Numerical Hydrodynamics - Solutions

ICTS Summer School on Gravitational-wave Astronomy

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Numerical Theory

Problem 1

Finite Differencing

1. Using Taylor Expansion,

$$\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=x_0} = \frac{\cancel{f(x_0)} + f'(x_0)\Delta x + \mathcal{O}(\Delta x)^2 - \cancel{f(x_0)}}{\Delta x} \tag{1}$$

$$= f'(x_0) + \mathcal{O}(\Delta x) \tag{2}$$

2. *There is a typo in the question : $f(x_0 - \Delta x)$ and not $f(x_0 - \Delta)$*

   Again, using Taylor expansion,

$$\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=x_0} = \frac{\cancel{f(x_0)} + f'(x_0)\Delta x + \cancel{\frac{f''(x_0)(\Delta x)^2}{2}} + \mathcal{O}(\Delta x)^3 - \cancel{f(x_0)} + f'(x_0)\Delta x - \cancel{\frac{f''(x_0)(\Delta x)^2}{2}} + \mathcal{O}(\Delta x)^3}{2\Delta x} \tag{3}$$

$$= f'(x_0) + \mathcal{O}(\Delta x)^2 \tag{4}$$

3. Let's Taylor expand the two sides,

$$\frac{y(x_0 + \Delta x) - y(x_0)}{\Delta x} + \mathcal{O}(\Delta x) = F(x_0, y(x_0)) + \mathcal{O}(\Delta x) \tag{5}$$

$$\implies y(x_0 + \Delta x) = y(x_0) + \Delta x F(x_0, y(x_0)) + \mathcal{O}(\Delta x)^2 \tag{6}$$

   Where, in the second step, we have multiplied by Δx and rearranged the terms.

4. Integrating the differential equation between x_0 and $x_0 + \Delta x$, we get,

$$y(x_0 + \Delta x) - y(x_0) = \int_{x_0}^{x_0+\Delta x} F(x, y)\mathrm{d}x \tag{7}$$

   In the infinitesimal limit, we assume that $F(x, y)$ within the integration limits $\approx F(x_0, y(x_0))$. Hence,

$$y(x_0 + \Delta x) - y(x_0) = F(x, y)\Delta x\mathrm{d}x \quad , \tag{8}$$

   which is the Euler method.

5. The advection equation for $q = q(x, t)$ with constant velocity v is,

$$\partial_t q - v\partial_x q = 0 \tag{9}$$

We can expand q about x_0 and t_0 as follows,

$$\partial_t q = \frac{q(x_0, t_0 + \Delta t) - q(x_0, t_0)}{\Delta t} + \mathcal{O}(\Delta t, \Delta x) \quad \text{and} \quad \partial_x q = \frac{q(x_0 + \Delta x, t_0) - q(x_0, t_0)}{\Delta x} + \mathcal{O}(\Delta t, \Delta x) \tag{10}$$

Now substituting these into Eq. 9,

$$\frac{q(x_0, t_0 + \Delta t) - q(x_0, t_0)}{\Delta t} - v\frac{q(x_0 + \Delta x, t_0) - q(x_0, t_0)}{\Delta x} + \mathcal{O}(\Delta t, \Delta x) = 0 \tag{11}$$

$$\implies q(x_0, t_0 + \Delta t) = q(x_0, t_0) + v\Delta t\frac{q(x_0 + \Delta x, t_0) - q(x_0, t_0)}{\Delta x} + \mathcal{O}(\Delta t, \Delta x) \tag{12}$$

6. We note that if there has to be communication between points that do not satisfy the CFL condition, the velocity required would be more than the speed of propagation v. This makes the scheme unstable, creating unphysical infinities.

