ICTS graduate course : Classical Electrodynamics

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Assignment 3 : **Bessel functions**

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Bessel functions : Recommended reading

References on Bessel functions include

- *A Treatise on the Theory of Bessel Functions* by G. N. Watson
  https://archive.org/details/ATreatiseOnTheTheoryOfBesselFunctions
  THE most detailed, canonical work on the subject. Somewhat dated and sometimes a boring read, but still very useful.

- *A treatise on Bessel functions and their applications to physics*
  by Gray A., Mathews G.B.
  https://archive.org/stream/treatiseonbessel00grayuoft/

- *Introduction to Bessel Functions* by Frank Bowman

- *Bessel functions and their applications* by Korenev G

- *Applied Bessel Functions* by Relton, F.E.

You would have noticed that our approach to Bessel functions in the class is a bit different from how the textbooks here motivate the subject. Unfortunately, there is no good textbook which does it this way, so you will have to make do with my notes for now. A mathematical treatment which is somewhat closer to this approach can be found in

- Chapter 6 of *Analysis of Spherical Symmetries in Euclidean Spaces* by Claus Müller
  https://link.springer.com/book/10.1007/978-1-4612-0581-4

Some good general references for special functions are

- *Special Functions & Their Applications* by N. N. Lebedev

- *Special Functions for Scientists and Engineers* by W. W. Bell

- *Special Functions* by George E. Andrews, Richard Askey, Ranjan Roy

Refer also to broader discussions in books like

- *Differential Equations, with Applications and Historical Notes* by G. Simmons
- *Mathematical Methods for Physicists* by George B. Arfken, Hans J. Weber

Detailed tables and results on Bessel functions can be found in

- Chapter 10 Bessel Functions of *NIST Digital Library of Mathematical Functions* by F. W. J. Olver, L. C. Maximon https://dlmf.nist.gov/10
- Tables Of Integrals, Series And Products by Gradshtein I. S., Ryzhik I. M. https://archive.org/details/GradshteinI.S.RyzhikI.M.TablesOfIntegralsSeriesAndProducts
- Chapter 9,10 and 11 of *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* edited by Milton Abramowitz and Irene A. Stegun
  1972 Edition :https://archive.org/details/AandS-mono600
  Modern edition :http://people.math.sfu.ca/~cbm/aands/

For more on axi-symmetric expansions in electron optics, see

- Chapter 7 of *Principles of Electron Optics, Volume 1: Basic Geometrical Optics* by Peter W. Hawkes, Erwin Kasper

1. **Gaussian averaging of Laplace eigenfunctions** : Consider a Gaussian distribution centred around a point $\vec{r}_0$ in space and with standard deviation R :

$$\psi_G^{(R)}(\vec{r}-\vec{r}_0) \equiv \frac{1}{(2\pi)^{d/2}R^d}\exp\left(-\frac{|\vec{r}-\vec{r}_0|^2}{2R^2}\right) \ . \tag{1}$$

Let $\Phi_\lambda(\vec{r})$ denote an eigen-function of Laplacian with an eigen-value λ. Then,

(a) Show that the Gaussian average of the function evaluates to

$$\langle\Phi_\lambda\rangle_{G[\vec{r}_0,R]} \equiv \int d^d\vec{r}\ \Phi_\lambda(\vec{r})\ \psi_G^{(R)}(\vec{r}-\vec{r}_0) = \Phi_\lambda(\vec{r}_0)\exp\left(\frac{\lambda}{2}R^2\right) \ . \tag{2}$$

Hint : Prove and then use the fact that $\frac{1}{R}\frac{\partial}{\partial R}\psi_G = \nabla^2\psi_G$ along with Green's second identity.

(b) Use the above result on Gaussian averaging to derive the following integral identities :

$$\int d^d\vec{r}\ J_0(d;kr)\ I_0(d;\kappa r)\ \psi_G^{(R)}(\vec{r}) = J_0(d;k\kappa R^2)\exp\left(\frac{1}{2}(\kappa^2-k^2)R^2\right) \ . \tag{3}$$

$$\int d^d\vec{r}\ I_0(d;\kappa_1 r)\ I_0(d;\kappa_2 r)\ \psi_G^{(R)}(\vec{r}) = I_0(d;\kappa_1\kappa_2 R^2)\exp\left(\frac{1}{2}(\kappa_1^2+\kappa_2^2)R^2\right) \ . \tag{4}$$

$$\int d^d\vec{r}\ J_0(d;k_1 r)\ J_0(d;k_2 r)\ \psi_G^{(R)}(\vec{r}) = I_0(d;k_1k_2 R^2)\exp\left(-\frac{1}{2}(k_1^2+k_2^2)R^2\right) \ . \tag{5}$$

Hint : Choose an appropriate $\vec{r}_0$ and use sphere averaging to get the additional Bessel function.

