Classical Mechanics: Assignment #6

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Problem 1

Liouville Theorem states that in a Hamiltonian system, the phase space density is constant in time.
Let our system consist of N points (q_k, p_k) in a $2N$ dimensional phase space.

Problem 2

Transformations of coordinates $(q, p, t) \to (Q, P, t)$ which preserves the form of Hamilton's equations are called canonical transformations. So, by definition,

$$\dot{p} = \frac{\partial H}{\partial q} \quad , \quad \dot{q} = -\frac{\partial H}{\partial p} \quad \text{and} \quad \dot{P} = \frac{\partial K}{\partial Q} \quad , \quad \dot{Q} = -\frac{\partial K}{\partial P}$$

The definition also implies that,

$$\delta(p\dot{q} - H) = 0 \quad \text{and} \quad \delta(P\dot{Q} - K) = 0$$
$$\lambda(p\dot{q} - H) = P\dot{Q} - K + \frac{\mathrm{d}F}{\mathrm{d}t}$$

We deal with the $\lambda = 1$ case. The $\frac{\mathrm{d}F}{\mathrm{d}t}$ term comes from the fact that Lagrangians are not unique and we can always add a total time derivative term without changing the equations of motion. If the above condition is satisfied, the transformation $(q, p, t) \to (Q, P, t)$ is guaranteed to be canonical, and the function F is called a generating function. We deal with four classes of generating functions case-by-case,

- $F = F_1(q, Q, t)$,

$$p\dot{q} - H = P\dot{Q} - K + \frac{\mathrm{d}F_1}{\mathrm{d}t} = P\dot{Q} - K + \frac{\partial F_1}{\partial q}\dot{q} + \frac{\partial F_1}{\partial Q}\dot{Q} + \frac{\partial F_1}{\partial t}$$

  As q and Q are independent, the coefficients should vanishh independently, such that $K = H + \frac{\partial F_1}{\partial t}$. This implies,

$$\frac{\partial F_1}{\partial q} = p \quad \text{and} \quad \frac{\partial F_1}{\partial Q} = -P$$

- $F = F_2(q, P, t) - QP$,

$$p\dot{q} - H = P\dot{Q} - K + \frac{\mathrm{d}F_2}{\mathrm{d}t} - \frac{\mathrm{d}(QP)}{\mathrm{d}t} = P\dot{Q} - K + \frac{\partial F_2}{\partial q}\dot{q} + \frac{\partial F_2}{\partial P}\dot{P} + \frac{\partial F_2}{\partial t} - P\dot{Q} - Q\dot{P}$$

$$\implies \frac{\partial F_2}{\partial q} = p \quad \text{and} \quad \frac{\partial F_2}{\partial P} = Q$$

- $F = F_3(p, Q, t) + qp$,

$$p\dot{q} - H = P\dot{Q} - K + \frac{\mathrm{d}F_3}{\mathrm{d}t} + \frac{\mathrm{d}(qp)}{\mathrm{d}t} = P\dot{Q} - K + \frac{\partial F_3}{\partial Q}\dot{Q} + \frac{\partial F_3}{\partial p}\dot{p} + \frac{\partial F_3}{\partial t} + p\dot{q} + q\dot{p}$$

$$\implies \frac{\partial F_3}{\partial Q} = -P \quad \text{and} \quad \frac{\partial F_2}{\partial p} = -q$$

- $F = F_4(p, P, t) + qp - QP$,

$$p\dot{q} - H = P\dot{Q} - K + \frac{\mathrm{d}F_4}{\mathrm{d}t} + \frac{\mathrm{d}(qp - QP)}{\mathrm{d}t} = P\dot{Q} - K + \frac{\partial F_4}{\partial P}\dot{P} + \frac{\partial F_4}{\partial p}\dot{p} + \frac{\partial F_4}{\partial t} + p\dot{q} + q\dot{p} - P\dot{Q} - Q\dot{P}$$

$$\implies \frac{\partial F_4}{\partial P} = Q \quad \text{and} \quad \frac{\partial F_4}{\partial p} = -q$$

Problem 3

We are given the Hamiltonian and generating function,

$$H = \frac{p^2}{2} + \frac{\omega^2 x^2}{2} + \alpha x^3 + \beta x p^2 \quad \text{and} \quad \phi = xP + ax^2 P + bP^3$$

$\phi = \phi(x, P)$. For ϕ to be a canonical transformation,

$$\begin{aligned}
\frac{\partial \phi}{\partial x} = p &\quad \text{and} \quad \frac{\partial \phi}{\partial P} = Q \\
\implies P + 2axP = p &\quad \text{and} \quad x + ax^2 + 3bP^2 = Q \\
\implies P + 2axP = p &\quad \text{and} \quad \frac{-1 \pm \sqrt{1 + 4a(Q - 3bP^2)}}{2a} = x
\end{aligned}$$

We know that,

$$p\dot{x} - H = P\dot{Q} - K + \frac{\mathrm{d}\phi}{\mathrm{d}t}$$
$$p\dot{x} - \frac{p^2}{2} + \frac{\omega^2 x^2}{2} + \alpha x^3 + \beta x p^2 = P\dot{Q} - K + (P + 2axP)\dot{x} + (x + ax^2 + 3bP^2)\dot{P}$$

Problem 4

We first note that,

$$y = x^2 \implies \dot{y} = 2x\dot{x}$$

and write down the Lagrangian and Hamiltonian of the system,

$$\begin{aligned}
L &= \frac{m\dot{x}^2}{2} + \frac{m\dot{y}^2}{2} - mgy \\
L &= \frac{m\dot{x}^2}{2} + 2mx^2\dot{x}^2 - mgx^2 \\
\implies p = m\dot{x} + 4mx^2\dot{x} &\implies \dot{x} = \frac{p}{m(1 + 4x^2)}
\end{aligned}$$

Problem 4 continued on next page. . .

