Classical Mechanics: Assignment #3

Due 5th October 2018

Aditya Vijaykumar

Problem 1

Part (a)
For $m = constant$

$$T = \frac{m\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}}{2}$$
$$\frac{\mathrm{d}T}{\mathrm{d}t} = m\dot{\vec{\mathbf{v}}} \cdot \vec{\mathbf{v}} = \vec{\mathbf{F}} \cdot \vec{\mathbf{v}}$$

If m varies with time,

$$mT = \frac{m^2\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}}{2}$$
$$\frac{\mathrm{d}(mT)}{\mathrm{d}t} = m^2\dot{\vec{\mathbf{v}}} \cdot \vec{\mathbf{v}} + m\dot{m}\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}$$
$$= (m\vec{\mathbf{v}}) \cdot (m\dot{\vec{\mathbf{v}}} + \dot{m}\vec{\mathbf{v}})$$
$$\frac{\mathrm{d}(mT)}{\mathrm{d}t} = \vec{\mathbf{p}} \cdot \vec{\mathbf{F}}$$

Part (b)
We know that,

$$M_1\frac{\mathrm{d}^2\vec{\mathbf{r_1}}}{\mathrm{d}t^2} + M_2\frac{\mathrm{d}^2\vec{\mathbf{r_2}}}{\mathrm{d}t^2} = \vec{\mathbf{F}}^{ext} + \vec{\mathbf{F}}^i_{12} + \vec{\mathbf{F}}^i_{21}$$

where $\vec{\mathbf{F}}^{ext}$ and $\vec{\mathbf{F}}^i$ are the external and interaction forces respectively. But we also know that,

$$M_1\frac{\mathrm{d}^2\vec{\mathbf{r_1}}}{\mathrm{d}t^2} + M_2\frac{\mathrm{d}^2\vec{\mathbf{r_2}}}{\mathrm{d}t^2} = M\frac{\mathrm{d}^2\vec{\mathbf{R}}}{\mathrm{d}t^2} = \vec{\mathbf{F}}^{ext}$$

Comparing the preceding equations, we get,

$$\vec{\mathbf{F}}^i_{12} + \vec{\mathbf{F}}^i_{21} = 0 \implies \vec{\mathbf{F}}^i_{12} = -\vec{\mathbf{F}}^i_{21}$$

This is the weak form of Newton's third law.
On similar lines,

$$I_1\vec{\mathbf{r}}_1 \times \dot{\vec{\mathbf{p}}}_1 + I_2\vec{\mathbf{r}}_2 \times \dot{\vec{\mathbf{p}}}_2 = \vec{\tau}^{ext} + \vec{\tau}^i_{12} + \vec{\tau}^i_{21} \quad \text{and} \quad I_1\vec{\mathbf{r}}_1 \times \dot{\vec{\mathbf{p}}}_1 + I_2\vec{\mathbf{r}}_2 \times \dot{\vec{\mathbf{p}}}_2 = I\vec{\mathbf{R}} \times \dot{\vec{\mathbf{p}}} = \vec{\tau}^{ext}$$

$$\implies \vec{\tau}^i_{12} + \vec{\tau}^i_{21} = 0$$
$$\vec{\mathbf{r}}_1 \times \vec{\mathbf{F}}^i_{12} + \vec{\mathbf{r}}_2 \times \vec{\mathbf{F}}^i_{21} = 0$$
$$\vec{\mathbf{r}}_1 \times \vec{\mathbf{F}}^i_{12} - \vec{\mathbf{r}}_2 \times \vec{\mathbf{F}}^i_{12} = 0$$
$$(\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2) \times \vec{\mathbf{F}}^i_{12} = 0$$

This means that the action-reaction pair acts along the line joining the two particles. This proves the strong form of the third law.

