Dynamical Systems: Homework #1

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Problem 1

$$\dot{x} = x^{1/3} \implies \frac{dx}{x^{1/3}} = dt \implies \frac{2}{3}\left(x^{2/3} - x_0^{2/3}\right) = t \implies x(t) = \left(x_0^{2/3} + \frac{3}{2}t\right)^{3/2} \tag{1}$$

For $x_0 = 0$, we have $x(t) = \left(\frac{3t}{2}\right)^{3/2}$. As the function $f(y) = y^{3/2}$ is only defined for $y \geq 0$, this solution has maximum interval of existence $t \in [0, \infty)$. For $x_0 = 0$, we also have the trivial solution $x(t) = 0$ which has maximum interval of existence $t \in (-\infty, \infty)$. So there are at least two distinct solutions for $x_0 = 0$. One can also imagine patching up the above solutions at origin and forming other possible solutions.

For the case where $x_0 \neq 0$, we are only left with [1]. As $x_0^{2/3} > 0, \forall x_0 \neq 0$, the maximal interval of existence for $x(t)$ is $\left[-\frac{2}{3}x_0^{2/3}, \infty\right)$.
If we try to solve the problem numerically, we only get the trivial solution $x(t) = 0$ for $x_0 = 0$.

Problem 2

Part (a)

$$\dot{x} = x(x^2 - 1)$$

For $x_0 = 0, 1, -1$, $x(t) = 0, 1, -1$ respectively is a solution for all times.

$$\begin{aligned}
\therefore \frac{dx}{x(x-1)(x+1)} &= dt \\
\therefore -\frac{dx}{x} + \frac{1}{2}\left[\frac{dx}{x+1} + \frac{dx}{x-1}\right] &= dt \\
\frac{1}{2}\log\left|\frac{x_0^2(x^2-1)}{x^2(x_0^2-1)}\right| &= t \\
\frac{(x^2-1)}{x^2} &= \pm\frac{x_0^2-1}{x_0^2}e^{2t} \\
x(t) &= \sqrt{\frac{1}{1 \pm \frac{x_0^2-1}{x_0^2}e^{2t}}}
\end{aligned}$$

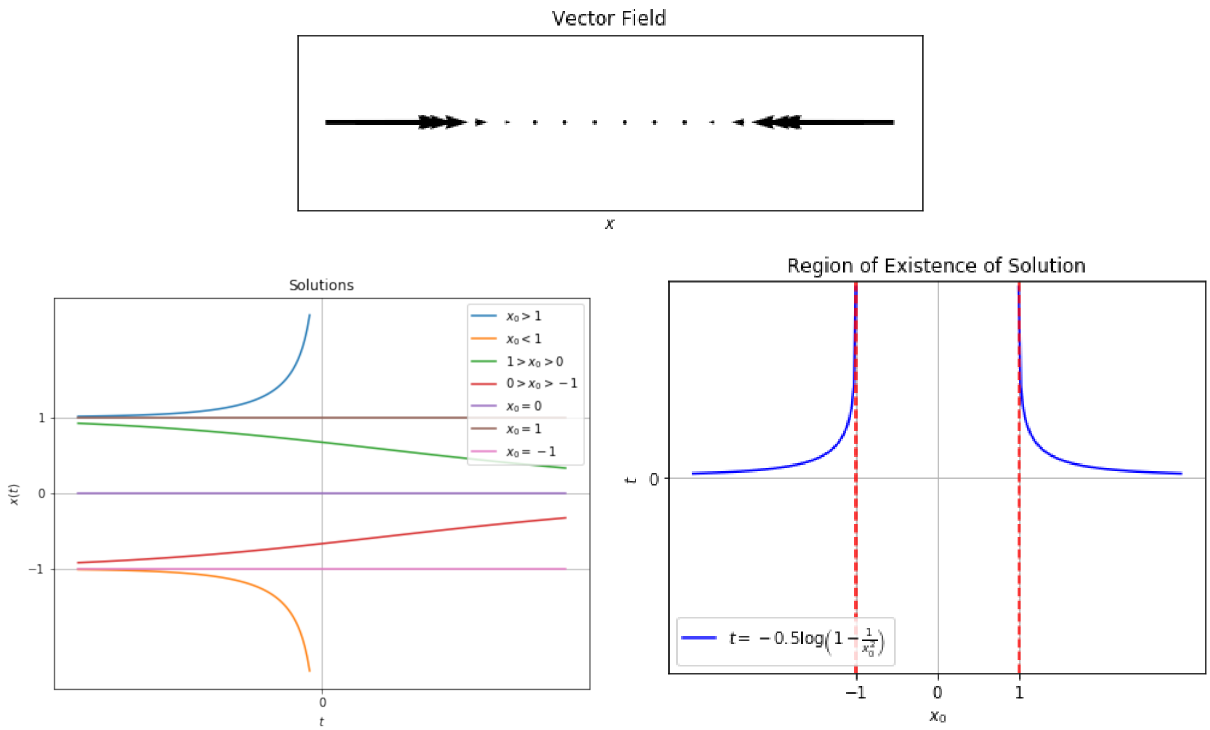
We take only the $-$ sign in the above expression as the $+$ sign does not satisfy the initial condition.

