Classical Mechanics: Assignment #5

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Problem 1

Problem 2

Let $f(q_i, t)$ be a function such that $L'(q_i, \dot{q}_i, t) = L(q_i, \dot{q}_i, t) + \frac{\mathrm{d}f}{\mathrm{d}t}$. We have seen that the equations of motion remain unchanged by this addition.
Let's first calculate the canonical momenta p_i',

$$p_i' = \frac{\partial L'}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{q}_i} + \frac{\partial}{\partial \dot{q}_i}\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right) = p_i + \frac{\partial}{\partial \dot{q}_i}\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right)$$

Let's calculate the Hamiltonian $H'(q_i, p_i, t)$,

$$\begin{aligned}
H'(q_i, p_i, t) &= \sum_i \left(p_i + \frac{\partial}{\partial \dot{q}_i}\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right)\right)\dot{q}_i - L' \\
&= \sum_i p_i \dot{q}_i - L + \sum_i \dot{q}_i \frac{\partial}{\partial \dot{q}_i}\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right) - \frac{\mathrm{d}f(q_i, t)}{\mathrm{d}t} \\
H'(q_i, p_i, t) &= H(q_i, p_i, t) + \sum_i \dot{q}_i \frac{\partial}{\partial \dot{q}_i}\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right) - \frac{\mathrm{d}f(q_i, t)}{\mathrm{d}t}
\end{aligned}$$

The equations of motion for H' are,

$$\frac{\partial H'}{\partial q_j} = \frac{\partial H}{\partial q_i} + \sum_i \dot{q}_j \frac{\partial^2}{\partial \dot{q}_i \partial q_i}\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right)$$

Part (b)
Given that,

$$\begin{aligned}
L &= -mc^2\sqrt{1 - \frac{v^2}{c^2}} - e\phi + \frac{e}{c}\vec{\mathbf{v}} \cdot \vec{\mathbf{A}} \\
&= -mc^2\sqrt{1 - \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{c^2}} - e\phi + \frac{e}{c}(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z) \\
&= -\frac{m(c^2 - \dot{x}^2 + \dot{y}^2 + \dot{z}^2)}{\sqrt{1 - \frac{v^2}{c^2}}} - e\phi + \frac{e}{c}(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z)
\end{aligned}$$

We first write the canonical momenta,

$$p_x = \frac{\partial L}{\partial \dot{x}} = -mc^2 \frac{\left(\frac{-2\dot{x}}{c^2}\right)}{2\sqrt{1 - \frac{v^2}{c^2}}} + \frac{e}{c}A_x = \frac{m\dot{x}}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{e}{c}A_x$$

Similarly,

$$p_y = \frac{m\dot{y}}{\sqrt{1 - \dfrac{v^2}{c^2}}} + \frac{e}{c}A_y \quad \text{and} \quad p_z = \frac{m\dot{z}}{\sqrt{1 - \dfrac{v^2}{c^2}}} + \frac{e}{c}A_z$$

From the formulae of the canonical momenta, one can see that,

$$\begin{aligned}
H &= p_x\dot{x} + p_y\dot{y} + p_z\dot{z} - L \\
&= \frac{m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)}{\sqrt{1 - \dfrac{v^2}{c^2}}} + \frac{e}{c}(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z) + \frac{m(c^2 - \dot{x}^2 + \dot{y}^2 + \dot{z}^2)}{\sqrt{1 - \dfrac{v^2}{c^2}}} + e\phi - \frac{e}{c}(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z) \\
H &= \frac{mc^2}{\sqrt{1 - \dfrac{v^2}{c^2}}} + e\phi = T + V
\end{aligned}$$

Problem 3

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Problem 4

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