Introduction to GR
Assignment - 3
Aditya Vijaykumar

Problems from D'Inverno

7.1 Let ϕ be a scalar density of weight $+1$.

Then $\phi' = J\phi$

Using eq (7.2) in the book,

$$\boxed{\nabla_c \phi = \partial_c \phi - \Gamma^d_{dc}\,\phi}$$

7.2 Given,

$$\nabla_a\left[\sqrt{-g}\,T^a\right] = \partial_a\left[\sqrt{-g}\,T^a\right]$$

$$\Rightarrow \nabla_a\left[\sqrt{-g}\right]T^a + \sqrt{-g}\,\nabla_a T^a = \partial_a\left(\sqrt{-g}\right)T^a + \sqrt{-g}\,\partial_a T^a$$

$$\Rightarrow \nabla_a\left[\sqrt{-g}\right]T^a + \cancel{\sqrt{-g}\,\partial_a T^a} + \sqrt{-g}\,\Gamma^a_{ac}T^c = \partial_a\left(\sqrt{-g}\right)T^a + \cancel{\sqrt{-g}\,\partial_a T^a}$$

$$\Rightarrow \nabla_a\left[\sqrt{-g}\right]T^a + \sqrt{-g}\,\Gamma^a_{ac}T^c = \partial_a\left[\sqrt{-g}\right]T^a$$

$$\Rightarrow \left\{\nabla_a\left[\sqrt{-g}\right] + \sqrt{-g}\,\Gamma^c_{ca}\right\} = \partial_a\left[\sqrt{-g}\right]T^a$$

$$\Rightarrow \boxed{\nabla_a\left[\sqrt{-g}\right] = \partial_a\left[\sqrt{-g}\right] - \sqrt{-g}\,\Gamma^c_{ca}}$$

which is indeed consistent with the answer of the previous problem.

7.6 We start off by considering the Euler-Lagrange equations written in a slightly modified way (in terms of $\mathcal{L}^2$), [eq. (7.40)],

$$\frac{d}{du}\left[\frac{\partial \mathcal{L}^2}{\partial \dot{x}^a}\right] - \frac{\partial \mathcal{L}^2}{\partial x^a} = 2\frac{\partial \mathcal{L}}{\partial \dot{x}^a}\frac{dL}{du}$$

Here, $\mathcal{L}^2 = g_{ab}\dot{x}^a\dot{x}^b = 2K$

Since we are looking for spacelike geodesics, we can always choose an affine parametrization, and $\frac{dL}{du} = 0$.

Substituting this fact and $2K = \mathcal{L}^2$,

$$2\frac{d}{du}\left[\frac{\partial K}{\partial \dot{x}^a}\right] - 2\frac{\partial K}{\partial x^a} = 0$$

$$\Longrightarrow \boxed{\frac{d}{du}\left[\frac{\partial K}{\partial \dot{x}^a}\right] - \frac{\partial K}{\partial x^a} = 0}$$

We can always choose s to be our affine parameter in this case,

$$ds^2 = 2K = g_{ab}\,dx^a dx^b > 0$$

$$\therefore \boxed{1 = g_{ab}\dot{x}^a x^b}$$

$\boxed{7.7}$ $$2K = g_{ab}\dot{x}^a\dot{x}^b$$
$$= e^\nu \dot{t}^2 - e^\lambda \dot{r}^2 - r^2\dot{\theta}^2 - r^2 \sin\theta\, \dot{\phi}^2$$

$$\frac{\partial K}{\partial \dot{t}} = e^\nu \dot{t} \quad ; \quad \frac{\partial K}{\partial \dot{r}} = -e^\lambda \dot{r}$$

$$\frac{\partial K}{\partial \dot{\theta}} = -r^2\dot{\theta} \quad ; \quad \frac{\partial K}{\partial \dot{\phi}} = -r^2 \sin\theta\, \dot{\phi}$$

$$\frac{\partial K}{\partial t} = \frac{e^\nu \dot{t}^2 \bar{\nu}}{2} - \frac{e^\lambda \bar{\lambda} \dot{r}^2}{2}$$

$$\frac{\partial K}{\partial r} = \frac{e^\nu \nu' \dot{t}^2}{2} - \frac{e^\lambda \lambda' \dot{r}^2}{2} - r\left[\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2\right]$$

$$\frac{\partial K}{\partial \theta} = -r^2 \sin\theta\cos\theta\, \dot{\phi}^2 \quad ; \quad \frac{\partial K}{\partial \phi} = 0$$

