Homework 3: Dynamical systems; Instructor: Amit Apte

Due Mon 25 Feb 2019, beginning of class

T: Theoretical; N: Numerical: T/N: Both theory and numerics; marks for each question in paranthesis (please email answers to numerical questions BEFORE the beginning of the class on the due date of the homework)

1. (T; 20) Suppose $\overline{\lim}\, f(t) = a \neq \pm\infty$, then what can you say about the following limits (they exist or not, what is the limit when it exists, etc.)? Prove the statements you make or find counter-examples.

   (a) $\lim[f(t) - a - \epsilon]$
   (b) $\overline{\lim}[f(t) - a - \epsilon]$
   (c) $\lim[f(t) - a + \epsilon]$
   (d) $\overline{\lim}[f(t) - a + \epsilon]$

   Of course, all the lim above are as $t \to \infty$, and $\epsilon > 0$.

   Suppose $x(t)$ denotes a solution of ODE $\dot{x} = A(t)x$, then

   (a) If $\chi[x] = \pm\infty$ for some solution $x(t)$, then, $\|A(t)\|$ is not bounded.
   (b) If $\|A(t)\|$ is not bounded, then $\chi[x] = \pm\infty$ for some solution $x(t)$.

2. (T; 20) If the eigenvalues of a $d \times d$ real matrix A are $u_k \pm iv_k$ (where some of the v_k may be zero, in case of real eigenvalues), for $k = 1, \ldots, p \leq d$, find the set of Lyapunov characteristic exponents of solutions of $\dot{x} = Ax$.

3. (T; 10) What is the most general form of a quadratic, time-independent (autonomous) Hamiltonian for a system with n degrees of freedom, using (a) positions-momenta variables, and using (b) action-angle variables? For the corresponding equations of motion, what are the Lyapunov characteristic exponents? Are Lyapunov characteristic exponents invariant under canonical transformations, either for (a) time-dependent or (b) time-independent transformations? (You may also want to think about the last question for a general, not-necessarily-quadratic, not-necessarily-integrable Hamiltonian.)

4. (T/N; 10, or 100) Are (some) mathematical theorems true? Are (at least some) mathematical statements about physical world true? In both the above cases, give examples in case of affirmative answer, or prove the negation. (In case of the second statement, if you do not believe that the physical world exists, then explain who typed up this homework!)