Classical Mechanics: Assignment #3

Due 21st September 2018

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Problem 3

The Lagrangian for this system can be written as,

$$L = \frac{1}{2} m_1 |\dot{\mathbf{r_1}}|^2 + \frac{1}{2} m_2 |\dot{\mathbf{r_2}}|^2 - V(\mathbf{r_1} - \mathbf{r_2})$$

We also know, from the question, that

$$\mathbf{R} = \frac{m_1 \mathbf{r_1} + m_2 \mathbf{r_2}}{m_1 + m_2} \quad \text{and} \quad \mathbf{r} = \mathbf{r_1} - \mathbf{r_2}$$

This leads us to,

$$\mathbf{r_1} = \mathbf{R} + \frac{m_2 \mathbf{r}}{m_1 + m_2} \quad \text{and} \quad \mathbf{r_2} = \mathbf{R} - \frac{m_1 \mathbf{r}}{m_1 + m_2}$$

$$|\dot{\mathbf{r_1}}|^2 = \left|\dot{\mathbf{R}}\right|^2 + \frac{m_2^2 |\dot{\mathbf{r}}|^2}{(m_1 + m_2)^2} + \frac{2m_2}{m_1 + m_2} \dot{\mathbf{R}} \cdot \dot{\mathbf{r}} \quad \text{and} \quad |\dot{\mathbf{r_2}}|^2 = \left|\dot{\mathbf{R}}\right|^2 + \frac{m_1^2 |\dot{\mathbf{r}}|^2}{(m_1 + m_2)^2} - \frac{2m_1}{m_1 + m_2} \dot{\mathbf{R}} \cdot \dot{\mathbf{r}}$$

Substituting into the expression for the Lagrangian, one gets,

$$L = \frac{M}{2} \left|\dot{\mathbf{R}}\right|^2 + \frac{\mu}{2} |\dot{\mathbf{r}}|^2 - V(\mathbf{r}) \quad \text{where} \quad M = m_1 + m_2 \quad , \quad \mu = \frac{m_1 m_2}{M}$$

Each component of $\dot{\mathbf{R}}$ will be conserved separately as all of them are cyclic coordinates. Using $\mathbf{R} = X\hat{\mathbf{x}} + Y\hat{\mathbf{y}} + Z\hat{\mathbf{z}}$ and $\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\hat{r}}$,

$$L = \frac{M}{2} (\dot{X}^2 + \dot{Y}^2) + \frac{\mu}{2} (\dot{r}^2 + r^2 \dot{\theta}^2) - V(r)$$

Part (a)

We can see from the form of the above Lagrangian that,

$$M\dot{X} = constant \quad , \quad M\dot{Y} = constant \quad , \quad \mu r^2 \dot{\theta} = constant$$

Consider the infinitesimal area swept by the vector $\mathbf{r}$,

$$dA = \frac{r^2 d\theta}{2} \implies \dot{A} = r^2 \frac{\dot{\theta}}{2} = constant = l$$

Hence, the radius vector sweeps equal areas in equal intervals of time.

Part (b)
If $m_2 \gg m_1$, $\mathbf{R} \approx \mathbf{r_2}$
Part (c)
The Euler-Lagrange equation for the coordinate r is given by,

$$\mu\ddot{r} = \mu r \frac{4l^2}{r^4} - \frac{k}{r^2} \implies \ddot{r} - \frac{4l^2}{r^3} + \frac{k}{\mu r^2} = 0$$

Multiplying by $\dot{r}$ and integrating with time, we get,

$$\dot{r}^2 + \frac{2l^2}{r^2} - \frac{k}{\mu r} = constant = E$$