Classical Mechanics: Assignment #1

Due 11th September 2018

Aditya Vijaykumar

Problem 1

Solution

The Lagrangian for the given system can be written as,

$$L = T + V = \frac{1}{2}mx^2\omega^2 + \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy$$

From the problem, we know that $y = k\left(\frac{x}{l}\right)^\alpha$, which means that $\dot{y} = k\alpha\frac{x^{\alpha-1}}{l^\alpha}\dot{x}$. Substituting these into the form of the Lagrangian and simplifying, we get,

$$L = \frac{1}{2}m\left(-2gk\left(\frac{x}{l}\right)^\alpha + \dot{x}^2\left(\frac{\alpha^2k^2x^{2\alpha-2}}{l^\alpha} + 1\right) + x^2\omega^2\right)$$

The equation of motion can be written as,

$$\alpha gkx^2\left(\frac{x}{l}\right)^\alpha + (\alpha-1)\alpha^2k^2\dot{x}^2\left(\frac{x}{l}\right)^{2\alpha} + \alpha^2k^2x\ddot{x}\left(\frac{x}{l}\right)^{2\alpha} - x^4\omega^2 + x^3\ddot{x} = 0$$

The equlibrium points will satisfy $\dot{x} = \ddot{x} = 0$. This means that the equilibrium point will be,

$$x_0 = \left(\frac{\omega^2 l^\alpha}{gk\alpha}\right)^{\frac{1}{\alpha-2}}$$

We substitute $x = x_0 + \epsilon$

Problem 2

content...

Problem 3

content...

Problem 4

Part (a)

The Schrodinger equation is given by

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi = i\hbar\frac{\partial\psi}{\partial t}$$

We choose ψ and ψ^* as our generalized coordinates, and (t, x) as the dependent coordinates. One can write the equations of motion in a compact form as follows,

$$\partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \psi)} \right) = \frac{\partial L}{\partial \psi} \quad \text{and} \quad \partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \psi^*)} \right) = \frac{\partial L}{\partial \psi^*}$$

where the index μ goes over (t, x).

Part (b)

Kinetic energy of the wire is zero. The Lagrangian can be written as,

$$L = -\int ds \; \rho g y = -\int \sqrt{dx^2 + dy^2} \; \rho g y = -\int dx y \sqrt{1 + y'^2} \; \rho g$$

Writing down the equation of motion for the Lagrangian density instead of the Lagrangian, one gets,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{yy'}{\sqrt{1+y'^2}}\right) - \sqrt{1+y'^2} = 0$$

$$\frac{yy'' + y'^2}{\sqrt{1+y'^2}} - \frac{yy'^2 y''}{1+y'^2} - \sqrt{1+y'^2} = 0$$

Expanding this out and simplifying a bit, one gets,

$$\frac{yy''}{(1+y'^2)^{\frac{3}{2}}} - \frac{1}{\sqrt{1+y'^2}} = 0 \implies \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{\sqrt{1+y'^2}}\right) = 0$$

$$\therefore \frac{y}{\sqrt{1+y'^2}} = \alpha \implies y = \alpha \cosh\left(\frac{x}{\alpha} + \beta\right)$$

One can get the constants α and β by imposing the end point conditions for the curve.

Part (c)

The distance metric on a sphere spherical polar coordinates is given by,

$$ds^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2) = d\theta^2[r^2(1 + \sin^2\theta \phi'^2)]$$

$$\therefore ds = d\theta \sqrt{r^2(1 + \sin^2\theta \phi'^2)}$$

From the ansatz $S = L d\tau$, we can identify that the Lagrangian $L = \sqrt{r^2(1 + \sin^2\theta \phi'^2)}$. For finding the *equations of motion*, it is fine and also easier to work with L^2 rather than L in this problem. Writing down the equations of motion for $\phi(\theta)$,

$$\frac{\mathrm{d} \sin^2\theta \phi'}{\mathrm{d}\theta} = 0 \implies \phi' = \alpha \csc^2(\theta) \implies \phi(\theta) = a \cot\theta + b$$

where α, a, b are constants. If the distance is to be found out between two points (ϕ_1, θ_1) and (ϕ_2, θ_2), then,

$$\phi_1 = a \cot\theta_1 + b \quad \text{and} \quad \phi_2 = a \cot\theta_2 + b$$

which gives,

$$a = \frac{\phi_1 - \phi_2}{\cot\theta_1 - \cot\theta_2} \quad \text{and} \quad b = \frac{\phi_1 \tan\theta_1 - \phi_2 \tan\theta_2}{\tan\theta_1 - \tan\theta_2}$$