Advanced General Relativity: Tutorial #3

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Problem 1

Part (a)

Given,

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \tag{1}$$

Using an affine parameter λ, we can rewrite these equations as,

$$\kappa = \left(\frac{ds}{d\lambda}\right)^2 = -\left(1 - \frac{2M}{r}\right)\dot{t}^2 + \left(1 - \frac{2M}{r}\right)^{-1} \dot{r}^2 + r^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) \tag{2}$$

where $\kappa = -1, 0, 1$ for timelike, null and spacelike geodesics respectively. Corresponding to the Killing vectors ∂_t and ∂_ϕ, we would have the following conserved quantities,

$$\tilde{E} = 2\left(1 - \frac{2M}{r}\right)\dot{t} \quad \text{and} \quad \tilde{L} = 2r^2 \sin^2\theta\dot{\phi} \tag{3}$$

Substituting $\dot{t}$ and $\dot{\phi}$ from (3) to (2) and assuming planar orbit ($\theta = \pi/2$), we get,

$$\kappa = -\left(1 - \frac{2M}{r}\right)\frac{\tilde{E}^2}{4\left(1 - \frac{2M}{r}\right)^2} + \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 + r^2\frac{\tilde{L}^2}{4r^4\sin^4\theta} \tag{4}$$

$$= -\left(1 - \frac{2M}{r}\right)^{-1}\frac{\tilde{E}^2}{4} + \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 + \frac{\tilde{L}^2}{4r^2} \tag{5}$$

$$\kappa - \frac{2M\kappa}{r} = -\frac{\tilde{E}^2}{4} + \dot{r}^2 + \frac{\tilde{L}^2}{4r^2}\left(1 - \frac{2M}{r}\right) \tag{6}$$

$$\therefore \frac{\dot{r}^2}{2} - \frac{1}{2}\left(\kappa + \frac{\tilde{E}^2}{4}\right) + \frac{M\kappa}{r} + \frac{\tilde{L}^2}{8r^2} - \frac{\tilde{L}^2 M}{4r^3} = 0 \tag{7}$$

Defining $E = \frac{\tilde{E}}{2}$ and $L = \frac{\tilde{L}}{2}$, we have,

$$\frac{\dot{r}^2}{2} + V_{eff}(r) = 0 \tag{8}$$

where $V_{eff}(r) = -\frac{E^2 + \kappa}{2} + \frac{M\kappa}{r} + \frac{L^2}{2r^2} - \frac{L^2 M}{r^3}$

Part (b)

Substituting $\kappa = 0$ and $\dot{r} = 0$ in (8), and further using (3) we have,

$$E^2 = \frac{L^2}{r^2}\left(1 - \frac{2M}{r}\right) \implies \left(\frac{dt}{d\phi}\right)^2 = r^2\left(1 - \frac{2M}{r}\right)^{-1} \tag{9}$$

Now, writing down the radial geodesic for the Schwarzschild metric, we have,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[2\left(1-\frac{2M}{r}\right)^{-1}\dot{r}\right] = -\frac{2M}{r^2}\dot{t}^2 - \left(1-\frac{2M}{r}\right)^{-2}\left(\frac{2M}{r^2}\right)\dot{r}^2 + 2r(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) \tag{10}$$

Using $\dot{r} = 0$, $\theta = \pi/2$, we get,

$$\frac{2M}{r^2}\dot{t}^2 = 2r\dot{\phi}^2 \implies \left(\frac{\mathrm{d}t}{\mathrm{d}\phi}\right)^2 = \frac{r^3}{M} \tag{11}$$

Comparing with (9), we have,

$$r^2\left(1-\frac{2M}{r}\right)^{-1} = \frac{r^3}{M} \implies 1-\frac{2M}{r} = \frac{M}{r} \implies r = 3M \tag{12}$$

Hence, there is a null circular orbit at $r = 3M$. To decide stability, one needs to check the value of $\frac{\partial^2 V_{eff}}{\partial r^2}$ at $r = r_p = 3M$,

$$\left.\frac{\partial^2 V_{eff}}{\partial r^2}\right|_{r=r_p} = \frac{3L^2}{r_p^4} - 12\frac{L^2 M}{r_p^5} = -\frac{L^2}{(3M)^4} < 0 \tag{13}$$

Hence, this orbit is unstable.

Parts (c) and (d)

Radial null geodesics will have $\kappa = 0$ and $L = 0$. Hence, using (8),

$$\dot{r}^2 = E^2 \tag{14}$$

$$\implies \frac{\mathrm{d}r}{\mathrm{d}\lambda} = -E \quad \text{for ingoing rays} \tag{15}$$

$$\implies \int_{r_0}^{2M} \mathrm{d}r = -E\int_0^{\lambda_0} \mathrm{d}\lambda \tag{16}$$

$$\implies \lambda_0 = \frac{r_0 - 2M}{E} \implies \quad \text{finite} \tag{17}$$

Also, using (2), and assuming null radial orbits, we get,

$$\frac{\dot{t}^2}{\dot{r}^2} = \left(\frac{\mathrm{d}t}{\mathrm{d}r}\right)^2 = \left(1-\frac{2M}{r}\right)^{-2} \tag{18}$$

$$\implies t = \pm\int \mathrm{d}r\left(1-\frac{2M}{r}\right)^{-1} + C \tag{19}$$

$$\implies t = \pm\left[r - r_0 + 2M\log\left(\frac{r-2M}{r_0-2M}\right)\right] + t_0 \tag{20}$$

Sketch the rays

Problem 2

The area of a black hole $A \propto M^2$. The area theorem hence implies,

$$M_3^2 \geq M_1^2 + M_2^2 \tag{21}$$

Consider the two numbers M_1^2 and M_2^2. Let's write down the $AM \geq GM$ inequality *ie.*,

$$\frac{M_1^2 + M_2^2}{2} \geq M_1 M_2 \tag{22}$$

$$\implies M_1^2 + M_2^2 \geq \frac{M_1^2 + M_2^2}{2} + M_1 M_2 \tag{23}$$

$$\implies M_1^2 + M_2^2 \geq \frac{(M_1 + M_2)^2}{2} \tag{24}$$

Combining (21) and the last inequality, we have,

$$M_3^2 \geq \frac{(M_1 + M_2)^2}{2} \implies \frac{M_3}{M_1 + M_2} \geq \frac{1}{\sqrt{2}} \tag{25}$$

Hence,

$$\eta \leq 1 - \frac{1}{\sqrt{2}} \tag{26}$$