Advanced Quantum Mechanics: Assignment #2

Due on 4th October, 2018

Aditya Vijaykumar

Problem 1

Let's use the follwing convention ($|l,m\rangle$)

$$|2,2\rangle = \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} \quad \text{and} \quad |2,1\rangle = \begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix} \quad \text{and} \quad |2,0\rangle = \begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix} \quad \text{and} \quad |2,-1\rangle = \begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix} \quad \text{and} \quad |2,-2\rangle = \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}$$

We know that,

$$J_3 |l,m\rangle = m\,|l,m\rangle \quad \text{and} \quad J_\pm |l,m\rangle = \sqrt{(l \mp m)(l \pm m + 1)}\,|l, m \pm 1\rangle$$

$$J_3 |2,2\rangle = 2\,|2,2\rangle \ , \ J_3 |2,1\rangle = |2,1\rangle \ , \ J_3 |2,0\rangle = 0 \ , \ J_3 |2,-1\rangle = -\,|2,-1\rangle \ , \ J_3 |2,-2\rangle = -2\,|2,-2\rangle$$

Hence, we can see that,

$$J_3 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}$$

We also note that,

$$J_+ |2,2\rangle = 0 \ , \ J_+ |2,1\rangle = 2\,|2,2\rangle \ , \ J_+ |2,0\rangle = \sqrt{6}\,|2,1\rangle \ , \ J_+ |2,-1\rangle = \sqrt{6}\,|2,0\rangle \ , \ J_+ |2,-2\rangle = 2\,|2,-1\rangle$$

$$J_- |2,2\rangle = 2\,|2,1\rangle \ , \ J_- |2,1\rangle = \sqrt{6}\,|2,0\rangle \ , \ J_- |2,0\rangle = \sqrt{6}\,|2,-1\rangle \ , \ J_- |2,-1\rangle = 2\,|2,-2\rangle \ , \ J_- |2,-2\rangle = 0$$

So,

$$J_+ = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad J_- = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

Hence,

$$J_1 = \frac{J_+ + J_-}{2} = \frac{1}{2}\begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & \sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & \sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix} \quad \text{and} \quad J_2 = \frac{J_+ - J_-}{2i} = \frac{i}{2}\begin{pmatrix} 0 & -2 & 0 & 0 & 0 \\ 2 & 0 & -\sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & -\sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & -2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

Problem 2

Given,

$$U = \exp\left(-i\frac{\sigma_3\alpha}{2}\right)\exp\left(-i\frac{\sigma_2\beta}{2}\right)\exp\left(-i\frac{\sigma_3\gamma}{2}\right)$$

Consider the trace of U,

$$\begin{aligned}
\operatorname{tr} U &= \langle 0|U|0\rangle + \langle 1|U|1\rangle \\
&= \langle 0|\exp\left(-i\frac{\sigma_3\alpha}{2}\right)\exp\left(-i\frac{\sigma_2\beta}{2}\right)\exp\left(-i\frac{\sigma_3\gamma}{2}\right)|0\rangle + \langle 1|\exp\left(-i\frac{\sigma_3\alpha}{2}\right)\exp\left(-i\frac{\sigma_2\beta}{2}\right)\exp\left(-i\frac{\sigma_3\gamma}{2}\right)|1\rangle \\
\operatorname{tr} U &= e^{-i\left(\frac{\alpha+\gamma}{2}\right)}\langle 0|\exp\left(-i\frac{\sigma_2\beta}{2}\right)|0\rangle + e^{i\left(\frac{\alpha+\gamma}{2}\right)}\langle 1|\exp\left(-i\frac{\sigma_2\beta}{2}\right)|1\rangle
\end{aligned}$$

