Classical Mechanics: Assignment #5

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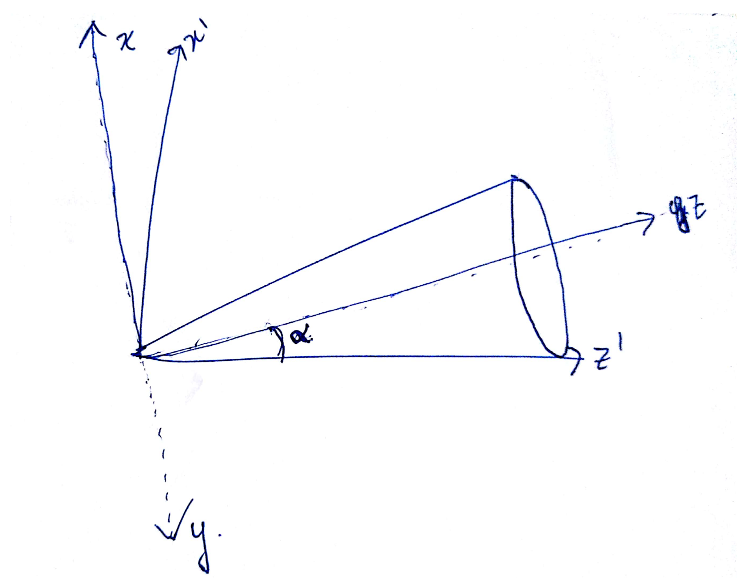
The Euler Equations of motion for rotation about principal axes with moments of inertia I_1, I_2, I_3 and torques N_1, N_2, N_3 are given by,

$$I_1\dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3) = N_1 \tag{1}$$
$$I_2\dot{\omega}_2 - \omega_3\omega_1(I_3 - I_1) = N_2 \tag{2}$$
$$I_3\dot{\omega}_3 - \omega_1\omega_2(I_1 - I_2) = N_3 \tag{3}$$

Problem 1

Part (a)

The moment of inertia tensor in the xyz coordinate system is given by,



$$I = \frac{3mh^2}{5}\begin{bmatrix} 1+\dfrac{\tan^2\alpha}{4} & 0 & 0 \\ 0 & 1+\dfrac{\tan^2\alpha}{4} & 0 \\ 0 & 0 & \dfrac{\tan^2\alpha}{2} \end{bmatrix}$$

As I is a tensor, we rotate the current I to the get the moment of inertia I' in the $x'y'z'$ frame,

$$I' = R(\alpha) I R^T(\alpha)$$

$$I' = \frac{3mh^2}{5}\begin{bmatrix} \left(\frac{\tan^2\alpha}{4}+1\right)\cos^2\alpha + \frac{1}{2}\sin^2\alpha\tan^2\alpha & 0 & \frac{1}{2}\sin^2\alpha\tan\alpha - \cos\alpha\sin\alpha\left(\frac{\tan^2\alpha}{4}+1\right) \\ 0 & \frac{\tan^2\alpha}{4}+1 & 0 \\ \frac{1}{2}\sin^2\alpha\tan\alpha - \cos\alpha\sin\alpha\left(\frac{\tan^2\alpha}{4}+1\right) & 0 & \left(\frac{\tan^2\alpha}{4}+1\right)\sin^2\alpha + \frac{\sin^2\alpha}{2} \end{bmatrix}$$

The angular velocity is along the instantaneous axis of rotation, which in this case is the new z axis. The cone traces out a circle of its slant height $l = \dfrac{h}{\cos\alpha}$. Hence the angular velocity is just,

$$\Omega = \frac{2\pi l}{\tau h \tan\alpha} = \frac{2\pi}{\tau\sin\alpha}$$

Hence, the angular velocity vector $\vec{\mathbf{\Omega}} = \begin{bmatrix} 0 & 0 & \Omega \end{bmatrix}$. The angular momentum $\vec{\mathbf{L}}$ is given by,

$$\vec{\mathbf{L}} = I'\vec{\mathbf{\Omega}} = \begin{bmatrix} -\frac{h^3\pi\rho(5\cos(2\alpha)+3)\tan^2(\alpha)}{20\tau} \\ 0 \\ \frac{h^3\pi\rho(5\cos(2\alpha)+7)\tan^3(\alpha)}{20\tau} \end{bmatrix}$$

and the Kinetic energy K is,

$$K = \vec{\mathbf{\Omega}}^T\vec{\mathbf{L}} = \frac{\pi^2 h^3\rho(5\cos(2\alpha)+7)\tan^2(\alpha)}{10\tau^2}$$

