Fluid Mechanics: Assignment #4

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Acknowledgements -

Problem 1

In ideal 2D flow, $\boldsymbol{\nabla} \cdot \vec{\mathbf{u}} = 0$. This means,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \implies u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}$$

Hence, $\boldsymbol{\nabla}\psi = \frac{\partial \psi}{\partial x}\hat{\mathbf{x}} + \frac{\partial \psi}{\partial y}\hat{\mathbf{y}} = -v\hat{\mathbf{x}} + u\hat{\mathbf{y}}$. By definition $\vec{\mathbf{u}} = \boldsymbol{\nabla}\phi = u\hat{\mathbf{x}} + v\hat{\mathbf{y}}$. Now we can do the calculations required in the problem,

- $\boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla}\phi = (-v\hat{\mathbf{x}} + u\hat{\mathbf{y}}) \cdot (u\hat{\mathbf{x}} + v\hat{\mathbf{y}}) = -vu + uv = 0.$
- $-\boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\phi = -(-v\hat{\mathbf{x}} + u\hat{\mathbf{y}}) \times (u\hat{\mathbf{x}} + v\hat{\mathbf{y}}) = -(-v^2 - u^2)\hat{\mathbf{z}} = |\vec{\mathbf{u}}|^2\hat{\mathbf{z}}$
- $|\boldsymbol{\nabla}\psi|^2 = u^2 + v^2 \quad \text{and} \quad |\boldsymbol{\nabla}\phi|^2 = u^2 + v^2 \implies |\boldsymbol{\nabla}\psi|^2 = |\boldsymbol{\nabla}\phi|^2$
- $-\hat{\mathbf{z}} \times \boldsymbol{\nabla}\psi = -\hat{\mathbf{z}} \times (-v\hat{\mathbf{x}} + u\hat{\mathbf{y}}) = u\hat{\mathbf{x}} + v\hat{\mathbf{y}} = \boldsymbol{\nabla}\phi$

Problem 2

- For point source, $\vec{\mathbf{u}} = \dfrac{q_s}{2\pi r}\hat{\mathbf{r}}$, where q_s is the source strength. In spherical polar coordinates, $\vec{\mathbf{u}} = \frac{\partial \phi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\hat{\theta}$. This means

$$\phi = \int \frac{\partial \phi}{\partial r} dr + \int \frac{1}{r}\frac{\partial \phi}{\partial \theta} d\theta = \frac{q_s}{2\pi} \ln r + constant$$

  For lines of constant ϕ,

$$\begin{aligned} \frac{q_s}{2\pi} \ln r &= C \\ \implies \frac{q_s}{2\pi r}\frac{\mathrm{d}r}{\mathrm{d}x} &= 0 \\ \implies \frac{q_s}{2\pi r^2}\left(2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x}\right) &= 0 \implies \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y} = m \end{aligned}$$

  As velocity is radial, the streamlines are also radial straight lines passing through the origin. The slope of such straight lines is $\dfrac{y}{x} = -\dfrac{1}{m}$. Hence the streamlines and lines of constant ψ are perpendicular.

- For point vortex, $\vec{\mathbf{u}} = \dfrac{\Gamma}{2\pi r}\hat{\theta}$. This means,

$$\phi = \int \frac{\partial \phi}{\partial r} dr + \int \frac{1}{r}\frac{\partial \phi}{\partial \theta} d\theta = \frac{\Gamma\theta}{2\pi r} + constant$$

For lines of constant ϕ,

$$\begin{aligned}
\frac{\Gamma\theta}{2\pi r} &= C_1 \\
\implies r\frac{\mathrm{d}\theta}{\mathrm{d}x} - \theta\frac{\mathrm{d}r}{\mathrm{d}x} &= 0 \\
\implies \frac{1}{r}\left(x\frac{\mathrm{d}y}{\mathrm{d}x} - y\right) - \frac{\theta}{r}\left(x + y\frac{\mathrm{d}y}{\mathrm{d}x}\right) &= 0 \\
\implies \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{y + \theta x}{x - \theta y}
\end{aligned}$$

where $\theta = \tan^{-1}\dfrac{y}{x}$

Problem 3

Given $A = \begin{bmatrix} -1 & p \\ 0 & -2 \end{bmatrix}$. The eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = -2$ with corresponding (normalized) eigenvectors are $v_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ and $v_2 = \begin{bmatrix} \dfrac{-p}{\sqrt{1+p^2}} & \dfrac{1}{\sqrt{1+p^2}} \end{bmatrix}^T$. So, the resultant vector is,

$$\begin{aligned}
v &= v_1 e^{\lambda_1 t} + v_2 e^{\lambda_2 t} \\
&= \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t} + \frac{1}{\sqrt{1+p^2}} \begin{bmatrix} -p \\ 1 \end{bmatrix} e^{-2t} \\
v &= \begin{bmatrix} e^{-t} - \frac{p}{\sqrt{1+p^2}} e^{-2t} \\ \frac{1}{\sqrt{1+p^2}} e^{-2t} \end{bmatrix} \\
|v|^2 &= \left(e^{-t} - \frac{p}{\sqrt{1+p^2}} e^{-2t}\right)^2 + \left(\frac{1}{\sqrt{1+p^2}} e^{-2t}\right)^2 \\
&= e^{-2t} + e^{-4t} - \frac{2p}{\sqrt{1+p^2}} e^{-3t} \\
\frac{\mathrm{d}|v|^2}{\mathrm{d}t} &= -2e^{-2t} - 4e^{-4t} + \frac{6p}{\sqrt{1+p^2}} e^{-3t}
\end{aligned}$$

For the resultant to grow, $\frac{\mathrm{d}|v|^2}{\mathrm{d}t} > 0$.

$$\begin{aligned}
\implies -2e^{-2t} - 4e^{-4t} + \frac{6p}{\sqrt{1+p^2}} e^{-3t} &> 0 \\
\implies \frac{p}{\sqrt{1+p^2}} &> \frac{e^{t} + 2e^{-t}}{3}
\end{aligned}$$

Problem 3 continued on next page. . .

The function $\dfrac{e^t + 2e^{-t}}{3}$ has a minimum value of $\dfrac{2\sqrt{2}}{3}$. Hence, for the resultant to grow for some finite time,

$$
\begin{aligned}
\frac{p}{\sqrt{1+p^2}} &> \frac{2\sqrt{2}}{3} \\
\implies \frac{p^2}{1+p^2} &> \frac{8}{9} \\
\implies p^2 &> 8 \\
\implies p &> 2\sqrt{2}
\end{aligned}
$$

Hence, the resultant will grow for some finite time if $p > 2\sqrt{2}$.

Problem 4