Electromagnetism: Pset #1

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Problem 1

Part (a)

- $\delta^i_i = \delta^1_1 + \delta^2_2 + \delta^3_3 = 3.$
- $\delta^{ij}\epsilon_{ijk} = \epsilon_{iik} = 0$
- $\epsilon^{ijk}\epsilon_{mjk} = 3(\delta^i_m\delta^j_j - \delta^j_m\delta^i_j) = 9\delta^i_m - 3\delta^i_m = 6\delta^i_m$
- $\epsilon^{ijk}\epsilon_{ijk} = \delta^i_i\delta^j_j - \delta^j_i\delta^i_j = 3^2 - 3 = 6$

Part (b)

$$\begin{aligned}
B^i = \epsilon^{ijk}\partial_j A_k &\implies \epsilon_{ijk}B^k = \epsilon_{ijk}\epsilon^{kab}\partial_a A_b \\
&\implies \epsilon_{ijk}B^k = \epsilon_{ijk}\epsilon^{abk}\partial_a A_b \\
&\implies \epsilon_{ijk}B^k = (\delta^a_i\delta^b_j - \delta^b_i\delta^a_j)\partial_a A_b \\
&\implies \epsilon_{ijk}B^k = \partial_i A_j - \partial_j A_i
\end{aligned}$$

Part (c)

$$\begin{aligned}
[\boldsymbol{\nabla}\times(\vec{\mathbf{A}}\times\vec{\mathbf{B}})]_i &= \epsilon_{ijk}\partial_j(\vec{\mathbf{A}}\times\vec{\mathbf{B}})_k \\
&= \epsilon_{ijk}\partial_j\epsilon_{kab}A_aB_b \\
&= \epsilon_{ijk}\epsilon_{abk}\partial_j A_aB_b \\
&= (\delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja})\partial_j A_aB_b \\
&= \partial_j A_iB_j - \partial_j A_jB_i \\
&= A_i\partial_j B_j + B_j\partial_j A_i - B_i\partial_j A_j - A_j\partial_j B_i \\
[\boldsymbol{\nabla}\times(\vec{\mathbf{A}}\times\vec{\mathbf{B}})]_i &= [\vec{\mathbf{A}}(\boldsymbol{\nabla}\cdot\vec{\mathbf{B}})]_i + [(\vec{\mathbf{B}}\cdot\boldsymbol{\nabla})\vec{\mathbf{A}}]_i - [\vec{\mathbf{B}}(\boldsymbol{\nabla}\cdot\vec{\mathbf{A}})]_i - [(\vec{\mathbf{A}}\cdot\boldsymbol{\nabla})\vec{\mathbf{B}}]_i
\end{aligned}$$

Hence Proved.

$$\begin{aligned}
[\vec{\mathbf{A}}\times(\boldsymbol{\nabla}\times\vec{\mathbf{B}})]_i &= \epsilon_{ijk}A_j(\boldsymbol{\nabla}\times\vec{\mathbf{B}})_k \\
&= \epsilon_{ijk}A_j\epsilon_{kab}\partial_a B_b \\
&= (\delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja})A_j\partial_a B_b \\
[\vec{\mathbf{A}}\times(\boldsymbol{\nabla}\times\vec{\mathbf{B}})]_i &= A_b\partial_i B_b - A_j\partial_j B_i \\
[\vec{\mathbf{B}}\times(\boldsymbol{\nabla}\times\vec{\mathbf{A}})]_i &= B_b\partial_i A_b - B_j\partial_j A_i
\end{aligned}$$

