

Numerical Hydrodynamics - Solutions

ICTS Summer School on Gravitational-wave Astronomy

Aditya Vijaykumar

Numerical Theory

Problem 1

Finite Differencing

1. Using Taylor Expansion,

$$\left. \frac{df}{dx} \right|_{x=x_0} = \frac{\cancel{f(x_0)} + f'(x_0)\Delta x + \mathcal{O}(\Delta x)^2 - \cancel{f(x_0)}}{\Delta x} \quad (1)$$

$$= f'(x_0) + \mathcal{O}(\Delta x) \quad (2)$$

2. *There is a typo in the question : $f(x_0 - \Delta x)$ and not $f(x_0 - \Delta)$*

Again, using Taylor expansion,

$$\left. \frac{df}{dx} \right|_{x=x_0} = \frac{\cancel{f(x_0)} + f'(x_0)\Delta x + \frac{f''(x_0)(\Delta x)^2}{2} + \mathcal{O}(\Delta x)^3 - \cancel{f(x_0)} + f'(x_0)\Delta x - \frac{f''(x_0)(\Delta x)^2}{2} + \mathcal{O}(\Delta x)^3}{2\Delta x} \quad (3)$$

$$= f'(x_0) + \mathcal{O}(\Delta x)^2 \quad (4)$$

3. Let's Taylor expand the two sides,

$$\frac{y(x_0 + \Delta x) - y(x_0)}{\Delta x} + \mathcal{O}(\Delta x) = F(x_0, y(x_0)) + \mathcal{O}(\Delta x) \quad (5)$$

$$\implies y(x_0 + \Delta x) = y(x_0) + \Delta x F(x_0, y(x_0)) + \mathcal{O}(\Delta x)^2 \quad (6)$$

Where, in the second step, we have multiplied by Δx and rearranged the terms.

4. Integrating the differential equation between x_0 and $x_0 + \Delta x$, we get,

$$y(x_0 + \Delta x) - y(x_0) = \int_{x_0}^{x_0 + \Delta x} F(x, y) dx \quad (7)$$

In the infinitesimal limit, we assume that $F(x, y)$ within the integration limits $\approx F(x_0, y(x_0))$. Hence,

$$y(x_0 + \Delta x) - y(x_0) = F(x, y) \Delta x \quad , \quad (8)$$

which is the Euler method.

5. The advection equation for $q = q(x, t)$ with constant velocity v is,

$$\partial_t q - v \partial_x q = 0 \quad (9)$$

We can expand q about x_0 and t_0 as follows,

$$\partial_t q = \frac{q(x_0, t_0 + \Delta t) - q(x_0, t_0)}{\Delta t} + \mathcal{O}(\Delta t, \Delta x) \quad \text{and} \quad \partial_x q = \frac{q(x_0 + \Delta x, t_0) - q(x_0, t_0)}{\Delta x} + \mathcal{O}(\Delta t, \Delta x) \quad (10)$$

Now substituting these into Eq. 9,

$$\frac{q(x_0, t_0 + \Delta t) - q(x_0, t_0)}{\Delta t} - v \frac{q(x_0 + \Delta x, t_0) - q(x_0, t_0)}{\Delta x} + \mathcal{O}(\Delta t, \Delta x) = 0 \quad (11)$$

$$\implies q(x_0, t_0 + \Delta t) = q(x_0, t_0) + v \Delta t \frac{q(x_0 + \Delta x, t_0) - q(x_0, t_0)}{\Delta x} + \mathcal{O}(\Delta t, \Delta x) \quad (12)$$

6. We note that if there has to be communication between points that do not satisfy the CFL condition, the velocity required would be more than the speed of propagation v . This makes the scheme unstable, creating unphysical infinities.

