Advanced Statistical Mechanics: Assignment #2

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Aditya Vijaykumar

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Problem 1

The two-particle Virial Coefficient b_2 is given by,

$$\begin{aligned} b_2 &= \int \mathrm{d}^d\vec{\mathbf{q}}_1 \mathrm{d}^d\vec{\mathbf{q}}_2 U(\vec{\mathbf{q}}_1 - \vec{\mathbf{q}}_2) \\ &= AS_{d-1}^2 \int \mathrm{d}q_1\, \mathrm{d}q_2\, q_1^{d-1} q_2^{d-1} \frac{1}{|q_1 - q_2|^\sigma} \end{aligned}$$

Problem 2

Do Part (a)

Part (b)

Let's denote the Vandermonde determinant by D_n,

$$D_n = \begin{vmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{vmatrix}$$

We prove the required statement by using row and column operations. We first use $R_n \to R_n - R_{n-1}, n = 2, 3 \dots, n$. We then have,

$$D_n = \begin{vmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 0 & x_2 - x_1 & x_2^2 - x_1^2 & \dots & x_2^{n-1} - x_1^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & x_n - x_1 & x_n^2 - x_1^2 & \dots & x_n^{n-1} - x_1^{n-1} \end{vmatrix}$$

We now proceed to make the topmost row elements 0, save for the first element. We use $C_n \to C_n - x_1 C_{n-1}, n = 2, 3 \dots, n$,

$$
\begin{aligned}
D_n &= \begin{vmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & x_2 - x_1 & x_2(x_2 - x_1) & \dots & x_2^{n-2}(x_2 - x_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & x_n - x_1 & x_n(x_n - x_1) & \dots & x_n^{n-2}(x_n - x_1) \end{vmatrix} \\
&= \begin{vmatrix} x_2 - x_1 & x_2(x_2 - x_1) & \dots & x_2^{n-2}(x_2 - x_1) & \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_n - x_1 & x_n(x_n - x_1) & \dots & x_n^{n-2}(x_n - x_1) & \end{vmatrix} \\
&= \left(\prod_{i=2}^{n} x_i - x_1\right) \begin{vmatrix} 1 & x_2 & x_2^2 & \dots & x_2^{n-2} \\ 1 & x_3 & x_3^2 & \dots & x_3^{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-2} \end{vmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\implies D_n &= \left(\prod_{i=2}^{n} x_i - x_1\right) D_{n-1} \\
D_n &= \left(\prod_{i=2}^{n} x_i - x_1\right)\left(\prod_{i=3}^{n-1} x_i - x_2\right) D_{n-2} \\
&= \left(\prod_{i=2}^{n} x_i - x_1\right)\left(\prod_{i=3}^{n-1} x_i - x_2\right) \dots D_2 \\
D_n &= \prod_{1 \le j < i \le n} x_i - x_j
\end{aligned}
$$

Hence Proved.

Problem 4

For N particles in a harmonic trap,

$$
\psi(x_1, x_2, \dots x_N) = \frac{1}{\sqrt{N!}} \det \phi_j(x_j) = \frac{1}{N!} \det A_{ij}
$$

$$
\text{where} \quad \phi_i(x) = \left(\frac{a^2}{\pi}\right)^{1/4} \frac{1}{\sqrt{2^i i!}} H_i(ax) e^{-a^2x^2/2} \quad , \quad a^2 = \frac{m\omega}{\hbar}
$$

The probability density is,

$$
P(\{x_i\}) = \frac{1}{N!} \det A^T A = \frac{1}{N!} \det K
$$

We define the average number density as,

$$
\begin{aligned}
\langle \rho(x) \rangle &= \sum_i \left\langle \frac{1}{N} \delta(x - x_i) \right\rangle \\
&= \frac{1}{N} \sum_i \int \prod_j \mathrm{d}x_j \, \delta(x - x_i) P(\{x_k\})
\end{aligned}
$$

Problem 4 continued on next page. . .

Let's have a closer look at the integral above. The integral over the delta function with replace the x_i in $P(\{x_k\})$ with x. Expanding the sum over i will give us N terms, each having one argument replaced by x. But we know that $P(x_1, x_2) = P(x_2, x_1)$, and hence we can relabel the terms in the summation, and get the following expression,

$$
\begin{aligned}
\langle \rho(x) \rangle &= \int \prod_{j=1}^{N-1} \mathrm{d}x_j \, P(x, x_1, x_2, \ldots, x_{N-1}) \\
&= \int \prod_{j=1}^{N-1} \mathrm{d}x_j \, \frac{1}{N!} \det K
\end{aligned}
$$