

# Advanced Quantum Mechanics: Assignment #1

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## Problem 1

### Solution

We solve each part separately.

#### Part 1 - Commutators

We expand out each term as follows,

$$[A, [B, C]] = ABC - ACB - BCA + CBA$$

$$[C, [A, B]] = CAB - CBA - ABC + BAC$$

$$[B, [C, A]] = BCA - BAC - CAB + ACB$$

Adding the three expressions above, we arrive at the expression

$$[A, [B, C]] + [C, [A, B]] + [B, [C, A]] = 0.$$

Hence Proved.

#### Part 2 - Poisson Brackets

$\{.,.\}$  denotes Poisson Bracket,  $X_y = \frac{\partial X}{\partial y}$  and  $X_{yz} = \frac{\partial^2 X}{\partial y \partial z}$ . We expand each term as follows,'

$$\{A, \{B, C\}\} = \{A, B_q C_p - B_p C_q\}$$

$$= A_q B_{pq} C_p + A_q B_q C_{pp} + A_p B_{pq} C_p + A_p B_q C_{pq} - A_q B_{pp} C_q - A_q B_p C_{pq} - A_p B_{pq} C_q - A_p B_p C_{qq}$$

$$\{C, \{A, B\}\} = C_q A_{pq} B_p + C_q A_q B_{pp} + C_p A_{pq} B_p + C_p A_q B_{pq} - C_q A_{pp} B_q - C_q A_p B_{pq} - C_p A_{pq} B_q - C_p A_p B_{qq}$$

$$\{B, \{C, A\}\} = B_q C_{pq} A_p + B_q C_q A_{pp} + B_p C_{pq} A_p + B_p C_q A_{pq} - B_q C_{pp} A_q - B_q C_p A_{pq} - B_p C_{pq} A_q - B_p C_p A_{qq}$$

Adding the three expressions above, we arrive at the expression

$$\{A, \{B, C\}\} + \{C, \{A, B\}\} + \{B, \{C, A\}\} = 0$$

Hence Proved.

## Problem 2

### Solution

$$\begin{aligned}
[AB, CD] &= A[B, CD] + [A, CD]B \\
&= A[B, C]D + AC[B, D] + C[A, D]B + [A, C]DB \\
&= A(\{B, C\} - 2CB)D + AC(2BD - \{B, D\}) + C(2AD - \{A, D\}) + (\{A, C\} - 2CA)DB \\
&= A\{B, C\}D - 2ACBD + 2ACBD - AC\{B, D\} + 2CADB - C\{A, D\}B + \{A, C\}DB - 2CADB \\
&= -AC\{B, D\} + A\{B, C\}D - C\{A, D\}B + \{A, C\}DB \\
&= -AC\{D, B\} + A\{C, B\}D - C\{D, A\}B + \{C, A\}DB
\end{aligned}$$

Hence Proved.

## Problem 3

### Solution

$$\vec{\sigma} \cdot \vec{n} = \begin{pmatrix} n_z & n_x - in_y \\ n_x + in_y & -n_z \end{pmatrix}$$

The corresponding equation for eigenvalues of this matrix is,

$$\lambda^2 - n_x^2 - n_y^2 - n_z^2 = 0$$

which gives as eigenvalues  $\lambda = \pm \sqrt{n_x^2 + n_y^2 + n_z^2}$ . Substituting these values in the eigenvalue equation  $(\vec{\sigma} \cdot \vec{n})X = \lambda X$ , we get the following eigenvectors

$$\begin{pmatrix} \frac{-\sqrt{n_x^2 + n_y^2 + n_z^2} + n_z}{n_x + in_y} \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} \frac{\sqrt{n_x^2 + n_y^2 + n_z^2} + n_z}{n_x + in_y} \\ 1 \end{pmatrix}$$