We have used the notation $\bar{a} = \frac{\partial a}{\partial t}$ and $a' = \frac{\partial a}{\partial r}$

For t : $$e^\nu\left[\bar{\nu}\dot{t} + \nu'\dot{r}\right]\dot{t} + e^\nu \ddot{t} = \frac{e^\nu \dot{t}^2 \bar{\nu}}{2} - \frac{e^\lambda \bar{\lambda} \dot{r}^2}{2}$$

$$\therefore \boxed{\ddot{t} = -\frac{\bar{\nu}}{2}\dot{t}^2 - \nu'\dot{r}\dot{t} - \frac{e^\lambda \bar{\lambda} \dot{r}^2 e^{-\nu}}{2}}$$

For r : $$-e^\lambda\left[\ddot{r} + \bar{\lambda}\dot{t}\dot{r} + \lambda'\dot{r}^2\right] = \frac{e^\nu \nu' \dot{t}^2}{2} - \frac{e^\lambda \lambda' \dot{r}^2}{2} - r\left[\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2\right]$$

$$\therefore \boxed{\ddot{r} = -\frac{e^{\nu-\lambda}\nu'\dot{t}^2}{2} - \frac{\lambda'\dot{r}^2}{2} - \bar{\lambda}\dot{t}\dot{r} + re^{-\lambda}\left[\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2\right]}$$

For θ : $-r^2\ddot{\theta} - 2r\dot{r}\dot{\theta} = -r^2\sin\theta\cos\theta\,\dot{\phi}^2$

$$\Rightarrow \boxed{\ddot{\theta} = -\frac{2\dot{r}}{r}\dot{\theta} + \sin\theta\cos\theta\,\dot{\phi}^2}$$

For ϕ : $-\left[2r\dot{r}\sin\theta\,\dot{\phi} + 2r^2\cos\theta\,\dot{\phi}\dot{\theta} + r^2\sin\theta\,\ddot{\phi}\right] = 0$

$$\Rightarrow \boxed{\ddot{\phi} = -\frac{2}{r}\dot{r}\dot{\phi} - 2\cot\theta\,\dot{\phi}\dot{\theta}}$$

Now we read off the Γ's,

$\Gamma^{\phi}_{r\phi} = \Gamma^{\phi}_{\phi r} = +\frac{1}{r}$; $\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = +\cot\theta$

$\Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} = +\frac{1}{r}$; $\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta$

$\Gamma^{r}_{tt} = \frac{+e^{\nu-\lambda}\nu'}{2}$; $\Gamma^{r}_{rr} = +\frac{\lambda'}{2}$; $\Gamma^{r}_{rt} = \Gamma^{r}_{tr} = +\frac{\bar{\lambda}}{2}$

$\Gamma^{r}_{\theta\theta} = -re^{-\lambda}$; $\Gamma^{r}_{\phi\phi} = -re^{-\lambda}\sin^2\theta$

$\Gamma^{t}_{tt} = +\frac{\bar{\nu}}{2}$; $\Gamma^{t}_{rt} = \Gamma^{t}_{tr} = +\frac{\nu'}{2}$

$\Gamma^{t}_{rr} = \frac{+e^{\lambda}\bar{\lambda}}{2}e^{-\nu}$

All the Γ's not mentioned above are zero.

7.10 Any Killing vector must satisfy

$$\nabla_a X_b + \nabla_b X_a = 0$$

We are working in flat cartesian 3-space, Hence all the Γ's are zero identically.

$$\Longrightarrow \nabla_b X_a = \partial_b X_a$$

$$\Longrightarrow \partial_a X_b + \partial_b X_a = 0$$

$$\therefore \boxed{\partial_c \partial_a X_b + \partial_c \partial_b X_a = 0} \longrightarrow (1)$$

Permuting indices, we get,

$$\longrightarrow \partial_b \partial_c X_a + \partial_b \partial_a X_c = 0 \longrightarrow (2)$$

$$\longrightarrow \partial_a \partial_b X_c + \partial_a \partial_c X_b = 0 \longrightarrow (3)$$

(1) + (2) − (3) gives,

$$\boxed{\partial_b \partial_c X_a = 0}$$

$$\Longrightarrow X_a = \omega_{ab} x^b + t_a$$

$$\therefore \boxed{X^a = \omega^a{}_b x^b + t^a}$$

where $\omega^a{}_b$ and t^a are (tensor) constants of integration.