We note that,

$$\begin{aligned}
\exp\left(-i\frac{(\hat{\mathbf{n}}\cdot\vec{\sigma})\beta}{2}\right) &= \sum_{n=0}^{\infty}\frac{1}{2n!}\left(-i\frac{\beta}{2}\right)^{2n}(\hat{\mathbf{n}}\cdot\vec{\sigma})^{2n} + \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(-i\frac{\beta}{2}\right)^{2n+1}(\hat{\mathbf{n}}\cdot\vec{\sigma})^{2n+1} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{2n!}\left(\frac{\beta}{2}\right)^{2n} - \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{\beta}{2}\right)^{2n+1}(\hat{\mathbf{n}}\cdot\vec{\sigma}) \impliedby (\hat{\mathbf{n}}\cdot\vec{\sigma})^{2n} = 1 \\
\exp\left(-i\frac{(\hat{\mathbf{n}}\cdot\vec{\sigma})\beta}{2}\right) &= \cos\frac{\beta}{2} - i\sin\frac{\beta}{2}\hat{\mathbf{n}}\cdot\vec{\sigma} \\
\implies \langle 0|\exp\left(-i\frac{\sigma_2\beta}{2}\right)|0\rangle &= \langle 1|\exp\left(-i\frac{\sigma_2\beta}{2}\right)|1\rangle = \cos\frac{\beta}{2} \quad \text{and} \quad \operatorname{tr}\left[\exp\left(-i\frac{(\hat{\mathbf{n}}\cdot\vec{\sigma})\beta}{2}\right)\right] = 2\cos\frac{\beta}{2}
\end{aligned}$$

Hence, we get,

$$\begin{aligned}
\operatorname{tr} U &= \cos\frac{\beta}{2}\left(e^{i\left(\frac{\alpha+\gamma}{2}\right)} + e^{-i\left(\frac{\alpha+\gamma}{2}\right)}\right) \\
\cos\frac{\theta}{2} &= \cos\frac{\beta}{2}\cos\frac{\alpha+\gamma}{2}
\end{aligned}$$

θ is given by the above equation.

Problem 3

We know that,

$$J_1 = \frac{J_+ + J_-}{2} \quad \text{and} \quad J_2 = \frac{J_+ - J_-}{2i} \quad \text{and} \quad J_\pm|l,m\rangle = \sqrt{(l\mp m)(l\pm m+1)}\,|l,m\pm 1\rangle$$

As successive action of the form $J_\pm^\alpha|l,m\rangle$ with integral $\alpha > 0$ takes a state to one with higher/lower m , we can see that $\langle l,m|J_\pm^\alpha|l,m\rangle = 0$. Lets consider $\langle J_1\rangle$,

$$\begin{aligned}
\langle J_1\rangle &= \langle l,m|J_1|l,m\rangle \\
&= \frac{1}{2}(\langle l,m|J_+|l,m\rangle + \langle l,m|J_-|l,m\rangle) \\
&= 0
\end{aligned}$$

Similarly for $\langle J_2\rangle$,

$$\begin{aligned}
\langle J_2\rangle &= \langle l,m|J_2|l,m\rangle \\
&= \frac{1}{2i}(\langle l,m|J_+|l,m\rangle - \langle l,m|J_-|l,m\rangle) \\
&= 0
\end{aligned}$$

Problem 3 continued on next page. . .

Consider $\langle J_1^2 \rangle$ and $\langle J_2^2 \rangle$,

$$\langle J_1^2 \rangle = \frac{1}{4}(\langle J_+^2 \rangle + \langle J_-^2 \rangle + \{J_+, J_-\}) = \frac{1}{4}\{J_+, J_-\} \quad \text{and}$$
$$\langle J_2^2 \rangle = \frac{1}{-4}(\langle J_+^2 \rangle + \langle J_-^2 \rangle - \{J_+, J_-\}) = \frac{1}{4}\{J_+, J_-\} \implies \langle J_1^2 \rangle = \langle J_2^2 \rangle$$