## Problem 4

### Solution

Let  $|\beta\rangle$  be an arbitrary state, and  $|\lambda_i\rangle$  be the eigenstates such that  $\sum_k |\lambda_k\rangle \langle \lambda_k| = 1$ . Consider the following,

$$\begin{aligned}
\prod_{i \neq j} \left( \frac{A - \lambda_i}{\lambda_j - \lambda_i} \right) |\beta\rangle &= \prod_{i \neq j} \left( \frac{A - \lambda_i}{\lambda_j - \lambda_i} \right) \sum_k |\lambda_k\rangle \langle \lambda_k | \beta \rangle \\
&= \sum_k \prod_{i \neq j} \left( \frac{\lambda_k - \lambda_i}{\lambda_j - \lambda_i} \right) |\lambda_k\rangle \langle \lambda_k | \beta \rangle
\end{aligned}$$

Lets look closer at the sum above. For  $k \neq j$ , the coefficient  $\frac{\lambda_k - \lambda_i}{\lambda_j - \lambda_i}$  in the sum will vanish for some  $i$ , rendering the whole product to be zero. So all that remains in the summation is the term corresponding to  $k = j$ . Hence,

$$\begin{aligned}
\prod_{i \neq j} \left( \frac{A - \lambda_i}{\lambda_j - \lambda_i} \right) |\beta\rangle &= \prod_{i \neq j} \left( \frac{\lambda_j - \lambda_i}{\lambda_j - \lambda_i} \right) |\lambda_j\rangle \langle \lambda_j | \beta \rangle \\
&= \prod_{i \neq j} (1) |\lambda_j\rangle \langle \lambda_j | \beta \rangle \\
&= |\lambda_j\rangle \langle \lambda_j | \beta \rangle \\
&= P_j |\beta\rangle
\end{aligned}$$

Hence,  $\prod_{i \neq j} \left( \frac{A - \lambda_i}{\lambda_j - \lambda_i} \right) = P_j$ , the Projection operator onto  $|\lambda_j\rangle$ .

## Problem 5

$F(\hat{x})$  and  $G(\hat{p})$  have regular series expansions. So, for some constants  $\alpha_i$  and  $\beta_i$ ,

$$F(\hat{x}) = \alpha_0 + \alpha_1 \hat{x} + \alpha_2 \hat{x}^2 + \dots$$

$$G(\hat{p}) = \beta_0 + \beta_1 \hat{p} + \beta_2 \hat{p}^2 + \dots$$

Consider  $[\hat{p}, \hat{x}^n]$ ,

$$\begin{aligned} [\hat{p}, \hat{x}^n] &= [\hat{p}, \hat{x}] \hat{x}^{n-1} + \hat{x} [\hat{p}, \hat{x}] \hat{x}^{n-2} + \hat{x}^2 [\hat{p}, \hat{x}] \hat{x}^{n-3} + \dots n \text{ terms} \\ &= -in \hat{x}^{n-1} \end{aligned}$$

Consider  $[\hat{p}, F(\hat{x})]$ .

$$\begin{aligned} [\hat{p}, F(\hat{x})] &= [\hat{p}, \alpha_0 + \alpha_1 \hat{x} + \alpha_2 \hat{x}^2 + \dots] \\ &= [\hat{p}, \alpha_0] + \alpha_1 [\hat{p}, \hat{x}] + \alpha_2 [\hat{p}, \hat{x}^2] + \dots \\ &= \sum_{j=0}^{\infty} \alpha_j [\hat{p}, \hat{x}^j] \\ &= -i \sum_{j=1}^{\infty} \alpha_j (j \hat{x}^{j-1}) \\ &= -i F'(\hat{x}) \end{aligned}$$

Similarly,  $[\hat{x}, \hat{p}^n] = in \hat{p}^{n-1}$ , and,

$$\begin{aligned} [\hat{x}, G(\hat{p})] &= \sum_{j=0}^{\infty} \beta_j [\hat{x}, \hat{p}^j] \\ &= i \sum_{j=1}^{\infty} \beta_j (j \hat{p}^{j-1}) \\ &= i G'(\hat{p}) \end{aligned}$$

