Classical Mechanics: Assignment #2

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Problem 1

Solution

The Lagrangian for the given system can be written as,

$$L = T + V = \frac{1}{2}mx^2\omega^2 + \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy$$

From the problem, we know that $y = k\left(\frac{x}{l}\right)^\alpha$, which means that $\dot{y} = k\alpha\frac{x^{\alpha-1}}{l^\alpha}\dot{x}$. Substituting these into the form of the Lagrangian and simplifying, we get,

$$L = \frac{1}{2}m\left(-2gk\left(\frac{x}{l}\right)^\alpha + \dot{x}^2\left(\frac{\alpha^2k^2x^{2\alpha-2}}{l^\alpha} + 1\right) + x^2\omega^2\right)$$

The equation of motion can be written as,

$$\boxed{\alpha gkx^2\left(\frac{x}{l}\right)^\alpha + (\alpha-1)\alpha^2k^2\dot{x}^2\left(\frac{x}{l}\right)^{2\alpha} + \alpha^2k^2x\ddot{x}\left(\frac{x}{l}\right)^{2\alpha} - x^4\omega^2 + x^3\ddot{x} = 0}$$

The equlibrium points will satisfy $\dot{x} = \ddot{x} = 0$. This means that the equilibrium point will be,

$$x_0 = \left(\frac{\omega^2l^\alpha}{gk\alpha}\right)^{\frac{1}{\alpha-2}}$$

We substitute $x(t) = x_0 + \epsilon y(t)$

$$\ddot{y} + y\frac{(\alpha-2)\omega^2}{\alpha^2k^2\left(\frac{x_0}{l}\right)^{2\alpha} + 1} = 0$$

For small oscillations, the coefficient of y in the above equation should be positive. Hence,

$$\frac{(\alpha-2)\omega^2}{\alpha^2k^2\left(\frac{x_0}{l}\right)^{2\alpha} + 1} > 0$$

$$\therefore \alpha - 2 > 0 \implies \boxed{\alpha > 2}$$

The frequency of oscillations ω_0 is simply the square root of the coefficient of y,

$$\boxed{\omega_0 = \sqrt{\frac{\alpha-2}{\alpha^2k^2\left(\frac{x_0}{l}\right)^{2\alpha} + 1}}\omega} \quad \text{where} \quad x_0 = \left(\frac{\omega^2l^\alpha}{gk\alpha}\right)^{\frac{1}{\alpha-2}}$$

Problem 2

Part (a)
We first write down the Lagrangian,

$$\begin{aligned} L &= T - V \\ &= \underbrace{\frac{M(R_1 - R_2)^2\dot{\theta}_1^2}{2}}_{\text{Centre of mass revolution}} + \underbrace{\frac{MR_2^2\dot{\theta_2}^2}{4}}_{\text{rotation}} + Mg(R_1 - R_2)\cos\theta_1 \end{aligned}$$

The equations of motion can now be written as,

$$\boxed{\ddot{\theta_2} = 0 \quad \text{and} \quad \ddot{\theta_1} = -\frac{g(R_1 - R_2)}{R_2^2}\sin\theta_1}$$

Part (b)
As there is rolling without slipping, the following constraint condition would hold,

$$R_2\dot{\theta_2} = (R_1 - R_2)\dot{\theta_1} \implies f = R_2\theta_2 - (R_1 - R_2)\theta_1 = 0$$

Note that differentiating this also gives,

$$R_2\ddot{\theta_2} = (R_1 - R_2)\ddot{\theta_1}$$

Part (c)
Putting in constraint conditions, the equations of motion get modified as,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \lambda\frac{\partial f}{\partial q}$$

which modifies the equations in this problem as,

$$MR_2^2\ddot{\theta_2} = \lambda R_2 \quad \text{and} \quad \frac{M(R_1 - R_2)^2}{2}\ddot{\theta_1} = -Mg(R_1 - R_2)\sin\theta_1 - \lambda(R_1 - R_2)$$

$$MR_2\ddot{\theta_2} = \lambda \quad \text{and} \quad \frac{M(R_1 - R_2)}{2}\ddot{\theta_1} = -Mg\sin\theta_1 - \lambda$$

Taking the ratio of these two and substituting,

$$\frac{2R_2}{R_1 - R_2}\frac{R_1 - R_2}{R_2} = \frac{\lambda}{-Mg\sin\theta_1 - \lambda}$$

$$|\lambda| = \frac{Mg\sin\theta_1}{3} = \quad \text{Constraint Force}$$

Problem 3

Part (a)
A bicycle really has just two degrees of freedom in the simplest sense,

- The angle associated to the pedalling motion
- The angle associated to the motion of the handle

Problem 3 continued on next page. . .

Part (b)
The arrangement has $M-1$ links and hence $M-2$ angles in between. There is also one degree of freedom associated with the rotation of the chain if considered as a rigid body. Hence, there are a total of $M-1$ degrees of freedom.

