Classical Mechanics: Assignment #6

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Problem 1

Liouville Theorem states that in a Hamiltonian system, the phase space density is constant in time.
Let our system consist of N points (q_k, p_k) in a $2N$ dimensional phase space.

Problem 2

Transformations of coordinates $(q, p, t) \to (Q, P, t)$ which preserves the form of Hamilton's equations are called canonical transformations. So, by definition,

$$\dot{p} = \frac{\partial H}{\partial q} \quad , \quad \dot{q} = -\frac{\partial H}{\partial p} \quad \text{and} \quad \dot{P} = \frac{\partial K}{\partial Q} \quad , \quad \dot{Q} = -\frac{\partial K}{\partial P}$$

The definition also implies that,

$$\delta(p\dot{q} - H) = 0 \quad \text{and} \quad \delta(P\dot{Q} - K) = 0$$
$$\lambda(p\dot{q} - H) = P\dot{Q} - K + \frac{\mathrm{d}F}{\mathrm{d}t}$$

We deal with the $\lambda = 1$ case. The $\frac{\mathrm{d}F}{\mathrm{d}t}$ term comes from the fact that Lagrangians are not unique and we can always add a total time derivative term without changing the equations of motion. If the above condition is satisfied, the transformation $(q, p, t) \to (Q, P, t)$ is guaranteed to be canonical, and the function F is called a generating function. We deal with four classes of generating functions case-by-case,

- $F = F_1(q, Q, t)$,

$$p\dot{q} - H = P\dot{Q} - K + \frac{\mathrm{d}F_1}{\mathrm{d}t} = P\dot{Q} - K + \frac{\partial F_1}{\partial q}\dot{q} + \frac{\partial F_1}{\partial Q}\dot{Q} + \frac{\partial F_1}{\partial t}$$

  As q and Q are independent, the coefficients should vanishh independently, such that $K = H + \frac{\partial F_1}{\partial t}$. This implies,

$$\frac{\partial F_1}{\partial q} = p \quad \text{and} \quad \frac{\partial F_1}{\partial Q} = -P$$

- $F = F_2(q, P, t) - QP$,

$$p\dot{q} - H = P\dot{Q} - K + \frac{\mathrm{d}F_2}{\mathrm{d}t} - \frac{\mathrm{d}(QP)}{\mathrm{d}t} = P\dot{Q} - K + \frac{\partial F_2}{\partial q}\dot{q} + \frac{\partial F_2}{\partial P}\dot{P} + \frac{\partial F_2}{\partial t} - P\dot{Q} - Q\dot{P}$$

$$\implies \frac{\partial F_2}{\partial q} = p \quad \text{and} \quad \frac{\partial F_2}{\partial P} = Q$$

- $F = F_3(p, Q, t) + qp$,

$$p\dot{q} - H = P\dot{Q} - K + \frac{\mathrm{d}F_3}{\mathrm{d}t} + \frac{\mathrm{d}(qp)}{\mathrm{d}t} = P\dot{Q} - K + \frac{\partial F_3}{\partial Q}\dot{Q} + \frac{\partial F_3}{\partial p}\dot{p} + \frac{\partial F_3}{\partial t} + p\dot{q} + q\dot{p}$$

$$\implies \frac{\partial F_3}{\partial Q} = -P \quad \text{and} \quad \frac{\partial F_2}{\partial p} = -q$$

- $F = F_4(p, P, t) + qp - QP$,

$$p\dot{q} - H = P\dot{Q} - K + \frac{\mathrm{d}F_4}{\mathrm{d}t} + \frac{\mathrm{d}(qp - QP)}{\mathrm{d}t} = P\dot{Q} - K + \frac{\partial F_4}{\partial P}\dot{P} + \frac{\partial F_4}{\partial p}\dot{p} + \frac{\partial F_4}{\partial t} + p\dot{q} + q\dot{p} - P\dot{Q} - Q\dot{P}$$

$$\implies \frac{\partial F_4}{\partial P} = Q \quad \text{and} \quad \frac{\partial F_4}{\partial p} = -q$$

Part (b)

