Classical Mechanics - Assignment 6
Due date: Nov 16, 2018

Instructor: Manas Kurkarni (H-204) - `manas.kulkarni@icts.res.in`
TA: Avijit Das (H-107) - `avijit.das@icts.res.in` and
Anugu Sumith Reddy (H-104) - `anugu.reddy@icts.res.in`

November 12, 2018

Note: Submit the assignment to any one of TA's office on/before the due date. For the numerical parts of the questions take print outs of the codes along with the plots, etc. and attach it in the correct place of your solution. If you have any doubt regarding the assignment problems or the topics discussed in the class, feel free to discuss with TAs or the instructor. For numerical parts use any of your favourite programming language and plotting software. Good luck!

Q1 $\boxed{\text{25 marks}}$

Liouville's theorem

(a) State and prove the Liouville's theorem. $\boxed{5}$

(b) Take the phase space of a one dimensional harmonic oscillator and consider a rectangle in this phase space. Try to find out the shape of it at some subsequent time and verify the Liouville's theorem explicitly. Write a code for this problem and verify your answer numerically. $\boxed{10}$

(c) Repeat the numerical part for a Duffing oscillator (without dissipation and drive of course). $\boxed{10}$

Q2 $\boxed{\text{30 marks}}$

Canonical transformation and Poisson Bracket

(a) Describe briefly the concept of the canonical transformation and generating functions. Derive the table of the generating functions discussed in class (Table 9.1 of Goldstein) and show that the those generating functions are related to each other through Legendre transforms. $\boxed{10}$

(b) prove that the transformation $\boxed{10}$

$$Q_1 = q_1, P_1 = p_1 - 2p_2$$
$$Q_2 = p_2, P_2 = -2q_1 - q_2 \tag{1}$$

is canonical and find a generating function. use the symplectic approach and the Poisson bracket invariance approach.

(c) Angular momentum is defined as $\vec{l} = \vec{r} \times \vec{p}$. Prove that $[l_x, l_y] = l_z$ for all cyclic permutations of l_x, l_y, l_z. Calculate all the Poisson brackets of the components of $\vec{r}$ and $\vec{p}$ with the components of the angular momentum (for example, $[x, l_z]$, $[p_x, l_z]$, etc.). $\boxed{10}$

Q3 $\boxed{\text{30 marks}}$

Canonical Transformations

(a) Consider the small oscillations of an anharmonic oscillator with Hamiltonian $H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 x^2 + \alpha x^3 + \beta x p^2$ under the assumption that $\alpha x \ll \omega^2$, $\beta x \ll 1$. Find the parameters a and b for the canonical transformation produced by the generating function $\Phi = xP + ax^2 P + bP^3$ such that the new Hamiltonian does not contain any anharmonic terms **of order** 3 **(for example, terms like** QP^2**,** Q^3**,** $P^2 Q$ **etc)**. Determine $x(t)$. $\boxed{10}$

(b) Discuss the physical meaning of the canonical transformations produced by the following generating functions : $\boxed{20}$

$$(i)\Phi(r, P) = (r.P) + (\delta a.P);$$
$$(ii)\Phi(r, P) = (r.P) + (\delta\psi.[r \times P]);$$
$$(iii)\Phi(q, P, t) = qP + \delta\tau H(q, p, t);$$
$$(iv)\Phi(r, P) = (r.P) + (r^2 + P^2)\delta a,$$

where r is the Cartesian radius vector while $\delta a, \delta\psi, \delta\tau$ and $\delta\alpha$ are infinitesimal parameters.

Q4 $\boxed{\text{30 marks}}$

Hamilton-Jacobi Theory

(a) A particle of mass m is constrained to move on a smooth parabolic wire, $y = x^2$ by under the action of gravity (along negative y axis). Write down the Hamilton-Jacobi equation for the system and find the general solution of the equation of motion. $\boxed{10}$

(b) A particle of unit mass moves in a field given by the potential depending on distance from origin r and z coordinate, $V(r, z) = \frac{k}{r} - Fz$, where k and F are constants. Prove that Hamilton-Jacobi equation is completely separable in parabolic co-ordinates (ξ, η, ψ) given in terms of cylindrical polar co-ordinates (ρ, ϕ, z) by the following relations $z = \frac{1}{2}(\xi^2 - \eta^2), \rho = \xi\eta, \psi = \phi$. $\boxed{20}$