Tutorial 1: ICTS Summer School

June 6, 2019

1. If ξ^a is a Killing vector in the space time and u^a is the tangent to a geodesic, prove that $\xi \, . \, u$ is a constant along the geodesic. As an illustration, consider a spherically symmetric and static space time which is described by the metric:

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\Omega^2 \tag{1}$$

This metric has following two Killing vectors (there are other two also),

$$\xi^a_{(t)} = \frac{\partial x^a}{\partial t}; \; \xi^a_{(\phi)} = \frac{\partial x^a}{\partial \phi} \tag{2}$$

Then, interpret the quantities $\xi_{(t)} \, . \, u$ and $\xi_{(\phi)} \, . \, u$ physically.

If ξ^a is a Killing vector of the space time and T^{ab} is the matter energy momentum tensor, prove that $J^a = T^{ab}\xi_b$ is a conserved current. Interpret J^a when ξ^a is a time like Killing vector.

2. Find the number of independent components of R_{abcd} in a general D dimensional space time.

Prove that $R_{abcd} = 0$ is the necessary as well as sufficient condition for space time to be flat.

3. Consider Einstein's equation:

$$R^{ab} - \frac{1}{2} R \, g^{ab} = 8\pi G \, T^{ab} \tag{3}$$

We expect these equations to be second order non linear partial differential equations. Show that (using Bianchi identity) that the following four Einstein's equations $G^{0a} = 8\pi \, G \, T^{0a}$ contain only first-order time derivatives. As a result, these four equations are not true dynamical equations; these are constraint equations which imply that the initial data can not be chosen arbitrarily.

4. Consider the four velocity u^a of a static observer, i.e. the observer hovering at fixed values of (r, θ, ϕ) in a Schwarzschild space time. Calculate the four acceleration $a^m = u^n \nabla_n u^m$ of such an observer. Show the the norm of the four acceleration $a(r) = (g_{mn} a^m a^n)^{1/2}$ diverges at the horizon.

You may note that we can define a finite quantity on the horizon as $\left(1 - \frac{2m}{r}\right) a(r)$. This is called the surface gravity of the horizon.

5. Show that if the rocket ship crosses the Schwarzschild radius, it will reach $r = 0$ in a finite proper time $\tau \leq \pi M$, no matter how the engines are fired.