

Electromagnetism: Pset #2

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Problem 1

Zangwill - Problem 24.1

Part (a) Given,

$$\begin{aligned}\frac{\partial F_i}{\partial \dot{r}_j} &= -\frac{\partial F_j}{\partial \dot{r}_i} \\ \implies \frac{\partial^2 F_i}{\partial \dot{r}_j \partial \dot{r}_k} &= -\frac{\partial^2 F_j}{\partial \dot{r}_i \partial \dot{r}_k} \\ \frac{\partial^2 F_i}{\partial \dot{r}_k \partial \dot{r}_j} &= -\frac{\partial^2 F_j}{\partial \dot{r}_k \partial \dot{r}_i} \\ -\frac{\partial^2 F_k}{\partial \dot{r}_i \partial \dot{r}_j} &= \frac{\partial^2 F_k}{\partial \dot{r}_j \partial \dot{r}_i} \\ \implies \frac{\partial^2 F_k}{\partial \dot{r}_j \partial \dot{r}_i} &= 0\end{aligned}$$

From Helmholtz first relation, we know that integrating the above equation should give us an object which is antisymmetric under the exchange of indices. We can take this into account by introducing our friendly neighbourhood antisymmetric object, namely the *Levi-Civita symbol*.

$$\implies \frac{\partial F_i}{\partial \dot{r}_j} = \epsilon_{ijk} Q_k(\mathbf{r}, t) \implies F_i = \epsilon_{ijk} Q_k(\mathbf{r}, t) \dot{r}_j + P(\mathbf{r}, t)$$

Hence proved.

Part (b) The second Helmholtz relation says,

$$\begin{aligned}\frac{\partial F_i}{\partial r_j} - \frac{\partial F_j}{\partial r_i} &= \frac{1}{2} \frac{d}{dt} \left(\frac{\partial F_i}{\partial \dot{r}_j} - \frac{\partial F_j}{\partial \dot{r}_i} \right) \\ \frac{\partial P_i}{\partial r_j} - \frac{\partial P_j}{\partial r_i} + \epsilon_{ijk} \left(\frac{\partial Q_k}{\partial r_j} + \frac{\partial Q_k}{\partial r_i} \right) &= \frac{d}{dt} (\epsilon_{ijk} Q_k(\mathbf{r}, t)) \\ \frac{\partial P_i}{\partial r_j} - \frac{\partial P_j}{\partial r_i} + \epsilon_{ijk} \left(\frac{\partial Q_k}{\partial r_j} + \frac{\partial Q_k}{\partial r_i} \right) &= \epsilon_{ijk} \left(\frac{\partial Q_k}{\partial t} + \frac{\partial Q_k}{\partial r_a} \dot{r}_a \right)\end{aligned}$$

As we can see, there are no terms involving \dot{r}_a on the LHS. Hence, the term involving \dot{r}_a on the RHS should be zero!

$$\implies \frac{\partial Q_k}{\partial r_a} \dot{r}_a = \vec{\nabla} Q = 0$$

Next, we multiply both sides of the equation with ϵ_{lmk} , and use the fact that $\epsilon_{lmk}\epsilon_{ijk} = (\delta_{li}\delta_{mj} - \delta_{lj}\delta_{mi})$,

$$\begin{aligned}\epsilon_{lmk}\left(\frac{\partial P_i}{\partial r_j} - \frac{\partial P_j}{\partial r_i}\right) + (\delta_{li}\delta_{mj} - \delta_{lj}\delta_{mi})\left(\frac{\partial Q_k}{\partial r_j} + \frac{\partial Q_k}{\partial r_i}\right) &= (\delta_{li}\delta_{mj} - \delta_{lj}\delta_{mi})\left(\frac{\partial Q_k}{\partial t}\right) \\ \epsilon_{lmk}\left(\frac{\partial P_i}{\partial r_j} - \frac{\partial P_j}{\partial r_i}\right) + \left(\frac{\partial Q_k}{\partial r_m} + \frac{\partial Q_k}{\partial r_l} - \frac{\partial Q_k}{\partial r_l} - \frac{\partial Q_k}{\partial r_m}\right) &= \left(\frac{\partial Q_k}{\partial t}\right) \\ \therefore \epsilon_{lmk}\left(\frac{\partial P_i}{\partial r_j} - \frac{\partial P_j}{\partial r_i}\right) &= \left(\frac{\partial Q_k}{\partial t}\right) \\ \therefore \vec{\nabla} \times P &= \frac{\partial Q}{\partial t}\end{aligned}$$

