Electromagnetism: Pset #3

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Problem 1

Consider, with $\psi_G^R(r) = \dfrac{1}{(2\pi)^{d/2}R^d}\exp\left(-\dfrac{r^2}{2R^2}\right)$,

$$\begin{aligned}
\frac{1}{R}\frac{\partial \psi_G^R}{\partial R} &= \frac{1}{R}\frac{1}{(2\pi)^{d/2}R^d}\exp\left(-\frac{r^2}{2R^2}\right)\left[\frac{-d}{R} - \frac{-2r^2}{2R^3}\right] \\
\frac{1}{R}\frac{\partial \psi_G^R}{\partial R} &= \frac{\psi_G^R(r)}{R^2}\left[-d + \frac{r^2}{R^2}\right]
\end{aligned}$$

Now consider,

$$\begin{aligned}
\nabla^2 \psi_G^R &= \frac{1}{r^{d-1}}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial \psi_G^R}{\partial r}\right) \\
&= \frac{\partial^2 \psi_G^R}{\partial r^2} + \frac{d-1}{r}\frac{\partial \psi_G^R}{\partial r} \\
&= -\frac{\psi_G^R}{R^2} + \frac{r^2}{R^4}\psi_G^R + \frac{d-1}{r}\left(\frac{-r}{R^2}\right)\psi_G^R \\
\nabla^2 \psi_G^R &= \frac{\psi_G^R(r)}{R^2}\left[-d + \frac{r^2}{R^2}\right] = \frac{1}{R}\frac{\partial \psi_G^R}{\partial R}
\end{aligned}$$

Consider now Green's vector field,

$$\begin{aligned}
\vec{\mathbf{G}}[\psi_G^R, \phi_\lambda] &= \psi_G^R \vec{\boldsymbol{\nabla}}\phi_\lambda - \phi_\lambda \vec{\boldsymbol{\nabla}}\psi_G^R \\
\implies \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{G}} &= \psi_G^R \nabla^2 \phi_\lambda - \phi_\lambda \nabla^2 \psi_G^R \\
\int \mathrm{d}^d\vec{\mathbf{r}}(\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{G}}) &= \int \mathrm{d}^d\vec{\mathbf{r}}(\psi_G^R \nabla^2 \phi_\lambda - \phi_\lambda \nabla^2 \psi_G^R) \\
\int_{S_{d-1}} \vec{\mathbf{G}} \cdot \mathrm{d}\vec{\mathbf{a}} &= \int \mathrm{d}^d\vec{\mathbf{r}}(\psi_G^R \nabla^2 \phi_\lambda - \phi_\lambda \nabla^2 \psi_G^R) \\
\implies 0 &= \int \mathrm{d}^d\vec{\mathbf{r}}(\psi_G^R \nabla^2 \phi_\lambda - \phi_\lambda \nabla^2 \psi_G^R)
\end{aligned}$$

From the above manipulations, we can see that,

$$\int \mathrm{d}^d\vec{\mathbf{r}}(\psi_G^R\nabla^2\phi_\lambda - \phi_\lambda\nabla^2\psi_G^R) = 0$$
$$\int \mathrm{d}^d\vec{\mathbf{r}}\left(\psi_G^R\lambda\phi_\lambda - \phi_\lambda\frac{1}{R}\frac{\partial\psi_G^R}{\partial R}\right) = 0$$
$$\implies \lambda\int \mathrm{d}^d\vec{\mathbf{r}}(\psi_G^R\phi_\lambda) = \frac{1}{R}\frac{\partial}{\partial R}\int \mathrm{d}^d\vec{\mathbf{r}}\phi_\lambda\psi_G^R$$
$$\implies \int \mathrm{d}^d\vec{\mathbf{r}}\phi_\lambda\psi_G^R = C\exp\left(\frac{\lambda R^2}{2}\right)$$

