Quantum Mechanics II
Assignment 2

Due Thursday, 20 September 2018

Problems:

1. Show that

$$U = \mathcal{T}\left[e^{-i\int_0^t H(x)dx}\right] \equiv \sum \frac{(-i)^n}{n!}\int_0^t dt_1 \ldots \int_0^t dt_n \mathcal{T}\left[H(t_1)\ldots H(t_n)\right]$$

is a unitary operator, i.e. that it satisfies $U^\dagger U = 1$.

2. We previously considered coherent states that satisfy $a|z\rangle = z|z\rangle$. Show that, in the case of the simple harmonic oscillator, these states can be written as

$$|z\rangle = N_z e^{a^\dagger z}|0\rangle$$

where $|0\rangle$ is the ground-state of the oscillator and N_z is a normalization constant that you need to find. Write the coherent state as

$$|z\rangle = \sum_n f(n)|n\rangle$$

where $|n\rangle$ are the normalized number/energy eigenstates of the oscillator and $f(n)$ is a function that you need to find.

3. Find the ground state correlation function

$$C_1(t) = \langle 0|x(t)x(0)|0\rangle$$

Also find the *unequal* time commutators

$$C_2(t) = \langle 0|[x(t), p(0)]|0\rangle$$

and

$$C_3(t) = \langle 0|[p(t), x(0)]|0\rangle$$

4. (a) Use the spectrum of the simple-harmonic oscillator to evaluate its *partition function*

$$Z(\beta) = \mathrm{Tr}(e^{-\beta H})$$

(b) Now, we will rederive the same result using *path integral* techniques. You need to evaluate

$$Z(\beta) = \int_{x(0)=x(\beta)} e^{-S_E}\mathcal{D}x$$

for the simple-harmonic oscillator. Recall the the usual Lagrangian in real time is $L = \frac{m}{2}\dot{x}^2 - \frac{1}{2}m\omega^2 x^2$. You need to rotate this to Euclidean time before performing the path-integral. To evaluate the path-integral it is useful to write the possible functions with the necessary periodicity as

$$x(t) = \sum_n \frac{x_n}{\sqrt{\beta}} e^{\frac{2\pi i n \tau}{\beta}}$$

where τ is Euclidean time. You may find the following identity useful

$$\sinh z = z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{(n\pi)^2}\right)$$