1: Locality and the Lieb-Robinson Bounds

Locality is a property of any system under consideration, and it manifests itself in the Hamiltonian of the system. For example, the Ising Model is local, in the sense that each particle (spin) interacts only with the spins adjacent to it in a one dimensional lattice. We could in principle also have theories (Hamiltonians) that exhibit locality upto a bound defined by an exponential decay or any other function.

It is hence apparent that to define locality, one needs to define the notion of a distance on the lattice. We could, most intuitively, define distance as $\mathrm{dist}(i,j) = |i-j|$, assuming open boundary conditions. We can in principle choose different metrics on the lattice, but our choice of metric should be such that the Hamiltonian remains local for the whole system. For example, in case of periodic boundary conditions the distance as above will not make the Hamiltonian local and we would need to define distance as $\mathrm{dist}(i,j) = \mathrm{min}_n|i-j+nN|$.

By defining the distance metric on the lattice, we can also define the diameter of a set, and also the notion of distance betweem two sets.

$$\mathrm{diam}(A) = \mathrm{max}_{i,j\in A}\mathrm{dist}(i,j). \tag{1.1}$$

$$\mathrm{dist}(A,B) = \mathrm{min}_{i\in A, j\in B}\mathrm{dist}(i,j). \tag{1.2}$$

Diameter of a set gives the range of distance upto which a the Hamiltonian is effective, thereby describing locality of the system.

NOTE: The lattice need not be a regular one, and can be also thought of as the vertices of a graph.

NOTE: Heisenberg time evolution ie. $O(t) \equiv \exp(iHt)O\exp(-iHt)$ is considered everywhere.

Theorem 1.1. *Suppose for all sites i, the following holds:*

$$\sum_{X\ni i} \|H_X\||X|\exp[\mu\,\mathrm{diam}(X)] \leq s \tag{1.3}$$

for some finite positive constants μ and s. Let A_X, B_Y be operators supported on sets X, Y, respectively. Then, if $\mathrm{dist}(X,Y) > 0$*,*

$$\begin{aligned}\|[A_X(t), B_Y]\| &\leq 2\|A_X\|\|B_Y\|\sum_{i\in X}\exp[-\mu\,\mathrm{dist}(i,Y)]\left[e^{2s|t|}-1\right] \\ &\leq 2\|A_X\|\|B_Y\||X|\exp[-\mu\,\mathrm{dist}(X,Y)]\left[e^{2s|t|}-1\right].\end{aligned} \tag{1.4}$$

Note that the summation in the assumption means the sum over all sets X which contain a particular site i satisfies the assumption (1.3).

Suppose A_X is some operator. Let $B_l(X)$ denote the ball of radius l about set X. That is, $B_l(X)$ is the set of sites i, such that $\mathrm{dist}(i,X) \leq l$. If we define

$$A_X^l(t) = \int \mathrm{d}U U A_X(t) U^\dagger, \tag{1.5}$$

where the integral is over unitaries supported on the set of sites $\Lambda \setminus B_l(X)$ with the Haar measure. Then, A_X^l is supported on $B_l(X)$. Since $UA_X(t)U^\dagger = A_X(t) + U[A_X(t), U^\dagger]$, we have

$$\|A_X^l(t) - A_X(t) \leq \int \mathrm{d}U\|[A_X, U]\|. \tag{1.6}$$

Using the Lieb-Robinson bound (1.4) to bound the right-hand side of the above equation, we see that $A_X^l(t)$ is exponentially close to A_X if l is sufficiently large compared to $2st/\mu$. Thus, to exponential accuracy, we can approximate a time-evolved operator such as $A_X(t)$ by an operator supported on the set $B_l(X)$. That is, the "leakage" of the operator outside the light-cone is small.

1.1 Proof of the Lieb-Robinson Bound

Let A be supported on X and B be supported on Y. Let $\epsilon = t/N$ where N is large, and lets define $t_n = nt/N = n\epsilon$. Then we have

$$\|[A(t), B]\| - \|[A(0), B]\| = \sum_{n=0}^{N-1} \|[A(t_{n+1}), B]\| - \|[A(t_n), B]\| \tag{1.7}$$

We proceed to evaluate the summand on the right hand side.

$$\begin{aligned} \|[A(t_{n+1}), B]\| - \|[A(t_n), B]\| &= \|[A(\epsilon), B(-t_n)]\| - \|[A, B(-t_n)]\| \\ &= \left\|[A + i\epsilon[H_\Lambda, A] + \mathcal{O}(\epsilon^2), B(-t_n)]\right\| - \|[A, B(-t_n)]\| \\ &\leq \|[A + i\epsilon[I_X, A], B(-t_n)]\| - \|[A, B(-t_n)]\| + \mathcal{O}(\epsilon^2) \end{aligned} \tag{1.8}$$

In the last step, we have used the triangle inequality, and have defined $I_X = \sum_{Z:Z\cap X\neq\emptyset} H_Z$