Problem 2

Modified Equation

1. The first order upwind scheme is given by,

$$q(x_0, t_0 + \Delta t) = q(x_0, t_0) + \frac{v\Delta t}{\Delta x}[q(x_0 - \Delta x, t_0) - q(x_0, t_0)] \tag{13}$$

We Taylor-expand all terms that have Δ's in them,

$$\cancel{q(x_0, t_0)} + \partial_t q \ \Delta t = \cancel{q(x_0, t_0)} + \frac{v\Delta t}{\Delta x}\left[\cancel{q(x_0, t_0)} - \partial_x q \ \Delta x + \partial_{xx} q \ \frac{\Delta x^2}{2} - \cancel{q(x_0, t_0)}\right] \tag{14}$$

$$\partial_t q = \frac{v}{\Delta x}\left[-\partial_x q \ \Delta x + \partial_{xx} q \ \frac{\Delta x^2}{2}\right] \tag{15}$$

$$\partial_t q + v\partial_x q = \frac{v\Delta x}{2}\left(1 - \frac{v\Delta t}{\Delta x}\right)\partial_{xx} q, \tag{16}$$

where is going from the first to the second step, we have equated the coefficients of Δt on both sides of the equation. The term $\beta = \frac{v\Delta x}{2}\left(1 - \frac{v\Delta t}{\Delta x}\right)$ acts as an effective numerical viscosity, and hence *diffusing* the numerics leading to numerical errors.

2. The violation of the CFL condition leads to the viscosity β turning negative. In a physical situation, this would be lead to the *opposite* of diffusion as we know it; this means that there will be energy added into the system, and smooth features in the solution will eventually go to sharp features and giving infinities.

3. Substituting $f(x - vt) = q$ in the advection equation and using the chain rule of differentiation,

$$-v \ q'(\eta) + vq'(\eta) = 0 \tag{17}$$

Hence, all $q(\eta) = f(x - vt)$ is a solution of the advection equation.

4. Substituting $f(x/t) = q(\xi)$ in the advection equation and using the chain rule of differentiation,

$$-\frac{x}{t^2}q'(\xi) + \frac{v}{t}q'(\xi) = 0 \implies \frac{x}{t} = \xi = v \tag{18}$$

Problem 2 continued on next page. . .

5. Substituting $f\left(\dfrac{x-vt}{t^\alpha}\right) = q(\eta)$ in the modified equation and using the chain rule of differentiation,

$$\left(-\frac{\alpha x}{t^{\alpha+1}} + \frac{v(\alpha-1)}{t^\alpha}\right)q'(\eta) + \frac{v}{t^\alpha}q'(\eta) = \beta\frac{v^2}{t^{2\alpha}}q''(\eta) \tag{19}$$

$$\implies -\alpha\frac{x-vt}{t^{\alpha+1}}q' = \beta\frac{v^2}{t^{2\alpha}}q'' \implies \beta\frac{v^2}{t^{2\alpha}}q'' + \alpha\frac{\eta}{t}q' = 0 \tag{20}$$

For $\alpha = 1/2$, the equation,

$$\beta\frac{v^2}{t}q'' + \frac{\eta}{2t}q' = 0 \implies \beta v^2 q'' + \frac{\eta}{2}q' = 0 \quad \text{as} \quad t > 0 \tag{21}$$

Assuming $q'(\eta) = h(\eta)$,

$$\beta v^2 h' + \frac{\eta}{2}h = 0 \implies h(\eta) = q'(\eta) = C_1 \exp\left(-\frac{\eta}{2\beta v^2}\right) \tag{22}$$

$$\implies q(\eta) = C_1 \int \mathrm{d}\eta \exp\left(-\frac{\eta}{2\beta v^2}\right) \quad . \tag{23}$$

The Gaussian integral is in fact the error function.

6.

7.

Problem 3

Phase errors and neutron stars

1. In Part 3 of Problem 2, we proved that $q = f(x - vt)$ is always a solution to the advection equation. Hence $q(x,t) = \exp\left[i\gamma(x - vt)\right]$ is a solution to the advection equation. As we are given the initial data $q(x,t) = \exp(i\ell x)$, we conclude that $\gamma = \ell$ for this problem.

2. Substituting the form of the solution into the second order difference formula,

$$\partial_x q = \frac{\exp[i\ell(x_k - vt)]}{2\Delta x}\{\exp(i\ell\Delta x) - \exp(-i\ell\Delta x)\} \tag{24}$$

$$= \frac{\exp[i\ell(x_k - vt)]}{\Delta x} i\sin(\ell\Delta x) \tag{25}$$

3. Substituting q_e and $q_{m,\Delta x}$,

$$\frac{q_e(x,t) - q_{m,\Delta x}(x,t)}{q_e(x,t)} = 1 - \exp[i\ell(v - v_m(\ell))T] \tag{26}$$