(c) We will now give an alternate derivation of the last identity (the previous two would then follow by analytic continuation).

Next, use the convolution property of Fourier transforms to show that the following relation holds for arbitrary spherically symmetric functions $\{f,g\}$ and their transform pairs $\{\overline{f},\overline{g}\}$:

$$\begin{aligned}\int \frac{d^d\vec{k}'}{(2\pi)^d}\overline{f}(k')\ \overline{g}(|\vec{k}-\vec{k}'|) &= \int d^d\vec{r}\ J_0(d;kr)\ f(r)\ g(r) \\ &= \int d^d\vec{r}\ J_0(d;kr)\ \ g(r)\int\frac{d^d\vec{k}'}{(2\pi)^d}\ J_0(d;k'r)\overline{f}(k') \ .\end{aligned} \tag{6}$$

Finally, take $g(r)=\psi_G^{(R)}(\vec{r})$ and argue that the required identity follows.

(d) Show that for $\Delta > 0$, the following identity holds

$$\int_0^\infty \frac{dR}{R^{2\Delta+1}} \ (2\pi)^{d/2} R^d \ \psi_G^{(R)}(\vec{r}) = 2^{\Delta-1} \frac{\Gamma(\Delta)}{r^{2\Delta}} \ . \tag{7}$$

Show that for $\frac{d}{2} > \Delta$, we have

$$\int_0^\infty \frac{dR}{R^{2\Delta+1}} \ (2\pi)^{d/2} R^d \ \exp\left(-\frac{1}{2}k^2 R^2\right) = 2^{d-\Delta-1} \pi^{\frac{d}{2}} \frac{\Gamma(\frac{d}{2}-\Delta)}{k^{d-2\Delta}} \ . \tag{8}$$

Show that for $\frac{d}{2} > \Delta > 0$, these two results can be combined with the result on Gaussian average to give

$$\boxed{\int d^d\vec{r} \ \frac{\Gamma(\Delta)}{|\vec{r}-\vec{r}_0|^{2\Delta}} \ \Phi_{-k^2}(\vec{r}) \ = 2^{d-2\Delta} \pi^{\frac{d}{2}} \frac{\Gamma(\frac{d}{2}-\Delta)}{k^{d-2\Delta}} \Phi_{-k^2}(\vec{r}_0) \ ,} \tag{9}$$

where $\Phi_{-k^2}(\vec{r})$ is the Laplace eigen-function with eigen-value $-k^2$.

2. **Series for Bessel functions and Bessel raising/lowering relations:**

(a) Use the Taylor expansion of Bessel I functions about $x = 0$, i.e.,

$$I_0(d;x) = \sum_{m=0}^{\infty} \frac{\Gamma(\frac{d}{2})}{m! \ \Gamma(\frac{d}{2}+m)} \left(\frac{x}{2}\right)^{2m} \tag{10}$$

to prove the following equalities

$$\frac{x}{d}\frac{d}{dx} I_0(d;x) = \frac{x^2}{d^2} I_0(d+2;x) = I_0(d-2;x) - I_0(d;x) \ . \tag{11}$$

Note that amounts to a direct proof of dimension raising/lowering relations and it also implies that these functions satisfy the appropriate differential equation. Similar relations for Bessel J functions also follow via the replacement $x \mapsto ix$.

(b) Can you prove the above equalities from the Schlafli's contour Integral for the Bessel function (See notes for a derivation)

$$I_0(d;x) = \oint_{\mathcal{C}[-\infty-i0,0+,-\infty+i0]} \frac{dz}{2\pi i} \frac{\Gamma(\frac{d}{2})}{z^{\frac{d}{2}}} \exp\left\{z + \frac{x^2}{4z}\right\} \ ? \tag{12}$$

(c) Check that the asymptotic expansion of Bessel I functions near $x = \infty$, i.e.,

$$I_0(d;x) \approx \frac{e^x}{|\mathbb{S}^{d-1}|} \ \left(\frac{2\pi}{x}\right)^{\frac{d-1}{2}} \sum_{n=0}^{\infty} \frac{(-)^n \Gamma\left(\frac{d-1}{2}+n\right)}{(2x)^n n! \ \Gamma\left(\frac{d-1}{2}-n\right)} \ , \tag{13}$$

also satisfies these equalities. Argue from there that similar relations for asymptotic series of Bessel J functions.