$$\begin{aligned}
[\vec{\mathbf{A}} \times (\boldsymbol{\nabla} \times \vec{\mathbf{B}})]_i + [\vec{\mathbf{B}} \times (\boldsymbol{\nabla} \times \vec{\mathbf{A}})]_i &= A_b \partial_i B_b - A_j \partial_j B_i + B_b \partial_i A_b - B_j \partial_j A_i \\
&= \partial_i A_b B_b - A_j \partial_j B_i - B_j \partial_j A_i \\
[\vec{\mathbf{A}} \times (\boldsymbol{\nabla} \times \vec{\mathbf{B}})]_i + [\vec{\mathbf{B}} \times (\boldsymbol{\nabla} \times \vec{\mathbf{A}})]_i &= [\boldsymbol{\nabla}(\vec{\mathbf{A}} \cdot \vec{\mathbf{B}})]_i - [(\vec{\mathbf{A}} \cdot \boldsymbol{\nabla})\vec{\mathbf{B}}]_i - [(\vec{\mathbf{B}} \cdot \boldsymbol{\nabla})\vec{\mathbf{A}}]_i \\
[\boldsymbol{\nabla}(\vec{\mathbf{A}} \cdot \vec{\mathbf{B}})]_i &= [\vec{\mathbf{A}} \times (\boldsymbol{\nabla} \times \vec{\mathbf{B}})]_i + [\vec{\mathbf{B}} \times (\boldsymbol{\nabla} \times \vec{\mathbf{A}})]_i + [(\vec{\mathbf{A}} \cdot \boldsymbol{\nabla})\vec{\mathbf{B}}]_i + [(\vec{\mathbf{B}} \cdot \boldsymbol{\nabla})\vec{\mathbf{A}}]_i
\end{aligned}$$

Hence Proved.

$$\begin{aligned}
[\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \vec{\mathbf{A}})]_i &= \epsilon_{ijk} \partial_j (\boldsymbol{\nabla} \times \vec{\mathbf{A}})_k \\
&= \epsilon_{ijk} \partial_j \epsilon_{kab} \partial_a A_b \\
&= \epsilon_{ijk} \epsilon_{abk} \partial_j \partial_a A_b \\
&= (\delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja}) \partial_j \partial_a A_b \\
&= \partial_j \partial_i A_j - \partial_j \partial_j A_i \\
[\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \vec{\mathbf{A}})]_i &= [\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \vec{\mathbf{A}})]_i - [\nabla^2 \vec{\mathbf{A}}]_i
\end{aligned}$$

Hence Proved.

Problem 2

Part (a)
Consider $(\vec{\mathbf{A}} \cdot \vec{\mathbf{dl}}) \wedge (\vec{\mathbf{B}} \cdot \vec{\mathbf{dl}})$,

$$\begin{aligned}
(\vec{\mathbf{A}} \cdot \vec{\mathbf{dl}}) \wedge (\vec{\mathbf{B}} \cdot \vec{\mathbf{dl}}) &= (A_x \,\mathrm{d}x + A_y \,\mathrm{d}y + A_z \,\mathrm{d}z) \wedge (B_x \,\mathrm{d}x + B_y \,\mathrm{d}y + B_z \,\mathrm{d}z) \\
&= (A_x B_y - A_y B_x) \,\mathrm{d}x \,\mathrm{d}y + (A_y B_z - A_z B_y) \,\mathrm{d}y \,\mathrm{d}z + (A_z B_x - A_x B_z) \,\mathrm{d}z \,\mathrm{d}x \\
&= (A_x B_y - A_y B_x) \,\mathrm{d}x \,\mathrm{d}y + (A_y B_z - A_z B_y) \,\mathrm{d}y \,\mathrm{d}z + (A_z B_x - A_x B_z) \,\mathrm{d}z \,\mathrm{d}x \\
&= (\vec{\mathbf{A}} \times \vec{\mathbf{B}}) \cdot \mathrm{d}\vec{\mathbf{a}}
\end{aligned}$$

$$\begin{aligned}
(\vec{\mathbf{A}} \cdot \vec{\mathbf{dl}}) \wedge (\vec{\mathbf{B}} \cdot \vec{\mathbf{dl}}) \wedge (\vec{\mathbf{C}} \cdot \vec{\mathbf{dl}}) &= (A_x \,\mathrm{d}x + A_y \,\mathrm{d}y + A_z \,\mathrm{d}z) \wedge (B_x \,\mathrm{d}x + B_y \,\mathrm{d}y + B_z \,\mathrm{d}z) \wedge (C_x \,\mathrm{d}x + C_y \,\mathrm{d}y + C_z \,\mathrm{d}z) \\
&= (A_x B_y C_z - A_x B_z C_y + A_y B_z C_x - A_y B_x C_z + A_z B_x C_y - A_z B_y C_x) \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z \\
&= [(A_x B_y - A_y B_z) C_z - (A_x B_z - A_z B_x) C_y + (A_y B_z - A_z B_y) C_x] \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z \\
&= (\vec{\mathbf{A}} \times \vec{\mathbf{B}}) \cdot \vec{\mathbf{C}} \,\mathrm{d}\forall
\end{aligned}$$

