Electromagnetism: Pset #3

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Problem 1

Consider, with $\psi_G^R(r) = \dfrac{1}{(2\pi)^{d/2} R^d} \exp\left(-\dfrac{r^2}{2R^2}\right)$,

$$\frac{1}{R}\frac{\partial \psi_G^R}{\partial R} = \frac{1}{R}\frac{1}{(2\pi)^{d/2} R^d}\exp\left(-\frac{r^2}{2R^2}\right)\left[\frac{-d}{R} - \frac{-2r^2}{2R^3}\right]$$

$$\frac{1}{R}\frac{\partial \psi_G^R}{\partial R} = \frac{\psi_G^R(r)}{R^2}\left[-d + \frac{r^2}{R^2}\right]$$

Now consider,

$$\nabla^2 \psi_G^R = \frac{1}{r^{d-1}}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial \psi_G^R}{\partial r}\right)$$

$$= \frac{\partial^2 \psi_G^R}{\partial r^2} + \frac{d-1}{r}\frac{\partial \psi_G^R}{\partial r}$$

$$= -\frac{\psi_G^R}{R^2} + \frac{r^2}{R^4}\psi_G^R + \frac{d-1}{r}\left(\frac{-r}{R^2}\right)\psi_G^R$$

$$\nabla^2 \psi_G^R = \frac{\psi_G^R(r)}{R^2}\left[-d + \frac{r^2}{R^2}\right] = \frac{1}{R}\frac{\partial \psi_G^R}{\partial R}$$

Consider now Green's vector field,

$$\vec{\mathbf{G}}[\psi_G^R, \phi_\lambda] = \psi_G^R \vec{\boldsymbol{\nabla}}\phi_\lambda - \phi_\lambda \vec{\boldsymbol{\nabla}}\psi_G^R$$

$$\implies \vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{G}} = \psi_G^R \nabla^2\phi_\lambda - \phi_\lambda \nabla^2\psi_G^R$$

$$\int \mathrm{d}^d\vec{\mathbf{r}}(\vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{G}}) = \int \mathrm{d}^d\vec{\mathbf{r}}(\psi_G^R\nabla^2\phi_\lambda - \phi_\lambda\nabla^2\psi_G^R)$$

$$\int_{S_{d-1}} \vec{\mathbf{G}}\cdot\mathrm{d}\vec{\mathbf{a}} = \int \mathrm{d}^d\vec{\mathbf{r}}(\psi_G^R\nabla^2\phi_\lambda - \phi_\lambda\nabla^2\psi_G^R)$$

$$\implies 0 = \int \mathrm{d}^d\vec{\mathbf{r}}(\psi_G^R\nabla^2\phi_\lambda - \phi_\lambda\nabla^2\psi_G^R)$$

From the above manipulations, we can see that,

$$\int d^d\vec{\mathbf{r}}(\psi_G^R \nabla^2 \phi_\lambda - \phi_\lambda \nabla^2 \psi_G^R) = 0$$

$$\int d^d\vec{\mathbf{r}}\left(\psi_G^R \lambda \phi_\lambda - \phi_\lambda \frac{1}{R}\frac{\partial \psi_G^R}{\partial R}\right) = 0$$

$$\implies \lambda \int d^d\vec{\mathbf{r}}(\psi_G^R \phi_\lambda) = \frac{1}{R}\frac{\partial}{\partial R}\int d^d\vec{\mathbf{r}}\phi_\lambda \psi_G^R$$

$$\implies \int d^d\vec{\mathbf{r}}\phi_\lambda \psi_G^R = C\exp\left(\frac{\lambda R^2}{2}\right)$$

Good, so now we have got an expression for the gaussian average. All that is left is to figure out the parameter C. For this we note that the Gaussian distribution approaches a Dirac delta function as $R \to 0$, and then write,

$$\int d^d\vec{\mathbf{r}}\phi_\lambda \lim_{R\to 0}\psi_G^R(\vec{\mathbf{r}} - \vec{\mathbf{r}}_0) = C\lim_{R\to 0}\exp\left(\frac{\lambda R^2}{2}\right)$$

$$\int d^d\vec{\mathbf{r}}\phi_\lambda \delta(\vec{\mathbf{r}} - \vec{\mathbf{r}}_0) = C$$

$$\implies C = \phi_\lambda(\vec{\mathbf{r}}_0)$$

$$\implies \boxed{\int d^d\vec{\mathbf{r}}\; \phi_\lambda(\vec{\mathbf{r}})\psi_G^R(\vec{\mathbf{r}} - \vec{\mathbf{r}}_0) = \phi_\lambda(\vec{\mathbf{r}}_0)\exp\left(\frac{\lambda R^2}{2}\right)}$$