In both the above expressions, we have used,

$$m = \rho\frac{\pi r^2 h}{3} = \rho\frac{\pi h^3\tan^2\alpha}{3}$$

Part (b)

The coordinates of the centre of mass of the rod are given by,

$$x = l\sin\theta \quad \text{and} \quad y = \alpha + l\cos\theta \implies \dot{x}^2 + \dot{y}^2 = \dot{\alpha}^2 + l^2\dot{\theta}^2 - 2l\dot{\alpha}\dot{\theta}\sin\theta$$

Let's write down the expressions for Kinetic Energy T and Potential Energy V,

$$\begin{aligned} T &= \frac{Mv_{CM}^2}{2} + \frac{I_{CM}\dot{\theta}^2}{2} \\ &= \frac{M(\dot{x}^2+\dot{y}^2)}{2} + \frac{Ml^2\dot{\theta}^2}{6} \\ T &= \frac{M(\dot{\alpha}^2 + l^2\dot{\theta}^2 - 2l\dot{\alpha}\dot{\theta}\sin\theta)}{2} + \frac{Ml^2\dot{\theta}^2}{6} \\ V &= -Mgy_{CM} + \frac{k\alpha^2}{2} \\ &= -Mg(\alpha + l\cos\theta) + \frac{k\alpha^2}{2} \end{aligned}$$

We now write down the Lagrangian,

$$L = \frac{M(\dot{\alpha}^2 + l^2\dot{\theta}^2 - 2l\dot{\alpha}\dot{\theta}\sin\theta)}{2} + \frac{Ml^2\dot{\theta}^2}{6} + Mg(\alpha + l\cos\theta) - \frac{k\alpha^2}{2}$$

The equations of motion are,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(M\dot{\alpha} - l\dot{\theta}\sin\theta\Big) = Mg - k\alpha \implies \ddot{\alpha} - \frac{l\ddot{\theta}\sin\theta + l\dot{\theta}^2\cos\theta}{M} - g + \frac{k\alpha}{M} = 0 \quad \text{and}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(Ml^2\dot{\theta} - l\dot{\alpha}\sin\theta + \frac{Ml^2\dot{\theta}}{3}\right) = -Mgl\sin\theta \implies \frac{4l\ddot{\theta}}{3} - \frac{\ddot{\alpha}\sin\theta + \dot{\alpha}\dot{\theta}\cos\theta}{M} + g\sin\theta = 0$$

Problem 2

Let $f(q_i, t)$ be a function such that $L'(q_i, \dot{q}_i, t) = L(q_i, \dot{q}_i, t) + \frac{\mathrm{d}f}{\mathrm{d}t}$. We have seen that the equations of motion remain unchanged by this addition.
Let's first calculate the canonical momenta p_i',

$$p_i' = \frac{\partial L'}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{q}_i} + \frac{\partial}{\partial \dot{q}_i}\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right) = p_i + \frac{\partial}{\partial \dot{q}_i}\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right) = p_i + \frac{\partial}{\partial \dot{q}_i}\left(\sum_j \frac{\partial f}{\partial q_j}\dot{q}_j\right) = p_i + \frac{\partial f}{\partial q_i}$$

Let's calculate the Hamiltonian $H'(q_i, p_i, t)$,

$$\begin{aligned}
H'(q_i, p_i, t) &= \sum_i \left(p_i + \frac{\partial f}{\partial q_i}\right)\dot{q}_i - L' \\
&= \sum_i p_i\dot{q}_i - L + \sum_i \dot{q}_i \frac{\partial f}{\partial q_i} - \frac{\mathrm{d}f}{\mathrm{d}t} \\
&= H(q_i, p_i, t) + \sum_i \dot{q}_i \frac{\partial f}{\partial q_i} - \sum_i \dot{q}_i \frac{\partial f}{\partial q_i} \\
H'(q_i, p_i, t) &= H(q_i, p_i, t)
\end{aligned}$$

As the Hamiltonians are essentially the same, the equations of motion are same as well.

