Theory and Numerics of PDEs: Assignment #1

Due on 7th September, 2021

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Problem 1

Question: Show $y(t) = \sum_{n\geq 0} \dfrac{a_0}{n!} t^n$ converges.
Let,

$$x_n = \frac{a_0}{n!} t^n \tag{1}$$

$$\frac{x_{n+1}}{x_n} = \frac{t}{n+1} \tag{2}$$

which goes to 0 as $n \to \infty$. So this passes the ratio test for convergence.

Problem 2

Question: Use $y = \sum_{n\geq 0} a_n t^n$ to solve $y'' + y = 0$.
Substituting, we get,

$$\sum_{n\geq 0} a_n t^{n-2} + a_n t^n = 0 \implies \sum_{n\geq 2} a_{n+2} t^{n-2} + \sum_{n\geq 0} a_n t^n = 0 \tag{3}$$

So we just need two constants, and the solution is fully specified. If $a_0 = A$ and $a_1 = B$,

$$y = A\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \dots\right) + B\left(x - \frac{t^3}{3!} + \frac{t^5}{5!}\right) = A\cos(t) + B\sin(t) \tag{4}$$

Problem 3

Question: Consider

$$t^2 y'' - 4ty' + 6y = 0 \quad ; \quad y(1) = 1 \quad ; \quad y'(1) = 2 \quad . \tag{5}$$

1. Show that, $y(t) = t^2$ is a solution to the differential equation.
2. Use Picard iteration to solve the differential equation.

Solution:
For $y = t^2$, $y' = 2t$ and $y'' = 2$. Hence, the LHS of (5) becomes,

$$y'' - \frac{4}{t} y' + \frac{6}{t^2} y = 0 \tag{6}$$

We write two iterates, for y' and y.

$$y_1' = 2 + \int_1^t \mathrm{d}t \left[\frac{8}{t} - \frac{6}{t^2}\right] = 2 + 8\log t + \frac{6}{t} - 6 \tag{7}$$

$$y_1 = 1 + \int_2^t (1)\,\mathrm{d}t = 1 + 2t - 2 = 2t - 1 \tag{8}$$

Problem 4

Question: Consider

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y(1-y) \quad , \quad y(0) = 1/2 \tag{9}$$

1. Show that,

$$y(t) = \frac{e^t}{e^t + 1} \tag{10}$$

   solves the differential equation.

2. Use Picard iterates to find the solution. Compare this solution to the true Taylor series.

3. How many iterates do we need to obtain the true Taylor coefficient for t^k.

Solution:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{e^t}{(e^t+1)^2} \quad ; \quad y(1-y) = \frac{e^t}{e^t+1}\frac{1}{e^t+1} = \frac{e^t}{(e^t+1)^2} \tag{11}$$

$$\implies \frac{\mathrm{d}y}{\mathrm{d}t} = y(1-y) \tag{12}$$

Also, $y(0) = 1/2$. So the solution satisfies both the differential equation as well as the initial condition. Let ϕ_k be the k^{th} Picard iterate. We start off with $\phi_0 = 1/2$. Then,

$$\phi_1 = \frac{1}{2} + \int_0^t \frac{1}{2}\frac{1}{2}\,\mathrm{d}s = \frac{1}{2} + \frac{t}{4} \tag{13}$$

$$\phi_2 = \frac{1}{2} + \int_0^t \left(\frac{1}{2} + \frac{s}{4}\right)\left(\frac{1}{2} - \frac{s}{4}\right)\mathrm{d}s = \frac{1}{2} + \frac{s}{4} - \frac{t^3}{48} \tag{14}$$

$$\phi_3 = \frac{1}{2} + \int_0^t \mathrm{d}t \left(\frac{1}{2} + \frac{s}{4} - \frac{s^3}{48}\right)\left(\frac{1}{2} - \frac{s}{4} + \frac{s^3}{48}\right) = \frac{t}{4} - \frac{t^3}{48} + \frac{t^5}{480} - \frac{t^7}{16128} \tag{15}$$