Advanced Quantum Mechanics: Assignment #5

Due on 20th November, 2018

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(**Acknowledgements** - I would like to thank Chandramouli Chowdhury, Biprarshi Chakraborty and Junaid Majeed for discussions.)

Problem 1

Problem 2

We don't need to apply any perturbation theory in this problem, and it can be solved exactly. The Hamiltonian is $H = \lambda S_1 \cdot S_2 = \lambda(S^2 - S_1^2 - S_2^2)$. We consider the action of the Hamiltonian on the singlet state $|00\rangle = \dfrac{|+-\rangle - |-+\rangle}{\sqrt{2}}$ and $|10\rangle = \dfrac{|+-\rangle + |-+\rangle}{\sqrt{2}}$. We know $S^2 |00\rangle = 0$ and $S^2 |10\rangle = |10\rangle$. Initially the system is in $|+-\rangle = \dfrac{|00\rangle + |10\rangle}{\sqrt{2}}$. Then we know, by the usual rules of time-evolution,

$$\begin{aligned}
|\psi_f(t)\rangle = e^{iHt} |+-\rangle &= \frac{e^{i\lambda t/4}}{\sqrt{2}} |10\rangle + \frac{e^{-i3\lambda t/4}}{\sqrt{2}} |00\rangle \\
&= \left(\frac{e^{i\lambda t/4} + e^{-i3\lambda t/4}}{2}\right) |+-\rangle + \left(\frac{e^{i\lambda t/4} - e^{-i3\lambda t/4}}{2}\right) |-+\rangle \\
\implies |\langle +-|\psi_f(t)\rangle|^2 &= \left|\left(\frac{e^{i\lambda t/4} + e^{-i3\lambda t/4}}{2}\right)\right|^2 = \frac{1+\cos\lambda t}{2} = P(|+-\rangle) \\
\implies |\langle -+|\psi_f(t)\rangle|^2 &= \left|\left(\frac{e^{i\lambda t/4} - e^{-i3\lambda t/4}}{2}\right)\right|^2 = \frac{1-\cos\lambda t}{2} = P(|-+\rangle) \\
\implies |\langle ++|\psi_f(t)\rangle|^2 &= 0 = P(|++\rangle) \\
\implies |\langle --|\psi_f(t)\rangle|^2 &= 0 = P(|--\rangle)
\end{aligned}$$

where $P(|\rangle)$ denotes probability of initial state to be in state $|\rangle$.

Problem 3

From (5.7.17) of Sakurai, we have the relations, (with $|i, t_0; t\rangle = \sum c_n(t) |n\rangle$)

$$c_n^0(t) = \delta_{ni} \quad , \quad c_n^1(t) = -i\int_{t_0}^{t} e^{i\omega_{ni}t'} V_{ni}(t')dt' \quad , \quad c_n^2(t) = -\sum_m \int_{t_0}^{t} dt' \int_{t_0}^{t'} e^{i\omega_{nm}t'} V_{nm}(t') e^{i\omega_{mi}t''} V_{mi}(t'')dt''$$

For our problem, we have $V = \lambda\delta(x - vt)$. We insert $1 = \int dx \left|x\right\rangle \left\langle x\right|$ such that $V_{ni}(t) = \int V(t) u_i^*(x) u_n(x) dx$. We have initial state $u_i(x)$ and final state $u_f(x)$. Hence, we can write the above coefficients as,

$$\begin{aligned} c_f^1(t) &= -i \int_{-\infty}^{\infty} dx \int_0^t dt' e^{i(E_i - E_f)t'} \delta(x - vt') u_i^*(x) u_n(x) \\ &= -i \int_{-\infty}^{\infty} dx e^{i(E_i - E_f)x/v} u_i^*(x) u_n(x) \end{aligned}$$

Hence the probability is just $\left|c_f^1\right|^2$

Problem 4

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Problem 5

We first note for $\left|\psi_I(t)\right\rangle = \sum_n c_n(t) \left|\alpha_n\right\rangle$

$$\begin{aligned} i\frac{\partial \left|\psi_I\right\rangle}{\partial t} &= i\frac{\partial (e^{iH_0 t}\left|\psi_S\right\rangle)}{\partial t} \\ &= i\left[e^{iH_0 t}\frac{\partial \left|\psi_S\right\rangle}{\partial t} + iH_0 e^{iH_0 t}\left|\psi_S\right\rangle\right] \\ &= -e^{iH_0 t}(H_0 + V)\left|\psi_S\right\rangle - H_0 e^{iH_0 t}\left|\psi_S\right\rangle \\ &= e^{iH_0 t} V \left|\psi_S\right\rangle \\ i\frac{\partial \left|\psi_I\right\rangle}{\partial t} &= V_I \left|\psi_I\right\rangle \\ i\frac{\partial \left\langle \alpha_n | \psi_I\right\rangle}{\partial t} &= \left\langle \alpha_n \middle| V_I \middle| \psi_I\right\rangle \\ \dot{c_n} &= -i \left\langle \alpha_n \middle| V_I \middle| \psi_I\right\rangle \\ \dot{c_n} &= -i \left\langle \alpha_n \middle| V \middle| \alpha_m\right\rangle e^{i(E_n - E_m)t} c_m \end{aligned}$$

So for the given problem, we have

$$\left|\psi_I(t)\right\rangle = c_1(t)\left|1\right\rangle + c_2(t) e^{iEt}\left|2\right\rangle$$
$$\dot{c_1} = -iV_{11}c_1 - iV_{12}e^{-iEt}c_2 = -i\gamma e^{i(\omega - E)t}c_2 \quad \text{and} \quad \dot{c_2} = -iV_{21}e^{iEt}c_1 - iV_{22}c_2 = -i\gamma e^{i(E-\omega)t}c_1$$

To solve the above equations, we make the substitution $c_1 = b_1 e^{i\Delta t}$ and $c_2 = b_2 e^{-i\Delta t}$, where $2\Delta = \omega - E$. We then have the equations in terms of b's,

$$i\dot{b_1} = \Delta b_1 + \gamma b_2 \quad \text{and} \quad i\dot{b_2} = \gamma b_1 - \Delta b_2$$

These are coupled equations, and we can solve these by making the substitution $b_1 = Ae^{i\Omega t}$ and $b_2 = Be^{i\Omega t}$. We then have,

$$-A\Omega = \Delta A + \gamma B \quad \text{and} \quad -B\Omega = \gamma A - \Delta B$$
$$\text{For non-trivial solutions,} \quad -\frac{\gamma}{\Delta + \Omega} = \frac{\Delta - \Omega}{\gamma} \implies \Omega = \pm\sqrt{\gamma^2 + \Delta^2} = \pm\Omega_0$$
$$\implies c_1 = A_1 e^{i(\Delta + \Omega_0)t} + A_2 e^{i(\Delta - \Omega_0)t} \quad \text{and} \quad c_2 = B_1 e^{i(-\Delta + \Omega_0)t} + B_2 e^{i(-\Delta - \Omega_0)t}$$

Problem 5 continued on next page...

We are told that at $t = 0$, the system is in state $|1\rangle \implies c_1(0) = 0, c_2(0) = 1 \implies A_1 = -A_2, B_1 = 1 - B_2$. We also know for a fact that $|c_1(0)|^2 + |c_2(0)|^2 = 1$. Using all these facts, we can write

$$4A_1^2 \cos^2 \Omega t + 4(B_1^2 - B_1) \sin^2 \Omega t + 1 = 1$$