{\partial F_1}{\partial q} = p \quad \text{and} \quad \frac{\partial F_1}{\partial Q} = -P$$

- $F = F_2(q, P, t) - QP$,

$$p\dot{q} - H = P\dot{Q} - K + \frac{\mathrm{d}F_2}{\mathrm{d}t} - \frac{\mathrm{d}(QP)}{\mathrm{d}t} = P\dot{Q} - K + \frac{\partial F_2}{\partial q}\dot{q} + \frac{\partial F_2}{\partial P}\dot{P} + \frac{\partial F_2}{\partial t} - P\dot{Q} - Q\dot{P}$$

$$\implies \frac{\partial F_2}{\partial q} = p \quad \text{and} \quad \frac{\partial F_2}{\partial P} = Q$$

- $F = F_3(p, Q, t) + qp$,

$$p\dot{q} - H = P\dot{Q} - K + \frac{\mathrm{d}F_3}{\mathrm{d}t} + \frac{\mathrm{d}(qp)}{\mathrm{d}t} = P\dot{Q} - K + \frac{\partial F_3}{\partial Q}\dot{Q} + \frac{\partial F_3}{\partial p}\dot{p} + \frac{\partial F_3}{\partial t} + p\dot{q} + q\dot{p}$$

$$\implies \frac{\partial F_3}{\partial Q} = -P \quad \text{and} \quad \frac{\partial F_2}{\partial p} = -q$$

- $F = F_4(p, P, t) + qp - QP$,

$$p\dot{q} - H = P\dot{Q} - K + \frac{\mathrm{d}F_4}{\mathrm{d}t} + \frac{\mathrm{d}(qp - QP)}{\mathrm{d}t} = P\dot{Q} - K + \frac{\partial F_4}{\partial P}\dot{P} + \frac{\partial F_4}{\partial p}\dot{p} + \frac{\partial F_4}{\partial t} + p\dot{q} + q\dot{p} - P\dot{Q} - Q\dot{P}$$

$$\implies \frac{\partial F_4}{\partial P} = Q \quad \text{and} \quad \frac{\partial F_4}{\partial p} = -q$$

Problem 3

We are given the Hamiltonian and generating function,

$$H = \frac{p^2}{2} + \frac{\omega^2 x^2}{2} + \alpha x^3 + \beta x p^2 \quad \text{and} \quad \phi = xP + ax^2 P + bP^3$$

$\phi = \phi(x, P)$. For ϕ to be a canonical transformation,

$$\frac{\partial \phi}{\partial x} = p \quad \text{and} \quad \frac{\partial \phi}{\partial P} = Q$$

$$\implies P + 2axP = p \quad \text{and} \quad x + ax^2 + 3bP^2 = Q$$

$$\implies P = \frac{p}{1 + 2ax} \quad \text{and} \quad Q = x + ax^2 + \frac{3bp^2}{(1 + 2ax)^2}$$

We know that,

$$p\dot{x} - H = P\dot{Q} - K + \frac{\mathrm{d}\phi}{\mathrm{d}t}$$

$$p\dot{x} - \frac{p^2}{2} + \frac{\omega^2 x^2}{2} + \alpha x^3 + \beta x p^2 = P\dot{Q} - K + (P + 2axP)\dot{x} + (x + ax^2 + 3bP^2)\dot{P}$$