Since $\partial_b X_a = -\partial_a X_b$, $\omega_{ba} = -\omega_{ab}$

Hence $\omega^a{}_b$ is antisymmetric and has only 3 independent components. t^a also has three independent components, giving 6 constants of integration in total.

Moreover, we can see that these transformations can be split into 3 translations (along x, y, z), rotations around three axes.

$$\therefore X^a = \lambda_1 (\partial_x) + \lambda_2 (\partial_y) + \lambda_3 (\partial_z)$$
$$+ \lambda_4 [y \partial_x - x \partial_y] + \lambda_5 [x \partial_z - z \partial_x]$$
$$+ \lambda_6 [z \partial_y - y \partial_z]$$

$\boxed{7.11}$ If X^a and Y^a are Killing vectors,

$$\nabla_a X_b + \nabla_b X_a = 0 \longrightarrow (1)$$
$$\nabla_a Y_b + \nabla_b Y_a = 0 \longrightarrow (2)$$

$\lambda \times (1) + \mu \times (2)$ gives,

$$\nabla_a [\lambda X_b + \mu Y_b] + \nabla_b [\lambda X_a + \mu Y_a] = 0$$

which shows that $[\lambda X_b + \mu Y_b]$ is a Killing vector.

$\boxed{7.14}$ $$(\nabla_b \nabla_a - \nabla_a \nabla_b) X_c = R_{cdba} X^d \rightarrow (1)$$

$$\Rightarrow g^{bc} \nabla_b \nabla_a X_c - g^{bc} \nabla_a \nabla_b X_c = R_{cdca} X^d$$

If X_c is Killing vector, $\nabla_b X_c = -\nabla_c X_b$

ie. $\nabla_c X_b$ is antisymmetric

Then $$\underbrace{g^{bc}}_{\text{symmetric}} \nabla_a \underbrace{[\nabla_b X_c]}_{\text{antisymmetric}} = 0$$

$$\therefore g^{bc} \nabla_b \nabla_a X_c = R_{ad} X^d \longrightarrow (2)$$

Consider (1) again

$$(\nabla_b \nabla_a - \nabla_a \nabla_b) X_c = R_{cdba} X^d$$

$$\therefore \nabla_b \nabla_a X_c = \nabla_a \nabla_b X_c + R_{cdba} X^d$$
$$= \nabla_a \nabla_b X_c - (R_{cbad} + R_{cadb}) X^d$$
$$= -R_{cadb} X^d + (\nabla_a \nabla_b X_c - R_{cbad} X^d)$$

(2) also means $g^{bc}\left[\nabla_b \nabla_a X_c - R_{cdba} X^d\right] = 0$

which in turn implies $\nabla_b \nabla_a X_c = R_{cdba} X^d = R_{cbad} X^d$.

$$\therefore \nabla_b \nabla_a X_c = -R_{cadb} X^d$$

$$\therefore \nabla_c \nabla_b X_a = -R_{abdc} X^d$$

$$\boxed{\nabla_c \nabla_b X_a = +R_{abcd} X^d}$$

Problems not from d'Inverno

Problem 1 If λ is non-affine, and $t^\alpha = \frac{dx^\alpha}{d\lambda}$

$$\frac{d}{d\lambda}\left[t^\alpha \xi_\alpha\right] = t^\beta \nabla_\beta \left[t^\alpha \xi_\alpha\right]$$

$$= t^\beta \xi_\alpha \nabla_\beta t^\alpha + \underbrace{t^\beta t^\alpha}_{\text{symmetric}} \cancel{\underbrace{\nabla_\beta \xi_\alpha}_{\text{antisymmetric}}}^{\,0}.$$

$$\frac{d}{d\lambda}\left[t^\alpha \xi_\alpha\right] = t^\beta \xi_\alpha \nabla_\beta t^\alpha$$

$$\frac{d}{d\lambda}\left[t^\alpha \xi_\alpha\right] = \kappa t^\alpha \xi_\alpha \implies \frac{d}{d\lambda}\left[p\right] = \kappa p$$

$$\int \frac{dp}{p} = \int \kappa(\lambda)\, d\lambda \implies \boxed{p \propto \exp\left[\int \kappa(\lambda)\, d\lambda\right]}$$

where κ is $\left(\frac{d^2\tau}{d\lambda^2}\right) \Big/ \left(\frac{d\tau}{d\lambda}\right)^2$.