Part (b)

$$\implies \int d^d\vec{\mathbf{r}}\; \phi_\lambda(\vec{\mathbf{r}})\frac{1}{(2\pi)^{d/2}R^d}\exp\left(-\frac{|\vec{\mathbf{r}} - \vec{\mathbf{r}}_0|^2}{2R^2}\right) = \phi_\lambda(\vec{\mathbf{r}}_0)\exp\left(\frac{\lambda R^2}{2}\right)$$

$$\implies \int d^d\vec{\mathbf{r}}\; \phi_\lambda(\vec{\mathbf{r}})\frac{1}{(2\pi)^{d/2}R^d}\exp\left(-\frac{r^2}{2R^2}\right)\exp\left(-\frac{r_0^2}{2R^2}\right)\exp\left(\frac{\vec{\mathbf{r}}_0 \cdot \vec{\mathbf{r}}}{R^2}\right) = \phi_\lambda(\vec{\mathbf{r}}_0)\exp\left(\frac{\lambda R^2}{2}\right)$$

$$\implies \int d^d\vec{\mathbf{r}}\; \phi_\lambda(\vec{\mathbf{r}})\psi_G^R(\vec{\mathbf{r}})\exp\left(-\frac{r_0^2}{2R^2}\right)\exp\left(\frac{\vec{\mathbf{r}}_0 \cdot \vec{\mathbf{r}}}{R^2}\right) = \phi_\lambda(\vec{\mathbf{r}}_0)\exp\left(\frac{\lambda R^2}{2}\right)$$

Taking averages over $\vec{\mathbf{r}}_0$ and noting that $\kappa = \dfrac{r}{R^2}$,

$$\int d^d\vec{\mathbf{r}}\; \phi_\lambda(\vec{\mathbf{r}})\psi_G^R(\vec{\mathbf{r}})\exp\left(-\frac{r_0^2}{2R^2}\right)I_0\left(d;\frac{r}{R^2}r_0\right) = \langle\phi_\lambda(\vec{\mathbf{r}}_0)\rangle\exp\left(\frac{\lambda R^2}{2}\right)$$

Now, let's make the substitution, $r_0 = \kappa R^2$ and $\phi_\lambda(r) = J_0(d;kr) \implies \lambda = -k^2$

$$\int d^d\vec{\mathbf{r}}\; J_0(d;kr)\psi_G^R(\vec{\mathbf{r}})\exp\left(-\frac{\kappa^2 R^2}{2}\right)I_0(d;\kappa r) = \langle J_0(d;kr_0)\rangle\exp\left(-\frac{k^2 R^2}{2}\right)$$

$$\int d^d\vec{\mathbf{r}}\; J_0(d;kr)I_0(d;\kappa r)\psi_G^R(\vec{\mathbf{r}}) = J_0(d;k\kappa R^2)\exp\left(\frac{(\kappa^2 - k^2)R^2}{2}\right)$$

Similarly, taking $\phi_\lambda(r) = I_0(d;\kappa' r) \implies \lambda = \kappa'^2$,

$$\int d^d\vec{\mathbf{r}}\; I_0(d;\kappa' r)\psi_G^R(\vec{\mathbf{r}})\exp\left(-\frac{\kappa^2 R^2}{2}\right)I_0(d;\kappa r) = \langle I_0(d;\kappa' r_0)\rangle\exp\left(\frac{\kappa'^2 R^2}{2}\right)$$

$$\int d^d\vec{\mathbf{r}}\; I_0(d;\kappa' r)I_0(d;\kappa r)\psi_G^R(\vec{\mathbf{r}}) = I_0(d;\kappa'\kappa R^2)\exp\left(\frac{(\kappa^2 + \kappa'^2)R^2}{2}\right)$$

