Dynamical Systems: Homework #1

Due on 14th January, 2019

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(**Acknowledgements** - I would like to thank Divya Jagannathan for discussions.)

Problem 1

Part (a)

Consider first the set E_u. Any vector in this said would be given by $V = \sum_i C_j w_j$, where w_j is a generalized eigenvector of this set. We know,

$$(A - \lambda I)^m w_j = 0$$

for some m. This also means that,

$$(A - \lambda I) w_j = W_j \implies A w_j = \lambda w_j + W_j$$

where $W_j \in \ker((A - \lambda I)^{m-1})$. Hence $A w_j \in E_u$. This also means that $A^k w_j \in E_u$ for any whole number k, which in turn means that, in general, $\sum_k c_k A^k w_j \in E_u$.
Consider,

$$e^{At} V = \sum_{i,k} C_j \frac{t^k A^k}{k!} w_j \in E_u$$

Hence Proved that E_u is an invariant subspace. The argument follows similarly for E_s, E_c.
If $\alpha_j, \beta_j, \gamma_j$ are all eigenvectors as defined below, the most general solution of the system in given by,

$$\sum_j C_j w_j = \sum_{j, \alpha_j \in E_u} M_j \alpha_j + \sum_{j, \beta_j \in E_s} N_j \beta_j + \sum_{j, \gamma_j \in E_c} K_j \gamma_j$$
$$\implies R^d = E_u \oplus E_s \oplus E_c$$

Part (b)

For $x_0 = \sum_j C_j w_j, w_j \in E_s$, $e^{At} x_0 = \sum_j C_j w_j e^{a_j t} e^{i b_j t}, a_j < 0$. Hence, we can say,

$$\lim_{t \to \infty} e^{At} x_0 = \sum_j C_j w_j \lim_{t \to \infty} e^{a_j t} e^{i b_j t} = 0$$

$$\lim_{t \to -\infty} \left| e^{At} x_0 \right| = \lim_{t \to -\infty} \left| \sum_j C_j w_j e^{a_j t} e^{i b_j t} \right| = \infty$$

Part (c)

For $x_0 = \sum_j C_j w_j, w_j \in E_u$, $e^{At}x_0 = \sum_j C_j w_j e^{a_j t} e^{ib_j t}, a_j > 0$. Hence, we can say,

$$\lim_{t\to-\infty} e^{At}x_0 = \sum_j C_j w_j \lim_{t\to-\infty} e^{a_j t} e^{ib_j t} = 0$$

$$\lim_{t\to\infty} \left|e^{At}x_0\right| = \lim_{t\to\infty} \left|\sum_j C_j w_j e^{a_j t} e^{ib_j t}\right| = \infty$$

Part (d)
For $x_0 = \sum_j C_j w_j, w_j \in E_s$, $e^{At}x_0 = \sum_j C_j w_j e^{a_j t} e^{ib_j t}, a_j < 0$. Hence, we can say,

Problem 2

Part (a)

Part (b)
As we are working with linear systems, any linear combination of the solutions will be linear too. We know that the columns of the fundamental matrix X are solutions to the ODE, and that they span the solution space. Right multiplying X with a constant, non-singular matrix C will give us a matrix which will have linear combination of the columns of X (ie the solutions to the ODE) according to the entries in C. Due to the condition of non-singularity, this new matrix will also have linearly independent columns. Hence, the new matrix $Y(t) = X(t)C$ will have columns which are solutions to the ODE and also span the solution space. Hence Proved.
Left multiplying X with a constant, non-singular matrix B will give us a matrix which will have linear combination of the *rows* of X (which are not the solutions to the ODE) according to the entries in B. This means that BX will not, in general, be a fundamental matrix for the system. Of course, BX can be a fundamental matrix if the rows of X are indeed solutions to the system, ie. if $X^T = X$.

Part (c)
From the fact that X_1, X_2 are fundamental matrices and from the previous part, we can make the following statement,

$$X_2(t) = X_1(t)C$$

where C is a non-singular matrix.
Consider,

$$\begin{aligned}
X_2(t+\omega) &= X_2(t)B_2 \\
X_1(t+\omega)C &= X_2(t)B_2 \\
X_1(t)B_1C &= X_1(t)CB_2 \\
\implies B_1 &= CB_2C^{-1}
\end{aligned}$$

where inverses could be taken in the last step only because the matrices are known to be nonsingular. Hence proved that B_1, B_2 are similar.