Good, so now we have got an expression for the gaussian average. All that is left is to figure out the parameter C. For this we note that the Gaussian distribution approaches a Dirac delta function as $R\to 0$, and then write,

$$\int \mathrm{d}^d\vec{\mathbf{r}}\phi_\lambda \lim_{R\to 0}\psi_G^R(\vec{\mathbf{r}}-\vec{\mathbf{r}}_0) = C\lim_{R\to 0}\exp\left(\frac{\lambda R^2}{2}\right)$$
$$\int \mathrm{d}^d\vec{\mathbf{r}}\phi_\lambda\delta(\vec{\mathbf{r}}-\vec{\mathbf{r}}_0) = C$$
$$\implies C = \phi_\lambda(\vec{\mathbf{r}}_0)$$

$$\implies \boxed{\int \mathrm{d}^d\vec{\mathbf{r}}\ \phi_\lambda(\vec{\mathbf{r}})\psi_G^R(\vec{\mathbf{r}}-\vec{\mathbf{r}}_0) = \phi_\lambda(\vec{\mathbf{r}}_0)\exp\left(\frac{\lambda R^2}{2}\right)}$$

Part (b)

$$\implies \int \mathrm{d}^d\vec{\mathbf{r}}\ \phi_\lambda(\vec{\mathbf{r}})\frac{1}{(2\pi)^{d/2}R^d}\exp\left(-\frac{|\vec{\mathbf{r}}-\vec{\mathbf{r}}_0|^2}{2R^2}\right) = \phi_\lambda(\vec{\mathbf{r}}_0)\exp\left(\frac{\lambda R^2}{2}\right)$$
$$\implies \int \mathrm{d}^d\vec{\mathbf{r}}\ \phi_\lambda(\vec{\mathbf{r}})\frac{1}{(2\pi)^{d/2}R^d}\exp\left(-\frac{r^2}{2R^2}\right)\exp\left(-\frac{r_0^2}{2R^2}\right)\exp\left(\frac{\vec{\mathbf{r}}_0\cdot\vec{\mathbf{r}}}{R^2}\right) = \phi_\lambda(\vec{\mathbf{r}}_0)\exp\left(\frac{\lambda R^2}{2}\right)$$
$$\implies \int \mathrm{d}^d\vec{\mathbf{r}}\ \phi_\lambda(\vec{\mathbf{r}})\psi_G^R(\vec{\mathbf{r}})\exp\left(-\frac{r_0^2}{2R^2}\right)\exp\left(\frac{\vec{\mathbf{r}}_0\cdot\vec{\mathbf{r}}}{R^2}\right) = \phi_\lambda(\vec{\mathbf{r}}_0)\exp\left(\frac{\lambda R^2}{2}\right)$$

Taking averages over $\vec{\mathbf{r}}_0$ and noting that $\kappa = \dfrac{r}{R^2}$,

$$\int \mathrm{d}^d\vec{\mathbf{r}}\ \phi_\lambda(\vec{\mathbf{r}})\psi_G^R(\vec{\mathbf{r}})\exp\left(-\frac{r_0^2}{2R^2}\right)I_0\left(d;\frac{r}{R^2}r_0\right) = \langle\phi_\lambda(\vec{\mathbf{r}}_0)\rangle\exp\left(\frac{\lambda R^2}{2}\right)$$

Now, let's make the substitution, $r_0 = \kappa R^2$ and $\phi_\lambda(r) = J_0(d;kr) \implies \lambda = -k^2$

$$\int \mathrm{d}^d\vec{\mathbf{r}}\ J_0(d;kr)\psi_G^R(\vec{\mathbf{r}})\exp\left(-\frac{\kappa^2R^2}{2}\right)I_0(d;\kappa r) = \langle J_0(d;kr_0)\rangle\exp\left(-\frac{k^2R^2}{2}\right)$$
$$\int \mathrm{d}^d\vec{\mathbf{r}}\ J_0(d;kr)I_0(d;\kappa r)\psi_G^R(\vec{\mathbf{r}}) = J_0(d;k\kappa R^2)\exp\left(\frac{(\kappa^2-k^2)R^2}{2}\right)$$

