Classical Mechanics: Assignment #4

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Problem 1

Part (a)
The formal definition of the functional derivative is given by,

$$\frac{\delta F[q(x)]}{\delta q(y)} = \lim_{\epsilon \to 0} \frac{F[q(y) + \epsilon \delta(x-y)] - F[q(y)]}{\epsilon}$$

Using familiar notions from calculus, we can write the following
Consider the variation of the action $S = \int L(q, \dot{q}, t) dt$,

$$\begin{aligned}
\delta S &= \int \delta L dt \\
&= \int \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) dt \\
&= \int \left(\frac{\partial L}{\partial q} \delta q + \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}} \delta q \right) - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}} \right) \delta q \right) dt \\
&= \int \left(\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}} \right) \right) \delta q dt + \left. \frac{\partial L}{\partial \dot{q}} \delta q \right|_{x_1}^{x_2}
\end{aligned}$$

As the variation at the end points in zero, the second term vanishes. The variation of the action δS should also be zero, and the only way this can happen is if,

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}} \right) = 0$$

which is the Euler-Lagrange equation.
Part (b)
The Lagrangian for the linear harmonic chain can be written as follows,

$$L = \sum_n \frac{1}{2} m \dot{x}_i^2 - \frac{1}{2} k (x_i - x_{i-1})^2$$

where the x_i's are the displacements from the mean positions of the respective particles. Lets change our notations such that $\phi_i = x_i$. Hence,

$$L = \sum_n \frac{1}{2} m \dot{\phi}_i^2 - \frac{1}{2} k (\phi_i - \phi_{i-1})^2$$

In the limit of separation between successive ϕ_i's $\to 0$ and $n \to \infty$, the potential terms becomes a spatial derivative. The whole expression can be written as,

$$L = \int dx \left(\frac{1}{2} m \dot{\phi}^2(x,t) - \frac{1}{2} k \phi'^2(x,t) \right)$$

The term in the parenthesis is called the *Lagrangian Density* $\mathcal{L}$. Obtaining the equations of motion is fairly straighforward by,

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)-\frac{\partial\mathcal{L}}{\partial\phi}=0$$
$$m\ddot{\phi}-k\phi''=0 \implies \quad \text{wave equation}$$

Problem 2

The idea is to write the equations of motion of this system in a combined matrix form as $\ddot{X}=(M^{-1}V)X$. M and V can be written as follows,

$$M=\begin{pmatrix} m_1 & & \\ & m_2 & \\ & & m_3 \end{pmatrix} \quad \text{and} \quad V=\begin{pmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{pmatrix}$$

The normal mode frequencies are given by the eigenvalues of the matrix $M^{-1}V$. The eigenvalues are given by,

$\omega_1=0$

$$\omega_2=\frac{\sqrt{\frac{k_2m_1m_2^2+k_1m_3m_2^2+k_1m_1m_3m_2+k_2m_1m_3m_2-\sqrt{m_2^2((k_1(m_1+m_2)m_3+k_2m_1(m_2+m_3))^2-4k_1k_2m_1m_2m_3(m_1+m_2+m_3))}}{m_1m_2^2m_3}}}{\sqrt{2}}$$

$$\omega_3=\frac{\sqrt{\frac{k_2m_1m_2^2+k_1m_3m_2^2+k_1m_1m_3m_2+k_2m_1m_3m_2+\sqrt{m_2^2((k_1(m_1+m_2)m_3+k_2m_1(m_2+m_3))^2-4k_1k_2m_1m_2m_3(m_1+m_2+m_3))}}{m_1m_2^2m_3}}}{\sqrt{2}}$$

Now that we have obtained the normal mode frequencies, let's consider a few special cases,

- $m_1=m_2=m_3=m$, $k_1=k_2=k \to \omega_1=0, \omega_2=\sqrt{\frac{k}{m}}, \omega_3=\sqrt{\frac{3k}{m}}$
- $m_1=m_3=m$, $k_1=k_2=k \to \omega_1=0, \omega_2=\sqrt{\frac{k}{m}}, \omega_3=\sqrt{\frac{k(2m+m_2)}{mm_2}}$

Let's calculate the normal mode frequencies for CO_2. $m_O=2.66\times10^{-26}$ kg and $m_C=1.99\times10^{-26}$ kg and $k=840$ N/m. This gives $\omega_1=0, \omega_2=1.78\times10^{14}$ s^{-1} and $\omega_3=3.41\times10^{14}$ s^{-1}.