Problem 2

Modified Equation

1. The first order upwind scheme is given by,

$$q(x_0, t_0 + \Delta t) = q(x_0, t_0) + \frac{v \Delta t}{\Delta x} [q(x_0 - \Delta x, t_0) - q(x_0, t_0)] \quad (13)$$

We Taylor-expand all terms that have Δ 's in them,

$$\cancel{q(x_0, t_0)} + \partial_t q \Delta t = \cancel{q(x_0, t_0)} + \frac{v \Delta t}{\Delta x} \left[\cancel{q(x_0, t_0)} - \partial_x q \Delta x + \partial_{xx} q \frac{\Delta x^2}{2} - \cancel{q(x_0, t_0)} \right] \quad (14)$$

$$\partial_t q = \frac{v}{\Delta x} \left[-\partial_x q \Delta x + \partial_{xx} q \frac{\Delta x^2}{2} \right] \quad (15)$$

$$\partial_t q + v \partial_x q = \frac{v \Delta x}{2} \left(1 - \frac{v \Delta t}{\Delta x} \right) \partial_{xx} q, \quad (16)$$

where is going from the first to the second step, we have equated the coefficients of Δt on both sides of the equation. The term $\beta = \frac{v \Delta x}{2} \left(1 - \frac{v \Delta t}{\Delta x} \right)$ acts as an effective numerical viscosity, and hence *diffusing* the numerics leading to numerical errors.

2. The violation of the CFL condition leads to the viscosity β turning negative. In a physical situation, this would lead to the *opposite* of diffusion as we know it; this means that there will be energy added into the system, and smooth features in the solution will eventually go to sharp features and giving infinities.
3. Substituting $f(x - vt) = q$ in the advection equation and using the chain rule of differentiation,

$$-v q'(\eta) + v q'(\eta) = 0 \quad (17)$$

Hence, all $q(\eta) = f(x - vt)$ is a solution of the advection equation.

4. Substituting $f(x/t) = q(\xi)$ in the advection equation and using the chain rule of differentiation,

$$-\frac{x}{t^2} q'(\xi) + \frac{v}{t} q'(\xi) = 0 \implies \frac{x}{t} = \xi = v \quad (18)$$

5. Substituting $f\left(\frac{x-vt}{t^\alpha}\right) = q(\eta)$ in the modified equation and using the chain rule of differentiation,

$$\left(-\frac{\alpha x}{t^{\alpha+1}} + \frac{v(\alpha-1)}{t^\alpha}\right)q'(\eta) + \frac{v}{t^\alpha}q'(\eta) = \beta \frac{1}{t^{2\alpha}}q''(\eta) \quad (19)$$

$$\implies -\alpha \frac{x-vt}{t^{\alpha+1}}q' = \beta \frac{1}{t^{2\alpha}}q'' \implies \beta \frac{1}{t^{2\alpha}}q'' + \alpha \frac{\eta}{t}q' = 0 \quad (20)$$

For $\alpha = 1/2$, the equation,

$$\beta \frac{1}{t}q'' + \frac{\eta}{2t}q' = 0 \implies \beta q'' + \frac{\eta}{2}q' = 0 \quad \text{as } t > 0 \quad (21)$$

Assuming $q'(\eta) = h(\eta)$,

$$\beta h' + \frac{\eta}{2}h = 0 \quad (22)$$

$$\implies \frac{dh}{h} = -\frac{\eta}{2\beta}d\eta \quad (23)$$

$$\implies h = C_1 \exp\left(-\frac{\eta^2}{4\beta}\right) \quad (24)$$

$$\implies q = C_1 \sqrt{\pi\beta} \operatorname{erf}\left(\frac{\eta}{2\sqrt{\beta}}\right) + C_2 \quad (25)$$

$$= C_1 \sqrt{\pi\beta} \operatorname{erf}\left(\frac{x-vt}{\sqrt{4\beta t}}\right) + C_2 \quad (26)$$

$$q = C_1 \sqrt{\pi\beta} \left[1 - \operatorname{erfc}\left(\frac{x-vt}{\sqrt{4\beta t}}\right)\right] + C_2 \quad (27)$$

$$q = C_3 \operatorname{erfc}\left(\frac{x-vt}{\sqrt{4\beta t}}\right) + C_4 \quad (28)$$

6. We should have the following limits in mind from the definition of erfc – $\operatorname{erfc}(\infty) = 0$ and $\operatorname{erfc}(-\infty) = 2$. We are given that $q(x, 0) = 2$ for $x < 0$ and $q(x, 0) = 0$ for $x > 0$. For $x > 0$, $\eta \rightarrow \infty$ as $t \rightarrow 0$ and $x < 0$, $\eta \rightarrow -\infty$ as $t \rightarrow 0$. This fixes the value of $C_3 = 1$ and $C_4 = 0$. Hence, $q(x, t) = \operatorname{erfc}\left(\frac{x-vt}{\sqrt{4\beta t}}\right)$ solves the modified equation exactly.