**Hence Proved**

$$\begin{aligned} [\hat{x}^2, \hat{p}^2] &= \hat{x} [\hat{x}, \hat{p}^2] + [\hat{x}, \hat{p}^2] \hat{x} \\ &= 2i \{x, p\} \end{aligned}$$

## Problem 6

**Solution**

**Part (a)**

The normalized coherent states are given by,

$$\begin{aligned} |z\rangle &= \frac{1}{\sqrt[4]{\pi} \exp\left(\frac{\alpha^2}{2}\right)} \exp\left(-\frac{x^2}{2} + \alpha x + i\beta x\right) \\ &= \frac{1}{\sqrt[4]{\pi}} \exp\left(-\frac{(x - \alpha)^2}{2} + i\beta x\right) \end{aligned}$$

where  $z = \alpha + i\beta$ ,  $\hat{a}|z\rangle = z|z\rangle$ . Hence,

$$\begin{aligned}\langle z'|z\rangle &= \int_{-\infty}^{\infty} \frac{1}{\sqrt[4]{\pi} \exp\left(\frac{\alpha^2}{2}\right)} \frac{1}{\sqrt[4]{\pi} \exp\left(\frac{\alpha'^2}{2}\right)} \exp\left(-x^2 + (\alpha + \alpha')x + i(\beta - \beta')x\right) \\ &= \exp\left(-\frac{\alpha^2}{4} - \frac{1}{2}i\alpha'\beta' + \frac{1}{2}i\beta\alpha' + \frac{\alpha\alpha'}{2} - \frac{(\alpha')^2}{4} - \frac{1}{2}i\alpha\beta' + \frac{i\alpha\beta}{2} - \frac{\beta^2}{4} + \frac{\beta\beta'}{2} - \frac{(\beta')^2}{4}\right) \\ &= \exp\left(-\frac{(\alpha - \alpha')^2}{4} - \frac{(\beta - \beta')^2}{4} + \frac{\alpha\alpha'(\beta - \beta')}{2}i\right)\end{aligned}$$

Here,  $z = \alpha + i\beta$  and  $z' = \alpha' + i\beta'$

### Part (b)

Consider  $\langle x'|x\rangle = \delta(x - x')$ . The completeness relation is of the form  $\int d^2z f(z) |z\rangle\langle z| = 1$ . Using this identity, we insert the complete states in  $\langle x|x\rangle$  as follows,

$$\begin{aligned}&\int d^2z f(z) \langle x'|z\rangle \langle z|x\rangle = \delta(x - x') \\ &\int d\alpha d\beta (-2i)f(\alpha, \beta) \frac{1}{\sqrt[4]{\pi}} \exp\left(-\frac{(x' - \alpha)^2}{2} + i\beta x'\right) \frac{1}{\sqrt[4]{\pi}} \exp\left(-\frac{(x - \alpha)^2}{2} - i\beta x\right) = \int d\beta \exp(i\beta(x - x')) \\ &\int d\beta \left\{ d\alpha (-2i)f(\alpha, \beta) \frac{1}{\sqrt[4]{\pi}} \exp\left(-\frac{(x' - \alpha)^2}{2} + i\beta x'\right) \frac{1}{\sqrt[4]{\pi}} \exp\left(-\frac{(x - \alpha)^2}{2} - i\beta x\right) - \exp(i\beta(x - x')) \right\} = 0\end{aligned}$$

This relation should hold for all  $x'$ , specifically for  $x = x'$ . Hence,

$$\int d\beta \left\{ \int d\alpha (-2i)f(\alpha, \beta) \frac{1}{\sqrt[4]{\pi}} \exp\{-(x - \alpha)^2\} - 1 \right\} = 0$$

For this to hold for all  $x$ , the term in curly brackets should be zero.

$$\int d\alpha (-2i)f(\alpha, \beta) \frac{1}{\sqrt[4]{\pi}} \exp\{-(x - \alpha)^2\} = 1 \quad (1)$$

At this point, we note that,

$$\int d\alpha \frac{1}{\sqrt{\pi}} \exp\{-(x - \alpha)^2\} = 1$$

By comparing preceding two equations, we can claim that  $(-2i)f(\alpha, \beta) = 1$  ie.  $f(\alpha, \beta) = \frac{i}{2}$  is **one** possibility and the corresponding completeness relation is

$$\int d^2z |z\rangle\langle z| = 1$$

Note that this is **a** completeness relation and not **the** completeness relation. In principle, any  $f(\alpha, \beta)$  that satisfies (1) can be included in the completeness relation.

