Electromagnetism: Pset #2

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Problem 1

Consider, with $\psi_G^R(r) = \dfrac{1}{(2\pi)^{d/2} R^d} \exp\left(-\dfrac{r^2}{2R^2}\right)$,

$$\begin{aligned}
\frac{1}{R}\frac{\partial \psi_G^R}{\partial R} &= \frac{1}{R}\frac{1}{(2\pi)^{d/2} R^d} \exp\left(-\frac{r^2}{2R^2}\right)\left[\frac{-d}{R} - \frac{-2r^2}{2R^3}\right] \\
\frac{1}{R}\frac{\partial \psi_G^R}{\partial R} &= \frac{\psi_G^R(r)}{R^2}\left[-d + \frac{r^2}{R^2}\right]
\end{aligned}$$

Now consider,

$$\begin{aligned}
\nabla^2 \psi_G^R &= \frac{1}{r^{d-1}}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial \psi_G^R}{\partial r}\right) \\
&= \frac{\partial^2 \psi_G^R}{\partial r^2} + \frac{d-1}{r}\frac{\partial \psi_G^R}{\partial r} \\
&= -\frac{\psi_G^R}{R^2} + \frac{r^2}{R^4}\psi_G^R + \frac{d-1}{r}\left(\frac{-r}{R^2}\right)\psi_G^R \\
\nabla^2 \psi_G^R &= \frac{\psi_G^R(r)}{R^2}\left[-d + \frac{r^2}{R^2}\right] = \frac{1}{R}\frac{\partial \psi_G^R}{\partial R}
\end{aligned}$$

Consider now Green's vector field,

$$\begin{aligned}
\vec{\mathbf{G}}[\psi_G^R, \phi_\lambda] &= \psi_G^R \vec{\boldsymbol{\nabla}}\phi_\lambda - \phi_\lambda \vec{\boldsymbol{\nabla}}\psi_G^R \\
\implies \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{G}} &= \psi_G^R \nabla^2 \phi_\lambda - \phi_\lambda \nabla^2 \psi_G^R \\
\int \mathrm{d}^d\vec{\mathbf{r}}(\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{G}}) &= \int \mathrm{d}^d\vec{\mathbf{r}}(\psi_G^R \nabla^2 \phi_\lambda - \phi_\lambda \nabla^2 \psi_G^R) \\
\int_{S_{d-1}} \vec{\mathbf{G}} \cdot \mathrm{d}\vec{\mathbf{a}} &= \int \mathrm{d}^d\vec{\mathbf{r}}(\psi_G^R \nabla^2 \phi_\lambda - \phi_\lambda \nabla^2 \psi_G^R) \\
\implies 0 &= \int \mathrm{d}^d\vec{\mathbf{r}}(\psi_G^R \nabla^2 \phi_\lambda - \phi_\lambda \nabla^2 \psi_G^R)
\end{aligned}$$

From the above manipulations, we can see that,

$$\int \mathrm{d}^d\vec{\mathbf{r}}(\psi_G^R\nabla^2\phi_\lambda - \phi_\lambda\nabla^2\psi_G^R) = 0$$

$$\int \mathrm{d}^d\vec{\mathbf{r}}\left(\psi_G^R\lambda\phi_\lambda - \phi_\lambda\frac{1}{R}\frac{\partial\psi_G^R}{\partial R}\right) = 0$$

$$\implies \lambda\int \mathrm{d}^d\vec{\mathbf{r}}(\psi_G^R\phi_\lambda) = \frac{1}{R}\frac{\partial}{\partial R}\int \mathrm{d}^d\vec{\mathbf{r}}\phi_\lambda\psi_G^R$$

$$\implies \int \mathrm{d}^d\vec{\mathbf{r}}\phi_\lambda\psi_G^R = C\exp\left(\frac{\lambda R^2}{2}\right)$$

Good, so now we have got an expression for the gaussian average. All that is left is to figure out the parameter C. For this we note that the Gaussian distribution approaches a Dirac delta function as $R \to 0$, and then write,

$$\int \mathrm{d}^d\vec{\mathbf{r}}\phi_\lambda \lim_{R\to 0}\psi_G^R(\vec{\mathbf{r}} - \vec{\mathbf{r}}_0) = C\lim_{R\to 0}\exp\left(\frac{\lambda R^2}{2}\right)$$

$$\int \mathrm{d}^d\vec{\mathbf{r}}\phi_\lambda\delta(\vec{\mathbf{r}} - \vec{\mathbf{r}}_0) = C$$

$$\implies C = \phi_\lambda(\vec{\mathbf{r}}_0)$$

$$\implies \boxed{\int \mathrm{d}^d\vec{\mathbf{r}}\ \phi_\lambda(\vec{\mathbf{r}})\psi_G^R(\vec{\mathbf{r}} - \vec{\mathbf{r}}_0) = \phi_\lambda(\vec{\mathbf{r}}_0)\exp\left(\frac{\lambda R^2}{2}\right)}$$

