Numerical Hydrodynamics - Solutions

ICTS Summer School on Gravitational-wave Astronomy

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Numerical Theory

Problem 1

Finite Differencing

1. Using Taylor Expansion,

$$\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=x_0} = \frac{\cancel{f(x_0)} + f'(x_0)\Delta x + \mathcal{O}(\Delta x)^2 - \cancel{f(x_0)}}{\Delta x} \tag{1}$$

$$= f'(x_0) + \mathcal{O}(\Delta x) \tag{2}$$

2. *There is a typo in the question : $f(x_0 - \Delta x)$ and not $f(x_0 - \Delta)$*

   Again, using Taylor expansion,

$$\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=x_0} = \frac{\cancel{f(x_0)} + f'(x_0)\Delta x + \cancel{\frac{f''(x_0)(\Delta x)^2}{2}} + \mathcal{O}(\Delta x)^3 - \cancel{f(x_0)} + f'(x_0)\Delta x - \cancel{\frac{f''(x_0)(\Delta x)^2}{2}} + \mathcal{O}(\Delta x)^3}{2\Delta x} \tag{3}$$

$$= f'(x_0) + \mathcal{O}(\Delta x)^2 \tag{4}$$

3. Let's Taylor expand the two sides,

$$\frac{y(x_0 + \Delta x) - y(x_0)}{\Delta x} + \mathcal{O}(\Delta x) = F(x_0, y(x_0)) + \mathcal{O}(\Delta x) \tag{5}$$

$$\implies y(x_0 + \Delta x) = y(x_0) + \Delta x F(x_0, y(x_0)) + \mathcal{O}(\Delta x)^2 \tag{6}$$

   Where, in the second step, we have multiplied by Δx and rearranged the terms.

4. Integrating the differential equation between x_0 and $x_0 + \Delta x$, we get,

$$y(x_0 + \Delta x) - y(x_0) = \int_{x_0}^{x_0+\Delta x} F(x, y)\mathrm{d}x \tag{7}$$

   In the infinitesimal limit, we assume that $F(x, y)$ within the integration limits $\approx F(x_0, y(x_0))$. Hence,

$$y(x_0 + \Delta x) - y(x_0) = F(x, y)\Delta x\mathrm{d}x \quad , \tag{8}$$

   which is the Euler method.

5. The advection equation for $q = q(x, t)$ with constant velocity v is,

$$\partial_t q - v\partial_x q = 0 \tag{9}$$

We can expand q about x_0 and t_0 as follows,

$$\partial_t q = \frac{q(x_0, t_0 + \Delta t) - q(x_0, t_0)}{\Delta t} + \mathcal{O}(\Delta t, \Delta x) \quad \text{and} \quad \partial_x q = \frac{q(x_0 + \Delta x, t_0) - q(x_0, t_0)}{\Delta x} + \mathcal{O}(\Delta t, \Delta x) \tag{10}$$

Now substituting these into Eq. 9,

$$\frac{q(x_0, t_0 + \Delta t) - q(x_0, t_0)}{\Delta t} - v\frac{q(x_0 + \Delta x, t_0) - q(x_0, t_0)}{\Delta x} + \mathcal{O}(\Delta t, \Delta x) = 0 \tag{11}$$

$$\implies q(x_0, t_0 + \Delta t) = q(x_0, t_0) + v\Delta t\frac{q(x_0 + \Delta x, t_0) - q(x_0, t_0)}{\Delta x} + \mathcal{O}(\Delta t, \Delta x) \tag{12}$$

6.

Problem 2

Modified Equation

1. The first order upwind scheme is given by,

$$q(x_0, t_0 + \Delta t) = q(x_0, t_0) + \frac{v\Delta t}{\Delta x}[q(x_0 - \Delta x, t_0) - q(x_0, t_0)] \tag{13}$$

We Taylor-expand all terms that have Δ's in them,

$$\cancel{q(x_0, t_0)} + \partial_t q\ \Delta t = \cancel{q(x_0, t_0)} + \frac{v\Delta t}{\Delta x}\left[\cancel{q(x_0, t_0)} - \partial_x q\ \Delta x + \partial_{xx} q\ \frac{\Delta x^2}{2} - \cancel{q(x_0, t_0)}\right] \tag{14}$$

$$\partial_t q = \frac{v}{\Delta x}\left[-\partial_x q\ \Delta x + \partial_{xx} q\ \frac{\Delta x^2}{2}\right] \tag{15}$$

$$\partial_t q + v\partial_x q = \frac{v\Delta x}{2}\left(1 - \frac{v\Delta t}{\Delta x}\right)\partial_{xx} q, \tag{16}$$

where is going from the first to the second step, we have equated the coefficients of Δt on both sides of the equation. The term $\beta = \frac{v\Delta x}{2}\left(1 - \frac{v\Delta t}{\Delta x}\right)$ acts as an effective numerical viscosity, and hence *diffusing* the numerics leading to numerical errors.