Taking the limit $\Delta x \to 0$ and retaining only upto next-to-leading order,

$$\frac{q_e(x,t) - q_{m,\Delta x}(x,t)}{q_e(x,t)} \approx -i\sin[\ell(v - v_m(l))T] \tag{27}$$

$$\therefore e_m(\ell) = \left|\frac{q_e(x,t) - q_{m,\Delta x}(x,t)}{q_e(x,t)}\right| \approx \ell(v - v_m(l))T \tag{28}$$

4. Substituting the expression for v_m in the result of the previous part,

$$e_{2\mathrm{cd}}(l) = \ell vT\left(1 - \frac{\sin(\ell\Delta x)}{\ell\Delta x}\right) \approx \ell vT\frac{(l\Delta x)^2}{6} \tag{29}$$

5. Given $p = \dfrac{2\pi}{\ell \Delta x}$ and $\nu = \dfrac{\ell v T}{2\pi}$, so $\ell \Delta x = \dfrac{2\pi}{p}$ and $T = \dfrac{2\pi\nu}{\ell v}$. Substituting these in the expression for $e_{\rm 2cd}(l)$, we get the required result,
$$e_{\rm 2cd} = \frac{\pi\nu}{3}\left(\frac{2\pi}{p}\right)^2 \quad . \tag{30}$$
We also get the minimum points per wavelength for a required phase error to be,
$$p_{\rm 2cd} = 2\pi\sqrt{\frac{\nu\pi}{3e_{\rm 2cd}}} \tag{31}$$

6. We know that $\ell = \dfrac{2\pi}{\lambda}$ and that $f = \dfrac{v}{\lambda}$ where f is the frequency. This means that $\nu = \dfrac{vlT}{2\pi} = fT \approx 30$, where we have substituted $f = 30$ kHz and $T = 10$ ms.

7. Subsituting $e_{\rm 2cd} = 0.01$ (1% error) and $\nu \approx 30$ in the expression for $p_{\rm 2cd}$, we get $p_{\rm 2cd} \approx 350$.

8. $\Delta x = \dfrac{2\pi}{\ell p} = \dfrac{\lambda}{p} = \dfrac{25 \text{ km}}{350} \approx 70$ m.

9. If $e_{\rm 2cd} = 0.1$ (10% error), $p_{\rm 2cd} \approx 110$ and hence $\Delta x \approx 220$.

10. As we did in Part 2, we substitute $q(x,t) = \exp[i\ell(x - vt)]$ in the expression for the fourth order central differencing. We get,
$$\begin{aligned}
\partial_x q &= \frac{-\exp[i\ell(x-vt)]}{12\Delta x}\{-\exp(2i\ell\Delta x) + 8\exp(i\ell\Delta x) - 8\exp(-i\ell\Delta x) + \exp(-2i\ell\Delta x)\} && (32)\\
&= \frac{-\exp[i\ell(x-vt)]}{12\Delta x}\{-\exp(2i\ell\Delta x) + 8\exp(i\ell\Delta x) - 8\exp(-i\ell\Delta x) + \exp(-2i\ell\Delta x)\} && (33)\\
&= \frac{\exp[i\ell(x-vt)]}{6\Delta x}(i)\{8\sin(\ell\Delta x) - \sin(2\ell\Delta x)\} && (34)\\
\therefore v_{\rm 4cd} &= \{8\sin(\ell\Delta x) - \sin(2\ell\Delta x)\}\frac{v}{6\ell\Delta x} && (35)
\end{aligned}$$
Again, making the small ℓ approximation, we get,
$$v_{\rm 4cd} = v\left(1 - \frac{\{\ell\Delta x\}^4}{30}\right) \implies e_{\rm 4cd} = \ell v T\frac{\{\ell\Delta x\}^4}{30} = \frac{\pi\nu}{15}\left(\frac{2\pi}{p}\right)^4 \tag{36}$$

