Calculus: Homework #2

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Numerical Models

Problem 1

The Conservation Laws

1. The problem is to find orthonormal tetrad $e^a_{(m)}$ *ie*,

$$g^{ab} = e^a_{(m)} e^b_{(n)} \delta^{mn} \tag{1}$$

for Minkowski spacetime in both Cartesian and spherical coordinates.

Problem 2

The perfect fluid

1. We have,

$$T_{ab} = \rho_0 h u_a u_b + p g_{ab} \tag{2}$$

A generic tensor T_{ab} in four dimensions would have 16 independent degrees of freedom. But we notice that T_{ab} here is symmetric under the exchange $a \to b$ and hence this leaves us with 10 independent degrees of freedom. Furthermore, we have the constraint $\boldsymbol{\nabla}_a T^{ab} = 0$ which gives us 4 additional constraints reducing the independent degrees of freedom to 6. Then, using the the equation of state $p = p(\rho_0, \epsilon)$, we can eliminate one more degree of freedom, and hence T_{ab} has five degrees of freedom.

Problem 3

Action Principles

Numerical Theory

Problem 1

Finite Differencing

1. Using Taylor Expansion,

$$\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=x_0} = \frac{\cancel{f(x_0)} + f'(x_0)\Delta x + \mathcal{O}(\Delta x)^2 - \cancel{f(x_0)}}{\Delta x} \tag{3}$$

$$= f'(x_0) + \mathcal{O}(\Delta x) \tag{4}$$

2. *There is a typo in the question : $f(x_0 - \Delta x)$ and not $f(x_0 - \Delta)$*

   Again, using Taylor expansion,

$$\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=x_0} = \frac{\cancel{f(x_0)} + f'(x_0)\Delta x + \cancel{\frac{f''(x_0)(\Delta x)^2}{2}} + \mathcal{O}(\Delta x)^3 - \cancel{f(x_0)} + f'(x_0)\Delta x - \cancel{\frac{f''(x_0)(\Delta x)^2}{2}} + \mathcal{O}(\Delta x)^3}{2\Delta x} \tag{5}$$

$$= f'(x_0) + \mathcal{O}(\Delta x)^2 \tag{6}$$

3. Let's Taylor expand the two sides,

$$\frac{y(x_0 + \Delta x) - y(x_0)}{\Delta x} + \mathcal{O}(\Delta x) = F(x_0, y(x_0)) + \mathcal{O}(\Delta x) \tag{7}$$

$$\implies y(x_0 + \Delta x) = y(x_0) + \Delta x F(x_0, y(x_0)) + \mathcal{O}(\Delta x)^2 \tag{8}$$

   Where, in the second step, we have multiplied by Δx and rearranged the terms.

4.

5. The advection equation for $q = q(x,t)$ with constant velocity v is,

$$\partial_t q - v\partial_x q = 0 \tag{9}$$

   We can expand q about x_0 and t_0 as follows,

$$\partial_t q = \frac{q(x_0, t_0 + \Delta t) - q(x_0, t_0)}{\Delta t} + \mathcal{O}(\Delta t, \Delta x) \quad \text{and} \quad \partial_x q = \frac{q(x_0 + \Delta x, t_0) - q(x_0, t_0)}{\Delta x} + \mathcal{O}(\Delta t, \Delta x) \tag{10}$$

   Now substituting these into Eq. 9,

$$\frac{q(x_0, t_0 + \Delta t) - q(x_0, t_0)}{\Delta t} - v\frac{q(x_0 + \Delta x, t_0) - q(x_0, t_0)}{\Delta x} + \mathcal{O}(\Delta t, \Delta x) = 0 \tag{11}$$

$$\implies q(x_0, t_0 + \Delta t) = q(x_0, t_0) + v\Delta t\frac{q(x_0 + \Delta x, t_0) - q(x_0, t_0)}{\Delta x} + \mathcal{O}(\Delta t, \Delta x) \tag{12}$$

Problem 2

Modified Equation

Problem 2 continued on next page. . .