We know,

$$\begin{aligned} \langle J^2 \rangle &= l(l+1) \\ \langle J_1^2 \rangle + \langle J_2^2 \rangle + \langle J_3^2 \rangle &= l(l+1) \\ 2\langle J_1^2 \rangle + m^2 &= l(l+1) \\ \langle J_1^2 \rangle = \langle J_2^2 \rangle &= \frac{l(l+1) - m^2}{2} \end{aligned}$$

Problem 4

Let's use the follwing convention ($|l, m\rangle$)

$$|1,1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad |1,0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |1,-1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

We know that $J_\pm |1, m\rangle = \sqrt{(1 \mp m)(1 \pm m + 1)}\, |1, m \pm 1\rangle$, which means,

$$J_+ |1,1\rangle = 0 \quad \text{and} \quad J_+ |1,0\rangle = \sqrt{2}\,|1,1\rangle \quad \text{and} \quad J_+ |1,-1\rangle = \sqrt{2}\,|1,0\rangle$$
$$J_- |1,1\rangle = \sqrt{2}\,|1,0\rangle \quad \text{and} \quad J_- |1,0\rangle = \sqrt{2}\,|1,-1\rangle \quad \text{and} \quad J_- |1,-1\rangle = 0$$

Using the above relations, one can write J_+ and J_- as follows,

$$J_+ = \sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad J_- = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \implies J_2 = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

Thus, we have obtained J_2. Let's also note the following,

$$J_2^2 = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad J_2^3 = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = J_2$$

$$J_2^4 = J_2^3 J_2 = J_2^2$$

We can see a pattern above, which can be written in a concise form as,

$$J_2^{2n-1} = J_2 \quad \text{and} \quad J_2^{2n} = J_2^2$$

Problem 4 continued on next page. . .

where $n = 1, 2, 3, \ldots$. Consider $e^{-iJ_2\beta}$,

$$\begin{aligned}
e^{-iJ_2\beta} &= \sum_{n=0}^{\infty} \frac{(-i)^n \beta^n J_2^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{(-i)^{2n} \beta^{2n} J_2^{2n}}{2n!} + \sum_{n=1}^{\infty} \frac{(-i)^{2n-1} \beta^{2n-1} J_2^{2n-1}}{(2n-1)!} \\
&= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n \beta^{2n} J_2^2}{2n!} + i \sum_{n=1}^{\infty} \frac{(-1)^n \beta^{2n-1} J_2}{(2n-1)!} \\
&= 1 + (1 - \cos\beta) J_2^2 - i J_2 \sin\beta
\end{aligned}$$

Hence proved.

Problem 5

Throughout the problem, we have assumed all relative phases to be 0,
We denote each representation by different subscripts, an example of which is shown below,

$$|2,2\rangle = |1,1\rangle_1 \otimes |1,1\rangle_2$$

To get other states in $j = 2$, we define the J_- operator as follows, and apply it successively,

$$\begin{aligned}
J_- &= J_-^{(1)} \otimes I + I \otimes J_-^{(2)} \\
\text{where} \quad J_- |j,m\rangle &= \sqrt{j(j+1) - m(m-1)}\, |j, m-1\rangle
\end{aligned}$$

Let's first consider $|2,1\rangle$,

$$\begin{aligned}
2\,|2,1\rangle &= J_- |2,2\rangle \\
&= J_-^{(1)} |1,1\rangle_1 \otimes |1,1\rangle_2 + |1,1\rangle_1 \otimes J_-^{(2)} |1,1\rangle_2 \\
&= \sqrt{2}\, |1,0\rangle_1 \otimes |1,1\rangle_2 + \sqrt{2}\, |1,1\rangle_1 \otimes |1,0\rangle_2 \\
|2,1\rangle &= \frac{|1,0\rangle_1 \otimes |1,1\rangle_2 + |1,1\rangle_1 \otimes |1,0\rangle_2}{\sqrt{2}}
\end{aligned}$$