Part (b)

$$\implies \int \mathrm{d}^d\vec{\mathbf{r}}\ \phi_\lambda(\vec{\mathbf{r}})\frac{1}{(2\pi)^{d/2}R^d}\exp\left(-\frac{|\vec{\mathbf{r}} - \vec{\mathbf{r}}_0|^2}{2R^2}\right) = \phi_\lambda(\vec{\mathbf{r}}_0)\exp\left(\frac{\lambda R^2}{2}\right)$$

$$\implies \int \mathrm{d}^d\vec{\mathbf{r}}\ \phi_\lambda(\vec{\mathbf{r}})\frac{1}{(2\pi)^{d/2}R^d}\exp\left(-\frac{r^2}{2R^2}\right)\exp\left(-\frac{r_0^2}{2R^2}\right)\exp\left(\frac{\vec{\mathbf{r}}_0\cdot\vec{\mathbf{r}}}{R^2}\right) = \phi_\lambda(\vec{\mathbf{r}}_0)\exp\left(\frac{\lambda R^2}{2}\right)$$

$$\implies \int \mathrm{d}^d\vec{\mathbf{r}}\ \phi_\lambda(\vec{\mathbf{r}})\psi_G^R(\vec{\mathbf{r}})\exp\left(-\frac{r_0^2}{2R^2}\right)\exp\left(\frac{\vec{\mathbf{r}}_0\cdot\vec{\mathbf{r}}}{R^2}\right) = \phi_\lambda(\vec{\mathbf{r}}_0)\exp\left(\frac{\lambda R^2}{2}\right)$$

Taking averages over $\vec{\mathbf{r}}_0$ and noting that $\kappa = \dfrac{r}{R^2}$,

$$\int \mathrm{d}^d\vec{\mathbf{r}}\ \phi_\lambda(\vec{\mathbf{r}})\psi_G^R(\vec{\mathbf{r}})\exp\left(-\frac{r_0^2}{2R^2}\right)I_0\left(d;\frac{r}{R^2}r_0\right) = \langle\phi_\lambda(\vec{\mathbf{r}}_0)\rangle\exp\left(\frac{\lambda R^2}{2}\right)$$

Now, let's make the substitution, $r_0 = \kappa R^2$ and $\phi_\lambda(r) = J_0(d;kr) \implies \lambda = -k^2$

$$\int \mathrm{d}^d\vec{\mathbf{r}}\ J_0(d;kr)\psi_G^R(\vec{\mathbf{r}})\exp\left(-\frac{\kappa^2R^2}{2}\right)I_0(d;\kappa r) = \langle J_0(d;kr_0)\rangle\exp\left(-\frac{k^2R^2}{2}\right)$$

$$\int \mathrm{d}^d\vec{\mathbf{r}}\ J_0(d;kr)I_0(d;\kappa r)\psi_G^R(\vec{\mathbf{r}}) = J_0(d;k\kappa R^2)\exp\left(\frac{(\kappa^2 - k^2)R^2}{2}\right)$$

Similarly, taking $\phi_\lambda(r) = I_0(d;\kappa' r) \implies \lambda = \kappa'^2$,

$$\int \mathrm{d}^d\vec{\mathbf{r}}\ I_0(d;\kappa' r)\psi_G^R(\vec{\mathbf{r}})\exp\left(-\frac{\kappa^2R^2}{2}\right)I_0(d;\kappa r) = \langle I_0(d;\kappa' r_0)\rangle\exp\left(\frac{\kappa'^2R^2}{2}\right)$$

$$\int \mathrm{d}^d\vec{\mathbf{r}}\ I_0(d;\kappa' r)I_0(d;\kappa r)\psi_G^R(\vec{\mathbf{r}}) = I_0(d;\kappa'\kappa R^2)\exp\left(\frac{(\kappa^2 + \kappa'^2)R^2}{2}\right)$$

