Homework 1: Dynamical systems; Instructor: Amit Apte

Due Wed 30 Jan 2019, beginning of class

T: Theoretical; N: Numerical: T/N: Both theory and numerics; marks for each question in paranthesis (please email answers to numerical questions BEFORE the beginning of the class on the due date of the homework)

1. (T, 40; Ref: section 1.9 "Stability theory" of Perko) A set $S \subset \mathbb{R}^d$ is called an invariant set for the ODE $\dot{x} = f(x)$ if for any initial condition $x_0 \in S$, the solution stays in the set $x(t) \in S$ for all t (for which the solution is defined).

   Let $w_j = u_j + iv_j$ be generalized eigenvectors with eigenvalue $\lambda_j = a_j + ib_j$ of a $d \times d$ matrix A (i.e. $(A - \lambda_j I_d)^m w_j = 0$ for some $m > 0$), with the notation that if $b_j = 0$, then $v_j = 0$. Define three subspaces of $\mathbb{R}^d$ as

   $$E^u = \mathrm{span}\{u_j, v_j | a_j > 0\}\,, \quad E^c = \mathrm{span}\{u_j, v_j | a_j = 0\}\,, \quad E^s = \mathrm{span}\{u_j, v_j | a_j < 0\}\,.$$

   (a) Prove that each of the above are invariant subspaces for the ODE $\dot{x} = Ax$ and that they span the whole space, i.e., $\mathbb{R}^d = E^u \oplus E^c \oplus E^s$.

   (b) Prove that for all $x_0 \in E^s$, $\lim_{t\to\infty} e^{At}x_0 = 0$, and $\lim_{t\to-\infty} |e^{At}x_0| = \infty$ (if $x_0 \neq 0$), and further,

   $$m|t^k|e^{-\alpha t}|x_0| \leq |e^{At}x_0| \leq Me^{-\beta t}|x_0|\,,$$

   for all t, for some $m, M, \alpha, \beta > 0$ and non-negative integer k. Hence, E^s is called the stable subspace.

   (c) State and prove the equivalent statement for initial conditions in the "unstable subspace" $x_0 \in E^u$.

   (d) What would be the corresponding statements about the norm $|e^{At}x_0|$ as $t \to \pm\infty$ for the the initial conditions in the "centre subspace" $x_0 \in E^c$?

2. (T, 20) Some of the steps I did not complete in the class:

   (a) Prove that if the initial condition $X(t_0)$ of the matrix equation $\dot{X} = A(t)X$ is non-degenerate, then the solution $X(t)$ is non-degenerate for all times t. Along the way, as a sub-problem, derive and solve the ODE satisfied by $y(t) = \det X(t)$.

   (b) Prove that, if $X(t)$ and $Y(t)$ are two fundamental matrix solutions, then there is a constant non-singular matrix C such that $Y(t) = X(t)C$.

   (Does there always exist some B) / (Can there exist some B) / (Is it true that for all B) that $BX(t)$ a fundamental matrix. (Apologies for the grammatically ill-formed sentence - hope you understand!)

   (c) For a time periodic linear ODE $\dot{X} = A(t)X$, if two fundamental matrices satisfy $X_i(t+\omega) = X_i(t)B_i$ for $i = 1, 2$, prove that B_i are similar.

3. (N, 30) Solve the linearized Lorenz system with various initial conditions (i.e. for different initial condition of the trajectory and different initial conditions for the linearized equations as well), for both the chaotic and periodic cases (using the parameter values used in the previous homework).

   Plot the norm of the solution of the linearized equation as a function time (possibly on a semilog plot), and interpret the results.

Does the norm of the solution of the linearized equation grow with time (exponentially?), decay with time (exponentially?), or stay “constant” (or bounded from above and below)?

Recall that for the system that we discussed in the class, the perturbations tangent to the orbit (i.e. with initial condition of the linearized equation being tangent to the orbit) did not grow or decay with time. Is similar behaviour observed for Lorenz system (either for chaotic or the periodic case)? Interprete the results.

In the chaotic case, can you find any initial conditions of the linearized equation that decay? Or those that stay bounded?

4. (T/N, 10, or 100) Are scientific theories true? If so, prove. If not, exemplify.