## Problem 7

### Solution

**Part (a)**

We know that  $\langle z|\hat{a}|z\rangle = z$  and  $\langle z|\hat{a}^\dagger|z\rangle = z^*$ . Adding these two up, we get,

$$\sqrt{2}\langle z|\hat{x}|z\rangle = z + z^* = 2\operatorname{Re}\{z\}$$

$$\langle z|\hat{x}|z\rangle = \sqrt{2}\operatorname{Re}\{z\}$$

Similarly, subtracting the two, we get,

$$\sqrt{2}i\langle z|\hat{p}|z\rangle = z - z^* = 2i\operatorname{Im}\{z\}$$

$$\langle z|\hat{p}|z\rangle = \sqrt{2}\operatorname{Im}\{z\}$$

**Part (b)**

$$\hat{a}^2 = \frac{1}{2}(\hat{x}^2 - \hat{p}^2 + i\{x, p\}) \quad (2)$$

$$\hat{a}^{\dagger 2} = \frac{1}{2}(\hat{x}^2 - \hat{p}^2 - i\{x, p\}) \quad (3)$$

$$\hat{a}\hat{a}^\dagger = \frac{1}{2}(\hat{x}^2 + \hat{p}^2 - i[x, p]) \quad (4)$$

$$\hat{a}^\dagger\hat{a} = \frac{1}{2}(\hat{x}^2 + \hat{p}^2 + i[x, p]) \quad (5)$$

Adding the above equations, we see,

$$\begin{aligned} 2\langle z|\hat{x}^2|z\rangle &= \langle z|\hat{a}^2 + \hat{a}^{\dagger 2} + \{a, a^\dagger\}|z\rangle \\ &= \langle z|\hat{a}^2|z\rangle + \langle z|\hat{a}^{\dagger 2}|z\rangle + \langle z|[a, a^\dagger]|z\rangle + 2\langle z|a^\dagger a|z\rangle \\ &= z^2 + z^{*2} + 1 + 2zz^* \\ &= 1 + 4\operatorname{Re}(z)^2 \\ \langle z|\hat{x}^2|z\rangle &= \frac{1}{2} + 2\operatorname{Re}(z)^2 \end{aligned}$$

(1) + (2) - (3) - (4) gives,

$$\begin{aligned} -2\langle z|\hat{p}^2|z\rangle &= \langle z|\hat{a}^2 + \hat{a}^{\dagger 2} - \{a, a^\dagger\}|z\rangle \\ &= -4\operatorname{Im}\{z\}^2 - 1 \\ \langle z|\hat{p}^2|z\rangle &= \frac{1}{2} + 2\operatorname{Im}\{z\}^2 \end{aligned}$$

Substituting all required values above, we get,

$$\Delta x = \sqrt{\frac{1}{2}}; \Delta p = \sqrt{\frac{1}{2}}; \Delta x \Delta p = \frac{1}{2}$$

As is evident, this saturates the uncertainty relation  $\Delta x \Delta p \geq \frac{1}{2}$ .

