Advanced Quantum Mechanics: Assignment #5

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Problem 1

Part (a)
We first note that,

$$\begin{aligned}
\lambda e^{-t/\tau} \langle n|x^2|m\rangle &= \lambda e^{-t/\tau} \langle n|\frac{(a_+ + a_-)^2}{2m\omega}|m\rangle \\
&= \lambda e^{-t/\tau} \langle n|\frac{a_+^2 + a_-^2 + a_+a_- + a_-a_+}{2m\omega}|m\rangle \\
&= \lambda e^{-t/\tau} \frac{1}{2m\omega}\left[\sqrt{(m+1)(m+2)}\delta_{n,m+2} + \sqrt{m(m-1)}\delta_{n,m-2} + (2m-1)\delta_{n,m}\right]
\end{aligned}$$

As is evident from above, a state $|m\rangle$ can transition into $|m\rangle, |m+2\rangle, |m-2\rangle$ and no other states under a potential with spatial dependence that goes as x^2. In general, the k-th order coefficient $c_n^k(t)$ will have k terms of the form $\langle .|x^2|.\rangle$. If we start out with ground state $|0\rangle$, the final state will have contributions from the following states order by order

$$\begin{aligned}
\mathcal{O}(\lambda) &\to |0\rangle, |2\rangle \\
\mathcal{O}(\lambda^2) &\to |0\rangle, |2\rangle, |4\rangle \\
\mathcal{O}(\lambda^3) &\to |0\rangle, |2\rangle, |4\rangle, |6\rangle \\
\therefore \mathcal{O}(\lambda^k) &\to |0\rangle, |2\rangle, |4\rangle, \dots |2k\rangle
\end{aligned}$$

Hence, we see that the $|n\rangle$ as mentioned in the question should be such that n is even, and the leading order contribution to the probability will $\sim (\lambda^{n/2})^2 \sim \lambda^n$.

Part (b)
As described above, upto $\mathcal{O}(\lambda^2)$ in probability (ie upto $\mathcal{O}(\lambda)$ in the coefficients), $|2\rangle$ is the only excited state that can be reached. From (5.7.17) of Sakurai, we have the relations, (with $|i, t_0; t\rangle = \sum c_n(t)\,|n\rangle$)

$$c_n^0(t) = \delta_{ni} \quad , \quad c_n^1(t) = -i\int_{t_0}^{t} e^{i\omega_{ni}t'} V_{ni}(t')dt'$$

Let's calculate $c_n^1(t)$,

$$
\begin{aligned}
c_n^1(t) &= -i\lambda \int_0^t e^{in\omega t'} \langle n|x^2|0\rangle \, e^{-t'/\tau} dt' \\
&= \frac{-i\lambda}{2m\omega}(\sqrt{2}\delta_{n,2} + \delta_{n,0}) \int_0^t e^{in\omega t'} e^{-t'/\tau} dt' \\
c_n^1(t) &= \frac{-i\lambda}{2m\omega}(\sqrt{2}\delta_{n,2} + \delta_{n,0}) \frac{e^{in\omega t} e^{-t/\tau} - 1}{in\omega - 1/\tau} \\
\implies c_2^1(t) &= \frac{-i\lambda}{\sqrt{2}m\omega} \frac{e^{2i\omega t} e^{-t/\tau} - 1}{2i\omega - 1/\tau} \implies \left|c_2^1(t)\right|^2 = \frac{\lambda^2}{2m^2\omega^2} \frac{e^{-2t/\tau} + 1 - 2e^{-t/\tau}\cos 2\omega t}{4\omega^2 + 1/\tau^2}
\end{aligned}
$$

$\left|c_2^1\right|^2$ is the required probability.

