Introduction to General Relativity: Assignment #1

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Note *— All problem numbers refer to the exercises numbers in d'Inverno.*

Problem 1

Exercise 5.2

Given: $x^a = (x, y, z)$ and $x'^a = (r, \theta, \phi)$. The relation between the coordinates is the following:

$$x = r\cos\phi\sin\theta \quad , \quad y = r\sin\phi\sin\theta \quad , \quad z = r\cos\theta \tag{1}$$

$$r = \sqrt{x^2+y^2+z^2} \quad , \quad \cos\theta = \frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{z}{r} \quad , \quad \tan\phi = \frac{y}{x} \tag{2}$$

The transformation matrix is given by:

$$\left[\tfrac{\partial x^a}{\partial x'^b}\right] = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \end{bmatrix}^T = \begin{bmatrix} \cos\phi\sin\theta & \sin\phi\sin\theta & \cos\theta \\ r\cos\phi\cos\theta & r\sin\phi\cos\theta & -r\sin\theta \\ -r\sin\phi\sin\theta & r\cos\phi\sin\theta & 0 \end{bmatrix}^T \tag{3}$$

$$J = \left|\left[\tfrac{\partial x^a}{\partial x'^b}\right]\right| = r^2\sin\theta \tag{4}$$

$$\left[\frac{\partial x'^a}{\partial x^b}\right] = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} & \frac{\partial \theta}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{bmatrix} = \begin{bmatrix} \dfrac{x}{\sqrt{x^2+y^2+z^2}} & \dfrac{y}{\sqrt{x^2+y^2+z^2}} & \dfrac{z}{\sqrt{x^2+y^2+z^2}} \\ \dfrac{xz}{\sqrt{x^2+y^2}(x^2+y^2+z^2)} & \dfrac{yz}{\sqrt{x^2+y^2}(x^2+y^2+z^2)} & -\dfrac{\sqrt{x^2+y^2}}{x^2+y^2+z^2} \\ \dfrac{-y}{x^2+y^2} & \dfrac{x}{x^2+y^2} & 0 \end{bmatrix} \tag{5}$$

$$J' = 1/J = \frac{1}{\sqrt{(x^2+y^2+z^2)(x^2+y^2)}} \tag{6}$$

For finite x, y, z, J' is never zero. It is infinite for $(x, y, z) = (0, 0, a)$, where a is any real number.

Problem 2

Exercise 5.5

Given: Y^a and Z^a are contravariant vectors, meaning they obey the contravariant transformation laws.

$$\implies Y'^a = \frac{\partial x'^a}{\partial x^b} Y^b \quad \text{and} \quad Z'^a = \frac{\partial x'^a}{\partial x^b} Z^b \tag{7}$$

Let $T^{ab} = Y^a Z^b$. Note that we haven't yet proved that T^{ab} is a tensor — to prove this, it would suffice to show $T'^{ab} = Y'^a Z'^b$, where the primed quantities denote objects transformed as per the contravariant transformation laws. Now consider $Y'^a Z'^b$

$$Y'^a Z'^b = \frac{\partial x'^a}{\partial x^c} Y^c \frac{\partial x'^b}{\partial x^d} Z^d \tag{8}$$
$$= \frac{\partial x'^a}{\partial x^c} \frac{\partial x'^b}{\partial x^d} Y^c Z^d \tag{9}$$
$$= \frac{\partial x'^a}{\partial x^c} \frac{\partial x'^b}{\partial x^d} T^{cd} \tag{10}$$
$$Y'^a Z'^b = T'^{ab} \tag{11}$$

Hence proved.

Problem 3

Exercise 5.8
Consider applying the transformation laws to δ^a_b and using properties of the Kronecker delta,

$$\frac{\partial x'^a}{\partial x^c} \frac{\partial x^d}{\partial x'^b} \delta^c_d = \frac{\partial x'^a}{\partial x^c} \frac{\partial x^c}{\partial x'^b} \tag{12}$$
$$= \frac{\partial x'^a}{\partial x'^b} \tag{13}$$
$$= \delta'^a_b \tag{14}$$

This proves that the Kronecker delta does indeed have tensor character.

