Chapter 15 : Experimental Tests of General Relativity

In contrast to special relativity, there are very few tests of the general theory. Is this true anymore?

The main reason is that gravitational fields in our vicinity are weak.

The very first tests were:

1. Precession of Mercury's perihelion.
2. Bending of light (gravitational lensing)
3. Gravitational Redshift

Other tests:

4. The delay of light in a gravitational field
5. Orbital motion of PSR 1913 + 16 (Hulse - Taylor)
6. Observation of gravitational waves from Compact Binary Mergers (2015 - present).

Classical Kepler Motion

A mass m moves under the influence of an inverse square law force with centre of attraction at O. The force is :

$$\boxed{F = \frac{-m\mu}{r^2}\hat{r}} \implies \boxed{m\ddot{r} = \frac{-m\mu}{r^2}\hat{r}}$$

Angular momentum $\vec{L} = \vec{r} \times (m\dot{\vec{r}})$ and hence,

$$\frac{dL}{dt} = \dot{\vec{r}} \times (m\dot{\vec{r}}) + \vec{r} \times (m\ddot{\vec{r}})$$

$$= 0$$

ie. angular momentum is conserved and

$$\boxed{\vec{L} = m\vec{h}}$$; $\vec{h}$ is constant

Classical Kepler Motion

Angular momentum conservation also implies the motion will be restricted to a plane.

To make things convenient, we'll work with polar coordinates (R, ϕ). With this, the EoM is,

$$\boxed{\left(\ddot{R} - R\dot{\phi}^2\right)\hat{R} + \frac{1}{R}\frac{d}{dt}\left[R^2\dot{\phi}\right]\hat{\phi} = \frac{-\mu}{R^2}\hat{R}}$$

Dotting with $\hat{\phi}$ immediately shows,

$$R^2\dot{\phi} = h$$

which is indeed conservation of angular momentum.

Classical Kepler Motion

Dotting with $\hat{R}$ gives,

$$\boxed{\ddot{R} - R\dot{\phi}^2 = \frac{-\mu}{R^2}}$$

Substituting $R = \dfrac{1}{u}$,

$$\boxed{\frac{d^2u}{d\phi^2} + u = \frac{\mu}{h^2}}$$

Binet's equation

which has solution,

$$\boxed{u = \frac{\mu}{h^2} + C\cos(\phi - \phi_0)}$$

Classical Kepler Motion

By substituting back $u = 1/R$, and $l = \frac{h^2}{\mu}$, $e = \frac{Ch^2}{\mu}$,

$$\boxed{\frac{l}{R} = 1 + e\cos(\phi - \phi_0)}$$

We immediately notice that this is the polar equation for a conic section!

l : semi-latus rectum

e : eccentricity

ϕ_0 : orientation relative to x-axis

If $e \in (0,1)$, conic section is an ellipse, and point of closest approach is called perihelion.

Classical Kepler Motion

The motion of a test mass in the field of a massive body is called the one-body problem.

One can map the two-body problem to the one-body problem in the following way,

$$\vec{F}_{12} = m\ddot{r} = -\frac{m\mu}{r^2}\hat{r}$$

where,

m : reduced mass $= \dfrac{m_1 m_2}{(m_1 + m_2)}$

$$\mu = (m_1 + m_2)(G)$$

Advance of Mercury's perihelion

Lets look at the one-body problem, looking at the Schwarzschild solution. (ie. spherically symmetric field)

The "Lagrangian" is,

$$2K = \left(1 - 2m/r\right)\dot{t}^2 - \left(1 - 2m/r\right)^{-1}\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\,\dot{\phi}^2 = 1$$

The equations are:

t eqn : $\dfrac{d}{d\tau}\left[\left(1 - 2m/r\right)\dot{t}\right] = 0$

θ eqn : $\dfrac{d}{d\tau}\left[r^2\dot{\theta}\right] - r^2\sin\theta\cos\theta\,\dot{\phi}^2 = 0$

ϕ eqn : $\dfrac{d}{d\tau}\left(r^2\sin\theta\,\dot{\phi}\right) = 0$

+1 eqn of the "Lagrangian"

$\boxed{\text{Advance of Mercury's perihelion}}$

Lets look if planar orbits are possible. Let's fix $\theta = \pi/2$, and $\dot{\theta} = 0$. Putting this in the θ eqn, we see that the higher derivatives are zero.