Now, let's make the substitution, $r_0 = ik_2 R^2$ and $\phi_\lambda(r) = J_0(d; k_1 r) \implies \lambda = -k_1^2$,

$$\int d^d \vec{r}\, J_0(d; k_1 r) \psi_G^R(\vec{r}) \exp\left(\frac{k_2^2 R^2}{2}\right) I_0(d; ik_2 r) = \langle J_0(d; ik_1 k_2 R^2)\rangle \exp\left(-\frac{k_1^2 R^2}{2}\right)$$

$$\int d^d \vec{r}\, J_0(d; k_1 r) J_0(d; k_2 r) \psi_G^R(\vec{r}) = I_0(d; k_1 k_2 R^2) \exp\left(-\frac{(k_1^2 + k_2^2) R^2}{2}\right)$$

Part (c)

$$\int d^d \vec{r} J_0(d; kr) g(r) f(r) = \int d^d \vec{r} J_0(d; kr) g(r) \int \frac{d^d k}{(2\pi)^d} J_0(d; k'r) \bar{f}(k)$$

$$= \int d^d \vec{r} J_0(d; kr) \int \frac{d^d(\vec{k} - \vec{k}')}{(2\pi)^d} J_0\left(d; \left|\vec{k} - \vec{k}'\right| r\right) \bar{g}\left(\left|\vec{k} - \vec{k}'\right|\right) \int \frac{d^d k}{(2\pi)^d} J_0(d; k'r) \bar{f}(k)$$

Part (d)
Consider,

$$I_1 = \int_0^\infty \frac{dR}{R^{2\Delta+1}} \exp\left(-\frac{r^2}{2R^2}\right)$$

Substitute $R^2 = \frac{r^2}{2u} \implies du = -\frac{r^2}{R^3} dR \implies dR = -\frac{r\, du}{(2u)^{3/2}}$,

$$I_1 = \int_0^\infty \frac{r\, du}{(2u)^{3/2}} \left(\frac{2u}{r^2}\right)^{\Delta+1/2} \exp(-u)$$

$$= \frac{2^{\Delta-1}}{r^{2\Delta}} \int_0^\infty du\, u^{\Delta-1} \exp(-u)$$

$$I_1 = \frac{2^{\Delta-1}}{r^{2\Delta}} \Gamma(\Delta) \quad ; \qquad \text{for} \quad \Delta > 0$$

Now consider,

$$I_2 = \int_0^\infty \frac{dR}{R^{2\Delta+1}} (2\pi)^{d/2} R^d \exp\left(-\frac{k^2 R^2}{2}\right)$$

Substitute $\frac{k^2 R^2}{2} = u \implies du = k^2 R\, dR\,, R = \left(\frac{2u}{k^2}\right)^{1/2}$,

$$I_2 = \int_0^\infty \frac{du}{k^2} \frac{1}{R^{2\Delta+2}} (2\pi)^{d/2} \left(\frac{2u}{k^2}\right)^{d/2} \exp(-u)$$

$$= \int_0^\infty \frac{du}{k^2} \left(\frac{k^2}{2u}\right)^{\Delta+1} (2\pi)^{d/2} \left(\frac{2u}{k^2}\right)^{d/2} \exp(-u)$$

$$= \frac{2^{d-\Delta-1} \pi^{d/2}}{k^{2\Delta-d}} \int_0^\infty du\, u^{d/2-\Delta-1} \exp(-u)$$

$$I_2 = \frac{2^{d-\Delta-1} \pi^{d/2}}{k^{2\Delta-d}} \Gamma(d/2 - \Delta) \quad ; \qquad \text{for} \quad \frac{d}{2} > \Delta$$

