Thesis Notes - Aditya Vijaykumar **Semester I 2017-18**

1: Path Integrals and Miscellaneous Topics

1.1 Preliminaries

Here are some things that need to be kept in mind :-

$$X|x\rangle = x|x\rangle$$

$$\langle x|x'\rangle = \delta(x-x')$$

$$\int dx|x\rangle\langle x| = 1$$

$$P|p\rangle = p|p\rangle$$

$$\langle p|x\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int dp\ e^{-ipx/\hbar} = \langle x|p\rangle^*$$

$$f(x) = \langle x|f\rangle = \int dp\langle x|p\rangle\langle p|f\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int dp\ e^{-ipx/\hbar}\langle p|f\rangle$$

1.2 Ordering and Conventions

In the Hamiltonian way of looking at things, we just promote the classical observables to operators while making the classical $\rightarrow$ quantum transition. But there is a slight problem - $[x_{cl}, p_{cl}] = 0$, but $[x_{op}, p_{op}] \neq 0$. Hence, we need to worry about the order in which we write the p's and x's.

There is no unique well-defined principle to do this. Two conventions are defined below :-

- **Normal Ordering -** One just puts all p's on the left of all the x's. For example.

$$px^2 \xrightarrow{NO} px^2$$

$$xpx \xrightarrow{NO} px^2$$

$$x^2p \xrightarrow{NO} px^2$$

- **Weyl Ordering** - One symmetrizes the product, and weights them equally.

$$px \xrightarrow{WO} \frac{px+xp}{2}$$

$$px^2 \xrightarrow{WO} \frac{px^2+xpx+x^2p}{3}$$

$$x^m p^n \xrightarrow{WO} \frac{(\dots)}{\binom{m+n}{m}}$$

How is the normal ordered Hamiltonian related to the classical Hamiltonian? We consider the matrix elements of H^{NO}, $\langle x'|H^{NO}|x\rangle$.

$$\langle x'|H^{NO}|x\rangle = \int \langle x'|p\rangle\langle p|H^{NO}|x\rangle dp = \int dp\ e^{-\frac{ip(x-x')}{\hbar}} H(p,x)$$

Here $H(p,x)$ is just the classical Hamiltonian.

To understand Weyl ordering, we must probably consider (due to similarity in the terms)

$$e^{\alpha x+\beta p} = \sum_{N=0}^{\infty} \frac{1}{N!}(\alpha x+\beta p)^N = \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{m,n=0}^{m+n=N} \alpha^m \beta^n (x^m p^n)_{symm} = \sum_{m,n=0}^{m+n=N} \frac{\alpha^m \beta^n}{m!n!}(x^m p^n)_{WO}$$

Expanding $e^{\alpha x/2}e^{\beta p}e^{\alpha x/2}$ using the Baker-Campbell-Hausdorff formula, and the fact that $[x,p]=i\hbar$,

$$e^{\alpha x/2}e^{\beta p}e^{\alpha x/2} = e^{\alpha x/2+\beta p+\frac{\alpha\beta}{2}[x,p]}e^{\alpha x/2} = e^{\alpha x+\beta p+\frac{\alpha\beta}{2}[x,p]+\frac{\alpha\beta}{2}[p,x]} = e^{\alpha x+\beta p}$$

Now consider,

$$\begin{aligned}\langle x_k|e^{\alpha x+\beta p}|x_{k-1}\rangle &= \langle x_k|e^{\alpha x/2}e^{\beta p}e^{\alpha x/2}|x_{k-1}\rangle = e^{\frac{\alpha}{2}(x_k+x_{k+1})}\langle x_k|e^{\beta p}|x_{k-1}\rangle \\ &= \int \frac{dp}{2\pi\hbar} e^{\frac{ip}{\hbar}(x_k-x_{k-1})} e^{\frac{\alpha}{2}(x_k+x_{k+1})+\beta p}\end{aligned}$$