**Problem 8****Solution****Part (a)**

$$|\psi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$$

The density matrix of system 1 is obtained by tracing over the degrees of freedom of system 2.  $\rho_A = \text{Tr}_B(|\psi\rangle\langle\psi|)$

$$\begin{aligned}
\rho_A &= \langle 0_B|\psi\rangle \langle\psi|0_B\rangle + \langle 1_B|\psi\rangle \langle\psi|1_B\rangle \\
&= (a_{00}|0\rangle + a_{10}|1\rangle)(a_{00}^*\langle 0| + a_{10}^*\langle 1|) + (a_{11}|1\rangle + a_{01}|0\rangle)(a_{11}^*\langle 1| + a_{01}^*\langle 0|) \\
&= (|a_{00}|^2 + |a_{01}|^2)|0\rangle\langle 0| + (a_{11}^*a_{01} + a_{10}^*a_{00})|0\rangle\langle 1| + (a_{11}a_{01}^* + a_{10}a_{00}^*)|1\rangle\langle 0| + (|a_{11}|^2 + |a_{10}|^2)|1\rangle\langle 1| \\
&= \begin{pmatrix} |a_{00}|^2 + |a_{01}|^2 & a_{11}^*a_{01} + a_{10}^*a_{00} \\ a_{11}a_{01}^* + a_{10}a_{00}^* & |a_{11}|^2 + |a_{10}|^2 \end{pmatrix}
\end{aligned}$$

Similarly, one can find  $\rho_B$

$$\begin{aligned}
\rho_B &= \langle 0_A|\psi\rangle \langle\psi|0_A\rangle + \langle 1_A|\psi\rangle \langle\psi|1_A\rangle \\
&= (a_{00}|0\rangle + a_{01}|1\rangle)(a_{00}^*\langle 0| + a_{01}^*\langle 1|) + (a_{11}|1\rangle + a_{10}|0\rangle)(a_{11}^*\langle 1| + a_{10}^*\langle 0|) \\
&= (|a_{00}|^2 + |a_{10}|^2)|0\rangle\langle 0| + (a_{11}^*a_{10} + a_{01}^*a_{00})|0\rangle\langle 1| + (a_{11}a_{10}^* + a_{01}a_{00}^*)|1\rangle\langle 0| + (|a_{11}|^2 + |a_{01}|^2)|1\rangle\langle 1| \\
&= \begin{pmatrix} |a_{00}|^2 + |a_{10}|^2 & a_{11}^*a_{10} + a_{01}^*a_{00} \\ a_{11}a_{10}^* + a_{01}a_{00}^* & |a_{11}|^2 + |a_{01}|^2 \end{pmatrix}
\end{aligned}$$

### Part (b)

We can clearly see that  $\rho_B$  can be obtained from  $\rho_A$  by merely switching  $a_{10}$  and  $a_{01}$ . We calculate the eigenvalues in Mathematica. For  $\rho_A$ , the eigenvalues  $\lambda_1$  and  $\lambda_2$  are given by,

$$\begin{aligned}
2\lambda_1 &= |a_{00}|^2 + |a_{11}|^2 + |a_{01}|^2 + |a_{10}|^2 - [|a_{00}|^4 + |a_{01}|^4 + (|a_{10}|^2 + |a_{11}|^2)^2 \\
&\quad + 2a_{01}^*(-a_{01}a_{10} + 2a_{00}a_{11})a_{10}^* + a_{01}|a_{11}|^2) + 2a_{00}^*(a_{00}|a_{01}|^2 + a_{00}|a_{10}|^2 + 2(a_{01}a_{10} - a_{00}a_{11}a_{11}^*))]^{1/2}
\end{aligned}$$

$$\begin{aligned}
2\lambda_2 &= |a_{00}|^2 + |a_{11}|^2 + |a_{01}|^2 + |a_{10}|^2 + [|a_{00}|^4 + |a_{01}|^4 + |a_{10}|^4 + |a_{11}|^4 \\
&\quad + 2\{|a_{10}a_{11}|^2 + a_{01}^*(a_{10}^*(-a_{10}a_{01} + 2a_{00}a_{11}) + a_{01}|a_{11}|^2) + a_{00}^*(a_{00}|a_{01}|^2 + a_{00}|a_{10}|^2) + a_{11}^*(2a_{01}a_{10} - a_{00}a_{11})\}]^{1/2}
\end{aligned}$$

As earlier mentioned, the eigenvalues of  $\rho_B$  can be obtained by interchanging  $a_{10}$  and  $a_{01}$ .