Quantum Mechanics II
Assignment 1

Due Tuesday, 28 August 2018

Instructions:

1. Although some problems are from the textbook, some others are slight variations of problems at the end of the chapter. So, please read the problem carefully.
2. You are free to discuss this problem set with your classmates. However, you must write your own answers and not simply reproduce the answer that someone else has worked out. Please also note the names of your collaborators with on the front page of your assignment.
3. This assignment should be submitted directly to Soumyadeep Chaudhuri by the due date. Everyone is allowed exactly *one late pass*, which allows them to submit *one* assignment in the semester after the due date. But, apart from this exception, no late assignments will be accepted or graded. Please use this freedom judiciously.

Problems:

1. Show that commutators in quantum mechanics and Poisson brackets in classical mechanics both obey the *Jacobi identity.*
$$[A,[B,C]]+[C,[A,B]]+[B,[C,A]]=0.$$
$$\{A,\{B,C\}_{\text{P.B.}}\}_{\text{P.B.}}+\{C,\{A,B\}_{\text{P.B.}}\}_{\text{P.B.}}+\{B,\{C,A\}_{\text{P.B.}}\}_{\text{P.B.}}=0.$$
2. Show that
$$[AB,CD]=-AC\{D,B\}+A\{C,B\}D-C\{D,A\}B+\{C,A\}DB$$
where the anti-commutator is defined by $\{A,B\}\equiv AB+BA$.
3. Let $\vec{n}=n_x\hat{x}+n_y\hat{y}+n_z\hat{z}$. We consider the sigma matrices
$$\sigma_z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\quad\sigma_x=\begin{pmatrix}0&1\\1&0\end{pmatrix}\quad\sigma_y=\begin{pmatrix}0&-i\\i&0\end{pmatrix}$$
Now consider $\vec{\sigma}\cdot\vec{n}\equiv n_z\sigma_z+n_x\sigma_x+n_y\sigma_y$. Find the eigenvalues and eigenvectors of $\vec{\sigma}\cdot\vec{n}$.
4. Let A be an observable whose spectral decomposition is $A=\sum_i\lambda_iP_i$. What is the significance of
$$\prod_{i\neq j}\frac{A-\lambda_i}{\lambda_j-\lambda_i},$$
where note that the product runs only over i and j is held fixed.

5. Show that if F and G are functions with a regular power series expansion

$$[\hat{x}, G(\hat{p})] = iG'(\hat{p}); \quad [\hat{p}, F(x)] = -iF'(\hat{x}).$$

Now also evaluate

$$[\hat{x}^2, \hat{p}^2] = ?$$

and compare this with the result above.

6. In class, we introduced the *creation* and *annihilation* operators

$$\hat{a} = \frac{1}{\sqrt{2}} (\hat{x} + i\hat{p}) \, ; \quad \hat{a}^\dagger = \frac{1}{\sqrt{2}} (\hat{x} + i\hat{p}) \, ;$$

We also introduced the coherent states that satisfied $\hat{a}|z\rangle = z|z\rangle$ for complex z.

   (a) Find the inner-product $\langle z'|z\rangle$.
   (b) Write down a completeness relation of the form $1 = \int d^2z f(z)|z\rangle\langle z|$ and find $f(z)$.

7. We again consider the coherent states described above.

   (a) Evaluate $\langle z|\hat{x}|z\rangle$ and $\langle z|\hat{p}|z\rangle$.
   (b) Evaluate $\Delta x = \sqrt{\langle z|\hat{x}^2|z\rangle - (\langle z|\hat{x}|\rangle z)^2}$ and $\Delta p = \sqrt{\langle z|\hat{p}^2|z\rangle - (\langle z|\hat{p}|\rangle z)^2}$ and verify that they satisfy the uncertainty relation $\Delta x \Delta p \geq \frac{1}{2}$.

8. Consider a system that consists of two two-state systems. The states of each individual system are spanned by $|0\rangle$ and $|1\rangle$. We now consider a state of the combined system given by

$$|\Psi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle.$$

   (a) Find the density matrix of system 1 and the eigenvalues of this density matrix.
   (b) Find the density matrix of system 2 and the eigenvalues of this density matrix. Compare your answer with the answer for system 1 above.