Similarly, taking $\phi_\lambda(r) = I_0(d;\kappa' r) \implies \lambda = \kappa'^2$,

$$\int \mathrm{d}^d\vec{\mathbf{r}}\ I_0(d;\kappa' r)\psi_G^R(\vec{\mathbf{r}})\exp\left(-\frac{\kappa^2R^2}{2}\right)I_0(d;\kappa r) = \langle I_0(d;\kappa' r_0)\rangle\exp\left(\frac{\kappa'^2R^2}{2}\right)$$
$$\int \mathrm{d}^d\vec{\mathbf{r}}\ I_0(d;\kappa' r)I_0(d;\kappa r)\psi_G^R(\vec{\mathbf{r}}) = I_0(d;\kappa'\kappa R^2)\exp\left(\frac{(\kappa^2+\kappa'^2)R^2}{2}\right)$$

Problem 1 continued on next page. . .

Now, let's make the substitution, $r_0 = ik_2R^2$ and $\phi_\lambda(r) = J_0(d; k_1 r) \implies \lambda = -k_1^2$,

$$\int \mathrm{d}^d\vec{\mathbf{r}}\ J_0(d;k_1r)\psi_G^R(\vec{\mathbf{r}})\exp\left(\frac{k_2^2R^2}{2}\right)I_0(d;ik_2r) = \left\langle J_0(d;ik_1k_2R^2)\right\rangle \exp\left(-\frac{k_1^2R^2}{2}\right)$$

$$\int \mathrm{d}^d\vec{\mathbf{r}}\ J_0(d;k_1r)J_0(d;k_2r)\psi_G^R(\vec{\mathbf{r}}) = I_0(d;k_1k_2R^2)\exp\left(-\frac{(k_1^2+k_2^2)R^2}{2}\right)$$

Part (c)

Part (d)

Consider,

$$I_1 = \int_0^\infty \frac{\mathrm{d}R}{R^{2\Delta+1}}\exp\left(-\frac{r^2}{2R^2}\right)$$

Substitute $R^2 = \dfrac{r^2}{2u} \implies \mathrm{d}u = -\dfrac{r^2}{R^3}\,\mathrm{d}R \implies \mathrm{d}R = -\dfrac{r\,\mathrm{d}u}{(2u)^{3/2}}$,

$$\begin{aligned} I_1 &= \int_0^\infty \frac{r\,\mathrm{d}u}{(2u)^{3/2}}\left(\frac{2u}{r^2}\right)^{\Delta+1/2}\exp(-u) \\ &= \frac{2^{\Delta-1}}{r^{2\Delta}}\int_0^\infty \mathrm{d}u\,u^{\Delta-1}\exp(-u) \\ I_1 &= \frac{2^{\Delta-1}}{r^{2\Delta}}\Gamma(\Delta) \quad ; \qquad \text{for} \quad \Delta > 0 \end{aligned}$$

Now consider,

$$I_2 = \int_0^\infty \frac{\mathrm{d}R}{R^{2\Delta+1}}(2\pi)^{d/2}R^d\exp\left(-\frac{k^2R^2}{2}\right)$$

Substitute $\dfrac{k^2R^2}{2} = u \implies \mathrm{d}u = k^2R\,\mathrm{d}R\,, R = \left(\dfrac{2u}{k^2}\right)^{1/2}$,

$$\begin{aligned} I_2 &= \int_0^\infty \frac{\mathrm{d}u}{k^2}\frac{1}{R^{2\Delta+2}}(2\pi)^{d/2}\left(\frac{2u}{k^2}\right)^{d/2}\exp(-u) \\ &= \int_0^\infty \frac{\mathrm{d}u}{k^2}\left(\frac{k^2}{2u}\right)^{\Delta+1}(2\pi)^{d/2}\left(\frac{2u}{k^2}\right)^{d/2}\exp(-u) \\ &= \frac{2^{d-\Delta-1}\pi^{d/2}}{k^{2\Delta-d}}\int_0^\infty \mathrm{d}u\,u^{d/2-\Delta-1}\exp(-u) \\ I_2 &= \frac{2^{d-\Delta-1}\pi^{d/2}}{k^{2\Delta-d}}\Gamma(d/2-\Delta) \quad ; \qquad \text{for} \quad \frac{d}{2} > \Delta \end{aligned}$$