Problem 2

Let R be the radius of the disc. The generalized coordinates for the motion are the planar coordinates x, y and angular coordinate θ of the disc. For rolling, we have,

$$R\dot{\theta} = v$$

Let's assume that the velocity vector makes an angle ϕ with the positive x-axis. We then have,

$$\dot{x} = v\cos\phi \quad \text{and} \quad \dot{y} = v\sin\phi \implies \dot{x} = R\dot{\theta}\cos\phi \quad \text{and} \quad \dot{y} = R\dot{\theta}\sin\phi$$
$$\therefore dx - Rd\theta\cos\phi = 0 \quad \text{and} \quad dy - Rd\theta\sin\phi = 0$$
$$\therefore dx + dy - R(\cos\phi + \sin\phi)d\theta = 0$$

It is straightforward to see that the above equations are specific instances of an equation of the form,

$$\sum_{i=1}^{n} g_i(x_1, x_2, \ldots, x_n)dx_i = 0$$

For the constraint to be holonomic there should be an integrating factor $f = f(x, y, \theta, \phi)$ which satisfies,

$$\frac{\partial f g_i}{\partial x_j} = \frac{\partial f g_j}{\partial x_i}$$

Let's say $f(x, y, \theta, \phi) = X(x)Y(y)\Theta(\theta)\Phi(\phi)$. Consider the following,

$$\frac{\partial f g_x}{\partial \theta} = \frac{\partial f g_\theta}{\partial x}$$
$$\frac{1}{f}\frac{\partial f}{\partial \theta} = -\frac{1}{f}R\sin\phi\frac{\partial f}{\partial x}$$
$$\frac{1}{\Theta}\frac{\partial \Theta}{\partial \theta} = -\frac{1}{X}R\sin\phi\frac{\mathrm{d}X}{\mathrm{d}x}$$

The RHS is a function of x multiplied by $\sin\phi$, while the LHS is purely a function of θ. They can never be equal, and hence an integrating factor f never exists.

Problem 3

Part (a)

Let r, θ, ϕ be the generalized coordinates in their usual polar form, and l_0 be the equilibrium length of the spring. The Lagrangian of the problem L can be written as,

$$L = \frac{m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2)}{2} + mgr\cos\theta - \frac{k(r - l_0)^2}{2}$$

The equations of motion are,

$$m\ddot{r} = mr\dot{\theta}^2 + mr\sin^2\theta\dot{\phi}^2 + mg\cos\theta - k(r - l_0)$$
$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = mr^2\sin\theta\cos\theta\dot{\phi}^2 - mgr\sin\theta$$
$$mr^2\sin^2\theta\ddot{\phi} + 2mr\sin^2\theta\dot{r}\dot{\phi} + 2mr^2\sin\theta\cos\theta\dot{\theta}\dot{\phi} = 0$$

Constraining the motion in a plane implies using $\phi = constant \implies \dot{\phi} = \ddot{\phi} = 0$. Is constraining possible? - yes, one just needs to give it initial velocity in the plane. Our equations then reduce to,

$$m\ddot{r} = mr\dot{\theta}^2 + mg\cos\theta - k(r - l_0)$$
$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = -mgr\sin\theta$$

Problem 3 continued on next page. . .

The equilibrium positions can be found by substituting all time derivatives of r and θ as zero. This gives the equilibrium $r_0 = l_0 + \dfrac{mg}{k}$ and $\theta_0 = 0$. We need to solve the above for small stretching in r and small angular displacements θ. Let's substitute $r = r_0 + \epsilon x$ and $\theta = \epsilon\alpha$ in the equations. We get,

$$m\epsilon\ddot{x} = m(r_0 + \epsilon x)\epsilon^2\dot{\alpha}^2 + mg\left(1 - \frac{\alpha^2}{2}\epsilon^2 + \dots\right) - k(\epsilon x + r_0 - l_0)$$

$$m(r_0 + \epsilon x)^2\epsilon\ddot{\alpha} + 2m\epsilon^2(r_0 + \epsilon x)\dot{x}\dot{\alpha} = -mg(r_0 + \epsilon x)\left(\epsilon\alpha + \dots\right)$$

Using only $\mathcal{O}(\epsilon)$ terms,

$$\begin{aligned}\ddot{x} &= -\frac{k}{m}x\\ \ddot{\alpha} &= -\frac{g}{r_0}\alpha\end{aligned}$$

Part (b)

The Lagrangian is given as,

$$L = e^{\gamma t}\left(\frac{m\dot{q}^2}{2} - \frac{kq^2}{2}\right)$$

Writing down the equations of motion for hte generalized coordinate q,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\gamma t}m\dot{q}\right) &= -e^{\gamma t}kq\\ \implies e^{\gamma t}(\gamma m\dot{q} + m\ddot{q}) &= -e^{\gamma t}kq\\ \implies \ddot{q} + \gamma\dot{q} + \frac{k}{m}q &= 0\end{aligned}$$

This is the equation of motion for a damped harmonic oscillator.