Now, let's make the substitution, $r_0 = ik_2R^2$ and $\phi_\lambda(r) = J_0(d; k_1 r) \implies \lambda = -k_1^2$,

$$\int \mathrm{d}^d\vec{\mathbf{r}}\ J_0(d; k_1 r)\psi_G^R(\vec{\mathbf{r}})\exp\left(\frac{k_2^2R^2}{2}\right)I_0(d; ik_2 r) = \langle J_0(d; ik_1k_2R^2)\rangle \exp\left(-\frac{k_1^2R^2}{2}\right)$$

$$\int \mathrm{d}^d\vec{\mathbf{r}}\ J_0(d; k_1 r)J_0(d; k_2 r)\psi_G^R(\vec{\mathbf{r}}) = I_0(d; k_1k_2R^2)\exp\left(-\frac{(k_1^2+k_2^2)R^2}{2}\right)$$

Part (c)

Part (d)

Consider,

$$I = \int_0^\infty \frac{\mathrm{d}R}{R^{2\Delta+1}}\exp\left(-\frac{r^2}{2R^2}\right)$$

Substitute $R^2 = \dfrac{r^2}{2u} \implies \mathrm{d}u = -\dfrac{r^2}{R^3}\,\mathrm{d}R \implies \mathrm{d}R = -\dfrac{r\,\mathrm{d}u}{(2u)^{3/2}}$,

$$\begin{aligned} I &= \int_0^\infty \frac{r\,\mathrm{d}u}{(2u)^{3/2}}\left(\frac{2u}{r^2}\right)^{\Delta+1/2}\exp(-u) \\ &= \frac{2^{\Delta-1}}{r^\Delta}\int_0^\infty \mathrm{d}u\, u^{\Delta-1}\exp(-u) \\ &= \frac{2^{\Delta-1}}{r^\Delta}\Gamma(\Delta) \quad ; \qquad \text{for} \quad \Delta > 1 \end{aligned}$$

Problem 2

Part (a)

We know,

$$\begin{aligned} I_0(d; x) &= \sum_{m=0}^\infty \frac{\Gamma(d/2)}{m!\,\Gamma(d/2+m)}\frac{x^{2m}}{2^{2m}} \\ \frac{x^2}{d^2}I_0(d+2; x) &= \sum_{m=0}^\infty \frac{\Gamma(d/2+1)}{m!\,\Gamma(d/2+m+1)}\frac{x^{2m+2}}{2^{2m}} \\ &= \sum_{m=0}^\infty \frac{\Gamma(d/2)\cdot d/2}{m!\,\Gamma(d/2+m+1)}\frac{x^{2m+2}}{2^{2m+2}}\frac{2^2}{d^2} \\ \frac{x^2}{d^2}I_0(d+2; x) &= \frac{2}{d}\sum_{m=0}^\infty \frac{\Gamma(d/2)}{m!\,\Gamma(d/2+m+1)}\frac{x^{2m+2}}{2^{2m+2}} \end{aligned}$$

Let's proceed and take the derivative,

$$\begin{aligned} \frac{x}{d}\frac{\mathrm{d}}{\mathrm{d}x}I_0(d; x) &= \frac{x}{d}\sum_{m=0}^\infty \frac{\Gamma(d/2)}{m!\,\Gamma(d/2+m)}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^{2m}}{2^{2m}}\right) \\ &= \frac{1}{d}\sum_{m=1}^\infty \frac{\Gamma(d/2)}{m!\,\Gamma(d/2+m)}\frac{(2m)x^{2m}}{2^{2m}} \\ &= \frac{1}{d}\sum_{m=0}^\infty \frac{\Gamma(d/2)}{(m+1)!\,\Gamma(d/2+m+1)}\frac{2(m+1)x^{2m+2}}{2^{2m+2}} \quad \Longleftarrow (m \to m+1) \\ \frac{x}{d}\frac{\mathrm{d}}{\mathrm{d}x}I_0(d; x) &= \frac{2}{d}\sum_{m=0}^\infty \frac{\Gamma(d/2)}{m!\,\Gamma(d/2+m+1)}\frac{x^{2m+2}}{2^{2m+2}} \end{aligned}$$

Problem 2 continued on next page. . .

