Fluid Mechanics: Assignment #3

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Acknowledgements -

Problem 1

Part (a)
We first write down the unsteady state Bernoulli equation,

$$\frac{\partial \phi_1}{\partial t} + \frac{P_{atm}}{\rho} + \frac{v_1^2}{2} + gz = \frac{\partial \phi_2}{\partial t} + \frac{P_{atm}}{\rho} + \frac{v_2^2}{2}$$
$$\frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial z} + g = \frac{\partial v_2}{\partial t} + v_2 \frac{\partial v_2}{\partial z}$$

where we have taken a partial derivative with respect to y in going from the first to the second step. The continuity equation tells us,

$$A_1 v_1 = A v_2 \implies A_1 \frac{\partial v_1}{\partial t} = A \frac{\partial v_2}{\partial t}$$

If we assume, $A_1 \gg A$, From the geometry of the cone, we have,

$$\begin{aligned} r_1 &= r_2 + z \tan\alpha \\ \pi r_1^2 &= \pi r_2^2 + \pi z^2 \tan^2\alpha + 2\pi r_2 z \tan\alpha \\ A_1 &= A + \pi z^2 \tan^2\alpha + 2\sqrt{A\pi} z \tan\alpha \end{aligned}$$

From this and the continuity equation, we get,

$$\begin{aligned} v_2 &= \left(1 + \frac{\pi z^2 \tan^2\alpha}{A} + 2\sqrt{\frac{A}{\pi}} z \tan\alpha\right) v_1 \\ \frac{\partial v_2}{\partial t} &= \left(1 + \frac{\pi z^2 \tan^2\alpha}{A} + 2\sqrt{\frac{A}{\pi}} z \tan\alpha\right) \frac{\partial v_1}{\partial t} + \left(\frac{2\pi z \tan^2\alpha}{A} + 2\sqrt{\frac{A}{\pi}} \tan\alpha\right) v_1^2 \\ &\approx \left(\frac{2\pi z \tan^2\alpha}{A} + 2\sqrt{\frac{A}{\pi}} \tan\alpha\right) v_1^2 \end{aligned}$$

Part (b)
From steady state Bernoulli equation,

$$\frac{v_1^2}{2} + gz = \frac{u_1^2}{2} \quad \text{and} \quad \frac{v_2^2}{2} + gz = \frac{u_2^2}{2}$$

From continuity equation,

$$A_1 v_1 = A u_1 \quad \text{and} \quad A_2 v_2 = A u_2$$

Problem 2

We first write the Bernoulli equation for between the point where water leaves the tap ($z_1 = 0$) and a point distance h below ($z_2 = -h$),

$$\frac{P_0}{\rho} + \frac{v_1^2}{2} = \frac{P_0}{\rho} + \frac{v_2^2}{2} - gh \implies \frac{v_2^2}{v_1^2} = 1 + \frac{2gh}{v_1^2}$$

The continuity equation gives,

$$\pi r_1^2 v_1 = \pi r_2^2 v_2 \implies \frac{v_2}{v_1} = \frac{r_1^2}{r_2^2}$$

Using the above two equations, we get,

$$\frac{r_1^4}{r_2^4} = 1 + \frac{2gh}{v_1^2} \implies \boxed{\frac{R_0^4}{r^4} = 1 + \frac{2gH}{v_0^2}}$$

where r is the cross-sectional radius at height H below the tap, and R_0 and v_0 and the cross-sectional radius and velocity of the water the moment it leaves the tap.

Problem 3

We work in cylindrical coordinates. The assumption of laminar flow $\implies u_r = u_\phi = 0$. The assumption of axisymmetry $\implies u_z = u_z(r, z)$ The continuity condition $\boldsymbol{\nabla} \cdot \vec{\mathbf{u}} = 0$ gives,

$$\frac{\partial u_z}{\partial z} = 0 \implies u_z = u_z(r)$$

We now proceed and write the Navier-Stokes equation in cylindrical coordinates component-wise,

$$\begin{aligned} 0 &= -\frac{1}{\rho}\frac{\partial P}{\partial r} \\ 0 &= -\frac{1}{\rho r}\frac{\partial P}{\partial \phi} \\ 0 &= -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) \end{aligned}$$

We can see from the first two equations that $P = P(z)$. In the third equation, since the first term on the RHS depends only on z and the second term depends only on r, we say that each of the terms should be constants. We get,

$$\begin{aligned} \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u_z}{\mathrm{d}r}\right) &= \frac{1}{\mu}\frac{\mathrm{d}P}{\mathrm{d}z} = constant \\ \frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u_z}{\mathrm{d}r}\right) &= \frac{r}{\mu}\frac{\mathrm{d}P}{\mathrm{d}z} \\ \implies r\frac{\mathrm{d}u_z}{\mathrm{d}r} &= \frac{r^2}{2\mu}\frac{\mathrm{d}P}{\mathrm{d}z} + A \\ \implies \frac{\mathrm{d}u_z}{\mathrm{d}r} &= \frac{r}{2\mu}\frac{\mathrm{d}P}{\mathrm{d}z} + \frac{A}{r} \\ \implies u_z &= \frac{r^2}{4\mu}\frac{\mathrm{d}P}{\mathrm{d}z} + A\ln r + B \end{aligned}$$

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We need the flow to be well-defined at $r = 0$. As it stands, for non-zero A, the flow will not be well-defined for $r = 0$, which is undesirable. Hence, $A = 0$.
If R is the radius of the pipe, and the pipe is not moving, we get $u_z(R) = 0$, which means,

$$0 = \frac{R^2}{4\mu}\frac{\mathrm{d}P}{\mathrm{d}z} + B \implies B = -\frac{R^2}{4\mu}\frac{\mathrm{d}P}{\mathrm{d}z}$$

So the final answer is,

$$u_z = \frac{1}{4\mu}\frac{\mathrm{d}P}{\mathrm{d}z}(r^2 - R^2)$$

Problem 4