Also, as the Kronecker delta is given by $\frac{\partial x^a}{\partial x^b}$ in a given frame, it has components $\mathrm{diag}(1,1,1,1)$ in all frames. This means it is a *constant tensor*.

Problem 4

Exercise 5.9
Consider differentiating $\frac{\partial \phi}{\partial x'^a}$ wrt x'^c,

$$\frac{\partial}{\partial x'^c}\left\{ \frac{\partial \phi}{\partial x'^a} \right\} = \frac{\partial x^d}{\partial x'^c} \frac{\partial}{\partial x^d} \left\{ \frac{\partial x^b}{\partial x'^a} \frac{\partial \phi}{\partial x^b} \right\} \tag{15}$$
$$= \frac{\partial x^d}{\partial x'^c} \frac{\partial x^b}{\partial x'^a} \frac{\partial^2 \phi}{\partial x^b \partial x^d} + \frac{\partial x^d}{\partial x'^c} \frac{\partial^2 x^b}{\partial x'^a \partial x^d} \frac{\partial \phi}{\partial x^b} \tag{16}$$
$$= \frac{\partial x^d}{\partial x'^c} \frac{\partial x^b}{\partial x'^a} \frac{\partial^2 \phi}{\partial x^b \partial x^d} + \frac{\partial^2 x^b}{\partial x'^a \partial x'^c} \frac{\partial \phi}{\partial x^b} \tag{17}$$

If $\frac{\partial^2 \phi}{\partial x'^c \partial x'^a}$ is to transform as a tensor, the second term on the RHS of the last equation should be zero. One can see that though one can construct specific cases in which this term is zero, it is not possible to do so in general. Hence proved that $\frac{\partial^2 \phi}{\partial x^a \partial x^b}$ is not a tensor.

Problem 5

Exercise 5.13
Consider $Y_c = \delta^b_a X^a_{bc}$,

$$\begin{aligned} Y'_c &= \delta'^b_a X'^a_{bc} && (18)\\ &= \delta^b_a \times \frac{\partial x'^a}{\partial x^v}\frac{\partial x^w}{\partial x'^b}\frac{\partial x^y}{\partial x'^c} X^v_{wy} && (19)\\ &= \frac{\partial x'^a}{\partial x^v}\frac{\partial x^w}{\partial x'^a}\frac{\partial x^y}{\partial x'^c} X^v_{wy} && (20)\\ &= \frac{\partial x^w}{\partial x^v}\frac{\partial x^y}{\partial x'^c} X^v_{wy} && (21)\\ &= \delta^w_v \frac{\partial x^y}{\partial x'^c} X^v_{wy} && (22)\\ Y'^c &= \frac{\partial x^y}{\partial x'^c} X^v_{vy} = \frac{\partial x^y}{\partial x'^c} Y_y && (23) \end{aligned}$$

Hence, we have seen that Y_c transforms as a covariant vector.

Problem 6

Exercise 5.14

$$\delta^a_a = \delta^1_1 + \delta^2_2 + \ldots + \delta^n_n = 1 + 1 + \ldots (n \text{ times}) = n \quad (24)$$

$$\delta^a_b \delta^b_a = \delta^a_a = n \quad (25)$$

Problem 7

Exercise 5.16

Part (i)
Given: In $\mathbb{R}^2$, $x^a = (x, y)$ and $x'^a = (R, \phi)$.

$$R = \sqrt{x^2+y^2} \quad \text{and} \quad \tan\phi = \frac{y}{x} \quad (26)$$

$$\left[\frac{\partial x'^a}{\partial x^b}\right] = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{bmatrix} \quad (27)$$

$$\boxed{\text{if} \quad X^a = (1,0) \quad \text{then} \quad X'^a = (1,0)} \quad (28)$$

Part (ii)
We have,

$$\nabla f \cdot \mathbf{i} = \frac{\partial f}{\partial x} \quad , \quad \nabla f \cdot \mathbf{j} = \frac{\partial f}{\partial y} \quad , \quad \nabla f \cdot \hat{\mathbf{R}} = \frac{\partial f}{\partial R} \quad , \quad \nabla f \cdot \hat{\phi} = \frac{1}{R}\frac{\partial f}{\partial \phi} \quad (29)$$

Problem 7 continued on next page. . .