From Gaussian averaging, we know that,

$$\int \mathrm{d}^d\vec{\mathbf{r}}\ \phi_{-k^2}(\vec{\mathbf{r}})\psi_G^R(\vec{\mathbf{r}}-\vec{\mathbf{r}}_0) = \phi_{-k^2}(\vec{\mathbf{r}}_0)\exp\left(-\frac{k^2R^2}{2}\right)$$

$$\int \mathrm{d}^d\vec{\mathbf{r}}\ \phi_{-k^2}(\vec{\mathbf{r}})\int_0^\infty \frac{\mathrm{d}R}{R^{2\Delta+1}}(2\pi)^{d/2}R^d\psi_G^R(\vec{\mathbf{r}}-\vec{\mathbf{r}}_0) = \phi_{-k^2}(\vec{\mathbf{r}}_0)\int_0^\infty \frac{\mathrm{d}R}{R^{2\Delta+1}}(2\pi)^{d/2}R^d\exp\left(-\frac{k^2R^2}{2}\right)$$

$$\int \mathrm{d}^d\vec{\mathbf{r}}\ \phi_{-k^2}(\vec{\mathbf{r}})\frac{2^{\Delta-1}}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}_0|^{2\Delta}}\Gamma(\Delta) = \phi_{-k^2}(\vec{\mathbf{r}}_0)\frac{2^{d-\Delta-1}\pi^{d/2}}{k^{2\Delta-d}}\Gamma(d/2-\Delta)$$

$$\implies \int \mathrm{d}^d\vec{\mathbf{r}}\ \phi_{-k^2}(\vec{\mathbf{r}})\frac{\Gamma(\Delta)}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}_0|^{2\Delta}} = \phi_{-k^2}(\vec{\mathbf{r}}_0)\frac{2^{d-2\Delta}\pi^{d/2}}{k^{2\Delta-d}}\Gamma(d/2-\Delta)$$

Problem 2

Part (a)
We know,

$$
\begin{aligned}
I_0(d;x) &= \sum_{m=0}^{\infty} \frac{\Gamma(d/2)}{m!\,\Gamma(d/2+m)} \frac{x^{2m}}{2^{2m}} \\
\frac{x^2}{d^2} I_0(d+2;x) &= \sum_{m=0}^{\infty} \frac{\Gamma(d/2+1)}{m!\,\Gamma(d/2+m+1)} \frac{x^{2m+2}}{2^{2m}} \\
&= \sum_{m=0}^{\infty} \frac{\Gamma(d/2)\cdot d/2}{m!\,\Gamma(d/2+m+1)} \frac{x^{2m+2}}{2^{2m+2}} \frac{2^2}{d^2} \\
\frac{x^2}{d^2} I_0(d+2;x) &= \frac{2}{d} \sum_{m=0}^{\infty} \frac{\Gamma(d/2)}{m!\,\Gamma(d/2+m+1)} \frac{x^{2m+2}}{2^{2m+2}}
\end{aligned}
$$

Let's proceed and take the derivative,

$$
\begin{aligned}
\frac{x}{d} \frac{\mathrm{d}}{\mathrm{d}x} I_0(d;x) &= \frac{x}{d} \sum_{m=0}^{\infty} \frac{\Gamma(d/2)}{m!\,\Gamma(d/2+m)} \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{x^{2m}}{2^{2m}} \right) \\
&= \frac{1}{d} \sum_{m=1}^{\infty} \frac{\Gamma(d/2)}{m!\,\Gamma(d/2+m)} \frac{(2m)x^{2m}}{2^{2m}} \\
&= \frac{1}{d} \sum_{m=0}^{\infty} \frac{\Gamma(d/2)}{(m+1)!\,\Gamma(d/2+m+1)} \frac{2(m+1)x^{2m+2}}{2^{2m+2}} \quad \Longleftarrow \quad (m \to m+1) \\
\frac{x}{d} \frac{\mathrm{d}}{\mathrm{d}x} I_0(d;x) &= \frac{2}{d} \sum_{m=0}^{\infty} \frac{\Gamma(d/2)}{m!\,\Gamma(d/2+m+1)} \frac{x^{2m+2}}{2^{2m+2}}
\end{aligned}
$$