1. The first order upwind scheme is given by,

$$q(x_0, t_0 + \Delta t) = q(x_0, t_0) + \frac{v\Delta t}{\Delta x}[q(x_0 - \Delta x, t_0) - q(x_0, t_0)] \tag{13}$$

We Taylor-expand all terms that have Δ's in them,

$$\cancel{q(x_0, t_0)} + \partial_t q \ \Delta t = \cancel{q(x_0, t_0)} + \frac{v\Delta t}{\Delta x}\left[\cancel{q(x_0, t_0)} - \partial_x q \ \Delta x + \partial_{xx} q \ \frac{\Delta x^2}{2} - \cancel{q(x_0, t_0)}\right] \tag{14}$$

$$\partial_t q = \frac{v}{\Delta x}\left[-\partial_x q \ \Delta x + \partial_{xx} q \ \frac{\Delta x^2}{2}\right] \tag{15}$$

$$\partial_t q + v\partial_x q = \frac{v\Delta x}{2}\left(1 - \frac{v\Delta t}{\Delta x}\right)\partial_{xx} q, \tag{16}$$

where is going from the first to the second step, we have equated the coefficients of Δt on both sides of the equation. The term $\beta = \frac{v\Delta x}{2}\left(1 - \frac{v\Delta t}{\Delta x}\right)$ acts as an effective numerical viscosity, and hence *diffusing* the numerics leading to numerical errors.

2. The violation of the CFL condition leads to the viscosity β turning negative, hence the solution will grow exponentially and will blow up.

3. Substituting $f(x - vt) = q$ in the advection equation and using the chain rule of differentiation,

$$-v \ q'(\eta) + vq'(\eta) = 0 \tag{17}$$

Hence, all $q(\eta) = f(x - vt)$ is a solution of the advection equation.

4. Substituting $f(x/t) = q(\xi)$ in the advection equation and using the chain rule of differentiation,

$$-\frac{x}{t^2}q'(\xi) + \frac{v}{t}q'(\xi) = 0 \implies \frac{x}{t} = \xi = v \tag{18}$$

5. Substituting $f\left(\frac{x - vt}{t^\alpha}\right) = q(\eta)$ in the modified equation and using the chain rule of differentiation,

$$\left(-\frac{\alpha x}{t^{\alpha+1}} + \frac{v(\alpha - 1)}{t^\alpha}\right)q'(\eta) + \frac{v}{t^\alpha}q'(\eta) = \beta\frac{v^2}{t^{2\alpha}}q''(\eta) \tag{19}$$

$$\implies -\alpha\frac{x - vt}{t^{\alpha+1}}q' = \beta\frac{v^2}{t^{2\alpha}}q'' \implies \beta\frac{v^2}{t^{2\alpha}}q'' + \alpha\frac{\eta}{t}q' = 0 \tag{20}$$

Write in terms of error functions

6. ifhas

Problem 3

1. In Part 3 of Problem 2, we proved that $q = f(x - vt)$ is always a solution to the advection equation. Hence $q(x, t) = \exp[i\gamma(x - vt)]$ is a solution to the advection equation. As we are given the initial data $q(x, t) = \exp(i\ell x)$, we conclude that $\gamma = \ell$ for this problem.

2. Substituting the form of the solution into the second order difference formula,

$$\partial_x q = \frac{\exp[i\ell(x_k - vt)]}{2\Delta x}\{\exp(i\ell\Delta x) - \exp(-i\ell\Delta x)\} \tag{21}$$

$$= \frac{\exp[i\ell(x_k - vt)]}{\Delta x}i\sin(\ell\Delta x) \tag{22}$$

Problem 3 continued on next page...

3. Substituting q_e and $q_{m,\Delta x}$,

$$\frac{q_e(x,t) - q_{m,\Delta x}(x,t)}{q_e(x,t)} = 1 - \exp[i\ell(v - v_m(\ell))T] \tag{23}$$

   Taking the limit $\Delta x \to 0$ and retaining only upto next-to-leading order,

$$\frac{q_e(x,t) - q_{m,\Delta x}(x,t)}{q_e(x,t)} \approx -i\sin[\ell(v - v_m(l))T] \tag{24}$$

$$\therefore e_m(\ell) = \left|\frac{q_e(x,t) - q_{m,\Delta x}(x,t)}{q_e(x,t)}\right| \approx \ell(v - v_m(l))T \tag{25}$$

4. Substituting the expression for v_m in the result of the previous part,

$$e_m(l) = \ell vT\left(1 - \frac{\sin(\ell\Delta x)}{\ell\Delta x}\right) \approx \ell vT\frac{(l\Delta x)^2}{6} \tag{26}$$