Part (b)
Given that,

$$\begin{aligned}
L &= -mc^2\sqrt{1 - \frac{v^2}{c^2}} - e\phi + \frac{e}{c}\vec{\mathbf{v}} \cdot \vec{\mathbf{A}} \\
&= -mc^2\sqrt{1 - \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{c^2}} - e\phi + \frac{e}{c}(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z) \\
&= -\frac{m(c^2 - \dot{x}^2 + \dot{y}^2 + \dot{z}^2)}{\sqrt{1 - \frac{v^2}{c^2}}} - e\phi + \frac{e}{c}(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z)
\end{aligned}$$

We first write the canonical momenta,

$$p_x = \frac{\partial L}{\partial \dot{x}} = -mc^2\frac{\left(\frac{-2\dot{x}}{c^2}\right)}{2\sqrt{1 - \frac{v^2}{c^2}}} + \frac{e}{c}A_x = \frac{m\dot{x}}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{e}{c}A_x$$

Similarly,

$$p_y = \frac{m\dot{y}}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{e}{c}A_y \quad \text{and} \quad p_z = \frac{m\dot{z}}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{e}{c}A_z$$

Problem 2 continued on next page. . .

Let's first evaluate $\gamma = \dfrac{1}{\sqrt{1-\dfrac{v^2}{c^2}}}$.

$$\gamma = \frac{1}{\sqrt{1-\dfrac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\dfrac{\sum_{i=x,y,z}\dot{x}_i^2}{c^2}}}$$

From the formulae of the canonical momenta, one can see that,

$$\begin{aligned}
H &= p_x\dot{x} + p_y\dot{y} + p_z\dot{z} - L \\
&= \frac{p_x\left(p_x - \frac{eA_x}{c}\right)}{\gamma m} + \frac{p_y\left(p_y - \frac{eA_y}{c}\right)}{\gamma m} + \frac{p_z\left(p_z - \frac{eA_z}{c}\right)}{\gamma m} - \frac{e\left(\frac{A_x\left(p_x - \frac{eA_x}{c}\right)}{\gamma m} + \frac{A_y\left(p_y - \frac{eA_y}{c}\right)}{\gamma m} + \frac{A_z\left(p_z - \frac{eA_z}{c}\right)}{\gamma m}\right)}{c} + e\phi + mc^2\gamma \\
&= \frac{-2ceA_xp_x - 2ceA_yp_y - 2ceA_zp_z + e^2(A_x^2 + A_y^2 + A_z^2) + c^2(p_x^2 + p_y^2 + p_z^2)}{c^2\gamma m} + e\phi + mc^2\gamma \\
H &= \frac{c^2\sum_{i=x,y,z}(p_i - \frac{e}{c}A_i)^2}{c^2\gamma m} + e\phi + mc^2\gamma \\
H &= \sqrt{p^2c^2 + m^2c^4} + e\phi
\end{aligned}$$

where $p^2 = \sum_{i=x,y,z}(p_i - \dfrac{e}{c}A_i)^2$. This is $\neq T + V$. The Hamilton equations of motion can then be written as,

$$\dot{x}_i = \frac{p_i - \dfrac{e}{c}A_i}{\sqrt{p^2c^2 + m^2c^4}} \quad \text{and} \quad \dot{p}_i = \frac{e}{c}\frac{\left(p_i - \dfrac{e}{c}A_i\right)\frac{\partial A_i}{\partial x_i}}{\sqrt{p^2c^2 + m^2c^4}} - e\frac{\partial \phi}{\partial x_i}$$

Problem 3

From the torque-free Euler equations of motion, one has,

$$\begin{aligned}
I_1\dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3) = 0 \\
I_2\dot{\omega}_2 - \omega_3\omega_1(I_3 - I_1) = 0 \\
I_3\dot{\omega}_3 - \omega_1\omega_2(I_1 - I_2) = 0
\end{aligned}$$

Let's say that the body is undergoing steady motion. This means,

$$\omega_2\omega_3(I_2 - I_3) = \omega_3\omega_1(I_3 - I_1) = \omega_1\omega_2(I_1 - I_2) = 0$$