Part (c)
Due to *homogeneity and isotropy of space* Lagrangian of a free particle should be,

- *Invariant under Rotation* - Hence, $L(v,x)$, where v and x are the absolute value of the velocity and position vectors.
- *Invariant under Translation* - This means that L cannot depend on x at all. Hence $L = L(v)$.

Part (d)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L'}{\partial v'}\right) = \frac{\mathrm{d}}{\mathrm{d}t}(2v') = \frac{\mathrm{d}}{\mathrm{d}t}v' = 0$$

which is the equation of motion for free particle in the original frame. Hence, $L' = v'^2$ is a possible choice.

Part (e)

$$L'(v+V_0) = L(v) + \frac{\mathrm{d}F(x,t)}{\mathrm{d}t} = L'(v) + V_0\left.\frac{\mathrm{d}L'}{\mathrm{d}v_c}\right|_{v_c=v} + \dots$$

As $L(v) = L'(v)$, we have $\frac{\mathrm{d}F(x,t)}{\mathrm{d}t} = V_0\left.\frac{\mathrm{d}L'}{\mathrm{d}v_c}\right|_{v_c=v}$. As F does not depend on v, it's first derivative can depend only linearly on v. Hence,

$$\frac{\mathrm{d}L'}{\mathrm{d}v} \sim v \implies L \sim v^2$$

Problem 4

Part (a)
The Schrodinger equation is given by

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi = i\hbar\frac{\partial\psi}{\partial t} \quad \text{and} \quad -\frac{\hbar^2}{2m}\frac{\partial^2\psi^*}{\partial x^2} + V\psi^* = -i\hbar\frac{\partial\psi^*}{\partial t}$$

We choose ψ and ψ^* as our generalized coordinates, and (t,x) as the dependent coordinates. One should be able to write the equations of motion in a compact form as follows,

$$\partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu\psi)}\right) = \frac{\partial L}{\partial\psi} \quad \text{and} \quad \partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu\psi^*)}\right) = \frac{\partial L}{\partial\psi^*}$$

where the index μ goes over (t,x). Let's analyze every term in the Schrodinger Equation and figure out what corresponding term in the Lagrangian will give rise to that term,

- The first term on the LHS is a double x derivative and will come from some single derivative term of the form $L_1 = \psi'\psi'^*$
- The second term on the LHS has no derivatives and will come from a term of the form $L_2 = \psi\psi^*$
- The RHS is a single t derivative and will come from some term of the form $\dot{\psi}\psi^*$. To make it symmetric, let's consider $L_3 = -\dot{\psi}\psi^* + \dot{\psi}^*\psi$

So our final Lagrangian will be of the form $L = a_1 L_1 + a_2 L_2 + a_3 L_3$. Substituting our ansatz, we find our constants, and then the final Lagrangian can be written as,

$$\boxed{L = -\frac{\hbar^2}{2m}\psi'\psi'^* + V\dot{\psi}\psi^* + i\hbar(-\dot{\psi}\psi^* + \dot{\psi}^*\psi)}$$

Part (b)

Kinetic energy of the wire is zero. The Lagrangian can be written as,

$$L = -\int ds\ \rho g y = -\int \sqrt{dx^2 + dy^2}\ \rho g y = -\int dx y\sqrt{1 + y'^2}\ \rho g$$

Writing down the equation of motion for the Lagrangian density instead of the Lagrangian, one gets,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{yy'}{\sqrt{1+y'^2}}\right) - \sqrt{1+y'^2} = 0$$

$$\frac{yy'' + y'^2}{\sqrt{1+y'^2}} - \frac{yy'^2y''}{1+y'^2} - \sqrt{1+y'^2} = 0$$

Expanding this out and simplifying a bit, one gets,

$$\frac{yy''}{(1+y'^2)^{\frac{3}{2}}} - \frac{1}{\sqrt{1+y'^2}} = 0 \implies \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{\sqrt{1+y'^2}}\right) = 0$$

$$\therefore \frac{y}{\sqrt{1+y'^2}} = \alpha \implies \boxed{y = \alpha\cosh\left(\frac{x}{\alpha} + \beta\right)}$$

One can get the constants α and β by imposing the end point conditions for the curve.