Problem 3

The Lagrangian for this system can be written as,

$$L=\frac{1}{2}m_1|\dot{\mathbf{r_1}}|^2+\frac{1}{2}m_2|\dot{\mathbf{r_2}}|^2-V(\mathbf{r_1}-\mathbf{r_2})$$

We also know, from the question, that

$$\mathbf{R}=\frac{m_1\mathbf{r_1}+m_2\mathbf{r_2}}{m_1+m_2} \quad \text{and} \quad \mathbf{r}=\mathbf{r_1}-\mathbf{r_2}$$

This leads us to,

$$\mathbf{r_1}=\mathbf{R}+\frac{m_2\mathbf{r}}{m_1+m_2} \quad \text{and} \quad \mathbf{r_2}=\mathbf{R}-\frac{m_1\mathbf{r}}{m_1+m_2}$$

$$|\dot{\mathbf{r_1}}|^2 = \left|\dot{\mathbf{R}}\right|^2 + \frac{m_2^2|\dot{\mathbf{r}}|^2}{(m_1+m_2)^2} + \frac{2m_2}{m_1+m_2}\dot{\mathbf{R}}\cdot\dot{\mathbf{r}} \quad \text{and} \quad |\dot{\mathbf{r_2}}|^2 = \left|\dot{\mathbf{R}}\right|^2 + \frac{m_1^2|\dot{\mathbf{r}}|^2}{(m_1+m_2)^2} - \frac{2m_1}{m_1+m_2}\dot{\mathbf{R}}\cdot\dot{\mathbf{r}}$$

Substituting into the expression for the Lagrangian, one gets,

$$L = \frac{M}{2}\left|\dot{\mathbf{R}}\right|^2 + \frac{\mu}{2}|\dot{\mathbf{r}}|^2 - V(\mathbf{r}) \quad \text{where} \quad M = m_1 + m_2 \quad , \quad \mu = \frac{m_1 m_2}{M}$$

Each component of $\dot{\mathbf{R}}$ will be conserved separately as all of them are cyclic coordinates. Using $\mathbf{R} = X\hat{\mathbf{x}} + Y\hat{\mathbf{y}} + Z\hat{\mathbf{z}}$ and $\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\hat{r}}$,

$$L = \frac{M}{2}(\dot{X}^2 + \dot{Y}^2) + \frac{\mu}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

Part (a)
We can see from the form of the above Lagrangian that,

$$M\dot{X} = constant \quad , \quad M\dot{Y} = constant \quad , \quad \mu r^2\dot{\theta} = constant$$

Consider the infinitesimal area swept by the vector $\mathbf{r}$,

$$dA = \frac{r^2 d\theta}{2} \implies \dot{A} = r^2\frac{\dot{\theta}}{2} = constant = l$$

Hence, the radius vector sweeps equal areas in equal intervals of time.

Part (b)
If $m_2 \gg m_1$, $\mathbf{R} \approx \mathbf{r_2}$, $\mu \approx m_1$ and the mass m_2 does not move. Using energy conservation (and the fact that the centre of mass does not move),

$$\frac{\mu}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = E \implies \dot{r}^2 + \frac{4l^2}{r^2} = \frac{2}{\mu}(E - V(r))$$

$r(t)$ will be given by the solution of this differential equation.

Part (c)
The Euler-Lagrange equation for the coordinate r is given by,

$$\mu\ddot{r} = \mu r\frac{4l^2}{r^4} - \frac{k}{r^2} \implies \ddot{r} - \frac{4l^2}{r^3} + \frac{k}{\mu r^2} = 0$$

Multiplying by $\dot{r}$ and integrating with time, we get,

$$\dot{r}^2 + \frac{2l^2}{r^2} - \frac{k}{\mu r} = constant = E$$