7.

Problem 3

Phase errors and neutron stars

1. In Part 3 of Problem 2, we proved that $q = f(x - vt)$ is always a solution to the advection equation. Hence $q(x, t) = \exp[i\gamma(x - vt)]$ is a solution to the advection equation. As we are given the initial data $q(x, t) = \exp(i\ell x)$, we conclude that $\gamma = \ell$ for this problem.
2. Substituting the form of the solution into the second order difference formula,

$$\partial_x q = \frac{\exp[i\ell(x_k - vt)]}{2\Delta x} \{\exp(i\ell\Delta x) - \exp(-i\ell\Delta x)\} \quad (29)$$

$$= \frac{\exp[i\ell(x_k - vt)]}{\Delta x} i \sin(\ell\Delta x) \quad (30)$$

3. Substituting q_e and $q_{m,\Delta x}$,

$$\frac{q_e(x, t) - q_{m,\Delta x}(x, t)}{q_e(x, t)} = 1 - \exp[i\ell(v - v_m(\ell))T] \quad (31)$$

Taking the limit $\Delta x \rightarrow 0$ and retaining only upto next-to-leading order,

$$\frac{q_e(x, t) - q_{m,\Delta x}(x, t)}{q_e(x, t)} \approx -i \sin[\ell(v - v_m(l))T] \quad (32)$$

$$\therefore e_m(\ell) = \left| \frac{q_e(x, t) - q_{m,\Delta x}(x, t)}{q_e(x, t)} \right| \approx \ell(v - v_m(l))T \quad (33)$$

4. Substituting the expression for v_m in the result of the previous part,

$$e_{2cd}(l) = \ell v T \left(1 - \frac{\sin(\ell \Delta x)}{\ell \Delta x} \right) \approx \ell v T \frac{(\ell \Delta x)^2}{6} \quad (34)$$

5. Given $p = \frac{2\pi}{\ell \Delta x}$ and $\nu = \frac{\ell v T}{2\pi}$, so $\ell \Delta x = \frac{2\pi}{p}$ and $T = \frac{2\pi \nu}{\ell v}$. Substituting these in the expression for $e_{2cd}(l)$, we get the required result,

$$e_{2cd} = \frac{\pi \nu}{3} \left(\frac{2\pi}{p} \right)^2. \quad (35)$$

We also get the minimum points per wavelength for a required phase error to be,

$$p_{2cd} = 2\pi \sqrt{\frac{\nu \pi}{3e_{2cd}}} \quad (36)$$

6. We know that $\ell = \frac{2\pi}{\lambda}$ and that $f = \frac{v}{\lambda}$ where f is the frequency. This means that $\nu = \frac{vT}{2\pi} = fT \approx 30$, where we have substituted $f = 30$ kHz and $T = 10$ ms.

7. Substituting $e_{2cd} = 0.01$ (1% error) and $\nu \approx 30$ in the expression for p_{2cd} , we get $p_{2cd} \approx 350$.

8. $\Delta x = \frac{2\pi}{\ell p} = \frac{\lambda}{p} = \frac{25 \text{ km}}{350} \approx 70 \text{ m}$.