Using this in the ϕ eqn, we get,

$$\boxed{\frac{d}{d\tau}\left(r^2\dot{\phi}\right) = 0 \implies r^2\phi = h}$$

Similarly, r eqn gives,

$$\boxed{\left(1 - 2m/r\right)\dot{t} = k}$$

Advance of Mercury's perihelion

Now, putting this all in the "Lagrangian",

$$k^2\left(1-\frac{2m}{r}\right)^{-1} + \dot{r}^2\left(1-\frac{2m}{r}\right)^{-1} - r^2\dot{\phi}^2 = 1$$

Substituting $r = 1/u$,

$$\boxed{\frac{d^2u}{d\phi^2} + u = \frac{m}{h^2} + 3mu^2}$$

corresponds to Binet's equation modulo $3mu^2$. This term is pretty small.

We'll solve this using perturbation analysis using,

$$\boxed{\epsilon = \frac{3m^2}{h^2}}$$

Advance of Mercury's perihelion

Then,

$$u'' + u = \frac{m}{h^2} + \epsilon \frac{h^2 u^2}{m}$$

Assume $u = u_0 + \epsilon u_1 + O(\epsilon^2)$, we get,

$$\boxed{\begin{aligned} u_0 &= \frac{m}{h^2}\left(1 + e\cos\phi\right) \\ u_1 &= A + B\phi \sin\phi + C\cos 2\phi \end{aligned}}$$

where

$$\boxed{A = \frac{m}{h^2}\left(1 + \tfrac{1}{2}e^2\right) \; ; \; B = \frac{me}{h^2} \; ; \; C = \frac{-me^2}{6h^2}}$$

Advance of Mercury's perihelion

u_0 is just the Newtonian result. Let's look at correction u_1

$$u_1 = A + B\phi \sin\phi + C\cos 2\phi$$

In this term, $\phi \sin\phi$ is a continuosly growing term, and hence are the most relevant.

$$\therefore \quad u = \frac{m}{h^2}\left(1 + e\cos\phi + \epsilon e \phi \sin\phi\right)$$

$$\therefore \quad u \approx \frac{m}{h^2}\left[1 + e\cos\left\{\phi(1-\epsilon)\right\}\right]$$

The new period is,

$$\frac{2\pi}{1-\epsilon} \approx 2\pi(1+\epsilon)$$

Advance of Mercury's perihelion

This means that the planet will travel in an ellipse, but the axis of the ellipse will rotate by an amount $2\pi\epsilon$ between two successive perihelia. In non-relativistic units,

$$\boxed{2\pi\epsilon \approx \frac{24\pi^3 a^2}{c^2 T^2 (1-e^2)}}$$

There is precession even in the Newtonian n-body problem, but there still was a 43 arcseconds per century discrepancy!

General Relativity explained this discrepancy.

Bending of Light

For null geodesics, the relativistic version of Binet's equation is,

$$\frac{d^2u}{d\phi^2} + u = 3mu^2 \longrightarrow ①$$

In the limit of special relativity,

$$\frac{d^2u}{d\phi^2} + u = 0 \longrightarrow ②$$

$$\Longrightarrow \boxed{u = \frac{1}{D}\sin(\phi - \phi_0)}$$

This is actually just the equation for a straight line as from ϕ to ϕ_0.

① can be thought of as a perturbation around ②.

Bending of Light

So then, we want to seek an approximate solution

$$u = u_0 + 3m u_1$$

Substituting this approximation, we get,

$$u_1'' + u_1 = u_0^2 = \frac{\sin^2\phi}{D^2}$$

$\phi_0 = 0$ Assumption

And the general solution is,

$$u \approx \frac{\sin\phi}{D} + \frac{m}{D^2}\left[1 + C\cos\phi + \cos^2\phi\right]$$