Zangwill - Problem 24.2

Part (a)

$$\begin{aligned}H(r, p) &= \frac{p^2}{2m} + g(r) + p^2 f(r) = \left(\frac{1+2fm}{2m}\right)p^2 + g(r) \\ \Rightarrow \dot{r} &= \frac{\partial H}{\partial p} = \frac{p}{m} + 2pf(r) \\ \Rightarrow p &= \frac{m\dot{r}}{1+2fm} \\ \therefore L(r, \dot{r}) &= \frac{m\dot{r}^2}{1+2fm} - \frac{m\dot{r}^2}{2(1+2fm)} - g(r) = \frac{m\dot{r}^2}{2(1+2fm)} - g(r)\end{aligned}$$

Part (b)

We write the Euler Lagrange equations for this Lagrangian,

$$\begin{aligned}\frac{d}{dt}\left(\frac{m\dot{r}}{1+2fm}\right) + \frac{\partial g}{\partial r} &= 0 \\ \frac{m\ddot{r}}{1+2fm} - \frac{m\dot{r}}{(1+2fm)^2}\dot{r}\frac{df}{dr} + \frac{dg}{dr} &= 0 \\ m\ddot{r} - \frac{m\dot{r}^2}{(1+2fm)}\frac{df}{dr} + (1+2fm)\frac{dg}{dr} &= 0\end{aligned}$$

As one can see from the above equation, the force depends on \ddot{r}, \dot{r}, f, g .

Zangwill - Problem 24.4

Let's first work with $c = 1$. We shall restore the factors of c at the end.

$$\begin{aligned}S &= -m \int ds - g \int ds \phi(\vec{r}(s)) \\ &= -m \int dt \frac{ds}{dt} - g \int dt \frac{ds}{dt} \phi(\vec{r}(s)) \\ S &= \int -\frac{m}{\gamma} dt + \int \frac{-g\phi}{\gamma} dt\end{aligned}$$

From the above form of action S , the Lagrangian is evidently,

$$\begin{aligned}L &= -\frac{m+g\phi}{\gamma} \quad \text{where} \quad \gamma = \frac{1}{\sqrt{1-\vec{v}\cdot\vec{v}}} = \frac{1}{\sqrt{1-\dot{\vec{r}}\cdot\dot{\vec{r}}}} \\ \therefore \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{r}}}\right) &= \frac{\partial L}{\partial \vec{r}} \\ \frac{d(m+g\phi)\gamma\dot{\vec{r}}}{dt} &= -\frac{g\vec{\nabla}\phi}{\gamma}\end{aligned}$$

Restoring the factors of c ,

$$\frac{d(m + g\phi/c)\gamma\vec{r}}{dt} = -\frac{gc\vec{\nabla}\phi}{\gamma}$$

This differs from the electric field force equation, and has an extra term on the left (second term).

Zangwill - Problem 24.11

Zangwill - Problem 24.12

$$L = \vec{j} \cdot \vec{A} - \rho\phi - \frac{1}{2}\epsilon_0(\vec{E}^2 - c^2\vec{B}^2) - \epsilon_0\vec{E} \cdot (\vec{\nabla}\phi + \dot{\vec{A}}) - \epsilon_0c^2\vec{B} \cdot (\vec{\nabla} \times \vec{A})$$

$$\frac{\partial L}{\partial \vec{E}} = -\epsilon_0\vec{E} - \epsilon_0(\vec{\nabla}\phi + \dot{\vec{A}}) \quad , \quad \frac{\partial L}{\partial \dot{\vec{E}}} = 0 \implies \vec{E} = -\vec{\nabla}\phi - \dot{\vec{A}}$$