We first use the Poisson Bracket invariance approach. We are given,

$$Q_1 = q_1 \quad , \quad Q_2 = p_2 \quad , \quad P_1 = p_1 - 2p_2 \quad , \quad P_2 = -2q_1 - q_2$$

Consider $\{Q_1, Q_2\}$,

$$\{Q_1, Q_2\} = \sum_{i=1}^{2} \frac{\partial Q_1}{\partial q_i}\frac{\partial Q_2}{\partial p_i} - \frac{\partial Q_1}{\partial p_i}\frac{\partial Q_2}{\partial q_i} = 0$$

$$\{P_1, P_2\} = \sum_{i=1}^{2} \frac{\partial P_1}{\partial q_i}\frac{\partial P_2}{\partial p_i} - \frac{\partial P_1}{\partial p_i}\frac{\partial P_2}{\partial q_i} = -\frac{\partial P_1}{\partial p_1}\frac{\partial P_2}{\partial q_1} - \frac{\partial P_1}{\partial p_2}\frac{\partial P_2}{\partial q_2} = 2 - 2 = 0$$

$$\{Q_1, P_2\} = \sum_{i=1}^{2} \frac{\partial Q_1}{\partial q_i}\frac{\partial P_2}{\partial p_i} - \frac{\partial Q_1}{\partial p_i}\frac{\partial P_2}{\partial q_i} = \frac{\partial Q_1}{\partial q_1}\frac{\partial P_2}{\partial p_1} = 0$$

$$\{Q_2, P_1\} = \sum_{i=1}^{2} \frac{\partial Q_2}{\partial q_i}\frac{\partial P_1}{\partial p_i} - \frac{\partial Q_2}{\partial p_i}\frac{\partial P_1}{\partial q_i} = -\frac{\partial Q_2}{\partial p_2}\frac{\partial P_1}{\partial q_2} = 0$$

$$\{Q_1, P_1\} = \sum_{i=1}^{2} \frac{\partial Q_1}{\partial q_i}\frac{\partial P_1}{\partial p_i} - \frac{\partial Q_1}{\partial p_i}\frac{\partial P_1}{\partial q_i} = \frac{\partial Q_1}{\partial q_1}\frac{\partial P_1}{\partial p_1} = 1$$

$$\{Q_2, P_2\} = \sum_{i=1}^{2} \frac{\partial Q_2}{\partial q_i}\frac{\partial P_2}{\partial p_i} - \frac{\partial Q_2}{\partial p_i}\frac{\partial P_2}{\partial q_i} = -\frac{\partial Q_2}{\partial p_2}\frac{\partial P_2}{\partial q_2} = 1$$

Hence, as $\{Q_i, P_j\} = \delta_{ij}, \{Q_i, Q_j\} = 0, \{P_i, P_j\} = 0$, the transformation is canonical. We now use the symplectic approach. If we denote $X = \begin{bmatrix} Q_1 & Q_2 & P_1 & P_2 \end{bmatrix}^T, x = \begin{bmatrix} q_1 & q_2 & p_1 & p_2 \end{bmatrix}^T$, then $X = Mx$ where M is the transformation matrix. From the definitions of the X, we can see that,

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ -2 & -1 & 0 & 0 \end{pmatrix}$$

For the transformation to be a canonical transformation, $M^T J M = J$, where,

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Problem 2 continued on next page...

$$
\begin{aligned}
M^TJM &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ -2 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ -2 & -1 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 2 & 0 & 1 \\ 0 & 0 & -2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ -2 & -1 & 0 & 0 \end{pmatrix} \\
M^TJM &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = J
\end{aligned}
$$

Hence, it is a canonical transformation.

