ICTS graduate course : Classical Electrodynamics

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Assignment 2 : **Lagrangian formalism for EM**

25th Jan 2019

Lagrangian Formalism : Recommended reading

References on Lagrangian formalism and Noether theorem include

- Chapter 8, 9 of *Classical Electrodynamics* by
  Julian Schwinger, Lester L. DeRaad, Kimball A. Milton, Wu-yang Tsai
- Chapter 24 of *Modern Electrodynamics* by Andrew Zangwill
- Chapter 8 of *Particles and Quantum Fields* by Hagen Kleinert
  http://users.physik.fu-berlin.de/~kleinert/b6/psfiles/Chapter-7-conslaw.pdf
- Chapter 2 of *Field Quantization* by Greiner, Rienhardt
  https://link.springer.com/chapter/10.1007/978-3-642-61485-9_2
  (Accessible inside ICTS)
- Chapter 13 of *Classical Mechanics* by Herbert Goldstein and Charles P. Poole & John Safko
- Chapter 7,8 of *The Quantum Theory of Fields: Volume 1, Foundations* by Steven Weinberg : The discussion here is typical Weinberg - clear and comprehensive once you get used to the notation.

Some good general references for reviewing Classical mechanics are

- *Mechanics* by Lev D Landau and E.M. Lifshitz
  (Course of Theoretical Physics Vol 1)
  https://archive.org/details/Mechanics_541
- *The Variational Principles of Mechanics* by Cornelius Lanczos

A more formal and symmetry oriented approach can be found in

- *Classical Dynamics: A Modern Perspective* by E. C. G. Sudarshan, N. Mukunda

Some of the mathematical aspects are explored in

- *Symplectic techniques in physics* by Victor Guillemin and Shlomo Sternberg
- *Mathematical Methods of Classical Mechanics* by V. I. Arnold
- *Introduction to Mechanics and Symmetry* by Jerrold E. Marsden, Tudor S. Ratiu https://link.springer.com/book/10.1007/978-0-387-21792-5 (Accessible inside ICTS)

For Darwin Lagrangian, refer to

- Sec 12.6 of Jackson,
- Chapter 33 of Schwinger et. al.,
- *Classical theory of Fields* by L.D.Landau and E.M. Lifshitz § 65 (Chapter 8). Their section § 106 (Chapter 12) describes the Einstein-Infeld-Hoffman Lagrangian for General Relativity. https://archive.org/details/TheClassicalTheoryOfFields
- A nice treatment can be found in the following paper by our own Amit Apte and his colloborators :
  Todd B. Krause, A. Apte, and P.J. Morrison :
  A unified approach to the Darwin approximation
  https://home.icts.res.in/~apte/homepage_files/09-KrauseA07.pdf

For more on Born-Infeld theory, see

- *A First Course in String Theory* by Barton Zwiebach

For more on Heisenberg-Euler Effective Lagrangians see

- *Heisenberg-Euler Effective Lagrangians : Basics and Extensions* by Gerald V. Dunne https://arxiv.org/abs/hep-th/0406216

1. **Zangwill Ch.24 Exercises :** 24.1, 24.2, 24.4, 24.11, 24.12.

2. **Darwin Lagrangian for electromagnetism** : The Darwin Lagrangian for set of particles with masses m_a, velocities $\vec{u}_a$ and charges q_a is given by

$$\begin{aligned} L_D \equiv & \sum_a \left\{ \frac{1}{2} m_a \vec{u}_a^2 + \frac{1}{8c^2} m_a \vec{u}_a^4 \right\} - \sum_a \sum_{b \neq a} \frac{q_a q_b}{8\pi\epsilon_0 r_{ab}} \\ & + \sum_a \sum_{b \neq a} \frac{q_a q_b}{16\pi\epsilon_0 c^2 r_{ab}} \left\{ \vec{u}_a \cdot \vec{u}_b + (\vec{u}_a \cdot \hat{r}_{ab})(\vec{u}_b \cdot \hat{r}_{ab}) \right\} \end{aligned} \tag{1}$$

Here

$$r_{ab} \equiv |\vec{r}_a - \vec{r}_b|, \quad \hat{r}_{ab} \equiv \frac{\vec{r}_a - \vec{r}_b}{|\vec{r}_a - \vec{r}_b|} \ . \tag{2}$$

This is the Lagrangian for charged particles derived from Maxwell theory ignoring the effects of electromagnetic radiation (more generally ignoring all $O(c^{-4})$ effects). As we will see below, this Lagrangian can reproduce *almost* all of electrodynamics except for electromagnetic waves.