From Gaussian averaging, we know that,

$$\int \mathrm{d}^d \vec{\mathbf{r}}\ \phi_{-k^2}(\vec{\mathbf{r}}) \psi_G^R(\vec{\mathbf{r}} - \vec{\mathbf{r}}_0) = \phi_{-k^2}(\vec{\mathbf{r}}_0) \exp\left(-\frac{k^2 R^2}{2}\right)$$

$$\int \mathrm{d}^d \vec{\mathbf{r}}\ \phi_{-k^2}(\vec{\mathbf{r}}) \int_0^\infty \frac{\mathrm{d}R}{R^{2\Delta+1}} (2\pi)^{d/2} R^d \psi_G^R(\vec{\mathbf{r}} - \vec{\mathbf{r}}_0) = \phi_{-k^2}(\vec{\mathbf{r}}_0) \int_0^\infty \frac{\mathrm{d}R}{R^{2\Delta+1}} (2\pi)^{d/2} R^d \exp\left(-\frac{k^2 R^2}{2}\right)$$

$$\int \mathrm{d}^d \vec{\mathbf{r}}\ \phi_{-k^2}(\vec{\mathbf{r}}) \frac{2^{\Delta-1}}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}_0|^{2\Delta}} \Gamma(\Delta) = \phi_{-k^2}(\vec{\mathbf{r}}_0) \frac{2^{d-\Delta-1} \pi^{d/2}}{k^{2\Delta-d}} \Gamma(d/2 - \Delta)$$

$$\implies \int \mathrm{d}^d \vec{\mathbf{r}}\ \phi_{-k^2}(\vec{\mathbf{r}}) \frac{\Gamma(\Delta)}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}_0|^{2\Delta}} = \phi_{-k^2}(\vec{\mathbf{r}}_0) \frac{2^{d-2\Delta} \pi^{d/2}}{k^{2\Delta-d}} \Gamma(d/2 - \Delta)$$

Problem 2

Part (a)
We know,

$$I_0(d; x) = \sum_{m=0}^\infty \frac{\Gamma(d/2)}{m!\ \Gamma(d/2 + m)} \frac{x^{2m}}{2^{2m}}$$

$$\frac{x^2}{d^2} I_0(d+2; x) = \sum_{m=0}^\infty \frac{\Gamma(d/2+1)}{m!\ \Gamma(d/2 + m + 1)} \frac{x^{2m+2}}{2^{2m}}$$

$$= \sum_{m=0}^\infty \frac{\Gamma(d/2) \cdot d/2}{m!\ \Gamma(d/2 + m + 1)} \frac{x^{2m+2}}{2^{2m+2}} \frac{2^2}{d^2}$$

$$\frac{x^2}{d^2} I_0(d+2; x) = \frac{2}{d} \sum_{m=0}^\infty \frac{\Gamma(d/2)}{m!\ \Gamma(d/2 + m + 1)} \frac{x^{2m+2}}{2^{2m+2}}$$

Let's proceed and take the derivative,

$$\frac{x}{d} \frac{\mathrm{d}}{\mathrm{d}x} I_0(d; x) = \frac{x}{d} \sum_{m=0}^\infty \frac{\Gamma(d/2)}{m!\ \Gamma(d/2 + m)} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^{2m}}{2^{2m}}\right)$$

$$= \frac{1}{d} \sum_{m=1}^\infty \frac{\Gamma(d/2)}{m!\ \Gamma(d/2 + m)} \frac{(2m)x^{2m}}{2^{2m}}$$

$$= \frac{1}{d} \sum_{m=0}^\infty \frac{\Gamma(d/2)}{(m+1)!\ \Gamma(d/2 + m + 1)} \frac{2(m+1)x^{2m+2}}{2^{2m+2}} \quad \impliedby \quad (m \to m+1)$$

$$\frac{x}{d} \frac{\mathrm{d}}{\mathrm{d}x} I_0(d; x) = \frac{2}{d} \sum_{m=0}^\infty \frac{\Gamma(d/2)}{m!\ \Gamma(d/2 + m + 1)} \frac{x^{2m+2}}{2^{2m+2}}$$

4

Let's now consider the third part,

$$I_0(d-2;x) - I_0(d,x) = \sum_{m=0}^{\infty} \left[\frac{\Gamma(d/2-1)}{\Gamma(d/2+m-1)} - \frac{\Gamma(d/2)}{\Gamma(d/2+m)} \right] \frac{x^{2m}}{m!\, 2^{2m}}$$

$$= \sum_{m=1}^{\infty} \left[\frac{\Gamma(d/2-1)}{\Gamma(d/2+m-1)} - \frac{\Gamma(d/2)}{\Gamma(d/2+m)} \right] \frac{x^{2m}}{m!\, 2^{2m}}$$

