Notes on Cosmology

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Chapter 1

Kinematics, Dynamics and Geometry

For most of the 20th century, cosmologists agreed that the universe was homogenous and isotropic ie. there is no single *preferred* point or direction in the universe. This is called the *Cosmological Principle* in popular terms. On the face of it, this is a very powerful assumption - it allows us to make predictions about the universe sitting at where we are on the earth. It remained an intelligent guess until the age of data took over cosmology.

Data now shows that the universe is indeed homogenous and isotropic, but only on large enough scales (greater than 100 Mpc). Hence, if we coarse grain the universe on a scale smaller than 100 Mpc, we will start seeing inhomogeneities like galaxies, galaxy clusters etc. This was not entirely unexpected. So we decided to live with it.

But then there is a twist. Theory suggests that the universe continues to be homogenous and isotropic for scales bigger than that of the observable universe (3000 Mpc and beyond), but inhomogeneities start to creep in at distance scales much larger than 3000 Mpc. Answering questions on these scales in indeed very tough, not only because one can't seem to pose the question in a mathematically precise fashion, but also because one can't imagine verifying these predictions empirically.

Nevertheless, as travellers in the conquest for truth, it is imperative for us to learn what our forefathers thought of the cosmos, and hopefully avoid making the same mistakes again.

1.1 Hubble Law

Hubble postulated that, in a homogenous, isotropic expanding universe, the relative velocities of observers obey the Hubble Law.

$$\vec{v}_{B,A} = H(t)\vec{r}_{AB}$$

Is the Hubble law in agreement with the homogenous and isotropic assumption? Let's check,

$$\vec{v}_{B,A} = H(t)\vec{r}_{AB} \; ; \; \vec{v}_{C,B} = H(t)\vec{r}_{BC}$$

$$\text{Hence, } \vec{v}_{C,A} = H(t)(\vec{r}_{AB} + \vec{r}_{BC}) = H(t)\vec{r}_{AC}$$

which is what we should have expected. In fact, one can show explicitly that Hubble Expansion Law is *the only* law that is comaptible with homogeneity and isotropic expansion.

One could go ahead and write the Law as a differential equation as follows

$$\dot{\vec{r}}_{AB} = H(t)\vec{r}_{AB}$$

Integrating this,

$$r_{AB} = r_0 \exp \int H(t)dt = a(t)r_0$$

where $\exp \int H(t)dt = a(t)$ is called the *scale factor*. r_0 is the separation between A and B at some given instant of time (taken to be $t = 0$ without loss of generality), and r_{AB} denotes the distance between them after time t has elapsed.

It is also straightforward to write the Hubble constant in terms of the scale factor and its time derivative,

$$H = \frac{\dot{a}}{a}$$

Obviously, the Hubble Law would only hold on cosmological length scales. The relative motion of the sun and the earth, for example, is not governed by this law, but by the inhomogeneities in the gravitational field.

1.2 Dynamics of Newtonian Dust

Consider a universe filled with *dust* particles. What dust really means is matter which exerts negligible pressure compared to its energy density ϵ. Lets also assume that gravity is a weak force and that the particles are not

very far away from each other so as to avoid exceeding the speed of light. Now consider a sphere expanding about the origin, with its radius being given by $R(t) = a(t)\chi_c$. As the total mass is M, the energy density ϵ can be written as

$$\epsilon(t) = \frac{3M}{4\pi R(t)^3} = \frac{3M}{4\pi R_0^3} \left(\frac{a_0}{a(t)}\right)^3 = \epsilon_0 \left(\frac{a_0}{a(t)}\right)^3$$

Taking the time derivative,

$$\dot{\epsilon}(t) = \epsilon_0 \left(\frac{a_0}{a(t)}\right)^3 \left(\frac{-3\dot{a}}{a}\right) = -3H\epsilon(t)$$

We can also write down Newton's second law for the sphere as follows,

$$\ddot{R} = -\frac{GM}{R^2} = -G\frac{4\pi}{3}\epsilon(t)R$$

Dividing by R_0 we get,

$$\therefore \ \ddot{a} = -\frac{4\pi G}{3}\epsilon a$$

We have essentially got the evolution equations for the energy density and the scale factor in a Newtonian dust setting. Substituting for ϵ in the equation for $a(t)$,

$$\ddot{a} = -\frac{4\pi G\epsilon_0}{3}\left(\frac{a_0^3}{a^2}\right)$$

Multiplying by $\dot{a}$ and integrating we get,

$$\frac{\dot{a}^2}{2} - \frac{4\pi G\epsilon_0}{3}\left(\frac{a_0^3}{a}\right) = \text{constant} = E$$

The form of this energy-like equation for the scale factor is similar to the energy equation for a rocket launched into orbit. If one provides the rocket too little kinetic energy, it will fall back to earth. Similarly, the fate of this Newtonian dust universe, ie. whether it keeps expanding or collapses unto itself depends on the sign of the energy E. Writing the equation in terms of the Hubble constant,

$$H^2 - \frac{2E}{a^2} = \frac{8\pi G}{3}\epsilon$$

Setting $E = 0$ in the above equation, one can derive the critical density,

$$\epsilon^{cr} = \frac{3H^2}{8\pi G}$$

Substituting this back in the other equation,

$$\epsilon^{cr}(1 - \Omega(t)) = \frac{3E}{4\pi G}\frac{\epsilon}{a^2}$$

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where $\Omega(t) = \frac{\epsilon^{cr}}{\epsilon}$ is called the *cosmological parameter*. *Omega(t)* varies with time, but because the sign of E is fixed, the quantity $(1 - \Omega(t))$ does not change sign. Hence, we just need to measure the current value of the cosmological parameter to determine the sign of E in the universe.

As is conveyed in popular talks, the sign of E is linked to the spatial geometry of the Universe in general relativity. This spatial curvature has the sign opposite to that of E.

Let's imagine we probe the value of Ω_0. The possible scenarios are :-

- $\epsilon_0 > \epsilon^{cr}$ ie $\Omega > 1$ and $E < 0$. The spatial curvature will be positive (closed universe). The scale factor reaches some maximum value and the universe recollapses.

- $\epsilon_0 < \epsilon^{cr}$ ie $\Omega < 1$ and $E > 0$. The spatial curvature will be negative (open universe). The universe exapnds hyperbolically.

- $\epsilon_0 = \epsilon^{cr}$ ie $\Omega = 1$ and $E = 0$. The spatial curvature will be zero (flat universe). The scale factor reaches some maximum value and the universe recollapses.

In all three cases, extrapolating the universe backwards, we face an initial singularity. Note that we are still working with dust-dominated universe (and also Newtonian theory, duh); the analyses we have done are only toy-model-like. Nonetheless, we hope we have gained some physical intuition from the same.

Lets just play with our toy-model for one last time. In a flat universe,

$$\frac{\dot{a}^2 a}{2} = \frac{4}{9} \left(\frac{\mathrm{d}a^{3/2}}{\mathrm{d}t} \right)^2 = \frac{4\pi G \epsilon_0 a_0^3}{3} = \text{constant}$$

Hence, $a \propto t^{2/3}$, $H = \frac{2}{3t}$. We can then write the age of universe expression as,

$$t_0 = \frac{2}{3H_0} \text{ and also } \epsilon(t) = \frac{1}{6\pi G t^2}$$