Part (c)

$$l_i = \epsilon_{ijk} x_j p_k \quad \text{using the Einstein summation convention}$$

We now note the following,

$$
\begin{aligned}
\{l_i, l_j\} &= \epsilon_{iab}\epsilon_{jmn}\{x_a p_b, x_m p_n\} \\
&= \epsilon_{iab}\epsilon_{jmn}\{x_a p_b, x_m p_n\} \\
&= \epsilon_{iab}\epsilon_{jmn}(\{x_a, p_n\}x_m p_b + \{p_b, x_m\}x_a p_n) \\
&= \epsilon_{iab}\epsilon_{jmn}(\delta_{an} x_m p_b - \delta_{bm} x_a p_n) \\
&= \epsilon_{inb}\epsilon_{jmn} x_m p_b - \epsilon_{iam}\epsilon_{jmn} x_a p_n \\
&= -\epsilon_{ibn}\epsilon_{jmn} x_m p_b + \epsilon_{ima}\epsilon_{jmn} x_a p_n \\
&= -(\delta_{ij}\delta_{bm} - \delta_{im}\delta_{jb}) x_m p_b + (\delta_{ij}\delta_{an} - \delta_{aj}\delta_{in}) x_a p_n \\
&= -\delta_{ij} x_b p_b + x_i p_j + \delta_{ij} x_a p_a - x_j p_i \\
&= +x_i p_j - x_j p_i \\
\{l_i, l_j\} &= \epsilon_{ijk} l_k
\end{aligned}
$$

$$
\begin{aligned}
\{x_i, l_j\} &= \epsilon_{jmn}\{x_i, x_m p_n\} \\
&= \epsilon_{jmn} x_m \{x_i, p_n\} \\
&= \epsilon_{jmn} x_m \delta_{in} \\
\{x_i, l_j\} &= \epsilon_{ijm} x_m
\end{aligned}
$$

$$
\begin{aligned}
\{p_i, l_j\} &= \epsilon_{jmn}\{p_i, x_m p_n\} \\
&= \epsilon_{jmn} p_n \{p_i, x_m\} \\
&= -\epsilon_{jmn} p_n \delta_{im} \\
\{p_i, l_j\} &= \epsilon_{ijn} p_n
\end{aligned}
$$

Problem 3

We are given the Hamiltonian and generating function,

$$H = \frac{p^2}{2} + \frac{\omega^2 x^2}{2} + \alpha x^3 + \beta x p^2 \quad \text{and} \quad \phi = xP + ax^2P + bP^3$$

Problem 3 continued on next page...

$\phi = \phi(x, P)$. For ϕ to be a canonical transformation,

$$\begin{aligned} \frac{\partial \phi}{\partial x} = p \quad &\text{and} \quad \frac{\partial \phi}{\partial P} = Q \\ \implies P + 2axP = p \quad &\text{and} \quad x + ax^2 + 3bP^2 = Q \\ \implies P\sqrt{-12abP^2 + 4aQ + 1} = p \quad &\text{and} \quad \frac{\sqrt{-12abP^2 + 4aQ + 1} - 1}{2a} = x \end{aligned}$$

where we have only considered the x root with positive sign before the discriminant. Then,

$$\begin{aligned} K(Q,P) = &\frac{\alpha\left(\sqrt{-12abP^2 + 4aQ + 1} - 1\right)^3}{8a^3} + \frac{\omega^2\left(\sqrt{-12abP^2 + 4aQ + 1} - 1\right)^2}{8a^2} \\ &+ \frac{\beta P^2\left(-12abP^2 + 4aQ + 1\right)\left(\sqrt{-12abP^2 + 4aQ + 1} - 1\right)}{2a} + \frac{1}{2}P^2\left(-12abP^2 + 4aQ + 1\right) \end{aligned}$$

Expanding the above upto third order, we have,

$$\begin{aligned} K(Q,P) = &Q^3\left(P^2\left(-30a^2b\omega^2 - 2a^2\beta + 36\alpha ab\right) - a\omega^2 + \alpha\right) + Q^2\left(P^2\left(9ab\omega^2 + 3a\beta - 9\alpha b\right) + \frac{\omega^2}{2}\right) \\ &+ P^2Q\left(2a - 3b\omega^2 + \beta\right) + \frac{P^2}{2} \end{aligned}$$

As anharmonic terms of third order should not be present, we can see from above that,

$$-a\omega^2 + \alpha = 0 \quad \text{and} \quad 2a - 3b\omega^2 + \beta = 0 \implies a = \frac{\alpha}{\omega^2} \quad \text{and} \quad b = \frac{1}{3\omega^2}\left(\frac{2\alpha}{\omega^2} + \beta\right)$$

