Advanced Quantum Mechanics: Assignment #2

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Problem 1

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Problem 2

We know that the energy eigenstates of the harmonic oscillator form a basis. Hence, the required coherent state $|z\rangle$ can be written in terms of these eigenstates as,

$$|z\rangle = \sum_{n=0}^{\infty} c_n |n\rangle = \sum_{n=0}^{\infty} c_n \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle$$

Substituting this in the equation for coherent state $a|z\rangle = z|z\rangle$,

$$\begin{aligned}
\sum_{n=0}^{\infty} c_n a |n\rangle &= \sum_{n=0}^{\infty} z c_n |n\rangle \\
\sum_{n=1}^{\infty} c_n \sqrt{n} |n-1\rangle &= \sum_{n=0}^{\infty} z c_n |n\rangle \\
\sum_{n=0}^{\infty} c_{n+1} \sqrt{n+1} |n\rangle &= \sum_{n=0}^{\infty} z c_n |n\rangle \\
\therefore c_{n+1}\sqrt{n+1} &= z c_n
\end{aligned}$$

We have effectively derived a recursion relation for the coefficients c_n. If we start off with $c_n = \alpha$,

$$c_1 = z\alpha \quad , \quad c_2 = \frac{z^2\alpha}{\sqrt{2}} \quad , \quad c_3 = \frac{z^3\alpha}{\sqrt{3\cdot 2}} \quad , \quad \ldots \quad , \quad c_n = \frac{z^n\alpha}{\sqrt{n!}}$$

So, our coherent state can now be written as,

$$\begin{aligned}
|z\rangle &= \alpha \sum_{n=0}^{\infty} \frac{(za^\dagger)^n}{n!} |0\rangle \\
&= \alpha e^{a^\dagger z} |0\rangle
\end{aligned}$$

Problem 3

We know that,

$$x(0) = \frac{a + a^\dagger}{\sqrt{2m\omega}} \quad , \quad p(0) = \frac{\sqrt{m\omega}(a - a^\dagger)}{\sqrt{2}i} \quad , \quad x(t) = e^{iHt}x(0)e^{-iHt} \quad , \quad p(t) = e^{iHt}p(0)e^{-iHt}$$

From this, we note the following,

$$
\begin{aligned}
x(t)\left|0\right\rangle &= e^{iHt}x(0)e^{-iHt}\left|0\right\rangle \\
&= e^{-i\omega t/2}e^{iHt}x(0)\left|0\right\rangle \\
&= \frac{e^{-i\omega t/2}}{\sqrt{2m\omega}}e^{iHt}\left|1\right\rangle \\
x(t)\left|0\right\rangle &= \frac{e^{i\omega t}}{\sqrt{2m\omega}}\left|1\right\rangle \implies \left\langle 0\right| x(t) = \frac{e^{-i\omega t}}{\sqrt{2m\omega}}\left\langle 1\right|
\end{aligned}
$$

$$
\begin{aligned}
\text{Similarly,}\quad p(t)\left|0\right\rangle &= e^{iHt}p(0)e^{-iHt}\left|0\right\rangle \\
&= e^{-i\omega t/2}e^{iHt}p(0)\left|0\right\rangle \\
&= -\frac{e^{-i\omega t/2}\sqrt{m\omega}}{\sqrt{2}i}e^{iHt}\left|1\right\rangle \\
p(t)\left|0\right\rangle &= -\frac{e^{i\omega t}\sqrt{m\omega}}{\sqrt{2}i}\left|1\right\rangle \implies \left\langle 0\right| p(t) = \frac{e^{-i\omega t}\sqrt{m\omega}}{\sqrt{2}i}\left\langle 1\right|
\end{aligned}
$$