Let's perform the transformation $s = e^{\gamma t}q \implies \dot{s} = e^{\gamma t}(\gamma q + \dot{q}) = \gamma s + e^{\gamma t}\dot{q}$. Inverting these, we have the following,

$$\begin{aligned}q &= e^{-\gamma t}s\\ \dot{q} &= e^{-\gamma t}(\dot{s} - \gamma s)\end{aligned}$$

Substituting this back into the expression for L,

$$L = e^{-\gamma t}\left(\frac{m\dot{s}^2}{2} + \frac{(m\gamma^2 - k)s^2}{2} - m\gamma s\dot{s}\right)$$

Writing the equations of motion for s,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-\gamma t}(m\dot{s} - m\gamma s)\right) &= -e^{-\gamma t}((k - m\gamma^2)s - m\gamma\dot{s})\\ m\ddot{s} - m\gamma\dot{s} - \gamma(m\dot{s} - m\gamma s) &= (k - m\gamma^2)s - m\gamma\dot{s}\\ \ddot{s} - \gamma\dot{s} + \left(2\gamma^2 - \frac{k}{m}\right)s &= 0\end{aligned}$$

Problem 4

Part (a)

As given, we take $y = at + bt^2 \implies \dot{y} = a + 2bt$. The Lagrangian L can be written as follows,

$$\begin{aligned}L &= \frac{m\dot{y}^2}{2} - mgy = \frac{m(a + 2bt)^2}{2} - mg(at + bt^2)\\ &= \frac{ma^2}{2} + (2mab - mga)t + (2mb^2 - mgb)t^2\end{aligned}$$

Let's evaluate $\int Ldt$,

$$\begin{aligned}
\int_0^{t_0} Ldt &= \int_0^{t_0} [\frac{ma^2}{2} + (2mab - mga)t + (2mb^2 - mgb)t^2]dt \\
&= \frac{ma^2}{2}t_0 + \frac{2mab - mga}{2}t_0^2 + \frac{2mb^2 - mgb}{3}t_0^3 \\
&= \frac{ma^2}{2}\sqrt{\frac{2y_0}{g}} + \frac{2mab - mga}{2}\frac{2y_0}{g} + \frac{2mb^2 - mgb}{3}\frac{2y_0}{g}\sqrt{\frac{2y_0}{g}} \\
&= 0 \impliedby \left(a = 0 \quad \text{and} \quad b = \frac{g}{2}\right)
\end{aligned}$$

Hence Proved.

Part (b) Given, $L = L(q_i, \dot{q}_i, \ddot{q}_i, t)$, and we know that $S = \int_{t_i}^{t_f} L(q_i, \dot{q}_i, \ddot{q}_i, t)dt$. Variation of the action can be written as,

$$\begin{aligned}
\delta S &= \int_{t_i}^{t_f} \delta L dt = 0 \\
&= \int_{t_i}^{t_f} \sum_i \left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i + \frac{\partial L}{\partial \ddot{q}_i}\delta \ddot{q}_i\right)dt \\
&= \int_{t_i}^{t_f} \sum_i \left(\frac{\partial L}{\partial q_i}\delta q_i + \left(\frac{\partial L}{\partial \dot{q}_i} - \frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial L}{\partial \ddot{q}_i}\right)\delta \dot{q}_i + \frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial L}{\partial \ddot{q}_i}\delta \dot{q}_i\right)\right)dt \\
&= \int_{t_i}^{t_f} \sum_i \left[\left\{\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial L}{\partial \dot{q}_i} - \frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial L}{\partial \ddot{q}_i}\right)\right\}\delta q_i + \frac{\mathrm{d}}{\mathrm{dt}}\left(\left(\frac{\partial L}{\partial \dot{q}_i} - \frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial L}{\partial \ddot{q}_i}\right)\delta q_i\right) + \frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial L}{\partial \ddot{q}_i}\delta \dot{q}_i\right)\right]dt
\end{aligned}$$

