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Notes on Gravity as a Quantum Theory

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1 All is Classical, All is Quantum

1.1 The Classical Field

$\phi(\vec{\mathbf{x}}, t)$ gives the value of a classical field at every point in spacetime. The simplest classical field is the *real scalar field*, which is characterized only by real numbers. The Klein-Gordon equation governs a free massive scalar field.

$$\frac{\partial^2 \phi}{\partial t^2} - \sum_{x_j} \frac{\partial^2 \phi}{\partial x_j^2} + m^2 \phi = 0$$

An interesting part about the free scalar field is that one can describe it as an infinite set of decoupled harmonic oscillators. Put this field into a box of length L and volume $V = L^3$, and having periodic boundary conditions. One can Fourier decompose this as,

$$\phi(\vec{\mathbf{x}}, t) = \frac{1}{\sqrt{V}} \sum_{\vec{\mathbf{k}}} \phi_{\mathbf{k}}(t) \exp\Big(i\vec{\mathbf{k}} \cdot \vec{\mathbf{x}}\Big) \text{ where } k_x = \frac{2\pi n_x}{L}, \ldots$$

Substituting this into the first equation, we find that the harmonic oscillators get nicely decoupled into an infinite set of ODEs of the form,

$$\ddot{\phi}_{\mathbf{k}} + (k^2 + m^2)\phi_{\mathbf{k}} = 0$$

which is basically the harmonic oscillator equation with frequency $\omega_k = \sqrt{k^2 + m^2}$.The energy of oscillators in simply equal to the sum of individual energies of the oscillators,

$$E = \sum_{\mathbf{k}} \left[\frac{1}{2}\dot{\phi}_{\mathbf{k}}^{\,2} + \frac{1}{2}\omega_k^2 \phi_{\mathbf{k}}^2\right]$$

Equivalently, when $V \to \infty$ and k is a continuous variable, the summation is just replaced by an integral over all k,

$$\phi(\mathbf{x}, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}} \phi_{\mathbf{k}}(t)$$

1.2 Quantizing Fields

As mentioned earlier, a field can be thought of as a collection of decoupled harmonic oscillators. We quantize each field $\phi_{\mathbf{k}}$ as a separate harmonic oscillator. We identify the position and momentum as operators $\hat{\phi_{\mathbf{k}}}$ and $\hat{\pi_{\mathbf{k}}}$. The commutation relations for the harmonic oscillator as $V \to \infty$ can now be written as,

$$\left[\hat{\phi_{\mathbf{k}}}(t), \hat{\pi_{\mathbf{k}'}}(t)\right] = i\delta(\mathbf{k} + \mathbf{k}')$$

The vacuum state is the state corresponding to the lowest energy configuration. One can clearly see that the commutation relations cannot be satisfied for the most intuitive low energy configuration *ie.* $\phi(\mathbf{x}, t) = 0$, implying that the vacuum state is really something non-trivial. But since, for a free field, all the $\phi_{\mathbf{k}}$ are decoupled, we can write the vacuum state wave functional as the product of all wavefunctions, each describing the ground state of the harmonic oscillator with the wavenumber $\mathbf{k}$. Again, for large volume, one can write,

$$\psi[\phi] \propto \exp\left(-\frac{1}{2}\int d^3\mathbf{k}|\phi_{\mathbf{k}}|^2\omega_{\mathbf{k}}\right)$$

Consider the integral inside the exponential,

$$\begin{aligned}
\int d^3\mathbf{k}|\phi_{\mathbf{k}}|^2\omega_{\mathbf{k}} &= \int d^3\mathbf{k}\phi_{\mathbf{k}}\phi^*_{\mathbf{k}}\sqrt{k^2+m^2} \\
&= \int d^3\mathbf{x}d^3\mathbf{y}\phi(\mathbf{x})\phi(\mathbf{y})\int d^3\mathbf{k}e^{i\mathbf{k}(\mathbf{y}-\mathbf{x})}\sqrt{k^2+m^2} \\
&= \int d^3\mathbf{x}d^3\mathbf{y}\phi(\mathbf{x})\phi(\mathbf{y})K(\mathbf{x},\mathbf{y})
\end{aligned}$$

where $K(\mathbf{x}, \mathbf{y})$ is called the kernel.

