Advanced Stat Mech - Assignment 2

Due date: February 4, 2019

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Note: Please submit the assignment to any one of TA's office on/before the due date. For the numerical parts, use your preferred programming language, plotting software, etc. and attach the print outs at the correct places of your solution. If you have any doubt regarding the assignment problems or the topics discussed in the class, feel free to discuss with the TA's or the instructor. Good luck!

Q1 20 marks

Thermodynamic limit

Consider a uniform sphere (radius R) of particles in d-dimension having an interaction potential between two particles given by $U(r) = A/r^\sigma$. You may also consider a hard core repulsion between them such that the above potential applies only for $r > a$. Find the energy of the system per unit volume and establish a relationship involving d and σ for which the thermodynamic limit exists. Explicitly discuss the case $d = 3$. What happens for screened Coulomb potential and logarithmic potential?

Q2 25 marks

Debye screening

In the last assignment we discussed about the Vlasov equation. Consider a neutral mixture of N ions of charge $+e$, mass m_+, and N electrons of charge $-e$ and mass m_-, in a volume $V = N/n_0$.

(a) Write down the Vlasov equations for this two-component system involving the one particle densities $f_\pm$, effective potential Φ_{eff}, etc. Write down an expression for Φ_{eff} in terms of Coulomb potential, $f_\pm$, etc. 5

(b) Assume that the one-particle densities have the stationary forms $f_\pm = g_\pm(\vec{p})n_\pm(\vec{q})$. Show that the effective potential satisfies the equation, 10

$$\nabla^2 \Phi_{eff} = 4\pi\rho_{ext} + 4\pi e(n_+(\vec{q}) - n_-(\vec{q})), \tag{1}$$

where ρ_{ext} is the external charge density. Further assuming that the densities relax to the equilibrium Boltzmann weights $n_\pm = n_0 \exp[\pm\beta e \Phi_{eff}(\vec{q})]$ rewrite the previous equation. Linearize the exponential to obtain the Debye equation,

$$\nabla^2 \Phi_{eff} = 4\pi\rho_{ext} + \Phi_{eff}/\lambda^2. \tag{2}$$

Give the expression for the Debye screening length λ.

(c) Show that the Debye equation has the general solution $\Phi_{eff}(\vec{q}) = \int d^3\vec{q}' G(\vec{q}-\vec{q}')\rho_{ext}(\vec{q}')$, where $G(\vec{q}) = \exp(-|\vec{q}|/\lambda)/|\vec{q}|$ is the screened Coulomb potential. 10

Q3 $\boxed{\text{20 marks}}$

Orthogonal polynomials and the Vandermonde determinant

(a) Show that the determinant made of any set of orthogonal polynomials (successive order) is proportional to the Vandermonde determinant. $\boxed{10}$

(b) Show that Vandermonde determinant is $\prod_{i<j}(x_i - x_j)$. $\boxed{10}$

Q4 $\boxed{\text{25 marks}}$

Fermions trapped in 1d harmonic potential at T = 0

Redo the problem discussed in class and fill up all the steps in details. In particular, find the probability density $P(\{x_i\})$, average density $\rho(x)$, $P[\rho(x) = \rho_0(x)]$ and its saddle point approximation at large N limit, etc.

Q5 $\boxed{\text{20 marks}}$

One dimensional Coulomb gas

In class we obtained the density for N fermions trapped in 1-d harmonic trap at T=0 for large N. The problem was reduced to that of a classical gas with interaction consisting of two parts: (i) attractive harmonic potential and (ii) a logarithmic repulsion. Now consider another situation of a classical gas where the repulsion is Coulombic instead of logarithmic. This means that the probability distribution function(pdf) for the particle to be in x_i and $x_i + dx_i$ for $i = 1, 2, ..., N$ is given by $P[\{x_i\}] = \frac{1}{Z_N} e^{-E(\{x_i\})}$ where,

$$E(\{x_i\}) = \frac{N^2}{2} \sum_{i=1}^{N} x_i^2 - N \sum_{i \neq j} |x_i - x_j| \tag{3}$$

$$Z_N = \int_{-\infty}^{\infty} dx_1 ... \int_{-\infty}^{\infty} dx_N P[\{x_i\}] \tag{4}$$

(a) Introduce a macroscopic density $\rho(x) = \frac{1}{N} \sum_{i=1}^{N} \delta(x - x_i)$ and write the probability in terms of ρ by keeping the leading order. You should get something like $\boxed{5}$

$$P(\rho[x]) = \frac{1}{Z_N} exp\left(-N^{\alpha} E[\rho(x)]\right) \delta\left(\int \rho(x)dx - 1\right).$$

Find the value of α and the expression for $E[\rho(x)]$.

(b) Now using the integral representation of δ-function in μ $\left(\delta(z) \sim \int d\mu e^{-N^{\alpha}\mu z}\right)$ and saddle point approximation, obtain the condition for the density ρ^* that corresponds to the maximum probability. Now use this condition and the normalization of density to find ρ^* and the region $[-B, B]$ over which ρ^* is non zero. $\boxed{10}$

(c) Let us now consider that we put a hard wall at $x = w$ where $-B \leq w \leq B$ which now means that $x_1 \leq w, ..., x_N \leq w$. One can argue that the introduction of wall at w changes the density profile only at $x = w$ and the density at other positions remains same as the one without wall obtained in (b). This means $\rho_w^*(x) = \rho^*(x) + C\delta(x - w)$ where C is function of w and $\rho^*(x)$ is obtained in (b). Given this, find C. Plot $\rho_w^*(x)$. What will be $\rho^*(x)$ if w is outside [-B, B] region? $\boxed{5}$