Using the fact that $A + i\epsilon[I_X, A] = e^{i\epsilon I_X} A e^{-i\epsilon I_X} + \mathcal{O}(\epsilon^2)$, we could further use triangle inequalities and write

$$\begin{aligned} \|[A + i\epsilon[I_X, A], B(-t_n)]\| &\leq \left\|[e^{i\epsilon I_X} A e^{-i\epsilon I_X}, B(-t_n)]\right\| + \mathcal{O}(\epsilon^2) \\ &\leq \left\|[A, e^{-i\epsilon I_X} B(-t_n) e^{i\epsilon I_X}]\right\| + \mathcal{O}(\epsilon^2) \\ &\leq \|[A, B(-t_n) - i\epsilon[I_X, B(-t_n)]]\| + \mathcal{O}(\epsilon^2) \\ &\leq \|[A, B(-t_n)]\| + \epsilon \|[A, [I_X, B(-t_n)]]\| + \mathcal{O}(\epsilon^2). \end{aligned} \tag{1.9}$$

Therefore (1.8) becomes,

$$\|[A(t_{n+1}), B]\| - \|[A(t_n), B]\| \leq \epsilon \|[A, [I_X, B(-t_n)]]\| + \mathcal{O}(\epsilon^2) \tag{1.10}$$

Consider some M, N

$$\begin{aligned} \|[M, N]\| &= \|MN - NM\| \\ &\leq 2\|MN\| \\ &\leq 2\|M\|\,\|N\| \end{aligned} \tag{1.11}$$

Using this, we write

$$\begin{aligned} \|[A(t_{n+1}), B]\| - \|[A(t_n), B]\| &\leq 2\epsilon \|A\| \, \|[I_X(t_n), B]\| + \mathcal{O}(\epsilon^2) \\ &\leq \sum_{Z:Z\cap X\neq\emptyset} 2\epsilon \|A\| \, \|[H_Z(t_n), B]\| + \mathcal{O}(\epsilon^2) \\ &\leq \sum_{Z:Z\cap X\neq\emptyset} 2\epsilon \|H_Z\| \, \|[A(t_n), B]\| + \mathcal{O}(\epsilon^2) \end{aligned} \tag{1.12}$$

where the last expression can be obtained by writing the Hamiltonian as a time evolution, and then doing some algebra. Substituting this in (1.7), we get

$$\|[A(t),B]\| - \|[A(0),B]\| \leq 2\,\|H_Z\| \times \sum_{n=0}^{N-1} \sum_{Z:Z\cap X\neq\emptyset} \epsilon \times \|[A(t_n),B]\| + \mathcal{O}(\epsilon^2) \tag{1.13}$$

In the limit $\epsilon \downarrow 0$ (ie. $N \uparrow \infty$), we can change the summation to an integration as follows,

$$\|[A(t),B]\| - \|[A(0),B]\| \leq \sum_{Z:Z\cap X\neq\emptyset} 2\|H_Z\| \int_0^{|t|} ds\, \|[A(s),B]\| \tag{1.14}$$

We define

$$c := \sup_{A\in\mathcal{A}_X} \frac{\|[A(t),B]\|}{\|A\|}, \tag{1.15}$$

where $\mathcal{A}_X$ is the set of observables supported on the set X. Then we have

$$C_B(X,t) \leq C_B(X,0) + \sum_{Z:Z\cap X\neq\emptyset} 2\|H_Z\| \int_0^{|t|} ds\, C_B(Z,s) \tag{1.16}$$

By definition of $C_B(X,t)$, we can say that $C_B(X,0) = 0$ because $dist(X,Y) > 0$. Hence,

$$C_B(X,t) \leq \sum_{Z:Z\cap X\neq\emptyset} 2\|H_Z\| \int_0^{|t|} ds\, C_B(Z,s) \tag{1.17}$$

Now we use (1.16) iteratively on the above equation,

$$\begin{aligned} C_B(X,t) &\leq 2 \sum_{Z_1:Z_1\cap X\neq\emptyset} \|H_{Z_1}\| \int_0^{|t|} ds_1\, C_B(Z_1,s_1) \\ &\leq 2 \sum_{Z_1:Z_1\cap X\neq\emptyset} \|H_{Z_1}\| \int_0^{|t|} ds_1\, C_B(Z_1,0) \\ &+ 2^2 \sum_{Z_1:Z_1\cap X\neq\emptyset} \|H_{Z_1}\| \sum_{Z_2:Z_2\cap Z_1\neq\emptyset} \|H_{Z_2}\| \int_0^{|t|} ds_1 \int_0^{|s_1|} ds_2\, C_B(Z_2,s_2) \end{aligned} \tag{1.18}$$

We can prove using definition of $C_B(X,t)$ that $C_B(Z,0) \leq 2\|B\|$ if $Z\cap Y \neq \emptyset$, and $C_B(Z,0) = 0$ otherwise. Using this we can further write,

$$\begin{aligned} C_B(X,t) &\leq 2\|B\|(2|t|) \sum_{Z_1:Z_1\cap X\neq\emptyset, Z_1\cap Y\neq\emptyset} \|H_{Z_1}\| \\ &+ 2\|B\|\frac{(2|t|)^2}{2!} \sum_{Z_1:Z_1\cap X\neq\emptyset} \|H_{Z_1}\| \sum_{Z_2:Z_2\cap Z_1\neq\emptyset, Z_2\cap Y\neq\emptyset} \|H_{Z_2}\| \\ &+ 2\|B\|\frac{(2|t|)^3}{3!} \sum_{Z_1:Z_1\cap X\neq\emptyset} \|H_{Z_1}\| \sum_{Z_2:Z_2\cap Z_1\neq\emptyset} \|H_{Z_2}\| \sum_{Z_3:Z_3\cap Z_2\neq\emptyset, Z_3\cap Y\neq\emptyset} \|H_{Z_3}\| + \cdots \end{aligned} \tag{1.19}$$