

Classical Mechanics: Assignment #2

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Problem 1

Solution

The Lagrangian for the given system can be written as,

$$L = T + V = \frac{1}{2}mx^2\omega^2 + \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$$

From the problem, we know that $y = k\left(\frac{x}{l}\right)^\alpha$, which means that $\dot{y} = k\alpha\frac{x^{\alpha-1}}{l^\alpha}\dot{x}$. Substituting these into the form of the Lagrangian and simplifying, we get,

$$L = \frac{1}{2}m\left(-2gk\left(\frac{x}{l}\right)^\alpha + \dot{x}^2\left(\frac{\alpha^2k^2x^{2\alpha-2}}{l^\alpha} + 1\right) + x^2\omega^2\right)$$

The equation of motion can be written as,

$$\boxed{\alpha g k x^2 \left(\frac{x}{l}\right)^\alpha + (\alpha - 1)\alpha^2 k^2 \dot{x}^2 \left(\frac{x}{l}\right)^{2\alpha} + \alpha^2 k^2 x \ddot{x} \left(\frac{x}{l}\right)^{2\alpha} - x^4 \omega^2 + x^3 \ddot{x} = 0}$$

The equilibrium points will satisfy $\dot{x} = \ddot{x} = 0$. This means that the equilibrium point will be,

$$x_0 = \left(\frac{\omega^2 l^\alpha}{g k \alpha}\right)^{\frac{1}{\alpha-2}}$$

We substitute $x(t) = x_0 + \epsilon y(t)$

$$\ddot{y} + y \frac{(\alpha - 2)\omega^2}{\alpha^2 k^2 \left(\frac{x_0}{l}\right)^{2\alpha} + 1} = 0$$

For small oscillations, the coefficient of y in the above equation should be positive. Hence,

$$\frac{(\alpha - 2)\omega^2}{\alpha^2 k^2 \left(\frac{x_0}{l}\right)^{2\alpha} + 1} > 0$$
$$\therefore \alpha - 2 > 0 \implies \boxed{\alpha > 2}$$

The frequency of oscillations ω_0 is simply the square root of the coefficient of y ,

$$\boxed{\omega_0 = \sqrt{\frac{\alpha - 2}{\alpha^2 k^2 \left(\frac{x_0}{l}\right)^{2\alpha} + 1}} \omega} \quad \text{where} \quad x_0 = \left(\frac{\omega^2 l^\alpha}{g k \alpha}\right)^{\frac{1}{\alpha-2}}$$

Problem 2

Part (a)

We first write down the Lagrangian,

$$L = T - V$$

$$= \underbrace{\frac{M(R_1 - R_2)^2 \dot{\theta}_1^2}{2}}_{\text{Centre of mass revolution}} + \underbrace{\frac{MR_2^2 \dot{\theta}_2^2}{4}}_{\text{rotation}} + Mg(R_1 - R_2) \cos \theta_1$$

The equations of motion can now be written as,

$$\ddot{\theta}_2 = 0 \quad \text{and} \quad \ddot{\theta}_1 = -\frac{g(R_1 - R_2)}{R_2^2} \sin \theta_1$$

Part (b)

As there is rolling without slipping, the following constraint condition would hold,

$$R_2 \dot{\theta}_2 = (R_1 - R_2) \dot{\theta}_1 \implies f = R_2 \theta_2 - (R_1 - R_2) \theta_1 = 0$$

Note that differentiating this also gives,

$$R_2 \ddot{\theta}_2 = (R_1 - R_2) \ddot{\theta}_1$$

Part (c)

Putting in constraint conditions, the equations of motion get modified as,

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = \lambda \frac{\partial f}{\partial q}$$

which modifies the equations in this problem as,

$$MR_2^2 \ddot{\theta}_2 = \lambda R_2 \quad \text{and} \quad \frac{M(R_1 - R_2)^2}{2} \ddot{\theta}_1 = -Mg(R_1 - R_2) \sin \theta_1 - \lambda(R_1 - R_2)$$

$$MR_2 \ddot{\theta}_2 = \lambda \quad \text{and} \quad \frac{M(R_1 - R_2)}{2} \ddot{\theta}_1 = -Mg \sin \theta_1 - \lambda$$

Taking the ratio of these two and substituting,

$$\frac{2R_2}{R_1 - R_2} \frac{R_1 - R_2}{R_2} = \frac{\lambda}{-Mg \sin \theta_1 - \lambda}$$

$$|\lambda| = \frac{Mg \sin \theta_1}{3} = \text{Constraint Force}$$

Problem 3

Part (a)

A bicycle really has just two degrees of freedom in the simplest sense,

- The angle associated to the pedalling motion
- The angle associated to the motion of the handle

Part (b)

The arrangement has $M - 1$ links and hence $M - 2$ angles in between. There is also one degree of freedom associated with the rotation of the chain if considered as a rigid body. Hence, there are a total of $M - 1$ degrees of freedom.

Part (c)

Due to *homogeneity and isotropy of space* Lagrangian of a free particle should be,

- *Invariant under Rotation* - Hence, $L(v, x)$, where v and x are the absolute value of the velocity and position vectors.
- *Invariant under Translation* - This means that L cannot depend on x at all. Hence $L = L(v)$.

Part (d)

$$\frac{d}{dt} \left(\frac{\partial L'}{\partial v'} \right) = \frac{d}{dt} (2v') = \frac{d}{dt} v' = 0$$

which is the equation of motion for free particle in the original frame. Hence, $L' = v'^2$ is a possible choice.