Bending of Light

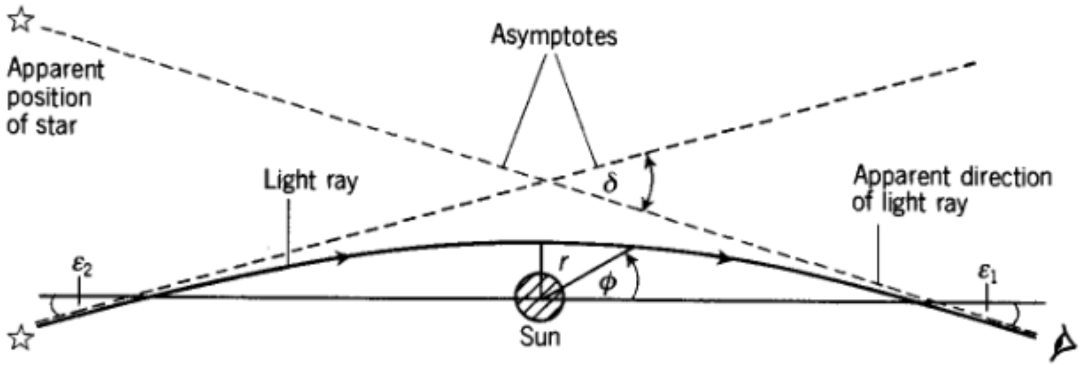


Fig. 15.6 Deflection of light in a gravitational field.

We note that as $r \to \infty$, $u \to 0$, For $u \to 0$,

let's take angles $-\epsilon_1$, $\pi + \epsilon_2$, then,

$$0 = \frac{-\epsilon_1}{D} + \frac{m}{D^2}\left[2 + c\right]$$

$$0 = \frac{-\epsilon_2}{D} + \frac{m}{D^2}\left[2 - c\right]$$

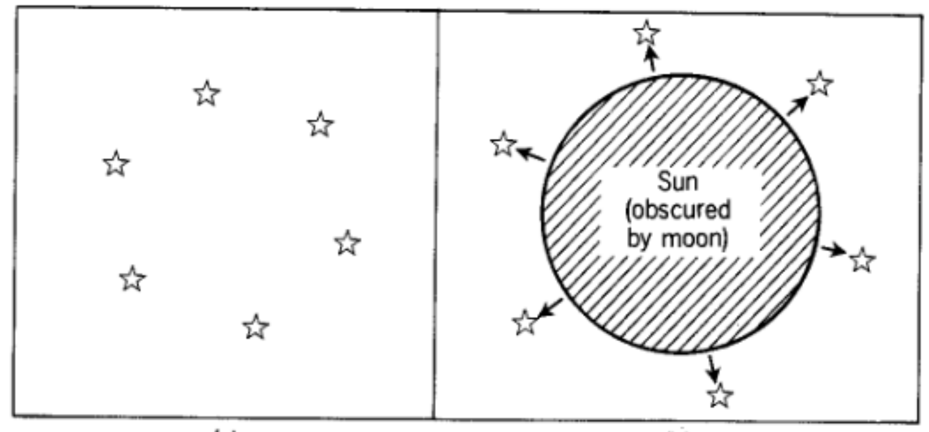
$$\Rightarrow$$

$$\delta = \epsilon_1 + \epsilon_2 = \frac{4m}{D}$$

$$\Rightarrow \delta = \frac{4GM}{c^2 D}$$

Bending of Light

Deflection produced by a light ray that just grazes the sun is 1.75 arcsec



(a) (b)

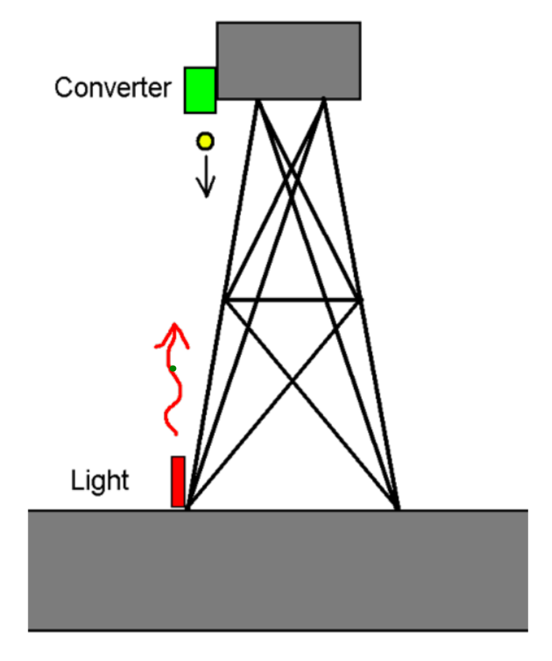
If one compares positions of stars when they are behind the sun and when they are not, one can measure this deflection. Eddington and team did, in 1919.

Lensing is now a very important probe of physics.

Gravitational Redshift

Consider a thought experiment of an endless chain running between earth and sun,

This experiment results in the construction of a perpetual motion machine, hence contradictory!



