Fluid Mechanics: Assignment #1

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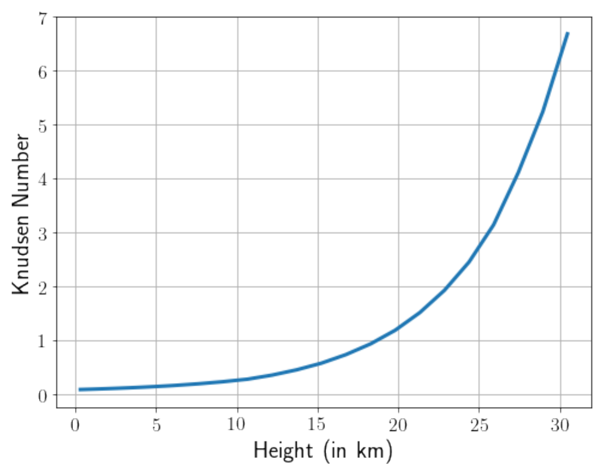
Problem 1

The *Knudsen Number* is given by,

$$\mathrm{Kn} = \frac{\lambda}{L} = \frac{k_B T}{\sqrt{2}\pi d^2 p L}$$

where L is the characteristic length scale, and T and p are the temperature and pressure respectively. We can take some approximations for the the variations of T and p, but we note that data for the variation of T and p is also available publicly `http://www.hyvac.com/tech_support/atmosphere%20vs%20pressure%202.htm`. We import that data and use it to do our calculations. We also assume some standard values for all the other parameters.



Problem 2

$$\frac{\mathrm{d}C}{\mathrm{d}\eta} = \kappa \exp\left(\frac{-\eta^2}{4D}\right)$$

where $\eta = x/\sqrt{t}$. Performing the indefinite integral and substituting for η, one gets,

$$C(\eta) = \sqrt{D\pi}\kappa \ \mathrm{erf}\left(\frac{\eta}{2\sqrt{D}}\right) + \alpha = \sqrt{D\pi}\kappa \ \mathrm{erf}\left(\frac{x}{2\sqrt{D}\sqrt{t}}\right) + \alpha$$

where α is some integration constant. We now go ahead and impose boundary conditions

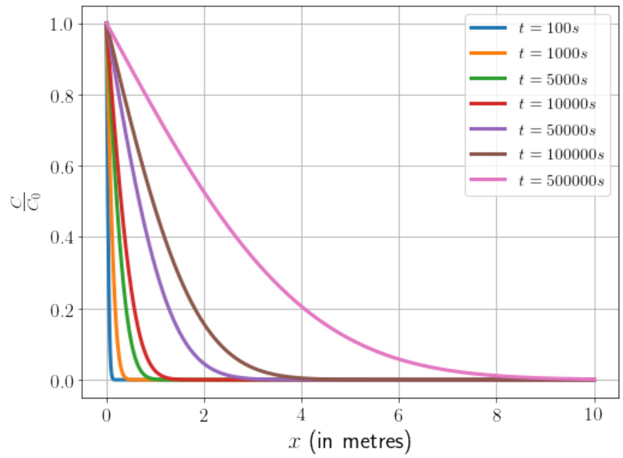
- The concentration at the flower ($x = 0$) is assumed to be a constant C_0 at all times. As $\mathrm{erf}(0) = 0$, $\alpha = C_0$

- At $t = 0$, any x would have zero concentration. As $\text{erf}(x \to \infty) = 1$, We get the condition that $\sqrt{D\pi}\kappa + C_0 = 0$ ie. $\kappa = -C_0/\sqrt{D\pi}$

Therefore, the final solution is,

$$C(x,t) = C_0 \left[1 - \text{erf}\left(\frac{x}{2\sqrt{D}\sqrt{t}}\right)\right]$$

The function is plotted below for different t using standard value of diffusivity $D \approx 10^{-5}\ m^2/s$,



Problem 3

Part (a)

The continuity equation ($\nabla \cdot \vec{v} = 0$) in cylindrical coordinates is given by,

$$\frac{1}{\rho}\frac{\partial v_\rho}{\partial \rho} + \frac{1}{\rho}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$$

For flows symmetric about the ϕ direction, this can be written as,

$$\frac{1}{\rho}\frac{\partial v_\rho}{\partial \rho} + \frac{\partial v_z}{\partial z} = 0$$