Problem 4

Vacuum part 1

Problem 5

Vacuum part 2

1. The conservation law is,
$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{q})}{\partial x} = 0 \implies \frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{q})}{\partial \mathbf{q}}\frac{\partial \mathbf{q}}{\partial x} = 0 \implies \frac{\partial \mathbf{q}}{\partial t} + J\frac{\partial \mathbf{q}}{\partial x} = 0 \quad . \tag{37}$$
where $J = \frac{\partial \mathbf{f}(\mathbf{q})}{\partial \mathbf{q}}$. Substituting $S^{-1}\mathbf{w} = \mathbf{q}$ subject to $SJS^{-1} = \Lambda$, where Λ is the diagonal matrix containing the eigenvalues of J,
$$S^{-1}\frac{\partial \mathbf{w}}{\partial t} + JS^{-1}\frac{\partial \mathbf{w}}{\partial x} = 0 \implies \frac{\partial \mathbf{w}}{\partial t} + SJS^{-1}\frac{\partial \mathbf{w}}{\partial x} = 0 \implies \frac{\partial \mathbf{w}}{\partial t} + \Lambda\frac{\partial \mathbf{w}}{\partial x} = 0 \tag{38}$$

2.

Problem 6

Well Balancing

1. This can be easily seen by substituting the form $q = Ce^x$ in the advection equation with source.

2. The advection equation with source is given by,

$$\partial_t q + \partial_x q = q \tag{39}$$

Using forward differencing for $\partial_t q$ and backward differencing for $\partial_x q$ we have,

$$\frac{q(x_0, t_0 + \Delta t) - q(x_0, t_0)}{\Delta t} + \frac{q(x_0 - \Delta x, t_0) - q(x_0, t_0)}{-\Delta x} = q(x_0, t_0) \tag{40}$$

$$\implies q(x_0, t_0 + \Delta t) = q(x_0.t_0) - \frac{\Delta t}{\Delta x}(q(x_0, t_0) - q(x_0 - \Delta x, t_0)) + \Delta t q(x_0, t_0) \tag{41}$$

3. For equilibrium solutions within this scheme, $q(x, t_1) = q(x, t_2) = q(x, \bar{t})$, $\forall t_1, t_2$. Applying this requirement to the scheme,

$$\cancel{q(x_0, \bar{t})} = \cancel{q(x_0, \bar{t})} - \frac{\Delta t}{\Delta x}(q(x_0, \bar{t}) - q(x_0 - \Delta x, \bar{t})) + \Delta t q(x_0, \bar{t}) \tag{42}$$

This translates to the requirement (for small Δx),

$$\frac{q(x_0, \bar{t}) - q(x_0 - \Delta x, \bar{t})}{\Delta x} = q'(x_0, t_0) = q(x_0, \bar{t}) \quad , \tag{43}$$

which is not true in general. Hence this scheme does not preserve equilibrium.

We also see the term $q(x_0, t_0) - q(x_0 - \Delta x, t_0)$ is $\mathcal{O}(\Delta x)$ and hence the next term in the scheme would be $\mathcal{O}(\Delta x \Delta t)$

4. Consider the scheme,

$$q(x_0, t_0 + \Delta t) = q(x_0, t_0) - \frac{\Delta t}{\Delta x}\big(q(x_0, t_0) - e^{\Delta x} q(x_0 - \Delta x, t_0)\big) \quad . \tag{44}$$

Applying the equilibrium requirement translates to,

$$q(x_0, t_0) = e^{\Delta x} q(x_0 - \Delta x, t_0) \quad , \tag{45}$$

which is satisfied for small Δx. Hence, this scheme does preserve the equilibrium.

5. Consider expanding the last term upto $\mathcal{O}(\Delta x)$,

$$\frac{\Delta t}{\Delta x}\big(q(x_0, t_0) - e^{\Delta x} q(x_0 - \Delta x, t_0)\big) = \frac{\Delta t}{\Delta x}[(\Delta x) q(x_0, t_0)] = \Delta t q(x_0, t_0) \quad , \tag{46}$$

which is the last term in the original scheme.

Problem 7

Shocks

1. Let us imagine integrating in an interval $\{x_1, x_2\}$ in x and $\{t_1, t_2\}$ in t which has the shock. We then write,

$$\int_{x_1}^{x_2} \int_{t_1}^{t_2} \mathrm{d}x \mathrm{d}t \partial_t q = - \int_{t_1}^{t_2} \int_{x_1}^{x_2} \mathrm{d}t \mathrm{d}x \partial_x f(q) \tag{47}$$

$$\Delta x (q_L - q_R) = -\Delta t (f_R - f_L) \tag{48}$$

$$\therefore V_s (q_R - q_L) = f_R - f_L \tag{49}$$

Problem 7 continued on next page. . .