9. If $e_{2cd} = 0.1$ (10% error), $p_{2cd} \approx 110$ and hence $\Delta x \approx 220$.

10. As we did in Part 2, we substitute $q(x, t) = \exp[i\ell(x - vt)]$ in the expression for the fourth order central differencing. We get,

$$\partial_x q = \frac{-\exp[i\ell(x - vt)]}{12\Delta x} \{-\exp(2i\ell\Delta x) + 8\exp(i\ell\Delta x) - 8\exp(-i\ell\Delta x) + \exp(-2i\ell\Delta x)\} \quad (37)$$

$$= \frac{-\exp[i\ell(x - vt)]}{12\Delta x} \{-\exp(2i\ell\Delta x) + 8\exp(i\ell\Delta x) - 8\exp(-i\ell\Delta x) + \exp(-2i\ell\Delta x)\} \quad (38)$$

$$= \frac{\exp[i\ell(x - vt)]}{6\Delta x} (i) \{8\sin(\ell\Delta x) - \sin(2\ell\Delta x)\} \quad (39)$$

$$\therefore v_{4cd} = \{8\sin(\ell\Delta x) - \sin(2\ell\Delta x)\} \frac{v}{6\ell\Delta x} \quad (40)$$

Again, making the small ℓ approximation, we get,

$$v_{4cd} = v \left(1 - \frac{\{\ell\Delta x\}^4}{30} \right) \implies e_{4cd} = \ell v T \frac{\{\ell\Delta x\}^4}{30} = \frac{\pi \nu}{15} \left(\frac{2\pi}{p} \right)^4 \quad (41)$$

Problem 4**Vacuum part 1**

1. Given,

$$\partial_t \mathbf{q} + \partial_x \mathbf{f}(\mathbf{q}) = 0 \implies \partial_t \mathbf{q} + \mathbf{f}'(\mathbf{q}) \partial_x \mathbf{q} = 0 \quad (42)$$

Changing variables to $\xi = x/t$, we get,

$$\left(-\frac{x}{t^2}\right) \partial_\xi \mathbf{q} + \frac{1}{t} \mathbf{f}'(\mathbf{q}) \partial_\xi \mathbf{q} = 0 \implies \partial_\xi \mathbf{q} (\mathbf{f}'(\mathbf{q}) - \xi I) = 0 \quad (43)$$

This tells us that a self-similar solution will exist if ξ is an eigenvalue of $J = \mathbf{f}'(\mathbf{q})$.

2. The equation of state is given by,

$$p = \rho e(\gamma - 1) = (\gamma - 1) \left(E - \frac{\rho v^2}{2} \right) = (\gamma - 1) \left(E - \frac{S^2}{2\rho} \right) \quad (44)$$

Using these, one can write,

$$\mathbf{f}(\mathbf{q}) = \left[S \quad \frac{S^2}{\rho} \frac{3-\gamma}{2} + (\gamma-1)E \quad \frac{ES}{\rho} \gamma - (\gamma-1) \left(\frac{S^3}{2\rho^2} \right) \right]^T \quad (45)$$

$J = \frac{\partial \mathbf{f}}{\partial \mathbf{q}}$ is given by,

$$J = \begin{bmatrix} 0 & 1 & 0 \\ -(3-\gamma) \frac{S^2}{2\rho^2} & (3-\gamma) \frac{S}{\rho} & \gamma-1 \\ -\frac{ES}{\rho^2} \gamma + \frac{S^3}{\rho^3} (\gamma-1) & \frac{E}{\rho} \gamma - (\gamma-1) \left(\frac{3S^2}{2\rho^2} \right) & \gamma \frac{S}{\rho} \end{bmatrix} \quad (46)$$

Using Mathematica to diagonalize J and substituting $S = \rho v$ and E in terms of v , we get the eigenvalues v and $v \pm c_s$, where $c_s^2 = \frac{\gamma p}{\rho}$

- 3.
- 4.
- 5.
- 6.
- 7.

Problem 5**Vacuum part 2**

1. The conservation law is,

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{q})}{\partial x} = 0 \implies \frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{q})}{\partial \mathbf{q}} \frac{\partial \mathbf{q}}{\partial x} = 0 \implies \frac{\partial \mathbf{q}}{\partial t} + J \frac{\partial \mathbf{q}}{\partial x} = 0 \quad (47)$$

where $J = \frac{\partial \mathbf{f}(\mathbf{q})}{\partial \mathbf{q}}$. Substituting $S^{-1} \mathbf{w} = \mathbf{q}$ subject to $SJS^{-1} = \Lambda$, where Λ is the diagonal matrix containing the eigenvalues of J ,

$$S^{-1} \frac{\partial \mathbf{w}}{\partial t} + JS^{-1} \frac{\partial \mathbf{w}}{\partial x} = 0 \implies \frac{\partial \mathbf{w}}{\partial t} + SJS^{-1} \frac{\partial \mathbf{w}}{\partial x} = 0 \implies \frac{\partial \mathbf{w}}{\partial t} + \Lambda \frac{\partial \mathbf{w}}{\partial x} = 0 \quad (48)$$

- 2.