Consider spherical coordinates first,

$$\begin{aligned}
(\vec{\mathbf{A}} \cdot \vec{\mathbf{dl}}) \wedge (\vec{\mathbf{B}} \cdot \vec{\mathbf{dl}}) &= (A_r \mathrm{d}r + A_\theta r \mathrm{d}\theta + A_\phi r \sin\theta \mathrm{d}\phi) \wedge (B_r \mathrm{d}r + B_\theta r \mathrm{d}\theta + B_\phi r \sin\theta \mathrm{d}\phi) \\
&= (A_r B_\theta - A_\theta B_r) r \,\mathrm{d}r \,\mathrm{d}\theta + (A_\theta B_\phi - A_\phi B_\theta) r^2 \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\phi + (-A_\phi B_r + A_r B_\phi) r \sin\theta \,\mathrm{d}\phi \,\mathrm{d}r
\end{aligned}$$

Comparing with the expression for $(\vec{\mathbf{A}} \times \vec{\mathbf{B}}) \cdot \mathrm{d}\vec{\mathbf{a}}$, one gets,

$$\mathrm{d}\vec{\mathbf{a}} = r^2 \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\phi \,\hat{\mathbf{r}} + r \sin\theta \,\mathrm{d}\phi \,\mathrm{d}r \,\hat{\theta} + r \,\mathrm{d}r \,\mathrm{d}\theta \,\hat{\phi}$$

Similarly,

$$\begin{aligned}
(\vec{\mathbf{A}} \cdot \vec{\mathbf{dl}}) \wedge (\vec{\mathbf{B}} \cdot \vec{\mathbf{dl}}) \wedge (\vec{\mathbf{C}} \cdot \vec{\mathbf{dl}}) &= (\vec{\mathbf{A}} \cdot \vec{\mathbf{dl}}) \wedge (\vec{\mathbf{B}} \cdot \vec{\mathbf{dl}}) \wedge (C_r \mathrm{d}r + C_\theta r \mathrm{d}\theta + C_\phi r \sin\theta \mathrm{d}\phi) \\
&= [C_\phi (A_r B_\theta - A_\theta B_r) + C_r (A_\theta B_\phi - A_\phi B_\theta) + C_\theta (-A_\phi B_r + A_r B_\phi)] r^2 \sin\theta \,\mathrm{d}r \,\mathrm{d}\theta \,\mathrm{d}\phi
\end{aligned}$$

$$\therefore \mathrm{d}\forall = r^2 \sin\theta \,\mathrm{d}r \,\mathrm{d}\theta \,\mathrm{d}\phi$$

We repeat the same exercise for cylindrical coordinates,

$$\begin{aligned}(\vec{\mathbf{A}} \cdot \vec{\mathbf{dl}}) \wedge (\vec{\mathbf{B}} \cdot \vec{\mathbf{dl}}) &= (A_s \,\mathrm{d}s + A_\phi s \,\mathrm{d}\phi + A_z \,\mathrm{d}z) \wedge (B_s \,\mathrm{d}s + B_\phi s \,\mathrm{d}\phi + B_z \,\mathrm{d}z) \\ &= (A_s B_\phi - A_\phi B_s) s \,\mathrm{d}s \,\mathrm{d}\phi + (A_z B_s - A_s B_z) \,\mathrm{d}s \,\mathrm{d}z + (A_\phi B_z - A_z B_\phi) s \,\mathrm{d}\phi \,\mathrm{d}z\end{aligned}$$