Part (b)

Part (a) $\epsilon = -t^\alpha t_\alpha$

$$\frac{d\epsilon}{d\lambda} = \frac{D\epsilon}{d\lambda} = -2\frac{Dt^\alpha}{d\lambda} t_\alpha$$

$$= -2\kappa t^\alpha t_\alpha$$

$$\frac{d\epsilon}{d\lambda} = 2\epsilon\kappa$$

$$\therefore \epsilon \propto \exp\left[\int 2\kappa(\lambda)\, d\lambda\right]$$

Problem continued on next page

Part (c) $\quad \mathcal{L}_b g_{\alpha\beta} = 2c g_{\alpha\beta} \quad , \quad q = b_\alpha u^\alpha$

$$\frac{Dq}{d\lambda} = b_\alpha \frac{Du^\alpha}{d\lambda} + u^\alpha \frac{Db_\alpha}{d\lambda}$$

$$\frac{dq}{d\lambda} = b_\alpha (\kappa t^\alpha) + u^\alpha u^\beta \nabla_\beta b_\alpha$$

Since, $\mathcal{L}_b g_{\alpha\beta} = \nabla_\alpha b_\beta + \nabla_\beta b_\alpha = 2c g_{\alpha\beta}$

Then, $u^\alpha u^\beta \nabla_\beta b_\alpha = u^\alpha u^\beta \nabla_\alpha b_\beta$

$$= u^\alpha u^\beta (2c g_{\alpha\beta}) - u^\alpha u^\beta (\nabla_\beta b_\alpha)$$

$$2u^\alpha u^\beta \nabla_\beta b_\alpha = -2c$$

$$\Longrightarrow \quad u^\alpha u^\beta \nabla_\beta b_\alpha = -c$$

And so, $\dfrac{dq}{d\lambda} = \kappa b_\alpha t^\alpha - c = \kappa q - c$

If $\lambda = \tau$, $\kappa(\lambda) = 0$, and hence

$$\frac{dq}{d\lambda} = -c \Longrightarrow \boxed{q = -c\lambda + C_2}$$

Problem 3

$$\nabla_\alpha \xi^\beta = \partial_\alpha \xi^\beta + \Gamma^\beta_{\alpha\delta}\, \xi^\delta$$

ξ's have nonzero components only in θ, ϕ dirn.

$$\therefore\ \nabla_\alpha \xi^\beta = \partial_\alpha\left[\xi^\theta \delta^\beta_\theta + \xi^\phi \delta^\beta_\phi\right] + $$

$$+\left(\Gamma^\theta_{\alpha\delta}\, \delta^\beta_\theta + \Gamma^\phi_{\alpha\delta}\, \delta^\beta_\phi\right)\left(\xi^\theta \delta^\alpha_\theta + \xi^\phi \delta^\alpha_\phi\right)$$

$$= \left(\delta^\theta_\alpha\, \partial_\theta + \delta^\phi_\alpha \partial_\phi\right) \times \left[\xi^\theta \delta^\beta_\theta + \xi^\phi \delta^\beta_\phi\right]$$

$$+ \left(\Gamma^\theta_{\alpha\delta}\, \delta^\beta_\theta + \Gamma^\phi_{\alpha\delta}\, \delta^\beta_\phi\right)\left(\xi^\theta \delta^\alpha_\theta + \xi^\phi \delta^\alpha_\phi\right)$$

Now the task is to find non-trivial $\Gamma^\theta_{\alpha\beta}, \Gamma^\phi_{\alpha\beta}$

For θ : $$2\frac{d}{d\tau}\left(r^2 \dot\theta\right) = 2r^2 \sin\theta\cos\theta\, \dot\phi^2$$

$$2r^2 \ddot\theta + 4 r \dot r \dot\theta = 2r^2 \sin\theta \cos\theta\, \dot\phi^2$$

$$\Rightarrow \boxed{\Gamma^\theta_{r\theta} = \frac{1}{r}, \quad \Gamma^\theta_{\phi\phi} = \sin\theta\cos\theta}$$

For ϕ : $$2\frac{d}{d\tau}\left(r^2 \sin^2\theta\, \dot\phi\right) = 0$$

$$\Rightarrow r^2 \sin^2\theta\, \ddot\phi + 2r \sin^2\theta\, \dot r \dot\phi + 2r^2 \sin\theta\cos\theta\, \dot\theta\dot\phi = 0$$

$$\Rightarrow \boxed{\Gamma^\phi_{r\phi} = \frac{1}{r}, \quad \Gamma^\phi_{\theta\phi} = \cot\theta}$$