$$x(t) = \frac{x_0}{|x_0|}\sqrt{\frac{1}{1-\frac{x_0^2-1}{x_0^2}e^{2t}}}$$

where the $\frac{x_0}{|x_0|}$ has been multiplied to take care of sign. For the problem to have the above solution, one must have,

$$1-\frac{x_0^2-1}{x_0^2}e^{2t} > 0 \implies t < -\frac{1}{2}\log\frac{x_0^2-1}{x_0^2} \quad \text{if} \quad x_0^2 > 1$$
$$\text{and} \quad t \in (-\infty, \infty) \quad \text{if} \quad x_0^2 < 1$$

Figure 1: Q2 (a)



Part (b)

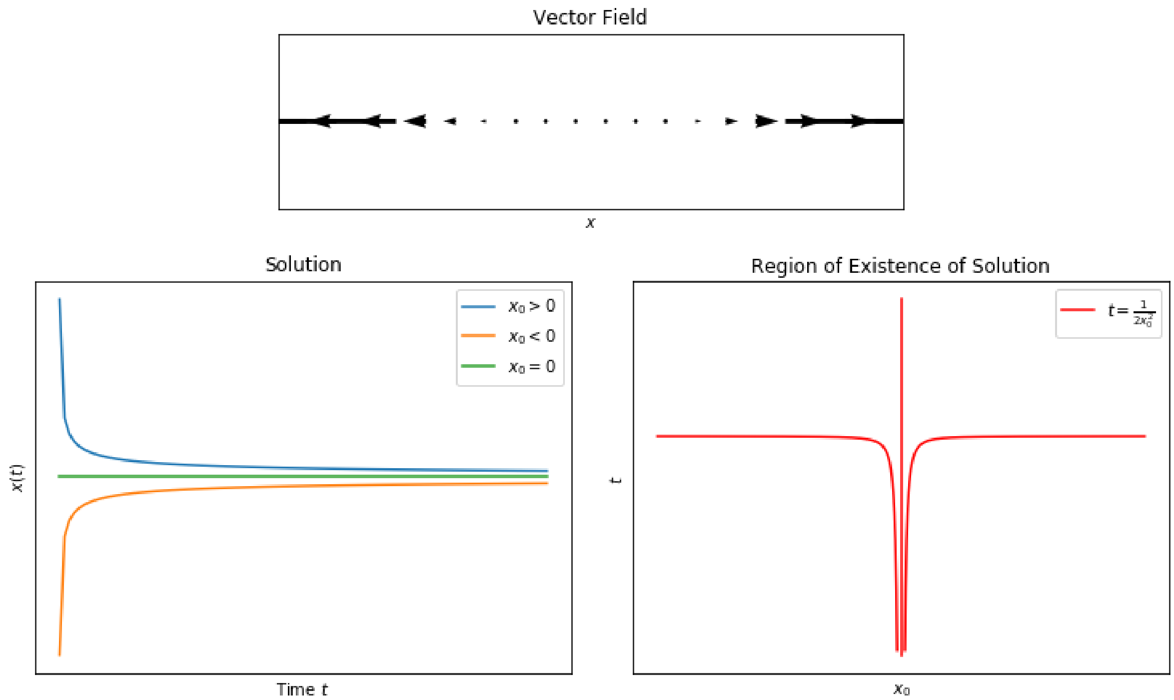
$$\dot{x} = -x^3$$

The above equation has the trivial solution $x(t) = 0; \forall t$ for $x_0 = 0$. We proceed to find solutions for other values of x_0.