The analogous Lagrangian in General relativity (named after Einstein-Infeld-Hoffman) similarly describes the physics ignoring gravitational waves which is one useful starting point when describing early in-spiral stages of black hole mergers.

(a) Compute the canonical momentum of this Lagrangian and write it in the form

$$\vec{p}_a \equiv \frac{\partial L}{\partial \vec{u}_a} = \frac{1}{c^2} \left(m_a c^2 + \frac{1}{2} m_a \vec{u}_a^2 \right) \vec{u}_a + q_a \vec{A}_a = \vec{p}_a^{\,kin} + q_a \vec{A}_a \tag{3}$$

to identify the vector potential $\vec{A}_a$ seen by the a^{th} particle as

$$\vec{A}_a \equiv \sum_{b \neq a} \frac{q_b}{8\pi\epsilon_0 c^2 r_{ab}} (\vec{u}_b + \hat{r}_{ab}(\vec{u}_b \cdot \hat{r}_{ab})) \ . \tag{4}$$

Note that the kinetic energy of the particle contributes to the inertia of a particle in Darwin's theory.

(b) Using Euler-Lagrange equations, show that the total canonical momentum $\sum_a \vec{p}_a$ is conserved. This means that, in Darwin theory, whatever kinetic momentum is lost by the particles is stored as a 'potential momentum' among particles (as opposed to being radiated away as electromagnetic waves).

(c) Show that

$$\frac{\partial L}{\partial \vec{r}_a} \equiv \vec{\nabla}_a L = -q_a \ \frac{\partial}{\partial \vec{r}_a} (\phi_a - \vec{u}_a \cdot \vec{A}_a) \tag{5}$$

where ϕ_a is the Coulomb potential

$$\phi_a \equiv \sum_{b\neq a} \frac{q_b}{4\pi\epsilon_0 r_{ab}} \ . \tag{6}$$

As usual in Lagrangian mechanics, $\vec{\nabla}_a$ here implies differentiation with respect to $\vec{r}_a$ keeping all other $\vec{r}_{b\neq a}$, the velocities $\vec{u}_b$ and explicit time t (if any) constant.

(d) Show that the equations of motion for the particle can be rewritten into a Lorentz force expression

$$\frac{d}{dt}\vec{p}_a^{\,kin} = q_a(\vec{E}_a + \vec{u}_a \times \vec{B}_a) \tag{7}$$

where the electric and magnetic fields felt by the particle have the form

$$\begin{aligned}
\vec{E}_a &\equiv -\vec{\nabla}_a\phi_a - \partial_t\vec{A}_a \\
&= \frac{1}{4\pi\epsilon_0}\sum_{b\neq a}\frac{q_b\hat{r}_{ab}}{r_{ab}^2}\left\{1 - \frac{\vec{u}_b^2}{c^2} + \frac{3}{2}\frac{(\vec{u}_b\times\hat{r}_{ab})^2}{c^2}\right\} \\
&\qquad - \frac{\mu_0}{8\pi}\sum_{b\neq a}\frac{q_b}{r_{ab}}\left\{\frac{d\vec{u}_b}{dt} + \hat{r}_{ab}\left(\frac{d\vec{u}_b}{dt}\cdot\hat{r}_{ab}\right)\right\} \ , \\
\vec{B}_a &\equiv \vec{\nabla}_a \times \vec{A}_a = \frac{\mu_0}{4\pi}\sum_{b\neq a}\frac{q_b\vec{u}_b\times\hat{r}_{ab}}{r_{ab}^2} \ .
\end{aligned} \tag{8}$$