Let's now consider the third part,

$$
\begin{aligned}
I_0(d-2;x) - I_0(d,x) &= \sum_{m=0}^{\infty}\left[\frac{\Gamma(d/2-1)}{\Gamma(d/2+m-1)} - \frac{\Gamma(d/2)}{\Gamma(d/2+m)}\right]\frac{x^{2m}}{m!\;2^{2m}} \\
&= \sum_{m=1}^{\infty}\left[\frac{\Gamma(d/2-1)}{\Gamma(d/2+m-1)} - \frac{\Gamma(d/2)}{\Gamma(d/2+m)}\right]\frac{x^{2m}}{m!\;2^{2m}} \\
&= \sum_{m=0}^{\infty}\left[\frac{\Gamma(d/2-1)}{\Gamma(d/2+m)} - \frac{\Gamma(d/2)}{\Gamma(d/2+m+1)}\right]\frac{x^{2m+2}}{(m+1)!\;2^{2m}} \\
&= \sum_{m=0}^{\infty}\left[\frac{\Gamma(d/2)}{\Gamma(d/2+m+1)}\frac{(d/2+m)}{(d/2-1)} - \frac{\Gamma(d/2)}{\Gamma(d/2+m+1)}\right]\frac{x^{2m+2}}{(m+1)!\;2^{2m}} \\
&= \sum_{m=0}^{\infty}\left[\frac{2(m+1)}{d-2}\right]\frac{\Gamma(d/2)}{\Gamma(d/2+m+1)}\frac{x^{2m+2}}{(m+1)!\;2^{2m}} \\
&= \frac{2}{d-2}\sum_{m=0}^{\infty}\frac{\Gamma(d/2)}{\Gamma(d/2+m+1)}\frac{x^{2m+2}}{m!\;2^{2m}} \\
\frac{d-2}{d}(I_0(d-2;x) - I_0(d,x)) &= \frac{2}{d}\sum_{m=0}^{\infty}\frac{\Gamma(d/2)}{m!\;\Gamma(d/2+m+1)}\frac{x^{2m+2}}{2^{2m+2}}
\end{aligned}
$$

$$
\implies \boxed{\frac{x}{d}\frac{\mathrm{d}}{\mathrm{d}x}I_0(d;x) = \frac{x^2}{d^2}I_0(d+2;x) = \frac{d-2}{d}[I_0(d-2;x) - I_0(d,x)]}
$$

Part (b)
Schafli's contour integral is given by,

$$
I_0(d;x) = \oint_{\mathcal{C}}\frac{\mathrm{d}z}{2\pi i}\frac{\Gamma(d/2)}{z^{d/2}}\exp\left(z+\frac{x^2}{4z}\right)
$$

where the contour $\mathcal{C} = C[-\infty - i0, 0+, -\infty + i0]$.

$$
\begin{aligned}
\frac{x}{d}\frac{\mathrm{d}}{\mathrm{d}x}I_0(d;x) &= \oint_{\mathcal{C}}\frac{x}{d}\frac{\mathrm{d}z}{2\pi i}\frac{\Gamma(d/2)}{z^{d/2}}\frac{x}{2z}\exp\left(z+\frac{x^2}{4z}\right) \\
&= \frac{x^2}{d^2}\oint_{\mathcal{C}}\frac{\mathrm{d}z}{2\pi i}\frac{(d/2)\cdot\Gamma(d/2)}{z^{(d+2)/2}}\exp\left(z+\frac{x^2}{4z}\right) \\
&= \frac{x^2}{d^2}\oint_{\mathcal{C}}\frac{\mathrm{d}z}{2\pi i}\frac{\Gamma((d+2)/2)}{z^{(d+2)/2}}\exp\left(z+\frac{x^2}{4z}\right) \\
\implies \frac{x}{d}\frac{\mathrm{d}}{\mathrm{d}x}I_0(d;x) &= \frac{x^2}{d^2}I_0(d+2;x)
\end{aligned}
$$

$$
\begin{aligned}
I_0(d-2;x) - I_0(d,x) &= \oint_{\mathcal{C}}\frac{\mathrm{d}z}{2\pi i}\left[\frac{\Gamma(d/2-1)}{z^{d/2-1}} - \frac{\Gamma(d/2)}{z^{d/2}}\right]\exp\left(z+\frac{x^2}{4z}\right) \\
&=
\end{aligned}
$$

Part (c)
We are given,

$$
\begin{aligned}
I_0(d;x) &\approx \frac{e^x}{|S^{d-1}|}\left(\frac{2\pi}{x}\right)^{\frac{d-1}{2}}\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(\frac{d-1}{2}+n)}{(2x)^n n!\;\Gamma(\frac{d-1}{2}-n)} \\
\implies \frac{x}{d}\frac{\mathrm{d}}{\mathrm{d}x}I_0(d;x) &\approx \frac{x}{d}\left[\frac{e^x}{|S^{d-1}|}\left(\frac{2\pi}{x}\right)^{\frac{d-1}{2}}\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(\frac{d-1}{2}+n)}{(2x)^n n!\;\Gamma(\frac{d-1}{2}-n)}\right]
\end{aligned}
$$