$$= \sum_{m=0}^{\infty} \left[\frac{\Gamma(d/2-1)}{\Gamma(d/2+m)} - \frac{\Gamma(d/2)}{\Gamma(d/2+m+1)} \right] \frac{x^{2m+2}}{(m+1)!\, 2^{2m}}$$

$$= \sum_{m=0}^{\infty} \left[\frac{\Gamma(d/2)}{\Gamma(d/2+m+1)} \frac{(d/2+m)}{(d/2-1)} - \frac{\Gamma(d/2)}{\Gamma(d/2+m+1)} \right] \frac{x^{2m+2}}{(m+1)!\, 2^{2m}}$$

$$= \sum_{m=0}^{\infty} \left[\frac{2(m+1)}{d-2} \right] \frac{\Gamma(d/2)}{\Gamma(d/2+m+1)} \frac{x^{2m+2}}{(m+1)!\, 2^{2m}}$$

$$= \frac{2}{d-2} \sum_{m=0}^{\infty} \frac{\Gamma(d/2)}{\Gamma(d/2+m+1)} \frac{x^{2m+2}}{m!\, 2^{2m}}$$

$$\frac{d-2}{d}(I_0(d-2;x) - I_0(d,x)) = \frac{2}{d} \sum_{m=0}^{\infty} \frac{\Gamma(d/2)}{m!\, \Gamma(d/2+m+1)} \frac{x^{2m+2}}{2^{2m+2}}$$

$$\implies \boxed{\frac{x}{d}\frac{\mathrm{d}}{\mathrm{d}x} I_0(d;x) = \frac{x^2}{d^2} I_0(d+2;x) = \frac{d-2}{d}[I_0(d-2;x) - I_0(d,x)]}$$

Part (b)

Schafli's contour integral is given by,

$$I_0(d;x) = \oint_{\mathcal{C}} \frac{\mathrm{d}z}{2\pi i} \frac{\Gamma(d/2)}{z^{d/2}} \exp\left(z + \frac{x^2}{4z} \right)$$

where the contour $\mathcal{C} = C[-\infty - i0, 0+, -\infty + i0]$.

$$\frac{x}{d}\frac{\mathrm{d}}{\mathrm{d}x} I_0(d;x) = \oint_{\mathcal{C}} \frac{x}{d} \frac{\mathrm{d}z}{2\pi i} \frac{\Gamma(d/2)}{z^{d/2}} \frac{x}{2z} \exp\left(z + \frac{x^2}{4z} \right)$$

$$= \frac{x^2}{d^2} \oint_{\mathcal{C}} \frac{\mathrm{d}z}{2\pi i} \frac{(d/2)\cdot \Gamma(d/2)}{z^{(d+2)/2}} \exp\left(z + \frac{x^2}{4z} \right)$$

$$= \frac{x^2}{d^2} \oint_{\mathcal{C}} \frac{\mathrm{d}z}{2\pi i} \frac{\Gamma((d+2)/2)}{z^{(d+2)/2}} \exp\left(z + \frac{x^2}{4z} \right)$$

$$\implies \frac{x}{d}\frac{\mathrm{d}}{\mathrm{d}x} I_0(d;x) = \frac{x^2}{d^2} I_0(d+2;x)$$

$$I_0(d;x) = \oint_{\mathcal{C}} \frac{\mathrm{d}z}{2\pi i} \frac{\Gamma(d/2)}{z^{d/2}} \exp\left(z + \frac{x^2}{4z} \right)$$

$$= H - \oint_{\mathcal{C}} \frac{\mathrm{d}z}{2\pi i} \int \mathrm{d}z \frac{\Gamma(d/2)}{z^{d/2}} \exp\left(z + \frac{x^2}{4z} \right)\left(1 - \frac{x^2}{4z^2} \right)$$

$$= H + \oint_{\mathcal{C}} \frac{\mathrm{d}z}{2\pi i} \frac{\Gamma(d/2)}{z^{d/2-1}(d/2-1)} \exp\left(z + \frac{x^2}{4z} \right)\left(1 - \frac{x^2}{4z^2} \right)$$