$$\therefore \vec{\mathrm{d}\mathbf{a}} = s \,\mathrm{d}s \,\mathrm{d}\phi \,\hat{\mathbf{z}} + \mathrm{d}s \,\mathrm{d}z \,\hat{\phi} + s \,\mathrm{d}\phi \,\mathrm{d}z \,\hat{\mathbf{s}}$$

$$(\vec{\mathbf{A}} \cdot \vec{\mathbf{dl}}) \wedge (\vec{\mathbf{B}} \cdot \vec{\mathbf{dl}}) \wedge (\vec{\mathbf{C}} \cdot \vec{\mathbf{dl}}) = [C_z(A_s B_\phi - A_\phi B_s) + C_\phi(A_z B_s - A_s B_z) + C_s(A_\phi B_z - A_z B_\phi)] s \,\mathrm{d}s \,\mathrm{d}\phi \,\mathrm{d}z$$

$$\therefore \mathrm{d}\forall = s \,\mathrm{d}s \,\mathrm{d}\phi \,\mathrm{d}z$$

Part (b)

$$\begin{aligned}(\mathrm{d}x + \mathrm{d}y - \mathrm{d}z) \wedge (\mathrm{d}x + \mathrm{d}y + \mathrm{d}z) &= \mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}x \wedge \mathrm{d}z + \mathrm{d}y \wedge \mathrm{d}x + \mathrm{d}y \wedge \mathrm{d}z - \mathrm{d}z \wedge \mathrm{d}x - \mathrm{d}z \wedge \mathrm{d}y \\ &= \mathrm{d}x \,\mathrm{d}y - \mathrm{d}z \,\mathrm{d}x - \mathrm{d}x \,\mathrm{d}y + \mathrm{d}y \,\mathrm{d}z - \mathrm{d}z \,\mathrm{d}x + \mathrm{d}y \,\mathrm{d}z \\ &= 2(\mathrm{d}y \,\mathrm{d}z - \mathrm{d}z \,\mathrm{d}x)\end{aligned}$$

$$\begin{aligned}[(x-y) \,\mathrm{d}x + (x+y) \,\mathrm{d}y + z \,\mathrm{d}z] \wedge [(x-y) \,\mathrm{d}x + (x+y) \,\mathrm{d}y] &= (x^2 - y^2) \,\mathrm{d}x \wedge \mathrm{d}y + (x^2 - y^2) \,\mathrm{d}y \wedge \mathrm{d}x \\ &\quad + z(x-y) \,\mathrm{d}z \wedge \mathrm{d}x + z(x+y) \,\mathrm{d}z \wedge \mathrm{d}y \\ &= z(x-y) \,\mathrm{d}z \,\mathrm{d}x - z(x+y) \,\mathrm{d}y \,\mathrm{d}z\end{aligned}$$

Part (c)

Let ω be the 2-form in R^4. The general expression for ω is,

$$\omega = \sum_{i \neq j} a_{ij} \,\mathrm{d}x^i \wedge \mathrm{d}x^j$$

We need to look for a 2-form that satisfies $\omega \wedge \omega = 0$,

$$\begin{aligned}\omega \wedge \omega &= \sum_{i \neq j} a_{ij} \,\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \sum_{m \neq n} a_{mn} \,\mathrm{d}x^m \wedge \mathrm{d}x^n \\ &= (a_{12} - a_{21}) \,\mathrm{d}x_1 \wedge \mathrm{d}x_2 + (a_{23} - a_{32}) \,\mathrm{d}x_2 \wedge \mathrm{d}x_3 + (a_{34} - a_{43}) \,\mathrm{d}x_3 \wedge \mathrm{d}x_4 + (a_{24} - a_{42}) \,\mathrm{d}x_2 \wedge \mathrm{d}x_4 \\ &\quad + (a_{13} - a_{31}) \,\mathrm{d}x_1 \wedge \mathrm{d}x_3 + (a_{14} - a_{41}) \,\mathrm{d}x_1 \wedge \mathrm{d}x_4 \\ &= b_{12} \,\mathrm{d}x_1 \wedge \mathrm{d}x_2 + b_{23} \,\mathrm{d}x_2 \wedge \mathrm{d}x_3 + b_{34} \,\mathrm{d}x_3 \wedge \mathrm{d}x_4 + b_{13} \,\mathrm{d}x_1 \wedge \mathrm{d}x_3 + b_{14} \,\mathrm{d}x_1 \wedge \mathrm{d}x_4 + b_{24} \,\mathrm{d}x_2 \wedge \mathrm{d}x_4\end{aligned}$$