(d) As we know, the Bessel functions in higher dimensions are given by iterated derivatives of the type

$$\frac{1}{x}\frac{d}{dx}$$

acting on Bessel functions in lower dimensions. It is often useful to derive a simplified form of this iterated derivative, where all derivatives are moved to the right. To this end, use induction to prove that, for an arbitrary function $f(x)$, we have

$$\left(\frac{1}{x}\frac{d}{dx}\right)^{\ell} f(x) = \frac{1}{x^{\ell}}\sum_{n=0}^{\ell-1}\frac{(-)^n\Gamma\left(\ell+n\right)}{(2x)^n n!\ \Gamma\left(\ell-n\right)}\left(\frac{d}{dx}\right)^{\ell-n} f(x)\ . \tag{14}$$

Comment : Note the remarkable similarity between this series and the asymptotic series of the Bessel functions ! This similarity becomes more manifest if we write

$$\left(\frac{1}{x}\frac{d}{dx}\right)^{\ell} f(x) = \frac{1}{x^{\ell}}\sum_{n=0}^{\infty}\frac{(-)^n\Gamma\left(\ell+n\right)}{(2x)^n n!\ \Gamma\left(\ell-n\right)}\left(\frac{d}{dx}\right)^{\ell-n} f(x)\ , \tag{15}$$

which is valid since $|\Gamma\left(\ell-n\right)| = \infty$ for $n \geq \ell$. If we take this as a formal definition for fractional powers of $\frac{1}{x}\frac{d}{dx}$, the asymptotic expansion of Bessel I and J functions can be written in the form

$$\begin{aligned}
I_0(d;x) &\approx \frac{(2\pi)^{\frac{d-1}{2}}}{|\mathbb{S}^{d-1}|}\left(\frac{1}{x}\frac{d}{dx}\right)^{\frac{d-1}{2}} e^x = \frac{2^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)}{\sqrt{2\pi}}\left(\frac{1}{x}\frac{d}{dx}\right)^{\frac{d-1}{2}}\frac{e^x}{2}\ , \\
J_0(d;x) &\approx \frac{2(2\pi)^{\frac{d-1}{2}}}{|\mathbb{S}^{d-1}|}\left(-\frac{1}{x}\frac{d}{dx}\right)^{\frac{d-1}{2}}\cos\left(x-\frac{\pi}{4}(d-1)\right) \\
&= \frac{2^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)}{\sqrt{2\pi}}\left(-\frac{1}{x}\frac{d}{dx}\right)^{\frac{d-1}{2}}\cos\left(x-\frac{\pi}{4}(d-1)\right)\ .
\end{aligned} \tag{16}$$

3. **Near axis expansion of axi-symmetric potentials :** In this question, we will look at a way to reconstruct a axi-symmetric potential (or more generally a harmonic function) from its values at z-axis. This is quite a useful technique in magnetostatics (to interpolate magnetostatic potential and get a model of the magnetic field), electron optics used in electron microscopes and particle accelerators.

(a) One way to reconstruct the potential is by analytic continuation. Given the potential at the z-axis $V(z, \varrho = 0) = g(z)$, we can analytically continue and write down an integral

$$V(z,\varrho) = \int_0^{2\pi}\frac{d\alpha}{2\pi}\ g(z - i\varrho\cos\alpha)\ .$$

This manifestly gives a function which satisfies $V(z, \varrho = 0) = g(z)$. Show that this also solves Laplace's equation (this is a simplified variant of the drop test question).

(b) By expanding in ϱ, get a series which gives the harmonic function in terms of the potential at z axis and its repeated z-derivatives. **Hint** : Your expansion should start off as

$$V(z, \varrho) = V(z, 0) - \frac{\varrho^2}{4}\frac{\partial^2}{\partial z^2}V(z, 0) + \frac{\varrho^4}{64}\frac{\partial^4}{\partial z^4}V(z, 0) + \dots$$

(c) Show that in fact the whole series can be summarised by usig the formal operator

$$V(z, \varrho) = J_0\left(d = 2; \varrho\frac{\partial}{\partial z}\right) V(z, 0) \ .$$

Could you give an explanation for why this should have been anticipated ? Show that the corresponding electric fields have the expansion

$$E_z(z, \varrho) = J_0\left(d = 2; \varrho\frac{\partial}{\partial z}\right) E_z(z, 0) \ , \qquad E_\varrho(z, \varrho) = \frac{\varrho}{2} J_0\left(d = 4; \varrho\frac{\partial}{\partial z}\right) \frac{\partial}{\partial z} E_z(z, 0) \ .$$

(d) Say we take a potential which on z-axis varies as a simple power of z. Then, the corresponding axi-symmetric extension to all space is given by a polynomial

$$V_\ell(z, \varrho) = J_0\left(\varrho\frac{\partial}{\partial z}\right) \frac{z^\ell}{\ell!}$$

Compute the explicit form of this polynomial for arbitrary ℓ. Evaluate explicitly the cases $\ell = 1, 2$. Can you recognise these polynomials ?

Assignment feedback : Please take time give your feedback on this assignment

1. Time taken to finish the assignment :
2. Make up an exam question on the topic of the assignment :
3. How many other students you collaborated with ?
4. How much was the class useful for solving the problems of the assignment ?
5. Was assignment useful to understand topics covered in the class ?
6. How much useful were the tutorials/tutor for the last assignment ?