Problem 6

Well Balancing

1. This can be easily seen by substituting the form $q = Ce^x$ in the advection equation with source.
2. The advection equation with source is given by,

$$\partial_t q + \partial_x q = q \quad (49)$$

Using forward differencing for $\partial_t q$ and backward differencing for $\partial_x q$ we have,

$$\frac{q(x_0, t_0 + \Delta t) - q(x_0, t_0)}{\Delta t} + \frac{q(x_0 - \Delta x, t_0) - q(x_0, t_0)}{-\Delta x} = q(x_0, t_0) \quad (50)$$

$$\implies q(x_0, t_0 + \Delta t) = q(x_0, t_0) - \frac{\Delta t}{\Delta x} (q(x_0, t_0) - q(x_0 - \Delta x, t_0)) + \Delta t q(x_0, t_0) \quad (51)$$

3. For equilibrium solutions within this scheme, $q(x, t_1) = q(x, t_2) = q(x, \bar{t})$, $\forall t_1, t_2$. Applying this requirement to the scheme,

$$\cancel{q(x_0, \bar{t})} = \cancel{q(x_0, \bar{t})} - \frac{\Delta t}{\Delta x} (q(x_0, \bar{t}) - q(x_0 - \Delta x, \bar{t})) + \Delta t q(x_0, \bar{t}) \quad (52)$$

This translates to the requirement (for small Δx),

$$\frac{q(x_0, \bar{t}) - q(x_0 - \Delta x, \bar{t})}{\Delta x} = q'(x_0, t_0) = q(x_0, \bar{t}) \quad , \quad (53)$$

which is not true in general. Hence this scheme does not preserve equilibrium.

We also see the term $q(x_0, t_0) - q(x_0 - \Delta x, t_0)$ is $\mathcal{O}(\Delta x)$ and hence the next term in the scheme would be $\mathcal{O}(\Delta x \Delta t)$

4. Consider the scheme,

$$q(x_0, t_0 + \Delta t) = q(x_0, t_0) - \frac{\Delta t}{\Delta x} (q(x_0, t_0) - e^{\Delta x} q(x_0 - \Delta x, t_0)) \quad . \quad (54)$$

Applying the equilibrium requirement translates to,

$$q(x_0, t_0) = e^{\Delta x} q(x_0 - \Delta x, t_0) \quad , \quad (55)$$

which is satisfied for small Δx . Hence, this scheme does preserve the equilibrium.

5. Consider expanding the last term upto $\mathcal{O}(\Delta x)$,

$$\frac{\Delta t}{\Delta x} (q(x_0, t_0) - e^{\Delta x} q(x_0 - \Delta x, t_0)) = \frac{\Delta t}{\Delta x} [(\Delta x) q(x_0, t_0)] = \Delta t q(x_0, t_0) \quad , \quad (56)$$

which is the last term in the original scheme.

Problem 7

Shocks

1. Let us imagine integrating in an interval $\{x_1, x_2\}$ in x and $\{t_1, t_2\}$ in t which has the shock. We then write,

$$\int_{x_1}^{x_2} \int_{t_1}^{t_2} dx dt \partial_t q = - \int_{t_1}^{t_2} \int_{x_1}^{x_2} dt dx \partial_x f(q) \quad (57)$$

$$\Delta x (q_L - q_R) = -\Delta t (f_R - f_L) \quad (58)$$

$$\therefore V_s (q_R - q_L) = f_R - f_L \quad (59)$$

2.

$$\partial_t(q^n) + \frac{n}{n+1} \partial_x(q^{n+1}) = 0 \implies nq^{n-1} \partial_t q + nq^n \partial_x q = 0 \implies \partial_t q + q \partial_x q = 0, \quad (60)$$

which is the Burgers equation.