$$= H + \oint_{\mathcal{C}} \frac{\mathrm{d}z}{2\pi i} \frac{\Gamma(d/2-1)}{z^{d/2-1}} \exp\left(z + \frac{x^2}{4z} \right) - \frac{x^2}{4} \oint_{\mathcal{C}} \frac{\mathrm{d}z}{2\pi i} \frac{\Gamma(d/2+1)}{z^{d/2+1}(d/2-1)(d/2)} \exp\left(z + \frac{x^2}{4z} \right)$$

$$= H + \oint_{\mathcal{C}} \frac{\mathrm{d}z}{2\pi i} \frac{\Gamma(d/2-1)}{z^{d/2-1}} \exp\left(z + \frac{x^2}{4z} \right) - \frac{x^2}{d(d-2)} \oint_{\mathcal{C}} \frac{\mathrm{d}z}{2\pi i} \frac{\Gamma(d/2+1)}{z^{d/2+1}} \exp\left(z + \frac{x^2}{4z} \right)$$

$$I_0(d;x) = H + I_0(d-2;x) - \frac{x^2}{d(d-2)} I_0(d+2;x)$$

5

where H is given by,

$$H = \exp\left(z + \frac{x^2}{4z}\right) \oint_{\mathcal{C}} \frac{dz}{2\pi i} \frac{\Gamma(d/2)}{z^{d/2}} = 0$$

as the contour $\mathcal{C}$ does not include the pole at the origin. Hence, we have proved,

$$I_0(d;x) = I_0(d-2;x) - \frac{x^2}{d(d-2)} I_0(d+2;x) \implies \frac{x^2}{d^2} I_0(d+2;x) = \frac{d-2}{d}[I_0(d-2;x) - I_0(d,x)]$$

Part (c)
We are given,

$$I_0(d;x) \approx \frac{e^x}{|S^{d-1}|}\left(\frac{2\pi}{x}\right)^{\frac{d-1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\frac{d-1}{2}+n)}{(2x)^n n! \, \Gamma(\frac{d-1}{2}-n)}$$

$$\implies \frac{x}{d}\frac{d}{dx} I_0(d;x) \approx \frac{x}{d}\left[\frac{e^x}{|S^{d-1}|}\left(\frac{2\pi}{x}\right)^{\frac{d-1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\frac{d-1}{2}+n)}{(2x)^n n! \, \Gamma(\frac{d-1}{2}-n)}\right]$$

Problem 3

Part (a)
Given,

$$V(z,\rho) = \int_0^{2\pi} \frac{d\alpha}{2\pi} g(z - i\rho\cos\alpha)$$

$$\nabla^2 V(z,\rho) = \int_0^{2\pi} \frac{d\alpha}{2\pi} \nabla^2 g(z - i\rho\cos\alpha)$$

$$= \int_0^{2\pi} \frac{d\alpha}{2\pi}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial(g(z - i\rho\cos\alpha))}{\partial\rho}\right) + \frac{\partial^2 g(z - i\rho\cos\alpha)}{\partial z^2}\right]$$

$$= \int_0^{2\pi} \frac{d\alpha}{2\pi}\left[\frac{1}{\rho}\frac{\partial\{\rho g'(t)(-i\cos\alpha)\}}{\partial\rho} + g''(t)\right] \quad ; \quad t = z - i\rho\cos\alpha$$

$$= \int_0^{2\pi} \frac{d\alpha}{2\pi}\left[-\frac{ig'(t)\cos\alpha}{\rho} - g''(t)\cos^2\alpha + g''(t)\right]$$

$$= \int_0^{2\pi} \frac{d\alpha}{2\pi}\left[-\frac{ig'(t)\cos\alpha}{\rho} + g''(t)\sin^2\alpha\right]$$

Now we proceed to integrate the first term by parts,

$$\nabla^2 V(z,\rho) = -\frac{ig'(t)\sin\alpha}{\rho}\bigg|_0^{2\pi} + \int_0^{2\pi} \frac{d\alpha}{2\pi}\left[ig''(t)i\sin\alpha\sin\alpha + g''(t)\sin^2\alpha\right]$$

$$\implies \nabla^2 V(z,\rho) = 0$$

Hence, we have proved that V solves Laplace equation.