Now we need to find $\dot{x}$. From Hamilton's equation of motion we have,

$$\dot{x} = \frac{\partial H}{\partial p} = p(1 + 2\beta x) =$$

Part (b)

- $\phi(\vec{\mathbf{r}}, \vec{\mathbf{P}}) = (\vec{\mathbf{r}} \cdot \vec{\mathbf{P}}) + (\delta\vec{\mathbf{a}} \cdot \vec{\mathbf{P}})$ This looks like $F_2(q, P)$. From the results of Problem 2, we can then write,

$$\frac{\partial \Phi}{\partial r} = p_r = P_r \quad , \quad \frac{\partial \Phi}{\partial \theta} = p_\theta = 0 \quad , \quad \frac{\partial \Phi}{\partial \phi} = p_\phi = 0$$
$$\frac{\partial \Phi}{\partial P_r} = Q_r = r + \delta a_x \quad , \quad \frac{\partial \Phi}{\partial P_\theta} = Q_\theta = \delta a_\theta \quad , \quad \frac{\partial \Phi}{\partial P_\phi} = Q_\phi = \delta a_\phi$$

  as $r + \delta a = Q$, it is evident that the transformation is a translation by constant, as the momentum remains the same but the coordinates get shifted by a constant amount.

- $\Phi(\vec{\mathbf{r}}, \vec{\mathbf{P}}) = (\vec{\mathbf{r}} \cdot \vec{\mathbf{P}}) + (\delta\psi \cdot \vec{\mathbf{r}} \times \vec{\mathbf{P}})$
- $\Phi = qP + \delta\tau H(q, p, t)$ This looks like F_2 again. We write,

$$\begin{aligned} \frac{\partial \Phi}{\partial q} &= P + \delta\tau(-\dot{p}) = p \quad \text{and} \\ \frac{\partial \Phi}{\partial P} &= q + \delta\tau\frac{\partial H}{\partial P} \\ &= q + \delta\tau\frac{\partial H}{\partial P} \\ &= q + \delta\tau\left(\frac{\partial H}{\partial p}\frac{\partial p}{\partial P} + \frac{\partial H}{\partial q}\frac{\partial q}{\partial P}\right) \\ &= q + \delta\tau(\dot{q}(1 - \delta\tau\dot{p})) \\ Q &\approx q + \delta\tau\dot{q} \end{aligned}$$

$$\therefore Q \approx q + \delta\tau\dot{q} \quad \text{and} \quad P \approx p + \delta\tau\dot{p}$$

So the canonical transformation just corresponds to time translation by parameter $d\tau$.

- $\Phi = \vec{\mathbf{r}} \cdot \vec{\mathbf{P}} + (r^2 + P^2)\delta a$

$$\frac{\partial \Phi}{\partial r} = P_r + 2r\delta a \implies P_r = p_r - 2r\delta a$$
$$\frac{\partial \Phi}{\partial P} = r + 2P\delta a \implies Q = r + 2P\delta a$$

This is equivalent to rotation in the phase space by amount $2\delta a$

Problem 4

We first note that,

$$y = x^2 \implies \dot{y} = 2x\dot{x}$$

and write down the Lagrangian and Hamiltonian of the system,

$$L = \frac{m\dot{x}^2}{2} + \frac{m\dot{y}^2}{2} - mgy$$
$$L = \frac{m\dot{x}^2}{2} + 2mx^2\dot{x}^2 - mgx^2$$
$$\implies p = m\dot{x} + 4mx^2\dot{x} \implies \dot{x} = \frac{p}{m(1+4x^2)}$$

Thus, we can write the Hamiltonian as,

$$H(x,p) = \frac{p^2}{m(1+4x^2)} - \frac{m}{2}(1+4x^2)\frac{p^2}{m^2(1+4x^2)^2} + mgx^2$$
$$H(x,p) = \frac{p^2}{2m(1+4x^2)} + mgx^2$$

