

# Advanced Statistical Mechanics: Assignment #2

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## Problem 1

The two-particle Virial Coefficient  $b_2$  is given by,

$$\begin{aligned} b_2 &= \int d^d \vec{q}_1 d^d \vec{q}_2 U(\vec{q}_1 - \vec{q}_2) \\ &= AS_{d-1}^2 \int dq_1 dq_2 q_1^{d-1} q_2^{d-1} \frac{1}{|q_1 - q_2|^\sigma} \end{aligned}$$

## Problem 2

**Do Part (a)**

**Part (b)**

Let's denote the Vandermonde determinant by  $D_n$ ,

$$D_n = \begin{vmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{vmatrix}$$

We prove the required statement by using row and column operations. We first use  $R_n \rightarrow R_n - R_{n-1}, n = 2, 3, \dots, n$ . We then have,

$$D_n = \begin{vmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 0 & x_2 - x_1 & x_2^2 - x_1^2 & \dots & x_2^{n-1} - x_1^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & x_n - x_1 & x_n^2 - x_1^2 & \dots & x_n^{n-1} - x_1^{n-1} \end{vmatrix}$$

We now proceed to make the topmost row elements 0, save for the first element. We use  $C_n \rightarrow C_n - x_1 C_{n-1}, n = 2, 3, \dots, n$ ,

$$\begin{aligned}
D_n &= \begin{vmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & x_2 - x_1 & x_2(x_2 - x_1) & \dots & x_2^{n-2}(x_2 - x_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & x_n - x_1 & x_n(x_n - x_1) & \dots & x_n^{n-2}(x_n - x_1) \end{vmatrix} \\
&= \begin{vmatrix} x_2 - x_1 & x_2(x_2 - x_1) & \dots & x_2^{n-2}(x_2 - x_1) \\ \vdots & \vdots & \ddots & \vdots \\ x_n - x_1 & x_n(x_n - x_1) & \dots & x_n^{n-2}(x_n - x_1) \end{vmatrix} \\
&= \left( \prod_{i=2}^n x_i - x_1 \right) \begin{vmatrix} 1 & x_2 & x_2^2 & \dots & x_2^{n-2} \\ 1 & x_3 & x_3^2 & \dots & x_3^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-2} \end{vmatrix}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow D_n &= \left( \prod_{i=2}^n x_i - x_1 \right) D_{n-1} \\
D_n &= \left( \prod_{i=2}^n x_i - x_1 \right) \left( \prod_{i=3}^{n-1} x_i - x_2 \right) D_{n-2} \\
&= \left( \prod_{i=2}^n x_i - x_1 \right) \left( \prod_{i=3}^{n-1} x_i - x_2 \right) \dots D_2 \\
D_n &= \prod_{1 \leq j < i \leq n} x_i - x_j
\end{aligned}$$

Hence Proved.

## Problem 4

For  $N$  particles in a harmonic trap,

$$\begin{aligned}
\psi(x_1, x_2, \dots, x_N) &= \frac{1}{\sqrt{N!}} \det \phi_j(x_j) = \frac{1}{N!} \det A_{ij} \\
\text{where } \phi_i(x) &= \left( \frac{a^2}{\pi} \right)^{1/4} \frac{1}{\sqrt{2^i i!}} H_i(ax) e^{-a^2 x^2/2}, \quad a^2 = \frac{m\omega}{\hbar}
\end{aligned}$$

The probability density is,

$$P(\{x_i\}) = \frac{1}{N!} \det A^T A = \frac{1}{N!} \det K$$

We define the average number density as,

$$\begin{aligned}
\langle \rho(x) \rangle &= \sum_i \left\langle \frac{1}{N} \delta(x - x_i) \right\rangle \\
&= \frac{1}{N} \sum_i \int \prod_j dx_j \delta(x - x_i) \frac{1}{N!} \det K
\end{aligned}$$