Part (b)
Consider,

$$g(z - i\rho\cos\alpha) = \sum_{n=0}^{\infty} g^{(n)}(z)(-i)^n \cos^n\alpha \frac{\rho^n}{n!}$$

$$\int_0^{2\pi} \frac{d\alpha}{2\pi} g(z - i\rho\cos\alpha) = \sum_{n=0}^{\infty} g^{(n)}(z)(-i)^n \frac{\rho^n}{n!} \int_0^{2\pi} \frac{d\alpha}{2\pi} \cos^n\alpha$$

6

$$\int_0^{2\pi} \frac{d\alpha}{2\pi} \cos^n \alpha = 0 \quad \text{for} \quad n = 1, 3, 5 \ldots \quad ; \quad \int_0^{2\pi} \frac{d\alpha}{2\pi} \cos^n \alpha = \frac{1}{2^n}{}^n C_{n/2} \quad \text{for} \quad n = 0, 2, 4, \ldots$$

$$\implies V(g, z) = \sum_{m=0}^{\infty} g^{(2m)}(z)(-1)^m \frac{\rho^{2m}}{(2m)!} \frac{1}{2^{2m}} {}^{2m}C_m$$

But, $g(z) = V(z, 0) \implies g^{(n)}(z) = V^n(z, 0)$. Hence, we write,

$$\boxed{V(g, z) = V(z, 0) - \frac{\rho^2}{4} V^{(2)}(z, 0) + \frac{\rho^4}{64} V^{(4)}(z, 0) + \ldots}$$

Part (c)

$$V(g, z) = \sum_{m=0}^{\infty} (-1)^m \frac{\rho^{2m}}{(2m)!} \frac{1}{2^{2m}} {}^{2m}C_m \frac{\partial^{2m}}{\partial z^{2m}} V(z, 0)$$

$$= \sum_{m=0}^{\infty} (-1)^m \frac{1}{(m!)^2} \left(\frac{1}{2} \rho \frac{\partial}{\partial z} \right)^{2m} V(z, 0)$$

$$V(g, z) = J_0 \left(d = 2; \rho \frac{\partial}{\partial z} \right) V(z, 0)$$

Consider,

$$E_z(z, \rho) = -\frac{\partial V(z, \rho)}{\partial z} = -\sum_{m=0}^{\infty} (-1)^m \frac{1}{(m!)^2} \left(\frac{1}{2} \rho \frac{\partial}{\partial z} \right)^{2m} \frac{\partial V(z, 0)}{\partial z} = J_0 \left(d = 2; \rho \frac{\partial}{\partial z} \right) E_z(z, 0)$$

$$E_\rho(z, \rho) = -\frac{\partial V(z, \rho)}{\partial \rho} = -\rho \frac{1}{\rho} \frac{\partial}{\partial \rho} J_0 \left(d = 2; \rho \frac{\partial}{\partial z} \right) V(z, 0) = -\frac{\rho}{2} J_0 \left(d = 4; \rho \frac{\partial}{\partial z} \right) \frac{\partial^2 V}{\partial z^2} = \frac{\rho}{2} J_0 \left(d = 4; \rho \frac{\partial}{\partial z} \right) \frac{\partial E_z}{\partial z}$$

Part (d)

$$V_l(z, \rho) = J_0 \left(\rho \frac{\partial}{\partial z} \right)$$

$$= \sum_{m=0}^{\infty} (-1)^m \frac{1}{(m!)^2} \left(\frac{1}{2} \rho \frac{\partial}{\partial z} \right)^{2m} \frac{z^l}{l!}$$

$$= \sum_{m=0}^{\infty} (-1)^m \frac{1}{(m!)^2} \frac{\rho^{2m}}{2^{2m}} \frac{l!}{(l - 2m)!} \frac{z^{l-2m}}{l!} \quad \text{if} \quad l \geq 2m \quad \text{else} \quad = 0$$

$$V_l(z, \rho) = \sum_{m=0}^{\infty} (-1)^m \frac{1}{(m!)^2} \frac{\rho^{2m}}{2^{2m}} \frac{z^{l-2m}}{(l - 2m)!}$$

$$\boxed{V_1(z, \rho) = z \quad \text{and} \quad V_2(\rho, z) = \frac{z^2}{2} - \frac{\rho^2}{4}}$$