Chapter 7 → Integration and Variation

7.1) Tensor Densities

A tensor density of weight W transforms like an ordinary tensor, except that there will now be a W^{th} power of the Jacobian.

$$J = \left| \frac{\partial x^a}{\partial x'^b} \right|$$

$$I'^{a\ldots}{}_{b\ldots} = J^W \frac{\partial x'^a}{\partial x^c} \cdots \frac{\partial x^d}{\partial x'^b} \, I^{c\ldots}{}_{d\ldots}$$

$$\nabla_c I^{a\ldots}{}_{b\ldots} = \text{usual terms of tensor cov. derivative} + \left(- W \Gamma^d_{dc} I^a_b \ldots \right)$$

In general, for a tensor density of weight

$$\nabla_c I^a = \partial_c I^a + \Gamma^a_{bc} I^b - W \Gamma^b_{bc} I^a$$

$$\Rightarrow \nabla_a I^a = \partial_a I^a + \Gamma^a_{ba} I^b - W \Gamma^b_{ba} I^a$$

If $W = +1 \Rightarrow \boxed{\nabla_a I^a = \partial_a I^a}$

The Levi - Civita Symbol.

The Levi - Civita alternating symbol / tensor is a tensor whose components are $+1, -1$ if the $(abcd)$ in ϵ_{abcd} is an even, odd permutation of 0123 respectively. ie. $\epsilon_{0123} = \epsilon_{1230} = 1$, $\epsilon_{0123} = -\epsilon_{0213} = +1$

All the other components are zero.

Some Identities:

$$\epsilon^{abcd}\epsilon_{efgh} = \delta^{abcd}_{efgh}$$ [where δ is so defined st. $\delta^{aacd}_{efgh} = 0$]

$$\epsilon^{abcd}\epsilon_{efgd} = \delta^{abc}_{efg} \quad ; \quad \epsilon^{abcd}\epsilon_{efcd} = 2\delta^{ab}_{ef} \quad ; \quad \epsilon^{abcd}\epsilon_{abcd} = 4!$$

$$\epsilon^{abcd}\epsilon_{ebcd} = 3!\,\delta^{a}_{e}$$

The Metric Determinant

$$g_{ab}'(x') = \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b} g_{cd}(x) \implies \underline{g' = J^2 g}$$

This means that the metric determinant is a scalar density of weight 2.

In GR, we'll be working with metrics of negative signature and hence g will be negative

$$(-g')^{1/2} = J\,(-g)^{1/2}$$ [which makes $\sqrt{-g}$ a scalar density of weight 1]

This also means that if T^a is some vector,

$$\nabla_a\left(\sqrt{-g}\,T^a\right) = \partial_a\left(\sqrt{-g}\,T^a\right)$$

Finding the determinant of the Metric

Consider a square matrix $A = (a_{ij})$. Let its determinant be given by a, and the cofactor of a_{ij} be given by A_{ij}. Then, we have,

$$a = \sum_{i,j} a_{ij} A^{ij}$$

$$\frac{\partial a}{\partial a_{ij}} = A^{ij} \implies \frac{\partial a}{\partial x^k} = \frac{\partial a}{\partial a_{ij}} \frac{\partial a_{ij}}{\partial x^k} = A^{ij} \frac{\partial a_{ij}}{\partial x^k}$$

But we know that the inverse of matrix A is a matrix with components b_{ij} given by,

$$b_{ij} = \frac{A^{ji}}{a}$$

$$\implies \frac{\partial a}{\partial x^k} = a b_{ji} \frac{\partial a_{ij}}{\partial x^k} \text{ ; Now substituting } g_{ab} \text{ for } a_{ij}.$$

$$\partial_c g = g\, g_{ab} \frac{\partial g^{ab}}{\partial x^c} = g\, g_{ab}\, \partial_c g^{ab}$$

But we also know that

$$\partial_c g_{ab} = \Gamma^d_{ac} g_{bd} + \Gamma^d_{bc} g_{ad}$$

$$\implies \partial_c g = g g^{ab} \left[\Gamma^d_{ac} g_{bd} + \Gamma^d_{bc} g_{ad}\right]$$

$$\underline{= g \Gamma^d_{dc} + g \Gamma^d_{dc} = 2g \Gamma^d_{dc}}$$

$$\nabla_c g = \partial_c g - 2g \Gamma^a_{ac} \quad (\text{since } g \text{ is a scalar density of weight } 2)$$
$$= 0$$

$$\partial_c(\sqrt{-g}) = \frac{1}{2\sqrt{-g}} \partial_c(-g) = 0$$

$$\implies \partial_c\left[\sqrt{-g}\, T^{a\cdots}_{b\cdots}\right] = \sqrt{-g}\, \partial_c T^{a\cdots}_{b\cdots} \text{ and } \nabla_c\left[\sqrt{-g}\, T^{a\cdots}_{b\cdots}\right] = \sqrt{-g}\, \nabla_c T^{a\cdots}_{b\cdots}$$