$$\frac{\partial L}{\partial \vec{B}} = \epsilon_0c^2\vec{B} - \epsilon_0c^2\vec{\nabla} \times \vec{A} \quad , \quad \frac{\partial L}{\partial \dot{\vec{B}}} = 0 \implies \vec{B} = \vec{\nabla} \times \vec{A}$$

$$\frac{\partial L}{\partial \phi} = -\rho - \epsilon_0\frac{\partial \vec{E} \cdot \vec{\nabla}\phi}{\partial \phi} = -\rho + \epsilon_0\vec{\nabla} \cdot \vec{E} \quad , \quad \frac{\partial L}{\partial \dot{\phi}} = 0 \implies \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\frac{\partial L}{\partial \vec{A}} = \vec{j} - \epsilon_0c^2\frac{\partial \vec{B} \cdot (\vec{\nabla} \times \vec{A})}{\partial \vec{A}} = \vec{j} - \epsilon_0c^2\frac{\partial (\vec{A} \cdot (\vec{\nabla} \times \vec{B}) + \vec{\nabla} \cdot \vec{A} \times \vec{B})}{\partial \vec{A}} = \vec{j} - \epsilon_0c^2\vec{\nabla} \times \vec{B}$$

$$\frac{\partial L}{\partial \dot{\vec{A}}} = -\epsilon_0\vec{E} \implies \vec{\nabla} \times \vec{B} = \frac{\vec{j}}{\epsilon_0c^2} + \frac{1}{c^2}\dot{\vec{E}}$$

The first two equations can be rewritten as $\vec{\nabla} \cdot \vec{B} = 0$ and $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$. Hence we have all the Maxwell equations.

Primary Constraints?

Problem 2

Part (a)

Given that,

$$\begin{aligned} L_D &= \sum_a \left(\frac{1}{2}m_a\vec{u}_a^2 + \frac{1}{8c^2}m_a\vec{u}_a^4 \right) + \sum_a \sum_{b \neq a} \left[-\frac{q_aq_b}{8\pi\epsilon_0r_{ab}} + \frac{q_aq_b}{16\pi\epsilon_0c^2r_{ab}}(\vec{u}_a \cdot \vec{u}_b + (\vec{u}_a \cdot \hat{r}_{ab})(\vec{u}_b \cdot \hat{r}_{ab})) \right] \\ \implies \frac{\partial L_D}{\partial \vec{u}_a} &= \left(m_a\vec{u}_a + \frac{1}{2c^2}m_a\vec{u}_a^2\vec{u}_a \right) + \sum_{b \neq a} \left[\frac{q_aq_b}{16\pi\epsilon_0c^2r_{ab}}(2\vec{u}_b + 2\hat{r}_{ab}(\vec{u}_b \cdot \hat{r}_{ab})) \right] \\ \frac{\partial L_D}{\partial \vec{u}_a} &= \underbrace{\sum_{b \neq a} \left[\frac{q_b}{8\pi\epsilon_0c^2r_{ab}}(\vec{u}_b + \hat{r}_{ab}(\vec{u}_b \cdot \hat{r}_{ab})) \right]}_{\vec{p}_a^{kin}} + \underbrace{q_a \sum_{b \neq a} \left[\frac{q_b}{8\pi\epsilon_0c^2r_{ab}}(\vec{u}_b + \hat{r}_{ab}(\vec{u}_b \cdot \hat{r}_{ab})) \right]}_{\vec{A}_a} \end{aligned}$$

which is the required form.