But, by using expressions of $\hat{\mathbf{R}}$ and $\hat{\phi}$ in the $\mathbf{i}, \mathbf{j}$ coordinates, we have,

$$\nabla f \cdot \hat{\mathbf{R}} = \cos\phi \frac{\partial f}{\partial x} + \sin\phi \frac{\partial f}{\partial y} \quad \text{and} \quad \nabla f \cdot \hat{\phi} = -\sin\phi \frac{\partial f}{\partial x} + \cos\phi \frac{\partial f}{\partial y} \tag{30}$$

Using these equations, we have,

$$\frac{\partial}{\partial R} = \cos\phi \frac{\partial}{\partial x} + \sin\phi \frac{\partial}{\partial y} \quad \text{and} \quad \frac{1}{R}\frac{\partial}{\partial \phi} = -\sin\phi \frac{\partial}{\partial x} + \cos\phi \frac{\partial}{\partial y} \tag{31}$$

Part (iii)
In the cartesian coordinate system,

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \tag{32}$$

In the polar coordinate system,

$$\begin{align}
X' &= \left(\frac{\partial R}{\partial x}x + \frac{\partial R}{\partial y}y\right)\frac{\partial}{\partial R} + \left(\frac{\partial \phi}{\partial x}x + \frac{\partial \phi}{\partial y}y\right)\frac{\partial}{\partial \phi} \tag{33}\\
&= \left(\frac{x}{R}x + \frac{y}{R}y\right)\frac{\partial}{\partial R} + \left(\frac{-y}{R^2}x + \frac{x}{R^2}y\right)\frac{\partial}{\partial \phi} \tag{34}\\
&= \left(\frac{x^2+y^2}{R}\right)\frac{\partial}{\partial R} \tag{35}\\
&= R\frac{\partial}{\partial R} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} = X \tag{36}\\
& \tag{37}
\end{align}$$

Substituting $X^a = (1, 0)$,

$$X = \frac{\partial}{\partial x} \tag{38}$$

Part (iv)
If $Y^a = (0, 1)$, then using the result from *part (i)*, $Y'^a = (1, 0)$ and $Y = \frac{\partial}{\partial y}$
If $Z^a = (-y, x)$, then $Z'^a = (0, 1)$ and $Z = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$

Part (v)

$$[X, Y] = \frac{\partial}{\partial x}\frac{\partial}{\partial y} - \frac{\partial}{\partial y}\frac{\partial}{\partial x} = 0 \tag{39}$$

$$\begin{align}
[Y, Z] &= \frac{\partial}{\partial y}\left\{-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right\} - \left\{-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right\}\left\{\frac{\partial}{\partial y}\right\} \tag{40}\\
&= -\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\frac{\partial}{\partial x} + x\frac{\partial^2}{\partial y^2} + y\frac{\partial}{\partial x}\frac{\partial}{\partial y} - x\frac{\partial^2}{\partial y^2} \tag{41}\\
&= -\frac{\partial}{\partial x} = -X \tag{42}
\end{align}$$

$$
\begin{aligned}
[X, Z] &= \frac{\partial}{\partial x}\left\{-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right\} - \left\{-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right\}\left\{\frac{\partial}{\partial x}\right\} && (43)\\
&= -y\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial y} + x\frac{\partial}{\partial x}\frac{\partial}{\partial y} + y\frac{\partial^2}{\partial x^2} - x\frac{\partial}{\partial x}\frac{\partial}{\partial y} && (44)\\
&= \frac{\partial}{\partial y} = Y && (45)
\end{aligned}
$$