Problem 2

We don't need to apply any perturbation theory in this problem, and it can be solved exactly. The Hamiltonian is $H = \lambda S_1 \cdot S_2 = \lambda(S^2 - S_1^2 - S_2^2)$. We consider the action of the Hamiltonian on the singlet state $|00\rangle = \dfrac{|+-\rangle - |-+\rangle}{\sqrt{2}}$ and $|10\rangle = \dfrac{|+-\rangle + |-+\rangle}{\sqrt{2}}$. We know $H\,|00\rangle = -3\lambda/4\,|00\rangle$ and $S^2\,|10\rangle = \lambda/4\,|10\rangle$. Initially the system is in $|+-\rangle = \dfrac{|00\rangle + |10\rangle}{\sqrt{2}}$. Then we know, by the usual rules of time-evolution,

$$
\begin{aligned}
|\psi_f(t)\rangle = e^{iHt}\,|+-\rangle &= \frac{e^{i\lambda t/4}}{\sqrt{2}}\,|10\rangle + \frac{e^{-i3\lambda t/4}}{\sqrt{2}}\,|00\rangle \\
&= \left(\frac{e^{i\lambda t/4} + e^{-i3\lambda t/4}}{2}\right)|+-\rangle + \left(\frac{e^{i\lambda t/4} - e^{-i3\lambda t/4}}{2}\right)|-+\rangle \\
\implies |\langle +-|\psi_f(t)\rangle|^2 &= \left|\left(\frac{e^{i\lambda t/4} + e^{-i3\lambda t/4}}{2}\right)\right|^2 = \frac{1+\cos\lambda t}{2} = P(|+-\rangle) \\
\implies |\langle -+|\psi_f(t)\rangle|^2 &= \left|\left(\frac{e^{i\lambda t/4} - e^{-i3\lambda t/4}}{2}\right)\right|^2 = \frac{1-\cos\lambda t}{2} = P(|-+\rangle) \\
\implies |\langle ++|\psi_f(t)\rangle|^2 &= 0 = P(|++\rangle) \\
\implies |\langle --|\psi_f(t)\rangle|^2 &= 0 = P(|--\rangle)
\end{aligned}
$$

where $P(|\rangle)$ denotes probability of initial state to be in state $|\rangle$.

Problem 3

Part (a)

From (5.7.17) of Sakurai, we have the relations, (with $|i, t_0; t\rangle = \sum c_n(t)\,|n\rangle$)

$$
c_n^0(t) = \delta_{ni} \quad , \quad c_n^1(t) = -i \int_{t_0}^t e^{i\omega_{ni}t'} V_{ni}(t')dt'
$$

For our problem, we have $V = \lambda\delta(x - vt)$. We insert $1 = \int dx\,|x\rangle\langle x|$ such that $V_{ni}(t) = \int V(t)u_i^*(x)u_n(x)dx$. We have initial state $u_i(x)$ and final state $u_f(x)$. Hence, we can write the above coefficients as,

$$
\begin{aligned}
c_f^1(t) &= -i\lambda \int_{-\infty}^{\infty} dx \int_0^t dt' e^{i(E_i - E_f)t'} \delta(x - vt') u_i^*(x) u_f(x) \\
&= -i\lambda \int_{-\infty}^{\infty} dx e^{i(E_i - E_f)x/v} u_i^*(x) u_f(x)
\end{aligned}
$$

Problem 3 continued on next page. . .

Hence the probability is just $\left|c_f^1\right|^2$

Part (b)

We now write,

$$\delta(x - vt) = \frac{1}{2\pi v}\int_{-\infty}^{\infty} d\omega e^{i\omega(x/v - t)}$$

$$\begin{aligned}
\therefore c_f^1(t) &= -i\lambda \int_{-\infty}^{\infty} dx \int_0^t dt' e^{i(E_i - E_f)t'} \frac{1}{2\pi v}\int_{-\infty}^{\infty} d\omega e^{i\omega(x/v - t')} u_i^*(x) u_f(x) \\
&= \frac{-i\lambda}{2\pi v}\int_{-\infty}^{\infty} dx \int_0^t dt' e^{i(E_{if} - \omega)t'} \int_{-\infty}^{\infty} d\omega e^{i\omega x/v} u_i^*(x) u_f(x) \\
&= \frac{-i\lambda}{2\pi v}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} d\omega \delta(E_{fi} - \omega) e^{i\omega x/v} u_i^*(x) u_f(x) 4
\end{aligned}$$

Integrating the above will give us the same expression as that is Part (a). We notice that there is this $\delta(E_{fi} - \omega)$ term, which basically ensures energy conservation.