As the variation of q_i and $\dot{q}_i$ at the endpoints is zero, the total derivative terms vanish. Accounting for the fact that all q_i's are independent, one can write the equation of motion as,

$$\boxed{\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial L}{\partial \dot{q}_i} + \frac{\mathrm{d}^2}{\mathrm{dt}^2}\frac{\partial L}{\partial \ddot{q}_i} = 0}$$

Taking $L = -\frac{m}{2}q\ddot{q} - \frac{k}{2}q^2$, we can write,

$$kq + \frac{m}{2}\ddot{q} = 0 \implies \ddot{q} + \frac{2k}{m}q = 0$$

This turns out to be the equation of the simple harmonic oscillator.

Problem 5

In all the parts, time translation is an implied conserved quantity, and energy is the corresponding conserved quantity.

Part (a) - The potential energy in this case will only be a function of z. Hence, the x and y momenta will be conserved.

Part (b) - Let's consider the half-plane in a way that only the part with $x > 0$ has uniform mass distribution. Here, only y-translations are symmetries and Hence, only p_y will be conserved.

Part (c) - An infinite cylinder possesses z-translation and z-rotation symmetry, and hence p_z and L_z will be conserved.

Part (d) - A finite cylinder only has z-rotation symmetry. So, only L_z is the conserved quantity.

Part (e) - The infinite right elliptical cylinder only has z-translation symmetry. So, only p_z is the conserved quantity.

Problem 5 continued on next page. . .

Part (f) - The dumbell only has z-translation symmetry. So, only p_z is the conserved quantity.
Part (g) - The infinite helical solenoid has$z$-translation and z-rotation symmetries, and hence p_z and L_z are the conserved quantities.

Problem 6

The Lagrangian for the problem is,

$$L = \frac{m(\dot{r}^2 + r^2\dot{\theta}^2)}{2} - V(r)$$

The equations of motion for this Lagrangian, with $V(r) = -V_0 e^{-\lambda^2 r^2}$, are,

$$\begin{aligned} mr^2\dot{\theta} &= constant = L_0 \quad \text{and} \\ m\ddot{r} &= mr\dot{\theta}^2 + (2\lambda^2 r)V(r) \\ \implies m\ddot{r} &= \frac{L_0^2}{mr^3} + (2\lambda^2 r)V(r) \end{aligned}$$

For stable circular orbit, $\dot{r} = \ddot{r} = 0$. Let r_0 be radius of stable circular orbit. We can see that r_0 will be given by the root of the equation,

$$\frac{L_0^2}{mr_0^3} - 2\lambda^2 r_0 V_0 e^{-\lambda^2 r_0^2} = 0 \implies L_0^2 = 2\lambda^2 m r_0^4 V_0 e^{-\lambda^2 r_0^2}$$

Note that L_0^2 has a functional dependence on r_0 ($\sim r_0^4 e^{-\lambda^2 r_0^2}$). This functional dependence has a maxima at $r_0^2 = \frac{2}{\lambda^2}$, where the value of the function is $\frac{4}{\lambda^2 e^2}$. L_0^2 should be lesser than this maximum value for it to be realizable. Hence,

$$L_0^2 \leq \frac{8mV_0}{e^2} \implies L_0 \leq \frac{\sqrt{8mV_0}}{e}$$

So, L_0 cannot exceed $\frac{\sqrt{8mV_0}}{e}$.

Problem 7

The radius of the circle r and the angle covered around the circle θ are the generalized coordinates. The Cartesian coordinates of the particle are,

$$x = r\cos\theta, y = r\sin\theta, z = r\cot\alpha \implies v^2 = \dot{r}^2\csc^2\alpha + r^2\dot{\theta}^2$$

The Lagrangian L can be written as,

$$L = \frac{m(\dot{r}^2\csc^2\alpha + r^2\dot{\theta}^2)}{2} - mgr\cot\alpha$$

The equations of motion are,

$$\begin{aligned} r^2\dot{\theta} &= constant = L_0 \\ \ddot{r}\csc^2\alpha = r\dot{\theta}^2 - g\cot\alpha &\implies \ddot{r}\csc^2\alpha = \frac{L_0^2}{r^3} - g\cot\alpha \end{aligned}$$

Part (b)
If $r = r_0$, $\ddot{r} = 0$ and,

$$L_0^2 = r_0^4\omega^2 = gr_0^3\cot\alpha \implies \boxed{\omega = \sqrt{\frac{g\cot\alpha}{r_0}}} \implies L_0 = r_0^3 g\cot\alpha$$

Problem 7 continued on next page. . .

