

Advanced General Relativity: Tutorial #3

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Tutorial Instructors : Aditya Vijaykumar, Md Arif Shaikh

Problem 1

Part (a)

Given,

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \quad (1)$$

Using an affine parameter λ , we can rewrite these equations as,

$$\kappa = \left(\frac{ds}{d\lambda}\right)^2 = -\left(1 - \frac{2M}{r}\right) \dot{t}^2 + \left(1 - \frac{2M}{r}\right)^{-1} \dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) \quad (2)$$

where $\kappa = -1, 0, 1$ for timelike, null and spacelike geodesics respectively. Corresponding to the Killing vectors ∂_t and ∂_ϕ , we would have the following conserved quantities,

$$\tilde{E} = 2\left(1 - \frac{2M}{r}\right) \dot{t} \quad \text{and} \quad \tilde{L} = 2r^2 \sin^2 \theta \dot{\phi} \quad (3)$$

Substituting \dot{t} and $\dot{\phi}$ from (3) to (2) and assuming planar orbit ($\theta = \pi/2$), we get,

$$\kappa = -\left(1 - \frac{2M}{r}\right) \frac{\tilde{E}^2}{4\left(1 - \frac{2M}{r}\right)^2} + \left(1 - \frac{2M}{r}\right)^{-1} \dot{r}^2 + r^2 \frac{\tilde{L}^2}{4r^4 \sin^4 \theta} \quad (4)$$

$$= -\left(1 - \frac{2M}{r}\right)^{-1} \frac{\tilde{E}^2}{4} + \left(1 - \frac{2M}{r}\right)^{-1} \dot{r}^2 + \frac{\tilde{L}^2}{4r^2} \quad (5)$$

$$\kappa - \frac{2M\kappa}{r} = -\frac{\tilde{E}^2}{4} + \dot{r}^2 + \frac{\tilde{L}^2}{4r^2} \left(1 - \frac{2M}{r}\right) \quad (6)$$

$$\therefore \frac{\dot{r}^2}{2} - \frac{1}{2} \left(\kappa + \frac{\tilde{E}^2}{4} \right) + \frac{M\kappa}{r} + \frac{\tilde{L}^2}{8r^2} - \frac{\tilde{L}^2 M}{4r^3} = 0 \quad (7)$$

Defining $E = \frac{\tilde{E}}{2}$ and $L = \frac{\tilde{L}}{2}$, we have,

$$\frac{\dot{r}^2}{2} + V_{eff}(r) = 0 \quad (8)$$

where $V_{eff}(r) = -\frac{E^2 + \kappa}{2} + \frac{M\kappa}{r} + \frac{L^2}{2r^2} - \frac{L^2 M}{r^3}$

Part (b)

Substituting $\kappa = 0$ and $\dot{r} = 0$ in (8), and further using (3) we have,

$$E^2 = \frac{L^2}{r^2} \left(1 - \frac{2M}{r}\right) \implies \left(\frac{dt}{d\phi}\right)^2 = r^2 \left(1 - \frac{2M}{r}\right)^{-1} \quad (9)$$

Now, writing down the radial geodesic for the Schwarzschild metric, we have,

$$\frac{d}{d\tau} \left[2 \left(1 - \frac{2M}{r} \right)^{-1} \dot{r} \right] = -\frac{2M}{r^2} \dot{t}^2 - \left(1 - \frac{2M}{r} \right)^{-2} \left(\frac{2M}{r^2} \right) \dot{r}^2 + 2r(\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) \quad (10)$$

Using $\dot{r} = 0$, $\theta = \pi/2$, we get,

$$\frac{2M}{r^2} \dot{t}^2 = 2r\dot{\phi}^2 \implies \left(\frac{dt}{d\phi} \right)^2 = \frac{r^3}{M} \quad (11)$$

Comparing with (9), we have,

$$r^2 \left(1 - \frac{2M}{r} \right)^{-1} = \frac{r^3}{M} \implies 1 - \frac{2M}{r} = \frac{M}{r} \implies r = 3M \quad (12)$$

Hence, there is a null circular orbit at $r = 3M$. To decide stability, one needs to check the value of $\frac{\partial^2 V_{eff}}{\partial r^2}$ at $r = r_p = 3M$,

$$\left. \frac{\partial^2 V_{eff}}{\partial r^2} \right|_{r=r_p} = \frac{3L^2}{r_p^4} - 12 \frac{L^2 M}{r_p^5} = -\frac{L^2}{(3M)^4} < 0 \quad (13)$$

Hence, this orbit is unstable.

Part (c)

Radial null geodesics will have $\kappa = 0$ and $L = 0$. Hence,

$$\dot{r}^2 = E^2 = \left(1 - \frac{2M}{r} \right)^2 \dot{t}^2 \quad (14)$$

$$\implies \frac{dr}{dt} = \pm \left(1 - \frac{2M}{r} \right) \quad (15)$$

$$a \quad (16)$$

Part (d)

Problem 2

The area of a black hole $A \propto M^2$. The area theorem hence implies,

$$M_3^2 \geq M_1^2 + M_2^2 \quad (17)$$

Consider the two numbers M_1^2 and M_2^2 . Let's write down the $AM \geq GM$ inequality *ie.*,

$$\frac{M_1^2 + M_2^2}{2} \geq M_1 M_2 \quad (18)$$

$$\implies M_1^2 + M_2^2 \geq \frac{M_1^2 + M_2^2}{2} + M_1 M_2 \quad (19)$$

$$\implies M_1^2 + M_2^2 \geq \frac{(M_1 + M_2)^2}{2} \quad (20)$$

Combining (17) and the last inequality, we have,

$$M_3^2 \geq \frac{(M_1 + M_2)^2}{2} \implies \frac{M_3}{M_1 + M_2} \geq \frac{1}{\sqrt{2}} \quad (21)$$

Hence,

$$\eta \leq 1 - \frac{1}{\sqrt{2}} \quad (22)$$