Variational Method for Geodesics

We start off by defining action, $\mathcal{L}[x^a, \dot{x}^a, u]$ which is a functional.

$$\mathcal{L} = [g_{ab}(x)\, \dot{x}^a \dot{x}^b]$$

$$\int_{P_1}^{P_2} \mathcal{L}\, du = \int_{P_1}^{P_2} ds = S$$

Euler-Lagrange equations say $\rightarrow$ $\frac{\partial \mathcal{L}}{\partial x^a} - \frac{d}{du}\left[\frac{\partial \mathcal{L}}{\partial \dot{x}^a}\right] = 0$

We need to massage this a bit

$$2\mathcal{L}\left[\frac{\partial \mathcal{L}}{\partial x^a} - \frac{d}{du}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^a}\right)\right] = 0$$

$$\Rightarrow \left(\frac{\partial \mathcal{L}^2}{\partial x^a} - \frac{d}{du}\left[\frac{\partial \mathcal{L}^2}{\partial \dot{x}^a}\right]\right) + \left(2\frac{\partial \mathcal{L}}{\partial \dot{x}^a}\frac{d\mathcal{L}}{du}\right) = 0$$

$$\Rightarrow \frac{\partial \mathcal{L}^2}{\partial x^a} - \frac{d}{du}\left[\frac{\partial \mathcal{L}^2}{\partial \dot{x}^a}\right] = \frac{\partial[g_{bc}\dot{x}^b\dot{x}^c]}{\partial x^a} - \frac{d}{du}\left[\frac{\partial[g_{bc}\dot{x}^b\dot{x}^c]}{\partial \dot{x}^a}\right]$$

$$\Rightarrow \quad = \partial_a g_{bc}\dot{x}^b\dot{x}^c - 2\frac{d}{du}\left[g_{ac}\dot{x}^c\right]$$

$$= \partial_a g_{bc}\dot{x}^b\dot{x}^c - 2g_{ac}\ddot{x}^c - 2\partial_b g_{ac}\dot{x}^b\dot{x}^c$$

$$= -2g_{ac}\ddot{x}^c - \dot{x}^b\dot{x}^c\{bc, a\}$$

$$2\frac{\partial \mathcal{L}}{\partial \dot{x}^a}\frac{d\mathcal{L}}{du} = 2\frac{\partial[g_{bc}\dot{x}^b\dot{x}^c]^{1/2}}{\partial \dot{x}^a}\left\{\frac{d}{du}\left[\frac{ds}{du}\right]\right\}$$

$$= 2(g_{bc}\dot{x}^b\dot{x}^c)^{-1/2}\{g_{ad}\dot{x}^d\}\frac{d^2s}{du^2}$$

$$= 2\frac{\ddot{s}}{\dot{s}}g_{ad}$$

$$\Rightarrow \boxed{\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = \frac{\ddot{s}}{\dot{s}}\dot{x}^a}$$

If $u = s \Rightarrow \ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0$

One can almost always choose a different affine parametrization such that $\bar{s} = \alpha s + \beta$. Note that in case of null geodesics, we can no longer do this, and then,

$$\ddot{x}^a + \Gamma^a_{bc} \dot{x}^b \dot{x}^c = \lambda(u)\, \dot{x}^a$$

where $\lambda(u)$ is a function of parameter u, and $\dot{x}^a$ satisfies

$$g_{ab} \dot{x}^a \dot{x}^b = 0$$

$\boxed{\text{Isometries}}$

If g_{ab} is inva under $x \to x'^a$, $g_{ab}(y) = g'_{ab}(y)$

The transformation is called an isometry.

We know, $$g_{ab}(x) = \frac{\partial x'^c}{\partial x^a} \frac{\partial x'^d}{\partial x^b} g'_{cd}(x')$$

This is isometry if, $$g_{ab}(x) = \frac{\partial x'^c}{\partial x^c} \frac{\partial x'^d}{\partial x^b} g_{cd}(x')$$

Considering infinitesimal coordinate transformation,

$$x'^a = x^a + \epsilon X^a$$

$$\frac{\partial x'^a}{\partial x^b} = \delta^a_b + \epsilon \partial_b X^a$$

$$g_{ab}(x) = \left(\delta^c_a + \epsilon \partial_a X^c\right)\left(\delta^d_b + \epsilon \partial_b X^d\right)\left(g_{cd}(x) + \epsilon X^e \partial_e g_{cd}(x)\right)$$

$$= g_{ab}(x) + \epsilon\left[\partial_a X^c \delta^d_b g_{cd} + \partial_b X^d \delta^c_a g_{cd} + X^e \partial_e g_{cd} \delta^c_a \delta^d_b\right]$$

$$= g_{ab} + \epsilon\Big[\underbrace{\partial_a X^c g_{bc} + \partial_b X^d g_{ad} + X^e \partial_e g_{ab}}_{\Downarrow\ 0}\Big] + O(\epsilon^2)$$

$$\Longrightarrow L_X g_{ab} = 0 \quad \text{ie} \quad \nabla_b X_a + \nabla_a X_b = 0$$

This is Killing equation, X is Killing vector.