Here we have used $\mu_0 \equiv (\epsilon_0 c^2)^{-1}$ to write the relativistic correction terms. Note that the partial derivative ∂_t is taken keeping $\vec{r}_a$ constant, viz., $\partial_t \equiv \frac{d}{dt} - \vec{u}_a\cdot\vec{\nabla}$. It is customary to split the electric field into a velocity independent Coulomb field and a velocity dependent 'Faraday field' which arises through induction :

$$\begin{aligned}
\vec{E}_a &\equiv \vec{E}_a^C + \vec{E}_a^F \\
\vec{E}_a^C &\equiv -\vec{\nabla}_a\phi_a = \frac{1}{4\pi\epsilon_0}\sum_{b\neq a}\frac{q_b\hat{r}_{ab}}{r_{ab}^2} \\
\vec{E}_a^F &\equiv -\partial_t\vec{A}_a \\
&= \frac{\mu_0}{4\pi}\sum_{b\neq a}\frac{q_b\hat{r}_{ab}}{r_{ab}^2}\left\{-\vec{u}_b^2 + \frac{3}{2}(\vec{u}_b\times\hat{r}_{ab})^2\right\} \\
&\qquad - \frac{\mu_0}{8\pi}\sum_{b\neq a}\frac{q_b}{r_{ab}}\left\{\frac{d\vec{u}_b}{dt} + \hat{r}_{ab}\left(\frac{d\vec{u}_b}{dt}\cdot\hat{r}_{ab}\right)\right\} \ .
\end{aligned} \tag{9}$$

(e) By these definitions, check that these electric and magnetic fields satisfy the sourceless Maxwell equations

$$\vec{\nabla}_a \times \vec{E}_a + \partial_t\vec{B}_a = 0 \ , \qquad \text{and} \qquad \vec{\nabla}_a\cdot\vec{B}_a = 0 \ . \tag{10}$$

Compare the above expressions against the exact expressions for electromagnetic fields of particles moving with uniform velocity (which can be derived via Lorentz transformations as we will discuss later in the course) :

$$\begin{aligned}\vec{E}_a &= \frac{1}{4\pi\epsilon_0}\sum_{b\neq a}\frac{q_b\hat{r}_{ab}}{r_{ab}^2}\frac{1-\frac{\vec{u}_b^2}{c^2}}{\left\{1-\frac{(\vec{u}_b\times\hat{r}_{ab})^2}{c^2}\right\}^{3/2}}\ ,\\ \vec{B}_a &= \frac{\mu_0}{4\pi}\sum_{b\neq a}\frac{q_b\vec{u}_b\times\hat{r}_{ab}}{r_{ab}^2}\frac{1-\frac{\vec{u}_b^2}{c^2}}{\left\{1-\frac{(\vec{u}_b\times\hat{r}_{ab})^2}{c^2}\right\}^{3/2}}\ .\end{aligned} \tag{11}$$

Check that Darwin theory reproduces the same expression to $O(c^{-4})$ when accelerations are set to zero. Later in the course, hopefully, we will also compute the acceleration terms in Maxwell theory and compare it against the acceleration terms above.

(f) Show that the vector potential coming out of Darwin's Lagrangian is automatically in Coulomb gauge, i.e, $\vec{\nabla}_a\cdot\vec{A}_a = 0$. (Hint : can you write it as a curl ?)

Since we have pure Coulomb scalar potential, this immediately implies that the electric field satisfies Gauss law

$$\vec{\nabla}_a\cdot\vec{E}_a = -\vec{\nabla}_a^2\phi_a - \partial_t(\vec{\nabla}_a\cdot\vec{A}_a) = -\vec{\nabla}_a^2\phi_a = \vec{\nabla}_a\cdot\vec{E}_a^C = \frac{1}{\epsilon_0}\sum_b q_b\delta^3(r_{ab}) \tag{12}$$

where we are implicitly assuming a limit where the charge q_a is smeared out and $q_a\to 0$. More precisely, one can take many test charges with negiligible charge at rest and compute the flux. The result is that the non-Coulomb part of $\vec{E}_a$ (coming from $-\partial_t\vec{A}_a$) does not contribute to this flux. The Coulomb part reproduces the usual Gauss law as usual.

