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Notes on Gravity as a Quantum Theory

Aditya Vijaykumar

International Centre for Theoretical Sciences, Bengaluru, India.

E-mail: aditya.vijaykumar@icts.res.in

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1 All is Classical, All is Quantum

1.1 The Classical Field

$\phi(\vec{x}, t)$ gives the value of a classical field at every point in spacetime. The simplest classical field is the *real scalar field*, which is characterized only by real numbers. The Klein-Gordon equation governs a free massive scalar field.

$$\frac{\partial^2 \phi}{\partial t^2} - \sum_{x_j} \frac{\partial^2 \phi}{\partial x_j^2} + m^2 \phi = 0$$

An interesting part about the free scalar field is that one can describe it as an infinite set of decoupled harmonic oscillators. Put this field into a box of length L and volume $V = L^3$, and having periodic boundary conditions. One can Fourier decompose this as,

$$\phi(\vec{x}, t) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} \phi_{\vec{k}}(t) \exp(i\vec{k} \cdot \vec{x}) \text{ where } k_x = \frac{2\pi n_x}{L}, \dots$$

Substituting this into the first equation, we find that the harmonic oscillators get nicely decoupled into an infinite set of ODEs of the form,

$$\ddot{\phi}_{\vec{k}} + (k^2 + m^2)\phi_{\vec{k}} = 0$$

which is basically the harmonic oscillator equation with frequency $\omega_k = \sqrt{k^2 + m^2}$. The energy of oscillators is simply equal to the sum of individual energies of the oscillators,

$$E = \sum_{\vec{k}} \left[\frac{1}{2} \dot{\phi}_{\vec{k}}^2 + \frac{1}{2} \omega_k^2 \phi_{\vec{k}}^2 \right]$$

Equivalently, when $V \rightarrow \infty$ and k is a continuous variable, the summation is just replaced by an integral over all k ,

$$\phi(\mathbf{x}, t) = \int \frac{d^3 \mathbf{k}}{(2\pi)^{3/2}} e^{i\mathbf{k} \cdot \mathbf{x}} \phi_{\mathbf{k}}(t)$$

1.2 Quantizing Fields

As mentioned earlier, a field can be thought of as a collection of decoupled harmonic oscillators. We quantize each field $\phi_{\mathbf{k}}$ as a separate harmonic oscillator. We identify the position and momentum as operators $\hat{\phi}_{\mathbf{k}}$ and $\hat{\pi}_{\mathbf{k}}$. The commutation relations for the harmonic oscillator as $V \rightarrow \infty$ can now be written as,

$$\left[\hat{\phi}_{\mathbf{k}}(t), \hat{\pi}_{\mathbf{k}'}(t) \right] = i\delta(\mathbf{k} + \mathbf{k}')$$

The vacuum state is the state corresponding to the lowest energy configuration. One can clearly see that the commutation relations cannot be satisfied for the most intuitive low energy configuration *ie.* $\phi(\mathbf{x}, t) = 0$, implying that the vacuum state is really something non-trivial. But since, for a free field, all the $\phi_{\mathbf{k}}$ are decoupled, we can write the vacuum state wave functional as the product of all wavefunctions, each describing the ground state of the harmonic oscillator with the wavenumber \mathbf{k} . Again, for large volume, one can write,

$$\psi[\phi] \propto \exp\left(-\frac{1}{2} \int d^3\mathbf{k} |\phi_{\mathbf{k}}|^2 \omega_{\mathbf{k}}\right)$$

Consider the integral inside the exponential,

$$\begin{aligned} \int d^3\mathbf{k} |\phi_{\mathbf{k}}|^2 \omega_{\mathbf{k}} &= \int d^3\mathbf{k} \phi_{\mathbf{k}} \phi_{\mathbf{k}}^* \sqrt{k^2 + m^2} \\ &= \int d^3\mathbf{x} d^3\mathbf{y} \phi(\mathbf{x}) \phi(\mathbf{y}) \int d^3\mathbf{k} e^{i\mathbf{k}(\mathbf{y}-\mathbf{x})} \sqrt{k^2 + m^2} \\ &= \int d^3\mathbf{x} d^3\mathbf{y} \phi(\mathbf{x}) \phi(\mathbf{y}) K(\mathbf{x}, \mathbf{y}) \end{aligned}$$

where $K(\mathbf{x}, \mathbf{y})$ is called the kernel.