2. The violation of the CFL condition leads to the viscosity β turning negative. In a physical situation, this would be lead to the *opposite* of diffusion as we know it; this means that there will be energy added into the system, and smooth features in the solution will eventually go to sharp features and giving infinities.

3. Substituting $f(x - vt) = q$ in the advection equation and using the chain rule of differentiation,

$$-v\ q'(\eta) + vq'(\eta) = 0 \tag{17}$$

Hence, all $q(\eta) = f(x - vt)$ is a solution of the advection equation.

4. Substituting $f(x/t) = q(\xi)$ in the advection equation and using the chain rule of differentiation,

$$-\frac{x}{t^2}q'(\xi) + \frac{v}{t}q'(\xi) = 0 \implies \frac{x}{t} = \xi = v \tag{18}$$

5. Substituting $f\left(\frac{x - vt}{t^\alpha}\right) = q(\eta)$ in the modified equation and using the chain rule of differentiation,

$$\left(-\frac{\alpha x}{t^{\alpha+1}} + \frac{v(\alpha - 1)}{t^\alpha}\right)q'(\eta) + \frac{v}{t^\alpha}q'(\eta) = \beta\frac{v^2}{t^{2\alpha}}q''(\eta) \tag{19}$$

$$\implies -\alpha\frac{x - vt}{t^{\alpha+1}}q' = \beta\frac{v^2}{t^{2\alpha}}q'' \implies \beta\frac{v^2}{t^{2\alpha}}q'' + \alpha\frac{\eta}{t}q' = 0 \tag{20}$$

For $\alpha = 1/2$, the equation,

$$\beta\frac{v^2}{t}q'' + \frac{\eta}{2t}q' = 0 \implies \beta v^2 q'' + \frac{\eta}{2}q' = 0 \quad \text{as} \quad t > 0 \tag{21}$$

Assuming $q'(\eta) = h(\eta)$,

$$\beta v^2 h' + \frac{\eta}{2}h = 0 \implies h(\eta) = q'(\eta) = C_1 \exp\left(-\frac{\eta}{2\beta v^2}\right) \tag{22}$$

$$\implies q(\eta) = C_1 \int \mathrm{d}\eta \exp\left(-\frac{\eta}{2\beta v^2}\right) \quad . \tag{23}$$

The Gaussian integral is in fact the error function.

6.

7.

Problem 3

Phase errors and neutron stars

1. In Part 3 of Problem 2, we proved that $q = f(x - vt)$ is always a solution to the advection equation. Hence $q(x,t) = \exp[i\gamma(x - vt)]$ is a solution to the advection equation. As we are given the initial data $q(x,t) = \exp(i\ell x)$, we conclude that $\gamma = \ell$ for this problem.

2. Substituting the form of the solution into the second order difference formula,

$$\partial_x q = \frac{\exp[i\ell(x_k - vt)]}{2\Delta x}\{\exp(i\ell\Delta x) - \exp(-i\ell\Delta x)\} \tag{24}$$

$$= \frac{\exp[i\ell(x_k - vt)]}{\Delta x} i\sin(\ell\Delta x) \tag{25}$$

3. Substituting q_e and $q_{m,\Delta x}$,

$$\frac{q_e(x,t) - q_{m,\Delta x}(x,t)}{q_e(x,t)} = 1 - \exp[i\ell(v - v_m(\ell))T] \tag{26}$$

Taking the limit $\Delta x \to 0$ and retaining only upto next-to-leading order,

$$\frac{q_e(x,t) - q_{m,\Delta x}(x,t)}{q_e(x,t)} \approx -i\sin[\ell(v - v_m(l))T] \tag{27}$$

$$\therefore e_m(\ell) = \left|\frac{q_e(x,t) - q_{m,\Delta x}(x,t)}{q_e(x,t)}\right| \approx \ell(v - v_m(l))T \tag{28}$$

4. Substituting the expression for v_m in the result of the previous part,

$$e_{2\mathrm{cd}}(l) = \ell vT\left(1 - \frac{\sin(\ell\Delta x)}{\ell\Delta x}\right) \approx \ell vT\frac{(l\Delta x)^2}{6} \tag{29}$$

5. Given $p = \dfrac{2\pi}{\ell\Delta x}$ and $\nu = \dfrac{\ell vT}{2\pi}$, so $\ell\Delta x = \dfrac{2\pi}{p}$ and $T = \dfrac{2\pi\nu}{\ell v}$. Substituting these in the expression for $e_{2\mathrm{cd}}(l)$, we get the required result,

$$e_{2\mathrm{cd}} = \frac{\pi\nu}{3}\left(\frac{2\pi}{p}\right)^2 \quad . \tag{30}$$

We also get the minimum points per wavelength for a required phase error to be,

$$p_{2\mathrm{cd}} = 2\pi\sqrt{\frac{\nu\pi}{3e_{2\mathrm{cd}}}} \tag{31}$$

Problem 3 continued on next page...