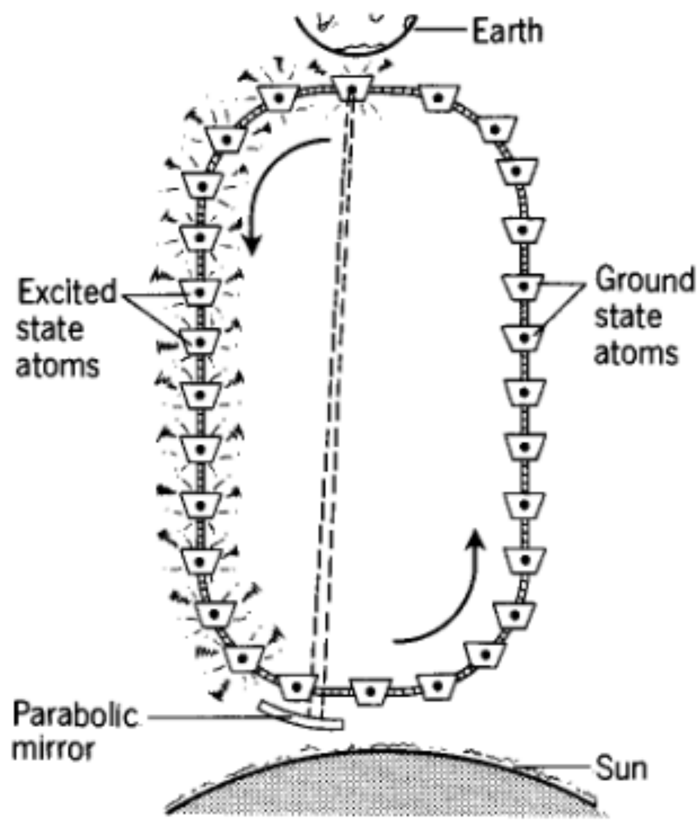


Fig. 15.10 A gravitational *perpetuum mobile*?

Gravitational Redshift

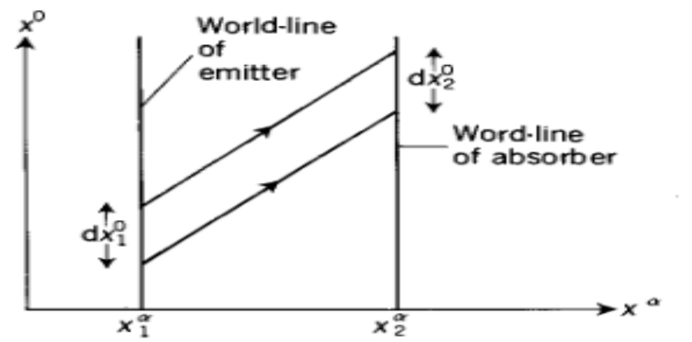


Fig. 15.11 Emission and reception of successive wave crests of a signal.

Redshift in static spacetime; Let $x^a = (x^0, x^\alpha)$

Consider two observers with atomic clocks with worldlines x_1^α, x_2^α respectively. The metric cannot depend on x^0's.

Assume that the first observer is sending radiation to the second. The first observer measures $d\tau$ ie. separation between successive wave crests. Second observer measures $\alpha d\tau$.

Gravitational Redshift

$$d\tau^2 = g_{00}(x_1^\alpha)(dx_1^0)^2$$

$$\alpha^2 d\tau^2 = g_{00}(x_2^\alpha)(dx_2^0)^2$$

Since spacetime is static, $dx_2^0 = dx_1^0$. Hence

$$\alpha = \left(\frac{g_{00}(x_2^\alpha)}{g_{00}(x_1^\alpha)}\right)^{1/2}$$

The second observer will hence measure frequency,

$$\bar{\nu}_0 = \nu_0 \left[\frac{g_{00}(x_1^\alpha)}{g_{00}(x_2^\alpha)}\right]^{1/2} \implies \bar{\nu}_0 < \nu_0 \text{ if } g_{00}(x_1^\alpha) < g_{00}(x_2^\alpha)$$

means that frequency will be redshifted

Gravitational Redshift

$$\frac{\Delta\nu}{\nu} = \frac{\bar{\nu}_0 - \nu_0}{\nu_0} = \frac{\phi_1 - \phi_2}{c^2}$$ in weak field

This has been confirmed to very good precision by terrestrial expts, such as the ones by Pound and Rebka, and numerous other follow-up's.