The vacuum energy density is just the sum of all ground state energies,

$$\frac{E_o}{V} = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{\omega_k}{2}$$

Okay, now this is a very interesting expression for the energy. We see that because $\omega_k = \sqrt{k^2 + m^2}$, we can see that this integral diverges as k^4 . If quantum gravity is assumed to be modelled as a scalar field, and we put a cutoff for our integration at let's say the Planckian scale, we see that the vacuum energy density is of the order unity in Planck units, which in turn corresponds to a mass density of $10^{94} g/cm^3$. The mass of the *entire* observable universe is $10^{55} g$! One can try to resolve this problem by *positing* that vacuum energy does not contribute to gravity, or by using some supersymmetric variants of such theories.

1.3 Vacuum Fluctuations

The fluctuation in the quantum field can be written as,

$$\delta\phi_{\mathbf{k}} = \sqrt{\langle |\phi_{\mathbf{k}}|^2 \rangle - \langle \phi_{\mathbf{k}} \rangle^2} = \sqrt{\langle |\phi_{\mathbf{k}}|^2 \rangle}$$

We know that

$$\phi_{\mathbf{k}} = \frac{a_{\mathbf{k}} + a_{-\mathbf{k}}}{\sqrt{2\omega_k}}$$

which means that

$$|\phi_{\mathbf{k}}^2| = \frac{(a_{\mathbf{k}} + a_{-\mathbf{k}})(a_{\mathbf{k}} + a_{-\mathbf{k}})}{2\omega_k}$$

Taking the ground state expectation value of this expression, one obtains that $\delta\phi_{\mathbf{k}} \sim \omega_k^{-1/2}$. What if we measure the average value of a field over space? Lets consider a cubical box of side L and define the average value ϕ_L as follows,

$$\phi_L = \frac{1}{L^3} \int_{-L/2}^{-L/2} dx \int_{-L/2}^{-L/2} dy \int_{-L/2}^{-L/2} dz \phi(\mathbf{x})$$

We again calculate fluctuations in this average value by the formula $\delta\phi_L = \sqrt{\langle\phi_L^2\rangle}$.

$$\begin{aligned} \phi_L &\sim \frac{1}{L^3} \int_{-L/2}^{-L/2} dx \int_{-L/2}^{-L/2} dy \int_{-L/2}^{-L/2} dz \int \phi_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}} d^3\mathbf{k} \\ &\sim \frac{1}{L^3} \int \frac{1}{k_x k_y k_z} \sin \frac{k_x L}{2} \sin \frac{k_y L}{2} \sin \frac{k_z L}{2} \phi_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}} d^3\mathbf{k} \end{aligned}$$

Let us say that $f_x = \frac{1}{k_x} \sin \frac{k_x L}{2}$ and so forth for y and z . $f_x \rightarrow L/2$ for small k_x and 0 for large k_x . So the contribution to the integral can be taken to be from small k , and hence is of the order L^3 . So $\delta\phi_L = \sqrt{\langle\phi_L^2\rangle} \sim [(\delta\phi_{\mathbf{k}})^2/L^3]^{1/2}$.

1.4 Gravity Can Create Particles?

Consider a single harmonic oscillator with the following features.

$$\underbrace{\ddot{q}(t) + \omega^2 q = 0}_{t < 0 \text{ and } t > T} ; \underbrace{\ddot{q}(t) - \Omega^2 q = 0}_{0 < t < T}$$

The solution, obviously, is

$$q(t) = \underbrace{q_1 \sin(\omega t)}_{t < 0 \text{ (assume)}} ; \underbrace{Ae^{\Omega t} + Be^{-\Omega t}}_{0 < t < T} ; \underbrace{q_2 \sin(\omega t + \alpha)}_{t > T}$$

Matching $q(t)$ and $\dot{q}(t)$ at $t = 0, T$, we get the condition,

$$\tan(\omega T + \alpha) = \frac{\omega}{\Omega} ; q_2 \sin(\omega T + \alpha) = Ae^{\Omega T} ; A = \frac{q_1}{2} \frac{\omega}{\Omega}$$

Hence,

$$q_2 \approx \frac{1}{2} q_1 \sqrt{1 + \frac{\omega^2}{\Omega^2}} e^{\Omega T}$$

We see that the final state has a much larger energy as compared to the initial state, which we, in turn, interpret as the creation of many particles the time interval $[0, T]$. Can

we look at how many particles are produced, approximately? The exact relation for the amplitudes (valid at all times) is

$$q_2 = q_1 \sqrt{1 + \frac{\omega^2}{\Omega^2}} \sinh \Omega_0 T$$

The oscillator energies $\propto q^2$. If we take the initial state to be the ground state, it is straightforward to see that the number of particles n produced is

$$n = \frac{q_2^2}{q_1^2} = \left(1 + \frac{\omega^2}{\Omega^2}\right) \sinh^2 \Omega_0 T$$

Hmm, can something of this sort happen in gravity? Let's hope.