Let's now consider the third part,

$$
\begin{aligned}
I_0(d-2;x) - I_0(d,x) &= \sum_{m=0}^{\infty} \left[\frac{\Gamma(d/2-1)}{\Gamma(d/2+m-1)} - \frac{\Gamma(d/2)}{\Gamma(d/2+m)} \right] \frac{x^{2m}}{m!\,2^{2m}} \\
&= \sum_{m=1}^{\infty} \left[\frac{\Gamma(d/2-1)}{\Gamma(d/2+m-1)} - \frac{\Gamma(d/2)}{\Gamma(d/2+m)} \right] \frac{x^{2m}}{m!\,2^{2m}} \\
&= \sum_{m=0}^{\infty} \left[\frac{\Gamma(d/2-1)}{\Gamma(d/2+m)} - \frac{\Gamma(d/2)}{\Gamma(d/2+m+1)} \right] \frac{x^{2m+2}}{(m+1)!\,2^{2m}} \\
&= \sum_{m=0}^{\infty} \left[\frac{\Gamma(d/2)}{\Gamma(d/2+m+1)} \frac{(d/2+m)}{(d/2-1)} - \frac{\Gamma(d/2)}{\Gamma(d/2+m+1)} \right] \frac{x^{2m+2}}{(m+1)!\,2^{2m}} \\
&= \sum_{m=0}^{\infty} \left[\frac{2(m+1)}{d-2} \right] \frac{\Gamma(d/2)}{\Gamma(d/2+m+1)} \frac{x^{2m+2}}{(m+1)!\,2^{2m}} \\
&= \frac{2}{d-2} \sum_{m=0}^{\infty} \frac{\Gamma(d/2)}{\Gamma(d/2+m+1)} \frac{x^{2m+2}}{m!\,2^{2m}} \\
\frac{d-2}{d}(I_0(d-2;x) - I_0(d,x)) &= \frac{2}{d} \sum_{m=0}^{\infty} \frac{\Gamma(d/2)}{m!\,\Gamma(d/2+m+1)} \frac{x^{2m+2}}{2^{2m+2}}
\end{aligned}
$$

$$
\Longrightarrow \boxed{\frac{x}{d} \frac{\mathrm{d}}{\mathrm{d}x} I_0(d;x) = \frac{x^2}{d^2} I_0(d+2;x) = \frac{d-2}{d}[I_0(d-2;x) - I_0(d,x)]}
$$

Problem 2 continued on next page. . .

Part (b)
Schafli's contour integral is given by,

$$I_0(d;x) = \oint_{\mathcal{C}} \frac{\mathrm{d}z}{2\pi i} \frac{\Gamma(d/2)}{z^{d/2}} \exp\left(z + \frac{x^2}{4z}\right)$$

where the contour $\mathcal{C} = C[-\infty - i0, 0+, -\infty + i0]$.

$$\begin{aligned}
\frac{x}{d}\frac{\mathrm{d}}{\mathrm{d}x} I_0(d;x) &= \oint_{\mathcal{C}} \frac{x}{d}\frac{\mathrm{d}z}{2\pi i} \frac{\Gamma(d/2)}{z^{d/2}} \frac{x}{2z} \exp\left(z + \frac{x^2}{4z}\right) \\
&= \frac{x^2}{d^2} \oint_{\mathcal{C}} \frac{\mathrm{d}z}{2\pi i} \frac{(d/2)\cdot\Gamma(d/2)}{z^{(d+2)/2}} \exp\left(z + \frac{x^2}{4z}\right) \\
&= \frac{x^2}{d^2} \oint_{\mathcal{C}} \frac{\mathrm{d}z}{2\pi i} \frac{\Gamma((d+2)/2)}{z^{(d+2)/2}} \exp\left(z + \frac{x^2}{4z}\right) \\
\implies \frac{x}{d}\frac{\mathrm{d}}{\mathrm{d}x} I_0(d;x) &= \frac{x^2}{d^2} I_0(d+2;x)
\end{aligned}$$