The stream function ψ should be such that,

$$d\psi = \frac{\partial \psi}{\partial \rho}d\rho + \frac{\partial \psi}{\partial z}dz = Md\rho + Ndz \implies \frac{\partial M}{\partial z} = \frac{\partial N}{\partial \rho}$$

Comparing the forms of the last two equations, one can infer,

$$M = \frac{\partial \psi}{\partial \rho} = \rho v_z \text{ and } N = \frac{\partial \psi}{\partial z} = \rho v_\rho$$

Therefore, the stream function ψ is given by,

$$\psi = \int \rho v_z d\rho + \int \rho v_\rho dz$$

The streamlines for a cone are plotted in the figure below.

Part (b)

The velocity fields at the respective locations of the stream functions as shown in the figure will be $v_1 =$

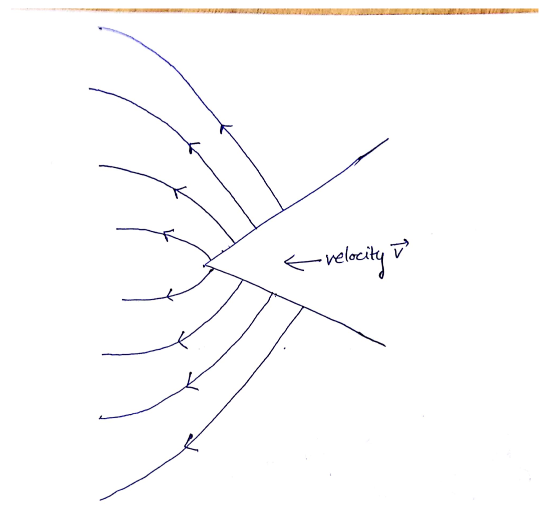


Figure 1: Flow past moving cone, assuming slip

$\boldsymbol{\nabla} \times \vec{\psi_1}$ and $v_2 = \boldsymbol{\nabla} \times \vec{\psi_2}$, where $\vec{\psi_1} = (0, 0, \psi_1)$ and $\vec{\psi_2} = (0, 0, \psi_2)$. The streamlines for this flow will meet somewhere in between (because there cannot be two directions of velocity at a single point). For example, such a flow can be generated by two moving infinite plates inclined at an acute angle, and having the no-slip boundary condition.

Problem 4

$$\boldsymbol{\nabla} u = e_{ij} + \Omega_{ij}$$

We can do the above decomposition of $\boldsymbol{\nabla} u$ into a symmetric and antisymmetric tensor. Ω_{ij} **represents pure rotation**. The symmetric part can be written and the divergence of some ϕ.
Let the principle axes in the problem be x, y and z (different from the coordinate axes). For some constants a, b, c, ϕ can be written as (in three dimensions),

$$\begin{aligned}
\phi &= \frac{1}{2}(ax^2 + by^2 + cz^2) \\
&= \frac{1}{2}\left[\frac{a+b+c}{3}(x^2+y^2+z^2) + \frac{2a-b-c}{3}x^2 + \frac{2b-a-c}{3}y^2 + \frac{2c-b-a}{3}z^2\right] \\
&= \frac{1}{2}\left[\alpha(x^2+y^2+z^2) + \alpha_1 x^2 + \alpha_2 y^2 + \alpha_3 z^2\right]
\end{aligned}$$

We note that $\alpha_1 + \alpha_2 + \alpha_3 = 0$. Then,

$$\begin{aligned}
\phi &= \frac{1}{2}\left[\alpha(x^2+y^2+z^2) + \alpha_1 x^2 + \alpha_2 y^2 - (\alpha_1+\alpha_2) z^2\right] \\
&= \frac{1}{2}\left[\alpha(x^2+y^2+z^2) + \alpha_1 (x^2 - z^2) + \alpha_2 (y^2 - z^2)\right]
\end{aligned}$$

We see that the first term is symmetric in the coordinates, and hence we remark that it is the pure expansion/contraction term. Similarly, the other two terms are antisymmetric under exchange of coordinates, hence they are pure strain terms. One can make a rotation of coordinates such that

$$x^2 - y^2 \to xy \text{ and } y^2 - z^2 \to yz$$

Problem 4 continued on next page...

One can now make the interpretation of the last two terms as shear strain terms in the rotated coordinates. **So, we have successfully decomposed the e_{ij} into shear strain + pure expansion/contraction.**