Thus, to conclude, Darwin theory already includes three of the four Maxwell equations along with Lorentz force equation exactly !

(g) We will now turn to the last Maxwell equation : Compute (with due care for δ functions) the curl of the Darwin magnetic field to show that

$$\vec{\nabla}_a\times\vec{B}_a = \frac{\mu_0}{4\pi}\sum_{b\neq a}\left\{\frac{q_b}{r_{ab}^3}(3\hat{r}_{ab}(\vec{u}_b\cdot\hat{r}_{ab})-\vec{u}_b)+\frac{8\pi}{3}q_b\vec{u}_b\delta^3(r_{ab})\right\} \tag{13}$$

Next compute the time derivative of the Coulomb electric field (carefully again) to show that

$$\frac{1}{c^2}\partial_t\vec{E}_a^C = -\frac{1}{c^2}\nabla_a(\partial_t\phi_a) = \frac{\mu_0}{4\pi}\sum_{b\neq a}\left\{\frac{q_b}{r_{ab}^3}(3\hat{r}_{ab}(\vec{u}_b\cdot\hat{r}_{ab})-\vec{u}_b)-\frac{4\pi}{3}q_b\vec{u}_b\delta^3(r_{ab})\right\} \tag{14}$$

So that we get the Darwin's version of the last Maxwell equation

$$\vec{\nabla}_a \times \vec{B}_a - \frac{1}{c^2}\partial_t \vec{E}^C_a = \mu_0 \sum_{b\neq a} q_b \vec{u}_b \delta^3(r_{ab}) \tag{15}$$

It crucially differs from Maxwell theory in that only the Coulomb part of electric field contributes to the displacement current. The usual charge current should be again be understood in an appropriate $q_a \to 0$ limit. We can actually the error made by Darwin theory by computing

$$\begin{aligned}\vec{\nabla}_a \times \vec{B}_a - \frac{1}{c^2}\partial_t \vec{E}_a - \mu_0 \sum_{b\neq a} q_b \vec{u}_b \delta^3(r_{ab}) &= -\frac{1}{c^2}\partial_t \vec{E}^F_a = -\frac{1}{c^2}\partial_t^2 \vec{A}_a \\ &= -\frac{1}{8\pi\epsilon_0 c^4}\frac{\partial^2}{\partial t^2}\sum_{b\neq a}\frac{q_b}{r_{ab}}(\vec{u}_b + \hat{r}_{ab}(\vec{u}_b\cdot\hat{r}_{ab}))\ .\end{aligned} \tag{16}$$

Darwin approximation is valid as long is this term is small - which happens in many cases because of c^{-4} factor in the front.

3. **Born-Infeld Lagrangian** : Many electrodynamic models arising as a part of string theory have a Lagrangian density of the following form (called Born Infeld form)

$$\mathcal{L}_{BI} = \frac{B_0^2}{\mu_0} - \frac{B_0}{\mu_0}\sqrt{B_0^2 + \vec{B}^2 - \frac{1}{c^2}\vec{E}^2 - \frac{(\vec{E}\cdot\vec{B})^2}{(cB_0)^2}}\ . \tag{17}$$

This reproduces the familiar Maxwell Lagrangian density when $B_0 \to \infty$. In string theory, this action describes electrodynamic response of certain membrane like objects called D-Branes.

To be complete, the vibrational modes of the membrane appropriately coupled to the electric and magnetic fields should appear in the Lagrangian density above. The open string then appears as a charged solitonic spike solution of this full theory. BI theory is also an interesting avenue for finding exact solutions/scattering amplitudes etc, and a toy model for non-linear optics.

