Summer School on Gravitational Wave Astronomy 2019
ICTS Bangalore
Tutorial - 3

1. The Schwarzschild metric is given by,

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \; d\phi^2\right),$$

We can learn a lot about the geometry by considering geodesics in the spacetime.

a) Using the constants of motion E and L associated with Killing vectors ∂_t and ∂_ϕ, and the norm of the tangent vector to affinely-parameterized null, spacelike, or timelike geodesics denoted by $\kappa = 0, 1, -1$ respectively, show that the radial component of the geodesic equation can be recast in terms of a 1-d motion in an effective potential defined by $\dot{r}^2/2 + V_{\text{eff}}(r) = 0$, where:

$$V_{\text{eff}}(r) = -\frac{E^2 + \kappa}{2} + \frac{\kappa\, M}{r} + \frac{L^2}{2r^2} - \frac{L^2 M}{r^3}.$$

b) Show that there is a null circular orbit at $r = r_p$ and the value of r_p at which such an orbit exists. Determine whether it is stable or unstable.

c) Now consider *radial null geodesics* in the spacetime (i.e. those with $L = 0$). Sketch the light cones (delimited by ingoing and outgoing null geodesics) in the (t, r) plane.

d) Show that the event horizon at $r = 2M$ is reached by ingoing light rays in a finite affine parameter.

2. According to Hawking's theorem in a collision of two black holes the total surface area must not decrease. Consider two black holes of masses M_1 and M_2 collides to form another black hole of mass M_3. This process generates some gravitational waves which can be used to obtain work. The efficiency of this process is defined as

$$\eta = 1 - \frac{M_3}{M_1 + M_2}. \tag{1}$$

Show that the area theorem implies an upper limit on this efficiency, find the upper bound.