Now we note $\xi^\alpha_{(1)} = (0, 0, \sin\phi, \cot\theta\cos\phi)$

$$\xi^\alpha_{(2)} = (0, 0, -\cos\phi, \cot\theta\sin\phi)$$

$$\nabla_\phi \xi^\theta_{(1)} = \partial_\phi \xi^\theta_{(1)} + \Gamma^\theta_{\phi\phi} \xi^\phi_{(1)}$$

$$= \sin\phi + \sin\theta\cos\theta \cot\theta \sin\phi$$

$$= \sin\phi \left[1 + \sin\theta\cos\theta\right]$$

$$\nabla_\phi \xi^\phi_{(1)} = -\cot\theta\sin\phi + \Gamma^\phi_{\mu\phi}\xi^\mu + \Gamma^\phi_{\theta\phi}\xi^\theta$$

$$= -\cot\theta\sin\phi + \cot\theta\sin\phi = 0$$

$$\nabla_\theta \xi^\theta_{(1)} = 0 + \Gamma^\theta_{\theta\mu}\xi^\mu = 0$$

$$\nabla_\theta \xi^\phi_{(1)} = -\csc^2\theta\cos\phi + \Gamma^\phi_{\theta\phi}\left(\cot\theta\cos\phi\right)$$

$$= -\csc^2\theta\cos\phi + \cot^2\theta\cos\phi$$

$$\nabla_\theta \xi^\phi_{(1)} = -\cos\phi$$

Since $g_{\alpha\beta}$ is diagonal, and none of the terms of type $\delta^\alpha_\beta \nabla_\beta \xi^\alpha$ are nonzero, we conclude that

$$g_{\theta\theta}\nabla_\phi \xi^\theta + g_{\phi\phi}\nabla_\theta \xi^\phi = 0$$

for ξ to be Killing vector

$$\sin\phi\left[1 + \sin\theta\cos\theta\right] + \sin^2\theta\left[-\cos\phi\right]$$

Similarly,

$$\nabla_\theta \xi^\theta_{(2)} \quad 0 + \Gamma^\theta_{\theta\mu}\xi^\mu = 0$$

$$\nabla_\phi \xi^\phi_{(2)} = \cot\theta\cos\phi + \cot\theta\cos\phi(-1)$$

$$= 0$$

$$\nabla_\theta \xi^\phi_{(2)} = -\csc^2\theta\sin\phi + \Gamma^\phi_{\theta\phi}\xi^\phi$$

$$= -\csc^2\theta\sin\phi + \cot^2\theta\sin\phi$$

$$= -\sin\phi$$

$$\nabla_\phi \xi^\theta_{(2)} = \sin\phi + \Gamma^\theta_{\phi\phi}(\cot\theta\sin\phi)$$

$$= \sin\phi - \sin\theta\cos\theta\cot\theta\sin\phi$$

$$= \sin\phi\left[1 - \cos^2\theta\right]$$

$$g_{\phi\phi}(-\sin\phi) + g_{\theta\theta}(1-\cos^2\theta)$$

$$= -\sin^2\theta\sin\phi + \sin\phi\left[1-\cos^2\theta\right]$$

$$= \sin\phi\left[-\sin^2\theta - \cos^2\theta + 1\right] = 0$$

Hence $\xi^\alpha_{(2)}$ is a Killing vector.

Problem 4

Given metric $g_{\alpha\beta}$, vector potential A^α.

Equations of motion → $u^\beta \nabla_\beta u_\alpha = e F_{\alpha\beta} u^\beta$

and $F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha$, $\mathcal{L}_\xi g_{\alpha\beta} = \mathcal{L}_\xi A_\alpha = 0$

To Prove: $(u_\alpha + e A_\alpha)\xi^\alpha$ is constant along the worldline. ie $u^\beta \nabla_\beta [\xi^\alpha (u_\alpha + eA_\alpha)] = 0$

The Lie Derivative conditions give,

$$\xi^\nu \nabla_\nu g_{\alpha\beta} = 0 \quad \text{and} \quad \xi^\nu \nabla_\nu A_\alpha + A_\alpha \nabla_\nu \xi^\nu = 0$$