$$\omega \wedge \omega = 0 \implies b_{12} b_{34} - b_{13} b_{24} + b_{14} b_{23} = 0$$

The above expression reminds us of the determinant of a 3×3 matrix where,

$$\begin{aligned}b_{ij} = x_i y_j - x_j y_i &\implies a_{ij} = x_i y_j \quad \text{(by comparison)} \\ \therefore \omega &= \sum_{i \neq j} x_i y_j \,\mathrm{d}x^i \wedge \mathrm{d}x^j \\ &= \sum_{i \neq j} x_i \,\mathrm{d}x^i \wedge y_j \,\mathrm{d}x^j \\ \omega &= \sum_{i,j} x_i \,\mathrm{d}x^i \wedge y_j \,\mathrm{d}x^j\end{aligned}$$

Hence proved that ω can be written as the wedge product of two 1-forms.

Problem 3

Part (a)
Given,

$$
\begin{aligned}
\Omega &= \frac{1}{p!}\Omega_{i_1 i_2 \ldots i_m \ldots i_p}\, \mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \ldots \wedge \mathrm{d}x^{i_m} \ldots \mathrm{d}x^{i_p} \\
&= \frac{1}{p!}\Omega_{i_m i_2 \ldots i_1 \ldots i_p}\, \mathrm{d}x^{i_m} \wedge \mathrm{d}x^{i_2} \ldots \wedge \mathrm{d}x^{i_1} \ldots \mathrm{d}x^{i_p} \\
&= -\frac{1}{p!}\Omega_{i_m i_2 \ldots i_1 \ldots i_p}\, \mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \ldots \wedge \mathrm{d}x^{i_m} \ldots \mathrm{d}x^{i_p} \\
\implies \Omega_{i_m i_2 \ldots i_1 \ldots i_p} &= -\Omega_{i_1 i_2 \ldots i_m \ldots i_{p+1}}\, \mathrm{d}x^{i_1} \implies \quad \text{antisymmetric}
\end{aligned}
$$

$$
\begin{aligned}
d\Omega &= \frac{1}{p!} d\Omega_{i_1 i_2 \ldots i_m \ldots i_p}\, \mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \ldots \wedge \mathrm{d}x^{i_m} \ldots \mathrm{d}x^{i_p} \\
&= \frac{1}{p!}\left(\partial_k \Omega_{i_1 i_2 \ldots i_m \ldots i_p}\, \mathrm{d}x^{i_k}\right) \wedge \mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \ldots \wedge \mathrm{d}x^{i_m} \ldots \mathrm{d}x^{i_p} \\
&= \frac{1}{p!}\left(\partial_k \Omega_{i_1 i_2 \ldots i_m \ldots i_{p+1}}\, \mathrm{d}x^{i_k}\right) \wedge \mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \ldots \wedge \mathrm{d}x^{i_m} \ldots \mathrm{d}x^{i_{p+1}}
\end{aligned}
$$

Where in the last step, we have assumed that x_{i_k} is not in the wedge products to the right of the brackets. Now, we need to shift the $\mathrm{d}x^{i_k}$ to it's position in the serial order. This will require us to do neighbouring flips $k-1$ times. Then,

$$
d\Omega = (-1)^{(k-1)p}\frac{1}{p!}\partial_k \Omega_{i_1 i_2 \ldots i_m \ldots i_{p+1}}\, \mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \ldots \wedge \mathrm{d}x^{i_m} \ldots \mathrm{d}x^{i_{p+1}}
$$