Now consider the quantities to be calculated,

$$C_1(t) = \langle 0|x(t)x(0)|0\rangle = \frac{e^{-i\omega t}}{\sqrt{2m\omega}}\frac{1}{\sqrt{2m\omega}} = \boxed{\frac{e^{-i\omega t}}{2m\omega}}$$

$$C_2(t) = \langle 0|x(t)p(0)|0\rangle - \langle 0|p(0)x(t)|0\rangle = -\frac{e^{-i\omega t}}{2i} - \frac{e^{i\omega t}}{2i} = \boxed{i\cos\omega t}$$

$$C_3(t) = \langle 0|p(t)x(0)|0\rangle - \langle 0|x(0)p(t)|0\rangle = \frac{e^{-i\omega t}}{2i} + \frac{e^{i\omega t}}{2i} = \boxed{-i\cos\omega t}$$

Problem 4

Part (a)

$$
\begin{aligned}
Z(\beta) = \mathrm{Tr}\, e^{-\beta H} &= \sum_{n=0}^{\infty} \langle n|e^{-\beta H}|n\rangle \\
&= \sum_{n=0}^{\infty} e^{-\beta(n+\frac{1}{2})\omega} \\
&= \frac{e^{-\beta\omega/2}}{1-e^{-\beta\omega}}
\end{aligned}
$$

Part (b)
Given that,

$$x(\tau) = \sum_n x_n e^{\frac{2\pi i n\tau}{\beta}} = \sum_n (a_n + ib_n)e^{\frac{2\pi i n\tau}{\beta}}$$

As we require $x(\tau)$ to be real, it follows from above that $x_n = x_n^*$. Taking $\omega_n = \frac{2\pi n}{\beta}$

$$-S_E = \int_0^\beta d\tau\left(\frac{m\dot{x}^2}{2} - \frac{m\omega^2 x^2}{2}\right)$$

Problem 4 continued on next page. . .

$$
\begin{aligned}
&= \sum_{p,q} \left(\frac{m x_p x_q \omega_p \omega_q}{2} - \frac{m\omega^2 x_p x_q}{2} \right) \int_0^\beta e^{i(\omega_q + \omega_p)\tau} d\tau \\
&= \sum_{p,q} \left(\frac{m x_p x_q \omega_p \omega_q}{2} - \frac{m\omega^2 x_p x_q}{2} \right) \beta \delta_{p,-q} \\
&= \sum_{q} \left(\frac{m x_{-q} x_q \omega_{-q} \omega_q}{2} - \frac{m\omega^2 x_{-q} x_q}{2} \right) \beta \\
&= \frac{m}{2} \sum_q \beta \big(-x_q^* x_q \omega_q^2 - \omega^2 x_q^* x_q \big) \\
-S_E &= -\frac{m\beta}{2} \sum_q x_q^* x_q \big(\omega_q^2 + \omega^2 \big)
\end{aligned}
$$

The required path integral to be done is,

$$
\begin{aligned}
Z_c(\beta) &= N \int \mathcal{D}x \exp -\frac{m\beta}{2} \sum_q x_q^* x_q \big(\omega_q^2 + \omega^2\big) \quad \text{where } N \text{ is some normalization} \\
&= N \int \mathcal{D}x \exp \left(-m\beta \sum_{q=1}^{\infty} x_q^* x_q \big(\omega_q^2 + \omega^2\big) - \frac{m\beta x_0^2 \omega^2}{2} \right) \\
&= N \int dx_0 \exp \left(-\frac{m\beta x_0^2 \omega^2}{2} \right) \times \prod_{q=1}^{\infty} \int dx_q dx_q^* \exp -m\beta x_q^* x_q \big(\omega_q^2 + \omega^2\big) \\
&= N \sqrt{\frac{2\pi}{m\beta\omega^2}} \times \prod_{q=1}^{\infty} \frac{2\pi}{m\beta(\omega_q^2 + \omega^2)}
\end{aligned}
$$

One can also calculate the quantity $Z_{free}(\beta)$ which corresponds to $\omega = 0$. Following the steps above, that comes out to be,

$$
Z_{free}(\beta) = N \prod_{q=1}^{\infty} \frac{2\pi}{m\beta\omega_q^2}
$$

Our final partition function is,

$$
Z(\beta) = \frac{Z_c(\beta)}{Z_{free}(\beta)} = \sqrt{\frac{2\pi}{m\beta\omega^2}} \times \prod_{q=1}^{\infty} \frac{\omega_q^2}{\omega_q^2 + \omega^2}
$$