Part (b)

Part (c)

Consider,

$$\begin{aligned}
\frac{\partial r_{ab}}{\partial \vec{r}_a} &= \frac{\partial \sqrt{(\vec{r}_a - \vec{r}_b) \cdot (\vec{r}_a - \vec{r}_b)}}{\partial \vec{r}_a} = \frac{2}{2r_{ab}} (\vec{r}_a - \vec{r}_b) = \hat{r}_{ab} \quad \text{and} \\
\frac{\partial \hat{r}_{ab}}{\partial \vec{r}_a} &= \frac{\partial (\vec{r}_{ab}/r_{ab})}{\partial \vec{r}_a} = \frac{1}{r_{ab}} - \frac{\vec{r}_{ab} \cdot \hat{r}_{ab}}{r_{ab}^2} = 0 \\
\Rightarrow \frac{\partial L_D}{\partial \vec{r}_a} &= \sum_{b \neq a} \left[-\frac{q_a q_b}{8\pi\epsilon_0 r_{ab}^2} \hat{r}_{ab} + \frac{q_a q_b \hat{r}_{ab}}{16\pi\epsilon_0 c^2 r_{ab}^2} (\vec{u}_a \cdot \vec{u}_b + (\vec{u}_a \cdot \hat{r}_{ab})(\vec{u}_b \cdot \hat{r}_{ab})) \right] \\
-q_a \frac{\partial}{\partial \vec{r}_a} (\phi_a - \vec{u}_a \cdot \vec{A}_a) &= \sum_{b \neq a} \left[-\frac{q_a q_b}{4\pi\epsilon_0 r_{ab}^2} \hat{r}_{ab} + \frac{q_a q_b \hat{r}_{ab}}{8\pi\epsilon_0 c^2 r_{ab}^2} (\vec{u}_a \cdot \vec{u}_b + (\vec{u}_a \cdot \hat{r}_{ab})(\vec{u}_b \cdot \hat{r}_{ab})) \right] \\
\Rightarrow \frac{\partial L_D}{\partial \vec{r}_a} &= -q_a \frac{\partial}{\partial \vec{r}_a} (\phi_a - \vec{u}_a \cdot \vec{A}_a) = -q_a \vec{\nabla}_a \phi_a + q_a \vec{\nabla}_a (\vec{u}_a \cdot \vec{A}_a)
\end{aligned}$$

Part (d)

We write our equations of motion,

$$\begin{aligned}
\frac{d}{dt} \left(\frac{\partial L_D}{\partial \vec{u}_a} \right) &= \frac{\partial L_D}{\partial \vec{r}_a} \\
\frac{d\vec{p}_a^{kin}}{dt} + q_a \dot{\vec{A}}_a &= -q_a \vec{\nabla}_a \phi_a + q_a \vec{\nabla}_a (\vec{u}_a \cdot \vec{A}_a)
\end{aligned}$$

Using $\vec{\nabla}(\vec{a} \cdot \vec{b}) = (\vec{a} \cdot \vec{\nabla})\vec{b} + (\vec{b} \cdot \vec{\nabla})\vec{a} + \vec{a} \times \vec{\nabla} \times \vec{b} + \vec{b} \times \vec{\nabla} \times \vec{a}$,

$$\begin{aligned}
\frac{d\vec{p}_a^{kin}}{dt} + q_a \partial_t \vec{A}_a + q_a (\vec{u}_a \cdot \vec{\nabla}_a) \vec{A}_a &= -q_a \vec{\nabla}_a \phi_a + q_a ((\vec{u}_a \cdot \vec{\nabla}_a) \vec{A}_a + (\vec{A}_a \cdot \vec{\nabla}_a) \vec{u}_a + \vec{u}_a \times \vec{\nabla} \times \vec{A}_a + \vec{A}_a \times \vec{\nabla} \times \vec{u}_a) \\
\frac{d\vec{p}_a^{kin}}{dt} + q_a \partial_t \vec{A}_a &= -q_a \vec{\nabla}_a \phi_a + q_a ((\vec{u}_a \cdot \vec{\nabla}_a) \vec{A}_a + (\vec{A}_a \cdot \vec{\nabla}_a) \vec{u}_a + \vec{u}_a \times \vec{\nabla} \times \vec{A}_a + \vec{A}_a \times \vec{\nabla} \times \vec{u}_a)
\end{aligned}$$