# Classical Mechanics - Assignment 7

# Endterm Preparatory

# Due date: Dec 5, 2018

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***Note:*** Submit the assignment to any one of TA's office on/before the due date. For the numerical parts of the questions take print outs of the codes along with the plots, etc. and attach it in the correct place of your solution. If you have any doubt regarding the assignment problems or the topics discussed in the class, feel free to discuss with TAs or the instructor. For numerical parts use any of your favourite programming language and plotting software. Good luck!

**Q1** $\boxed{\text{25 marks}}$

**Rigid body kinematics**

(a) Derive the generators $M_i$'s of infinitesimal rotations (Goldstein eq. 4.80) and show that they satisfy the commutation relations $\boxed{15}$

$$[M_i, M_j] = \epsilon_{ijk} M_k.$$

(b) A projectile is fired horizontally along Earth's surface. Show that to a first approximation the angular deviation from the direction of fire resulting from the Coriolis effect varies linearly with time at a rate $\omega \cos\theta$, where $\omega$ is the angular frequency of Earth's rotation and $\theta$ is the co-latitude, the direction of deviation being to the right in the northern hemisphere. $\boxed{10}$

**Q2** $\boxed{\text{25 marks}}$

**Rigid body dynamics**

(a) A wheel rolls down a flat inclined surface that makes an angle $\alpha$ with the horizontal. The wheel is constrained so that its plane is always perpendicular to the inclined plane, but it may rotate about the axis normal to the surface. Obtain the solution for the two-dimensional motion of the wheel, using Lagrange's equations and the method of undetermined multipliers. $\boxed{15}$

(b) Apply Euler's equations to the problem of the heavy symmetrical top, expressing $\omega_i$ in terms of the Euler angles. Show that the two integrals of motion, (Goldstein Eqs. 5.53 and 6.54), can be obtained directly from Euler's equations in this form. $\boxed{10}$

**Q3** $\boxed{\text{25 marks}}$

**Hamiltonian**

(a) If the canonical variables are not all independent, but are connected by auxiliary conditions of the form $\psi_k(q_i, p_i, t) = 0$, show that the canonical equations of motion can be written $\boxed{15}$

$$\frac{\partial H}{\partial p_i} + \sum_k \lambda_k \frac{\partial \psi_k}{\partial p_i} = \dot{q}_i, \quad \frac{\partial H}{\partial q_i} + \sum_k \lambda_k \frac{\partial \psi_k}{\partial q_i} = -\dot{p}_i,$$

where $\lambda_k$ are the undetermined Lagrange multipliers. The formulation of Hamiltonian equations in which t is a canonical variable is a case in point, since a relation exists between $p_{n+1}$ and the other canonical variables: $H(q_1, ..., q_{n+1}; p_1, ..., p_n) + p_{n+1} = 0$. Show that as a result of these circumstances the $2n+2$ Hamilton's equations of this formulation can be reduced to the $2n$ ordinary Hamilton's equations plus $\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$ and the relation $\lambda = dt/d\theta$.

(b) The Lagrangian for a system of one degree of freedom can be written as $L = \frac{m}{2}(\dot{q}^2 \sin^2(\omega t) + \dot{q}q\omega \sin(2\omega t) + q^2\omega^2)$. What is the corresponding Hamiltonian? Is it conserved? Introduce a new coordinate defined by $Q = q\sin(\omega t)$. Find the Lagrangian in terms of the new coordinate and the corresponding Hamiltonian. Is $H$ conserved? $\boxed{10}$

**Q4** $\boxed{\text{25 marks}}$

**Poisson brackets**

(a) Set up the problem of the spherical pendulum in the Hamiltonian formulation, using spherical polar coordinates for the $q_i$. Evaluate directly in terms of these canonical variables the following Poisson brackets $[L_i, L_j]$. Why is it that $p_\theta, p_\psi$ can be used as canonical momenta, although they are perpendicular components of the angular momentum? $\boxed{15}$

(b) Show from the Poisson bracket condition for conserved quantities that the Laplace-Runge-Lenz vector $\vec{A} = \vec{p} \times \vec{L} - mk\vec{r}/r$ is a constant of motion for the Kepler problem. Also verify that $[A_i, L_i] = \epsilon_{ijk}A_k$ where $\vec{L}$ is the angular momentum of the system. $\boxed{10}$

**Q5** $\boxed{\text{25 marks}}$

**Rigid Body Dynamics** Consider a top made of a uniform disk of radius $R$, connected to the origin by a

top point.png

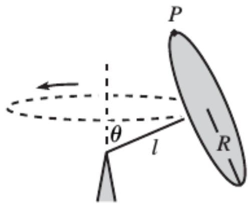


Figure 1:

massless stick (which is glued perpendicular to the disk) of length $l$. Paint a dot on the top at its highest point, and label this as point $P$ (see figure above). You wish to set up uniform circular precession, with the stick making a constant angle $\theta$ with the vertical ($\theta$ can be chosen to be any angle between zero and $pi$ ), and with $P$ always being the highest point on the top.

(a) What is the frequency of precession? $\boxed{10}$

(b) What relation between $R$ and $l$ must be satisfied for this motion to be possible? $\boxed{15}$

**Q6** $\boxed{\text{25 marks}}$

**Hamilton-Jacobi Method** Given a hamiltonian system with

$$T = \frac{1}{2}(q_1^2 + q_2^2)(\dot{q}_1^2 + \dot{q}_2^2), \;\; V = \frac{K}{q_1^2 + q_2^2}$$

Assume $K > 0$.

(a) Use Hamilton-Jacobi method to solve for path in configuration space. $\boxed{15}$

(b) Show that $\boxed{10}$

$$\int \frac{dq_1}{\sqrt{q_1^2 + a^2}} - \int \frac{dq_2}{\sqrt{q_2^2 - b^2}} = constant,$$

For some constants $a, b$.

**Q7** $\boxed{\text{15 marks}}$

**Hamilton-Jacobi Method**

(a) Use Hamilton-Jacobi Method to solve for the motion of the uniform disk of mass, $m$ and radius, $r$ which rolls without slipping down a plane that is inclined at angle $\gamma$ with the horizontal. Assume $\theta(0) = 0$, $\dot{\theta}(0) = 0$, where $\theta$ is the rotation angle. Evaluate the integral and solve for $\theta$ as a function of $t$. $\boxed{15}$

**Q8** $\boxed{\text{25 marks}}$

**Action-Angle Variables** A particle of mass $m$ moves in a one dimensional potential $V = \frac{1}{2}(kq^2 + \frac{\lambda}{q^2})$, where $k, \lambda, q > 0$.

(a) Find the energy in terms of action J. $\boxed{10}$

(b) Show that the time period is independent of amplitude. $\boxed{10}$

(c) Calculate $q$ as function of time $t$. $\boxed{5}$