6. Substituting $v =$ for $\ell = 2$ (corresponding to the fundamental mode)
7. Subsituting $e_{2\text{cd}} = 0.01$ (1% error) and $\nu \approx 30$ in the expression for $p_{2\text{cd}}$, we get $p_{2\text{cd}} \approx 350$.
8. $\Delta x = \dfrac{2\pi}{\ell p}$
9.
10.

Problem 4

Vacuum part 1

Problem 5

Vacuum part 2

1. The conservation law is,

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{q})}{\partial x} = 0 \implies \frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{q})}{\partial \mathbf{q}}\frac{\partial \mathbf{q}}{\partial x} = 0 \implies \frac{\partial \mathbf{q}}{\partial t} + J\frac{\partial \mathbf{q}}{\partial x} = 0 \quad . \tag{32}$$

where $J = \frac{\partial \mathbf{f}(\mathbf{q})}{\partial \mathbf{q}}$. Substituting $S^{-1}\mathbf{w} = \mathbf{q}$ subject to $SJS^{-1} = \Lambda$, where Λ is the diagonal matrix containing the eigenvalues of J,

$$S^{-1}\frac{\partial \mathbf{w}}{\partial t} + JS^{-1}\frac{\partial \mathbf{w}}{\partial x} = 0 \implies \frac{\partial \mathbf{w}}{\partial t} + SJS^{-1}\frac{\partial \mathbf{w}}{\partial x} = 0 \implies \frac{\partial \mathbf{w}}{\partial t} + \Lambda\frac{\partial \mathbf{w}}{\partial x} = 0 \tag{33}$$

2.

Problem 6

Well Balancing

1. This can be easily seen by substituting the form $q = Ce^x$ in the advection equation with source.
2. The advection equation with source is given by,

$$\partial_t q + \partial_x q = q \tag{34}$$

Using forward differencing for $\partial_t q$ and backward differencing for $\partial_x q$ we have,

$$\frac{q(x_0, t_0 + \Delta t) - q(x_0, t_0)}{\Delta t} + \frac{q(x_0 - \Delta x, t_0) - q(x_0, t_0)}{-\Delta x} = q(x_0, t_0) \tag{35}$$

$$\implies q(x_0, t_0 + \Delta t) = q(x_0 . t_0) - \frac{\Delta t}{\Delta x}(q(x_0, t_0) - q(x_0 - \Delta x, t_0)) + \Delta t q(x_0, t_0) \tag{36}$$

3.

Problem 7

Shocks

1.

2.
$$\partial_t(q^n) + \frac{n}{n+1}\partial_x(q^{n+1}) = 0 \implies nq^{n-1}\partial_t q + nq^n\partial_x q = 0 \implies \partial_t q + q\partial_q q = 0 \ , \tag{37}$$
which is the Burgers equation.

3. ds

Problem 8

Telescoping

1. The first order upwind scheme for $\partial_t q + v\partial_x q = 0$ is given by,
$$q(x_0, t_0 + \Delta t) = q(x_0, t_0) + \frac{v\Delta t}{\Delta x}[q(x_0 - \Delta x, t_0) - q(x_0, t_0)] \tag{38}$$
Our equation $\partial_t q + \partial_x f(q) = 0$ differs in the ∂_x part, and hence one would have to discretize $f(q)$ in x. Using the definitions given in the problem, one can write,
$$q_i^{n+1} = q_i^n = \frac{\Delta t}{\Delta x}\left(f^n_{i-1/2} - f^n_{i+1/2}\right) \tag{39}$$

2.

Problem 9

Monotonicity

Problem 10

Stiffness

1. Given,
$$\frac{\mathrm{d}q}{\mathrm{d}t} = -\frac{1}{\eta}q \tag{40}$$
Using Euler method,
$$\frac{q^{n+1} - q^n}{\Delta t} = -\frac{1}{\eta}q^n \implies q^{n+1} = \left(1 - \frac{\Delta t}{\eta}\right)q^n \tag{41}$$

2. The general solution of the ODE is $q(t) = Ce^{-t/\eta}$. If $q(0) = 1$, $q(t) = e^{-t/\eta} \implies \lim_{t\to\infty} q(t) = 0$.

3. From the form in eq. 41, we can write an ansatz solution for q^n to be,
$$q^N = C\left(1 - \frac{\Delta t}{\eta}\right)^N \ . \tag{42}$$
The above form of the equation suggests that $\lim_{N\to\infty} q^N = 0 \iff \Delta t \le \eta$, meaning for all other values of the timestep Δt, the solution will blow up and go to infinity for large time.

Problem 10 continued on next page. . .

4. Applying backward difference to the original ODE gives,

$$\frac{q^n - q^{n-1}}{\Delta t} = -\frac{1}{\eta} q^n \implies q^n = \left(1 + \frac{\Delta t}{\eta}\right)^{-1} q^{n-1} \tag{43}$$

An ansatz solution for the above discretized form is,

$$q^N = C\left(1 + \frac{\Delta t}{\eta}\right)^{-N} . \tag{44}$$

As one can see, $\lim_{N\to\infty} q^N = 0 \quad , \quad \forall \Delta t.$