$$\begin{aligned}
I_0(d-2;x) - I_0(d,x) &= \oint_{\mathcal{C}} \frac{\mathrm{d}z}{2\pi i} \left[\frac{\Gamma(d/2-1)}{z^{d/2-1}} - \frac{\Gamma(d/2)}{z^{d/2}}\right] \exp\left(z + \frac{x^2}{4z}\right) \\
&=
\end{aligned}$$

Part (c)
We are given,

$$\begin{aligned}
I_0(d;x) &\approx \frac{e^x}{|S^{d-1}|}\left(\frac{2\pi}{x}\right)^{\frac{d-1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\frac{d-1}{2}+n)}{(2x)^n n!\ \Gamma(\frac{d-1}{2}-n)} \\
\implies \frac{x}{d}\frac{\mathrm{d}}{\mathrm{d}x} I_0(d;x) &\approx \frac{x}{d}\left[\frac{e^x}{|S^{d-1}|}\left(\frac{2\pi}{x}\right)^{\frac{d-1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\frac{d-1}{2}+n)}{(2x)^n n!\ \Gamma(\frac{d-1}{2}-n)}\right]
\end{aligned}$$

Problem 3

Part (a)
Given,

$$\begin{aligned}
V(z,\rho) &= \int_0^{2\pi} \frac{\mathrm{d}\alpha}{2\pi} g(z - i\rho\cos\alpha) \\
\nabla^2 V(z,\rho) &= \int_0^{2\pi} \frac{\mathrm{d}\alpha}{2\pi} \nabla^2 g(z - i\rho\cos\alpha) \\
&= \int_0^{2\pi} \frac{\mathrm{d}\alpha}{2\pi} \left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial(g(z-i\rho\cos\alpha))}{\partial\rho}\right) + \frac{\partial^2 g(z-i\rho\cos\alpha)}{\partial z^2}\right] \\
&= \int_0^{2\pi} \frac{\mathrm{d}\alpha}{2\pi} \left[\frac{1}{\rho}\frac{\partial\{\rho g'(t)(-i\cos\alpha)\}}{\partial\rho} + g''(t)\right] \quad ; \quad t = z - i\rho\cos\alpha \\
&= \int_0^{2\pi} \frac{\mathrm{d}\alpha}{2\pi} \left[-\frac{ig'(t)\cos\alpha}{\rho} - g''(t)\cos^2\alpha + g''(t)\right] \\
&= \int_0^{2\pi} \frac{\mathrm{d}\alpha}{2\pi} \left[-\frac{ig'(t)\cos\alpha}{\rho} + g''(t)\sin^2\alpha\right]
\end{aligned}$$

Problem 3 continued on next page...

Now we proceed to integrate the first term by parts,

$$\begin{aligned}\nabla^2 V(z,\rho) &= -\left.\frac{ig'(t)\sin\alpha}{\rho}\right|_0^{2\pi} + \int_0^{2\pi}\frac{\mathrm{d}\alpha}{2\pi}\left[ig''(t)i\sin\alpha\sin\alpha + g''(t)\sin^2\alpha\right]\\ \implies \nabla^2 V(z,\rho) &= 0\end{aligned}$$

Hence, we have proved that V solves Laplace equation.