The vacuum energy density is just the sum of all ground state energies,

$$\frac{E_o}{V} = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{\omega_k}{2}$$

Okay, now this is a very interesting expression for the energy. We see that because $\omega_k = \sqrt{k^2+m^2}$, we can see that this integral diverges as k^4. If quantum gravity is assumed to be modelled as a scalar field, and we put a cutoff for our integration at let's say the Planckian scale, we see that the vacuum energy density is of the order unity in Planck units, which in turn corresponds to a mass density of $10^{94}g/cm^3$. The mass of the *entire* observable universe is $10^{55}g$! One can try to resolve this problem by *positing* that vacuum energy does not contribute to gravity, or by using some supersymmetric variants of such theories.

1.3 Vacuum Fluctuations

The fluctuation in the quantum field can be written as,

$$\delta\phi_{\mathbf{k}} = \sqrt{\left\langle|\phi_{\mathbf{k}}|^2\right\rangle - \langle\phi_{\mathbf{k}}\rangle^2} = \sqrt{\left\langle|\phi_{\mathbf{k}}|^2\right\rangle}$$

We know that

$$\phi_{\mathbf{k}} = \frac{a_{\mathbf{k}} + a_{-\mathbf{k}}}{\sqrt{2\omega_k}}$$

which means that

$$|\phi_{\mathbf{k}}^2| = \frac{(a_{\mathbf{k}} + a_{-\mathbf{k}})(a_{\mathbf{k}} + a_{-\mathbf{k}})}{2\omega_k}$$

Taking the ground state expectation value of this expression, one obtains that $\delta\phi_{\mathbf{k}} \sim \omega_k^{-1/2}$. What if we measure the average value of a field over space? Lets consider a cubical box of side L and define the average value ϕ_L as follows,

$$\phi_L = \frac{1}{L^3}\int_{-L/2}^{-L/2} dx \int_{-L/2}^{-L/2} dy \int_{-L/2}^{-L/2} dz\ \phi(\mathbf{x})$$

We again calculate fluctuations in this average value by the formula $\delta\phi_L = \sqrt{\langle\phi_L^2\rangle}$.

$$\begin{aligned}\phi_L &\sim \frac{1}{L^3}\int_{-L/2}^{-L/2} dx \int_{-L/2}^{-L/2} dy \int_{-L/2}^{-L/2} dz\ \int \phi_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}} d^3\mathbf{k} \\ &\sim \frac{1}{L^3}\int \frac{1}{k_x k_y k_z} \sin\frac{k_x L}{2} \sin\frac{k_y L}{2} \sin\frac{k_z L}{2}\ \phi_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}} d^3\mathbf{k}\end{aligned}$$

Let us say that $f_x = \frac{1}{k_x}\sin\frac{k_x L}{2}$ and so forth for y and z. $f_x \to L/2$ for small k_x and 0 for large k_x. So the contribution to the integral can be taken to be from small k, and hence is of the order L^3. So $\delta\phi_L = \sqrt{\langle\phi_L^2\rangle} \sim [(\delta\phi_{\mathbf{k}})^2/L^3]^{1/2}$.

1.4 Gravity Can Create Particles?

Consider a single harmonic oscillator with the following features.

$$\underbrace{\ddot{q}(t) + \omega^2 q = 0}_{t<0 \text{ and } t>T}\ ;\ \underbrace{\ddot{q}(t) - \Omega^2 q = 0}_{0<t<T}$$

The solution, obviously, is

$$q(t) = \underbrace{q_1 \sin(\omega t)}_{t<0 \text{ (assume)}}\ ;\ \underbrace{Ae^{\Omega t} + Be^{-\Omega t}}_{0<t<T}\ ;\ \underbrace{q_2 \sin(\omega t + \alpha)}_{t>T}$$

Matching $q(t)$ and $\dot{q}(t)$ at $t = 0, T$, we get the condition,

$$\tan(\omega T + \alpha) = \frac{\omega}{\Omega}\ ;\ q_2 \sin(\omega T + \alpha) = Ae^{\Omega T}\ ;\ A = \frac{q_1}{2}\frac{\omega}{\Omega}$$

Hence,

$$q_2 = \frac{1}{2} q_1 \sqrt{1 + \frac{\omega^2}{\Omega^2}} e^{\Omega T}$$

DO the exercise here