Thus, we can write the Hamiltonian as,

$$H(x,p) = \frac{p^2}{m(1+4x^2)} - \frac{m}{2}(1+4x^2)\frac{p^2}{m^2(1+4x^2)^2} + mgx^2$$
$$H(x,p) = \frac{p^2}{2m(1+4x^2)} + mgx^2$$

The Hamilton-Jacobi equation is given by,

$$\frac{1}{2m(1+4x^2)}\left(\frac{\partial S}{\partial x}\right)^2 + mgx^2 + \frac{\partial S}{\partial t} = 0$$

Substituting $S = W(x) - Et$, we get,

$$\frac{1}{2m(1+4x^2)}\left(\frac{\mathrm{d}W}{\mathrm{d}x}\right)^2 + mgx^2 - E = 0 \implies \frac{\mathrm{d}W}{\mathrm{d}x} = \sqrt{2m(E-mgx^2)(1+4x^2)}$$
$$\implies S = \int dx\sqrt{2m(E-mgx^2)(1+4x^2)} - Et$$

We know that $\frac{\partial S}{\partial E} = \alpha t + \beta$ for constants α and β. Hence the equation of motion is,

$$\sqrt{\frac{m(1+4x^2)}{2(E-mgx^2)}} - E = \alpha t + \beta$$

Part (b)

We first note that,

$$z = \frac{\xi^2-\eta^2}{2} \quad , \quad \rho = \eta\xi \quad , \quad \psi = \phi \implies \dot{z} = \xi\dot{\xi} - \eta\dot{\eta} \quad , \quad \dot{\rho} = \eta\dot{\xi} + \xi\dot{\eta} \quad , \quad \dot{\phi} = \dot{\psi}$$

We first write down the Lagrangian and canonical momenta,

$$L = \frac{m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2)}{2} - \frac{k}{\sqrt{\rho^2+z^2}} + Fz$$
$$= \frac{m(\eta^2\dot{\xi}^2 + \xi^2\dot{\eta}^2 + 2\eta\xi\dot{\eta}\dot{\xi} + \eta^2\xi^2\dot{\psi}^2 + \xi^2\dot{\xi}^2 - 2\xi\dot{\xi}\eta\dot{\eta} + \eta^2\dot{\eta}^2)}{2} - \frac{k}{\sqrt{\left(\frac{\xi^2-\eta^2}{2}\right)^2 + \eta^2\xi^2}} + F\frac{\xi^2-\eta^2}{2}$$
$$L = m\frac{(\eta^2+\xi^2)(\dot{\xi}^2+\dot{\eta}^2) + \eta^2\xi^2\dot{\psi}^2}{2} - \frac{2k}{\eta^2+\xi^2} + F\frac{\xi^2-\eta^2}{2}$$
$$\implies p_\xi = m(\eta^2+\xi^2)\dot{\xi} \quad , \quad p_\eta = m(\eta^2+\xi^2)\dot{\eta} \quad , \quad p_\psi = m\eta^2\xi^2\dot{\psi}$$
$$\implies H = \frac{p_\xi^2 + p_\eta^2}{2m(\eta^2+\xi^2)} + \frac{p_\psi^2}{2m\eta^2\xi^2} + \frac{2k}{\eta^2+\xi^2} - F\frac{\xi^2-\eta^2}{2}$$

Let's apply the transformations given in the problem We can now write down the Hamilton-Jacobi equation as,

$$\frac{\partial S}{\partial t} + \frac{1}{2m(\eta^2+\xi^2)}\left[\left(\frac{\partial S}{\partial \xi}\right)^2 + \left(\frac{\partial S}{\partial \eta}\right)^2\right] + \frac{1}{2m\eta^2\xi^2}\left(\frac{\partial S}{\partial \psi}\right)^2 + \frac{2k}{\eta^2+\xi^2} - F\frac{\xi^2-\eta^2}{2} = 0$$

Multiply the equation by $2m(\eta^2+\xi^2)$,

$$2m(\eta^2+\xi^2)\frac{\partial S}{\partial t} + \left(\frac{\partial S}{\partial \xi}\right)^2 + \left(\frac{\partial S}{\partial \eta}\right)^2 + \left(\frac{1}{\eta^2} + \frac{1}{\xi^2}\right)\left(\frac{\partial S}{\partial \psi}\right)^2 + 4mk - Fm(\xi^4 - \eta^4) = 0$$