Part (e)

$$L'(v + V_0) = L(v) + \frac{dF(x, t)}{dt} = L'(v) + V_0 \frac{dL'}{dv_c} \Big|_{v_c=v} + \dots$$

As $L(v) = L'(v)$, we have $\frac{dF(x, t)}{dt} = V_0 \frac{dL'}{dv_c} \Big|_{v_c=v}$. As F does not depend on v , it's first derivative can depend only linearly on v . Hence,

$$\frac{dL'}{dv} \sim v \implies L \sim v^2$$

Problem 4**Part (a)**

The Schrodinger equation is given by

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V\psi = i\hbar \frac{\partial \psi}{\partial t} \quad \text{and} \quad -\frac{\hbar^2}{2m} \frac{\partial^2 \psi^*}{\partial x^2} + V\psi^* = -i\hbar \frac{\partial \psi^*}{\partial t}$$

We choose ψ and ψ^* as our generalized coordinates, and (t, x) as the dependent coordinates. One should be able to write the equations of motion in a compact form as follows,

$$\partial_\mu \left(\frac{\partial L}{\partial (\partial_\mu \psi)} \right) = \frac{\partial L}{\partial \psi} \quad \text{and} \quad \partial_\mu \left(\frac{\partial L}{\partial (\partial_\mu \psi^*)} \right) = \frac{\partial L}{\partial \psi^*}$$

where the index μ goes over (t, x) . Let's analyze every term in the Schrodinger Equation and figure out what corresponding term in the Lagrangian will give rise to that term,

- The first term on the LHS is a double x derivative and will come from some single derivative term of the form $L_1 = \psi' \psi'^*$
- The second term on the LHS has no derivatives and will come from a term of the form $L_2 = \psi \psi^*$
- The RHS is a single t derivative and will come from some term of the form $\dot{\psi} \psi^*$. To make it symmetric, let's consider $L_3 = -\dot{\psi} \psi^* + \dot{\psi}^* \psi$

So our final Lagrangian will be of the form $L = a_1 L_1 + a_2 L_2 + a_3 L_3$. Substituting our ansatz, we find our constants, and then the final Lagrangian can be written as,

$$L = -\frac{\hbar^2}{2m} \psi' \psi'^* + V \psi \psi^* + i\hbar(-\dot{\psi} \psi^* + \dot{\psi}^* \psi)$$

Part (b)

Kinetic energy of the wire is zero. The Lagrangian can be written as,

$$L = - \int ds \rho g y = - \int \sqrt{dx^2 + dy^2} \rho g y = - \int dx y \sqrt{1 + y'^2} \rho g$$

Writing down the equation of motion for the Lagrangian density instead of the Lagrangian, one gets,

$$\begin{aligned} \frac{d}{dx} \left(\frac{yy'}{\sqrt{1+y'^2}} \right) - \sqrt{1+y'^2} &= 0 \\ \frac{yy'' + y'^2}{\sqrt{1+y'^2}} - \frac{yy'^2 y''}{1+y'^2} - \sqrt{1+y'^2} &= 0 \end{aligned}$$

Expanding this out and simplifying a bit, one gets,

$$\begin{aligned} \frac{yy''}{(1+y'^2)^{\frac{3}{2}}} - \frac{1}{\sqrt{1+y'^2}} &= 0 \implies \frac{d}{dx} \left(\frac{y}{\sqrt{1+y'^2}} \right) = 0 \\ \therefore \frac{y}{\sqrt{1+y'^2}} &= \alpha \implies \boxed{y = \alpha \cosh \left(\frac{x}{\alpha} + \beta \right)} \end{aligned}$$

One can get the constants α and β by imposing the end point conditions for the curve.

Part (c)

The distance metric on a sphere spherical polar coordinates is given by,

$$\begin{aligned} ds^2 &= r^2(d\theta^2 + \sin^2 \theta d\phi^2) = d\theta^2 [r^2(1 + \sin^2 \theta \phi'^2)] \\ \therefore ds &= d\theta \sqrt{r^2(1 + \sin^2 \theta \phi'^2)} \end{aligned}$$

From the ansatz $S = Ld\tau$, we can identify that the Lagrangian $L = \sqrt{r^2(1 + \sin^2 \theta \phi'^2)}$. For finding the *equations of motion*, it is fine and also easier to work with L^2 rather than L in this problem. Writing down the equations of motion for $\phi(\theta)$,

$$\frac{d \sin^2 \theta \phi'}{d\theta} = 0 \implies \phi' = \alpha \csc^2(\theta) \implies \boxed{\phi(\theta) = a \cot \theta + b}$$

where α, a, b are constants. If the distance is to be found out between two points (ϕ_1, θ_1) and (ϕ_2, θ_2) , then,

$$\phi_1 = a \cot \theta_1 + b \quad \text{and} \quad \phi_2 = a \cot \theta_2 + b$$

which gives,

$$a = \frac{\phi_1 - \phi_2}{\cot \theta_1 - \cot \theta_2} \quad \text{and} \quad b = \frac{\phi_1 \tan \theta_1 - \phi_2 \tan \theta_2}{\tan \theta_1 - \tan \theta_2}$$

Problem 5

The Lagrangian for a charged particle in a magnetic field is given by,

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + q(A_x\dot{x} + A_y\dot{y})$$

Part (a)

The magnetic field \mathbf{B} is given by $\mathbf{B} = \nabla \times \mathbf{A}$. We can easily see that the addition of any term of the form $\nabla\lambda$, where λ is a scalar, to the vector potential gives the same \mathbf{B} .

Part (b)

Taking $\mathbf{A} = \mathbf{A}(x)$, we write the Euler-Lagrange equations for the coordinate y ,

$$\frac{d(m\dot{y} + qA_y)}{dt} = 0 \implies m\dot{y} + qA_y = \text{constant}$$