We can permute the indices $p+1$ times and add them, so as to get,

$$
\begin{aligned}
(p+1)\mathrm{d}\Omega &= \frac{1}{p!}(\mathrm{d}\Omega)_{i_1 i_2 \ldots i_m \ldots i_{p+1}}\, \mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \ldots \wedge \mathrm{d}x^{i_m} \ldots \mathrm{d}x^{i_{p+1}} \\
\mathrm{d}\Omega &= \frac{1}{(p+1)!}(\mathrm{d}\Omega)_{i_1 i_2 \ldots i_m \ldots i_{p+1}}\, \mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \ldots \wedge \mathrm{d}x^{i_m} \ldots \mathrm{d}x^{i_{p+1}}
\end{aligned}
$$

As the wedge products are antisymmetric under the exchange of indices, the only way that LHS can be invariant is if $(\mathrm{d}\Omega)_{i_1 i_2 \ldots i_m \ldots i_{p+1}}$ is also antisymmetric in the exchange of indices.

Problem 4

Part (a)

$$
\begin{aligned}
\int_C x^3\, \mathrm{d}x + \int_C \left(\frac{x^3}{3} + xy^2\right) \mathrm{d}y &= \int_{x^2+y^2 \le 4} \left[-\frac{\partial x^3}{\partial y} + \frac{\partial}{\partial x}\left(\frac{x^3}{3} + xy^2\right)\right] \mathrm{d}x\, \mathrm{d}y \\
&= \int_{x^2+y^2 \le 4} \left[x^2 + y^2\right] \mathrm{d}x\, \mathrm{d}y \\
&= \int_0^2 r^3\, \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta \\
&= 8\pi
\end{aligned}
$$

Part (b)
The generalized Stokes Theorem tells us,

$$\int_\Gamma \mathrm{d}\omega = \int_{\partial\Gamma} \omega$$
$$\int_\Gamma \mathrm{d}(z^2\,\mathrm{d}x \wedge \mathrm{d}y) = \int_{\partial\Gamma} z^2\,\mathrm{d}x \wedge \mathrm{d}y$$
$$\int_\Gamma (2z\,\mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}y) = \int_{\partial\Gamma} z^2\,\mathrm{d}x \wedge \mathrm{d}y$$
$$\int_0^1 \rho\,\mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\phi \int_0^1 2z\,\mathrm{d}z = \int_{\partial\Gamma} z^2\,\mathrm{d}x \wedge \mathrm{d}y$$
$$\int_{\partial\Gamma} z^2\,\mathrm{d}x \wedge \mathrm{d}y = \pi$$

Part (c)
For spherical coordinates,

$$\mathrm{d}\vec{\mathbf{a}} = r^2 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\,\hat{\mathbf{r}} + r\sin\theta\,\mathrm{d}\phi\,\mathrm{d}r\,\hat{\theta} + r\,\mathrm{d}r\,\mathrm{d}\theta\,\hat{\phi}$$

$$\begin{aligned}\mathrm{d}(\vec{\mathbf{A}}\cdot\vec{\mathbf{dl}}) &= \mathrm{d}(A_r\mathrm{d}r + A_\theta r\mathrm{d}\theta + A_\phi r\sin\theta\mathrm{d}\phi)\\ &= -\partial_\theta A_r\,\mathrm{d}r\,\mathrm{d}\theta + \partial_\phi A_r\,\mathrm{d}\phi\,\mathrm{d}r + (r\partial_r A_\theta + A_\theta)\,\mathrm{d}r\,\mathrm{d}\theta - r\partial_\phi A_\theta\,\mathrm{d}\theta\,\mathrm{d}\phi - \sin\theta(r\partial_r A_\phi + A_\phi)\,\mathrm{d}\phi\,\mathrm{d}r\\ &\quad + r(A_\phi\cos\theta + \partial_\theta\sin\theta)\,\mathrm{d}\theta\,\mathrm{d}\phi\\ &= (-\partial_\theta A_r + (r\partial_r A_\theta + A_\theta))\,\mathrm{d}r\,\mathrm{d}\theta + [\partial_\phi A_r - \sin\theta(r\partial_r A_\phi + A_\phi)]\,\mathrm{d}\phi\,\mathrm{d}r + [r(A_\phi\cos\theta + \partial_\theta\sin\theta) - r\partial_\phi A_\theta]\,\mathrm{d}\theta\,\mathrm{d}\phi\end{aligned}$$

