

Advanced Quantum Mechanics: Assignment #2

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Aditya Vijaykumar

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Problem 1

Before we start looking at the the actual problem itself, let's consider a simpler problem - that of proving $e^x e^{-x} = 1$. Lets Taylor expand the same and look at terms order by order,

$$\begin{aligned} e^x e^{-x} &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots\right) \\ &= 1 + (x - x) + \left(\frac{x^2}{2} + \frac{x^2}{2} - x^2\right) + \left(\frac{x^3}{6} - \frac{x^3}{6} + \frac{x^3}{2} - \frac{x^3}{2}\right) + \dots \end{aligned}$$

We note that terms cancel out order by order.

$$\begin{aligned} U &= \sum_n \frac{(-i)^n}{n!} \int_0^t dt_1 \dots \int_0^t dt_n \mathcal{T}[H(t_1) \dots H(t_n)] \\ &= 1 - i \int_0^t dt_1 H(t_1) - \frac{1}{2} \int_0^t \int_0^t dt_1 dt_2 [\theta(t_1 - t_2) H(t_1) H(t_2) + \theta(t_2 - t_1) H(t_2) H(t_1)] + \dots \\ U^\dagger &= 1 + i \int_0^t dt_1 H(t_1) - \frac{1}{2} \int_0^t \int_0^t dt_1 dt_2 [\theta(t_2 - t_1) H(t_1) H(t_2) + \theta(t_1 - t_2) H(t_2) H(t_1)] + \dots \\ \therefore U^\dagger U &= 1 + \left(-i \int_0^t dt_1 H(t_1) + i \int_0^t dt'_1 H(t'_1)\right) + \int_0^t \int_0^t dt_1 dt_2 [H(t_1) H(t_2)] \\ &\quad \left(-\frac{1}{2} \int_0^t \int_0^t dt_1 dt_2 [\theta_{12} H(t_1) H(t_2) + \theta_{21} H(t_2) H(t_1)] - \frac{1}{2} \int_0^t \int_0^t dt'_1 dt'_2 [\theta_{2'1'} H(t'_1) H(t'_2) + \theta_{1'2'} H(t'_2) H(t'_1)]\right) \\ &\quad + \dots \end{aligned}$$

The terms in the first parenthesis just cancel each other. As the primed variables are just dummies one can ignore the primes and write the terms in the second parenthesis in terms of t_1 and t_2 . We can rewrite that particular term as follows (using the fact that $\theta_{12} + \theta_{21} = 1$),

$$\begin{aligned} &-\frac{1}{2} \int_0^t \int_0^t dt_1 dt_2 [\theta_{12} H(t_1) H(t_2) + \theta_{21} H(t_2) H(t_1) + \theta_{21} H(t_1) H(t_2) + \theta_{12} H(t_2) H(t_1)] \\ &= -\frac{1}{2} \int_0^t \int_0^t dt_1 dt_2 [H(t_1) H(t_2) + H(t_2) H(t_1)] = -\int_0^t \int_0^t dt_1 dt_2 [H(t_1) H(t_2)] \end{aligned}$$

This shows that the terms cancel. This will hold at higher orders too, albeit with more complicated θ functions.

Problem 2

We know that the energy eigenstates of the harmonic oscillator form a basis. Hence, the required coherent state $|z\rangle$ can be written in terms of these eigenstates as,

$$|z\rangle = \sum_{n=0}^{\infty} c_n |n\rangle = \sum_{n=0}^{\infty} c_n \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle$$

Substituting this in the equation for coherent state $a|z\rangle = z|z\rangle$,

$$\begin{aligned} \sum_{n=0}^{\infty} c_n a |n\rangle &= \sum_{n=0}^{\infty} z c_n |n\rangle \\ \sum_{n=1}^{\infty} c_n \sqrt{n} |n-1\rangle &= \sum_{n=0}^{\infty} z c_n |n\rangle \\ \sum_{n=0}^{\infty} c_{n+1} \sqrt{n+1} |n\rangle &= \sum_{n=0}^{\infty} z c_n |n\rangle \\ \therefore c_{n+1} \sqrt{n+1} &= z c_n \end{aligned}$$

We have effectively derived a recursion relation for the coefficients c_n . If we start off with $c_n = \alpha$,

$$c_1 = z\alpha \quad , \quad c_2 = \frac{z^2\alpha}{\sqrt{2}} \quad , \quad c_3 = \frac{z^3\alpha}{\sqrt{3 \cdot 2}} \quad , \quad \dots \quad , \quad c_n = \frac{z^n\alpha}{\sqrt{n!}}$$

So, our coherent state can now be written as,

$$\begin{aligned} |z\rangle &= \alpha \sum_{n=0}^{\infty} \frac{(za^\dagger)^n}{n!} |0\rangle \\ &= \alpha e^{a^\dagger z} |0\rangle \end{aligned}$$

Problem 3

We know that,

$$x(0) = \frac{a + a^\dagger}{\sqrt{2m\omega}} \quad , \quad p(0) = \frac{\sqrt{m\omega}(a - a^\dagger)}{\sqrt{2}i} \quad , \quad x(t) = e^{iHt}x(0)e^{-iHt} \quad , \quad p(t) = e^{iHt}p(0)e^{-iHt}$$