Problem 4

The ground state wavefunction for a hydrogen-like atom is given by,

$$|0_Z\rangle = \frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_0}$$

We need to find $|\langle 0_2|0_1\rangle|^2$,

$$\begin{aligned}
\langle 0_2|0_1\rangle &= \frac{1}{\pi}\left(\frac{2}{a_0^2}\right)^{3/2} \int_0^{2\pi} d\phi \int_0^{\pi} -d(\cos\theta) \int_0^{\infty} dr r^2 e^{-3r/a_0} \\
&= \frac{1}{\pi}\left(\frac{2}{a_0^2}\right)^{3/2} (4\pi)\left(\frac{2a_0^3}{27}\right) \\
\langle 0_2|0_1\rangle &= \sqrt{8}\frac{8}{27} \\
\implies |\langle 0_2|0_1\rangle|^2 &\approx 0.7
\end{aligned}$$

So, the probability is close to 0.7.

Problem 5

Part (a)

We first note for $|\psi_I(t)\rangle = \sum_n c_n(t)\,|\alpha_n\rangle$

$$\begin{aligned}
i\frac{\partial\,|\psi_I\rangle}{\partial t} &= i\frac{\partial(e^{iH_0t}\,|\psi_S\rangle)}{\partial t} \\
&= i\left[e^{iH_0t}\frac{\partial\,|\psi_S\rangle}{\partial t} + iH_0 e^{iH_0t}\,|\psi_S\rangle\right] \\
&= -e^{iH_0t}(H_0 + V)\,|\psi_S\rangle - H_0 e^{iH_0t}\,|\psi_S\rangle \\
&= e^{iH_0t} V\,|\psi_S\rangle
\end{aligned}$$

Problem 5 continued on next page. . .

$$
\begin{aligned}
i\frac{\partial \left|\psi_I\right\rangle}{\partial t} &= V_I \left|\psi_I\right\rangle \\
i\frac{\partial \left\langle \alpha_n|\psi_I\right\rangle}{\partial t} &= \left\langle \alpha_n|V_I|\psi_I\right\rangle \\
\dot{c_n} &= -i\left\langle \alpha_n|V_I|\psi_I\right\rangle \\
\dot{c_n} &= -i\left\langle \alpha_n|V|\alpha_m\right\rangle e^{i(E_n-E_m)t}c_m
\end{aligned}
$$

So for the given problem, we have

$$\left|\psi_I(t)\right\rangle = c_1(t)\left|1\right\rangle + c_2(t)e^{iEt}\left|2\right\rangle$$
$$\dot{c_1} = -iV_{11}c_1 - iV_{12}e^{-iEt}c_2 = -i\gamma e^{i(\omega-E)t}c_2 \quad \text{and} \quad \dot{c_2} = -iV_{21}e^{iEt}c_1 - iV_{22}c_2 = -i\gamma e^{i(E-\omega)t}c_1$$

To solve the above equations, we make the substitution $c_1 = b_1e^{i\Delta t}$ and $c_2 = b_2e^{-i\Delta t}$, where $2\Delta = \omega - E$. We then have the equations in terms of b's,

$$i\dot{b_1} = \Delta b_1 + \gamma b_2 \quad \text{and} \quad i\dot{b_2} = \gamma b_1 - \Delta b_2$$

These are coupled equations, and we can solve these by making the substitution $b_1 = Ae^{i\Omega t}$ and $b_2 = Be^{i\Omega t}$. We then have,

$$-A\Omega = \Delta A + \gamma B \quad \text{and} \quad -B\Omega = \gamma A - \Delta B$$
$$\text{For non-trivial solutions,} \quad -\frac{\gamma}{\Delta+\Omega} = \frac{\Delta-\Omega}{\gamma} \implies \Omega = \pm\sqrt{\gamma^2+\Delta^2} = \pm\Omega_0$$
$$\implies c_1 = A_1e^{i(\Delta+\Omega_0)t} + A_2e^{i(\Delta-\Omega_0)t} \quad \text{and} \quad c_2 = B_1e^{i(-\Delta+\Omega_0)t} + B_2e^{i(-\Delta-\Omega_0)t}$$

