1: Teleportation

1.1 Classical Message Sending

How does one send a message classically? Let's suppose Alice and Bob have a bit each, and are far away from each other. Each bit can take value 0 or 1, but lets say that both bits have the same value. Alice and Bob could have met at some time in the past, created their bits, assigned them the same value and then moved on to their respective present positions.

Alice has one more bit with her - and she wishes to send the state of this bit to Bob. Lets call the bit the *teleportee* bit. She proceeds to compare both bits in her possession. If both the bits in her possession are of the same value she sends Bob a classical message reading *same*; if the bits have different value the message reads *different*. Bob, on looking at the message, flips the value of the bit if the message reads *different* and does not if it reads *same*. So, whatever the initial value of Bob's bit, he always ends up with his bit in the same state as that of the teleportee bit.

Now lets suppose Eve intercepts the message sent from Alice to Bob. Obviously, she cannot know anything about the original state of the teleportee solely from this information. But, in principle, she can probe the brain waves or the gravitational waves around Alice and in the environment to know exactly what the state of the bit was. So, in a sense, we could say that the information did in fact exist in the environment between Alice and Bob and we can measure it in principle.

1.2 Quantum Teleportation

Lets consider the quantum analog of the above protocol. Instead of bits, we consider qubits. Lets say, without loss of generality, Alice and Bob begin in the normalized entangled state

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

Alice and Bob are currently far away from each other. Alice has a *teleportee* qubit in the unit normalized state

$$|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$$

We can write the combined state of the three qubits as follows, with the first qubit denoting the teleportee, second denoting Alice's and third denoting Bob's.

$$\frac{\alpha}{\sqrt{2}}|000\rangle + \frac{\alpha}{\sqrt{2}}|011\rangle + \frac{\beta}{\sqrt{2}}|100\rangle + \frac{\beta}{\sqrt{2}}|111\rangle \tag{1.2.1}$$

We shall now measure the two qubits in Alice's possession (ie. first and second qubits) in the Bell Basis. The normalized Bell basis vectors are given by

$$|B_1\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle; |B_2\rangle = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle; |B_3\rangle = \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle; |B_4\rangle = \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle$$

We relate the computational basis to the Bell basis as

$$|00\rangle = \frac{|B_1\rangle + |B_3\rangle}{\sqrt{2}}; |10\rangle = \frac{|B_2\rangle + |B_4\rangle}{\sqrt{2}}; |01\rangle = \frac{|B_2\rangle - |B_4\rangle}{\sqrt{2}}; |11\rangle = \frac{|B_1\rangle - |B_3\rangle}{\sqrt{2}}$$

Substituting these expressions in 1.2.1, one can write the combined state as

$$\frac{|B_1\rangle(\alpha|0\rangle + \beta|1\rangle) + |B_2\rangle(\alpha|1\rangle + \beta|0\rangle) + |B_3\rangle(\alpha|0\rangle - \beta|1\rangle) + |B_4\rangle(\alpha|1\rangle - \beta|0\rangle)}{2}$$

We can clearly see from this state that any measurement in the bell basis will give $|B_1\rangle, |B_2\rangle, |B_3\rangle$ or $|B_4\rangle$ with equal probability (25 percent). After the measurement, Bob's qubit is left in the state $\alpha|0\rangle + \beta|1\rangle$, $\alpha|1\rangle + \beta|0\rangle$, $\alpha|0\rangle - \beta|1\rangle$, and $\alpha|1\rangle + \beta|0\rangle$ respectively.

Alice can, through a classical message, send the result of her measurement to Bob. Bob can then, depending on the classical message, apply the appropriate unitary operator on his state to take it to $(\alpha|0\rangle + \beta|1\rangle)$, the original state of the teleportee qubit. Thus, the teleportation has been completed successfully.

A few comments about this protocol :-

- Lets say Eve intercepts the classical message sent by Alice. Obviously, she cannot know anything about the original state of the teleportee solely from this information. But now, even if she probes the environment, she cannot extract any information about the qubits. This is because the entangled state that we started out with is maximally entangled. By the monogamy principle of entanglement, these qubits can now not be entangled with anything else.
- This also implies that the teleportee qubit got teleported to Bob with no information about it existing anywhere in between Alice and Bob. This tenet of non-locality of information is crucial to quantum mechanics.
- The process of measuring the qubits destroys the existing entanglement between the qubits.

1.3 Teleportation through Multiple Entangled States

We considered teleportation when Alice and Bob have one qubit each, entangled maximally. But what if they have 2 qubits each? Or in general, N qubits each (ie. a total of $2N$ qubits)? How does one teleport a single qubit through these states? One protocol is described here.

Alice selects one qubit out of her N qubits. Lets call that qubit A, and the qubit to be teleported T. She measures the AT system in the Bell basis, and as described above sends a classical message to Bob containing the result of her measurement. Bob needs to act on his system with a unitary operator to get a 1-qubit subsystem of his set of qubits into the state of the teleportee qubit, and all is well.

Some things to note and confirm :-

- The unitary operator that Bob acts as a part of this protocol is equivalent to the operator he would have used in the $N = 1$ case on one of the qubits, tensored with the identity operator on rest of the qubits. This operator will obviously be dependent on the classical message sent.

- (confirm) Bob needs to know on which of his qubits he should act the $N = 1$ case operator. This qubit should be the one which was originally entangled with the qubit A (the entanglement is destroyed by the process of measurement). So technically, the classical message sent must have some sort of identity of A which can be used by Bob to decide which qubit to act the unitary operator on.
- The degree of entanglement of the system is reduced by this protocol because the entanglement between one set of qubits is destroyed. If initially the entanglement degree was Nk corresponding to N entangled pairs, it now becomes $(N-1)k$.