Classical Mechanics: Assignment #3

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Problem 1

Problem 2

Part (b)
The Lagrangian is given as,

$$L = e^{\gamma t}\left(\frac{m\dot{q}^2}{2} - \frac{kq^2}{2}\right)$$

Writing down the equations of motion for hte generalized coordinate q,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\gamma t} m\dot{q}\right) &= -e^{\gamma t} kq \\
\implies e^{\gamma t}(\gamma m\dot{q} + m\ddot{q}) &= -e^{\gamma t} kq \\
\implies \ddot{q} + \gamma\dot{q} + \frac{k}{m}q &= 0
\end{aligned}$$

This is the equation of motion for a damped harmonic oscillator.
Let's perform the transformation $s = e^{\gamma t} q \implies \dot{s} = e^{\gamma t}(\gamma q + \dot{q}) = \gamma s + e^{\gamma t}\dot{q}$. Inverting these, we have the following,

$$\begin{aligned}
q &= e^{-\gamma t} s \\
\dot{q} &= e^{-\gamma t}(\dot{s} - \gamma s)
\end{aligned}$$

Substituting this back into the expression for L,

$$L = e^{-\gamma t}\left(\frac{m\dot{s}^2}{2} + \frac{(m\gamma^2 - k)s^2}{2} - m\gamma s\dot{s}\right)$$

Writing the equations of motion for s,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-\gamma t}(m\dot{s} - m\gamma s)\right) &= -e^{-\gamma t}((k - m\gamma^2)s - m\gamma\dot{s}) \\
m\ddot{s} - m\gamma\dot{s} - \gamma(m\dot{s} - m\gamma s) &= (k - m\gamma^2)s - m\gamma\dot{s} \\
\ddot{s} - \gamma\dot{s} + \left(2\gamma^2 - \frac{k}{m}\right)s &= 0
\end{aligned}$$

Problem 3

Part (b)
Given, $L = L(q_i, \dot{q}_i, \ddot{q}_i, t)$, and we know that $S = \int_{t_i}^{t_f} L(q_i, \dot{q}_i, \ddot{q}_i, t)dt$. Variation of the action can be written as,

$$
\begin{aligned}
\delta S &= \int_{t_i}^{t_f} \delta L dt = 0 \\
&= \int_{t_i}^{t_f} \sum_i \left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i + \frac{\partial L}{\partial \ddot{q}_i}\delta \ddot{q}_i \right) dt \\
&= \int_{t_i}^{t_f} \sum_i \left(\frac{\partial L}{\partial q_i}\delta q_i + \left(\frac{\partial L}{\partial \dot{q}_i} - \frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial L}{\partial \ddot{q}_i} \right)\delta \dot{q}_i + \frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial L}{\partial \ddot{q}_i}\delta \dot{q}_i \right) \right) dt \\
&= \int_{t_i}^{t_f} \sum_i \left[\left\{ \frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial L}{\partial \dot{q}_i} - \frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial L}{\partial \ddot{q}_i} \right) \right\}\delta q_i + \frac{\mathrm{d}}{\mathrm{dt}}\left(\left(\frac{\partial L}{\partial \dot{q}_i} - \frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial L}{\partial \ddot{q}_i} \right)\delta q_i \right) + \frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial L}{\partial \ddot{q}_i}\delta \dot{q}_i \right) \right] dt
\end{aligned}
$$

As the variation of q_i and $\dot{q}_i$ at the endpoints is zero, the total derivative terms vanish. Accounting for the fact that all q_i's are independent, one can write the equation of motion as,

$$
\boxed{\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial L}{\partial \dot{q}_i} + \frac{\mathrm{d}^2}{\mathrm{dt}^2}\frac{\partial L}{\partial \ddot{q}_i} = 0}
$$

Taking $L = -\frac{m}{2}q\ddot{q} - \frac{k}{2}q^2$, we can write,

$$
-kq + \frac{m}{2}\ddot{q} = 0 \implies \ddot{q} - \frac{2k}{m}q = 0
$$

Problem 4

Problem 5

Problem 6

Problem 7

The radius of the circle r and the angle covered around the circle θ are the generalized coordinates. The Lagrangian L can be written as,

$$
L = \frac{m\dot{r}^2}{2} + \frac{m\dot{\theta}^2 r^2}{2} - mgr\cot\alpha
$$

The equations of motion are,

$$
\begin{aligned}
r^2\dot{\theta} &= constant = L_0 \\
\ddot{r} = r\dot{\theta}^2 - g\cot\alpha &\implies \ddot{r} = \frac{L_0^2}{r^3} - g\cot\alpha
\end{aligned}
$$

Part (b)
If $r = r_0$, $\ddot{r} = 0$ and,

$$
L_0^2 = r_0^4\omega^2 = gr_0^3\cot\alpha \implies \boxed{\omega = \sqrt{\frac{g\cot\alpha}{r_0}}} \implies L_0 = r_0^3 g\cot\alpha
$$

Problem 7 continued on next page...

Part (c)
Substituting $r = r_0 + \epsilon x$, $\epsilon \ll 1$ into the equation of motion for r,

$$\begin{aligned}
\epsilon \ddot{x} &= \frac{L_0^2}{(r_0 + \epsilon x)^3} - g \cot \alpha \\
&= \frac{L_0^2}{r_0^3}\left(1 - \frac{3\epsilon x}{r_0} + \dots\right) - g \cot \alpha \\
&= \frac{L_0^2}{r_0^3}\left(- \frac{3\epsilon x}{r_0} + \dots\right)
\end{aligned}$$

Choosing only the term first order in ϵ,

$$\ddot{x} = -\frac{3g \cot \alpha}{r_0} x \implies \boxed{\Omega = \sqrt{\frac{3g \cot \alpha}{r_0}}}$$

Check if this is really correct

Problem 8

Let θ_1 and θ_2 be the angles that the sticks make with the vertical. Each stick is of length $2l$. One can write the Lagrangian L of the system as follows (considering moments of inertia about the joint)