Homework 1: Dynamical systems; Instructor: Amit Apte

Due Mon 14 Jan 2017, beginning of class

T: Theoretical; N: Numerical: T/N: Both theory and numerics; marks for each question in paranthesis (please email answers to numerical questions BEFORE the beginning of the class on the due date of the homework)

1. (T, 20) The mandatory problem in any course related to ODE: show that the following problem has multiple solutions. Find as many as you can (but at least two) and sketch a plot of these solutions, i.e., plot $x(t)$ vs. t.
$$\dot{x} = x^{1/3}\,, \qquad x(0) = 0\,. \tag{1}$$
If the initial condition is changed to $x(0) = x_0 \neq 0$, does it have unique solution? If so, prove it and find the unique solution, and if not, find multiple solutions. If you try to solve the above equation numerically, what conclusions can you draw?

2. (T, 10+10+20) Find the flow of each of the following differential equations and find the maximal interval of existence as a function of the initial condition. Sketch the vector field. Plot the sketches of the solutions for a few representative values of the initial conditions. Show the "region of existence of solutions" Ω for each of these sets of equations.
   (a) $\dot{x} = x(x^2 - 1)\,, \qquad x(0) = x_0\,.$
   (b) $\dot{x} = -x^3\,, \qquad x(0) = x_0\,.$
   (c) A Hamiltonian system (It may help to first prove that the "energy" $E = x_2^2/2 + x_1^2/2 - x_1^4/4$ is conserved and use the first equation below to define an ODE for x_1 which can be integrated to find the solution, but you can use other methods to analyse the system.)
$$\dot{x}_1 = x_2\,, \qquad x_1(0) = x_{01}\,, \tag{2}$$
$$\dot{x}_2 = -x_1 + x_1^3\,, \qquad x_2(0) = x_{02}\,. \tag{3}$$

3. (N, 30) Solve the following two ODE with two "nearby" initial conditions x_{0A} and x_{0B}, for example with $\|x_{0B} - x_{0A}\| = 10^{-10}$ or a similarly small number, and find their trajectories $x_A(t)$ and $x_B(t)$. Plot $\|x_B(t) - x_A(t)\|$ vs. time. Does the distance between them grow exponentially (approximately)? Show a few illustrative plots of the solutions, of the growth of the distance (semilog plots may be clearer). The main idea is for you to explain to me what you have understood from playing around with these systems!
   (a) "Chaotic" Lorenz system with $\sigma = 10, \beta = 8/3, \rho = 28$.
$$\dot{x}_1 = \sigma(x_2 - x_1)\,, \tag{4}$$
$$\dot{x}_2 = x_1(\rho - x_3) - x_2\,, \tag{5}$$
$$\dot{x}_3 = x_1 x_2 - \beta x_3\,. \tag{6}$$
   (b) "Periodic" Lorenz system with $\sigma = 10, \beta = 8/3, \rho = 350$.

4. (T/N, 10, or 100) Is science a religion? If so, how? If not, why?

Of course, we will come back to the Lorenz system again. The above problem is just for you to get comfortable with numerical solutions of ODE. Ref: (i) about periodic orbits: `http://blogs.mathworks.com/cleve/2014/04/28/periodic-solutions-to-the-lorenz-equations/`; (ii) a detailed study: `http://web.math.ucsb.edu/~jhateley/paper/lorenz.pdf`; (iii) anything you may want to know about Lorenz system: `http://link.springer.com/book/10.1007%2F978-1-4612-5767-7`