Consider contracting EOMs with ξ^α

$$\xi^\alpha u^\beta \nabla_\beta u_\alpha = e \xi^\alpha F_{\alpha\beta} u^\beta$$
$$= e\left[\xi^\alpha \nabla_\alpha A_\beta - \xi^\alpha \nabla_\beta A_\alpha\right] u^\beta$$

$$\xi^\alpha u^\beta \left[\nabla_\beta (u_\alpha + eA_\alpha)\right] = e \xi^\alpha (\nabla_\alpha A_\beta) u^\beta$$

$$\therefore u^\beta \nabla_\beta \left[\xi^\alpha (u_\alpha + eA_\alpha)\right] - u^\beta (u_\alpha + eA_\alpha) \nabla_\beta \xi^\alpha = e \xi^\alpha (\nabla_\alpha A_\beta) u^\beta$$

$$\therefore u^\beta \nabla_\beta \left[\xi^\alpha (u_\alpha + eA_\alpha)\right] = \underbrace{u^\beta u_\alpha \nabla_\beta \xi^\alpha}_{0} + e u^\beta \underbrace{\left[A_\alpha \nabla_\beta \xi^\alpha + \xi^\alpha \nabla_\alpha A_\beta\right]}_{0}$$

$u^\beta u_\alpha$ is symmetric while $\nabla_\beta \xi^\alpha$ is antisymmetric.

Due to Killing eqn.

Hence, $$\boxed{u^\beta \nabla_\beta \left[\xi^\alpha (u_\alpha + eA_\alpha)\right] = 0}$$

ie. $\xi^\alpha (u_\alpha + eA_\alpha)$ is constant on the worldline. $$\boxed{\frac{d\left[\xi^\alpha (u_\alpha + eA_\alpha)\right]}{d\lambda} = 0}$$

Problem 5

$\tilde{E} = -u_\alpha \xi^\alpha_{(t)}$; $\tilde{L} = u_\alpha \xi^\alpha_{(\phi)}$

Also $u^\alpha = \gamma\left(\xi^\alpha_{(t)} + \Omega \xi^\alpha_{(\phi)}\right)$

$$\xi^\alpha_{(t)} u_\alpha = \gamma\left[g_{\alpha\beta}\xi^\beta_{(t)}\xi^\alpha_{(t)} + \Omega g_{\alpha\beta}\xi^\alpha_{(t)}\xi^\beta_{(\phi)}\right]$$

$$\tilde{E} = -\gamma\left[g_{tt} + \Omega g_{t\phi}\right]$$

Also $\tilde{L} = \gamma\left[g_{t\phi} + \Omega g_{\phi\phi}\right]$

But $u^\alpha u_\alpha = -1 \Rightarrow \underbrace{\gamma^2\left(\xi^\alpha_{(t)} + \Omega \xi^\alpha_{(\phi)}\right)\left(g_{\alpha\beta}\xi^\beta_{(t)} + \Omega g_{\alpha\beta}\xi^\beta_{(\phi)}\right)}_{=-1}$

$$\Rightarrow \gamma^2\left[g_{tt} + \Omega^2 g_{\phi\phi} + g_{t\phi}\Omega + g_{t\phi}\Omega\right] = -1$$

Hence, we get $-\tilde{E} + \Omega\tilde{L} = \gamma\left[g_{tt} + 2\Omega g_{t\phi} + \Omega^2 g_{\phi\phi}\right]$

$\therefore -\tilde{E} + \Omega\tilde{L} = -1/\gamma \quad \Rightarrow -\frac{1}{\gamma^2}\delta\gamma = -\delta\tilde{E} + \Omega\delta\tilde{L}$

$\Rightarrow \frac{\delta\gamma}{\gamma} = -\gamma\left(-\delta\tilde{E} + \Omega\delta\tilde{L}\right)$

Now $\delta u^\alpha = \delta\gamma \frac{u^\alpha}{\gamma}$

$$\delta u^\alpha = -\gamma\left(-\delta\tilde{E} + \Omega\delta\tilde{L}\right)u^\alpha$$

$$\therefore u_\alpha \delta u^\alpha = \gamma\left(-\delta\tilde{E} + \Omega\delta\tilde{L}\right)$$

But, $\delta\left[u_\alpha u^\alpha\right] = 0 = 2u_\alpha \delta u^\alpha$

$$\Rightarrow -\delta\tilde{E} + \Omega\delta\tilde{L} = 0$$

$$\Rightarrow \boxed{\delta\tilde{E} = \Omega\delta\tilde{L}}.$$