   (a) Show that the constitutive relations in this theory take the form

$$c\vec{D} = \eta_E\left(\vec{E} + \alpha\, c\vec{B}\right)\ ,\quad \vec{H} = \eta_E(c\vec{B} - \alpha\vec{E})\ . \tag{18}$$

   where

$$\begin{aligned}\alpha &\equiv \frac{\vec{E}\cdot\vec{B}}{cB_0^2}\ ,\\ \eta_E &\equiv \frac{1}{\mu_0 c}\frac{B_0}{\sqrt{B_0^2 + \vec{B}^2 - \frac{1}{c^2}\vec{E}^2 - \frac{(\vec{E}\cdot\vec{B})^2}{(cB_0)^2}}} \equiv \frac{1}{\mu_0 c}\frac{1}{\sqrt{1-\alpha^2 + \frac{1}{B_0^2}\left(\vec{B}^2 - \frac{1}{c^2}\vec{E}^2\right)}}\ .\end{aligned} \tag{19}$$

Check that the expressions above satisfy the non-trivial relation

$$(\mu_0 c)^2 \vec{D} \cdot \vec{H} = \vec{E} \cdot \vec{B} \ ,$$

which gives an alternate expression for α

$$\alpha \equiv (\mu_0 c)^2 \frac{\vec{D} \cdot \vec{H}}{c B_0^2} \ , \tag{20}$$

(b) Say we define

$$\eta_H \equiv \frac{\mu_0 c}{\sqrt{1 - \alpha^2 + \left(\frac{\mu_0 c}{B_0}\right)^2 \left(\vec{D}^2 - \frac{1}{c^2}\vec{H}^2\right)}} \ . \tag{21}$$

Derive then the relation

$$\eta_E \eta_H (1 + \alpha^2) = 1 \ .$$

Use this to invert the above constitutive relations solving for $\vec{E}, \vec{B}$ in terms of $\vec{D}, \vec{H}$. Show that the Born-Infeld theory exhibits the electric-magnetic duality, viz., it is invariant under the exchange

$$\frac{1}{\mu_0 c} \vec{B} \leftrightarrow \vec{D} \ , \quad \frac{1}{\mu_0 c} \ \vec{E} \leftrightarrow -\vec{H}$$

which exchanges the 2-forms

$$\frac{1}{\mu_0 c} \mathbb{F} \leftrightarrow \mathbb{G} \ .$$

In fact, BI theory is invariant under duality *rotations*

$$\begin{pmatrix} \mathbb{G} \\ \frac{1}{\mu_0 c}\mathbb{F} \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \mathbb{G} \\ \frac{1}{\mu_0 c}\mathbb{F} \end{pmatrix} .$$

Bonus question : Can you show this ?

(c) Compute the energy density, energy current density, momentum density and momentum current density(viz., stress tensor) of this theory. Do they satisfy the symmetry relations

$$\overleftrightarrow{\Sigma} = \overleftrightarrow{\Sigma}^T \ , \quad \vec{\rho}_p = \frac{1}{c^2} \vec{J}_E \ ?$$

Are they duality invariant ?

(d) Say there is a point charge coupled to this theory via usual $\vec{J} \cdot \vec{A} - \rho\phi$ coupling. What is the electric field of this charge ?

(e) What is the total energy contained in the electrostatic field of a point charge in this theory ? How does it scale with the charge ?

(f) To return back to the real world, the leading quantum corrections to Maxwell Lagrangian due to quantum fluctuations of electrons were computed by Euler-Kockel and Euler-Heisenberg as

$$\Delta\mathcal{L}_{EKH} = \frac{2}{45}\frac{\hbar}{m_e^4 c^3}\left(\frac{e^2}{4\pi}\right)^2\left\{\left(\vec{B}^2 - \frac{1}{c^2}\vec{E}^2\right)^2 + 7\frac{1}{c^2}(\vec{E}\cdot\vec{B})^2\right\}, \tag{22}$$

where e, m_e denotes electronic charge and mass. Can you use it to compute the leading correction to the dielectric constant and magnetic permeability of vacuum in presence of electric and magnetic fields ? At what numerical values of electric fields does the permittivity of the vacuum change appreciably ?

Assignment feedback : Please take time give your feedback on this assignment

1. Time taken to finish the assignment :
2. Make up an exam question on the topic of the assignment :
3. How many other students you collaborated with ?
4. How much was the class useful for solving the problems of the assignment ?
5. Was assignment useful to understand topics covered in the class ?
6. How much useful were the tutorials/tutor for the last assignment ?