3. Using the Rankine-Hugoniot conditions for the n -dependent conservation law,

$$V_s(q_R^n - q_L^n) = \frac{n}{n+1} (q_R^{n+1} - q_L^{n+1}) \implies V_s = \frac{n}{n+1} \frac{q_R^{n+1} - q_L^{n+1}}{q_R^n - q_L^n}, \quad (61)$$

which would generically depend on n .

Problem 8

Telescoping

1. The first order upwind scheme for $\partial_t q + v \partial_x q = 0$ is given by,

$$q(x_0, t_0 + \Delta t) = q(x_0, t_0) + \frac{v \Delta t}{\Delta x} [q(x_0 - \Delta x, t_0) - q(x_0, t_0)] \quad (62)$$

Our equation $\partial_t q + \partial_x f(q) = 0$ differs in the ∂_x part, and hence one would have to discretize $f(q)$ in x . Using the definitions given in the problem, one can write,

$$q_i^{n+1} = q_i^n = \frac{\Delta t}{\Delta x} (f_{i-1/2}^n - f_{i+1/2}^n) \quad (63)$$

2. Integrating the conservation law within the given limits,

$$\int_{x_{i-1/2}}^{x_{i+1/2}} dx (q(x, t^{n+1}) - q(x, t^n)) = \int_{t^n}^{t^{n+1}} dt (f(x_{i+1/2}, t) - f(x_{i-1/2}, t)) \quad (64)$$

$$\implies \hat{q}_i^{n+1} = \hat{q}_i^n + \frac{\Delta t}{\Delta x} (f_{i-1}^n - f_{i+1/2}^n) \quad (65)$$

3. One can see that for the two schemes to match,

$$f_{i-1/2}^n - f_{i+1/2}^n = f_{i-1}^n - f_i^n \implies f_{i-1/2}^n = f_{i-1}^n \quad \text{and} \quad f_{i+1/2}^n = f_i^n \quad (66)$$

Also, consider,

$$f_{i+1/2}^n - f_i^n = f_i'^n \Delta x + f_i'^n \frac{\Delta x^2}{2} \quad (67)$$

$$f_{i-1/2}^n - f_{i-1}^n = f_{i-1}'^n \Delta x + f_{i-1}'^n \frac{\Delta x^2}{2} \quad (68)$$

4.

Problem 9

Monotonicity

Problem 10**Stiffness**

1. Given,

$$\frac{dq}{dt} = -\frac{1}{\eta}q \quad (69)$$

Using Euler method,

$$\frac{q^{n+1} - q^n}{\Delta t} = -\frac{1}{\eta}q^n \implies q^{n+1} = \left(1 - \frac{\Delta t}{\eta}\right)q^n \quad (70)$$

2. The general solution of the ODE is $q(t) = Ce^{-t/\eta}$. If $q(0) = 1$, $q(t) = e^{-t/\eta} \implies \lim_{t \rightarrow \infty} q(t) = 0$.

3. From the form in eq. 71, we can write an ansatz solution for q^n to be,

$$q^N = C \left(1 - \frac{\Delta t}{\eta}\right)^N. \quad (71)$$

The above form of the equation suggests that $\lim_{N \rightarrow \infty} q^N = 0 \iff \Delta t \leq \eta$, meaning for all other values of the timestep Δt , the solution will blow up and go to infinity for large time.

4. Applying backward difference to the original ODE gives,

$$\frac{q^n - q^{n-1}}{\Delta t} = -\frac{1}{\eta}q^n \implies q^n = \left(1 + \frac{\Delta t}{\eta}\right)^{-1} q^{n-1} \quad (72)$$

An ansatz solution for the above discretized form is,

$$q^N = C \left(1 + \frac{\Delta t}{\eta}\right)^{-N}. \quad (73)$$

As one can see, $\lim_{N \rightarrow \infty} q^N = 0$, $\forall \Delta t$.