Part (c)

The distance metric on a sphere spherical polar coordinates is given by,

$$ds^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2) = d\theta^2[r^2(1 + \sin^2\theta\phi'^2)]$$

$$\therefore ds = d\theta\sqrt{r^2(1+\sin^2\theta\phi'^2)}$$

From the ansatz $S = Ld\tau$, we can identify that the Lagrangian $L = \sqrt{r^2(1+\sin^2\theta\phi'^2)}$. For finding the *equations of motion*, it is fine and also easier to work with L^2 rather than L in this problem. Writing down the equations of motion for $\phi(\theta)$,

$$\frac{\mathrm{d}\sin^2\theta\phi'}{\mathrm{d}\theta} = 0 \implies \phi' = \alpha\csc^2(\theta) \implies \boxed{\phi(\theta) = a\cot\theta + b}$$

where α, a, b are constants. If the distance is to be found out between two points (ϕ_1, θ_1) and (ϕ_2, θ_2), then,

$$\phi_1 = a\cot\theta_1 + b \quad \text{and} \quad \phi_2 = a\cot\theta_2 + b$$

which gives,

$$a = \frac{\phi_1 - \phi_2}{\cot\theta_1 - \cot\theta_2} \quad \text{and} \quad b = \frac{\phi_1\tan\theta_1 - \phi_2\tan\theta_2}{\tan\theta_1 - \tan\theta_2}$$

Problem 5

The Lagrangian for a charged particle in a magnetic field is given by,

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + q\mathbf{A}\cdot\mathbf{v} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + q(A_x\dot{x} + A_y\dot{y})$$

Part (a)

The magnetic field $\mathbf{B}$ is given by $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$. We can easily see that the addition of any term of the form $\boldsymbol{\nabla}\lambda$, where λ is a scalar, to the vector potential gives the same $\mathbf{B}$. This freedom is called the *gauge freedom* and the choice of a particular λ is called *choosing a gauge*.

Part (b)

As $\mathbf{B} = constant$ and $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A} = \left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}\right)\hat{\mathbf{z}} = -\frac{\partial A_y}{\partial x}\hat{\mathbf{z}}$

$$A_y = Bx + \alpha$$

where α is a constant. There is infinite freedom in the choice for A_x, and we can take it to be $A_x = 0$. This choice of gauge is called the *Landau Gauge.* The manifest symmetries from the form of the Lagrangian are,

- x-translation - we make the transformation $x' = x + a$, $y' = y$. We have,

  $$\delta L = q(A_x(x+a) - A_x(x))\dot{y} = qB\dot{y} = a\frac{\mathrm{d}(qBy)}{\mathrm{d}t}$$

  The corresponding conserved quantity is, hence,

  $$\frac{\partial L}{\partial \dot{x}'}\frac{\partial x'}{\partial a} + \frac{\partial L}{\partial \dot{y}'}\frac{\partial y'}{\partial a} - qBy = \boxed{m\dot{x} - qBy = constant}$$

- y-translation - $x' = x$, $y' = y + a$. $\delta L = 0$. Hence, $\boxed{m\dot{y} + qBx = constant}$.
- z-rotation - $x' = x\cos\theta - y\sin\theta$, $y' = x\sin\theta + y\cos\theta$. $\delta L = 0$. Hence,

  $$\left.\left(\frac{\partial L}{\partial \dot{x}'}\frac{\partial x'}{\partial \theta} + \frac{\partial L}{\partial \dot{y}'}\frac{\partial y'}{\partial \theta}\right)\right|_{\theta=0} = \boxed{-m\dot{x}y + (m\dot{y} + qBx)(x) = constant}$$

  This is essentially the angular momentum $xp_y - yp_x$.

Part (c)

The symmetric gauge is given by $\mathbf{A} = \frac{B}{2}(-y, x)$. The manifest symmetries from the form of the Lagrangian are :-

- x-translation - $x' = x + a$, $y' = y$. $\delta L = a\frac{\mathrm{d}(q\frac{B}{2}y)}{\mathrm{d}t}$. Hence,

  $$m\dot{x} - \frac{qBy}{2} - \frac{qBy}{2} = \boxed{m\dot{x} - qBy = constant}$$

- y-translation - $x' = x$, $y' = y + a$. $\delta L = a\frac{\mathrm{d}(-q\frac{B}{2}x)}{\mathrm{d}t}$. Hence, $\boxed{m\dot{y} + qBx = constant}$.
- z-rotation - $x' = x\cos\theta - y\sin\theta$, $y' = x\sin\theta + y\cos\theta$. $\delta L = 0$. Hence,

  $$\begin{aligned}\left.\left(\frac{\partial L}{\partial \dot{x}'}\frac{\partial x'}{\partial \theta} + \frac{\partial L}{\partial \dot{y}'}\frac{\partial y'}{\partial \theta}\right)\right|_{\theta=0} &= \left(m\dot{x} - \frac{qBy}{2}\right)(-y) + \left(m\dot{y} + \frac{qBx}{2}\right)(x) \\ &= \boxed{-my\dot{x} + mx\dot{y} + \frac{qB}{2}(x^2 + y^2) = constant}\end{aligned}$$

  This is also the angular momentum $xp_y - yp_x$.

Part (e)

As is evident from above, symmetries and conserved quantities do not depend on choice of gauge.