$$\dot{x} = -x^3 \implies -\frac{dx}{x^3} = dt \implies \frac{1}{2x^2}\bigg|_{x_0}^{x} = t \implies \frac{1}{x^2} = \frac{1}{x_0^2} + 2t$$
$$\implies x = +\frac{x_0}{\sqrt{1+2x_0^2 t}}$$

The above solution will exist for $t > -\frac{1}{2x_0^2}; x_0 \neq 0$. The region of existence is also plotted below.

Figure 2: Q2 (b)



Part (c)
Given that

$$\begin{aligned}
E &= \frac{x_2^2}{2} + \frac{x_1^2}{2} - \frac{x_1^4}{4} \\
\dot{E} &= x_2\dot{x}_2 + x_1\dot{x}_1 - \dot{x}_1 x_1^3 \\
&= -x_1x_2 + x_1^3x_2 + x_1x_2 - x_2x_1^3 = 0 \\
E &= constant
\end{aligned}$$

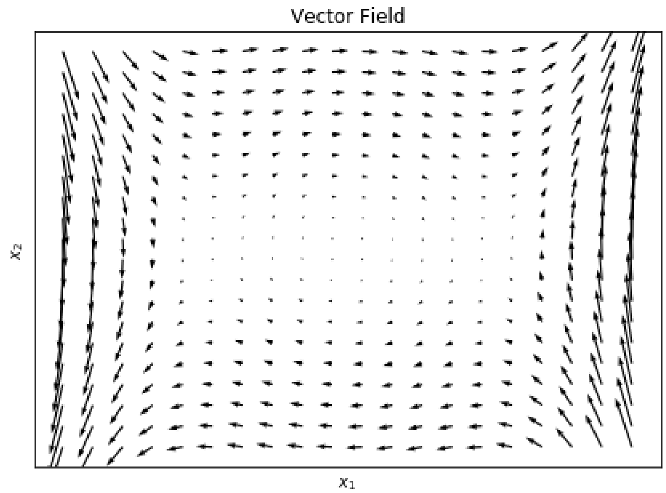
Hence the system as defined in the question is a Hamiltonian system. Now we can proceed to analytically solve this equation,

$$\begin{aligned}
\frac{\dot{x}_1^2}{2} + \frac{x_1^2}{2} - \frac{x_1^4}{4} &= E \\
\implies \dot{x}_1 &= \sqrt{2E + \frac{x_1^4}{2} - x_1^2} \\
\implies x &= \int dx_1 \sqrt{2E + \frac{x_1^4}{2} - x_1^2}
\end{aligned}$$

Problem 3

Solved completely in the Jupyter Notebook.

Figure 3: Q2 (c)



Problem 4

One definition of religion is [1] :-
A religion involves a communal, transmittable body of teachings and prescribed practices about an ultimate, sacred reality or state of being ***that calls for reverence or awe****, a body which guides its practitioners into what it describes as a saving, illuminating or emancipatory relationship to this reality through a personally transformative life of prayer, ritualized meditation, and/or moral practices like repentance and personal regeneration.*

One can argue whether this definition is really general, but I think to first order religions like Hinduism, Islam and Christianity agree with this definition.

If one is pedantic about it, science, unlike religion, is only a recently coined term and has been in vogue only since the 19th Century[2]. The most accepted definition of a scientific hypothesis was given by Karl Popper in 1959, w*hich, in short, states that scientific hypotheses should in-principle be falsifiable. There is no reference made to an absolute authority of science*, which then is the most striking difference between science and religion.

Note that this is not to say there aren't any similarities between science and religion - science communities also have cults like religion.

[1]Dictionary of Philosophy of Religion, Taliaferro & Marty 2010
[2]https://plato.stanford.edu/entries/religion-science/#WhatScieWhatReli