Comparing with $\nabla \times \vec{\mathbf{A}} \cdot \mathrm{d}\vec{\mathbf{a}}$, we get,

$$\nabla\times\vec{\mathbf{A}} = \frac{1}{\sin\theta}[(A_\phi\cos\theta + \partial_\theta\sin\theta) - \partial_\phi A_\theta]\hat{\mathbf{r}} + \frac{1}{r\sin\theta}[\partial_\phi A_r - \sin\theta(r\partial_r A_\phi + A_\phi)]\hat{\theta} + \frac{1}{r}(-\partial_\theta A_r + (r\partial_r A_\theta + A_\theta))\hat{\phi}$$

$$\begin{aligned}\mathrm{d}(\vec{\mathbf{B}}\cdot\vec{\mathrm{d}\mathbf{a}}) &= \mathrm{d}(B_r r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi + B_\theta r\sin\theta\,\mathrm{d}\phi\,\mathrm{d}r + B_\phi r\,\mathrm{d}r\,\mathrm{d}\theta)\\ &= [\partial_r(B_r r^2\sin\theta) + \partial_\theta(B_\theta r\sin\theta) + \partial_\phi(B_\phi r)]r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi\\ \implies \nabla\cdot\vec{\mathbf{B}} &= \partial_r(B_r r^2\sin\theta) + \partial_\theta(B_\theta r\sin\theta) + \partial_\phi(B_\phi r)\end{aligned}$$

For cylindrical coordinates,

$$\mathrm{d}\vec{\mathbf{a}} = s\,\mathrm{d}s\,\mathrm{d}\phi\,\hat{\mathbf{z}} + \mathrm{d}s\,\mathrm{d}z\,\hat{\phi} + s\,\mathrm{d}\phi\,\mathrm{d}z\,\hat{\mathbf{s}}$$

$$\begin{aligned}\mathrm{d}(\vec{\mathbf{A}}\cdot\vec{\mathbf{dl}}) &= \mathrm{d}(A_s\mathrm{d}s + A_\phi s\mathrm{d}\phi + A_z\mathrm{d}z)\\ &= (A_\phi + s\partial_s A_\phi - \partial_\phi A_s)\,\mathrm{d}s\,\mathrm{d}\phi + (s\partial_z A_\phi - \partial_\phi A_z)\,\mathrm{d}\phi\,\mathrm{d}z + (\partial_z A_s - \partial_s A_z)\,\mathrm{d}z\,\mathrm{d}s\end{aligned}$$

Comparing with $\nabla \times \vec{\mathbf{A}} \cdot \mathrm{d}\vec{\mathbf{a}}$, we get,

$$\nabla\times\vec{\mathbf{A}} = \frac{1}{s}(s\partial_z A_\phi - \partial_\phi A_z)\hat{\mathbf{s}} + (\partial_z A_s - \partial_s A_z)\hat{\phi} + \frac{1}{2}(A_\phi + s\partial_s A_\phi - \partial_\phi A_s)\hat{\mathbf{z}}$$

$$\begin{aligned}\mathrm{d}(\vec{\mathbf{B}}\cdot\vec{\mathrm{d}\mathbf{a}}) &= \mathrm{d}(B_z s\,\mathrm{d}s\,\mathrm{d}\phi + B_\phi\mathrm{d}s\mathrm{d}z + B_s s\,\mathrm{d}\phi\,\mathrm{d}z)\\ &= [\partial_z B_z + \frac{1}{s}\partial_\phi B_\phi + \frac{1}{s}\partial_s(B_s s)]s\mathrm{d}s\mathrm{d}z\mathrm{d}\phi\\ \implies \nabla\cdot\vec{\mathbf{B}} &= \partial_z B_z + \frac{1}{s}\partial_\phi B_\phi + \frac{1}{s}\partial_s(B_s s)\end{aligned}$$