We are told that at $t = 0$, the system is in state $\left|1\right\rangle \implies c_1(0) = 1, c_2(0) = 0 \implies A_1 = 1 - A_2, B_1 = -B_2$. We also know that $\dot{c}_2(0) = -i\gamma c_1(0)$ and $\dot{c}_1(0) = -i\gamma c_2(0)$ which means,

$$-i(1-A_1)\Omega_0 + iA_1\Omega_0 + i\Delta = 0 \implies A_1 = \frac{\Omega_0-\Delta}{2\Omega_0} \quad \text{and} \quad A_2 = -\frac{\Omega_0-\Delta}{2\Omega_0}$$
$$2iB_1\Omega_0 = -i\gamma \implies B_1 = -\frac{\gamma}{2\Omega_0} \quad \text{and} \quad B_2 = 1+\frac{\gamma}{2\Omega_0}$$
$$\implies c_1 = \frac{\Omega_0-\Delta}{2\Omega_0}e^{i(\Delta+\Omega_0)t} - \frac{\Omega_0-\Delta}{2\Omega_0}e^{i(\Delta-\Omega_0)t} \quad \text{and} \quad c_2 = -\frac{\gamma}{2\Omega_0}e^{i(-\Delta+\Omega_0)t} + \left(1+\frac{\gamma}{2\Omega_0}\right)e^{i(-\Delta-\Omega_0)t}$$

where $\Delta = \dfrac{\omega-E}{2}$ and $\Omega_0 = \sqrt{\gamma^2+\Delta^2}$
. We calculate $\left|c_2(t)\right|^2$ using Mathematica, and we get,

$$\left|c_2(t)\right|^2 = \frac{\gamma^2\sin^2(t\Omega_0)}{\Omega_0^2} \quad \text{and} \quad \left|c_1(t)\right|^2 = 1 - \frac{\gamma^2\sin^2(t\Omega_0)}{\Omega_0^2}$$

Part (b)
To prove to all orders in perturbation, we consider $\omega = E \implies \Omega_0 = \gamma \implies \left|c_2(t)\right|^2 = \sin^2(t\gamma)$. From the Dyson series, we have,

$$c_2^n(t) = \left\langle 2\right|(-i)^n\int_0^t dt' \ldots \int_0^{t^{(n-1)}} dt^n V_I(t')\ldots V_I(t^{(n)})\left|1\right\rangle$$

Inserting a complete set of state before each V_I

$$c_2^n(t) = \left\langle 2\right|(-i)^n\int_0^t dt' \ldots \int_0^{t^{(n-1)}} dt^n V_I(t')(\left|1\right\rangle\left\langle 1\right| + \left|2\right\rangle\left\langle 2\right|)\ldots(\left|1\right\rangle\left\langle 1\right| + \left|2\right\rangle\left\langle 2\right|)V_I(t^{(n)})\left|1\right\rangle$$

Problem 5 continued on next page...

There is a constraint that the initial state should be $|1\rangle$ and the final state should be $|2\rangle$. Since, $|1\rangle \to |2\rangle$ and $|2\rangle \to |1\rangle$ at each application of V_I, terms with even n will vanish. For terms with odd n, let's first consider $n = 3$

$$\begin{aligned}
c_2^3(t) &= (-i)^3 \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt''' \langle 2|V_I(t')|1\rangle \langle 1|V_I(t'')|2\rangle \langle 2|V_I(t''')|1\rangle \\
&= (-i)^3 \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt''' \langle 2|V_I(t')|1\rangle \langle 1|V_I(t'')|2\rangle \langle 2|V_I(t''')|1\rangle \\
&= (-i\gamma)^3 \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt''' e^{iEt'} \langle 2|V(t')|1\rangle e^{iEt''} \langle 1|V(t'')|2\rangle e^{iEt''} \langle 2|V(t''')|1\rangle \\
&= (-i\gamma)^3 \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt''' e^{iEt'} e^{-iEt'} e^{iEt''} e^{-iEt''} e^{iEt''} e^{-iEt''} \\
&= (-i\gamma)^3 \frac{t^3}{3!}
\end{aligned}$$

We see that because $\omega = E$, all the matrix elements become independent of t, and we get a very simple answer for $c_2^n = (-i\gamma)^n \dfrac{t^n}{n!}$. Hence,

$$c_2(t) = \sum_{\text{odd n}} (-i\gamma)^n \frac{t^n}{n!} = -i \sin t\gamma \implies |c_2(t)|^2 = \sin^2 t\gamma$$

Hence the formula is verified.