For $|2,0\rangle$,

$$\begin{aligned}
\sqrt{6}\,|2,0\rangle &= J_- |2,1\rangle \\
2\sqrt{3}\,|2,0\rangle &= J_-^{(1)} |1,0\rangle_1 \otimes |1,1\rangle_2 + J_-^{(1)} |1,1\rangle_1 \otimes |1,0\rangle_2 + |1,0\rangle_1 \otimes J_-^{(2)} |1,1\rangle_2 + |1,1\rangle_1 \otimes J_-^{(2)} |1,0\rangle_2 \\
&= \sqrt{2}(|1,-1\rangle_1 \otimes |1,1\rangle_2 + |1,-1\rangle_1 \otimes |1,0\rangle_2 + |1,0\rangle_1 \otimes |1,0\rangle_2 + |1,1\rangle_1 \otimes |1,-1\rangle_2) \\
|2,0\rangle &= \frac{|1,-1\rangle_1 \otimes |1,1\rangle_2 + 2\,|1,0\rangle_1 \otimes |1,0\rangle_2 + |1,1\rangle_1 \otimes |1,-1\rangle_2}{\sqrt{6}}
\end{aligned}$$

For $|2,-1\rangle$,

$$\begin{aligned}
\sqrt{6}\,|2,-1\rangle &= J_- |2,0\rangle \\
6\,|2,-1\rangle &= J_-^{(1)} |1,-1\rangle_1 \otimes |1,1\rangle_2 + 2J_-^{(1)} |1,0\rangle_1 \otimes |1,0\rangle_2 + J_-^{(1)} |1,1\rangle_1 \otimes |1,-1\rangle_2 + \\
&\quad |1,-1\rangle_1 \otimes J_-^{(2)} |1,1\rangle_2 + 2\,|1,0\rangle_1 \otimes J_-^{(2)} |1,0\rangle_2 + |1,1\rangle_1 \otimes J_-^{(2)} |1,-1\rangle_2 \\
&= 3\sqrt{2}(|1,-1\rangle_1 \otimes |1,0\rangle_2 + |1,0\rangle_1 \otimes |1,-1\rangle_2) \\
|2,-1\rangle &= \frac{|1,-1\rangle_1 \otimes |1,0\rangle_2 + |1,0\rangle_1 \otimes |1,-1\rangle_2}{\sqrt{2}}
\end{aligned}$$

For $|2,-2\rangle$,

$$\begin{aligned}
2\,|2,-2\rangle &= J_-\,|2,-1\rangle \\
2\sqrt{2}\,|2,-2\rangle &= J_-^{(1)}\,|1,-1\rangle_1 \otimes |1,0\rangle_2 + J_-^{(1)}\,|1,0\rangle_1 \otimes |1,-1\rangle_2 + |1,-1\rangle_1 \otimes J_-^{(2)}\,|1,0\rangle_2 + |1,0\rangle_1 \otimes J_-^{(2)}\,|1,-1\rangle_2 \\
&= \sqrt{2}(|1,-1\rangle_1 \otimes |1,-1\rangle_2 + |1,-1\rangle_1 \otimes |1,-1\rangle_2) \\
|2,-2\rangle &= |1,-1\rangle_1 \otimes |1,-1\rangle_2
\end{aligned}$$

We now consider $j = 1$. The highest weight state can be written as,

$$|1,1\rangle = a\,|1,0\rangle_1 \otimes |1,1\rangle_2 + b\,|1,1\rangle_1 \otimes |1,0\rangle_2$$

Consider acting on this state with the operator $J_+ = J_+^{(1)} \otimes I + I \otimes J_+^{(2)}$,

$$\begin{aligned}
aJ_+^{(1)}\,|1,0\rangle_1 \otimes |1,1\rangle_2 + b\,|1,1\rangle_1 \otimes J_+^{(2)}\,|1,0\rangle_2 = 0 \\
\implies a + b = 0
\end{aligned}$$