2.
$$\partial_t(q^n) + \frac{n}{n+1}\partial_x(q^{n+1}) = 0 \implies nq^{n-1}\partial_t q + nq^n\partial_x q = 0 \implies \partial_t q + q\partial_q q = 0 \ , \tag{50}$$
which is the Burgers equation.

3. Using the Rankine-Hugoniot conditions for the n-dependent conservation law,
$$V_s(q_R^n - q_L^n) = \frac{n}{n+1}\left(q_R^{n+1} - q_L^{N+1}\right) \implies V_s = \frac{n}{n+1}\frac{q_R^{n+1} - q_L^{n+1}}{q_R^n - q_L^n} \quad , \tag{51}$$
which would generically depend on n.

Problem 8

Telescoping

1. The first order upwind scheme for $\partial_t q + v\partial_x q = 0$ is given by,
$$q(x_0, t_0 + \Delta t) = q(x_0, t_0) + \frac{v\Delta t}{\Delta x}[q(x_0 - \Delta x, t_0) - q(x_0, t_0)] \tag{52}$$
Our equation $\partial_t q + \partial_x f(q) = 0$ differs in the ∂_x part, and hence one would have to discretize $f(q)$ in x. Using the definitions given in the problem, one can write,
$$q_i^{n+1} = q_i^n = \frac{\Delta t}{\Delta x}\left(f_{i-1/2}^n - f_{i+1/2}^n\right) \tag{53}$$

2. Integrating the conservation law within the given limits,
$$\int_{x_{i-1/2}}^{x_{i+1/2}} \mathrm{d}x \left(q(x, t^{n+1}) - q(x, t^n)\right) = \int_{t^n}^{t^{n+1}} \mathrm{d}t \left(f(x_{i+1/2}, t) - f(x_{i-1/2}, t)\right) \tag{54}$$
$$\implies \hat{q}_i^{n+1} = \hat{q}_i^n + \frac{\Delta t}{\Delta x}(f_{i-1}^n - f_{i+1/2}^n) \tag{55}$$

3. One can see that for the two schemes to match,
$$f_{i-1/2}^n - f_{i+1/2}^n = f_{i-1}^n - f^i \implies f_{i-1/2}^n = f_{i-1}^n \quad \text{and} \quad f_{i+1/2}^n = f_i^n \tag{56}$$
Also, consider,
$$f_{i+1/2}^n - f_i^n = f_i'^n\Delta x + f_i'^n\frac{\Delta x^2}{2} \tag{57}$$
$$f_{i-1/2}^n - f_{i-1}^n = f_i'^n\Delta x + f_i'^n\frac{\Delta x^2}{2} \tag{58}$$

4.

Problem 9

Monotonicity

Problem 10

Stiffness

1. Given,
$$\frac{\mathrm{d}q}{\mathrm{d}t} = -\frac{1}{\eta}q \tag{59}$$
Using Euler method,
$$\frac{q^{n+1} - q^n}{\Delta t} = -\frac{1}{\eta}q^n \implies q^{n+1} = \left(1 - \frac{\Delta t}{\eta}\right)q^n \tag{60}$$

2. The general solution of the ODE is $q(t) = Ce^{-t/\eta}$. If $q(0) = 1$, $q(t) = e^{-t/\eta} \implies \lim_{t\to\infty} q(t) = 0$.

3. From the form in eq. 60, we can write an ansatz solution for q^n to be,
$$q^N = C\left(1 - \frac{\Delta t}{\eta}\right)^N . \tag{61}$$
The above form of the equation suggests that $\lim_{N\to\infty} q^N = 0 \iff \Delta t \leq \eta$, meaning for all other values of the timestep Δt, the solution will blow up and go to infinity for large time.

4. Applying backward difference to the original ODE gives,
$$\frac{q^n - q^{n-1}}{\Delta t} = -\frac{1}{\eta}q^n \implies q^n = \left(1 + \frac{\Delta t}{\eta}\right)^{-1} q^{n-1} \tag{62}$$
An ansatz solution for the above discretized form is,
$$q^N = C\left(1 + \frac{\Delta t}{\eta}\right)^{-N} . \tag{63}$$
As one can see, $\lim_{N\to\infty} q^N = 0 \quad , \quad \forall \Delta t$.