Dynamical Systems: Homework #1

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Problem 1

$$\dot{x} = x^{1/3} \implies \frac{dx}{x^{1/3}} = dt \implies \frac{2}{3}\left(x^{2/3} - x_0^{2/3}\right) = t \implies x(t) = \left(x_0^{2/3} + \frac{3}{2}t\right)^{3/2} \tag{1}$$

For $x_0 = 0$, we have $x(t) = \left(\frac{3t}{2}\right)^{3/2}$. As the function $f(y) = y^{3/2}$ is only defined for $y \geq 0$, this solution has maximum interval of existence $t \in [0, \infty)$. For $x_0 = 0$, we also have the trivial solution $x(t) = 0$ which has maximum interval of existence $t \in (-\infty, \infty)$. So there are at least two distinct solutions for $x_0 = 0$. One can also imagine patching up the above solutions at origin and forming other possible solutions.

For the case where $x_0 \neq 0$, we are only left with [1]. As $x_0^{2/3} > 0, \forall x_0 \neq 0$, the maximal interval of existence for $x(t)$ is $[0, \infty)$.

Problem 2

Part (a)

$$\dot{x} = x(x^2 - 1)$$

For $x_0 = 0, 1, -1$, $x(t) = 0, 1, -1$ respectively is a solution.

$$\begin{aligned}
\therefore \frac{dx}{x(x-1)(x+1)} &= dt \\
\therefore -\frac{dx}{x} + \frac{1}{2}\left[\frac{dx}{x+1} + \frac{dx}{x-1}\right] &= dt \\
\therefore \frac{1}{2}\log\frac{x+1}{x-1} - \log x &= -\frac{1}{2}\log\frac{x_0+1}{x_0-1} + \log x_0 + t \\
\therefore \frac{1}{2}\log\frac{x+1}{x^2(x-1)} &= -\frac{1}{2}\log\frac{x_0^2(x_0+1)}{x_0-1} + t
\end{aligned}$$

For the problem to have the above solution, one must have,

$$\frac{x_0+1}{x_0-1} > 0 \implies x_0 > 1 \quad \text{or} \quad x_0 < -1$$