

Fluid Mechanics: Assignment #4

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Problem 1

In ideal 2D flow, $\nabla \cdot \vec{u} = 0$. This means,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \implies u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}$$

Hence, $\nabla \psi = \frac{\partial \psi}{\partial x} \hat{x} + \frac{\partial \psi}{\partial y} \hat{y} = -v \hat{x} + u \hat{y}$. By definition $\vec{u} = \nabla \phi = u \hat{x} + v \hat{y}$. Now we can do the calculations required in the problem,

- $\nabla \psi \cdot \nabla \phi = (-v \hat{x} + u \hat{y}) \cdot (u \hat{x} + v \hat{y}) = -vu + uv = 0$.
- $-\nabla \psi \times \nabla \phi = -(-v \hat{x} + u \hat{y}) \times (u \hat{x} + v \hat{y}) = -(-v^2 - u^2) \hat{z} = |\vec{u}|^2 \hat{z}$
- $|\nabla \psi|^2 = u^2 + v^2 \quad \text{and} \quad |\nabla \phi|^2 = u^2 + v^2 \implies |\nabla \psi|^2 = |\nabla \phi|^2$
- $-\hat{z} \times \nabla \psi = -\hat{z} \times (-v \hat{x} + u \hat{y}) = u \hat{x} + v \hat{y} = \nabla \phi$

Problem 2

- For point source, $\vec{u} = \frac{q_s}{2\pi r} \hat{r}$, where q_s is the source strength. In spherical polar coordinates, $\vec{u} = \frac{\partial \phi}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial \phi}{\partial \theta} \hat{\theta}$. This means

$$\phi = \int \frac{\partial \phi}{\partial r} dr + \int \frac{1}{r} \frac{\partial \phi}{\partial \theta} d\theta = \frac{q_s}{2\pi} \ln r + \text{constant}$$

For lines of constant ϕ ,

$$\begin{aligned} \frac{q_s}{2\pi} \ln r &= C \\ \implies \frac{q_s}{2\pi r} \frac{dr}{dx} &= 0 \\ \implies \frac{q_s}{2\pi r^2} \left(2x + 2y \frac{dy}{dx} \right) &= 0 \implies \frac{dy}{dx} = -\frac{x}{y} = m \end{aligned}$$

As velocity is radial, the streamlines are also radial straight lines passing through the origin. The slope of such straight lines is $\frac{y}{x} = -\frac{1}{m}$. Hence the streamlines and lines of constant ψ are perpendicular.

- For point vortex, $\vec{u} = \frac{\Gamma}{2\pi r} \hat{\theta} = \frac{-\Gamma y}{2\pi(x^2 + y^2)} \hat{x} + \frac{\Gamma x}{2\pi(x^2 + y^2)} \hat{y}$. This means,

$$\phi = \frac{\Gamma}{2\pi} \tan^{-1} \frac{y}{x} \quad \text{and} \quad \psi = -\frac{\Gamma}{2\pi} \ln r$$

For lines of constant ϕ ,

$$\begin{aligned} \frac{\Gamma}{2\pi} \tan^{-1} \frac{y}{x} &= C_1 \\ \implies \frac{y}{x} &= \text{constant} = m_1 \\ \implies \frac{dy}{dx} &= m_1 \end{aligned}$$

For lines of constant ψ ,

$$\begin{aligned} \frac{\Gamma}{2\pi} \ln r &= C_2 \\ \implies r &= \text{constant} \\ \implies x^2 + y^2 &= \text{constant} \\ \implies \frac{dy}{dx} &= -\frac{x}{y} = -\frac{1}{m_1} \end{aligned}$$

Hence proved.

Problem 3

Given $A = \begin{bmatrix} -1 & p \\ 0 & -2 \end{bmatrix}$. The eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = -2$ with corresponding (normalized)

eigenvectors are $v_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ and $v_2 = \begin{bmatrix} \frac{-p}{\sqrt{1+p^2}} & \frac{1}{\sqrt{1+p^2}} \end{bmatrix}^T$. So, the resultant vector is,

$$\begin{aligned} v &= v_1 e^{\lambda_1 t} + v_2 e^{\lambda_2 t} \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t} + \frac{1}{\sqrt{1+p^2}} \begin{bmatrix} -p \\ 1 \end{bmatrix} e^{-2t} \\ v &= \begin{bmatrix} e^{-t} - \frac{p}{\sqrt{1+p^2}} e^{-2t} \\ \frac{1}{\sqrt{1+p^2}} e^{-2t} \end{bmatrix} \\ |v|^2 &= \left(e^{-t} - \frac{p}{\sqrt{1+p^2}} e^{-2t} \right)^2 + \left(\frac{1}{\sqrt{1+p^2}} e^{-2t} \right)^2 \\ &= e^{-2t} + e^{-4t} - \frac{2p}{\sqrt{1+p^2}} e^{-3t} \\ \frac{d|v|^2}{dt} &= -2e^{-2t} - 4e^{-4t} + \frac{6p}{\sqrt{1+p^2}} e^{-3t} \end{aligned}$$

For the resultant to grow, $\frac{d|v|^2}{dt} > 0$.