From this, we note the following,

$$\begin{aligned} x(t)|0\rangle &= e^{iHt}x(0)e^{-iHt}|0\rangle \\ &= e^{-i\omega t/2}e^{iHt}x(0)|0\rangle \\ &= \frac{e^{-i\omega t/2}}{\sqrt{2m\omega}}e^{iHt}|1\rangle \\ x(t)|0\rangle &= \frac{e^{i\omega t}}{\sqrt{2m\omega}}|1\rangle \implies \langle 0|x(t) = \frac{e^{-i\omega t}}{\sqrt{2m\omega}}\langle 1| \end{aligned}$$

$$\begin{aligned} \text{Similarly, } p(t)|0\rangle &= e^{iHt}p(0)e^{-iHt}|0\rangle \\ &= e^{-i\omega t/2}e^{iHt}p(0)|0\rangle \\ &= -\frac{e^{-i\omega t/2}\sqrt{m\omega}}{\sqrt{2}i}e^{iHt}|1\rangle \\ p(t)|0\rangle &= -\frac{e^{i\omega t}\sqrt{m\omega}}{\sqrt{2}i}|1\rangle \implies \langle 0|p(t) = \frac{e^{-i\omega t}\sqrt{m\omega}}{\sqrt{2}i}\langle 1| \end{aligned}$$

Now consider the quantities to be calculated,

$$C_1(t) = \langle 0|x(t)x(0)|0\rangle = \frac{e^{-i\omega t}}{\sqrt{2m\omega}} \frac{1}{\sqrt{2m\omega}} = \boxed{\frac{e^{-i\omega t}}{2m\omega}}$$

$$C_2(t) = \langle 0|x(t)p(0)|0\rangle - \langle 0|p(0)x(t)|0\rangle = -\frac{e^{-i\omega t}}{2i} - \frac{e^{i\omega t}}{2i} = \boxed{i \cos \omega t}$$

$$C_3(t) = \langle 0|p(t)x(0)|0\rangle - \langle 0|x(0)p(t)|0\rangle = \frac{e^{-i\omega t}}{2i} + \frac{e^{i\omega t}}{2i} = \boxed{-i \cos \omega t}$$

Problem 4

Part (a)

$$\begin{aligned} Z(\beta) &= \text{Tr } e^{-\beta H} = \sum_{n=0}^{\infty} \langle n|e^{-\beta H}|n\rangle \\ &= \sum_{n=0}^{\infty} e^{-\beta(n+\frac{1}{2})\omega} \\ &= \frac{e^{-\beta\omega/2}}{1 - e^{-\beta\omega}} = \frac{2}{\sinh \frac{\beta\omega}{2}} \end{aligned}$$

Part (b)

Given that,

$$x(\tau) = \sum_n x_n e^{\frac{2\pi i n \tau}{\beta}}$$

As we require $x(\tau)$ to be real, it follows from above that $x_n = x_n^*$. Taking $\omega_n = \frac{2\pi n}{\beta}$

$$\begin{aligned} S_E &= \int_0^\beta d\tau \left(\frac{m\dot{x}^2}{2} + \frac{m\omega^2 x^2}{2} \right) \\ &= \sum_{p,q} \left(-\frac{mx_p x_q \omega_p \omega_q}{2} + \frac{m\omega^2 x_p x_q}{2} \right) \int_0^\beta e^{i(\omega_q + \omega_p)\tau} d\tau \\ &= \sum_{p,q} \left(-\frac{mx_p x_q \omega_p \omega_q}{2} + \frac{m\omega^2 x_p x_q}{2} \right) \beta \delta_{p,-q} \\ &= \sum_q \left(-\frac{mx_{-q} x_q \omega_{-q} \omega_q}{2} + \frac{m\omega^2 x_{-q} x_q}{2} \right) \beta \\ &= \frac{m}{2} \sum_q \beta (x_q^* x_q \omega_q^2 + \omega^2 x_q^* x_q) \\ -S_E &= -\frac{m\beta}{2} \sum_q x_q^* x_q (\omega_q^2 + \omega^2) \end{aligned}$$

The required path integral to be done is,

$$\begin{aligned}
Z(\beta) &= N \int \mathcal{D}x \exp -\frac{m\beta}{2} \sum_q x_q^* x_q (\omega_q^2 + \omega^2) \quad \text{where } N \text{ is some normalization} \\
&= N \int \mathcal{D}x \exp \left(-m\beta \sum_{q=1}^{\infty} x_q^* x_q (\omega_q^2 + \omega^2) - \frac{m\beta x_0^2 \omega^2}{2} \right) \\
&= N \int dx_0 \exp \left(-\frac{m\beta x_0^2 \omega^2}{2} \right) \times \prod_{q=1}^{\infty} \int dx_q dx_q^* \exp -m\beta x_q^* x_q (\omega_q^2 + \omega^2) \\
&= N \sqrt{\frac{2\pi}{m\beta\omega^2}} \times \prod_{q=1}^{\infty} \frac{2\pi}{m\beta(\omega_q^2 + \omega^2)} \\
&= N \sqrt{\frac{2\pi}{m\beta\omega^2}} \times \prod_{q=1}^{\infty} \frac{2\pi}{m\beta \frac{4q^2\pi^2}{\beta^2} \left(1 + \left(\frac{\beta\omega}{2q\pi} \right)^2 \right)} \\
&= N \sqrt{\frac{2\pi}{m\beta\omega^2}} \times \frac{\beta\omega}{2\pi \sinh \frac{\beta\omega}{2}} \times \prod_{q=1}^{\infty} \frac{\beta}{2\pi m q^2} \\
&= N \sqrt{\frac{\beta}{2\pi m}} \times \prod_{q=1}^{\infty} \frac{\beta}{2\pi m q^2} \times \frac{1}{\sinh \frac{\beta\omega}{2}} \\
&= N' \times \frac{1}{\sinh \frac{\beta\omega}{2}}
\end{aligned}$$

As N' does not depend on ω , we can drop it altogether for computation purposes. This gives us the required result.