Part (c)

We consider perturbations along the surface of the cone ie $l = r_0 \csc\alpha + \epsilon x$. This in turn corresponds to a radial perturbation of the form $r = r_0 + \epsilon x \sin\alpha$, $\epsilon \ll 1$. Substituting this into the equation of motion for r,

$$\begin{aligned}
\epsilon \ddot{x} \csc\alpha &= \frac{L_0^2}{(r_0 + \epsilon x \sin\alpha)^3} - g\cot\alpha \\
&= \frac{L_0^2}{r_0^3}\left(1 - \frac{3\epsilon x \sin\alpha}{r_0} + \dots\right) - g\cot\alpha \\
\epsilon \ddot{x} \csc\alpha &= \frac{L_0^2}{r_0^3}\left(-\frac{3\epsilon x \sin\alpha}{r_0} + \dots\right)
\end{aligned}$$

Choosing only the term first order in ϵ,

$$\ddot{x} = -\frac{3g\cot\alpha}{r_0}(\sin^2\alpha)x \implies \boxed{\Omega = \sqrt{\frac{3g\cot\alpha}{r_0}}\sin\alpha}$$

For $\Omega = \omega$, we can see that,

$$\sin\alpha = \frac{1}{\sqrt{3}} \implies \alpha = \sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

Problem 8

Let θ_1 and θ_2 be the angles that the sticks make with the vertical. Each stick is of length $2r$. θ_1 (lower stick) and θ_2 (upper stick) are the generalized coordinates. The position coordinates of the lower and upper masses are,

$$(x_1, y_1) = (r\sin\theta_1, r\cos\theta_1) \quad \text{and} \quad (x_2, y_2) = (2r\sin\theta_1 + r\sin\theta_2, 2r\cos\theta_1 + r\cos\theta_2)$$

$$\implies v_1^2 = r^2\dot{\theta}_1^2 \quad \text{and} \quad v_2^2 = 4r^2\dot{\theta}_1^2 + r^2\dot{\theta}_2^2 + 4r^2\cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2$$

The Lagrangian can be written as,

$$L = \frac{mr^2}{2}\dot{\theta}_1^2 + \frac{m(4r^2\dot{\theta}_1^2 + r^2\dot{\theta}_2^2 + 4r^2\cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2)}{2} - mgr\cos\theta_1 - mg(2r\cos\theta_1 + r\cos\theta_2)$$

The equations of motion are,

$$\begin{aligned}
mr^2\ddot{\theta}_1 + 4mr^2\ddot{\theta}_1 + 2mr^2\cos(\theta_1 - \theta_2)\ddot{\theta}_2 + 2mr^2\cos(\theta_1 - \theta_2)\dot{\theta}_2(\dot{\theta}_2 - \dot{\theta}_1) = \\
-2mr^2\sin(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2 + mgr\sin\theta_1 + 2mgr\sin\theta_1
\end{aligned}$$

$$mr^2\ddot{\theta}_2 + 2mr^2\cos(\theta_1 - \theta_2)\ddot{\theta}_1 + 2mr^2\cos(\theta_1 - \theta_2)\dot{\theta}_1(\dot{\theta}_2 - \dot{\theta}_1) = 2mr^2\sin(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2 + mgr\sin\theta_2$$

In the above equations of motion, we substitute, $\theta_1 = 0, \theta_2 = \epsilon \ll 1, \dot{\theta}_1 = \dot{\theta}_2 = 0$,

$$5mr^2\ddot{\theta}_1 + 2mr^2\ddot{\theta}_2 = 0 \quad \text{and}$$
$$mr^2\ddot{\theta}_2 + 2mr^2\ddot{\theta}_1 = mgr\epsilon$$

$$\boxed{\ddot{\theta}_1 = -\frac{2g\epsilon}{r} \quad \text{and} \quad \ddot{\theta}_2 = \frac{5g\epsilon}{r}}$$