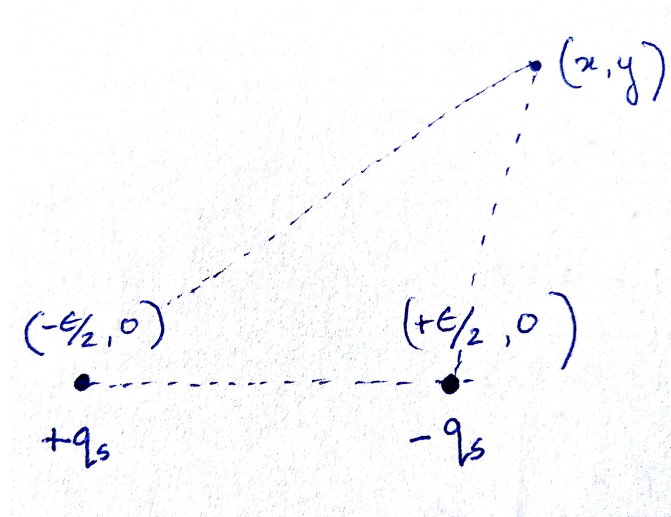
$$\begin{aligned} \implies -2e^{-2t} - 4e^{-4t} + \frac{6p}{\sqrt{1+p^2}} e^{-3t} &> 0 \\ \implies \frac{p}{\sqrt{1+p^2}} &> \frac{e^t + 2e^{-t}}{3} \end{aligned}$$

The function $\frac{e^t + 2e^{-t}}{3}$ has a minimum value of $\frac{2\sqrt{2}}{3}$. Hence, for the resultant to grow for some finite time,

$$\begin{aligned}\frac{p}{\sqrt{1+p^2}} &> \frac{2\sqrt{2}}{3} \\ \Rightarrow \frac{p^2}{1+p^2} &> \frac{8}{9} \\ \Rightarrow p^2 &> 8 \\ \Rightarrow p &> 2\sqrt{2}\end{aligned}$$

Hence, the resultant will grow for some finite time if $p > 2\sqrt{2}$.

Problem 4



Let's first find the potential. As described in Problem 1, $\phi = \frac{q_s}{2\pi} \ln r$. So for this problem,

$$\begin{aligned}\phi &= -\frac{q_s}{4\pi} \ln[(x - \epsilon/2)^2 + y^2] + \frac{q_s}{4\pi} \ln[(x + \epsilon/2)^2 + y^2] \\ &= \frac{q_s}{4\pi} \ln \frac{(x + \epsilon/2)^2 + y^2}{(x - \epsilon/2)^2 + y^2} \\ &= \frac{q_s}{4\pi} \ln \frac{1 + \epsilon x/r^2}{1 - \epsilon x/r^2} \\ &= \frac{q_s}{4\pi} \ln(1 + 2\epsilon x/r^2) \\ \phi &= \frac{q_s \epsilon \cos \theta}{2\pi r}\end{aligned}$$

Hence, the velocity profile is,

$$\vec{u} = -\frac{q_s \epsilon \cos \theta}{2\pi r^2} \hat{r} - \frac{q_s \epsilon \sin \theta}{2\pi r^2} \hat{\theta}$$

For the streamlines, we note that for a single point source at origin,

$$\begin{aligned}u &= \frac{q_s x}{2\pi(x^2 + y^2)} \quad \text{and} \quad v = \frac{q_s y}{2\pi(x^2 + y^2)} \\ \therefore \psi &= \frac{q_s}{2\pi} \tan^{-1} \frac{y}{x}\end{aligned}$$

Hence, for this problem,

$$\begin{aligned}
 \psi &= -\frac{q_s}{2\pi} \tan^{-1} \frac{y}{x - \epsilon/2} + \frac{q_s}{2\pi} \tan^{-1} \frac{y}{x + \epsilon/2} \\
 &= \frac{q_s}{2\pi} \left(\tan^{-1} \frac{y}{x + \epsilon/2} - \tan^{-1} \frac{y}{x - \epsilon/2} \right) \\
 &= \frac{q_s}{2\pi} \tan^{-1} \frac{y(x - \epsilon/2 - x - \epsilon/2)}{x^2 + y^2} \\
 &= \frac{q_s}{2\pi} \tan^{-1} \frac{-y\epsilon}{r^2} \\
 \psi &= -\frac{q_s}{2\pi} \frac{\sin \theta}{r}
 \end{aligned}$$

$$\psi = -\frac{q_s}{2\pi} \frac{\sin \theta}{r} = \text{constant} \implies \frac{\sin \theta}{r} = \frac{y}{x^2 + y^2} = \frac{1}{2C} \implies x^2 + (y - C)^2 = C^2$$

which is the equation of a circle with centre at $(0, C)$ and radius C .