Hence, we make a choice $a = \frac{1}{\sqrt{2}}, b = -\frac{1}{\sqrt{2}}$,

$$\therefore |1,1\rangle = \frac{1}{\sqrt{2}}\,|1,0\rangle_1 \otimes |1,1\rangle_2 - \frac{1}{\sqrt{2}}\,|1,1\rangle_1 \otimes |1,0\rangle_2$$

We apply a similar procedure as outlined previously for $|1,0\rangle$

$$\begin{aligned}
\sqrt{2}\,|1,0\rangle &= J_-\,|1,1\rangle \\
2\,|1,0\rangle &= J_-^{(1)}\,|1,0\rangle_1 \otimes |1,1\rangle_2 - J_-^{(1)}\,|1,1\rangle_1 \otimes |1,0\rangle_2 + |1,0\rangle_1 \otimes J_-^{(2)}\,|1,1\rangle_2 - |1,1\rangle_1 \otimes J_-^{(2)}\,|1,0\rangle_2 \\
&= \sqrt{2}(|1,-1\rangle_1 \otimes |1,1\rangle_2 - |1,0\rangle_1 \otimes |1,0\rangle_2 + |1,0\rangle_1 \otimes |1,0\rangle_2 - |1,1\rangle_1 \otimes |1,-1\rangle_2) \\
|1,0\rangle &= \frac{|1,-1\rangle_1 \otimes |1,1\rangle_2 - |1,1\rangle_1 \otimes |1,-1\rangle_2}{\sqrt{2}}
\end{aligned}$$

For $|1,-1\rangle$,

$$\begin{aligned}
\sqrt{2}\,|1,-1\rangle &= J_-\,|1,0\rangle \\
2\,|1,-1\rangle &= J_-^{(1)}\,|1,-1\rangle_1 \otimes |1,1\rangle_2 - J_-^{(1)}\,|1,1\rangle_1 \otimes |1,-1\rangle_2 + |1,-1\rangle_1 \otimes J_-^{(2)}\,|1,1\rangle_2 - |1,1\rangle_1 \otimes J_-^{(2)}\,|1,-1\rangle_2 \\
|1,-1\rangle &= \frac{-\,|1,0\rangle_1 \otimes |1,-1\rangle_2 + |1,-1\rangle_1 \otimes |1,0\rangle_2}{\sqrt{2}}
\end{aligned}$$

For $j = 0$,

$$\begin{aligned}
|0,0\rangle &= a\,|1,1\rangle \otimes |1,-1\rangle + b\,|1,-1\rangle \otimes |1,1\rangle + c\,|1,0\rangle \otimes |1,0\rangle \\
J_-\,|0,0\rangle &= aJ_-^{(1)}\,|1,1\rangle \otimes |1,-1\rangle + bJ_-^{(1)}\,|1,-1\rangle \otimes |1,1\rangle + cJ_-^{(1)}\,|1,0\rangle \otimes |1,0\rangle \\
&\quad + a\,|1,1\rangle \otimes J_-^{(1)}\,|1,-1\rangle + b\,|1,-1\rangle \otimes J_-^{(1)}\,|1,1\rangle + c\,|1,0\rangle \otimes J_-^{(1)}\,|1,0\rangle \\
0 &= a\,|1,0\rangle \otimes |1,-1\rangle + c\,|1,-1\rangle \otimes |1,0\rangle + b\,|1,-1\rangle \otimes |1,0\rangle + c\,|1,0\rangle \otimes |1,-1\rangle
\end{aligned}$$

$$\implies b + c = 0 \quad \text{and} \quad a + c = 0$$

We choose $a = \frac{1}{\sqrt{3}}, b = \frac{1}{\sqrt{3}}, c = -\frac{1}{\sqrt{3}}$. Hence,

$$|0,0\rangle = \frac{|1,1\rangle \otimes |1,-1\rangle + |1,-1\rangle \otimes |1,1\rangle - |1,0\rangle \otimes |1,0\rangle}{\sqrt{3}}$$