The Hamilton-Jacobi equation is given by,

$$\frac{1}{2m(1+4x^2)}\left(\frac{\partial S}{\partial x}\right)^2 + mgx^2 + \frac{\partial S}{\partial t} = 0$$

Substituting $S = W(x) - Et$, we get,

$$\frac{1}{2m(1+4x^2)}\left(\frac{\mathrm{d}W}{\mathrm{d}x}\right)^2 + mgx^2 - E = 0 \implies \frac{\mathrm{d}W}{\mathrm{d}x} = \sqrt{2m(E - mgx^2)(1+4x^2)}$$
$$\implies S = \int dx \sqrt{2m(E - mgx^2)(1+4x^2)} - Et$$

We know that $\frac{\partial S}{\partial E} = \alpha t + \beta$ for constants α and β. Hence the equation of motion is,

$$\sqrt{\frac{m(1+4x^2)}{2(E - mgx^2)}} - E = \alpha t + \beta$$

Part (b)
We first note that,

$$z = \frac{\xi^2 - \eta^2}{2} \quad , \quad \rho = \eta\xi \quad , \quad \psi = \phi \implies \dot{z} = \xi\dot{\xi} - \eta\dot{\eta} \quad , \quad \dot{\rho} = \eta\dot{\xi} + \xi\dot{\eta} \quad , \quad \dot{\phi} = \dot{\psi}$$

We first write down the Lagrangian and canonical momenta,

$$
\begin{aligned}
L &= \frac{m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2)}{2} - \frac{k}{\sqrt{\rho^2 + z^2}} + Fz \\
&= \frac{m(\eta^2\dot{\xi}^2 + \xi^2\dot{\eta}^2 + 2\eta\xi\dot{\eta}\dot{\xi} + \eta^2\xi^2\dot{\psi}^2 + \xi^2\dot{\xi}^2 - 2\xi\dot{\xi}\eta\dot{\eta} + \eta^2\dot{\eta}^2)}{2} - \frac{k}{\sqrt{\left(\frac{\xi^2 - \eta^2}{2}\right)^2 + \eta^2\xi^2}} + F\frac{\xi^2 - \eta^2}{2} \\
L &= m\frac{(\eta^2 + \xi^2)(\dot{\xi}^2 + \dot{\eta}^2) + \eta^2\xi^2\dot{\psi}^2}{2} - \frac{2k}{\eta^2 + \xi^2} + F\frac{\xi^2 - \eta^2}{2} \\
\implies p_\xi &= m(\eta^2 + \xi^2)\dot{\xi} \quad , \quad p_\eta = m(\eta^2 + \xi^2)\dot{\eta} \quad , \quad p_\psi = m\eta^2\xi^2\dot{\psi} \\
\implies H &= \frac{p_\xi^2 + p_\eta^2}{2m(\eta^2 + \xi^2)} + \frac{p_\psi^2}{2m\eta^2\xi^2} + \frac{2k}{\eta^2 + \xi^2} - F\frac{\xi^2 - \eta^2}{2}
\end{aligned}
$$

Let's apply the transformations given in the problem We can now write down the Hamilton-Jacobi equation as,

$$
\frac{\partial S}{\partial t} + \frac{1}{2m(\eta^2 + \xi^2)}\left[\left(\frac{\partial S}{\partial \xi}\right)^2 + \left(\frac{\partial S}{\partial \eta}\right)^2\right] + \frac{1}{2m\eta^2\xi^2}\left(\frac{\partial S}{\partial \psi}\right)^2 + \frac{2k}{\eta^2 + \xi^2} - F\frac{\xi^2 - \eta^2}{2} = 0
$$

Multiply the equation by $2m(\eta^2 + \xi^2)$,

$$
2m(\eta^2 + \xi^2)\frac{\partial S}{\partial t} + \left(\frac{\partial S}{\partial \xi}\right)^2 + \left(\frac{\partial S}{\partial \eta}\right)^2 + \left(\frac{1}{\eta^2} + \frac{1}{\xi^2}\right)\left(\frac{\partial S}{\partial \psi}\right)^2 + 4mk - Fm(\xi^4 - \eta^4) = 0
$$