If I_1, I_2, I_3 are all different, the above condition can be satisfied if and only if two of the ω's are zero. Hence, if the body is indeed undergoing uniform motion, it can do so only along one of the principal axes. (In general, one cannot reduce an overall non-uniform motion to uniform motion along one axis).
Let ω_1 be the non-zero angular velocity. We apply a perturbation, such that $\omega_1' = \omega_1 + \epsilon\Omega_1, \omega_2' = \epsilon\Omega_2, \omega_3' = \epsilon\Omega_3$. Substituting this in the Euler equations (upto first order in ϵ),

$$\begin{aligned}
I_1\dot{\Omega}_1 &= 0 \\
I_2\dot{\Omega}_2 &= \omega_1\Omega_3(I_3 - I_1) \\
I_3\dot{\Omega}_3 &= \omega_1\Omega_2(I_1 - I_2)
\end{aligned}$$

Problem 3 continued on next page. . .

The last two equations can be differentiated with respect to t and can be written as follows,

$$\ddot{\Omega}_2 = -\frac{\omega_1^2}{I_2 I_3}(I_1 - I_2)(I_1 - I_3)\Omega_2$$
$$\ddot{\Omega}_3 = -\frac{\omega_1^2}{I_2 I_3}(I_1 - I_3)(I_1 - I_2)\Omega_3$$

We see that the Ω_1 and Ω_2 will oscillate with the same frequency $\alpha = \sqrt{\frac{\omega_1^2}{I_2 I_3}(I_1 - I_2)(I_1 - I_3)}$. The motion will be stable only when $\alpha > 0 \implies I_1 > I_2 \quad \text{and} \quad I_1 > I_3$ or $I_1 < I_2 \quad \text{and} \quad I_1 < I_3$.

Problem 4

The sides are given to be $2a, 2b, 2c$ along x, y, z. Let's first calculate the moments of inertia,

$$\begin{aligned}
I_{zz} &= \rho \int_{-a}^{a}\int_{-a}^{a}\int_{-b}^{b} dxdydz(x^2+y^2) \\
&= \rho(2b)\int_{-a}^{a} dx \left(x^2 y + \frac{y^2}{3}\right)\bigg|_{-a}^{a} \\
&= \rho(2b)\left(\frac{x^3}{3}(2a) + x\frac{2a}{3}\right)\bigg|_{-a}^{a} \\
&= \frac{M}{8a^2 b}(2b)\frac{8a^4}{3} \\
I_{zz} &= \frac{2Ma^2}{3}
\end{aligned}$$

Similarly, $I_{yy} = I_{xx} = \dfrac{M(a^2+b^2)}{3}$. Let's now calculate I_{xy},

$$\begin{aligned}
I_{xy} &= -\rho \int_{-a}^{a}\int_{-a}^{a}\int_{-b}^{b} dxdydz\ xy \\
I_{xy} &= 0
\end{aligned}$$

Similarly, $I_{yz} = I_{zx} = 0$. Hence I_{jj}'s are the moment of inertia about principal axes. The system has no forces acting on it. We can then write down the Euler equations as follows, with $I_{xx} = I_1, I_{yy} = I_2, I_{zz} = I_3$

$$\begin{aligned}
\dot{\omega}_1 - \omega_2\omega_3\left(1 - \frac{2a^2}{a^2+b^2}\right) &= 0 \\
\dot{\omega}_2 - \omega_3\omega_1\left(\frac{2a^2}{a^2+b^2} - 1\right) &= 0 \\
\dot{\omega}_3 = 0 &\implies \omega_3 = constant
\end{aligned}$$

Taking the time derivative of the first equation and substituting for $\dot{\omega}_2$ from the second equation, we get,

$$\ddot{\omega}_1 = -\left[\omega_3\left(1 - \frac{2a^2}{a^2+b^2}\right)\right]^2 \omega_1 \implies \quad \text{periodic motion with frequency} \quad \omega = \omega_3\left|1 - \frac{2a^2}{a^2+b^2}\right|$$

As the problem is symmetric about the x and y axes, the motion will be periodic about the y axis too.