Part (b)
Consider,

$$g(z - i\rho\cos\alpha) = \sum_{n=0}^{\infty} g^{(n)}(z)(-i)^n\cos^n\alpha\frac{\rho^n}{n!}$$

$$\int_0^{2\pi}\frac{\mathrm{d}\alpha}{2\pi}g(z - i\rho\cos\alpha) = \sum_{n=0}^{\infty} g^{(n)}(z)(-i)^n\frac{\rho^n}{n!}\int_0^{2\pi}\frac{\mathrm{d}\alpha}{2\pi}\cos^n\alpha$$

$$\int_0^{2\pi}\frac{\mathrm{d}\alpha}{2\pi}\cos^n\alpha = 0 \quad \text{for} \quad n = 1,3,5\ldots \quad ; \quad \int_0^{2\pi}\frac{\mathrm{d}\alpha}{2\pi}\cos^n\alpha = \frac{1}{2^n}{}^nC_{n/2} \quad \text{for} \quad n = 0,2,4,\ldots$$

$$\implies V(g,z) = \sum_{m=0}^{\infty} g^{(2m)}(z)(-1)^m\frac{\rho^{2m}}{(2m)!}\frac{1}{2^{2m}}{}^{2m}C_m$$

But, $g(z) = V(z,0) \implies g^{(n)}(z) = V^n(z,0)$. Hence, we write,

$$\boxed{V(g,z) = V(z,0) - \frac{\rho^2}{4}V^{(2)}(z,0) + \frac{\rho^4}{64}V^{(4)}(z,0) + \ldots}$$

Part (c)

$$\begin{aligned}V(g,z) &= \sum_{m=0}^{\infty}(-1)^m\frac{\rho^{2m}}{(2m)!}\frac{1}{2^{2m}}{}^{2m}C_m\frac{\partial^{2m}}{\partial z^{2m}}V(z,0)\\ &= \sum_{m=0}^{\infty}(-1)^m\frac{1}{(m!)^2}\left(\frac{1}{2}\rho\frac{\partial}{\partial z}\right)^{2m}V(z,0)\\ V(g,z) &= J_0\left(d=2;\rho\frac{\partial}{\partial z}\right)V(z,0)\end{aligned}$$

Consider,

$$E_z(z,\rho) = -\frac{\partial V(z,\rho)}{\partial z} = -\sum_{m=0}^{\infty}(-1)^m\frac{1}{(m!)^2}\left(\frac{1}{2}\rho\frac{\partial}{\partial z}\right)^{2m}\frac{\partial V(z,0)}{\partial z} = J_0\left(d=2;\rho\frac{\partial}{\partial z}\right)E_z(z,0)$$

$$E_\rho(z,\rho) = -\frac{\partial V(z,\rho)}{\partial\rho} = -\rho\frac{1}{\rho}\frac{\partial}{\partial\rho}J_0\left(d=2;\rho\frac{\partial}{\partial z}\right)V(z,0) = -\frac{\rho}{2}J_0\left(d=4;\rho\frac{\partial}{\partial z}\right)\frac{\partial^2 V}{\partial z^2} = \frac{\rho}{2}J_0\left(d=4;\rho\frac{\partial}{\partial z}\right)\frac{\partial E_z}{\partial z}$$

Part (d)

Problem 3 continued on next page. . .

$$
\begin{aligned}
V_l(z,\rho) &= J_0\left(\rho\frac{\partial}{\partial z}\right) \\
&= \sum_{m=0}^{\infty}(-1)^m\frac{1}{(m!)^2}\left(\frac{1}{2}\rho\frac{\partial}{\partial z}\right)^{2m}\frac{z^l}{l!} \\
&= \sum_{m=0}^{\infty}(-1)^m\frac{1}{(m!)^2}\frac{\rho^{2m}}{2^{2m}}\frac{l!}{(l-2m)!}\frac{z^{l-2m}}{l!} \quad \text{if} \quad l \geq 2m \quad \text{else} \quad = 0 \\
V_l(z,\rho) &= \sum_{m=0}^{\infty}(-1)^m\frac{1}{(m!)^2}\frac{\rho^{2m}}{2^{2m}}\frac{z^{l-2m}}{(l-2m)!}
\end{aligned}
$$

$$
\boxed{V_1(z,\rho) = z \quad \text{and} \quad V_2(\rho,z) = \frac{z^2}{2} - \frac{\rho^2}{4}}
$$